Lecture 7: Curves in Space (§11.1, 13.1)

Goals:

1. Fluently and accurately apply the terminology of parametric curves (parametric curve, parametric equations, parameter, parameter interval, initial point, terminal point).

2. Analyze the parametrization for a curve or path and sketch (or describe) the curve or path, including a description of the motion performed.

3. Interpret vector-valued functions as representing the position of a moving particle.

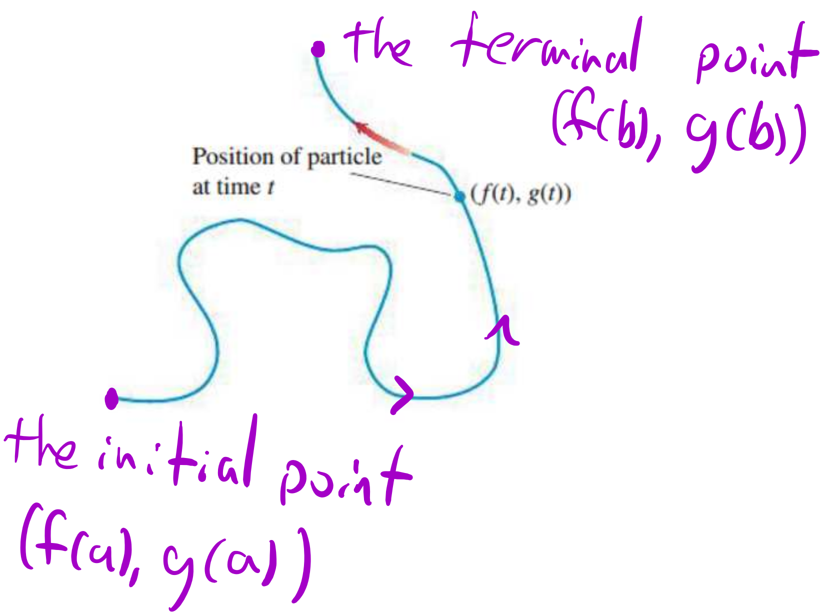
Parametrization of plane curves

If x and y are given as continuous functions

$$x = f(t) \text{ and } y = g(t),$$

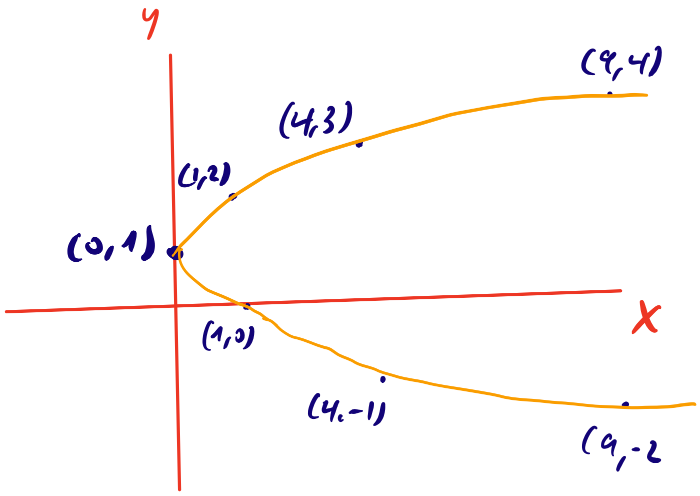
over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

If $I = [a, b]$ is a closed interval, the point $(f(a), g(a))$ is the **initial point** of the curve and the point $(f(b), g(b))$ is the **terminal point**.



Example. Sketch the curve defined by the parametric equation $x = t^2$, $y = t + 1$ for $I = (-\infty, \infty)$.

