Math 230-1 Notes

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Contents

Chapter 1

12.1: Three-Dimensional Coordinate Systems

1.1 Reminders

- There are **two** MyLab Math assignments that are due on **Sunday, April 2nd, 2023**
  f
  - Three-Dimensional Coordinate Systems
  - Vectors
- The **first written homework** is going to be due on **Friday, April 12, 2023**.
- Remember to re-write notes in LaTeX for every class!

1.2 Objectives

- Be able to understand and visualize the three-dimensional coordinate plane.
- Be able to draw basic objects in the three-dimensional coordinate plane
- Become fluent in the various attributes of the three-dimensional coordinate plane
- Be able to define a graph in set-builder notation

1.3 Motivation

In former calculus classes (and former math classes in general), we have learned how to graph different objects in two-dimensional space. We have also learned about a particular way of graphing objects (or interpreting coordinates), and that has been through coordinate grids.

However, in multivariable calculus, we want to move out of these two-dimesional coordinate systems. Instead we want to be able to understand graphs in the third-dimensional coordinate system in order to tackle more complex problems.

1.4 What we already know

1.4.1 Cartesian Coordinate Systems

The Cartesian Coordinate system, which is also known as the **rectangular coordinate system** is a coordinate system in which we locate our points based on their position in relation to the origin of the graph, based on the x and y axes.

- For example, given the following point:

$$(4, 5)$$

what exactly comes to mind?

  - We shift the point 4 positive units along the x-axis from the origin.
  - We shfit the point 5 positive units along the y-axis from the origin.

1.4.2 Two-dimensional coordinate systems $\mathbb{R}^2$

Recall that real numbers are numbers that can be used to express one-dimensional quantities.

- This basically includes every single number that can be plotted on the number line.

We denote real numbers in mathematics with the following symbol:

$$\mathbb{R}$$

$\mathbb{R}$ represents all **real, one-dimensional quantities.** We can, of course, think of this as all of the numbers and points that exist on the number line, since the number line only contains one-dimension, the scalar x. By comparision, whenever we see the following notation:

$$\mathbb{R}^2$$

This means that are observing all **real, two-dimensional quantities**. When we say, "real, two-dimensional quantities," we are referring to all of the points that exist in the xy-plane, or the two-dimensional Cartesian coordinate system.

By this logic, then, we know that if $\mathbb{R}^2$ represents a pair of real numbers, we know that

$$\mathbb{R}^3$$

represents a triple of all real numbers, in the form of (x, y, z).

1.4.3 Terminology in $\mathbb{R}^2$

Definition 1.1: A

es

Axes represent the way that a point can "move" in a coordinate plane.

- In the one-dimensional coordinate system, we can only move along the x-axis: x.
- In the two-dimensional coordinate system, we can move along both the x-axis and the y-axis: (x, y).

Definition 1.2: Q

adrants

Quadrants define the different possible areas of the coordinate system a point can exist on, which are based on the signs of both the x and y values.

- For example, there are four quadrants in the xy-plane, including
  - Quadrant I: $(+, +)$
  - Quadrant II: $(-, +)$
  - Quadrant III: $(-, -)$
  - Quadrant IV:n $(+, -)$

1.5 The Three-Dimensional Coordinate Plane $\mathbb{R}^3$

By what we know about the one-dimensional coordinate system $\mathbb{R}$ as x and the two-dimensional coordinate system $\mathbb{R}^2$ as (x, y), we must think of the three-dimensional coordinate system $\mathbb{R}^3$ as (x, y, z).

1.5.1 Terminology in $\mathbb{R}^3$

Definition 1.3: A

es

This is literally a copy of what we have in $\mathbb{R}$ and $\mathbb{R}^2$, insofar that we have the number of dimensions corresponding to the exponent of $\mathbb{R}$. Obviously, in this case, since we are working in $\mathbb{R}^3$, we now have three dimensions to work with, the x-axis, the y-axis, and the z-axis.

Definition 1.4: O

tants

Similarly to what we had in the two-dimensional coordinate plane, we can distinguish what general area a point in three dimensions is going to occupy.

- There is no good way to define which octant is which, but we can visualize it as the xy-plane quadrants, but just duplicated for all positive values of z and all negative values of z.

Definition 1.5: P

anes

Planes are objects that occupy all real-numbers in two dimensions.

There are three planes in $\mathbb{R}^3$

- xy-plane
  - We can think of this as all points in which $z = 0$.
  - All points that satisfy $(x, y, 0)$, where x and y are real numbers.
- yz-plane
  - All points in which $x = 0$
  - Any coordinates that satisfy $(0, y, z)$ where y and z are real numbers.
- xz-plane
  - All points in which $y = 0$
  - Any coordinates that satisfy $(x, 0, z)$, where x and z are real numbers.

1.6 Set-Builder Notation

1.6.1 What are sets?

Sets are just collections of different objects in mathematics.

- We can think of sets as containing integers, variables, etc...

They are generally notated using **curly braces**.

1.6.2 Examples of Sets

$$\{1, 2, 3, \dots\}$$

$$\{a, b, c, \dots\}$$

But, how do we define what kinds of objects we are putting into our set?

1.6.3 Set Builder Notation

Set Builder Notation is a type of mathematical notation that allows us to describe what kinds of objects are in our sets and the properties of such objects.

The generally follow the following format

$$\{ \textit{ variable(s)} : \textit{condition(s) that define the variable(s) } \}$$

1.6.4 Common Symbols in Set Builder Notation

Symbol 1.

$$\mathbb{R},\ \mathbb{R}^2,\ \mathbb{R}^3, \ldots\ \mathbb{R}^N$$

Definition: t

he set of real numbers in N dimensions

Symbol 2.

$$\in$$

Definition: '

'is an element of " or "in" or "belongs to "

Symbol 3.

$$: \ \& \ |$$

Definition: '

'such that"

1.6.5 Non-mathematical examples of Set Builder Notation

Example.

$$\{x : x \textit{ is a left-handed guitar player}\}$$

Definition: T

he set of x such that x is a left-handed guitar player.

Example.

$$\{y \mid y\textit{'s name is Randy Truong}\}$$

Definition: T

he set of y such that y's name is Randy Truong.

1.6.6 Mathematical Examples of Set Builder Notation

Example.

$$\{(x, y, z) \in \mathbb{R}^3 : y = 0, z = 0\}$$

Definition: T

he set of all ordered triples (x, y, z) such that y is equal to 0 and z is equal to 0.

Example.

$$\{(t, 0, 0) : t \in \mathbb{R}\}$$

Definition: T

he set of all ordered triples $(t, 0, 0)$ such that t is an element of real numbers (or is a real number).

1.7 Drawing basic objects (points, lines, planes) in R^3

Whenever we want to draw things in three dimensions, there are a few things that we need to consider first.

1.7.1 Drawing the Coordinate System and Right-Hand Rule

Whenever we draw the three-dimensional coordinate system, we must remember that there is a particular way in which we draw the system. The best way to visualize this is to use the **right-hand rule**

- Our arm represents the y-axis, while our fingers represent the x-axis.
- Our thumb is always going to be pointing towards the z-axis.
- Make sure that whenever we are drawing a coordinate system that we are just rotating the system, rather than just "mirroring" it.

Otherwise, whenever we actually plot our points and actually draw things in three dimensions, we need to follow this algorithm:

1. Think of what happens to the object at the origin or think of the shape in two dimensions.
2. Shift the object accordingly based on the third dimension.

1.8 Distance Between Two Points in Three-Dimensional Spacde

1.8.1 Distance in $\mathbb{R}^2$

Formula 1. Distance in $\mathbb{R}^2$

$$\text{let } d \text{ be distance}$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1.8.2 Distance in $\mathbb{R}^3$

Formula 2. Distance in $\mathbb{R}^3$

$$\text{let } d \text{ be distance}$$
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2 + (z_2 - z_1)^2}$$

Chapter 2

12.2: Vectors

2.1 Reminders

TODO

2.2 Objectives

1. Explain the difference between a vector and a point
2. Express the vector in **component form** and compute its magnitude
3. Perform elementary vector algebra using vector addition and scalar multiplication
4. Produce a unit vector with a specificed direction
5. Compute the midpoint of a line segment

2.3 Motivation

TODO

2.4 Scalar Who?

TODO

2.5 Vector Who?

One of the key **datatypes** that we utilize in **multivariable calculus** is the idea of the **vector**, which we can intuit as a **directed line segment**.

Definition 2.1: V

ctors

The vector represented by the directed line segment $\vec{AB}$ has an **initial point** A and a **terminal point** B and its **length** is denoted by $|\vec{AB}|$.
Two vectors are **equal** if and only if they have the same length and direction, regardless of the intial point.
Layman's Definition.

Definition 2.2: C

mponent Form of Two-Dimensional Vectors

If v is a **two-demensional** vector in the plane equal to the vector with initial point at the origin and terminal point $\langle v_1, v_2 \rangle$, then the **component form** of v is

$$\vec{v} = \langle v_1, v_2 \rangle$$

Layman's Definition.

Definition 2.3: C

mponent Form of Three-Dimensional Vectors

If v is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point $\langle v_1, v_2, v_3 \rangle$, then the **component form** of v is

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

Layman's Definition.

Definition 2.4: C

mponent Form

Given an **initial point** A and an **terminal point** B, the **component form** of the vector from point A to point B or $\overrightarrow{AB}$ is equal to

$$\overrightarrow{AB} = \langle B_1 - A_1,\ B_2 - A_2,\ B_3 - A_3 \rangle$$

Layman's Definition.
We can essentially think of the **component form** of a vector as the pure "movement" from an **initial point** A to a **terminal point** B. That it, it is going to be the difference in x, y, z positions between both points.

- We can almost think of this as the **slope** between two points, since the component vector is just indicating the movement from one point to another in all three directions.

2.5.1 Vector Magnitude

Recall whenever we tried to solve for the distance between two points in $\mathbb{R}^2$

- We would apply the **Pythagorean Theorem** and arrive at a formula like this

$$\text{let } a \text{ and } b \text{ be two points and let } c \text{ be the distance from point } a \text{ to point } b$$

$$c = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$$

We are actually able to apply this same principle to vectors in both $\mathbb{R}^2$ and $\mathbb{R}^3$. Whenever we are notating the **magnitude** between two vectors, we just apply the **absolute value** symbols.

$$\text{let } |\overrightarrow{AB}| \text{ and } ||\overrightarrow{AB}|| \text{ be the magnitude of the vector } \overrightarrow{AB}$$

Definition 2.5: D

stance Between Two Vectors in $\mathbb{R}^2$

$$||\overrightarrow{AB}|| = \sqrt{(B_1 - A_1)^2 + (B_2 - A_2)^2}$$

Definition 2.6: D

stance Between Two Vectors in $\mathbb{R}^3$

$$||\overrightarrow{AB}|| = \sqrt{(B_1 - A_1)^2 + (B_2 - A_2)^2 + (B_3 - A_3)^2}$$

2.5.2 Proof of the Distance Between Two Vectors in $\mathbb{R}^3$

2.5.3 Equivalency between Vectors

TODO

2.6 Position Vectors

Definition 2.7: P

sition Vectors

Layman's Definition.

2.7 Basic Vector Operations

Definition 2.8: B

sic Vector Operations

Let $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ be vectors with a scalar k

1. **Vector Addition:**
$$u + v = \langle u_1 + v_1,\ u_2 + v_2,\ u_3 + v_3 \rangle$$

2. **Scalar Multiplication:**
$$k \cdot u = \langle k \cdot u_1,\ k \cdot u_2,\ k \cdot u_3 \rangle$$

2.8 Interpreting Basic Vector Operations

2.8.1 Parallelogram Law of Additon

Chapter 3

12.3: The Dot Product

3.1 Reminders

TODO

3.2 Objectives

1. Compute the dot product of two vectors
2. COmpute the angle between two vectors in terms of the dot product
3. Algebraically determine when two vectors are **orthogonal** and be able to geometrically define what **orthongonality** refers to
4. Perform elementary vector algebra using properties of vector addition, scalar multiplication, and the dot product
5. Algebraically compute (and geometrically explain/describe) the projection of a given vector onto another, non-zero vector
6. Solve elemntary problems involving effective force and work using vector projections

3.3 Motivation

TODO

3.4 The Algebraic Definition of a Dot Product

Definition 3.1: D

t Product (Algebraic Definition)

The dot product $u \cdot v$ or "u dot v " of vectors $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ is equivalent to the **scalar**

$$u \cdot v$$

$$\Rightarrow \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle$$

$$\Rightarrow u_1 v_1 + u_2 v_2 + u_3 v_3$$

3.4.1 Dot Product Examples

3.5 The Geometric Definition of a Dot Product

Definition 3.2: D

t Product (Geometric Definition)

The dot product $u \cdot v$ is given by $u \cdot v = ||\vec{u}||||\vec{v}|| \cos \theta$

3.6 Uses for the Dot Product

Evidently, the dot product is very **powerful** because it can be both portrayed algebraically and geometrically. However, most notably, we are able to apply the dot product as a means of **detecting whether or not two given vectors u and v are perpendicular or orthogonal** as well as determining the **magnitude of a vector**

3.6.1 Perpendicularity and Orthogonality

Definition 3.3: O

thgonal Vectors

Two vectors $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ are orthogonal **if and only if**

$$u \cdot v = 0$$

3.6.2 Magnitude of a Vector Using Dot Product

Based on the algebraic properties of the dot product, we are able to evaluate the length of a vector using exclusively a dot product...

