Chapter 7

Introduction to linear regression

Linear regression is a very powerful statistical technique. Many people have some familiarity with regression just from reading the news, where graphs with straight lines are overlaid on scatterplots. Linear models can be used for prediction or to evaluate whether there is a linear relationship between two numerical variables.

Figure 7.1 shows two variables whose relationship can be modeled perfectly with a straight line. The equation for the line is

$$y = 5 + 57.49x$$

Imagine what a perfect linear relationship would mean: you would know the exact value of y just by knowing the value of x. This is unrealistic in almost any natural process. For example, if we took family income x, this value would provide some useful information about how much financial support y a college may offer a prospective student. However, there would still be variability in financial support, even when comparing students whose families have similar financial backgrounds.

Linear regression assumes that the relationship between two variables, x and y, can be modeled by a straight line:

$$y = \beta_0 + \beta_1 x \tag{7.1}$$

β_0, β_1
Linear model parameters

where β_0 and β_1 represent two model parameters (β is the Greek letter *beta*). These parameters are estimated using data, and we write their point estimates as b_0 and b_1. When we use x to predict y, we usually call x the explanatory or **predictor** variable, and we call y the response.

It is rare for all of the data to fall on a straight line, as seen in the three scatterplots in Figure 7.2. In each case, the data fall around a straight line, even if none of the observations fall exactly on the line. The first plot shows a relatively strong downward linear trend, where the remaining variability in the data around the line is minor relative to the strength of the relationship between x and y. The second plot shows an upward trend that, while evident, is not as strong as the first. The last plot shows a very weak downward trend in the data, so slight we can hardly notice it. In each of these examples, we will have some uncertainty regarding our estimates of the model parameters, β_0 and β_1. For instance, we might wonder, should we move the line up or down a little, or should we tilt it more or less?

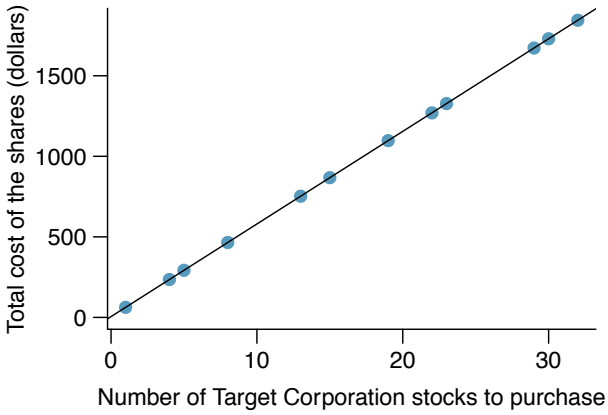


Figure 7.1: Requests from twelve separate buyers were simultaneously placed with a trading company to purchase Target Corporation stock (ticker `TGT`, April 26th, 2012), and the total cost of the shares were reported. Because the cost is computed using a linear formula, the linear fit is perfect.

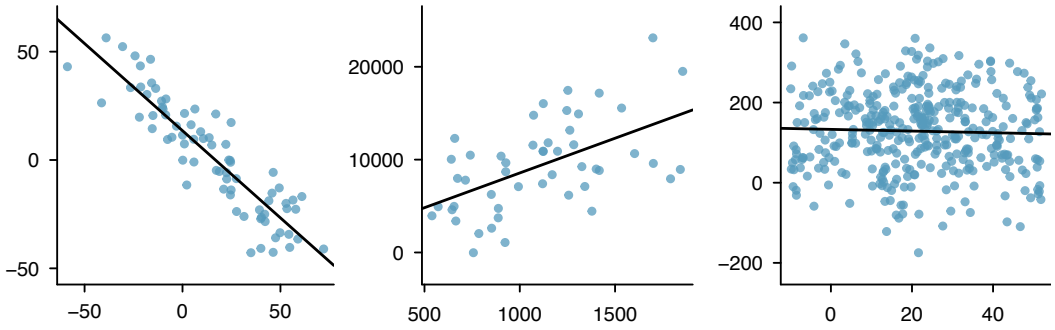


Figure 7.2: Three data sets where a linear model may be useful even though the data do not all fall exactly on the line.

As we move forward in this chapter, we will learn different criteria for line-fitting, and we will also learn about the uncertainty associated with estimates of model parameters.

We will also see examples in this chapter where fitting a straight line to the data, even if there is a clear relationship between the variables, is not helpful. One such case is shown in Figure 7.3 where there is a very strong relationship between the variables even though the trend is not linear. We will discuss nonlinear trends in this chapter and the next, but the details of fitting nonlinear models are saved for a later course.

7.1 Line fitting, residuals, and correlation

It is helpful to think deeply about the line fitting process. In this section, we examine criteria for identifying a linear model and introduce a new statistic, *correlation*.

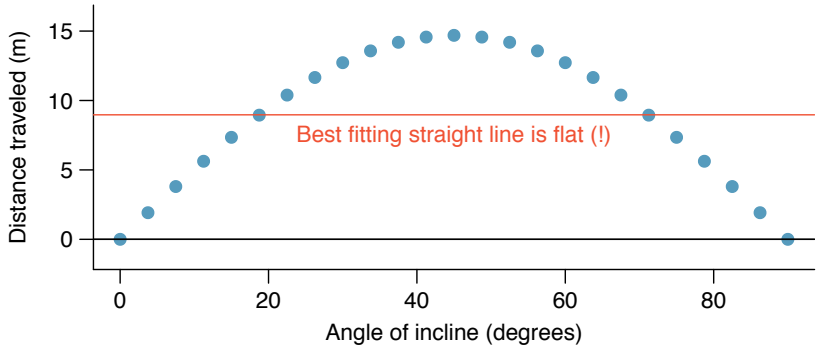


Figure 7.3: A linear model is not useful in this nonlinear case. These data are from an introductory physics experiment.

7.1.1 Beginning with straight lines

Scatterplots were introduced in Chapter 1 as a graphical technique to present two numerical variables simultaneously. Such plots permit the relationship between the variables to be examined with ease. Figure 7.4 shows a scatterplot for the head length and total length of 104 brushtail possums from Australia. Each point represents a single possum from the data.

The head and total length variables are associated. Possums with an above average total length also tend to have above average head lengths. While the relationship is not perfectly linear, it could be helpful to partially explain the connection between these variables with a straight line.

Straight lines should only be used when the data appear to have a linear relationship, such as the case shown in the left panel of Figure 7.6. The right panel of Figure 7.6 shows a case where a curved line would be more useful in understanding the relationship between the two variables.

Caution: Watch out for curved trends
We only consider models based on straight lines in this chapter. If data show a nonlinear trend, like that in the right panel of Figure 7.6, more advanced techniques should be used.

7.1.2 Fitting a line by eye

We want to describe the relationship between the head length and total length variables in the possum data set using a line. In this example, we will use the total length as the predictor variable, x, to predict a possum's head length, y. We could fit the linear relationship by eye, as in Figure 7.7. The equation for this line is

$$\hat{y} = 41 + 0.59x \tag{7.2}$$

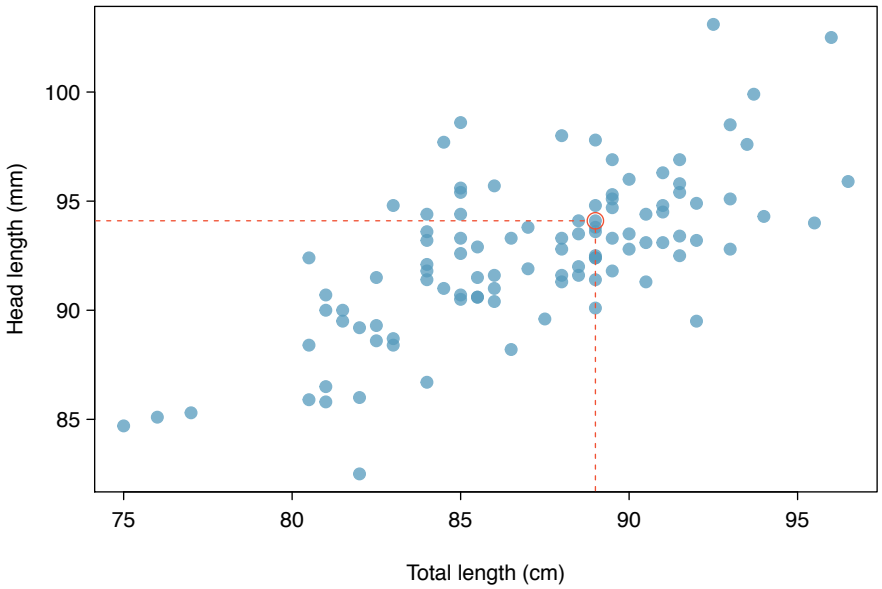


Figure 7.4: A scatterplot showing head length against total length for 104 brushtail possums. A point representing a possum with head length 94.1mm and total length 89cm is highlighted.

Figure 7.5: The common brushtail possum of Australia.

Photo by wollombi on Flickr: www.flickr.com/photos/wollombi/58499575

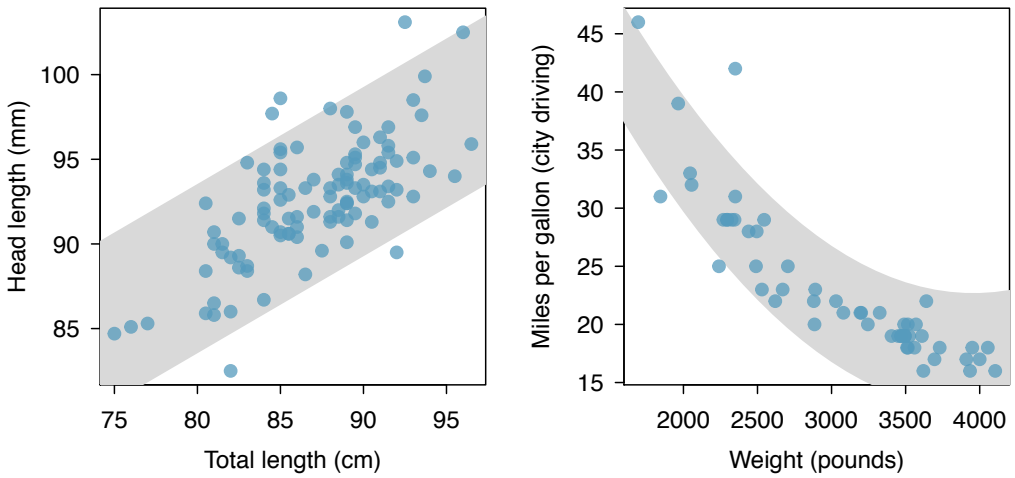


Figure 7.6: The figure on the left shows head length versus total length, and reveals that many of the points could be captured by a straight band. On the right, we see that a curved band is more appropriate in the scatterplot for `weight` and `mpgCity` from the `cars` data set.

We can use this line to discuss properties of possums. For instance, the equation predicts a possum with a total length of 80 cm will have a head length of

$$\begin{aligned}\hat{y} &= 41 + 0.59 \times 80 \\ &= 88.2\end{aligned}$$

A "hat" on y is used to signify that this is an estimate. This estimate may be viewed as an average: the equation predicts that possums with a total length of 80 cm will have an average head length of 88.2 mm. Absent further information about an 80 cm possum, the prediction for head length that uses the average is a reasonable estimate.

7.1.3 Residuals

Residuals are the leftover variation in the data after accounting for the model fit:

$$\text{Data} = \text{Fit} + \text{Residual}$$

Each observation will have a residual. If an observation is above the regression line, then its residual, the vertical distance from the observation to the line, is positive. Observations below the line have negative residuals. One goal in picking the right linear model is for these residuals to be as small as possible.

Three observations are noted specially in Figure 7.7. The observation marked by an "$\times$" has a small, negative residual of about -1; the observation marked by "+" has a large residual of about +7; and the observation marked by "$\triangle$" has a moderate residual of about -4. The size of a residual is usually discussed in terms of its absolute value. For example, the residual for "$\triangle$" is larger than that of "$\times$" because $|-4|$ is larger than $|-1|$.

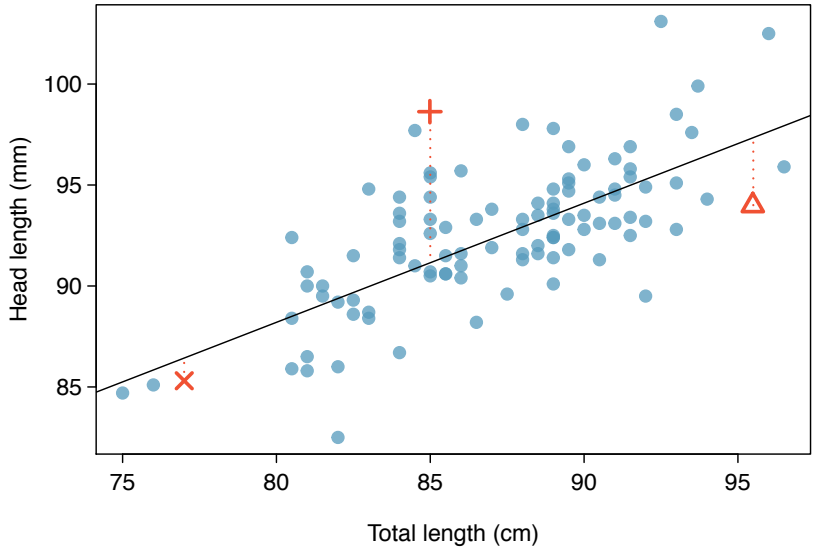


Figure 7.7: A reasonable linear model was fit to represent the relationship between head length and total length.

