Lecture 4: The cross product (§12.4)

Goals:

1. Algebraically compute the cross product of two given vectors using determinants.
2. Geometrically interpret the magnitude and direction of the cross product of two given vectors.
3. Perform elementary vector algebra using properties of vector addition, scalar multiplication, the dot product, and the cross product.

In this lecture we focus on the following operation:

Definition. (intuition) The **cross product** $\vec{u} \times \vec{v}$ of vectors $\vec{u}$ and $\vec{v}$ is the vector

$$\vec{u} \times \vec{v} := |\vec{u}||\vec{v}|\sin(\theta) \cdot \vec{n},$$

where $\vec{n}$ is the unit normal vector perpendicular to the plane spanned by $\vec{u}$ and $\vec{v}$, chosen according to the right-hand rule.

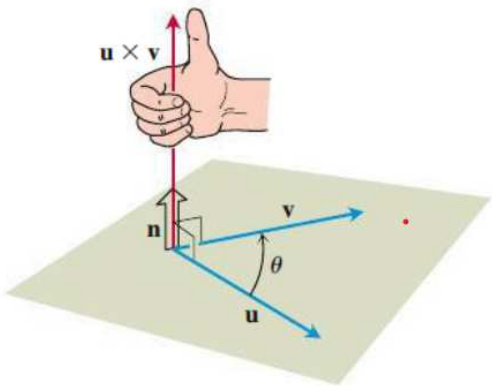


FIGURE 12.28 The construction of **u** × **v**.

1. Note that $\vec{u} \times \vec{v} = \vec{0}$ if $\vec{u}$ and $\vec{v}$ are parallel, or if $\vec{u}$ or $\vec{v}$ are the zero vector.

2. Further note that $|\vec{u} \times \vec{v}|$ is the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

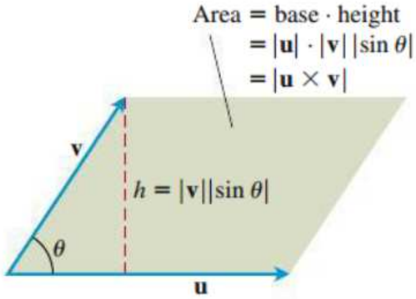


FIGURE 12.31 The parallelogram determined by **u** and **v**.

Here are a few properties of cross product:

Let $\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}$ be vectors, and let r, s be scalars. Then:

1. $(r\vec{u}) \times (s\vec{v}) = rs\, \vec{u} \times \vec{v}$
2. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
3. $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$
4. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
5. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} + (\vec{u} \cdot \vec{v})\vec{w}$

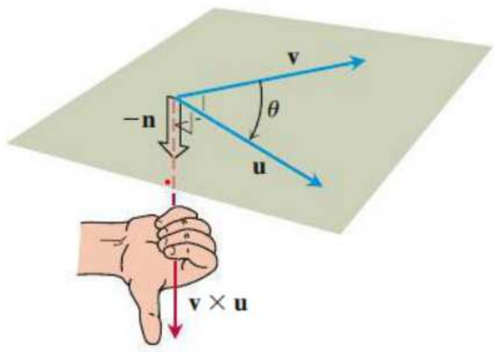
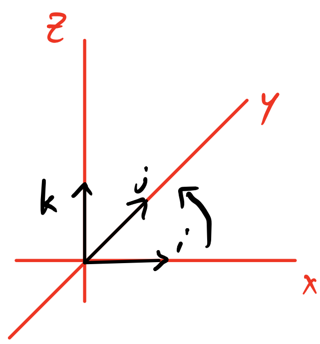
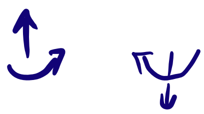


FIGURE 12.29 The construction of $\mathbf{v} \times \mathbf{u}$.

Example. Compute

1. i × j= k
2. k × i= j
3. k × j= −i
4. j × j= 0





We now give an explicit formula to the cross product. We first define determinants:

Definition.

1. The determinant of a 2×2-matrix is calculated as follows

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

2. The determinant of a 3×3-matrix is calculated as follows

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \cdot \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} =$$

$$= a_1(b_2c_2 - b_3c_3) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

We can now define the cross product more explicitly:

Definition (Cross product as a determinant)**.** Let $\overrightarrow{u} = \langle u_1, u_2, u_3 \rangle$ and $\overrightarrow{v} = \langle v_1, v_2, v_3 \rangle$. Then

$$\overrightarrow{u} \times \overrightarrow{v} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = i \cdot \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - j \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + k \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= i(u_2v_3 - u_3v_2) - j(u_1v_3 - u_3v_1) + k(u_1v_2 - u_2v_1)$$

$$\langle u_1, u_2, u_3 \rangle \times \langle v_1, v_2, v_3 \rangle = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle$$

Example. Let $\overrightarrow{u} = \langle 2, 1, 1\rangle$ and $\overrightarrow{v} = \langle -4, 3, 1\rangle$. Find $\overrightarrow{u} \times \overrightarrow{v}$? How about Find $\overrightarrow{u} \times (\overrightarrow{u} + \overrightarrow{v})$?

$$\vec{u}\times\vec{v} = \langle 2,1,1\rangle \times \langle -4,3,1\rangle = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} =$$

