Phase Lag Compensator Design Using Bode Plots

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CONTENTS

LIST OF FIGURES

These notes are lecture notes prepared by Prof. Guy Beale for presentation in ECE 421, *Classical Systems and Control Theory*, in the Electrical and Computer Engineering Department, George Mason University, Fairfax, VA. Additional notes can be found at: http://teal.gmu.edu/~gbeale/examples.html.

I. INTRODUCTION

The purpose of phase lag compensator design in the frequency domain generally is to satisfy specifications on steady-state accuracy and phase margin. There may also be a specification on gain crossover frequency or closed-loop bandwidth. A phase margin specification can represent a requirement on relative stability due to pure time delay in the system, or it can represent desired transient response characteristics that have been translated from the time domain into the frequency domain.

The overall philosophy in the design procedure presented here is for the compensator to adjust the system's Bode magnitude curve to establish a gain-crossover frequency, without disturbing the system's phase curve at that frequency and without reducing the zero-frequency magnitude value. In order for phase lag compensation to work in this context, the following two characteristics are needed:

- the uncompensated Bode phase curve must pass through the correct value to satisfy the phase margin specification at some acceptable frequency;
- the Bode magnitude curve (after the steady-state accuracy specification has been satisfied) must be above 0 db at the frequency where the uncompensated phase shift has the correct value to satisfy the phase margin specification (otherwise no compensation other than additional gain is needed).

If the compensation is to be performed by a single-stage compensator, then it must also be possible to drop the magnitude curve down to 0 db at that frequency without using excessively large component values. Multiple stages of compensation can be used, following the same procedure as shown below. Multiple stages are needed when the amount that the Bode magnitude curve must be moved down is too large for a single stage of compensation. More is said about this later.

The gain crossover frequency and closed-loop bandwidth for the lag-compensated system will be lower than for the uncompensated plant (after the steady-state error specification has been satisfied), so the compensated system will respond more slowly in the time domain. The slower response may be regarded as a disadvantage, but one benefit of a smaller bandwidth is that less noise and other high frequency signals (often unwanted) will be passed by the system. The smaller bandwidth will also provide more stability robustness when the system has unmodeled high frequency dynamics, such as the bending modes in aircraft and spacecraft. Thus, there is a trade-off between having the ability to track rapidly varying reference signals and being able to reject high-frequency disturbances.

The design procedure presented here is basically graphical in nature. All of the measurements needed can be obtained from accurate Bode plots of the uncompensated system. If data arrays representing the magnitudes and phases of the system at various frequencies are available, then the procedure can be done numerically, and in many cases automated. The examples and plots presented here are all done in MATLAB, and the various measurements that are presented in the examples are obtained from the various data arrays.

The primary references for the procedures described in this paper are [1]–[3]. Other references that contain similar material are [4]–[11].

II. DESIGN PROCEDURE

A. Compensator Structure

The basic phase lag compensator consists of a gain, one pole, and one zero. Based on the usual electronic implementation of the compensator [3], the specific structure of the compensator is:

$$
\begin{aligned}
G_{c\_lag}(s) &= K_c \left[\frac{1}{\alpha} \cdot \frac{(s+z_c)}{(s+p_c)}\right] \\
&= K_c \frac{(s/z_c+1)}{(s/p_c+1)} = K_c \frac{(\tau s+1)}{(\alpha\tau s+1)}
\end{aligned} \tag{1}
$$

with

$$
z_c > 0, \quad p_c > 0, \quad \alpha \triangleq \frac{z_c}{p_c} > 1, \quad \tau = \frac{1}{z_c} = \frac{1}{\alpha p_c} \tag{2}
$$

Figure 1 shows the Bode plots of magnitude and phase for a typical lag compensator. The values in this example are $K_c = 1$, $p_c = 0.4$, and $z_c = 2.5$, so $\alpha = 2.5/0.4 = 6.25$. Changing the gain merely moves the magnitude curve by $20 * \log_{10}|K_c|$. The major characteristics of the lag compensator are the constant attenuation in magnitude at high frequencies and the zero phase shift at high frequencies. The large negative phase shift that is seen at intermediate frequencies is undesired but unavoidable. Proper design of the compensator requires placing the compensator pole and zero appropriately so that the benefits of the magnitude attenuation are obtained without the negative phase shift causing problems. The following paragraphs show how this can be accomplished.

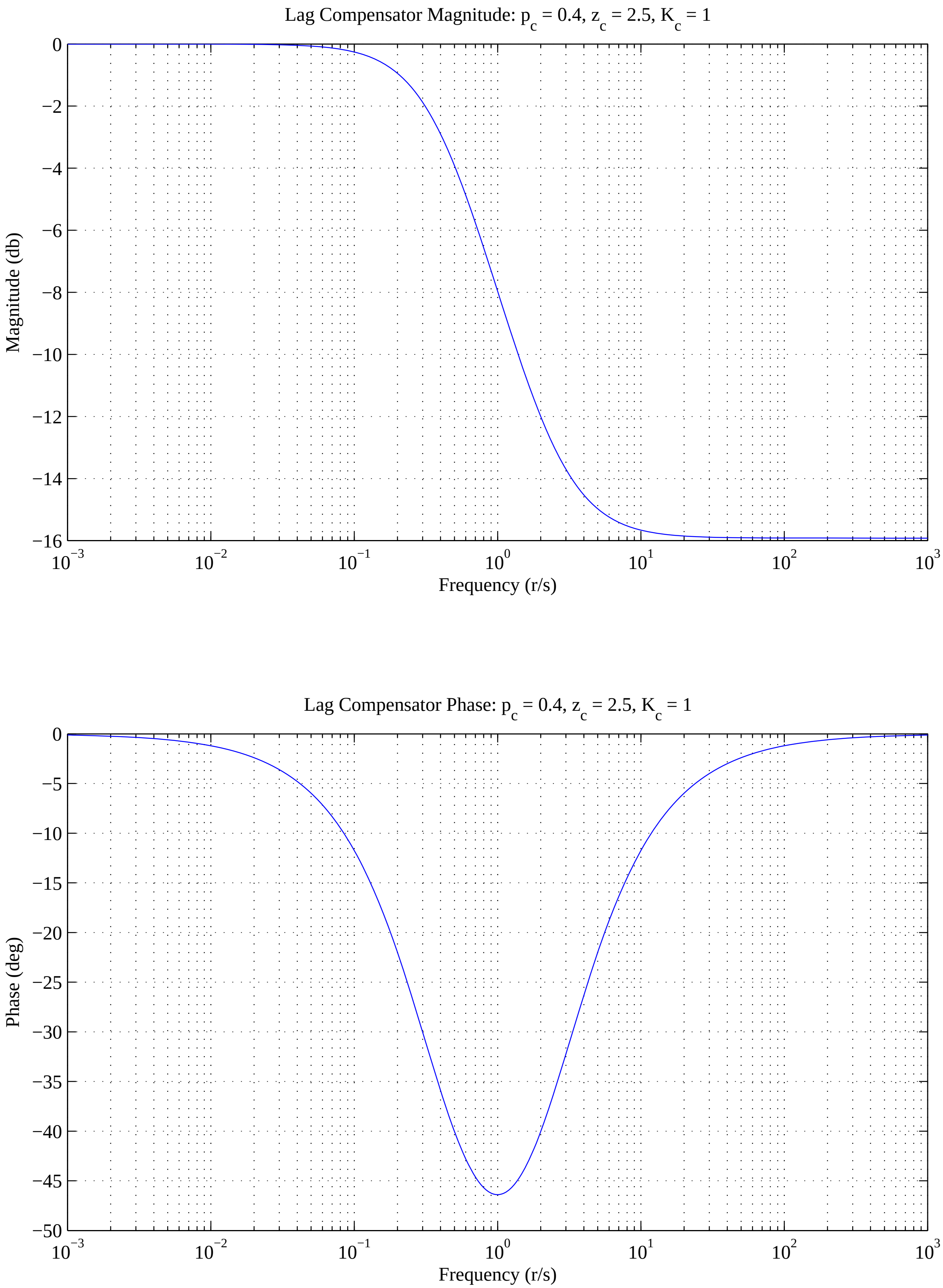

Fig. 1. Magnitude and phase plots for a typical lag compensator.

