Math 230-1

Previous

$f(x) = \sin x$

$f'(x) = \cos x$

$\shortparallel$

$\frac{df}{dx}$

Now

$f(x,y) = x^2 y$

Partial derivation

$\frac{df}{dx} = 2xy$

$\frac{df}{dy} = x^2$

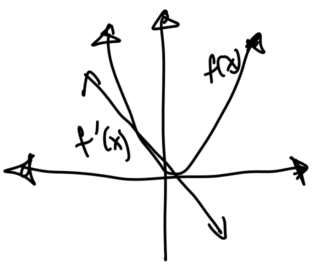
Topics:

2nd half:
- limits
- derivation
- optimization
- gradients

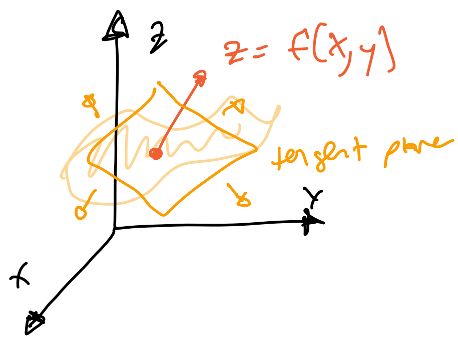
1st half:
- 3d space
- lines, planes
- surfaces

Previous

$y = x^2$

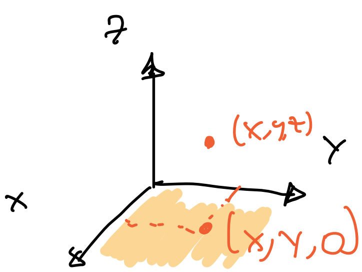


Now



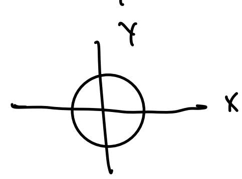
3d-space $\mathbb{R}^3$

$\mathbb{R}$ = real line



2d

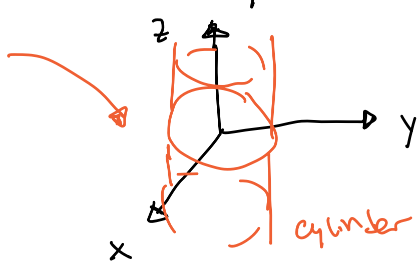
$x^2+y^2=1$



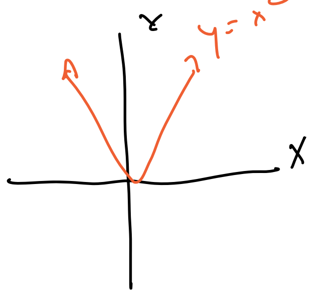
circle

3d

$x^2+y^2=1$

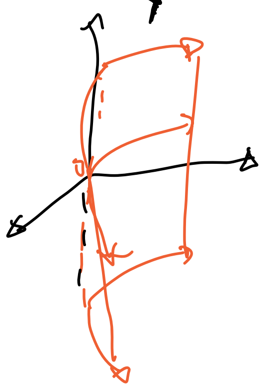


2d

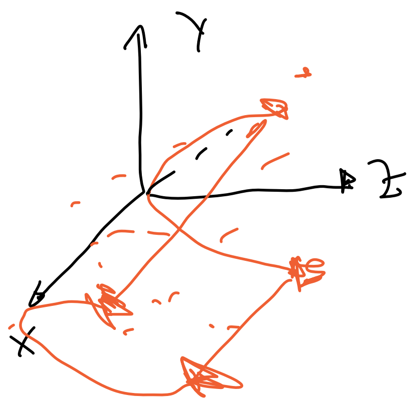


3d

$y = x^2$



$z = y^2$

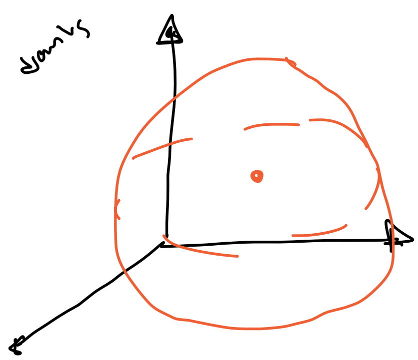


Spheres

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$$

center (a, b, c) and radius R

i suck at drawing



$$\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = R$$

distance from (x, y, z) to (a, b, c)

September 23, 2022

① Warmup

Find an equation that characterizes points (x,y,z) whose distance to $(1,2,1)$ is same as distance to $(2,1,1)$.

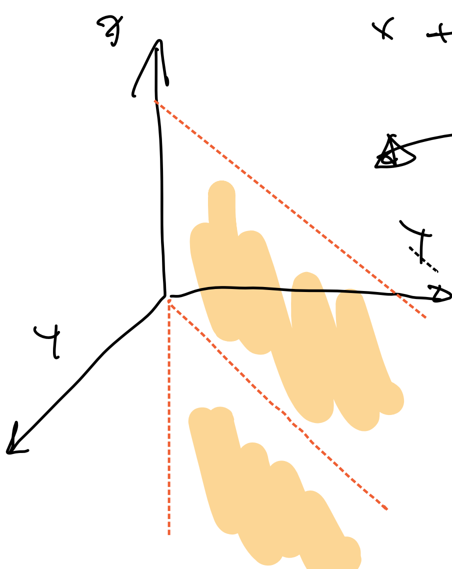
$$(x-1)^2+(y-2)^2+(z-1)^2 = (x-2)^2+(y-1)^2+(z-1)^2$$

$$x^2-2x+1+y^2-4y+4 = x^2-4x+4+y^2-2y+1$$

$$-2x-4y = -4x-2y$$

$$x+2y = 2x+y$$

$$y = x$$



② Warmup

$$x^2 - 4x + y^2 - 6y + z^2 = -12$$

Find point (x, y, z) satisfying which is closest to xz-plane.

$$(x-2)^2 - 4 + (y-3)^2 - 9 + z^2 = -12$$

$$(x-2)^2 + (y-3)^2 + z^2 = 1$$

radius $= 1$

xz plane $\to$ closest to $y = 0$ center: $(2, 3, 0)$

so $(2, 2, 0)$

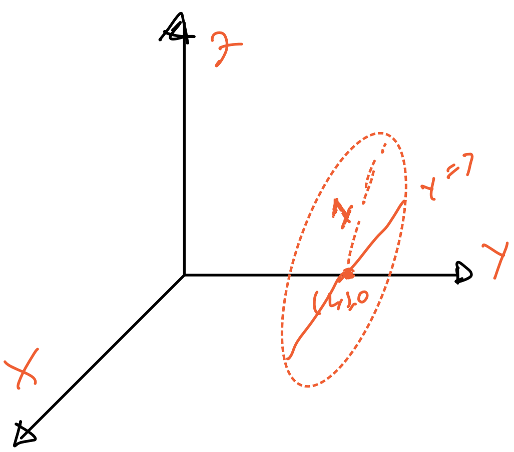
③ Warmup

$(x-2)^2 + (y-3)^2 + z^2 = 1$, $y = 3$

what points satisfy both

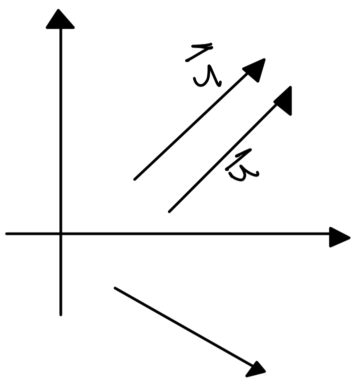
circle with radius 1 with center

$(2, 3, 0)$ where all y values are 3.



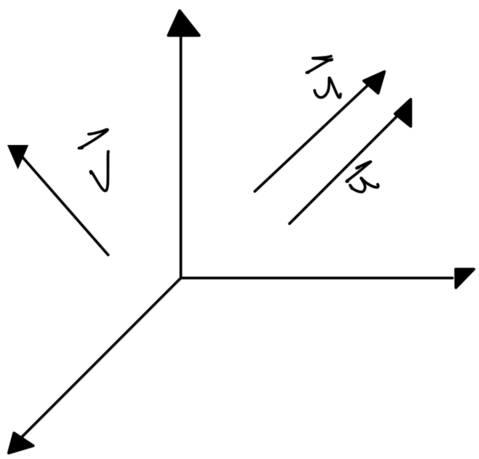
Vectors: A vector is an object that captures a direction and a magnitude / length.

2 dimensional example:



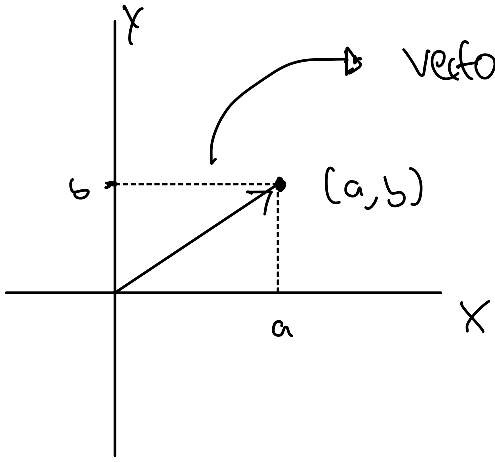
if direction and length of vectors are equal, they are the same vectors.

3 dimensional:



Algebraically:

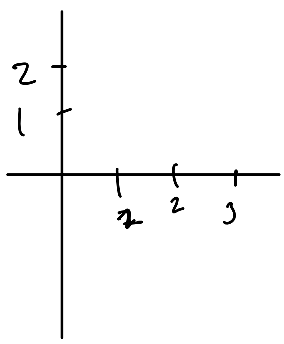
vector $\langle a, b \rangle$



Example:

Vector from (1,2) to (3,1)

$$\langle 3-1,\ 1-2 \rangle = \langle 2, -1 \rangle$$



length:

$$\sqrt{2^2 + (-1)^2} = \sqrt{5}$$

☆ Vector of length 1 pointing from (1,2) towards (3,1)

length on $\sqrt{5}$ the vector: $\langle 2, -1 \rangle$ → multiply by $\frac{1}{\sqrt{5}}$

$$\left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$$

So $\left\langle \frac{2\sqrt{5}}{5}, \frac{-\sqrt{5}}{5} \right\rangle$.

★ Length 10

$$\left\langle \frac{20\sqrt{5}}{5}, -\frac{10\sqrt{5}}{5} \right\rangle$$

Note (mylab): $\overrightarrow{PQ}$ component form:

point Q − point P

mylab vector angle questions were hard

↳ practice those again

Sepember 26, 2022

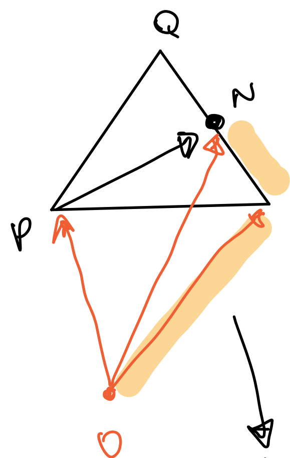
Warmup HARD!

Given the triangle with 3 vertices

$P = (1,1,1)$, $Q = (2,-1,2)$, $R = (-1,3,4)$

find vector of length 3

pointing from P towards the midpoint of segment QR.



① 1st step

Midpoint between (N)

$(2,-1,2)$ and

$(-1,3,4)$

$\vec{OR} + \frac{1}{2}\vec{RQ}$

$\vec{RQ} = \langle 2-(-1), -1-3, 2-4 \rangle =$

$\langle 3,-4,-2 \rangle$

$\vec{OR}$ $\langle -1,3,4 \rangle + \frac{1}{2}\langle 3,-4,-2 \rangle$

$= \langle -1,3,4 \rangle + \langle \frac{3}{2},-2,-1 \rangle$

$\langle \frac{1}{2}, 1, 3 \rangle = \vec{ON}$

② 2nd step $\overrightarrow{ON} - \overrightarrow{OP} = \overrightarrow{PN}$

$\langle \frac{1}{2}, 1, 3 \rangle - \langle 1, 1, 1 \rangle = \langle \frac{-1}{2}, 0, 2 \rangle$

Length: $\sqrt{\left(-\frac{1}{2}\right)^2 + 0^2 + 2^2} = \frac{\sqrt{17}}{2}$ unit vector $= \frac{2}{\sqrt{17}}$

$\frac{3 \times 2}{\sqrt{17}} \times \langle \frac{-1}{2}, 0, 2 \rangle =$

$\langle \frac{-3}{2\sqrt{17}}, 0, \frac{12}{\sqrt{17}} \rangle$

DOT PRODUCT

$i = \langle 1, 0, 0 \rangle$, $j = \langle 0, 1, 0 \rangle$, $k = \langle 0, 0, 1 \rangle$

Given $\vec{u} = \langle a, b, c \rangle = ai + bj + ck$
and $\vec{v} = \langle x, y\ z \rangle$

the dot product of $\vec{u}$ and $\vec{v}$ is $\vec{u} \cdot \vec{v} = ax + by + cz$

Ex! $\langle 1, 3, -4 \rangle \cdot \langle 2, 1, 0 \rangle =$

$2 + 3 + 0 = 5$

Geometric meaning of dot product

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$

where θ is angle between $\vec{u}$ and $\vec{v}$.

"law of cosines"

Ex: $\vec{u} \cdot \vec{v} = 5 = \langle 1, 3, -4 \rangle \cdot \langle 2, 1, 0 \rangle =$

$|\vec{u}|\,|\vec{v}| \cdot \cos\theta$

$\theta \rightarrow$ has to be $0 < \theta < 90$ because the dot product is positive.

Fact: $\vec{u} \cdot \vec{v} > 0 \rightarrow$ angle < 90

$\vec{u} \cdot \vec{v} < 0 \rightarrow$ angle > 90

Angle between $\vec{u}$ and $\vec{v}$:

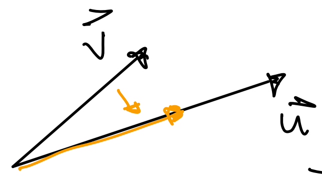
$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|} = \frac{5}{\sqrt{26} \cdot \sqrt{5}}$$ $\Rightarrow$ find θ using arccos

If $\vec{u}$, $\vec{v}$ are nonzero, $\vec{u} \cdot \vec{v} = 0$ when $\underline{\cos\theta = 0}$

$\theta = 90 / \frac{\pi}{2}$

Fact! $\vec{u}$ and $\vec{v}$ are perpendicular to each other when $\vec{u} \cdot \vec{v} = 0$.

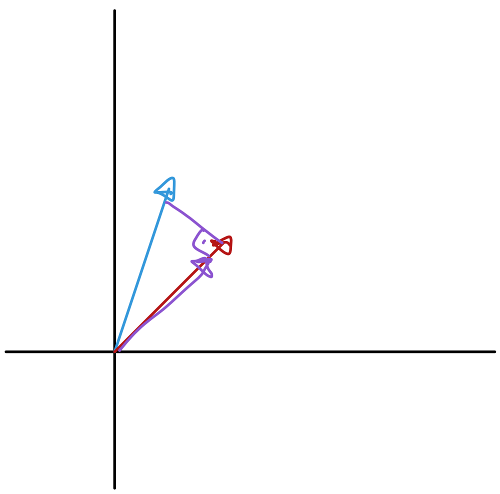
Vector Projections (orthogonal projection) watch a vid on this



$$\text{projection of } \vec{v} \text{ onto } \vec{u} = \underbrace{\left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\right)}_{\text{scaler projection}} \cdot \vec{u}$$

Example

Find vector projection of $\vec{v}$ onto $\vec{u}$



$\vec{v} = \langle 1, 3 \rangle$

$\vec{u} = \langle 2, 2 \rangle$

projection!

$$= \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \cdot \vec{u}$$

$\vec{u} \cdot \vec{v}$

$$\left(\frac{2+6}{4+4}\right) \cdot \vec{u}$$

$\vec{u} = \langle 2, 2 \rangle$

*

Projection of 2j $\langle 0, 2\rangle$ onto $\vec{u}$

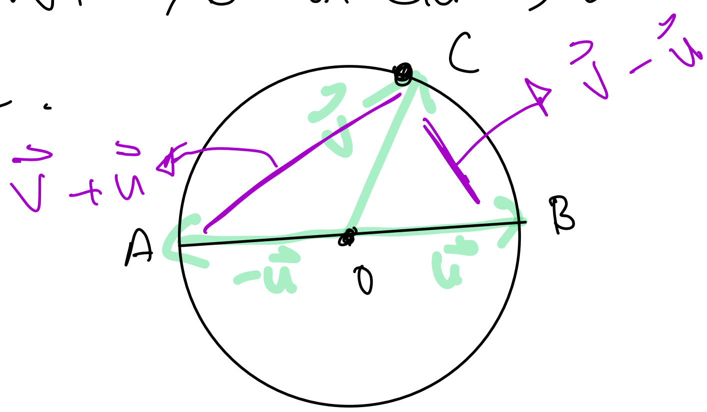
$$\left(\frac{0+4}{4+4}\right)\vec{u} = \frac{1}{2}\vec{u} = \frac{1}{2}\langle 2, 2\rangle = \langle 1, 1\rangle$$

28 Sept 2022

Warm up

i don't get this one

① Given A, B on diameter, show angle at C is $\pi/2$.



Verify $\vec{v} - \vec{u}$ orthogonal.

$$(\vec{v} - \vec{u}) \cdot (\vec{v} + \vec{u}) = \vec{v} \cdot \vec{v} + \cancel{\vec{v} \cdot \vec{u}} - \cancel{\vec{u} \cdot \vec{v}} - \vec{u} \cdot \vec{u}$$

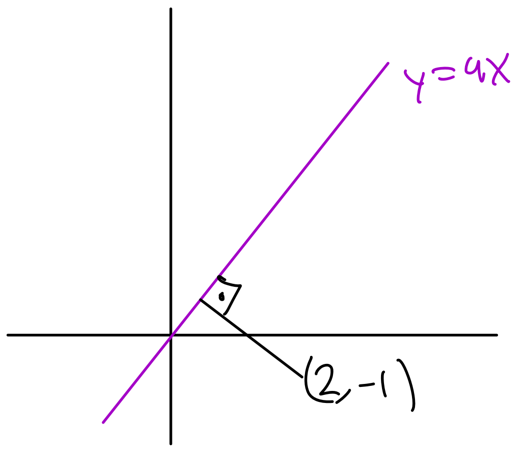
$$= \vec{v} \cdot \vec{v} - \vec{u} \cdot \vec{u}$$

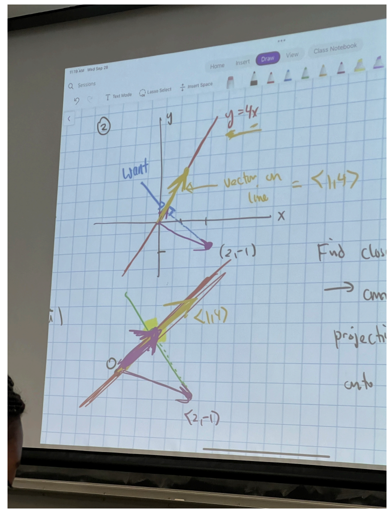
$$= \vec{v}^2 - \vec{u}^2$$

$= 0$ since $\vec{u} = \vec{v}$ (both radius)

$$\vec{u} \cdot \vec{u} = |\vec{u}| \cdot |\vec{u}| \cos\theta$$

2 Find the point on the line $y = 4x$ closest to $(-2, 1)$





Find the closest point → compute projection of $\langle 2,-1\rangle$ onto $\langle 1,4\rangle$.

$$\left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \left(\frac{2-4}{1+16}\right)\langle 1,4\rangle =$$

$$\left\langle \frac{-2}{17}, \frac{-8}{17} \right\rangle$$

Cross product

Ex/ $\vec{u} = 2\vec{i} - 3\vec{j} - \vec{k}$

$= \langle 2, -3, -1 \rangle$

$\vec{v} = \vec{i} + 2\vec{j} - 2\vec{k}$

$= \langle 1, 2, -2 \rangle$

The cross product of $\vec{u}$ and $\vec{v}$ is :

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & -1 \\ 1 & 2 & -2 \end{vmatrix} = \langle 8, 3, 7 \rangle$$

$$= (6 - (-2))\,\vec{i} - (-4 - (-1))\,\vec{j} + (4 - (-3))\,\vec{k}$$

$$= 8\vec{i} + 3\vec{j} + 7\vec{k}$$

Note:

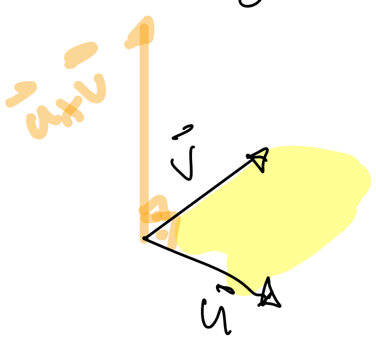
$\langle 8, 3, 7\rangle \cdot \langle 2, -3, -1\rangle = 16-5-7 = 0$

$\langle 8, 3, 7\rangle \cdot \langle 1, 2, -2\rangle = 8+6-14 = 0$

— Geometrically

→ $\vec{u} \times \vec{u}$ is orthogonal to $\vec{u}$ and $\vec{v}$.

