第六次機率與統計作業

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一、手寫部分：

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$\boxed{7.12}$

1° $\begin{cases} Y_1 = X_1 + X_2 \\ Y_2 = \frac{X_1}{(X_1+X_2)} \end{cases} \Rightarrow \begin{cases} Y_1 = X_1 + X_2 \\ Y_2(X_1+X_2) = X_1 \end{cases} \Rightarrow \begin{cases} Y_1 = X_1 + X_2 \\ 0 = (1-Y_2)X_1 - X_2 Y_2 \end{cases} \Rightarrow \begin{cases} Y_1 Y_2 = X_1 Y_2 + X_2 Y_2 \quad -① \\ 0 = (1-Y_2)X_1 - X_2 Y_2 \quad -② \end{cases}$

①+②

$\Rightarrow Y_1 Y_2 = X_1 - X_1 Y_2 + X_1 Y_2 \Rightarrow X_1 = Y_1 Y_2 \quad -③$

Substitute ③ to ①

$\Rightarrow Y_1 \cancel{Y_2} = Y_1 Y_2^{\cancel{2}} + X_2 \cancel{Y_2} \Rightarrow X_2 = Y_1 - Y_1 Y_2$

$(e^{y_1 y_2})(e^{y_1 - y_1 y_2})$

$= e^{y_1}$

By Jacobian,

$$J = \begin{vmatrix} y_2 & y_1 \\ 1-y_2 & -y_1 \end{vmatrix} = \cancel{-y_1 y_2} - y_1 \cancel{+ y_1 y_2} = -y_1$$

Therefore, we can write

$g(y_1, y_2) = F(x_1, x_2) \cdot |J| \quad -(*)$

Since X_1 & X_2 are independent

$\therefore (*) = f(x_1) \cdot f(x_2) \cdot |J| = e^{-x_1} \cdot e^{-x_2} \cdot y_1 = y_1 e^{-y_1} = g(y_1, y_2)$, for $y_1 > 0$ & $0 < y_2 < 1$;

also, by the property of $f(x)$, then for otherwise, $g(y_1, y_2) = 0$.

2° To prove Y_1, Y_2 independent, apply $g(y_1, y_2) = h(y_1) \cdot h(y_2)$:

$h(y_1) = \int_0^1 g(y_1, y_2)\, dy_2 = \int_0^1 y_1 e^{-y_1}\, dy_2 = y_1 e^{-y_1}$

$h(y_2) = \int_0^\infty g(y_1, y_2)\, dy_1 = \int_0^\infty y_1 e^{-y_1}\, dy_1 = \Gamma(2) = 1$

$\therefore$ we can get $h(y_1) \times h(y_2) = y_1 e^{-y_1} \cdot 1 = y_1 e^{-y_1} = g(y_1, y_2)$

Therefore, Y_1, Y_2 are independent #

$\boxed{7.14}$

By the probability distribution of the random variable $Y = X^2$

$\Rightarrow$ change to $X = \pm\sqrt{Y}$, for $0 < y < 1$

And by the property of $f(x)$, we can get

for $0 < x_1 < 1$: $y = x_1^2 \Rightarrow x_1 = \sqrt{y}$

for $-1 < x_2 \leq 0$: $y = x_2^2 \Rightarrow x_2 = -\sqrt{y}$

Then,

$J_1 = x_1 \frac{dx_1}{dy} = \frac{1}{2\sqrt{y}}$

$J_2 = x_2 \frac{dx_2}{dy} = -\frac{1}{2\sqrt{y}}$

Therefore, we can get the probability distribution

$g(y) = f(x_1)|J_1| + f(x_2)|J_2|$

$= \frac{1+\sqrt{y}}{2} \times \left|\frac{1}{2\sqrt{y}}\right| + \frac{1-\sqrt{y}}{2} \times \left|\frac{1}{-2\sqrt{y}}\right|$

$= \frac{1}{2\sqrt{y}} \left(\frac{(1+\sqrt{y}) + (1-\sqrt{y})}{2} \right) = \frac{1}{2\sqrt{y}} \times 1$

$= \frac{1}{2\sqrt{y}}$, for $0 < y < 1$

$\therefore g(y) = \begin{cases} \frac{1}{2\sqrt{y}}, & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$ #

7.18

1° By moment generating function,

$$M_X(t) = E(e^{tX}) = \sum_{x=1}^{\infty} e^{tx} g(x) = \sum_{x=1}^{\infty} e^{tx} p q^{x-1} = pe^t \sum_{x=1}^{\infty} e^{tx} q^{x-1} = pe^t \sum_{x=0}^{\infty} e^{tx} q^{x} = pe^t \sum_{x=0}^{\infty} (qe^t)^x$$

$$= pe^t \cdot \frac{1}{1-qe^t} = \underline{\frac{pe^t}{1-qe^t}} \ \#$$

Also, if $\sum_{x=0}^{\infty} (qe^t)^x$ has value (not ∞), then

$$|qe^t| < 1 \Rightarrow e^t < \frac{1}{q} \Rightarrow t < \ln(q^{-1}) \Rightarrow \underline{t < -\ln q} \ \#$$

