Lecture 15: The Chain Rule (§14.4)

Goals: Compute the derivative of a composition of functions using the chain rule.

Recall- chain rule:

Given functions $f(t)$ and $g(x)$ then

$$\frac{d}{dt}g(f(t)) = g'(f(t)) \cdot f'(t).$$

Example. $\frac{d}{dt}\sin(t^2) = \cos(t^2) \cdot 2t$

Intuition: Write s instead of f

$s = s(t)$

at $s(t)$

at t

$$\frac{d}{dt} f(s(t)) = \frac{df}{ds} \cdot \frac{ds}{dt}$$

We have also seen a version of the chain rule for vector functions; given $\overrightarrow{r}(t) = \langle x(t), y(t), z(t)\rangle$, and $t = f(s)$. Then

$$\frac{d}{ds}\overrightarrow{r}(f(s)) = \overrightarrow{r}'(f(\cancel{t}\,s)) \cdot f'(\cancel{t}\,s).$$

Today we are going to discuss the chain rule in a more general context. For example:

1. $f(x, y)$ with $x = x(t)$ and $y = y(t)$. What is $\frac{df}{dt}$?
2. $f(x, y)$ with $x = x(u, v)$ and $y = y(u, v)$. What are $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$?

In the notation above, variables x and y are called **intermediate variables**, while the variables t, u, v are **independent variables**.

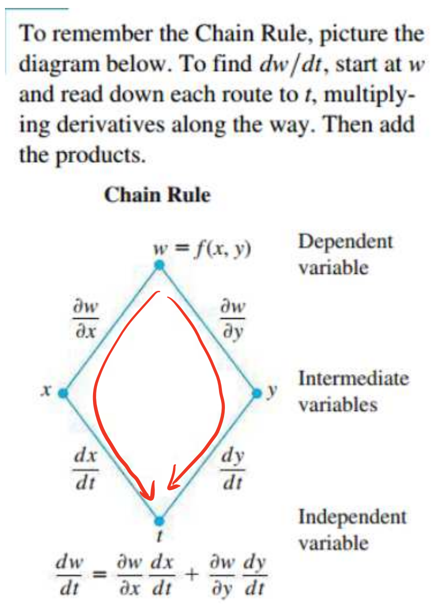
Multivariable chain rule (version 1) Suppose $w = f(x, y)$ is "nice" and that $x = x(t), y = y(t)$ are differentiable. Then

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}.$$

In other words

$$w'(t) = \frac{\partial f}{\partial x}\Big|_{(x(t),y(t))} \cdot x'(t) + \frac{\partial f}{\partial y}\Big|_{(x(t),y(t))} \cdot y'(t)$$

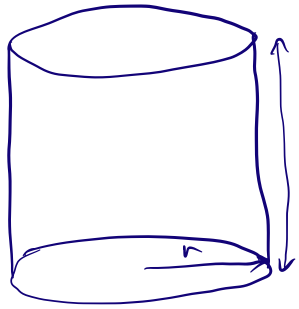
Dependency diagram:



Example. The radius of a cylinder is increasing at a rate of 3m/sec, and the height is increasing at a rate of 2m/sec. What is the rate of change of the volume of the cylinder assuming the radius of the cylinder is 2m and the height is 4m?

$\boxed{V = V(r, h) = \pi r^2 \cdot h}$

$\frac{dr}{dt} = 3$



$h = h(t)$

$r = r(t)$

$\frac{\partial V}{\partial r} = 2\pi r h, \quad \frac{\partial V}{\partial h} = \pi r^2$

$\frac{dh}{dt} = 2$

$r = 2, \ h = 4$

$\frac{dV}{dt} = \frac{\partial V}{\partial r} \cdot \frac{dr}{dt} + \frac{\partial V}{\partial h} \cdot \frac{dh}{dt} = 2\pi r h \cdot 3 + \pi r^2 \cdot 2 = 2\pi \cdot 2 \cdot 4 \cdot 3 + \pi \cdot 2^2 \cdot 2 = 48\pi + 8\pi = 56\pi$

