Lecture 3: The dot product (§12.3)

Goals:

1. Compute the dot product of two vectors.
2. Compute the angle between two vectors in terms of the dot product.
3. Algebraically determine when two given vectors are orthogonal, and geometrically explain what this means.
4. Perform elementary vector algebra using properties of vector addition, scalar multiplication, and the dot product.
5. Algebraically compute (and geometrically explain) the vector projection of a given vector onto a given nonzero vector.
6. Solve elementary problems involving effective force and work using vector projection.

In this lecture we focus on the following operation:

Definition. The **dot product** $\overrightarrow{u} \cdot \overrightarrow{v}$ of vectors $\overrightarrow{u} = \langle u_1, u_2, u_3 \rangle$ and $\overrightarrow{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\overrightarrow{u} \cdot \overrightarrow{v} := u_1 v_1 + u_2 v_2 + u_3 v_3.$$

The dot product can be interpreted geometrically in terms of the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$:

Theorem 1. *The angle $\theta \in [0, \pi]$ between two non-zero vectors $\overrightarrow{u} = \langle u_1, u_2, u_3 \rangle$ and $\overrightarrow{v} = \langle v_1, v_2, v_3 \rangle$ is given by*

$$\theta = \arccos\left(\frac{\overrightarrow{u} \cdot \overrightarrow{v}}{|\overrightarrow{u}| \cdot |\overrightarrow{v}|}\right) \textit{ so that } \cos(\theta) = \frac{\overrightarrow{u} \cdot \overrightarrow{v}}{|\overrightarrow{u}| \cdot |\overrightarrow{v}|}.$$

Remark. Note that while $c \cdot \overrightarrow{v}$ and $\overrightarrow{u} + \overrightarrow{v}$ are **vectors**, the output of a dot product $\overrightarrow{u} \cdot \overrightarrow{v}$ is a **scalar**.

Example.

1. Find the dot product of $\overrightarrow{v} = \mathrm{i} - \mathrm{j} + 2\mathrm{k}$ and $\overrightarrow{u} = 2\mathrm{i} + \mathrm{j} + \mathrm{k}$? Find the angle between $\overrightarrow{u}$ and $\overrightarrow{v}$.

$\vec{u} \cdot \vec{v} = \langle 2, 1, 1 \rangle \cdot \langle 1, -1, 2 \rangle = 2 \cdot 1 - 1 \cdot 1 + 1 \cdot 2 = 3$

$|\vec{u}| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$ $\qquad \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} = \cos\theta$

$|\vec{v}| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}$

$\dfrac{1}{2} = \dfrac{3}{6} = \dfrac{3}{\sqrt{6} \cdot \sqrt{6}} = \cos\theta$

$\boxed{\theta = \dfrac{\pi}{3}}$

2. Let $\overrightarrow{v}$, $\overrightarrow{u}$ be vectors. Is it always true that $(\overrightarrow{v} \cdot \overrightarrow{u}) \cdot (\overrightarrow{v} \cdot \overrightarrow{u}) = (\overrightarrow{v} \cdot \overrightarrow{v}) \cdot (\overrightarrow{u} \cdot \overrightarrow{u})$? NO!

<u>Example</u> $\quad \vec{u} = i = \langle 1, 0, 0 \rangle \qquad \vec{u} \cdot \vec{u} = 1$

$\vec{v} = j = \langle 0, 1, 0 \rangle \qquad \vec{v} \cdot \vec{v} = 1$

$\vec{v} \cdot \vec{u} = 0$

$(\vec{v} \cdot \vec{u}) \cdot (\vec{v} \cdot \vec{u}) = 0$, $(\vec{v} \cdot \vec{v}) \cdot (\vec{u} \cdot \vec{u}) = 1$

3. Let $\overrightarrow{u} = \langle u_1, u_2, u_3 \rangle$ and $\overrightarrow{v} = \langle v_1, v_2, v_3 \rangle$ be vectors. Compute $|\overrightarrow{v} - \overrightarrow{u}|$.

