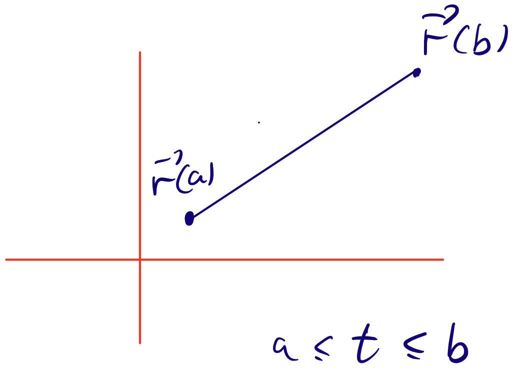
Lecture 10: Arc Length (§13.3)

Goals:

1. Compute the length of a space curve.
2. Reparametrize a space curve with respect to arc length.
3. Compute the unit tangent vector of a space curve.

Length of a curve

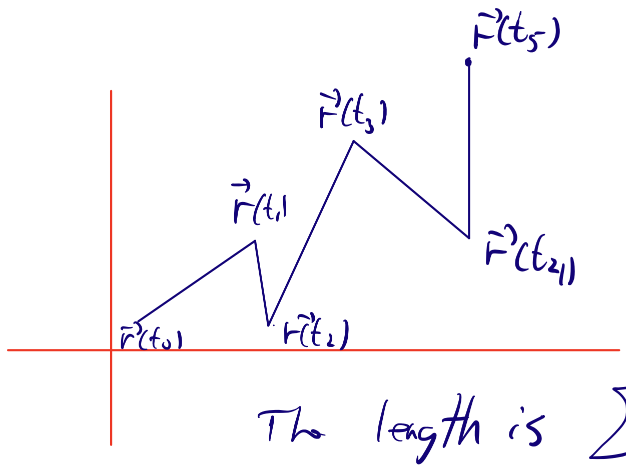
1 Toy example



the length is $|\vec{r}(b) - \vec{r}(a)|$

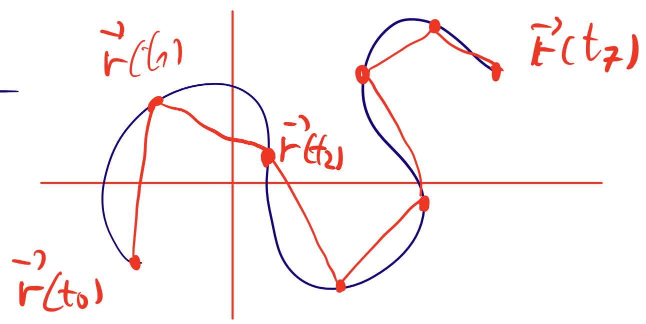
$a \leq t \leq b$

Example 2



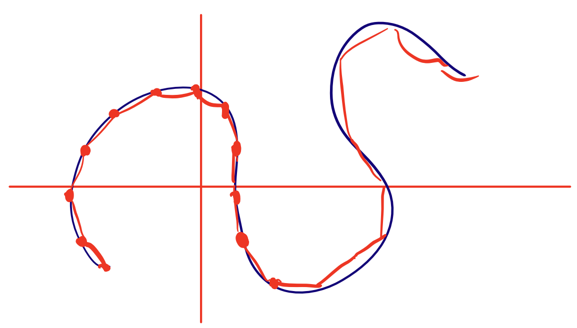
The length is $\sum_{i=1}^{5} \vec{r}(t_i) - \vec{r}(t_{i-1})$

More generally



$\sum_{i=1}^{7} |\vec{r}(t_i) - \vec{r}(t_{i-1})|$

approximate the length



A finer division of the time interval would give a better approximation

Let $\overrightarrow{r}(t) = \langle x(t), y(t), z(t)\rangle$ be a continuously differentiable curve (i.e. $\overrightarrow{r}'(t)$ is continuous). Assume that as t goes from a to b, every point $\overrightarrow{r}(t)$ is traced exactly once. Then the **length** of $\overrightarrow{r}(t)$ from $t = a$ to $t = b$ is:

$$L = \int_a^b |\overrightarrow{r}'(t)|\, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt.$$

Example. A glider is soaring upward along the helix $\overrightarrow{r}(t) = \langle \cos(t), \sin(t), t\rangle$.