B. Outline of the Procedure

The following steps outline the procedure that will be used to design the phase lag compensator to satisfy steady-state error and phase margin specifications. Each step will be described in detail in the subsequent sections.

1) Determine if the System Type N needs to be increased in order to satisfy the steady-state error specification, and if necessary, augment the plant with the required number of poles at $s = 0$. Calculate K_c to satisfy the steady-state error.
2) Make the Bode plots of $G(s) = K_c G_p(s)/s^{(N_{req}-N_{sys})}$.
3) Design the lag portion of the compensator:
   a) determine the frequency where $G(j\omega)$ would satisfy the phase margin specification if that frequency were the gain crossover frequency;
   b) determine the amount of attenuation that is required to drop the magnitude of $G(j\omega)$ down to 0 db at that same frequency, and compute the corresponding α;
   c) using the value of α and the chosen gain crossover frequency, compute the lag compensator's zero z_c and pole p_c.
4) If necessary, choose appropriate resistor and capacitor values to implement the compensator design.

C. Compensator Gain

The first step in the design procedure is to determine the value of the gain K_c. In the procedure that I will present, the gain is used to satisfy the steady-state error specification. Therefore, the gain can be computed from

$$K_c = \frac{e_{ss\_plant}}{e_{ss\_specified}} = \frac{K_{x\_required}}{K_{x\_plant}} \tag{3}$$

where e_{ss} is the steady-state error for a particular type of input, such as step or ramp, and K_x is the corresponding error constant of the system. Defining the number of open-loop poles of the system $G(s)$ that are located at $s = 0$ to be the System Type N, and restricting the reference input signal to having Laplace transforms of the form $R(s) = A/s^q$, the steady-state error and error constant are (assuming that the closed-loop system is bounded-input, bounded-output stable)

$$e_{ss} = \lim_{s \to 0} \left[\frac{A s^{N+1-q}}{s^N + K_x} \right] \tag{4}$$

where

$$K_x = \lim_{s \to 0} \left[s^N G(s) \right] \tag{5}$$

For $N = 0$, the steady-state error for a step input $(q = 1)$ is $e_{ss} = A/\left(1 + K_x\right)$. For $N = 0$ and $q > 1$, the steady-state error is infinitely large. For $N > 0$, the steady-state error is $e_{ss} = A/K_x$ for the input type that has $q = N + 1$. If $q < N + 1$, the steady-state error is 0, and if $q > N + 1$, the steady-state error is infinite.

The calculation of the gain in (3) assumes that the given system $G_p(s)$ is of the correct Type N to satisfy the steady-state error specification. If it is not, then the compensator must have one or more poles at $s = 0$ in order to increase the overall System Type to the correct value. Once this is recognized, the compensator poles at $s = 0$ can be included with the plant $G_p(s)$ during the rest of the design of the lag compensator. The values of K_x in (5) and of K_c in (3) would then be computed based on $G_p(s)$ being augmented with these additional poles at the origin.

Example 1: As an example, consider the situation where a steady-state error of $e_{ss\_specified} = 0.05$ is specified when the reference input is a unit ramp function $(q = 2)$. This requires an error constant $K_{x\_required} = 1/0.05 = 20$. Assume that the plant is $G_p(s) = 200/\left[(s + 4)(s + 5)\right]$, which is Type 0. Then the compensator must have one pole at $s = 0$ in order to satisfy this specification. When $G_p(s)$ is augmented with this compensator pole at the origin, the error constant of $G_p(s)/s$ is $K_x = 200/\left(4 \cdot 5\right) = 10$, so the steady-state error for a ramp input is $e_{ss\_plant} = 1/10 = 0.1$. Therefore, the compensator requires a gain having a value of $K_c = 0.1/0.05 = 20/10 = 2$. ♦

Once the compensator design is completed, the total compensator will have the transfer function

$$G_{c\_lag}(s) = \frac{K_c}{s^{(N_{req}-N_{sys})}} \cdot \frac{(s/z_c + 1)}{(s/p_c + 1)} \tag{6}$$

where N_{req} is the total required number of poles at $s = 0$ to satisfy the steady-state error specification, and N_{sys} is the number of poles at $s = 0$ in $G_p(s)$. In the above example, $N_{req} = 2$ and $N_{sys} = 1$.

D. Making the Bode Plots

The next step is to plot the magnitude and phase as a function of frequency ω for the series combination of the compensator gain (and any compensator poles at $s = 0$) and the given system $G_p(s)$. This transfer function will be the one used to determine the values of the compensator's pole and zero and to determine if more than one stage of compensation is needed. The magnitude $|G(j\omega)|$ is generally plotted in decibels (db) vs. frequency on a log scale, and the phase $\angle G(j\omega)$ is plotted in degrees vs. frequency on a log scale. At this stage of the design, the system whose frequency response is being plotted is

$$G(s) = \frac{K_c}{s^{(N_{req} - N_{sys})}} \cdot G_p(s) \tag{7}$$

If the compensator does not have any poles at the origin, the gain K_c just shifts the plant's magnitude curve by $20 * \log_{10}|K_c|$ db at all frequencies. If the compensator does have one or more poles at the origin, the slope of the plant's magnitude curve also is changed by -20 db/decade at all frequencies for each compensator pole at $s = 0$. In either case, satisfying the steady-state error sets requirements on the zero-frequency portion of the magnitude curve, so the rest of the design procedure will manipulate the magnitude and phase curves without changing the magnitude curve at zero frequency. The plant's phase curve is shifted by $-90° (N_{req} - N_{sys})$ at all frequencies, so if the plant $G_p(s)$ has the correct System Type, then the compensator does not change the phase curve at this point in the design.

The remainder of the design is to determine $(s/z_c + 1)/(s/p_c + 1)$. The values of z_c and p_c will be chosen to satisfy the phase margin and crossover frequency specifications. Note that at $\omega = 0$, the magnitude $|(j\omega/z_c + 1)/(j\omega/p_c + 1)| = 1 \Rightarrow 0$ db and the phase $\angle(j\omega/z_c + 1)/(j\omega/p_c + 1) = 0$ degrees. Therefore, the low-frequency parts of the curves just plotted will be unchanged, and the steady-state error specification will remain satisfied. The Bode plots of the complete compensated system $G_{c\_lag}(j\omega)G_p(j\omega)$ will be the sum, at each frequency, of the plots made in this step of the procedure and the plots of $(j\omega/z_c + 1)/(j\omega/p_c + 1)$.

E. Gain Crossover Frequency

Now we will find the frequency that will become the gain crossover frequency for the compensated system. The gain crossover frequency is defined to be that frequency ω_x where $|G(j\omega_x)| = 1$ in absolute value or $|G(j\omega_x)| = 0$ in db. The purpose of the lag compensator's pole–zero combination is to drop the magnitude of the transfer function $G(j\omega)$, defined in (7), down to 0 db at the appropriate frequency to satisfy the phase margin specification.

