Math 230-1 Notes

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Contents

Chapter 1

12.1: Three-Dimensional Coordinate Systems

1.1 Reminders

- There are **two** MyLab Math assignments that are due on **Sunday, April 2nd, 2023**
  - Three-Dimensional Coordinate Systems
  - Vectors
- The **first written homework** is going to be due on **Friday, April 12, 2023**.
- Remember to re-write notes in LaTeX for every class!

1.2 Objectives

- Be able to understand and visualize the three-dimensional coordinate plane.
- Be able to draw basic objects in the three-dimensional coordinate plane
- Become fluent in the various attributes of the three-dimensional coordinate plane
- Be able to define a graph in set-builder notation

1.3 Motivation

In former calculus classes (and former math classes in general), we have learned how to graph different objects in two-dimensional space. We have also learned about a particular way of graphing objects (or interpreting coordinates), and that has been through coordinate grids.

However, in multivariable calculus, we want to move out of these two-dimesional coordinate systems. Instead we want to be able to understand graphs in the third-dimensional coordinate system in order to tackle more complex problems.

1.4 What we already know

1.4.1 Cartesian Coordinate Systems

The Cartesian Coordinate system, which is also known as the **rectangular coordinate system** is a coordinate system in which we locate our points based on their position in relation to the origin of the graph, based on the x and y axes.

- For example, given the following point:

$$(4, 5)$$

  what exactly comes to mind?

  - We shift the point 4 positive units along the x-axis from the origin.
  - We shfit the point 5 positive units along the y-axis from the origin.

1.4.2 Two-dimensional coordinate systems $\mathbb{R}^2$

Recall that real numbers are numbers that can be used to express one-dimensional quantities.

- This basically includes every single number that can be plotted on the number line.

We denote real numbers in mathematics with the following symbol:

$$\mathbb{R}$$

$\mathbb{R}$ represents all **real, one-dimensional quantities.** We can, of course, think of this as all of the numbers and points that exist on the number line, since the number line only contains one-dimension, the scalar x. By comparision, whenever we see the following notation:

$$\mathbb{R}^2$$

This means that are observing all **real, two-dimensional quantities**. When we say, “real, two-dimensional quantities,” we are referring to all of the points that exist in the xy-plane, or the two-dimensional Cartesian coordinate system.

By this logic, then, we know that if $\mathbb{R}^2$ represents a pair of real numbers, we know that

$$\mathbb{R}^3$$

represents a triple of all real numbers, in the form of (x, y, z).

1.4.3 Terminology in $\mathbb{R}^2$

Definition 1. Axes

Axes represent the way that a point can "move" in a coordinate plane.

- In the one-dimensional coordinate system, we can only move along the x-axis: x.
- In the two-dimensional coordinate system, we can move along both the x-axis and the y-axis: (x, y).

Definition 2. Quadrants

Quadrants define the different possible areas of the coordinate system a point can exist on, which are based on the signs of both the x and y values.

- For example, there are four quadrants in the xy-plane, including
  - Quadrant I: $(+, +)$
  - Quadrant II: $(-, +)$
  - Quadrant III: $(-, -)$
  - Quadrant IV: $(+, -)$

1.5 The Three-Dimensional Coordinate Plane $\mathbb{R}^3$

By what we know about the one-dimensional coordinate system $\mathbb{R}$ as x and the two-dimensional coordinate system $\mathbb{R}^2$ as (x, y), we must think of the three-dimensional coordinate system $\mathbb{R}^3$ as (x, y, z).

1.5.1 Terminology in $\mathbb{R}^3$

Definition 3. Axes

This is literally a copy of what we have in $\mathbb{R}$ and $\mathbb{R}^2$, insofar that we have the number of dimensions corresponding to the exponent of $\mathbb{R}$. Obviously, in this case, since we are working in $\mathbb{R}^3$, we now have three dimensions to work with, the x-axis, the y-axis, and the z-axis.

Definition 4. Octants

Similarly to what we had in the two-dimensional coordinate plane, we can distinguish what general area a point in three dimensions is going to occupy.

- There is no good way to define which octant is which, but we can visualize it as the xy-plane quadrants, but just duplicated for all positive values of z and all negative values of z.

