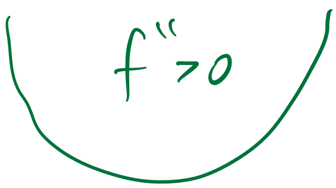
Lecture 23: Extreme Values and Saddle Points (§14.7)

Goal:

1. Define the terms **absolute maximum value, absolute minimum value, local maximum value, local minimum value, critical point, and saddle point**.

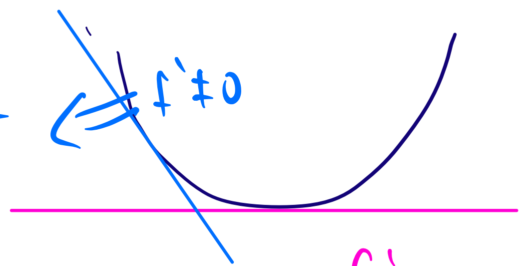
2. Locate all critical points of a function using the First Derivative Test for Local Extreme Values.

3. Use the Second Derivative Test to classify a critical point of a function as local minimum, local maximum, or saddle point.



the function is convex

the point is not a local min/max



$f'=0$ at a local minimum/maximum

$f'=0$ and $f''>0 \Rightarrow$ local min

$f'=0$ and $f''<0 \Rightarrow$ local max

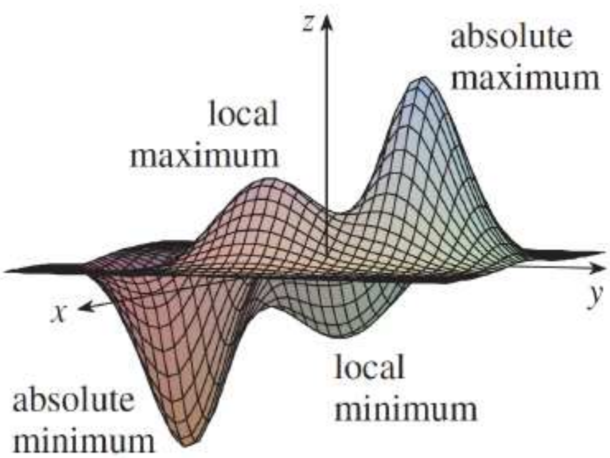
Derivative tests for local extreme values

Definition.

1. A function $f(x, y)$ has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is sufficiently close to (a, b).

2. $f(x, y)$ has a **local minimum** at (a, b) if $f(x, y) \geq f(a, b)$ $\cancel{(x, y)}$ is sufficiently close to (a, b).

3. If either one of the above inequalities holds for **every** (x, y) in the domain of f, then (a, b) is an absolute maximum or absolute minimum.



Theorem. ***First derivative test for local extreme values:*** *If $f(x, y)$ has a local maximum or a local minimum at an interior point (a, b) of its domain, and $f_x, f_y|_{(a,b)}$ exist, then*

$$f_x|_{(a,b)} = f_y|_{(a,b)} = 0.$$

$f_x = f_y = 0 \iff \nabla f = 0 \iff D_{\vec{u}} f = 0$

$\forall \vec{u}$

Corollary. *The tangent plane at a local maximum or minimum is horizontal.*

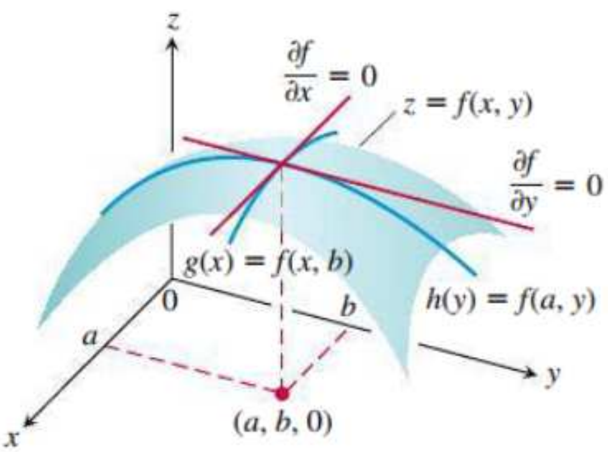


FIGURE 14.44 If a local maximum of f occurs at $x = a$, $y = b$, then the first partial derivatives $f_x(a, b)$ and $f_y(a, b)$ are both zero.

