Lecture 12: Tangential and Normal Components of Acceleration (§13.5)

Goals:

1. Compute the tangential and normal scalar components of acceleration of a space curve.
2. Compute the binormal vector of a curve.

Recall: Curvature and normal vectors for space curves

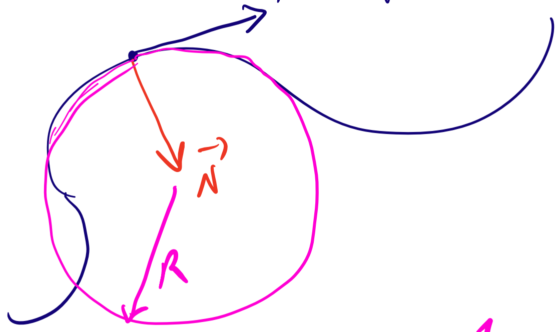
Let $\overrightarrow{r}(t) = \langle \overrightarrow{x}(t), \overrightarrow{y}(t), \overrightarrow{z}(t) \rangle$ be a curve in space. Then: $\vec{T} = \frac{\vec{v}}{\vec{v}}$

$\vec{r}(t)$

$\vec{r}(t(s))$ - unit speed parametrization

$$\overrightarrow{T}(t) = \frac{\overrightarrow{r}'(t)}{|\overrightarrow{r}'(t)|}.$$

$$\overrightarrow{N}(t) = \frac{\overrightarrow{T}'(t)}{\left|\overrightarrow{T}'(t)\right|}.$$



$k = \frac{1}{R}$

$$\kappa := \left|\frac{d\overrightarrow{T}}{ds}\right| = \frac{1}{|\overrightarrow{r}'(t)|} \cdot \left|\overrightarrow{T}'(t)\right|.$$

Another formula for the curvature:

$$\kappa := \frac{|\overrightarrow{r}'(t) \times \overrightarrow{r}''(t)|}{|\overrightarrow{r}'(t)|^3} = \frac{|\overrightarrow{v}(t) \times \overrightarrow{a}(t)|}{|\overrightarrow{v}(t)|^3}.$$

When a particle is moving in space, with position vector $\overrightarrow{r}(t) = \langle \overrightarrow{x}(t), \overrightarrow{y}(t), \overrightarrow{z}(t) \rangle$, at any given moment t, there is a more natural coordinate system than the usual coordinate system $\mathrm{i}, \mathrm{j}, \mathrm{k}$:

- The direction of the particle, indicated by the unit tangent vector

$$\overrightarrow{T}(t) = \frac{\overrightarrow{r}'(t)}{|\overrightarrow{r}'(t)|}. \qquad = \frac{\vec{v}}{|\vec{v}|}$$

- The direction in which the curve is turning, indicated by the unit normal vector

$$\overrightarrow{N}(t) = \frac{\overrightarrow{T}'(t)}{\left|\overrightarrow{T}'(t)\right|}.$$

- The **binomial vector**

$$\overrightarrow{B}(t) := \overrightarrow{T}(t) \times \overrightarrow{N}(t)$$

  which is a normal vector to the plane defined by $\overrightarrow{T}(t), \overrightarrow{N}(t)$, called **the osculating plane**.

Together, $\overrightarrow{T}, \overrightarrow{N}$,and $\overrightarrow{B}$ define a moving right handed vector frame, called the **Frenet frame**, or the **TNB frame**.

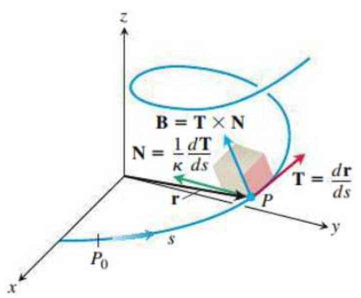


FIGURE 13.23 The **TNB** frame of mutually orthogonal unit vectors traveling along a curve in space.

Tangential and normal component of acceleration:

Any vector can be expressed in the TNB coordinate frame. In particular, when an object accelerates (e.g. by gravity) we often need to know how much of the acceleration occurs in the direction of motion.

Tangential component of the acceleration: a_T is the scalar component of $\overrightarrow{a}$ in direction of $\overrightarrow{T}$ (i.e. $\text{proj}_{\overrightarrow{T}}\overrightarrow{a} = a_T\overrightarrow{T}$).

Normal component of the acceleration: a_N is the scalar component of $\overrightarrow{a}$ in direction of $\overrightarrow{N}$ (i.e. $\text{proj}_{\overrightarrow{N}}\overrightarrow{a} = a_N\overrightarrow{N}$).

Example. What can we say about a_B?

$\vec{r} = \vec{r}(t)$

$\vec{v} = \vec{r}\,' = |\vec{v}|\cdot\vec{T}$

$\vec{a} = \vec{v}\,' = |\vec{v}|'\cdot\vec{T} + |\vec{v}|\cdot\vec{T}\,' =$

$= |\vec{v}|'\cdot\vec{T} + |\vec{v}|\cdot|\vec{T}\,'|\cdot\dfrac{\vec{T}\,'}{|\vec{T}\,'|} =$

$= |\vec{v}|'\cdot\vec{T} + |\vec{v}|^2\cdot\dfrac{|\vec{T}\,'|}{|\vec{v}|}\cdot\vec{N}$

$|\vec{v}|'\cdot\vec{T} + |\vec{v}|^2\cdot k\cdot\vec{N} = |\vec{v}|'\cdot\vec{T} + \dfrac{|\vec{v}|^2}{R}\cdot\vec{N}$

$\vec{T} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\vec{r}\,'}{|\vec{r}\,'|}$

$\vec{N} = \dfrac{\vec{T}\,'}{|\vec{T}\,'|}$

$k = \dfrac{|\vec{T}\,'|}{|\vec{v}|}$

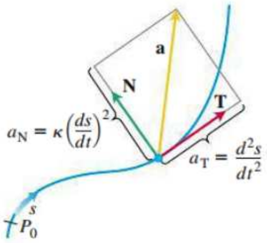


FIGURE 13.25 The tangential and normal components of acceleration. The acceleration **a** always lies in the plane of **T** and **N** and is orthogonal to **B**.