Since the goal of the lag compensator is to adjust the magnitude curve as needed without shifting the phase curve, the frequency that is chosen for the compensated gain crossover frequency is that frequency where the phase shift of the system given in (7) has the correct value to satisfy the phase margin specification. Generally, the specified phase margin is increased $5°$–$10°$ to account for the fact that the lag compensator is not ideal, and the phase curve of the system in (7) is actually shifted by a small amount at the chosen frequency.

Therefore, the compensated gain crossover frequency $\omega_{x\_compensated}$ is that frequency where

$$\angle\left[\frac{K_c}{s^{(N_{req} - N_{sys})}} \cdot G_p(s)\right] = -180° + PM_{specified} + 10° \tag{8}$$

where $PM_{specified}$ is the specification on phase margin (in degrees), and a safety factor of $10°$ is being used.

Example 2: For example, if the specified phase margin is $40°$, then $\omega_{x\_compensated}$ is chosen to be that frequency where the phase shift in (8) evaluates to $-180 + 40 + 10 = -130°$. This $\omega_{x\_compensated}$ is the frequency where the magnitude of the compensated system $|G_{c\_lag}(j\omega)\, G_p(j\omega)| = 1 \Rightarrow 0$ db, with $G_{c\_lag}(s)$ given by (6). ♦

F. Determination of α

Once the compensated gain crossover frequency has been determined, the amount of attenuation that the lag compensator must provide at that frequency can be evaluated. For all frequencies greater than about $4 * z_c$, the magnitude of the $(j\omega/z_c + 1)/(j\omega/p_c + 1)$ part of the lag compensator is $-20 * \log_{10}(\alpha)$ db. Therefore, high frequencies (relative to z_c) will be attenuated by $-20 * \log_{10}(\alpha)$ db. Thus, α can be determined by evaluating the magnitude of $G(j\omega)$ in (7) at the frequency $\omega_{x\_compensated}$. If $|G(j\omega_{x\_compensated})|$ is expressed in db, then the value of α is computed from

$$\alpha = 10^{\left(\frac{|G(j\omega_{x\_compensated})|_{db}}{20}\right)} \tag{9}$$

and if $|G(j\omega_{x\_compensated})|$ is expressed as an absolute value, then the value of α is computed from

$$\alpha = |G(j\omega_{x\_compensated})|_{absolute\_value} \tag{10}$$

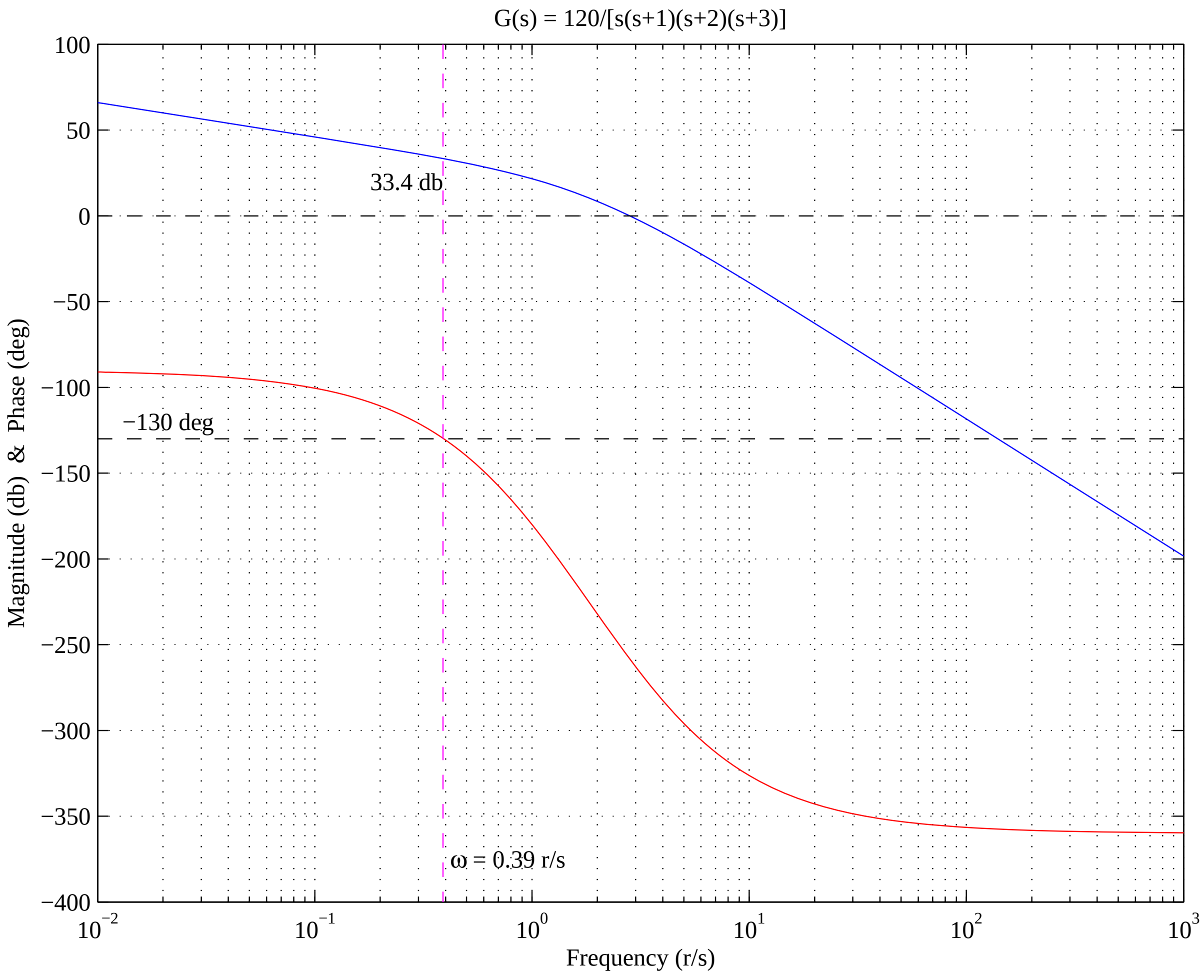

Fig. 2. Bode plots for G(s) in Example 3.

Example 3: As an example, consider the following specifications and plant model.

- $PM_{specified} \geq 40°$;
- $e_{ss\_specified} = 0.05$ for a ramp input.

$$G_p(s) = \frac{60}{(s+1)(s+2)(s+3)} \tag{11}$$

Since the plant is Type 0 and the steady-state error specification is for a ramp input, the compensator must have one pole at $s = 0$. The compensator must also have a gain $K_c = 2$ in order to satisfy the steady-state error specification, since $K_{x\_plant} = 10$, and $K_{x\_required} = 20$. Thus, the system $G(s)$ for this example is

$$G(s) = \frac{120}{s(s+1)(s+2)(s+3)} \tag{12}$$

Once the Bode plots of $G(s)$ are made, shown in Fig. 2, expression (8) tells us that we should search for the frequency where the phase shift is $-130°$. This occurs at approximately 0.39 r/s; this frequency is $\omega_{x\_compensated}$. At this frequency, the magnitude of $G(s)$ can be read directly from the graph, and its value is $|G(j\omega_{x\_compensated})| = 33.4$ db. Converting this to an absolute value using (9) gives us the value $\alpha = 46.6$. Therefore, the compensator's pole–zero combination will be related by the ratio $z_c/p_c = \alpha = 46.6$. ♦

The value of $\alpha = 46.6$ in this example is generally considered to be too large for a single stage of lag compensation. The values of the resistors and capacitors needed to implement the compensator increase with the value of α, as does the maximum amount of negative phase shift that the compensator produces. Many references state that $\alpha \leq 10$ should be used for the lag compensator to prevent excessively large component values and to limit the amount of undesired phase shift.

