Math 230-1 Notes

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Contents

Chapter 1

12.1: Three-Dimensional Coordinate Systems

1.1 Reminders

- There are **two** MyLab Math assignments that are due on **Sunday, April 2nd, 2023**
  - Three-Dimensional Coordinate Systems
  - Vectors
- The **first written homework** is going to be due on **Friday, April 12, 2023**.
- Remember to re-write notes in LaTeX for every class!

1.2 Objectives

- Be able to understand and visualize the three-dimensional coordinate plane.
- Be able to draw basic objects in the three-dimensional coordinate plane
- Become fluent in the various attributes of the three-dimensional coordinate plane
- Be able to define a graph in set-builder notation

1.3 Motivation

In former calculus classes (and former math classes in general), we have learned how to graph different objects in two-dimensional space. We have also learned about a particular way of graphing objects (or interpreting coordinates), and that has been through coordinate grids.

However, in multivariable calculus, we want to move out of these two-dimesional coordinate systems. Instead we want to be able to understand graphs in the third-dimensional coordinate system in order to tackle more complex problems.

1.4 What we already know

1.4.1 Cartesian Coordinate Systems

The Cartesian Coordinate system, which is also known as the **rectangular coordinate system** is a coordinate system in which we locate our points based on their position in relation to the origin of the graph, based on the x and y axes.

- For example, given the following point:

$$(4, 5)$$

  what exactly comes to mind?

  - We shift the point 4 positive units along the x-axis from the origin.
  - We shfit the point 5 positive units along the y-axis from the origin.

1.4.2 Two-dimensional coordinate systems $\mathbb{R}^2$

Recall that real numbers are numbers that can be used to express one-dimensional quantities.

- This basically includes every single number that can be plotted on the number line.

We denote real numbers in mathematics with the following symbol:

$$\mathbb{R}$$

$\mathbb{R}$ represents all **real, one-dimensional quantities.** We can, of course, think of this as all of the numbers and points that exist on the number line, since the number line only contains one-dimension, the scalar x. By comparision, whenever we see the following notation:

$$\mathbb{R}^2$$

This means that are observing all **real, two-dimensional quantities**. When we say, "real, two-dimensional quantities," we are referring to all of the points that exist in the xy-plane, or the two-dimensional Cartesian coordinate system.

By this logic, then, we know that if $\mathbb{R}^2$ represents a pair of real numbers, we know that

$$\mathbb{R}^3$$

represents a triple of all real numbers, in the form of (x, y, z).

1.4.3 Terminology in $\mathbb{R}^2$

Definition 1. Axes

Axes represent the way that a point can "move" in a coordinate plane.

- In the one-dimensional coordinate system, we can only move along the x-axis: x.
- In the two-dimensional coordinate system, we can move along both the x-axis and the y-axis: (x, y).

Definition 2. Quadrants

Quadrants define the different possible areas of the coordinate system a point can exist on, which are based on the signs of both the x and y values.

- For example, there are four quadrants in the xy-plane, including
  - Quadrant I: $(+, +)$
  - Quadrant II: $(-, +)$
  - Quadrant III: $(-, -)$
  - Quadrant IV: $(+, -)$

1.5 The Three-Dimensional Coordinate Plane $\mathbb{R}^3$

By what we know about the one-dimensional coordinate system $\mathbb{R}$ as x and the two-dimensional coordinate system $\mathbb{R}^2$ as (x, y), we must think of the three-dimensional coordinate system $\mathbb{R}^3$ as (x, y, z).

1.5.1 Terminology in $\mathbb{R}^3$

Definition 3. Axes

This is literally a copy of what we have in $\mathbb{R}$ and $\mathbb{R}^2$, insofar that we have the number of dimensions corresponding to the exponent of $\mathbb{R}$. Obviously, in this case, since we are working in $\mathbb{R}^3$, we now have three dimensions to work with, the x-axis, the y-axis, and the z-axis.

Definition 4. Octants

Similarly to what we had in the two-dimensional coordinate plane, we can distinguish what general area a point in three dimensions is going to occupy.

- There is no good way to define which octant is which, but we can visualize it as the xy-plane quadrants, but just duplicated for all positive values of z and all negative values of z.

Definition 5. Planes

Planes are objects that occupy all real-numbers in two dimensions.

There are three planes in $\mathbb{R}^3$

- xy-plane
  - We can think of this as all points in which $z = 0$.
  - All points that satisfy $(x, y, 0)$, where x and y are real numbers.
- yz-plane
  - All points in which $x = 0$
  - Any coordinates that satisfy $(0, y, z)$ where y and z are real numbers.
- xz-plane
  - All points in which $y = 0$
  - Any coordinates that satisfy $(x, 0, z)$, where x and z are real numbers.