Definition 5. Planes

Planes are objects that occupy all real-numbers in two dimensions.

There are three planes in $\mathbb{R}^3$

- xy-plane
  - We can think of this as all points in which $z = 0$.
  - All points that satisfy $(x, y, 0)$, where x and y are real numbers.
- yz-plane
  - All points in which $x = 0$
  - Any coordinates that satisfy $(0, y, z)$ where y and z are real numbers.
- xz-plane
  - All points in which $y = 0$
  - Any coordinates that satisfy $(x, 0, z)$, where x and z are real numbers.

1.6 Set-Builder Notation

1.6.1 What are sets?

Sets are just collections of different objects in mathematics.

- We can think of sets as containing integers, variables, etc...

They are generally notated using **curly braces**.

1.6.2 Examples of Sets

$$\{1, 2, 3, \dots\}$$

$$\{a, b, c, \dots\}$$

But, how do we define what kinds of objects we are putting into our set?

1.6.3 Set Builder Notation

Set Builder Notation is a type of mathematical notation that allows us to describe what kinds of objects are in our sets and the properties of such objects.

The generally follow the following format

$$\{\ \textit{variable(s)} : \textit{condition(s) that define the variable(s)}\ \}$$

1.6.4 Common Symbols in Set Builder Notation

Symbol 1.

$$\mathbb{R},\ \mathbb{R}^2,\ \mathbb{R}^3, \ldots\ \mathbb{R}^N$$

Meaning. the set of real numbers in N dimensions

Symbol 2.

$$\in$$

Meaning. "is an element of " or "in" or "belongs to "

Symbol 3.

$$:\ \&\ |$$

Meaning. "such that"

1.6.5 Non-mathematical examples of Set Builder Notation

Example.

$$\{x : x \text{ is a left-handed guitar player}\}$$

Meaning. The set of x such that x is a left-handed guitar player.

Example.

$$\{y \mid y\text{'s name is Randy Truong}\}$$

Meaning. The set of y such that y's name is Randy Truong.

1.6.6 Mathematical Examples of Set Builder Notation

Example.

$$\{(x, y, z) \in \mathbb{R}^3 : y = 0, z = 0\}$$

Meaning. The set of all ordered triples (x, y, z) such that y is equal to 0 and z is equal to 0.

Example.

$$\{(t, 0, 0) : t \in \mathbb{R}\}$$

Meaning. The set of all ordered triples $(t, 0, 0)$ such that t is an element of real numbers (or is a real number).

1.7 Drawing basic objects (points, lines, planes) in R^3

Whenever we want to draw things in three dimensions, there are a few things that we need to consider first.

1.7.1 Drawing the Coordinate System and Right-Hand Rule

Whenever we draw the three-dimensional coordinate system, we must remember that there is a particular way in which we draw the system. The best way to visualize this is to use the **right-hand rule**

- Our arm represents the y-axis, while our fingers represent the x-axis.
- Our thumb is always going to be pointing towards the z-axis.
- Make sure that whenever we are drawing a coordinate system that we are just rotating the system, rather than just "mirroring" it.

Otherwise, whenever we actually plot our points and actually draw things in three dimensions, we need to follow this algorithm:

1. Think of what happens to the object at the origin or think of the shape in two dimensions.
2. Shift the object accordingly based on the third dimension.

1.8 Distance Between Two Points in Three-Dimensional Spacde

1.8.1 Distance in $\mathbb{R}^2$

Formula 1. Distance in $\mathbb{R}^2$

$$\text{let } d \text{ be distance}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1.8.2 Distance in $\mathbb{R}^3$

Formula 2. Distance in $\mathbb{R}^3$

$$\text{let } d \text{ be distance}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2 + (z_2 - z_1)^2}$$

Chapter 2

12.2: Vectors

2.1 Objectives

2.2 Motivation

Chapter 3

12.3: The Dot Product

3.1 Objectives

3.2 Motivation

Chapter 4

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11.2 Motivation

Chapter 12

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12.2 Motivation

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Chapter 17

14.8: Lagrange Multipliers

17.1 Objectives

17.2 Motivation

Chapter 18

14.8: Lagrange Multipliers (Part 2)

18.1 Objectives

18.2 Motivation