Course introduction, Newtonian mechanics

Sunday, August 13, 2023 12:34 PM

The purpose of this course is to show you that starting with one or two laws you can deduce so many different formulas and equations.

In this course we will be shown how to follow the reasoning behind how we got the formulas and ideas we use to solve elementary physics problems.

Our first topic will be Newtonian Mechanics.

Newtonian mechanics along with all of physics works in two parts, given the present, we want to predict the future.

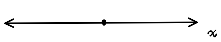
The present means that we will pick a part of what we need in order to predict the future. We see this type of deduction when having to catch a ball. But what did we need to know about the ball in order to catch it? It's implicit that the color of the ball, the shape of the ball, the emotions of the ball, are irrelevant. We are only interested in the initial velocity and position of the ball in order to predict the future, which is its final velocity and position.

The two parts is also referred to as

1) Kinematics, a complete description of the present, a list of what you have to know about a system right now
2) Dynamics, tells us why a thrown object goes up or why it goes down, how things change and why things change

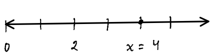
Let's take the simplest possible example of kinematics.

An entity that moves along a single line.



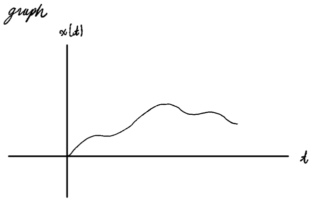
What constitutes the present?

Add position markers, and time



Position markers are in meters, time in t sec

We describe what the object does w/ a time versus space graph

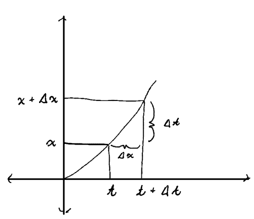


The object is not going up and down, it is changing position over time.

Two elementary results of this graph are

Velocity $\vec{v} = \dfrac{x(t_2) - x(t_1)}{t_2 - t_1}$ & acceleration $\vec{a}$

But a more important result are the fundamental ideas from calculus

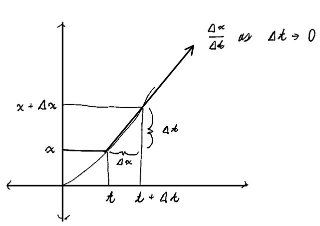


$v = \dfrac{\Delta x}{\Delta t}$, a derivative ↓

We look at a position now, and a position later, and change the "later" infinitely close to now, we can ascertain the velocity at the now.

$$v(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

$\dfrac{dx}{dt}$ or $\dfrac{\Delta x}{\Delta t}$ is a tangential line to the function on our graph



Once we can take a derivative, we can take any amount of derivatives as we please.

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

So if $x(t) = t^n$, then $\dfrac{dx}{dt} = nt^{n-1}$.

Lets specialize the rest of this class to a very specific set of problems.

$a(t) = a$ is constant, $g = 9.8\ m/s^2$

So given a constant acceleration, can we find what the position is?

$x(t) = ?$

What we need to do is find a function whose second derivative is "a". We know this as integration, integration at its fundamental root is guessing. So taking two derivatives with respect to t is a result of what function?

$x(t) = \frac{a}{2}t^2$ Is this the most general answer or just one answer?

This is only one answer, so lets add a constant.

$$x(t) = \frac{a}{2}t^2 + C$$

But we can add one more thing that is eliminated after taking two derivatives with respect to time.

$x(t) = \frac{a}{2}t^2 + C + bt$ And here we have the most general position for a particle that has a constant acceleration.

But we know that a particle can also move up and down, so we simply change the name of the function.

$$y(t) = \frac{a}{2}t^2 + C + bt$$

So we now have general formulas for a position of a particle with a constant acceleration, but we do not have any meaning for what C is or what bt is. So let us do some physicist -like analysis to give these terms meaning.

Given $x(t) = \frac{a}{2}t^2 + C + bt$ Recognize that only being given acceleration is not enough to define position.

Lets say a = gravity $= -g$ which affects the y-position

$y(t) = -\frac{g}{2}t^2 + C + bt$, notice that we still dont have meaning for C and bt

To find C, lets put t which is time in seconds, equal to 0. (t = 0)

$$x(0) = \frac{a}{2}(0)^2 + C + b(0)$$

$x(0) = C$ We now observe that C is the initial position of the particle.

C can be rewritten as x_0.

To find the meaning of bt let us first find the first derivative of x(t) with respect to time, then let t = 0.

$$\frac{dx}{dt} = v(t) = at + b$$

$\left.\dfrac{dx}{dt}\right|_{t=0} = v(0) = b \Downarrow v_0$ We now observe that b is the initial velocity of the particle.

So we can now define x(t) with more meaning.

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2$$

Now lets discuss another celebrated formula that was a result of the fundamentals of calculus.

In our expressions we keep switching from x(t) to y(t) so as to reinforce the notion that our function can be whatever we need it to be.

$$y(t) = y_0 + v_0 t + \frac{1}{2}at^2$$

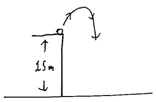
The second formula attempts to relate the final velocity to the initial velocity, as well as the distance traveled without the need for time.

First $\frac{dy}{dt} = v(t) = v_0 + at$, $t = \frac{v - v_0}{a}$, lets plug this time into our first equation. By doing this we get

$$v_f^2 = v_0^2 + 2a(x - x_0)$$

Now with such an intuitive understanding of these two formulas and their meanings, it's pointless to only memorize formulas. Instead, understand the logic behind them.

Let's solve a problem now. On top of a 15 meter high building an object is thrown with an initial speed of 10 m/s.



$V_0 = 10\ m/s$

We can claim now with our current understanding is that we can predict the motion of this object.

$y_0 = 15\ m \qquad a = -g = -10\ m/s$

$V_0 = 10\ m/s$ (rounded up)

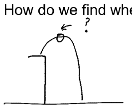
$y(t) = 15 + 10t - 5t^2$

Now be careful when using this formula, understand that t cannot continue forever, as the object will stop falling after it hits the ground. Look out for the assumptions that are implicit in physics problems.

How do we find the velocity of this object?

$V(t) = \frac{dy}{dt} = 10 - 10t$

How do we find when the object reaches its maximum height after being thrown?



By critical thinking, we know that the highest point is where the object is not going up or down. In other words, the objects velocity is 0.

Let's say the time for this point is at t*.

$10t^* - 10 = 0$

$10t^* = 10$

$t^* = 1$, now we plug this time into our position formula

$y(1) = 15 + 10 - 5 = 20m$

Now let us find when the object hits the ground, and what speed it will hit the ground at.

$y = 0$

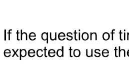
$0 = 15 + 10t - 5t^2$

$t^2 - 2t - 3 = 0$, here we have a quadratic equation so we use the quadratic formula

$t = \frac{2 \pm \sqrt{4 + 12}}{12} = \frac{2 \pm 4}{2} = -1, 3$ The object hits the ground at these times.

We get two solutions, obviously t = 3 is the right answer but why do we get two solutions?

This is because the motion of the object can be looked at as a parabola.



The mathematical expression doesn't know that there is supposed to be a building in the path of the object. This is an extra solution provided by the mathematics.

This is the beauty of physics. This is how we understand that there are particles, but also antiparticles. We find laws of motion in mathematical form, and the results we get are always have a meaning. This shows that despite us having a specific goal in mind, we can find an infinite amount of relevant information.

Let's now utilize the second equation we deduced.

If the question of time for the problem we are working with is not given, then we know that we are expected to use the second equation.

$(V_f)^2 = (V_0)^2 + 2a(x_f - x_0)$?

$0^2 = (V_0)^2 + 2(-g)(y - y_0)$, physicists tend to symbolically solve first

$y - y_0 = \frac{(V_0)^2}{2g} = \frac{100}{20} = 5m \rightarrow$ add to initial height to get max height

We can also get the speed when the object reaches the ground.

Let's derive the second equation using calculus.

We know $\frac{dv}{dt} = a$, multiply both sides by v

$v \frac{dv}{dt} = av$

$\frac{d}{dt}\left(\frac{v^2}{2}\right) = av = a\frac{dx}{dt}$?, the dt's cancel because dt gets infinitesimally small

So $d\left(\frac{v^2}{2}\right) = a\,dx$ so we add all changes (take integral)

$\Downarrow$

$\Delta\left(\frac{v^2}{2}\right) = a\,\Delta x$, this only holds true as the change gets small

Stay aware of when we can write equations and manipulate them in this manner.

$\int d\left(\frac{v^2}{2}\right) = a\int dx$

$\frac{v_f^2}{2} - \frac{v_0^2}{2} = a(x - x_0) \rightarrow v_f^2 = v_0^2 + 2a(x_f - x_0)$

physicists tend to symbolically solve first

$y - y_0 = \frac{(v_0)^2}{2g} = \frac{100}{20} = 5m \rightarrow$ add to initial height to get max height

We can also get the speed when the object reaches the ground.

$v_f^2 = v_0^2 + 2(-g)(0-25)$

Vectors in multiple dimensions

Sunday, August 13, 2023 4:00 PM

Last lecture we discussed the most simple case of particle motion, with a constant acceleration. We used two principles, kinematics and dynamics to construct two equations that allow us to analyze a particles motion and position.

One of these functions being…

$x(t) = x_0 + v_0 t + \frac{1}{2} a t^2$ an easy check for the validity of this equation is to take two derivatives and observes that the particle will have a constant acceleration left

We also deduced the different meanings of the symbols in these equations.

$x_0 \to$ initial position

$v_0 \to$ initial velocity

$a \to$ constant acceleration

Taking the derivative of position gives the velocity, thus taking the derivative of this formula is…

$v(t) = v_0 + at$

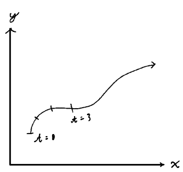
Remember that acceleration is simply the rate in which velocity is changing. This is a trivial observation when you write out the units…

SI : $m/s^2 = \frac{m/s}{s}$ means m/s changes every second

With this equation for velocity we can make observations that if you give the time, you can get velocity. Conversely, if we know the velocity, we can also find the time of the object. Therefore we can solve for t and plug this value of t back into our first equation and get…

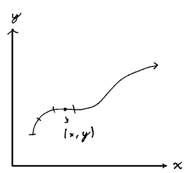
$v^2 = v_0^2 + 2a(x_f - x_0)$

Let us now work in two dimensions of motion. This is illustrated as…



This is not a graph with x dependent on time or y dependent on time, it's the change in position of the particle. Time can be represented as tick marks. Time is measured as a parameter along the motion of a particle, it's not explicitly shown.

How do we apply kinematics to this two dimension scenario?



You pick a point (x, y), that we are at.

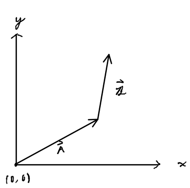
What is x, and what is y? How do we get a pair of numbers that represent position. This single entity is called a vector.

The simplest context where we are dealing with vectors is when we are looking at a real space of x and y, we move from a starting point in different increments at different times.

assuming we are told that we move from this start maybe 5 kilometers in one day first, and then 4 kilometers on the second day…

You cannot determine how far you are from your original starting point. We are missing the direction in which we move.

In one dimension this is solved by simply including a negative sign to indicate the opposite from the positive direction. With two dimensions, it is not as simple, as there are an infinite number of directions one can travel 5 km and then 4 km.



We use vectors to describe motion in a two dimension plane.

$\vec{A}$ is a description of what we did on the first day.

$\vec{B}$ of the second.

A vector is an arrow that has a beginning and end.

The magnitude of a vector is the length of the vector, and direction is at what angle does the length go.

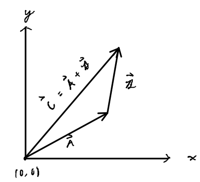
A vector is notated with an arrow at the top to indicate that it has a direction. If the arrow is omitted then we are discussing the magnitude of the vector, which has no direction.

It's then natural to notate $\vec{A} + \vec{B}$, as this means do $\vec{A}$ then $\vec{B}$

What if we want to notate the entire trip as one vector?

$\vec{A} + \vec{B} = \vec{C}$

$\vec{A} + \vec{B} = \vec{C}$ is illustrated as



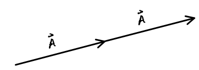
The addition of vectors is commutative.

$\vec{A} + \vec{B} = \vec{B} + \vec{C}$

$(\vec{0})$

Let us also define the zero vector. If a zero vector is added to another vector the vector will remain unchanged. You also cannot see this vector on an x and y illustration. This is also called the null vector.

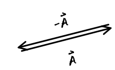
Consider the following vector.



The notation for this vector is $\vec{A} + \vec{A} = 2\vec{A}$

So what about $-\vec{A}$?

$\vec{A} - \vec{A} = \vec{0}$

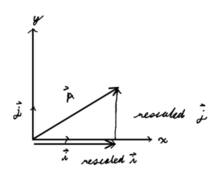


A minus vector is the same vector but flipped over going the opposite direction.

This is the same thing as multiplying the vector by -1. If we can multiply by a factor of -1 we can multiply by any negative factor.

What works with regular numbers usually naturally works with vectors.

Now let us come to an important concept. There are two special vectors called unit vectors.



Using, any vector can be rewritten as a re-scaled $\vec{i}$ & $\vec{j}$.

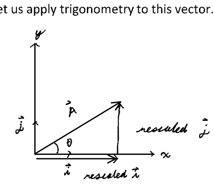
rescaled $\vec{i} = A_x \cdot \vec{i}$

rescaled $\vec{j} = A_y \cdot \vec{j}$

So we can then write our vector as…

$\vec{A} = A_x \vec{i} + A_y \vec{j}$

Let us apply trigonometry to this vector.



Dropping the arrow from a vector indicates that we are discussing the magnitude/length of the vector.

$A =$ length of $\vec{A}$

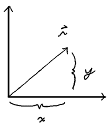
From pythagoreans theorem, we know

$A = \sqrt{A_x^2 + A_y^2} \longrightarrow$ Length of vector

also, $\tan\theta = \frac{A_y}{A_x} \to$ Orientation of vector

So given a pair of numbers A_x & A_y, it's as good as being given the arrow itself.

So,



$\vec{r} = \vec{i}\,x + \hat{j}\,y$

So, $\vec{A} + \vec{B} = \vec{C}$ can be expressed w/o any illustrations.

$$\vec{i} A_x + \vec{j} A_y + \vec{i} B_x + \vec{j} B_y = \vec{i}\,(A_x + B_x) + \vec{j}\,(A_y + B_y)$$

This means that when you add two vectors, you simply add the components.

$$\vec{i}\,(A_x + B_x) + \vec{j}\,(A_y + B_y) = \vec{i} C_x + \vec{j} C_y$$

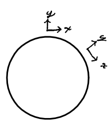
Therefore it follows $C_x = A_x + B_y$ & $C_y = A_y + B_y$

Additionally, if $\vec{A} = \vec{B}$ is a shorthand for $A_x = B_x$ & $A_y = B_y$

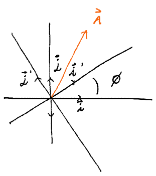
A_x & A_y are known as the components of the vector $\vec{A}$.

Now let us discuss the fact that a set of axes is not absolute. They can very easily be different.

Earth



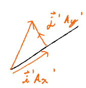
Let' s analyze how we can deal with somebody proposing a new set of axes relative to another set.



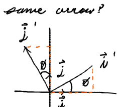
$\vec{A}$ can be written in terms of $\vec{i}$ & $\vec{j}$ as well as $\vec{i}'$ & $\vec{j}'$.

How can we relate the components of $\vec{A}$ in our two sets of axes?

$\vec{A} = \vec{i}' A_x' + \vec{j}' A_y'$ →



How much $\vec{i}$ & $\vec{j}$ do we need to express the same arrow?



$\vec{i}' = \vec{i}\cos\phi + \vec{j}\sin\phi$ ☆ ☆ WOAH!

$\vec{j}' = -\vec{i}\sin\phi + \vec{j}\cos\phi$

If we plug these into $\vec{A} = \vec{i}' A_x' + \vec{j}' A_y'$

$$= \vec{i}\,(A_x'\cos\phi - A_y'\sin\phi) + \vec{j}\,(A_y'\cos\phi + A_x'\sin\phi) = \vec{i} A_x + \vec{j} A_y$$

Therefore we can now make the relation of...

$A_x = A_x'\cos\phi - A_y'\sin\phi$

$A_y = A_y'\sin\phi + A_y'\cos\phi$

The entire purpose of this exercise was to make the remark that we can pick any basis vectors that are any two perpendicular directions. These will be related to any different set of perpendicular basis vectors related by an angle phi. An independent arrow can then be described by both basis vectors using the above relation. (assume in this exercise that our desires basic vectors are non-prime)

Let us now ask the opposite question, how can we use the prime numbers to get our original numbers.

$A_x' = ?$

$A_y' = ?$

View this question in the same manner in which you view...

$3x + 2y = 4$

$4x + 6y = 6$

$A_x = A_x'\cos\phi - A_y'\sin\phi$

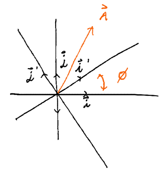
$A_y = A_y'\sin\phi + A_y'\cos\phi$

cos ∅ & sin ∅ are just numbers that can be eliminated

cos ∅ can = $\frac{1}{2}$ sin ∅ can equal $\frac{\sqrt{2}}{2}$

Although it may not be obvious at first, you simply treat the equations as a pair of simultaneous equations.

However you can also notice that in order to go from the prime basis vectors to our basis vectors, you move in a clockwise direction. So to undo this we simply need to move the same angle in a counterclockwise direction.



So, $A_x' = A_x\cos\phi + A_x\sin\phi$

$A_y' = -A_y\sin\phi + A_y\cos\phi$

(due to $\sin(-\phi) = -\sin\phi$, $\cos(-\phi) = \cos(\phi)$

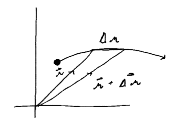
So there is another important message... when going from one set of axes to another set of axes, the components of the vector change. There is however, one quantity that remains the same despite a change in axes, and that is the length of the vector.

Therefore $A_x^2 + A_y^2 = A_x'^2 + A_y'^2$

In other words, the length of the vector is invariant.

The old understanding is that a vector has an arrow and a direction. But now, our understanding is that is a pair of numbers that transforms with the relations we revealed in our earlier exercises.

So, can we manufacture more vectors, given one vector?



$\vec{r}(t) = \vec{i}\,x(t) + \vec{j}\,y(t)$

$\vec{r}(t + \Delta t) = \vec{i}\,(x(t) + \Delta x) + \vec{j}\,(y(t) + \Delta y)$

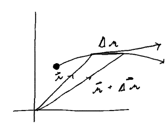
$= \vec{r}(t) + \underbrace{\vec{i}\,\Delta x + \vec{j}\,\Delta y}_{\Delta\vec{r}}$

↓ the change itself is a vector

Therefore the velocity vector is $\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \vec{i}\frac{dx}{dt} + \vec{j}\frac{dy}{dt}$

So we can take derivatives of a vector with respect to time, which results in another vector.

Just imagine the change in r going closer to the original starting point, this will give the tangent of the arc.



Also, $\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d\vec{v}}{dt}$

Just know that taking the derivative of a vector gives another vector. So with one vector we have now manufactured more vectors.

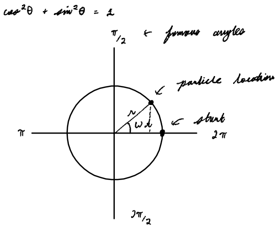
Let's do a concrete problem to see where we can apply these derivatives.

$r(t) = r\,(\vec{i}\cos wt + \vec{j}\sin wt)$

So what is going on as time passes? What is this particle doing?

We can tell that finding the length of the vector will give 1 because

$\cos^2\theta + \sin^2\theta = 1$



As t increases, the angle also increases.

Let us get a feeling for what W is.

How long does it take for the particle to do a full circle?

T = time period

Circle = 2π rad = $360°$ Radians are just another way to measure a circle.

$2\pi \approx 6$

We will measure angles in radians, this will prove useful at a later time.

Therefore $WT = 2\pi$ gives $W = \frac{2\pi}{T}$ ← Angular velocity

$\frac{1}{T} = f$ so W also $= 2\pi f$ f is Hz meaning "revolutions / s"

How fast is this particle moving?