2° Find the mean of X:

$$\mu_X = E(X) = \frac{dM_X(t)}{dt}\Big|_{t=0} = \frac{d}{dt}\left(\frac{pe^t}{1-qe^t}\right)\Big|_{t=0} = \frac{pe^t(1-qe^t) - pe^t(-qe^t)}{(1-qe^t)^2}\Big|_{t=0}$$

$$= \frac{pe^t}{(1-qe^t)^2}\Big|_{t=0} = \frac{p}{(1-q)^2} \xlongequal{\text{by } q=1-p} \frac{p}{p^2} = \underline{\frac{1}{p}} \ \#$$

$(p-1)^2$
$= p^2 - 2p + 1$

3° Find the variance of X:

$$Var(X) = E(X^2) - E(X)^2 = \frac{d^2M_X(t)}{dt^2}\Big|_{t=0} - \left(\frac{1}{p}\right)^2 = \frac{d}{dt}\left(\frac{pe^t}{(1-qe^t)^2}\right)\Big|_{t=0} - \frac{1}{p^2}$$

$$= \frac{pe^t(1-qe^t)^2 - pe^t \cdot 2(1-qe^t)\cdot(-qe^t)}{(1-qe^t)^4}\Big|_{t=0} - \frac{1}{p^2}$$

$\cancel{p} - \cancel{p} + 2p^2 - p^3$

$$= \frac{pe^t - pq^2e^t}{(1-qe^t)^4}\Big|_{t=0} - \frac{1}{p^2} = \frac{p-pq^2}{(1-q)^4} - \frac{1}{p^2} \xlongequal{\text{by } q=1-p} \frac{p-p(1-p)^2}{p^4} - \frac{1}{p^2} = \frac{\cancel{2}p^2 - p^3 - \cancel{p^2}}{p^4}$$

$$= \frac{1-p}{p} \xlongequal{\text{by } p=1-q} \underline{\frac{q}{p}} \ \#$$

7.22

From Exercise 7.21, we know the moment generating function of random variable X having chi-squared distribution with ν degrees of freedom is $M_X(t) = (1-2t)^{-\nu/2}$

1° The mean of random variable X:

$$\mu_X = \frac{dM_X(t)}{dt}\Big|_{t=0} = \frac{d}{dt}\left((1-2t)^{-\frac{\nu}{2}}\right) = \left(+\frac{\nu}{\cancel{2}} \cdot (1-2t)^{-\frac{\nu}{2}-1} \cdot (\cancel{-2})\right)\Big|_{t=0} = \underline{\nu} \ \#$$

2° The variance of random variable X:

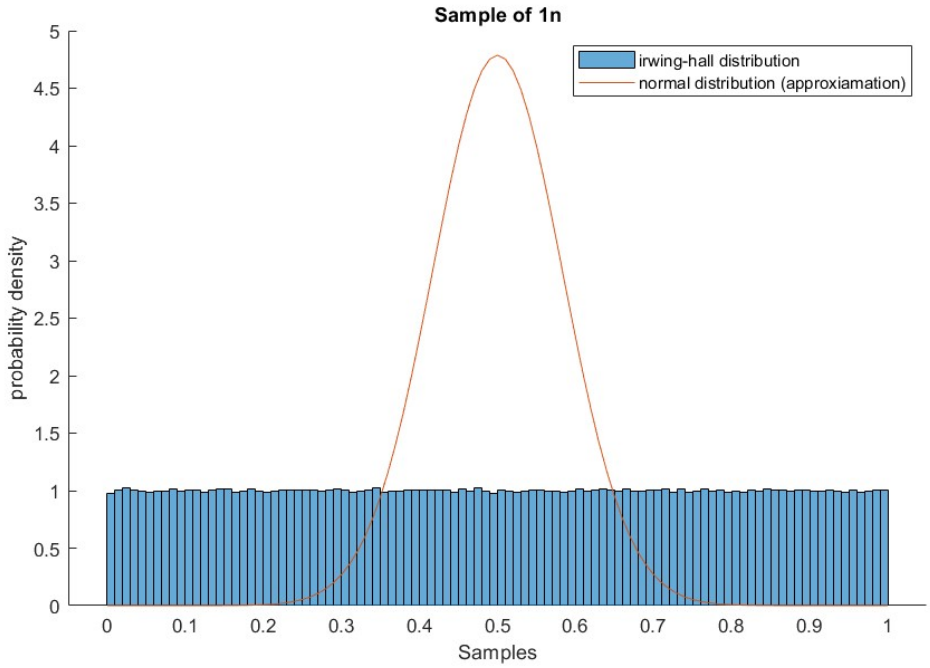
$$\sigma_X^2 = E(X^2) - E(X)^2 = \frac{d^2M_X(t)}{dt^2}\Big|_{t=0} - \nu^2 = \frac{d}{dt}\left(\nu(1-2t)^{-\frac{\nu}{2}-1}\right)\Big|_{t=0} - \nu^2$$