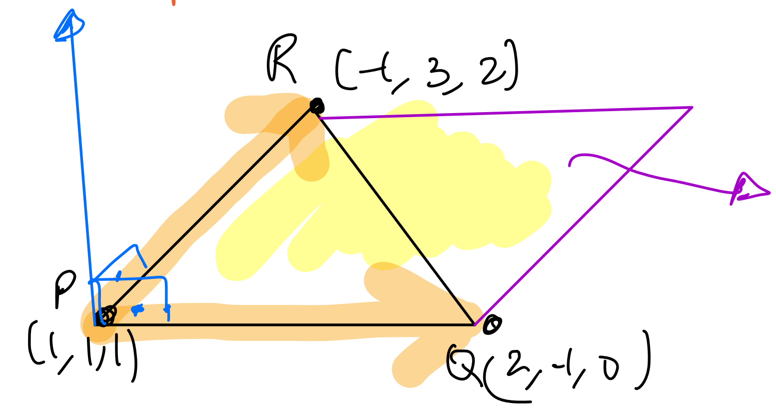
→ right hand rule



direction of cross product is given right hand rule thumb

→ $|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}|\sin\theta =$ area of parallelogram

Example:



think like half of parallelogram and cross product.

→ Find area of triangle.

$$= \frac{1}{2}\left|\vec{PQ} \times \vec{PR}\right| = \frac{1}{2}\sqrt{0^2+1^2+(-1)^2} = \frac{\sqrt{5}}{2}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & -1 \\ -2 & 2 & 1 \end{vmatrix} = (-2+2)\vec{i} - (1-2)\vec{j} + (2-4)\vec{k} =$$

$$\vec{j} - 2\vec{k} = \langle 0, 1, -2 \rangle$$

Ex 2

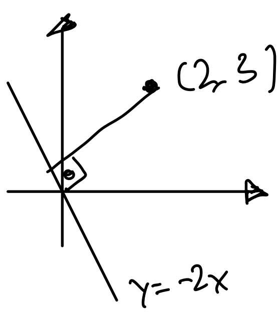
(Telegondcn sef; ir)

Find vector length 2 that's orthogonal to

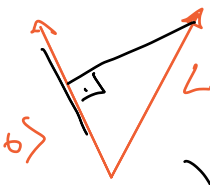
1st

Sept 30

Use a cross product to find the distance from $(2,3)$ to the line $y=-2x$.



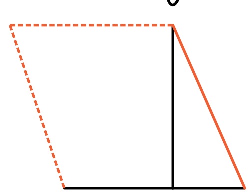
Hint look at parallelogram whose height is desired distance.



$\langle 2,3\rangle \to \langle 2,3,0\rangle$

can pick any number that satisfies $y=-2x$ $\langle -3,6\rangle$

$\langle -3,6,0\rangle$



area = base × height

so distance = $\dfrac{\text{area of parallelogram}}{\text{base}}$

1st) cross product of edges

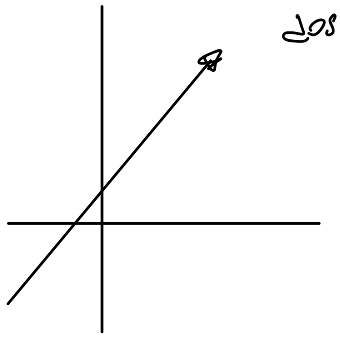
$$\langle 2,3,0\rangle \times \langle -3,6,0\rangle = \begin{vmatrix} 2 & 3 & 0 \\ -3 & 6 & 0 \end{vmatrix} = \langle 0,0,21\rangle$$

2nd) area of parallelogram $|\text{cross product}| = |\langle 0,0,21\rangle| = 21$

3rd) distance from $(2,3)$ to $y=-2x$ is height. $\dfrac{21}{|\langle -3,6\rangle|} = \dfrac{21}{\sqrt{45}}$

Lines!

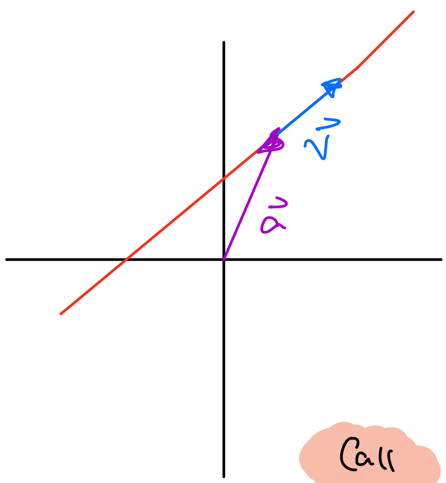
2d) $y = mx + b$



describes in $\mathbb{R}^3$ not a line

so can't describe lines using a single equation in terms of x, y, z

instead:



$\vec{a} + \vec{v}$ ends in the line

$\vec{a} + 2\vec{v}$ ends in the line

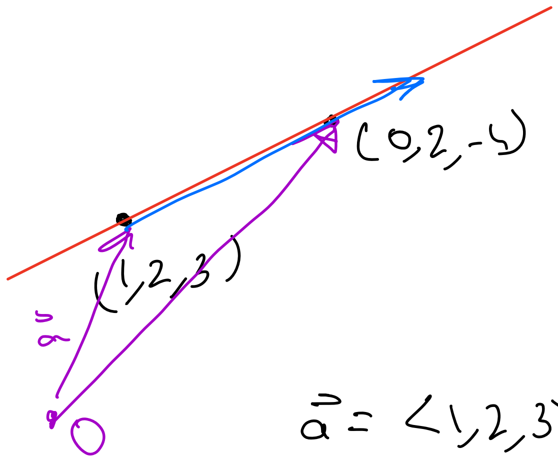
$\vec{a} - \vec{v}$ ends in the line

GENERALLY: $\vec{a} + t\vec{v}$ ends in the line.

Call:

$\vec{r}(t) = \vec{a} + t\vec{v}$ the vector equation of the line passing through endpoint of $\vec{a}$ with direction $\vec{v}$

Ex: Describe the line passing through $(1,2,3)$ and $(0,2,-4)$.



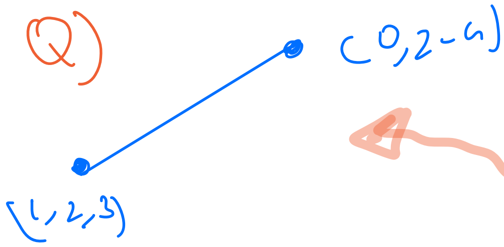
$$\vec{a} = \langle 1,2,3 \rangle$$

$$\vec{v} = \langle 0,2,-4 \rangle - \langle 1,2,3 \rangle = \langle -1, 0, -7 \rangle$$

$$\vec{r}(t) = \vec{a} + t\vec{v} = \langle 1,2,3 \rangle + t\langle -1, 0, -7 \rangle = \langle 1-t,\ 2,\ 3-7t \rangle$$

$$\boxed{\begin{aligned} x &= 1-t \\ y &= 2 \\ z &= 3-7t \end{aligned}} \quad \boxed{-\infty < t < \infty}$$

Q)



Parametric Equation

$x = 1 - t$

$y = 2$

$z = 3 - 7t$

$? \leq t \leq ?$

(we only want the points)

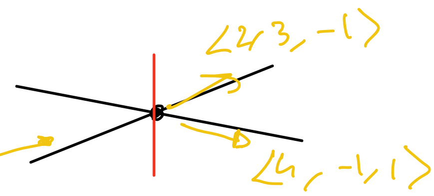
$0 \leq t \leq 1$

Example Given lines with p. eqs. that intersect at (1,2,1) Find the line perpendicular to both passing through (1,2,1)

$x = 1 + 4t$
$y = 2 - t$
$z = 1 + t$

and

$x = 1 - 2t$
$y = 2 + 3t$
$z = 1 - t$



Need: • point on line: (1,2,1)

• direction 1st dir X 2nd dir

$$\langle 4, -1, 1 \rangle \times \langle -2, 3, -1 \rangle$$

$$\begin{vmatrix} 4 & -1 & 1 \\ -2 & 3 & -1 \end{vmatrix} =$$

$$\langle 1 - 3, -(-4 - (-2)), (12 - 2) \rangle = \langle -2, 2, 10 \rangle$$

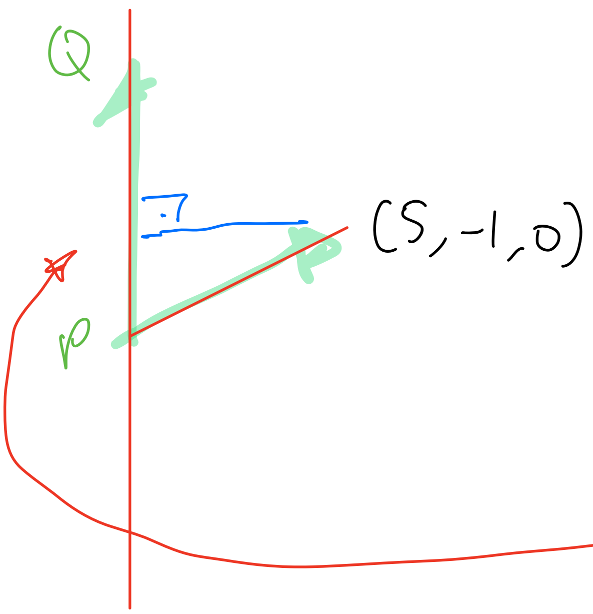
So perpendicular line has vector:

$$\vec{r}(t) = \langle 1, 2, 1 \rangle + t \langle -2, 2, 10 \rangle$$
$$= \langle 1 - 2t, 2 + 2t, 1 + 10t \rangle$$

Parametric equations:

$$\boxed{\begin{aligned} x &= 1 - 2t \\ y &= 2 + 2t \\ z &= 1 + 10t \end{aligned}}$$

Find distance from (5, -1, 0) to the line we just found.



take $P = (1,2,1)$ $t=0$

$Q = (3,0,-9)$ $t=-1$

$$\vec{PQ} = \langle 3,0,-9\rangle - \langle 1,2,1\rangle$$
$$= \langle 2,-2,-10\rangle$$

$$\vec{PR} = \langle 5,-1,0\rangle - \langle 1,2,1\rangle$$
$$= \langle 4,-3,-1\rangle$$

distance

$$= \frac{\text{area of parallelogram}}{\text{base}}$$

$$= \frac{|\vec{PQ} \times \vec{PR}|}{|PQ|}$$

The direction of the cross product is the unit vector that points in the same direction as the cross product. Use the formula below to find the direction of the vector $\mathbf{a} \times \mathbf{b}$. Note that a vector with length 0 does not have a direction.

$$\frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|}$$

Oct 3

Warmup Find the distance from $(1,2,3)$ to the line that is perpendicular to the lines

$$\begin{aligned} 1+2t &= x \\ 2-t &= y \\ 3+t &= z \end{aligned} \quad \text{and} \quad \begin{aligned} x &= 5-3t \\ y &= 5-t \\ z &= -2+2t \end{aligned}$$

and passes through their intersection.

1) Find the point of intersection.

$$\begin{aligned} 1+2t &= 5-3t \\ 2-t &= 5-t \\ 3+t &= -2+2t \end{aligned}$$

No solution

instead solve:

$$1+2t_1 = 5-3t_2$$

$$2-t_1 = 5-t_2$$

$$3+t_1 = -2+2t_2$$

$t_1 = -3 + t_2 \rightarrow t_1 = -3+2$

$t_1 = -1$

$3+(-3+t_2) = -2+2t_2$

$t_2 = -2+2t_2$

$2 = t_2$

Intersection:
$(-1, 3, 2)$

2) Step 2: Find desired direction

$\langle 2,-1,1\rangle \times \langle -3,-1,2\rangle$

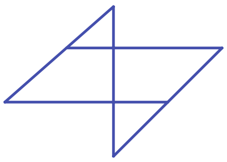
$$\begin{vmatrix} 2 & -1 & 1 \\ -3 & -1 & 2 \end{vmatrix} = \langle -1,-7,5\rangle$$

▽▽▽

Step3 Perpendicular line

$$\langle -1, 3, 2\rangle + t\langle -1, 7, -5\rangle$$

$$\langle -1 - t, \ 3 - 7t, 2 - 5t\rangle$$



$x = -1 - t$

$y = 3 - 7t$

$z = 2 - 5t$

4th) Find distance from $(5, 1, 1)$ to the line.

$$\text{distance} = \frac{\text{area}}{\text{base}} = \frac{|\overline{QP} \times \overline{QR}|}{|QR|}$$

$P = (5, 1, 1)$

$Q = (-1, 3, 2)$

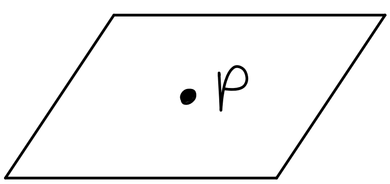
$R = (-2, -4, -3)$

$$\vec{QP} \times \vec{QR} = \begin{vmatrix} 6 & -2 & -8 \\ 4 & -7 & -5 \end{vmatrix}$$

$$= \langle 3, -31, -44 \rangle$$

$$\frac{\sqrt{3^2 + 31^2 + 44^2}}{\sqrt{1^2 + 7^2 + 5^2}}$$

PLANES!



To describe a plane, need:

- a point P on the plane
- a normal vector $\vec{n}$ (perpendicular)

Vector Equation of Plane

$$\bar{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

normal

arbitrary pt

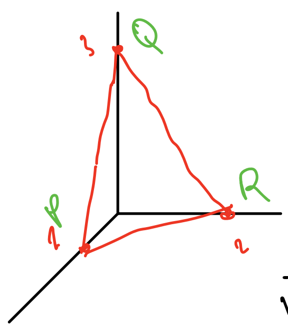
(x, y, z)

We can also extract "standard equation"

Ex

Find an equation for the plane passing through $(1, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 3)$.

P R Q



point on plane: $(0, 2, 0)$ can pick any

• normal vector.

$$\vec{n} = \vec{QP} \times \vec{QR}$$

$$\begin{vmatrix} i & j & k \\ 1 & 0 & -3 \\ 0 & 2 & -3 \end{vmatrix} = \langle 6, 3, 2 \rangle$$

$$\langle 6, 3, 2 \rangle \cdot \langle x-0,\ y-2,\ z-0 \rangle = 0$$

$$6x + 3y - 6 + 2z = 0$$ → Standard eq

$$6x + 3y + 2z = 6$$

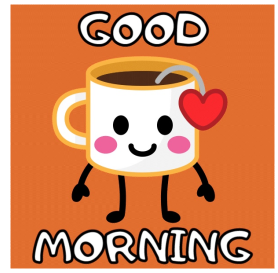
Ex

$$4x - 6y + 3z = 11$$

Find a normal vector to this plane.

$$\langle 4, -6, 3 \rangle$$

Oct 5

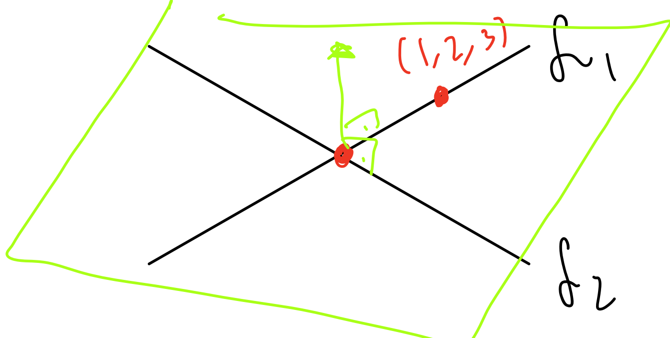


Warmup

① Find the distance from $(-3, 1, 5)$ to the plane containing lines

$$x = 1 + 2t \qquad\qquad x = 5 - 3t$$
$$y = 2 - t \quad \text{and} \quad y = 5 + t$$
$$z = 3 + t \qquad\qquad z = -2 + 2t$$

1st step Find a point on the plane and a normal vector.



normal vector: $\langle 2, -1, 1 \rangle \times \langle -3, 1, 2 \rangle =$

$$\begin{vmatrix} i & j & k \\ 2 & -1 & 1 \\ -3 & 1 & 2 \end{vmatrix} = \vec{n} = \langle -3, -7, -1 \rangle$$

2nd) Equation of plane:

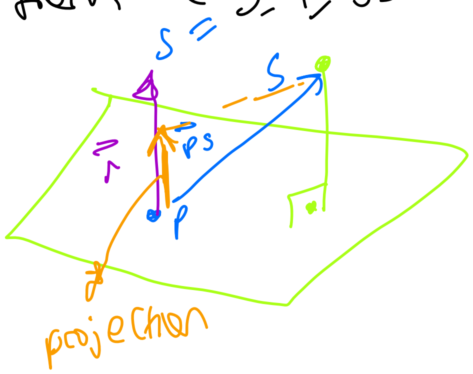
$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$$

$$\langle -3, -7, -1 \rangle \cdot \langle x-1, y-2, z-3 \rangle =$$

$$-3(x-1) - 7(y-2) - (z-3) = 0$$

3rd) Find the distance:

from $(-3, 1, 5)$ to the plane



distance = $\left| \text{projection of } \vec{PS} \text{ to } \vec{n} \right|$

Projection of $\vec{PS}$ onto $\vec{n}$:

$\vec{PS} = \langle -4, -1, 2 \rangle$

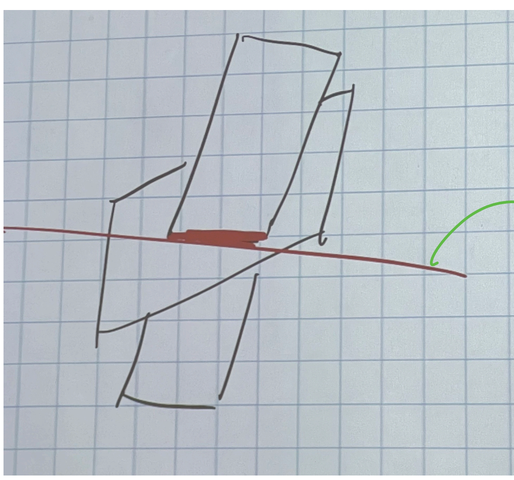
$\vec{n} = \langle -3, -7, -1 \rangle$

|

$$\left(\frac{\vec{PS}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\right)\cdot\vec{n} = \left(\frac{12+7-2}{9+45+1}\right)\langle -3,-7,-1\rangle$$

$$= \frac{17}{55}\langle -3,-7,-1\rangle$$

3) Find parametric equations for the line of intersection of $3x+2y-z=1$ and
$2x-y+2z=-8$



Find this line

Need: Point on line
direction vector

Look for a point $y=0$.

$$-3x - z = 1 \longrightarrow z = -3x - 1$$

$$2x - 2z = -8$$

$$2x - 2(-3x - 1) = -8$$

$$8x = -10$$

$$x = -\frac{10}{8} = -\frac{5}{4}$$

$$\longrightarrow z = -3\left(-\frac{5}{4}\right) - 1$$

$$\frac{15}{4} - 1 = \frac{11}{4} = z$$

a point on intersection $\left(-\frac{5}{4}, 0, \frac{11}{4}\right)$

direction vector = cross product of normal vectors.

$$= \langle -3, 2, -1 \rangle \times \langle 2, -1, -2 \rangle$$

$$\begin{vmatrix} i & j & k \\ -3 & 2 & -1 \\ 2 & -1 & -2 \end{vmatrix} = \langle -5, -8, -1 \rangle$$

line of intersection:

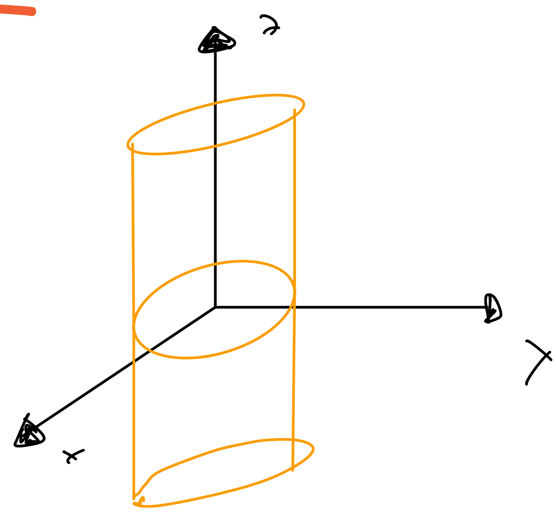
$$-5/4 - 5t = x$$

$$0 - 8t = y$$

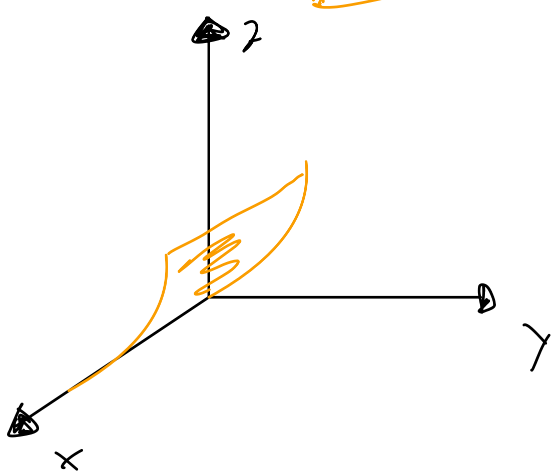
$$11/4 - t = z$$

OTHER SURFACES!