$V = \frac{\text{dist}}{\text{time}} = \frac{2\pi r}{T}$, $\frac{2\pi}{T} = W$ so $V = Wr$

$\vec{v}(t) = r'(t) = \frac{d}{dt}[r\,(\vec{i}\cos wt + \vec{j}\sin wt) = r\,(\vec{i}\,(-w\sin wt) + \vec{j}\,(w\cos wt))$

$\vec{a}(t) = \vec{v}'(t) = -w^2 r^2$ meaning acceleration is always pointing to the center.

$A_y = A_y' \sin\theta + A_y' \cos\theta$

$\cos\theta$ & $\sin\theta$ are just numbers that can be eliminated

$\cos\theta$ can $= \frac{1}{2}$ $\sin\theta$ can equal $\frac{\sqrt{2}}{2}$

$v = \frac{}{time} = \frac{}{T}$, $\frac{2\pi}{T} = W$ so $v = Wr$

$\vec{v}(t) = r'(t) = \frac{d}{dt}[r(\vec{i}\cos Wt + \vec{j}\sin Wt) = r(\vec{i}(-W\sin Wt) + \vec{j}(W\cos Wt))$

$\vec{a}(t) = \vec{v}'(t) = -W^2 r^2$, meaning acceleration is always pointing to the center.

$a = W^2 r = \frac{(Wr)^2}{r} = \frac{v^2}{r}$ ☆ very important !

↑ ↖ centripetal acceleration

magnitude of $\vec{a}$.

Remember that this acceleration is pointing towards the center, since we are traveling in a circle.

Let's now get into a second class of problems.

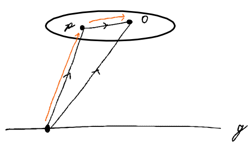
At any instant, if your motion follows a curve, you can approximate the centripetal acceleration using circles.

Another miscellaneous result...

p = plane

o = object

g = ground



The position of the object with respect to the ground, is the same as the position of the object with respect to the plane summed with the position of the plane with respect to the ground.

$\vec{r}_{og} = r_{op} + r_{pg}$, if we take derivatives

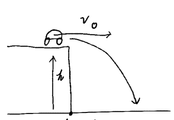
$\vec{v}_{og} = \vec{v}_{op} + \vec{v}_{pg}$

That means if you are going in a plane and you throw something off of it, it appear only to fall up and down. But from the ground it appears that the object has horizontal velocity and vertical velocity.

Consider a particle with $\vec{a}$, what is its location ?

In one dimension, $r = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2$

Observe the following...



where does the car hit the ground?

(0, 0) ← origin

In this situation we are working with two equations, one along x and one along y.

$x = x_0 + v_0 t$

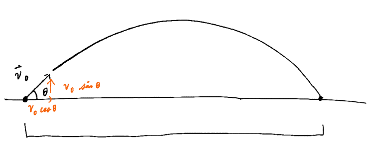
$y = h - \frac{1}{2}gt^2$ ← no initial velocity vertically

It's pretty obvious what we need to do to find when the car hits the ground.

t^* = ground hit time

$0 = h - \frac{1}{2}g(t^*)^2$, just solve for t^*

Consider ...



r (range)

$x = (v_0\cos\theta)t$ $y = (v_0\sin\theta)t - \frac{1}{2}gt^2$

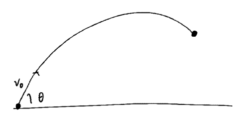
$r = ?$ range is how long we travel x until we hit ground

Plug $t = t^* = \frac{2v_0\sin\theta}{g}$ into x.

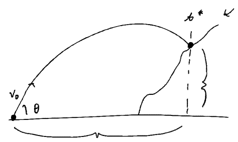
$$x = \frac{(v_0\cos\theta)\,2v_0\sin\theta}{g} = \frac{v_0^2}{g}2\sin\theta\cos\theta = \frac{v_0^2}{g}\sin(2\theta) = r$$

Do NOT memorize this formula, you simply do not have enough brain power to memorize things as there are an infinite number of variations.

Some elaboration on this solution, using critical thinking, obviously we want to try and hit the farthest spot. You can point straight to the spot along the ground, which doesn't work since the particle will get no air time. But you can also maximize time by shooting the particle straight up at an angle of 90 degrees. So intuition tells us that the most efficient angle to get our maximum range is between 0 and 90 degrees. 90 degrees is the maximum angle we can plug in for maximum time, if we plug that into our formula we notice that the most efficient angle for maximum range is 45 degrees!



find v_0 to hit a desired spot



same way as other problem

$x = (v_0 \cos\theta) t \qquad y = (v_0 \sin\theta) t - \frac{1}{2} g t^2$

$r = ?$ range is how long we travel x until we hit ground

$0 = h - \frac{1}{2} g (t^*)^2 = h - \left(\frac{1}{2} g t^*\right) t^*$

So $t^* = 0$ & $\dfrac{2 v_0 \sin\theta}{g}$

$\uparrow$

disregard initial time

Newton's Laws of motion

Friday, May 12, 2023 8:55 AM

We are going to talk about Newtons laws.

Until Physics 2, everything will be discussed in terms of Newtons laws.

Law 1, law of inertia.

The law of inertia says that if a body has no forces acting on it, then it will remain at rest, if it had a velocity to begin with, then it will maintain that velocity.

Which also means every-body will remain at rest or in uniform motion in a straight line. Which means a body will maintain its velocity if it is not acted upon by another force.

Galileo and Newton discovered that you don't need a force for a body to have a constant velocity.

However, this first law is not valid for everybody.

Imagine you are on an airplane, as the plane accelerates down the runway, anything on the floor will slide towards the back of the plane, even though we left the object at rest it appears to suddenly gain a velocity. This is an example of a person for whom the laws of inertia does not apply.

For the first law to apply, you need to be an inertial observer, meaning that in your system, objects left at rest must stay at rest.

The main point is that constant velocity is obtainable for free for an inertial observer. Another thing is that if you are able to find one inertial observer, you are able to manufacture an infinite amount of other inertial observers.

If the law of inertia is valid for me, it should be valid for the others in the room. Imagine you are observing a chalk on the ground and a train passes you at a constant velocity. For you the chalk is at a constant velocity, but this is also true for the person passing you in the train. The difference is simply that there is a constant velocity that needs to be added for the person on the train. For us watching the chalk on the ground, it appears at rest. For the person on the train, both us and the chalk appears to be moving backwards.

There is an infinite amount of inertial observers that differ by constant velocities.

In other words, constant velocities mean inertial observation, acceleration is non-inertial observation.

It's important to note that the Earth does possess an acceleration. However this is approximated to be inertial. (good enough basically)

Summary: constant velocity requires nothing.

Now let us discuss Law 2 of Inertia.

$F = m\,a$

a units : m/s^2

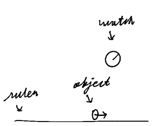
m units : kg

$m \cdot a = \frac{kg \cdot m}{s^2} = N$ (newton)

Okay so now that we have this law, what can we do with it? How do we prove that this law is correct? What exactly does this equation measure?

Lets start by understanding how we can measure acceleration.

We can measure acceleration by using a watch, and a ruler.



What do we do to measure instantaneous acceleration?

First find the initial and final velocity over a specific time period, then find the change in those velocities over the same time period.

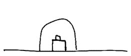
In the real world it's impossible to change the change in time close enough to 0, thankfully we can do this measurement in the world of mathematics. This is the meaning of the limit in calculus.

$\Delta x \to 0 \qquad \Delta t \to 0$

But, acceleration is pretty easy to measure, what we want is to verify whether or not Newton's second law stands correct.

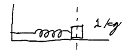
Let's discuss what the mass of an object is.

Let us take an arbitrary mass, a simple chunk or chunks of objects and denote that as one kilogram. Lets put that into a glass case.



We also have an elephant.

The one kilogram is a matter of convention, but our goal is to measure the "mass" of the elephant. A spring exerts an unknown force, and we also do not have a definition for force. Let's hook the kilogram to the spring, and then the elephant to the spring.



We don't know the force that the spring exerts onto the kilogram, but we know that

$a_{kg} = \frac{"F"}{1kg}$

$a_E = \frac{"F"}{M_E}$ Do the same for the elephant.

The force from the spring is unknown, but it is a consistent force between the two measurements.

If we divide both of our accelerations with each other...

$\frac{a_{kg}}{a_E} = \frac{M_E}{M_{kg}}$

The goal with this weird mechanism was to have a mechanism that exerts a consistent force. Using this system we can observe that the accelerations influence on the elephant and on the mass is an inverse relationship. $a_{kg} \cdot \frac{1}{a_E} = M_E \cdot \frac{1}{M_{kg}}$

If the acceleration of the elephant were to be one-hundredth of that of the mass in this mechanism, then the mass of the elephant would be 100 kilograms.

We can actually find more subtleties in this mechanism.

How do we know that the exertion that the spring exerts onto the kilogram is the same amount of exertion it exerts onto the elephant?

While we don't have a definition of force we can check this by having the mechanism pull the kilogram multiple times and measuring its acceleration as consistent.

This reasoning is important to understand because everything written down onto the paper to express solutions to a problem is a symbol of a measured quantity. You should always understand how to measure anything. If you don't know how to measure what you are working with, you're not doing physics.

This mechanism tells us that mass is how much an object is resistant to acceleration in response to an applied force.

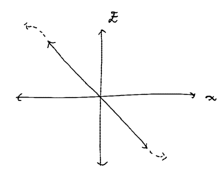
Newton tells us that forces cause acceleration, but acceleration is not the same on different objects. Certain objects have less accelerations than others, which means that they have a larger mass.

Using the inverse relationships, if a body has ten times the acceleration of a kilogram, then it is one tenth the size of that kilogram.

From now on, we can find the mass of any object using this mechanism.

We can now measure how much force is exerted from the spring in this mechanism.

If we pull the mass by 1 centimeter, then 1.1 centimeter, and 1.2 centimeters, and find the result of mass times acceleration. Let's graph the distance the spring is pulled versus the force exerted by the spring.



The dotted arrows means that for unmodest deformations the spring may have different exertion of force.

Let's focus on modest deformations.

We then get a formula of

$F = -\underbrace{k}_{\text{force constant}}x$

What is the purpose of the negative sign? We know that this if the force exerted by the spring exerted by the mass. If we pull the string away from the wall, then the spring will exert a negative force to drag it back to its equilibrium (spring is neither compressed nor stretched), also known as its natural length.

This means that if we compress the spring to the left, the force exerted is positive and pushing the spring out away from the wall.

Now we can take many different springs and calibrate the force exerted by them with our mechanism.

Consider two equations...

$F = m\,a \qquad F = -kx$

Why does there seem to be two definitions to force? Which one of these is newtons law?

Newtons law with force equaling mass times acceleration is universally true, while the spring force is only a measurement of the spring force.

Newtonian dynamics has two parts, if you find the forces acting on the body, without going into what caused the force, then you can use Newton's second law.

$\underbrace{F}_{\text{cause}} = \underbrace{m\,a}_{\text{result effect}}$

$F = ma$ doesn't tell us what forces will act on a body.

Newton does not tell us what force a spring exerts when its pulled by a certain amount. That is another part of our purpose as physicists. We need to constantly find out what forces act on a body. In every context, in which I place a body, I will have to know what are the forces acting on it. We find them by experimenting, put other bodies and observe how they react and then find out what the force or forces acting on a body are. Then and only then you can deduce a particular law in the same manner we deduced...

$$F = -kx$$

For example for a body near the surface of the earth, we can use...

$$F_g = -mg$$

Every problem you find a different force that is acting on the body that has a different origin.

$F_g = -mg$ can be verified because $a = \frac{-mg}{m} = -g$

This is a remarkable property of the gravitational force, since mass cancels out. The pull of the earth is itself proportional to the inertia of the object. When you divide by m, m cancels, everything falls at the same rate on the surface of the earth. There is actually a property of gravitational fields everywhere, even in outer space, but there is some residual fields between all the planets and all the stars in the universe, that the force on a body is proportional to the mass of the body. So when you divide by the mass to get the acceleration you get the same answer. Everything acts the same way in the gravitational field.

There are two quantities that end up as equal.

One is inertial mass which is how resistant you are to a change in velocity which is independent of gravitational forces, and the other is gravitational mass, which is the attraction to an object in space.

They are proportional and equal by units, and that turns out to be a part of a bigger picture in general relativity (this is a GREAT equivalence).

Again, physicists put bodies in various circumstances and deduce different circumstances.

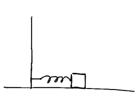
Another force you might find... You put a chunk of wood on a table and you try to move it at constant speed. Then you find that you have to apply minimum force. We are moving at constant velocity. That means the force you're applying is cancelled by another force, which has got to be the force of friction. So force of friction is yet another force.

$$F_f$$

Newton said F = ma, but didn't tell you what value F has in a given context. He just said whenever there's an acceleration, it's going to be due to some forces and it's your job to find what the forces are. To find the force, what you will do is, suppose somebody says, "Hey, I've got a new force. Every time I go near the podium, I find I'm drawn to it." Okay, that's a new force. The word gets around and we want to measure the force. What do I do? I stand near this podium. I'm drawn to it. I cannot stop. I tie a spring to my back and I anchor it to the wall and see how much the spring stretches before the two forces balance. Then I know that kx is equal to the force this is exerting at this separation. I move a little closer and I find the stretching is a different number. Maybe the force is getting stronger. That's how by either balancing the unknown force with a known force or by simply measuring the acceleration as I fall towards this podium and multiplying by mass, you can find the force that exerts on me. It's not a cyclical and useless definition. It's a very interesting interplay and that's the foundation of all of mechanics. We are constantly looking for values of F and we're constantly looking for responses or bodies to a known force.

Let's observe a complete Newtonian problem.

A mass is attached to a spring. It is pulled by a certain amount x, and is released. What is it going to do?



Newtons 2nd law gives...

$$F = ma$$

$$-kx = m\frac{d^2x}{dt^2}$$

Suddenly, you have a mathematically complete problem. Mathematically complete problem is that you can find the function x(t) by saying that the second derivative of the function is equal to -k over m times the function. We don't have to worry about how you solve it, but it's problem in mathematics and the answer will be--surprise, it's going to be oscillating back and forth and that'll come out of the wash. This is how you formulate problems.

$F_T = ma$

↓

total force acting on a body

Newtons third law:

$$F_{12} = -F_{21}$$

If body one is acting on a body two, then this force is equivalent to the negative of the force of body two acting on body 1.

What does it require to be a successful mechanic, to do all the mechanics problems? You got to be good at writing down the forces acting on a body. That's what it's all going to boil down to. Here is my advice to you. Do not forget the existing forces and do not make up your own forces. I've seen both happen. Right now at this point in our course, whenever you have a problem where there is some body and someone says, "Write all the forces on it", what you have to do is very simple. Every force, with one exception, can be seen as a force due to direct contact with the body. Either a rope is pulling, a rope is pushing it, you are pushing it, you are pulling it. That's a contact on the body. If nothing is touching the body, there are no forces on it, with one exception which is, of course, gravity. Gravity is one force that acts on a body without the source of the force actually touching it. That's it. Do not draw any more forces. People do draw other forces. When a body is going around a circle, they say that's some centrifugal force acting. There is no such thing. Be careful. Whenever there is a force, it can be traced back to a tangible material cause, which is all the time a force of contact, with the exception of gravity. Okay, so with that, if you write the right forces, you will be just fine. You will be able to solve all the problems we have in mechanics.

Here is our first mechanics problem.

Here is some object, it's 5 kilograms and I apply 10 Newtons. Someone says, "What's the acceleration?" Imagine that this is in outer space where there is no gravity for now. Motion is only along the x axis.

$$F = ma$$

$\boxed{5kg} \to 10N$ $\quad a = \frac{10}{5} = 2\ m/s^2$

Then, the next problem is a little more interesting. Here I got 3 kg and I got 2 kg and I'm pushing with 10 Newtons and I want to know what happens.

$\xrightarrow{10N}\boxed{3}\boxed{2}$ $\quad F = ma$

Let's use a free-body diagram.

$\xrightarrow{10N}\boxed{3}$ any other acting forces?

$\xrightarrow{10N}\boxed{3} \leftarrow F_{23}$ ⇓ f

$\overset{f}{\Rightarrow}\boxed{2}$ This is the ONLY acting force on this body.

Now we apply newtons 2nd law for both bodies.

$$10 - f = 3a$$
$$f = 2a$$

Acceleration is the same for both of these bodies.

$$10 - f = 3a$$
$$+ \quad f = 2a$$
$$10 = 5a$$
$$a = 2$$

so $f = 2(2)$

$f = 4N$

so $\xrightarrow{10N}\boxed{3}\xleftarrow{4N}$

so $\frac{6N}{3} = 2 = a$.

$\boxed{3}$—$\boxed{2}\to 10N$ massless rope

$\boxed{3}\to \quad \overset{F}{\longleftrightarrow}$ A massless rope will always have equal and opposite forces on both ends.

Therefore newtons third law gives...

$\overset{F}{\leftarrow}\boxed{2}\to 10N$

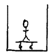
Now we can apply newtons second law.

$$F = 3a$$
$$+\ 10 - F = 2a$$
$$10 = 5a$$
$$a = 2$$

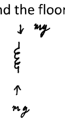
$F = 3(2)$

$F = 6N$

I'm going to give the last class a problem which is pretty interesting, which is what happens to you when you have an elevator. Here is a weighing machine and that's you standing on the elevator.



We're going to ask, "What's the needle showing at different times?" First, take the case in the elevator is on the ground floor of some building and completely addressed. Then, let's look at the spring. The spring is getting squashed because you are pushing down and the floor is pushing up. You are pushing down with the weight mg, and the floor has got to be pushing up with the mg, because the spring is not going anywhere.



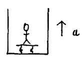
$F = -kx$

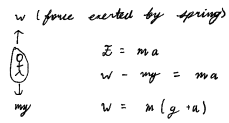
$mg = -k(-x)$

$x = \frac{mg}{k}$

Notice that force is exerted at both ends of a spring.

Now, what happens if the elevator is accelerating upwards with an acceleration a?





$F = ma$

$w - mg = ma$

$w = m(g + a)$

This means that the more you accelerate upwards, the heavier you feel.

You feel heavy and it reads more because the poor spring not only has to support you from falling through the floor, but also accelerate you counter to what gravity wants to do. That's why it is g plus a.

As you come to the top of the building, the elevator has to decelerate, so that it loses its positive velocity and comes to rest. So a will be negative and w, in fact, will be less than mg.

$w = m(g - |a|)$ If acceleration were to be equal to gravity, you would feel weightless.

Newton's laws continued and inclined planes

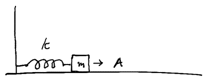
Sunday, October 8, 2023 11:21 AM

Let us remember Newton's second law…

$F = m \cdot a$

Force mass acceleration

Here is a table and here's a spring and here's the mass m. There's a force constant k. I want to pull it by some amount A, and let it go. So that's the knowledge of the present. The question is, when I let it go, what's this guy going to do?



That's the typical physics problem. It can get more and more complex. You can replace the mass by a planet; you can replace the spring by the Sun, which is attracting the planet; you can put many planets, you can make it more and more complicated. But they all boil down to a similar situation. I know some information now and I want to be able to say what'll happen next.

The second law of Newton only gives $F = ma$

The first thing you have to know in order to use the laws of Newton, you have to separately know the left-hand side. You have to know what force is going to act on a body. You cannot simply say, "Oh, I know the force on the body, it is m times a; ma is not a force acting on a body; a is the response to a force; you got to have some other means of finding the force.

I study the spring. And I've learned, by studying the spring, that the force it exerts is some number k, called a force constant, times the amount by which I pull it. If I start off the mass in a position where the spring is neither expanded nor contracted, that's what we like to call x = 0. So I pulled it to x = A.

$$F = ma$$
$$\downarrow$$
$$-kx = ma$$
$$-kx = m\frac{d^2x}{dt^2}$$

You now go from a physical law, which is really a postulate. There is no way to derive F = ma. You cannot just think about it and get it. That means don't even try to derive it. It just summarizes everything we know in terms of some new terms, but it cannot be deduced.

On the other hand, the fate of this mass can now be deduced by applying Newton's law to…

$$-kx = m\frac{d^2x}{dt}$$

The time derivative of this unknown function is not a given number but the unknown function itself; in other words, x itself is a function of time. This is called a differential equation.

$$-kx(t) = m \cdot \frac{d^2x}{dt^2}$$

We can solve this equation by guessing. In fact, the only way to solve a differential equation is by guessing the answer; there is no other way. You can make a lot of guesses and every time it works you keep a little table; then you publish it, called Table of Integrals.

