Phase Lead Compensator Design Using Bode Plots

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These notes are lecture notes prepared by Prof. Guy Beale for presentation in ECE 421, *Classical Systems and Control Theory*, in the Electrical and Computer Engineering Department, George Mason University, Fairfax, VA. Additional notes can be found at: http://teal.gmu.edu/~gbeale/examples.html.

I. INTRODUCTION

As with phase lag compensation, the purpose of phase lead compensator design in the frequency domain generally is to satisfy specifications on steady-state accuracy and phase margin. There may also be a specification on gain crossover frequency or closed-loop bandwidth. A phase margin specification can represent a requirement on relative stability due to pure time delay in the system, or it can represent desired transient response characteristics that have been translated from the time domain into the frequency domain.

The overall philosophy in the design procedure presented here is for the compensator to adjust the system's Bode phase curve to establish the required phase margin at the existing gain-crossover frequency, ideally without disturbing the system's magnitude curve at that frequency and without reducing the zero-frequency magnitude value. The unavoidable shift in the gain crossover frequency is a function of the amount of phase shift that must be added to satisfy the phase margin requirement. In order for phase lead compensation to work in this context, the following two characteristics are needed:

- the Bode magnitude curve (after the steady-state accuracy specification has been satisfied) must pass through 0 db in some acceptable frequency range;
- the uncompensated phase shift at the gain crossover frequency must be more negative than the value needed to satisfy the phase margin specification (otherwise, no compensation is needed).

If the compensation is to be performed by a single-stage compensator, then the amount that the phase curve needs to be moved up at the gain crossover frequency in order to satisfy the phase margin specification must be less than $90°$, and is generally restricted to a maximum value in the range $55°$–$65°$. Multiple stages of compensation can be used, following the same procedure as shown below, and are needed when the amount that the Bode phase curve must be moved up exceeds the available phase shift for a single stage of compensation. More is said about this later.

The gain crossover frequency and bandwidth for the lead-compensated system will be higher than for the plant (even when the steady-state error specification is satisfied), so the system will respond more rapidly in the time domain. The faster response may be an advantage in many applications, but a disadvantage of a wider bandwidth is that more noise and other high frequency signals (often unwanted) will be passed by the system. A smaller bandwidth also provides more stability robustness when the system has unmodeled high frequency dynamics, such as the bending modes in aircraft and spacecraft. Thus, there is a trade-off between having the ability to track rapidly varying reference signals and being able to reject high-frequency disturbances.

The design procedure presented here is basically graphical in nature. All of the measurements needed can be obtained from accurate Bode plots of the uncompensated system. If data arrays representing the magnitudes and phases of the system at various frequencies are available, then the procedure can be done numerically, and in many cases automated. The examples and plots presented in this paper are all done in MATLAB, and the various measurements that are presented in the examples are obtained from the relevant data arrays.

The primary references for the procedures described in this paper are [1]–[3]. Other references that contain similar material include [4]–[11].

II. DESIGN PROCEDURE

A. Compensator Structure

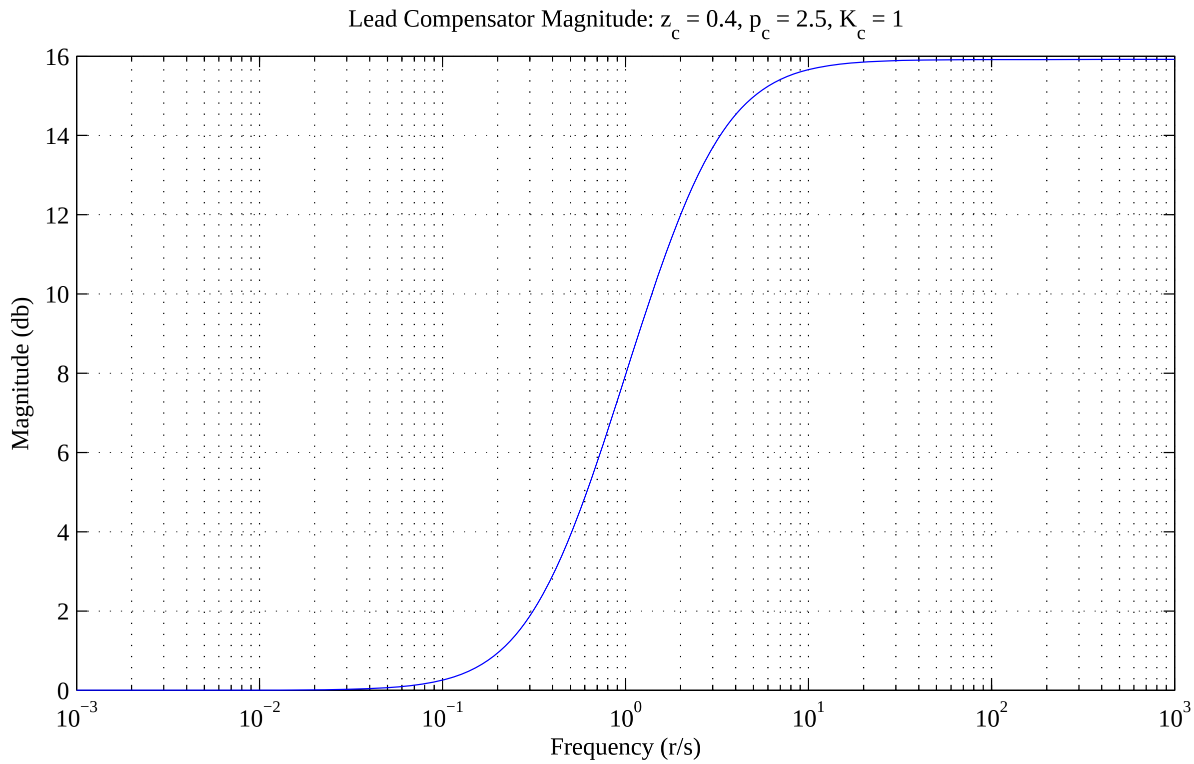
The basic phase lead compensator consists of a gain, one pole, and one zero. Based on the usual electronic implementation of the compensator [3], the specific structure of the compensator is:

$$\begin{aligned} G_{c\_lead}(s) &= K_c \left[\frac{1}{\alpha} \cdot \frac{(s+z_c)}{(s+p_c)} \right] \\ &= K_c \frac{(s/z_c+1)}{(s/p_c+1)} = K_c \frac{(\tau s+1)}{(\alpha \tau s+1)} \end{aligned} \tag{1}$$

with

$$z_c > 0, \quad p_c > 0, \quad \alpha \triangleq \frac{z_c}{p_c} < 1, \quad \tau = \frac{1}{z_c} = \frac{1}{\alpha p_c} \tag{2}$$

Figure 1 shows the Bode plots of magnitude and phase for a typical lead compensator. The values in this example are $K_c = 1$, $p_c = 2.5$, and $z_c = 0.4$, so $\alpha = 0.4/2.5 = 0.16$. Changing the gain merely moves the magnitude curve by $20 * \log_{10} |K_c|$. The major characteristic of the lead compensator is the positive phase shift in the intermediate frequencies. The maximum phase shift occurs at the frequency $\omega = \omega_{\max}$, which is the geometric mean of z_c and p_c. The shift in the magnitude curve that is seen at intermediate and high frequencies is undesired but unavoidable. Proper design of the compensator requires placing the compensator pole and zero appropriately so that the benefits of the positive phase shift are obtained and the magnitude shift is accounted for. The following paragraphs show how this can be accomplished.



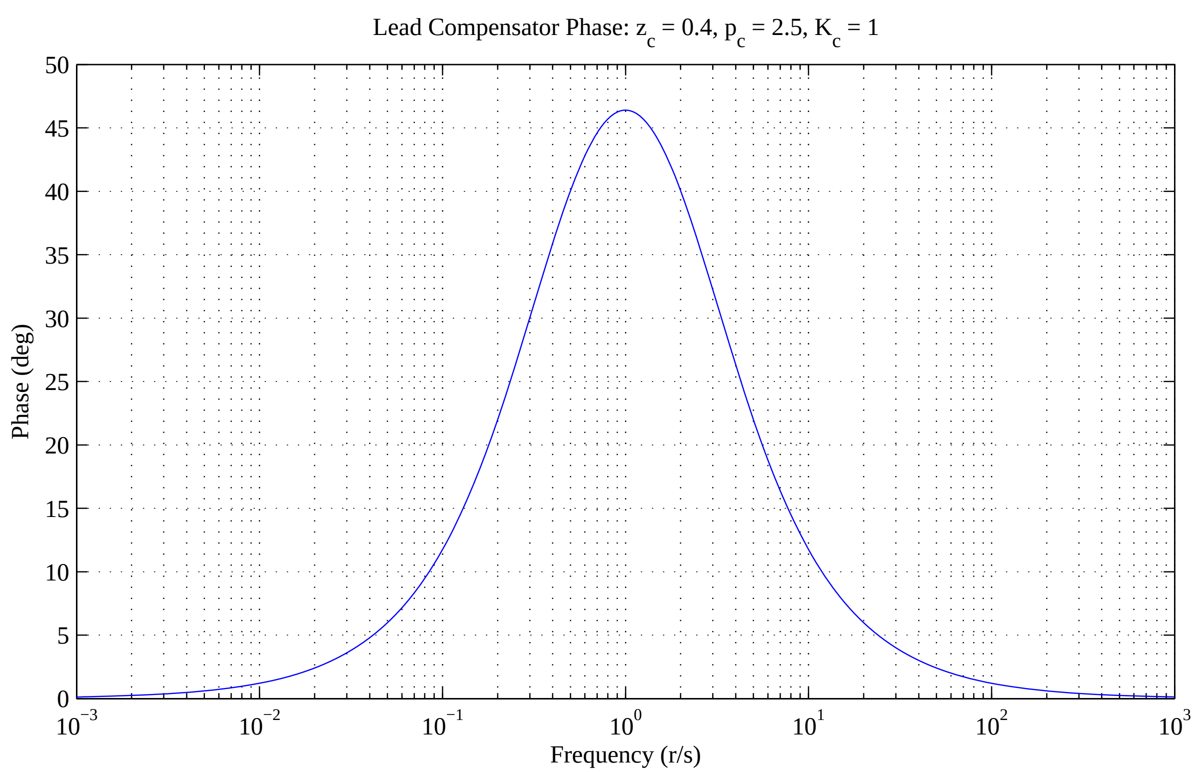


Fig. 1. Magnitude and phase plots for a typical lead compensator.