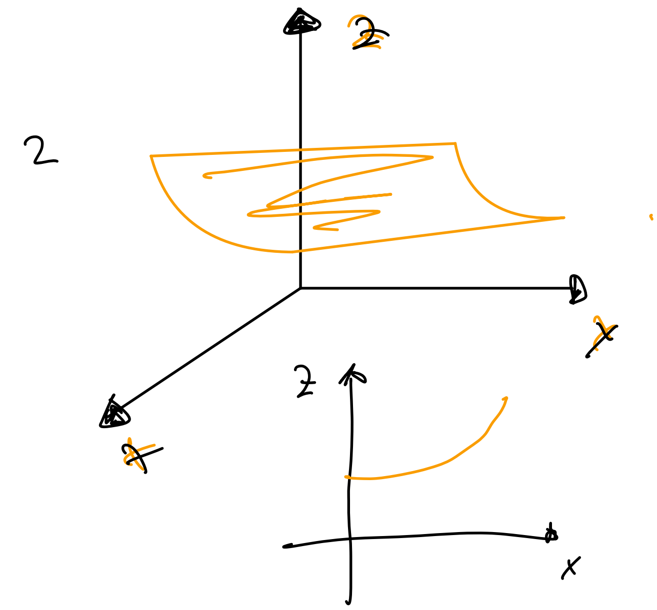
EX) $x^2+y^2=1$



EX) $z=y^2$



Ex) $z = e^x$



Quadric Surfaces

- conic sections

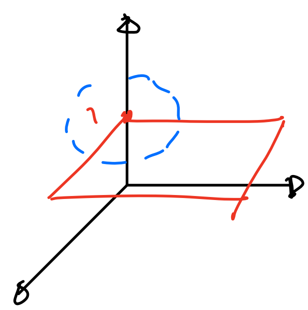
Ex) $x^2 + y^2 = z^2$

Use cross-sections (intersections with planes) to build up the surface

cross section of $z = 1$

equation $x^2+y^2=1$

circle



cross section of $z = 2$

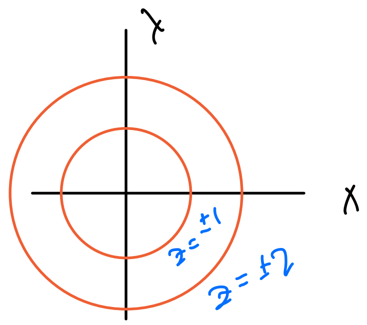
equation $x^2+y^2=4$

circle

cross section of $z = -1$

↳ equation $x^2+y^2=1$

circle



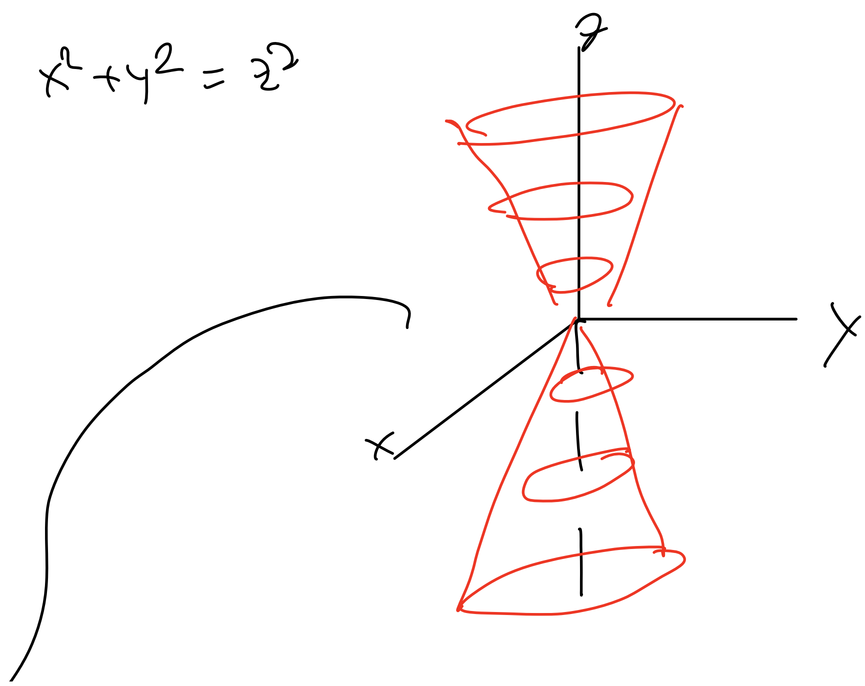
at $z=k$: $x^2+y^2=k^2$

as $|z|$ increases, radius increases.

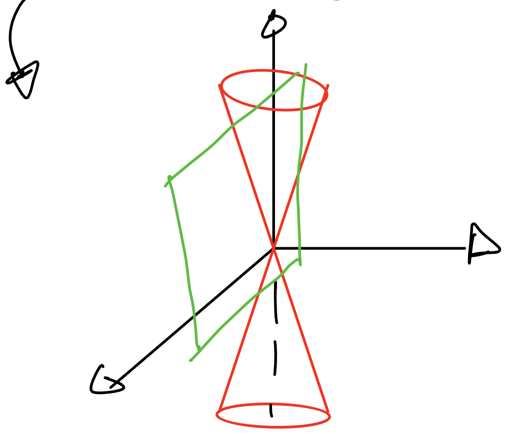
at $z=0$: $x^2+y^2=0$

$\left.\begin{array}{l} x=0 \\ y=0 \end{array}\right\}$ origin

surface $x^2+y^2=z^2$



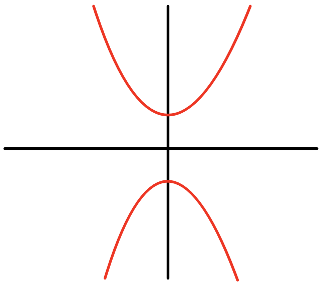
double cone



Cross section at $y=1$

$$x^2+1 = z^2$$

$$z^2-x^2=1 \quad \text{hyperbola}$$



Oct 7

⇒ Quadratic surfaces are given by equations involving all of x, y, z all with 2nd powers

Ex: $z^2 = x^2 + y^2$ "double cone"

Can visualize surface by considering cross-sections, which are intersections with planes. x = constant, y = constant or z = constant.

Warmup

Sketch cross-sections of the surface $z = 2x^2 + 3y^2$ at $z = 0, 1, 2, -1$

and then sketch the surface.

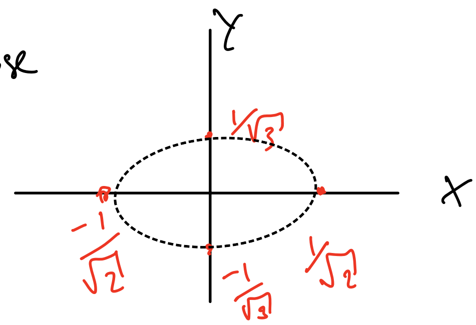
What about cross-sections $x = k$?

$z=0$ $\quad 0 = 2x^2 + 3y^2$

↳ point $(0, 0)$

$z = 1$ $\quad 1 = 2x^2 + 3y^2$

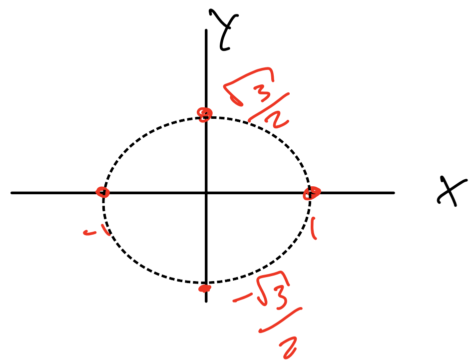
ellipse



|

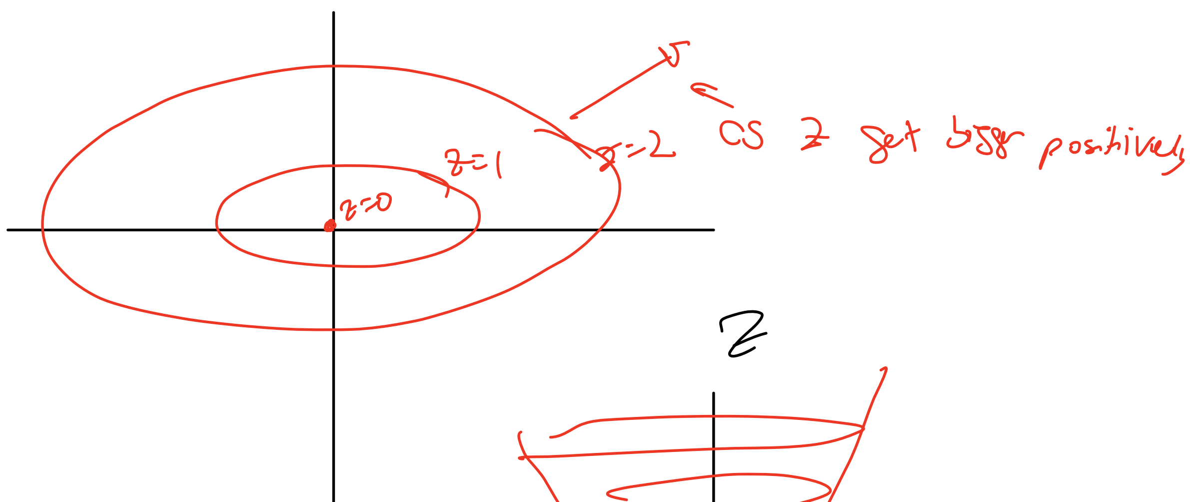
$z=2$ $\quad 2 = 2x^2+3y^2$

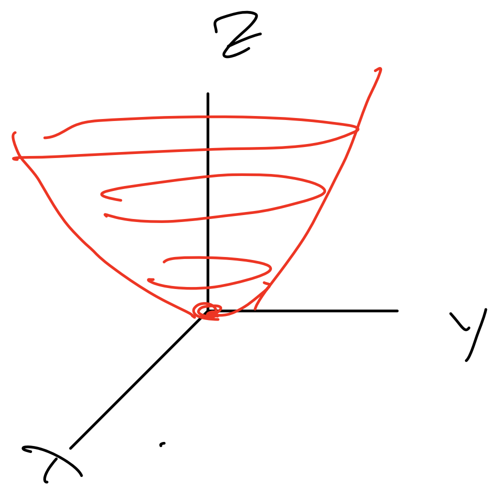
ellipse



$z=-1$ $\quad -1 = 2x^2+3y^2$

empty cross section

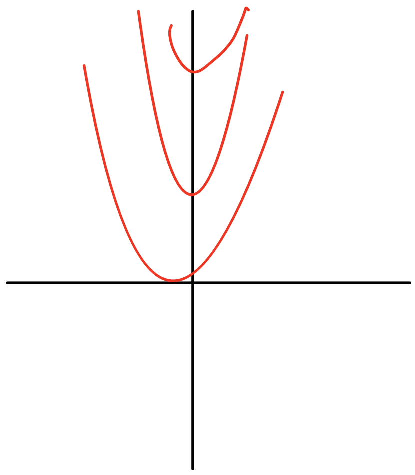


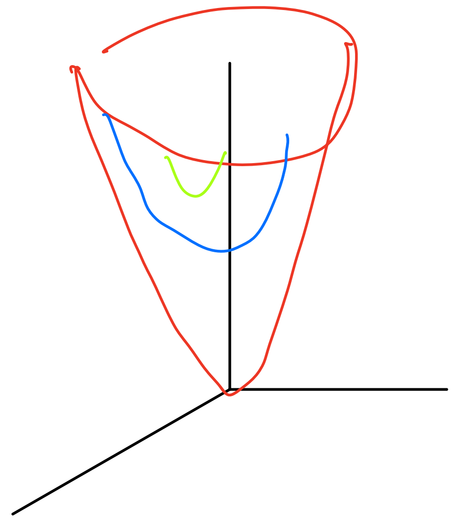


$z = 2x^2 + 3y^2$

cross section $x = k$

$z = 2k^2 + 3y^2$

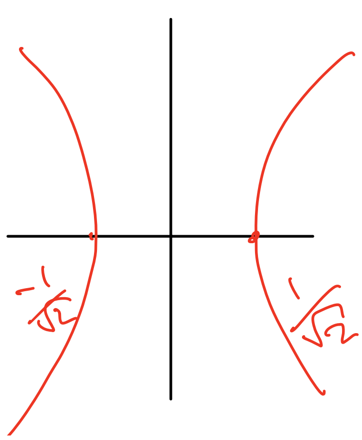




Exemple 2

$z = 2x^2 - 3y^2$

$z = 1 \qquad 1 = 2x^2 - 3y^2$



$z=2 \quad 2x^2-2y^2$

hyperbola

crossing x-axis

$z=0 \qquad 0=2x^2-3y^2$

$3y^2=2x^2$ pair of lines

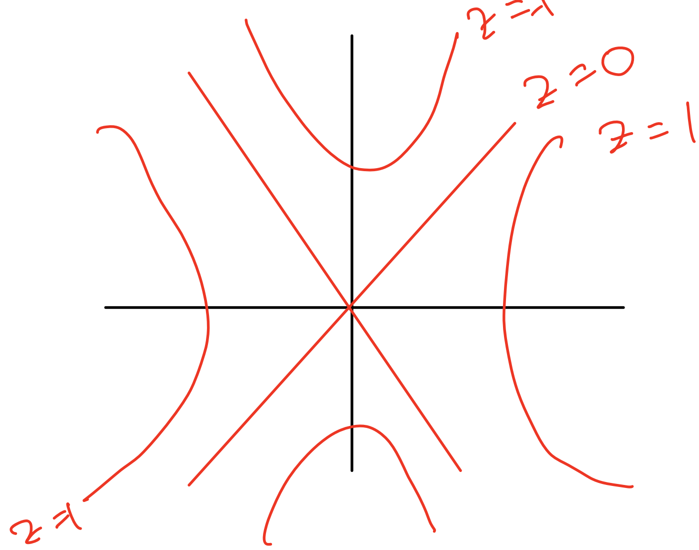
$y^2=x^2$

$y=\pm x$

$z=-1 \qquad -1=2x^2-3y^2$

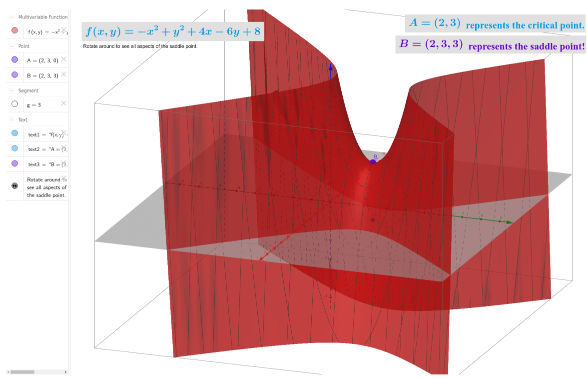
$1=-2x^2+3y^2$

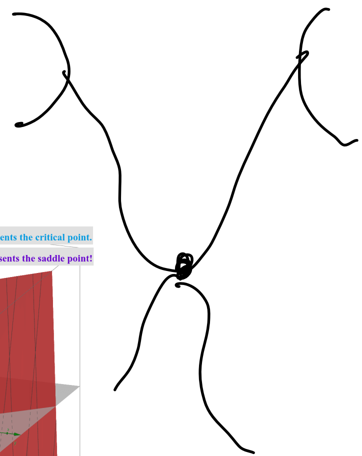
hyperbola crossing y-axis



hyperbolic paraboloid

(saddle)





Example 3

$x^2+y^2-z^2=1$

$z=0$ $\quad x^2+y^2=1$

circle

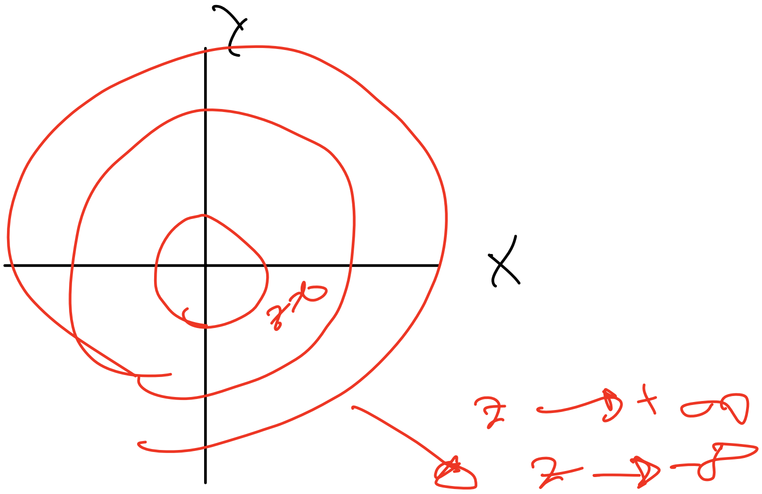
$z=1$ $\quad x^2+y^2=2$

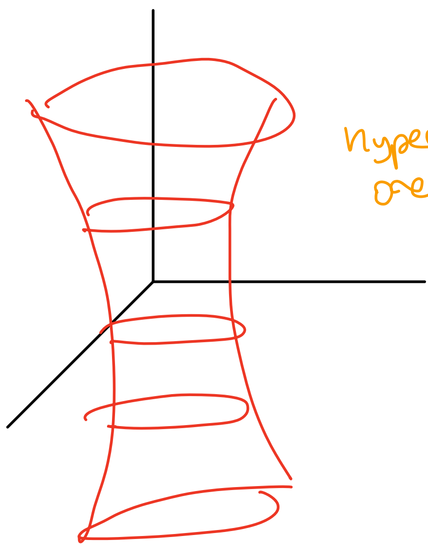
$z=2$ $\quad x^2+y^2=5$

$z=k$ $\quad x^2+y^2-k^2=1$

$x^2+y^2=\underbrace{1+k^2}_{\text{positive}}$

always circle





hyperboloid of one sheet.