Definition 3.4: M

gnitude Using Dot Product

Recall that in order to find the magnitude of a vector v, we used the following formula

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

If we find the dot product of a vector with itself, however, we also find that

$$\vec{v} \cdot \vec{v} = v_1^2 + v_2^2 + v_3^2 = |\vec{v}|$$

$$|\vec{v}|^2 = v_1^2 + v_2^2 + v_3^2$$

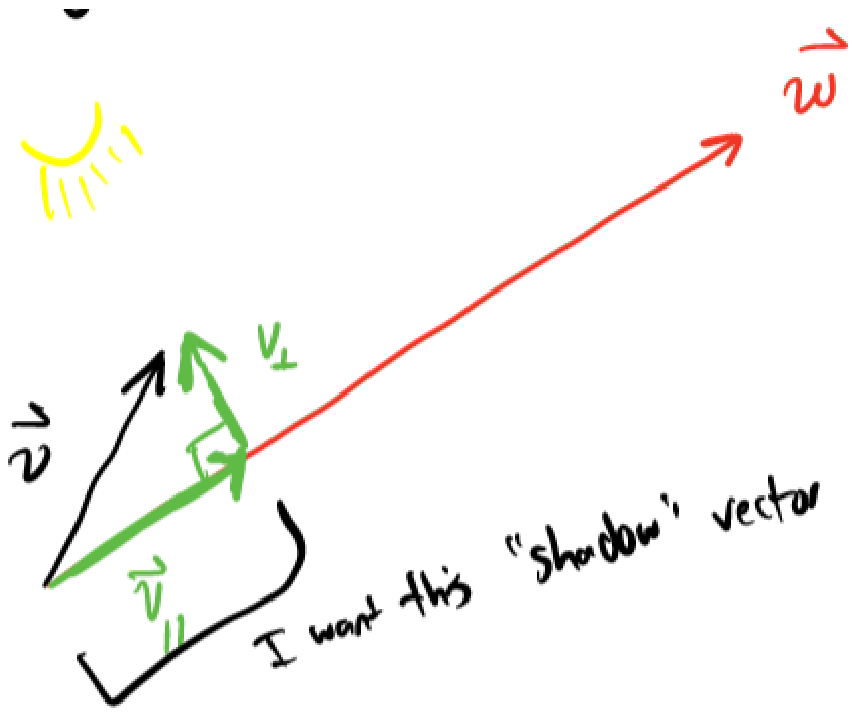
From this formula alone, we can see that there must be some correlation between the dot product as well as length. For the dot product with a vector with itself, we can find the magnitude of that vector in the direction of that vector...

But what else can dot products do?

3.7 Orthogonal Projection

3.7.1 Introduction

If we are given two vectors $\vec{v}$ and $\vec{w}$, how could we go about finding the "shadow" that vector $\vec{v}$ makes onto $\vec{w}$?



Given the above figure, we are trying to find the vector that is parallel (goes in the same direction as vector $\vec{w}$).

- Notice how, much like in anything that is related to **multivariable calculus**, that we can generally form a **triangle**.
- In this case, we are able to create a triangle based on our original vector $\vec{v}$ that has a component that is perpendicular to $\vec{w}$ but also parallel to $\vec{w}$. This parallel vector will the basis of this concept of **orthogonal projection.**

  Let us declare some variables first:

  $$\text{let } \overrightarrow{v_{||}} \text{ be a leg of } \vec{v} \text{ that is parallel to } \vec{w}$$

  $$\text{let } \overrightarrow{v_{\perp}} \text{ be a leg of } \vec{v} \text{ that is perpendicular or } \textbf{orthogonal} \text{ to } \vec{w}$$

  Given these two variables as well as their relationships between $\vec{v}$ and $\vec{w}$, we can make some assertions.

1. First, we know that $\vec{v}_{||}$, since it is parallel to $\vec{w}$ must be a **scaled down version of** $\vec{w}$. This would make it a **scalar multiple of** $\vec{w}$. Therefore, if we let a be some scalar, we know that

   $$\overrightarrow{v_{||}} = a \cdot \vec{w}$$

2. Second, we know that $\vec{v}_{\perp}$ is **orthogonal** to vector $\vec{w}$, therefore we know that the dot product between $\vec{v}_{\perp}$ and $\vec{w}$ must be equal to 0, since according to the definition of a dot product, two vectors are orthogonal if and only if their dot product is equal to 0.

   $$\overrightarrow{v_{\perp}} \cdot \vec{w} = 0$$

Using all of this information, we are able to answer the question, what is the value of a? How can we evaluate for the leg of the vector $\vec{v}$ that is parallel to vector $\vec{w}$?

3.7.2 Orthogonal Projection Proof

$$\vec{v} \cdot \vec{w} = \overrightarrow{v_{||}} \cdot \vec{w} + \overrightarrow{v_{\perp}} \cdot w$$

Since we know that $\overrightarrow{v_{\perp}} \cdot \vec{w}$ are perpendicular to each other, then we know that they equal 0.

$$\Rightarrow \vec{v} \cdot \vec{w} = \overrightarrow{v_{||}} \cdot \vec{w}$$

Now, we are able to replace $\overrightarrow{v_{||}}$ with the fact that we know that $\overrightarrow{v_{||}} = a \cdot \vec{w}$ or that the parallel component of vector $\vec{v}$ is just a scaled version of vector $\vec{w}$.

$$\Rightarrow \vec{v} \cdot \vec{w} = (a \cdot \vec{w}) \cdot \vec{w}$$

By applying vector properties, we know that we can just **move the parentheses** around.

$$\Rightarrow \vec{v} \cdot \vec{w} = a \cdot (\vec{w} \cdot \vec{w})$$

$$\Rightarrow a = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}$$

$$\Rightarrow \overrightarrow{v_{||}} = a \cdot w$$

$$\Rightarrow \overrightarrow{v_{||}} = \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right) \cdot \vec{w}$$

From this, we know that $\vec{v_{||}}$, which we also know as **projection of vector** v **onto vector** w is given by the following formula:

$$proj_w v = \overrightarrow{v_{||}} = \left(\frac{v \cdot w}{w \cdot w}\right) \cdot \vec{w}$$

3.7.3 Vector Projection

Definition 3.5: V

ctor Projection (or the Vector Component)

When given two vectors $\vec{u}$ and $\vec{v}$, the vector projection of u onto v is the vector produced when one vector u is resolved into two component vectors, one that is parallel to the second vector v and one that is perpendicular to the second vector v. The vector that is parallel to the "target" vector is the **vector projection**.

3.7.4 Vector Projection Intuition

Based on the formula, recall how the vector projection can be separated into two different terms: the a scalar and the normalization of the vector being projected onto v.

$$proj_v u = \left(\frac{u \cdot v}{||\vec{v}||^2}\right)\vec{v}$$

$$\Rightarrow proj_v u = \left(\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||}\right) \cdot \left(\frac{\vec{v}}{||\vec{v}||}\right)$$

$$\Rightarrow proj_v u = (\text{some scalar}) \cdot (\text{the unit vector or the } \textbf{direction} \text{ of } \vec{v})$$

The idea here is that we are creating a vector that is parallel (or technically overlapping) the vector being projected on. Although the vector is not necessarily the same magnitude of the vector we are projecting, we can safely say that the second term will always be the **direction/unit vector of the vector being project onto**.

3.7.5 The Angle Between Two Vectors

e Angle Between Two Vectors (Theorem 1) The angle θ between two nonzero vectors $u = \langle u_1, u_2, u_3 \rangle$ and $v = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{||\vec{u}|||\vec{v}||}\right)$$

$$\Rightarrow \cos^{-1}\left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{||\vec{u}|||\vec{v}||}\right)$$

3.7.6 The Proof of the Angle Between Two Vectors

TODO

3.8 Dot Product Properties and Vector Properties

Properties of Dot Product and Vectors.
Let $u = \langle u_1, u_2, u_3 \rangle$, $v = \langle v_1, v_2, v_3 \rangle$, and c be a scalar

1. $u \cdot v = v \cdot u$
2. $ku \cdot v = u \cdot kv = k(u \cdot v)$
3. $u \cdot (v + w) = u \cdot u \cdot w$
4. $u \cdot u = ||\vec{u}||^2$
5. $0 \cdot u = 0$

3.9 Proofs of the Properties of Dot Products

TODO

3.9.1 Scalar Component and the Scalar Projection

Definition 3.6: S

alar Components/ Scalar Projection

Notation.

$$comp_v u,\ \text{or the "scalar component of u onto v"}$$

Formula.

$$comp_v u = \frac{u \cdot v}{||v||}$$

Explanation.

Let $comp_v u$ be the **scalar component of u onto v**

The magnitude of a vector that is projected onto another vector. If we divide the projection fromula into two terms: the magnitude of the resulting vector and the direction of the resulting vector (which is parallel to the vector being projected onto), we get the following formula:

$$proj_v u = (\text{magnitude of resulting vector}) \times (\text{direction of } \vec{v})$$

$$\Rightarrow proj_v u = \left(\frac{u \cdot v}{||\vec{v}||}\right) \cdot \left(\frac{v}{||v||}\right)$$

$$comp_v u = \left(\frac{u \cdot v}{||v||}\right)$$

3.9.2 Scalar Component Intuition

TODO

3.10 Summary

In this chapter, we learned about the idea of the **Dot Product**, as well as both of its algebraic and geometric representations. Most importantly, however, we leanred how to apply the dot product to various different sitautions.

1.

3.10.1 Important Formulas

Formulas Involving Dot Product

1. Algebraic Formulation of Dot Product

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

2. Geometric Formulation of Dot Product

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \cdot ||\vec{v}|| \cdot \cos\theta$$

3. Dot Product to Angle Formula

$$\cos\theta = \frac{u \cdot v}{||\vec{u}|| \cdot ||\vec{v}||}$$

4. Vector Projection Formula

$$proj_v u = \frac{u \cdot v}{v} \cdot \frac{v}{||\vec{v}||}$$

5. Scalar Projection (or Scalar Component) Formula

$$comp_v u = \frac{u \cdot v}{||\vec{v}||}$$

Chapter 4

12.4: The Cross Product

4.1 Reminders

TODO

4.2 Objectives

1. Compute the cross product of two given vectors using **determinants**
2. Geometrically interpret the magnitude and direction of the cross product of two given vectors
3. Perform elementary vector operations
   - Vector Addition
   - Scalar Multiplication
   - Dot Product
   - Cross Product

4.3 Recall

Recall that in the last lecture, we discussed the **dot product**, which had both an **algebraic** definition as well as a **geometric** definition

- Whenever we were finding the algebraic dot product, we were just multiplying the components and finding their sum
  $$\vec{v} \cdot \vec{u} \;=\; v_1u_1 + v_2u_2 + v_3 + u_3$$
  We learned that this scalar actually represented a lot more than it let on. In fact, the **scalar that results from a dot product** actually represents the product of both vectors' magnitudes as well as the cosine of the angle between them:
  $$\vec{v} \cdot \vec{u} \;=\; ||\vec{v}|| \cdot ||\vec{u}|| \cdot \cos\theta$$
  where, of course, θ represents the angle between the vectors v and u.

  We also see the angle between two vectors θ represented by the following formula
  $$\cos\theta = \frac{\vec{v} \cdot \vec{u}}{||\vec{v}||||\vec{u}||}$$

- We also learned about **projection**, which is the idea of taking a vector v and then imposing it onto another vector u. Imagine that we were just "flattening" a vector onto another, preserving its length/magnitude while maintaining another vector's direction.

$$proj_v u = \frac{\vec{v} \cdot \vec{u}}{\vec{v} \cdot \vec{v}} \vec{v}$$

which can also be represented as

$$\Rightarrow \left(\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||} \right) \cdot \left(\frac{1}{||\vec{v}||} \right) \vec{v}$$

where the first term $\frac{\vec{v} \cdot \vec{u}}{||\vec{v}||}$ represents the **scalar component of v** and the second term $\left(\frac{1}{||\vec{v}||} \right) \vec{v}$ represents the **unit vector of** $\vec{v}$.