B. Outline of the Procedure

The following steps outline the procedure that will be used to design the phase lead compensator to satisfy steady-state error and phase margin specifications. Each step will be described in detail in the subsequent sections.

1) Determine if the System Type N needs to be increased in order to satisfy the steady-state error specification, and if necessary, augment the plant with the required number of poles at $s = 0$. Calculate K_c to satisfy the steady-state error.
2) Make the Bode plots of $G(s) = K_c G_p(s)/s^{(N_{req}-N_{sys})}$.
3) Design the lead portion of the compensator:
   a) determine the amount of phase shift in $G(j\omega)$ at the gain crossover frequency and calculate the uncompensated phase margin $PM_{uncompensated}$;
   b) calculate the values for $\phi_{\max}$ and α that are required to raise the phase curve to the value needed to satisfy the phase margin specification;
   c) determine the value for the final gain crossover frequency;
   d) using the value of α and the final gain crossover frequency, compute the lead compensator's zero z_c and pole p_c.
4) If necessary, choose appropriate resistor and capacitor values to implement the compensator design.

C. Compensator Gain

The first step in the design procedure is to determine the value of the gain K_c. In the procedure that I will present, the gain is used to satisfy the steady-state error specification. Therefore, the gain can be computed from

$$K_c = \frac{e_{ss\_plant}}{e_{ss\_specified}} = \frac{K_{x\_required}}{K_{x\_plant}} \tag{3}$$

where e_{ss} is the steady-state error for a particular type of input, such as step or ramp, and K_x is the corresponding error constant of the system. Defining the number of open-loop poles of the system $G(s)$ that are located at $s = 0$ to be the System Type N, and restricting the reference input signal to having Laplace transforms of the form $R(s) = A/s^q$, the steady-state error and error constant are (assuming that the closed-loop system is bounded-input, bounded-output stable)

$$e_{ss} = \lim_{s \to 0} \left[\frac{A s^{N+1-q}}{s^N + K_x} \right] \tag{4}$$

where

$$K_x = \lim_{s \to 0} \left[s^N G(s) \right] \tag{5}$$

For $N = 0$, the steady-state error for a step input $(q = 1)$ is $e_{ss} = A/\left(1 + K_x\right)$. For $N = 0$ and $q > 1$, the steady-state error is infinitely large. For $N > 0$, the steady-state error is $e_{ss} = A/K_x$ for the input type that has $q = N + 1$. If $q < N + 1$, the steady-state error is 0, and if $q > N + 1$, the steady-state error is infinite.

The calculation of the gain in (3) assumes that the given system $G_p(s)$ is of the correct Type N to satisfy the steady-state error specification. If it is not, then the compensator must have one or more poles at $s = 0$ in order to increase the overall System Type to the correct value. Once this is recognized, the compensator poles at $s = 0$ can be included with the plant $G_p(s)$ during the rest of the design of the lead compensator. The values of K_x in (5) and of K_c in (3) would then be computed based on $G_p(s)$ being augmented with these additional poles at the origin.

Example 1: As an example, consider the situation where a steady-state error of $e_{ss\_specified} = 0.05$ is specified when the reference input is a unit ramp function $(q = 2)$. This requires an error constant $K_{x\_required} = 1/0.05 = 20$. Assume that the plant is $G_p(s) = 200/\left[(s+4)(s+5)\right]$, which is Type 0. Then the compensator must have one pole at $s = 0$ in order to satisfy this specification. When $G_p(s)$ is augmented with this compensator pole at the origin, the error constant of $G_p(s)/s$ is $K_x = 200/\left(4 \cdot 5\right) = 10$, so the steady-state error for a ramp input is $e_{ss\_plant} = 1/10 = 0.1$. Therefore, the compensator requires a gain having a value of $K_c = 0.1/0.05 = 20/10 = 2$. ♦

Once the compensator design is completed, the total compensator will have the transfer function

$$G_{c\_lead}(s) = \frac{K_c}{s^{(N_{req}-N_{sys})}} \cdot \frac{(s/z_c + 1)}{(s/p_c + 1)} \tag{6}$$

where N_{req} is the total required number of poles at $s = 0$ to satisfy the steady-state error specification, and N_{sys} is the number of poles at $s = 0$ in $G_p(s)$. In the above example, $N_{req} = 2$ and $N_{sys} = 1$.

D. Making the Bode Plots

The next step is to plot the magnitude and phase as a function of frequency ω for the series combination of the compensator gain (and any compensator poles at $s = 0$) and the given system $G_p(s)$. This transfer function will be the one used to determine the values of the compensator's pole and zero and to determine if more than one stage of compensation is needed. The magnitude $|G(j\omega)|$ is generally plotted in decibels (db) vs. frequency on a log scale, and the phase $\angle G(j\omega)$ is plotted in degrees vs. frequency on a log scale. At this stage of the design, the system whose frequency response is being plotted is

$$G(s) = \frac{K_c}{s^{(N_{req} - N_{sys})}} \cdot G_p(s) \tag{7}$$

If the compensator does not have any poles at the origin, the gain K_c just shifts the plant's magnitude curve by $20 * \log_{10} |K_c|$ db at all frequencies. If the compensator does have one or more poles at the origin, the slope of the plant's magnitude curve also is changed by -20 db/decade at all frequencies for each compensator pole at $s = 0$. In either case, satisfying the steady-state error sets requirements on the zero-frequency portion of the magnitude curve, so the rest of the design procedure will manipulate the phase curve without changing the magnitude curve at zero frequency. The plant's phase curve is shifted by $-90^\circ (N_{req} - N_{sys})$ at all frequencies, so if the plant $G_p(s)$ has the correct System Type, then the compensator does not change the phase curve at all at this point in the design.

The remainder of the design is to determine $(s/z_c + 1) / (s/p_c + 1)$. The values of z_c and p_c will be chosen to satisfy the phase margin specification. Note that at $\omega = 0$, the magnitude $|(j\omega/z_c + 1) / (j\omega/p_c + 1)| = 1 \Rightarrow 0$ db and the phase $\angle (j\omega/z_c + 1) / (j\omega/p_c + 1) = 0$ degrees. Therefore, the low-frequency parts of the curves just plotted will be unchanged, and the steady-state error specification will remain satisfied. The Bode plots of the complete compensated system $G_{c\_lead}(j\omega)G_p(j\omega)$ will be the sum, at each frequency, of the plots made in this step of the procedure and the plots of $(j\omega/z_c + 1) / (j\omega/p_c + 1)$.

E. Uncompensated Phase Margin

Since the purpose of the lead compensator is to move the phase curve upwards in order to satisfy the phase margin specification, we need to determine how much positive phase shift is required. The first step in this determination is to evaluate the phase margin of the given system in (7). The uncompensated phase margin is the vertical distance between -180° and the phase curve of $G(j\omega)$ measured at the gain crossover frequency. The gain crossover frequency is defined to be that frequency ω_x where $|G(j\omega_x)| = 1$ in absolute value or $|G(j\omega_x)| = 0$ in db. This frequency can easily be found on the graphs made in the previous step. The uncompensated phase margin is

$$PM_{uncompensated} = 180^\circ + \angle G(j\omega_x) \tag{8}$$

If the phase curve is above -180° at ω_x (less negative than -180°), then the phase margin is positive, and if the phase curve is below -180° at ω_x, the phase margin is negative. A positive value for the uncompensated phase margin means that the given system is supplying some of the specified phase margin itself. However, if the uncompensated phase margin is negative, then the lead compensator will need to provide additional phase shift, since it not only has to satisfy all the phase margin specification, but must also make up for the deficit in phase margin of the system $G(s)$.