$x^2+y^2-z^2=1$

$y=3$

$x^2+9-z^2=1$

$8=-x^2+z^2$

hyperbola

Example: $-x^2-y^2+z^2=1$

$z=2$ $\quad -x^2-y^2+4=1$

$3=x^2-y^2$

circle

$z=1$ $\quad -x^2-y^2+1=1$

$x^2+y^2=0$

point

$z = k$ $\quad -x^2 - y^2 - k^2 = 1$

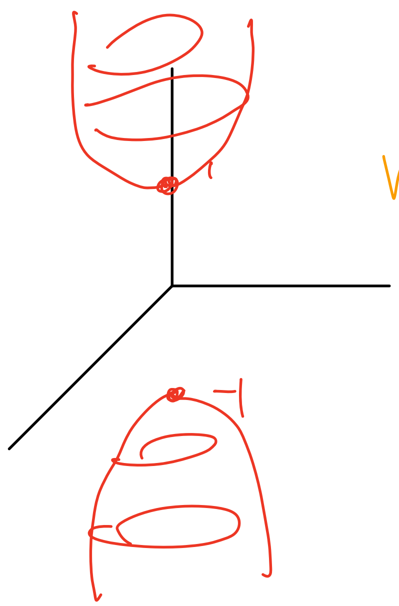
$x^2 + y^2 = k^2 - 1$

$-1 < k < 1$ ~~φ~~ empty

$k = \pm 1$

$k < -1$

or $k > 1$ circles



hyperboloid of 2 sheets

$-x^2-y^2+z^2=1$

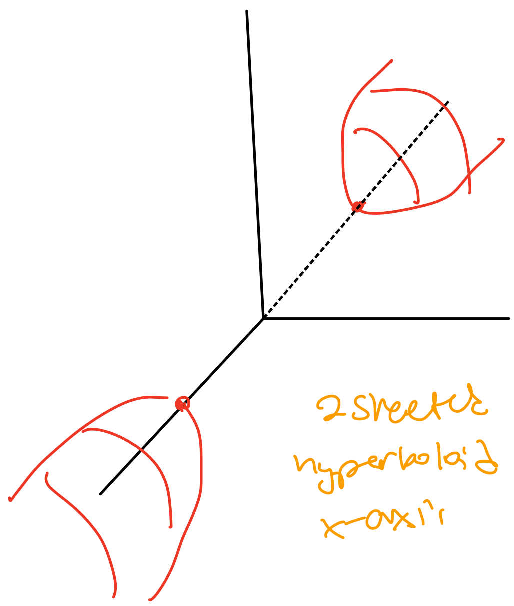
2 sheeted hyperboloid along z axis.

$x^2-y^2-z^2=1$

$x=k$ $\quad k^2-y^2-z^2=1$

$k^2-1=y^2+z^2$

2-sheeted hyperboloid along z-axis

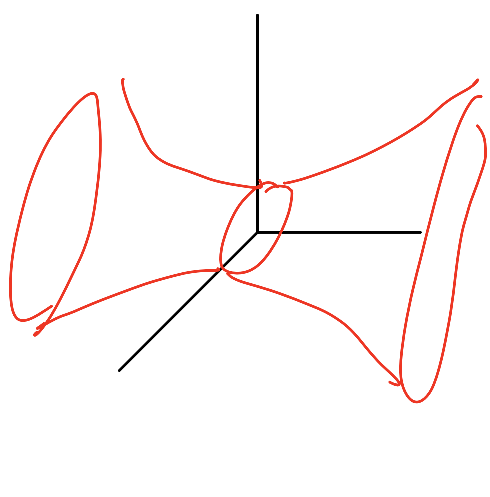


2 sheeted hyperboloid along x-axis

Ex

$x^2-y^2+z^2=1$

$y=k$ $\quad x^2+z^2=1+k^2$



$x^2 - 2x - x^2 -$

Oct 10

Warmup

Identify / sketch the surface

$x^2-2y^2+3z^2=k$

where k is some constant.

(different k might give different types of surfaces)

Q) Which cross sections give ellipsis?

A) $y=$ constant

Take $y=d$

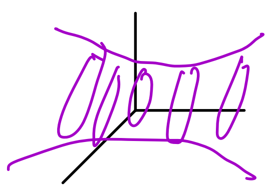
$x^2-2d^2+3z^2=k$

$\Longleftrightarrow$

$x^2+3z^2=k+2d^2$

1st) $k>0$

x^2+3z^2 = positive so an ellipse.



hyperboloid of one sheet.

k $d=0$

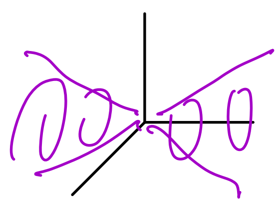
x^2+3z^2

$x=0$, $z=0$

$d\neq 0$ ellipse

$2y^2=x^2+3z^2$

$x^2-2y^2+3z^2=0$



double cone

cross section at $y = d$

$x^2 - 2d^2 + 3z^2 = k$

$x^2 + 3z^2 = k + 2d^2$

at $y = 0$ $\underline{-2}$ $\downarrow$ <0

$x^2 + 3z^2 = k$

* if $k + 2d^2 < 0$

  get nothing

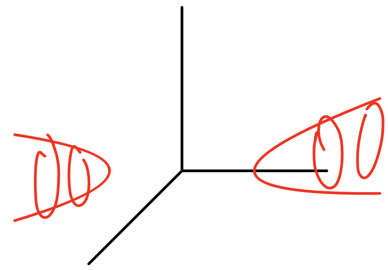
* if $k + 2d^2 = 0$

  $x^2 + 3z^2 = 0$

  point

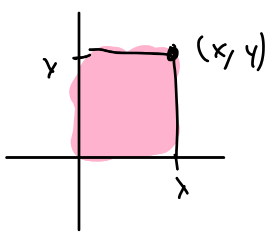
* if $k + 2d^2 > 0$

  ellipse.

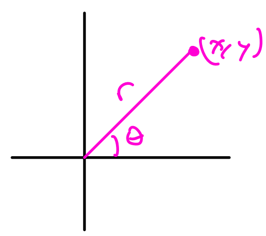


POLAR COORDINATES!

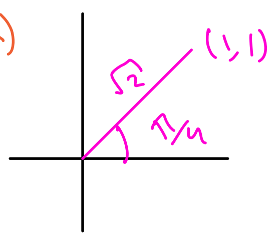
Rectangular Coordinates



Polar coordinates



Ex)



$r = \sqrt{2}$

$\theta = \frac{\pi}{4}$

Conversions:

Given (r, θ), then

then $x = r\cos\theta$

$y = r\sin\theta$

Given (x, y), then

$r^2 = x^2 + y^2$

$\theta = \arctan \frac{y}{x}$

$\tan\theta = y/x$

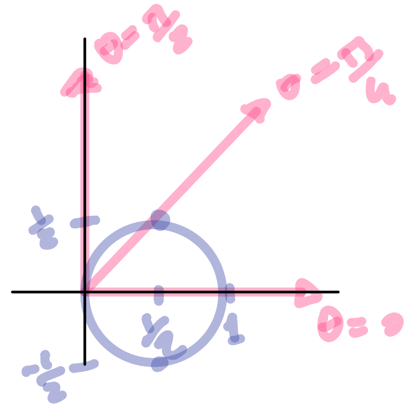
$$r = \frac{7}{y - 2x}$$

$y - 2x = 7$

$7 + 2x$

Ex) Curve with polar equation $r = \cos\theta$. Sketch this curve.

θ	r	(x, y)
0	1	$(1, 0)$
$\pi/4$	$\sqrt{2}/2$	$(1/2, 1/2)$
$\pi/2$	0	$(0, 0)$
$3\pi/4$	$-\sqrt{2}/2$	$(-\frac{1}{2}, \frac{1}{2})$
π	-1	$(1, 0)$



$$x = \frac{-\sqrt{2}}{2}\cos\frac{3\pi}{4} = \frac{-\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}\right) = \frac{1}{2}$$

$$y = \frac{-\sqrt{2}}{2}\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{1}{2}$$

Now find the rectangular equation for that curve:

multiply by r:

$r^2 = r\cos\theta$

$x^2 + y^2 = x$

$x^2 - x + y^2 = 0$

$(x - \frac{1}{2})^2 - \frac{1}{4} + y^2 = 0$

$(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$

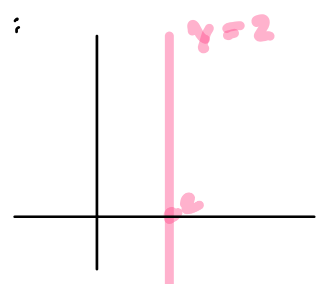
circle center $(\frac{1}{2}, 0)$

radius $\frac{1}{2}$

Ex) $r = \dfrac{2}{\sin\theta}$ Convert to rectangular:

$r\sin\theta = 2$

$y = 2$



$r^2\cos^2\theta + r^2\sin^2\theta - 7r\sin\theta + 49 = 49$

$\underbrace{r^2\cos^2\theta + r^2\sin^2\theta} - 7r\sin\theta = 0$

$r^2 - 7r\sin\theta = 0$

$r^2 = 7r\sin\theta$

$r = 7\sin\theta$

$r^2(\cancel{\cos}\cos^2 + \sin^2\theta$

r^2

Oct 12

Warmup: Find rectangular/cartesian equations for the curves with the following polar equations and sketch the curves for $0 \le \theta \le \pi$.

① $r = 2\sin\theta$

② $r = 1 - 2\cos\theta$

1/ Multiply by r

$r^2 = 2r\sin\theta$

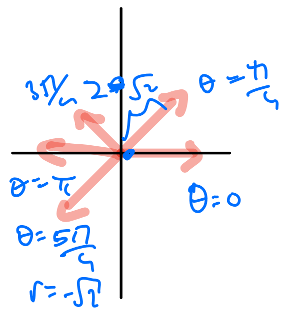
$x^2+y^2 = 2y$

$x^2 - 2y = 0$

$x^2 + (y-1)^2 - 1 = 0$

$x^2 + (y-1)^2 = 1$

$r = 2\sin\theta$

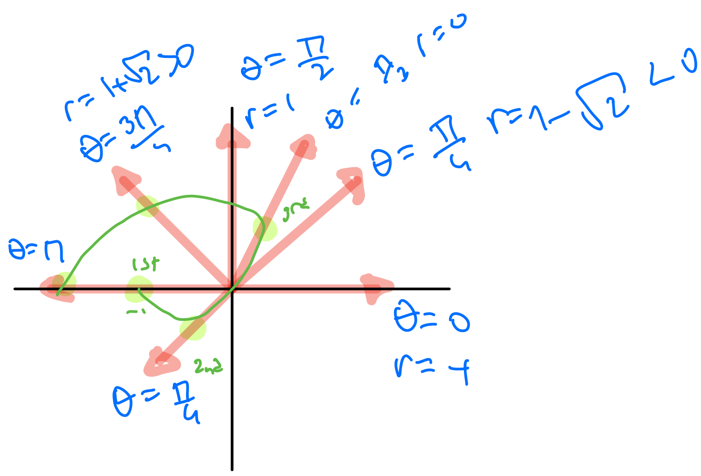


2/ $r = 1 - 2\cos\theta$

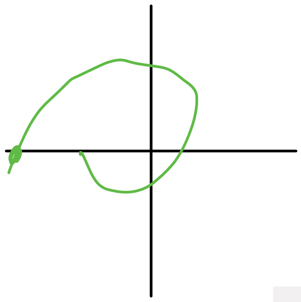
$r^2 = r - 2r\cos\theta$

$x^2+y^2 = \sqrt{x^2+y^2} - 2x$

cartesian



What if we take $\pi \leq \theta \leq 2\pi$?



CURVES!

Previously:

$x^2+y^2=1$

or

$2\sin\theta$

give a single equation

Now:

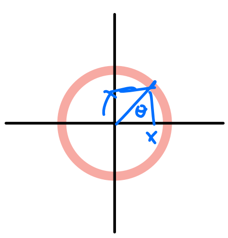
Recall that for lines we use parametric equations

$$\left.\begin{aligned} x &= \\ y &= \\ z &= \end{aligned}\right\}\text{ depending on a parameter}$$

Ex

Parametric equation for a circle $x^2+y^2=1$

What $x=$? What y?

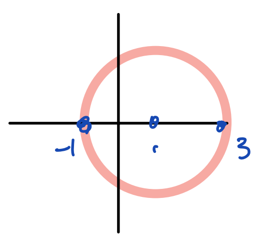


$x=\cos\theta$

$y=\sin\theta$

$0\leq\theta\leq 2\pi$

$(x-1)^2+y^2=4$



want parametric equations

$x=$

$y=$

$x = \cos t$

$y = 2\sin t$

$x^2 + y^2 = y$

so

$x = 2\cos t + 1$

$y = 2\sin t$

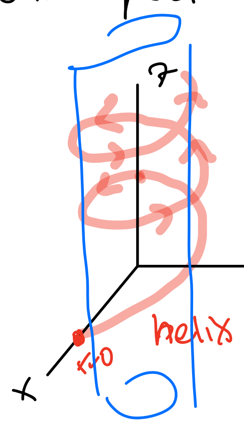
Ex

Sketch the curve in $\mathbb{R}^3$ with parametric equations

$\left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\} x^2 + y^2 = 1$

$z = t$

$0 \leq t \leq 4\pi$



$x^2 + y^2 = 1$

[cylinder]

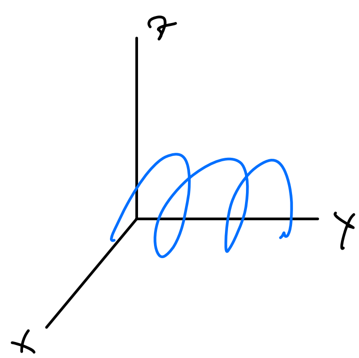
Ex

$x = \cos t$

$y = t$

$z = \sin t$

$x^2 + z^2 = 1$

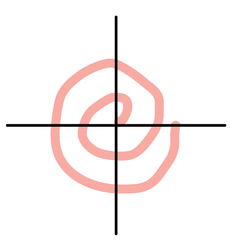


Ex

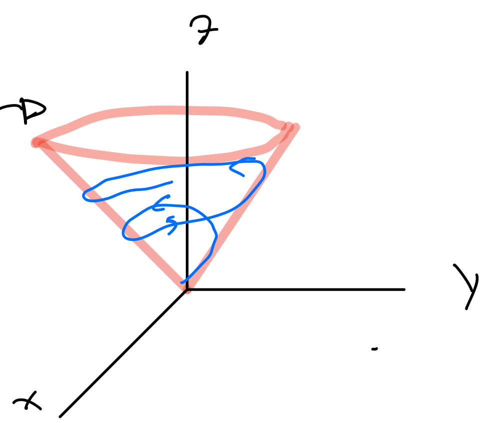
$x = t\cos t$

$y = t\sin t$

$z = t$



$x^2 + y^2 = z^2$



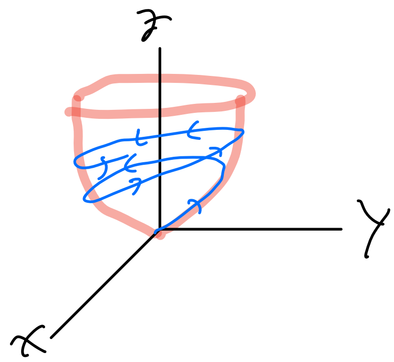
EX

$x = t\cos t$

$y = t\sin t$

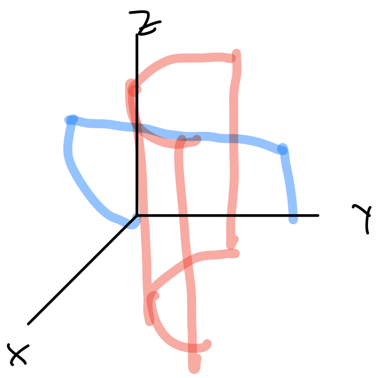
$z = t^2$

$x^2 + y^2 = z$



EX

intersection of $y = x^2$ and $z = x^3$.



Ex

Find parametric eq for this intersection

$x = t$

$y = t^2$

$z = t^3$

9/21	W	Three-Dimensional Coordinate Systems	12.1
9/23	F	Vectors	12.2
9/26	M	The Dot Product	12.3
9/28	W	The Cross Product	12.4
9/30	F	Lines in Space	12.5
10/3	M	Planes in Space	12.5
10/5	W	Conic Sections	11.6
10/7	F	Cylinders and Quadric Surfaces	12.6
10/10	M	Polar Coordinates	11.3

Lecture 11 : More on Curves

Find parametric equations for the curve $y=e^x$

$x = t^3$ → orientation

$y = e^{t^3}$

or

$x = t$

$y = e^t$

② $x = 1 - \sin\theta$

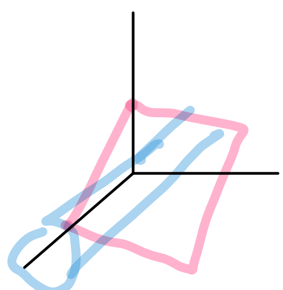
$y = \cos\theta$

$z = \sin\theta$

describe the curve as the intersection of two surfaces.

$y^2 + z^2 = 1$

cylinder.



$x + z = 1$

$y^2 + z^2 = 1$

$x + z = 1$

RECALL

curves can be described using parametric equations

$x =$

$y =$

$z =$

curve "traced out" by pts $(x(t), y(t), z(t))$ as t varies.

$x = \cos t \quad y = \sin t$ for unit circle

WARM UP

① Find parametric equations for the curve $y = e^x$ in $\mathbb{R}^2$

If $x = t^3$ If $x = t$

$y = e^{t^3}$ $y = e^t$

② Describe the curve in $\mathbb{R}^3$ with parametric equations

$x = 1 - \sin\theta \quad y = \cos\theta \quad z = \sin\theta \quad 0 \leq \theta \leq 2\pi$

Describe this curve as intersection of two surfaces

$y^2 + z^2 = 1$

given curve on cylinder with this equation

$x = 1 - \sin\theta$

$x = 1 - z$

$x+z=1$

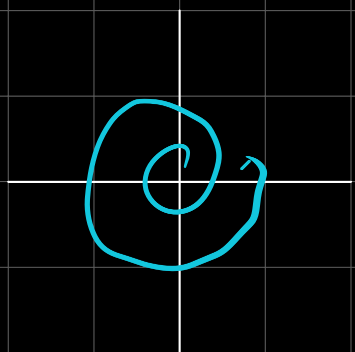
given curve is on this plane

The given curve is the intersection of the cylinder $y^2+z^2=1$ and plane $x+z=1$

③ What $x=$, $y=$, for the curve with polar equation $r=\theta$

$x = r\cos\theta = \theta\cos\theta$

$y = r\sin\theta = \theta\sin\theta$



Ex.

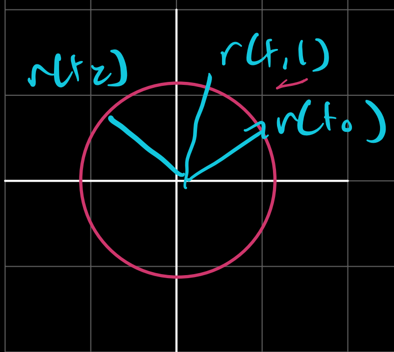
$x = 1 - \sin t \quad y = \cos t \quad z = \sin t$

$\vec{r}(t) = \langle 1-\sin t, \cos t, \sin t \rangle$

vector-valued function: function that outputs a vector

Ex.