4.4 Motivation

In the former lectures, we have learned about **vectors** as well as numerous operations to perform ont hem

- Vector Addition/Subtraction $\longrightarrow$ vectors
- Scalar Multiplication $\longrightarrow$ vector
- Dot Product $\longrightarrow$ scalar

What if there was such an operation that we were able to **multiply** two vectors together?

4.5 Cross Product

Definition 4.1: C

oss Product

The cross product of two vecturs $\vec{v}$ and $\vec{u}$, $(\vec{v} \times \vec{u})$, is the geometrically defined by the following vector:

$$\vec{v} \times \vec{u} := ||\vec{v}||||\vec{u}|| \cdot \sin \theta \cdot \vec{n}$$

where vector $\vec{n}$ is the **normal unit vector** perpendicular to the plane spanned by $\vec{v}$ and $\vec{u}$, which is chosen accordingly by the right hand rule.

Book Definition. The cross product $u \times v$ or "u cross v" is the vector

$$u \times v = (|u||v| \sin \theta) n$$

Layman Definition.
The cross product is the reuslting vector $\vec{n}$, which is a vector that is orthogonal toe the plane of which $\vec{v}$ and $\vec{u}$ occupy, and of which whose magnitude is determined by the product of the magnitude of $\vec{v}$ and $\vec{u}$, and the value of the angle betwen the two vectors that span the plane.

Definition 4.2: P

rallel Vectors

Nonzero vectors u and v are parallel **if and only if** $u \times v = 0$

TODO: Insert Picture from Tablet

4.5.1 Observations of the Cross Product

TODO

4.6 Properties of the Cross Product

Definition 4.3: P

opeties of the Cross Product

If u, v, and w are any vectors and r, s are any scalars, then

1. $ru \times sv = rs(u \times v)$
2. $u \times (v + w) = u \times v + u \times w$
3. $v \times u = -(u \times v)$
4. $(v + w) \times u = v \times u + w \times u$
5. $0 \times u = 0$
6. $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$

4.7 Parallelograms and the Magnitude of a Cross Product

Definition 4.4: A

ea of a Parallelogram bounded by u and v

Given that n is a unit vector, we can think of the magnitude of $u \times v$ as

$$|u \times v| = |u||v||\sin\theta||n| = |u||v|\sin\theta$$

4.8 Algebraically Evaluating the Cross Product

Definition 4.5: C

lculating the cross Product as a Determinant

If $u = u_1 i + u_2 j + u_3 k$ and $v = v_1 i + v_2 j + v_3 k$, then

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\Rightarrow u \times v = \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} i - \begin{pmatrix} u_1 & v_3 \\ v_1 & v_3 \end{pmatrix} j + \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} k$$

Chapter 5

12.5: Lines in Space (04/07/23 - 04/10/23)

5.1 Reminders

-

5.2 Objectives

In this section, we want to be able to do the following:

- Be able to write equations for lines and line segments in $\mathbb{R}^3$ space using scalar and vector products

5.3 Motivation

Recall in the last lessons we have been learninge exclusively about vectors as well as how we are able to manipulate them in order to get different objects in space.

- For example, we have learned that the **Dot Product** is a function that takes in two vectors $\vec{v}$ and $\vec{u}$ and returns some scalar.

$$\vec{v} \cdot \vec{u}$$

Much like anything in multivariable calculus, we actually have to consider that a lot of the concepts we learn are **multi-dimensional**. In this case, we must understand that the **dot product** has both a **algebraic** and **geometric** definition.

Algebraically, we can think of the dot product as the following equation:

$$\text{let } \vec{v} = \langle v_1, v_2, v_3 \rangle$$

$$\text{let } \vec{u} = \langle u_1, u_2, u_3 \rangle$$

$$\Rightarrow \vec{v} \cdot \vec{u} = v_1 u_1 + v_2 u_2 + v_3 u_3$$

Geometrically, we can think of the **dot product** as the **angle between two vectors, but scaled based on the magnitude of the vectors.**

$$\text{let } \vec{v} = \langle v_1, v_2, v_3 \rangle$$

$$\text{let } \vec{u} = \langle u_1, u_2, u_3 \rangle$$

$$\Rightarrow \vec{v} \cdot \vec{u} = ||\vec{v}|| \cdot ||\vec{u}|| \cdot \cos\theta$$

Of course, there are a few implications and use cases of the dot product both algebraically and geometrically

- If $\vec{v} \cdot \vec{u} = 0$, then we know that the vectors v and u are **orthogonal or perpendicular**, which makes sense, since if we were to find some value $\cos\theta = 0$, then we would have $\theta = \frac{\pi}{2}$, which is of course, a right angle.
-

TODO: Include Graphics that Visualizes this relationship

5.3.1 Lines in $\mathbb{R}^2$

Definition 5.1: L

nes in $\mathbb{R}^2$

Layman's Definition.

5.3.2 Intuition Behind Lines in $\mathbb{R}^2$

5.3.3 Lines in $\mathbb{R}^3$

Definition 5.2: L

nes in $\mathbb{R}^3$

A **vector equation for the line** L through the point P_0 where $P_0(x_0, y_0, z_0)$ and parallel to v is given by

$$\vec{r}(t) = r_0 + t \cdot \vec{v} \quad \text{where } -\infty < t < \infty$$

where r represents the position vector of a point $P(x, y, z)$ on the line L from the point P_0 and r_0 is the position vector of $P_0(x_0, y_0, z_0)$. We also let $\vec{v}$ be some vector that is **parallel** to our desired direction. **Layman's Definition.**

5.3.4 Intuition Behind $\mathbb{R}^3$ Lines

TODO

5.4 Parametric Equations of a Line

Given any vector equation of a line, we notice that everything can be put into terms of each of the individual directions x, y, z.

Definition 5.3: P

rametric Equations of a Line

The **standard parameterization of the line** through $P_0(x_0, y_0, z_0)$ parallel to $v = v_1 i + v_2 j + v_3 k$ is

$$x = x_0 + t \cdot v_1 \quad \text{where} \quad -\infty < t < \infty$$

$$y = y_0 + t \cdot v_2 \quad \text{where} \quad -\infty < t < \infty$$

$$z = z_0 + t \cdot v_3 \quad \text{where} \quad -\infty < t < \infty$$

Layman's Definition.

5.5 Solving Lines in Space Problems

5.5.1 Distance from a Point to a Line in Space

Definition 5.4: D

stance from a Point S to a Line Through P Parallel to v

$$d = \frac{|\overrightarrow{PS} \times \vec{v}|}{|\vec{v}|}$$

5.5.2 2-D Lines versus 3-D Lines

Whenever we were defining lines in $\mathbb{R}^2$, we were always thinking of these lines as an a set of points with two defining characteristics:

- Some point P_0 that the line intersected
- Some slope or direction m that the line went in

With the intuition that lines in two-dimensional space were defined by the points they intersectd as well as their **slope**, which we can think of as the **change in x and the change in y over time**, we can apply the same general principals to vectors in three-dimensional space.

5.5.3 3-D Lines

Three dimensional lines, by comparison, are also defined by some point that the line goes through as well as a direction in which the line continues infinitely. Instead of having a slope, however, we like to think of the "slope" of a three-dimensional line as a "parallel" vector, that doesn't necessarily represent the actual line, but the **behavior** of our current line.

A three-dimensional line is defined by the following terms:

- An **initial point** P_0 or just P.
- A **vector** that defines the line's **direction** and **behavior**, $\overrightarrow{P_0P}$ or $\overrightarrow{PQ}$, where P_0 and represents the initial point and P and Q represent **any point on the line**.

5.5.4 3-D Line Summary

Essentially, much like hte **two-dimensional line**, a **three-dimensional line** is defined by some point P that the line goes through, as well as a **directional vector** that starts from that initial point P and extends to any point Q on the line $\overrightarrow{PQ}$.

Chapter 6

12.5: Planes in Space (04/07/23 - 04/10/23)

6.1 Reminders

6.1.1 MATH_226 Reminders (as of Saturday, April 22, 2023)

1. **Written Homework 4: Power Series** is due on **Monday, April 24, 2023.**

6.1.2 MATH_230 Reminders (as of Saturday, April 22, 2023)

1.

6.2 Objectives

1. Determine vector and component equations
2. Produce non-zero vectors normal to a given plane
3. Compute the distance from a point to a plane in space
4. Determine whether two given planes coincide, intersect in a line, or are parallel

6.3 Motivation

In the last seciton, we learned about **lines in space**, how they're portrayed in two dimensions, as well as how they're portrayed in three dimensions. We analyzed the relationship between lines in space as ewll as

6.4 Planes in R^3

6.4.1 Vector Equation of a Plane

Definition 6.1: V

ctor Equation of a Plane

The plane through $P_0(x_0, y_0, z_0)$ that is normal to $n = Ai + Bj + Ck$ is given by the equation

$$n \cdot \overrightarrow{P_0P} = 0$$

Layman's Definition.

6.4.2 Component Equation of a Plane

Definition 6.2: C

mponent Equation of a Plane

The plane through $P_0(x_0, y_0, z_0)$ that is normal to $n = Ai + Bj + Ck$ is given by the equation

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Layman's Definition.

6.4.3 Simplified Component Equation of a Plane

Definition 6.3: S

mplified Component Equation of a Plane

The plane through $P_0(x_0, y_0, z_0)$ that is normal to $n = Ai + Bj + Ck$ is given by the equation

$$Ax + By + Cz = D$$

Layman's Definition.

6.4.4 Intuition Behind Planes

TODO

6.5 Types of Problems and How to Solve Them

6.5.1 Determining the Intersection Between Two Planes

Algorithm. *Line of Intersection Between Two Planes*

Given two planes P_1 and P_2

1. Determine what the normal vectors of P_1 and P_2 are
2. Find the vector parallel to the vector that is orthogonal to the normal vectors of P_1 and P_2
3. Evaluate a point that exists on the line by literally plugging in any value into the system of equatiosn formed by the two different planes.
   (a) The idea here is that, since a line expands infinitely in both directions, that we must be able to find some point in which the x coordinate is 4, for example. There is no rhyme or reason to anything here, literally just plug in some numbers and you will most likely be fine.
4. Write equation using the direction vector (which is the vector that is parallel to the vector normal to the normal vectors of P_1 and P_2).

6.5.2 Line of Intersection Between Two Planes Intuition.

Whenever we are determining the line of intersection between two planes, it is important to remember how this geometrically works in relation to what we know about planes.

1. Recall that planes are represented by two different components

(a) Some point that exists on the plane

(b) Some vector $\vec{n}$ that is parallel to the vector that is normal to the plane

2. Using this knowledge, we are able to make some implications

   (a) First, it is important for us to remember that whenever two planes are parallel, then their normal vectors must also be parallel to each other

   (b) Likewise, if two planes were to intersect, that must mean that their normal vectors **must also be intersecting**. We can use this information to actually find the direction in which the two vectors intersect (imagine that the two vectors are intersecting in two-dimensional space and that we are trying to find the vector that goes "through" that point of intersection).

   (c) Of course, in order to find this point of intersection, we think of the cross product, since through the cross product, we are able to find a vector that is normal to the plane spanned by the two constituent vectors, which, in this case, are the two vectors that are normal to the planes.

   (d) This normal vector will serve as the direction vector for our line of intersection.

6.5.3 Finding the Equation of a Plane Given Three Points

Definition 6.5.1: Algorithm: Finding the Equation of a Plane Given Three Points

1. Create two vectors from the three given points $\overrightarrow{PQ}$ and $\overrightarrow{PO}$

2. Find the cross product between the two newly created vectors $\overrightarrow{PQ}$ and $\overrightarrow{PO}$

$$\overrightarrow{PQ} \times \overrightarrow{PO}$$

3. Let the result of the cross product be the vector parallel to the normal vector spanned by the plane $\vec{n}$

4. Create the equation using any given point plus the vector parallel to the normal vector $\vec{n}$

6.5.4 Intuition for Finding the Equation of a Plane Given Three Points.

When finding the equation of a plane given three points, its important to consider what exactly we need to define a plane.

1. First, we need a vector that is parallel to the vector that is normal to the plane $\vec{n}$

2. Second, we just need a point that exists on the plane

When we are given three points we are able to meet all of these different requirements.

1. With three points, we are able to create two vectors that could exist on the plane, and then we are able to find their **cross product** and let this result be the vector that is parallel to the vector that is normal to the plane $\vec{n}$.

2. The only thing that remians is just using a point on the plane, and we are literally given three points of potential comparison.