Example 2: Consider the transfer function $G(s) = 5/[s(s+1)(s+2)(s+3)]$. This represents the system in (7). (Later, in Example 6, we will assume that $G_p(s) = 2/[(s+1)(s+2)(s+3)]$ and that the compensator provides $2.5/s$ in order to satisfy the steady-state error specification.) The Bode plots for this system are shown in Fig. 2. The gain crossover frequency is $\omega_x = 0.65$ r/s. At that frequency, the phase shift of $G(j\omega)$ is $\angle G(j\omega) = -153.2^\circ$. Therefore, the uncompensated phase margin is $PM_{uncompensated} = 180^\circ + (-153.2^\circ) = 26.8^\circ$. ♦

F. Determination of $\phi_{\max}$ and α

Given the value of the uncompensated phase margin from the previous step, we can now determine the amount of positive phase shift that the lead compensator must provide. The compensator must move the phase curve of $G(j\omega)$ at $\omega = \omega_x$ upward from its current value to the value needed to satisfy the phase margin specification. As with the lag compensator, a safety factor will be added to this required phase shift. Thus, the amount of phase shift that the lead compensator needs to provide at $\omega = \omega_x$ is

$$\phi_{\max} = PM_{specified} + 10^\circ - PM_{uncompensated} \tag{9}$$

The notation $\phi_{\max}$ is used to signify that the phase shift provided at $\omega = \omega_x$ is the maximum phase shift produced by the lead compensator at any frequency. A safety factor of 10° is included in (9). In many applications, that will be enough. However, there are cases where more phase shift is needed from the compensator in order to satisfy the phase margin specification. This may require the use of multiple stages of compensation. More will be said about this later in this section, in Section II-I, and in Section III.

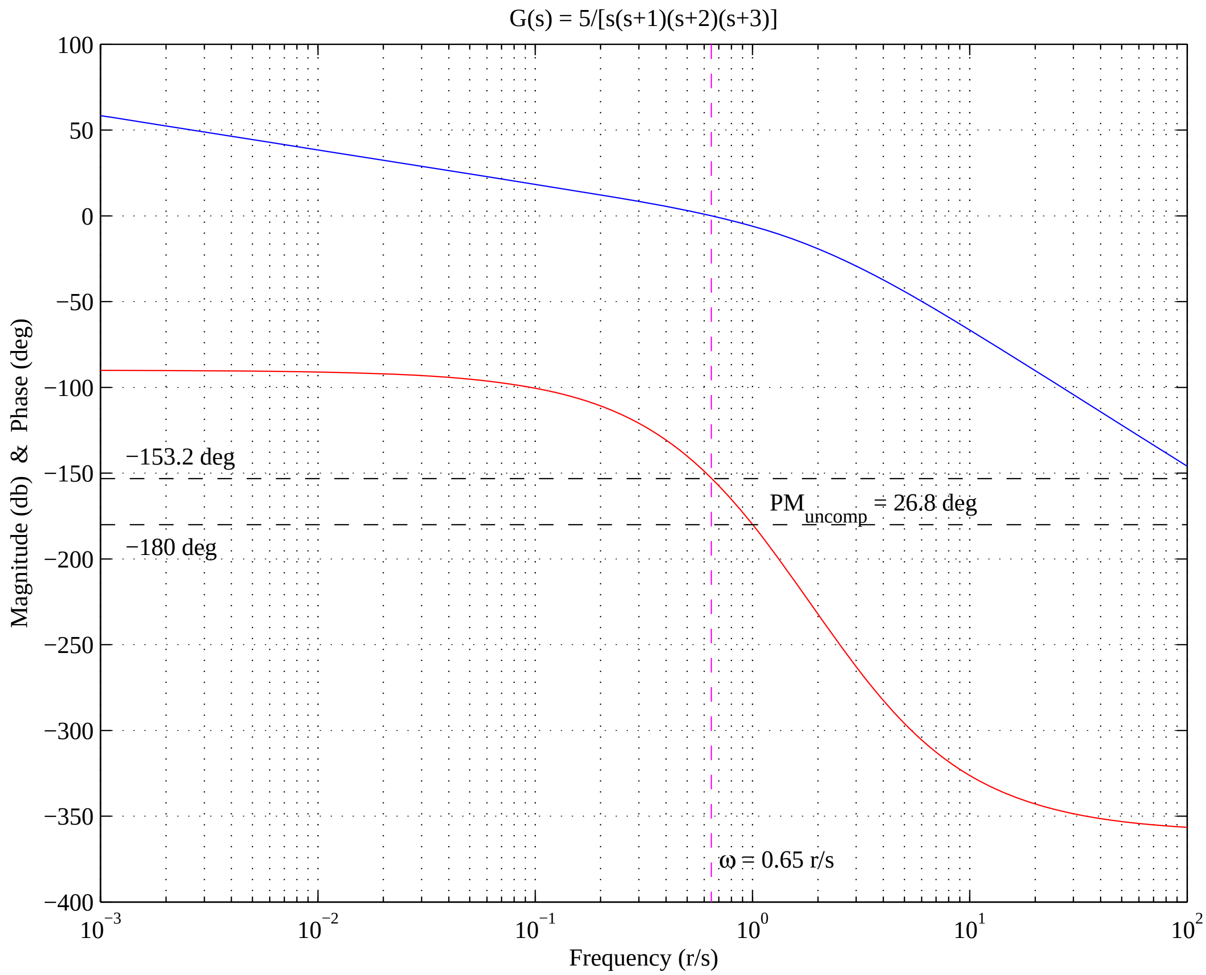


Fig. 2. Bode plots for the system in Example 2.

Once $\phi_{\max}$ is known, we can compute the value of α. Figure 3 shows the polar plot representation of the typical lead compensator whose Bode plots were given in Fig. 1. The radius of the semicircular polar plot is $(1/\alpha - 1)/2$, and the center of the plot is at $s = (1/\alpha + 1)/2$. The largest angle produced by the compensator is the angle of the line from the origin that is tangent to the polar plot. This angle is

$$\sin(\phi_{\max}) = \frac{1-\alpha}{1+\alpha} \tag{10}$$

so the value of α is computed from

$$\alpha = \frac{1-\sin(\phi_{\max})}{1+\sin(\phi_{\max})} \tag{11}$$

Example 3: As an example, consider the system described in Example 2, and assume that the specified phase margin is $PM_{specified} = 50°$. Then $\phi_{\max} = 50° + 10° - 26.8° = 33.2°$. The corresponding value of α is

$$\alpha = \frac{1-\sin(33.2°)}{1+\sin(33.2°)} = 0.292 \tag{12}$$

Therefore, the compensator's pole–zero combination will be related by the ratio $z_c/p_c = \alpha = 0.292$. ♦

The value of $\phi_{\max} = 33.2°$ in this example is quite acceptable for a single stage of lead compensation. Similar to a lag compensator, the values of the resistors and capacitors needed to implement the compensator are functions of the compensator's parameters. Specifically, the range of component values increases as $\phi_{\max}$ increases. However, with a lead compensator, there is an additional and more important restriction. The compensator defined in (1) can provide no more than $+90°$ phase shift regardless of the separation between the pole and zero, since there is only a single zero. For $z_c/p_c > 0$, the maximum phase shift is less than $90°$, and there is a corresponding minimum value of α. Many references state that $\alpha \geq 0.1$ should be used for the lead compensator to prevent excessively large component values and to limit the amount of undesired shift in the

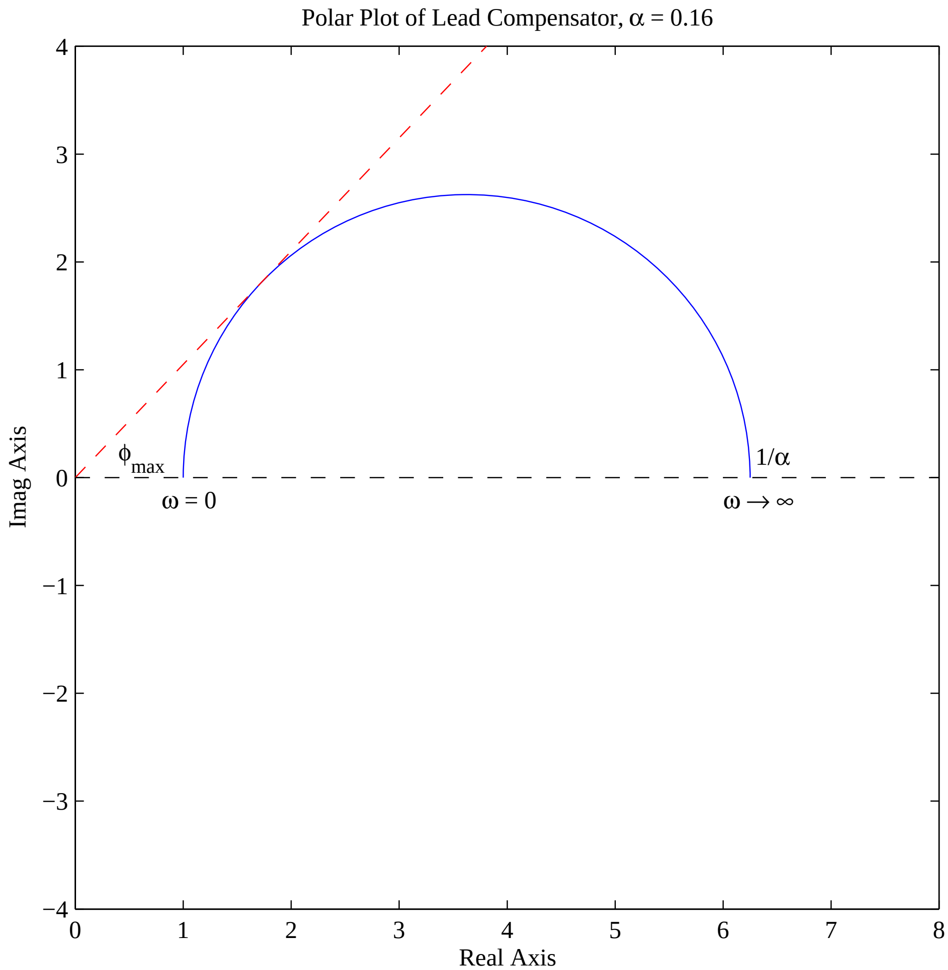


Fig. 3. Polar plot for phase lead compensator with $K_c = 1$, $\alpha = 0.16$.

magnitude curve of $G(s)$ due to the compensator. The value $\alpha = 0.1$ corresponds to a maximum phase shift $\phi_{\max} \approx 55°$, which can be implemented by a single stage of lead compensation. A minimum allowed value of $\alpha = 0.05$ corresponds to a maximum phase shift $\phi_{\max} \approx 65°$, and I feel that this is also acceptable.