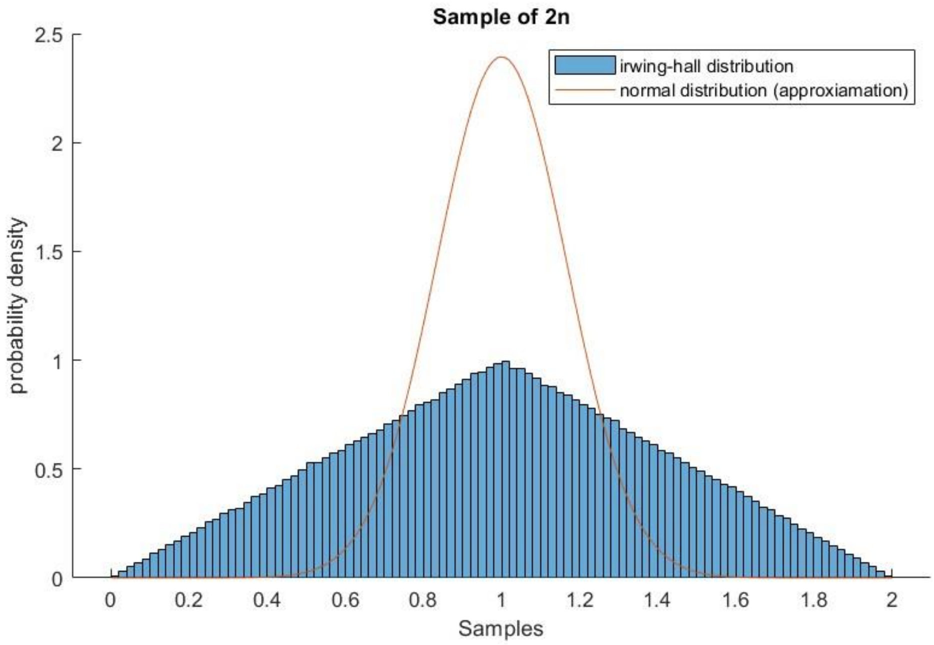
$$= \left(\nu\left(-\frac{\nu}{2}-1\right)(1-2t)^{-\frac{\nu}{2}-2}\cdot(-2)\right)\Big|_{t=0} - \nu^2 = \cancel{\nu^2} + 2\nu - \cancel{\nu^2} = \underline{2\nu} \ \#$$

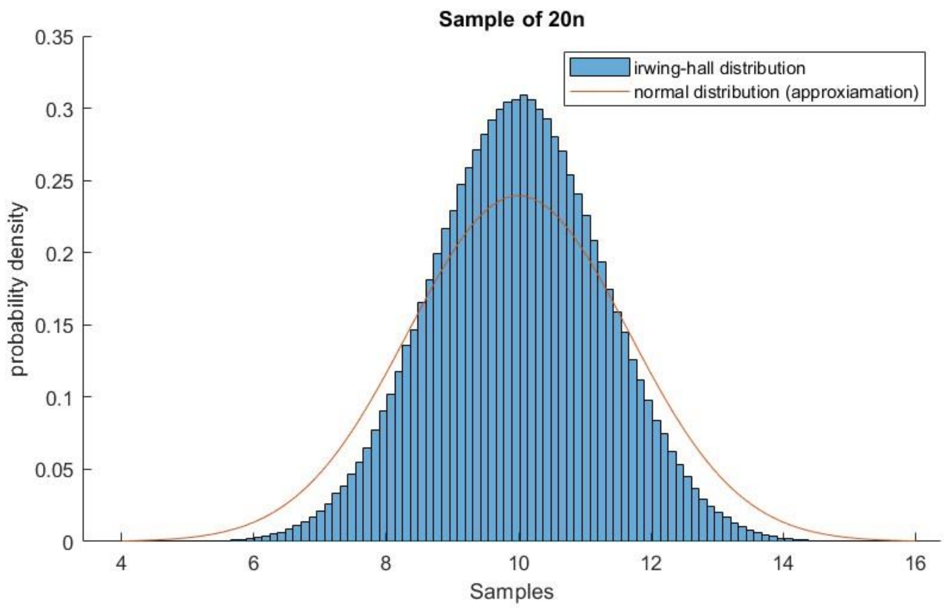
$\therefore$ we show that the mean and variance of the chi-squared distribution with ν degrees of freedom are, respectively, ν and 2ν #

二、matlab 部分：

HW7_1b：







從三個結果圖來說，irwing-hall distribution 在 n = 1 時確實是 uniform distribution；n = 2 時也是題目說的 triangular distribution；而 n = 20 的 case 確實有點像 normal distribution。其中，在經過 normal distribution approximation 後，可以發現到，在 irwing-hall distribution 在 n = 1 及 n = 2 的時候的 irwing-hall distribution hostogram 和 normal distribution approximation plot 的波形相差很多，代表著失真率大，其中尤以 n = 1 的 case 最為嚴重。相反的，而在 n = 20 時，由於 n 有著一定的大小，因此和 normal distribution approximation plot 相比時，他們波形比率相差的並不大，也就是說這個 case 是可以被 normal distribution 做近似的，和作業 pdf 中的說法完全吻合