Lecture 21: Applications (§10.9)

Goal:

1. Produce polynomial approximations to standard functions, and estimate the error in these approximations.

2. Explain several standard mathematical phenomena facts using Taylor polynomials and Taylor's Formula.

Recall:

Taylor's Theorem: Let f be a function that admits derivatives up to order $(n+1)$. Write $f(x) = P_n(x) + R_n(x)$. Then

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1},$$

for some c between a and x. $P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

The remainder estimation Theorem: Assume there exists $M > 0$ such that $\left|f^{(n+1)}(c)\right| \leq M$, for any c between x and a. Then:

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}.$$

Example. Find the Taylor polynomial of smallest order generated by $\frac{1}{1-x}$ at $x = 0$, that can approximate $\frac{1}{1-x}$ up to an accuracy of 10^{-4}, on each of the following intervals:

1. $[0, 0.1]$.

$P_n(x) = 1 + x + x^2 + \dots + x^n$

2. ~~$[0.3, 0.4]$~~.

$f(x) = \frac{1}{1-x}$

$f'(x) = \frac{1}{(1-x)^2}$

$f''(x) = \frac{2}{(1-x)^3}$

$\vdots$

$f^{k}(x) = \frac{k!}{(1-x)^{k+1}}$ — $k = n+1$

$\frac{f^{(k)}(0)}{k!} = 1$

$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = \frac{1}{(1-c)^{n+2}} x^{n+1}$

we consider the interval $[0, 0.1]$ so $|x| \leq 0.1$

$c \leq 0.1$

$1 - c \geq 0.9$

$\frac{1}{1-c} \leq \frac{1}{0.9}$

$R_n(x) \leq \frac{1}{0.9^{n+2}} \cdot 0.1^{n+1} = \frac{1}{0.9} \cdot \left(\frac{0.1}{0.9}\right)^{n+1} < 1.2 \left(\frac{1}{9}\right)^{n+1}$

For which n $R_n < 10^{-4}$?

$R_1 < 1.2 \cdot \frac{1}{9}^2 > \frac{1}{10}^4$

$R_2 < 1.2\, \frac{1}{9}^3 > \frac{1}{10}^4$

$R_3 \leq 1.2\, \frac{1}{9}^4 > \frac{1}{10}^4$

$R_4 < 1.2 \cdot \frac{1}{9}^5 < \frac{1}{10}^4$ ✓

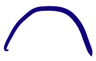
$$\frac{1}{1-x} = \underbrace{1+x+x^2+x^3+x^4}_{P_4(x)} + \underset{\wedge\ 10^{-4}}{R_4(x)} \qquad \text{At } [0, 0.1]$$

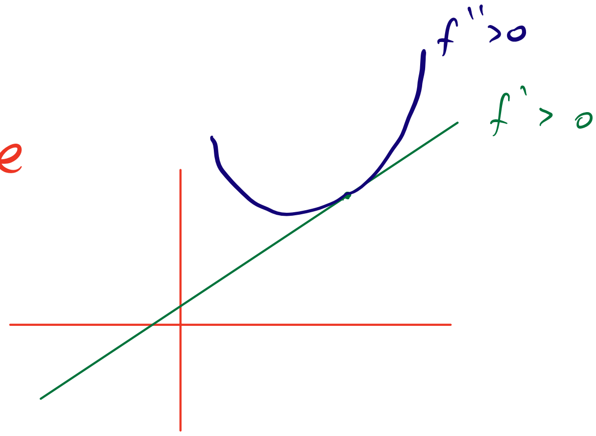
Convexity and concavity. The second derivative test

Consider the linear approximation of a function $f(x)$ near $x = b$:

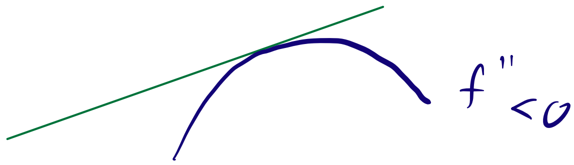
$$f(x) \approx f(b) + f'(b)(x - b).$$

Example. How does the graph $y = f(x)$ look with respect to the tangent line near $x = b$ if:

1. $f''(b) < 0$?  concave
2. $f''(b) > 0$?  convex

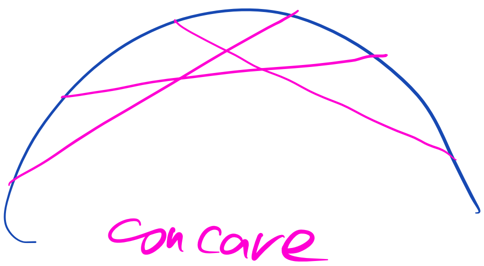


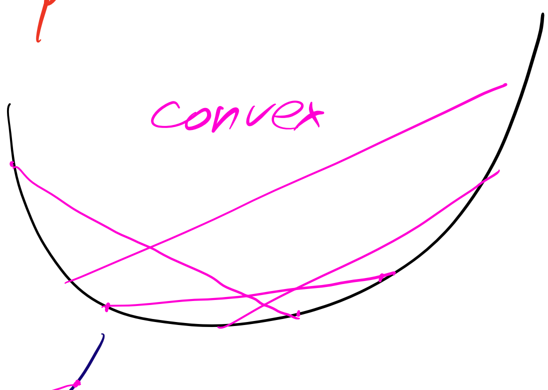
$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2}_{P_2(x)\ \text{parabola}} + \underset{\uparrow\ \text{small}}{R_2}$$



the parabola is open upwards if $f''(a) > 0$

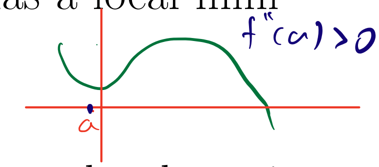
A function is convex if any segment joining points on the graph lies above the graph