$|\vec{u} - \vec{v}| = |\langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$

$|\vec{u} - \vec{v}|^2 = (u_1 - v_1)^2 + (u_2 - v_2^2) + (u_3 - v_3)^2 =$

$= u_1^2 + v_1^2 - 2u_1v_1 + u_2^2 + v_2^2 - 2u_2v_2 + u_3^2 + v_3^2 - 2u_3v_3 =$

$= (u_1^2 + u_2^2 + u_3^2) + (v_1^2 + v_2^2 + v_3^2) - 2(u_1v_1 + u_2v_2 + u_3v_3)$

$= |\vec{u}|^2 + |\vec{v}|^2 - 2\,\vec{u} \cdot \vec{v}$

$|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2\,\vec{u} \cdot \vec{v}$

We can now prove Theorem 1:

Theorem. *The angle $\theta \in [0, \pi]$ between two non-zero vectors $\overrightarrow{u} = \langle u_1, u_2, u_3 \rangle$ and $\overrightarrow{v} = \langle v_1, v_2, v_3 \rangle$ is given by*

$$\cos(\theta) = \frac{\overrightarrow{u} \cdot \overrightarrow{v}}{|\overrightarrow{u}| \cdot |\overrightarrow{v}|}.$$

We need to show that

$$\boxed{\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos}$$

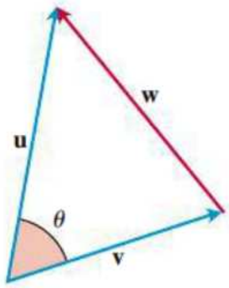


FIGURE 12.21 The parallelogram law of addition of vectors gives $\mathbf{w} = \mathbf{u} - \mathbf{v}$.

Proof: We will compute the length of $\vec{w}$ in two ways and compare.

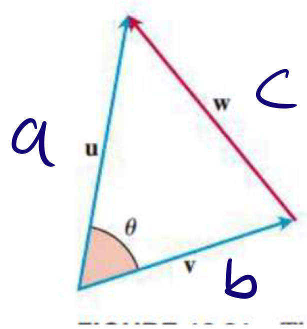
1) By the cosine theorem

$$|\vec{w}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}| \cdot |\vec{v}| \cdot \cos\theta$$

$a = |\vec{u}|$

$b = |\vec{v}|$

$c = |\vec{w}|$



2) By the exercise above:

$$|\vec{w}|^2 = |\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2\,\vec{u} \cdot \vec{v}$$

comparing 1 and 2 we get

$$|\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}| \cdot |\vec{v}| \cos\theta =$$
$$= |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u} \cdot \vec{v}$$

$\Rightarrow \quad \vec{u}\cdot\vec{v} = |\vec{u}|\cdot|\vec{v}|\cos\theta$ QED.

Orthogonal vectors

Definition. Two vectors $\vec{u}$ and $\vec{v}$ are **orthogonal**, if $\vec{u}\cdot\vec{v} = 0$. Equivalently, the angle between $\vec{u}$ and $\vec{v}$ is $\pi/2$ radians (90°).

Example. The vectors $\vec{u} = 3\mathrm{i}-2\mathrm{j}+\mathrm{k}$ and $\vec{v} = 2\mathrm{j}+4\mathrm{k}$ are orthogonal.

$\vec{u}\cdot\vec{v} = \langle 3,-2,1\rangle\cdot\langle 0,2,4\rangle = 3\cdot 0 - 2\cdot 2 + 1\cdot 4 = 0$

Here are a few properties of dot product:

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors, and c be a scalar. Then:

1) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$
2) $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$
3) $(c\vec{u})\cdot\vec{v} = \vec{u}\cdot(c\vec{v}) = c\,\vec{u}\cdot\vec{v}$
4) $\boxed{\vec{v}\cdot\vec{v} = |\vec{v}|^2}$ $\qquad \langle v_1,v_2,v_3\rangle\cdot\langle v_1,v_2,v_3\rangle = v_1^2+v_2^2+v_3^2 = |\vec{v}|^2$
5) $\vec{0}\cdot\vec{v} = 0$ $\qquad \vec{0}$ is orthogonal to any vector

Orthogonal projection

Definition. The **orthogonal projection of $\vec{u}$ onto another vector $\vec{v}$**, denoted $\mathrm{proj}_{\vec{v}}\vec{u}$, is the unique scalar multiple $\vec{w} = c\vec{v}$ such that $\vec{u} - \vec{w}$ is orthogonal to $\vec{v}$.