If this limit is used, and $\alpha > 10$, then multiple stages of compensation are required. An easy way to accomplish this is to design identical compensators (that will be implemented in series), so that each stage of the compensator attenuates the magnitude of $G(j\omega)$ by the same amount. Since the magnitude of a product of transfer functions is the product of the individual magnitudes, the value of α for each of the stages is

$$\alpha_{stage} = \sqrt[n_{stage}]{\alpha_{total}} \tag{13}$$

where n_{stage} is the number of stages to be used in the compensator, given by

$$n_{stage} = \left\{ \begin{array}{ll} 2, & 10 < \alpha_{total} \leq 100 \\ 3, & 100 < \alpha_{total} \leq 1000 \\ \vdots & \vdots \\ n, & 10^{n-1} < \alpha_{total} \leq 10^{n} \end{array} \right\} \tag{14}$$

Example 4: If the value $\alpha = 46.6$ in Example 3 is assumed to be too large, two stages of compensation can be used. Each stage would be given an attenuation of α_{stage}=$\sqrt{46.6} = 6.83$. When the compensator is implemented, the gain $K_c = 2$ could also be divided evenly between the two stages in the same fashion, with $K_{c\_stage} = \sqrt{2} = 1.414$. ♦

G. Determination of z_c and p_c

The last step in the design of the transfer function for the lag compensator is to determine the values of the pole and zero. We have already determined their ratio α, so only one of those terms is a free variable. We will choose to place the compensator zero and then compute the pole location from $p_c = z_c/\alpha$. Figure 1 is the key to deciding how to place z_c. In that example, $z_c = 2.5$, and at the frequency $\omega = 25$ r/s (one decade above the frequency corresponding to the value of z_c), the magnitude is constant at -15.9 db, and the phase shift of the compensator is $-4.7°$. Because of the nature of the tangent of an angle, the phase shift for a single-stage lag compensator will never be more negative than $-5.7°$ at a frequency one decade above the location of the compensator zero. In determining the compensated gain crossover frequency in (8), we added $10°$ to the specified phase margin to account for phase shift from the compensator at $\omega_{x\_compensated}$. Therefore, if the value of the compensator zero is chosen to correspond to the frequency one decade below $\omega_{x\_compensated}$, then the phase margin will always be satisfied. Smaller values of z_c can also be used. Lower limits on the values of z_c and p_c are governed by the values of the corresponding electronic components used to implement the compensator.[1] The compensator's zero and pole are computed from

$$z_c \leq \frac{\omega_{x\_compensated}}{10}, \qquad p_c = \frac{z_c}{\alpha} \tag{15}$$

Example 5: Continuing from Example 3, with $\omega_{x\_compensated} = 0.39$ r/s and $\alpha = 46.6$, acceptable values for the compensator's zero and pole are $z_c = 0.39/10 = 0.039$ and $p_c = 0.039/46.6 = 8.37 \cdot 10^{-4}$. The complete compensator for these examples is

$$G_{c\_lag}(s) = \frac{2\,(s/0.039 + 1)}{s\,(s/8.37 \cdot 10^{-4} + 1)} = \frac{0.0429\,(s + 0.039)}{s\,(s + 8.37 \cdot 10^{-4})} \tag{16}$$

If the compensator is split into two stages, as in Example 4, then $z_c = 0.39/10 = 0.039$ and $p_c = 0.039/6.83 = 5.71 \cdot 10^{-3}$ can be used for the zero and pole for each stage of the compensator.

Placing the zero at $z_c = 0.039$ and using two stages of compensator uses up all of the $10°$ safety factor included in expression (8). To assure that the phase margin specification will be satisfied when using multiple stages of lag compensation, I use the following technique for determining the compensator's zero and pole:

$$z_c = \frac{\omega_{x\_compensated}}{10 n_{stage}}, \qquad p_c = \frac{z_c}{\alpha_{stage}} \tag{17}$$

which for this example gives $z_c = 1.95 \cdot 10^{-2}$ and $p_c = 2.86 \cdot 10^{-3}$, and the total compensator is

$$\begin{aligned} G_{c\_lag}(s) &= \frac{2\left(s/1.95 \cdot 10^{-2} + 1\right)^2}{s\left(s/2.86 \cdot 10^{-3} + 1\right)^2} = \frac{1}{s} \cdot \left[\frac{1.414\left(s/1.95 \cdot 10^{-2} + 1\right)}{\left(s/2.86 \cdot 10^{-3} + 1\right)}\right]^2 \\ &= \frac{1}{s} \cdot \left[\frac{0.207\left(s + 1.95 \cdot 10^{-2}\right)}{\left(s + 2.86 \cdot 10^{-3}\right)}\right]^2 \end{aligned} \tag{18}$$

with $n_{stage} = 2$ and $\alpha_{stage} = 6.83$. ♦

The lag compensator is ideal in the sense that any amount of magnitude attenuation can be theoretically provided by a single stage of compensation since $\alpha = \left|G\left(j\omega_{x-compensated}\right)\right|_{abs\_val} > 1$. The effects of negative phase shift from the compensator can be made negligible by making z_c suitably small. The need for multiple stages of compensation is dictated by implementation considerations in the choice of electronic component values.

[1]The textbook by Ogata [3] has a table in Chapter 5 that shows op amp circuits for various types of compensators.

III. DESIGN EXAMPLE

A. Plant and Specifications

The plant to be controlled is described by the transfer function

$$\begin{aligned} G_p(s) &= \frac{280\,(s+0.5)}{s\,(s+0.2)\,(s+5)\,(s+70)} \\ &= \frac{2\,(s/0.5+1)}{s\,(s/0.2+1)\,(s/5+1)\,(s/70+1)} \end{aligned} \tag{19}$$

This is a Type 1 system, so the closed-loop system will have zero steady-state error for a step input, and a non-zero, finite steady-state error for a ramp input (assuming that the closed-loop system is stable). As shown in the next section, the error constant for a ramp input is $K_{x-plant} = 2$. At low frequencies, the plant has a magnitude slope of -20 db/decade, and at high frequencies the slope is -60 db/decade. The phase curve starts at $-90°$ and ends at $-270°$.

The specifications that must be satisfied are:

- steady-state error for a ramp input $e_{ss\_specified} \leq 0.02$;
- phase margin $PM_{specified} \geq 45°$.

These specifications do not impose any explicit requirements on the gain crossover frequency or on the type of compensator that should be used. It may be possible to use either lag or lead compensation for this problem, or a combination of the two, but we will use the phase lag compensator design procedure described above. The following paragraphs will illustrate how the procedure is applied to design the compensator for this system that will allow the specifications to be satisfied.

B. Compensator Gain

The given plant is Type 1, and the steady-state error specification is for a ramp input, so the compensator does not need to have any poles at $s = 0$. Only the gain K_c needs to be computed for steady-state error. The steady-state error for a ramp input for the given plant is

$$\begin{aligned} K_{x\_plant} &= \lim_{s\to 0}\left[s\cdot\frac{280\,(s+0.5)}{s\,(s+0.2)\,(s+5)\,(s+70)}\right] \\ &= \lim_{s\to 0}\left[s\cdot\frac{2\,(s/0.5+1)}{s\,(s/0.2+1)\,(s/5+1)\,(s/70+1)}\right] \\ &= 2 \end{aligned} \tag{20}$$

$$e_{ss\_plant} = \frac{1}{K_x} = 0.5 \tag{21}$$

Since the specified value of the steady-state error is 0.02, the required error constant is $K_{x\_required} = 50$. Therefore, the compensator gain is

$$\begin{aligned} K_c &= \frac{e_{ss\_plant}}{e_{ss\_specified}} = \frac{0.5}{0.02} = 25 \\ &= \frac{K_{x\_required}}{K_{x\_plant}} = \frac{50}{2} = 25 \end{aligned} \tag{22}$$

This value for K_c will satisfy the steady-state error specification, and the rest of the compensator design will focus on the phase margin specification.