1.6 Set-Builder Notation

1.6.1 What are sets?

Sets are just collections of different objects in mathematics.

- We can think of sets as containing integers, variables, etc...

They are generally notated using **curly braces**.

1.6.2 Examples of Sets

$$\{1, 2, 3, \dots\}$$

$$\{a, b, c, \dots\}$$

But, how do we define what kinds of objects we are putting into our set?

1.6.3 Set Builder Notation

Set Builder Notation is a type of mathematical notation that allows us to describe what kinds of objects are in our sets and the properties of such objects.

The generally follow the following format

$$\{\ \textit{variable(s)} : \textit{condition(s) that define the variable(s)}\ \}$$

1.6.4 Common Symbols in Set Builder Notation

Symbol 1.
$$\mathbb{R},\ \mathbb{R}^2,\ \mathbb{R}^3, \ldots\ \mathbb{R}^N$$

Meaning. the set of real numbers in N dimensions

Symbol 2.
$$\in$$

Meaning. "is an element of " or "in" or "belongs to "

Symbol 3.
$$:\ \&\ |$$

Meaning. "such that"

1.6.5 Non-mathematical examples of Set Builder Notation

Example.
$$\{x : x \text{ is a left-handed guitar player}\}$$

Meaning. The set of x such that x is a left-handed guitar player.

Example.
$$\{y \mid y\text{'s name is Randy Truong}\}$$

Meaning. The set of y such that y's name is Randy Truong.

1.6.6 Mathematical Examples of Set Builder Notation

Example.
$$\{(x, y, z) \in \mathbb{R}^3 : y = 0, z = 0\}$$

Meaning. The set of all ordered triples (x, y, z) such that y is equal to 0 and z is equal to 0.

Example.
$$\{(t, 0, 0) : t \in \mathbb{R}\}$$

Meaning. The set of all ordered triples $(t, 0, 0)$ such that t is an element of real numbers (or is a real number).

1.7 Drawing basic objects (points, lines, planes) in R^3

Whenever we want to draw things in three dimensions, there are a few things that we need to consider first.

1.7.1 Drawing the Coordinate System and Right-Hand Rule

Whenever we draw the three-dimensional coordinate system, we must remember that there is a particular way in which we draw the system. The best way to visualize this is to use the **right-hand rule**

- Our arm represents the y-axis, while our fingers represent the x-axis.
- Our thumb is always going to be pointing towards the z-axis.
- Make sure that whenever we are drawing a coordinate system that we are just rotating the system, rather than just “mirroring” it.

Otherwise, whenever we actually plot our points and actually draw things in three dimensions, we need to follow this algorithm:

1. Think of what happens to the object at the origin or think of the shape in two dimensions.
2. Shift the object accordingly based on the third dimension.

1.8 Distance Between Two Points in Three-Dimensional Spacde

1.8.1 Distance in $\mathbb{R}^2$

Formula 1. Distance in $\mathbb{R}^2$

$$\text{let } d \text{ be distance}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1.8.2 Distance in $\mathbb{R}^3$

Formula 2. Distance in $\mathbb{R}^3$

$$\text{let } d \text{ be distance}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2 + (z_2 - z_1)^2}$$

Chapter 2

12.2: Vectors

2.1 Reminders

TODO

2.2 Objectives

1. Explain the difference between a vector and a point
2. Express the vector in **component form** and compute its magnitude
3. Perform elementary vector algebra using vector addition and scalar multiplication
4. Produce a unit vector with a specificed direction
5. Compute the midpoint of a line segment

2.3 Motivation

TODO

2.4 Scalar Who?

TODO

2.5 Vector Who?

TODO

2.5.1 Vector Notation

TODO

2.5.2 Vector Magnitude

TODO

2.5.3 Equivalency between Vectors

TODO

2.6 Position Vectors

TODO

2.7 Component Form of Vectors

TODO

2.8 Basic Vector Operations

TODO

2.8.1 Vector Addition and Subtraction

TODO

2.8.2 Scalar Multiplication

TODO

Chapter 3

12.3: The Dot Product

3.1 Reminders

TODO

3.2 Objectives

1. Compute the dot product of two vectors
2. COmpute the angle between two vectors in terms of the dot product
3. Algebraically determine when two vectors are **orthogonal** and be able to geometrically define what **orthongonality** refers to
4. Perform elementary vector algebra using properties of vector addition, scalar multiplication, and the dot product
5. Algebraically compute (and geometrically explain/describe) the projection of a given vector onto another, non-zero vector
6. Solve elemntary problems involving effective force and work using vector projections

3.3 Motivation

TODO

Chapter 4

12.4: The Cross Product

4.1 Reminders

TODO

4.2 Objectives

1. Compute the cross product of two given vectors using **determinants**
2. Geometrically interpret the magnitude and direction of the cross product of two given vectors
3. Perform elementary vector operations
   - Vector Addition
   - Scalar Multiplication
   - Dot Product
   - Cross Product

