Lecture 16: Directional Derivatives and Gradient Vectors (§14.5)

Goals:

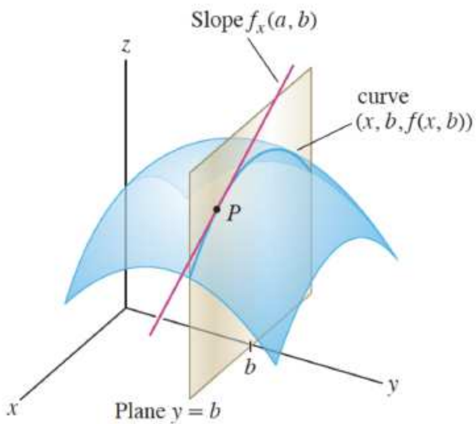
1. Interpret directional derivatives geometrically and in terms of directional rates of change.
2. Interpret the gradient vector in terms of direction yielding the maximum directional rate of change.
3. Exploit the connection between the gradient vector and directional derivatives to simplify compu- tation of directional derivatives.

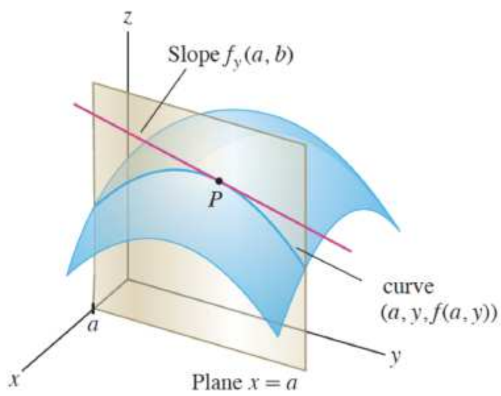
Recall: A function $f(x, y)$ of two variables has two partial derivatives:

$$f_x(a, b) := \frac{\partial f}{\partial x}|_{(a,b)} = \lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h}.$$
$$f_y(a, b) := \frac{\partial f}{\partial y}|_{(a,b)} = \lim_{h\to 0} \frac{f(a, b+h) - f(a, b)}{h},$$

which measure the rate of change of f in the x and y directions.

How about other directions?





Definition. The **derivative of** f **at** (a,b) **in the direction of the unit vector** $\overrightarrow{u} = \langle u_1, u_2\rangle$ is

$$D_{\overrightarrow{u}}(f)|_{(a.b)} := \lim_{s\to 0}\frac{f(a+su_1, b+su_2) - f(a,b)}{s}.$$

Example. Write $D_{\overrightarrow{u}}(f)|_{(a.b)}$ in terms of $\frac{\partial f}{\partial x}|_{(a,b)}$, $\frac{\partial f}{\partial y}|_{(a,b)}$ and $\overrightarrow{u}$.

write

$\vec{r}(s) = \langle \underbrace{a+su_1}_{x(s)}, \underbrace{b+su_2}_{y(s)} \rangle = \langle a,b\rangle + s\langle u_1,u_2\rangle$ $\qquad \vec{u} = \langle u_1, u_2\rangle$

Consider $f(\vec{r}(s))$ and diffreatiate

This will give the directional derivative.

By the Chain rall: $\frac{d}{ds} f(\vec{r}(s)) = \frac{\partial f}{\partial x}\cdot\frac{dx}{ds} + \frac{\partial f}{\partial f}\cdot\frac{dy}{ds} = \frac{\partial f}{\partial x}\cdot u_1 + \frac{\partial f}{\partial y}\cdot u_2$

$\qquad \|$

$D_{\vec{u}} f(a,b) = \frac{\partial f}{\partial x}(a,b)\cdot u_1 + \frac{\partial f}{\partial y}(a,b)\, u_2 = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle \cdot \langle u_1, u_2\rangle$

Definition. The **gradient of** $f(x,y)$ is the vector $\nabla f = \langle\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$. It value at a specific point (a,b) is

$$\nabla f(a,b) = \langle\frac{\partial f}{\partial x}|_{(a,b)}, \frac{\partial f}{\partial y}|_{(a,b)}\rangle.$$

In particular, we have seen that:

$$D_{\overrightarrow{u}}(f)|_{(a.b)} = \nabla f(a,b)\cdot\overrightarrow{u}.$$

If $f(x,y,z)$ is a function of 3 variables

$\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle$

and the directional derivative w.r.t. $\vec{u} = \langle u_1, u_2, u_3\rangle$

is : $\nabla f \cdot \vec{u} = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2 + \frac{\partial f}{\partial z} u_3$

Example. Find the directional derivative of $f(x,y) = \frac{1}{1+x^2+y^2}$ at point $P = (0,1)$ in the direction of $\langle 4, 3\rangle$.

The gradient of f is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle \frac{-2x}{(1+x^2+y^2)^2}, \frac{-2y}{(1+x^2+y^2)^2} \right\rangle$$

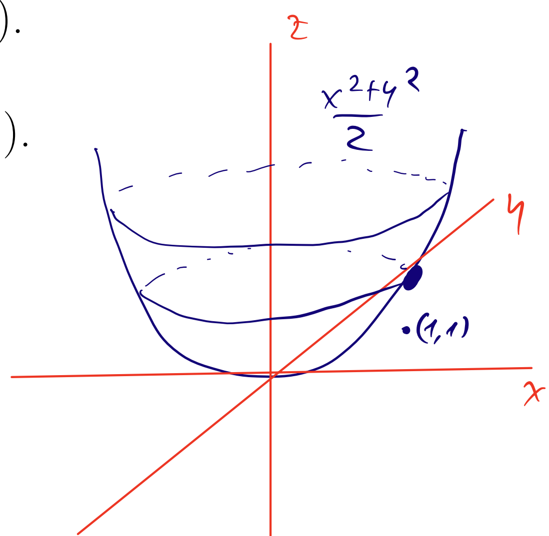
$$\nabla f(0,1) = \left\langle 0, -\frac{1}{2} \right\rangle$$

The unit vector in the direction of $\langle 4,3\rangle$ is $\vec{u} = \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$.