C. The Bode Plots

The magnitude and phase plots for $K_c G_p(s)$ are shown in Fig. 3. The dashed magnitude curve is for $G_p(s)$ and illustrates the effect that K_c has on the magnitude. Specifically, $|K_c G_p(j\omega)|$ is $20\log_{10}|25| \approx 28$ db above the curve for $|G_p(j\omega)|$ at all frequencies. The phase curve is unchanged when the steady-state error specification is satisfied since the compensator does not have any poles at the origin.

The horizontal dashed line at $-125°$ is included in the figure to indicate the phase shift that will satisfy the phase margin specification $(+10°)$ when the frequency at which this phase shift occurs is made the gain crossover frequency. The vertical dashed line indicates that frequency.

Note that the gain crossover frequency of $K_c G_p(s)$ is larger than that for $G_p(s)$; the crossover frequency has moved to the right in the graph. The closed-loop bandwidth will have increased in a similar manner. The phase margin has decreased due to K_c, so satisfying the steady-state error has made the system less stable; in fact, increasing K_c in order to decrease the steady-state error can even make the closed-loop system unstable. Maintaining stability and achieving the desired phase margin is the task of the pole–zero combination in the compensator.

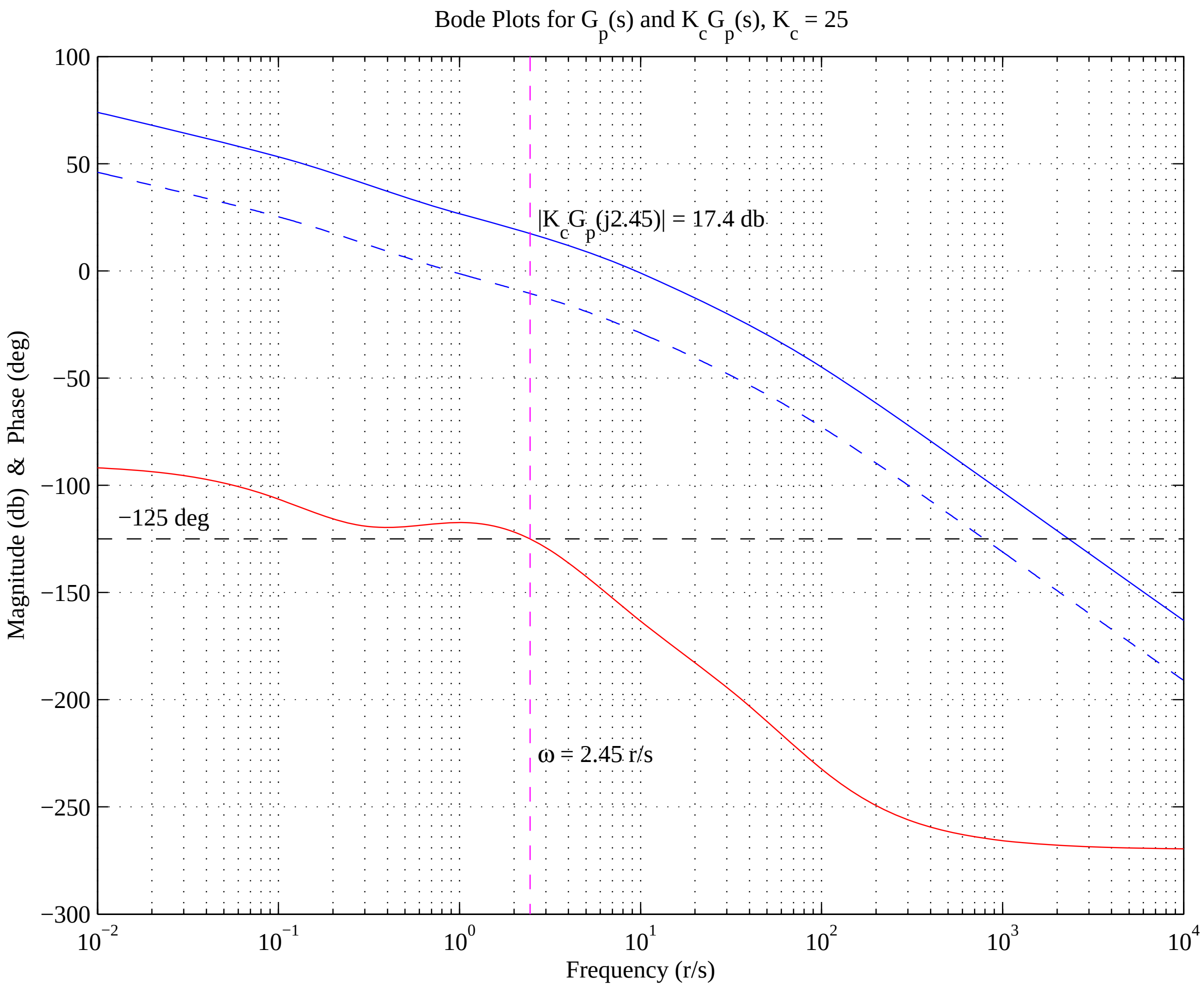

Fig. 3. Bode plots for the plant after the steady-state error specification has been satisfied.

Our ability to graphically determine values for $\omega_{x\_compensated}$ and α obviously depends on the accuracy and resolution of the Bode plots of $|K_cG_p(j\omega)|$. High resolution plots like those obtained from MATLAB allow us to obtain reasonably accurate measurements. Rough, hand-drawn sketches would yield much less accurate results and might be used only for first approximations to the design. Being able to access the actual numerical data allows for even more accurate results than the MATLAB-generated plots. The procedure that I use when working in MATLAB generates the data arrays for frequency, magnitude, and phase from the following instructions:

```
w = logspace(N1,N2,1+100*(N2-N1));
[mag,ph] = bode(num,den,w);
semilogx(w,20*log10(mag),w,ph),grid
```

where `N1`$= \log_{10}(\omega_{\min})$, `N2` $= \log_{10}(\omega_{\max})$, and `num`, `den` are the numerator and denominator polynomials, respectively, of $K_cG_p(s)$. For this example,

`N1`$= -2$;
`N2`$= 4$;
`num`$= 25 * 280 * \left[\begin{array}{cc} 1 & 0.5 \end{array}\right]$;
`den`$=$`conv`$\left(\left[\begin{array}{ccc} 1 & 0.2 & 0 \end{array}\right]\right.$, `conv`$\left(\left[\begin{array}{cc} 1 & 5 \end{array}\right], \left[\begin{array}{cc} 1 & 70 \end{array}\right]\right))$;

The data arrays `mag`, `ph`, and `w` can be searched to obtain the various values needed during the design of the compensator.

D. Gain Crossover Frequency

As mentioned above, the compensated gain crossover frequency is selected to be the frequency at which the phase shift $\angle K_cG_p(j\omega) = -125°$. This value of phase shift was computed from (8), which allows the phase margin specification to be satisfied, taking into account the non-ideal characteristics of the compensator.

