Lecture 2: Vectors (§12.2)

Goals:

1. Explain the difference between a point and a vector.
2. Express a vector in component form and compute its magnitude.
3. Perform elementary vector algebra using properties of vector addition and scalar multiplication
4. Produce a unit vector with specified direction.
5. Compute the midpoint of a line segment.

Some reminders:

- MyLab homeworks- Wednesday lecture assignment due Sunday at 11:59 PM.
- ~~There is a link to my annotated notes at my page.~~

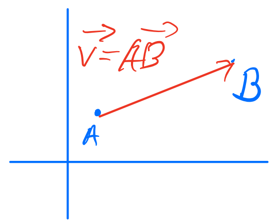
Vectors

To describe **mass**, **length** and **time** we only need to state a number and a unit of measure (e.g. 6 ft, 3 seconds). Such objects are called **scalars**.

On the other hand, to describe **force** or **velocity**, we need both magnitude, and direction. Such objects are called **vectors**.

Definition.

1. We write $\vec{v} = \overrightarrow{AB}$ to denote a vector with **initial point** A and **terminal point** B.
2. The length of the vector $\vec{v}$ is denoted by $\left|\overrightarrow{AB}\right|$.
3. We say that two vectors are **equal** of they have the same length and direction.



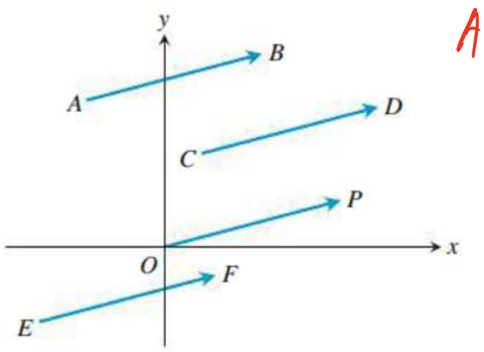


FIGURE 12.9 The four arrows in the plane (directed line segments) shown here have the same length and direction. They therefore represent the same vector, and we write $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF}$.

$\overrightarrow{AB} = \overrightarrow{CD} = \vec{v}$

$|\vec{v}| = |\overrightarrow{AB}|$

length

Definition. For each vector $\overrightarrow{PQ}$, there is one vector $\overrightarrow{v}$ which is equal to $\overrightarrow{PQ}$, whose initial point is the origin $(0,0,0)$. It is called the **position vector of** $\overrightarrow{PQ}$, and it can be represented using only its terminal point (v_1, v_2, v_3). We write $\overrightarrow{v} = \langle v_1, v_2, v_3 \rangle$ for its **component form**. The numbers v_1, v_2, v_3 are called the **components of** $\overrightarrow{v}$.

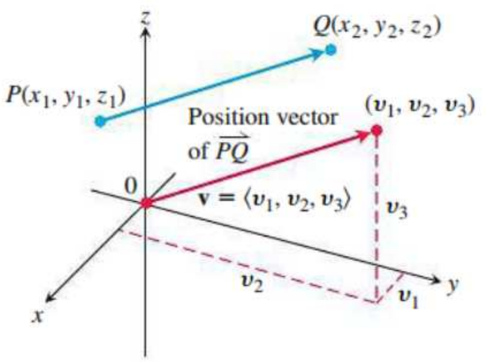
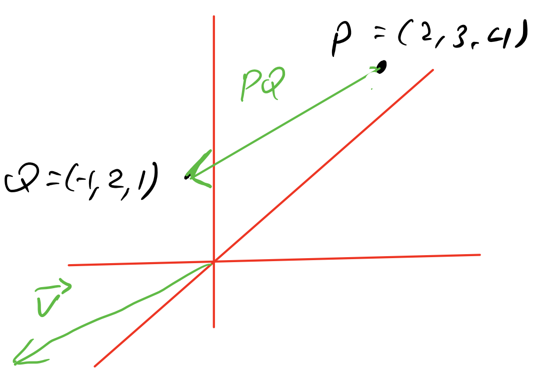


FIGURE 12.10 A vector $\overrightarrow{PQ}$ in standard position has its initial point at the origin. The directed line segments $\overrightarrow{PQ}$ and $\mathbf{v}$ are parallel and have the same length.