4.3 Recall

Recall that in the last lecture, we discussed the **dot product**, which had both an **algebraic** definition as well as a **geometric** definition

- Whenever we were finding the algebraic dot product, we were just multiplying the components and finding their sum

$$\vec{v} \cdot \vec{u} \; = \; v_1 u_1 + v_2 u_2 + v_3 + u_3$$

  We learned that this scalar actually represented a lot more than it let on. In fact, the **scalar that results from a dot product** actually represents the product of both vectors' magnitudes as well as the cosine of the angle between them:

$$\vec{v} \cdot \vec{u} \;=\; ||\vec{v}|| \cdot ||\vec{u}|| \cdot \cos\theta$$

where, of course, θ represents the angle between the vectors v and u.

We also see the angle between two vectors θ represented by the following formula

$$\cos\theta = \frac{\vec{v} \cdot \vec{u}}{||\vec{v}||||\vec{u}||}$$

- We also learned about **projection**, which is the idea of taking a vector v and then imposing it onto another vector u. Imagine that we were just "flattening" a vector onto another, preserving its length/magnitude while maintaining another vector's direction.

  $$proj_v u = \frac{\vec{v} \cdot \vec{u}}{\vec{v} \cdot \vec{v}}\vec{v}$$

  which can also be represented as

  $$\Rightarrow \left(\frac{\vec{u} \cdot \vec{v}}{||\vec{v}||}\right) \cdot \left(\frac{1}{||\vec{v}||}\right)\vec{v}$$

  where the first term $\frac{\vec{v} \cdot \vec{u}}{||\vec{v}||}$ represents the **scalar component of v** and the second term $\left(\frac{1}{||\vec{v}||}\right)\vec{v}$ represents the **unit vector of** $\vec{v}$.

4.4 Motivation

In the former lectures, we have learned about **vectors** as well as numerous operations to perform ont hem

- Vector Addition/Subtraction $\rightarrow$ vectors
- Scalar Multiplication $\rightarrow$ vector
- Dot Product $\rightarrow$ scalar

What if there was such an operation that we were able to **multiply** two vectors together?

4.5 Cross Product

Definition 6. Cross Product

The cross product of two vecturs $\vec{v}$ and $\vec{u}$, $(\vec{v} \times \vec{u})$, is the geometrically defined by the following vector:

$$\vec{v} \times \vec{u} := ||\vec{v}||||\vec{u}|| \cdot \sin\theta \cdot \vec{n}$$

where vector $\vec{n}$ is the **normal unit vector** perpendicular to the plane spanned by $\vec{v}$ and $\vec{u}$, which is chosen accordingly by the right hand rule.
Layman Definition.
The cross product is the reuslting vector $\vec{n}$, which is a vector that is orthogonal toe the plane of which $\vec{v}$ and $\vec{u}$ occupy, and of which whose magnitude is determined by the product of the magnitude of $\vec{v}$ and $\vec{u}$, and the value of the angle betwen the two vectors that span the plane.

TODO: Insert Picture from Tablet

4.5.1 Observations of the Cross Product

TODO

Chapter 5

12.5: Lines and Planes in Space (04/07/23 - 04/10/23)

5.1 TODOs

- Finish writing lesson for lines in space
- Finish writing lesson for planes in space
- Finish OHW

5.2 Lines in Space (04/07/23)

TODO

5.3 Reminders

5.4 Objectives

In this section, we want to be able to do the following:

- Be able to write equations for lines and line segments in $\mathbb{R}^3$ space using scalar and vector products

5.5 Motivation

Recall in the last lessons we have been learninge exclusively about vectors as well as how we are able to manipulate them in order to get different objects in space.

- For example, we have learned that the **Dot Product** is a function that takes in two vectors $\vec{v}$ and $\vec{u}$ and returns some scalar.

$$\vec{v} \cdot \vec{u}$$

  Much like anything in multivariable calculus, we actually have to consider that a lot of the concepts we learn are **multi-dimensional**. In this case, we must understand that the **dot product** has both a **algebraic** and **geometric** definition.