1. How long is the glider's path from $t = 0$ to $t = 2\pi$?

2. Repeat Item (1) for a different parametrization of the helix $\overrightarrow{r}(t) = \langle \cos(4t), \sin(4t), 4t\rangle$, for $0 \leq t \leq \frac{1}{2}\pi$.

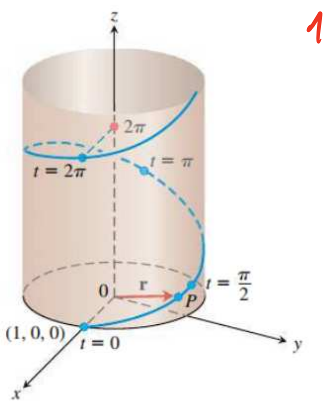


FIGURE 13.13 The helix in Example 1, $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$.

1) The length is:

$$\int_0^{2\pi} |\vec{r}'(t)|dt = \int_0^{2\pi} |\langle -\sin t, \cos t, 1\rangle| dt =$$

$$= \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\, dt = \int_0^{2\pi} \sqrt{2}\, dt = \underline{\underline{2\sqrt{2}\pi}}$$

2) This time the length is:

$$\int_0^{\pi/2} |\vec{r}'(t)|dt = \int_0^{\pi/2} |\langle -4\sin 4t, 4\cos 4t, 4\rangle| dt = \int_0^{\pi/2} \sqrt{16(\sin 4t)^2 + 16(\cos 4t)^2 + 16}\, dt$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{32}\, dt = \underline{\underline{2\sqrt{2}\pi}}$$

Example. Consider $\vec{r}(t) = \langle \cos(10t), \sin(10t) \rangle$ for $0 \leq t \leq 2\pi$.

The length is

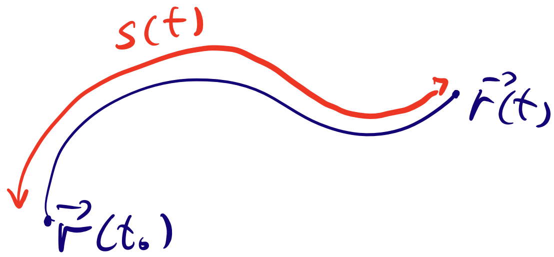
$$\int_0^{2\pi} |\vec{r}\,'(t)|\,dt = \int_0^{2\pi} \sqrt{100(\sin \underbrace{10t}_{\alpha})^2 + 100(\cos \underbrace{10t}_{\alpha})^2}\, dt = \int_0^{2\pi} 10 = \underline{\underline{20\pi}}$$

$(\sin\alpha)^2 + (\cos\alpha)^2 = 1$

Arc length parametrization

Given a curve $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, fix a starting time t_0. Then the following scalar function

$$s(t) = \int_{t_0}^{t} |\vec{r}\,'(\tau)|\, d\tau,$$



is called an **arc length parameter** for the curve (we use τ for the variable of integration, since t is used as the upper limit).

$s = s(t)$

$t = t(s)$

$\vec{r}(t(s))$ is a unit speed (w.r.t s) parametrization

The arc length parameter can be used to find a parametrization with unit speed, as follows:

Example. Consider the curve C parametrized by

$$\vec{r}(t) = \langle \cos(2t), \sin(2t), \frac{1}{3}(2t-4)^{\frac{3}{2}} \rangle,$$

for $t \geq 2$. Find a parametrization of C with unit speed.

