Lecture 26: Lagrange Multipliers (§14.8)

Goal:

1. Apply the Method of Lagrange Multipliers to determine the points on a constraint curve at which a function of two variables may have a local maximum or minimum value (relative to its values on the curve).

2. Apply the Method of Lagrange Multipliers to determine the points on a constraint surface at which a function of three variables may have a local maximum or minimum value (relative to its values on the curve).

Constrained optimization. A constrained optimization problem is: minimize (or maximize) a function $f(x, y, z)$ subject to a constraint $g(x, y, z) = 0$.

For example, to find the distance from the origin to the plane $x + 2y + 3z = 4$, we minimize $d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ subject to the constraint $g(x, y, z) = 0$, where $g(x, y, z) = x + 2y + 3z - 4$.

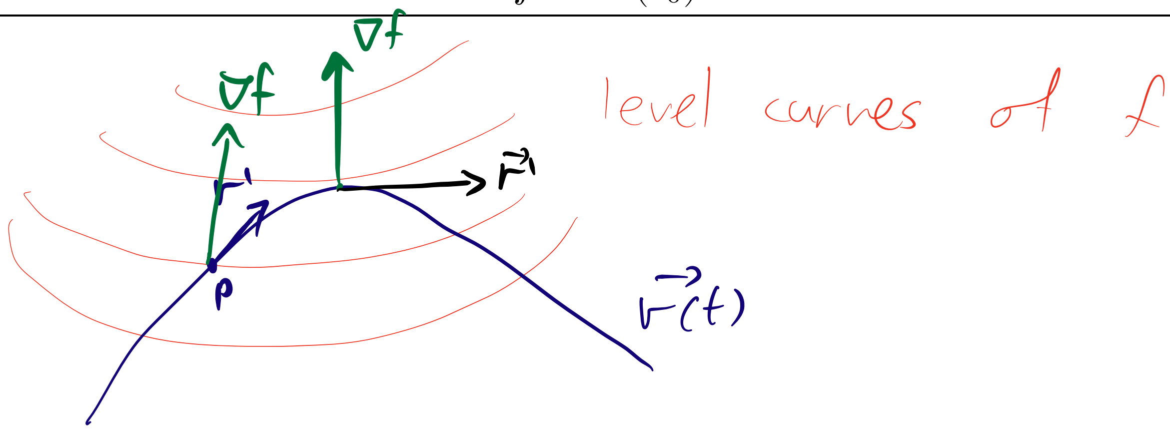
The method of Lagrange multipliers

The Orthogonal Gradient Theorem: Suppose $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve C:

$$\overrightarrow{r}(t) = \langle x(t), y(t), z(t) \rangle.$$

If $P_0 = \overrightarrow{r}(t_0)$ is a point on C where f has a local max or min relative to its values along C, then