From the phase curve plotted in Fig. 3, this value of phase shift occurs approximately midway between 2 r/s and 3 r/s. Since the midpoint frequency on a log scale is the geometric mean of the two end frequencies, an estimate of the desired frequency is $\omega = \sqrt{2 \cdot 3} = 2.45$ r/s. Searching the MATLAB data arrays storing the frequency and phase information gives the value $\omega = 2.4531$ r/s (using 100 equally-spaced values per decade of frequency), so the estimate obtained from the graph is very accurate in this example.

E. Calculating α

The lag compensator must attenuate $|K_c G_p(j\omega)|$ so that it has the value 0 db at frequency $\omega = 2.45$ r/s. From the magnitude curve in Fig. 3, it is easy to see that $|K_c G_p(j2.45)|_{db}$ is between 15 db and 20 db. An accurate measurement of the graph would provide a value of 17.4 db for the magnitude. The corresponding value of α is calculated by using (9), and the result is $\alpha = 7.435$, which can be easily implemented with a single stage of compensation. A search of the MATLAB data array storing the magnitudes (as absolute values) yields 7.435 as the element corresponding to $\omega = 2.45$ r/s, so equation (10) is verified.

F. Compensator Zero and Pole

Now that we have values for $\omega_{x\_compensated}$ and α, we can determine the values for the compensator's zero and pole from (15). The values that we obtain are $z_c = 0.245$ and $p_c = 0.033$. The final compensator for this example is

$$\begin{aligned} G_{c\_lag}(s) &= \frac{25\,(s/0.245+1)}{(s/0.033+1)} = \frac{25\,(4.08s+1)}{(30.3s+1)} \\ &= \frac{3.37\,(s+0.245)}{(s+0.033)} \end{aligned} \tag{23}$$

G. Evaluation of the Design

The frequency response magnitude and phase of the compensated system $G_{c\_lag}(s)G_p(s)$ are shown in Fig. 4. The magnitude is attenuated at high frequencies, passing through 0 db at approximately 2.45 r/s, so the compensated gain crossover frequency has been established at the desired value. The compensated phase curve is seen to be nearly back to the uncompensated curve at $\omega_{x\_compensated}$. The lag compensator is contributing approximately $-4.9°$ at that frequency, so the phase margin specification is satisfied.

To illustrate the effects of the compensator on closed-loop bandwidth, the magnitudes of the closed-loop systems are plotted in Fig. 5. The smallest bandwidth occurs with the plant $G_p(s)$. Including the compensator gain $K_c > 1$ increases the bandwidth and the size of the resonant peak. Significant overshoot in the time-domain step response should be expected from the closed-loop system with $K_c = 25$. Including the entire lag compensator reduces the bandwidth and the resonant peak, relative to that with $K_c G_p(s)$. The step response overshoot in the lag-compensated system should be similar to the uncompensated system, but the settling time will be less due to the larger bandwidth.

The major difference in the time-domain responses between the uncompensated system and the lag-compensated system is in the ramp response. The steady-state error is reduced by a factor of 25 due to K_c. The closed-loop step and ramp responses are shown in Fig. 6. A closed-loop steady-state error $e_{ss} = 0.02$ is achieved both for $K_c G_p(s)$ and $G_{c\_lag}(s)G_p(s)$. The faster of those two responses is from $K_c G_p(s)$.

To see some of the effects of the value of the compensator's zero, a second lag compensator was designed. The same specifications were used; the only difference between the two designs is that the zero of the new compensator was placed two decades in frequency below the compensated gain crossover frequency, rather than one decade. The value of the compensator pole also changed as a result. The new compensator is

$$\begin{aligned} G_{c\_lag\_2}(s) &= \frac{25\,(s/0.0245+1)}{(s/0.0033+1)} = \frac{25\,(40.8s+1)}{(303s+1)} \\ &= \frac{3.37\,(s+0.0245)}{(s+0.0033)} \end{aligned} \tag{24}$$

This new compensator reduces the percent overshoot to a step input and provides additional phase margin. The added phase margin is due to the fact that the compensator's phase shift is virtually zero at a frequency two decades above z_c. Both of these characteristics are good. The drawbacks to using the second compensator are the increased values of the electronic components (a problem particularly with the capacitors) and the increase in the settling time of the ramp response. The time it takes the unit ramp response to reach a value 0.02 less than the input (e_{ss}) increases from approximately 6.3 seconds with $G_{c\_lag}(s)$ to 316 seconds with $G_{c\_lag\_2}(s)$. Trade-offs such as this typically have to be made during the design of any control system.

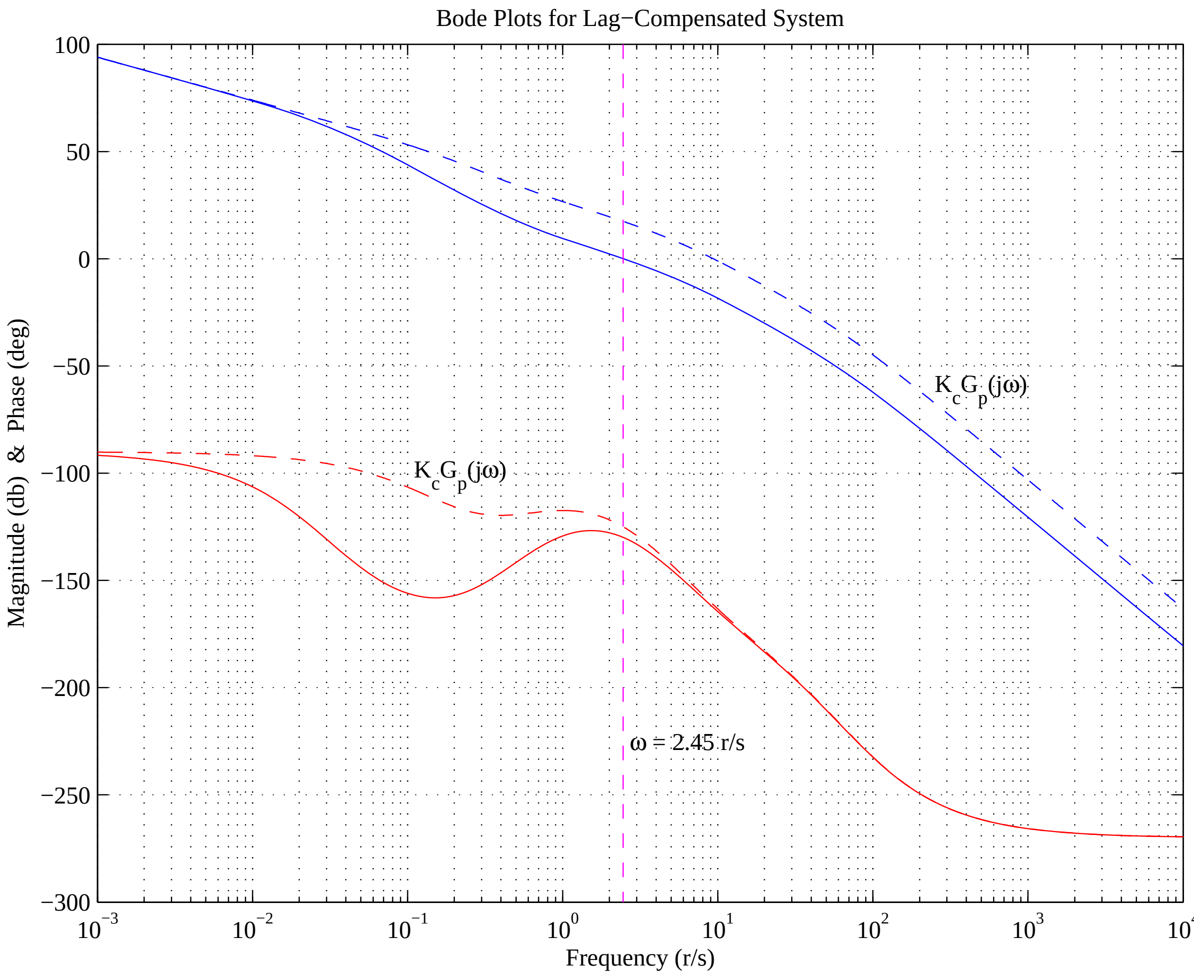

Fig. 4. Bode plots for the compensated system.

