Lecture 18: Linear Approximation and Differentials (§14.6)

Goals:

1. Estimate the change of a differentiable function in a specific direction, and relate this estimate to the differential of the function.
2. Use differentials to analyze the change in a multivariate quantity as its inputs change.

Linear approximation in one variable

Given a single variable function $f(x)$, and a point a, the tangent line $L(x)$ of f at a, among all possible lines passing at a, gives the best approximation for the curve $(x, f(x))$ in a small neighborhood of a.

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

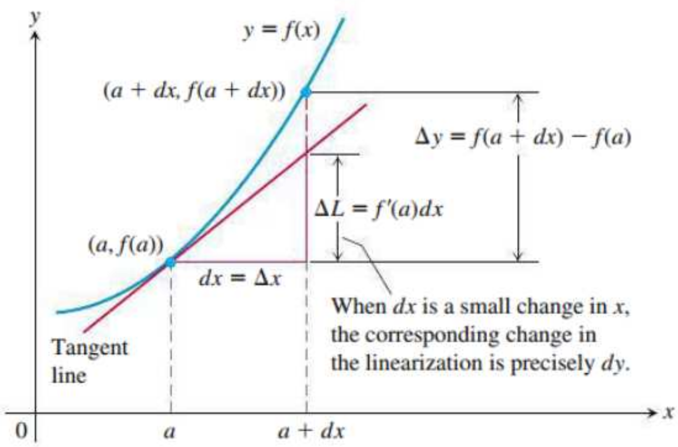


FIGURE 3.44 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

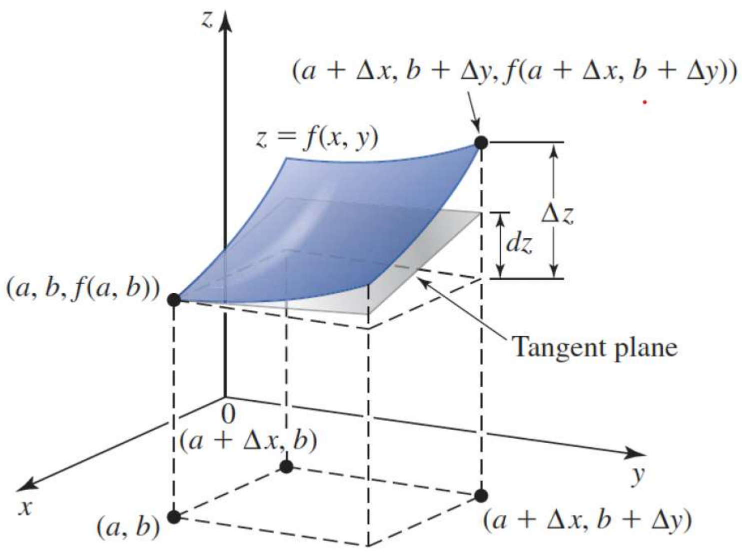
Linear approximation in two variable

1. The **tangent line $L(x)$ at a, approximates the graph of a one-variable function $f(x)$ near a**; Similarly,
2. The **tangent plane approximates the graph $\{x, y, f(x, y)\}$ of a two-variable function $f(x, y)$ near (a, b).**

Example. The surface of Earth is approximately a sphere of radius $\sim$ 6500km, but when we are walking outside, it seems like we are walking on a plane (the tangent plane to our current position).

Consider the surface $z = f(x, y)$ and $P = (a, b, f(a, b))$ a point on the surface. Recall that the tangent plane to the surface at P is

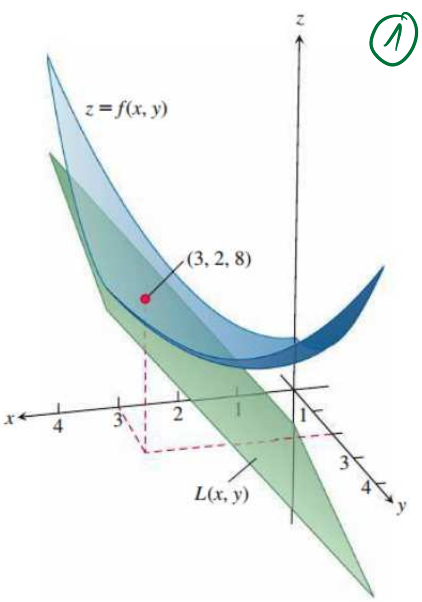
$$z - f(a, b) = \frac{\partial f}{\partial x}|_{(a,b)} \cdot (x - a) + \frac{\partial f}{\partial y}|_{(a,b)}(y - b).$$



Definition. The **linearization** of a function $f(x,y)$ at a point (a,b) is
$$L(x,y) = f(a,b) + \frac{\partial f}{\partial x}|_{(a,b)} \cdot (x-a) + \frac{\partial f}{\partial y}|_{(a,b)}(y-b).$$
The approximation $f(x,y) \approx L(x,y)$ is the **standard linear approximation of f at** (a,b).

Example.

1. Find the linearization of $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ at $(3,2)$.
2. Use it to estimate $f(3.05, 2.1)$.