If $\phi_{\max}$ computed from (9) is greater than the maximum allowed value $(55°$ or $65°)$, then multiple stages of compensation are required. An easy way to accomplish this is to design identical compensators (that will be implemented in series), so that each stage of the compensator provides the same amount of phase shift. Since the phase shift of a product of transfer functions is the sum of the individual phase shifts, the value of $\phi_{\max}$ for each of the stages is

$$\phi_{\max-stage} = \frac{\phi_{\max-total}}{n_{stage}} \tag{13}$$

where n_{stage} is the number of stages to be used in the compensator, given by

$$n_{stage} = \left\{ \begin{array}{cc} 2, & 55° < \phi_{\max} \leq 110° \\ 3, & 110° < \phi_{\max} \leq 165° \\ \vdots & \vdots \\ n, & (n-1)\,55° < \phi_{\max} \leq n55° \end{array} \right\}, \tag{14}$$

$\phi_{\max-total}$ is the value of $\phi_{\max}$ computed from (9), and $55°$ has been used for the maximum allowed value of $\phi_{\max}$. Using a maximum value for $\phi_{\max}$ of $65°$ could be used in an obvious fashion when determining the value of n_{stage}.

Once the value of $\phi_{\max-stage}$ is computed from (13), the corresponding value of α_{stage} is computed from (11). Thus, the steps to be taken at this point in the design procedure are:

- determine $\phi_{\max-total}$ from (9);
- determine n_{stage} from (14) — if $\phi_{\max-total}$ is less than the maximum allowed value, $n_{stage} = 1$;

- determine $\phi_{\max-stage}$ from (13);
- determine α_{stage} from (11).

Example 4: Consider the same system that was used in Example 3. However, assume that the specified phase margin is increased to $PM_{specified} = 85°$. Then $\phi_{\max-total} = 85° + 10° - 26.8° = 68.2°$. If the maximum allowed value of phase shift per stage is $55°$, then this value of $\phi_{\max-total}$ is too large for a single stage of compensation. From (14), the number of stages required is $n_{stage} = 2$. Therefore, $\phi_{\max-stage} = 68.2°/2 = 34.1°$, and the corresponding α is $\alpha_{stage} = (1 - \sin(34.1°)) / (1 + \sin(34.1°)) = 0.282$. ♦

G. Compensated Gain Crossover Frequency

At this stage in the design, we know how much phase shift the compensator must provide and the ratio z_c/p_c. These computations were based on the assumption that the gain crossover frequency does not change from that of $G(s)$ in (7). Now we must account for the non-ideal nature of the lead compensator. The maximum phase shift $\phi_{\max}$ occurs at the frequency $\omega = \omega_{\max}$, and it is clear from Fig. 1 that the magnitude curve of the compensator is greater than 0 db at that frequency. Specifically, $|(j\omega_{\max}/z_c + 1) / (j\omega_{\max}/p_c + 1)| = 10\log_{10}(1/\alpha)$ db.

This compensator magnitude at $\omega = \omega_{\max}$ will change the gain crossover frequency to a higher frequency, with the amount of change depending on α. We would still like the phase shift $\phi_{\max}$ to occur at the gain crossover frequency to satisfy the phase margin specification, but now we have to find the new gain crossover frequency for the total compensated system $G_{c-lead}(s)G_p(s)$.

Since the compensator will shift the magnitude upwards by $10\log_{10}(1/\alpha)$ db at $\omega = \omega_{\max}$, we will choose the compensated gain crossover frequency to be that frequency where $|G(j\omega)| = -10\log_{10}(1/\alpha) = 10\log_{10}(\alpha)$ db. The effect of the compensator will be to produce a magnitude of 0 db at $\omega = \omega_{\max}$. Therefore, the frequency at which the maximum phase shift is produced by the compensator will be the frequency at which the phase margin is defined, that is, $\omega_{\max} = \omega_{x-compensated}$. This frequency can be obtained approximately by inspection of the Bode magnitude plot or more accurately by searching the magnitude and frequency data arrays.

Example 5: Consider the system and specifications from Examples 2 and 3. The uncompensated gain crossover frequency is $\omega = 0.65$ r/s. The compensator must provide $33.2°$ phase shift to satisfy the phase margin specification, with a corresponding $\alpha = 0.292$. At the frequency of maximum phase shift, the compensator's magnitude (not including K_c) is $10\log_{10}(1/0.292)$ db $= 5.35$ db. Therefore, the compensated gain crossover frequency will be chosen to be the frequency where $|G(j\omega)|$ is -5.35 db. This frequency is $\omega \approx 0.957$ r/s. ♦

H. Determination of z_c and p_c

The last step in the design of the transfer function for the lead compensator is to determine the values of the pole and zero. Having already determined their ratio α and the value of $\omega_{x-compensated}$, there are no decisions to be made at this point in the design. Only simple calculations are needed to compute z_c and p_c.

As mentioned in Section II-A, the frequency $\omega_{\max}$ is the geometric mean of z_c and p_c; that is, $\omega_{\max} = \sqrt{z_c p_c}$. Since $\omega_{\max} = \omega_{x-compensated}$ by design, the compensator's zero and pole are computed from

$$z_c = \omega_{x-compensated}\sqrt{\alpha}, \qquad p_c = \frac{z_c}{\alpha} \tag{15}$$

Example 6: Continuing from Examples 2, 3, and 5, with $\omega_{x\_compensated} = 0.957$ r/s and $\alpha = 0.292$, the values for the compensator's zero and pole are $z_c = 0.957\sqrt{0.292} = 0.517$ and $p_c = 0.517/0.292 = 1.77$. The complete compensator for these examples is (remembering that $K_c/s^{(N_{req}-N_{sys})} = 2.5/s$ was assumed in Example 2)

$$G_{c\_lead}(s) = \frac{2.5\,(s/0.517 + 1)}{s\,(s/1.77 + 1)} = \frac{8.56\,(s + 0.517)}{s\,(s + 1.77)} \tag{16}$$

♦

I. Evaluating the Design – A Potential Problem

At this point, the design of the compensator should be complete. If the procedure has been followed correctly, then the steady-state error and phase margin specifications should be satisfied. However, evaluation of the results in Example 6 illustrates a potential problem that may be encountered when using the procedure.

When the frequency response of the transfer function

$$G_{c\_lead}(s)G_p(s) = \frac{8.56\,(s + 0.517)}{s\,(s + 1.77)} \cdot \frac{2}{(s+1)\,(s+2)\,(s+3)} \tag{17}$$

is evaluated, the gain crossover frequency is indeed $\omega = 0.957$ r/s as designed. However, the phase shift at that frequency is $-143.8°$, so that compensated phase margin is only $36.2°$, rather than the $50°$ that was specified. The reason for this is that the phase shift of $G(s)$ changes by $23.8°$ in the frequency interval from 0.65 r/s to 0.957 r/s, and only $10°$ safety factor was included in the calculation of $\phi_{\max}$ in (9).

One thought would be to increase the safety factor by an additional $13.8°$ and recalculate the compensator's parameters. However, the new value for α will change the compensated gain crossover frequency even more, and the new $\phi_{\max}$ may still not be large enough to satisfy the phase margin specification. A sort of "Catch-22" situation can occur, with the phase of $G(s)$ becoming more negative "faster" than the compensator can overcome.

With $G(s)$ defined as in (7), the safety factor (SF) in (9) must satisfy the following inequality in order for the phase margin specification to be satisfied.

$$SF \geq \angle G\left(j\omega_{x-uncompensated}\right) - \angle G\left(j\omega_{x-compensated}\right) \tag{18}$$

This inequality was not satisfied in Example 6, and so the specification was not satisfied. The trouble with (18) of course is that $\angle G\left(j\omega_{x-compensated}\right)$ is not known at the time it is needed. Because of the nonlinear relationship between $\phi_{\max}$ and $\omega_{x-compensated}$, the design of the compensator may have to be done in an iterative manner before an acceptable design is reached. Also, increasing the safety factor may produce a value for $\phi_{\max}$ that is too large for a single stage of compensation, so the order of the compensator may also increase.