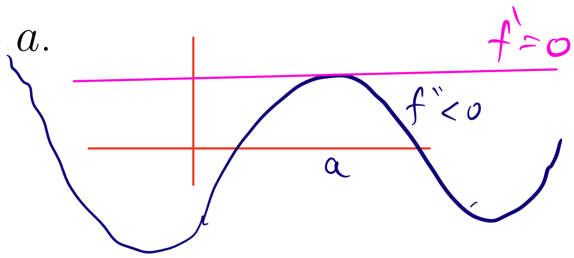


Example. (**Second Derivative Test for Local Exterma**). Show the following:

1. Suppose $f'(a) = 0$ and $f''(a) > 0$. Then $f(x)$ has a local minimum at a, i.e. $f(x) \lessgtr f(a)$ for x close to a.



2. Suppose $f'(a) = 0$ and $f''(a) < 0$. Then $f(x)$ has a local maximum at a, i.e. $f(x) \lessgtr f(a)$ for x close to a.

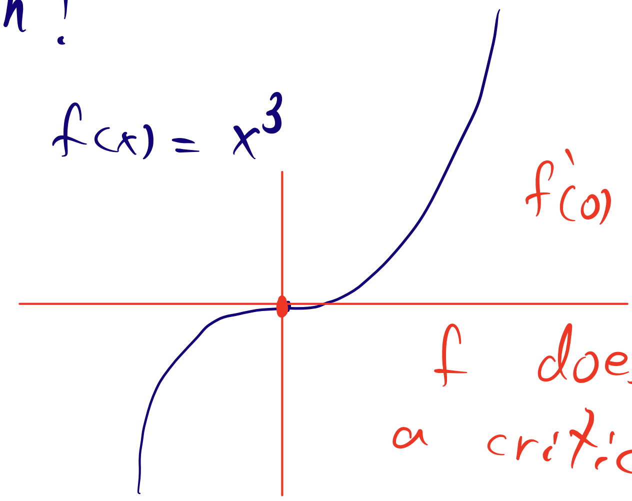


Example. Assume now that both $f'(a) = 0$ and $f''(a) = 0$. How can we know if it is a local maximum, local minimum, or neither?

Not much!

Example 1

$f(x) = x^3$



$f'(0) = f''(0) = 0$

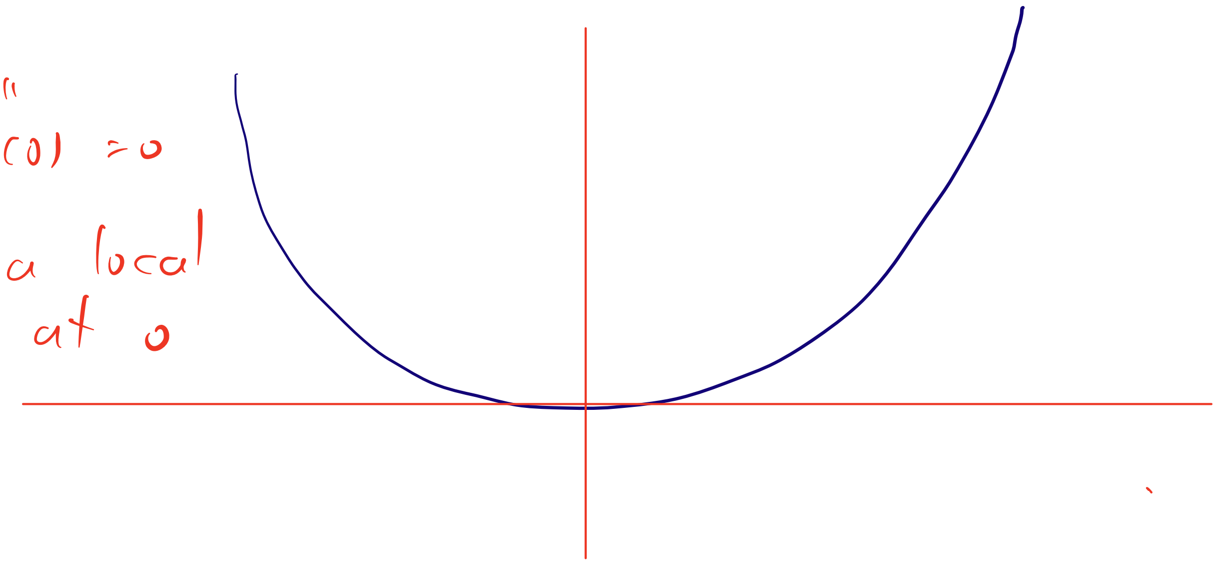
f does not have a critical point at 0.

Example 2

$f(x) = x^4$

$f'(0) = f''(0) = 0$

f has a local minimum at 0



Example. Find the Taylor polynomial of smallest order generated by $f(x) = \cos x$ at $\pi/6$ that can approximate $\cos x$ for any x between 28° and 32° with an error of less than 10^{-9}.