Residual: difference between observed and expected
The residual of the i^{th} observation (x_i, y_i) is the difference of the observed response (y_i) and the response we would predict based on the model fit $(\hat{y}_i)$:

$$e_i = y_i - \hat{y}_i$$

We typically identify $\hat{y}_i$ by plugging x_i into the model.

Example 7.3 The linear fit shown in Figure 7.7 is given as $\hat{y} = 41 + 0.59x$. Based on this line, formally compute the residual of the observation $(77.0, 85.3)$. This observation is denoted by "×" on the plot. Check it against the earlier visual estimate, -1.

We first compute the predicted value of point "×" based on the model:

$$\hat{y}_\times = 41 + 0.59x_\times = 41 + 0.59 \times 77.0 = 86.4$$

Next we compute the difference of the actual head length and the predicted head length:

$$e_\times = y_\times - \hat{y}_\times = 85.3 - 86.4 = -1.1$$

This is very close to the visual estimate of -1.

Exercise 7.4 If a model underestimates an observation, will the residual be positive or negative? What about if it overestimates the observation?[1]

[1] If a model underestimates an observation, then the model estimate is below the actual. The residual, which is the actual observation value minus the model estimate, must then be positive. The opposite is true when the model overestimates the observation: the residual is negative.

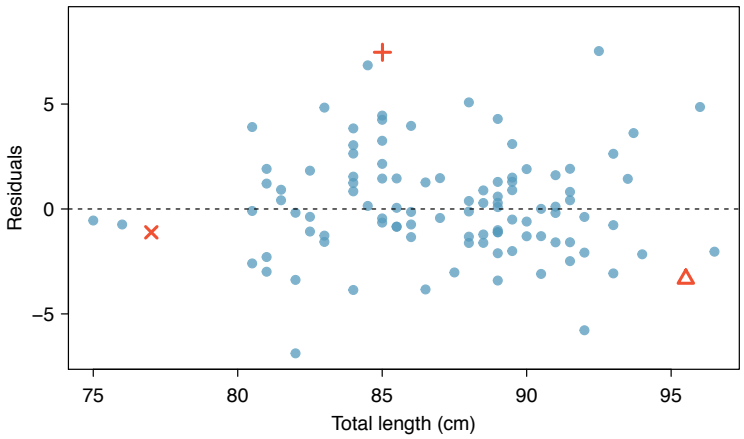


Figure 7.8: Residual plot for the model in Figure 7.7.

⊙ **Exercise 7.5** Compute the residuals for the observations $(85.0, 98.6)$ ("+" in the figure) and $(95.5, 94.0)$ ("△") using the linear relationship $\hat{y} = 41 + 0.59x$. [2]

Residuals are helpful in evaluating how well a linear model fits a data set. We often display them in a **residual plot** such as the one shown in Figure 7.8 for the regression line in Figure 7.7. The residuals are plotted at their original horizontal locations but with the vertical coordinate as the residual. For instance, the point $(85.0, 98.6)_+$ had a residual of 7.45, so in the residual plot it is placed at $(85.0, 7.45)$. Creating a residual plot is sort of like tipping the scatterplot over so the regression line is horizontal.

● **Example 7.6** One purpose of residual plots is to identify characteristics or patterns still apparent in data after fitting a model. Figure 7.9 shows three scatterplots with linear models in the first row and residual plots in the second row. Can you identify any patterns remaining in the residuals?

In the first data set (first column), the residuals show no obvious patterns. The residuals appear to be scattered randomly around the dashed line that represents 0.

The second data set shows a pattern in the residuals. There is some curvature in the scatterplot, which is more obvious in the residual plot. We should not use a straight line to model these data. Instead, a more advanced technique should be used.

The last plot shows very little upwards trend, and the residuals also show no obvious patterns. It is reasonable to try to fit a linear model to the data. However, it is unclear whether there is statistically significant evidence that the slope parameter is different from zero. The point estimate of the slope parameter, labeled b_1, is not zero, but we might wonder if this could just be due to chance. We will address this sort of scenario in Section 7.4.

[2] (+) First compute the predicted value based on the model:

$$\hat{y}_+ = 41 + 0.59x_+ = 41 + 0.59 \times 85.0 = 91.15$$

Then the residual is given by

$$e_+ = y_+ - \hat{y}_+ = 98.6 - 91.15 = 7.45$$

This was close to the earlier estimate of 7.

(△) $\hat{y}_\triangle = 41 + 0.59x_\triangle = 97.3$. $e_\triangle = y_\triangle - \hat{y}_\triangle = -3.3$, close to the estimate of -4.

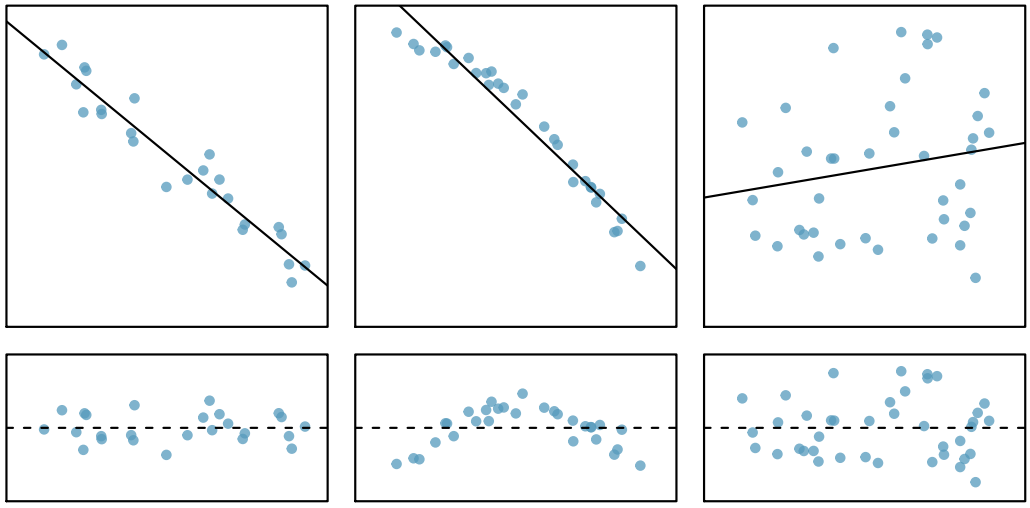


Figure 7.9: Sample data with their best fitting lines (top row) and their corresponding residual plots (bottom row).

7.1.4 Describing linear relationships with correlation

R
correlation

Correlation: strength of a linear relationship
Correlation, which always takes values between -1 and 1, describes the strength of the linear relationship between two variables. We denote the correlation by R.

We can compute the correlation using a formula, just as we did with the sample mean and standard deviation. However, this formula is rather complex,[3] so we generally perform the calculations on a computer or calculator. Figure 7.10 shows eight plots and their corresponding correlations. Only when the relationship is perfectly linear is the correlation either -1 or 1. If the relationship is strong and positive, the correlation will be near +1. If it is strong and negative, it will be near -1. If there is no apparent linear relationship between the variables, then the correlation will be near zero.

The correlation is intended to quantify the strength of a linear trend. Nonlinear trends, even when strong, sometimes produce correlations that do not reflect the strength of the relationship; see three such examples in Figure 7.11.

⊙ **Exercise 7.7** It appears no straight line would fit any of the datasets represented in Figure 7.11. Try drawing nonlinear curves on each plot. Once you create a curve for each, describe what is important in your fit.[4]

[3] Formally, we can compute the correlation for observations (x_1, y_1), (x_2, y_2), ..., (x_n, y_n) using the formula

$$R = \frac{1}{n-1}\sum_{i=1}^{n}\frac{x_i - \bar{x}}{s_x}\frac{y_i - \bar{y}}{s_y}$$

where $\bar{x}$, $\bar{y}$, s_x, and s_y are the sample means and standard deviations for each variable.

[4] We'll leave it to you to draw the lines. In general, the lines you draw should be close to most points and reflect overall trends in the data.

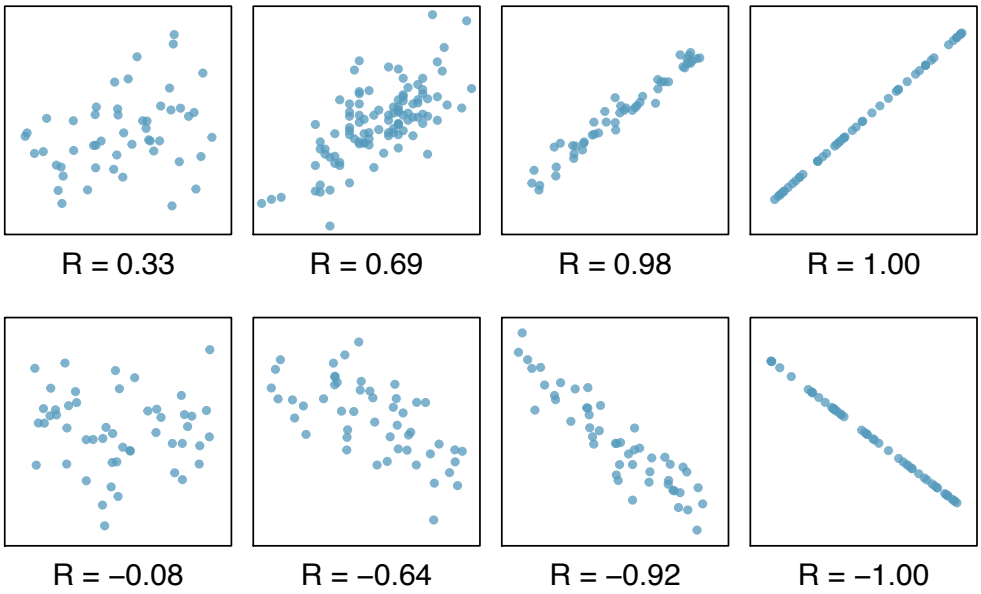


Figure 7.10: Sample scatterplots and their correlations. The first row shows variables with a positive relationship, represented by the trend up and to the right. The second row shows variables with a negative trend, where a large value in one variable is associated with a low value in the other.

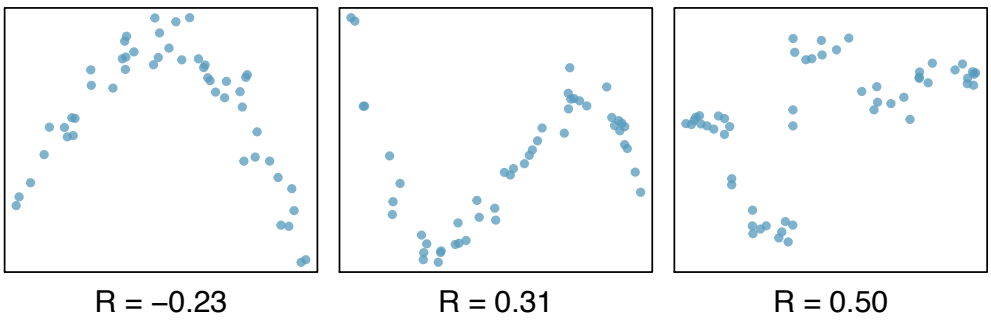


Figure 7.11: Sample scatterplots and their correlations. In each case, there is a strong relationship between the variables. However, the correlation is not very strong, and the relationship is not linear.

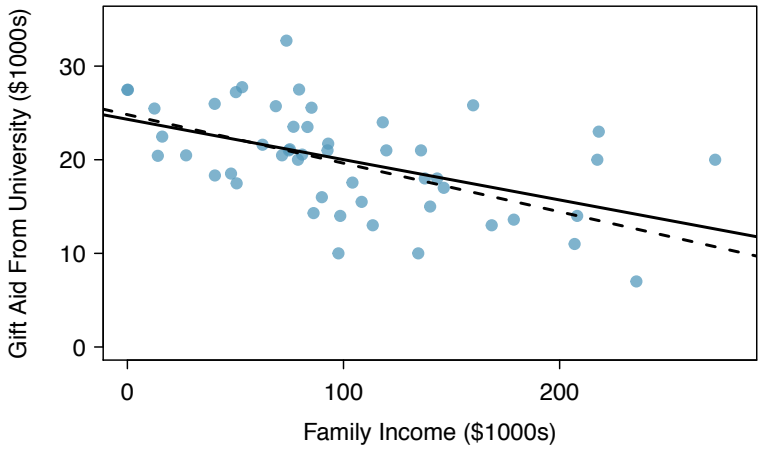


Figure 7.12: Gift aid and family income for a random sample of 50 freshman students from Elmhurst College. Two lines are fit to the data, the solid line being the *least squares line*.

7.2 Fitting a line by least squares regression

Fitting linear models by eye is open to criticism since it is based on an individual preference. In this section, we use *least squares regression* as a more rigorous approach.

This section considers family income and gift aid data from a random sample of fifty students in the 2011 freshman class of Elmhurst College in Illinois.[5] Gift aid is financial aid that does not need to be paid back, as opposed to a loan. A scatterplot of the data is shown in Figure 7.12 along with two linear fits. The lines follow a negative trend in the data; students who have higher family incomes tended to have lower gift aid from the university.

⊙ **Exercise 7.8** Is the correlation positive or negative in Figure 7.12?[6]

7.2.1 An objective measure for finding the best line

We begin by thinking about what we mean by "best". Mathematically, we want a line that has small residuals. Perhaps our criterion could minimize the sum of the residual magnitudes:

$$|e_1| + |e_2| + \cdots + |e_n| \tag{7.9}$$

which we could accomplish with a computer program. The resulting dashed line shown in Figure 7.12 demonstrates this fit can be quite reasonable. However, a more common practice is to choose the line that minimizes the sum of the squared residuals:

$$e_1^2 + e_2^2 + \cdots + e_n^2 \tag{7.10}$$

[5]These data were sampled from a table of data for all freshman from the 2011 class at Elmhurst College that accompanied an article titled *What Students Really Pay to Go to College* published online by *The Chronicle of Higher Education*: chronicle.com/article/What-Students-Really-Pay-to-Go/131435

[6]Larger family incomes are associated with lower amounts of aid, so the correlation will be negative. Using a computer, the correlation can be computed: -0.499.

