Lecture 24: Optimization (§14.7)

Goal:

1. State the Extreme Value Theorem for continuous functions on closed, bounded regions.
2. Identify all points in a closed and bounded region where a given continuous function may have an absolute maximum or absolute minimum value.
3. Determine the absolute maximum and absolute minimum values of a continuous function on a closed and bounded region, and identify the point(s) where these values occur.

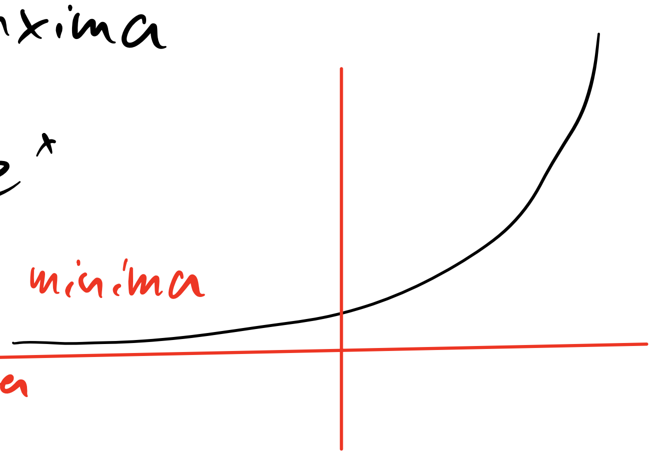
Last lecture we learn how to find local minimum and maximum of functions of several variables. For many real life applications, we are interested in the finding the **absolute** maximum or minimum. For examples:

- Finding the optimal release angle of a javelin in the Olympics.
- Navigation- finding the optimal route between two points A and B.
- Finding the optimal seats in a basketball game.
- Manufacturing an iPhone.
- Making the perfect Cappuccino.
- Maximizing your experience ("happiness function") when going to a restaurant.

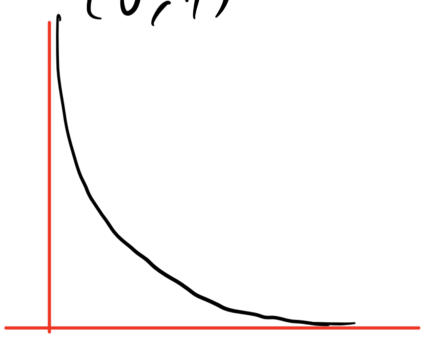
Remark A continuous may not have a global minima / maxima

examples: 1) $f(x) = e^x$

has no minima and maxima



2) $f(x) = \frac{1}{x}$ on $(0,1)$



Definition.

1. An **absolute minimum** of $f(x, y)$ is a point (a, b) in the domain D of f such that $f(a, b) \leq f(x, y)$ for all $(x, y) \in D$.

2. An **absolute maximum** of $f(x, y)$ is a point (a, b) in the domain of f such that $f(a, b) \geq f(x, y)$ for all $(x, y) \in D$.

A function $f(x, y)$ may not admit an absolute maximum or minimum.

Example. Let $f(x, y) = xy$. Then f has no absolute minimum or maximum.

<u>Theorem</u> On a bouded closed domain every continuous function attains an absolute minima and maxima.

How to find the absolute minima and maxima?

Suppose a (continuous) function $f(x, y)$ is defined on a **closed** and **bounded** domain D (otherwise, absolute extrema may not exist).

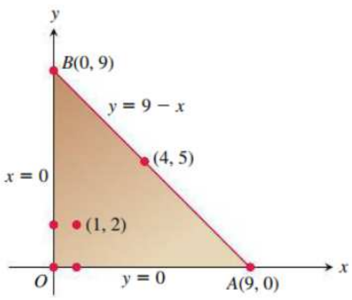
1. Find the values of f at the critical points in the interior of D.

2. Find the extreme values of f on the boundary of D.

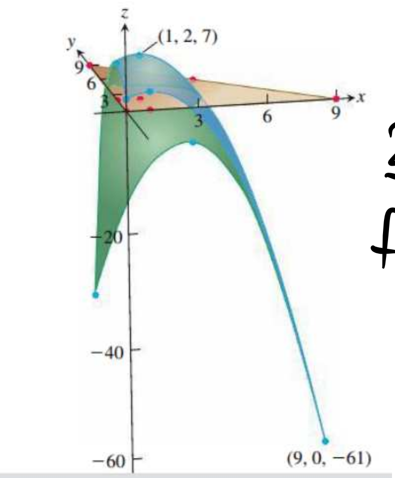
3. Compare the values in (1) and (2).

The absolute minimum and maximum values of f are precisely the smallest and largest numbers among the ones found above.

Example. Find the absolute maximum and minimum values of $f(x,y) = 2 + 2x + 4y - x^2 - y^2$ on a triangular region in the first quadrant by the lines $x = 0$, $y = 0$ and $y = 9 - x$.