H. Implementation of the Compensator

Ogata [3] presents a table showing analog circuit implementations for various types of compensators. The circuit for phase lag is the series combination of two inverting operational amplifiers. The first amplifier has an input impedance that is the parallel combination of resistor R_1 and capacitor C_1 and a feedback impedance that is the parallel combination of resistor R_2 and capacitor C_2. The second amplifier has input and feedback resistors R_3 and R_4, respectively.

Assuming that the op amps are ideal, the transfer function for this circuit is

$$\begin{aligned}\frac{V_{out}(s)}{V_{in}(s)} &= \frac{R_2R_4}{R_1R_3}\cdot\frac{(sR_1C_1+1)}{(sR_2C_2+1)} \\ &= \frac{R_2R_4}{R_1R_3}\cdot\frac{R_1C_1}{R_2C_2}\cdot\frac{(s+1/R_1C_1)}{(s+1/R_2C_2)}\end{aligned} \tag{25}$$

Comparing (25) with $G_{c\_lag}(s)$ in (1) shows that the following relationships hold:

$$\begin{aligned} K_c = \frac{R_2R_4}{R_1R_3}, \qquad \tau = R_1C_1, \qquad \alpha\tau = R_2C_2 \\ z_c = 1/R_1C_1, \qquad p_c = 1/R_2C_2, \qquad \alpha = \frac{z_c}{p_c} = \frac{R_2C_2}{R_1C_1}\end{aligned} \tag{26}$$

Equation (26) shows that decreasing the value of z_c and/or increasing the value of α leads to larger component values. Since space on printed circuit boards is generally restricted, upper limits are imposed on the values of the resistors and capacitors. Thus, decisions have to be made concerning the values of z_c and p_c and the number of stages of compensation that are used.

Equations (25) and (26) are the same as for a lag compensator. The only difference is that $\alpha < 1$ for a lead compensator and $\alpha > 1$ for a lag compensator, so the relative values of the components change.

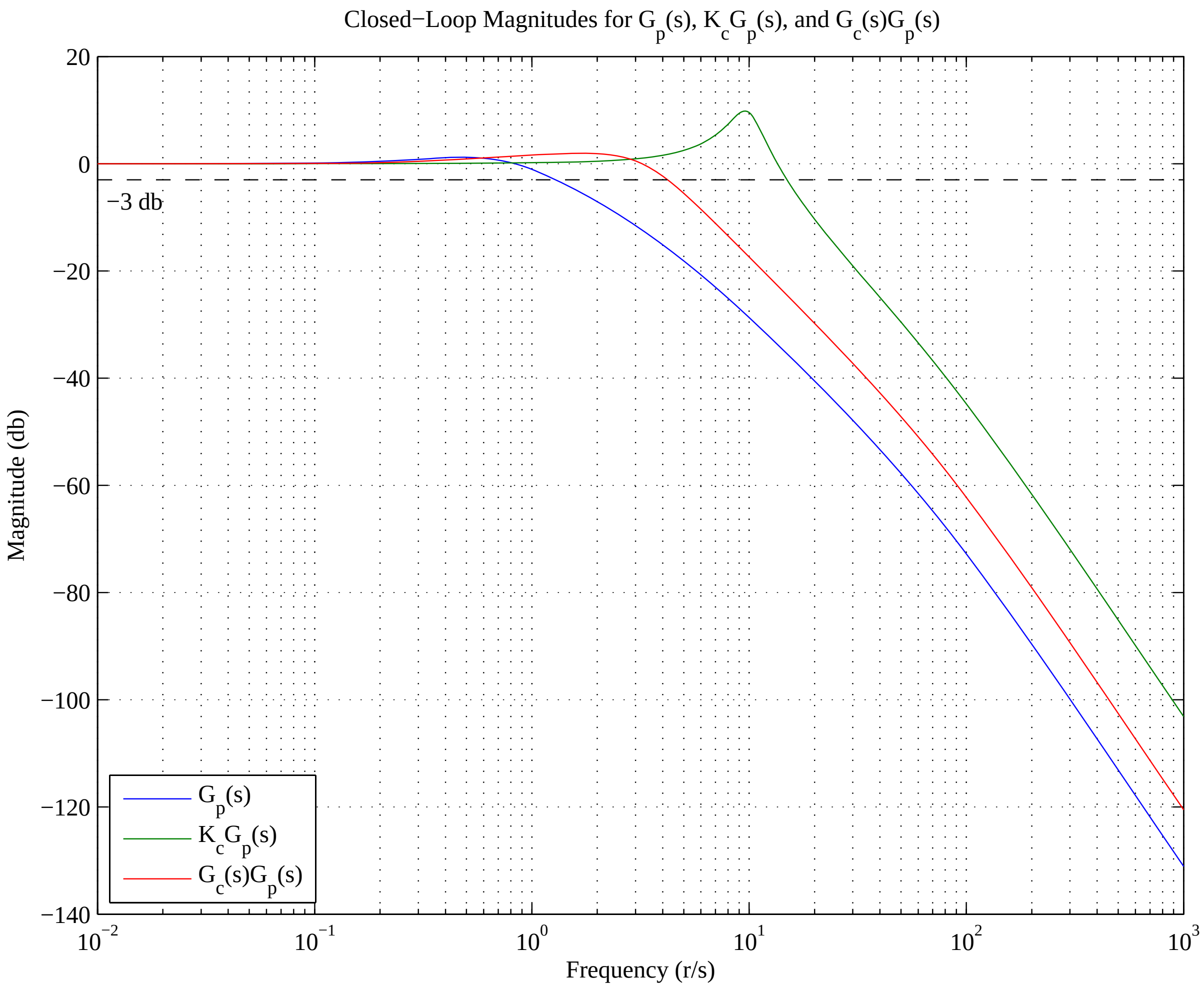

Fig. 5. Closed-loop frequency response magnitudes for the example.

To implement the compensator using the circuit in [3] note that there are 6 unknown circuit elements (R_1, C_1, R_2, C_2, R_3, R_4) and 3 compensator parameters (K_c, z_c, p_c). Therefore, three of the circuit elements can be chosen to have convenient values. To implement the original lag compensator $G_{c\_lag}(s)$ in this design example, we can use the following values

$$\begin{aligned} C_1 &= C_2 = 0.47\ \mu\text{F} \ = 4.7 \cdot 10^{-7}\ \text{F}, \qquad R_3 = 10\ \text{K}\Omega = 10^4\ \Omega \\ R_1 &= \frac{1}{z_c C_1} = 8.67\ \text{M}\Omega = 8.67 \cdot 10^6\ \Omega \\ R_2 &= \frac{1}{p_c C_2} = 64.4\ \text{M}\Omega = 6.44 \cdot 10^7\ \Omega \\ R_4 &= \frac{R_3 K_c}{\alpha} = 33.6\ \text{K}\Omega = 3.36 \cdot 10^4\ \Omega \end{aligned} \tag{27}$$

where the elements in the first row of (27) were specified and the remaining elements were computed from (26).