Example 7: Continuing the previous examples, assume that a safety factor of $30°$ is used in (9). The compensator must now provide a phase shift of $\phi_{\max}$=53.2°, and the corresponding $\alpha = 0.110$. The new gain crossover frequency for the compensated system will be the frequency where $|G(j\omega)| = -9.57$ db; this frequency is $\omega = 1.24$ r/s. The zero and pole for the new compensator are $z_c = 0.412$ and $p_c = 3.72$, and the compensator's transfer function is

$$G_{c\_lead\_2}(s) = \frac{2.5\left(s/0.412+1\right)}{s\left(s/3.72+1\right)} = \frac{22.6\left(s+0.412\right)}{s\left(s+3.72\right)} \tag{19}$$

The phase shift of $G_{c\_lead\_2}(j\omega)G_p(j\omega)$ at $\omega = 1.24$ r/s is $-142°$, so the compensated phase margin is only $38°$. The design goals have still not been satisfied. Increasing the safety factor further will lead to the need for two stages of compensation. ⧫

Example 8: If the safety factor is increased to $60°$, the required value for the compensator's phase shift is $\phi_{\max}$=83.2°. Since this is too large for a single stage of compensation, two stages will be used, each having the value $\phi_{\max-stage} = 83.2°/2 = 41.6°$. The corresponding value for α is $\alpha = 0.202$. The new gain crossover frequency for the compensated system will be the frequency where $|G(j\omega)| = -13.9$ db; this frequency is $\omega = 1.56$ r/s. The zero and pole for the new compensator are $z_c = 0.701$ and $p_c = 3.47$, and the compensator's transfer function is

$$G_{c\_lead\_3}(s) = \frac{2.5\left(s/0.701+1\right)^2}{s\left(s/3.47+1\right)^2} = \frac{61.4\left(s+0.701\right)^2}{s\left(s+3.47\right)^2} \tag{20}$$

The phase shift of $G_{c\_lead\_3}(j\omega)G_p(j\omega)$ at $\omega = 1.55$ r/s is $-129.5°$, so the compensated phase margin is $50.6°$. The phase margin specification has been satisfied with this two-stage design. Bode plots are shown in Fig. 4. ⧫

An alternative to increasing the safety factor in the lead compensator is to design a compensator that combines both lag and lead compensators. This is known as a lag-lead compensator. The lead portion of the compensator provides the positive phase shift at the uncompensated gain crossover frequency, and the lag portion takes care of the magnitude shift to keep the gain crossover frequency at the uncompensated value. This approach is described in more detail in my paper "*Lag-Lead Compensator Design Using Bode Plots*", as well as in the references.

Example 9: Using the initial design of the lead compensator from Example 6 in series with the lag compensator $G_{c-lag}(s) = \left(s/0.065+1\right)/\left(s/0.043+1\right) = 0.663\left(s+0.065\right)/\left(s+0.043\right)$ provides a gain crossover frequency for the total system at $\omega = 0.65$ r/s (the uncompensated value) with a phase margin of $56°$. The steady-state error is not changed by the lag compensator, so all specifications have been satisfied with this lag-lead design. ⧫

Not every system will suffer from the problem shown in this example; it depends on the phase characteristics of $G(s)$ at the gain crossover frequency. If the phase curve has a large negative slope at that frequency, the problem may exist. A rule of thumb to avoid having the problem is that the slope of the magnitude curve at the compensated gain crossover frequency should be -20 db/decade. If the slope is -40 db/decade at the crossover frequency, then that frequency interval with -40 db/decade slope should be both preceded and followed by frequency ranges where the slope is -20 db/decade. This rule of thumb was not satisfied in the examples presented earlier.

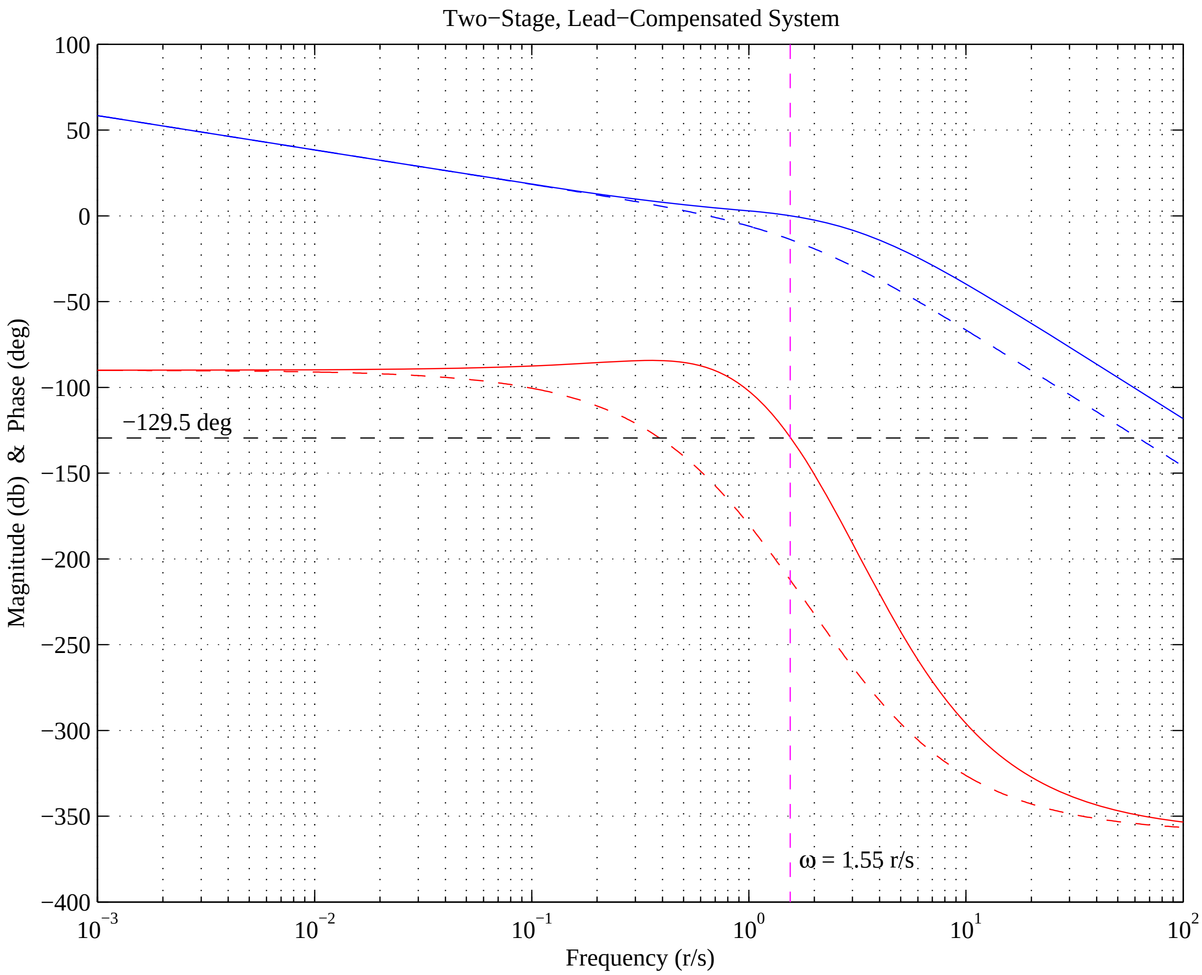


Fig. 4. Bode plots for the lead-compensated system in Example 8.

III. DESIGN EXAMPLE

A. Plant and Specifications

The plant to be controlled is described by the transfer function

$$\begin{aligned} G_p(s) &= \frac{280\,(s+0.5)}{s\,(s+0.2)\,(s+5)\,(s+70)} \\ &= \frac{2\,(s/0.5+1)}{s\,(s/0.2+1)\,(s/5+1)\,(s/70+1)} \end{aligned} \tag{21}$$

This is a Type 1 system, so the closed-loop system will have zero steady-state error for a step input, and a non-zero, finite steady-state error for a ramp input (assuming that the closed-loop system is stable). As shown in the next section, the error constant for a ramp input is $K_{x-plant} = 2$. At low frequencies, the plant has a magnitude slope of -20 db/decade, and at high frequencies the slope is -60 db/decade. The phase curve starts at $-90°$ and ends at $-270°$.

The specifications that must be satisfied by the closed-loop system are:

- steady-state error for a ramp input $e_{ss\_specified} \leq 0.02$;
- phase margin $PM_{specified} \geq 45°$.

These specifications do not impose any explicit requirements on the gain crossover frequency or on the type of compensator that should be used. It may be possible to use either lag or lead compensation for this problem, or a combination of the two, but we will use the phase lead compensator design procedure described above. The following paragraphs will illustrate how the procedure is applied to design the compensator for this system that will allow the specifications to be satisfied.

B. Compensator Gain

The given plant is Type 1, and the steady-state error specification is for a ramp input, so the compensator does not need to have any poles at $s = 0$. Only the gain K_c needs to be computed for steady-state error. The steady-state error for a ramp input for the given plant is

$$\begin{aligned} K_{x\_plant} &= \lim_{s\to 0}\left[s\cdot\frac{280\,(s+0.5)}{s\,(s+0.2)\,(s+5)\,(s+70)}\right] \\ &= \lim_{s\to 0}\left[s\cdot\frac{2\,(s/0.5+1)}{s\,(s/0.2+1)\,(s/5+1)\,(s/70+1)}\right] \\ &= 2 \end{aligned} \tag{22}$$

$$e_{ss\_plant} = \frac{1}{K_x} = 0.5 \tag{23}$$

Since the specified value of the steady-state error is 0.02, the required error constant is $K_{x\_required} = 50$. Therefore, the compensator gain is

$$\begin{aligned} K_c &= \frac{e_{ss\_plant}}{e_{ss\_specified}} = \frac{0.5}{0.02} = 25 \\ &= \frac{K_{x\_required}}{K_{x\_plant}} = \frac{50}{2} = 25 \end{aligned} \tag{24}$$

This value for K_c will satisfy the steady-state error specification, and the rest of the compensator design will focus on the phase margin specification.