The line that minimizes this **least squares criterion** is represented as the solid line in Figure 7.12. This is commonly called the **least squares line**. The following are three possible reasons to choose Criterion (7.10) over Criterion (7.9):

1. It is the most commonly used method.
2. Computing the line based on Criterion (7.10) is much easier by hand and in most statistical software.
3. In many applications, a residual twice as large as another residual is more than twice as bad. For example, being off by 4 is usually more than twice as bad as being off by 2. Squaring the residuals accounts for this discrepancy.

The first two reasons are largely for tradition and convenience; the last reason explains why Criterion (7.10) is typically most helpful.[7]

7.2.2 Conditions for the least squares line

When fitting a least squares line, we generally require

Linearity. The data should show a linear trend. If there is a nonlinear trend (e.g. left panel of Figure 7.13), an advanced regression method from another book or later course should be applied.

Nearly normal residuals. Generally the residuals must be nearly normal. When this condition is found to be unreasonable, it is usually because of outliers or concerns about influential points, which we will discuss in greater depth in Section 7.3. An example of non-normal residuals is shown in the second panel of Figure 7.13.

Constant variability. The variability of points around the least squares line remains roughly constant. An example of non-constant variability is shown in the third panel of Figure 7.13.

Be cautious about applying regression to data collected sequentially in what is called a **time series**. Such data may have an underlying structure that should be considered in a model and analysis. There are other instances where correlations within the data are important. This topic will be further discussed in Chapter 8.

⊙ **Exercise 7.11** Should we have concerns about applying least squares regression to the Elmhurst data in Figure 7.12?[8]

7.2.3 Finding the least squares line

For the Elmhurst data, we could write the equation of the least squares regression line as

$$\widehat{aid} = \beta_0 + \beta_1 \times family\_income$$

Here the equation is set up to predict gift aid based on a student's family income, which would be useful to students considering Elmhurst. These two values, β_0 and β_1, are the *parameters* of the regression line.

[7] There are applications where Criterion (7.9) may be more useful, and there are plenty of other criteria we might consider. However, this book only applies the least squares criterion.

[8] The trend appears to be linear, the data fall around the line with no obvious outliers, the variance is roughly constant. These are also not time series observations. Least squares regression can be applied to these data.

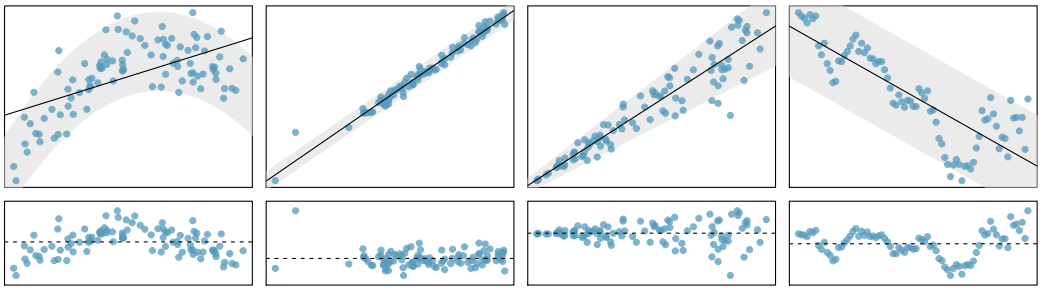


Figure 7.13: Four examples showing when the methods in this chapter are insufficient to apply to the data. In the left panel, a straight line does not fit the data. In the second panel, there are outliers; two points on the left are relatively distant from the rest of the data, and one of these points is very far away from the line. In the third panel, the variability of the data around the line increases with larger values of x. In the last panel, a time series data set is shown, where successive observations are highly correlated.

As in Chapters 4-6, the parameters are estimated using observed data. In practice, this estimation is done using a computer in the same way that other estimates, like a sample mean, can be estimated using a computer or calculator. However, we can also find the parameter estimates by applying two properties of the least squares line:

- The slope of the least squares line can be estimated by

$$b_1 = \frac{s_y}{s_x} R \tag{7.12}$$

  where R is the correlation between the two variables, and s_x and s_y are the sample standard deviations of the explanatory variable and response, respectively.

- If $\bar{x}$ is the mean of the horizontal variable (from the data) and $\bar{y}$ is the mean of the vertical variable, then the point $(\bar{x}, \bar{y})$ is on the least squares line.

b_0, b_1
Sample estimates of β_0, β_1

We use b_0 and b_1 to represent the point estimates of the parameters β_0 and β_1.

⊙ **Exercise 7.13** Table 7.14 shows the sample means for the family income and gift aid as \$101,800 and \$19,940, respectively. Plot the point (101.8, 19.94) on Figure 7.12 on page 324 to verify it falls on the least squares line (the solid line).[9]

	family income, in \$1000s ("x")	gift aid, in \$1000s ("y")
mean	$\bar{x} = 101.8$	$\bar{y} = 19.94$
sd	$s_x = 63.2$	$s_y = 5.46$
		$R = -0.499$

Table 7.14: Summary statistics for family income and gift aid.

[9] If you need help finding this location, draw a straight line up from the x-value of 100 (or thereabout). Then draw a horizontal line at 20 (or thereabout). These lines should intersect on the least squares line.

Exercise 7.14 Using the summary statistics in Table 7.14, compute the slope for the regression line of gift aid against family income.[10]

You might recall the **point-slope** form of a line from math class (another common form is *slope-intercept*). Given the slope of a line and a point on the line, (x_0, y_0), the equation for the line can be written as

$$y - y_0 = slope \times (x - x_0) \tag{7.15}$$

A common exercise to become more familiar with foundations of least squares regression is to use basic summary statistics and point-slope form to produce the least squares line.

TIP: Identifying the least squares line from summary statistics
To identify the least squares line from summary statistics:

- Estimate the slope parameter, b_1, using Equation (7.12).
- Noting that the point $(\bar{x}, \bar{y})$ is on the least squares line, use $x_0 = \bar{x}$ and $y_0 = \bar{y}$ along with the slope b_1 in the point-slope equation:

$$y - \bar{y} = b_1(x - \bar{x})$$

- Simplify the equation.

Example 7.16 Using the point $(101.8, 19.94)$ from the sample means and the slope estimate $b_1 = -0.0431$ from Exercise 7.14, find the least-squares line for predicting aid based on family income.

Apply the point-slope equation using $(101.8, 19.94)$ and the slope $b_1 = -0.0431$:

$$y - y_0 = b_1(x - x_0)$$
$$y - 19.94 = -0.0431(x - 101.8)$$

Expanding the right side and then adding 19.94 to each side, the equation simplifies:

$$\widehat{aid} = 24.3 - 0.0431 \times family\_income$$

Here we have replaced y with $\widehat{aid}$ and x with $family\_income$ to put the equation in context.

We mentioned earlier that a computer is usually used to compute the least squares line. A summary table based on computer output is shown in Table 7.15 for the Elmhurst data. The first column of numbers provides estimates for b_0 and b_1, respectively. Compare these to the result from Example 7.16.

[10] Apply Equation (7.12) with the summary statistics from Table 7.14 to compute the slope:

$$b_1 = \frac{s_y}{s_x} R = \frac{5.46}{63.2}(-0.499) = -0.0431$$

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	24.3193	1.2915	18.83	0.0000
family_income	-0.0431	0.0108	-3.98	0.0002

Table 7.15: Summary of least squares fit for the Elmhurst data. Compare the parameter estimates in the first column to the results of Example 7.16.

Example 7.17 Examine the second, third, and fourth columns in Table 7.15. Can you guess what they represent?

We'll describe the meaning of the columns using the second row, which corresponds to β_1. The first column provides the point estimate for β_1, as we calculated in an earlier example: -0.0431. The second column is a standard error for this point estimate: 0.0108. The third column is a t test statistic for the null hypothesis that $\beta_1 = 0$: $T = -3.98$. The last column is the p-value for the t test statistic for the null hypothesis $\beta_1 = 0$ and a two-sided alternative hypothesis: 0.0002. We will get into more of these details in Section 7.4.

Example 7.18 Suppose a high school senior is considering Elmhurst College. Can she simply use the linear equation that we have estimated to calculate her financial aid from the university?

She may use it as an estimate, though some qualifiers on this approach are important. First, the data all come from one freshman class, and the way aid is determined by the university may change from year to year. Second, the equation will provide an imperfect estimate. While the linear equation is good at capturing the trend in the data, no individual student's aid will be perfectly predicted.

7.2.4 Interpreting regression line parameter estimates

Interpreting parameters in a regression model is often one of the most important steps in the analysis.

Example 7.19 The slope and intercept estimates for the Elmhurst data are -0.0431 and 24.3. What do these numbers really mean?

Interpreting the slope parameter is helpful in almost any application. For each additional \$1,000 of family income, we would expect a student to receive a net difference of \$1,000 $\times$ (-0.0431) = $-$\$43.10 in aid on average, i.e. \$43.10 *less*. Note that a higher family income corresponds to less aid because the coefficient of family income is negative in the model. We must be cautious in this interpretation: while there is a real association, we cannot interpret a causal connection between the variables because these data are observational. That is, increasing a student's family income may not cause the student's aid to drop. (It would be reasonable to contact the college and ask if the relationship is causal, i.e. if Elmhurst College's aid decisions are partially based on students' family income.)

The estimated intercept $b_0 = 24.3$ (in \$1000s) describes the average aid if a student's family had no income. The meaning of the intercept is relevant to this application since the family income for some students at Elmhurst is \$0. In other applications, the intercept may have little or no practical value if there are no observations where x is near zero.

Interpreting parameters estimated by least squares
The slope describes the estimated difference in the y variable if the explanatory variable x for a case happened to be one unit larger. The intercept describes the average outcome of y if $x = 0$ *and* the linear model is valid all the way to $x = 0$, which in many applications is not the case.

7.2.5 Extrapolation is treacherous

When those blizzards hit the East Coast this winter, it proved to my satisfaction that global warming was a fraud. That snow was freezing cold. But in an alarming trend, temperatures this spring have risen. Consider this: On February 6^{th} it was 10 degrees. Today it hit almost 80. At this rate, by August it will be 220 degrees. So clearly folks the climate debate rages on.

Stephen Colbert
April 6th, 2010 [11]

Linear models can be used to approximate the relationship between two variables. However, these models have real limitations. Linear regression is simply a modeling framework. The truth is almost always much more complex than our simple line. For example, we do not know how the data outside of our limited window will behave.

Example 7.20 Use the model $\widehat{aid} = 24.3 - 0.0431 \times family\_income$ to estimate the aid of another freshman student whose family had income of \$1 million.

Recall that the units of family income are in \$1000s, so we want to calculate the aid for $family\_income = 1000$:

$$24.3 - 0.0431 \times family\_income = 24.3 - 0.0431 \times 1000 = -18.8$$

The model predicts this student will have -\$18,800 in aid (!). Elmhurst College cannot (or at least does not) require any students to pay extra on top of tuition to attend.

Applying a model estimate to values outside of the realm of the original data is called **extrapolation**. Generally, a linear model is only an approximation of the real relationship between two variables. If we extrapolate, we are making an unreliable bet that the approximate linear relationship will be valid in places where it has not been analyzed.

7.2.6 Using R^2 to describe the strength of a fit

We evaluated the strength of the linear relationship between two variables earlier using the correlation, R. However, it is more common to explain the strength of a linear fit using R^2, called **R-squared**. If provided with a linear model, we might like to describe how closely the data cluster around the linear fit.

The R^2 of a linear model describes the amount of variation in the response that is explained by the least squares line. For example, consider the Elmhurst data, shown in Figure 7.16. The variance of the response variable, aid received, is $s^2_{aid} = 29.8$. However, if we apply our least squares line, then this model reduces our uncertainty in predicting

[11] http://www.colbertnation.com/the-colbert-report-videos/269929/

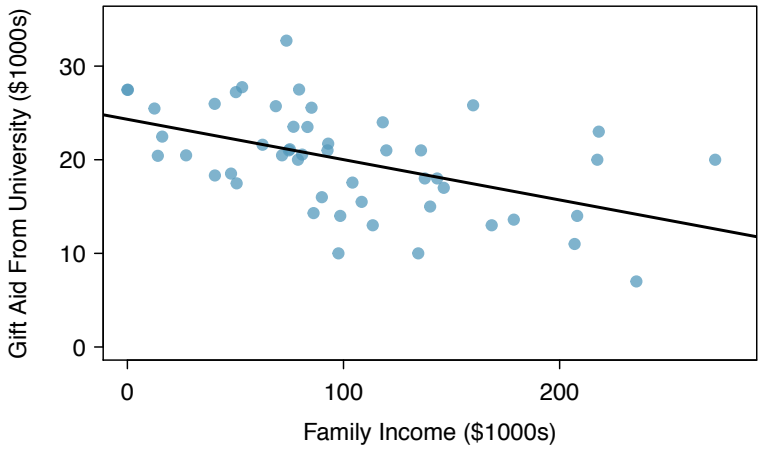


Figure 7.16: Gift aid and family income for a random sample of 50 freshman students from Elmhurst College, shown with the least squares regression line.

aid using a student's family income. The variability in the residuals describes how much variation remains after using the model: $s^2_{RES} = 22.4$. In short, there was a reduction of

$$\frac{s^2_{aid} - s^2_{RES}}{s^2_{aid}} = \frac{29.8 - 22.4}{29.8} = \frac{7.5}{29.8} = 0.25$$

or about 25% in the data's variation by using information about family income for predicting aid using a linear model. This corresponds exactly to the R-squared value:

$$R = -0.499 \qquad\qquad R^2 = 0.25$$

⊙ **Exercise 7.21** If a linear model has a very strong negative relationship with a correlation of -0.97, how much of the variation in the response is explained by the explanatory variable?[12]

7.2.7 Categorical predictors with two levels

Categorical variables are also useful in predicting outcomes. Here we consider a categorical predictor with two levels (recall that a *level* is the same as a *category*). We'll consider Ebay auctions for a video game, *Mario Kart* for the Nintendo Wii, where both the total price of the auction and the condition of the game were recorded.[13] Here we want to predict total price based on game condition, which takes values `used` and `new`. A plot of the auction data is shown in Figure 7.17.

