# Lecture 8: Curves in Space and their tangent (§13.1)

**Goals:**

1. Analyze a vector-valued function using limits, continuity, and the derivative.
2. When modeling the motion of a particle using a vector-valued function, interpret the derivative of a vector-valued function as the velocity vector of the particle.
3. Analyze the velocity vector of a particle to determine the speed, direction, and acceleration of the particle.
4. Fluently apply differentiation rules for vector-valued functions.
5. Show that the output of a vector-valued function of constant length is orthogonal to its derivative.

**Recall:**

A **vector-valued function** (or **vector function** or **vector parametrization** is a function that takes as input a real number $t$ and returns as an output a vector $\overrightarrow{r}(t)$ .

Any vector function in space can be written in terms of its components:

$$\overrightarrow{r}(t) = \langle f(t), g(t), h(t) \rangle$$

The $f(t), g(t)$ and $h(t)$ are called the **component functions** of $\overrightarrow{r}(t)$.

# Limits and continuity

Let $\overrightarrow{r}(t) = \langle f(t), g(t), h(t) \rangle$ be a vector function. Then:

1. The limit $\lim\limits_{t\to a} \overrightarrow{r}(t)$ exists and equals $\overrightarrow{s} = \langle s_1, s_2, s_3 \rangle$ if

$$\lim_{t\to a} f(t) = s_1 \quad \lim_{t\to a} g(t) = s_2 \quad \lim_{t\to a} h(t) = s_3.$$

2. The vector function $\overrightarrow{r}(t)$ is **continuous at a point** $t_0$ if $f(t), g(t)$, and $h(t)$ are continuous at $t_0$, i.e.

$$\lim_{t\to t_0} f(t) = f(t_0) \quad \lim_{t\to t_0} g(t) = g(t_0) \quad \lim_{t\to t_0} h(t) = h(t_0).$$

3. The vector function $\overrightarrow{r}(t)$ is **continuous** if $\overrightarrow{r}(t)$ is continuous at every point in its domain.

**Example.** Let $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$. Then

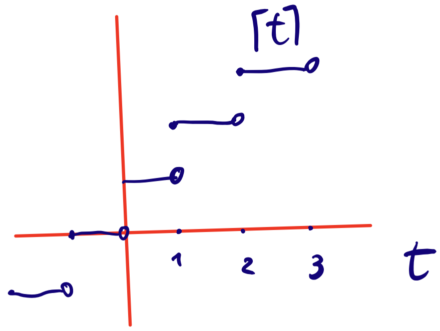
$$\lim_{t \to \frac{\pi}{4}} \vec{r}(t) = \langle \lim_{t \to \frac{\pi}{4}} \cos(t), \lim_{t \to \frac{\pi}{4}} \sin(t), \lim_{t \to \frac{\pi}{4}} t \rangle$$

$$= \langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4}, \frac{\pi}{4} \rangle = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\pi}{4} \rangle$$

**Example.** Let $\lceil t \rceil$ be the function which assign to each real number $t$ the smallest integer $n$ such that $t < n$. Is the following function continuous?

$$\vec{r}(t) = \langle \sin(\cos(t^2)), \sin(t), \lceil t \rceil \rangle$$

If not, what are the points of discontinuity?




the function $\lceil t \rceil$ is not continuous at $\ldots, -2, -1, 0, 1, 2, 3, \ldots$

Since both $x(t) = \sin(\cos(t^2))$ and $y(t) = \sin t$ are everywhere continuous we may look only at the 3rd component $z(t) = \lceil t \rceil$.

Hence the discontinuity point of $\vec{r}(t)$ are exactly the integers $\ldots, -2, -1, 0, 1, 2, \ldots$

