Lecture 25: Optimization (§14.7)

Recall:

How to find the absolute minima and maxima?

Suppose a (continuous) function $f(x, y)$ is defined on a **closed** and **bounded** domain D (otherwise, absolute extrema may not exist).

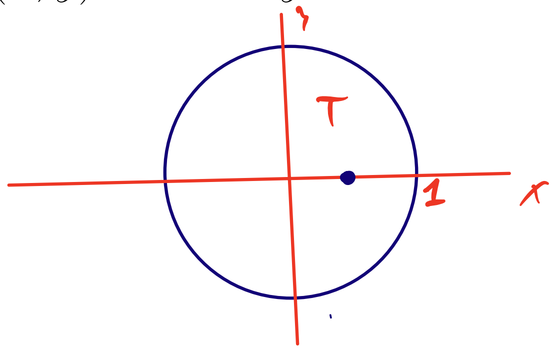
1. Find the values of f at the critical points in the interior of D.

2. Find the extreme values of f on the boundary of D.

3. Compare the values in (1) and (2).

The absolute minimum and maximum values of f are precisely the smallest and largest numbers among the ones found above.

Example. A flat circular plate of radius 1 centered at the origin has temperature at point (x, y) given by $T(x, y) = x^2 + 2y^2 - x$. Find the hottest and coldest points on the plate.



Step 1 (Interior)

$$\nabla T = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y} \right\rangle = \langle 2x-1, 4y \rangle$$

$\nabla T = 0$ only at $(\frac{1}{2}, 0)$. $\boxed{T(\frac{1}{2}, 0) = -\frac{1}{2}}$

Step 2 (Boundary) Its convinient to express the function in (r, θ) coordinate

$x = r\cos\theta$

$y = r\sin\theta$

$$\boxed{\begin{array}{l} r = \sqrt{x^2+y^2} \\ \theta = \arctan \frac{y}{x} \end{array}}$$

On the unit circle $r = 1$ so $(x,y) = (\cos\theta, \sin\theta)$, $0 \le \theta \le 2\pi$

$$T(\theta) = T(\cos\theta, \sin\theta) = \cos^2\theta + 2\sin^2\theta - \cos\theta$$

$$= 1 + \sin^2\theta - \cos\theta$$

$$T' = \frac{\partial T}{\partial \theta} = 2\sin\theta\cos\theta + \sin\theta = \sin\theta(2\cos\theta + 1)$$

$T' = 0$ if

$\sin\theta = 0$

i.e. $\theta = 0, \pi$

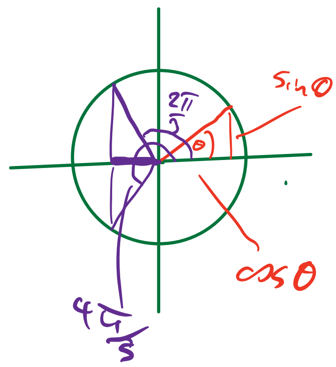
$(x,y) = (1,0), (-1,0)$

$\cos\theta = -\frac{1}{2}$

i.e.

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

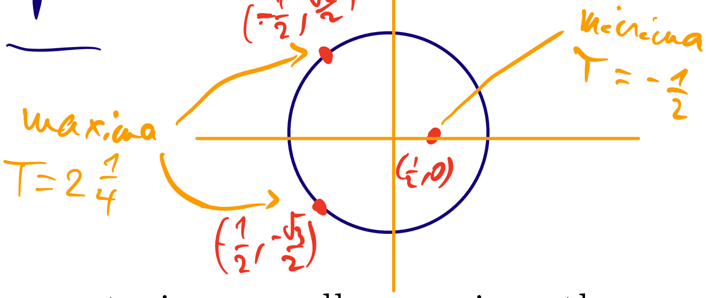
$(x,y) = (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$



	$(\frac{1}{2}, 0)$	$(1,0)$	$(-1,0)$	$(-\frac{1}{2}, \frac{\sqrt{3}}{2})$	$(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$
T	$-\frac{1}{4}$	0	2	$\frac{9}{4}$	$\frac{9}{4}$

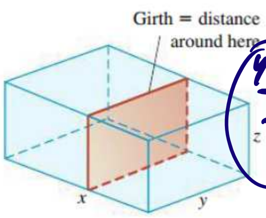
$T = x^2 + 2y^2 - x$

Constrained optimization



Solving extreme value theorems with constrains usually requires the method of Lagrange multipliers which we will learn next class. But in some cases, it can be solved directly.

Example. A delivery company accepts only rectangular boxes, whose sum of length and girth does not exceed 108cm. Find the dimensional of an acceptable box of largest volume.



$y+z$ the girth

The restriction is:

$$x+y+z+y+z \leq 108$$

$$x+2y+2z \leq 108$$

The volume function $V(x,y,z) = x \cdot y \cdot z$

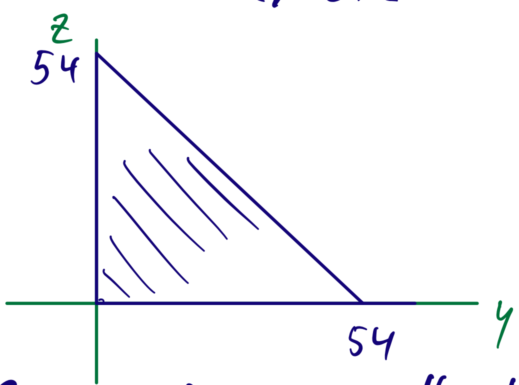
We may assume $x+2y+2z = 108$

we may express x in terms of y,z: $x = 108-2y-2z$

$$V(y,z) = V(108-2y-2z,\, y,z) = (108-2y-2z)yz = 108yz-2y^2z-2yz^2$$

$x,y,z \geq 0$ $\qquad 0 \leq y,z \leq 54$

$\underline{x = 2(54-(y+z))}$ $\qquad (y+z) \leq 54$



$$V(y,z) = 108yz-2y^2z-2yz^2$$

Remark
On the boundary $V=0$ so the maxima will be in the interior

$$\nabla V = \left\langle \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right\rangle = \langle 108z-4yz-2z^2,\, 108y-4yz-2y^2 \rangle$$

$$= \langle 2z(54-2y-z),\, 2y(54-2z-y) \rangle$$

$\nabla V = 0$ for some points on the boundary e.g $(0,0)$, $(0,54)$ but on these points we already made the remark that $V=0$

Suppose $y, z \neq 0$ but $\nabla V = 0$

then $54 - 2y - z = 54 - 2z - y = 0$

$\Rightarrow y = z$ and $54 - 3y = 0 \Rightarrow \underline{y = z = 18}$

$x = 108 - 2y - 2z = 108 - 4 \cdot 18 = 36$

$V(36, 18, 18) = 36 \cdot 18 \cdot 18 = 11664$

Reminder:

Theorem. ***Second Derivative Test for Local Extreme Values*** *Let $f(x, y)$ be a (nice) function, with a critical point (a, b). Then:*

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

1. *If $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) then f has a **local maximum** at (a, b).*

2. *If $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) then f has a **local minimum** at (a, b).*

3. *If $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b) then f has a **saddle point** at (a, b).*

4. *If $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b), then **the test is inconclusive**.*

Example. Find the points on the surface $z^2 - xy - 1 = 0$ that are closest to the origin.