C. The Bode Plots

The magnitude and phase plots for $K_cG_p(s)$ are shown in Fig. 5. The dashed magnitude curve is for $G_p(s)$ and illustrates the effect that K_c has on the magnitude. Specifically, $|K_cG_p\,(j\omega)|$ is $20\log_{10}|25| \approx 28$ db above the curve for $|G_p\,(j\omega)|$ at all frequencies. The phase curve is unchanged when the steady-state error specification is satisfied since the compensator does not have any poles at the origin. The horizontal and vertical dashed lines in the figure indicate the uncompensated gain crossover frequency (with K_c included) of 9.33 r/s and the phase shift of $-161.3°$ at that frequency.

Note that the gain crossover frequency of $K_cG_p(s)$ is larger than that for $G_p(s)$; the crossover frequency has moved to the right in the graph. The closed-loop bandwidth will have increased in a similar manner. The phase margin has decreased due to K_c, so satisfying the steady-state error has made the system less stable; in fact, increasing K_c in order to decrease the steady-state error can even make the closed-loop system unstable. Maintaining stability and achieving the desired phase margin is the task of the pole–zero combination in the compensator.

Our ability to graphically make the various measurements needed during the design obviously depends on the accuracy and resolution of the Bode plots of $|K_cG_p\,(j\omega)|$. High resolution plots like those obtained from MATLAB allow us to obtain reasonably accurate measurements. Rough, hand-drawn sketches would yield much less accurate results and might be used only for first approximations to the design. Being able to access the actual numerical data allows for even more accurate results than the MATLAB-generated plots. The procedure that I use when working in MATLAB generates the data arrays for frequency, magnitude, and phase from instructions such as the following:

```
w = logspace(N1,N2,1+100*(N2-N1));
[mag,ph] = bode(num,den,w);
semilogx(w,20*log10(mag),w,ph),grid
```

where `N1`$= \log_{10}(\omega_{\min})$, `N2` $= \log_{10}(\omega_{\max})$, and `num`, `den` are the numerator and denominator polynomials, respectively, of $K_cG_p\,(s)$. For this example,

```
N1= −3;
N2= 4;
num= 25 * 280 * [ 1   0.5 ];
den=conv([ 1   0.2   0 ], conv([ 1   5 ], [ 1   70 ]));
```

The data arrays `mag`, `ph`, and `w` can be searched to obtain the various values needed during the design of the compensator.

D. Uncompensated Phase Margin

From the Bode plots made in the previous step, we can see that the uncompensated gain crossover frequency is $\omega_{x-uncompensated} = 9.36$ r/s. The phase shift of the plant at that frequency and the uncompensated phase margin are

$$\begin{aligned} \angle K_cG_p\,(j\omega_{x-uncompensated}) &= -161.3^\circ \\ PM_{uncompensated} &= 180^\circ + (-161.3^\circ) = 18.7^\circ \end{aligned} \tag{25}$$

Since the uncompensated phase margin is positive, the closed-loop system formed by placing unity feedback around $K_cG_p\,(s)$ is stable, but the phase margin is smaller than the specified value, so compensation is required.

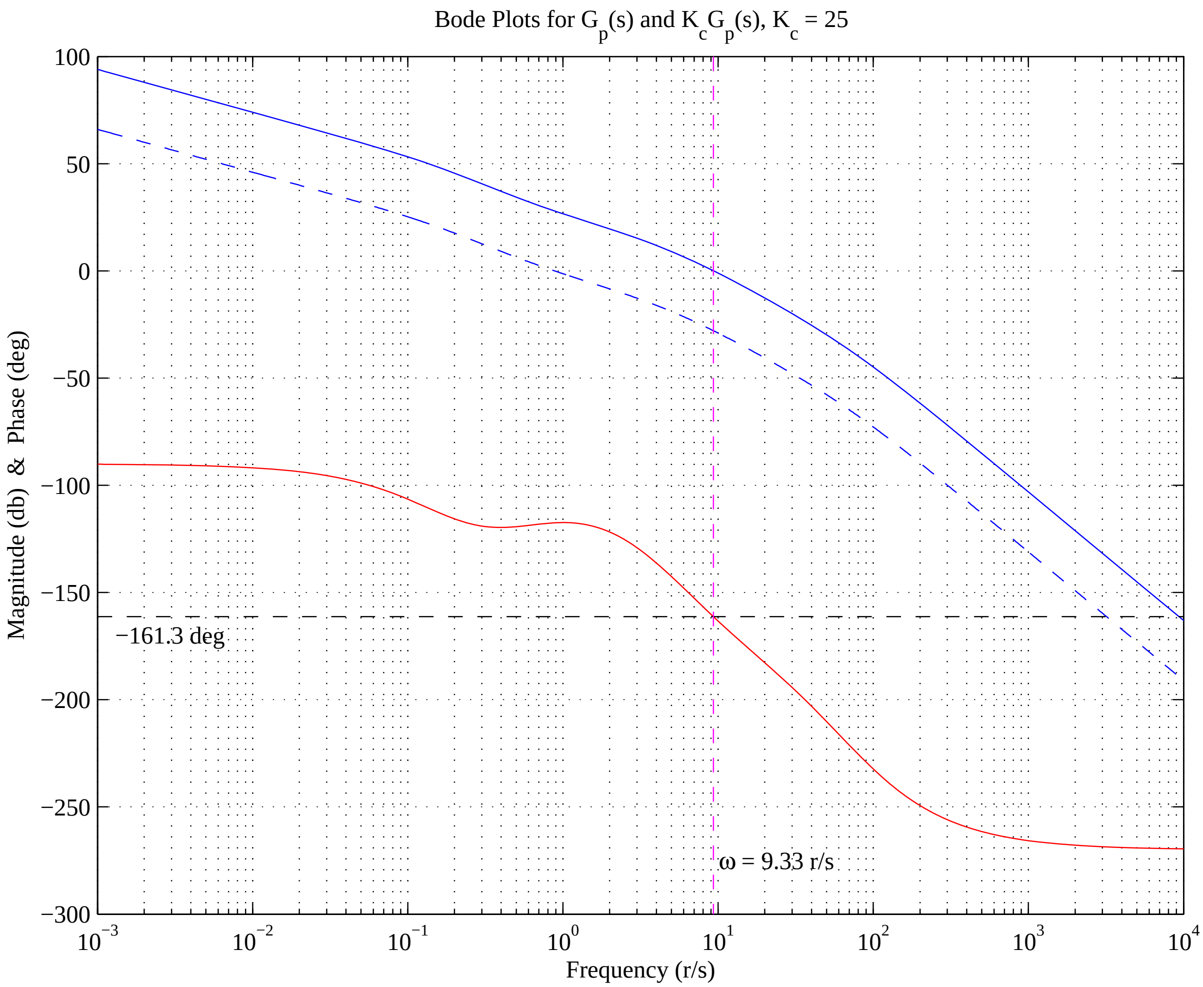


Fig. 5. Bode plots for the plant after the steady-state error specification has been satisfied.

E. Determination of $\phi_{\max}$ and α

The lead compensator will need to move the phase curve up at the gain crossover frequency by an amount

$$\phi_{\max} = 45^\circ + 10^\circ - 18.7^\circ = 36.3^\circ \tag{26}$$

Since this value of $\phi_{\max}$ is well below the limit of 55°, we can design a single-stage compensator. Hopefully, that will provide the correct phase margin for the compensated system. The value of α that corresponds to this $\phi_{\max}$ is

$$\alpha = \frac{1 - \sin(36.3^\circ)}{1 + \sin(36.3^\circ)} = 0.256 \tag{27}$$

F. Compensated Gain Crossover Frequency

This value of α will shift the magnitude curve at the frequency $\omega = \omega_{\max}$ by $10\log_{10}(1/0.256) = 5.92$ db. Therefore, the compensated gain crossover frequency will be chosen to be that frequency where $|K_c G_p(j\omega)| = -5.92$ db. From the Bode plots or from the MATLAB data arrays, this frequency is $\omega_{x-compensated} = 13.5$ r/s. Placing the frequency of maximum compensator phase shift $\omega_{\max}$ at this frequency adds the most positive phase shift possible to the plant at the frequency where the compensated phase margin will be defined.

G. Compensator Zero and Pole

Now that we have values for $\omega_{x\_compensated}$ and α, we can determine the values for the compensator's zero and pole from (15). These values are

$$\begin{aligned} z_c &= 13.5\sqrt{0.256} = 6.83 \\ p_c &= \frac{6.83}{0.256} = 26.7 \end{aligned} \tag{28}$$

The final compensator for this example is

$$G_{c\_lead}(s) = \frac{25\,(s/6.83+1)}{(s/26.7+1)} = \frac{25\,(0.146s+1)}{(0.038+1)} = \frac{97.7\,(s+6.83)}{(s+26.7)} \tag{29}$$

H. Evaluating the Design

When the frequency response magnitude and phase of the compensated system $G_{c\_lead}(s)G_p(s)$ are plotted, the gain crossover frequency is $\omega = 13.5°$ as expected. The phase shift of the compensated system at that frequency (from the data array) is $\angle G_{c\_lead}(j\omega)G_p(j\omega) = -135.5°$, so the phase margin is only $44.5°$. This is very close to the specified $45°$, and might be accepted in many applications. This is certainly much closer than the results in Examples 6 and 7. However, we will redesign the compensator so that the specifications will be strictly satisfied.