t	x	y
3	9	4
2	4	3
1	1	2
0	0	1
-1	1	0
-2	4	-1
-3	9	-2



$$y = t + 1, \quad t = y - 1$$

$$x = t^2 = (y-1)^2$$

$$\boxed{x = (y-1)^2}$$

TABLE 11.2 **Values of $x = t^2$ and $y = t + 1$ for selected values of t.**

t	x	y
−3	9	−2
−2	4	−1
−1	1	0
0	0	1
1	1	2
2	4	3
3	9	4

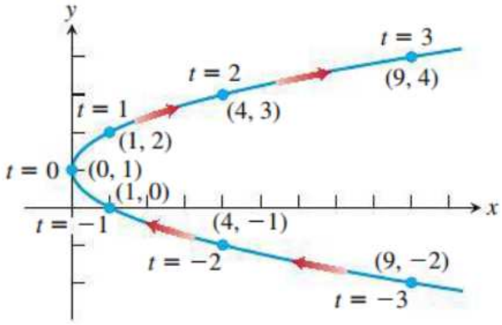


FIGURE 11.3 The curve given by the parametric equations $x = t^2$ and $y = t + 1$ (Example 2).

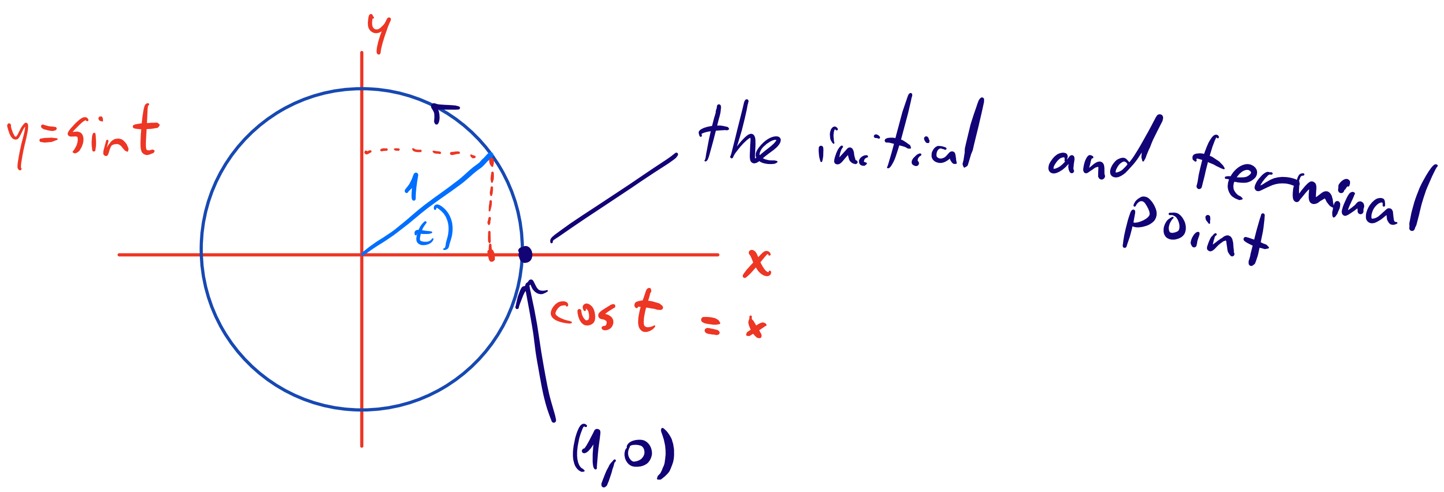
Another method of drawing a curve is to use algebraic manipulation to write y as a function of x, or vice versa (or via an implicit equation). This however doesn't provide us any information on the position $(f(t), g(t))$ of the curve as t varies, so this should be done as well.

Example. Sketch the parametrized curves:

1. $x = \cos(t)$, $y = \sin(t)$, $0 \leq t \leq 2\pi$.

2. $x = a\cos(t)$, $y = a\sin(t)$, $0 \leq t \leq 2\pi$.

