Lecture 13: Functions of Several Variables (§14.1)

Goals:

1. Identify the domain and range of a real-valued function of two or three real variables.
2. Identify the boundary and interior points of a simple set in the plane or in space, and whether or not the set is bounded.
3. Distinguish between a function f of several variables and its graph.
4. Given a function of two variables, determine and plot several of its level curves and describe its graph.
5. Given a function of three variables, determine several of its level surfaces.
6. Given a curve (surface), produce a function for which the curve (surface) is a level curve (level surface).

Functions of several variables

Definition. A **function** $f(x_1, ..., x_n)$ **in several variables** is a function that takes as an input n real numbers $x_1, ..., x_n$ and returns as output a real number.

- The **domain** D of f is the set of points on which it is defined.
- The image of the function $f(D)$ is called the **range** of f.

Example. Here are examples of a function of several variables:

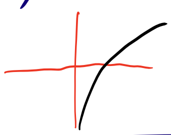
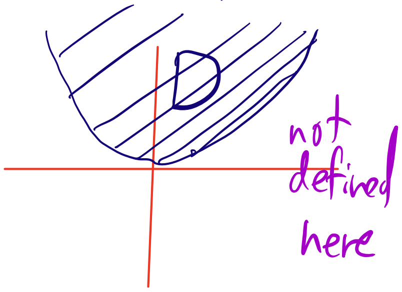
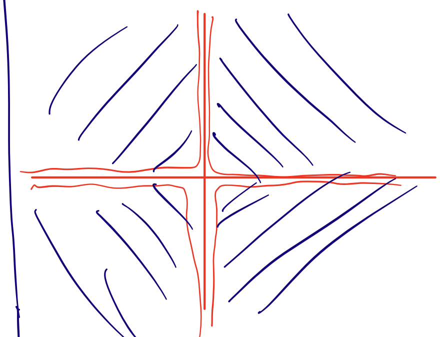
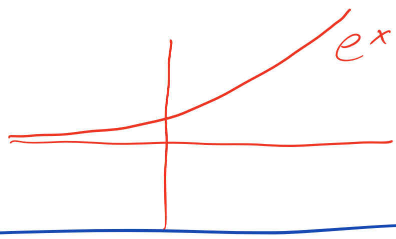
$f(x_1, x_2) = x_1^2 x_2 + 2x_1 + 3x_2$

1. $f(x, y) = x^2 y + 2x + 3y$. Domain $= \mathbb{R}^2$
2. The volume $V(l, w, h) = lwh$ of a box (l is the length, w is the width and h is the height). $l, w, h > 0$, the domain is $\{(l, w, h) : l, w, h > 0\} = (\mathbb{R}^{>0})^3$

   the range in $\mathbb{R}^{>0} = (0, \infty)$
3. Temperature $T(x, y, z, t)$, pressure.

Recall that for functions of one variable, the domain of f consisted of a union of intervals.

Example. Describe the domain and range of the following functions: $\ln(x)$, $\sqrt{y-x^2}$,$1/xy$, $e^{(e^{x+y})}$, $\frac{1}{x^2+y^2+z^2}$.

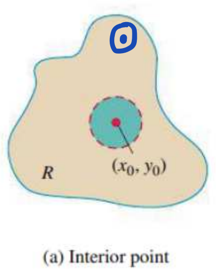
Function	Domain	Range
$\ln(x)$ 	$\{x : x>0\} = (0,\infty)$	$\mathbb{R} = (-\infty, \infty)$
$\sqrt{y-x^2}$	$\{(x,y) : y \geq x^2\}$ 	$\mathbb{R}^{\geq 0}$ or $[0,\infty)$
$\frac{1}{xy}$	$\{(x,y) : x \cdot y \neq 0\}$ 	$\mathbb{R} \setminus \{0\}$ or $(-\infty,0) \cup (0,\infty)$
$e^{e^{(x+y)}}$	$\mathbb{R}^2$	$\mathbb{R}^2 \xrightarrow{x+y} \mathbb{R} \xrightarrow{e^{(\cdot)}} (0,\infty) \xrightarrow{e^{(\cdot)}} \underline{\underline{(1,\infty)}}$ the range 
$\frac{1}{x^2+y^2+z^2}$	$\mathbb{R}^3 \setminus \{(0,0,0)\}$	$\mathbb{R}^{>0}$ or $(0,\infty)$

> **Definition.** A region is said to be
>
> - **open** if it consists entirely of interior points.
> - **closed** if it contains all of its boundary points.
> - **bounded** if it lies in a interval\disc\ball of finite radius.