6.5.5 Determining the Distance from a Point to a Plane

Definition 6.5.2: Algorithm: Distance from a Point to a Plane

1. Determine a position vector between the point in space and a point on the plane.
2. Using the position vector from the point in space and the point on the plane, find the **projection of the position vector onto the vector that is parallel to the normal vector of the plane** $\vec{n}$.

$$\text{let } P \text{ be a point in space}$$

$$\text{let a plane } M \text{ be defined as}$$

$$M = \vec{Q} + \vec{n}t$$

$$\overrightarrow{PQ} = \langle Q_1 - P_1, Q_2 - P_2, Q_3 - P_3 \rangle$$

$$proj_n \overrightarrow{PQ} = \frac{\overrightarrow{PQ} \cdot \vec{n}}{||\vec{n}||||\vec{n}||} \vec{n}$$

Layman's Definition.
The idea here is that, much like other distanc eproblems, we want to find any distance from the point to the plane, then we just want to **put it into terms of the normal vector**.

6.5.6 Intuition Behind Determining the Distance from a Point to a Plane

When we are given a distance question between anything– whether it be a point to a line, a line to a plane, or a plane to a plane, it is important to **always** consider the idea of **projectoin.** Projection will always be the basis of finding the distance, since, at the end of the day, we want to be able to take *any distance we find between the objects*, and project it onto the least possible distance– **the normal vector.**

Chapter 7

(TODO) 11.6: Conic Sections (04/12/23)

7.1 Reminders (as of 04/23/23)

7.1.1 MATH_230-1 Reminders

- **MyLab Math 9: Curves in Space and Their Tangents** is due **tonight, April 23, 2023**.
- **Midterm 1** is on **Tuesday, April 25, 2023**.

7.1.2 MATH_226-0 Reminders

- **MyLab Math 11: Manipulation of Series** is due on **Tuesday, April 25, 2023**.
- **Written Homework 4: Power Series** is due **tomorrow, Monday April 24, 2023**

7.2 Objectives

1. Be able to algebraically and geometrically understand the three conic sections
2. Be able to graph all three conic sections

7.3 Motivation

In order to

7.4 Circles

Definition 7.1: C

rcles

$$x^2 + y^2 = 1$$
$$(x-h)^2 + (y-k)^2 = r^2$$

7.5 Ellipses

Definition 7.2: E

lipses

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

7.6 Parabolas

Definition 7.3: P

rabolas

$$y = x^2$$
$$x = y^2$$

7.7 Hyperbolas

Definition 7.4: H

perbolas

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$
$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Chapter 8

(TODO) 12.6: Cylinders and Quadric Surfaces (04/14/23)

8.1 Reminders (as of 04/23/23)

8.1.1 MATH_230-1 Reminders

- **MyLab Math 9: Curves in Space and Their Tangents** is due **tonight, April 23, 2023**.
- **Midterm 1** is on **Tuesday, April 25, 2023**.

8.1.2 MATH_226-0 Reminders

- **MyLab Math 11: Manipulation of Series** is due on **Tuesday, April 25, 2023**.
- **Written Homework 4: Power Series** is due **tomorrow, Monday April 24, 2023**

8.2 Objectives

1. Sketch the graph of various cylinders
2. Graph the **six** quadric surfaces by hand
3. Understand the usefulness of the coordinate plane traces as well as how to find them

8.3 Motivation

8.4 The Six Types of Quadric Surfaces and Cylinders

8.4.1 Ellipsoid

Definition 8.1: E

lipsoid

TODO

8.4.2 Hyperboloid of One Sheet

Definition 8.2: H

perboloid of One Sheet

TODO

8.4.3 Hyperboloid of Two Sheets

Definition 8.3: H

perboloid of Two Sheets

TODO

8.4.4 Elliptic Paraboloid

Definition 8.4: E

liptic Paraboloid

TODO

8.4.5 Hyperbolic Paraboloid

Definition 8.5: H

perbolic Paraboloid

TODO

8.4.6 Elliptical Cone

Definition 8.6: E

liptical Cone

TODO

Chapter 9

(TODO) 11.3: Polar Coordinates (04/17/23)

9.1 Reminders (as of 04/23/23)

9.1.1 MATH_230-1 Reminders

- **MyLab Math 9: Curves in Space and Their Tangents** is due **tonight, April 23, 2023**.
- **Midterm 1** is on **Tuesday, April 25, 2023**.

9.1.2 MATH_226-0 Reminders

- **MyLab Math 11: Manipulation of Series** is due on **Tuesday, April 25, 2023**.
- **Written Homework 4: Power Series** is due **tomorrow, Monday April 24, 2023**

9.2 Objectives

1. Be able to differentiate between poloar coordinates and Cartesian coordinates
2. Be able to relate polar coordinates to Cartesian coordinates
3. Graph polar coordinate functions

9.3 Motivation

In the previous sections, we have learned about different ways to depict sets of coordinates in space

- Vectors
- Lines
- Planes

However, we have done all of this using the **Cartesian or Rectangular Coordinate System**, which is our system that determines movements in graphs through the x, y, and z dimensions. There are, of course, other ways to explore this movement, though, such as through the **Polar System**, which, instead of basing two-dimensional movement as movement in the x and y, dimensions, calculates movement as a function of magnitude r and angle θ. We will find that these concepts of r and θ can be applied to normal rectangular system equations and, in investigating the relationships between the Cartesian and Polar Systems, we are able to develop functions direclty out of rectangular movement.

9.4 Polar Coordinate Movement

Definition 9.1: P

lar Coordinate System

Fix an origin O (which we call the pole) and an create an **initial ray** from the pole O (which will just be a ray across the positive x-axis). We are able to locate any point P in the coordinate system by assigning it to a polar coordinate pair (r, θ) in which r gives the directed distance from O to P, and θ gives the directed angle from the initial ray to the ray $\overrightarrow{OP}$. We label this point P as $P(r, \theta)$.

9.5 Switching Between Polar Coordinates and Cartesian Coordinates

Definition 9.2: C

mmon Cartesian/Polar Translations

We are able to switch between polar and cartesian coordinates based on the following known transformations

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

9.5.1 Common Strategies for Switching Between Polar Coordinates

Whenever we are trying to convert Cartesian equations into Polar Equations, we will find that there are a number of strategies to convert an equation from one coordinate system to the next.

For example, what if you had to convert

$$r\sin\theta \text{ to Cartesian}$$

$$r^2\sin^2\theta \text{ to Cartesian}$$

$$y = x^2 \ \forall x \neq 0$$

9.5.2 Direct Substitution

Example ((1))**.** *Convert the following Polar Equation into a Cartesian Equatino*

$$r\sin\theta = 3$$

*When we approach this problem, it is important to **recall** the standard conversions we are given.*
Recall.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2 + y^2 = r^2$$

*From this, we can just **directly subsitute** some of these converisons.*

$$r\sin\theta = 3$$

$$\Rightarrow y = 3$$

9.5.3 Multiplying r on Both Sides

Example (2)**.** *Convert the following Polar Equation into a Cartesian Equation*

$$\sin\theta = r\cos^2\theta$$

Unfortunately here, we don't have a particularly easy way of substituting the knowledge we know. After all, all we can really think of doing is separating the term on the right into something we know, but then that would leave us with a very messy looking equation. THerefore, what we are able to do (or what we should consider) doing is **multiplying both sides by** r.

$$\Rightarrow \sin\theta = r\cos^2\theta$$

$$\Rightarrow r\cdot\sin\theta = r\cdot r\cos^2\theta$$

$$\Rightarrow r\sin\theta = r^2\cos^2 theta$$

From here, now we have a lot of information in which we are able to just **substitute what we know**.

$$\Rightarrow r\sin\theta = (r\cos\theta)^2$$

$$\Rightarrow y = x^2$$

9.5.4 Tricks in Summary

Evidently, in order to succeed in converting Polar and Cartesian Equations, we just have to look out for the following terms:

1. $r\cos\theta = x$
2. $r\sin\theta = y$
3. $\tan\theta = \frac{y}{x}$
4. $r^2 = x^2 + y^2$

Example (3 (harder))**.**

$$\tan^2\theta - 1 = \frac{6}{r}\tan\theta\sec\theta + \frac{4}{r}\sec\theta$$

In this problem, we have an even worse initial state than the previous problem, since we are confronted with a new term, **secant**. However, as with most problems in algebra and calculus, the best way to solve something you don't know is to *break it into something you do know.*

In this, case we are going to first **substitue all of the secants with reciprocals of** $\cos\theta$.

$$\Rightarrow \tan^2\theta - 1 = \frac{6}{r}\tan\theta\frac{1}{\cos\theta} + \frac{4}{r}\frac{1}{\cos\theta}$$

Then, we are just able to rewrite $\tan^2\theta$ as a functino of sine and cosine.

$$\Rightarrow \frac{\sin^2}{\cos^2} - 1 = \frac{6}{r}\tan\theta\frac{1}{\cos\theta} + \frac{4}{r\cos\theta}$$

$$\Rightarrow r\left(\frac{\sin^2\theta}{\cos^2\theta} - 1\right) = \left(\frac{6}{r}\tan\theta\frac{1}{\cos\theta} + \frac{4}{r\cos\theta}\right)\cdot r$$

$$\Rightarrow r\sin^2\theta - r = 6\tan\theta\cos\theta + 4\cos\theta$$

$$\Rightarrow r\sin^2\theta - r = 6\sin\theta + 4\cos\theta$$

$$\Rightarrow r^2\sin^2\theta - r^2 = 6r\sin\theta + 4r\cos\theta$$

$$\Rightarrow y^2 - x^2 - y^2 = 6y + 4x$$

$$\Rightarrow y^2 - 6y - x^2 - 4x = 0$$

$$\Rightarrow (y-3)^2 - 9 - (x+2)^2 + 4 = 0$$

$$\Rightarrow (y-3)^2 - (x+2)^2 = 5$$

Chapter 10

13.1: Curves in Space and their Tangents (04/19/23)

10.1 Reminders

10.1.1 MATH_230-1

1. **MyLab Math 8: Polar Coordinates** is going to be due **tomorrow, Thursday, April 19, 2023**.
2. **Written Homework 2** is due **tonight, Wednesday, April 19, 2023**.
3. **MyLab Math 9: Curves in Space and Their Tangents** is going to be due on **Sunday, April 23, 2023**.

10.1.2 MATH_226-0

1. **MyLab 10: Radius and Interval of Convergence** is going to be due on **Friday, April 19, 2023**.

10.2 Objectives

In this section, we want to be able to understand how to portray curves as they exist in space. Most importantly we want to be able to understand the concept of **parameterized equations** and **vector-valued functions** and how we are able to manipulate them and understand them in space.

1. Analyze a vector-valued function using limits, continuity, and the derivative
2. Interpret the derivative of a vector-valued function
    - Interpret the derivative of a vector-valued function in the context of particle motion
3. Analyze the velocity vectror of a particle to determine a particle's speed, direction, and accelration
4. Fluently apply differentiation rules to vector-valued functions
5. Show that the output of a vector-valued function of constant length is orthogonal to its derivative.

10.3 Recall

10.4 Motivation

10.5 Vector-Valued Functions and the Parameterization of Plane Curves

Definition 10.1: V

ctor-Valued Functions

A function on a domain set D that assigns a **vector** for every element in D.

$$r(t) = \overrightarrow{PQ} = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$$

$$\Rightarrow \langle f(t), g(t), h(t) \rangle$$

The functions $f(t)$, $g(t)$, and $h(t)$ are known as **component functions**, notice that this is because they are **components** of the vector (recall that the **component form** of a vector is just the vector in terms of the standard unit vectors).
For now, the domain of these functions will just be **real numbers**, while the actual graph of the function will represent a **curve in space**.

Layman's Definition.
Essentially, a **vector-valued function** is just a function that takes in some number $\in \mathbb{R}$ and outputs a **vector**. By contrast, **vector-valued functions** or **vector functions** are contrasted by **scalar functions**, which are functions on the domain set $D \in \mathbb{R}$ that have a range $\mathbb{R} \in \mathbb{R}$.

Definition 10.2: P

rametric Equations

If x and y are given as continuous functions

$$x = f(t); \;\; y = g(t)$$

over an interval I of t-values, then the set of points

$$(x, y) = (f(t), g(t))$$

defined by these equations is a **parametric curve**. From this, we know that the equations

$$x = f(t); \;\; y = g(t)$$

are **parametric equations** for this parametric curve.
Given the closed interval $I = [a, b]$ the point A(f(a), g(a)) is known as the **initial point** of the parametric curve, while point B(f(b), g(b)) is known as the **terminal point** of the parametric curve.

10.6 Methods for Sketching Parametric Curves

10.6.1 Method 1: Parametric Equation Tables

TODO

10.6.2 Method 2: Re-Writing Parametric Equations in Terms of x, y, z

TODO

Definition 10.3: C

mponent Functions

A **component function** is just a **component** of a vector $\vec{v}$ where, instead of being some constant, is a **function**.

$$\vec{v} = \langle f(t), g(t), h(t) \rangle$$

This is in contrast to vector $\vec{u}$, which has components that are constants.

$$\vec{u} = \langle 1, 2, 3 \rangle$$

Layman's Definition.
A component function is a component of a vector (representing the "movement"of a vector), except, instead of being a constant number, is just a function.