① $f(3,2) = 3^2 - 3\cdot 2 + \frac{1}{2}2^2 + 3 = 8$

$\frac{\partial f}{\partial x} = 2x - y$, $\frac{\partial f}{\partial x}(3,2) = 6-2 = 4$

$\frac{\partial f}{\partial y} = -x+y$, $\frac{\partial f}{\partial y}(3,2) = -1$

$L(x,y) = f(3,2) + \frac{\partial f}{\partial x}(x-3) + \frac{\partial f}{\partial y}(y-2) =$

$= 8 + 4(x-3) - 1(y-2)$

② $L(3.05, 2.1) = 8 + 4\cdot 0.05 - 1\cdot 0.1 = \underline{\underline{8.1}}$

this is an estimate of
$f(3.05, 2.1) = \underline{\underline{8.1025}}$

the mistake is 0.0025

Differentials

Definition. If we consider an infinitesimally small change from (a, b) to $(a + dx, b + dy)$, the resulting change in the linearization $L(x, y)$ of $z = f(x, y)$ is

$$dz = df = \frac{\partial f}{\partial x}|_{(a,b)}dx + \frac{\partial f}{\partial y}|_{(a,b)}dy.$$

We call df the **total differential** of f at (a, b).

Example. Let $z = f(x, y) = x^3y^2 + y$.

1. Find the differential dz.

2. Compute the value of dz if (x, y) changes from $(1, -2)$ to $(1.1, -1.9)$. Compare the result with $\triangle z$.

① $dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 3x^2y^2\, dx + (2x^3y + 1)\, dy$

② $\frac{\partial f}{\partial x}(1,-2) = 3 \cdot 1^2 \cdot (-2)^2 = 12$, $\frac{\partial f}{\partial y}(1,-2) = 2 \cdot (-2) + 1 = -3$

$dx = 0.1$, $dy = +0.1$

$dz = 12 \cdot 0.1 - 3\ 0.1 = 1.2 - 0.3 = 0.9$

Functions of three variables

Definition. The **linearization** of $f(x, y, z)$ at a point $P(a, b, c)$ is

$$L(x, y, z) = f(a, b, c) + \frac{\partial f}{\partial x}|_P \cdot (x - a) + \frac{\partial f}{\partial y}|_P \cdot (y - b) + \frac{\partial f}{\partial z}|_P \cdot (z - c).$$

The **total differential** is:

$$df = \frac{\partial f}{\partial x}|_P \cdot dx + \frac{\partial f}{\partial y}|_P \cdot dy + \frac{\partial f}{\partial z}|_P \cdot dz.$$

Example. Consider a rectangular box with sides $x = 3m$, $y = 2m$ and $z = 1m$. There's an increase of $0.01m$ in each of the sides of the box. Estimate the change in the volume $\triangle V$.

$V(x,y,z) = x \cdot y \cdot z$, $\frac{\partial V}{\partial x} = yz$, $\frac{\partial V}{\partial y} = xz$, $\frac{\partial V}{\partial z} = xy$

$V(3,2,1) = 6$, $\frac{\partial V}{\partial x} = 2$, $\frac{\partial V}{\partial y} = 3$, $\frac{\partial V}{\partial z} = 6$

$dx = dy = dz = 0.01$

$$dz = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = 2 \cdot 0.01 + 3 \cdot 0.01 + 6 \cdot 0.01 =$$

$$= 11 \cdot 0.01 = \underline{\underline{0.11}}$$

$$V(3.01, 2.01, 1.01) \approx 6 + dz = \underline{6}.11$$

our estimate

$\boxed{6.1106}$

the true result

the mistake is

0.0006

Change in a specific direction:

Example. Estimate how much the value of $f(x, y, z) = y\sin(x) + 2yz$ will change if the point $P(x, y, z)$ moves 0.1 units from $P_0(0, 1, 0)$ straight toward $P_1(2, 2, -2)$.

$(0,1,0)$

$\vec{v} = \overrightarrow{P_0P_1} = \langle 2, 1, -2 \rangle$

$(2,2,-2)$

the unit vector in the direction of $\vec{v}$ is

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \left\langle \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} \right\rangle$$

$$\langle dx, dy, dz \rangle = 0.1 \cdot \hat{v} = \left\langle \frac{2}{30}, \frac{-1}{30}, \frac{-2}{30} \right\rangle$$

$$dx = \frac{2}{30}, \quad dy = \frac{-1}{30}, \quad dz = \frac{-2}{30}$$

$$\frac{\partial f}{\partial x} = y\cos x = 1$$

$$\frac{\partial f}{\partial y} = \sin x + 2z = 0$$

$$\frac{\partial f}{\partial z} = 2y = 2$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= 1 \cdot \frac{2}{30} + 0 \cdot \frac{-1}{30} + 2\,\frac{-2}{30} = \underline{\underline{\frac{-1}{15}}}$$

A plane is given by $Ax + By + Cz = D$, $A^2+B^2+C^2=1$

$f(x,y,z)$ = the distance from (x,y,z) to the plane

$$f(x,y,z) = |Ax + By + Cz - D|$$

$$g(x,y,z) = f(x,y,z)^2 = (Ax + By + Cz - D)^2 =$$

$$= \cancel{A^2x^2} + \cancel{B^2y^2} + \cancel{C^2z^2} + \cancel{D^2} + 2ABxy + \cancel{2ACxz} + \cancel{2BCyz}$$

$$- (\cancel{2ADx} + \cancel{2BDy} + \cancel{2CDz})$$

$$\frac{\partial^2 g}{\partial x \partial y} = \frac{\partial^2\, 2ABxy}{\partial x \partial y} = 2AB$$