Lecture 17: Tangent Planes (§14.6)

Goals:

1. Explain the connection between the gradient vector of a function at a point and the level curve (surface) of the function through that point.

2. Compute the equation of a plane tangent to a graph of a function.

3. Compute the linearization of a function of two or three variables at a point.

4. Relate the linearization of a two-variable function at a point to the plane tangent to the graph of the function at a point.

Last lecture we discussed the gradient and directional derivatives of functions of two variables. The same thing holds for three variables.

Definition. The **derivative of** $f(x, y, z)$ **at** (x_0, y_0, z_0) **in the direction of the unit vector** $\overrightarrow{u} = \langle u_1, u_2, u_3 \rangle$ is

$$D_{\overrightarrow{u}}(f)|_{(x_0,y_0,z_0)} := \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2, z_0 + su_3) - f(x_0, y_0, z_0)}{s}.$$

The **gradient of** $f(x, y, z)$ is the vector $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle$. In particular:

$$D_{\overrightarrow{u}}(f)|_{(x_0,y_0,z_0)} = \nabla f(x_0, y_0, z_0) \cdot \overrightarrow{u}.$$

Example.

1. Find the directional derivative of $f(x, y, z) = x^3 - xy^2 - z$ at the point $(1, 1, 0)$ in the direction of vector $v = \langle 2, -3, 6 \rangle$.

2. In what directions, starting at the point $(1, 1, 0)$, f has zero change?

1) $|\vec{v}| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7$

Set $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \langle \frac{2}{7}, \frac{-3}{7}, \frac{6}{7} \rangle$ be the unit vector in the same direction.

$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle 3x^2 - y^2, -2xy, -1 \rangle$, $\nabla f(1,1,0) = \langle 2, -2, -1 \rangle$

$D_{\vec{u}}(f)(1,1,0) = \nabla f(1,1,0) \cdot \vec{u} = \langle 2, -2, -1 \rangle \cdot \langle \frac{2}{7}, \frac{-3}{7}, \frac{6}{7} \rangle = \frac{4}{7}$

2) The directional $D_{\vec{w}} f(1,1,0) = 0$ iff $\nabla f(1,1,0) \cdot \vec{w} = 0$, i.e. $\vec{w}$ is orthogonal to $\nabla f(1,1,0)$. $\boxed{2w_1 - 2w_2 - w_3 = 0}$

If $\vec{w} = \langle w_1, w_2, w_3 \rangle$

There is a circle of directions for which $D_{\vec{w}}(f) = 0$.

Tangent planes

Given a level curve $f(x, y) = c$ of a function $f(x, y)$, and a point (a, b) on this curve, we have seen that $\nabla f(a, b)$ is orthogonal to the tangent line of the curve passing through (a, b).

Example. Let $\overrightarrow{r}(t) = \langle x(t), y(t), z(t) \rangle$ be a curve, and let $f(x, y, z)$ be a function. Find a formula for $\frac{d}{dt} f(\overrightarrow{r}(t))$.

By the chain rule:

$$\frac{d}{dt} f(\vec{r}(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} = \underbrace{\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle}_{\nabla f} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle}_{\vec{r}\,' = \vec{v}}$$

$$= \nabla f \cdot \vec{r}\,'$$

Example. Assume that the curve $\overrightarrow{r}(t)$ lies inside a level ~~curve~~ set $f(x, y, z) = c$. What can we say on the gradient ∇f at any point $\overrightarrow{r}(t)$?

If $\vec{r}(t) \subseteq \{ f = c \}$ it means $f(\vec{r}(t)) = c$

$$0 = \frac{d}{dt} f(\vec{r}(t)) = \vec{\nabla} f \cdot \vec{r}\,'(t)$$

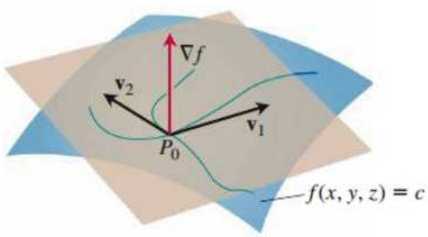


FIGURE 14.33 The gradient ∇f is orthogonal to the velocity vector of every smooth curve in the surface through P_0. The velocity vectors at P_0 therefore lie in a common plane, which we call the tangent plane at P_0.

Definition.