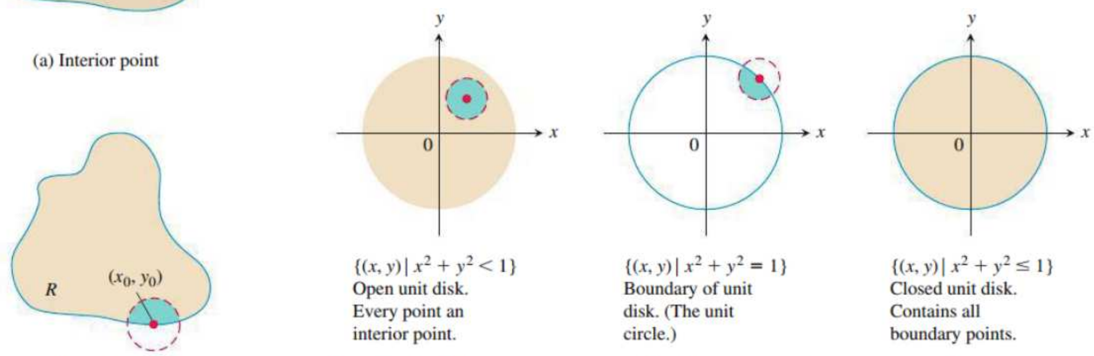
(a,b) open in $\mathbb{R}$ but not in $\mathbb{R}^2$

$(a,b)\cup(c,d)$ is also open in $\mathbb{R}$

$[a,b]$ is a closed interval in $\mathbb{R}$



(a) Interior point

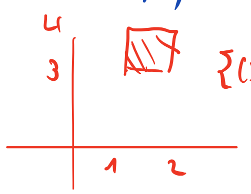


(b) Boundary point

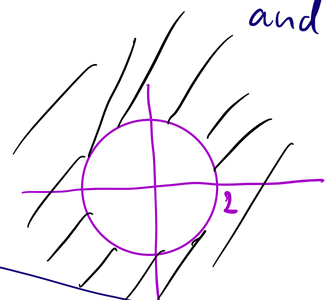
FIGURE 14.3 Interior points and boundary points of the unit disk in the plane.

Example. For each of the following, determined whether it is closed, open, bounded (couple of options are possible):

1. $(0, \infty)$. unbounded, open in $\mathbb{R}$ (but not open in $\mathbb{R}^2$)
2. $[1, 2)$. $= \{x\in\mathbb{R}: 1\leq x<2\}$, bounded, neither open nor closed
3. $[1, 2] \times [3, 4]$. $\{(x,y):\ x\in[1,2],\ y\in[3,4]\}$ bounded and closed



4. $x^2 + y^2 \geq 4$. $x^2+y^2\geq 4$ closed and unbounded



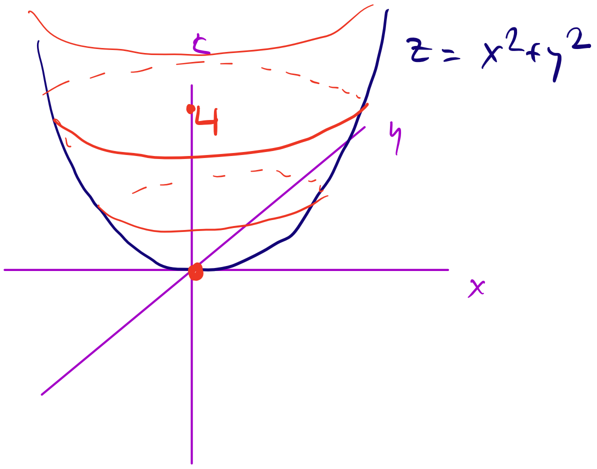
5. $x^2 + y^2 < 4$. open and bounded

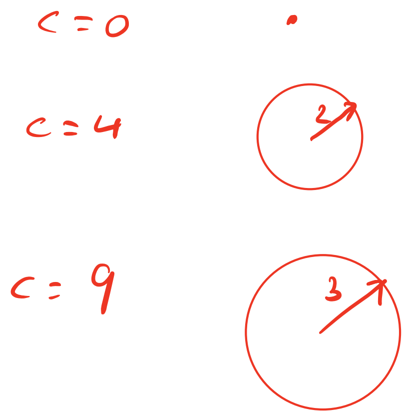
Definition. If $z = f(x, y)$ has domain D, then the graph of f is the set of all points of the form

$$\text{graph}(f) := \{x, y, f(x, y) : x, y \in D\}.$$

The set of all points where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** (or **contour curve**) . A **contour diagram** is a diagram in $\mathbb{R}^2$ of multiple labeled level curves.