①



② $(x(t), y(t)) = (a\cos t, a\sin t)$

$0 \leq t \leq 2\pi$

describes a motion on the circle of radius a

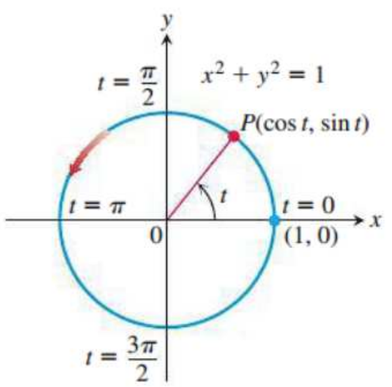
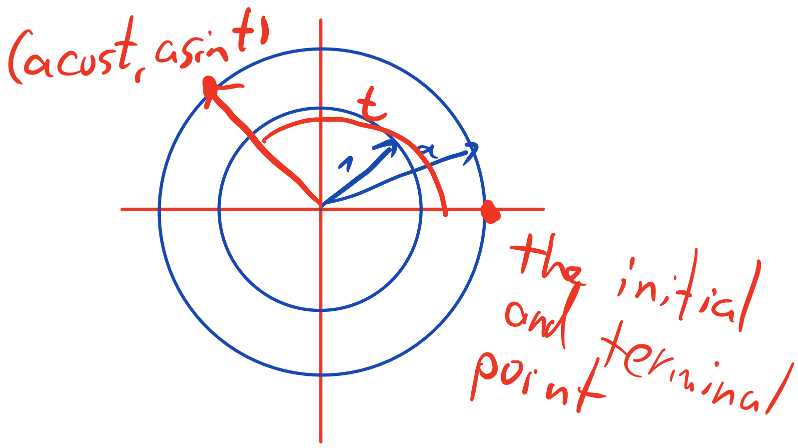


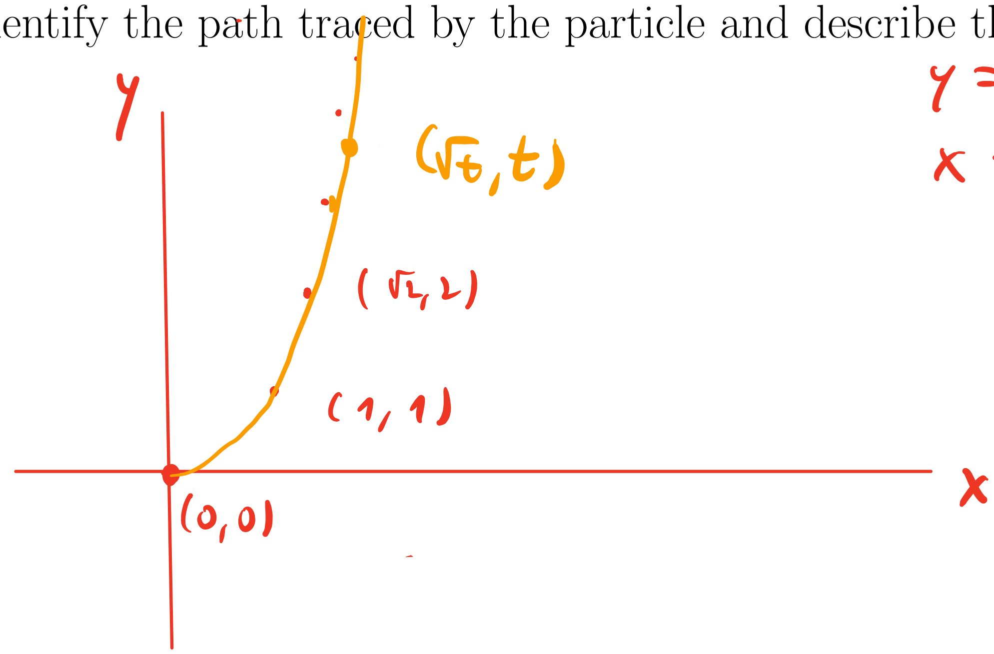
FIGURE 11.4 The equations $x = \cos t$ and $y = \sin t$ describe motion on the circle $x^2 + y^2 = 1$. The arrow shows the direction of increasing t (Example 3).



Example. The position $P(x, y)$ of a particle moving in the xy-plane is given by equations and parameter interval

$$x = |\sqrt{t}| \text{ and } y = t,\ t \geq 0.$$

Identify the path traced by the particle and describe the motion.



$y = t$

$x = \sqrt{t} = \sqrt{y}$

Remark. The points on the curve are precisely, $x = \sqrt{y}$ for $y \geq 0$. It might be tempting to write $x^2 = y$ and $y \geq 0$, instead of $x = \sqrt{y}$, but this set is much larger, since it contain also the negative values of x.

