Lecture 27: Lagrange Multipliers (§14.8)

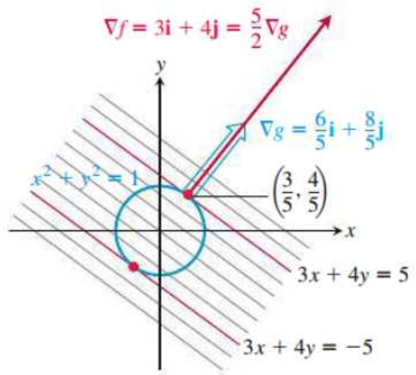
The Method of Lagrange multipliers

Suppose $f(x, y, z)$ and $g(x, y, z)$ are differentiable, and $\nabla g \neq 0$. To find the local maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$, we find x, y, z and λ, such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \quad \text{and} \quad g(x, y, z) = 0.$$

Note that such points (x, y, z) are only **candidates** for a local maximum or minimum.

Example. Find the maximum and minimum value of the function $f(x,y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.



Write the constraint as follows:

$g(x,y) = x^2+y^2-1 = 0.$

Then use LMT:

$\lambda \cdot \nabla g = \nabla f$ and $g = 0$

$$\nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right\rangle = \langle 2x, 2y \rangle$$

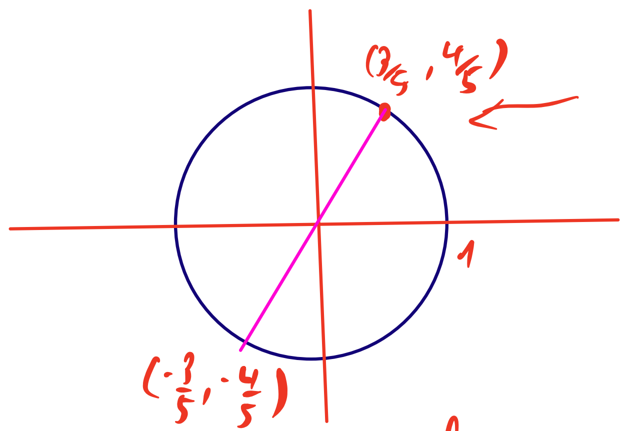
$$\nabla f = \langle 3, 4 \rangle$$

LMT: $\lambda \langle 2x, 2y \rangle = \langle 3, 4 \rangle \Rightarrow x = \frac{3}{2\lambda}, \quad y = \frac{4}{2\lambda}$

$g(x,y) = 0, \quad x^2 + y^2 - 1 = 0$

$\Rightarrow \left(\frac{3}{2\lambda}\right)^2 + \left(\frac{4}{2\lambda}\right)^2 = 1 \Rightarrow \underbrace{9 + 16}_{25} = (2\lambda)^2 \Rightarrow 2\lambda = \pm 5$

$\Rightarrow (x,y) = \left(\frac{3}{2\lambda}, \frac{4}{2\lambda}\right) = \pm\left(\frac{3}{5}, \frac{4}{5}\right)$



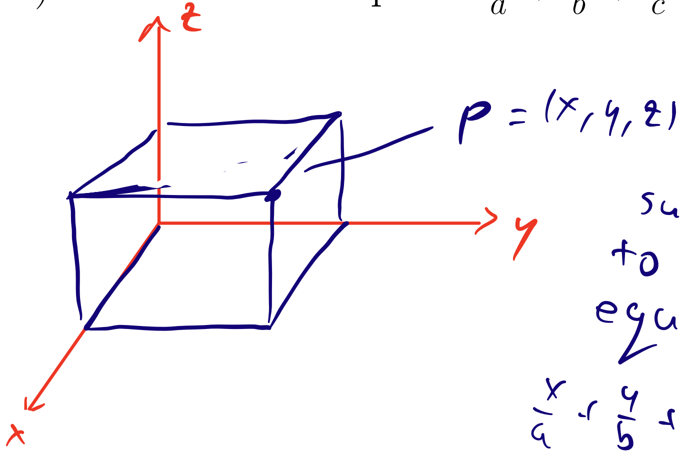
$f\left(\frac{3}{5}, \frac{4}{5}\right) = \frac{9}{5} + \frac{16}{5} = 5$

maxima

$f\left(-\frac{3}{5}, -\frac{4}{5}\right) = -5$

minima

Example. Find the volume of the largest closed rectangular box in the first octant having three faces in the coordinate planes and a vertex (outside of the coordinate planes) which is on the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, where $a, b, c > 0$.