$\vec{r}(t) = \langle \cos t, \sin t \rangle$ describes a unit circle



Ex. limit

$\vec{r}(t) = \langle 1-\sin t, \cos t, \sin t \rangle$

Compute

$\lim_{t \to \pi/3} \vec{r}(t)$

$= \lim_{t \to \pi/3} \langle 1-\sin t, \cos t, \sin t \rangle$

$= \langle 1-\sin(\pi/3), \cos(\pi/3), \sin(\pi/3) \rangle$

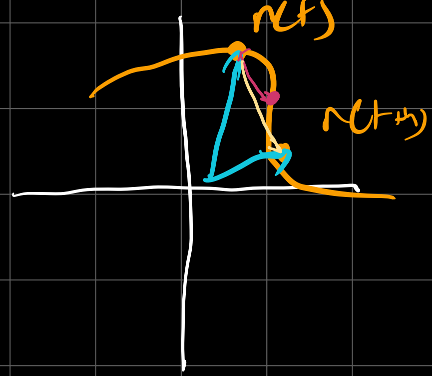
$= \langle 1-\sqrt{3}/2, 1/2, \sqrt{3}/2 \rangle$

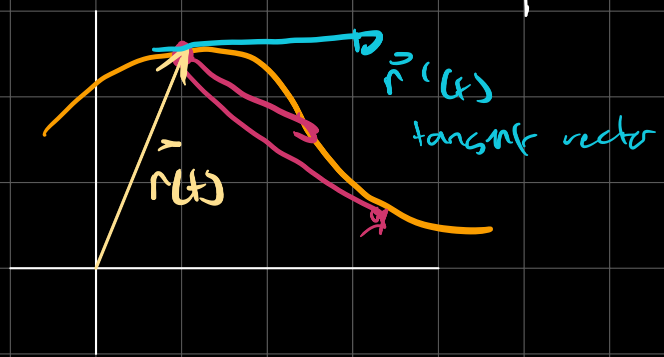
Ex. derivative

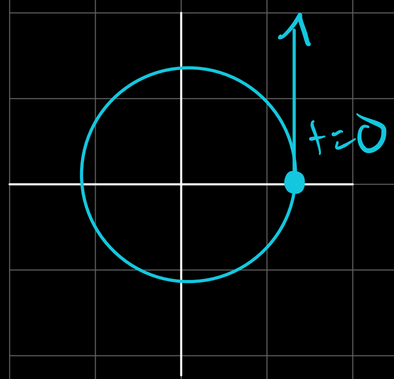
$\vec{r}(t) = \langle 1-\sin t, \cos t, \sin t \rangle$

$\vec{r}\,'(t) = \langle -\cos t, -\sin t, \cos t \rangle$

$$\lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$







$\vec{r}(0) = \langle -\sin 0, \cos 0 \rangle$
$= \langle 0, 1 \rangle$

$\vec{r}(\pi/2) = \langle -\sin \frac{\pi}{2}, \cos \pi/2 \rangle$
$\vec{r}(\pi/2) = \langle -\sqrt{2}/2, -\sqrt{2}/2 \rangle$

Ex. Given $\vec{r}(t) = \langle 1-\sin t, \cos t, \sin t \rangle$, find parametric equations for the tangent line at $(1,-1,0)$

Use $r'(t)$ as the direction vector.

point $(1,-1,0)$ is at $t=\pi$

$\vec{r}(t) = \langle -\cos t, -\sin t, \cos t \rangle$

$\vec{r}(\pi) = \langle -\cos \pi, -\sin \pi, \cos \pi \rangle$
$= \langle 1, 0, -1 \rangle$

tangent line at $(1,-1,0)$

$x = 1 + t$ $\qquad x = 1+t$

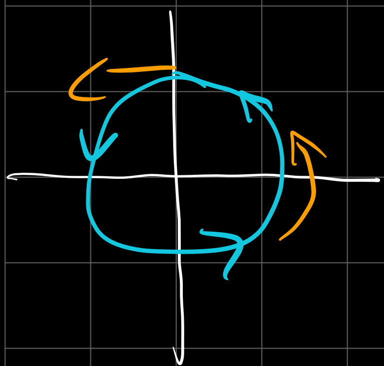
$y = -1 + 0t$ $\qquad y = -1$

$z = 0 - t$ $\qquad z = -t$

Ex. $\vec{r}(t) = \langle \cos t, \sin t \rangle$

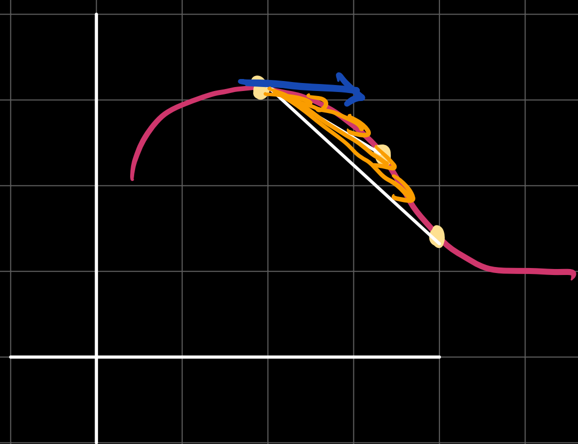
$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$ → velocity vector

$|\vec{r}\,(t)|$ = speed



SUMMARY

derivative of a vector is the vector tangent to the point at t



$$r'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

derivative is also a velocity vector

Given $\vec{r}(t) = \langle 1-\sin t, \cos t, \sin t \rangle$, find parametric equations for tangent line at $(1,-1,0)$.

1 Find t

$\to t = \pi$

2 Find $r'(t)$, use as direction vector

$r'(t) = \langle -\cos t, -\sin t, \cos t \rangle$

$r'(\pi) = \langle -\cos\pi, -\sin\pi, \cos\pi \rangle$

$= \langle 1, 0, -1 \rangle$

3 tangent line:

$x = 1 + t$

$y = -1$

$z = -t$

3 want $x =$

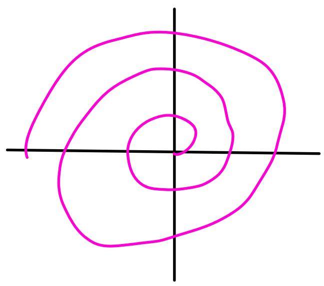
$y =$

for the curve with polar equation $r = \theta$

$x^2 + y^2 = r^2 = \theta^2$

$x = r\cos\theta = \theta\cos\theta$

$y = r\sin\theta = \theta\sin\theta$



a vector valued functions

$\vec{r}(t) = \langle 1 - \sin t, \cos t, \sin t \rangle$

Ex)

$\vec{r}(t) = \langle \cos t, \sin t \rangle$

describes a unit circle.

$\vec{r}(t)$ position vector

Ex) limit

$\vec{r}(t) = \langle 1 - \sin t, \cos t, \sin t \rangle$

Compute $\lim\limits_{t \to \pi/3} \vec{r}(t)$

$$\lim_{t \to \frac{\pi}{3}} \langle 1 - \sin t, \cos t, \sin t \rangle =$$

$$\langle 1 - \sin \tfrac{\pi}{3}, \cos \tfrac{\pi}{2}, \sin \tfrac{\pi}{2} \rangle$$

$$\langle 1 - \tfrac{\sqrt{3}}{2}, \tfrac{1}{2}, \tfrac{\sqrt{3}}{2} \rangle$$

Ex) derivative

$$\vec{r}(t) = \langle 1 - \sin t, \cos t, \sin t \rangle$$

$$\vec{r}\,'(t) = \langle -\cos t, -\sin t, \cos t \rangle$$

$$\lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

Ex) $\vec{r}(t) = \langle \cos t, \sin t \rangle$

$$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$$

$$\vec{r}\,'(0) = \langle -\sin 0, \cos 0 \rangle$$
$$= \langle 0, 1 \rangle$$

$$\vec{r}\,'\left(\tfrac{3\pi}{4}\right) = \langle -\sin \tfrac{3\pi}{4}, \cos \tfrac{3\pi}{4} \rangle$$

$$\left\langle -\tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2} \right\rangle$$

Ex) Given $\vec{r}(t) = \langle 1-\sin t, \cos t, \sin t \rangle$

find parametric equations for the tangent line

at $(1,-1,0)$.

$t = \pi$

use $\vec{r}\,'(\pi)$ as the direction vector.

$r'(\pi) = \langle -\cos\pi, -\sin\pi, \cos\pi \rangle$

$\langle 1, 0, -1 \rangle$

tangent line at $(1,-1,0)$

$x = 1 + t = 1 + t$

$y = -1 + 0t = -1$

$z = 0 - t = -t$

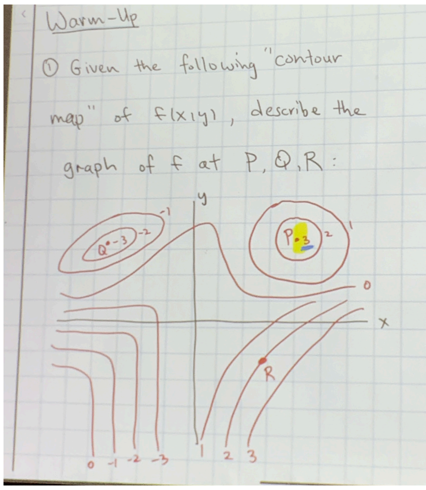
Ex) $\vec{r}(t) = \langle \cos t, \sin t \rangle$

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

"velocity vector"

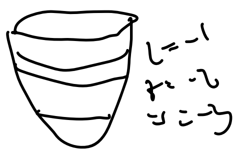
$|\vec{r}\,'(t)| =$ speed.

October 26



① $f(p) = 3$

$f(q) = -3$



near R: graph slopes upward in positive x direction but downward in positive y direction.

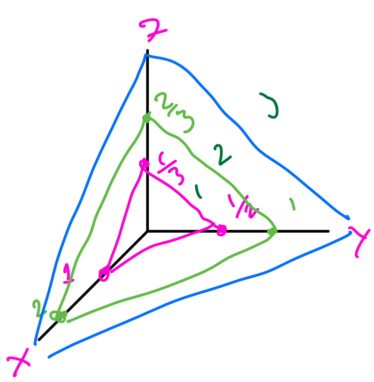
Level surface $s(x,y,z) = k$

$k = x+2y+z$

$k=1 \quad 1 = x+2y+3z$

$k=2 \quad 2 = x+2y+3z$

$k=3 \quad 3 = x+2y+3z$



Limits

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

if there is one

inputs

= the number that $f(x,y)$ approach as (x,y) approaches (a,b).

same with $f(x,y,z)$:

$\lim f(x,y,z) \to a,b,c$

EXAMPLE:

$$\lim_{(x,y)\to(1,2)} (xy + y^3)$$

since $x \to 1$, $y \to 2$

so $xy \to 1 \cdot 2 = 2$

and $y^3 \to 2^3$, thus

$xy + y^3 = 2 + 2^3 = 10$

Since $f(x,y) = xy + y^3$ is continuous, built from individual functions which are all continuous.

$$\lim_{(x,y)\to(1,2)} f(x,y) = f(1,2)$$

EXAMPLE

$$\lim_{(x,y)\to(1,3)} \frac{xy + x - y - 1}{x-1}$$

$3+1-3-1 = 0$

$1-1 = 0$

$\frac{0}{0}$

$$\lim_{(x,y)\to(1,3)} \frac{xy+x-y-2}{x-1}$$

-1

0

$\to \infty$

does not exist

$$= \lim_{(x,y)\to(1,3)} \frac{\cancel{(x-1)}(y+1)}{\cancel{(x-1)}} = 4$$

EXAMPLE:

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)$$

$$-x^2-y^2 \le x^2+y^2 \sin\left(\frac{1}{x^2+y^2}\right) \le x^2+y$$

0 $\qquad \downarrow$ $\qquad 0$

so 0

Similarly

Example:

$\lim_{(x,y)\to(0,0)} x^2 \sin \frac{1}{x^2}$

Sandwich theorem

$-x^2 \leq x^2 \sin\left(\frac{1}{x^2}\right) \leq x^2$

$0 \qquad$ so $0 \qquad 0$

my lab

$9 - 25x^2 - 25y^2 > 0$

$-x^2 - y^2 > -\frac{9}{25}$

$x^2 + y^2 < +\frac{9}{25}$

$16 - x^2 - y^2 = 0$

$16 - 4\cdot(\sqrt{3})^2 - 4(3\sqrt{3})^2 =$

$16 - 12 - 4\cdot 27 = -104$

$-104 = 16 - \cancel{4x^2} - 4x^2$

$-26 = u - x^2 - y^2$

$-30 = -x^2 - y^2$

$30 = x^2 + y^2$

$\sqrt{6 + 16 - 6} = 4$

$u = \sqrt{x + y^2 - 6}$

$16 = x + y^2 - 6$

$22 = x + y^2$

$\sqrt{2 + 3^2 - 2} = 3$

$3 = \sqrt{x + y^2 - 2}$

$9 = x + y^2 -$ $\quad$ $11 = x + y^2$

$$\sqrt{x+y} - \ln z$$

$$3 = \sqrt{x+y} - \ln z$$

October 28

Warmup

① Determine the value of c that makes

$$f(x,y) = \frac{x^3 - 3x - y^2 + 3y}{x+y-3}$$

$x+y-3 \neq 0$

$x+y-3=0$

cont. at $(1,2)$

① Need

$$\lim_{(x,y)\to(1,2)} f(x,y) = f(1,2)$$

in order for f to be continuous at $(1,2)$

Goal Compute

$$\lim_{(x,y)\to(1,2)} f(x,y) = \lim_{(x,y)\to(1,2)} \frac{x^2 - 3x - y^2 + 3y}{x+y-3}$$

$$\lim_{(x,y)\to(1,2)} \frac{(x+y-3)(x-y)}{(x+y-3)} = \lim_{(x,y)\to(1,2)} x-y$$

$$= 1-2 = -1$$

2 If

$$|f(x,y)| \leq \left[\ln(x^2+y^2+1)\right]^2 \qquad 0$$

Compute $\lim_{(x,y)\to(0,0)} e^{f(x,y)}$

By continuity of e^{power}, $\lim_{(x,y)\to(0,0)} e^{f(x,y)}$

$$= e^{\lim_{(x,y)\to(0,0)} f(x,y)}$$

Need

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

$$\underbrace{-\left[\ln(x^2+y^2+1)\right]^2}_{0} \le f(x,y) \le \underbrace{\left[\ln(x^2+y^2+1)\right]^2}_{0}$$

so

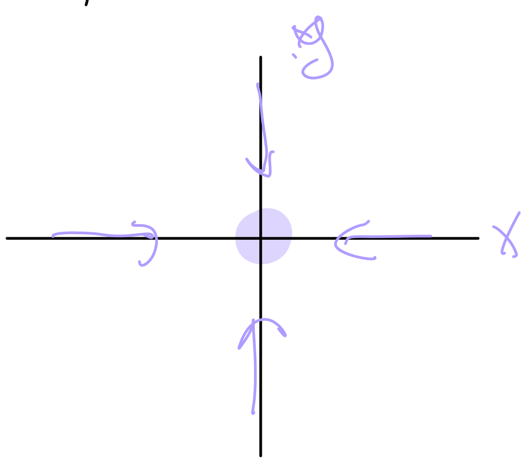
$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

by sandwich theorem.

EX Determine if

$$\lim_{(x,y)\to(0,0)} \frac{x+y}{x+2y}$$

exists.



if a limit exists, approaching along any curve should give the same limiting value.

Approach along $y=0$

$$\lim_{(x,0)\to(0,0)} \frac{x}{x} = \lim_{x\to 0} 1 = 1$$

Approaching along $x=0$

$$\lim_{(0,y)\to(0,0)} \frac{y}{2y} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$$

Since the curves give different candidate limits $\lim_{(x,y)\to(0,0)} \frac{x+y}{x+2y}$ DNE.

Example:

does $\lim_{(x,y)\to(0,0)} \frac{xy}{}$

GEÇİR

Example 2

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6+y^2}$$

Along $y=0 \to 0$

Along $x=0 \to 0$

Along $y=x$

$$\lim_{(x,x)\to(0,0)} \frac{x^4}{x^6+x^2} = \lim_{x\to 0} \frac{x^2}{x^4+1} = 0$$

Along $y=x^3$

$$\lim_{(x,x^3)\to(0,0)} \frac{x^6}{x^6+x} = \lim_{x\to 0} \frac{x^6}{2x^6} = \frac{1}{2}$$

Example

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$$

This is $=0$, but why?

Convert to polar:

$$\lim_{r\to 0} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2} =$$

θ can be anything

$$\lim_{r\to 0} r\left[\cos^3\theta + \sin^3\theta\right]$$

$-1 \leq \cos^3\theta \leq 1$

$$-2r \leq r\left[\cos^3\theta + \sin^3\theta\right] \leq 2r$$

so by sandwhich theorem

limit is 0.

Oct 31

Warmup

① Verify that

$$\lim_{\substack{(x,y)\to(0,1) \\ y = x+1}} \frac{x \ln y}{y-x-1}$$

does not exist.

→ Along $x=0$:

$$\lim_{(0,y)\to(0,1)} \frac{0}{y-1} = 0$$

→ Along $y=-x+1$:

$$\lim_{(x,-x+1)\to(0,1)} \frac{x\ln(-x+1)}{-2x} =$$

$$\lim_{x\to 0} -\frac{1}{2}\ln(-x+1) = 0$$

→ Along $y=1$

$=0$

→ Along $y=e^x$

$$\lim_{(x,e^x)\to(0,1)} \frac{x\ln e^x}{e^x-x-1}$$

$$= \lim_{x\to 0} \frac{x^2}{e^x-x-1}$$

L'hopital (since its single var)

again

$$= \lim_{x\to 0} \frac{2x}{e^x-1}$$

$$= \lim_{x\to 0} \frac{2}{e^x} = 2 \,//$$

(2) $$\lim_{(x,y)\to(0,0)} \frac{x-2xy+y^2}{\underbrace{\sqrt{x^2+y^2}}_{\text{polar coordinates!}}}$$

What if $\theta \to \frac{\pi}{2}$?

in polar coordinates:

$$\lim_{\substack{r\to 0\\ \theta \text{ anything}}} \frac{r\cos\theta - 2r^2\cos\theta\sin\theta + r^2\sin^2\theta}{r} =$$

$$\lim_{r\to 0} \cos\theta \; \underbrace{-2r\cos\theta\sin\theta + r\sin^2\theta}$$

$\cos\theta$ depends on θ

so does not exist

≥ -3

$$-3r \leq r(\underbrace{-2\cos\theta\sin\theta + \sin^2\theta}_{\leq 3}) \leq 3r$$

so $=0$ by sandwich theorem.