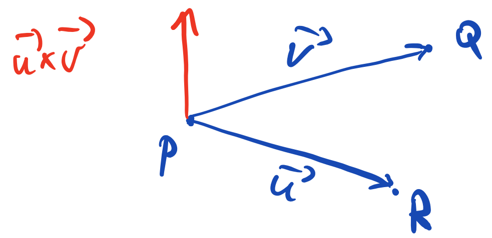
$$= i\cdot\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} - j\cdot\begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} + k\cdot\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} = -2i - 6j + 10k = \langle -2, -6, 10\rangle$$

$$\vec{u}\times(\vec{u}+\vec{v}) = \vec{u}\times\vec{u} + \vec{u}\times\vec{v} = \vec{u}\times\vec{v} = \langle -2, -6, 10\rangle$$

Example. Let $P(1, -1, 0)$, $Q(2, 1, -1)$, and $R(-1, 1, 2)$ be points. Find:

1. A unit vector perpendicular to the plane of P, Q and R (how many such unit vectors exist?)

2. The area if the triangle PQR.



$\vec{u} = \vec{PR} = \langle -2, 2, 2\rangle$

$\vec{v} = \vec{PQ} = \langle 1, 2, -1\rangle$

$$\vec{u}\times\vec{v} = \begin{vmatrix} i & j & k \\ -2 & 2 & 2 \\ 1 & 2 & -1 \end{vmatrix} = i(-2-4) - j(2-2) + k(-4-2) = -6i - 6k$$

$$= \langle -6, 0, -6\rangle$$

$$|\langle -6,0,-6\rangle| = \sqrt{36+0+36} = \sqrt{72} = 6\sqrt{2}$$

The normalized orthogonal vector is $\frac{1}{6\sqrt{2}}\langle -6,0,-6\rangle = \left\langle \frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right\rangle$

the area of $\triangle PQR$ is half the area of the parallelogram

Example. Is it true that $(\overrightarrow{u} \times \overrightarrow{v}) \times \overrightarrow{w} = \overrightarrow{u} \times (\overrightarrow{v} \times \overrightarrow{w})$?

NO!

Example $(i\times i)\times j = 0\times j = 0$

$i\times(i\times j) = i\times k = -j \neq 0$

i.e. $\frac{1}{2} \cdot |u \times v|$
$= \frac{1}{2} \cdot 6\sqrt{2} = 3\sqrt{2}$

Triple (or Box) product

Definition. Given vectors $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$, the **box product** is (the scalar) defined by

$$(\overrightarrow{u} \times \overrightarrow{v}) \cdot \overrightarrow{w}.$$

The absolute value $|(\overrightarrow{u} \times \overrightarrow{v}) \cdot \overrightarrow{w}| = |\overrightarrow{u} \times \overrightarrow{v}|\,|\overrightarrow{w}|\,|\cos(\theta)|$ of the box product can be seen geometrically as the volume of the parallelepiped, as below:

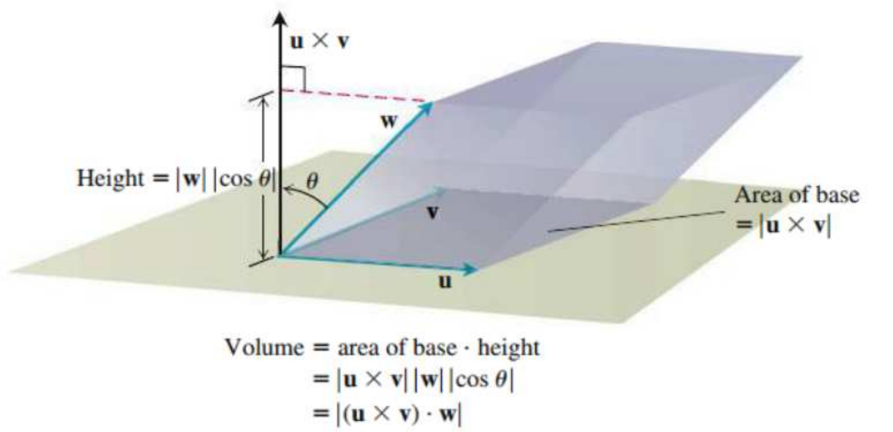


FIGURE 12.35 The number $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ is the volume of a parallelepiped.

The box product has the following nice formula:

$$(\overrightarrow{u} \times \overrightarrow{v}) \cdot \overrightarrow{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Example. Find the volume of the parallelepiped determined by $\overrightarrow{u} = \langle 1, 2, -1\rangle$, $\overrightarrow{v} = \langle -2, 0, 3\rangle$ and $\overrightarrow{w} = \langle 0, 7, -4\rangle$ (the coordinates are in meters).

Example. Is it always true that

$$(\overrightarrow{u} \times \overrightarrow{v}) \cdot \overrightarrow{w} = \overrightarrow{u} \cdot (\overrightarrow{v} \times \overrightarrow{w})?$$

Example. Using the dot product and cross product to describe the following:

1. A vector orthogonal to $\overrightarrow{u}$ and $\overrightarrow{v}$.
2. A vector orthogonal to $\overrightarrow{u} \times \overrightarrow{v}$ and $\overrightarrow{w}$.
3. The area of a triangle with vertices P, Q and R.