Lecture 22: Taylor's Formula for Two Variables (§14.9)

Goal: Compute the quadratic and cubic approximations of a function near a point, and estimate the error in the approximation.

Recall:

Definition. The **linearization** of a function $f(x, y)$ at a point (a, b) is

$$P_1(x, y) := L(x, y) = f(a, b) + f_x|_{(a,b)} \cdot (x - a) + f_y|_{(a,b)}(y - b).$$

It is the **first order Taylor polynomial** generated by $f(x, y)$ at (a, b).
The approximation $f(x, y) \approx P_1(x, y)$ is the **standard linear approximation of f at** (a, b).

Target : To obtain a second order and 3rd order etc Taylor polynomial for a multivariable function

Let $f(x, y)$ be a function (high-order differentiable). We would like to study its behavior near the point (a, b). Write $x = a + ht$ and $y = b + kt$ and consider

$$F(t) := f(a + ht, b + kt)$$

The 2nd order Taylor polynomial of F at $t=0$ is:

$$P_2(t) = \underline{F(0)} + \underline{F'(0)\cdot t} + \underline{\tfrac{1}{2}F''(0)\,t^2}$$

$$f(a+ht, b+kt) = F(t) \sim P_2(t) =$$

$$= \underline{f(a,b)} + \underline{\left(\frac{\partial f}{\partial x}(a,b)\,h + \frac{\partial f}{\partial y}(a,b)\cdot k\right)t} + \underline{\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(a,b)\,h^2 + 2\frac{\partial^2 f}{\partial x\partial y}hk + \frac{\partial^2 f}{\partial y^2}k^2\right)t^2}$$

Evaluating this at $t=1$ we get

$$f(a+h, b+k) \sim f(a,b) + \frac{\partial f}{\partial x}h + \frac{\partial f}{\partial y}k + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}h^2 + \frac{\partial^2 f}{\partial x\partial y}hk + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}k^2$$

Definition. The **second order Taylor polynomial** of $f(x,y)$ at (a,b) is

$$P_2(x,y) = f(a,b) + f_x|_{(a,b)} \cdot (x-a) + f_y|_{(a,b)}(y-b) + \frac{1}{2} f_{xx}|_{(a,b)} \cdot (x-a)^2 + f_{xy}|_{(a,b)}(x-a)(y-b) + \frac{1}{2} f_{yy}|_{(a,b)} \cdot (y-b)^2.$$

$P_2(x,y)$ has the same value, first order derivatives and second order partial derivatives as $f(x,y)$ at (a,b).

We say $f(x,y) \approx P_2(x,y)$ is the **quadratic approximation** of $f(x,y)$.

Example. Find the quadratic approximation of $f(x,y) = e^y \cos(x)$ near $(0,0)$.

$$P_2(x,y) = f(0,0) + f_x \cdot x + f_y \cdot y + \frac{1}{2} f_{xx}(0,0) x^2 + f_{xy}(0,0) xy + \frac{1}{2} f_{yy}(0,0) y^2$$

	at $(0,0)$
$f = e^y \cos x$	1
$f_x = -e^y \sin x$	0
$f_y = e^y \cos x$	1
$f_{xx} = -e^y \cos x$	-1
$f_{xy} = -e^y \sin x$	0
$f_{yy} = e^y \cos x$	1

$$\boxed{P_2(x,y) = 1 + y - \frac{1}{2}x^2 + \frac{1}{2}y^2}$$

Definition. The **third order Taylor polynomial** of $f(x,y)$ at (a,b) is

$$P_3(x,y) = P_2(x,y) + \frac{1}{3!}\left(f_{xxx}|_{(a,b)} \cdot (x-a)^3 + 3f_{xxy}|_{(a,b)}(x-a)^2(y-b)\right)$$
$$+ \frac{1}{3!}\left(3f_{xyy}|_{(a,b)} \cdot (x-a)(y-b)^2 + f_{yyy}|_{(a,b)} \cdot (y-b)^3\right).$$

We say $f(x,y) \approx P_3(x,y)$ is the **cubic approximation** of $f(x,y)$.