Since the first design was so close to being acceptable, the only change we will make is to add $5°$ to the amount of phase shift provided by the compensator. With this,

$$\phi_{\max} = 45° + 15° - 18.7° = 41.3° \tag{30}$$

and

$$\alpha = \frac{1-\sin(41.3°)}{1+\sin(41.3°)} = 0.205 \tag{31}$$

The new value of $\phi_{\max}$ is well within the limit for a single stage of compensation. The new compensated gain crossover frequency will be the frequency where $|K_c G_p(j\omega)| = 10\log_{10}(\alpha) = -6.9$ db. This frequency is $\omega = 14.2$ r/s. The compensator parameters are

$$\begin{aligned} z_c &= 14.2\sqrt{0.205} = 6.54 \\ p_c &= \frac{6.54}{0.205} = 31.9 \end{aligned} \tag{32}$$

and the final compensator is

$$\begin{aligned} G_{c\_lead\_2}(s) &= \frac{25\,(s/6.54+1)}{(s/31.9+1)} = \frac{25\,(0.153s+1)}{(0.031s+1)} \\ &= \frac{122.2\,(s+6.54)}{s+31.9} \end{aligned} \tag{33}$$

The frequency response of the newly compensated system $G_{c\_lead\_2}(s)G_p(s)$ is shown in Fig. 6. At the gain crossover frequency of 14.1 r/s, the phase shift is $-132°$, so the compensated phase margin is $48°$, and the specification is satisfied. Since the gain K_c has not changed, the steady-state error specification is still satisfied. Thus, the compensator in (33) is acceptable.

To illustrate the effects of the compensator on closed-loop bandwidth, the magnitudes of the closed-loop systems are plotted in Fig. 7. The smallest bandwidth occurs with the plant $G_p(s)$. Including the compensator gain $K_c > 1$ increases the bandwidth and the size of the resonant peak. Significant overshoot in the time-domain step response should be expected from the closed-loop system with $K_c = 25$. Including the entire lead compensator $G_{c\_lead\_2}(s)$ increases the bandwidth further but reduces the resonant peak, relative to that with $K_c G_p(s)$. The step response overshoot in the lead-compensated system should be similar to the uncompensated system, but the settling time will be much less due to the larger bandwidth.

The major difference in the time-domain step responses between the uncompensated system and the lead-compensated system is the settling time. The compensated system settles to its final value more than 20 times faster than the uncompensated system and 7 times faster than $K_c G_p(s)$. The step responses are shown in Fig. 8, where the bottom plot is a zoomed view of the top plot.

The steady-state error specification is for a ramp input. The steady-state error is reduced by a factor of 25 due to K_c. A closed-loop steady-state error $e_{ss} = 0.02$ is achieved for $K_c G_p(s)$, $G_{c\_lead}(s)G_p(s)$ and $G_{c\_lead\_2}(s)G_p(s)$. With the design approach presented here, once K_c is determined, then the steady-state error specification is satisfied for any of the subsequent designs.

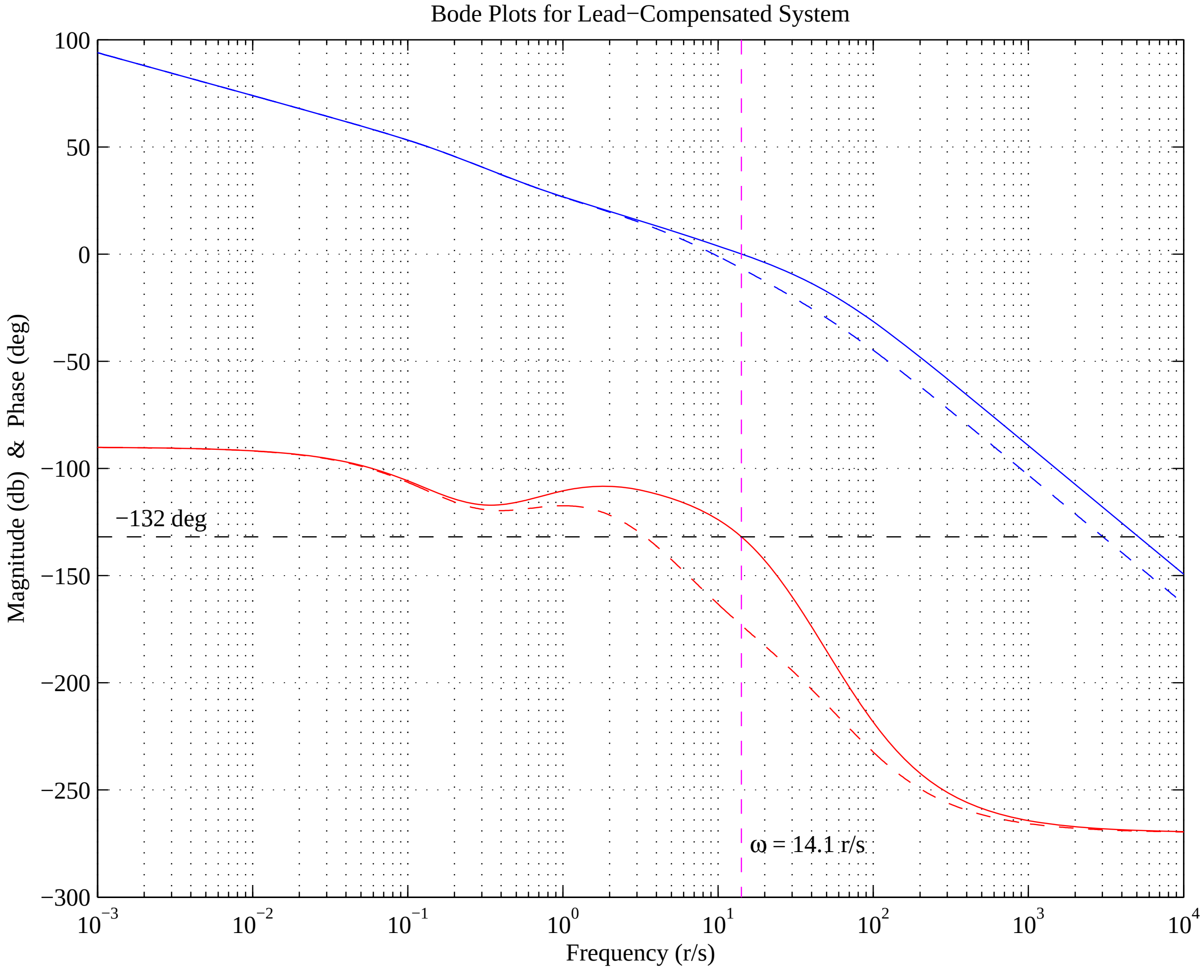


Fig. 6. Bode plots for the compensated system.

I. Implementation of the Compensator

Ogata [3] presents a table showing analog circuit implementations for various types of compensators. The circuit for phase lead is the series combination of two inverting operational amplifiers. The first amplifier has an input impedance that is the parallel combination of resistor R_1 and capacitor C_1 and a feedback impedance that is the parallel combination of resistor R_2 and capacitor C_2. The second amplifier has input and feedback resistors R_3 and R_4, respectively.

Assuming that the op amps are ideal, the transfer function for this circuit is

$$\begin{aligned}\frac{V_{out}(s)}{V_{in}(s)} &= \frac{R_2R_4}{R_1R_3} \cdot \frac{(sR_1C_1+1)}{(sR_2C_2+1)} \\ &= \frac{R_2R_4}{R_1R_3} \cdot \frac{R_1C_1}{R_2C_2} \cdot \frac{(s+1/R_1C_1)}{(s+1/R_2C_2)}\end{aligned} \tag{34}$$

Comparing (34) with $G_{c\_lead}(s)$ in (1) shows that the following relationships hold:

$$\begin{aligned} &K_c = \frac{R_2R_4}{R_1R_3}, \qquad \tau = R_1C_1, \qquad \alpha\tau = R_2C_2 \\ &z_c = 1/R_1C_1, \qquad p_c = 1/R_2C_2, \qquad \alpha = \frac{z_c}{p_c} = \frac{R_2C_2}{R_1C_1}\end{aligned} \tag{35}$$

Equations (34) and (35) are the same as for a lag compensator. The only difference is that $\alpha < 1$ for a lead compensator and $\alpha > 1$ for a lag compensator, so the relative values of the components change.