To incorporate the game condition variable into a regression equation, we must convert the categories into a numerical form. We will do so using an **indicator variable** called `cond_new`, which takes value 1 when the game is new and 0 when the game is used. Using this indicator variable, the linear model may be written as

$$\widehat{price} = \beta_0 + \beta_1 \times \texttt{cond\_new}$$

[12] About $R^2 = (-0.97)^2 = 0.94$ or 94% of the variation is explained by the linear model.
[13] These data were collected in Fall 2009 and may be found at openintro.org.

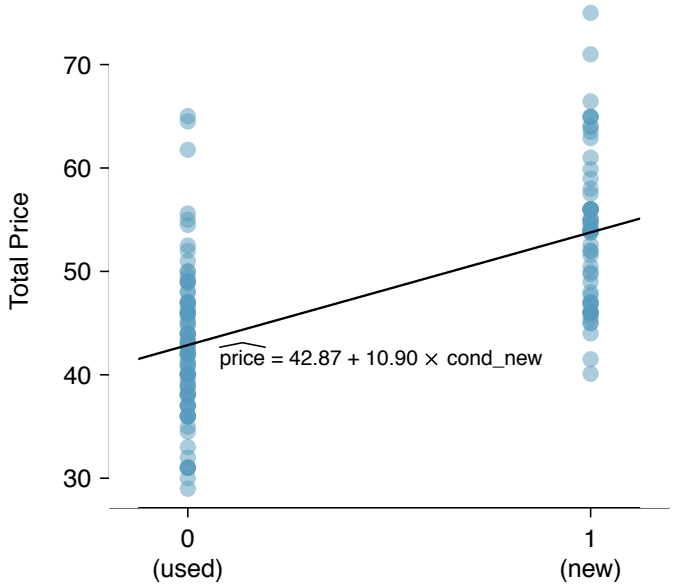


Figure 7.17: Total auction prices for the video game *Mario Kart*, divided into used ($x = 0$) and new ($x = 1$) condition games. The least squares regression line is also shown.

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	42.87	0.81	52.67	0.0000
cond_new	10.90	1.26	8.66	0.0000

Table 7.18: Least squares regression summary for the final auction price against the condition of the game.

The fitted model is summarized in Table 7.18, and the model with its parameter estimates is given as

$$\widehat{price} = 42.87 + 10.90 \times \texttt{cond\_new}$$

For categorical predictors with just two levels, the linearity assumption will always be satisfied. However, we must evaluate whether the residuals in each group are approximately normal and have approximately equal variance. As can be seen in Figure 7.17, both of these conditions are reasonably satisfied by the auction data.

Example 7.22 Interpret the two parameters estimated in the model for the price of *Mario Kart* in eBay auctions.

The intercept is the estimated price when `cond_new` takes value 0, i.e. when the game is in used condition. That is, the average selling price of a used version of the game is $42.87.

The slope indicates that, on average, new games sell for about $10.90 more than used games.

TIP: Interpreting model estimates for categorical predictors.
The estimated intercept is the value of the response variable for the first category (i.e. the category corresponding to an indicator value of 0). The estimated slope is the average change in the response variable between the two categories.

We'll elaborate further on this Ebay auction data in Chapter 8, where we examine the influence of many predictor variables simultaneously using multiple regression. In multiple regression, we will consider the association of auction price with regard to each variable while controlling for the influence of other variables. This is especially important since some of the predictors are associated. For example, auctions with games in new condition also often came with more accessories.

7.3 Types of outliers in linear regression

In this section, we identify criteria for determining which outliers are important and influential.

Outliers in regression are observations that fall far from the "cloud" of points. These points are especially important because they can have a strong influence on the least squares line.

Example 7.23 There are six plots shown in Figure 7.19 along with the least squares line and residual plots. For each scatterplot and residual plot pair, identify any obvious outliers and note how they influence the least squares line. Recall that an outlier is any point that doesn't appear to belong with the vast majority of the other points.

(1) There is one outlier far from the other points, though it only appears to slightly influence the line.

(2) There is one outlier on the right, though it is quite close to the least squares line, which suggests it wasn't very influential.

(3) There is one point far away from the cloud, and this outlier appears to pull the least squares line up on the right; examine how the line around the primary cloud doesn't appear to fit very well.

(4) There is a primary cloud and then a small secondary cloud of four outliers. The secondary cloud appears to be influencing the line somewhat strongly, making the least square line fit poorly almost everywhere. There might be an interesting explanation for the dual clouds, which is something that could be investigated.

(5) There is no obvious trend in the main cloud of points and the outlier on the right appears to largely control the slope of the least squares line.

(6) There is one outlier far from the cloud, however, it falls quite close to the least squares line and does not appear to be very influential.

Examine the residual plots in Figure 7.19. You will probably find that there is some trend in the main clouds of (3) and (4). In these cases, the outliers influenced the slope of the least squares lines. In (5), data with no clear trend were assigned a line with a large trend simply due to one outlier (!).

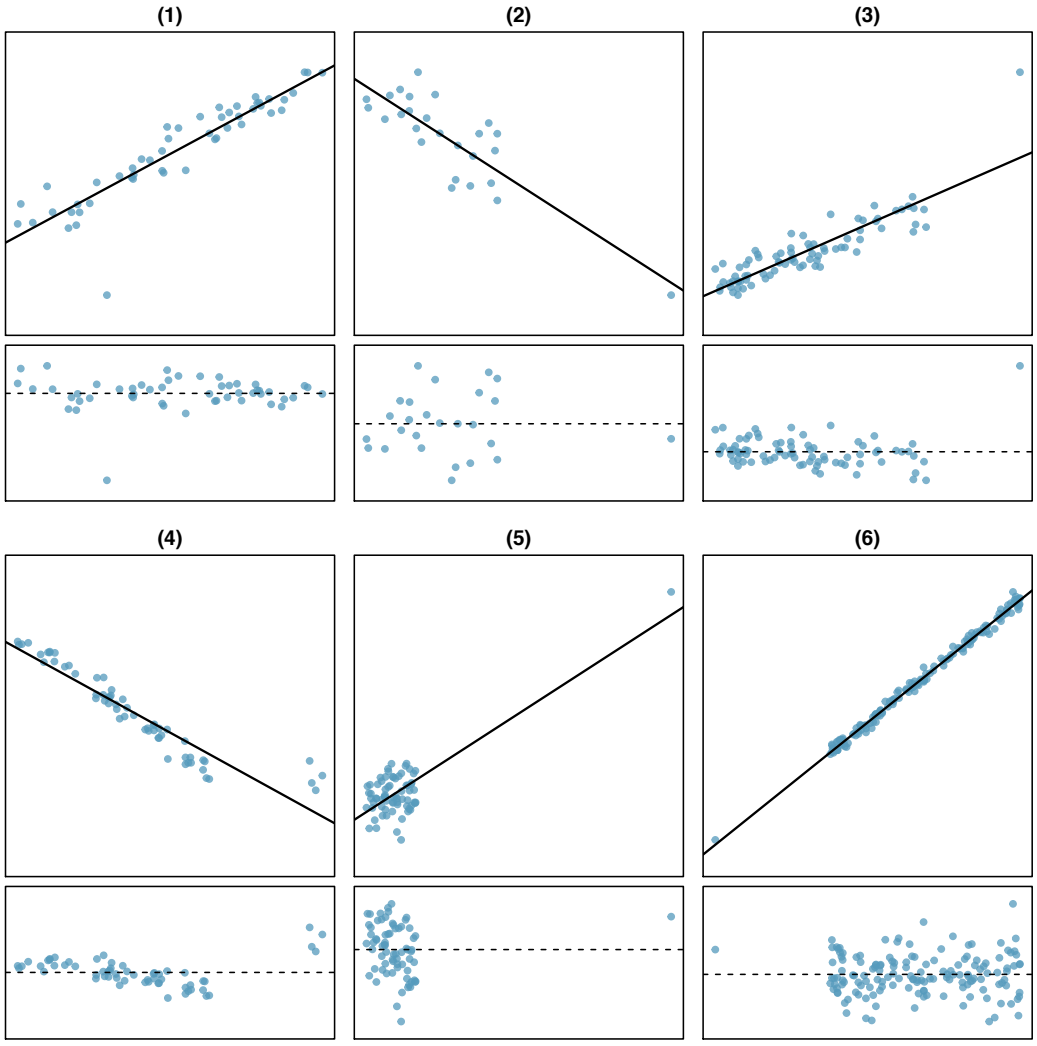


Figure 7.19: Six plots, each with a least squares line and residual plot. All data sets have at least one outlier.

Leverage
Points that fall horizontally away from the center of the cloud tend to pull harder on the line, so we call them points with **high leverage**.

Points that fall horizontally far from the line are points of high leverage; these points can strongly influence the slope of the least squares line. If one of these high leverage points does appear to actually invoke its influence on the slope of the line – as in cases (3), (4), and (5) of Example 7.23 – then we call it an **influential point**. Usually we can say a point is influential if, had we fitted the line without it, the influential point would have been unusually far from the least squares line.

It is tempting to remove outliers. Don't do this without a very good reason. Models that ignore exceptional (and interesting) cases often perform poorly. For instance, if a financial firm ignored the largest market swings – the "outliers" – they would soon go bankrupt by making poorly thought-out investments.

Caution: Don't ignore outliers when fitting a final model
If there are outliers in the data, they should not be removed or ignored without a good reason. Whatever final model is fit to the data would not be very helpful if it ignores the most exceptional cases.

Caution: Outliers for a categorical predictor with two levels
Be cautious about using a categorical predictor when one of the levels has very few observations. When this happens, those few observations become influential points.

7.4 Inference for linear regression

In this section we discuss uncertainty in the estimates of the slope and y-intercept for a regression line. Just as we identified standard errors for point estimates in previous chapters, we first discuss standard errors for these new estimates. However, in the case of regression, we will identify standard errors using statistical software.

7.4.1 Midterm elections and unemployment

Elections for members of the United States House of Representatives occur every two years, coinciding every four years with the U.S. Presidential election. The set of House elections occurring during the middle of a Presidential term are called midterm elections. In America's two-party system, one political theory suggests the higher the unemployment rate, the worse the President's party will do in the midterm elections.

To assess the validity of this claim, we can compile historical data and look for a connection. We consider every midterm election from 1898 to 2010, with the exception of those elections during the Great Depression. Figure 7.20 shows these data and the least-squares regression line:

$$\begin{aligned}&\%\text{ change in House seats for President's party}\\ &\quad = -6.71 - 1.00 \times (\text{unemployment rate})\end{aligned}$$

We consider the percent change in the number of seats of the President's party (e.g. percent change in the number of seats for Democrats in 2010) against the unemployment rate.

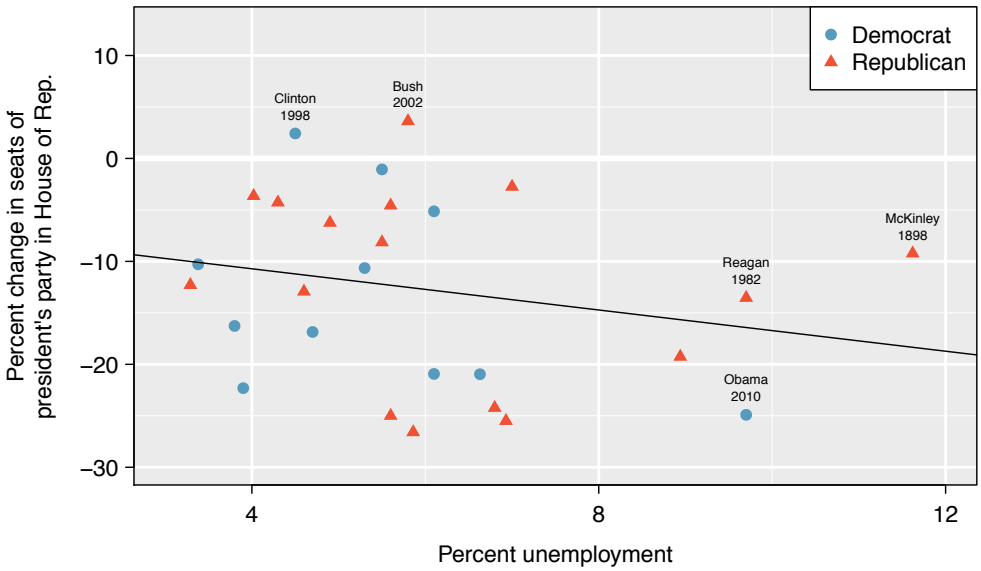


Figure 7.20: The percent change in House seats for the President's party in each election from 1898 to 2010 plotted against the unemployment rate. The two points for the Great Depression have been removed, and a least squares regression line has been fit to the data.