Definition. If the acceleration vector is written as

$$\overrightarrow{a} = a_T\overrightarrow{T} + a_N\overrightarrow{N},$$

then the **tangential** and **normal** scalar components of acceleration are:

$$a_T = \frac{d}{dt}|v(t)| \text{ and } a_N = \kappa\cdot|v(t)|^2.$$

Another way to compute a_T, a_N, is

$a_N = \vec{a}\cdot\vec{N}$

$$a_T = \overrightarrow{a}\cdot\overrightarrow{T} = \frac{\overrightarrow{r}''(t)\cdot r'(t)}{|r'(t)|} \text{ and } a_N = \sqrt{|\overrightarrow{a}|^2 - a_T^2}$$

Example. Compute everything!!!!!!! ($\overrightarrow{T}$, $\overrightarrow{N}$, $\overrightarrow{B}$, κ, a_T and a_N), for

$$\overrightarrow{r}(t) = \langle \cos(t), \sin(t), t \rangle.$$

$\vec{v} = \vec{r}\,' = \langle -\sin t, \cos t, 1 \rangle$ $\qquad |\vec{v}| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{2}}\langle -\sin t, \cos t, 1 \rangle$

$\vec{N} = \frac{\vec{T}\,'}{|\vec{T}\,'|}$, $\quad \vec{T}\,' = \frac{1}{\sqrt{2}}\langle -\cos t, -\sin t, 0 \rangle$, $\qquad \vec{N} = \langle -\cos t, -\sin t, 0 \rangle$

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} i & j & k \\ \frac{-\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \left\langle \frac{\sin t}{\sqrt{2}}, \frac{-\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$\vec{B} = \frac{1}{\sqrt{2}}\langle \sin t, -\cos t, 1 \rangle$

$K = \frac{|\vec{T}\,'|}{|\vec{v}|}$, $\quad \vec{T}\,' = \frac{1}{\sqrt{2}}\langle -\cos t, -\sin t, 0 \rangle$, $\quad |\vec{T}\,'| = \frac{1}{\sqrt{2}}$

$K = \frac{1}{2}$

$\vec{a} = \vec{v}\,' = \langle -\cos t, -\sin t, 0 \rangle$

$a_T = \vec{a} \cdot \vec{T} = \langle -\cos t, -\sin t, 0 \rangle \cdot \left\langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, 0 \right\rangle = 0$

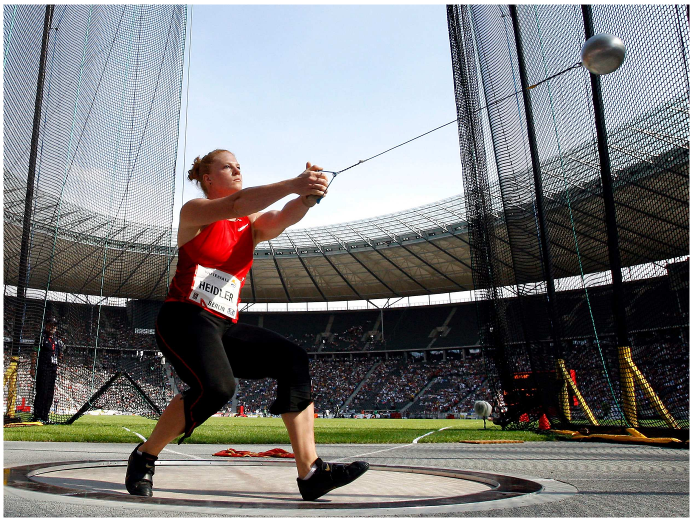
$a_T = 0$

$a_N = \vec{a} \cdot \vec{N} = \langle -\cos t, -\sin t, 0 \rangle \cdot \langle -\cos t, -\sin t, 0 \rangle = 1$

$a_N = 1$

$\vec{a} = a_N \cdot \vec{N} \iff$ constant speed

Example. Assume you are in the Olympics, in a hammer throw event, and you are rotating a metal ball, attached by a steel wire (of a fixed length R) to a grip, as in the picture. In order to keep the ball at constant speed v, you are holding the grip with a force of constant magnitude F. How much force you need to apply on the grip, in order to rotate the ball at a constant speed of $2v$?



$\vec{r}(t) = R\langle \cos \frac{v}{R}t, \sin \frac{v}{R}t \rangle$

$\vec{v} = \vec{r}\,' = v\langle -\sin(\frac{v}{R}t), \cos(\frac{v}{R}t) \rangle$

$\vec{a} = \vec{v}\,' = \vec{r}\,'' = \frac{v^2}{R}\langle -\cos(\frac{v}{R}t), -\sin(\frac{v}{R}t) \rangle$

$|\vec{a}| = \frac{v^2}{R}$

$F = \vec{a} \cdot m$

mass

If the speed is doubled then the force is multiplied by 4.