1. The **tangent plane** to the level surface $f(x, y, z) = c$ at a point $P_0(x_0, y_0, z_0)$ is the plane normal to $\nabla f(x_0, y_0, z_0)$ (assuming $\nabla f(x_0, y_0, z_0) \neq 0$), that is:

$$\frac{\partial f}{\partial x}|_{P_0} \cdot (x - x_0) + \frac{\partial f}{\partial y}|_{P_0} \cdot (y - y_0) + \frac{\partial f}{\partial z}|_{P_0} \cdot (z - z_0) = 0$$

$\nabla f \cdot \overrightarrow{P_0P} = 0, \quad P(x,y,z)$

2. The **normal line** of $f(x, y, z) = c$ at P_0 is the line through P_0, parallel to $\nabla f(x_0, y_0, z_0)$:

$\langle P_0 \rangle + t \cdot \nabla f(P_0)$

$$\langle x_0 + tf_x|_{P_0}, y_0 + tf_y|_{P_0}, z_0 + tf_z|_{P_0} \rangle.$$

Example. Find the tangent plane and normal line of the level surface $x^2 + y^2 + z - 9 = 0$ at the point $P_0(1, 2, 4)$.

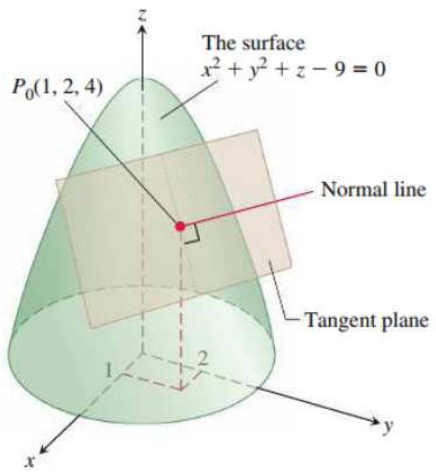


FIGURE 14.34 The tangent plane and normal line to this level surface at P_0 (Example 1).

$\nabla f = \langle 2x, 2y, 1 \rangle$

$\nabla f(1,2,4) = \langle 2, 4, 1 \rangle$

The tangent plane is

$\nabla f \cdot \overrightarrow{P_0P} = 0$

$2(x-1) + 4(y-2) + 1(z-4) = 0$

$2x + 4y + z - 14 = 0$

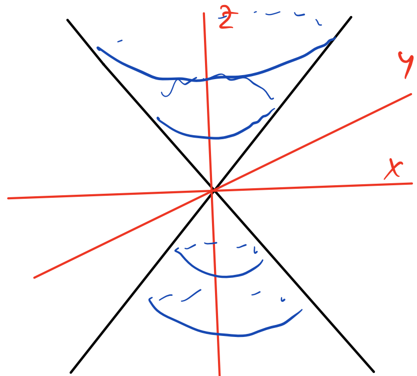
The normal line is given by

$\underbrace{\langle 1,2,4 \rangle}_{P_0} + t \cdot \underbrace{\langle 2,4,1 \rangle}_{\nabla f(P_0)} = \langle 1+2t, 2+4t, 4+t \rangle$

Example. (pathology) Consider the surface $x^2 + y^2 - z^2 = 0$. What happens at $(0, 0, 0)$?

$\nabla f = \langle 2x, 2y, -2z \rangle$

$\nabla f(0,0,0) = \vec{0}$



Tangent plane to surface $z = f(x, y)$

Suppose we have a surface given by the graph of function $z = f(x, y)$.

Set $F(x,y,z) = f(x,y) - z$

The surface $\{f(x,y) = z\}$ is the level set $\{F(x,y,z) = 0\}$

So we can find its tangent plane and normal line using $\nabla F = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \rangle$

Definition. The **tangent plane** to a surface $z = f(x, y)$ at a point $(x_0, y_0, f(x_0, y_0))$ is:

$\nabla F \cdot \overrightarrow{P_0 P}$

$$f_x|_{(x_0,y_0)} \cdot (x - x_0) + f_y|_{(x_0,y_0)} \cdot (y - y_0) - (z - z_0) = 0.$$

Example. Find the plane tangent to the surface $z = x\cos(y) - ye^x$ at $(0, 0, 0)$.

