Lecture 6: Planes in Space (§12.5)

Goals:

1. Determine vector and component equations for a given plane.
2. Product nonzero vector normal to a given plane.
3. Compute the distance from a given point to a given plane.
4. Determine whether two given planes coincide, intersect in a line, or are parallel.

An equation for a plane in space

A line can be characterize by an initial position and “direction” vector $\overrightarrow{v}$.

How can we characterize a plane?

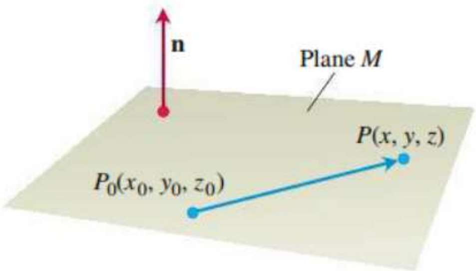

FIGURE 12.40 The standard equation for a plane in space is defined in terms of a vector normal to the plane: A point P lies in the plane through P_0 normal to $\mathbf{n}$ if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$.

The plane through $P_0(x_0, y_0, z_0)$ normal to $\overrightarrow{n} = \langle A, B, C \rangle$ is the set of points $P(x, y, z)$ satisfying:

- **Vector equation:** $\overrightarrow{n} \cdot \overrightarrow{P_0P} = 0.$

This can also be written as:

- **Component equation:**

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0, \text{ or,}$$

$$Ax + By + Cz = D,$$

with $D = Ax_0 + By_0 + Cz_0$

Question.

1. *What does it mean geometrically when $D = 0$?*

2. *In general, what is the geometric meaning of D?*

① $D=0$ means that the plane $Ax+By+Cz=\overset{0}{\overset{\shortparallel}{D}}$ passes through the origen.

② consider a point on the line $t\cdot\vec{n}$ which is also no the plane $\vec{n}\cdot\langle x,y,z\rangle = D \Rightarrow \vec{n}\cdot\langle t\vec{n}\rangle = D \Rightarrow t|\vec{n}|^2 = D \quad t=\frac{D}{|\vec{n}|^2}$

$t\cdot\vec{n} = \frac{D}{|\vec{n}|}\cdot\frac{\vec{n}}{|\vec{n}|} \Rightarrow$ the distance from the origen $=\frac{|D|}{|\vec{n}|}$.

Each plane normal to $\overrightarrow{n} = \langle A, B, C\rangle$ passes through a unique point on the line $t \cdot \overrightarrow{n}$. Explicitly, the plane $Ax + By + Cz = D$ passes through the point $\frac{D\overrightarrow{n}}{|\overrightarrow{n}|^2} = \frac{D}{|\overrightarrow{n}|} \cdot \frac{\overrightarrow{n}}{|\overrightarrow{n}|}$. Thus $\frac{D}{|\overrightarrow{n}|}$ measures how far is the plane from $(0, 0, 0)$ (in the direction of $\overrightarrow{n}$).

> Given a plane through $P_0(x_0, y_0, z_0)$ normal to $\overrightarrow{n} = \langle A, B, C\rangle$, we have:
>
> $$D = \overrightarrow{OP_0} \cdot \overrightarrow{n}.$$

Example 1. Find an equation for the plane through $P_0(-3, 0, 7)$ perpendicular to $\overrightarrow{n} = \langle 5, 2, -1\rangle$.

The equation of the plane is

$$5x + 2y - z = D$$

$$\vec{n} \cdot \langle x, y, z\rangle = D$$

To find D we avaluate the equation at P_0

$$D = 5\cdot(-3) + 2\cdot 0 - 7 = -15 - 7 = -22$$

$$D = \vec{n} \cdot \overrightarrow{OP_0} = \langle 5, 2, -1\rangle \langle -3, 0, 7\rangle$$

$$\boxed{5x + 2y - z = -22}$$

Example. Find an equation for a plane through $A(0, 0, 1)$, $B(2, 0, 0)$ and $C(0, 3, 0)$.

Lets start by finding a normal to that plane.

The normal $\vec{n}$ is ortogonal to $\overrightarrow{AB} = \langle 2, 0, -1\rangle$

and $\overrightarrow{AC} = \langle 0, 3, -1\rangle$

Let

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = \begin{array}{ccc} i & j & k \\ \langle 3, & +2, & 6\rangle \end{array}$$

The plane equation is $3x + 2y + 6z = D$

To find D we can consider the point A

$$\vec{n} \cdot \overrightarrow{OA} = \langle 3, 2, 6\rangle \cdot \langle 0, 0, 1\rangle = 6$$

$$\boxed{D = 6}$$

Lines of intersection

Two planes $A_1x + B_1y + C_1z = D_1$ and $A_2x + B_2y + C_2z = D_2$ are parallel if and only if their normals are parallel. In other words, there is some non zero scalar k such that $\overrightarrow{n}_1 = k \cdot \overrightarrow{n}_2$, with $\overrightarrow{n}_1 = (A_1, B_1, C_1)$ and $\overrightarrow{n}_2 = (A_2, B_2, C_2)$.