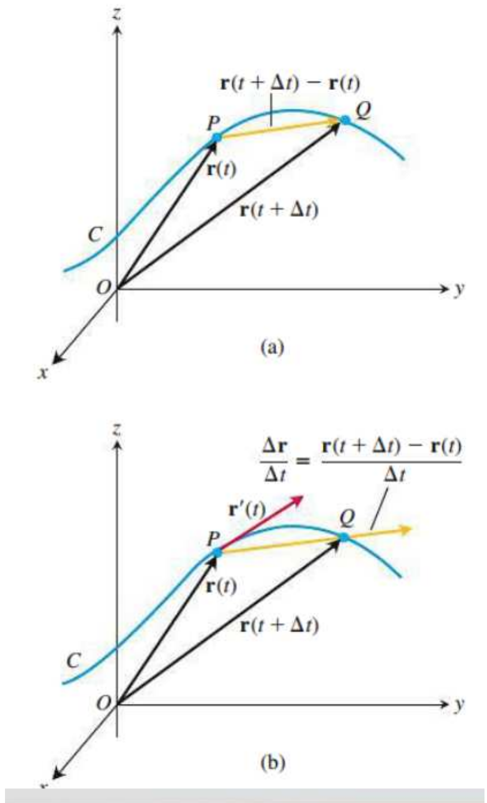
# Derivatives and motion

Let $\overrightarrow{r}(t) = \langle f(t), g(t), h(t) \rangle$ be a vector function. Then $\overrightarrow{r}(t)$ is **differentiable at** $t$ if each of its components $f(t), g(t)$, and $h(t)$ are differentiable at $t$. The derivative is following vector function:

$$\overrightarrow{r}'(t) := \frac{d}{dt}\overrightarrow{r}(t) = \lim_{h \to 0} \frac{\overrightarrow{r}(t+h) - \overrightarrow{r}(t)}{h} = \langle \frac{d}{dt}f(t), \frac{d}{dt}g(t), \frac{d}{dt}h(t) \rangle.$$

$$\lim_{h \to 0} \langle \frac{f(t+h)-f(t)}{h}, \frac{g(t+h)-g(t)}{h}, \frac{h(t+h)-h(t)}{h} \rangle = \langle \lim *, \lim *, \lim * \rangle$$

The vector $\overrightarrow{r}'(t_0)$ is **tangent** to the curve (defined by $\overrightarrow{r}(t)$) at $P = \overrightarrow{r}(t_0)$. The **tangent line** at $P$ is the line parallel to $\overrightarrow{r}'(t_0)$ which passes through $P$.



Let $\overrightarrow{r}(t) = \langle x(t), y(t), z(t)\rangle$ be a position vector of a particle moving along a curve is space. Then:

1. The **velocity vector** of the particle is the derivative of position: $\overrightarrow{v}(t) = \frac{d}{dt}\overrightarrow{r}(t)$. $= \vec{r}\,'(t)$

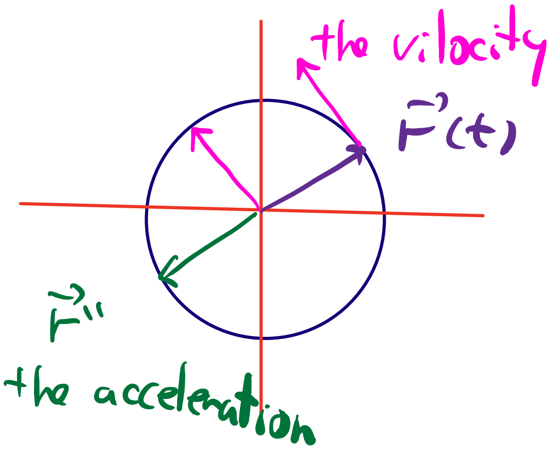
2. The **speed** is the magnitude of velocity Speed $= |\overrightarrow{v}(t)|$.

3. The **acceleration vector** is the derivative of velocity: $\overrightarrow{a}(t) = \frac{d}{dt}\overrightarrow{v}(t)$.

4. The unit vector $\frac{\overrightarrow{v}(t)}{|\overrightarrow{v}(t)|}$ is the **direction of motion** at time $t$.

**Example.** Find the velocity, speed and acceleration of the following particle:

$$\overrightarrow{r}(t) = \langle \cos(t), \sin(t)\rangle$$




The velocity:

$$\vec{r}\,'(t) = \frac{d}{dt}\vec{r}(t) = \left\langle \frac{d}{dt}\cos(t), \frac{d}{dt}\sin(t)\right\rangle$$

$$= \langle -\sin(t), \cos(t)\rangle$$

The speed is $|\vec{r}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$

The acceleration is : $\vec{r}\,''(t) = (\vec{r}\,'(t))' = \frac{d}{dt}\langle -\sin(t), \cos(t)\rangle$

$$= \langle -\cos(t), -\sin(t)\rangle$$

## Differentiation rules for vector functions

Let $\vec{u}(t)$, $\vec{v}(t)$ be differentiable vector functions, $\vec{C}$ a constant vector, $c$ any scalar, and $f(t)$ any differentiable function. Then:

1) $\frac{d}{dt}\vec{C} = \vec{0}$

2) $\frac{d}{dt}\left(c\vec{v}(t)\right) = c \cdot \frac{d}{dt}\vec{v}(t)$

3) $\frac{d}{dt}\left(f(t)\cdot\vec{v}(t)\right) = \frac{d}{dt}f(t)\cdot\vec{v}(t) + f(t)\cdot\frac{d}{dt}\vec{v}(t) = f'(t)\vec{v}(t) + f(t)\vec{v}'(t)$

4) $\frac{d}{dt}\left(\vec{u}(t)+\vec{v}(t)\right) = \frac{d}{dt}\vec{u}(t) + \frac{d}{dt}\vec{v}(t)$

5) $\frac{d}{dt}\left(\vec{u}(t)\cdot\vec{v}(t)\right) = \frac{d}{dt}\vec{u}(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\frac{d}{dt}\vec{v}(t) = \vec{u}'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}'(t)$

6) $\frac{d}{dt}\left(\vec{u}(t)\times\vec{v}(t)\right) = \frac{d}{dt}\vec{u}(t)\times\vec{v}(t) + \vec{u}(t)\times\frac{d}{dt}\vec{v}(t) = \vec{u}'\times\vec{v} + \vec{u}\times\vec{v}'$

7) $\frac{d}{dt}\left(\vec{u}(f(t))\right) = f'(t)\cdot\vec{u}'\left(f(t)\right)$

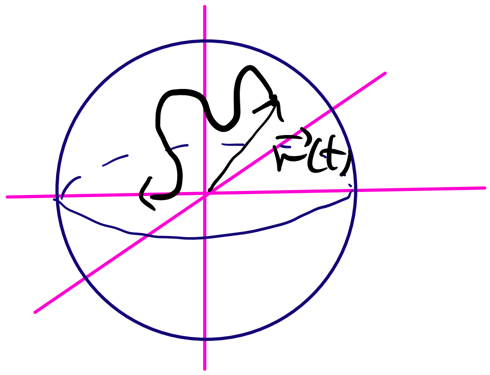
## Vector functions of constant length

We saw that for the curve $\overrightarrow{r}(t) = \langle \cos(t), \sin(t) \rangle$, the velocity vector $\overrightarrow{v}(t)$ is orthogonal to $\overrightarrow{r}(t)$ at any time $t$. This is part of a more general phenomenon, which occurs when the position vector $\overrightarrow{r}(t)$ has constant length $|\overrightarrow{r}(t)|$.

If $|\vec{r}(t)|$ is constant, say $|\vec{r}(t)| = c$

Then $\vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2 = c^2$

If we difrentiate, we get




$$\frac{d}{dt} \vec{r} \cdot \vec{r} = 0$$

$$\vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 0 \Rightarrow 2\vec{r}'(t) \cdot \vec{r}(t) = 0$$

$$\Rightarrow \vec{r}'(t) \cdot \vec{r}(t) = 0$$

**Exercise.** Assume that $\overrightarrow{r}(t) \cdot \frac{d}{dt}\overrightarrow{r}(t) = 0$. What can we say about $|\overrightarrow{r}(t)|$?

Arguing as above but in oposite order we deduce

that if $\vec{r} \cdot \vec{r}' = 0$

$$\Downarrow$$

$$|\vec{r}| = \text{const}$$

$$\vec{r}(t) \cdot \vec{r}'(t) = 0 = 2\vec{r}'(t) \cdot \vec{r}(t) = 0 \Rightarrow \frac{d}{dt}(\vec{r}(t) \cdot \vec{r}(t)) = 0$$



$$\Rightarrow \vec{r}(t) \cdot \vec{r}(t) = \text{const}$$

$$\Downarrow$$

$$|\vec{r}(t)|^2 = \text{const}$$