10.6.3 Examples of Vector-Valued Functions

Example (1)**.** *Graph the vector-valued function*

$$r(t) = \cos t\hat{i} + \sin t\hat{j} + t\hat{k}$$

Whenever we are looking at this function, we first just need to **break everything down into its components**.

Remember, that this is a **vector-valued function**, so we can think of each **component function** as being representative of what the vector does for a given t. In this case, we can contextualize the function like so

$$x = \cos t$$

$$y = \sin t$$

$$z = t$$

With some algebraic manipulation with the component function of x and the component function of y, we can see that our vector-valued function actually satisfies the equation of a circle if we ignore the behavior in the z-axis.

$$r(t) = \cos t\hat{i} + \sin t\hat{j}$$

$$\Rightarrow x = \cos t;\ y = \sin t$$

$$x^2 + y^2 = \cos^2 t + \sin^2 t$$

$$\Rightarrow \cos^2 t + \sin^2 t = 1$$

We can see that a little algebraic manipulation demonstrates that as t increases, the comopnent functinos in the x and y directions are just tracing a circle, which we deduced using the values of the component functions as well as using the trignometric identity $\sin^2\theta + \cos^2\theta = 1$.

Therefore, once we start factoring in the behavior of the function for the z-direction, we see, that the vector-valued function will be tracing out the shape of a circle, but will be increasing in the z direction at the same time, thereby creating a **helix shape**. The helix is just tracing out a **circular cylinder**, where a cylinder just represents any function that is extended in some third dimension.

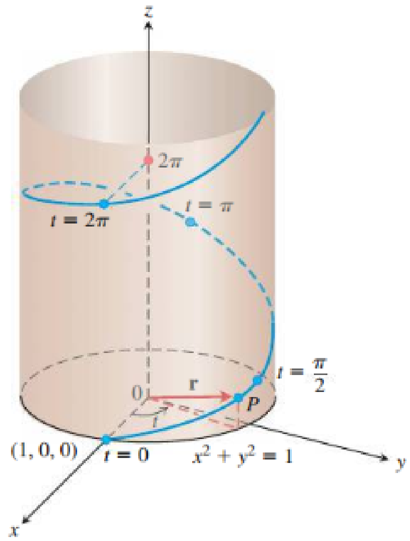


FIGURE 13.3 The upper half of the helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$ (Example 1).

Example (2)**.** *(Taken from Practice Midterm D) Let C be a conic in $\mathbb{R}^3$ defined by the following system of equations:*

$$\frac{(x-1)^2}{9} + \frac{(z-2)^2}{25} = 1$$
$$y = 3$$

1. *Describe C qualitatively: include what type of conic it is, what its center is, and how it is situated in $\mathbb{R}^3$.*
2. *Give a vector parameterization $r(t)$ for C. Include explicit bounds $a \leqslant t \leqslant b$ ensuring that the entire curve is parameterized.*

10.7 Parametric Equations of Conic Sections

We are able to actually generate functions of geometric shapes using parametric equations. Sure, we know that shapes such as circles, ellipses, and hyperbolas are defined by the following equations, respectively

$$x^2 + y^2 = c^2$$
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1^2$$
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1^2$$

But these aren't **functions**, as the movement of x, for example, isn't going to affect the movement in the y or the z direction. Therefore, how are we able to actually rewrite these familiar geomtetric formulas as **parametric equations?**

- We represent the movement of each dimension as through **polar coordinates**.
- Functions with polar coordinates generally just have some constant that represents the "radius" r, but are all dependent on the angle θ. Therefore, if we rewrite everything as a polar coordinate function, then we can thereby make a **parametric equation of any geometric shape**.

10.7.1 Parametric Equation of a Circle

Definition 10.4: P

rametric Form of a Circle

Recall.
The standard form of a circle is given by the equation

$$(x-h)^2 + (y-k)^2 = r^2$$

Movement in the x and y direction in polar coordinate form is defined by the following equations, where r represents the **magnitude or radius**.

$$x = r\cos t; y = r\sin t$$

Formula.
Therefore, the parametric form of a circle must be

$$F(t) = \langle x(t), y(t)\rangle$$

$$\langle r\cos t + h, r\sin t + k\rangle$$

10.7.2 Parametric Equations of Ellipses

Definition 10.7.1: Parametric Form of an Ellipse

Recall.
The standard form of an ellipse is defined by the following characteristics

1. Both terms of equation are positive

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Additionally, we can remember our simple polar coordinate formulas

$$x = r\cos\theta; y = r\sin\theta; \tan\theta = \frac{y}{x}$$

We can think of the the h and the k values as **offsets** to the sine and cosine functions.
Formula.
If we make the proper substitutions, we are able to arrive at the following formula

$$F(t) = \langle x(t), y(t)\rangle$$

$$x(t) = a\cos t + h$$

$$y(t) = b\sin t + k$$

10.7.3 Parametric Equations of Hyperbolas

Definition 10.5: P

rametric Form of a Horizontal Hyperbola

Recall.
The standard form of a hyperbola is defined by the following characteristics

1. Both terms are of unlike signs, ie: one is positive and the other is negative

and is given by the following equations:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Formula. Instead of utilizing the sine and cosine functions, however, the hyperbola utilizes parametric equations based off of functions of **secant** and **tangent**, where positive terms are functions of **secant** and negative terms are functions of **tangent.** The horizontal formula of a hyperbola is as follows

$$F(t) = \langle x(t), y(t) \rangle$$

$$x(t) = a \sec\theta + h$$

$$y(t) = b \tan\theta + k$$

Definition 10.6: P

rametric Form of a Vertical Hyperbola

Recall.
The standard form of a hyperbola is defined by the following characteristics

1. Both terms are of unlike signs, ie: one is positive and the other is negative

and is given by the following equations:

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Formula. Instead of utilizing the sine and cosine functions, however, the hyperbola utilizes parametric equations based off of functions of **secant** and **tangent**, where the positive term always gets the **secant function** and the negative terms always gets **tangent.**
The parametric equation of a vertical hyperbola is as follows

$$F(t) = \langle x(t), y(t) \rangle$$

$$x(t) = b \tan\theta + h$$

$$y(t) = a \sec\theta + k$$

10.7.4 Parametric Equations of Parabolas

Definition 10.7: P

rametric Form of a Parabola

Recall.
The standard form of a parabola possesses the following characteristics

1. One variable is squared and the other variable is not

and is given by the following equations:

$$y^2 = 4ax \text{ for a horizontal parabola}$$

or

$$x^2 = 4ay \text{ for a vertical parabola}$$

Formula.

10.8 Limits and Continuity

Now that we have essentially learned all of the basic operations of vectors, as well as have evaluated functions that return vectors, let us try to apply concepts from **calculus** to these functions.

Recall that the limit of a function essentially states that there exists some number δ, at which every value of the function for all values of the domain greater than δ will possess a corresponding ε distance from the limit.

Essentially, we know that a limit basically states that if the domain of a function gets aribtrariliy close to some value a, then the value of the function must approach some value L.

Definition 10.8: L

mits of a Vector-Valued Function

The limit $\lim_{t\to a} \vec{r}(t)$ exists and equals $\vec{s} = \langle s_1, s_2, s_3 \rangle$ if and only if

$$\lim_{t\to a} f(t) = s_1, \lim_{t\to a} g(t) = s_2, \lim_{t\to a} h(t) = s_3$$

Definition 10.9: C

ntinuity of a Vector-Valued Function at a Point

A vector-valued function $\vec{r}(t)$ is **continuous** at a point t_0 if $f(t)$, $g(t)$, and $h(t)$ are continuous at t_0 or

$$\lim_{t\to t_0} f(t) = f(t_0), \lim_{t\to t_0} g(t) = g(t_0), \lim_{t\to t_0} h(t) = h(t_0)$$

Definition 10.10: C

ntunity of a Vector-Valued Function as as Function

The vector-valued function $\vec{r}(t)$ is continuous as a function if $\vec{r}(t)$ is continuous at **every point** in its domain.

10.9 Derivatives of Vector-Valued Functions

Similarly to how we evaluated the limits of **vector-valued functions**, we can also evaluate the **derivatives of vector-valued** functions in a similar manner.

Much like limits, we evaluate derivatives of **vector-valued functions** by evaluating the derivatives of each **component function**.

Definition 10.11: D

rivatives of Vector-Valued Functions

The vector function $r(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ has a derivative (or is differentiable) at t if and only if f, g, and h have derivatives at t. Then, the derivative of the vector function is equal to

$$r'(t) = \frac{d}{dt}r = \lim_{\triangle\to 0}\frac{r(t+\triangle t) - r(t)}{\triangle t}$$

$$\Rightarrow \frac{df}{dt}\hat{i} + \frac{dg}{dt}\hat{j} + \frac{dh}{dt}\hat{k}$$

$$\Rightarrow \langle\frac{df}{dt}, \frac{dg}{dt}, \frac{dh}{dt}\rangle$$

Example ((1))**.**

10.9.1 Understanding Derivatives of Vector-Valued Functions

Definition 10.12: V

ctor-Valued Functions in Particle Motion

If r is the position vector of a particle moving along a smooth curve in space, then

$$v(t) = \frac{dr}{dt} = r'(t)$$

is the particle's $velocityvector$, which is the vector that is tangent to the curve. At any time t, the direction of v is known as the **direction of motion**, while the magnitude of v is the particle's **speed**, and the derivative

$$a = \frac{dv}{dt} = v'(t) = r''(t)$$

is the particle's **acceleration vector**.

1. Velocity $v(t)$ is the derivative of position $r(t)$
2. Speed is the magnitude of velocity $||v(t)||$
3. Accleration $a(t)$ is the derivative of velocity $\frac{dv}{dt} = \frac{d^2r}{dt^2}$
4. The unit vector $\frac{v}{||v||}$ is the direction of motion at time r.

Example ((4))**.** *Find the velocity, speed, and accleration of a particle whose motion in space is given by the position vector*

$$r(t) = 2\cos t\hat{i} + 2\sin t\hat{j} + 5\cos^2 t\hat{k}$$

Sketch the velocity vector of $v(\frac{7\pi}{4})$

10.10 Differentiation Rules for Vector-Valued Functions

Definition 10.13: D

fferentation Rules of Vector-Valued Functions

1. Constant Function Rule
2. Scalar Multiple Rule
3. Sum Rule
4. Difference Rule
5. Dot Product Rule
6. Cross Product Rule
7. Chain Rule

10.11 Integrals of Vector-Valued Functions

TODO

10.11.1 Understanding Integrals of Vector-Valued Functions

TODO

Chapter 11

13.3: Arc Length

11.1 Reminders

11.2 Objectives

11.3 Motivation

Chapter 12

14.1: Functions of Several Variables

12.1 Reminders

12.2 Objectives

12.3 Motivation

Chapter 13

14.3: Partial Derivatives

13.1 Reminders

13.2 Objectives

13.3 Motivation

Chapter 14

14.4: The Chain Rule

14.1 Reminders

14.2 Objectives

14.3 Motivation

Chapter 15

14.5: Gradient Vectors and Tangent Planes

15.1 Reminders (As of 05/20/23)

15.1.1 MATH_226

15.1.2 MATH_230

15.2 Objectives

15.3 Motivation

Chapter 16

14.5 (cont'd): Directional Derivatives

16.1 Reminders

16.1.1 MATH_226

16.1.2 MATH_230

16.2 Objectives

16.3 Motivation

Chapter 17

14.6: Tangent Planes and Linearization

17.1 Reminders

17.1.1 MATH_226

17.1.2 MATH_230

17.2 Objectives

17.3 Motivation

17.4 Review: Finding the Tangent Line to a Curve

Example 17.4.1 (Review: Find Tangent Line ot a Curve)
Find the tangent line to the curve

$$\frac{x^2}{9} + y^2 = 1$$

at the point $(0, 1)$

Solution: In single variable calculus, we would consider two different techniques:

- Solving for y, then solving for $y'(0)$
- Implicit differentiation (which is evaluating $\frac{dy}{dx}$ in terms of y)

In multivariable calculus, however, we take a different approach:

- We think of the function as a level curve of $f(x, y) = z$. The **gradient vector** of the function at that point will be perpendicular to the level curve.