Partial Derivative

ex: $f(x,y) = x^2 y$ at $(1,2)$

defn: the partial derivative of f with respect to x at (1,2)

is $\frac{df}{dx}(1,2) = \lim_{h \to 0} \frac{f(1+h,2) - f(1,2)}{h}$

$\quad\; \shortparallel$

$f_x(1,2)$.

i.e. the single variable derivative

of f with respect to holding y constant.

$$\frac{df}{dx}(1,2) = \lim_{h \to 0} \frac{(1+h)^2 \cdot 2 - 1^2 \cdot 2}{h} =$$

$$\frac{2(1+2h+h^2) - 2}{h} = \frac{4h + 2h^2}{h} =$$

$$\lim_{h \to 0} 4 + 2h = 4$$

$f(x,y) = x^2 y$ at $(1,2)$

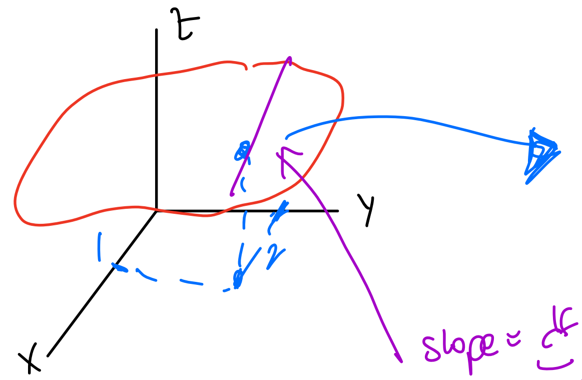
$$\frac{df}{dx}(x,y) = 2xy = 2 \cdot 1 \cdot 2 = 4$$

(

$$\frac{df}{dy}(1,2) = \lim_{h \to 0} \frac{f(1,2+h) - f(1,2)}{h}$$

x^2 evaluated at $(1,2)$

$= 1.$



$(1,2,f(1,2)) = (1,2,2)$

slope $= \frac{df}{dx}(1,2)$

slope $= \frac{df}{dy}(1,2)$ in y direction

EXAMPLE:

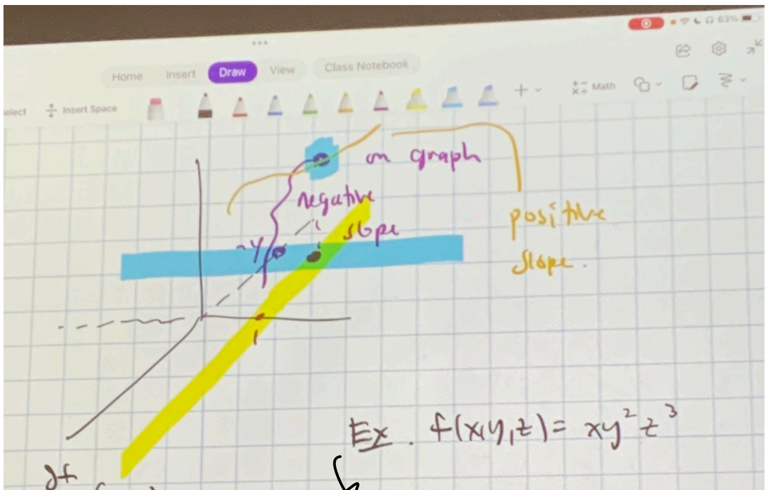
$$f(x,y) = x e^{xy^2}$$

$$\frac{df}{dx} = f_x = e^{xy^2} + xy^2 e^{xy^2}$$

$$\frac{df}{dy} = f_y = 2y \cdot x^2 \cdot e^{xy^2}$$

$$\frac{df}{dx}(-4,1) = e^{-4} - 4e^{-4} = -3e^{-4}$$

$$\frac{\partial f}{\partial y}(-4,1) = 32e^{-4}$$



$$\frac{\partial f}{\partial x} = f_x = y^2z^3$$

$$\frac{\partial f}{\partial y} = f_y = 2xyz^3$$

$$\frac{\partial f}{\partial z} = 3xy^2z^2$$

Second derivatives

$$f(x,y) = x^3y^2 + x$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 + 1$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = 6xy^2$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} = f_{xy} = 6yx^2$$

$$\frac{\partial f}{\partial y} = 2x^3y$$

November 2

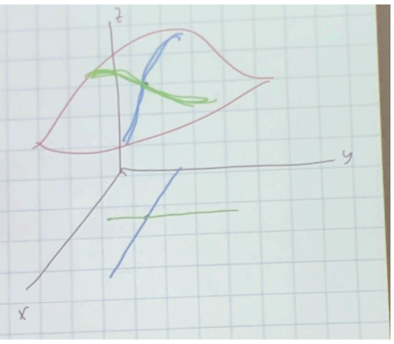
Warmup:

Find the slope of the tangent line to the piece of

$z = x^3y^4 + y\sin xy$

in the plane $y = \frac{\pi}{2}$ at $(1, \frac{\pi}{2})$ and to the piece

in $x=1$ at $\rightarrow$



in plane $y = \frac{\pi}{2}$ only vary x,

slope $= \frac{df}{dx}(1, \frac{\pi}{2})$

$z = x^3y^4 + y\sin xy$

$\frac{df}{dx} = f_x = 3x^2y^4 + y^2\cos(xy)$

slope in x dir:

$$\frac{df}{dx}\left(1,\frac{\pi}{2}\right) = 3\left(\frac{\pi}{2}\right)^4 + \left(\frac{\pi}{2}\right)^2 \cdot \cos\left(\frac{\pi}{2}\right) = \frac{3\pi^4}{16}$$

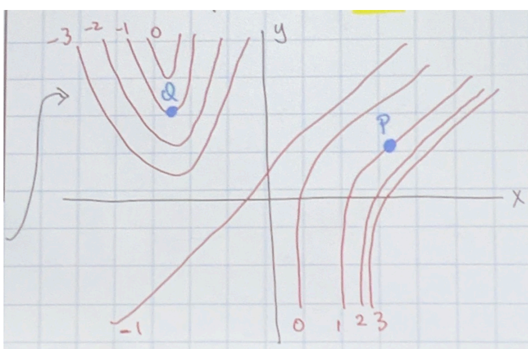
$$\frac{df}{dy} = f_y = 4y^3x^3 + \sin(xy) + y\cdot x\cos(xy)$$

slope in y dir:

$$\frac{df}{dy}\left(1,\frac{\pi}{2}\right) = 4\cdot\left(\frac{\pi}{2}\right)^3 + \sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2}\cdot\cos\left(\frac{\pi}{2}\right)$$

$$\frac{4\pi^3}{8} + 1$$

②



Given, what does

$f_x(p)$? $f_y(p)$

$f_x(q)$? $f_y(q)$

$$\frac{df}{dx}(p) > 0 \qquad \frac{df}{dx}(Q) = 0$$

$$\frac{df}{dy}(p) < 0 \qquad \frac{df}{dy}(Q) > 0$$

Example: $f(x,y) = x^3y^2 + x$

$$\frac{df}{dx} = 3x^2y^2 + 1$$

$$\frac{d^2f}{dx^2} = f_{xx} = 6xy^2$$

$$\frac{d^2f}{dydx} = f_{xy} = 6x^2y$$

$$\frac{df}{dy} = 2x^3y$$

$$\frac{d^2f}{dxdy} = f_{yx} = 6x^2y$$

$$\frac{d^2f}{dy^2} = f_{yy} = 2x^3$$

Example: $f(x,y,z) = xy^2z^3$

$$\frac{df}{dx} = y^2z^3 \rightarrow f_{xz} = 3y^2z^2$$

$$\frac{d^2f}{dzdx} = \frac{d^2f}{dxdy}$$

"mixed 2nd order derivatives"

$$\frac{df}{dy} = 2yz^3$$

$$\frac{df}{dz} = 3xy^2z^2 \rightarrow f_{zx} = 3y^2z^2$$

Clairut's Theorem

If $\frac{d^2 f}{dx dy}$ and $\frac{d^2 f}{dy dx}$ are continuous, then

$$\frac{d^2 f}{dx dy} = \frac{d^2 f}{dy dx}$$

Chain Rule

Recall $y = f(x)$
$x = g(t)$

Compute the derivative of $y = f(g(t))$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

outside   inside

Example: say $f(x, y) = x^2 y + e^{xy}$ and $x = 2st$, $y = s^2 - t^2$

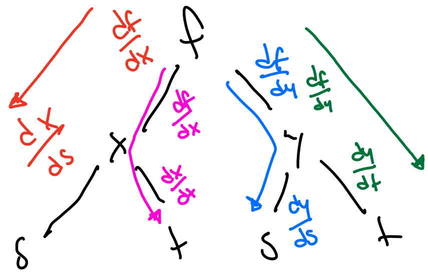
compute $\frac{df}{ds}$ and $\frac{df}{dt}$

one approach: write f in terms of s and t.

$$f(s, t) = (2st)^2 (s^2 - t^2) + e^{2st(s^2 - t^2)}$$

then compute $\frac{df}{ds}$, $\frac{df}{dt}$

Better approach:



$$\frac{df}{ds} = \frac{df}{dx} \cdot \frac{dx}{ds} + \frac{df}{dy} \cdot \frac{dy}{ds}$$

$$(2xy + ye^{xy})(2t) + (x^2 + xe^{xy})(2s)$$

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt}$$

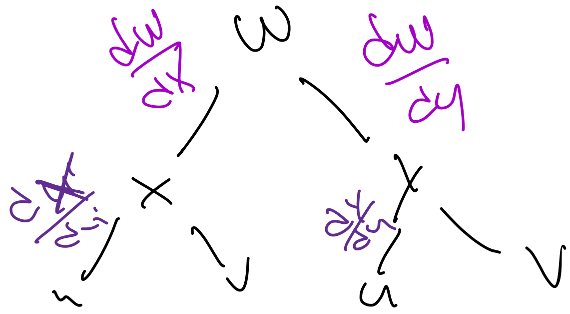
$$(2xy + ye^{xy})(2s) + (x^2 + e^{xy}) \cdot (-2t)$$

Example: $f(x, y, z) = xyz$

$x = st \qquad y = s + t \qquad z = s - t$

$$\frac{df}{ds} = \frac{df}{dx} \cdot \frac{dx}{ds} + \frac{df}{dy} \cdot \frac{dy}{ds} + \frac{df}{dz} \cdot \frac{dz}{ds}$$

$$yz \cdot t + xz \cdot 1 + xy \cdot 1$$



$w = xy + yz + xz$

$x = 3u + v$

$y = 3u - v$

$z = 4v$

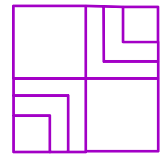
$$\frac{dw}{dx}\cdot\frac{dx}{du} + \frac{dw}{dy}\cdot\frac{dy}{du} + \frac{dw}{dy}\cdot\frac{dy}{du}$$

$\frac{dw}{dx} = y + z \qquad \frac{dx}{du} = 3 \quad \rightarrow \quad 3y + 3z$

$\frac{dw}{dy} = x + z \qquad \frac{dy}{du} = 3 \quad \rightarrow \quad 3x + 3z$

$3y + 3x + 6z$

$\frac{dw}{dz} = x + y \qquad \frac{dw}{du} = v$



$$\frac{dw}{du} = \frac{dw}{dx} \cdot \frac{dx}{du}$$

$$= (y+z)\,3$$

$$= 3(3u - v + uv)$$

$$= \boxed{9u - 3v + 3uv}$$

$$\frac{dw}{dy} \cdot \frac{dy}{du}$$

$$= (x+z)\,3$$

$$= 3(3u + v + uv)$$

$$= \boxed{9u + 3v + 3uv}$$

$$\frac{dw}{dz} \cdot \frac{dz}{du} =$$

$$v(x+y) =$$

$$v(3u \cancel{+v} + 3u \cancel{-v}) =$$

$$6uv$$

$$18u + 12uv$$

$\frac{dw}{dy} \cdot \frac{dy}{dv} =$

$(x+z)(-1) = -x - z$

$\frac{dw}{dx} \cdot \frac{dx}{dv} =$

$y - x + ux + uy$

$(y+z)(1) = y + z$

$(3u - v) - (3u + v)$

$-2v + u(3u + v) + u(3u - v)$

$\frac{dw}{dz} \cdot \frac{dz}{dv} =$

$-2v + 3u^2 + uv + 3u^2 - uv$

$(x+y)(u) = ux + uy$

$6u^2 - 2v$

mylab math

$18\left(\frac{-1}{4}\right) + 12\left(\frac{-1}{4}\right) \cdot (-2)$

$\frac{-18}{4} + \frac{24}{4} = \frac{6}{4} = \frac{3}{2}$

$6\left(\frac{1}{4}\right)^2 - 2(-2) = \frac{6}{16} + 4$

$\frac{3}{8} + 4 =$

$$q - r = 6x - 6y + 6z$$
$$-6x - 6y + 6z =$$
$$12z - 12y$$

$$\frac{3+32}{8} = \frac{40}{8} = \frac{5}{4}$$

$$r - p = 6x + 6y - 6z$$
$$-6x - 6y - 6z = -12z$$

$$p - q = 6x + 6y + 6z$$
$$-6x + 6y - 6z = 12y$$

$$\left(\frac{1}{q-r}\right)6 + \left(\frac{r-p}{(q-r)^2}\right)6 + \left(\frac{p-q}{(q-r)^2}\right)6$$

$$\frac{1}{2z-2y} + \frac{6\cdot(-12z)}{(12z-12y)^2} + \frac{(12y)\cdot 6}{(12z-12y)^2}$$

$$\frac{6(12z-12y)}{(12z-12y)^2} + \frac{-72z \quad +72y}{} =$$

$$\frac{du}{dp} = \frac{1}{(q-r)}$$

$$\underbrace{\frac{du}{dp}\cdot\frac{dp}{dy}}_{\left(\frac{1}{q-r}\right)\cdot 6} + \underbrace{\frac{du}{dq}\cdot\frac{dq}{dy}}_{\frac{r-p}{(q-r)^2}\cdot(-6)} + \frac{du}{dr}\cdot\frac{dr}{dy} =$$

$$+ \frac{p-q}{(q-r)^2}\cdot 6$$

$$72z - 72y + 72y + 72z$$

$$\frac{2\times 72y}{(12z-12y)^2} = \frac{2\cdot 72\cdot 4}{(12\cdot 2 - 12\cdot 4)^2} =$$

$$\frac{144y}{(12z-12y)^2} = \frac{y}{(z-y)^2}$$

$$\frac{3}{}$$

$u=0 \qquad v=2 \qquad y=uv$

$z= \sin(xy)+x \sin y \qquad x=2u^2+3v^2$

$$\frac{dz}{dx}\cdot\frac{dx}{du} + \frac{dz}{dy}\cdot\frac{dy}{du} = \frac{dz}{du}$$

$$\left(y\cos(xy) + \sin y\right)\cdot 4u =$$

$$uv\left(\cos\left(\left(2u^2+3v^2\right)\cdot uv\right)\right) \cdot 4u = 0$$

$$\left(x\cos(xy) + x\cos y\right)\cdot v = 0$$

$$\left(3\cdot 2^2\cdot\cos(0) + 3\cdot 2^2\cdot\cos(0)\right)\cdot 2 =$$

$$2^3\cdot 3\cdot 2 = 16\cdot 3 = 48$$

$$\frac{dw}{dr} = \frac{dw}{dy} \cdot \frac{dy}{dr} + \frac{dw}{dx} \cdot \frac{dx}{dr}$$

$$\frac{dz}{dy} \cdot e^{s} + \frac{dz}{dx} \cdot 9r^{2}$$

$$\frac{dz}{dy} + 9\frac{dz}{dx}$$

$$3 \quad -4 \cdot 9 = -33$$

-2 -4

3 -3

7

7

$f'(9) = -4$ $f'(7) = -3$

$-3\,[\quad$ $-3 \cdot (-3)$

$$(-3) \times (-4) \times (7) + (-3)(-3)(1)$$

$$12 \times 7 + 9$$

$f'(7) \times g_x \times g + f'(7) \cancel{\times} g_y$

$(-3)(-4) \times 9 + (-3)(3)$

117

$f'(7) \times g_y$

$-3 \cdot 3 = -9$

$T = T(x,y)$ satisfies $T_x(1,5) = 5$

$T_y(1,5) = -4$

$x = e^{3t-3}$

$y = 5t \ln t$

$3 \cdot e^{3t-3}$

$\frac{1}{t}$

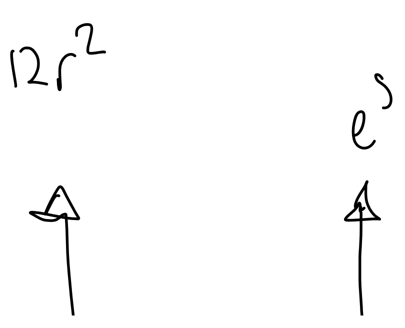
$T_x \cdot e^{3t-3} + T_y \frac{1}{t}$

when $t=1$

$T_x \cdot 3e^0 + t_y \cdot 1$

$3T_x + t_y$

$15 + (-4) = 11$



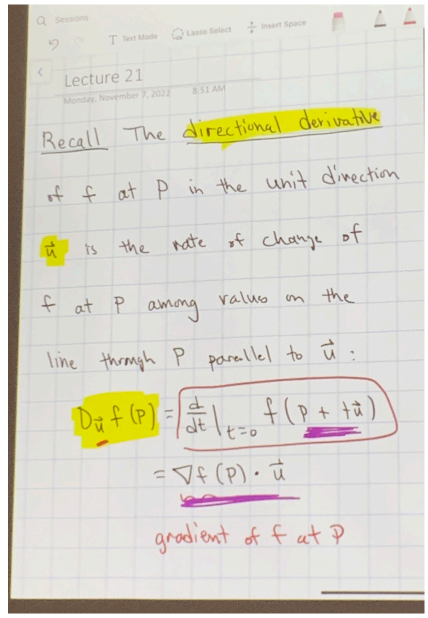
$$\frac{dz}{dr} = f'(w)\left[g_x \cdot \frac{dx}{dr} + g_y \cdot \frac{dy}{dr} \right]$$

$$-2\left[-4 \cdot 12r^2 + 3 \cdot 1\right] = 90$$

$$\frac{dz}{ds} = f'(w)\left[g_x \cdot \frac{dx}{ds} + g_y \cdot \frac{dy}{ds} \right]$$

$$-2\left[\underbrace{-4 \cdot (-2s^2)}_{0} + \underbrace{3 \cdot r \cdot e^s}_{3 \cdot 1 \cdot e^0}\right]$$

Nov 7


Lecture 21

Monday, November 7, 2022 8:51 AM

Recall The directional derivative of f at P in the unit direction $\vec{u}$ is the rate of change of f at P among values on the line through P parallel to $\vec{u}$:

$$\boxed{D_{\vec{u}} f(P) = \frac{d}{dt}\Big|_{t=0} f(P + t\vec{u})}$$
$$= \nabla f(P) \cdot \vec{u}$$

gradient of f at P

Warmup

Set $f(x,y,z) = x^2y - y^3z + xz^2$

and $\vec{v} = i + j + k = \langle 1,1,1\rangle$

Is the rate of change of f in the direction of $\vec{v}$ larger at $(1,1,1)$ or at $(1,-3,0)$?

$$D_i f(p) = \frac{df}{dx}(p)$$

$$D_j f(p) = \frac{df}{dy}(p)$$

i.e. Which of $D_{\vec{u}} f(1,1,1)$ or $D_{\vec{u}} f(1,-3,0)$ is larger, where

$$\vec{u} = \frac{\langle 1,1,1\rangle}{|\langle 1,1,1\rangle|} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle \ ?$$

$$\nabla f(x,y,z) = \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle =$$

$$\langle 2xy+z,\ x^2 - 3y^2 z,\ -y^3 + 2xz \rangle$$

$$D_u f(1,1,1) = \nabla f(1,1,1) \cdot \vec{u} =$$

$$\langle 3, -2, 1 \rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle =$$

$$\frac{3}{\sqrt{3}} - \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$D_u f(1,-3,0) = \nabla f(1,-3,0) \cdot \vec{u} =$$

$$\langle -6, 1, 27 \rangle \cdot \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle$$

$$= \frac{-6}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{27}{\sqrt{3}} = \frac{22}{\sqrt{3}}$$

so $D_u f(1,-3,0) > D_u f(1,1,1)$

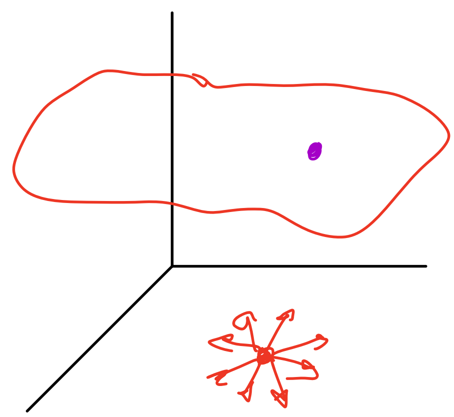
Example:

$f(x,y) = x^2y + y$

$\nabla f(x,y) = \langle 2xy, x^2+1 \rangle$

Q At $(1,2)$, is there a direction $\vec{u}$ such that

$D_{\vec{u}} f(1,2) = 10$?



A No

$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u}$

$= \langle 4,2 \rangle \cdot \vec{u} = 10$

Q What is the largest rate of change of f at $(1,2)$ among all directions?

