Lecture 14: Partial Derivatives (§14.3)

Goals:

1. Apply the limit definition of the derivative to compute the partial derivative of a function at a point with respect to a variable.
2. Apply differentiation rules for functions of a single real variable to compute the partial derivative of a function at a point with respect to a variable.
3. Interpret partial derivatives geometrically and in terms of directional rates of change.
4. Compute higher-order partial derivatives, and interchange the order in which the partial derivatives are computed where appropriate.

Partial derivatives of a function of two variables

The partial derivatives of a multivariable function are the rates of change with respect to each variable separately. A function $f(x,y)$ of two variables has two partial derivatives, one with respect to x and another with respect to y:

$$f_x(a,b) := \frac{\partial f}{\partial x}\Big|_{(a,b)} = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}.$$

$$f_y(a,b) := \frac{\partial f}{\partial y}\Big|_{(a,b)} = \lim_{h\to 0}\frac{f(a,b+h)-f(a,b)}{h}.$$

In other words, $f_x(a,b)$ is the derivative of $f(x,b)$ as a function of x alone, and $f_y(a,b)$ is the derivative of $f(a,y)$ as a function of y alone.

Example. Find the partial derivatives $f_x(1,1)$ and $f_y(1,1)$ for the function $f(x,y) = 5x + x^2y^2 - 3x^3y$.

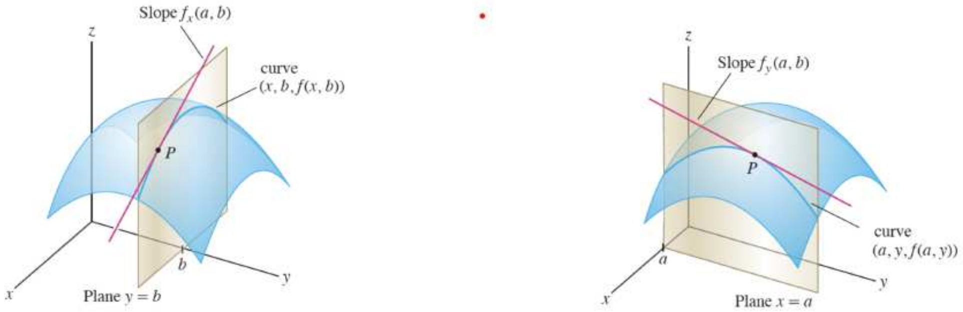
$f_x(x,y) = 5 + 2xy^2 - 9x^2y$, $\quad f_x(1,1) = 5+2-9 = -2$

$f(x,1) = 5x + x^2 - 3x^3$, $\quad f_x(x,1) = f'(x,1) = 5+2x-9$, $\quad f_x(1,1) = -2$

$f_y(x,y) = 2x^2y - 3x^3$, $\quad f_y(1,1) = 2-3 = -1$

Geometric interpretation

The partial derivative $f_x(a,b)$ is the slope of the tangent line to the curve $z = f(x,b)$. Similarly, $f_y(a,b)$ is the slope of the tangent line to the curve $z = f(a,y)$.



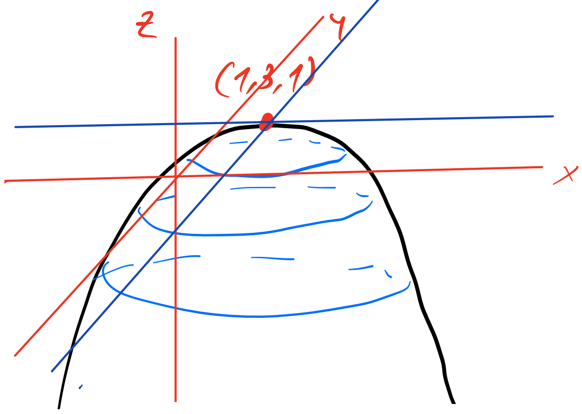
Example. Find the tangent lines to the surface $z = f(x, y) = 1 - (x-1)^2 - (y-3)^2$ at the point $(1, 3, 1)$ in the x- and y-directions.

$z = f(x,y) = 1-(x-1)^2-(y-3)^2$

$f(1,3) = 1 \Rightarrow (1,3,1)$ is indeed on the surface $\{(x,y,f(x,y))\}$

$f_x(x,y) = -2(x-1)$, $\quad f_x(1,3) = 0$

The tangent line in the x direction is $\boxed{(1,3,1)+t(1,0,0)}$



$f_y(x,y) = -2(x-3)$, $f_y(1,3) = 0$

The tangent at $(1,3,1)$ in the y direction is $\boxed{(1,3,1)+t(0,1,0)}$

Lets compute the tangent in the x direction at $(2,3,0)$. $f_x(x,y) = -2(x-1)$, $f_x(2,3) = -2$