Examining the data, there are no clear deviations from linearity, the constant variance condition, or in the normality of residuals (though we don't examine a normal probability plot here). While the data are collected sequentially, a separate analysis was used to check for any apparent correlation between successive observations; no such correlation was found.

⊙ **Exercise 7.24** The data for the Great Depression (1934 and 1938) were removed because the unemployment rate was 21% and 18%, respectively. Do you agree that they should be removed for this investigation? Why or why not?[14]

There is a negative slope in the line shown in Figure 7.20. However, this slope (and the y-intercept) are only estimates of the parameter values. We might wonder, is this convincing evidence that the "true" linear model has a negative slope? That is, do the data provide strong evidence that the political theory is accurate? We can frame this investigation into a one-sided statistical hypothesis test:

H_0: $\beta_1 = 0$. The true linear model has slope zero.

H_A: $\beta_1 < 0$. The true linear model has a slope less than zero. The higher the unemployment, the greater the loss for the President's party in the House of Representatives.

We would reject H_0 in favor of H_A if the data provide strong evidence that the true slope parameter is less than zero. To assess the hypotheses, we identify a standard error for the estimate, compute an appropriate test statistic, and identify the p-value.

[14]We will provide two considerations. Each of these points would have very high leverage on any least-squares regression line, and years with such high unemployment may not help us understand what would happen in other years where the unemployment is only modestly high. On the other hand, these are exceptional cases, and we would be discarding important information if we exclude them from a final analysis.

7.4.2 Understanding regression output from software

Just like other point estimates we have seen before, we can compute a standard error and test statistic for b_1. We will generally label the test statistic using a T, since it follows the t distribution.

We will rely on statistical software to compute the standard error and leave the explanation of how this standard error is determined to a second or third statistics course. Table 7.21 shows software output for the least squares regression line in Figure 7.20. The row labeled *unemp* represents the information for the slope, which is the coefficient of the unemployment variable.

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-6.7142	5.4567	-1.23	0.2300
unemp	-1.0010	0.8717	-1.15	0.2617
				$df = 25$

Table 7.21: Output from statistical software for the regression line modeling the midterm election losses for the President's party as a response to unemployment.

Example 7.25 What do the first and second columns of Table 7.21 represent?

The entries in the first column represent the least squares estimates, b_0 and b_1, and the values in the second column correspond to the standard errors of each estimate.

We previously used a t test statistic for hypothesis testing in the context of numerical data. Regression is very similar. In the hypotheses we consider, the null value for the slope is 0, so we can compute the test statistic using the T (or Z) score formula:

$$T = \frac{\text{estimate} - \text{null value}}{\text{SE}} = \frac{-1.0010 - 0}{0.8717} = -1.15$$

We can look for the one-sided p-value – shown in Figure 7.22 – using the probability table for the t distribution in Appendix B.2 on page 410.

Example 7.26 Table 7.21 offers the degrees of freedom for the test statistic T: $df = 25$. Identify the p-value for the hypothesis test.

Looking in the 25 degrees of freedom row in Appendix B.2, we see that the absolute value of the test statistic is smaller than any value listed, which means the tail area and therefore also the p-value is larger than 0.100 (one tail!). Because the p-value is so large, we fail to reject the null hypothesis. That is, the data do not provide convincing evidence that a higher unemployment rate has any correspondence with smaller or larger losses for the President's party in the House of Representatives in midterm elections.

We could have identified the t test statistic from the software output in Table 7.21, shown in the second row (unemp) and third column (t value). The entry in the second row and last column in Table 7.21 represents the p-value for the two-sided hypothesis test where the null value is zero. The corresponding one-sided test would have a p-value half of the listed value.

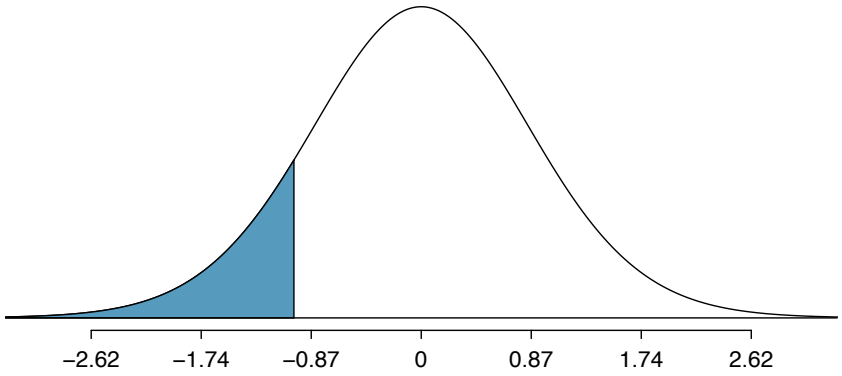


Figure 7.22: The distribution shown here is the sampling distribution for b_1, if the null hypothesis was true. The shaded tail represents the p-value for the hypothesis test evaluating whether there is convincing evidence that higher unemployment corresponds to a greater loss of House seats for the President's party during a midterm election.

Inference for regression

We usually rely on statistical software to identify point estimates and standard errors for parameters of a regression line. After verifying conditions hold for fitting a line, we can use the methods learned in Section 5.3 for the t distribution to create confidence intervals for regression parameters or to evaluate hypothesis tests.

Caution: Don't carelessly use the p-value from regression output

The last column in regression output often lists p-values for one particular hypothesis: a two-sided test where the null value is zero. If your test is one-sided and the point estimate is in the direction of H_A, then you can halve the software's p-value to get the one-tail area. If neither of these scenarios match your hypothesis test, be cautious about using the software output to obtain the p-value.

Example 7.27 Examine Figure 7.16 on page 330, which relates the Elmhurst College aid and student family income. How sure are you that the slope is statistically significantly different from zero? That is, do you think a formal hypothesis test would reject the claim that the true slope of the line should be zero?

While the relationship between the variables is not perfect, there is an evident decreasing trend in the data. This suggests the hypothesis test will reject the null claim that the slope is zero.

⊙ **Exercise 7.28** Table 7.23 shows statistical software output from fitting the least squares regression line shown in Figure 7.16. Use this output to formally evaluate the following hypotheses. H_0: The true coefficient for family income is zero. H_A: The true coefficient for family income is not zero.[15]

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	24.3193	1.2915	18.83	0.0000
family_income	-0.0431	0.0108	-3.98	0.0002
				$df = 48$

Table 7.23: Summary of least squares fit for the Elmhurst College data.

TIP: Always check assumptions
If conditions for fitting the regression line do not hold, then the methods presented here should not be applied. The standard error or distribution assumption of the point estimate – assumed to be normal when applying the t test statistic – may not be valid.

7.4.3 An alternative test statistic

We considered the t test statistic as a way to evaluate the strength of evidence for a hypothesis test in Section 7.4.2. However, we could focus on R^2. Recall that R^2 described the proportion of variability in the response variable (y) explained by the explanatory variable (x). If this proportion is large, then this suggests a linear relationship exists between the variables. If this proportion is small, then the evidence provided by the data may not be convincing.

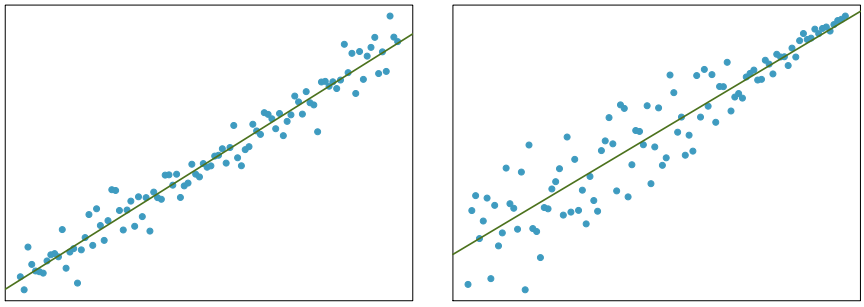
This concept – considering the amount of variability in the response variable explained by the explanatory variable – is a key component in some statistical techniques. The *analysis of variance (ANOVA)* technique introduced in Section 5.5 uses this general principle. The method states that if enough variability is explained away by the categories, then we would conclude the mean varied between the categories. On the other hand, we might not be convinced if only a little variability is explained. ANOVA can be further employed in advanced regression modeling to evaluate the inclusion of explanatory variables, though we leave these details to a later course.

[15] We look in the second row corresponding to the family income variable. We see the point estimate of the slope of the line is -0.0431, the standard error of this estimate is 0.0108, and the t test statistic is -3.98. The p-value corresponds exactly to the two-sided test we are interested in: 0.0002. The p-value is so small that we reject the null hypothesis and conclude that family income and financial aid at Elmhurst College for freshman entering in the year 2011 are negatively correlated and the true slope parameter is indeed less than 0, just as we believed in Example 7.27.

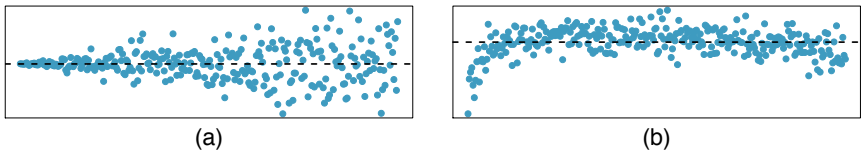
7.5 Exercises

7.5.1 Line fitting, residuals, and correlation

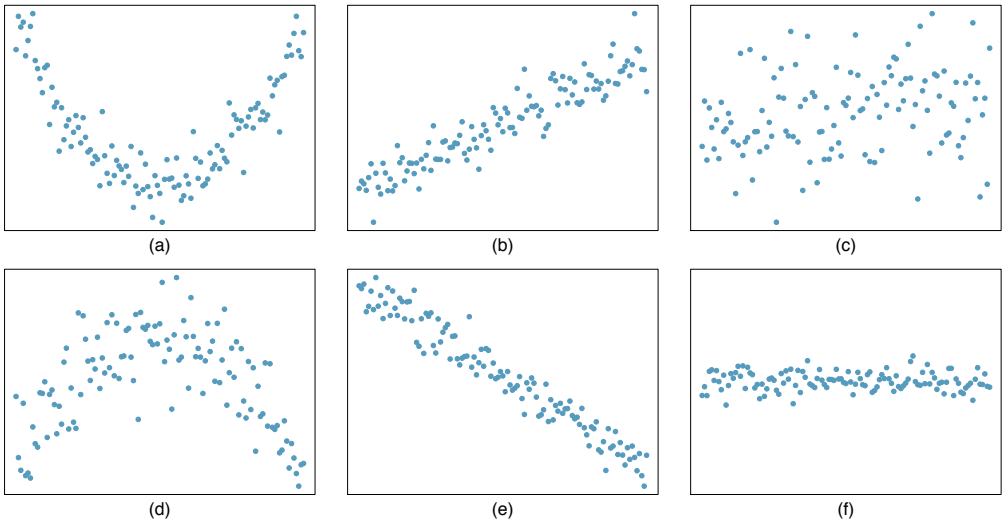
7.1 Visualize the residuals. The scatterplots shown below each have a superimposed regression line. If we were to construct a residual plot (residuals versus x) for each, describe what those plots would look like.



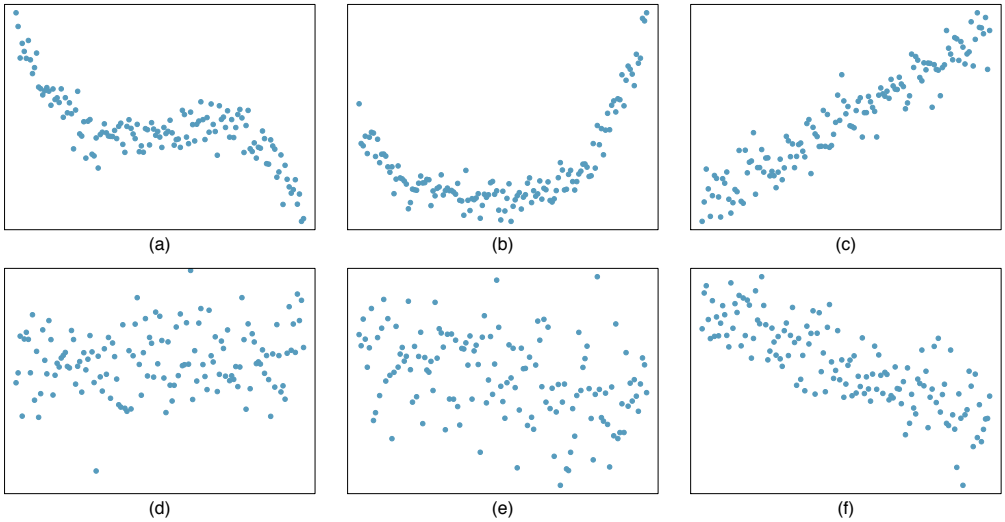
7.2 Trends in the residuals. Shown below are two plots of residuals remaining after fitting a linear model to two different sets of data. Describe important features and determine if a linear model would be appropriate for these data. Explain your reasoning.



7.3 Identify relationships, Part I. For each of the six plots, identify the strength of the relationship (e.g. weak, moderate, or strong) in the data and whether fitting a linear model would be reasonable.