Exercise. (complete at home) Find the cubic approximation of $f(x,y) = xe^y$ near $(0,0)$.

$$P_3(x,y) = f(0,0) + f_x \cdot x + f_y \cdot y + \frac{1}{2}(f_{xx}x^2 + 2f_{xy}xy + f_{yy}y^2) + \frac{1}{3!}(f_{xxx}x^3 + 3f_{xxy}x^2y + 3f_{xyy}xy^2 + f_{yyy}y^3)$$

	at $(0,0)$
$f = xe^y$	0
$f_x = e^y$	1
$f_y = xe^y$	0
$f_{xy} = e^y$	1
$f_{xx} = 0$	0
$f_{yy} = xe^y$	0
$f_{xxx} = 0$	0
$f_{xxy} = 0$	0
$f_{xyy} = e^y$	1
$f_{yyy} = xe^y$	0

$$\boxed{P_3(x,y) = x + xy + \frac{1}{2}xy^2}$$

BTW

we know that

$$e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \dots$$

Therefore

$$xe^y = x\left(1 + y + \frac{y^2}{2} + \frac{y^3}{3!} + \dots\right) = \underbrace{\boxed{x + xy + \frac{1}{2}xy^2}}_{P_3(x,y)} + \dots$$

Write $P_1(x,y), P_2(x,y), P_3(x,y)$ be the 1\2\3-order Taylor polynomials of $f(x,y)$ at (a,b). For each $i \in \{1,2,3\}$, write

$$f(x,y) = P_i(x,y) + R_i(x,y),$$

where $R_i(x,y)$ indicates the error term. Recall that for some $0 < c < 1$:

$$R_1(x,y) = \underline{\left(\frac{(x-a)^2}{2} f_{xx} + (x-a)(y-b) f_{xy} + \frac{(y-b)^2}{2} f_{yy}\right)} \Big|_{(a+c(x-a),b+c(y-b))}$$

The error formula:

1. If M_1 is an upper bound for $|f_{xx}|, |f_{xy}|$ and $|f_{yy}|$ on a rectangular region R containing (a,b), then for each $(x,y) \in R$

$$|R_1(x,y)| \leq \frac{1}{2} M_1 \left(|x-a| + |y-b|\right)^2.$$

$|x-a|^2 + 2|x-a||y-b| + |y-b|^2$

2. Similarly, if M_2 is an upper bound for $|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}|$, then for each $(x,y) \in R$:

$$|R_2(x,y)| \leq \frac{1}{3!} M_2 \left(|x-a| + |y-b|\right)^3.$$

3. Similarly, if M_3 is an upper bound for all fourth order derivatives, then for each $(x,y) \in R$:

$$|R_3(x,y)| \leq \frac{1}{4!} M_3 \left(|x-a| + |y-b|\right)^4.$$

$$|R_n(x,y)| \leq M_n/(n+1)! \, (|x-a| + |y-b|)^{n+1}$$

where M_n is a bound for all the $(n+1)$'th derivatives

Example. Find the quadratic approximation to $f(x, y) = \sin(x)\sin(y)$ near the origin. How accurate is the approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$.

	at $(0,0)$
$f = \sin x \sin y$	0
$f_x = \cos x \sin y$	0
$f_y = \sin x \cos y$	0
$f_{xx} = -\sin x \sin y$	0
$f_{xy} = \cos x \cos y$	1
$f_{yy} = -\sin x \sin y$	0

$P_2(x,y) = xy$

The error is R_2 bonded by $\frac{M_2}{3!}(|x|+|y|)^3 \leq \frac{1}{6} 0.2^3 = \frac{4}{3000}$

$|x|, |y| \leq 0.1$

Again we could use the trick from before if we want higher order approximations:

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

$\sin y = y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots$

$\sin x \sin y \sim \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\left(y - \frac{y^3}{3!} + \frac{y^5}{5!}\right)$

$= \underbrace{xy - \frac{xy^3}{6} - \frac{x^3y}{6} + \frac{x^3y^3}{36} + \frac{xy^5}{5!} + \frac{x^5y}{5!}}_{P_6}$