So the tangent is: $(2,3,0)+t(1,0,-2)$ | also can be written $(2,3,0)+s(5,0,-10)$

We can take partial derivatives of functions with more than two variables:

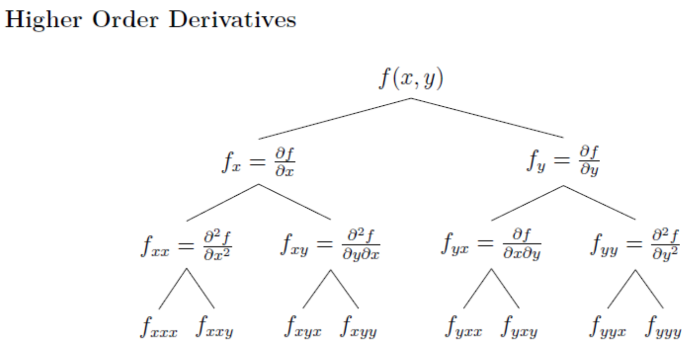
Example. Let $f(x, y, z) = \frac{e^{xy}}{z^2+x}$. Compute $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$.

$$\frac{\partial f}{\partial x} = \frac{ye^{xy}(z^2+x) - e^{xy}}{(z^2+x)^2}$$

$$\frac{\partial f}{\partial y} = \frac{xe^{xy}}{z^2+x}$$

$$\frac{\partial f}{\partial z} = \frac{-2z\, e^{xy}}{(z^2+x)^2}$$

Higher order partial derivatives



Example. Let $f(x,y) = x^4 + y^3 e^x$. Compute $f_{xx}, f_{xy}, f_{yx}, f_{yy}$.

$f_x = 4x^3 + y^3 e^x$, $f_y = 3y^2 e^x$

$f_{xx} = 12x^2 + y^3 e^x$ $\quad$ $f_{yy} = 6y e^x$

$f_{xy} = 3y^2 e^x$ $\quad$ $f_{yx} = 3y^2 e^x$

Note that $f_{xy} = f_{yx}$. Is it a coincidence?

The Mixed Derivative Theorem:

If $f(x,y)$ and its partials f_x, f_y, f_{xy} and f_{yx} are all defined and nice enough, then $f_{xy} = f_{yx}$. More generally, if a function is nice enough, the order of the partial derivatives does not matter, regardless of the number of variables and the order of the partial derivatives. Example

$$f_{xyxy} = f_{xxyy} = f_{yyxx} = f_{yxyx} = f_{xyyx} = f_{yxxy}.$$

Example. Compute $f_{yyxyxyyy}$ for

$$f(x, y) = xe^{e^y \sin(y)} + \sin(\cos(\sin(\cos(\text{whatever}(y))).$$

$f_{xxyyyyyy} = 0$

Example. Let $f(x, y) = -ye^{xy}$.

1. Find all first and second order derivatives.

2. Evaluate f_x, f_y, f_{xx}, f_{yy} at $(1, 1)$ and explain what they mean about the graph of f.

1) $f_x = -y^2 e^{xy}$, $\quad f_y = -e^{xy} - xe^{xy} = -(1+xy)e^{xy}$

$f_{xx} = -y^3 e^{xy}$, $\quad f_{yy} = xe^{xy} - x(1+xy)e^{xy}$

$$f_{xy} = f_{yx} = \frac{\partial f_x}{\partial y} = -2ye^{xy} - y^2 x e^{xy}$$

2) $f_x(1,1) = -e \qquad f_{xx}(1,1) = -e$

$f_y(1,1) = -2e \qquad f_{yy}(1,1) = -2e$

Fixing $y \equiv 1$ the function becomes:

$f(x,1) = -e^x \boxed{= g(x)}$

$f_x(1,1) = g'(1) = -e$, $f_{xx}(1,1) = g''(1) = -e$

Fixing $x=1$ we get $h(y)=-ye^y$ $h'=-e^y+y^2e^y$

$f_y(1,1)=h'(1)=0$, $f_{yy}=h''=-e^y+2ye^y+y^2e^y$, $f_{yy}(1,1)=$

Given a function $f(x,y,z)$ in three variables, its graph is $(x,y,z,f(x,y,z))$ is inside a four dimensional space, so we cannot sketch it effectively. Instead, we can have a good understanding of f by drawing its level surfaces.

Definition. The set of all points (x,y,z) where a function $f(x,y,z)$ has a constant value $f(x,y,z)=c$ is called a **level surface**.

Example. Describe the level surfaces of the function $\left(x^2+y^2+z^2\right)^2$.

Simplify the expression:

$(x-a)\cdot(x-b)\cdot(x-c)\cdot\ldots(x-z)=0$

$(x-a)\ldots(x-x)(x-y)(x-z)$