(a)



Step 1 (Finding critical points in the interior)

$\nabla f = (2-2x, 4-2y)$

$\nabla f = 0$ only at $\boxed{(1,2)}$

step 2 (Critical points on the boundary

2a on the segment $y=0,\ 0\le x\le 9$

$f(x,0) = 2+2x-x^2$, $f_x = 2-2x$

$f_x = 0$ only at $\boxed{(1,0)}$

2b Consider the segment $x=0,\ 0\le y\le 9$

$f_y = 4-2y$

$f_y = 0 \Rightarrow y = 2 \Rightarrow \boxed{(0,2)}$

2c On the 3rd segment $y = 9-x,\ 0\le x\le 9$

$f(x, 9-x) = 2+2x+4(9-x)-x^2-(9-x)^2 =$

$= 2+2x+36-4x-x^2-81+18x-x^2 =$

$= -43+16x-2x^2$ $\quad 0 = \frac{d}{dx}(-43+16x-2x^2) = 16-4x$

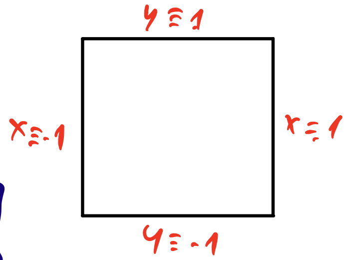
$\Rightarrow x=4,\ y=9-x=5$ $\quad \boxed{(4,5)}$

Step 2' (Corners) $(0,0)$, $(0,9)$, $(9,0)$

(1,2)	(1,0)	(0 2)	(4,5)	(0,0)	(0,9)	(9,0)
7	3	+6	−11	2	−43	−61

$2+2x+4y-x^2-y^2$

Example. Find the absolute maximum and minimum values of $f(x,y) = y^2 - x^2$ on the square $|x| \leq 1, |y| \leq 1$.



step 1 (Interior critical points)

$\nabla f = (-2x, 2y)$, $\nabla f = 0 \Rightarrow (x,y) = \boxed{(0,0)}$

Step 2a The boundary segment $x=1$, $-1 \leq y \leq 1$

$f(1,y) = y^2 - 1$, $\frac{\partial f}{\partial y} = 2y$ $\quad \frac{\partial f}{\partial y} = 0 \Rightarrow y = 0 \rightarrow \boxed{(1,0)}$

2b $x=-1$, $-1 \leq y \leq 1$, $f(-1,y) = y^2 - 1 \Rightarrow \boxed{(-1,0)}$

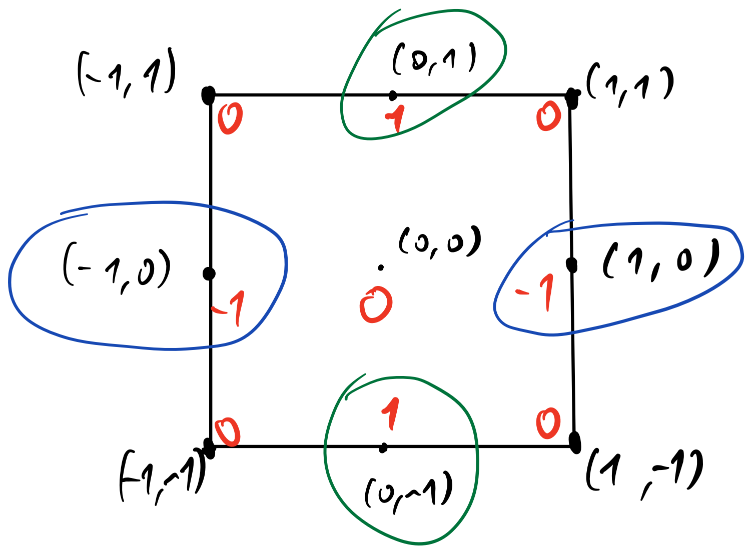
2c On the boundary segment $y = 1$, $-1 \leq x \leq 1$

$f(x,1) = 1 - x^2$, $f_x = -2x$, $f_x = 0 \Rightarrow x = 0 \rightarrow \boxed{(0,1)}$

2d On $y = -1$, $-1 \leq x \leq 1$

$f(x,y) = 1 - x^2 \longrightarrow \boxed{(0,-1)}$

2' (corners) $(-1,-1)$, $(-1,1)$, $(1,-1)$, $(1,1)$



$y^2 - x^2$

There are two maxima
$(0,1,1)$, $(0,-1,1)$

and two minima
$(1,0,-1)$, $(-1,0,-1)$

Example. Find the critical points of $f(x, y) = (x^2 + y^2)e^{-x}$, and determine whether they are minima, maxima or saddle points.