Lecture 20: Taylor's Formula (§10.9)

Goal: Apply Taylor's Formula to estimate the error incurred in estimating a function near a point with one of the Taylor polynomials generated by the function at that point.

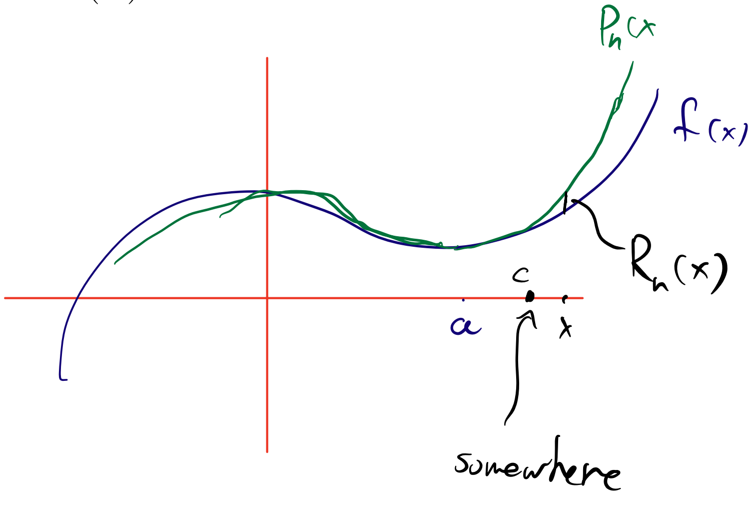
Recall:

Given a function $f(x)$ with derivatives up to order n, we defined its Taylor polynomial at $x = a$:

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

We further claimed that $f(x) \approx P_n(x)$. In this lecture we focus on estimating the error of such an approximation.

Write $f(x) = P_n(x) + R_n(x)$. Then $R_n(x)$ is called the **remainder of order** n.



Taylor's Theorem: Let f be a function that admits derivatives up to order $(n+1)$. Then

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some c between a and x.

In particular, we have:

The remainder estimation Theorem: Assume there exists $M > 0$ such that $\left|f^{(n+1)}(c)\right| \leq M$, for any c between x and a. Then:

$$|R_n(x)| \leq \frac{M}{(n+1)!}\left|x-a\right|^{n+1}.$$

Example. Give an estimate for the value sin(0.1) using the Taylor polynomial of order 3 generated by $f(x) = \sin x$ at $x = 0$. Give an effective upper bound for the error involved in this estimate.

$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin(x)$

$f^{(3)}(x) = -\cos(x)$

$f^{(4)}(x) = \sin(x)$

$f^{(5)}(x) = \cos(x)$

$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \dots$

$P_n(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

$P_1 = P_2$

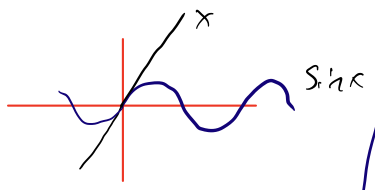
$0.1 = 1/10$

$P_3(x) = x - \frac{x^3}{3!}$, $P_3(0.1) = \frac{1}{10} - \frac{1}{6000} = 0.09983333$

The error is $R_3(0.1)$ $\quad x = 0.1$

$R_3(x) = \frac{f^{(4)}(c)}{4!}x^4 = \frac{\sin(c)}{4!} \cdot 0.1^4 \leq \frac{0.1^5}{4!} = \frac{0.1^5}{24}$

$|\sin x| \leq |x|$



$M = 0.1$

Sin is bounded by 0.1 on $[0, 0.1]$

Observe that $P_4(x) = P_3(x)$

therefore $R_4(x) = R_3(x)$

$\sin x = P_3(x) + R_3(x) = P_4(x) + R_4(x)$

Thus we can bound the error using the formula for $R_4(0.1)$

$R_4(x) = \frac{f^{(5)}(c)}{5!}(x-a)^5 = \frac{\cos(c)}{5!} 0.1^5 \leq \frac{0.1^5}{5!} = \frac{0.1^5}{120}$

$M = 1$

Example. Use the second order Taylor polynomial generated by $f(x) = \sqrt{x}$ at $x = 9$ to approximate $\sqrt{10}$. Give an effective upper bound for the error involved in this estimate.

$f(x) = x^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$

$f''(x) = -\frac{1}{4}x^{-\frac{3}{2}}$

$f^{(3)}(x) = \frac{3}{8}x^{-\frac{5}{2}}$

$a = 9 \quad , \quad x = 10$

$$f(x) = f(9) + f'(9)\cdot\overset{1}{(x-9)} + \frac{f''(9)}{2}\overset{1}{(x-9)}^2 + R_2$$

$$= 3 + \frac{1}{6} - \frac{1}{4}\cdot\frac{1}{27} = 3 + \frac{1}{6} - \frac{1}{108} = 3.16$$

$$R_2 = \frac{f^{(3)}(c)}{3!}1^3 = \frac{3}{8}\cdot\frac{1}{3!}c^{-\frac{5}{2}} \leq \frac{3}{8}\cdot\frac{1}{3!}\cdot\frac{1}{3^5}$$

$$= \frac{1}{8\cdot 6\cdot 81}$$

Example. Determine the degree of the Taylor polynomial of $f(x) = \ln x$ at at $x = 1$ that gives an estimate for $\ln(1.1)$ with an accuracy of 10^{-5}.

$f = \ln x$

$f' = \frac{1}{x}$

$f'' = \frac{-1}{x^2}$

$f''' = \frac{2}{x^3}$

$\vdots$

$f^{(k)}(x) = \frac{(-1)^{k-1}\, k!}{x^k}$

$P_n(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$

$R_n(x) = \frac{(-1)^n}{(1+n) \cdot c^{n+1}} \cdot (x-1)^{n+1}$

$|R_n(x)| \leq \frac{1}{n} \cdot 0.1^{n+1}$

$R_4(1.1) = \frac{1}{5} \cdot 0.1^5 < 10^{-5}$

But $R_3(1.1) = \frac{1}{4}\, 0.1^4 > 10^{-5}$

So we want to use $P_4(1.1)$

$P_4(1.1) = 0.1 - \frac{0.1^2}{2} + \frac{0.1^3}{3} - \frac{0.1^4}{4}$

Example. Find the Taylor polynomial of order n of $\frac{1}{1-x}$ at $x = 0$. For $n = 3$, give an upper bound for the magnitude of the error when $|x| \leq 0.1$.

$$P_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$$

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = \frac{1}{(1-c)^{n+2}} \cdot x^{n+1}$$

Example. Find the Taylor Polynomials of order n for $f(x) = 2x^2 - x + 1$ at $x = 1$.