To implement the compensator using the circuit in [3], note that there are 6 unknown circuit elements (R_1, C_1, R_2, C_2, R_3, R_4) and 3 compensator parameters (K_c, z_c, p_c). Therefore, three of the circuit elements can be chosen to have convenient

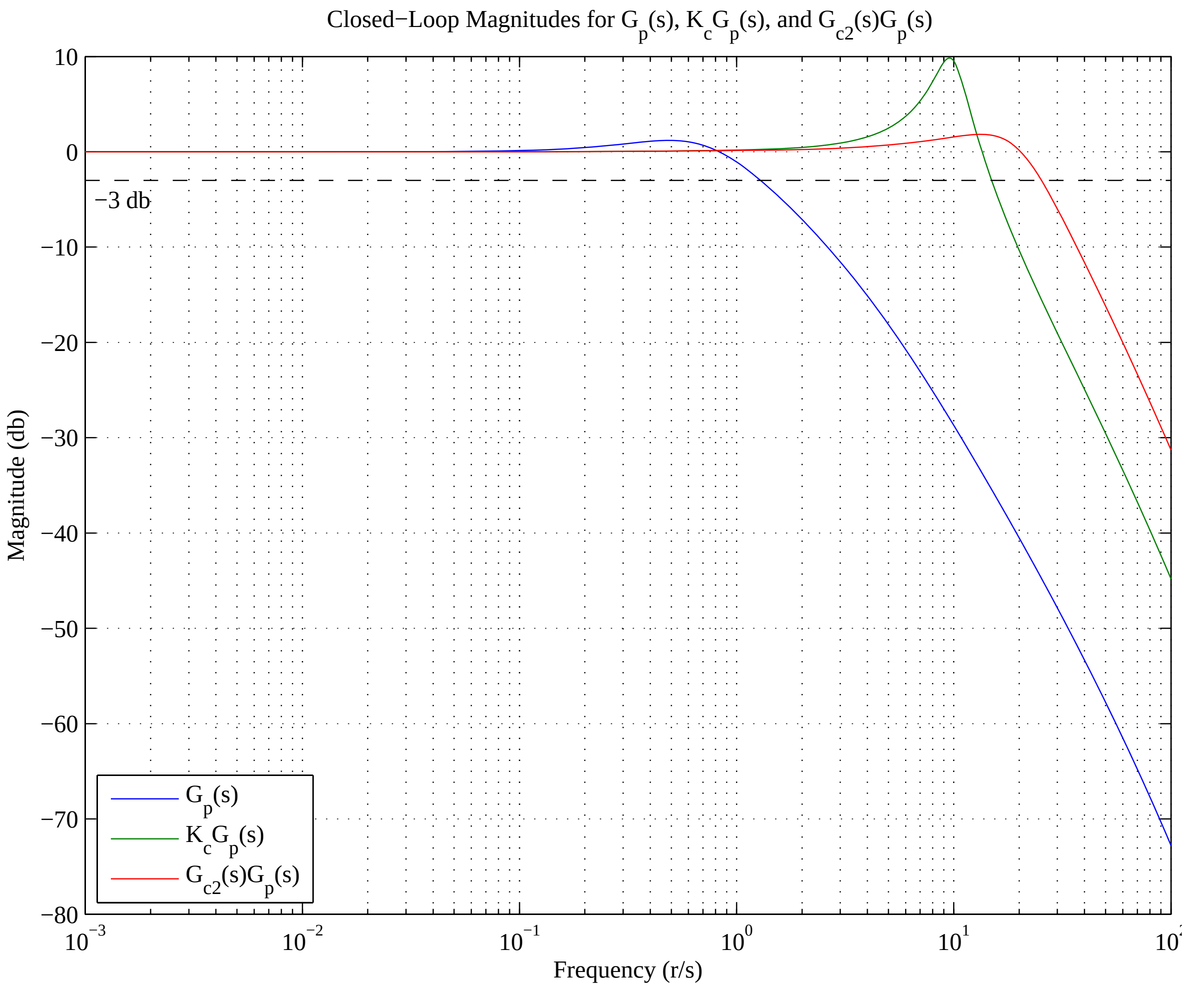


Fig. 7. Closed-loop frequency response magnitudes for the example.

values. To implement the final lead compensator $G_{c\_lead\_2}(s)$, we can use the following values

$$\begin{aligned} C_1 &= C_2 = 0.1\ \mu\text{F} \ = 10^{-7}\ \text{F}, \qquad R_3 = 10\ \text{K}\Omega = 10^4\ \Omega \\ R_1 &= \frac{1}{z_c C_1} = 1.53\ \text{M}\Omega = 1.53\cdot 10^6\ \Omega \\ R_2 &= \frac{1}{p_c C_2} = 313\ \text{K}\Omega = 3.13\cdot 10^5\ \Omega \\ R_4 &= \frac{R_3 K_c}{\alpha} = 1.22\ \text{M}\Omega = 1.22\cdot 10^6\ \Omega \end{aligned} \tag{36}$$

where the elements in the first row of (36) were specified and the remaining elements were computed from (35).

J. Summary

In this example, the phase lead compensator in (33) is able to satisfy both of the specifications of the system given in (21). In addition to satisfying the phase margin and steady-state error specifications, the lead compensator also produced a step response with much shorter settling time.

In summary, phase lead compensation can provide steady-state accuracy and necessary phase margin when the Bode phase plot can be moved up the necessary amount at the uncompensated gain crossover frequency. The philosophy of the lead compensator is to add positive phase shift at the crossover frequency without shifting the magnitude at that frequency. As we have seen in the examples, there is at least a small shift in the magnitude, and iteration of the design might be required.

The step response of the compensated system will be faster than that of the plant even with its gain set to satisfy the steady-state accuracy specification, and its phase margin will be larger than $K_cG_p(s)$. The following table provides a comparison between the systems in this example.

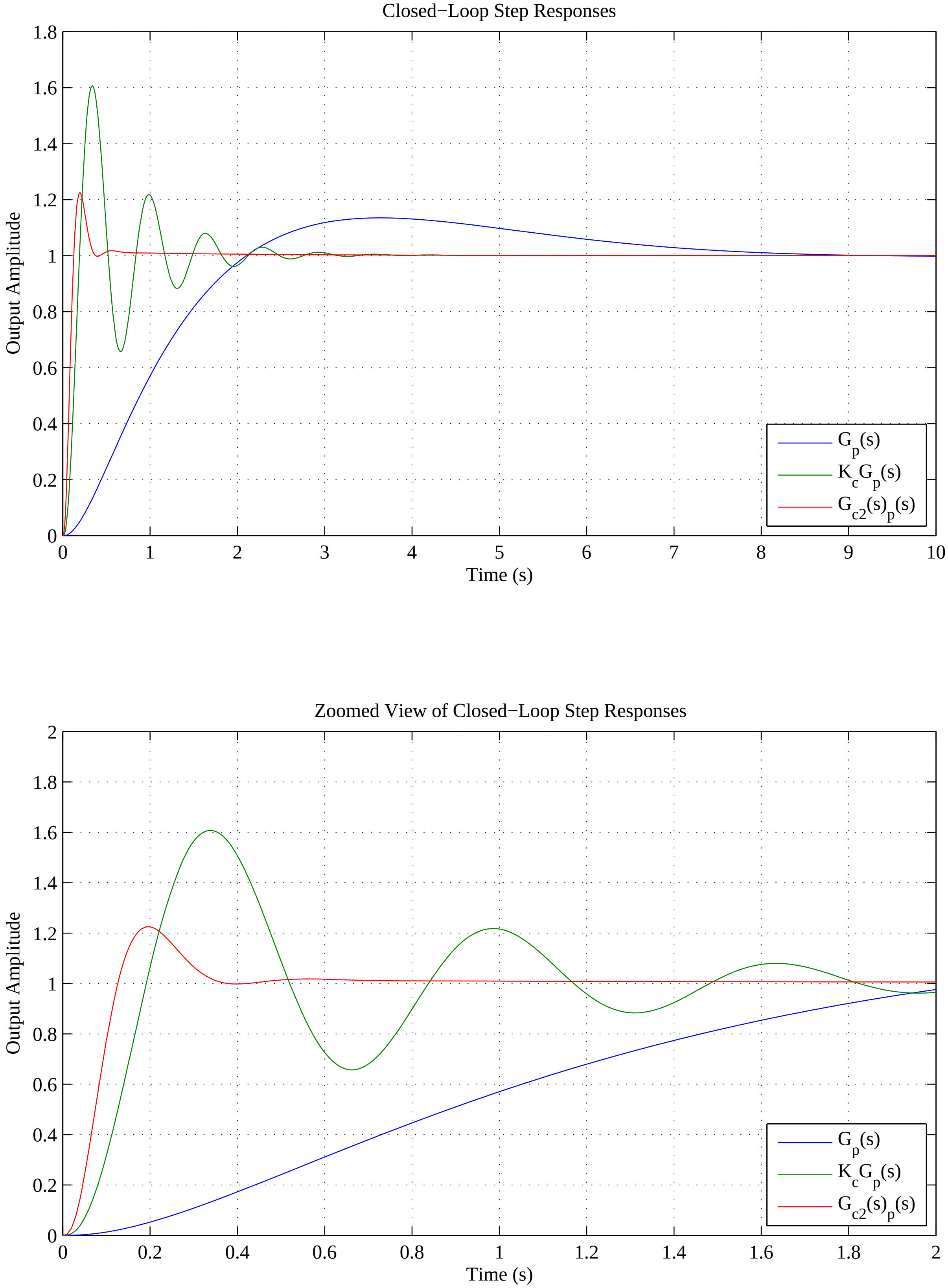


Fig. 8. Step responses for the closed-loop systems.

Characteristic	Symbol	$G_p(s)$	$K_c G_p(s)$	$G_{c\_lead}(s)G_p(s)$	$G_{c\_lead\_2}(s)G_p(s)$
steady-state error	e_{ss}	0.5	0.02	0.02	0.02
phase margin	PM	62.5°	18.7°	44.5°	48.0°
gain xover freq	ω_x	0.88 r/s	9.36 r/s	13.5 r/s	14.2 r/s
time delay	T_d	1.24 sec	0.035 sec	0.058 sec	0.059 sec
gain margin	GM	87.7	3.51	5.86	6.09
gain margin (db)	GM_{db}	38.9 db	10.9 db	15.4 db	15.7 db
phase xover freq	ω_ϕ	18.1 r/s	18.1 r/s	40.8 r/s	45.3 r/s
bandwidth	ω_B	1.29 r/s	14.9 r/s	23.8 r/s	25.4 r/s
percent overshoot	PO	13.5%	60.7%	26.7%	22.5%
settling time	T_s	7.52 sec	2.39 sec	0.36 sec	0.34 sec

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