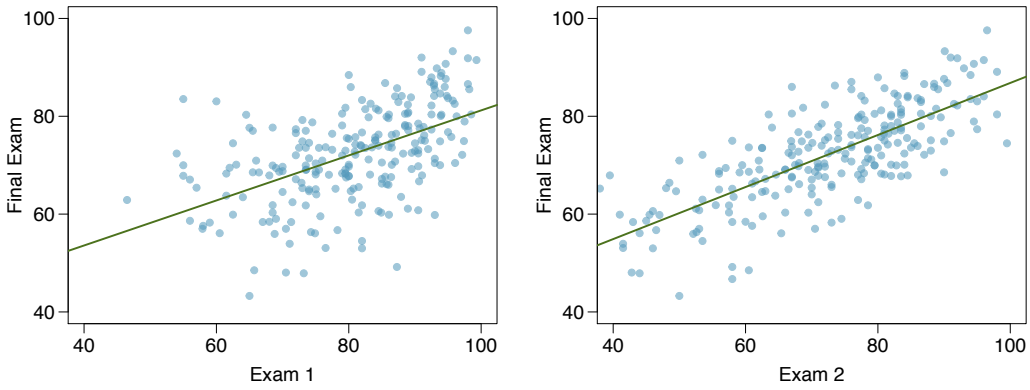
7.4 Identify relationships, Part I. For each of the six plots, identify the strength of the relationship (e.g. weak, moderate, or strong) in the data and whether fitting a linear model would be reasonable.



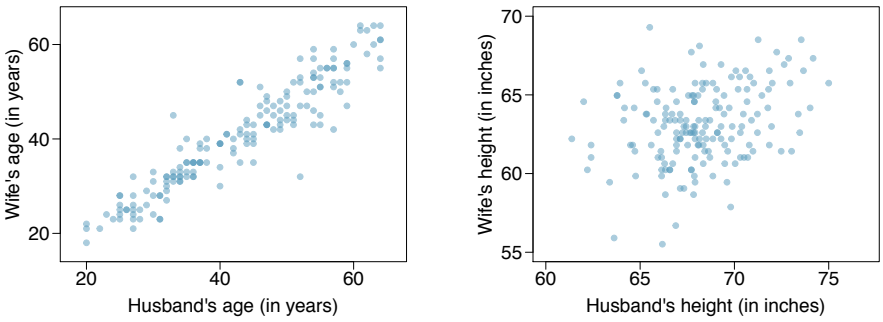
7.5 Exams and grades. The two scatterplots below show the relationship between final and mid-semester exam grades recorded during several years for a Statistics course at a university.

(a) Based on these graphs, which of the two exams has the strongest correlation with the final exam grade? Explain.

(b) Can you think of a reason why the correlation between the exam you chose in part (a) and the final exam is higher?



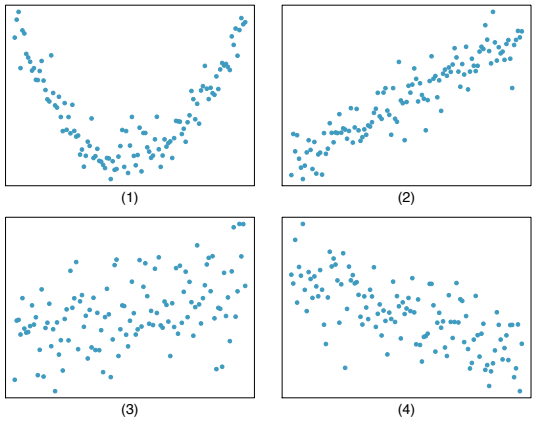
7.6 Husbands and wives, Part I. The Great Britain Office of Population Census and Surveys once collected data on a random sample of 170 married couples in Britain, recording the age (in years) and heights (converted here to inches) of the husbands and wives.[16] The scatterplot on the left shows the wife's age plotted against her husband's age, and the plot on the right shows wife's height plotted against husband's height.



(a) Describe the relationship between husbands' and wives' ages.
(b) Describe the relationship between husbands' and wives' heights.
(c) Which plot shows a stronger correlation? Explain your reasoning.
(d) Data on heights were originally collected in centimeters, and then converted to inches. Does this conversion affect the correlation between husbands' and wives' heights?

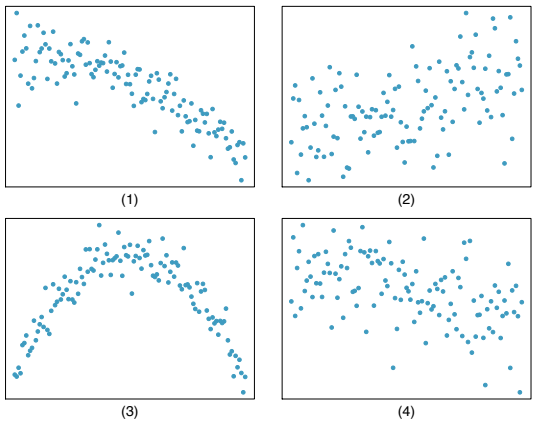
7.7 Match the correlation, Part I. Match the calculated correlations to the corresponding scatterplot.

(a) $R = -0.7$
(b) $R = 0.45$
(c) $R = 0.06$
(d) $R = 0.92$



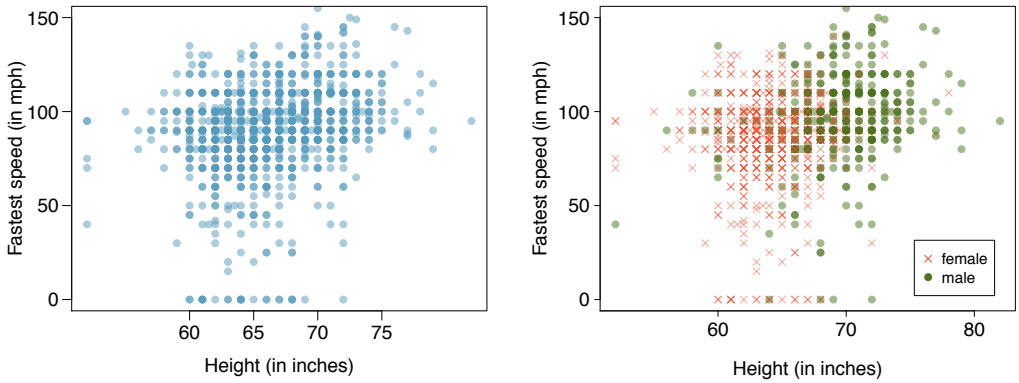
7.8 Match the correlation, Part II. Match the calculated correlations to the corresponding scatterplot.

(a) $R = 0.49$
(b) $R = -0.48$
(c) $R = -0.03$
(d) $R = -0.85$



[16] D.J. Hand. *A handbook of small data sets.* Chapman & Hall/CRC, 1994.

7.9 Speed and height. 1,302 UCLA students were asked to fill out a survey where they were asked about their height, fastest speed they have ever driven, and gender. The scatterplot on the left displays the relationship between height and fastest speed, and the scatterplot on the right displays the breakdown by gender in this relationship.

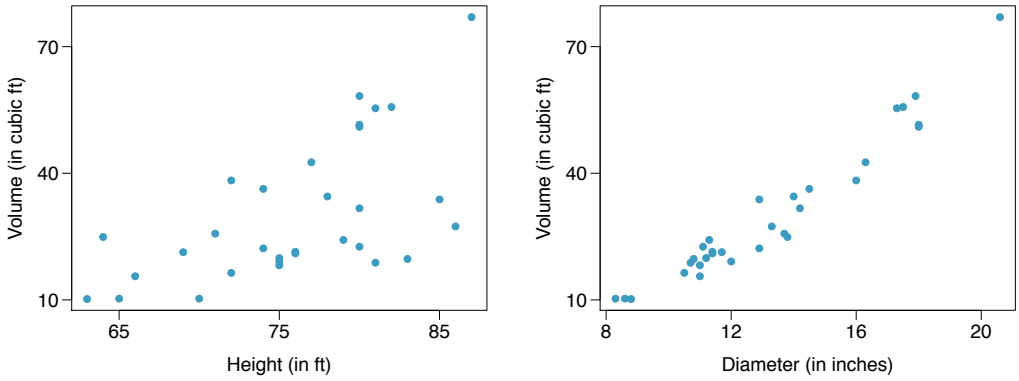


(a) Describe the relationship between height and fastest speed.

(b) Why do you think these variables are positively associated?

(c) What role does gender play in the relationship between height and fastest driving speed?

7.10 Trees. The scatterplots below show the relationship between height, diameter, and volume of timber in 31 felled black cherry trees. The diameter of the tree is measured 4.5 feet above the ground.[17]



(a) Describe the relationship between volume and height of these trees.

(b) Describe the relationship between volume and diameter of these trees.

(c) Suppose you have height and diameter measurements for another black cherry tree. Which of these variables would be preferable to use to predict the volume of timber in this tree using a simple linear regression model? Explain your reasoning.

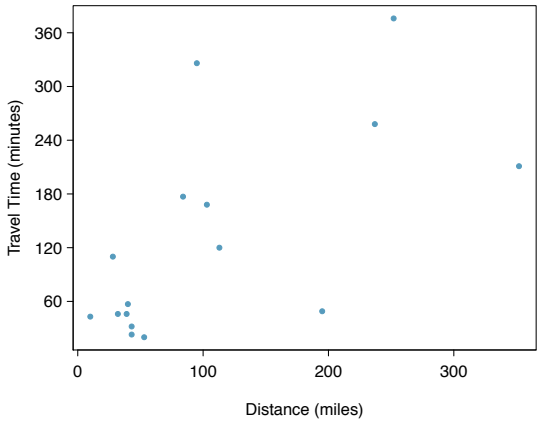
[17]Source: R Dataset, http://stat.ethz.ch/R-manual/R-patched/library/datasets/html/trees.html.

7.11 The Coast Starlight, Part I. The Coast Starlight Amtrak train runs from Seattle to Los Angeles. The scatterplot below displays the distance between each stop (in miles) and the amount of time it takes to travel from one stop to another (in minutes).

(a) Describe the relationship between distance and travel time.

(b) How would the relationship change if travel time was instead measured in hours, and distance was instead measured in kilometers?

(c) Correlation between travel time (in miles) and distance (in minutes) is $R = 0.636$. What is the correlation between travel time (in kilometers) and distance (in hours)?

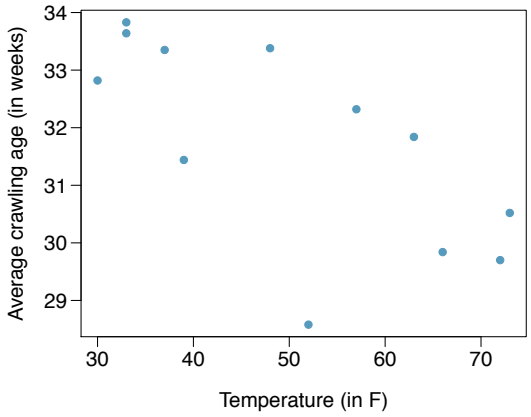


7.12 Crawling babies, Part I. A study conducted at the University of Denver investigated whether babies take longer to learn to crawl in cold months, when they are often bundled in clothes that restrict their movement, than in warmer months.[18] Infants born during the study year were split into twelve groups, one for each birth month. We consider the average crawling age of babies in each group against the average temperature when the babies are six months old (that's when babies often begin trying to crawl). Temperature is measured in degrees Fahrenheit (°F) and age is measured in weeks.

(a) Describe the relationship between temperature and crawling age.

(b) How would the relationship change if temperature was measured in degrees Celsius (°C) and age was measured in months?

(c) The correlation between temperature in °F and age in weeks was $R = -0.70$. If we converted the temperature to °C and age to months, what would the correlation be?

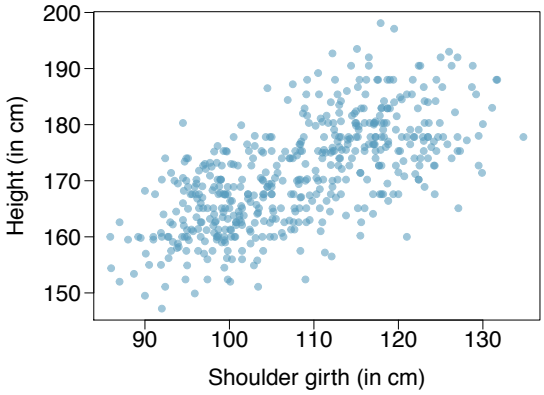


[18] J.B. Benson. "Season of birth and onset of locomotion: Theoretical and methodological implications". In: *Infant behavior and development* 16.1 (1993), pp. 69–81. ISSN: 0163-6383.

7.13 Body measurements, Part I. Researchers studying anthropometry collected body girth measurements and skeletal diameter measurements, as well as age, weight, height and gender for 507 physically active individuals.[19] The scatterplot below shows the relationship between height and shoulder girth (over deltoid muscles), both measured in centimeters.

(a) Describe the relationship between shoulder girth and height.

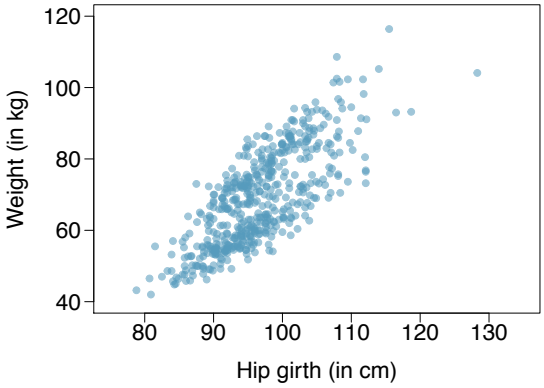
(b) How would the relationship change if shoulder girth was measured in inches while the units of height remained in centimeters?



7.14 Body measurements, Part II. The scatterplot below shows the relationship between weight measured in kilograms and hip girth measured in centimeters from the data described in Exercise 7.13.

(a) Describe the relationship between hip girth and weight.

(b) How would the relationship change if weight was measured in pounds while the units for hip girth remained in centimeters?