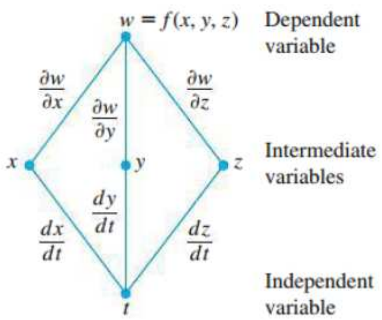
Multivariable chain rule (version 2) Suppose $w = f(x, y, z)$ and $x = x(t), y = y(t), z = z(t)$. Then

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}.$$

Dependency diagram:

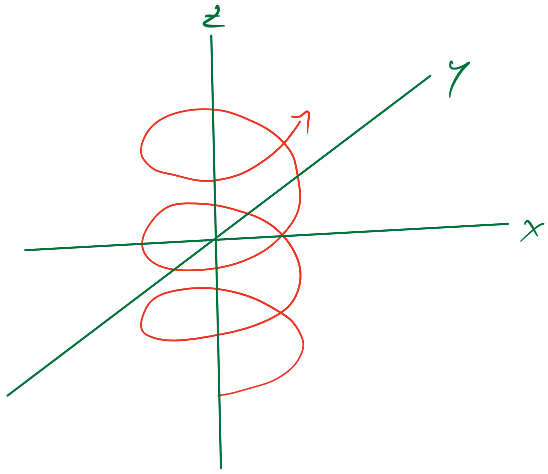
Here we have three routes from w to t instead of two, but finding dw/dt is still the same. Read down each route, multiplying derivatives along the way; then add.

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Example. A bee is flying around a room in a helix. Its position at time t is given by $r(t) = \langle \cos t, \sin t, t \rangle$. The temperature in the room at position (x, y, z) is $T(x, y, z) = xy + xz + yz$. What is the rate of change of the temperature experienced by the bee at time t?



$$\frac{dT}{dt} = \frac{\partial T}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial T}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial T}{\partial z} \cdot \frac{dz}{dt} =$$

$$\frac{\partial T}{\partial x} = y+z, \quad \frac{\partial T}{\partial y} = x+z, \quad \frac{\partial T}{\partial z} = x+y$$

$$x(t) = \cos t, \quad y(t) = \sin t, \quad z(t) = t$$

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t, \quad \frac{dz}{dt} = 1$$

$$= (y+z)\frac{dx}{dt} + (x+z)\frac{dy}{dt} + (x+y) \cdot \frac{dz}{dt} =$$

$$= -(\sin t + t)\sin t + (\cos t + t)\cos t + (\cos t + \sin t) \cdot 1$$

$$= -\sin^2 t + \cos^2 t + \cos t(1+t) + \sin t(1-t)$$

Multivariable chain rule (two independent variables):

1. Suppose $w = f(x, y)$ with $x = x(s, t)$ and $y = y(s, t)$. Then:

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

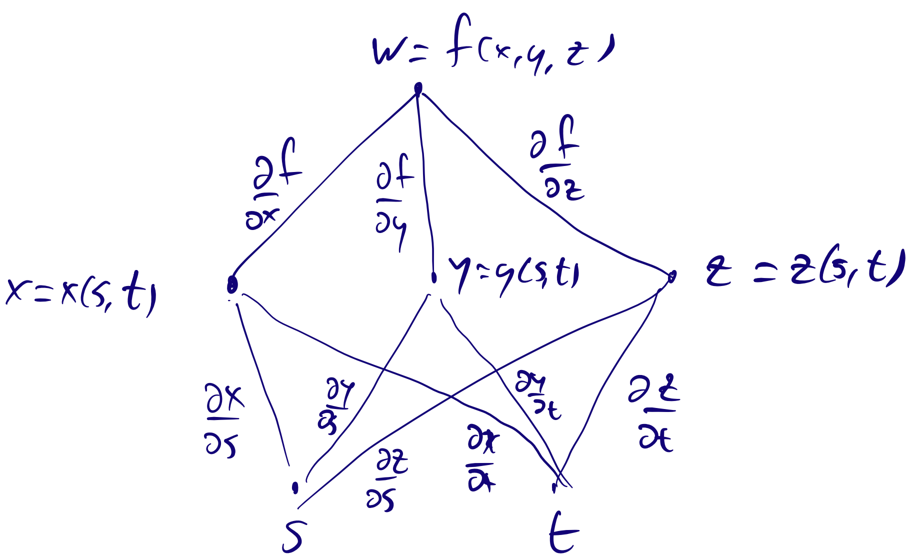
$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}.$$