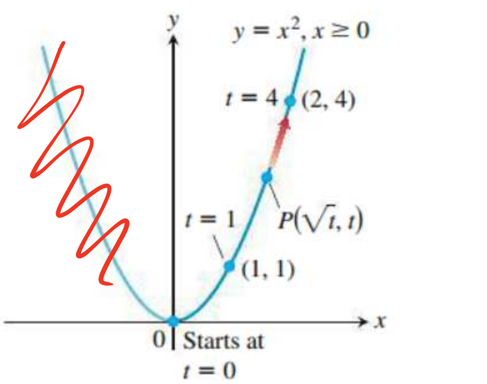


FIGURE 11.5 The equations $x = \sqrt{t}$ and $y = t$ and the interval $t \geq 0$ describe the path of a particle that traces the right-hand half of the parabola $y = x^2$ (Example 4).

Curves in space

A **vector-valued function** (or **vector function** or **vector parametrization** is a function that takes as input a real number t and returns as an output a vector $\overrightarrow{r}(t)$.

Any vector function in space can be written in terms of its components:

$$\overrightarrow{r}(t) = (f(t), g(t), h(t))$$

The $f(t), g(t)$ and $h(t)$ are called the **component functions** of $\overrightarrow{r}(t)$.

Example. Graph the following vector function:

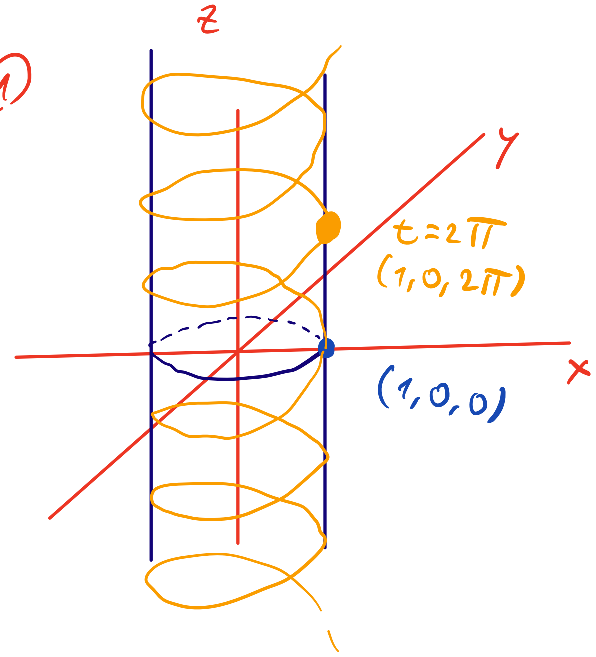
$-\infty < t < \infty$

1. $\overrightarrow{r}(t) = (\cos(t), \sin(t), t)$.

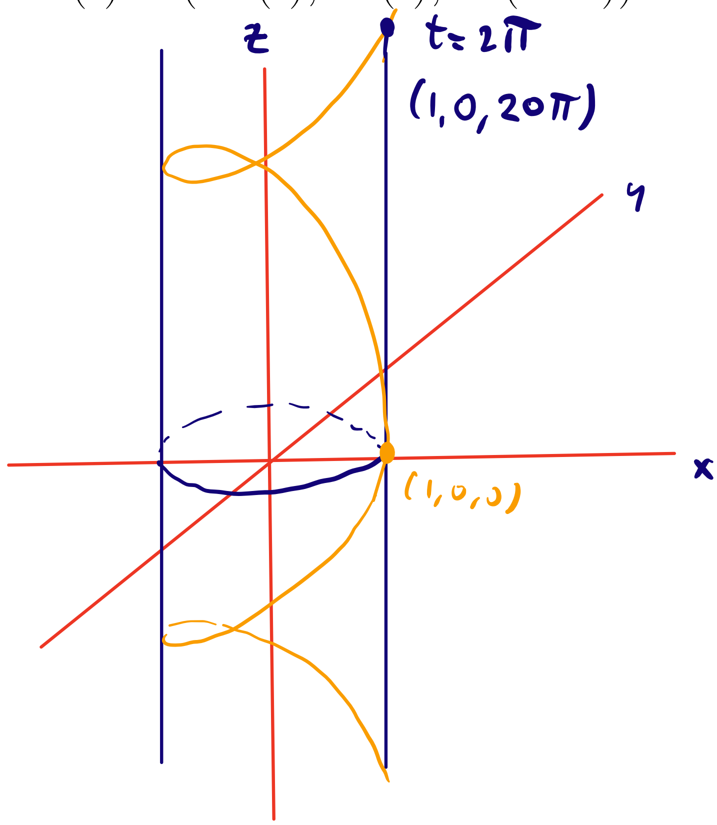
2. $\overrightarrow{r}(t) = (\cos(t), \sin(t), 10t)$.