Note:-
A gradient vector is given by the following form

$$\nabla f(x, y) = \left(\frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \right)$$

$$f(x, y) = \frac{x^2}{9} + y^2$$

$$\nabla f(x, y) = \left(\frac{\partial}{\partial x} \left(\frac{x^2}{9} + y^2 \right), \frac{\partial}{\partial y} \left(\frac{x^2}{9} + y^2 \right) \right)$$

$$\Rightarrow \nabla f(x,y) = \left(\left(\frac{2x}{9}\right), (2y)\right)$$

$$\Rightarrow \nabla f(x,y)\Big|_{(0,1)} = \left(\left(\frac{2(0)}{9}\right), (2(1))\right) = (0,2) = \nabla f(x,y)$$

Therefore, we know that the equation of the tangent line is given by

$$Ax + By = C$$

$$\Rightarrow F_x(x-h) + F_y(y-k) = 0$$

$$\Rightarrow 0\,(x-0) + 2\,(y-1) = 0$$

$$\Rightarrow y = 1$$

Note:-

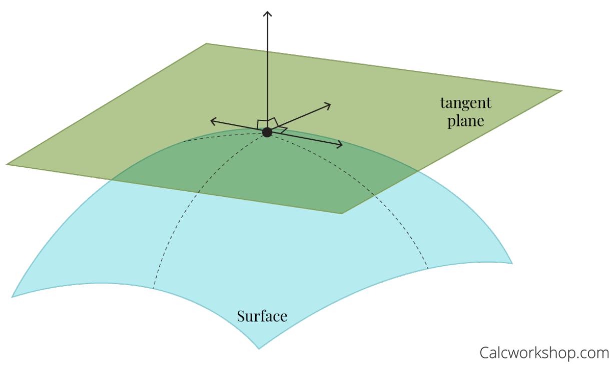
In order to find the tangent line to a given shape (in the former case, an ellipse), all we have to do is just find the **gradient vector** at the given point, then substitute the proper values in.

But what if we want to find the tangent *something* for a function with more than two variables?

17.5 Finding Tangent Planes

In the former example, we were working with exclusively two variables in $\mathbb{R}^2$, but what if we wanted to translate these techniques into $\mathbb{R}^3$?

Well, it is first important to remember that in three dimensions, we are working with surfaces, which means that if we were to find the tangent-*something* to a surface, we would be able to extend it in some third direction.



Notice how *all* of the values in the plane are tangent/perpendicular to the surface...

All of this culminates in the idea that, whenever we are working with a function of **three variables,** we are now working with not just a tangent line, but a tangent *plane*. If we wanted to evaluate the equation of this tangent plane, we apply the same general concept for finding the equation of the tangent line in $\mathbb{R}^2$.

Example 17.5.1 (Finding a Tangent *Plane*)

Find the tangent plane to

$$\frac{x^2}{9} - \frac{y^2}{16} + \frac{z^2}{25} = 1$$

at the point $(3,4,5)$

.

Much like in the previous example, we apply our understanding of gradients to finding the tangent plane.

Note:-

It is important to our understanding that the gradient represents the movement of the function in terms of *each of its variables*. The gradient in this case is effectively the derivative of a function in single-varialbe calculus. Although there are multiple rates of change within a multivariable function and a surface, the gradient represents the **best possible direction that represents the "slope".**

Anyways, the algorihtm for evaluating the tangent plane is just like evaluating the tangent line.

- We first evaluate the **gradient vector** of the function $f(x, y, z)$
- Then, we just plug the point at which we want to find the "slope"of into the gradient
- Finally, we want to apply the value of the gradient at a point as well as the original point into the standard formula of a plane.

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\Rightarrow \nabla f(x,y,z) = \left(\frac{\partial}{\partial x}\left(\frac{x^2}{9} - \frac{y^2}{16} + \frac{z^2}{25} - 1\right), \frac{\partial}{\partial y}\left(\frac{x^2}{9} - \frac{y^2}{16} + \frac{z^2}{25} - 1\right), \frac{\partial}{\partial z}\left(\frac{x^2}{9} - \frac{y^2}{16} + \frac{z^2}{25} - 1\right)\right)$$

$$\Rightarrow \nabla f(x,y,z) = \left(\frac{2x}{9}, \frac{-2y}{16}, \frac{2z}{25}\right)$$

$$\Rightarrow \nabla f(x,y,z)\Big|_{(3,4,5)} = \left(\frac{2(3)}{9}, \frac{-2(4)}{16}, \frac{2(5)}{25}\right)$$

$$\Rightarrow \nabla f(x,y,z) = \left(\frac{2}{3}, \frac{-1}{2}, \frac{2}{5}\right)$$

Note:-

Recall that the equation of a plane is given by

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$\Rightarrow \frac{2}{3}(x - 3) + \frac{-1}{2}(y - 4) + \frac{2}{5}(z - 5) = 0$$

17.6 Summary of Evaluating Tangent-*somethings*

Definition 17.6.1: How to Evaluate a Tangent Line to a Level Curve

Given a level curve $f(x, y) = c$, we are able to find the **tangent line** at (a, b) by evaluating

$$f_x(a,b)(x - a) + f_y(a,b)(y - b) = 0 \tag{17.1}$$

It is important to remember, though, that the general methodology to achieve (17.1) is the following:

1. Evaluate the gradient vector of the levle curve $\nabla f(x, y)$
2. Evaluate the value of the gradient vector at the given point (a, b)
3. Plug hte vlaues into the standard equatoin of a line

$$Ax + By = C$$

Definition 17.6.2: How to Evaluate a Tangent Plane

Given a level surface $f(x, y, z) = d$, we can find the **tangent plane** to the surface at a point (a, b, c) by evaluating

$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0 \tag{17.2}$$

In order to achieve the equation (17.2), you can also follow the algorithm:

1. Evaluate the **gradient vector** of the level curve $\nabla f(x, y, z)$
2. Evaluate the value of the gradient vector at the given point (a, b, c)
3. Plug the values of the gradient vector at a point as well as the given point into the standard form of a plane

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

17.7 Linearization in Single and Multivariable Calculus

In this next section, we are discussing **linearization of functions**, which should recall the idea of the **Taylor Series**.

- Recall that the Taylor Series is an infinite sum representation of a function (also known as the generating function). What makes the Taylor Series special is that we are able to derive this infinite sum representation by **only using the function itself**. The Taylor series is created by using a function and its derivatives. The more terms that we add to the Taylor Polynomial, the more accurate that the approximation is. The point at which the Taylor Series converges is the function itself (or at least, very close).

Note:-

The Taylor Series of a function $f(x)$ at some center of convergence a is given by the infinite sum:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n$$

which translates to

$$\Rightarrow f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f^{(3)}(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

"Linearization" is just another name for the **first-order Taylor expansion**. So, whenever we are evaluating the linearization of a function, we are essentially just finding the **first-order Taylor expansion** of the function, the terms of hte Taylor expansion up until the first derivative of the function, inclusive.

Let us first recall what linearization looks like in single variable calculus...

Definition 17.7.1: Linearization (Single Variable Calculus)

The linearization of a function $y = f(x)$ at some center $x = a$ is equal to

$$y = f(a) + \frac{f'(a)}{1!}(x - a)$$

which we can also denote as

$$L(x) = f(a) + \frac{f'(a)}{1!}(x - a)$$

such that $L(x)$ is just the linearization of the function $f(x)$.

17.7.1 Linearization in Multivariable Calculus

Definition 17.7.2: Linearization (Multivariable Calculus)

The linearization of $z = f(x, y)$ at a point (a, b) is

$$f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \tag{17.3}$$

Notice that this equation is also the same as

$$L(x,y) = f(a,b) + \frac{\partial f}{\partial x}(x-a) + \frac{\partial f}{\partial y}(y-b) \tag{17.4}$$

such that $f(a,b)$ is current value of the function at a point, $f_x(a,b)(x-a)$ is the partial derivative of the function with respect to x and $f_y(a,b)(y-b)$is the partial derivative of the function with respect to y.

Intuition.
Our intuition here is that we are finding the equation of the plane that is tangent to the surface at the given pint, and then we are solving for the z term.

Example 17.7.1 (Multivariable Linearization Example 1)
What's bigger: $\sqrt{9.1}$ or $\sqrt{27.2}$?

Solution: Since we are unable to actually compute the values of $\sqrt{9.1}$ and $\sqrt[3]{27.2}$, we can use **linearizaiton** in order to approximate the difference between $\sqrt{9.1}$ and $\sqrt{27.2}$.

We can use the following equation to evaluate the difference between $\sqrt{9.1}$ and $\sqrt{27.2}$

$$f(x,y) = \sqrt{x} - \sqrt[3]{y}$$

and we will center our approximation at the point

$$(a,b) = (9,27)$$

since it is reasonably close to what we are trying to approximate.

Note:-
Remember that the first-order Taylor expansion of a multivariable function is given by the following function

$$P_1(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \tag{17.5}$$

In order to evaluate f_x and f_y, let us just evaluate the **gradient vector** of $f(x,y)$ and then work with the components of the gradient vector.

$$\nabla f(x,y) = (f_x, f_y)$$

$$\Rightarrow \nabla f(x,y) = \left(\frac{\partial}{\partial x}\left(\sqrt{x} - \sqrt[3]{y}\right), \frac{\partial}{\partial y}\left(\sqrt{x} - \sqrt[3]{y}\right)\right)$$

$$\Rightarrow \nabla f(x,y) = \left(\frac{1}{2}x^{-\frac{1}{2}}, \frac{1}{3}y^{-\frac{2}{3}}\right)$$

$$\Rightarrow f_x = \frac{1}{2}x^{-\frac{1}{2}}, f_y = -\frac{1}{3}y^{-\frac{2}{3}}$$

Now, the linearization equation is as follows:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$\Rightarrow= f(9,27) + \frac{1}{2}9^{-\frac{1}{2}}(x-9) - \frac{1}{3}27^{-\frac{2}{3}}(y-27)$$

$$\Rightarrow = (0) + (\frac{1}{6})(x-9) - \frac{1}{27}(y-27)$$

Now, we can just substitute our original values $x = 9.1$ and $y = 27.2$

$$\rightarrow \left(\frac{1}{6}(9.1-9)\right) - \left(\frac{1}{27}\right)(27.2-27)$$

$$\Rightarrow \left(\frac{1}{6}\right)(0.1) + \left(\frac{1}{27}\right)(0.2)$$

$$\Rightarrow L(x) > 0$$

Now that's cool that we were able to find this value, but how can we actually be sure that we are correct? How can we be sure that our approximation didn't give us the wrong number? After all, it's possible that perhaps $\sqrt{9.1}$ is much smaller than we assumed it was, which would yield a **negative value** for $L(x)$, indicating that $\sqrt{9.1}$ is smaller.

Fortunately, there is a way for us to actually calculate *the interval with which our approximation is actually correct.*

17.8 Taylor's Formula (Separating Approximation from Error)

From Calculus 2 (or just Sequences and Series), we remember that Taylor actually accounts for the difference between the approximation of a finite Taylor Polynomial at a point and the actual value of a function at that same point. He essentially states that the Taylor Series $T(x)$(that is, the infinite sum), will always be composed of two parts:

- A finite polynomial approximation of the Taylor Expansion $P_n(x)$
- The difference between the finite Taylor polynomial at a point and the actual value of the functoin at that point, the remainder $R_n(x)$

such that n represents the order of the Taylor Polynomial.

$$T(x) = P_n(x) + R_n(x)$$

Definition 17.8.1: Taylor Error Estimation Theorem (Or Lagrange Estimation Theorem)

Given that a functoin $z = f(x, y)$ and all of its partial derivatives $f_x, f_y, f_{xx}, f_{xy}, f_{yy}$ are all *continuous* at and near a point (a, b). We know that the **next terms of the Taylor expansion**, given by the coefficients $|f_{xx}|, |f_{yy}|, |f_{xy}|$ will be, at most, some constant M. Therefore, the remainder R_n for the 1st Taylor expansion of a multivariable functoin is given by

$$R_1(x) = |f(x, y) - L(x, y)| \leq \frac{1}{2!}M(|x-a| + |y-b|)^2 \tag{17.6}$$

It is important to remember that (17.6) represents the *maximum possible value* that the next term in the Taylor expansion can be. Remember that the second Taylor expansion of a multivariable function will include terms that have coefficients of mixed partial derivatives $|f_{xx}|, |f_{xy}|, |f_{yy}|$. The only thing that we are doing is that we are calculating this "worst case scenario", in which all of these terms have the highest possible coefficient, which is one of those second-order partial derivatives.

Note:-

The error bound is *not* meant to be accurate, but rather, a **doomsday marker**.

Example 17.8.1 (Applying the Taylor Estimation Theorem to the Previous Example)
Given a function

$$f(x, y) = \sqrt{x} - \sqrt[3]{y}$$

that is linearized at the center $(a, b) = (9, 27)$, find the error bound of the linearization at the point $(9.1, 27.2)$.