People have tabulated them over hundreds of years. But how do they find them? They're going to find them in the way I'm going to describe to you now. You look at the equation and you guess the answer. Let's make our life simple by taking a case where…

$-k = 1, \quad m = 1,$ this gives

$$\frac{d^2x}{dt^2} = -x$$

lets guess $x(t) = \cos(t)$

$$\frac{dx}{dt} = -\sin(t)$$
$$\frac{d^2x}{dt^2} = -\cos(t)$$

There is something off in our guess however.

If I put t = 0, I get x = 1. Why should it be true that I pulled it by exactly one meter? I could have pulled it by 2 or 3 or 9 meters. I want to be able to tell how much I pulled it by at t = 0.

To fix this we can add a constant multiplier.

$$x(t) = 5\cos(t)$$

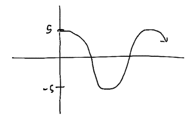
$\frac{d^2x}{dt^2} = -5\cos(t) = -x$ NOT $-5x$, The entire function itself with the constant multiplier is x.

Here's an answer that does everything you want it to do. At the initial time, it gives you 5 times cos 0, which is 5; if that's what you told me was the initial displacement. Initial velocity, if you take a dx/dt, it's proportional to sin (t) and a t = 0, that vanishes, that's correct; you pulled it and you let it go. So, the instant you released it, it had no velocity. And it satisfies Newton's laws, and that's your answer.

This is the paradigm. This is the example after which everything else is modeled.

The way F = ma is applied in real life is half the world is working on the left -hand side, finding the forces that act on bodies under various conditions. In this example, the force is due to a spring and by playing with a spring you can study the force it exerts the various displacements of various amounts by which you pull it, and in this example if you pull the spring by an amount x, it exerts the force -kx, and you can measure k and you put the label on your product and you sell the spring with a given k.

We can now plot the springs motion :



Now, will this spring oscillate forever? No.

We have missed something. There's another force acting on this mass, besides the spring; that's the force of friction that'll oppose the motion of the mass.

Either you can say something is wrong with Newton's laws, or you can say we've not applied Newton's laws properly because we haven't identified all the forces.

You can always push the frontiers of observation until you come to a situation where a law doesn't work.

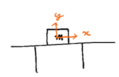
No one's found anything wrong with Newton's laws, provided you don't violate two conditions. You don't deal with objects moving at speeds comparable to that of light, and you don't deal with objects which are very, very tiny.

Let us start working with Newton's laws in a higher dimension.

In higher dimensions, we work with vectors. Therefore…

$$\vec{F} = m\vec{a}$$

Table on which this mass is sitting, doing absolutely nothing.



$\vec{F} = m\vec{a}$ means $\vec{i} F_x + \vec{j} F_y = m \vec{i} a_x + m \vec{j} a_y$

If two vectors are equal, then the x components have to match and the y parts also have to match.

So $F_x = m a_x$

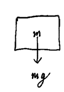
No known forces acting in the x.

so $F_x = m a_x$

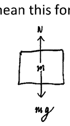
$0 = 0$

$F_y = m a_y$

lets draw a free-body diagram



If that's all you had, the block would fall through the table; it's not doing that, so the table we know is exerting a force. The standard name for that is N, and N stands for normal. And normal is a mathematical term for perpendicular. You say this vector is normal with that vector, you mean they are perpendicular, and here we mean this force is perpendicular to the table.



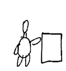
Therefore $F_y = m a_y$ gives $N - mg = ma$

$a = 0$ so $N - mg = 0$ therefore $N = mg$

Let us now introduce another force: friction.

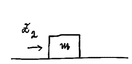
How do we learn there is a force of friction?

I try to push a podium, well, I don't know my own strength, but I'm going to imagine, I push it, it's not moving. But if I push hard enough it will move. What's happening before it moves? I'm applying a force and I'm getting nothing in return for it. So I know there is another force opposing what I do, and that's called a force of static friction.



no motion

[back to the box]



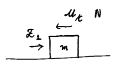
Static friction is not a fixed force; it's whatever it takes to keep it from moving. It will not be less than what I apply, because then it'll move; it cannot be more than what I apply because then it'll start moving backwards.

$f_s \le \mu_s N$

coefficient of static friction — normal force

In this case $N = mg$

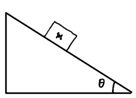
Once you surpass the maximum force resistance by static friction, the box will start sliding. After this occurs you will then have...



Additionally, $\mu_k < \mu_s$

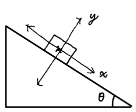
μ sub-static is a fixed number, .25. If it's .25 for static, it means if the normal force is some mg, up to a quarter of the weight is the frictional force it can apply to prevent its motion. Suppose I apply a force on a body, which is originally 1/10^(th) of its weight; it won't move; 2/10^(th) of its weight it won't move; a quarter of its weight it's a tie; .26 of its weight it'll start moving. Once it starts moving, frictional force will be .2 times its weight, not .25, because the kinetic friction is less.

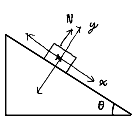
Let us now do the famous problem that sends people away from physics. The inclined plane.



what will this mass do?

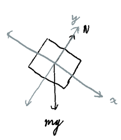
All right, we know it's going to slide down the hill, but we want to be more precise, and the whole purpose of Newton's Laws is to quantify things for which you already have an intuition. So the only novel thing about the inclined plane is that for the first time we are going to pick our x and y axes not along the usual directions, but along and perpendicular to the incline.

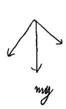




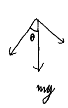
1) what are the contact forces
- no friction

2) non contact forces

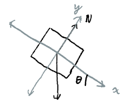




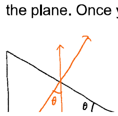
If



then

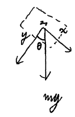


θ is the angle between the horizontal and the incline. Now, you've got to agree that if I draw perpendicular to the horizontal and I draw perpendicular to the incline, the angle between those two perpendiculars will be the same as the angle between the two lines I began with, because perpendicular means rotate by 90. If you rotate both lines by 90, the angle between the rotated lines is the same. The vertical is perpendicular to this horizontal, and this is the perpendicular to the plane. Once you got that, the rest is very easy.



(dotted lines are the box)

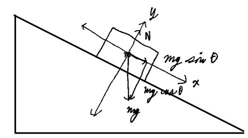
Therefore



$x : mg \sin\theta$
$y : mg \cos\theta$

Get used to this.

So now,



Applying newtons law in the x-direction : $mg\sin\theta = m a_x$

y-direction : $N - mg\cos\theta = m a_y$

We chose this set of axes at this particular angle so that there will be no motion in the y axis.

Therefore : $N - mg\cos\theta = m(0)$

$N = mg\cos\theta$

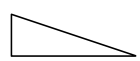
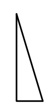
$\cancel{m}g\sin\theta = \cancel{m}a_x$

$a_x = g\sin\theta$ Thus we can identify that making theta small, acceleration will also get small.

Don't put in numbers till the very end. So, if you are told the incline is an angle of 37 degrees, g is 9.8, don't start putting numbers into the first equation.

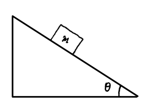
It's much better to work with symbols than with numbers till the end. Why is that?

You can also test your results. We know sin theta works considering that if we play around with the value of theta the acceleration reasonably increase or decreases.

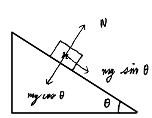
 has slower acceleration than 

What is missing in this formula that could have been there? Mass. The mass. It doesn't seem to matter what the object is. That's an interesting property of the result. Right? And that's because m cancelled out of the two sides of that equation. So, these are all interesting features you would not know if you kept numbers everywhere and you got a = .62 meters per second squared. To me, that number doesn't tell me as much. At the end, I want to know the number but the formula is very interesting.

Let's make the problem a little more complicated. Friction is now in this system.



Let the block be at rest and θ is an angle I can vary. Maybe there is a little thingy here, you can turn it back and forth. Let me crank up θ and see where it'll start slipping. In other words, there is a coefficient of static friction between the block and the plane, and I want to know, like parking a car in San Francisco, what angle can it take before it starts sliding down.



What is friction? NOT $\mu_s \cdot N$

In other words, if θ was very small and mg sin θ is very small, the frictional force will be in fact mg sin θ. So, it'll equal whatever it takes to keep it from slipping. But if I've cranked up the angle to that maximum angle, beyond which it cannot stay there, let's give a name to that angle that is sin θ*.

So $mg\sin\theta = \mu_s N$

$mg\sin\theta = \mu_s mg\cos\theta$

$\tan\theta \le \mu_s$ If we tilt our incline more than that then the block will topple over.

$\tan\theta^* = \mu_s$

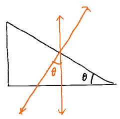
Shankar: You notice that I'm not writing every step, every time. Each time I deal with it I'm going to do some things faster. For example, I already resolved gravity to two parts, without going through the whole song and dance, because you know how to do that. Now, I put in the friction. So now, I've got μ sub kinetic not equal to 0, mass is moving.

$\mu_k \ne 0$, mass moving What is the acceleration now?

$mg\sin\theta - \mu_k N = m a_x$

$mg\sin\theta - \mu_k mg\cos\theta = m a_x$

the plane. Once you got that, the rest is very easy.



$\mu_k \neq 0$, mass moving What is the acceleration now?

$mg\sin\theta - \mu_k N = ma_x$

$mg\sin\theta - \mu_k mg\cos\theta = ma_x$

The question is, can you do this? Would you have thought about this? Do you agree that I'm not doing anything beyond Newton's laws when I do this?

If you can do this, you know everything I know about Newton's laws. This is all there is to Newton's laws and how to apply them.

$\cancel{m}g\sin\theta - \mu_k \cancel{m}g\cos\theta = \cancel{m}a_x$

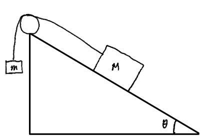
$g(\sin\theta - \mu_k\cos\theta) = a_x = a$

I want to study this in various limits. The first limit I take is no coefficient of kinetic friction, I get back to my sin θ. Well, let me try the following interesting limits. I crank up the coefficient of kinetic friction, more and more and more. Look what happens here: sin θ and cos θ are fixed numbers. If this becomes larger than some amount, than say 10, what happens to the expression? a is negative. And what does that mean? Student: It's going uphill. The block is moving uphill. And do you buy that? So, what's wrong? I mean, I took the equation, right? I told you all the time how you can study various limits. Something is wrong. It says, if kinetic friction is very large, the block will move uphill.

The correct answer is, in writing this forced law, I'm assuming the force of friction points to the left, uphill. That is correct only if the body is moving downhill. But if you got an answer where a was negative, therefore it's going uphill, the starting premise is wrong. So, one very useful lesson to learn here is when you apply formula, don't forget the conditions under which you derived it, and don't apply the result to a case where the answer does not belong to the domain that you assumed you're applying it to. So, the trouble with friction is the following. Friction is not a definite force with a definite magnitude or direction. The magnitude is definite, but direction is not definite. It is uphill if you're going downhill; it's downhill if you're going uphill. It's like the constant opposition to motion.

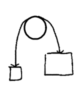
So in a formula, where the forces are what they are in any context, you can apply the formula in any limit. But if the frictional force assumed a downward motion, don't apply it when the block is going up.

This is a problem of two masses. There's a rope that goes over the pulley, let's call this m, let's call that M, and this angle again, it's θ, and no friction.

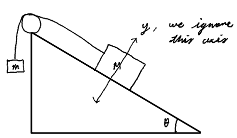


If the big mass M is bigger than the small mass m, is that enough to say it'll go downhill?

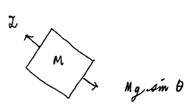
It depends on the angle because part of the mass is not helpful in going downhill.



In a case like this it is enough to say that the bigger mass will overcome the weight of the smaller mass, but in the incline plane not all of the mass is being pulled down by gravity.

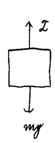


Lets look at the inclined plane direction.



gives $Mg\sin\theta - T = Ma$

And also the hanging mass.



$T - mg = ma$

For $Mg\sin\theta - T = Ma$ we are in ↖ direction

For $T - mg = ma$ we are in ↕ direction.

However a is a magnitude thus equivalent in both equations.

Assuming the rope is inelastic, if the small mass moves down one then the big mass will go up one inch. Therefore, if either of the masses go one inch in any direction, they will have the same velocity and acceleration which is why there is only one unknown acceleration for both masses.

$Mg\sin\theta - T = Ma$

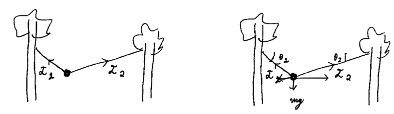
$+ (-mg + T = ma)$

The second thing to notice is: can I use this formula when m is bigger than M sin θ? In other words, can I use this formula when a becomes a negative? After all, I did all my thinking saying it's going down, but can I use it when it's going up?

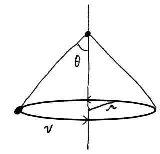
The answer is yes because there is no friction.

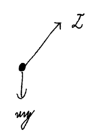
All these forces are fixed in magnitude and direction, and therefore once you got a formula for positive a, you can apply it to negative a, but you cannot do it when you have friction because in friction, you assume the direction of motion.

There's a tree here, there's a tree here, and that is--there are two ropes and there's a backpack hanging here with some weight. You are told what that angle is; you're told what that angle is. You are supposed to find the tension on these two ropes.

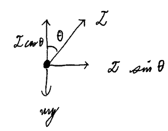


This is a string on which there is a mass and the mass is going around in a circle. This is like an amusement park, so that they have these dangling little, tiny, baby rockets and you sit there and start spinning, and instead of being vertical, it starts lifting up, and you want to know why it's lifting up. So, if this angle here is θ, and this is some mass m, and it's going around in a circle with velocity v, and the radius of the circle is say r; we want to find some relation between this angle θ and these other parameters in the problem. So what do I do?





This is all. Now we analyze a little more and apply $F = ma$.



Therefore $mg = T\cos\theta$ (vertical motion)

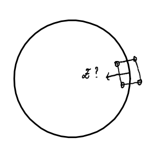
But the acceleration is circular, therefore for the horizontal motion we express

$$m\frac{v^2}{r} = T\sin\theta$$

So
$$\frac{mg = T\cos\theta}{m\frac{v^2}{r} = T\sin\theta} \Rightarrow \frac{v^2}{rg} = \tan\theta$$

Thus we have the relation between theta and the other parameters.

Here's another interesting problem. When you go on these roads, if you're going in a circle--so for this is your car, I'm looking at you from the top, you're going in a circle. Going in a circle, you're accelerating towards the center; someone's got to provide the force, that's what Newton says. That force is the friction between your tire and the road. In fact, it's a case of static friction. You might think it's kinetic because the car is moving, but it's not moving in this direction. So, it's a static friction that keeps you from slipping.

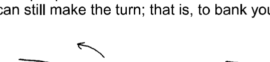


So $N\cdot\mu_s = \frac{mv^2}{r}$

$mg\cdot\mu_s = \frac{mv^2}{r}$

Without static friction your car would shoot outwards away from the center of the road.

So, what people have done is to find a clever way in which you don't have to have any friction and you can still make the turn; that is, to bank your road.



So we find $N\cos\theta = mg$

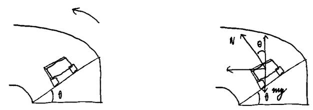
$$Mg\sin\theta - T = Ma$$
$$+\ (-mg + T = ma)$$
$$Mg\sin\theta - mg = Ma + ma$$
$$\frac{g(M\sin\theta - m)}{M+m} = a$$

Now does this formula make sense?

You notice that for it to be positive, and to go downhill, it's not enough if M is bigger than m. M sin θ should be bigger than m; that's because M sin θ is the part that's really pulling downhill. M cos θ is trying to ram it into the inclined plane, and that's being countered by the normal force.

Can I use this formula when a becomes a negative?

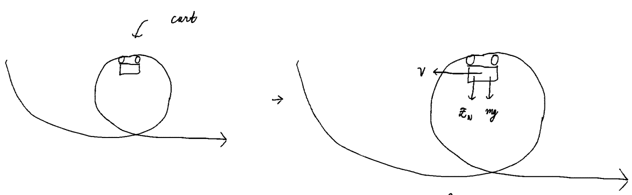
So, what people have done is to find a clever way in which you don't have to have any friction and you can still make the turn; that is, to bank your road.



So we find
$$N\cos\theta = mg$$
$$N\sin\theta = m\frac{v^2}{r}$$
Take the ratios and we get
$$\tan\theta = \frac{v^2}{rg}$$

What that means is the following. If you want the car to go around the bend, at a certain speed, 40 miles per hour or so many kilometers per hour, and the road itself is part of a circle of radius r; it doesn't have to be a full circle. At that instant it has to be a part of a circle. Then, if you bank your road at tan theta, you don't need any friction to make the turn. You got to understand how we beat the system. If your road instead of being flat is tilted, then even though the road can only exert a frictionless normal force, now part of the normal force is pointing to the center.

The last problem is a very famous, very important, and that's the loop-the-loop problem.



$$T + mg = m\frac{v^2}{r}$$
$$T = m\left(\frac{v^2}{r} - g\right)$$

So, what does that mean? If this T comes out negative, you're dead, because if it's negative, you want the track to exert an upward force and it cannot. In a real amusement park they have other T-brackets and so on to support you. But if you really believe in the laws of physics, you don't need any of that, you just got to make sure this number is positive.

For T to be positive, $v^2 \geq rg$

If you go faster than that, that is just fine; if you go faster than that, T will be some positive number. So, here's the interesting thing. Have we escaped the pull of gravity? How come we are not falling down? And the track is not helping us, it's also pushing us down.

In this example, it is definitely accelerating, but the acceleration is not added to zero velocity, in which case it'll pick up more and more speed; it's added to a huge horizontal velocity. So, in a tiny time, you give it a little velocity like this, your new velocity points at a new angle.

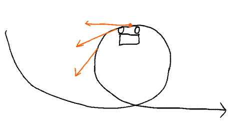
Therefore, downward acceleration does not mean coming closer to the ground, it only means your velocity vector is changing its direction. On the other hand, if you had no velocity vector, downward acceleration means what you think it is.

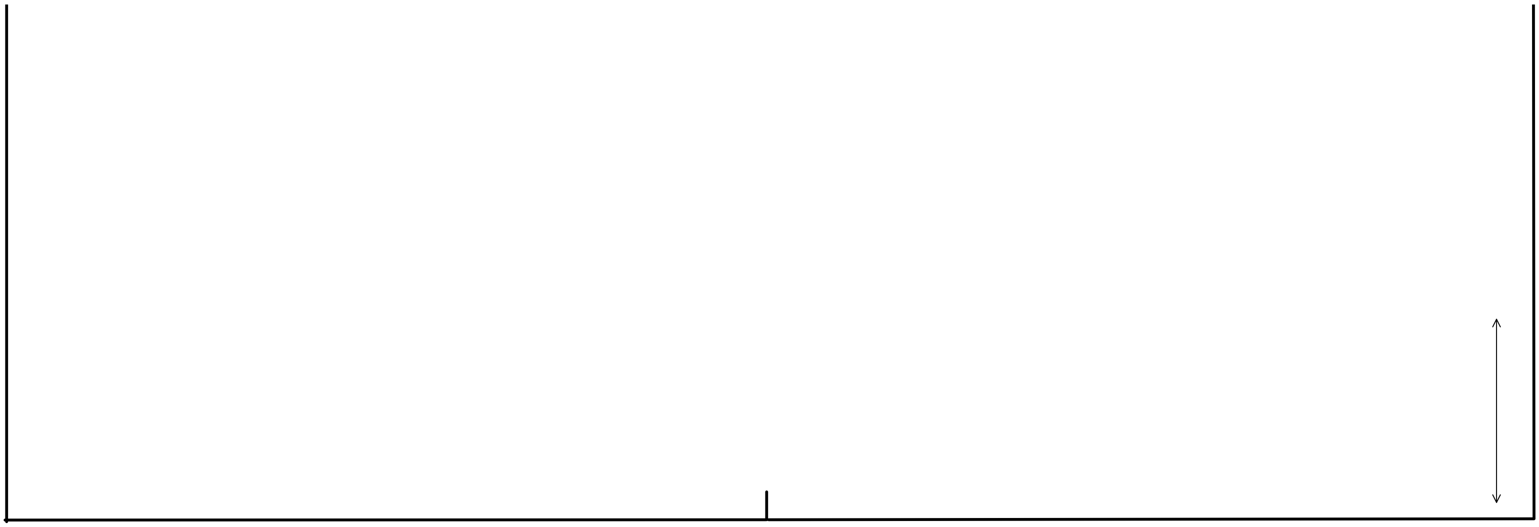
Visualizing this:

Tangential velocity (big arrow)

Small arrow is a miniscule velocity caused by the instantaneous centripetal acceleration.