3. $\overrightarrow{r}(t) = (\cos(t), \sin(t), \sin(100t))$.

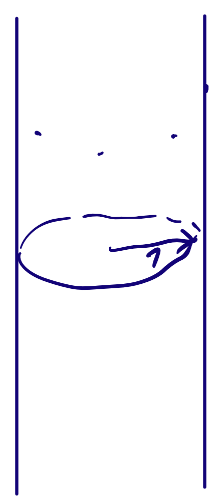
(1)

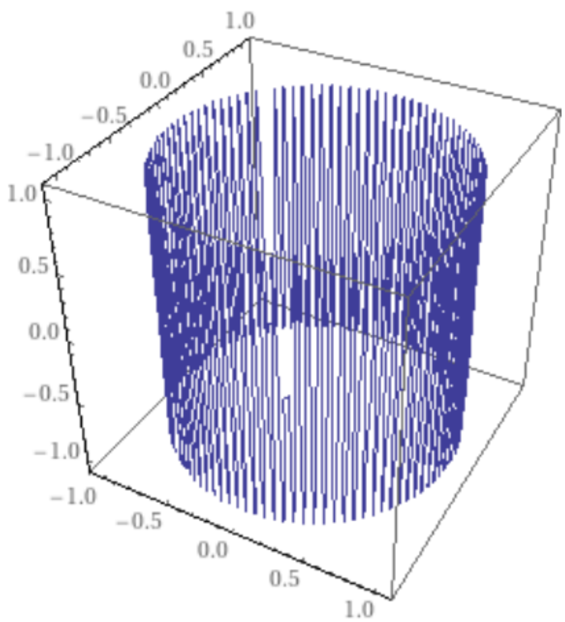


(2)



$x^2+y^2=1$





clarification: The following example is a joke, (sorry, Batman) please do not try to manipulate it.

Example. Use algebraic manipulation to find a parametrization for a curve whose components satisfy the following implicit equation:

$$\left((\frac{x}{7})^2 \cdot \sqrt{\frac{||x|-3|}{(|x|-3)}} + (\frac{y}{3})^2 \cdot \sqrt{\frac{|y+3\cdot\frac{\sqrt{33}}{7}|}{y+3\cdot\frac{\sqrt{33}}{7}}} - 1\right)$$
$$\cdot\left(|\frac{x}{2}| - ((3\cdot\frac{\sqrt{33}-7)}{112}) \cdot x^2 - 3 + \sqrt{1-(||x|-2|-1)^2} - y\right)$$
$$\cdot\left(3\cdot\sqrt{\frac{|(|x|-1)\cdot(|x|-0.75)|}{((1-|x|)\cdot(|x|-0.75))}} - 8\cdot|x| - y\right)$$
$$\cdot\left(3\cdot|x| + 0.75\cdot\sqrt{\frac{|(|x|-0.75)\cdot(|x|-0.5)|}{((0.75-|x|)\cdot(|x|-0.5))}} - y\right)$$
$$\cdot\left(2.25\cdot\sqrt{\frac{|(x-0.5)\cdot(x+0.5)|}{((0.5-x)\cdot(0.5+x))}} - y\right)$$
$$\cdot\left(\frac{\sqrt{360}}{7} + (\frac{3-|x|}{2})\cdot\sqrt{\frac{||x|-1|}{|x|-1}} - \frac{\sqrt{360}}{14}\cdot\sqrt{4-(|x|-1)^2} - y\right) = 0.$$