7.15 Correlation, Part I. What would be the correlation between the ages of husbands and wives if men always married woman who were

(a) 3 years younger than themselves?

(b) 2 years older than themselves?

(c) half as old as themselves?

7.16 Correlation, Part II. What would be the correlation between the annual salaries of males and females at a company if for a certain type of position men always made

(a) $5,000 more than women?

(b) 25% more than women?

(c) 15% less than women?

[19] G. Heinz et al. "Exploring relationships in body dimensions". In: *Journal of Statistics Education* 11.2 (2003).

7.5.2 Fitting a line by least squares regression

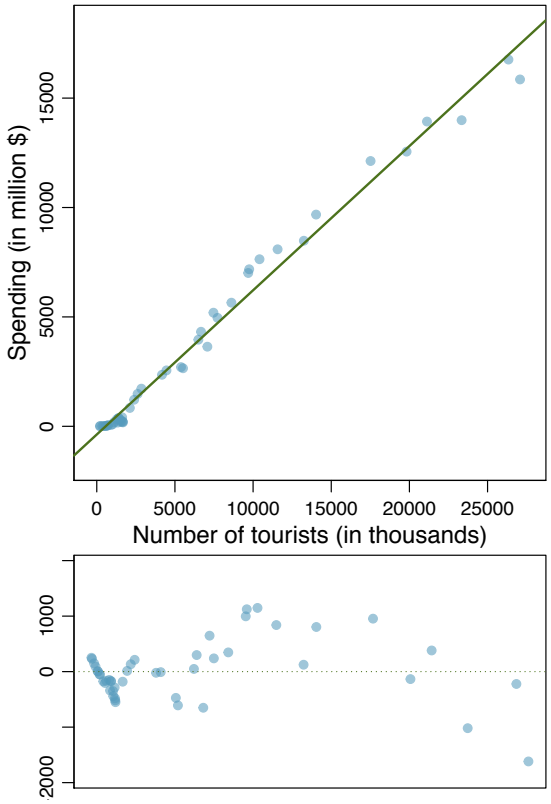
7.17 Tourism spending. The Association of Turkish Travel Agencies reports the number of foreign tourists visiting Turkey and tourist spending by year.[20] The scatterplot below shows the relationship between these two variables along with the least squares fit.

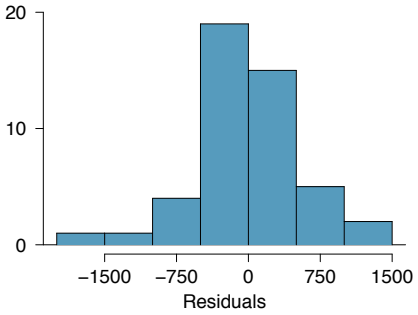
(a) Describe the relationship between number of tourists and spending.

(b) What are the explanatory and response variables?

(c) Why might we want to fit a regression line to these data?

(d) Do the data meet the conditions required for fitting a least squares line? In addition to the scatterplot, use the residual plot and histogram to answer this question.





[20] Association of Turkish Travel Agencies, Foreign Visitors Figure & Tourist Spendings By Years.

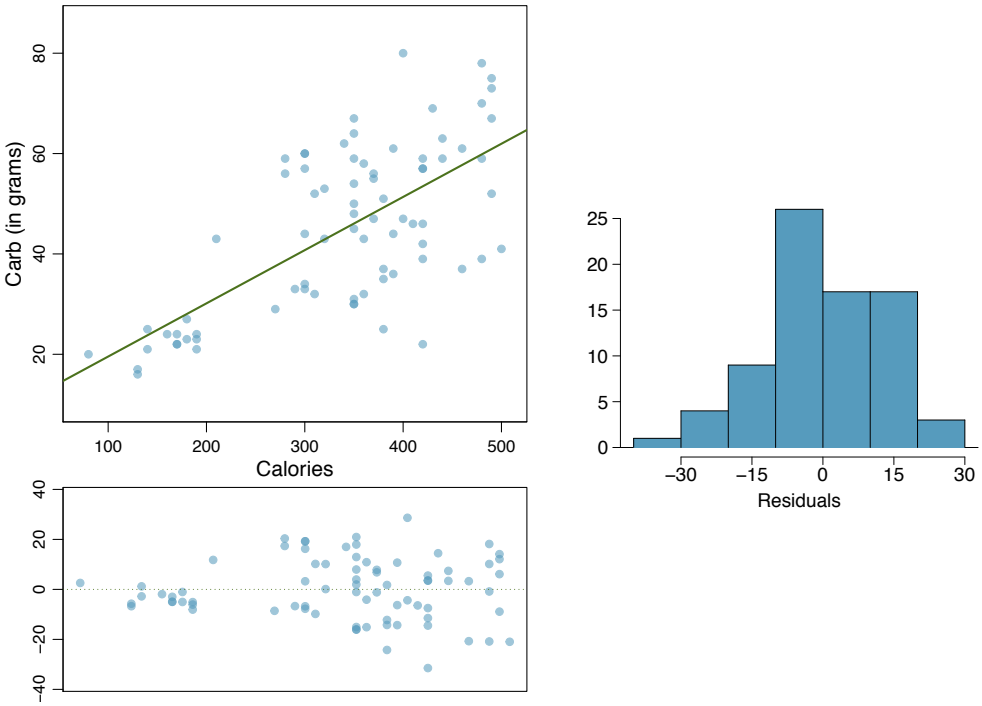
7.18 Nutrition at Starbucks, Part I. The scatterplot below shows the relationship between the number of calories and amount of carbohydrates (in grams) Starbucks food menu items contain.[21] Since Starbucks only lists the number of calories on the display items, we are interested in predicting the amount of carbs a menu item has based on its calorie content.

(a) Describe the relationship between number of calories and amount of carbohydrates (in grams) that Starbucks food menu items contain.

(b) In this scenario, what are the explanatory and response variables?

(c) Why might we want to fit a regression line to these data?

(d) Do these data meet the conditions required for fitting a least squares line?



7.19 The Coast Starlight, Part II. Exercise 7.11 introduces data on the Coast Starlight Amtrak train that runs from Seattle to Los Angeles. The mean travel time from one stop to the next on the Coast Starlight is 129 mins, with a standard deviation of 113 minutes. The mean distance traveled from one stop to the next is 107 miles with a standard deviation of 99 miles. The correlation between travel time and distance is 0.636.

(a) Write the equation of the regression line for predicting travel time.

(b) Interpret the slope and the intercept in this context.

(c) Calculate R^2 of the regression line for predicting travel time from distance traveled for the Coast Starlight, and interpret R^2 in the context of the application.

(d) The distance between Santa Barbara and Los Angeles is 103 miles. Use the model to estimate the time it takes for the Starlight to travel between these two cities.

(e) It actually takes the the Coast Starlight about 168 mins to travel from Santa Barbara to Los Angeles. Calculate the residual and explain the meaning of this residual value.

(f) Suppose Amtrak is considering adding a stop to the Coast Starlight 500 miles away from Los Angeles. Would it be appropriate to use this linear model to predict the travel time from Los Angeles to this point?

[21] Source: Starbucks.com, collected on March 10, 2011, http://www.starbucks.com/menu/nutrition.

7.20 Body measurements, Part III. Exercise 7.13 introduces data on shoulder girth and height of a group of individuals. The mean shoulder girth is 108.20 cm with a standard deviation of 10.37 cm. The mean height is 171.14 cm with a standard deviation of 9.41 cm. The correlation between height and shoulder girth is 0.67.

(a) Write the equation of the regression line for predicting height.

(b) Interpret the slope and the intercept in this context.

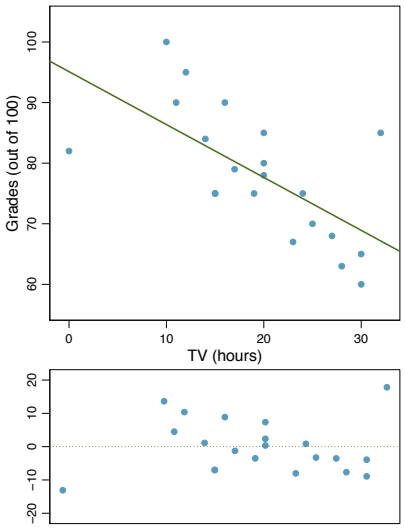
(c) Calculate R^2 of the regression line for predicting height from shoulder girth, and interpret it in the context of the application.

(d) A randomly selected student from your class has a shoulder girth of 100 cm. Predict the height of this student using the model.

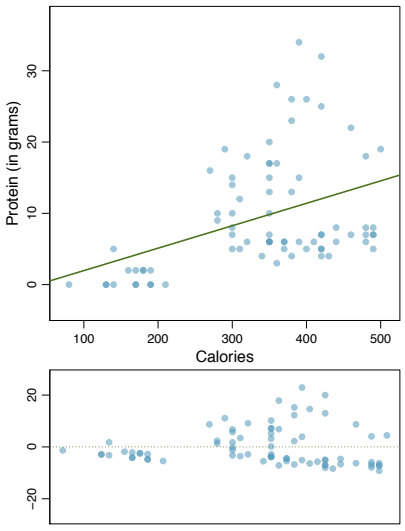
(e) The student from part (d) is 160 cm tall. Calculate the residual, and explain what this residual means.

(f) A one year old has a shoulder girth of 56 cm. Would it be appropriate to use this linear model to predict the height of this child?

7.21 Grades and TV. Data were collected on the number of hours per week students watch TV and the grade they earned in a biology class on a 100 point scale. Based on the scatterplot and the residual plot provided, describe the relationship between the two variables, and determine if a simple linear model is appropriate to predict a student's grade from the number of hours per week the student watches TV.



7.22 Nutrition at Starbucks, Part II. Exercise 7.18 introduced a data set on nutrition information on Starbucks food menu items. Based on the scatterplot and the residual plot provided, describe the relationship between the protein content and calories of these menu items, and determine if a simple linear model is appropriate to predict amount of protein from the number of calories.



7.23 Helmets and lunches. The scatterplot shows the relationship between socioeconomic status measured as the percentage of children in a neighborhood receiving reduced-fee lunches at school (`lunch`) and the percentage of bike riders in the neighborhood wearing helmets (`helmet`). The average percentage of children receiving reduced-fee lunches is 30.8% with a standard deviation of 26.7% and the average percentage of bike riders wearing helmets is 38.8% with a standard deviation of 16.9%.

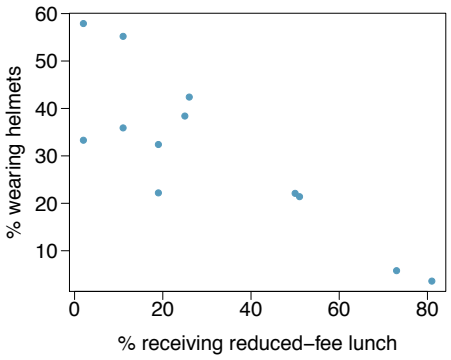
(a) If the R^2 for the least-squares regression line for these data is 72%, what is the correlation between `lunch` and `helmet`?

(b) Calculate the slope and intercept for the least-squares regression line for these data.

(c) Interpret the intercept of the least-squares regression line in the context of the application.

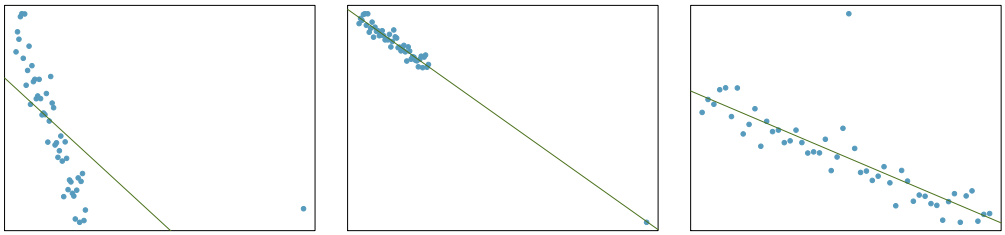
(d) Interpret the slope of the least-squares regression line in the context of the application.

(e) What would the value of the residual be for a neighborhood where 40% of the children receive reduced-fee lunches and 40% of the bike riders wear helmets? Interpret the meaning of this residual in the context of the application.

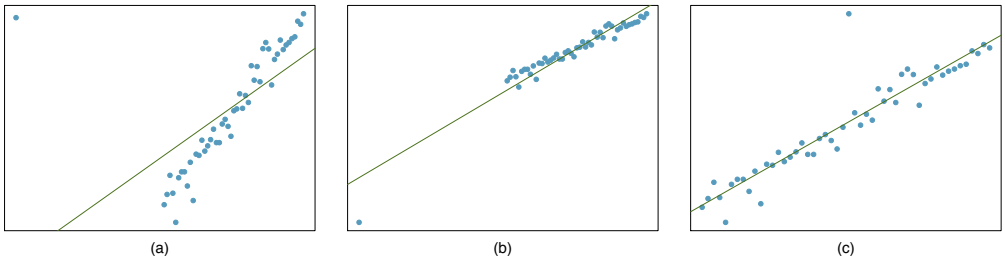


7.5.3 Types of outliers in linear regression

7.24 Outliers, Part I. Identify the outliers in the scatterplots shown below, and determine what type of outliers they are. Explain your reasoning.



7.25 Outliers, Part II. Identify the outliers in the scatterplots shown below and determine what type of outliers they are. Explain your reasoning.