So on the illustration the vector of velocity changes as such...





Work-energy theorem and law of conservation of energy

Monday, October 9, 2023 10:37 AM

NOTES ABOUT LOOP-THE-LOOP PROBLEM AND LAUNCHING SOMETHING TO ORBIT THE PLANET IS OMITTED. In short, since a small centripetal velocity is added to a large tangential velocity, only the direction of an object would change.

Let us now discuss energy.

Don't memorize the formulas and start plugging them in. Try to follow the logic by which you are driven to this notion, because then it'll be less luggage for you to carry in your head.

When a force acts on a body, what is its main effect?

Let's return to one dimension. We want to find the relation between how much speed is accumulated when a force acts on a body.

Assume F = constant

This gives $a = \frac{F}{m}$

Remember $v^2 = v_0^2 + 2ad$, our kinematic, was assumed to have a constant acceleration.

Now that we learned dynamics, we know acceleration has a cause, namely a certain force. So, I'm going to write here as you can imagine F/m in the place of a, but I'm going to make one more cosmetic change in notation.

$$v_2^2 = v_1^2 + 2 \cdot \frac{F}{m} \cdot d$$

Since we are interested in the effect of the force, we want to see what the force did to the velocity.

$$\frac{m}{2} v_2^2 - \frac{m}{2} v_1^2 = F d$$

This is a very important concept. It says, when the force acts on a body it changes the velocity, and it depends on how far the force has been acting, how many meters you've been pushing the object. It's clear that if you didn't push it at all, then even the forces acting, velocity hasn't had time to change. So, it seems to depend on how far the force acted, and the change is not simply in velocity but in velocity squared. That's what comes out; then, you realize that it's the most natural thing in the world to give this combination a name because that's what comes out of this. That combination is called kinetic energy, and I'm going to call it...

$k_2 - k_1 = F \cdot d$, thus $F \cdot d = W$

W units: $J = N/m$

Let us write $\Delta k = k_2 - k_1 = F \cdot \Delta x$ ← work energy theorem

The Work Energy Theorem says, "The change in energy is equal to the work done by all the forces."

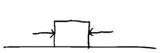
Now, what if there are 36 forces acting on the body? Which one should I use?

It's got to be the net force because Newton's law connects the net force to the acceleration.

If you and I have a tug of war and we cancel each other out to zero then there's no acceleration.

Note: F = summation of forces

Here, $W_{me} = F \cdot d$ & $W_{you} = -F \cdot d$



If the body is moving to the right, and I'm pushing to the right, then the work done by me is positive. And if you were pushing to the left and the body still moved to the right, the work done by you is negative. In other words, if you get your way, namely, things move the way you're pushing, the work done by you is positive. If people are pushing you counter to your will, in the opposite direction to your force, work done by you is negative.

$\Delta k = k_2 - k_1 = F \cdot \Delta x$, $F \cdot \Delta x = \Delta W$, ΔW

ΔW is maybe a result of many works, work done by me, work done you, work done by somebody else, some are positive, some are negative, you add them all up. So, that is the notion of work done by one of many forces acting on a body. And the total work done is the algebraic sum of the work done by all the forces.

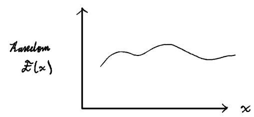
Let us imagine all of this occurs in a time Δt.

Then $\frac{\Delta k}{\Delta t} \Rightarrow \frac{dk}{dt} = F \frac{dx}{dt} = F \cdot v$ = Power (rate at which work is done)

Power units: J/s = Watts (W)

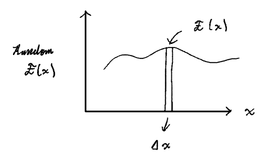
Assume $F \neq$ constant but $F = F(x)$

The force of a spring is one concrete example.



We can't apply $F d = \Delta W$ because $F \neq$ constant

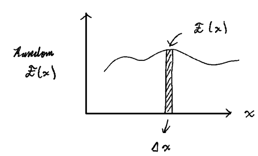
Using calculus we can



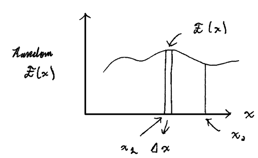
The usual trick in calculus is find me an interval in x, which is so narrow, and I'll make it as narrow as you insist, so that during that period I'll think that F is a constant and the constant value of F is just the function F at that time x, at that time where the location is x.

Over this interval, $\frac{F}{m}$ is "constant" thus $\Delta K = F(x) \cdot \Delta x$

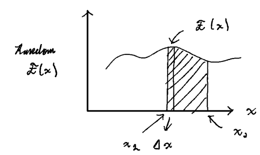
Geometrically, our work is the little rectangle whose base is the change in x and height is F(x).



But suppose we went from x_1 to x_2.



The work would then be the entire area under the curve.



Meaning our work would then be...

$\sum \Delta k = \sum F(x)\, dx$ which is $k_2 - k_1 = \int_{x_1}^{x_2} F(x)\, dx$

mystery function

$\int_{x_1}^{x_2} F(x)\, dx = G(x_2) - G(x_1)$

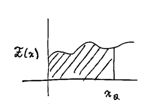
$\frac{dG}{dx} = F(x)$ G is a guessed function whose derivative is $F(x)$

So $F(x) = x^3$, $G(x) = \frac{x^4}{4} + C$, C doesn't change the function

But the beauty is that when you take this difference, I may not put the constant, and you may put the constant, but it's going to cancel out. So, most of the time, people don't bother with the constant. Sometimes it's very important to keep the constant in place -- it plays a special role -- but we don't want that here. We'll just call this the integral.

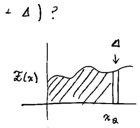
So why is this true?

Suppose we want,



$\int_0^{x_2} F(x)\, dx = G(x_2)$

What is $G(x_2 + \Delta)$?

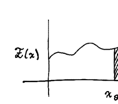


$\int_0^{x_2 + \Delta} F(x)\, dx$

What is the difference between the two?

$G(x_2 + \Delta) - G(x_2)$

$G(x_2 + \Delta) - G(x_2) = F(x_2) \cdot \Delta x$



Therefore $F(x_2) = \frac{dG}{dx}$

We can drop our subscript and find

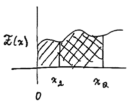
$F(x) = \frac{dG}{dx}$ Which is why we know what G is.

Some common situations are…

$F(x) = c$ so $G = cx$

$F(x) = x$ so $G = \frac{x^2}{2}$

Also,



means $K_2 - K_1 = \int_{x_1}^{x_2} F(x)\, dx = G(x_2) - G(x_1)$

$\equiv G_2 - G_1$

Remember, G is a definite function of x.

So now we have $K_2 - K_1 = G_2 - G_1$. Lets rewrite as

$K_2 - G_2 = K_1 - G_1$

So, the Work Energy Theorem says that, "If a force is acting on a body, a variable force associated with the force is the function G, a variable function of x, whose derivative is the given force." Then, at the beginning, if you take the difference K_1 minus G_1, it'll be the same as K_2 - G_2. Now, we don't like the form in which this is written. We have to make a little cosmetic change, and the cosmetic change is to introduce the function

$U(x) = -G(x)$, so now $F = -\frac{dU}{dx}$

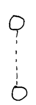
We can now write $\underbrace{K_2 + U_2}_{E_2} = \underbrace{K_1 + U_1}_{E_1}$

$E_2 = E_1$ is conservation of energy

So, what does conservation mean? Conservation of Energy in physics has a totally different meaning from conservation of energy in daily life. Here it means, when a body's moving under the effect of this force F(x), you agree it's gaining speed or losing speed or doing all kinds of things. So, as it moves around it's speeding up and slowing down, so its speed is definitely a variable, but a certain magic combination connected to the velocity of the object and to its location, U depends on x, that does not change with time. And it's very useful to know that does not change with time, because if you knew the value of E, even if E were to equal some fixed number E at one time you know E at all of the times.

Let's discuss a simple example.

So, we take a rock and we drop it. We know it's picking up speed; we know it's losing height. So, you may think maybe there is some combination of height and speed, which does not change in this exchange, and we find the combination by this rule now.



$F = -mg$ therefore $U(x) = mgy$ $\left(-\frac{dU}{dy} = F\right)$

Therefore $\frac{1}{2} m v_1^2 + mgy_1 = \frac{1}{2} m v_2^2 + mgy_2$

You take any random time you want. The sum of these two numbers doesn't change. And that's a very, very powerful result.

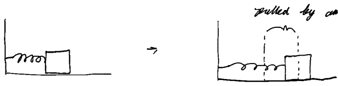
Instead of saying vaguely, "you lose and gain," ½mv^(2) is the part connected with the speed and mgy is the part connected with the height and that sum does not change with time.

Let's return to the mass and spring system.

For

we have $\frac{1}{2} m v^2 + \frac{1}{2} k x^2 = E$

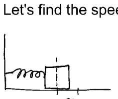
E must $=$ constant



pulled by amount A, then released

how fast is block moving?

Let's find the speed the block will move back for any given x.



$\frac{1}{2} m v_1^2 + \frac{1}{2} k x_1^2 = \frac{1}{2} m v_2^2 + \frac{1}{2} k x_2^2$

This is the truth; this is a constant that does not change with time. Now, to get some mileage out of this, you've got to know what the constant value is. Well, we know what it is in the beginning because when you pull this guy, at the instant we knew it had no velocity, and we knew it was sitting at the point A.

So $\frac{1}{2} m \cancel{v_1}^{\,0\;2} + \frac{1}{2} k x_1^2 = \frac{1}{2} m v_2^2 + \frac{1}{2} k x_2^2 = E$

Therefore $E = \frac{1}{2} k A^2$ $(x_1 = A)$, thus $E = \frac{1}{2} k A^2 = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$

$x_2 = x$

Then given x, $\frac{1}{2} k x^2$ is taken care of and we get

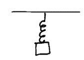
$v^2 = \frac{kA^2}{m}$

So, the power of the Law of Conservation of Energy is if you knew the initial energy, you don't have to go through Newton's laws, and you'll find it in all the problem sets and many exams. When a problem is given to you and you are told to find the velocity of something, you should try whenever possible to use the Law of Conservation of Energy rather than going to Newton's laws.

If you tell me the force of the spring is a function of x, I have done that integral in the Work Energy Theorem, once and for all. The integral of the spring force has been integrated once and for all and that's what the potential energy is doing. The potential energy comes from integrating that force.

(professor shankar pointed to how we came up with the work energy formula, specifically the special function of G)

Another problem:



$K_1 - K_2 = \int_{y_1}^{y_2} F(y)\, dy = \int_{y_1}^{y_2} F_g\, dy + \int_{y_1}^{y_2} F_s\, dy$

$= -(mgy_2 - mgy_1) - \left(\frac{1}{2} k y_2^2 - \frac{1}{2} k y_1^2\right)$

$= mgy_1 - mgy_2 + \frac{1}{2} k y_1^2 - \frac{1}{2} k y_2^2$

Negative because $U(x) = -G(x)$, and we defined $\int_1^2 F(x) = U(1) - U(2)$

Thus, $\frac{1}{2} m v^2 + mgy + \frac{1}{2} k y^2 = E$, E will not change.

So, this mass can bump up and down and go back and forth. If you knew the total of these three numbers at one instant, you know the value at any other instant. For example, if you pulled it from its equilibrium by .2 centimeters and you released it, and you say what's the speed when it's .1 centimeters away, it's very easy because you know the initial energy was all spring energy and gravitational energy. At any later time if you knew the y, if you knew the position, you know the potential energy; that's the whole point. Potential energy depends on where you are; kinetic energy depends on how fast you're moving. And some combination of where you are and how fast you're moving is invariant, does not change with time. So, if you tell me where you are, I will tell you how fast you're moving.

Note that solving for velocity gives a square root and thus it gives a negative and a positive answer. This agrees with the spring system considering a spring can either be pulling an attached mass towards its equilibrium, or pushing an attached mass away from its equilibrium.

Now lets talk about a "bad force" that messes with the law of conservation of energy. Friction.

Adding friction to the conservation of energy gives…

$K_2 - K_1 = \underbrace{\int_{x_1}^{x_2} F_s(x)\, dx}_{\frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2} + \int_{x_1}^{x_2} F_f\, dx$

?

Why don't we just integrate the force of friction and then turn that into a U and then we're done?

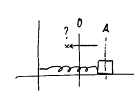
The force of friction is not a function of x.

You might say, "What do you mean?" I'm pushing this guy, I know how hard it's pushing me back, but I'm telling you now, "Push it the opposite way!" Then, you'll find the force of friction at the very same location is pointing in a different direction whereas the force of a spring is the force of the spring. It's -kx, whether you're going towards the spring or you're going away from the spring; try to think about that. Same with gravity.

Lets continue w/ our equation.

$K_2 + U_2 - K_1 + U_1 = \int_{x_1}^{x_2} F(x)\, dx$

Can I do the integral?" The answer is, if you just tell me x_1 and x_2, then I know that's the point you're making; I cannot just do an integral because I will have to know the whole story, but in limited cases I can do this.



The spring will not return to -A due to friction.

During its return to equilibrium, the force of friction is…

$-(\mu mg)$ ($\rightarrow$ direction)

So, for this part of the trip, I can do the integral. It is very easy to find out what it is; it's -mg times µ kinetic, times the distance traveled. Of course, it's a little more complicated now. You have to tell me where you want

x

This is the truth; this is a constant that does not change with time. Now, to get some mileage out of this, you've got to know what the constant value is. Well, we know what it is in the beginning because when you pull this guy, at the instant we knew it had no velocity, and we knew it was sitting at the point A.

... force of friction is...

$-(\mu mg)$ ($\rightarrow$ direction)

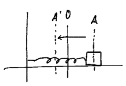
So, for this part of the trip, I can do the integral. It is very easy to find out what it is; it's -mg times µ kinetic, times the distance traveled. Of course, it's a little more complicated now. You have to tell me where you want your velocities.

Lets assume $x = 0$, we then have

$$\frac{1}{2} m v^2 + \frac{1}{2} k (0)^2 - \frac{1}{2} k A^2 = -(\mu mg) A$$

This is the difference in energies due to the conserved forces, the conserved energies, but the difference E_ 2 - E_1 is no longer 0 but is given by this number (right side of equation is friction). So, I can calculate it for this part.

If we ask, "How far does it go to the left?" Well, what do you do? Let's call the place where it goes to A prime. We will use the same formula except the distance the friction force acts on changes a little, as we must consider that the spring will go past the equilibrium a little, we will denote this distance as A prime.



So all we are left with now for the left side of the equation is potential energy, giving us the equation...

$$\underbrace{\frac{1}{2} k (A')^2}_{\text{final}} - \underbrace{\frac{1}{2} k (A)^2}_{\text{initial}} = -(\mu mg)(A + A')$$

You see that A prime now satisfies a quadratic equation; you can solve it.

But what is the point? The point is you cannot do an integral for the force of friction with the bounds only dependent on the endpoints. In our problem, depending on whether we are going left or right, there will be a different magnitude for velocity and the force of friction. Note that...

$\int F_f \, dx$, F_f = constant $= -\mu_K mg$ (always negative), therefore

$-\mu mg \int dx = -\mu mg x$, x = distance travelled.

So in summary...

$k_2 - k_1 = W = \int F \, dx$, express our good and bad forces

$k_2 - k_1 = W = \int F_g \, dx + \int F_f \, dx$, these forces only depend on location

$\int F_g \, dx = U_1 - U_2$

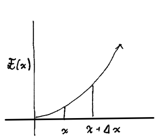
So $E_2 - E_1 = (k_2 + U_2) - (k_1 + U_1) = W_f$

Where $W_f = \int F_f \, dx$

And this integration requires us to divide the motion of the object into pieces. This is because of the fact that we start from A, then we return from A prime (see illustration). This is why friction is a "bad force" because depending on its location the magnitude of velocity and force of friction changes.

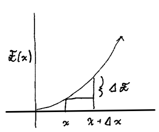
The point is, there is no universal formula for F(x) due to friction. So, in one dimension that's all it is, bringing the good and bad forces, those which lead to potential, those which don't; the difference is the good forces are functions of x, bad forces are functions of x and something else, so that at a given x it's possible to have two values for that function. Mathematically, we don't call it a function, that's why you cannot do the integral once and for all. So, this is the Law of Conservation of Energy.

Let us do some mathematical preparation for working in higher dimensions.



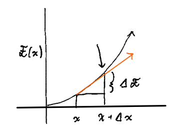
(Δx exagerrated)

What is the change in the function? (x to $x + \Delta x$)



Lets approximate the change

Its not an equality because we observe the tangential line split apart from the curve.



Lets do a concrete example.

$F(x) = x^2$

$F(x + \Delta x) = x^2 + 2x\Delta x + \Delta x^2$

$$\Delta F = \underbrace{2x}_{\frac{dF}{dx}} \Delta x + \underbrace{(\Delta x)^2}_{\text{this is what we are missing}}$$

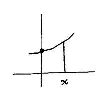
$$\Delta F = \left.\frac{dF}{dx}\right|_x \cdot \Delta x + O((\Delta x)^2)$$, O is a higher order of Δx

But since $\Delta x \to 0$, our approximation disregards $O((\Delta x)^2)$

Let us use this rule.

$F(x) = (1 + x)^n$ $\qquad F(0) = 1$

$$\Delta F = \left.\frac{dF}{dx}\right|_0 \cdot x \rightarrow$$



$$= n(1 + x)^{n-1}\Big|_0 \cdot x$$

$$= nx$$

So $(1 + x)^n = 1 + nx + \ldots$ ← small numbers, assuming x is very small

Let's do an example.

Relativity states: $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$ $\quad c$ = velocity of light, v = velocity of mass

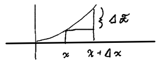
$$m = m_0 (1 - v^2/c^2)^{-1/2}$$

So now $x = -\dfrac{v^2}{c^2}$ & $n = -\dfrac{1}{2}$, we find $m = m_0 + \dfrac{m_0 v^2}{2c^2} + \ldots$

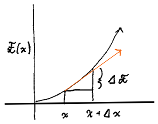
So we now have a formula that approximates the mass of an object. Apparently in relativity the mass of an object has a slight change if it is moving. The approximate formula can be used assuming that v and c are very small. The approximate formula is more friendly to use.

In a nutshell, we have now observed that the rate of change itself also has a rate of change, therefore there are higher derivatives of a function (big O!).

Professor Shankar then speedruns an explanation of the Taylor representation of a function. He focuses on the fact that after the first two terms in a Taylor representation, you can disregard the terms that come after.



Lets approximate the change.



$\Delta E = \left.\frac{dE}{dx}\right|_x \cdot \Delta x$, but this isnt an equality

Law of conservation of energy in higher dimensions

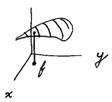
Tuesday, October 10, 2023 9:50 PM

$$\Delta E = \frac{dE}{dx}\Big|_x \cdot \Delta x + \ldots \quad \text{small corrections}$$

We are going to move the whole Work Energy Theorem and the Law of Conservation of Energy to two dimensions. So, when you go to two dimensions, you've got to ask yourself, "What am I looking for?" Well, in the end, I'm hoping I will get some relation like...

$$k_1 + U_1 = k_2 + U_2, \quad \text{but } U(x, y)$$

So, how do you visualize the function of two variables?

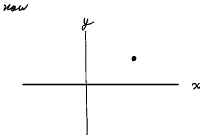


This like a canopy over a flat surface, the distance between the flat surface and the canopy being the value of the function.