  Algebraically, we can think of the dot product as the following equation:

$$\text{let } \vec{v} = \langle v_1, v_2, v_3 \rangle$$
$$\text{let } \vec{u} = \langle u_1, u_2, u_3 \rangle$$
$$\Rightarrow \vec{v} \cdot \vec{u} \ = \ v_1u_1 + v_2u_2 + v_3u_3$$

  Geometrically, we can think of the **dot product** as the **angle between two vectors, but scaled based on the magnitude of the vectors.**

$$\text{let } \vec{v} = \langle v_1, v_2, v_3 \rangle$$
$$\text{let } \vec{u} = \langle u_1, u_2, u_3 \rangle$$
$$\Rightarrow \vec{v} \cdot \vec{u} \ = \ ||\vec{v}|| \cdot ||\vec{u}|| \cdot \cos\theta$$

  Of course, there are a few implications and use cases of the dot product both algebraically and geometrically

  - If $\vec{v} \cdot \vec{u} = 0$, then we know that the vectors v and u are **orthogonal or perpendicular**, which makes sense, since if we were to find some value $\cos\theta = 0$, then we would have $\theta = \frac{\pi}{2}$, which is of course, a right angle.
  -

  TODO: Include Graphics that Visualizes this relationship

5.5.1 2-D Lines versus 3-D Lines

Whenever we were defining lines in $\mathbb{R}^2$, we were always thinking of these lines as an a set of points with two defining characteristics:

- Some point P_0 that the line intersected
- Some slope or direction m that the line went in

With the intuition that lines in two-dimensional space were defined by the points they intersectd as well as their **slope**, which we can think of as the **change in x and the change in y over time**, we can apply the same general principals to vectors in three-dimensional space.

5.5.2 3-D Lines

Three dimensional lines, by comparison, are also defined by some point that the line goes through as well as a direction in which the line continues infinitely. Instead of having a slope, however, we like to think of the "slope" of a three-dimensional line as a "parallel" vector, that doesn't necessarily represent the actual line, but the **behavior** of our current line.

A three-dimensional line is defined by the following terms:

- An **initial point** P_0 or just P.
- A **vector** that defines the line's **direction** and **behavior**, $\overrightarrow{P_0P}$ or $\overrightarrow{PQ}$, where P_0 and represents the initial point and P and Q represent **any point on the line**.

5.5.3 3-D Line Summary

Essentially, much like hte **two-dimensional line**, a **three-dimensional line** is defined by some point P that the line goes through, as well as a **directional vector** that starts from that initial point P and extends to any point Q on the line $\overrightarrow{PQ}$.

5.6 Planes in Space (04/10/23)

5.7 Reminders

TODo

5.8 Objectives

1. Determine vector and component equations
2. Produce non-zero vectors normal to a given plane
3. Compute the distance from a point to a plane in space
4. Determine whether two given planes coincide, intersect in a line, or are parallel

5.9 Motivation

TODO

Chapter 6

11.6: Conic Sections (04/12/23)

6.1 Reminders

6.2 Objectives

6.3 Motivation

Chapter 7

12.6: Cylinders and Quadric Surfaces (04/14/23)

7.1 Reminders

7.2 Objectives

1. Sketch the graph of various cylinders
2. Graph the **six** quadric surfaces by hand
3. Understand the usefulness of the coordinate plane traces as well as how to find them

7.3 Motivation

Chapter 8

11.3: Polar Coordinates (04/17/23)

8.1 Reminders

8.2 Objectives

1. Be able to differentiate between poloar coordinates and Cartesian coordinates
2. Be able to relate polar coordinates to Cartesian coordinates
3. Graph polar coordinate functions

8.3 Motivation

Chapter 9

13.1: Curves in Space and their Tangents (04/19/23)

9.1 Reminders

9.2 Objectives

9.3 Motivation

Chapter 10

13.3: Arc Length

10.1 Reminders

10.2 Objectives

10.3 Motivation

Chapter 11

14.1: Functions of Several Variables

11.1 Reminders

11.2 Objectives

11.3 Motivation

Chapter 12

14.3: Partial Derivatives

12.1 Reminders

12.2 Objectives

12.3 Motivation

Chapter 13

14.4: The Chain Rule

13.1 Reminders

13.2 Objectives

13.3 Motivation

Chapter 14

14.5: Gradient Vectors and Tangent Planes

14.1 Reminders

14.2 Objectives

14.3 Motivation

Chapter 15

14.5 (cont'd): Directional Derivatives

15.1 Reminders

15.2 Objectives

15.3 Motivation

Chapter 16

14.6: Tangent Planes and Linearization

16.1 Reminders

16.2 Objectives

16.3 Motivation

Chapter 17

10.9: Taylor's Formula

17.1 Reminders

17.2 Objectives

17.3 Motivation

Chapter 18

10.9: Taylor's Polynomials

18.1 Reminders

18.2 Objectives

18.3 Motivation

Chapter 19

14.7: Optimization

19.1 Reminders

19.2 Objectives

19.3 Motivation

Chapter 20

14.8: Lagrange Multipliers

20.1 Reminders

20.2 Objectives

20.3 Motivation

Chapter 21

14.8: Lagrange Multipliers (Part 2)

21.1 Reminders

21.2 Objectives

21.3 Motivation