Solution: TODO

Chapter 18

10.9: Taylor Polynomials

18.1 Reminders

18.2 Objectives

18.3 Motivation

18.4 Review of Single Variable Taylor Polynomials

Example 18.4.1 (Review Taylor Polynomials)
Given a generating function $f(x) = e^x$, we are able to approximate the value of this generating function with the following nth order Taylor polynomials. Let us find the Taylor Polynomials of $f(x)$ near the center $a = 0$

Note:-
A Taylor Polynomial is a *truncated* version of a Taylor Series. The nth order Taylor polynomial just represents the terms up until the term with the nth order derivative as its coefficient. The Taylor Series is given by the infinite sum

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$\Rightarrow f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}$$

Solution:

$$T_0(0) = e^0 \Rightarrow 1$$

$$T_1(0) = e^0 + \frac{e^0}{1!}x \Rightarrow 1 + \frac{1}{1!}x$$

$$T_2(0) = e^0 + \frac{e^0}{1!}x + \frac{e^0}{2!}x^2 \Rightarrow 1 + \frac{1}{1!}x + \frac{1}{2!}x^2$$

$$T_3(0) = e^0 + \frac{e^0}{1!}x + \frac{e^0}{2!}x^2 + \frac{e^0}{3!}x^3 \Rightarrow 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$$

From this, we know that the Taylor Series, based on the pattern of the terms in the Taylor Polynomials is given by the following summation:

$$\sum_{n=0}^{\infty} \frac{1}{n!}x^n$$

Note:-

It is ipmortant to remember that the higher the order of the Taylor Polynomial → the more accurate (and therefore the less the remainder) that the Taylor Polynomial will have.

Again, it is important to remember that this entire idea of Taylor Polynomials and Remainders is all given by the **Taylor Theorem**.

Definition 18.4.1: Taylor Theorem (Single Variable)

If f is n times differentiable at $x = 0$, then $|f(x) - P_n(x)| \cdot \frac{1}{x^n}$ will approach zero. That is, if a function is infinitely differentiable, then the higher order the Taylor Polynomial $P_n(x)$ is, the less difference between the Taylor Polynomial and the generating function $f(x)$.
From this, the error R_n for the nth Taylor Polynomial centered at $a = 0$is given by the following equation:

$$R_n(x) = |f(x_0) - P_n(x_0)| \leq \frac{1}{(n+1)!} M\, |x_0|^{n+1} \tag{18.1}$$

such that M is the maximum possible Taylor coefficient $f^{(n+1)}(x)$ between $-x_0$ and x_0. We can of course, actually apply this same principle to **multivariable functions**, the only caveat is that, as we add more terms to the Taylor polynomial, the more complex the equation is going to be.

- We are no longer just finding the second, third, or nth derivatives, but because we are working with multivariables, we have to also account for the repeating partial derivatives, such as f_{xx} and f_{yy} as well as **mixed partial derivatives** such as f_{xy}

18.5 Why use higher-order Taylor Polynomials?

The reason why we would even consider caring about these higher order (and more complex) Taylor Polynomials is that, as we add more terms, the more context we have on the behavior of the graph.

- We can think of adding more Taylor terms as adding more "hills" and "valleys"
- We can also think of the fact that the nth order derivatives of a function reveal more information about their graphical behavior.

So, what can $z = P_2(x, y)$ detect about the behavior of a function $f(x, y)$ that the first-order Taylor polynomial $z = P_1(x, y)$ cannot?

- **First-order Taylor polynomials** $P_1(x)$ → Increasing/Decreasing behavior
- **Second-order Taylor polynomials** $P_2(x)$ → Concavity behavior

Example 18.5.1 (Multivariable Second-Order Taylor Polynomials)
Given the function

$$f(x, y) = e^{x+y} + y^2$$

Find the second-order Taylor polynomial $P_2(x, y)$ about the center point $(0, 0)$.

Solution: TODO

Definition 18.5.1: Multivariable Error Estimation Theorem

The remainder of a multivariable nth order Taylor polynomial R_n is given by the following expression

$$R_n(x) \leq \frac{1}{(n+1)!} M_{n+1} \left(|x-a| + |y-b|\right)^{n+1} \tag{18.2}$$

Note:-

We now have a subscript for M, since we have to remember that M, unlike in single variable calculus, does not consist of a single possible value (that is, the next derivative), instead we have to consider *all possible coefficients* and combinations of the nth order partial derivatives of a function.

Example 18.5.2 (Error Bound of Multivariable Taylor Polynomial)

Chapter 19

14.6: Extreme Values and Saddle Points

19.1 Reminders

19.2 MATH_226

- written homework due friday
- final exam is in *13 days*

19.3 MATH_230

- written homework due tonight
- mylab math 17: extreme values and saddle points is due tomorrow
- exam is in *12 days*

19.4 Objectives

19.5 Motivation

19.6 Recapping Important Terminology

Over the last few sections, we have discussed some pretty useful concpets and ideas, including

- partial derivatives
- gradient vectors
- directional derivatives

Definition 19.6.1: Gradient Vectors

Given a function $z = f(x, y)$, the gradient of f(x,y) is equal to

$$\nabla f(x, y) = (f_x(x, y), f_y(x, y))$$

The gradient represents a data type that *stores* all first derivatives of a given function.

Along the same train of thought, we also have the concept of the **Hessian Matrix**.

Definition 19.6.2: Hessian Matrix

Given a function $z = f(x, y)$, the Hessian of $f(x, y)$ is equal to the 2x2 matrix

$$Hf(x, y) = Hess(f)(x, y) =$$

$$\begin{bmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{bmatrix}$$

which, of course, is just a data type that stores all of the second-order and second-order mixed partial derivatives of a function f.

19.6.1 More About Matrices

Definition 19.6.3: Diagonals of Matrices

In the following diagram, we observe the **main diagonal** as well as the **off diagonal** of a matrix. The easiest way to remember these is that the main diagonal is going from *left to right* whereas the off diagonal is going from *right to left*.

Given the 2x2 matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

the **main diagonal** is as follows: $\begin{bmatrix} a & \\ & d \end{bmatrix}$,

whereas the **off diagonal** is as follows: $\begin{bmatrix} & b \\ c & \end{bmatrix}$

Definition 19.6.4: Determinants

The determinant of a 2x2 matrix is equated by finding the difference between the product of the main diagonal terms as well as the product of the terms in the off diagonal. **Warning:** this formula only applies for *2x2 matrices* and does *not* apply to larger matrices.

$$det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= a \cdot d - b \cdot c$$

19.7 Extrema Analysis of Functions

In this section, we will be going over how we are able to examine extrema of functoin in both single variable as well as multivariable calculus. First, I am going to be recalling how we actually examine extrema in single variable calculus and what these techniques intuitively mean. Then, I will be *translating* these concepts into multivariable calculus.

19.7.1 Extrema in Single-Variable Calculus

Definition 19.7.1: Extrema (Single-Variable Calculus)

Extrema reflect the notable visual behavior of a graph. Whenever we analyze extrema, we are generlaly thinking of the following:

- Critical Points
- Concavity

Definition 19.7.2: Critical Point (Single-Variable Calculus)

A critical point represents the point at which the *first-order derivative* of a function is constant. We can visualize critical points as the "hills" or "valleys" of a function (or similarly, as the humps and lumps of a function).

Definition 19.7.3: Concavity (Single-Variable Calculus)

Concavity of a critical point demonstrates whether or not the current critical point is the greatest possible point of nearby points or the least possible value of nearby points. We can think of concavity as actually describing whether or not a critical point is a hill or a valley.

19.7.2 Extrema in Multi-Variable Calculus

19.8 Techniques for Evaluating for Extrema in Multivariable Calculus

Chapter 20

14.7: Optimization

20.1 Reminders

20.1.1 MATH_226

- written homework 9 (last one) is due on friday

20.1.2 MATH_230

- mlm 17 due tomorrow
- mlm 18 due tomorrow
- written homework 5 is due on wednesday

20.2 Objectives

20.3 Motivation

20.4 Evaluating Optimizatoin Problems in Single Variable Calculus

Definition 20.4.1: Optimization Problems

Recall that optimization is the entire idea of utilizing a function and its derivatives in order to find local maxima/minima as well as global/absolute maxima/minima.
A lot of optimization predates on the **Extreme Value Theorem**

Theorem 20.4.1 Extreme Value Theorem (Single Variable Calculus)

Given a continuous function f that exists over a **closed, bounded interval** I, we know that maxima and minima of that function must exist at either the **critical points** or the **endpoints** of the function.

We are actually able to apply these ideas to multivariable calculus. But first, let us recall how we actually evaluate a single variable optimizaiton problem, and what kinds of questions we are trying to ask whenever we do solve a single variable optimization problem.

Note:-

We have to translate the fact that we are working in multiple variables with our approach...

20.4.1 Solving a Single-Variable Optimizatoin problem

Whenever we are evaluating a single-varialbe optimization problem, we have to consider a few things first:

- What does hte graph look like?
- What do the endpoints look like?
- Where is the graph likely to peak or dip?

A lot of these questions can be answered by the idea of the **critical point**.

> **Note:-**
> Remember that a critical point is any point in which the **derivative** of a function $f'(x)$ is equal to 0. This corresponds to a graph in $\mathbb{R}^2$ reaching a "hill" or a "valley".

The general algorith mfor evaluating a single variable optimization problem is as follows:

1. Locate the endpoints of the grpah and evaluate the value of the functoin at the endpoints
2. Evaluate the **derivative** of the function.
3. Let the **derivative** of the functoin equal zero, then evaluate this equatoin. This will give you the **critical points** of the function.
4. Evaluate the **original function** at the critical points and examine their values. By this point you should have evaluated all possible places in which the graph has maxima or minima.
5. "Analyze the books" and compare values and locate which ones are the highest and which ones are the lowest.

> **Example 20.4.1** (Single-Variable Optimization Problem)
> Find the maxima/minima of the following function over the interval $I = [-10, 1]$
> $$f(x) = x^3 + 3x^2 - 7$$

Solution:

1. First, we want to locate all values of x in which there is a critical point. We do this by evaluating for the derivative of f.
$$f'(x) = 3x^2 + 6x = 0$$
$$\Rightarrow 3x(x+2) = 0$$
$$x = -2, 0$$
We can see here that there are critical points at $x = -2$ as well as $x = 0$. It is important to remember that $x = -2$ and $x = 0$ both exist within the desired interval $[-10, 1]$, which means that they are valid critical points.

2. Now that we know where the critcal points are, that is, from deriving the original function as well as looking towards the endpoints of the interval, we can now evaluate the values of the critical points.

> **Note:-**
> The critical points in $\mathbb{R}^2$ will represent the "hills" and "valleys" of the graph.

$$x = \{-10, -2, 0, 1\}$$

Now that we nkow which values of x could possibly be local maxima or minima, we want to plug in these critical values of x into the **original function**.

$$f(-10) = -693$$

$$f(-2) = 11$$
$$f(0) = 7$$
$$f(1) = 11$$

All of these values, again, represent **critical points** on the graph, so now, we can just compare the values and state where the **absolute maxima and minima are**.

3. Since -693 is the lowest, we know that the **global minimum** is at $(-10, -693)$. However, we notice that there is a **tie** between $x = -2, 1$, so we just state that there are two global maximums at $(-2, 11)$ and $(1, 11)$. Therefore, the global minimum is -693 and the global maximum is 11.

Now that we have reminded ourselves on how to optimize a single-variable function, let us begin working with **multivariable optimization**.

20.5 Translating the Optimization Algorithm to $\mathbb{R}^3$

Whenever we are approaching an optimization problem with mulntiple variables, we have to shift our conceptions of optimization and shift our methodology as well as our approach.

Note:-
We are working with the *xy-plane* and how "high" or "low" the values are on the z-axis.

- Derivative $\longrightarrow$ Gradient
- Critical Value (on x-axis) $\longrightarrow$ Critical Point (on xy-plane)
- Closed Interval $\Rightarrow$ Boundaries

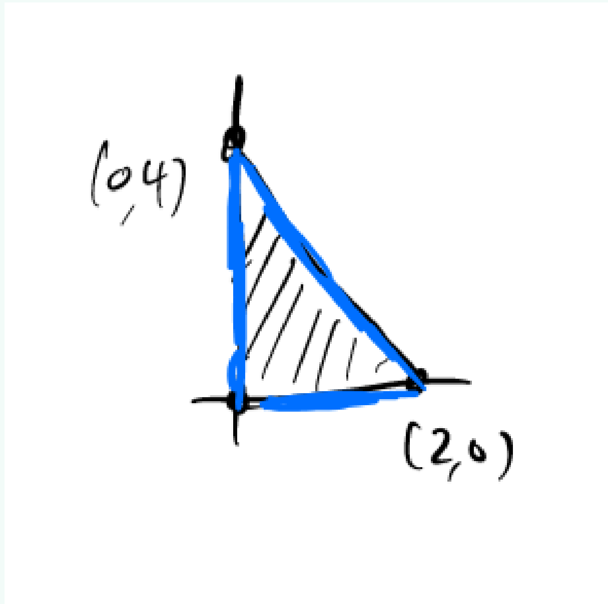
The most notable change about how to approach optimizing a multivariable surface is the fact that we are now working with **boundaries** instead of just intervals. This means that critical points no longer might just occur along a few points, but that entire **lines** or **curves** can contain maxima and minima. Whereas there was an extreme value theorem for single varaible calculus, there is now an extreme value theorem for multivariable calculus, too.