So, once you've got the notion of a function of two variables, if you're going to do calculus the next thing is what about the derivatives of the function. How does it change? Well, in the old days, it was dependent on x…

$$\lim_{\Delta x \to 0} \frac{\Delta E}{\Delta x} \to \frac{dE}{dx}$$

but now



The function comes towards us out of the paper.

Now I have a lot of options; in fact, an infinite number of options. I can move along x, I can move along y, I can move at some intermediate angle, we have to ask what do you want me to do when it comes time to take derivatives. So, it turns out, and you will see it proven amply as we go along, that you just have to think about derivatives on two principle directions which I will choose to be x and y.

Lets define one derivative $\lim_{\Delta x \to 0} \dfrac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \dfrac{\partial E}{\partial x}\Big|_y$ ← partial derivative

this curly d instead of the straight d tells you it's called a "partial derivative." Some people may want to make it very explicit by saying this is the derivative with subscript y. That means y is being held constant when x is varied, but we don't have to write that because we know we've got two coordinates. If I'm changing one, the other guy is y so we won't write that. (treat other variables as constants)

We can also find $\dfrac{\partial E}{\partial y}$

So given $f(x) = x^3 y^2 + y^2$

$$\frac{\partial f}{\partial x} = 3x^2 y^2, \quad \text{then} \quad \frac{\partial E}{\partial y} = 2x^3 y + 2y$$

And we know that we can take higher derivatives.

$$\frac{\partial^2 E}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial E}{\partial x}\right) = 6xy^2$$

$$\frac{\partial^2 E}{\partial^2 y} = 2x^3 + 2$$

What if we? $\dfrac{\partial}{\partial x}\left(\dfrac{\partial E}{\partial y}\right)$? (take cross derivative)

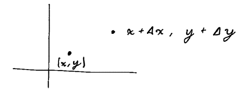
$$\frac{\partial}{\partial x}\left(\frac{\partial E}{\partial y}\right) = \frac{\partial^2 E}{\partial x \partial y} = 6x^2 y$$

Additionally, $\dfrac{\partial^2 E}{\partial y \, \partial x} = 6x^2 y$

If you've never seen this before, you will notice that the cross derivative, y followed by x and x followed by y, will come up being equal. That's a general property of any reasonable function. By reasonable, I mean you cannot call the mathematicians to help because they will always find something where this won't work, okay? But if you write down any function that you are capable of writing down with powers of x and powers of y and sines and cosines, it'll always be true that you could take the cross derivatives in either order and get the same answer.

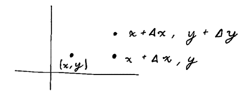
Let's get a feeling for why this is true.

Consider,



what is $f(x+\Delta x, y+\Delta y) - f(x,y) = ?$

Lets introduce an intermediate point



So $f(x+\Delta x, y+\Delta y) - f(x,y) = f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) + f(x+\Delta x, y) - f(x,y)$ basically add nothing

But $f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y)$ only appears to change y, therefore

$f(x+\Delta x, y+\Delta y) - f(x+\Delta x, y) = \dfrac{\partial x}{\partial y}\Delta y$, and also $\underbrace{f(x+\Delta x, y) - f(x,y)}_{\frac{\partial E}{\partial x}\Delta x}$

Therefore $\Delta E = \dfrac{\partial E}{\partial x}\Delta x + \dfrac{\partial E}{\partial y}\Delta y$

There is one thing you should be careful about which has to do with where the derivatives are really taken.

$$\text{So} \quad \Delta E = \underbrace{\frac{\partial E}{\partial x}\Big|_{x, y}}_{\text{starting point}} \Delta x + \underbrace{\frac{\partial E}{\partial y}\Big|_{x+\Delta x, y}}_{\text{intermediate point}}$$

So, you've got to fix that. And how do you fix that part? You argue that the derivative with respect to y is just another function of y; f is a function of x and y, its derivatives are functions of x and y. Everything is a function of x and y, and we are saying this derivative has been computed at x + Δx, instead of x, so it is going to be df/dy at xy + d^(2)f/dxdy times Δx.

$$\text{In notation,} \quad \frac{\partial E}{\partial x}\Big|_{x+\Delta x, y} \Delta y = \frac{\partial E}{\partial y}\Big|_{x,y} + \left(\frac{\partial^2 E}{\partial x \partial y}\right)\Delta x$$

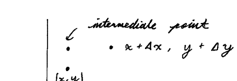
Translation: the derivative at the intermediate point is the derivative of the starting point plus the derivatives rate of change multiplied by the change in x. The derivative itself is changing. Putting this all together gives...

$$\frac{\partial E}{\partial x}\Delta x + \frac{\partial E}{\partial y}\Delta y + \frac{\partial^2 E}{\partial x \partial y}\Delta x \Delta y \quad (\text{we here mean derivative at starting point})$$

Now, you've got to realize that when you're doing these calculus problems, Δx is a tiny number, Δy is a tiny number, Δx, Δy, is tiny times tiny. So normally, we don't care about it, or if you want to be more accurate, of course, you should keep that term.

$$\underbrace{\frac{\partial E}{\partial x}\Delta x + \frac{\partial E}{\partial y}\Delta y}_{\approx \Delta f} + \underbrace{\frac{\partial^2 E}{\partial x \partial y}\Delta x \Delta y}_{\text{ambitious accuracy}}$$

But another person comes along and says, "You know what, I want to go like this."



That person will get $\dfrac{\partial E}{\partial x}\Delta x + \dfrac{\partial E}{\partial y}\Delta y + \dfrac{\partial^2 E}{\partial y \partial x}\Delta y \Delta x$

But the location of the intermediate point does not interfere with the way that our function changes. So we can say...

$$\frac{\partial^2 E}{\partial x \partial y}\Delta x \Delta y = \frac{\partial^2 E}{\partial y \partial x}\Delta y \Delta x \quad \text{b/c change in function must = change in function}$$

This is why we can make the claim that cross derivatives will tend to be equal when working with reasonable functions. But for this lecture...

$$\Delta E = \frac{\partial E}{\partial x}\Delta x + \frac{\partial E}{\partial y}\Delta y$$

The function is changing because the independent variables x and y are changing, and the change in the function is one part, which I blame on the changing x, and a second part, which I blame on the changing y and I add them. So, you're worried about the fact that we're moving in the plane and there are vectors.

So, let's now go back to our original goal, which was to derive something like the Law of Conservation of Energy in two dimensions instead of one.

Remember $\Delta K = \Delta W = F \Delta x$

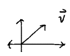
$$K_2 - K_1 = F\,dx = \int_{x_1}^{x_2} F\,dx = U(x_1) - U(x_2) \quad \text{where} \quad -\frac{dU}{dx} = F$$

You want to try the same thing in two dimensions; that's your goal. So, the first question is, "What should I use for the work done?" What expression should I use for the work done in two dimensions, because the force now is a vector; force is not one number, it's got an x part and a y part. My displacement has also got an x part and a y part and I can worry about what I should use.

I'm going to deduce the quantity I want to use for ΔW.

Remember for 1-d, $\frac{\Delta K}{\Delta t} = F \frac{\Delta x}{\Delta t}$, $\frac{dK}{dt} = \underbrace{F v}_{\text{power}}$

I'm going to look for dK/dt in two dimensions, of a body that's moving. What's the rate at which kinetic energy is changing when a body is moving? For that, I need a formula for kinetic energy.

$K = \frac{1}{2} m v^2 = \frac{1}{2} m (v_x^2 + v_y^2)$ because  $v = \sqrt{v_x^2 + v_y^2}$

So $\frac{dK}{dt} = \frac{m}{2}\left(2 v_x \cdot \frac{dv_x}{dt} + 2 v_y \cdot \frac{dv_y}{dt}\right) = \Big(\underbrace{m \frac{dv_x}{dt}}_{m a_x\,!}\Big) v_x + \Big(\underbrace{m \frac{dv_y}{dt}}_{m a_y\,!}\Big) v_y$

Therefore $\frac{dK}{dt} = F_x v_x + F_y v_y =$ Power in 2-d

Multiplying by delta t however gives...

$\Delta K = F_x \overset{v_x \cdot \Delta t}{dx} + F_y \overset{v_y \cdot \Delta t}{dy} = \Delta W \rightarrow$ tiny amount of work done by a force

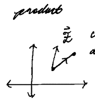
This combination is now guaranteed to have the property that if I call this the work done. Then, it has the advantage that the work done is in fact the change in kinetic energy.

Notice $F = \vec{i} F_x + \vec{j} F_y$, $\vec{v} = \vec{i} v_x + \vec{j} v_y$, & $d\vec{r} = \vec{i}\,dx + \vec{j}\,dy$

So, $\vec{A} = \vec{i} A_x + \vec{j} A_y$, $\vec{B} = \vec{i} B_x + \vec{j} B_y$

You can regard A as the force maybe, and B as the velocity. However, a very natural combination appears.

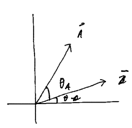
$A_x B_x + A_y B_y = \vec{A} \cdot \vec{B} \leftarrow$ dot product

Therefore $\Delta W = \vec{F}\,d\vec{r}$ since  $\vec{F}$ can have an independent vector $d\vec{r}$ $\quad d\vec{r} = \vec{i}\,dx + \vec{j}\,dy$

Let's get a feeling for what the dot product is.

$\vec{A} \cdot \vec{A} = A_x \cdot A_x + A_y \cdot A_y = |A|^2 = A^2$

$\vec{A} \cdot \vec{B} = ?$ lets derive using law of cosines



$A_x = A \cos\theta_A$

$A_y = A \sin\theta_A$

B_x, B_y

So $\vec{A} \cdot \vec{B} = A \cdot B \cdot (\cos\theta_A \cos\theta_B + \sin\theta_A \sin\theta_B)$

$= A \cdot B \cos(\theta_A - \theta_B)$

$\equiv A \cdot B \cos(\theta)$ Theta being the angle between A and B.

Law $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$

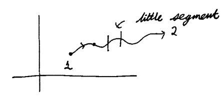
What can you say if you use a different set of axes?

What you're looking for is the length of A, which certainly doesn't change on your orientation, or the length of B, or the angle between them.

The angle between the vectors, it's an invariant property, something intrinsic to the two vectors, doesn't change, so the dot product is an invariant. This is a very important notion. When you learn relativity, you will find you have one observer saying something, another observer saying something. They will disagree on a lot of things, but there are few things they will all agree on. Those few things will be analog of A.B. So, it's very good to have this part of it very clear in your head.

So geometrically, $\Delta W = F\,dr \cos\theta = F\,d\vec{r}$

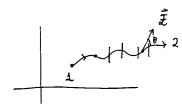
So, let me make a big trip, okay, let me make a trip in the xy plane made up of an whole bunch of little segments in each one of which I calculate this, and I add them all up.



Each segment calculate $F \cdot d\vec{r}$, add all segments up.

$\Delta W = \Delta K = K_2 - K_1 = \int \vec{F}\,d\vec{r}$

This is the notation we use in calculus. That just means, if you want to go from A to B along some path, you chop up the path into tiny pieces. Each tiny segment, if it's small enough to be approximated by a tiny vector dr, then take the dot product of that little dr, with the force at that point, which means length of F, times the length of the segment, times cosine of the angle; add them all up.



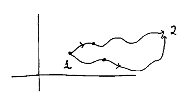
So, now maybe it'll be true, just like in one dimension, the integral of this function will be something that depends on the end points.

$\int \vec{F}\,d\vec{r} = U(1) - U(2)$, and if this is true, because we now have

$K_1 + U_1 = K_2 + U_2$. but it is not this simple.

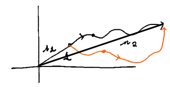
If we add friction, then this doesn't work.

It doesn't even work in 1D. In 1D, if at a point and end and a point, and I worry about friction, if it went straight from those two points there's amount of friction. If I just went back and forth 97 times and then I ended up with 97 times more friction. So, we agreed that if there's friction, this is not going to work. But the trouble with friction was, the force was not a function only of x and y. It depended on the direction of motion. But now, I grant you that the force is not a function of velocity. It's only a function of where you are. Can something still be wrong? Well, let me ask you the following question. Another person does this.



Does the orange path have more work? (yes)

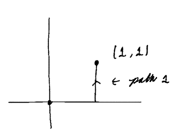
So, you see, in one dimension there's only one way to go from here to there. Just go, right? That way, when you write an integral you write the lower limit and the upper limit and you don't say anymore because it's only one way in 1D to go from x_1 to x_2. In two dimensions, there are thousands of ways to go from one point to another point. You can wander all over the place and you end up here. Therefore, this integral, even if I say the starting point is r_1 and the ending point is r_2, that this is r_1 and that's r_2, it's not adequate. What do you think I should attach to this integration? What other information should I give? What more should I specify?

$\Delta W \neq \int_{r_1}^{r_2} \vec{F}\,d\vec{r}$,  , so does $\int_{r_1}^{r_2} \vec{F}\,d\vec{r} = U(1) - U(2)$? (no)

Assume we have $F(r) = \vec{i}\, x^2 y^3 + \vec{j}\, x y^2$ (a random force)

Is it true for this force that the work done only depends on the end points, or does it depend in detail on how you go from the end points?

Lets find the work for this function for



horizontal : $d\vec{r} = \vec{i}\,dx$

So $\int F \cdot d\vec{r} = F_x\,dx = \int_0^1 x^2 y^3\,dx$, what about y^3 ?

Well, it turns out, throughout this horizontal segment y = 0, so this is gone. Basically, the point is very simple. When you move horizontally, you're working against horizontal forces doing work, but on the x axis when y is 0 there is no horizontal force; that's why there's nothing to do.

So $\int_0^1 x^2 y^2\,dx \rightarrow 0$

so the dot product is an invariant. This is a very important notion. When you learn relativity, you will find you have one observer saying something, another observer saying something. They will disagree on a lot of things, but there are few things they will all agree on. Those few things will be analog of A.B. So, it's very good to have this part of it very clear in your head.

o there is no horizontal force, that's why there's nothing to do.

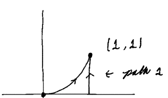
So $\int_0^1 x^2 y^2\, dx \to 0$

Now for $d\vec{r} = \vec{j}\, dy$ (second part of trip)

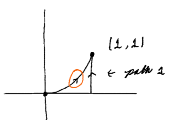
Therefore $\int_0^1 F_y\, dy = \int_0^1 x y^2\, dy = \frac{1}{3} J$

So, the work done in going first to the right and then to the top is 1/3.

I'm going to pick another way to go from 0,0 to 1,1, which is on this curve; this is the curve y = x^(2).



And I'm asking you, if I did the work done by the force along that segment, what is the integral of F.dr? So, let's take a tiny portion of that...



$$\Delta_{dx}\, dy \qquad dy = \left(\frac{dy}{dx}\right) dx$$

In other words, dx and dy are not independent if they're moving in a particular direction. I hope you understand that. You want to follow a certain curve. If you step to the right by some amount, you've got to step vertically by a certain amount so you're moving on that curve. That's why, when you calculate the work done, dx and dy are not independent.

So we really want $\int F_x\, dx + F_y\, dy \left(\frac{dy}{dx}\right) dx$

So, everything depends only on dx. But what am I putting inside the integral?

$F_x = x^2 y^3$, $F_y = x y^2$, but now $y = x^2$ so...

$$\int \Big(x^2 \underbrace{x^6}_{(y^3)^2} + x \cdot \underbrace{x^4}_{(y^2)^2} \cdot \underbrace{2x}_{\frac{dy}{dx}}\Big)\, dx = \int_0^1 (x^8 + 2x^6)\, dx = \frac{1}{9} + \frac{2}{7}$$

And $\frac{1}{9} + \frac{2}{7} \neq \frac{1}{3}$ So we have proved that if we took a random force, the work done is dependent on the path.

So, if you're looking for a conservative force. It's a force for which you can define a potential energy. It has the property of the work done in going from A to B, or 1 to 2, is independent of how you got from 1 to 2. And the one force that the class generated pretty much randomly is not a conservative force because the work done was path dependent.

You realize, that looks really miraculous because we just wrote down an arbitrary force that we all cooked up together with these exponents, and the answer depends on the path. And I guarantee you, if you just arbitrarily write down some force, it won't work. So, maybe there is no Law of Conservation of Energy in more than one dimension. So, how am I going to search for a force that will do the job? Are there at least some forces for which this will be true?

But you see, what I want is to ask, "Is there a machine that'll manufacture conservative forces?" and I'm going to tell you there is. I will show you a machine that'll produce a large number, an infinite number of conservative forces, and I'll show you how to produce that.

The trick is, instead of taking a force and finding if there's a potential that will come from it by doing integrals, let me assume there is a potential. Then, I will ask what force I can associate with the potential, and here is the answer.

1.) pick any $U(x, y)$

2.) The force must, $F_x = -\frac{dU}{dx}$, $F_y = -\frac{dU}{dy}$

The claim is now that this force will be a conservative force.

Suppose $U = x y^3$, $F_x = -\frac{dU}{dx} = -y^3$, $F_y = -\frac{dU}{dy} = -3xy^2$

The claim is, if you put I times F_x and J times F_y, the answer will not depend on how you go from start to finish. So, let me prove that to you. Here is the proof.

$\Delta U = \frac{\partial U}{} dx + \partial U$ ...

Maybe the way to think about it is, why do certain integrals not depend on how you took the path, right? Let me ask you a different question; forget about integrals. You are on top of some hilly mountain. We have a starting point, you have an ending point, okay. I started the starting point, and I walked to the ending point. At every portion of my walk, I keep track of how many feet I'm climbing; that's like my ΔU. I add them all up. At the end of the day, the height change will be the top of the mountain minus bottom of the mountain, the height of the mountain. You go on a different path, you don't go straight for the summit, you loop around and you coil and you wind down, you go up, you do this, and you also end up at the summit. If you kept track of how long you walked, it won't be the same as me. But if you also kept track of how many feet you climbed and you added them all up, what answer will you get? You'll get the same answer I got. Therefore, if what you were keeping track of was the height change in a function, then the sum of all the height changes will be simply the total height change, which is the height at the end minus height at the beginning. Therefore, starting with the height function, by taking its derivatives, if you manufacture a force, this will be none other than the fact of adding up the changes.

Which is why we get $U(1) - U(2) = \int \vec{F} \cdot d\vec{r} = K_2 + K_2$

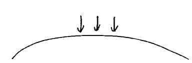
Not only is this a machine that generates conservative forces, my two step algorithm, pick a U and take its derivatives, every conservative force you get is necessarily obtained by taking derivatives with respect to x and y of some function U and that U will be the potential energy associated with that force. So, when that force alone acts on a body the kinetic plus that potential will not change.

A mechanical way to determine whether or not a force is conservative is it has to meet the following conditions.

$$\frac{\partial^2 U}{\partial y\, \partial x} = \boxed{\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}} = \frac{\partial^2 U}{\partial x\, \partial y}$$

So, I can summarize by saying the following thing. In two dimensions, there are indeed many, many forces for which the potential energy can be defined. But every one of them has an ancestor which is simply a function, not a vector, but a scalar function, an ordinary function of x and y. Then, the force is obtained by taking x and y derivatives of that function; the x derivative with a minus sign is called F_x, and the y derivative is called F_y.

Let's take the most popular example is the force of gravity near the surface of the Earth.



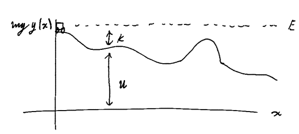
$F_g = -mg \cdot \vec{j}$

Then you can ask, "What is the potential U that led to this?"

$-\frac{\partial U}{\partial y} = -mg$ and $\frac{\partial U}{\partial x} = 0$, which checks out w/ $U = mgy$.

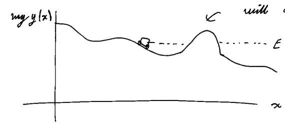
This means $\frac{1}{2} m v_1^2 + mg y_1 = \frac{1}{2} m v_2^2 + mg \cdot y_2$

So let us do one final example to think about. A roller coaster.



claim: $K + U = \text{constant} = E$

So if you start at...



will not go over hump, not enough energy

Falling over the edge to the left gives a purely conservative force. If the cart goes along the track, there is a normal force, therefore the proper expression is that...

$$K_2 - K_1 = \int \underbrace{\vec{F}_N}_{0} \cdot d\vec{r} + \int \vec{F}_g\, d\vec{r}$$

Therefore $K_2 - K_1 = U_1 - U_2$

The claim is, if you put I times F_x and J times F_y, the answer will not depend on how you go from start to finish. So, let me prove that to you. Here is the proof.

$$\Delta U = \frac{\partial U}{\partial x} dx + \frac{\partial U}{dy} = - \vec{F} \cdot d\vec{r}$$

$$\sum \Delta U = - \sum \vec{F} \cdot d\vec{r}$$

$$-\sum \Delta U = \sum \vec{F} \, d\vec{r}$$

$$U_1 - U_2 = \int \vec{F} \cdot d\vec{r}$$

I cooked up a force by design so that F.dr was a change in a certain function U. If I add all the F.drs, I'm going to get a change in the function U from start to finish and it's got to be U_1 - U_2.