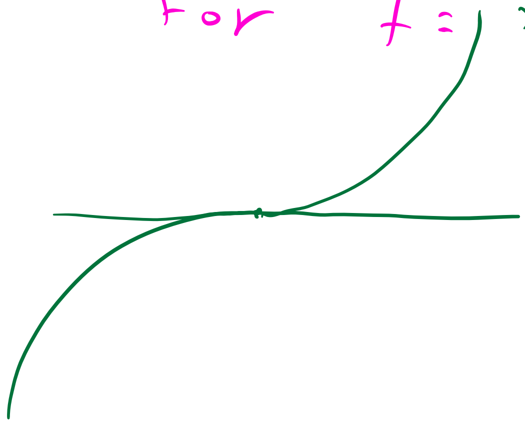
Definition. We say that a point (a, b) is a **critical point** if either:

1. $f_x|_{(a,b)} = f_y|_{(a,b)} = 0$, or

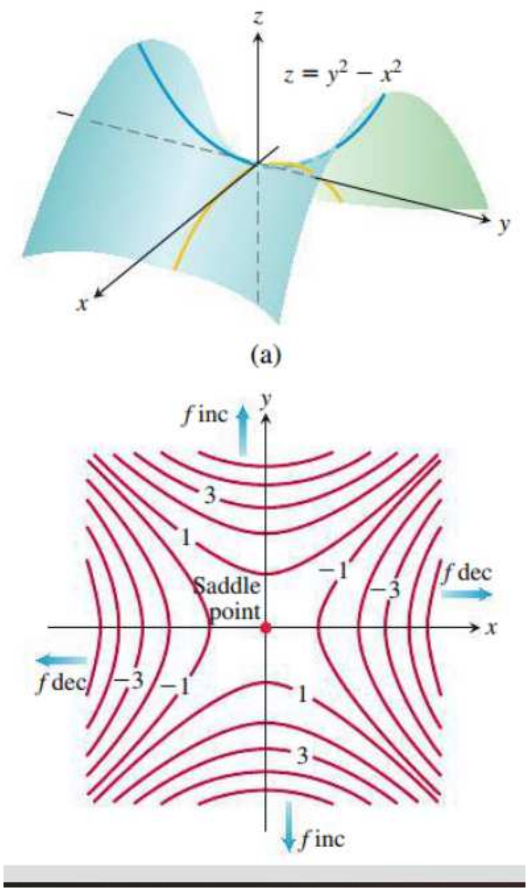
2. one or both of the partial derivatives of f do not exist at (a, b).

A critical point is a candidate for a local minimum and maximum, but may be neither.

For $f = x^3$ 0 is a critical point but not a local extamum



Example. Find the critical points of $f(x,y) = y^2 - x^2$. Determine whether it is a local minimum, local maximum, or neither.

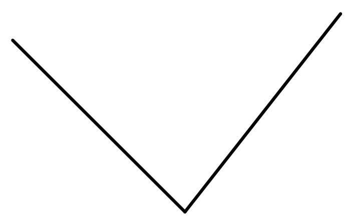


$f_x = -2x$, $f_y = 2y$

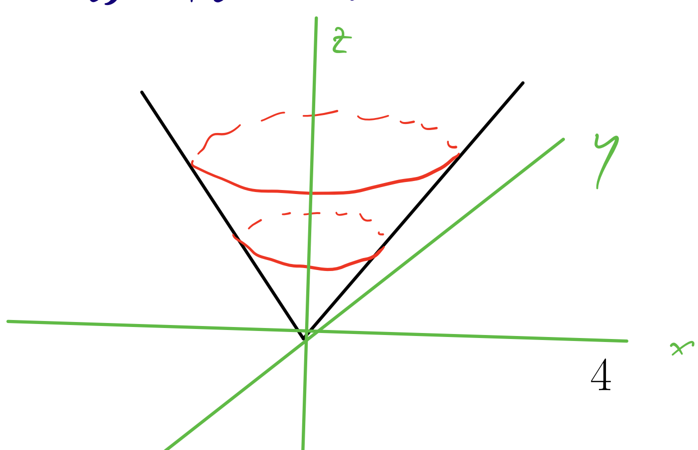
$\nabla f = 0 \iff (x,y) = (0,0)$

Example. Consider $f(x,y) = \sqrt{x^2 + y^2}$. Find the critical points and determine whether it is a local minimum, local maximum, or neither.

Recall the graph of $|x|$



The graph of $f(x,y) = \sqrt{x^2+y^2}$ looks as follows:



Example. Find the critical points of $f(x,y) = x^2y + 2xy^2$.

$$\nabla f = \langle f_x, f_y \rangle = \langle 2xy + 2y^2 , x^2 + 4xy \rangle$$

$$f_x = 0 \iff y(2x+2y) = 0$$

$$f_y = 0 \iff x(x+4y) = 0$$

<u>option 1</u> $x = 0$, then $y(0+2y) = 0 \Rightarrow y = 0$

$(0,0)$ is a critical point.

