Lecture 11: Curvature (§13.4)

Goals:

1. Compute the principal unit normal vector and curvature of a space curve.
2. Describe the relationship between curvature, osculating circles, and the plane spanned by the unit tangent and unit normal vectors to a space curve.

Unit tangent vector

We know that the velocity vector $\vec{v}(t)$ is tangent to the curve $\vec{r}(t)$ (at the point t) and hence the vector

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

is called the **unit tangent vector**.

If $s(t)$ is the arc length parameter for the curve, and let $\vec{r}(t(s))$ re-parametrization according to s. Recall that $\vec{r}(t(s))$ has unit speed, so:

$\vec{v} = \vec{r}\,' = \frac{d\vec{r}}{dt}$

$$\frac{d\vec{r}}{ds} = \frac{\vec{v}}{|\vec{v}|} = \vec{T}.$$

$\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|}$

Curvature of a plane curve

The curvature of a curve $\overrightarrow{r}(t)$ is a measurement of how fast does a curve change its direction. Let us first define it for unit speed curve.

Definition (for unit speed curve)**.** Let $\overrightarrow{r}(s)$ be a unit speed curve. Then the **curvature** is the rate of change of the (unit) tangent vector, i.e. the magnitude of the acceleration of $\overrightarrow{r}(s)$

$$\kappa := \left|\frac{d\overrightarrow{T}}{ds}\right| = \left|\frac{d^2\overrightarrow{r}}{ds^2}\right|.$$

$\vec{T} = \vec{r}\,' = \vec{v}$

$|\vec{v}| = 1$

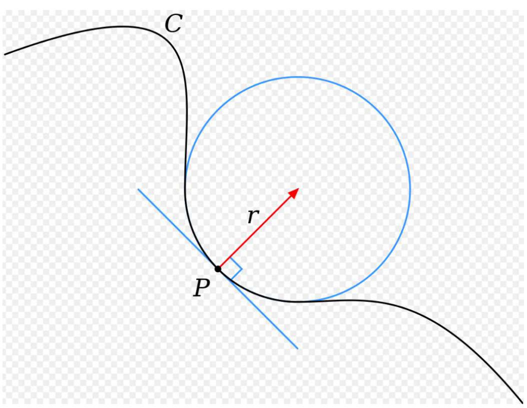
$\vec{a} = \vec{v}\,' = \vec{r}\,''$

$|\vec{a}| = k$

Geometric intuition of curvature: Osculating circles.

The circle of curvature, or osculating circle, at the point P (with $\kappa \neq 0$) is the circle in the plane of the curve which best "approximates" the curve in a small neighborhood of the point P. This circle is t**angent to the curve** at P, and its **center lies toward the inner side of the curve**.

The **curvature at P is precisely $\kappa = 1/R$, where** R is the radius of this circle.



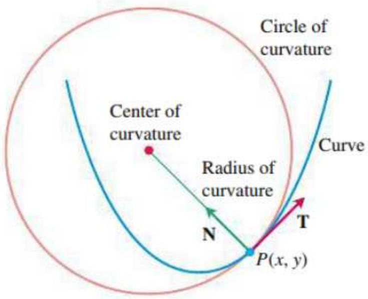


FIGURE 13.20 The center of the osculating circle at $P(x, y)$ lies toward the inner side of the curve.

Example 1. What is the curvature of a straight line $\overrightarrow{r}(t) = \overrightarrow{C} + t\overrightarrow{v}$?

$\vec{v} = \vec{r}'(t)$, $|\vec{v}|$ is the speed

$\vec{r}(t(s)) = \vec{C} + s \cdot \frac{\vec{v}}{|\vec{v}|} = \vec{C} + \frac{s}{|\vec{v}|}\vec{v}$

is a unit speed parametrisation

$s(t) = |\vec{v}| \cdot t$

$t(s) = \frac{s}{|\vec{v}|}$

(Suppose $|\vec{v}| = 1$)

$\frac{d\vec{r}}{ds} = \frac{\vec{v}}{|\vec{v}|}$, $\frac{d^2\vec{r}}{ds^2} = \vec{0}$,

$k = |\vec{r}''| = 0$

Example 2. Consider the circle of radius R, $\overrightarrow{r}(t) = \langle R\cos(t), R\sin(t)\rangle$. what is the curvature of $\overrightarrow{r}(t)$?

The unit speed parametrisation is $\vec{r}(t(s)) = \langle R\cos\frac{s}{R}, R\sin\frac{s}{R}\rangle$

$$\frac{d\vec{r}}{ds} = \langle -\sin\frac{s}{R}, \cos\frac{s}{R}\rangle, \quad \frac{d^2\vec{r}}{ds^2} = \langle -\frac{1}{R}\cos\frac{s}{R}, -\frac{1}{R}\sin\frac{s}{R}\rangle$$

$$k = \left|\frac{d^2\vec{r}}{ds^2}\right| = \frac{1}{R}$$

We would like now to define curvature for a general curve.

Definition. Let $\overrightarrow{r}(t)$ be a curve. Then the **curvature** is the following scalar function

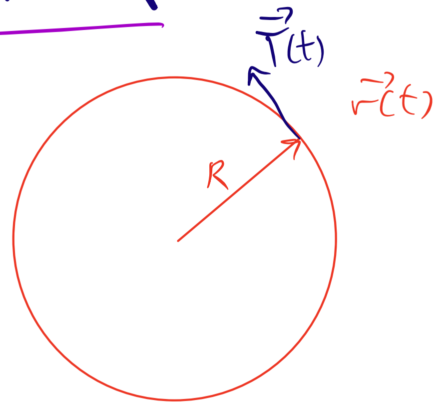
$$\kappa := \frac{1}{|\overrightarrow{r}'(t)|} \cdot \left|\frac{d\overrightarrow{T}}{dt}\right|,$$

where $\overrightarrow{T} = \frac{\overrightarrow{v}}{|\overrightarrow{v}|}$ is the unit tangent vector.