$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u}$

$|\nabla f(1,2)| \underbrace{|\vec{u}|}_{1} \cos\theta = |\nabla f(1,2)| \cos\theta$

$D_{\vec{u}} f(1,2)$ is maximized when $\cos\theta = 1$, ie $\theta = 0$

A max rate of change $= |\nabla f(1,2)| = \sqrt{20}$

direction in which $D_{\vec{u}} f(1,2) = \sqrt{20}$ is the given by $\nabla f(1,2) = \langle 4,2 \rangle$.

Summary

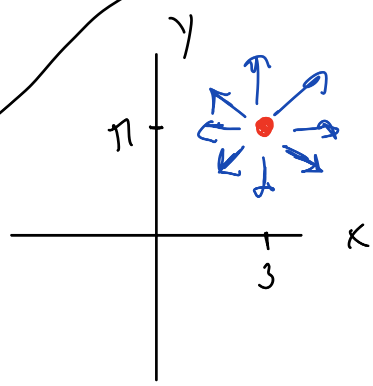
- $\nabla f(p)$ gives direction in which f increases most rapidly at p.
- $|\nabla f(p)|$ = max rate of change.

$$D_u f(p) = |\nabla f(p)| \cos\theta$$

Example:

$$T(x,y) = x^3y^2 + x\cos(xy)$$

↳ temperature at (x,y). At $(3,\pi)$, in which direction does T increase most rapidly?



$\nabla f(3,\pi)$

Answer

$$\nabla T = \langle 3x^2y^2 + \cos(xy) - xy\sin(xy),\ 2x^3y - x^2\sin(xy)\rangle$$

$$\nabla T(3,\pi) = \langle 27\pi^2 - 1,\ 54\pi\rangle$$

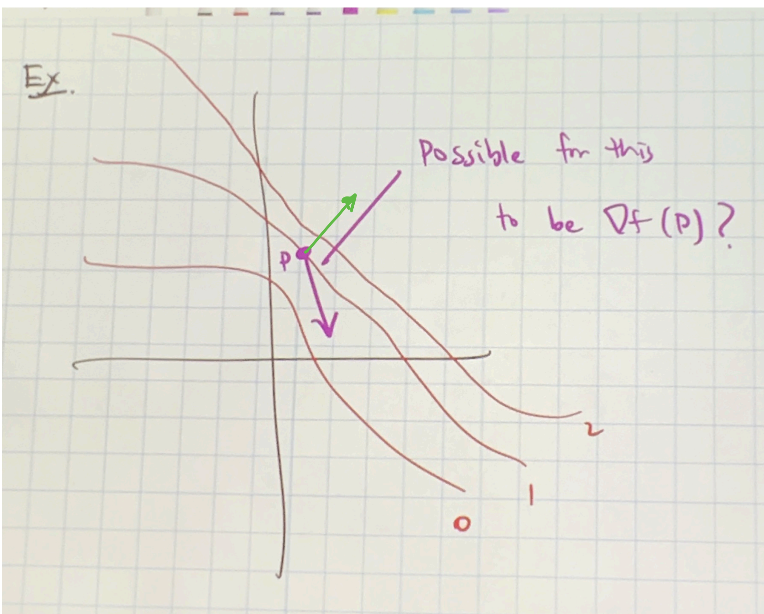
Q direction of most rapid decrease?

$D_{\vec{u}} f(3,\pi) = |\nabla f(3,\pi)| \cos\theta$

min of -1
when $\theta = \pi$

Ans $-\nabla f(3,\pi) = \langle -27\pi^2+1, -54\pi \rangle$

Example:

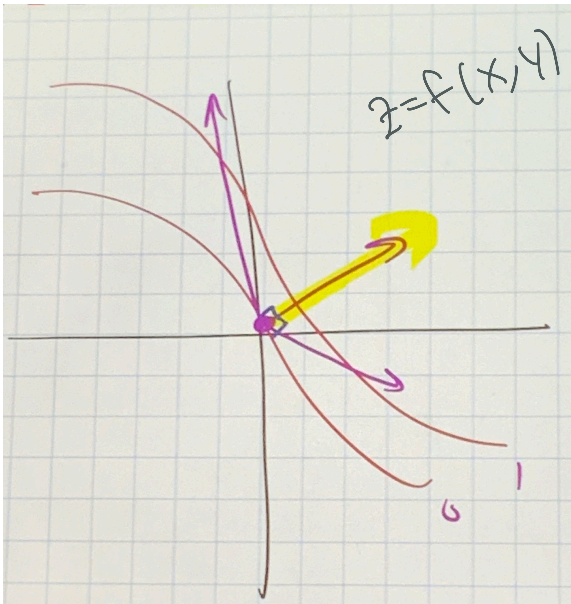


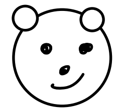
NO.

shows the possible arrow.

Final property of $\nabla f(p)$:

$\nabla f(p)$ is orthogonal to the level set containing P.





Example: $f(x,y)=x^2y+y^3$

Find the line passing through $(1,2)$ which is perpendicular to the level curve containing $(1,2)$.

Plug $(1,2)$ $\rightarrow$ $x^2y+y^3=10$

$$\nabla f = \langle 2xy,\ x^2+3y^2 \rangle$$

$$\nabla f(1,2) = \langle 4, 13 \rangle$$ = direction

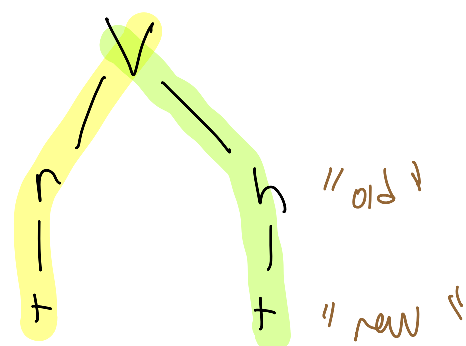
$$\bar{r}(t) = \langle 1,2 \rangle + t\langle 4,13 \rangle = \langle 1+4t,\ 2+13t \rangle$$

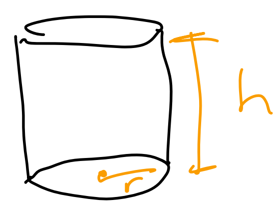
$$x = 1+4t$$
$$y = 2+13t$$

Nov 4 (watching the lecture i missed)

Warmup:

1. A cylindirical candle melts. Express the change in volume with respect to time in terms of change of radius and height.





$$V = \pi r^2 h$$

$$+ = \frac{dV}{dt}$$

$$\frac{dV}{dr} \cdot \frac{dr}{dt} + \frac{dV}{dr} \cdot \frac{dr}{dt} = \frac{dV}{dt}$$

$$2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}$$

ex question

if when $r=10$, $h=5$, we know $\frac{dV}{dt} = -1$

and $\frac{dr}{dt} = -2$

Find $\frac{dh}{dt}$.

② The temperature at (x, y, z) is $T(x, y, z)$

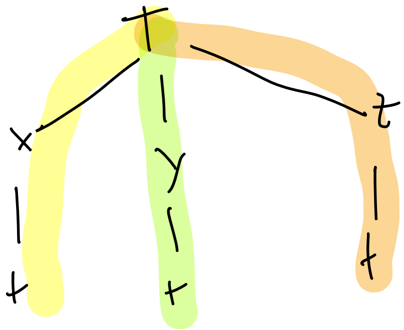
and specifically $T_x(0, 1, \frac{\pi}{2}) = 1$,

$T_y(0, 1, \frac{\pi}{2}) = 2$,

$T_z(0, 1, \frac{\pi}{2}) = -3$.

Find the rate at which T changes with respect to time at $T_y(0, 1, \frac{\pi}{2})$ when moving along the curve $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

finding $\frac{dT}{dt}$ at $(0, 1, \frac{\pi}{2})$



so:

$$\frac{dT}{dt} = \frac{dT}{dx}\cdot\frac{dx}{dt} + \frac{dT}{dy}\cdot\frac{dy}{dt} + \frac{dT}{dz}\cdot\frac{dz}{dt}$$

$$= \frac{dT}{dx}(-\sin t) + \frac{dT}{dy}\cos t + \frac{dT}{dz}\cdot 1$$

bc $z = t$

$$\left.\frac{dT}{dt}\right|_{t=\frac{\pi}{2}} = \frac{dT}{dx}\Big(\big(\underbrace{x(\tfrac{\pi}{2})}_{=0}, \underbrace{y(\tfrac{\pi}{2})}_{=1}, \underbrace{z(\tfrac{\pi}{2})}_{=\frac{\pi}{2}}\big)\Big)\left(-\sin\frac{\pi}{2}\right)$$

$$+ \frac{dT}{dy}\underbrace{\left(0, 1, \frac{\pi}{2}\right)}_{2}\cos\frac{\pi}{2}$$

$$+ \frac{dT}{dz}\underbrace{\left(0, 1, \frac{\pi}{2}\right)}_{-3}$$

$$1 \cdot \left(-\sin\frac{\pi}{2}\right) + 2\cos\frac{\pi}{2} + -3(1)$$

$$= -1$$

Recall

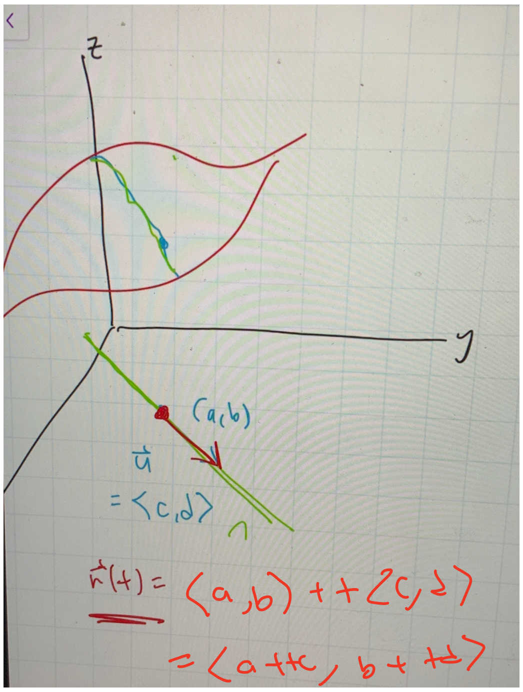
$\frac{df}{dx}$ = Slope / rate of change in x direction i

$\frac{df}{dy}$ = Slope / rate of change in y direction j

$i + j$?

definition: Given a unit direction vector $\vec{u}$, the directional derivative of $f(x,y)$ at (a,b) is in the direction of $\vec{u}$ is

$$D_{\vec{u}} f(a,b) = \lim_{t \to 0} \frac{f(a+ct, b+ct) - f(a,b)}{t}$$



$x = a + ct \quad y = b + td$

chain rule

$$\frac{d}{dt}\Big|_{t=0} f(a+ct, b+dt) = \frac{df}{dx}(a,b)\,c + \frac{df}{dy}(a,b)\,d = \left\langle \frac{df}{dx}(a,b), \frac{df}{dy}(a,b)\right\rangle \cdot \langle c,d\rangle$$

i.e. derivative $f(x,y)$
where $x = a + ct$
$y = b + dt$
at $t = 0$.

$$\begin{array}{ccc} & f & \\ x & & y \\ | & & | \\ t & & t \end{array}$$

SUMMARY

$$D_{\vec{u}} f(a,b) = \underbrace{\langle f_x(a,b), f_y(a,b)\rangle}_{\text{gradient vector of } f \text{ at } (a,b),\ \nabla f(a,b)} \cdot \vec{u}$$

Example:

Find the rate of change of $f(x,y) = x^2 + y^2 + y$

at $(2,1)$ in the direction of $\langle 1,1 \rangle = i + \hat{j}$.

$$\vec{u} = \frac{\langle 1,1 \rangle}{|\langle 1,1 \rangle|} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$D_{\vec{u}} f(2,1) = \nabla f(2,1) \cdot \vec{u} =$$

$$\frac{df}{dx} = 2xy^3$$

eval at $(2,1)$

$$\frac{df}{dy} = 3x^2y^2 + 1$$

$$\langle 4, 13 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$= \frac{4}{\sqrt{2}} + \frac{13}{\sqrt{2}} = \frac{17}{\sqrt{2}}$$

New rate of change in direction of $\langle 2, -5 \rangle = 2i - 5j$

$$\vec{u} = \frac{\langle 2, -5 \rangle}{|\langle 2, -5 \rangle|} = \left\langle \frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}} \right\rangle$$

$$\langle 4, 13 \rangle \cdot \left\langle \frac{2}{\sqrt{29}}, \frac{-5}{\sqrt{29}} \right\rangle =$$

$$\frac{8}{\sqrt{29}} \quad \frac{-65}{\sqrt{29}} = \frac{-57}{\sqrt{29}}$$

For $f(x, y, z)$;

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$$

Same formula:

$$D_{\vec{u}} f(a, b, c) = \nabla f(a, b, c) \cdot \vec{u}$$

applies

Ex $f(x,y) = x^2y^3 + xy$ at $(2,1)$

$$\nabla f(2,1) = \langle 4, 13 \rangle$$

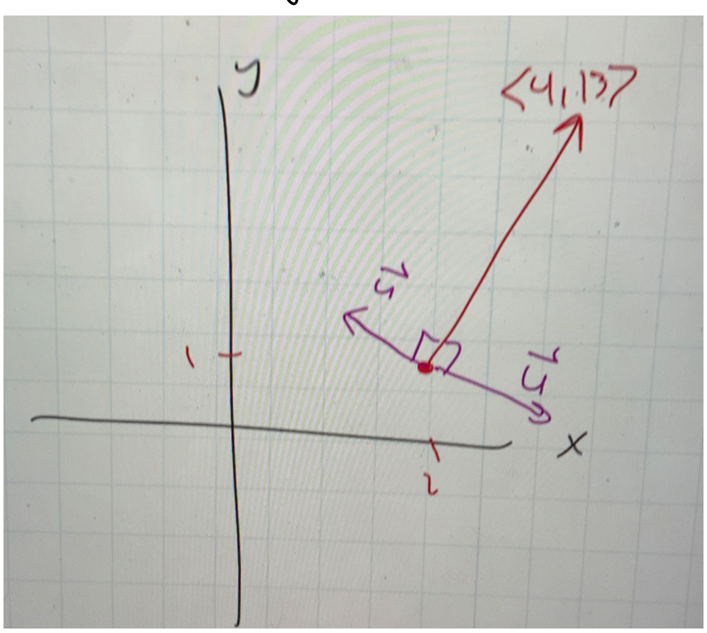
is there a direction $\vec{u}$ in $D_{\vec{u}} f(2,1) = 0$?

$\nabla f(2,1) \cdot \vec{u} = 0$

$\langle 4, 13 \rangle \cdot \vec{u} = 0$

$\vec{u}$ needs to be orthogonal to $\langle 4, 13 \rangle$.

↳ so yes, 2.



(Copy Pasting the warmup related to this topic

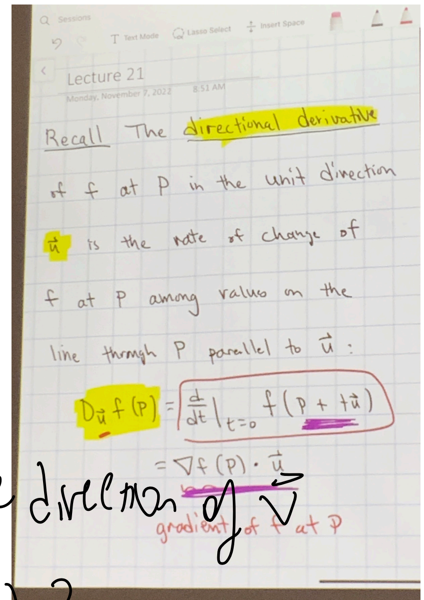
Warmup

Set $f(x,y,z) = x^2y - y^3z + xz^2$

and $\vec{v} = i + j + k = \langle 1,1,1\rangle$

Is the rate of change of f in the direction of $\vec{v}$

larger at $(1,1,1)$ or at $(1,-3,0)$?



Lecture 21

Monday, November 7, 2022 8:51 AM

Recall The directional derivative of f at P in the unit direction $\vec{u}$ is the rate of change of f at P among values on the line through P parallel to $\vec{u}$:

$$\boxed{D_{\vec{u}} f(P) = \frac{d}{dt}\Big|_{t=0} f(P + t\vec{u})}$$

$$= \nabla f(P) \cdot \vec{u}$$

gradient of f at P

$$D_i f(p) = \frac{df}{dx}(p)$$

$$D_j f(p) = \frac{df}{dy}(p)$$

i.e. Which of $D_{\vec{u}} f(1,1,1)$ or $D_{\vec{u}} f(1,-3,0)$ is

larger, where

$$\vec{u} = \frac{\langle 1,1,1\rangle}{|\langle 1,1,1\rangle|} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right\rangle \ ?$$

$$\nabla f(x,y,z) = \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle =$$

$$\langle 2xy + z,\ x^2 - 3y^2 z,\ -y^3 + 2xz \rangle$$

$$D_u f(1,1,1) = \nabla f(1,1,1) \cdot \vec{u} =$$

$$\langle 3, -2, 1 \rangle \cdot \langle \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}} \rangle =$$

$$\frac{3}{\sqrt{3}} - \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$D_u f(1,-3,0) = \nabla f(1,-3,0) \cdot \vec{u} =$$

$$\langle -6, 1, 27 \rangle \cdot \langle \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}} \rangle$$

$$= \frac{-6}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{27}{\sqrt{3}} = \frac{22}{\sqrt{3}}$$

so $D_u f(1,-3,0) > D_u f(1,1,1)$

Nov 9

Warmup: Find the direction in which $f(x,y) = xe^{xy} + y^2$ increases most rapidly at $(1,2)$ and the direction in which it increases at a rate of $\frac{\sqrt{2}}{2}$.

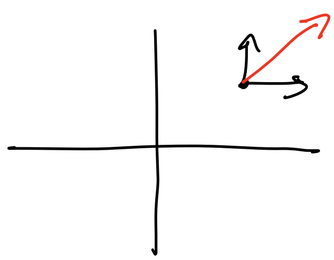
$\boxed{\text{largest rate of change at } (1,2)}$

$\nabla f\langle e^{xy} + xye^{xy}, x^2e^{xy} + 2y\rangle$

direction at $(1,2)$ in which f increases most rapidly

$= \nabla f(1,2)$

$\langle 3e^2, e^2 + 4\rangle$



direction in which

$D_{\vec{u}}f(1,2) = \frac{\sqrt{2}}{2}|\nabla f(1,2)|$

$\nabla f(1,2)\cdot\vec{u} = |\nabla f(1,2)|\cos\theta$

$= \frac{\sqrt{2}}{2}$

Need $\theta = \frac{\pi}{4}$

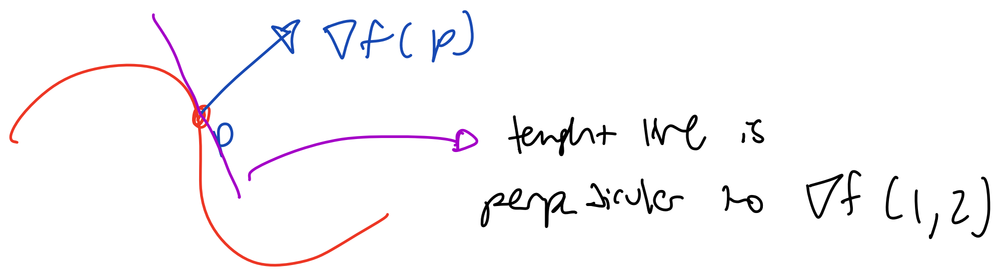
so directions obtained by rotating $\nabla f(1,2)$ by $\theta = \frac{\pi}{4}$

Warmup ②

Find an equation of the line tangent to $xe^{xy} + y^2 = e^2 + 4$ at $(1,2)$

$xe^{xy} + y^2 = e^2 + 4$

is the level curve of $f(x,y) = xe^{xy} + y^2$ containing $(1,2)$



(x,y) on the tangent line when $\underline{\nabla f(1,2) \cdot \langle x-1, y-2 \rangle = 0}$

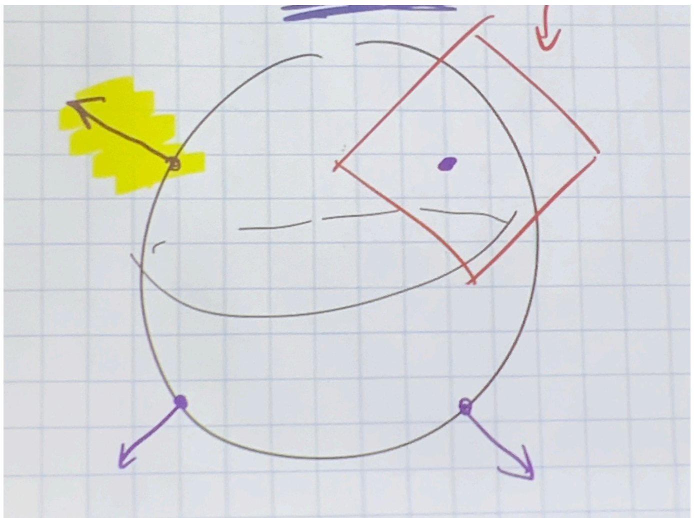
$\langle 3e^2, e^2+4 \rangle \cdot \langle x-1, y-2 \rangle = 0$

so $3e^2(x-1) + (e^2+4)(y-2) = 0$

is the desired tangent line.