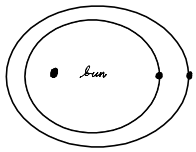
0

Therefore $K_2 - K_1 = U_1 - U_2$

Keplers Laws

Wednesday, October 11, 2023 12:52 PM

The situation was as follows at Newton's time. You know that Copernicus proposed that the way to think about our Solar System is to put the Sun at the center.



Tycho Brahe who was a rich guy who had his own lab and he studied the Solar System and Kepler was an assistant who carried on the work for 40 years. For 40 years he studied this problem and he published his answers.

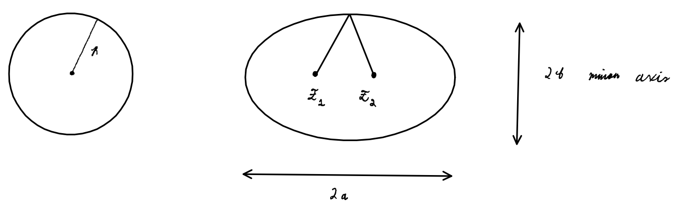
Three laws of kepler.

① Elliptical orbit

The first law is, "All planets go around the Sun on an elliptical orbit with the Sun as the focus."

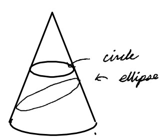
Circle vs elliptical

F_1, F_2 are focal points



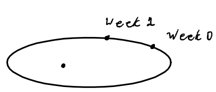
So, do you know how you can get an ellipse? Why the Greeks were studying ellipses? Where they come from?

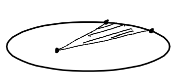
If you slice a cone parallel to the base, you'll get a circle; at an angle, you'll get the ellipse. So, that's the first statement.



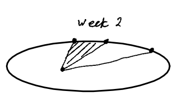
Second statement is the following:

Pick a planet and see where it goes after a week.





Shaded area is the area swept out by planet for first week.



area swept out after week 2.

And Kepler said the rate at which the area is swept out is…

$\frac{dA}{dt}$ is constant

The third result of his laws is…

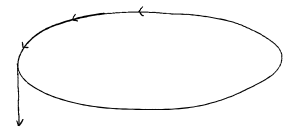
$\frac{T^2}{a^3}$ = same for all planets

T being the time it takes for a planet to orbit, a is the orbit size.

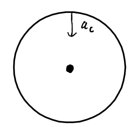
Where we stand right now is with a strong understanding of systems like an inclined plane, maybe with a pulley and maybe with a spring. Essentially, we are in the shoes of Newton when he was pondering the same thing. We are going to take a humungous leap from systems of these manner, and analyze the cosmological effects at work in our solar systems that keep us in orbit.

We are indeed missing one thing.

If you think it's forces that cause velocity, you look at a planet going around the Sun, where the velocity is pointing in different directions at different times and you have to ask what can possibly be propelling the planet in this loop, you don't get any definitive answer.

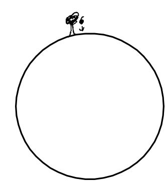


But calculating the acceleration shows that it is always pointing towards the center.



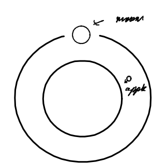
So, it is again here that Newton decided that the force that bends the planets around the Sun is the same as the force that bends the Moon around the Earth.

Then, here is the famous apple, also accelerating towards the Earth. And Newton decided that the cause behind both the forces is a new force, the force of gravity. The same thing that pulls the apple is what pulls the Moon.



So, we have to now ask, "What is the nature of that force?" We're going to write a formula for the force. First, let me look at the magnitude. The direction is very clear, it is directed towards the center of the Earth. So, I won't write that down. Later on I will write you a formula with vector symbols in it; so the vector will point towards the center of the Earth. But let's look at the magnitude.

So given



Near the Earth we all know the acceleration of a body is constant. Therefore, it can happen only if the force depends on the mass of the apple. This force is going to depend on various things. I'm going to write the formula, but one thing it depends on, it's got to be proportional to the mass of the apple. That's because then if you divide by m, to get a, the mass drops out and you understand the fact that everything near the Earth falls at the same rate. The only way that can happen is if the force depends on the mass of the apple.

So, $F = m$ for sure m being the mass of the apple.

An important thing to consider is that the apple is also exerting a very very small but not unimportant force on the Earth. Imagine the following: keep the Earth at the same mass, but increase the mass of the apple. In fact, increase the mass of the apple until the apple is at a much larger mass than the Earth, maybe around 10 times the size of Earth, it is obvious that the apple will have a gravitational force that pulls on the Earth that's now stronger than the Earth pulls on the apple. Because of this logic, we can add mass M for the mass of the Earth.

Therefore $F = m \cdot M$ M the mass of earth

The last thing we must consider is the distance. The distance being a variable that is subject to change therefore we add it as a function.

$F = m \cdot M \cdot f(r)$

a_A (acceleration of apple) $= M f(r_E)$

$a_{moon} = M f(r_{moon})$

Assume $r_{moon} = 240000$ miles & $r_E = 4000$ miles (radius of Earth)

If we take $\frac{a_A}{a_{moon}}$, ← M also cancels out, what is a_{moon}? It's $\frac{v^2}{r}$, v = orbital velocity

This is assuming that the orbit is in a circular manner.

Calculations show $\frac{a_A}{3600} = a_{moon}$, and also $60\, r_E = r_{moon}$.

Thus, it appears $\frac{a_A}{(r_E)^2} = a_{moon}$.

So, $F = \frac{Mm}{r^2}$ is a result of these ratios w/ $\frac{v^2}{r}$

Now are we done, or should I do something else to this?

The units don't match, right? So, if some kid called, Isaac comes to you and says, "I got this new law of gravitation", I would say "Go back and fix your units." The units don't match. But not only that, I think some of you had another objection, that if you took the mass of the Earth and mass of the Moon and the apple and divided by the radius, you'll get a number that won't agree with the force of gravity near the Earth as we know it. So the point is, when a function is proportional to all these things, there is always a proportionality constant. And the constants--One of the purposes of a constant is to make the dimensions come out right; the other is to make the values come out right.

constant = G, fixes our units & values.

So, $F = G\frac{Mm}{r^2}$.

$a_A = \frac{GM}{r^2}$ better $= 9.8\ m/s^2$

$G = (\text{number}) \cdot \frac{N \cdot m^2}{kg^2}$ = universal gravitational constant

I'm almost ready to write down now the great Newtonian law. The great Newtonian law says

$F = G\frac{Mm}{r^2}$ Where M and m are the masses of two objects that are at a distance r from each other.

But it's not yet a vector; this is just a number.

Lets say a mass feels a force towards the center

← moon

↑ earth, one vector pointing away from center is $\vec{r}$.

So $F = G\frac{Mm}{r^2} \cdot \vec{r}$, lets divide that by r.

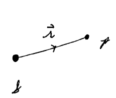
$F = G\frac{Mm}{r^2}\left(\frac{\vec{r}}{r}\right)$, but we want r to point towards the center, so

$F = -G\frac{Mm}{r^2}\left(\frac{\vec{r}}{r}\right)$, and now we have the law of universal gravitation.

It's called the Law of Universal Gravitation because it really is universal. You got to realize that. It's a tremendous leap of faith to believe that the laws that are operative near the surface of the Earth also apply to the Moon and also apply to the planets.

And luckily for us that leap of faith is justified because the laws of gravitation--In fact, the laws of physics that we deduce near the Earth seemed to work all over the entire universe. If you've seen pictures and nova of where we are in the scheme of things, we are in a very tiny part of the universe, sampling a tiny volume, over a tiny period of time. But we apply those equations to the far end of the galaxy, the far end of the universe, and going all the way back to the Big Bang. And we predict the future fate of the universe. So, we assume that the laws that we discover here, now, are valid everywhere and at all times. So far that seems to be true, and that's a big break for us, but for that you really couldn't do anything. If the laws you discovered today are not valid tomorrow, you cannot make any predictions. So, it's another unwritten benefit we got -- unspoken -- it's a great benefit that the laws you find are universal. Somebody had to stick his neck out and say, this is one of the first occasions.

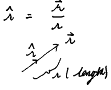
So now, you are going to apply this law to understand what Kepler did. Okay, so what do you have to do to do what Kepler did? So, we are going to consider the motion of a planet around the Sun.



We use $\vec{F} = m\vec{a}$

$-\frac{GMm}{r^2}\hat{r} = m\frac{d^2\vec{r}}{dt^2}$

$\hat{r} = \frac{\vec{r}}{r}$

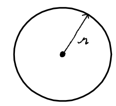


So if you want, if r is the position vector of an object, whose length is r, r-hat is a little guy whose length is 1, and which points in the radial direction. Radial direction being a direction pointing along the radius.

So we have $-\frac{GMm}{r^2}\hat{r} = m\frac{d^2\vec{r}}{dt^2}$

So, this is your equation. It's then a problem in calculus, to solve the equation, and everything Kepler said should

So, we are going to do the circle. We're going to take the orbit to be circular, and we're going to apply this force law to that and see what we can get. So, let's do that. We can ask ourselves, "Can a planet have a circular orbit of radius r?"



Can I find an orbit of a planet going around at a constant speed around the Sun, on a circle?

Let v = constant. We get $m\frac{v^2}{r}$.

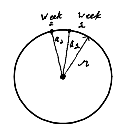
$m\frac{v^2}{r} = G\frac{Mm}{r^2}$ We can cancel things and say

The left-hand side is the effect, the right-hand side is the cause. If you're spinning a rock, spinning it in a circle, that also has an acceleration toward the center but the string is providing the force and you can work out the force of a string, of the string. Here, it's the unseen force of gravity from the Sun reaching out to a planet and pulling it in. And Newton tells you the formula, deduced from terrestrial and lunar observations, now applied to planets. Okay, so this is something that should make you very happy because this is a very simple application of F = ma, but you're describing planetary motion now.

$v^2 r = GM$

The equation says, yes, you can have a circular orbit of any radius you like, provided you move at the speed satisfying this equation. And as long as you satisfy the equation, I don't care if the thing you're spinning is a satellite or a potato, because the mass of the object drops out. So, the condition for orbit is independent of the object orbiting, again, due to this cancellation of the mass on both sides. All right, so this is all we can get out of Newton's laws.

Let's go back to Kepler. Is the orbit possible? Well, if you say an ellipse --a circle is a special case of the ellipse, you have shown circular orbits are possible. How about equal areas in equal time? It's obviously true in this problem because it's going at a constant speed.



We can tell $a_2 = a_1$

So, the only thing left to knock off is a third assumption, the third observation of Kepler involving the time period [T].

$V = \frac{2\pi r}{T}$, plug this in and we get

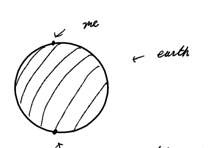
$\frac{4\pi^2 r^3}{T^2} = GM$

$\frac{T^2}{r^3} = \frac{4\pi^2}{GM}$ → third result

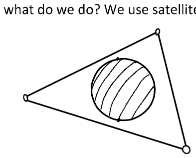
What did Newton do that went further than Kepler?

Kepler said T^(2)/a^(3) is the same thing, but he didn't tell you what the same thing is. What is the constant? Well, now we know that constant to be 4, known to all of us, π, known to all of us, M, mass of the Sun and G is the universal constant of gravitation.

Okay, so this thing has got enormous power. It tells you, if you find another planet, you find a new planet tomorrow, if you know how far it is, I'll tell you what a year is in that planet. I'll tell you how long the planet takes to go around the Sun, because T^(2)/r^(3) is known to me. So, if that planet is at a certain distance, I can tell you the time period, or if I see it going around the Sun and I know it's going to complete the revolution in 240 years, I can tell you how far it is; that's another power of the theory. Okay, so now you can do a variety of problems using this formula. They're all plug-ins, but I'll just mention one. And most problems are like that. Everything's going to be plugging into this formula. But one interesting example is the following.



How do I watch the boxing match? Use radio waves, but radio waves cannot travel through the earth, so what do we do? We use satellites, specifically...



Three satellites that form a triangle.

If you have three satellites, they can help you connect any point on the Earth to any other point by reflecting off the satellites. But the important thing is, the satellites better be where you think they are. If they're constantly moving around, it doesn't work.

So, what you really want is what's called geosynchronous satellites. They are satellites --Remember, if I look at the Earth from the top, it is spinning. This is the North Pole and they're spinning. Therefore, these satellites should be rotating around the Earth once every 24 hours. Then, if they're on top of your head today, they'll be on top of your head all the time. Right? So, the time period of the satellite is 24 hours, and the only question is at what altitude should I launch them, to go to this formula? Put in 24 hours, write it in seconds, you will get the radius. Then, once

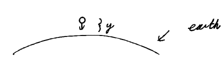
$\hat{r}$ ($\hat{r}$ length) pointing along the radius.

So we have $-\frac{GMm}{r^2}\hat{r} = m\frac{d^2\vec{r}}{dt^2}$

So, this is your equation. It's then a problem in calculus, to solve the equation, and everything Kepler said should come out of that. You should find that planets like to move in elliptical orbits. You should find the areas swept out as a constant. You should find the square of a time period is proportional to the cube of the orbit size.

So, what you really want is what's called geosynchronous satellites. They are satellites --Remember, if I look at the Earth from the top, it is spinning. This is the North Pole and they're spinning. Therefore, these satellites should be rotating around the Earth once every 24 hours. Then, if they're on top of your head today, they'll be on top of your head all the time. Right? So, the time period of the satellite is 24 hours, and the only question is at what altitude should I launch them, to go to this formula? Put in 24 hours, write it in seconds, you will get the radius. Then, once you got the radius, this equation will tell you at what velocity they should be put into orbit. That's it, that's how you put the geosynchronous satellite. That's a simple example of applying the formula.

Now, I'm going to come to the question of energy. Whenever you have a new force you can talk about the Law of Conservation of Energy. So, I want to ask myself, what's the potential energy I can associate with a gravitational force? I can ask myself, is it even a conservative force? So, let's take the problem in two stages.

Say  earth $\vec{F} = -mg\hat{j}$ potential energy $U = mgy$

Therefore $\frac{1}{2}mv^2 + mgh = E$ = constant

But you know this formula is an approximation near the Earth. You want to apply it globally. Before going to the global problem, let me remember one more thing…

$$g = \frac{GM}{r_E^{\,2}}$$

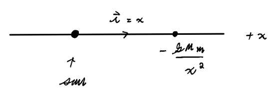
Side-note: there was one thing holding Newton back from publishing these findings. It was the fact that he had to prove that the Earth's gravitational force would come from at a point in its center. In order to prove this he would have to break the earth into little chunks, each of the chunks pulling things near the surface of the Earth, the sum of all of these chunks being the result that all things are pulled at a point in the center of the earth. These calculations required integral calculus, which didn't exist in his time, so he invented it.

I say I want to apply it not near the Earth but arbitrarily far.

Far from the Earth, $\vec{F} = \frac{-GMm}{r^2}\hat{r}$ We first check if this is a conservative force.

$$\vec{F} = \frac{-GMm}{r^2}\left(\frac{\vec{r}}{r}\right) = -GMm\left(\frac{\hat{i}x}{r^3} + \frac{\hat{j}y}{r^3}\right), \quad r = \sqrt{x^2+y^2}$$

I will repeat what he said; that's the correct answer. I must take the x component of this, take the y partial derivative, the y component of this, take an x partial derivatives. You guys should do that. When you take the derivative of something like 1/r^(3), you should first take the r derivative of r^(3), and then take the x derivative of r or the y derivative. Using the Chain Rule of calculus, you'll find the two agree. So, we know it's conservative. That's the power of the result. So, all you have to do is guess what potential could have led to this force.

So lets say  $+x$, $-\frac{dU}{dx} = \frac{-GMm}{x^2}$,

Therefore $U = -\frac{GMm}{r}$

So, what am I claiming? I'm claiming that if you got the Sun and you got some other planet moving around it, the following quantity will not change…

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant}$$

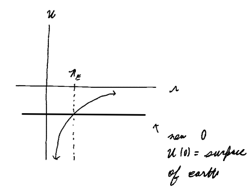
Now, take an object moving near the Earth. I gave you a different rule. I said near the Earth. What did I say the rule was?

$$\text{constant} = \frac{1}{2}mv^2 + mgh$$

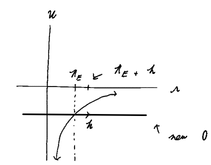
There seem to be two definitions of potential energy. They don't even agree in sign. So, what's going on? What's the way out of this problem I'm in?

The right answer is, when you define a potential, there is a freedom to add a constant to the definition of the potential. If you look at every theorem, everything I derived, including the stuff I've hidden under the board, U_1 - U_2 is what enters everywhere; the change in kinetic energy is U_1 - U_2. Therefore, one has a freedom in defining the potential to add a constant, and what has happened is that the person working with our first potential energy far from earth and the person working with the second potential near earth have not taken the same reference point. This person working near the Earth said, let me choose for convenience the potential at height equal to 0 to be 0, because that's a natural point for me. Potential is 0 at infinity because when you're doing celestial mechanics, here's one object, other objects of various distances, infinity is rather an interesting point. You go infinitely far from it, no potential energy. So, the two of them differ by a constant. That's the reason why. And the constant can easily overturn a quantity that's negative, and bring it up to positive values. So, except for that constant, once I add a certain constant, everything should work out.

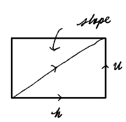
Let's look at the formula for near the surface of the earth.



lets measure as h



If we zoom in…



so $mg \cdot h$ is our formula

So going back to our universal potential energy…

$$U = -\frac{GMm}{r_E + h} = \frac{GMm}{r_E(1 + h/r_E)} = -\frac{GMm}{r_E}\cdot\left(1 + \frac{h}{r_E}\right)^{-1}$$

Remember $(1+x)^n = 1 + nx + \ldots$ if x is very small

x is the height over the radius of the Earth, which is obviously tiny, so I can approximate that and I get…

$$U = \underbrace{-\frac{GMm}{r_E}\cdot 1}_{\text{potential energy near earth}} + \underbrace{\frac{GMm}{r_E^{\,2}}\cdot h}_{\text{change in energy for } U(0)\to\infty}$$

$$\text{So } U(h) - U(0) = \frac{GMm}{r_E^{\,2}}h$$

It's best understood in terms of the graph we drew.

Now, for a circular orbit, what's the energy?

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

$$= \frac{1}{2}\frac{GMm}{r} - \frac{GMm}{r} = \frac{U}{2} = -K$$

$$\text{Meaning } \frac{1}{2}\frac{GMm}{r} - \frac{GMm}{r} = -\frac{GMm}{2r}$$

So, for a particle in a circular orbit, the kinetic energy is exactly half the magnitude of the potential energy and the total energy is negative.

Let's close with one very important result.

So for 

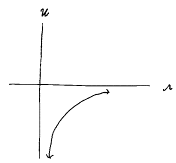
A celestial body moving around the sun, you can compute the total energy. If the total energy is negative, that object is bound to go on orbiting the Sun. It can never escape the infinity. If the total energy is positive, it can go to infinity.

So, why is it that if the total energy is negative, the body can never run off to infinity?

at infinity the potential energy is zero. What about kinetic energy? Some positive number, right? But its initial energy was negative, and you cannot change it. So if you got total negative energy, you cannot be found wandering around infinity, because at infinity if you're moving around, you got positive kinetic energy, no potential energy, and the energy that was originally negative, you have no business being there. So, objects with negative energy will never escape. So, if you saw a comet, you want to know if it'll come back again, find the kinetic plus potential. If it's positive, it won't come back; if it's negative it's trapped. That's the dividing line.

