Lecture 19: Taylor Polynomials (§10.8)

Goals:

1. Compute the Taylor polynomial of a given order generated by a given function at a given point.
2. Approximate functions with Taylor polynomials.

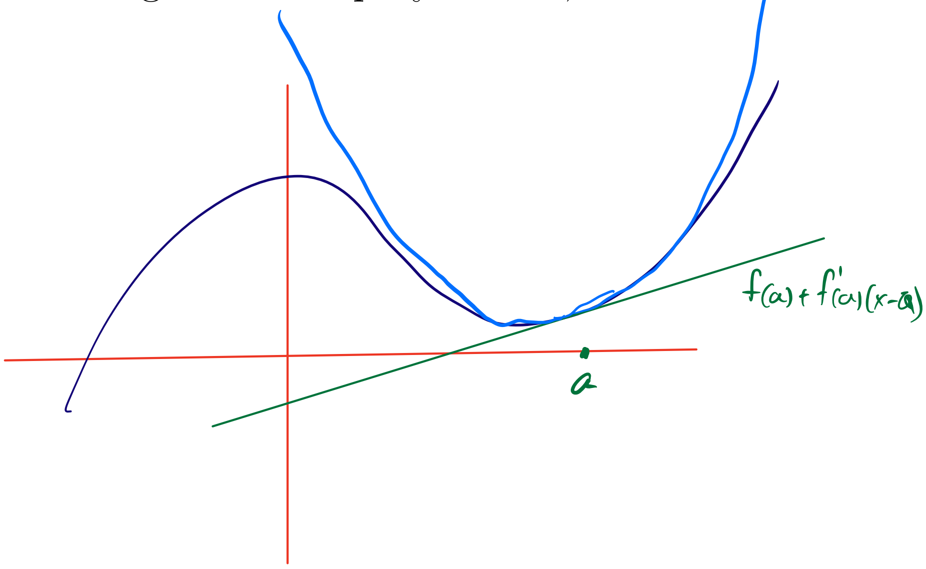
Taylor polynomials

Recall that the tangent line constitutes the best linear approximation of a function $f(x)$ near $x = a$:

$$f(x) \approx f(a) + f'(a)(x - a)$$

How can we improve on the linear approximation?