$$\vec{r}\,'(t) = \langle -2\sin 2t, 2\cos 2t, (2t-4)^{\frac{1}{2}} \rangle$$

$$s(t) = \int_2^t \vec{r}\,'(\tau)\,d\tau = \int_2^t \sqrt{\underbrace{(-2\sin 2t)^2 + (2\cos 2t)^2}_{4} + (2t-4)}\;dt$$

$$\int \sqrt{t} = \frac{2}{3} t^{\frac{3}{2}}$$

$$= \int_2^t \sqrt{2\tau}\,d\tau = 2\sqrt{2}\,\tau^{\frac{3}{2}}\Big|_2^t = \frac{2\sqrt{2}}{3} t^{\frac{3}{2}} - 8$$

$$s(t) = \frac{2\sqrt{2}}{3} t^{\frac{3}{2}} - 8$$

$$t(s) = \left(\frac{3}{2\sqrt{2}}(s+8)\right)^{\frac{2}{3}}$$

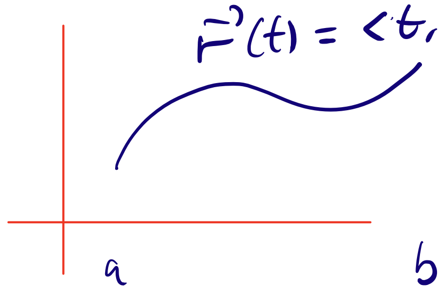
$$\boxed{\begin{aligned} s &= 2\sqrt{2}\,t^{\frac{3}{2}} - 8 \\ s+8 &= 2\sqrt{2}\,t^{\frac{3}{2}} \\ \frac{1}{2\sqrt{2}}(s+8) &= t^{\frac{3}{2}} \end{aligned}}$$

The unit speep parametrization of C is $\vec{r}(s) = \vec{r}(t(s)) = \vec{r}\left(\left(\frac{3}{2\sqrt{2}}(s+8)\right)^{\frac{2}{3}}\right)$

$$\vec{r}(t) = \langle \cos(2t), \sin(2t), \frac{1}{3}(2t-4)^{\frac{3}{2}} \rangle \simeq \vec{r}(t(s))$$

Example. Let $f(x)$ be a smooth scalar function. Write a formula for the length of the graph of $f(x)$ from $x = a$ to $x = b$.

$\vec{r}(t) = \langle t, f(t) \rangle$



$x = t \, , \quad y = f(t)$

The length of the graph from a to b is

$$\int_a^b |\vec{r}'(t)| \, dt = \int_a^b \sqrt{1 + f'(t)} \, dt$$

Unit tangent vector

We know that the velocity vector $\vec{v}(t)$ is tangent to the curve $\vec{r}(t)$ (at the point t) and hence the vector

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

is called the **unit tangent vector**.

If $s(t)$ is the arc length parameter for the curve, and let $\vec{r}(t(s))$ re-parametrization according to s. Recall that $\vec{r}(t(s))$ has unit speed so:

$$\frac{d\vec{r}}{ds} = \frac{\vec{v}}{|\vec{v}|} = \vec{T}.$$

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{\left| \frac{d\vec{r}}{dt} \right|} = \vec{T}$$

Example. Find the unit tangent vector of the curve

$$\vec{r}(t) = \langle 1 + 3\cos(t), 3\sin(t), t^2 \rangle.$$

$$\vec{r}\,'(t) = \langle -3\sin(t),\ 3\cos(t),\ 2t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2} = \sqrt{9+4t^2}$$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{1}{\sqrt{9+4t^2}} \langle -3\sin t,\ 3\cos t,\ 2t \rangle$$

Example. Find an arc length parametrization of

$$\vec{r}(t) = \langle \cos(e^t), \sin(e^t) \rangle.$$

Example. Give a good estimate to the unit tangent vector at $t = 5$

$$\vec{r}(t) = \langle e^{\cos(e^t)}, e^{(e^t)}, e^{(e^{(e^t)})} \rangle,$$

as t goes from 0 to 5.

Some riddles

Problem 1. (Monty Hall problem) Suppose you're on a game show, and you're given the choice of three doors: behind one door is a car; behind the others, goats. You pick a door, say No. 1, and the host, who knows what's behind the doors, opens another door, say No. 3, which has a goat. He then says to you, "Do you want to pick door No. 2 instead?" What will be your answer?

Problem 2. (In class)