subject to the equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$V(x,y,z) = xyz$$

$$g(x,y,z) = \frac{x}{a} + \frac{y}{b} + \frac{z}{c} - 1 = 0$$

the constraint

By LMT we should look at the equation $\nabla f = \lambda \nabla g$ and $g = 0$

$$\nabla g = \left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle = \left\langle \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \right\rangle$$

$$\nabla V = \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right\rangle = \langle yz, xz, xy \rangle$$

$$\nabla V = \lambda \nabla g \Rightarrow \langle yz, xz, xy \rangle = \lambda \left\langle \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \right\rangle$$

$$ayz = bxz = cxy = \lambda$$

$$\Downarrow \qquad \Downarrow$$

$$y = \frac{bx}{a} \qquad z = \frac{cx}{a}$$

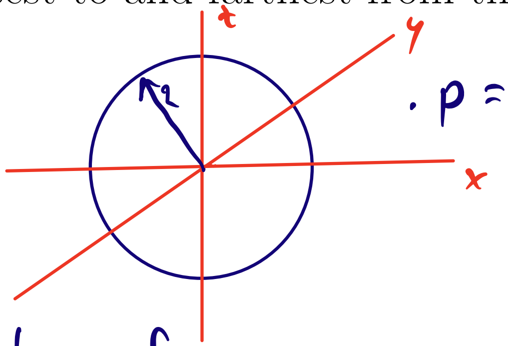
$$g(x,y,z) = 0 \Rightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \Rightarrow \frac{x}{a} + \frac{x}{a} + \frac{x}{a} = 1 \Rightarrow x = \frac{a}{3}$$

$$y = \frac{b}{3}$$

$$z = c$$

$\Rightarrow \quad V = xyz = \dfrac{abc}{27}$ $\qquad \overline{3}$

Example. Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.



$\cdot\, P = (3, 1, -1)$

$g(x,y,z) = x^2 + y^2 + z^2 - 4 = 0$

$\nabla g = \langle 2x, 2y, 2z \rangle$

distance from P

$\downarrow$

$d(x,y,z) = \sqrt{(x-3)^2 + (y-1)^2 + (z+1)^2}$

Write $\quad f(x,y,z) = d^2 = (x-3)^2 + (y-1)^2 + (z+1)^2$

$\nabla f = \langle 2(x-3), 2(y-1), 2(z+1) \rangle$

LMT $\Rightarrow \quad \nabla f = \lambda \nabla g \quad \Rightarrow \quad 2\langle x-3, y-1, z+1 \rangle = \lambda 2 \langle x, y, z \rangle$

$$\Rightarrow \quad \begin{aligned} x - 3 &= \lambda x \\ y - 1 &= \lambda y \\ z + 1 &= \lambda z \end{aligned} \quad \Rightarrow \quad \begin{aligned} (1-\lambda)x &= 3 \\ (1-\lambda)y &= 1 \\ (1-\lambda)z &= -1 \end{aligned} \quad \Rightarrow \quad \langle x,y,z \rangle = \beta \langle \overset{P}{\overset{\downarrow}{3,1,-1}} \rangle$$

$\beta = \dfrac{1}{1-\lambda}$

On the sphere

The constraint is $g(x,y,z) = 0$

$(3\beta)^2 + \beta^2 + \beta^2 = 4 \quad \Rightarrow \quad 11\beta^2 = 4$

$\beta = \pm\sqrt{\dfrac{4}{11}}$

The points are $\pm\sqrt{\dfrac{4}{11}}\,(3, 1, -1)$

$d(P, O) = \sqrt{3^2 + 1 + 1} = \sqrt{11}$ $\qquad$ the max and min distances are $\sqrt{11} + 2, \quad \sqrt{11} - 2$

Example. Find the absolute max/min values of $f(x, y) = 2x^2 + y^2 - 2y + 1$ on the domain $x^2 + y^2 \leq 4$.

step 1 (interior) $\nabla f = \langle 4x, 2y-2 \rangle = 0$

$x = 0, \ y = 1$

$\boxed{f(0,1) = 0}$ minima

step 2 (Boundary)

$\nabla g = \langle 2x, 2y \rangle$

LMT: $\langle 4x, 2y-2 \rangle = \lambda \langle 2x, 2y \rangle \xRightarrow{x \neq 0} \lambda = 2$

$2y - 2 = 4y \Rightarrow y = -1$

$x^2 + y^2 = 4 \Rightarrow x = \pm\sqrt{3}$

$\left.\begin{array}{l} f(\sqrt{3}, -1) = 18 + 4 = 22 \\ f(-\sqrt{3}, -1) = 22 \end{array}\right\}$ the maxima

If $x = 0 \Rightarrow y = \pm 2$

$\left[\begin{array}{l} f(0,2) = 1 \\ f(0,-2) = 9 \end{array}\right.$

Good luck on your final exam!