So, if you're near the Earth and you want to shoot something, you can pick it in any direction you want. Suppose you start shooting them up. They go to different heights, and one day you don't want it to come back. You can ask yourself, "How fast should I fire a gun so the bullet never comes back?"

other objects of various distances, infinity is rather an interesting point. You go infinitely far from it, no potential energy. So, the two of them differ by a constant. That's the reason why. And the constant can easily overturn a quantity that's negative, and bring it up to positive values. So, except for that constant, once I add a certain constant, everything should work out.

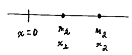
lets take  for $U(0)$ at ∞ (the celestial formula).

you want to know if it'll come back again, find the kinetic plus potential. If it's positive, it won't come back; if it's negative it's trapped. That's the dividing line.

So, if you're near the Earth and you want to shoot something, you can pick it in any direction you want. Suppose you start shooting them up. They go to different heights, and one day you don't want it to come back. You can ask yourself, "How fast should I fire a gun so the bullet never comes back?" Well, you should fire it so that it can go to infinity. When it goes to infinity, you want to just manage to get there; it's going to go to infinity, stagger, and fall down. Well, it's got no kinetic energy in infinity, it's got no potential energy in infinity. Therefore, its total energy should be 0. Therefore, what you should do--At the surface of the Earth your ½ mv^(2) - GMm/R_E should be 0, or these two numbers should be equal; or canceling the mass you will find some formula, which is v^(2) = GM/ R_E. So, depending on the mass of the Earth and gravitational constant radius of the Earth, you'll get a velocity. That's called "the escape velocity." When something is fired with escape velocity, it will just about make it to infinity. You want to be really sure it doesn't come back, give it a little extra speed. A little less, it'll orbit the Earth, it just won't go away.

Notation : $E = \frac{1}{2} m v^2 - \frac{GMm}{R_E^2}$, or $\frac{1}{2} m v^2 = \frac{GMm}{R_E^2}$ so $v^2 = \frac{2GM}{R_E}$

We are going to enlarge our domain to more than one body obeying Newton's laws. So, let me start with the simplest possible case: two bodies.



$$m \frac{d^2x}{dt^2} = m_1 \ddot{x}_1$$

two dots mean two derivatives

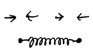
$$\Downarrow$$
$$F_1$$

Now, look at the body one and ask, "What are the forces acting on it?" Well, it could be the whole universe. But we're going to divide that into two parts.

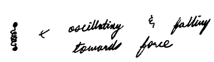
$F_1 = F_{12} + F_{1e}$ 'e' being external forces, outside world force like gravity

Similarly, $m_2 \ddot{x}_2 = F_{21} + F_{2e}$

For example, if the spring is compressed at this instant, it's trying to push them out; that means, really, 1 is trying to push 2 out, whereas that way 2 is trying to push 1 to the left, this way. That's an example of F_12 and F_21. The external force could be due to something extraneous to these two bodies.



So, one example is at the surface of the Earth. I take these two masses connected by a spring. Here is mass 1 and here's mass 2. I squash the spring. If there is no gravity, they will just go vibrating up and down but let them fall into the field of gravity, so they're also experiencing the mg due to gravity. So, they will both fall down and also oscillate a little with each other. They're all described by this equation.



$$m_2 \ddot{x}_2 = F_{21} + F_{2e}$$

And let us

$$+ m_1 \ddot{x}_1 = F_{12} + F_{1e}$$

$$m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = F_{1e} + F_{2e} + F_{12} + F_{21}$$

Whatever the underlying force, gravity, spring, anything, force on 1 due to 2, and force on 2 due to 1 will cancel, and everything I get today, the whole lecture is mainly about this one simple result, this cancellation.

$$m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = F_{1e} + F_{2e} + \cancel{F_{12}} + \cancel{F_{21}}$$

So now $m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = F_{1e} + F_{2e} = F_e$

F_e = total external force on two body system.

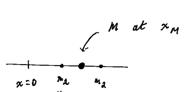
Lets introduce $X \equiv \dfrac{m_1 x_1 + m_2 x_2}{M}$, $M = m_1 + m_2$

We can write the equation as $M\ddot{X} = F_e$, this is the big equation

$$M\ddot{X} = F_e$$
$$\downarrow$$
$$(m_1 + m_2)\left[\frac{m_1 \ddot{x}_1 + m_2 \ddot{x}_2}{m_1 + m_2}\right]$$

, so M (which is $m_1 + m_2$) cancels itself out

So what have I done? I have introduced a fictitious entity, the center of mass. The center of mass is a location X, some kind of a weighted average of x_1 and x_2. By weighted, I mean if m_1 and m_2 are equal, then capital M will be two times that mass and you'll just get x_1 + x_2 over 2. The center of mass will sit right in between. But if m_1 is heavier, it'll be tilted towards m_1; if m_2 is heavier, it'll be tilted towards m_2. It's a weighted sum that gives a certain coordinate. There is nothing present at that location. There's nobody there. The center of mass is the location of a mathematical entity. It's not a physical entity. If you go there and say, "What's at the center of mass?" you typically won't find anything. And it behaves like a body.

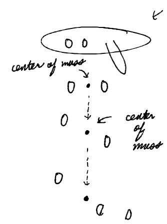


M is a little left if $m_1 > m_2$

M is a little right if $m_2 > m_2$

The key is however, that all of the internal forces have canceled out, so all that is left on the right side of the equation is the external forces.

So, if I can say this in words, what we have learned is that the advantage of introducing a quantity called "center of mass" is that it responds only to the total force; it doesn't care about internal forces.



Two masses thrown out

Even if they split away center of mass is still same.

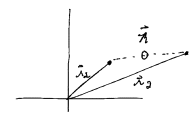
The dynamics of the center of mass remains unchanged.

Even if one of the masses splits into two the center of mass dynamics is still unchanged.

If there were no external forces, then the center of mass will behave like a particle with no force. If it's not moving to begin with, it won't move later. Or if it is moving to begin with, it'll maintain a constant velocity.

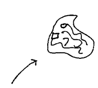
Here's another example. Now you can obviously generalize this to more than one dimension. If you're living in two dimensions, you will introduce an x coordinate and introduce a y coordinate and then you will have

$$M\ddot{\vec{r}} = F \qquad \vec{r} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{M}$$

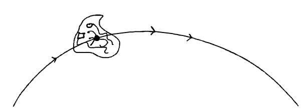


The above is the two-dimension version of our formula, with the location of the center of mass now being represented as a vector.

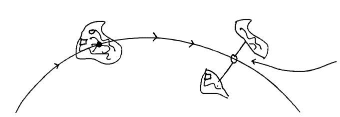
Another example, a mass with a bunch of chains, springs, strings, all jumbled up is thrown into the air.



If you apply our formula at every instant you will find that the mass will still have a motion with respect to the center of mass.



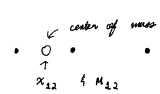
Even if the mass splits apart during the motion, the motion will still be with respect to the center of mass.



You should realize that if you've got several bodies, say three bodies, then I will define the center of mass to be

$$x_{\text{center of mass}} = \frac{\sum m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{m_1 x_1 + m_2 x_2 + \dots m_n x_n}{m_1 + m_2 + \dots + m_n} \quad (n = 3)$$

Given m_1, m_2, m_3, to find the center of mass for all of these objects ...



Then add $x_3\ m_3$ to M_{12} and we will get same result as formula

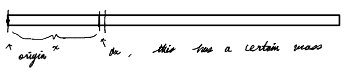
If you've got many bodies and you want the center of mass of all of them, you can take a subset of them, replace them by their center of mass, namely, all their mass sitting at their center of mass, replace the other half by their mass sitting at their center of mass, and finally find the center of mass of these two centers of mass, properly weighted, and that'll give you this result.

Things become more interesting if I give you not a set of discrete masses but discrete locations, namely, a countable set of masses.

rod of mass M, length L

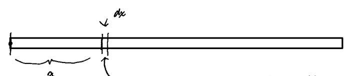
How do we find the center of mass?

To find the center of mass for this rod we need to adapt our definition.



I argue it's at a definite distance x from the coordinate's origin. And if you're nitpicking you will say, "What do you mean by definite distance?" It's got a width dx, so one part of it is at x, the other part is at x + dx so it doesn't have a definite coordinate. But if dx is going to 0, this argument will eventually be invalid. So dx goes to 0, the sliver has a definite location, which is just the x coordinate of where I put it.

Let us first find how much mass is sitting at dx, lets call it



δm , $\delta \to$ delta (greek)

$\downarrow$ how much mass

$$\underbrace{\delta m = \frac{M}{L}}_{\text{mass per unit length}} dx \leftarrow \text{width}$$

$$\text{Therefore} \quad x = \frac{\int_0^L x \frac{M}{L}\, dx}{M}$$

Therefore, the center of mass that I want is found by taking this sliver of that mass(mass of dx), multiplying by its coordinate and summing over all the slivers, which is what we do by the integral from 0 to L. Then, I should divide by the total mass, which is just M. You can see now if you do this calculation. I get 1/L; then, I get integral xdx from 0 to L, and that's going to be L^(2) over 2. So, if you do that you'll get

$$x = \frac{1}{L}\int_0^L x\, dx = \frac{L}{2}$$

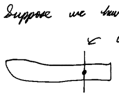
Therefore $x_{\text{center of mass}} = \frac{L}{2}$, the center

Note that this is under the assumption that the rod has a uniform mass.

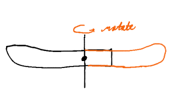
It must be clear to most people that the center of mass of this rod is at the center. But how do we argue that? How do you make it official? If you do the integral you will get the answer, but I want to short circuit the integral. And here is a trick. It's not going to work for arbitrary bodies. If I give you some crazy object

you cannot do anything

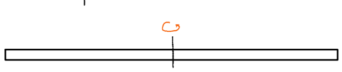
Suppose we have



, if we replace every x w $-x$ we get



$x \to -x$

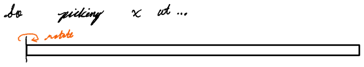


do the same thing for the rod

If the rod looks the same, then -x = x.

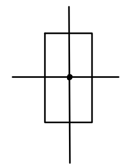
So, without doing any detailed calculation, you argue that the answer is X = 0. To do this, of course, you must cleverly pick your coordinates so that the symmetry of the body is evident.

So picking x at ...



We have another rod... one w/ non-zero width

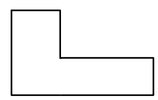
We can argue by symmetry that the center of mass for the rod has to be…



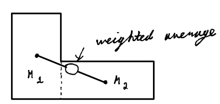
$-y = y$

$-x = y$

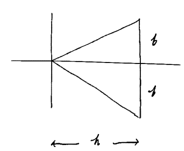
But what if we now have an object such as ?



What you should do is split these masses and find the center of mass for the individual masses, then sum their mass and position vectors in order to get the center of mass.



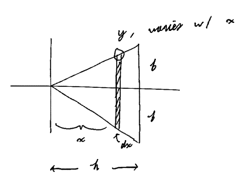
Let us analyze one more object.



mass of object = M

Obviously the object is symmetrical over the x axis. And intuition would tell us that the center of mass of this object would then be on the x axis.

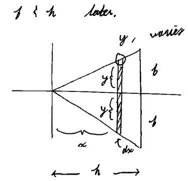
I cannot do further calculations of this type by saying where it is on the line, because it has no longer a symmetry in the x direction; it's symmetric on the y after flipping y, but I cannot take x to -x. For x we must do…



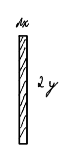
$x_{\text{center of mass}} = ?$

To find the center of mass of this triangle I can divide it into vertical strips which are parallel to each other, and find the center of mass by adding the weighted average of all these things. For that, I need to know what's the mass of the shaded region. So, let's call it δm.

$\delta m = \frac{M}{A}$ (area of shaded region in mass per unit area, A will be found in terms of b & h later.

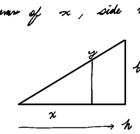


$dx \to 0$ makes slope look like a rectangle



$\delta m = \frac{M}{A} 2y\, dx$, lets write y in terms of x, side note: $\delta m \propto dx \to 0$, so dx's location is at x.

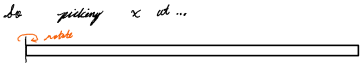
Similar triangles show: $\frac{y}{h} = \frac{x}{b}$



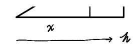
Ratios of corresponding sides must be equal to each other.

$$\delta m = \frac{M}{A}(2)\frac{bx}{h}\, dx$$

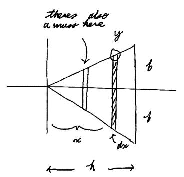
So picking x at ...



Does not really give anything.



$$\delta m = \frac{M}{A}(2)\frac{bx}{h}\,dx$$



So remember, I don't just integrate this over x; that would just give me the mass of the body. I should multiply it by a further x and then do the integral.

$$x_{\text{center of mass}} = \Big(\underbrace{\int \overset{\text{mass of each sliver}}{\delta m} \cdot \overset{x \text{ for span of the mass in question}}{x}\, dx}_{\text{weighted average}}\Big)\cdot\frac{1}{M}$$

$$= \int \frac{2Mbx}{Ah}\cdot x\,dx\cdot\frac{1}{M}$$

$$= \frac{2Mb}{Ah}\cdot\int_0^h x^2\,dx\cdot\frac{1}{M}$$

$$= \frac{2Mb}{Ah}\cdot\frac{h^3}{3}\cdot\frac{1}{M}$$

Now A = area of triangle $\cdot\, 2 = b\cdot h$

So $$x_{cm} = \frac{2Mb}{bh^2}\cdot\frac{h^3}{3}\cdot\frac{1}{M}$$

$$= \frac{2}{3}h$$

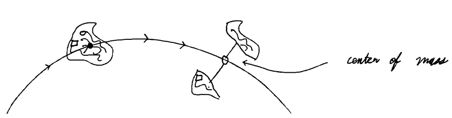
The center of mass of this mass is not halfway to the other end, but two-thirds of the way because it's top heavy; this side of it is heavier. This is the level of calculus you should be able to do in this course, be able to take some body, slice it up in some fashion, and find the location of the center of mass. You combine symmetry arguments with actual calculation. For this sliver, by symmetry, you know the center of mass is at the center, you don't waste your time, but then when you add these guys there is no further symmetry you can use; you have to do the actual work. So, what have I done so far? What I've done is point out to you that when you work with extended bodies, or more than one body, we can now treat the entire body, replace the body by a single point for certain purposes. The single point is called a center of mass, where it imagines all the mass concentrated at the center of mass. So, you have created a brand new entity which is fictitious. It has a mass equal to total mass. It has a location equal to the center of mass, and it moves in response to the total force. And it's not aware of internal forces, and that's what we want to exploit.

So let's now conduct a thorough analysis of...

$M\ddot{\vec{x}}$

Case 1: $F_e \neq 0$

This example



Case 2a: $F_e = 0$, meaning $M\ddot{\vec{x}} = 0$ so $M\dot{\vec{x}}$ = constant

Who is this MR.? What does it stand for? Well, it looks like the following. If you take a single particle of mass m, and velocity x., we use the symbol p, maybe I've never used it before in the course, and that's called the momentum.

$$m_1\dot{x} = p = mv$$

The momentum of a body is this peculiar combination of mass and velocity. In fact, in terms of momentum we may write Newton's law.

$$F = m\frac{dV}{dt} = \frac{d}{dt}(mv) = \frac{dp}{dt}$$

Sometimes, instead of saying force is mass times acceleration, people often say "force is the rate of change of momentum." The rate at which the momentum of a body is changing is the applied force.

If we go back to $m_1 + m_2\left[\frac{m_1\dot{x}_1 + m_2\dot{x}_2}{m_1+m_2}\right]$

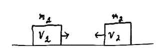
$$\underbrace{M\dot{x}}_{p \,\to\, \text{momentum of c.m.}} = m_1 + m_2\left[\frac{m_1\dot{x}_1 + m_2\dot{x}_2}{m_1+m_2}\right] = p_1 + p_2$$

So if $F_e = 0$, $p_1 + p_2$ does not change

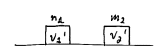
Take a collection of bodies. At a given instant everything is moving; it's got its own velocity and its momentum, add up all the momenta. If you had one dimension, just add the numbers. If in two dimensions, add the vectors; you get a total momentum. That total momentum does not change if there are no outside forces acting on it.

A classic example is two people are standing on ice. Their total momentum is 0 to begin with, and the ice is incapable of any force along the plane. It's going to support you vertically against gravity, but if it's frictionless it cannot do anything in the plane. Then, the claim is that if you and I are standing and we push against each other and we fly apart, my momentum has to be exactly the opposite of your momentum, because initially yours plus mine was 0; that cannot change because there are no external forces. If two particles are pushing against each other, they can only do so without changing the total.

Another important context...



After collision, we will have...



$$m_1 V_1 + m_2 V_2 = m_1 V_1' + m_2 V_2'$$

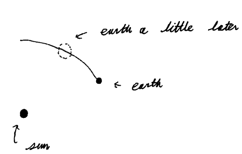
Here's a simple example; you can show that if mass 1 and that mass 2 are equal, and say mass 2 is at rest, that mass 1 comes and hits mass 2. You can show under certain conditions that mass 1 will come to rest and mass 2 will start moving with the speed of the--the target will move at the speed of the projectile. So, momenta of individual objects have changed. One was moving before, it is not moving; one was at rest, it's moving, but when you add up the total, it doesn't change. This is called the Law of Conservation of Momentum.

This is a super important finding that says...

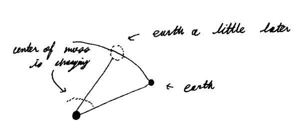
$$\text{If } \vec{F}_e = 0,\quad \underbrace{\vec{p}_1 + \vec{p}_2 + \ldots + \vec{p}_n}_{\text{initial}} = \underbrace{\vec{p}_1{}' + \vec{p}_2{}' + \ldots \vec{p}_n{}'}_{\text{final}}$$

It's very important you follow this statement and follow the conditions under which it's valid. There cannot be external forces. For example, in the collision of these two masses, if there is friction between the blocks and the table, you can imagine they collide and they both come to rest after a while. Originally, they had momentum and finally they don't. What happened? Well, here you have an explanation, namely, the force of friction was an external force acting on them. What I'm saying is that if the only force on each block is the one due to each other, then the total momentum will not change.

Case 2b: $\vec{F}_e = 0$ $\quad \dot{x} = 0$ (mass is at rest)



"Is this picture of the Sun sitting here and the Earth moving around acceptable or not in view of what I've said?" What is wrong if I just say the Sun remains here and the Earth goes in a circle, which is what we accepted last time?



The center of mass is moving without external forces which is incorrect from our analysis.

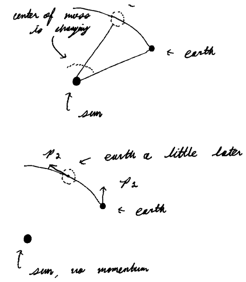
$$F = m\frac{dv}{dt} = \frac{d}{dt}(mv) = \frac{dp}{dt}$$

Sometimes, instead of saying force is mass times acceleration, people often say "force is the rate of change of momentum." The rate at which the momentum of a body is changing is the applied force.

So if you think about it this way, it looks like…

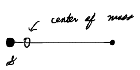
$M\vec{R}$ = momentum of center of mass

And we are told the momentum of the center of mass does not change if there are no external forces. But the momentum of the center of mass has a very simple interpretation in terms of the parts that make up the center of mass; let's see what it is.

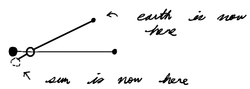


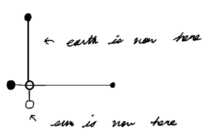
The center of mass is moving without external forces which is incorrect from our analysis.

So the center of mass must result in…

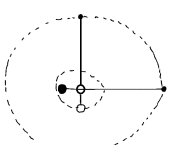


(assuming that the sun isn't so big as to make the center of mass the center of the sun)



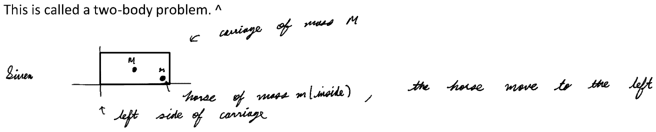


Therefore the orbits look like

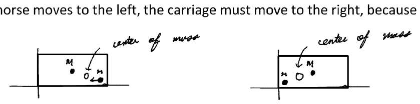


So, what will happen is, the Sun will go around on a circle of smaller radius, the planet will go around on a circle of a bigger radius, always around the center of mass.