2. Suppose $w = f(x, y, z)$ with $x = x(s, t)$, $y = y(s, t)$ and $z = z(s, t)$. Then:

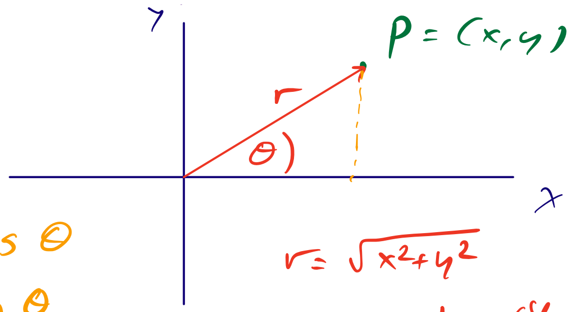
$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t}$$

Dependency diagrams:



Example. Given a point (x, y) on a plane, we can write it using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$. Suppose we have a function $f(x, y)$ on a plane. Write a formula for $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$.



$$r = \sqrt{x^2+y^2}$$

$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta \quad \Longleftarrow$$

$$\frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r} = \cos\theta\,\frac{\partial f}{\partial x} + \sin\theta\,\frac{\partial f}{\partial y}$$

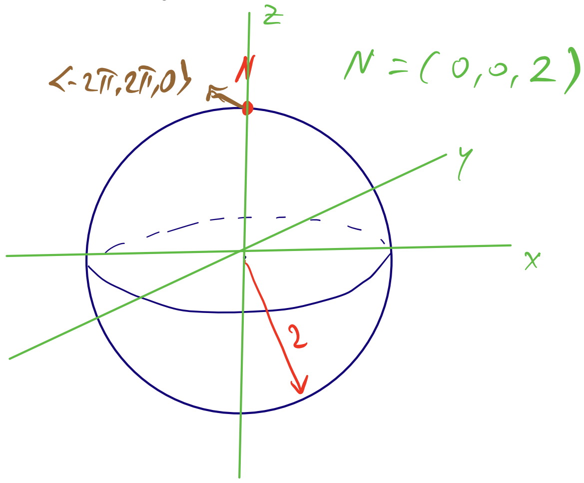
$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta} = -r\sin\theta\,\frac{\partial f}{\partial x} + r\cos\theta\,\frac{\partial f}{\partial y}$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

Example. Suppose the magnitude of pressure exerted on the surface of a sphere of radius 2 is given by

$$f(x, y, z) = z^2 e^{-(x^2+y^2)} \text{ pa}$$

Suppose an object moves around the sphere. As the object moves through the north pole, its instantaneous velocity is $\langle -2\pi, 2\pi, 0 \rangle$. What is the rate of change of pressure exerted on the object with respect to time as it moves through the north pole.



$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} =$$

$$\frac{\partial f}{\partial x} = -2xz^2 e^{-(x^2+y^2)}$$

$$\frac{\partial f}{\partial y} = -2yz^2 e^{-(x^2+y^2)}$$

$$\frac{\partial f}{\partial z} = 2z e^{-(x^2+y^2)}$$

$$= 0 \cdot (-2\pi) + 0(2\pi) + 4 \cdot 0 = 0$$

$$\langle x, y, z \rangle = \langle 0, 0, 2 \rangle$$

$$\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \langle -2\pi, 2\pi, 0 \rangle$$