Example: $x^2+y^2+z^2=1$ ← a level surface

Find tangent plane at $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$



of $f(x,y,z) = x^2+y^2+z^2$

so $\nabla f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

is perpendicular to sphere.

$$\nabla f = \langle 2x, 2y, 2z \rangle$$

$$\nabla f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left\langle \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}} \right\rangle$$

Tangent plane

$$\nabla f(p) \cdot \left(x - \frac{1}{\sqrt{3}}, y - \frac{1}{\sqrt{3}}, z - \frac{1}{\sqrt{3}}\right) = 0$$

$$\frac{2}{\sqrt{3}}\left(x - \frac{1}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}}\left(y - \frac{1}{\sqrt{3}}\right) + \frac{2}{\sqrt{3}}\left(z - \frac{1}{\sqrt{3}}\right) = 0$$

Example:

$xy + z^2 = 5$

Find a normal vector at (1, 1, 2)

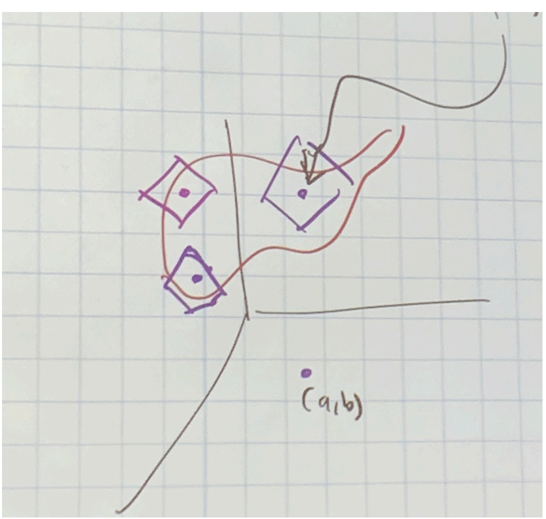
a level surface of $f(x,y,z) = xy + z^2$

normal vector $= \nabla f(1,1,2) = \langle 1, 1, 4 \rangle$

$\nabla f = \langle y, x, 2z \rangle$

Example:

Given $z = f(x,y)$, want tangent plane at $(a, b, f(a,b))$



View as $f(x,y) - z = 0$

$g(x,y,z)$

so $\nabla g = \langle f_x, f_y, -1 \rangle$ is perpendicular to $z = f(x,y)$.

tengent plate to

$z = f(x,y)$ at (a,b) $f(a,b)$

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - 1(z - f(a,b)) = 0$$

$\longrightarrow$ or $z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Exemple

Find the tengent plane to the graph of

$f(x,y) = x^2y^3 + x$ at $(2,1)$

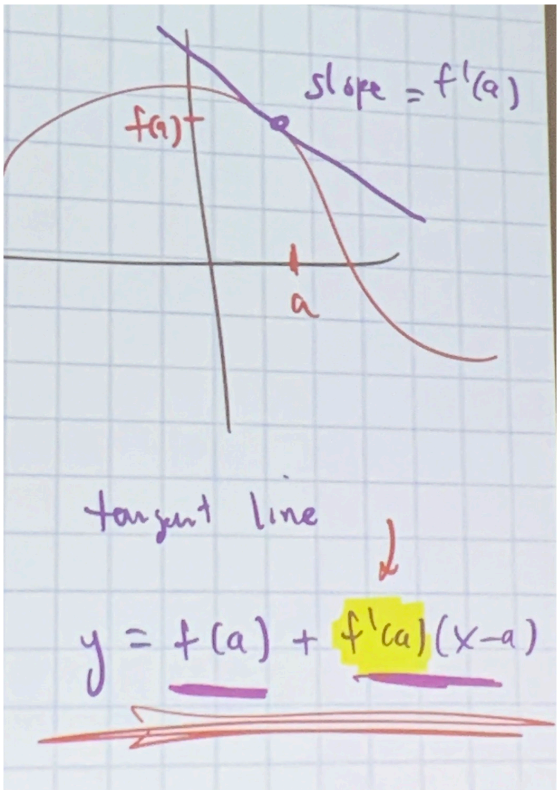
$f_x = 2xy^3 + 1$

$f_y = 3x^2y$

$z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$

$z = 6 + 5(x-2) + 12(y-1)$

Recall:



The tangent plane gives linear approximation to $f(x,y)$ at (a,b).

i.e. the plane that best approximates $f(x,y)$ near (a,b) among all planes,

Ex $f(x,y) = x^2y^3 + x$

write down the linear approximation to f at $(1,2)$.

$L(x,y) =$ tangent plane expression.

$= 6 + 5(x-2) + 12(y-1)$

Approximate the value of using tangent plane

$(1.1)^2(1.9)^2 + 1.1$

$= f(1.9, 1.1)$

$\approx 6 + 5(1.9-2) + 12(1.1-1)$

$= 6 + 5(-0.1) + 12(0.1)$

$= 6 - 0.5 + 1.2$

$= 6.7$

November 11, 2022

tangent plane to a surface $g(x,y,z)=k$ found using ∇g as a normal vector. In particular tangent plane to $z=f(x,y)$ at (a,b) is:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Point: this gives best linear approximation to f near (a,b)

Warmup: find the tangent plane to $z = y e^{3x}$ at $(0,1,1)$ and use it to approximate the value of $(0.5)e$

$$f(x,y) = x e^{3y}$$

$$f_x = 3y e^{3x}$$

$$f_y = 3e^{x}$$

tangent plane at $(0,1)$:

$$z = \underbrace{f(0,1)}_{1} + f_x(0,1)(x-0) + f_y(0,1)(y-1)$$

$$z = 1 + 3x + y - 1$$

$$z = 3x + y$$

$$f(0.1, 0.9) \approx 1 + 0.3 - 0.1 = 1.2$$

If we do approximate $f(x,y)$ with the tangent plane, how large is the error?

$$f(x,y) - (\text{tangent plane approx}) \leq \frac{1}{2} M \left[\Delta x + \Delta y \right]^2$$

where M is bound on the second derivatives of f.

Quadratic Approximations

Given $f(x,y)$ and (a,b), the 1st order Taylor polynomial of f at (a,b) is

$$f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

The second-order Taylor polynomial of f at (a,b) is

$$f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

same

$$+ \frac{1}{2}\left[f_{xx}(x-a)^2 + f_{xy}(x-a)(y-b) + f_{yx}(y-b)(x-a) + f_{yy}(y-b)^2\right]$$

$$= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{1}{2}\left[f_{xx}(x-a)^2 + f_{xy}(x-a)(y-b) + f_{yy}(y-b)^2\right]$$

Example

$f(x,y) = ye^{3x}$ at $(0,1)$

1st order $= 1 + 3x + (y-1)$

key pt

$f_x = 3ye^{3x}$ $\quad f_y = e^{3x}$

$f_{xx} = 9ye^{3x}$ $\quad f_{xy} = 3e^{3x}$ $\quad f_{yy} = 0$

2nd ord

$$1 + 3x + (y-1) + \frac{1}{2}\left[9(x-0)^2 + 3(x-0)(y-1) + 3(y-1)(x-0) + 0(y-1)^2\right]$$

$$= 1 + 3x + (y-1) + \frac{9}{2}x^2 + 3x(y-1)$$

Ex: $9xe^{2y}$

$f(0,0) = 0$

$f_x = 9e^{2y} = 9$

$f_y = 18xe^{2y} = 0$

$f_{xx} = 0$

$f_{xy} = 18e^{2y} = 18$

$f_{yy} = 36xe^{2y} = 0$

$$0 + 9x + \frac{1}{2}\left[2 \times 18 \times (x)(y)\right] =$$

$9x + 18xy$

Cubic:

$9x + 18xy + 36y^2x$

Cubic Approximation

1st and second order terms

$$1st + 2nd + \frac{1}{6}\left[f_{xxx}(x-a)^3 + 3f_{xxy}(x-a)^2(y-b)\right.$$
$$\left.+3f_{xyy}(y-b)^2(x-a) + f_{yyy}(\cancel{y}-b)\right]$$

Ex $y\sin x$

$f(0,0) = 0$

$f_x = y\cos x = 0$

$f_y = \sin x = 0$

$f_{xx} = -y\sin x = 0$

$f_{xxx} = -y\cos x = 0$

$f_{xy} = \cos x = 1$

$f_{xyy} = 0 \quad 0$

$f_{xxy} = -\sin x \quad 0$

$f_{yy} = 0 \quad 0$

$f_{yyy} = 0 \quad 0$

1st $= 0 +$

Ex $\frac{1}{1-2x-2y} = (1-2x-2y)^{-1}$

1st order

$f(0,0) = 1$

$f_x = 2(1-2x-2y)^{-2} = 2$ $\qquad 1+2x+2y$

$f_y = 2(1-2x-2y)^{-2} = 2$

2nd order

$f_{xx} = 8(1-2x-2y)^{-3} = 8$

$f_{xy} = 8 \qquad = 8$

$f_{yy} = \qquad = 8$

$1+2x+2y+\frac{1}{2}[8x^2+16xy+8y^2] =$

$1+2x+2y+4x^2+8xy+4y^2$

3rd

$f_{xxx} = \qquad 48$

48

48

$\frac{1}{6}\left[48x^3 + 48x^2y + 48y^2x + 48x^3\right]$

Excmple $9\cos x\cos y$

$f(0,0) = 9$

$f_x = -9\sin x\cos y = 0$

$f_y = -9\sin y\cos x = 0$

$f_{xx} = -9\cos x\cos y = -9$

$f_{yy} = -9\cos y\cos x = 9$

$f_{xy} = 9\sin x\sin x = 0$

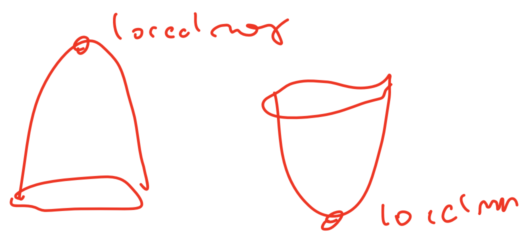
$f_{xxx} = 9\sin x\cos y = 0$

$f_{xyy} = 9\sin x\cos y$

$f_{xyy} = 0$

Optimization

$f(x,y) = 4x + 6y - 12 - x^2 - y^2$



$f(x,y) = -(x-2)^2 - (y-3)^2 + 1$

$@(2,3) = 1 \rightarrow$ local max.

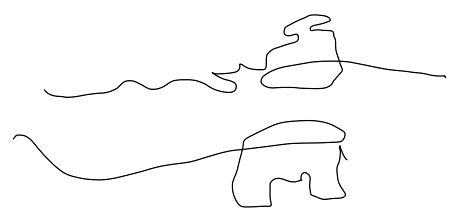
definition: A critical point of $f(x,y)$ is a point where $f_x = 0$ and $f_y = 0$.

EX

$f(x,y) = 4x + \cancel{4y} - 12 - x^2 - y^2$

$f_x = 4 - 2x = 0 \rightarrow x = 2$

$f_y = 0 - 2y = 0 \rightarrow y = 0$



EX $f(x,y) = x^2 - 2y^2 + 2x + 3$

$f_x = 2x + 2 = 0 \qquad x = -1$

$(-1, 0) \rightarrow$ only cp

$f_y = -4y = 0 \qquad y = 0$

$z = (x+1)^2 - 2y^2 + 2$ saddle point.

hyperbolic paraboloid / saddle

$f' < 0 \qquad f'' > 0$

$\cap$ max $\qquad \cup$ min

2nd Derivative Test

Set $D = f_{xx} \cdot f_{yy} - f_{xy}^2$

A critical point (a,b) of $f(x,y)$ is

* a local min if $D > 0$ and $f_{xx} > 0$

* a local maximum if $D > 0$ and $f_{xx} < 0$

* a saddle point if $D < 0$

Ex

$f(x,y) = x^2 - y^3 - x^2 y + y$

find and classify critical points

$f_x = 2x - 2xy = 0$ $\quad 2x(1-y) = 0$ so $x = 0$ or $y = 1$

$f_y = 3y^2 - x^2 + 1 = 0$

but!

if $x = 0$

$3y^2 + 1 = 0$

which is not possible.
so x can't be 0

when $y = 1$

$3 - x^2 + 1 = 0$

$x = \pm 2$ $\quad$ so 2 cp: $(2,1)$ and $(-2,1)$

seconds:

$f_{xx} = 2-2$

$f_{xy} = -2x$

$f_{yy} = 6y$

$(2,1)$: saddle bc $D<0$

$f_{xx} f_{yy} - (f_{xy})^2$

$0 \cdot 6 - (-4)^2 = -16$

$(-2,1)$:

$D = 0 \cdot 6 - 4^2 = -16$

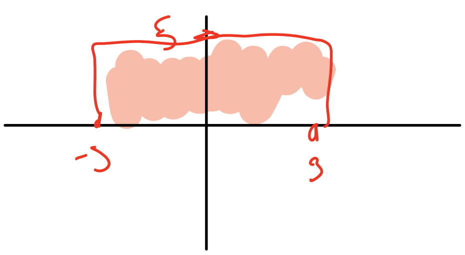
saddle bc $D<0$

November 16

Absolute Extrema

Ex $f(x,y) = x^2 + xy + y^2 - 6y$

Find absolute maximum and absolute minimum of $f(x,y)$ over the rectangle $-3 \leq x \leq 3$, $0 \leq y \leq 5$.

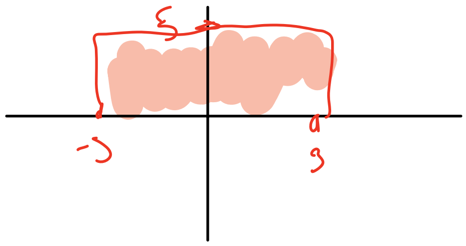


Recall abs max and min of $f(x)$ over $[a,b]$

→ extreme value theorem

Any continuous $f(x,y)$ and closed and bounded region has an absolute max and an absolute min.

$f(x,y) = x^2 + xy + x^2 - 6y$



1st Fine the critical points.

$f_x = 2x + y = 0$

$f_y = x + 2y - 6 = 0$

$x + 2(-2x) - 6 = 0$

$x = -2 \quad y = 4$

$(-2, 4)$

2nd Check the boundaries

$y=0$

$f(x,0) = x^2 \rightarrow$ derivative: $2x \quad x = 0$

$(0,0)$

$x=3$

$f(3,y) = 9 - 3y + y^2$ der: $-3 + 2y = 0$

$y = 2/3 \quad \left(3, \frac{3}{2}\right)$

$y=5$

$f(x,5) = x^2 + 5x - 5$ der: $2x + 5 = 0 \quad \left(-\frac{5}{2}, 5\right)$

$x = -\frac{5}{2}$

$x=-3$

$f(-3,y) = 9 - 9y + y^2$ der: $2y - 9 = 0 \quad \left(-3, \frac{9}{2}\right.$

$y = \frac{9}{2}$

3rd

Test all points:

$(-2,4), (0,0), (3, \frac{3}{2}), (\frac{-5}{2}, 5), (-3, \frac{-9}{2})$

$(-3,0), (3,0), (-3,5), (3,5)$

plug in:

$f(3,5) = 15$

$f(-2,4) = -12$

Ex

$f(x,y) = x^2 y$

over ellipse $3x^2 + 4y^2 \leq 12$

Find absolute max and min

1st Find CP

$f_x = 2xy = 0$

$f_y = x^2 = 0$

CP:

$(0, y)$

2nd crecle boundry

$3x^2+4y^2=12$

$x=\pm\sqrt{4-\frac{4}{3}y^2}$

left: $x=-\sqrt{4-\frac{4}{3}y^2}$ $f\left(-\sqrt{4-\frac{4}{3}y^2},\,y\right)$

$=\left(4-\frac{4}{3}y^2\right)y = 4y-\frac{4}{3}y^3$

der: $4-4y=0$ $y=\pm 1$

$x=-\sqrt{8/3}$

right: ^ same

3rd test

$\left(\sqrt{\frac{8}{3}},1\right)$, $\left(\sqrt{\frac{8}{3}},-1\right)$

$\left(-\sqrt{\frac{8}{3}},1\right)$, $\left(\sqrt{\frac{8}{3}},-1\right)$

max min

November 18

Wormup

① Find the abs. extrema of

$$f(x,y) = 1 + (x+1)^2 - 2(x+1)(y-1) - (y-1)^2$$

over triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$

1st
find cp:

$f_x = 2(x+1) - 2(y-1) = 0$ $\qquad (-1, 1)$

$f_y = -2(x+1) - 2(y-1) = 0$ $\qquad (0, 2)$

2nd
check boundry

$y=0$ $\quad f(x,0) = 1 + (x+1)^2 + 2(x+1) - 1$

$x^2 + 4x + 3$ $\quad$ der: $2x + 4 = 0$ $\quad x = -2$

$x=0$ $\quad f(0,y) = 1 + 1 - 2(y-1) - (y-1)^2 = -y^2 - 4y + 3$

$-2y - 4 = 0$

$y = -2$

$y = 1 - x$

$f(x, 1-x) = 1 + (x+1)^2 - 2(x+1)(-x) - (-x)^2$

$= 4x^2 + 4x + 2$

$x^2 - 4x + 3 = 0$

$0)$

$5x^2 + 4y^2$

$\frac{8}{}$

$f(1, \frac{\pi}{4})$

$(8 - 2)$

$\cos \frac{\pi}{4}$

6

$x + 2y + 6z = 0$

$$\frac{|x + 2y + 6(12 - x^2 - y^2)|}{\sqrt{1^2 + 2^2 + 6^2}} =$$

$$\frac{x + 2y\ 6(12 - x^2 - y^2)}{\sqrt{41}}$$

$$\frac{x + 2y + 72 - 6x^2 - 6y^2}{\sqrt{41}}$$

$f_x = 0$ $\quad 1 - 12x = 0 \quad x = \frac{1}{12}$

$f_y = 0 \quad 2 - 12y = \quad y = \frac{1}{6}$

$z = 12 - \left(\frac{1}{12}\right)^2 - \left(\frac{1}{6}\right)^2 = \frac{1723}{144}$

$2x + 5y + z = 10$

$x + y + z =$

$h = \frac{1024}{x^2} \quad 2x^3 - 1024 =$

$2x^3 = 1024$

$x^3 = 512$

$n > \frac{1024}{8}$ $x = 8$

10 $1, 1, -1$

$\nabla f = \langle 2(x-1), 2(y-1), 2(z+1) \rangle$

$\nabla g = \langle 2x, 2y, 2z \rangle$

$\frac{x-1}{y-1} = \frac{x}{y}$

$xy - x = xy - y$

$x = y = -z$

$x^2 + x^2 + (-x)^2 = 9$

$y = \pm\sqrt{3}$

$x^2 + y^2 + z^2 = 56$

$6x + 4y + 2z$