Hence $$D_{\vec{u}}(f)\Big|_{(0,1)} = \nabla f(0,1) \cdot \vec{u} = \left\langle 0, -\frac{1}{2} \right\rangle \cdot \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle = -\frac{3}{10}$$

Example. Let $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$. Find the directions in which:

1. f increases most rapidly at the point $(1,1)$.
2. f decreases most rapidly at the point $(1,1)$.
3. f has 0 change at the point $(1,1)$.



$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle x, y \rangle$$

$$(1,1) = \langle 1, 1\rangle$$

$$D_{\vec{u}}(f) = \nabla f \cdot \vec{u} = \langle 1,1\rangle \cdot \langle u_1, u_2\rangle = u_1 + u_2$$

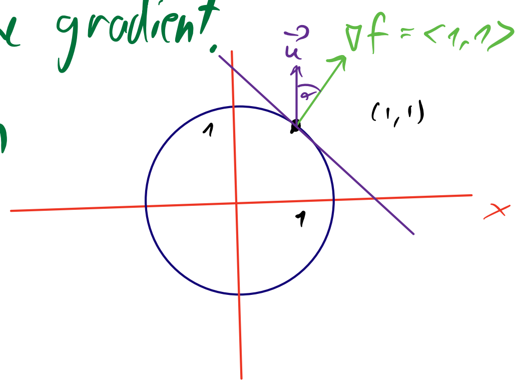
$$= |\nabla f| \cdot \cos\alpha = |\langle 1,1\rangle| \cdot |\vec{u}| \cdot \cos\alpha = \sqrt{2} \cdot \cos\alpha$$

project to the xy plane

1) when $\alpha = 0$, i.e. in the direction of the gradient.

2) When $\alpha = \pi$, i.e. the oposite direction to the gradient

3) $D_{\vec{u}} f = 0$ iff $\vec{u}$ is orthogonal to ∇f
$\alpha = \pm \frac{\pi}{2}$



Summarizing:

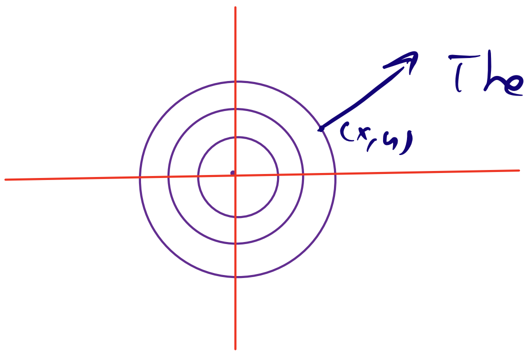
1. $f(x,y)$ increases most rapidly in the direction of ∇f.
2. $f(x,y)$ decreases most rapidly in the direction of $-\nabla f$.
3. Any direction $\overrightarrow{u}$ orthogonal to ∇f is a direction of zero change.

Gradients and level curves

Consider the previous example $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$.

In that example the level sets are circles $f(x,y) = c \iff x^2+y^2 = 2c$

(of radius $r = \sqrt{2c}$)



The gradient $\langle x,y \rangle$ is equals to the position

The fact that the gradient is orthogonal to the level set is general.

Using the gradient vector ∇f it is easy to find tangent lines to level curves.

Example. Find the equation for the tangent line to the ellipse

$$\frac{x^2}{4} + y^2 = 2,$$

at the point $(-2, 1)$.

Consider the function $f(x,y) = \frac{x^2}{4} + y^2$.
The ellipse is the level set $f = 2$.
The tangent to the ellipse (the level set) is orthogonal to ∇f.
$\nabla f = \langle \frac{x}{2}, 2y \rangle$, $\nabla f(-2,1) = \langle -1, 2 \rangle$
We need a vector orthogonal to $\nabla f = \langle -1, 2 \rangle$. E.g. $\vec{u} = \langle 2, 1 \rangle$

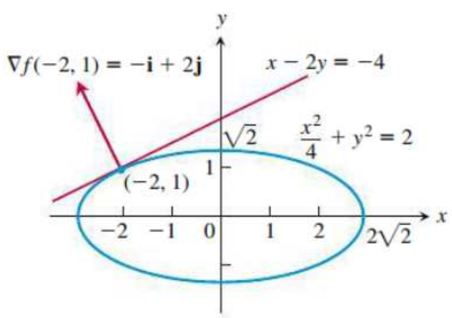


FIGURE 14.32 We can find the tangent to the ellipse $(x^2/4) + y^2 = 2$ by treating the ellipse as a level curve of the function $f(x, y) = (x^2/4) + y^2$ (Example 4).

Therefore the tangent line to the ellipse at $(-2,1)$ is given by:

$\langle -2, 1 \rangle + s\langle 2, 1 \rangle = \langle 2s-2, s+1 \rangle$.

Algebra rules for gradients:

Let $f(x,y)$, $g(x,y)$ be functions of 2 variables
$h(x)$ a function of 1 variable
and c a scalar.

1) $\nabla(f+g) = \nabla f + \nabla g$

2) $\nabla(cf) = c \cdot \nabla f$

3) $\nabla(f \cdot g) = f \nabla g + g \nabla f$

4) $\nabla h(f(x,y)) = h' \cdot \nabla f = h'(f(x,y)) \cdot \nabla f(x,y)$