If the planes are not parallel, they always intersect in a line.

Example.

1. Find a vector parallel to the line of intersection of the planes

$$3x - 6y - 2z = 15 \quad \text{and} \quad 2x + y - 2z = 5.$$

$\vec{n_1} = \langle 3, -6, -2 \rangle$ $\qquad$ $\vec{n_2} = \langle 2, 1, -2 \rangle$

2. Find parametric equations for this line of intersection.

(1) $\vec{n} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} i & j & k \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = \langle 14, +2, 15 \rangle$

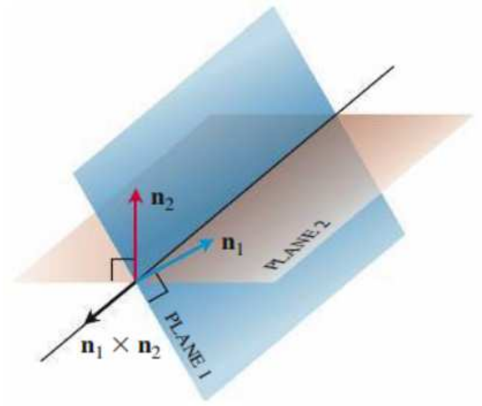

FIGURE 12.41 How the line of intersection of two planes is related to the planes' normal vectors (Example 8).

(2) We need to find a point on that line.

Look at $z = 0$

$3x - 6y = 15$ $\qquad$ $2x + y = 5$

$6x - 12y = 30$

$6x + 3y = 15$

$-15y = 15 \Rightarrow y = -1$

$x = 3$

$P_0 = (3, -1, 0)$

the line is given by

$\vec{OP_0} + t \cdot \vec{n} = \langle 3, -1, 0 \rangle + t \langle 14, 2, 15 \rangle$

$x = 3 + 14t, \; y = -1 + 2t, \; z = 15t$

Example. Find the point where the line

$$x = \frac{8}{3} + 2t, \qquad y = -2t, \qquad z = 1 + t,$$

the line equation

intersect the plane $3x + 2y + 6z = 6$.

$t = -1$

$$3 \cdot x(t) + 2 \cdot y(t) + 6 \cdot z(t) = 6$$

$$3\left(\frac{8}{3} + 2t\right) + 2(-2t) + 6(1+t) = 6$$

$$8 + 6t - 4t + 6 + 6t = 6$$

$$8t = -8 \Rightarrow t = -1$$

$(x(-1), y(-1), z(-1))$

$\left(\frac{8}{3} - 2, 2, 0\right)$

Warning: a line and a plane might intersect at infinitely many points! this happens if and only if the line is contained in the plane.

The distance from a point to a plane

Assume we are given a plane H in space with normal $\overrightarrow{n}$ passing through $\overrightarrow{P}$. Let S be any point in space. As in the case of a line, we define the distance between S and the plane H to be the minimal distance from S to any point Q on H. The closest "route" from S to H will be in the direction $\overrightarrow{n}$.

The distance from a point S to a plane through P with normal $\overrightarrow{n}$ is:

$$d = \left| \overrightarrow{PS} \cdot \frac{\overrightarrow{n}}{|\overrightarrow{n}|} \right|.$$

Example. Find the distance from $S(1,1,3)$ to the plane $3x+2y+6z = 6$.

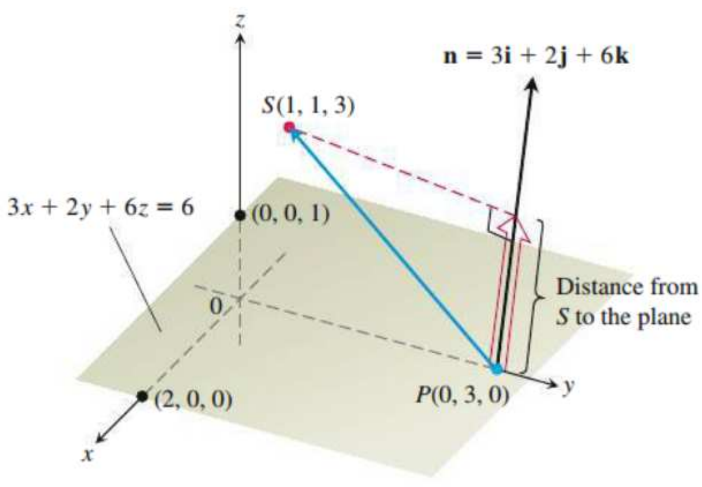

FIGURE 12.42 The distance from S to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto **n** (Example 11).

Exercise. Find an equation for a plane which contains the points $A(1,3,4)$, $B(2,3,0)$.

Exercise. Find an equation for a plane through $(1,-1,3)$ parallel to the plane $3x+y+z=7$.