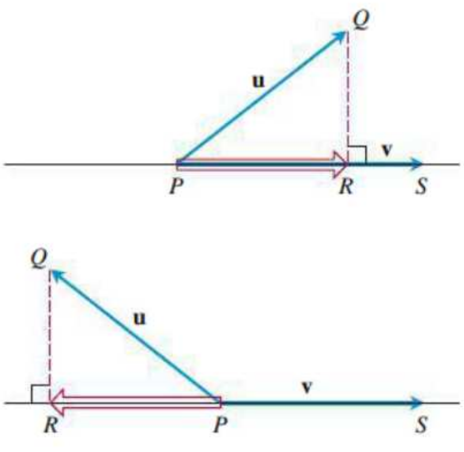


FIGURE 12.23 The vector projection of **u** onto **v**.

If $\vec{v}$ is a unit vector $|\vec{v}| = 1$

Then

$$\mathrm{Proj}_{\vec{v}}\vec{u} = (\vec{u}\cdot\vec{v})\vec{v}$$

the scalar component of $\vec{u}$ in the direction of $\vec{v}$

There is an explicit formula for $\mathrm{proj}_{\vec{v}}\vec{u}$:

We have:

unit vector

$$\mathrm{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}\right)\frac{\vec{v}}{|\vec{v}|}.$$

The **scalar component of $\vec{u}$ in the direction of $\vec{v}$** is defined as

$$\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|} = |\vec{u}|\cos\theta.$$

<u>Remark</u> If $\vec{V_1}, \vec{V_2}$ are on the same line ($\vec{V_2} = c\vec{V_1}$, $c \neq 0$)

then $\mathrm{Proj}_{\vec{V_1}}\vec{u} = \mathrm{Proj}_{\vec{V_2}}\vec{u}$.

Exercise. Find the vector projection of $\vec{u} = \langle 6, 3, 2\rangle$ onto $\vec{v} = \langle 1, -2, -2\rangle$, and the scalar component of $\vec{u}$ in the direction of $\vec{v}$.

$$\text{Proj}_{\vec v}\vec u = \underbrace{\frac{\vec u \cdot \vec v}{|\vec v|}}_{\text{the scalar component}} \cdot \frac{\vec v}{|\vec v|} = \frac{-4}{3}\cdot \frac{1}{3}\langle 1, -2, -2\rangle$$

$$= \underbrace{\frac{-4}{3}}_{\text{the scalar component}} \left\langle \frac{1}{3}, \frac{-2}{3}, -\frac{2}{3}\right\rangle$$

$\vec u \cdot \vec v = 6 - 6 - 4 = -4$

$|\vec v|^2 = 1 + 4 + 4 = 9$, $|\vec v| = 3$

Exercise. Show that $\vec{u} - \text{proj}_{\vec{v}}\vec{u}$ is indeed orthogonal to $\vec{v}$.

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Definition. The **work** done by a constant force $\overrightarrow{F}$ acting through a displacement $\overrightarrow{D} = \overrightarrow{PQ}$ is

$$W = \overrightarrow{F} \cdot \overrightarrow{D}.$$

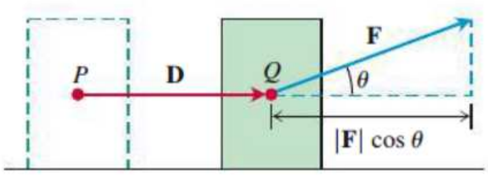
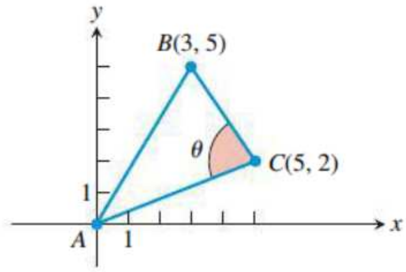


FIGURE 12.27 The work done by a constant force **F** during a displacement **D** is $(|\mathbf{F}| \cos\theta)|\mathbf{D}|$, which is the dot product $\mathbf{F} \cdot \mathbf{D}$.

Example. If $\left|\overrightarrow{F}\right| = 40N$, and $\left|\overrightarrow{D}\right| = 3\text{m}$ and $\theta = 60°$, then

$$W =$$

Exercise. Find the angle θ in the triangle ABC determined by the vertices $A = (0, 0)$, $B = (3, 5)$ and $C = (5, 2)$.



Exercise. Let $\overrightarrow{u}$, $\overrightarrow{v}_1$, $\overrightarrow{v}_2$ be vectors. And assume that $\overrightarrow{u} \cdot \overrightarrow{v}_1 = \overrightarrow{u} \cdot \overrightarrow{v}_2$. Is it true that $\overrightarrow{v}_1 = \overrightarrow{v}_2$?