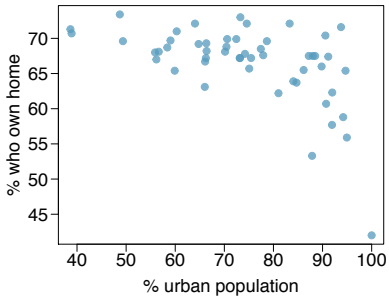


7.26 Crawling babies, Part II. Exercise 7.12 introduces data on the average monthly temperature during the month babies first try to crawl (about 6 months after birth) and the average first crawling age for babies born in a given month. A scatterplot of these two variables reveals a potential outlying month when the average temperature is about 53°F and average crawling age is about 28.5 weeks. Does this point have high leverage? Is it an influential point?

7.27 Urban homeowners, Part I. The scatterplot below shows the percent of families who own their home vs. the percent of the population living in urban areas in 2010.[22] There are 52 observations, each corresponding to a state in the US. Puerto Rico and District of Columbia are also included.

(a) Describe the relationship between the percent of families who own their home and the percent of the population living in urban areas in 2010.

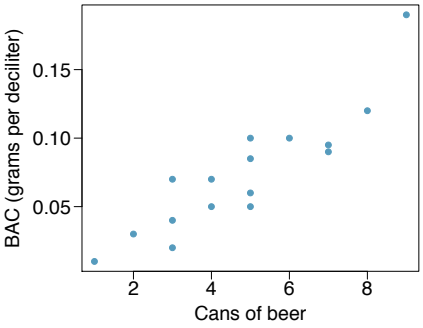
(b) The outlier at the bottom right corner is District of Columbia, where 100% of the population is considered urban. What type of outlier is this observation?



7.5.4 Inference for linear regression

In the following exercises, visually check the conditions for fitting a least squares regression line, but you do not need to report these conditions in your solutions.

7.28 Beer and blood alcohol content. Many people believe that gender, weight, drinking habits, and many other factors are much more important in predicting blood alcohol content (BAC) than simply considering the number of drinks a person consumed. Here we examine data from sixteen student volunteers at Ohio State University who each drank a randomly assigned number of cans of beer. These students were evenly divided between men and women, and they differed in weight and drinking habits. Thirty minutes later, a police officer measured their blood alcohol content (BAC) in grams of alcohol per deciliter of blood.[23] The scatterplot and regression table summarize the findings.



	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-0.0127	0.0126	-1.00	0.3320
beers	0.0180	0.0024	7.48	0.0000

(a) Describe the relationship between the number of cans of beer and BAC.

(b) Write the equation of the regression line. Interpret the slope and intercept in context.

(c) Do the data provide strong evidence that drinking more cans of beer is associated with an increase in blood alcohol? State the null and alternative hypotheses, report the p-value, and state your conclusion.

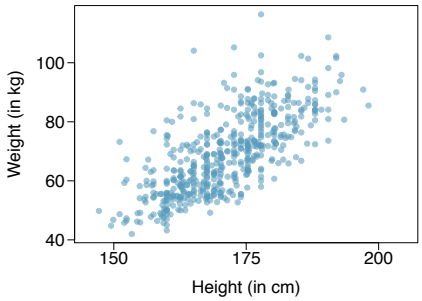
(d) The correlation coefficient for number of cans of beer and BAC is 0.89. Calculate R^2 and interpret it in context.

(e) Suppose we visit a bar, ask people how many drinks they have had, and also take their BAC. Do you think the relationship between number of drinks and BAC would be as strong as the relationship found in the Ohio State study?

[22] United States Census Bureau, 2010 Census Urban and Rural Classification and Urban Area Criteria and Housing Characteristics: 2010.

[23] J. Malkevitch and L.M. Lesser. *For All Practical Purposes: Mathematical Literacy in Today's World.* WH Freeman & Co, 2008.

7.29 Body measurements, Part IV. The scatterplot and least squares summary below show the relationship between weight measured in kilograms and height measured in centimeters of 507 physically active individuals.



	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-105.0113	7.5394	-13.93	0.0000
height	1.0176	0.0440	23.13	0.0000

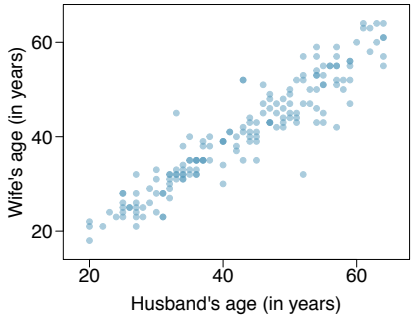
(a) Describe the relationship between height and weight.

(b) Write the equation of the regression line. Interpret the slope and intercept in context.

(c) Do the data provide strong evidence that an increase in height is associated with an increase in weight? State the null and alternative hypotheses, report the p-value, and state your conclusion.

(d) The correlation coefficient for height and weight is 0.72. Calculate R^2 and interpret it in context.

7.30 Husbands and wives, Part II. Exercise 7.6 presents a scatterplot displaying the relationship between husbands' and wives' ages in a random sample of 170 married couples in Britain, where both partners' ages are below 65 years. Given below is summary output of the least squares fit for predicting wife's age from husband's age.



	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	1.5740	1.1501	1.37	0.1730
age_husband	0.9112	0.0259	35.25	0.0000
				$df = 168$

(a) We might wonder, is the age difference between husbands and wives consistent across ages? If this were the case, then the slope parameter would be $\beta_1 = 1$. Use the information above to evaluate if there is strong evidence that the difference in husband and wife ages differs for different ages.

(b) Write the equation of the regression line for predicting wife's age from husband's age.

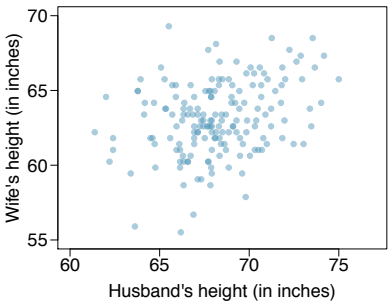
(c) Interpret the slope and intercept in context.

(d) Given that $R^2 = 0.88$, what is the correlation of ages in this data set?

(e) You meet a married man from Britain who is 55 years old. What would you predict his wife's age to be? How reliable is this prediction?

(f) You meet another married man from Britain who is 85 years old. Would it be wise to use the same linear model to predict his wife's age? Explain.

7.31 Husbands and wives, Part III. The scatterplot below summarizes husbands' and wives' heights in a random sample of 170 married couples in Britain, where both partners' ages are below 65 years. Summary output of the least squares fit for predicting wife's height from husband's height is also provided in the table.



	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	43.5755	4.6842	9.30	0.0000
height_husband	0.2863	0.0686	4.17	0.0000

(a) Is there strong evidence that taller men marry taller women? State the hypotheses and include any information used to conduct the test.

(b) Write the equation of the regression line for predicting wife's height from husband's height.

(c) Interpret the slope and intercept in the context of the application.

(d) Given that $R^2 = 0.09$, what is the correlation of heights in this data set?

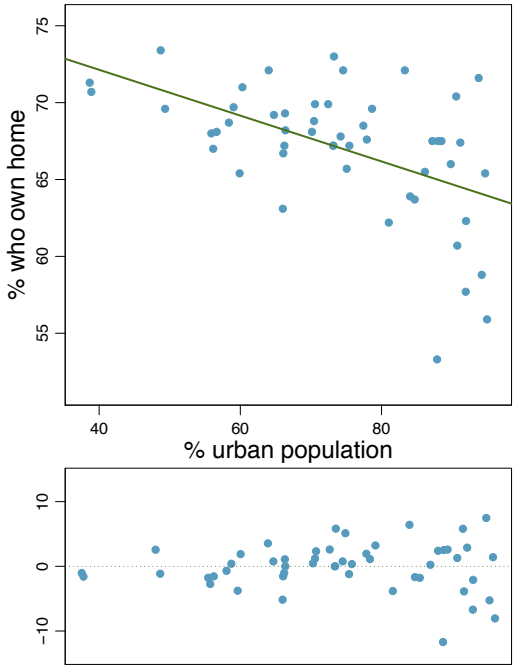
(e) You meet a married man from Britain who is 5'9" (69 inches). What would you predict his wife's height to be? How reliable is this prediction?

(f) You meet another married man from Britain who is 6'7" (79 inches). Would it be wise to use the same linear model to predict his wife's height? Why or why not?

7.32 Urban homeowners, Part II. Exercise 7.27 gives a scatterplot displaying the relationship between the percent of families that own their home and the percent of the population living in urban areas. Below is a similar scatterplot, excluding District of Columbia, as well as the residuals plot. There were 51 cases.

(a) For these data, $R^2 = 0.28$. What is the correlation? How can you tell if it is positive or negative?

(b) Examine the residual plot. What do you observe? Is a simple least squares fit appropriate for these data?



7.33 Babies. Is the gestational age (time between conception and birth) of a low birth-weight baby useful in predicting head circumference at birth? Twenty-five low birth-weight babies were studied at a Harvard teaching hospital; the investigators calculated the regression of head circumference (measured in centimeters) against gestational age (measured in weeks). The estimated regression line is

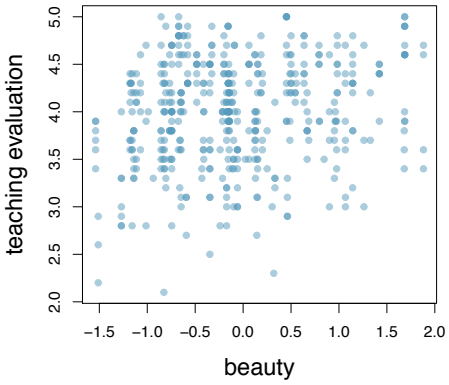
$$\widehat{head\_circumference} = 3.91 + 0.78 \times gestational\_age$$

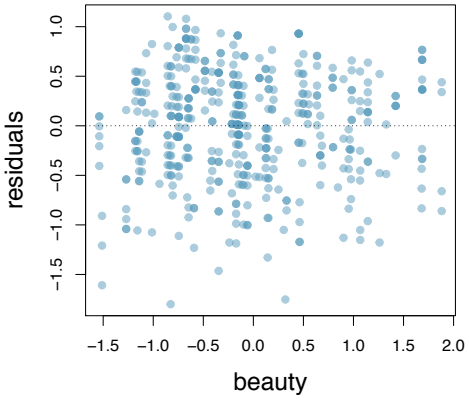
(a) What is the predicted head circumference for a baby whose gestational age is 28 weeks?

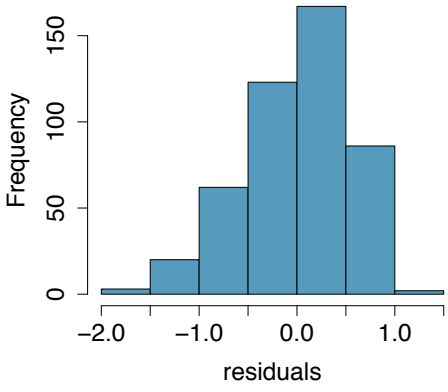
(b) The standard error for the coefficient of gestational age is 0.35, which is associated with $df = 23$. Does the model provide strong evidence that gestational age is significantly associated with head circumference?

7.34 Rate my professor. Some college students critique professors' teaching at RateMyProfessors.com, a web page where students anonymously rate their professors on quality, easiness, and attractiveness. Using the self-selected data from this public forum, researchers examine the relations between quality, easiness, and attractiveness for professors at various universities. In this exercise we will work with a portion of these data that the researchers made publicly available.[24]

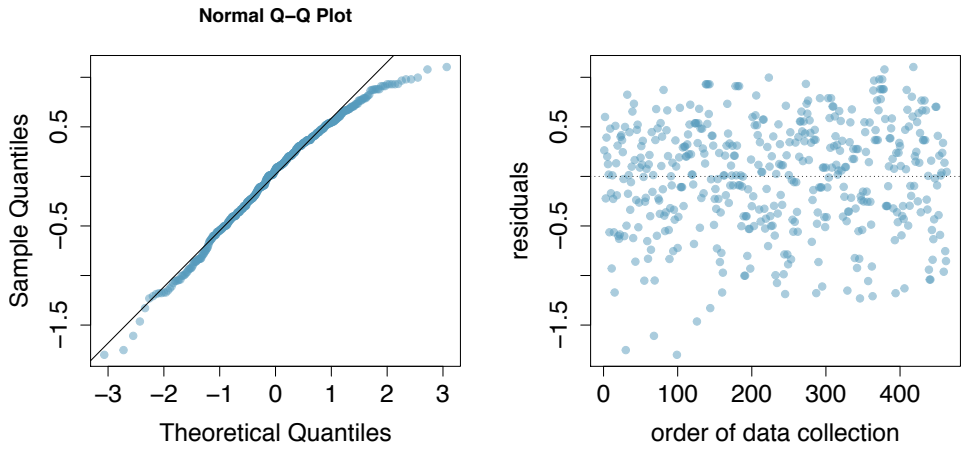
The scatterplot on the right shows the relationship between teaching evaluation score (higher score means better) and standardized beauty score (a score of 0 means average, negative score means below average, and a positive score means above average) for a sample of 463 professors. Given below are associated diagnostic plots. Also given is a regression output for predicting teaching evaluation score from beauty score.







[24] J. Felton et al. "Web-based student evaluations of professors: the relations between perceived quality, easiness and sexiness". In: *Assessment & Evaluation in Higher Education* 29.1 (2004), pp. 91–108.



	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	4.010	0.0255	157.21	0.0000
beauty		0.0322	4.13	0.0000

(a) Given that the average standardized beauty score is -0.0883 and average teaching evaluation score is 3.9983, calculate the slope. Alternatively, the slope may be computed using just the information provided in the model summary table.

(b) Do these data provide convincing evidence that the slope of the relationship between teaching evaluation and beauty is positive? Explain your reasoning.

(c) List the conditions required for linear regression and check if each one is satisfied for this model.