I. Summary

In this example, the phase lag compensator in (23) is able to satisfy both of the specifications of the system given in (19). In addition to satisfying the phase margin and steady-state error specifications, the lag compensator also produced a step response with shorter settling time.

In summary, phase lag compensation can provide steady-state accuracy and necessary phase margin when the Bode magnitude plot can be dropped down at the frequency chosen to be the compensated gain crossover frequency. The philosophy of the lag compensator is to attenuate the frequency response magnitude at high frequencies without adding additional negative phase shift at those frequencies. The step response of the compensated system will be slower than that of the plant with its gain set to satisfy the steady-state accuracy specification, but its phase margin will be larger. The following table provides a comparison between the systems in this example.

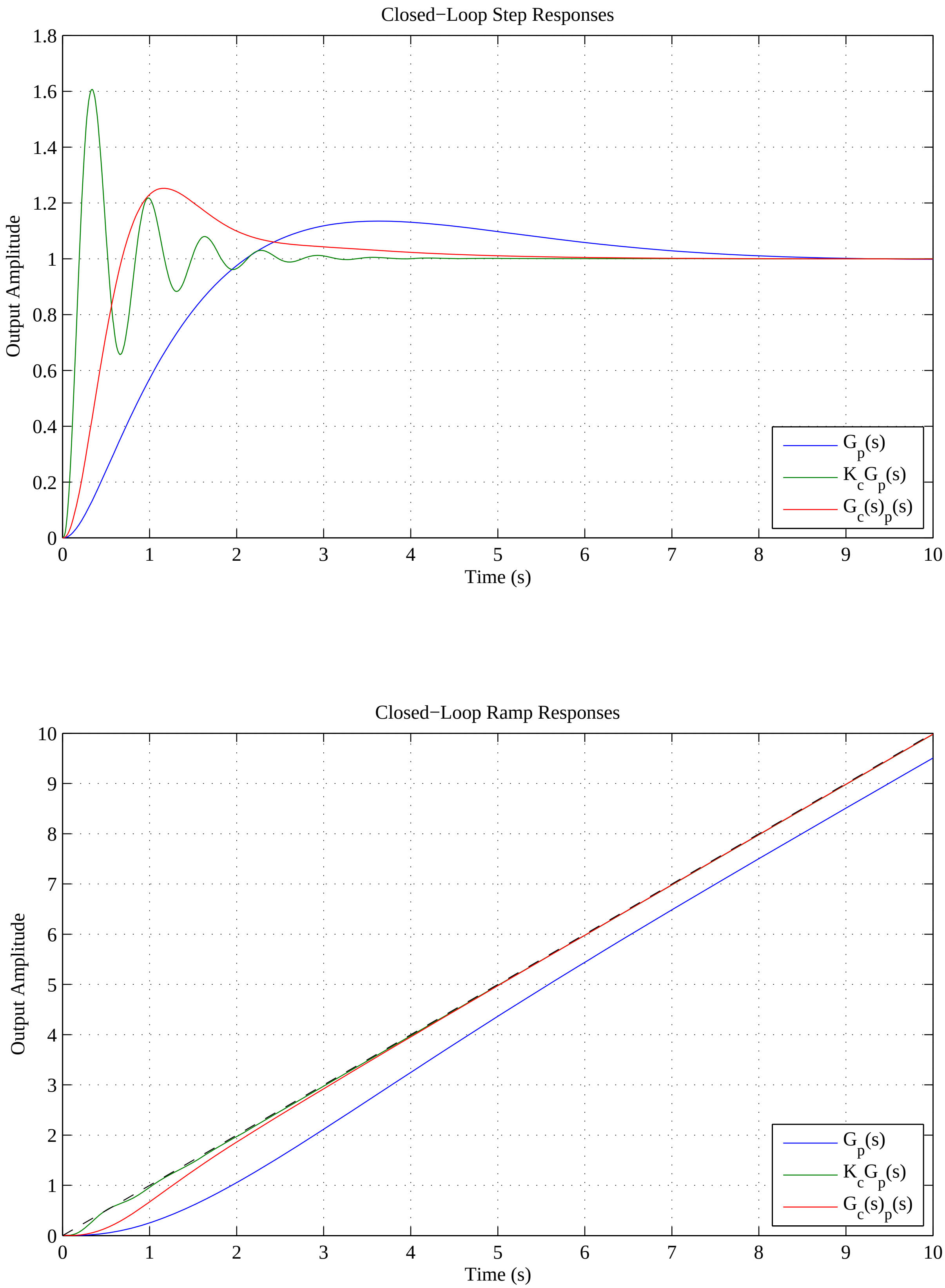

Fig. 6. Step and ramp responses for the closed-loop systems.

Characteristic	Symbol	$G_p(s)$	$K_c G_p(s)$	$G_{c\_lag}(s)G_p(s)$	$G_{c\_lag\_2}(s)G_p(s)$
steady-state error	e_{ss}	0.5	0.02	0.02	0.02
phase margin	PM	62.5°	18.7°	50°	54.5°
gain xover freq	ω_x	0.88 r/s	9.36 r/s	2.46 r/s	2.45 r/s
time delay	T_d	1.24 sec	0.035 sec	0.354 sec	0.387 sec
gain margin	GM	87.7	3.51	24.8	25.9
gain margin (db)	GM_{db}	38.9 db	10.9 db	27.9 db	28.3 db
phase xover freq	ω_ϕ	18.1 r/s	18.1 r/s	17.7 r/s	18.1 r/s
bandwidth	ω_B	1.29 r/s	14.9 r/s	4.21 r/s	4.12 r/s
percent overshoot	PO	13.5%	60.7%	25.3%	17.8%
settling time	T_s	7.52 sec	2.38 sec	4.24 sec	3.84 sec

References

[1] J.J. D'Azzo and C.H. Houpis, *Linear Control System Analysis and Design*, McGraw-Hill, New York, 4th edition, 1995.

[2] Richard C. Dorf and Robert H. Bishop, *Modern Control Systems*, Addison-Wesley, Reading, MA, 7th edition, 1995.

[3] Katsuhiko Ogata, *Modern Control Engineering*, Prentice Hall, Upper Saddle River, NJ, 4th edition, 2002.

[4] G.F. Franklin, J.D. Powell, and A. Emami-Naeini, *Feedback Control of Dynamic Systems*, Addison-Wesley, Reading, MA, 3rd edition, 1994.

[5] G.J. Thaler, *Automatic Control Systems*, West, St. Paul, MN, 1989.

[6] William A. Wolovich, *Automatic Control Systems*, Holt, Rinehart, and Winston, Fort Worth, TX, 3rd edition, 1994.

[7] John Van de Vegte, *Feedback Control Systems*, Prentice Hall, Englewood Cliffs, NJ, 3rd edition, 1994.

[8] Benjamin C. Kuo, *Automatic Controls Systems*, Prentice Hall, Englewood Cliffs, NJ, 7th edition, 1995.

[9] Norman S. Nise, *Control Systems Engineering*, John Wiley & Sons, New York, 3rd edition, 2000.

[10] C.L. Phillips and R.D. Harbor, *Feedback Control Systems*, Prentice Hall, Upper Saddle River, NJ, 4th edition, 2000.

[11] Graham C. Goodwin, Stefan F. Graebe, and Mario E. Salgado, *Control System Design*, Prentice Hall, Upper Saddle River, NJ, 2001.