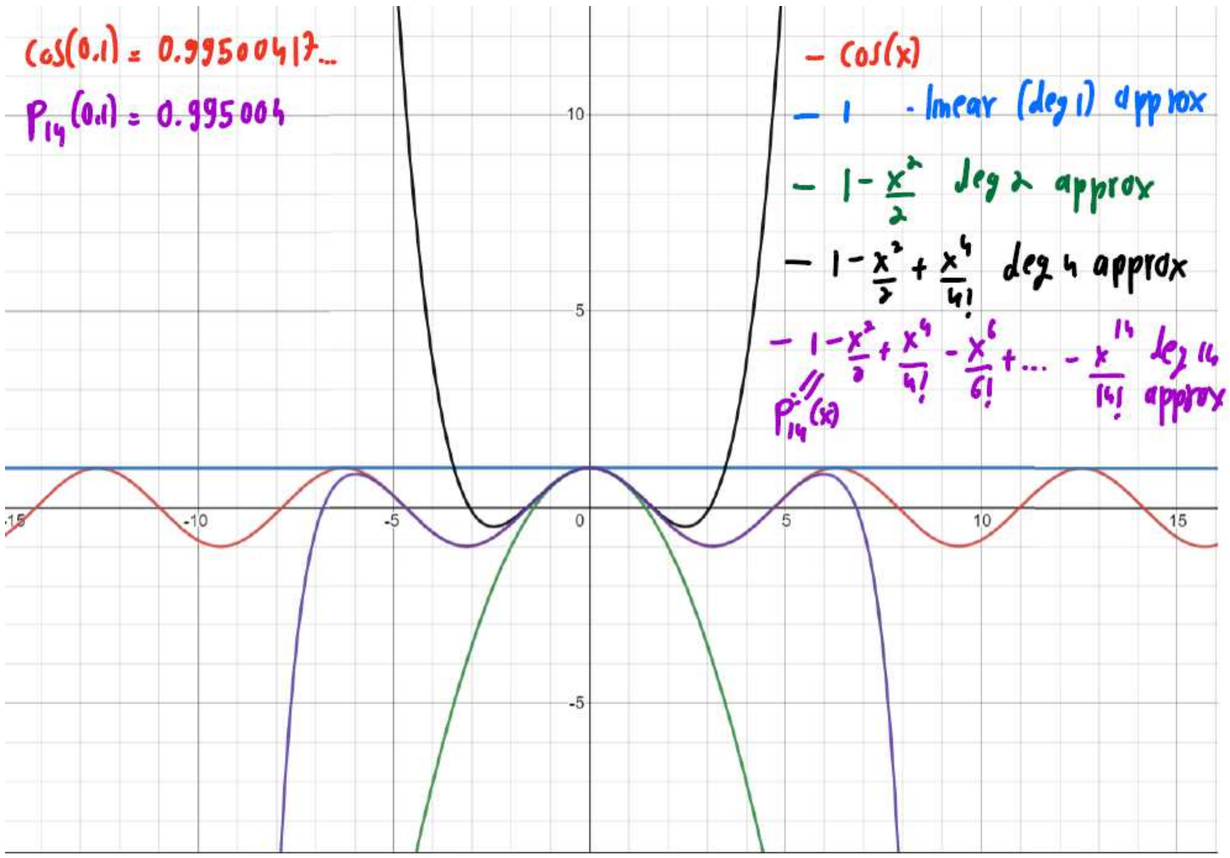
Instead of approximating f with linear functions, we can approximate f using quadratic polynomials, or more generally using polynomials of arbitrary degrees. The higher the degree of the polynomial, the better we can approximate $f(x)$.



Example. Consider $f(x) = \cos(x)$. The linear approximation $L(x)$ near 0 is

$$\cos(x) \simeq L(x) = \cos(0) + \cos'(0) \cdot x = 1.$$

Here are a few better approximations:



Polynomial approximations

Given a function $f(x)$ we would like to approximate it near $x = a$, by a polynomial $P_n(x) = b_0 + b_1(x-a) + \ldots + b_n(x-a)^n$ of degree n.

$b_2(x-a)^2$

How we calculate b_k, $k=0,\ldots,n$?

$b_0 = P_n(0) = f(a)$

$b_1 = P_n'(a) = f'(a)$

$b_2 = P_n''(a)/2$

$b_3 = \frac{P_n'''(a)}{3 \cdot 2}$

$\vdots$

$b_k = \frac{P^{(k)}(a)}{k!}$

We assume that

$P_n(0) = f(a)$

$P_n'(a) = f'(a)$

$P_n''(a) = f''(a)$

$\vdots$

$P_n^{(k)} = f^{(k)}$

$\vdots$

$P_n^{(n)} = f^{(n)}$

Definition. Let $f(x)$ be a function with derivatives of order k for $k = 1, \ldots, n$. Then

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

is called the **Taylor polynomial of order** n generated by f at $x = a$.

The approximation $f(x) \approx P_n(x)$, for x near a is called the **order** n **approximation of** $f(x)$.

$P_n(x) \sim f(x-a)$

Example. Find the Taylor polynomials of order n of $\cos(x)$ and $\sin(x)$ at 0.

1) $f(x) = \cos(x)$ at 0: 1

$f'(x) = -\sin(x)$: 0

$f''(x) = -\cos(x)$: -1

$f'''(x) = \sin(x)$: 0

$f^{(4)}(x) = \cos(x)$: 1

$\vdots$: 0

-1

$\vdots$

$\cos^{(k+4)}(x) = \cos^{(k)}(x)$

Recall that the k'th coeficient of $P_n(x)$ is $\dfrac{f^{(k)}(0)}{k!}$

For $\cos(x)$ at $a=0$ we get:

$$P_{12}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!}$$

2) $f(x) = \sin(x)$: $\sin(0) = 0$

$f' = \cos x$: $\cos(0) = 1$

$f'' = -\sin x$: $-\sin(0) = 0$

$f''' = -\cos x$: $-\cos(0) = -1$

$f^{(4)} = \sin x$

$\vdots$

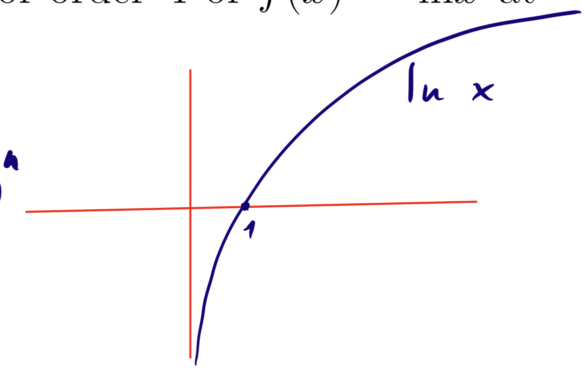
$f^{(k+4)} = f^{(k)}$

The Taylor polynomial of degree 12 of $\sin x$ is

$$P_{12}(x) = P_{11}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$$

$\ln(1.1) \sim P_n(0.1)$

Example. Find the Taylor polynomial of order 4 of $f(x) = \ln x$ at $x = 1$. Use it to to estimate $\ln(1.1)$.



$P_n(x) = b_0 + b_1(x-1) + b_2(x-1)^2 + \dots b_n(x-1)^n$

$b_k = \dfrac{\ln^{(k)}(1)}{k!}$

$b_k = \dfrac{(-1)^{k+1} \cdot (k-1)!}{k!} = \dfrac{(-1)^{k+1}}{k}$

$\ln(1) = 0$

$\ln'(x) = \frac{1}{x}, \quad \ln'(1) = 1$

$\ln''(x) = \frac{-1}{x^2}, \quad \ln''(1) = -1$

$\ln'''(x) = \frac{+2}{x^3}, \quad \ln'''(1) = +2$

$\ln^{(k)} = (-1)^{k+1} \cdot (k-1)!$

$\ln(1+x) \sim x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6}$

$\ln(1.1) \sim 0.1 - \frac{0.1^2}{2} \qquad = 0.1 - 0.005$

Example. Find the Taylor Polynomials of order n for $f(x) = 2x^2 - x + 1$ at $x = 1$.

$y = 1 + x$

$x = y - 1$

$\ln(y) \sim (y-1) - \frac{(y-1)^2}{2} + \frac{(y-1)^3}{3} - \frac{(y-1)^4}{4} + \dots$

$$\boxed{f(x) = 2x^2 + x + 1}$$

$P_n(x) = b_0 + b_1(x-1) + b_2(x-1)^2 + b_3(x-1)^3 + \dots b_n(x-1)^n$

$b_k = \dfrac{f^{(k)}(1)}{k!}$

$f = 2x^2 + x + 1 \qquad f(1) = 4$

$f' = 4x + 1 \qquad f'(1) = 5$

$f'' = 4 \qquad f''(1) = 4$

$f''' = 0$

$$P_n(x) = 4 + 5(x-1) + \frac{4}{2}(x-1)^2 = 4 + 5(x-1) + 2(x-1)^2 =$$
$$= 2x^2 + x + 1$$

Example. Find the Taylor polynomial of order n of $\frac{1}{1-x}$ at $x = 0$.

$f(x) = \frac{1}{1-x}$ $\qquad f(0) = 1$

$f' = \frac{1}{(1-x)^2}$, $\quad f'(0) = 1$

$f''(x) = \frac{2}{(1-x)^3}$, $\quad f''(0) = 2$

$\vdots$

$f^{(k)}(0) = k!$

$b_k = \frac{f^{(k)}}{k!} = 1$

$P_n(x) = 1 + x + x^2 + x^3 + \dots x^n$

$1 + q + q^2 + q^3 + \dots = \frac{1}{1-q}$