Remark. After you learn a new definition (even if it is an easy one), it is a good practice to think of a few examples to see you have the correct concept in mind.

Example.

1. Let $P(2,3,4)$ and $Q(-1,2,1)$. Find the position vector $\overrightarrow{v}$ of $\overrightarrow{PQ}$, its length, and its components.

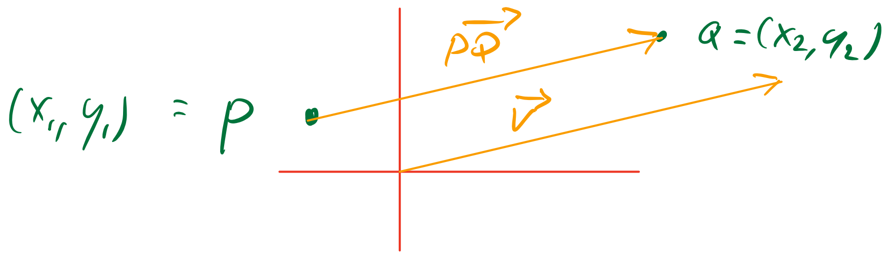


$\vec{v} = \langle -3, -1, -3 \rangle$

$|\vec{v}| = |\overrightarrow{PQ}| = \sqrt{(-3)^2 + (-1)^2 + 3^2}$

$= \sqrt{9+1+9} = \sqrt{19}$

2. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$. What is the component form of $\overrightarrow{PQ}$?



$$\vec{v} = \langle x_2 - x_1 , y_2 - y_1 \rangle$$

Convention: we call $\langle 0, 0, 0\rangle$ the **zero vector** (although it has no specific direction).

we also denote the zero vector by $\vec{0}$

Vector algebra operation

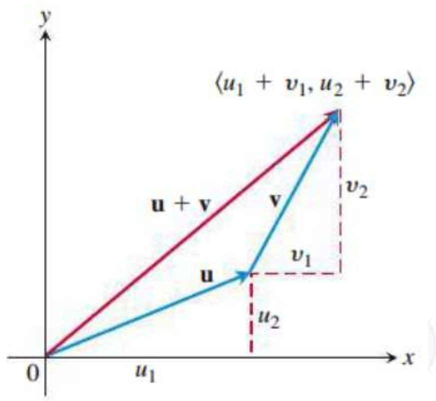
We now define **vector addition** and **scalar multiplication**. **Scalars** are just real numbers.

Definition. Let $\overrightarrow{u} = \langle u_1, u_2, u_3 \rangle$ and $\overrightarrow{v} = \langle v_1, v_2, v_3 \rangle$ be vectors and k be a scalar. Then we define a few operations:

1. **Addition:** $\overrightarrow{u} + \overrightarrow{v} := \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$.

2. **Scalar multiplication**: $k.\overrightarrow{v} = \langle kv_1, kv_2, kv_3 \rangle$.

3. **Difference:** $\overrightarrow{u} - \overrightarrow{v} := \overrightarrow{u} + (-\overrightarrow{v})$. $= \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$