<u>option 2</u> $x \neq 0 \Rightarrow x + 4y = 0 \Rightarrow x = -4y \Rightarrow y \neq 0$

and $2x + 2y \neq 0$

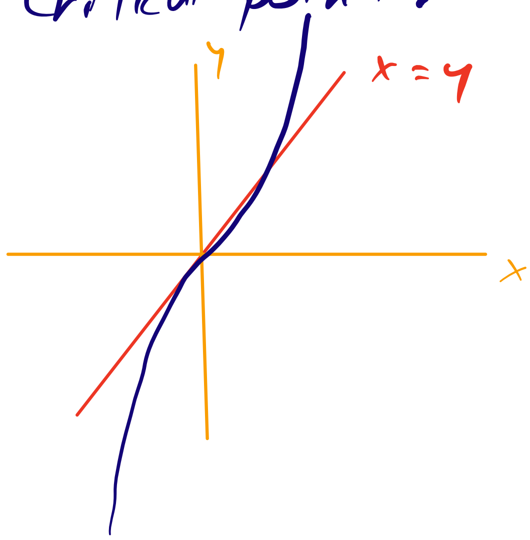
$\Rightarrow f_x = y(2x+2y) \neq 0$

Therefore $(0,0)$ is the only critical point.

Along the line $x = y$

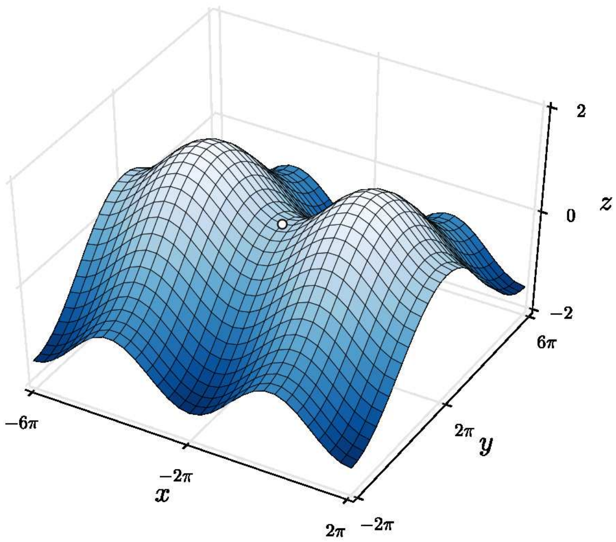
The function is

$f(x,x) = 3x^3$



hence $(0,0)$ is not a local extremum.

Definition. A critical point (a, b) is called a **saddle point** if $\nabla f|_{(a,b)} = 0$, but (a, b) is not a local extremum point. In practice, a saddle point is a point on $z = f(x, y)$ where the surface curves up in one direction and down in another.



Theorem. ***Second Derivative Test for Local Extreme Values*** *Let $f(x, y)$ be a (nice) function, with a critical point (a, b). Then:*

$\boxed{\nabla f = 0}$

1. *If $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) then f has a* ***local maximum*** *at (a, b).*

$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$

2. *If $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) then f has a* ***local minimum*** *at (a, b).*

3. *If $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b) then f has a* ***saddle point*** *at (a, b).*

4. *If $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b), then* ***the test is inconclusive****.*

Example. Find the critical points of $f(x, y) = (x^2 + y^2)e^{-x}$, and determine whether they are minima, maxima or saddle points.

$f_x = 2xe^{-x} - (x^2+y^2)e^{-x}$

$f_y = 2ye^{-x}$

$\nabla f = 0 \iff \langle f_x, f_y \rangle = \langle 0, 0 \rangle$

$\Rightarrow y = 0 \quad , \quad 2x - x^2 = 0 \Rightarrow x = 0, 2$

The critical points are $(0,0)$, $(2,0)$

$f_{xx} = 2e^{-x} - 2\cancel{x}e^{-x} \underset{-}{+} 2\cancel{x}e^{-x} + (x^2+y^2)e^{-x} = \underline{2e^{-x} + (x^2+y^2)e^{-x}}$

$f_{yy} = 2e^{-x}$

$f_{xy} = -2ye^{-x}$

At $(0,0)$ $\qquad f_{xx} = 2 > 0$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \Rightarrow \quad \Rightarrow \text{local minimum}$$

At $(2,0)$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6e^{-2} & 0 \\ 0 & 2e^{-2} \end{vmatrix} \left.\begin{matrix} = 12 > 0 \\ f_{xx} > 0 \end{matrix}\right\} \Rightarrow \text{local minimum}$$

There must be a mistake since a function cannot have only two local min

Example. Let $f(x, y) = 3y^2 - 2y^3 - 3x^2 + 6xy$. Find the critical points of $f(x, y)$ and determine whether they are minima, maxima or saddle points.

The green calculation is correct. W.r.t that

f_{xx} at $(0,0)$ the same

$$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$$

so this is a local minimum.

But at $(2,0)$ we get

$$\begin{vmatrix} -2e^{-2} & 0 \\ 0 & 2e^{-2} \end{vmatrix} = -4e^{-4} < 0$$

hence $(2,0)$ is a saddle point.