$\frac{\partial f}{\partial x} = \cos(y) - y \cdot e^x$, $\frac{\partial f}{\partial y} = -x\sin(y) - e^x$

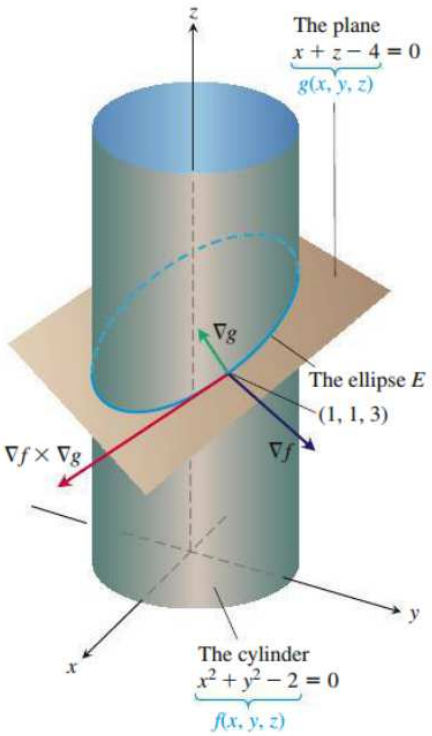
$\frac{\partial f}{\partial x}(0,0,0) = 1$, $\frac{\partial f}{\partial y}(0,0,0) = -1$

So the tangent plane is

$1 \cdot (x - 0) - 1(y - 0) - 1(z - 0) = 0$

$\Rightarrow$ $x - y - z = 0$

Example. The surfaces $f(x,y,z) = x^2 + y^2 - 2 = 0$ (a cylinder) and $g(x,y,z) = x + z - 4 = 0$ (a plane) intersect in an ellipse E. Find parametric equations for the line tangent to E at the point $P_0(1,1,3)$.



$\nabla f = \langle 2x, 2y, 0 \rangle$

$\nabla f(P_0) = \langle 2, 2, 0 \rangle$

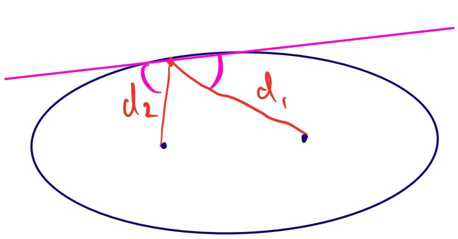
$\nabla g = \langle 1, 0, 1 \rangle$

$$\begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \langle \overset{i}{2}, \overset{j}{-2}, \overset{k}{-2} \rangle$$

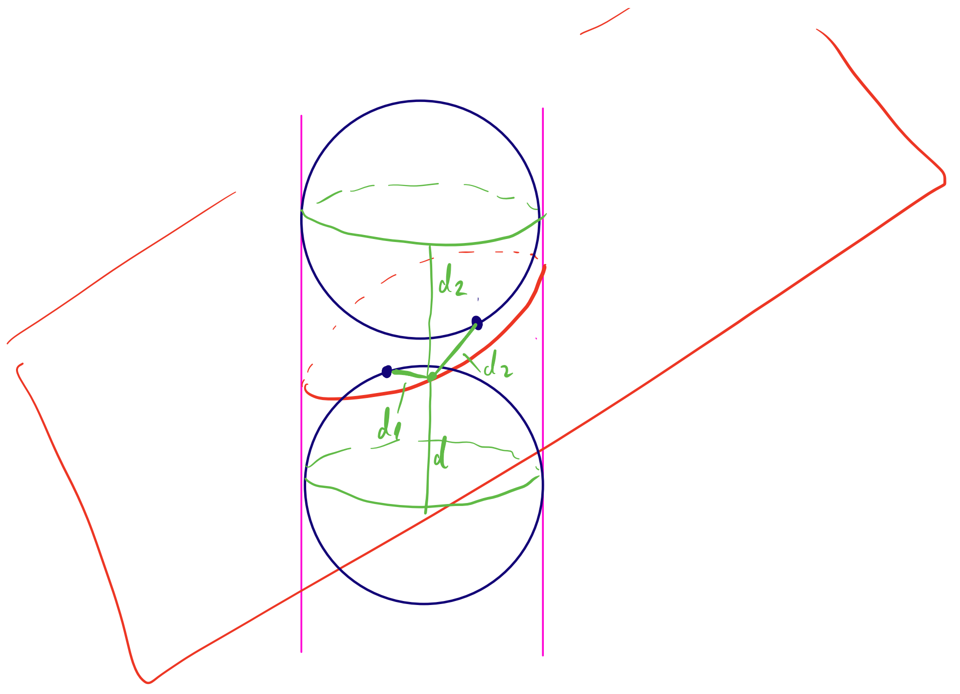
The line tangent to the ellipse at $(1,1,3)$ is

$$\langle 1, 1, 3 \rangle + t\langle 2, -2, -2 \rangle$$

An ellipse has two centers and a radius r and is defined by the set of points for which the sum of distances to the centers is 2r



$d_1 + d_2 = 2r$



d_2
d_2
d_1
d