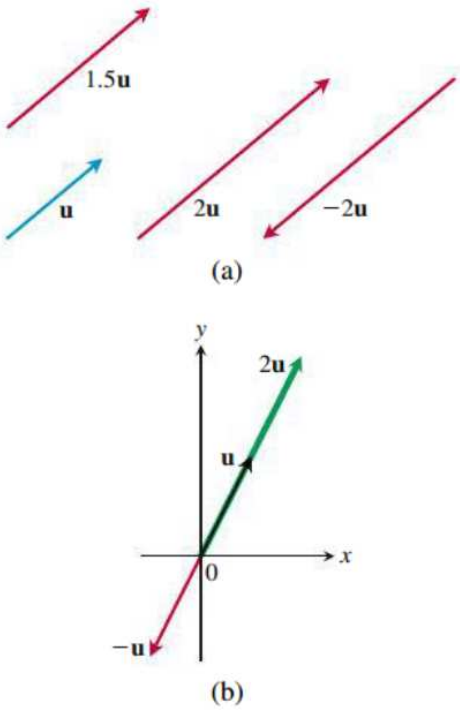


FIGURE 12.13 (a) Scalar multiples of **u**. (b) Scalar multiples of a vector **u** in standard position.

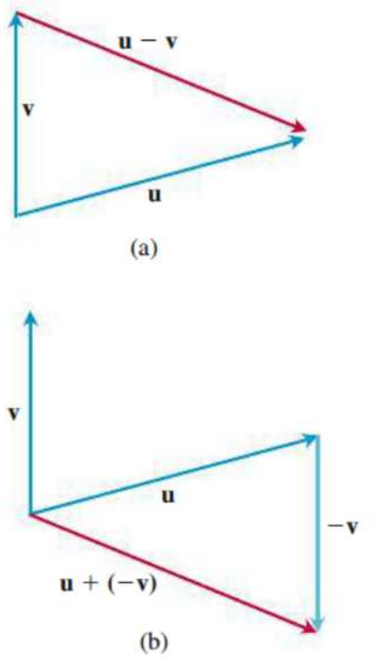


FIGURE 12.14 (a) The vector **u** − **v**, when added to **v**, gives **u**. (b) **u** − **v** = **u** + (−**v**).

Example 0.1.

1. Let $\overrightarrow{v} = \langle v_1, v_2, v_3 \rangle$ be a vector and k be a scalar (some number). What is the length of $k\overrightarrow{v}$? (in terms of $|\overrightarrow{v}|$).

$k\vec{v} = k\langle v_1, v_2, v_3\rangle = \langle kv_1, kv_2, kv_3\rangle$

$|k\vec{v}| = |\langle kv_1, kv_2, kv_3\rangle| = \sqrt{(kv_1)^2 + (kv_2)^2 + (kv_3)^2} = \sqrt{k^2v_1^2 + k^2v_2^2 + k^2v_3^2}$

$= \sqrt{k^2(v_1^2 + v_2^2 + v_3^2)} = \sqrt{k^2} \cdot \sqrt{v_1^2 + v_2^2 + v_3^2} = \underbrace{|k|}_{\text{absolute value}} \cdot \underbrace{|\vec{v}|}_{\text{length}}$

2. Let $\overrightarrow{u} = \langle -1, 3, 1\rangle$ and $\overrightarrow{v} = \langle 4, 7, 0\rangle$. find the components of

   (a) $2\overrightarrow{u} + 3\overrightarrow{v}$.

   (b) $\overrightarrow{u} - \overrightarrow{v}$.

(a) $2\vec{u} + 3\vec{v} = 2\langle -1, 3, 1\rangle + 3\langle 4, 7, 0\rangle =$

$= \langle -2, 6, 2\rangle + \langle 12, 21, 0\rangle =$

$= \langle 10, 27, 2\rangle$

(b) $\vec{u} - \vec{v} = \langle -1, 3, 1\rangle - \langle 4, 7, 0\rangle = \langle -5, -4, 1\rangle$

Here are a few properties of vector operations:

Let $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ be vectors, and a, b be scalars. Then:

1) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

2) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

3) $\vec{v} + \vec{0} = \vec{v}$

4) $\vec{v} - \vec{v} = \vec{0}$

5) $a(\vec{v} + \vec{u}) = a\vec{v} + a\vec{u}$

6) $(a+b)\vec{v} = a\vec{v} + b\vec{v}$

7) $a(b\vec{v}) = (ab)\vec{v}$

Unit vectors

Definition.