Let's get back to the example of a circle, this time with a non unit-speed parametrization. We should get the same curvature as we got before, $\frac{1}{R}$.

Example 3. Consider the circle of radius R, $\vec{r}(t) = \langle R\cos(ct), R\sin(ct)\rangle$. what is the curvature of $\vec{r}(t)$?

$\vec{r}\,' = \langle -cR\sin(ct), cR\cos(ct)\rangle$, $\underline{|\vec{r}\,'| = cR}$



$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\vec{r}\,'}{|\vec{r}\,'|} = \langle -\sin(ct), \cos(ct)\rangle$

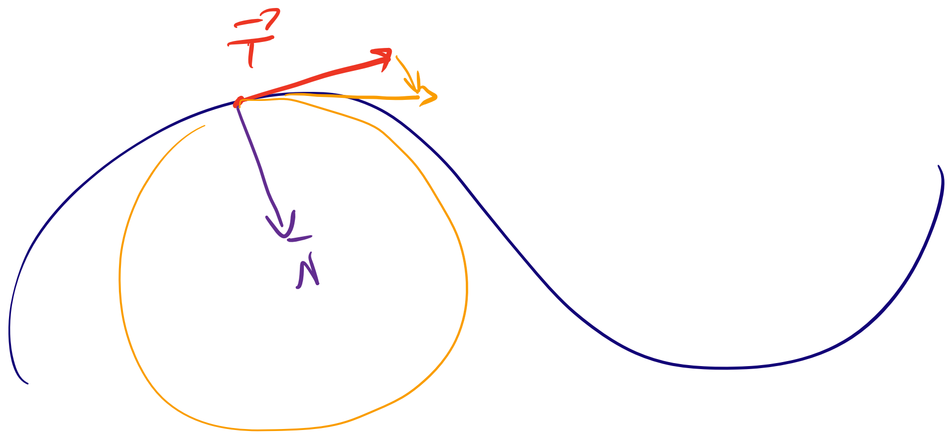
$\frac{d\vec{T}}{dt} = \langle -c\cdot\cos(ct), -c\cdot\sin(ct)\rangle$, $\left|\frac{d\vec{T}}{dt}\right| = c$

$k = \frac{1}{|\vec{r}\,'|}\cdot\left|\frac{d\vec{T}}{dt}\right| = \frac{c}{cR} = \frac{1}{R}$

Definition. Since $\left|\vec{T}(t)\right| = 1$, the vectors $\vec{T}\,'(t)$ and $\vec{T}(t)$ are orthogonal. In particular,

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{\left|\vec{T}\,'(t)\right|}$$

is a unit normal vector to the direction of motion.



$\vec{N}$ points to the center of the curvature circle

Curvature and normal vectors for space curves

Now assume that we have a curve $\overrightarrow{r}(t) = \langle \overrightarrow{x}(t), \overrightarrow{y}(t), \overrightarrow{z}(t) \rangle$ in space. The definitions for curvature, and normal vector are the same as in plane curves:

$$\kappa := \left| \frac{d\overrightarrow{T}}{ds} \right| = \frac{1}{|\overrightarrow{r}'(t)|} \cdot \left| \frac{d\overrightarrow{T}}{dt} \right| \text{ and } \overrightarrow{N}(t) = \frac{\overrightarrow{T}'(t)}{\left|\overrightarrow{T}'(t)\right|}$$

Here is a slightly more convenient formula for computing the curvature:

$$\kappa := \frac{|\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)|}{|\overrightarrow{r}'(t)|^3} = \frac{|\overrightarrow{v}(t) \times \overrightarrow{a}(t)|}{|\overrightarrow{v}(t)|^3}.$$

Example. Find a formula for a curvature of a plane curve $\overrightarrow{r}(t) = \langle \overrightarrow{x}(t), \overrightarrow{y}(t), 0 \rangle$. $r = \langle x, y, 0 \rangle$

$\vec{r}' = \langle x', y', 0 \rangle$, $|\vec{r}'| = \sqrt{x'^2 + y'^2}$

$\vec{r}'' = \langle x'', y'', 0 \rangle$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} i & j & k \\ x' & y' & 0 \\ x'' & y'' & 0 \end{vmatrix} = k \cdot (x'y'' - y'x'')$$

$$\boxed{K = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3} = \frac{|x'y'' - y'x''|}{(x'^2 + y'^2)^{\frac{3}{2}}}}$$

Example. Find the curvature $\kappa(t)$ of the curve $\overrightarrow{r}(t) = \langle \frac{t^2}{2}, t, \frac{t^3}{3} \rangle$.

$\vec{r}' = \langle t, 1, t^2 \rangle$, $|\vec{r}'| = \sqrt{t^2 + 1 + t^4}$

$\vec{r}'' = \langle 1, 0, 2t \rangle$

$$\vec{r}' \times \vec{r}'' = \begin{vmatrix} i & j & k \\ t & 1 & t^2 \\ 1 & 0 & 2t \end{vmatrix} = \begin{array}{ccc} i & j & k \\ \langle 2t, & -t^2, & -1 \rangle \end{array}$$

$$K = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{\sqrt{4t^2 + t^4 + 1}}{(t^2 + 1 + t^4)^{\frac{3}{2}}}$$

$K = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$

Example 4. Find the unit tangent vector $\overrightarrow{T}(t)$, unit normal vector $\overrightarrow{N}(t)$ and curvature $\kappa(t)$ of

$$\overrightarrow{r}(t) = \langle -4\sin(t), 4\cos(t), 3t \rangle.$$