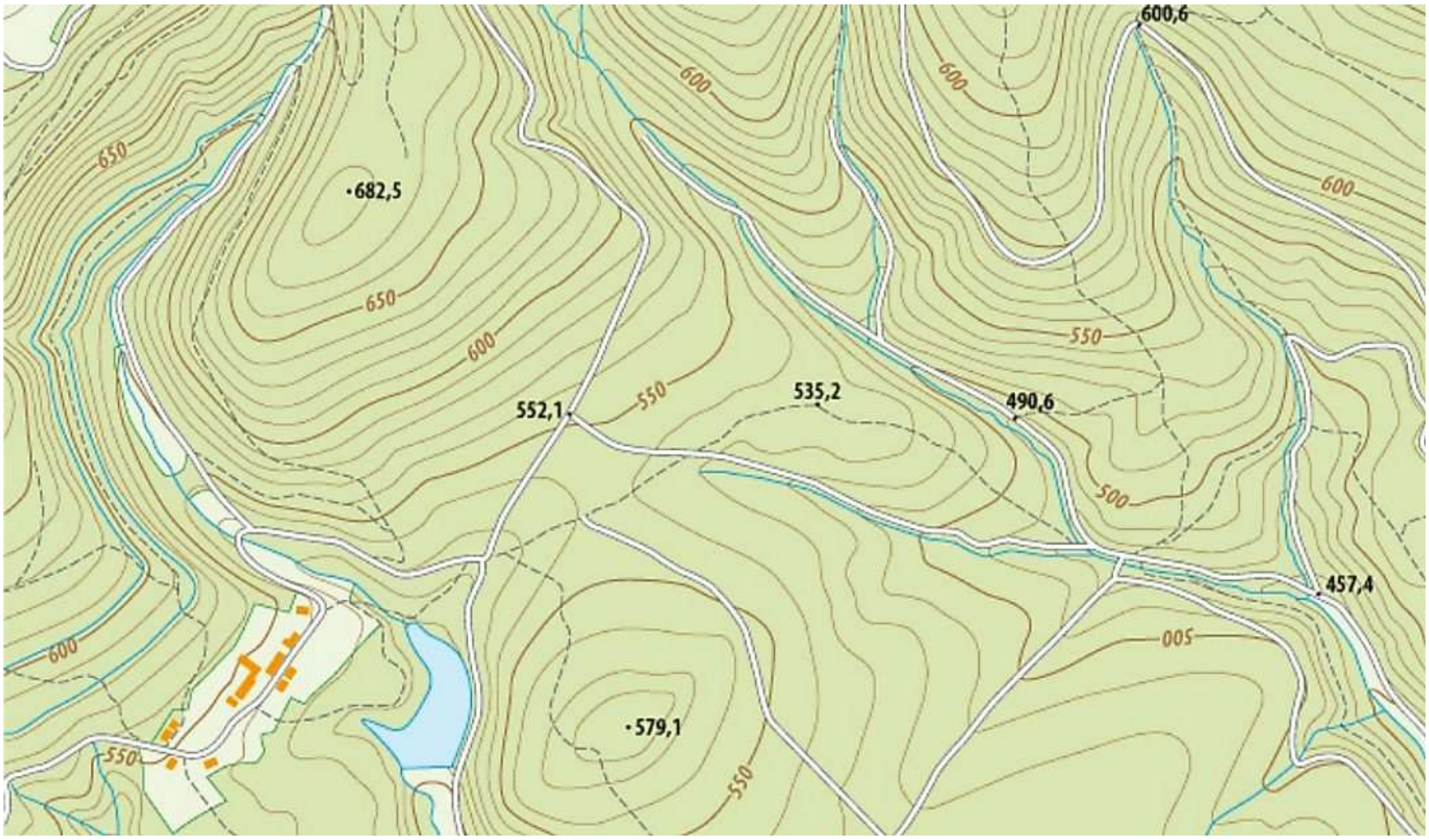
$$\nabla f \cdot \overrightarrow{r}'(t_0) = 0.$$



$$\frac{d}{dt} f(\vec{r}(t)) = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

by the chain rule

$$= \nabla f \cdot \vec{r}'$$



Example. Let $f(x, y) = (1 - x^2 - y^2)^2$. Find the minimal value of f along the curve $y = x$, for $x \in [-10, 10]$.

$\vec{r}(t) = \langle t, t \rangle$ is a parametrization of the curve $x = y$

$|t| \leq 10$

$\vec{r}'(t) = \langle 1, 1 \rangle$

$\nabla f(x,y) = \langle \underbrace{-4x(1-x^2-y^2)}_{\frac{\partial f}{\partial x}}, -4y(1-x^2-y^2) \rangle$

$r' \cdot \nabla f = 0 \iff -(4x+4y)(1-x^2-y^2) = 0$

Since we are restricted to $x = y$

we get $-8x(1-2x^2) = 0 \Rightarrow x = 0$ or $x = \pm \frac{1}{\sqrt{2}}$

The candidates are

$(0,0), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right), (-10,-10), (10,10)$ the end points

$f(x,y)=(1-x^2-y^2)^2$	$(0,0)$	$(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$	$(\frac{-1}{\sqrt{2}},\frac{-1}{\sqrt{2}})$	$(-10,-10)$	$(10,10)$
	1	0	0	199^2	199^2

maxima

minima

The Orthogonal Gradient Theorem can be generalized to surfaces. Suppose we have differentiable functions $f(x, y, z)$ and $g(x, y, z)$.

The Method of Lagrange multipliers

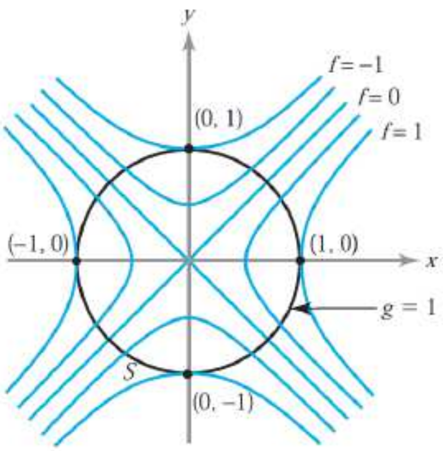
Suppose $f(x, y, z)$ and $g(x, y, z)$ are differentiable, and $\nabla g \neq 0$. To find the local maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$, we find x, y, z and λ, such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \quad \text{and} \quad g(x, y, z) = 0.$$

Note that such points (x, y, z) are only **candidates** for a local maximum or minimum.