Theorem 20.5.1 Extreme Value Theorem (Multivariable Calculus)
If a domain D is a closed and bounded subset of R^n, then f must obtain a global min and global maximum at either the **critical points** or the **boundary points** of D.

From this theorem, we can discern that the global maximums and minimums of a multivariable function are at the critical points of the function as well as the boundary points of a function. That is, the critical points might exist at the points of which the domain of the function (which are (x,y) pairs) are bounded.

Example 20.5.1 (Trivial Multivariable Optimization)

Find the absolute min/max of $f(x, y) = xy$ over the triangle bounded by $(0, 0), (2, 0)$ and $(0, 4)$. This also includes values within the triangle.



Much like the optimization problems of single variable calculus, we need to first consider where the possible maxima/minima could be.

- Critical Points
- (new!) Boundary Points

Let us start by finding the critical points of $f(x, y)$. Recall that $f(x, y)$ is equal to

$$f(x, y) = xy \tag{20.1}$$

Whenever we worked with single variable calculus, we evaluated the derivative of the function. However, since we are now working with a **multivariable function**, we now accordingly work with **gradients**.

$$\Rightarrow \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$$

$$\Rightarrow \nabla f = (\frac{\partial}{\partial x}(xy), \frac{\partial}{\partial y}(xy))$$

$$\Rightarrow \partial f = (y, x)$$

Now, much like how we evaluated the critical points by letting the derivative of a single variable function f equal to 0, we now let the **gradient vector** equal the **zero vector**.

$$\Rightarrow \partial f = (y, x) = (0, 0)$$

$$\Rightarrow y = 0, x = 0$$

Our critical point is at $y = 0, x = 0$ or at the point $(0, 0)$.

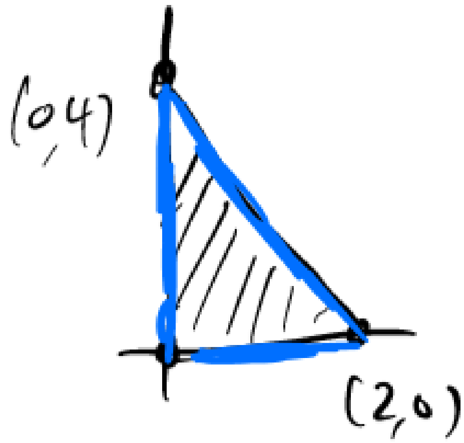
Note:-

Remember, we have ot make sure that our critical point is actually *inside* of the boundaries, much like how it is supposed to be inside of the interval.

Now that we have the **critical points** of the function $f(x, y)$, we now have to find the **boundary points** of $f(x, y)$

- The Extreme Value Theorem for Multivariable Functions states that a maxima/minima of a function can occur at either the critical points or the boundary points of a function. We are essentially equating the boundary points to the **endpoints**.

But how do we actually find the "boundary points"?

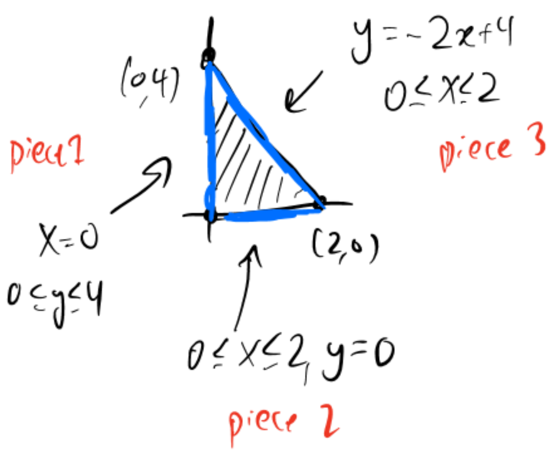


When we look at the graph, we observe that the triangle is bounded by the three points $(0, 0), (0, 4), (2, 0)$, so we can define our boundaries by finding the equations of the lines that contains these boundaries. We will from here on out refer to these boundaries interchangeably with the term **pieces**.

The piece from $(0, 4)$ to $(2, 0)$ is bounded by on the interval $0 \le x \le 2$ on the line $y = -2x + 4$.

The piece from $(0, 0)$ to $(2, 0)$ is bounded on the interval $0 \le x \le 2$ and is given by the function $y = 0$.

The piece from $(0, 4)$ to $(0, 0)$ is bounded oin the interval $0 \le y \le 4$ and is given by the function $x = 0$.



Now, as we evaluated the function f at its critical points x in single variable calculus, in multivariable calculus, we now evaluate the multivariable function $f(x, y)$ at its critical points as well as its boundary points.

For piece 1:

$$x = 0, 0 \le y \le 4$$

$$\Rightarrow f(0, y) = xy \Rightarrow (0)y = 0$$

For piece 2:

$$y = 0, 0 \le x \le 2$$

$$\Rightarrow f(x, 0) = xy = x(0) = 0$$

For piece 3:

$$y = -2x + 4, 0 \le x \le 2$$

$$f(x, -2x + 4) = xy \Rightarrow x(-2x + 4)$$

Note:-

We now have a single variable equation! Since the values of the boundary points on this line have different values, we must find the maxima/minima *of this boundary.* We just evaluate this equation the same exact way we would evaluate the maxima/minima of a single varialbe expression. We must derive this equation then let it equal 0.

$$\Rightarrow g(x) = -2x^2 + 4x = 0, x \in [0, 2]$$

$$\Rightarrow g'(x) = -4x + 4 = 0$$

$$\Rightarrow -4x = -4$$

$$\Rightarrow x = 1$$

The critical points of $g(x) = -2x^2 + 4x$ are $x \in \{0, 1, 2\}$. So, now we must evaluate the critical values of this boundary line $g(x)$.

For $x = 0$ of the boundary line $g(x)$,

$$g(0) = 0$$

For $x = 1$ of the boundayr line $g(x)$,

$$g(1) = -2 + 4 = 2$$

For $x = 2$ of the boundary line $g(x)$,

$$g(2) = 0$$

We observe that the maximum of the boundary line $g(x)$ is 2 at $x = 1$, while the minimum is 0 at $x = 0, 2$.

THis means that for the third piece, the minimum occurs at $(0, 4)$ (which is the minima of $g(x)$, $x = 0$ in the original function $f(x, y)$) as well as $(2, 0)$ (which is the minima of $g(x)$, $x = 2$ in the original function $f(x, y)$). The maxiama of piece 3 is $g(1) = -2(1) + 4(1) = 2$, which is $(1, 2)$.

Now, we can assess all of the points we have assembled from our critical values as well as our boundary points and determine where the maxima and minima are of our funciton.

- We know that the value of $f(x, y)$ is 0 for the first piece as well as the second piece, since the first piece is $f(0, y) \forall 0 \le y \le 4$ and the second piece is $f(x, 0) \forall 0 \le x \le 2$.

Let us now evaluate the boundary point $(1, 2)$.

$$f(x, y) = xy \Rightarrow (1)(2) = 2$$

The global maximum of $f(x, y)$ is $z = 2$, which is at $(1, 2)$.

Congrats, we just solved our first multivariable optimization problem! Here's more to come in the future... Let us now recap what exactly we did as well as the general strategy for evaluating multivarialbe optimizatoin problems.

1. First, we found the critical points of our multivariable functoin $f(x, y)$ by evaluating for the **gradient vector**. Whereas we had to let the derivative of our functoin equal 0 in single variable calculus, we let our **gradient vector** equal to the **zero vector**. We evaluate this to get the xy-coordinates of our critical points.

2. Now, we can evaluate these critical points and find the corresponding z values.
3. After evaluating the critical points as well as their values, we now find the **boundary points** by creating **pieces** from the constraints of the problem. In this case, we found lines that created the triangle we were looking for.
4. With these boundary lines, again, we want to evaluate the values of the functoin along hte points of these boundary lines. This might be very easy, as they might just be zero. However, oftentimes, we will have more complex boundaries.
   - With more complex boundaries, we will get a piece that has different values along its points. From this, we have to find the maxima and minima of this boundary line. By this point, we should have a single varialbe function, so we can just find the maxima/minima of this function using the tactics we already know.
   - With these maxima and minima, we can then **plug the maxima and minima of these pieces into the original multivariable function** and we can then discern there value in relation to the prevoius boundary points and critical values.
5. At this point, we should now have all of the values of the most important parts of the function. We can now just compare these values and discern what the maxima and minima are of the general funciton.

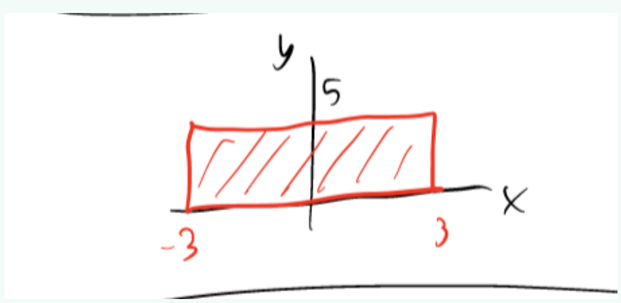
20.6 Philosophy of Optimization

Are we always going to be guarenteed an answer with these problems?

- Well, it's complicated. If the function is not closed (meaning that there aren't any endpoints), then it is possible that there aren't any maxima or minima. Furtheremore, if hte function wasn't continuous, then there also might not be any maxima or minima.
- The general criteria for achieving an ansewr from an optimization problem is as follows: we must have a **closed**, **bounded**, and **continuous** function.

  Let us try another example now

Example 20.6.1 (Slightly Harder Example + Introducing Parameterization)
Find the absolute extrema of
$$f(x,y) = x^2 + xy + y^2 - 6y$$
on the region $-3 \le x \le 3$, $0 \le y \le 5$.



/sl Our intution reminds us that, in essence, we must locate all the possible locations that the extrema can occur (the critical points and the boundary points), and then just evaluate the value of the function at these boundary points.

Chapter 21

14.8: Lagrange Multipliers

21.1 Reminders

21.1.1 MATH_226

- written homework due friday
- midterm in 14 days

21.1.2 MATH_230

- written homework due tonight
- extrema and saddle points mylab due thursday
- midterm in 13 days

21.2 Objectives

- Derive the intuition and understanding behind Lagrange Multipliers
- Be able to apply Lagrange Multipliers to any problem

21.3 Motivation

In the previous section, **Optimization**, we discussed different ways to evaluate the extrema of a surface.

- We realized that we have to not only consider the critical points, but also the *boundary points* of the level curves of a multivariable function

However, what if we wanted to anazlye the extrema of a surface on a very *specific subset* of the surface. Sure, we could always just pull out our optimization techniques, analyze the boundary points, make pieces, and evaluate the critical points within these boundaries, however, this isn't very *efficient*. Instead, let us examine a new method of analyzing extrema within a function using a technique called *Lagrange Multipliers*.

21.4 Introduction: Constrained Maxima and Minima

In order to begin, let us consider a trivial example in which we analyze where a minimum can be in a *constrained subset* of a graph, as well as strategies to actually finding this minimum.

> **Example 21.4.1** (Finding Extrema within a Constrained Subset)
> Find the point (x, y, z) on the plane $2x + y - z = 0$ that is *closest to the origin.*

Solution: The problem is essentially asking us to *minimize the distance from the origin to a point $\overrightarrow{OP}$, so we know that we are basically trying to find the smallest possible point P that satisfies*

$$\overrightarrow{OP} = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}$$

but also satisfies the fact that the point (x, y, z) exists on the plane

$$= 2x + y - z + 5 = 0 \tag{21.2}$$

What we can do here, is that we can put the equation *in terms of z.*

$$z = 2x + y + 5$$

which means that we can let f just be

$$= h(x, y) = f(x, y, 2x + y + 5) \tag{21.3}$$

$$= x^2 + y^2 + (2x + y + 5)^2 \tag{21.4}$$

21.5 General Method of Lagrange Multipliers

The main equation that we will utlize with Lagrange Multipliers is

$$\nabla f(P) = \lambda \nabla g(P)$$

Chapter 22

14.8: Lagrange Multipliers (Part 2)

22.1 Reminders

22.2 Objectives

22.3 Motivation