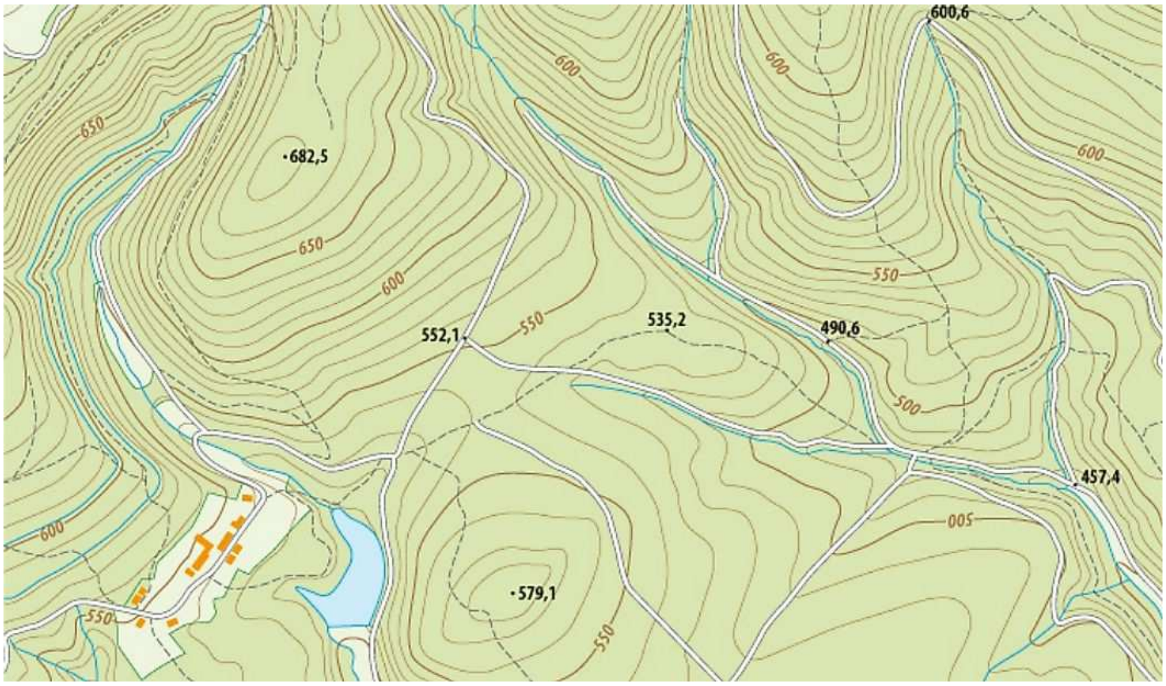
Example. Consider the function $f(x, y) = x^2 + y^2$. Draw the graph of $f(x, y)$, ~~and a~~ contour diagram for $c = 0, 4, 9, 16, 25$.





the level set $\{f(x,y) = c\}$

is a circle of radius $\sqrt{c}$


600,6
600
600
650
682,5
600
650
600
550
552,1
550
535,2
490,6
500
600
457,4
500
579,1
550
550

Given a function $f(x, y, z)$ in three variables, its graph is $(x, y, z, f(x, y, z))$ is inside a four dimensional space, so we cannot sketch it effectively. Instead, we can have a good understanding of f by drawing its level surfaces.

Definition. The set of all points (x, y, z) where a function $f(x, y, z)$ has a constant value $f(x, y, z) = c$ is called a **level surface**.

Example. Describe the level surfaces of the function $(x^2 + y^2 + z^2)^2$.

$f(x,y,z) = (x^2+y^2+z^2)^2$

the domain is $\mathbb{R}^3$, the range is $[0,\infty)$

For $c \geq 0$ we can compute the level set.

For $c=0$ it is the point $\{(0,0,0)\}$

For $c>0$

$(x^2+y^2+z^2)^2 = c$

$\{x^2+y^2+z^2 = \sqrt{c}\}$