1. A vector $\overrightarrow{v}$ of length 1 is called a **unit vector**.

2. The **standard unit vectors** are

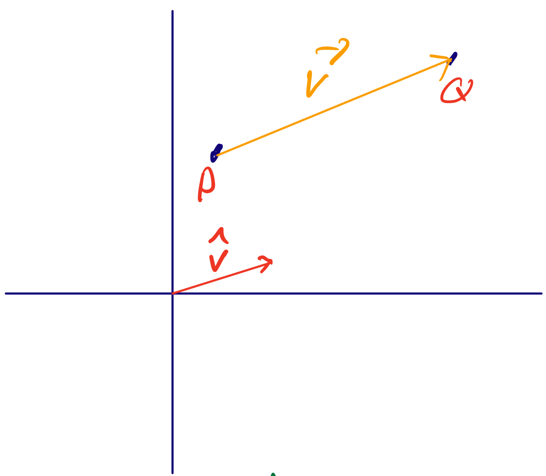
$$\mathrm{i} := \langle 1,0,0\rangle, \quad \mathrm{j} := \langle 0,1,0\rangle \quad \mathrm{k} := \langle 0,0,1\rangle.$$

Example 0.2. Write the vector $v = \langle 2,4,7\rangle$ as a sum of standard unit vectors.

$$\langle 2,4,7\rangle = 2\langle 1,0,0\rangle + 4\langle 0,1,0\rangle + 7\langle 0,0,1\rangle$$
$$= 2i + 4j + 7k$$

Given a vector $\overrightarrow{v} = \langle v_1, v_2, v_3\rangle = v_1 \cdot \mathrm{i} + v_2 \cdot \mathrm{j} + v_3 \cdot \mathrm{k}$, v_1 is called the i-**component** of $\overrightarrow{v}$, and similarly v_2 and v_3 are the j- and k-**components** of $\overrightarrow{v}$.

Example 0.3. Let $P_1(0,2,3)$ and $P_2(4,-1,3)$ be two points. Find a unit vector in the direction of $\overrightarrow{P_1P_2}$.



$$\vec{v} = \overrightarrow{P_1P_2} = \langle 4,-3,0\rangle$$
$$|\vec{v}| = \sqrt{4^2+(-3)^2+0^2} = \sqrt{16+9}$$
$$= \sqrt{25} = 5$$

the unit vector

$$\hat{v} = \frac{1}{5}\langle 4,-3,0\rangle = \left\langle \frac{4}{5}, \frac{-3}{5}, 0\right\rangle$$

More generally, given a vector $\vec{v}$, the unit vector with the same direction is given by $\hat{v} = \frac{1}{|\vec{v}|} \cdot \vec{v}$

Midpoint of line segment

Definition. The **midpoint** M of the line segment joining $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

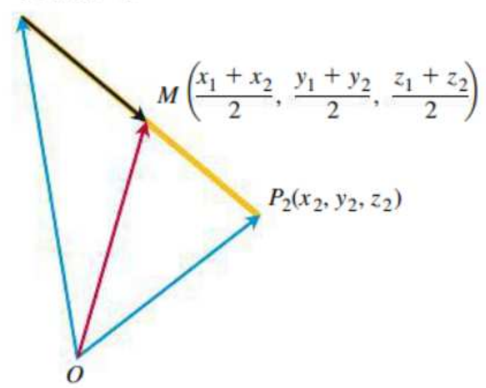


FIGURE 12.16 The coordinates of the midpoint are the averages of the coordinates of P_1 and P_2.

Exercise 0.4. If $\overrightarrow{v} = 3 \cdot \mathrm{i} - 4 \cdot \mathrm{j}$ is a velocity vector, express $\overrightarrow{v}$ as a product of its speed times its direction of motion.