Example. Let $f(x, y) = x^2 - y^2$ and let S be the circle of radius 1 centered at the origin. Find the maximum and minimum values and points of $f(x, y)$ on S.

We want to apply Lagrange multipliers theorem



Write $g(x,y) = x^2+y^2-1$

the constrain is $g(x,y)=0$

$\nabla g = \langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \rangle = \langle 2x, 2y \rangle$ $\boxed{\neq 0 \text{ on the circle}}$

$f(x,y) = x^2 - y^2$

$\nabla f = \langle 2x, -2y \rangle$

LMT $\Rightarrow$ we should look at the equation

$\nabla f = \lambda \nabla g$ $\quad \langle 2x, -2y \rangle = \lambda \langle 2x, 2y \rangle$

$g = 0$ $\quad \Downarrow$

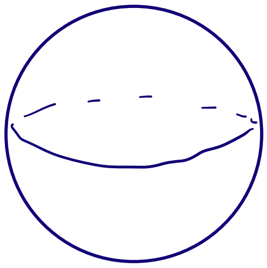
either $\lambda = 1$ and $y = 0$

or $\lambda = -1$ and $x = 0$

we get 4 candidates

	$(1,0)$	$(-1,0)$	$(0,1)$	$(0,-1)$
x^2-y^2	1	1	-1	-1

Example. Maximize $f(x,y,z) = x + z$ subject to the constraint $x^2 + y^2 + z^2 = 1$.



$g(x,y,z) = x^2+y^2+z^2-1$

$g(x,y,z)=0$ is the constraint

$\nabla g = \langle 2x, 2y, 2z \rangle$

$f(x,y,z) = x+z$

$\nabla f = \langle 1,0,1 \rangle$

By LMT we should consider the conditions

$$\nabla f = \lambda \nabla g$$
$$g = 0$$

$$\langle 1,0,1 \rangle = \lambda \langle 2x, 2y, 2z \rangle$$

$$\Rightarrow \quad y=0, \quad x=z$$

$$g(x,y,z) = x^2+y^2+z^2-1 = 0 \Rightarrow \quad x = z = \frac{\pm 1}{\sqrt{2}}$$

The candidates are

$$\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad \left(\frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$$

$f = x+z$ $\qquad \underline{\frac{2}{\sqrt{2}} = \sqrt{2}}$ $\qquad \underline{-\sqrt{2}}$

maxima $\qquad$ minima

For example, to find the distance from the origin to the plane $x + 2y + 3z = 4$, we minimize $d(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ subject to the constraint $g(x, y, z) = 0$, where $g(x, y, z) = x + 2y + 3z - 4$.

Take $f(x,y,z) = d(x,y,z)^2 = x^2+y^2+z^2$

Applying LMT we get

$$\nabla g = \lambda \nabla f$$

$$\nabla g = \langle 1, 2, 3 \rangle$$

$$\nabla f = \langle 2x, 2y, 2z \rangle$$

$$\lambda \nabla g = \nabla f \qquad \lambda \langle 1,2,3 \rangle = \langle 2x, 2y, 2z \rangle$$

$$\lambda = 2x \qquad x = \tfrac{1}{2}\lambda$$

$$2\lambda = 2y \quad , \quad y = \lambda$$

$$3\lambda = 2z \qquad z = \tfrac{3}{2}\lambda$$

$$\underline{g = 0}$$

$$x + 2y + 3z - 4 = 0$$

$$\tfrac{1}{2}\lambda + 2\lambda + 3 \cdot \tfrac{3}{2}\lambda = 4 \Rightarrow \lambda = \frac{4}{7}$$

$$\Rightarrow \langle x, y, z \rangle = \langle \tfrac{2}{7}, \tfrac{4}{7}, \tfrac{6}{7} \rangle$$

$$\Rightarrow d = \sqrt{x^2+y^2+z^2} = \sqrt{\frac{4}{49} + \frac{16}{49} + \frac{36}{49}} = \frac{\sqrt{56}}{7}$$