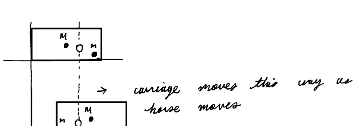
This is called a two-body problem. ^

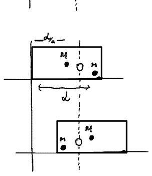


If the horse moves to the left, the carriage must move to the right, because

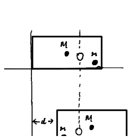


But we know the center of mass cannot move, therefore…

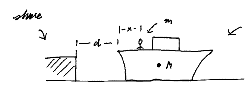




Take the weighted average of L/2 and L and find the x coordinate to get the center of mass.

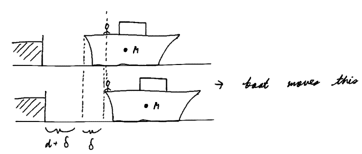


Let us say it has moved an unknown distance d, which is what you're trying to calculate. Then, compute the center of mass. When you do that, remember that the center of the carriage is L/2 + d from this origin. The horse is at the distance d from the origin. Equate the center as a mass and you will get an equation that the only unknown will be d, and you solve for d and it'll tell



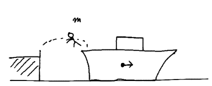
Boat of mass M and we have a mass of m. Distance d from the shore. We stand at a distance x from the edge of the boat. We want to get off the boat. How do we get off?

Get as close to the shore as possible before we jump onto the shore.



You are certainly at the edge of the boat, but the boat has moved a little extra distance δ, and you have to find that δ. You find it by the same trick. You find the center of mass of you and the boat, preferably with this as the origin. You can use any origin you want for center of mass, it's not going and it's not going according to anybody, but it's convenient to pick the shore as your origin, find the weighted sum of your location and your mass, the boat's location and boat's mass. At the end of the day, put yourself on the left hand of the boat, and let's say it has moved a distance δ, so the real distance now is d + δ. That's where you are. That plus L/2 is where the center of the boat is. Now, find the new center of mass and equate them and you will find how much the boat would have moved, and that means you have to jump a distance d + δ.

Next we leap into the air…

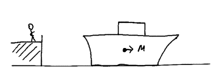


Since momentum cannot change, the boat will have to move to the right after we jump off of it.

The momentum cannot change. Originally, the momentum was 0, nobody was moving, but suddenly you're moving, the boat has to move the other way. Of course, it doesn't move with the same velocity, or the same speed; it moves with the same momentum.

$$mV = Mv$$

Now we landed on the shore, and our momentum is 0.

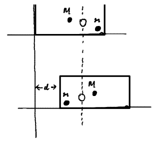


The boat will not stop just because you hit the shore. The boat will keep moving because there's no force on the boat; it's going to keep moving. The question is, "How come I suddenly have momentum in my system when I had no momentum before?" It's because the F external has now come into play. Previously, it was just you and the boat and you couldn't change your total momentum. But the ground is now pushing you, and it's obviously pushing you to the right because you were flying to the left and you were stopped. So, your combined system, you and the boat, have a rightward force acting for the time it took to stop you; it's that momentum that's carried by the boat. A better way to say this is, you and the boat exchange momenta, you push the boat to the right, you move to the left, and your momentum is killed by the shore. The boat, no reason to change, and keeps going. So, can you calculate how fast the boat is moving?

If I only told you that I jumped and landed on the shore, that's not enough to predict how fast the boat is moving. But if I told you my velocity when I was airborne, then of course I know my momentum and you can find the boat momentum and that's the velocity it will retain forever. So, you need more information than simply saying, "I jumped to the shore." It depends with what velocity I left the boat and landed on the ground. If I leap really hard, the boat will go really fast the opposite way.

Let's go to another class of problems…

A rocket is something everybody understands but it's a little more complicated than you think. Everyone



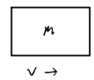
Let us say it has moved an unknown distance d, which is what you're trying to calculate. Then, compute the center of mass. When you do that, remember that the center of the carriage is L/2 + d from this origin. The horse is at the distance d from the origin. Equate the center as a mass and you will get an equation that the only unknown will be d, and you solve for a d and it'll tell you how much it moves.

Find the center of mass before, find the center of mass after, equate them and that linear equation will have one unknown, which is the d by which the carriage has moved and you can solve for it.

than simply saying, "I jumped to the shore." It depends with what velocity I left the boat and landed on the ground. If I leap really hard, the boat will go really fast the opposite way.

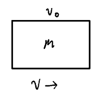
Let's go to another class of problems...

A rocket is something everybody understands but it's a little more complicated than you think. Everyone knows you blow up a balloon, you let it go, the balloon goes one way, the air goes the other way, action and reaction are equal, even lay people know that. Or, if you stand on a frozen lake and you take a gun and you fire something, but then the bullet goes one way and you go the opposite way, again, because of conservation of momentum. The rocket is a little more subtle and I just want to mention a few aspects of it.

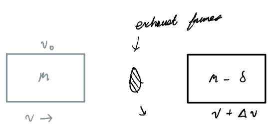


Rocket of mass M w velocity $\vec{v}$.

What rockets do is they emit gasses, and the gasses have a certain exhaust velocity. That velocity is called v_0. In magnitude, it's pointing away from the rocket, and it has a fixed value relative to the rocket, not relative to the ground.



A short time later...



There is a change in mass because fuel is expended in the exhaust.

velocity of fumes $= v - v_0$ ← important point

momentum of exhaust $= \delta(v - v_0)$

Its velocity with respect to the rocket is pointing to the left of v_0, but the rocket itself is going to the right at speed v. So, the speed as seen from the ground will be v - v_0. So, the Law of Conservation of Momentum will say...

$$Mv = (M - \delta)(v + \Delta v) + \delta(v - v_0)$$

$$\boxed{Mv} = \boxed{Mv} - \boxed{v\delta} + M\Delta v - \overset{\text{ignore}}{\underbrace{\boxed{\delta(\Delta v)}}_{\text{two infitesimals}}} + \boxed{\delta v} - v_0\delta$$

So $M \cdot \Delta v = v_0 \delta$

$$\frac{\Delta v}{v_0} = \frac{\delta}{M}$$

This is the relation between the change in the velocity of the rocket; the velocity of the exhaust gases seen by the rocket, the amount emitted in the small time divided by the mass at that instant. But in the sense of calculus, what is the change [dM] in the mass of the rocket? If M is the mass of the rocket, what would you call this a change, the mass of the rocket, in this short time?

But $dM = -\delta$, since change in mass is ⊖

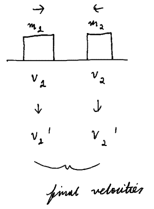
So, $\frac{\Delta v}{v_0} = \frac{\delta}{M} = -\frac{dM}{m}$, and $\int \frac{\Delta v}{v_0} = \frac{\delta}{M} = -\int \frac{dM}{M}$.

We get $\frac{v_f - v_0}{v_0} = \log \frac{m_0}{m_f}$, meaning $v_f = v_0 + v_0 \log \frac{M_0}{M_f}$

This is just to show you how we can apply the Law of Conservation of Momentum.

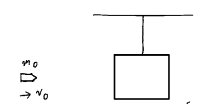
That is a formula that tells you the velocity of the rocket at any instant, if you knew the mass at that instant. The rocket will pick up speed and its mass will keep going down, and the log of the mass before to the mass after times v_0 is the change in the velocity of the rocket.

Our last topic will be the subject of collisions.

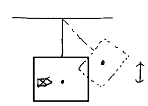


So here, there are two conditions you need because you're trying to find two unknowns, right? We want two unknowns, I need two equations.

You cannot use the Law of Conservation of Energy in an inelastic collision. In fact, I ask you to check if two bodies--Take two bodies identical with opposite velocities; the total momentum is 0. They slam, they sit together as a lump. They've got no kinetic energy in the end. In the beginning, they both had kinetic energy. So, kinetic energy is not conserved in a totally inelastic collision, in an elastic collision it is.



Bullet impacts into block



When can I use Conservation of Mechanical Energy? When can I not? A bullet driving into a chunk of wood, you better know you cannot use Conservation of Kinetic Energy. But once the combination is going up, trading kinetic for potential, you can.

"Look, I don't care about what happened in between; finally, I've got a certain energy, M + m times g times h, that's my potential energy, not kinetic." Initially, I had ½ m_0^(2). I equate these two guys and I found v 0; that would be wrong. That's wrong because you cannot use the Law of Conservation of Energy in this process when I tell you that it's a totally inelastic collision in the middle. Because, what'll happen is, some energy will go into heating up the block; it might even catch fire if the bullet's going too fast. But you can use the Law of Conservation of Momentum all the time in the first collision to deduce that M + m times some intermediate velocity is the incoming momentum. You understand that? From that, you can find the velocity v with which this composite thing, block and bullet, will start moving. Once they start moving, it's like a pendulum with the initial momentum, or energy. It can climb up to the top and convert the potential to kinetic, or kinetic to potential. There is no loss of energy in that process. Therefore, if you extract this velocity and took ½ (M + m) times this velocity squared, you may in fact equate that to (M + m)gh.

$\frac{1}{2} m_0 (v_0)^2 \neq (M + m) gh$ (total inelastic collision in the middle)

use $(M + m) v_1 = m v_0$

then $\frac{1}{2} (M + m) v_1^2 = (M + m) gh$

$\underbrace{v_1 \quad v_2}_{\text{final velocities}}$

We want two unknowns, I need two equations.

An equation that always true: $m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$

You need a second equation to solve for the two unknowns, and that's where there are two extreme cases for which I can give you the second equation.

Totally inelastic equation where the masses stick together...

$v_1' = v_2' = v'$

In this case we can write

$m_1 v_1 + m_2 v_2 = m_1 v' + m_2 v'$

$m_1 v_1 + m_2 v_2 = (m_1 + m_2) v'$

So $v' = \dfrac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$

The other category is called totally elastic where kinetic energy is conserved...

$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$

We get $\left.\begin{aligned} v_1' &= \frac{m_1 - m_2}{m_1 + m_2} v_1 + \frac{2 m_2}{m_1 + m_2} v_2 \\ v_2' &= \frac{m_2 - m_1}{m_1 + m_2} v_2 + \frac{2 m_1}{m_1 + m_2} v_1 \end{aligned}\right\}$ one dimension

What you carry in your head is there's enough data to solve this, because I will tell you the two bodies, I'll tell you their masses, I'll tell you the initial velocities; so plug in the numbers you get the final velocity.

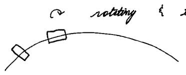
Rotations, Part 1: Dynamics of Rigid bodies

Friday, October 13, 2023 12:15 PM

So, a rigid body is something that will not bend, will not change its shape when you apply forces to it, and one example could be a dime, a coin, or a meter stick. Of course, nobody is absolutely rigid. You can always bend anything, but we'll take the approximation that we have in our hands, a completely rigid body. A technical definition of a rigid body is that if you pick a couple of points, the distance between them does not change during the motion of the body.

The dynamics of rigid bodies, in three dimensions, is fairly complicated, because what I want you to imagine now is not a point mass but some object with some shape. You take this; you throw it up in the air; you can see it does something fairly complicated. In fact, we can characterize what it does using two expressions. I think you can guess what they are. One is, if this had been a point mass, then the point mass will just go back and forth or up and down. It will do what's called "translations." It'll go from a place to another place, maybe on a straight line, maybe on a curve. But as it goes, the point mass has no further information; you've got to tell me where it is, and how fast it's moving. That's the whole story. But if you have a body like this, it's not enough to tell me where it is, because if I throw this up in the air you can imagine that as it travels along some parabolic path, it is translating AND rotating.

↷ rotating & translating

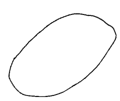


What we want to do is to break the problem into two parts. We want to focus on a body which is only rotating but not translating. Once we've sharpened our skills, we will then put back the ability of the body to translate. So we are going to focus initially on a body that cannot translate.

Real bodies can actually not only wobble and twist and turn, they can also vibrate and they can move their arms around. That's much more difficult, so we won't go there, and we'll try to take the simplest rigid body.

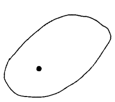
A one-dimensional plane is not enough, because it can only move side to side. This is an instance to where we cannot explain a concept with only one dimension.

So, I'm going to imagine rigid bodies, which are living in the plane of the paper. Now, one way to do that is to take a piece of metal, just cut out a shape, and that's a rigid body.

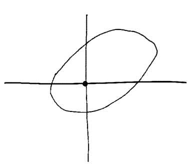


And of course this has a slight thickness, because anything you make of matter has some thickness, at least one atom thick, but imagine that thickness is negligible so it's like a piece of metal, a piece of foil or something, but it's rigid, and this is the rigid body we're going to study. This rigid body can move in the plane of the paper and rotate, but we're going to grab it at one point.

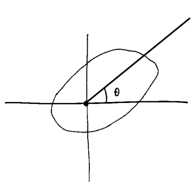
We grab it at one point and ask what the piece of metal can do. (imagine taking a skewer and sticking this object down onto the paper at the point).



at $t = 0$,



A little later, the body will rotate.



metal rotates by θ

So, if I give you θ, I think you'll have to think about it and you have to agree I have told you where the body is. You can reconstruct everything, right, because that point is nailed; it cannot move. You take the body; you turn it by an angle θ. In other words, here is a question. Suppose you're not in this room; you're in another room and I want to tell you what the rigid body is doing. What information do I have to communicate to you, without seeing the picture? I claim, if I tell you how the line was drawn in the body, which is something you do once and for all, a ruled line, then I say the ruled line makes an angle θ with the x axis. I believe you can imagine in another room what this body is doing, and that's the meaning of saying, "I've given complete information." Of course, that's not the complete dynamical information, because it also depends on how fast it's rotating. But to simply say the analog of where the body is in one dimension is this angle θ. So, what we are going to set up, you'll find, is an analogy between one dimensional translation and two dimensional rotations. You'll find the analogy is very helpful.

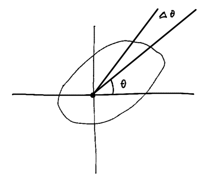
We will add on to this analogy procedurally as we keep explaining these concepts. The first part being...

In one dimension, when a body is moving, x tells you where it is, right?

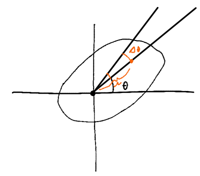
x θ ← tells us what the body is doing

↑

tells us where we are



Let's discuss why physicists prefer radians.



If the rigid body is going around in a circle, then our point r will also go in a circle.

How big is the segment(arc length) that it has traveled after it has rotated by theta?

Full circle $= 2\pi r$

$$\Delta s = 2\pi r \frac{\Delta\theta^\circ}{360} = \left(\frac{\pi}{180}\Delta\theta\right) r$$

If you measured θ in degrees, this is what you have to do to convert an angular traverse to an actual distance along the tangent to the circle. The linear distance you travel is connected with the angle by which you rotate, by the distance r from the center, and this number here. So, people say, "Look, why don't we make life easy by calling θ times π /180 as a new angle."

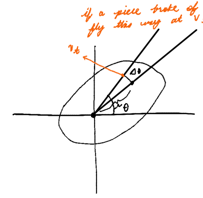
lets say $\Delta\theta = \frac{\theta^\circ}{180}\pi$, so that $\Delta s = \Delta\theta\, r$

But now, when you do this, you are measuring it in radians, so you've got to have an idea how big is a radian. Well, a full circle, if you go a full circle; the distance you travel should be 2πr so a full circle will Δθ = 2π. 360 degrees is an arbitrarily picked human number. There's nothing natural about the number.

You have to know a few popular angles in radian measure. I'll be using only radians most of the time. You've got to know a circle is 2π; π is a half a circle, and π/2 is a quarter of a circle and when you are by 90 degrees that is π/2; that's something you should know. Once you write it this way--Let's get another result that's very useful.

$\frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t}$, $\frac{\Delta s}{\Delta t} = v_{tangential}$, $\frac{\Delta\theta}{\Delta t}$ = angular velocity $= \omega$ (omega)

If a piece broke off it would fly this way at v_t



So, you can calculate angular velocity for various things. For example, what's the angular velocity of the Moon as it goes around the Earth?

$$\omega = \frac{2\pi}{28 \times 24 \times 3600}\ \text{rad/s}$$

So, the angular velocity of a blade that's spinning with frequency f is 2πf.

$\omega = 2\pi f$

So let's add this to our analogy...

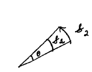
x	θ
$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$

ω is similar to v.

That is not part of my dictionary but it's a very useful result to know. The tangential velocity is the angular velocity times the distance of that point from the center of the rotation.

$v_t = \omega r$ ↖ distance from center point of rotation

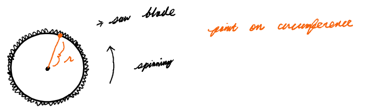
Also note that points closer to the center point of rotation travel less distance as the angle changes, therefore we say the linear velocity increases from the center.

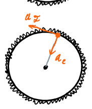


$s_2 > s_1$

Let's take a problem where the angular velocity is itself changing. Fine, that's the analog of saying there is a rate of change of velocity...

x	θ
$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$
$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$	$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$ (alpha is an analog to a)





$v_t = \omega r$

$a_c = \frac{v_T^2}{r} = \omega^2 r$

$a_T = \alpha r$ → tangential acceleration where angular speed is changing

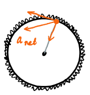
So, the acceleration of a point on the edge or anywhere else can have two components. The tangential one will be there only if you speed up or slow down the rotating disk, but the radial one will always be there as long as it's rotating. Okay, once you've got this dictionary, you can have a whole bunch of analogous quantities. For example, if you focus on problems where the body has a constant angle of acceleration α, what can you say?

x	θ
$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$
$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
(a constant) $x = x_0 + v_0 t + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega t + \frac{1}{2}\alpha t^2$ (α constant)

This is because theta would be behaving just like x in our kinematics.

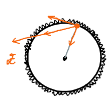
All right, now I'm going to pursue the analogy and ask what is the kinetic energy of a rigid body, when it's rotating?

Suppose this little speck here is you, okay? You happen to be stuck on a rotating chainsaw blade, and you're clinging to it for dear life. What force do you have to apply to stay onto that rotating blade? This may not happen to you and me but if you're in organized crime this is not a very unusual situation. You can find yourself intermingled with all kinds of machinery, and you want to know what do I do to stay on.



$\vec{a}_{net} = \vec{a}_c + \vec{a}_T$

Your acceleration towards the center has to be paid for, and if the blade is picking up speed, that's got to be paid for too. That's the net acceleration; that, times your mass will be the force that has to be applied to you.



$F = m(a_{net})$

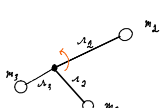
A chainsaw blade, spinning at 3,600 rpm, is decelerated at some rate α. How long does it take to come to rest?

x	θ
$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$
$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$
(a constant) $x = x_0 + v_0 t + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega t + \frac{1}{2}\alpha t^2$ (α constant)
$v = v + at$	$\omega = \omega_0 + \alpha t$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

So, this is the analogy, so we don't have to reinvent all of these things. They just come from the fact that things which are mathematically the same have the same mathematical results. They obey the same equation; they have the same results. All right, now let's find the kinetic energy of a rigid body.

All right, now let's find the kinetic energy of a rigid body.

It's got mass, and if it's spinning, all the little atoms making up the body are moving, and they've got their own ½ mv^(2). And we want to ask, "What is the total ½ mv^(2) summed over all the particles?" For that purpose, it's convenient to take the following simpler rigid body.



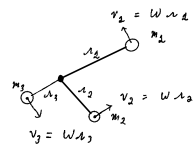
Three massless rigid rods with masses attached to the end of it. Imagine the center point is pinned to the paper therefore the masses can only spin around the center point.

For angular velocity, $+\omega$ for counter clockwise. $-\omega$ for clockwise

So, what's the kinetic energy of this object?

$$K = \frac{1}{2}\sum_{i=1}^{n} m_i v_i^2$$

Now, what's the velocity of each object?



therefore $K = \frac{1}{2}\sum_{i=1}^{3} m_i \omega_i r_i^2$

This is not a new kinetic energy; this is what we know to be kinetic energy, just ½ mv^(2) for every piece that makes up the rigid body. For convenience, I've taken the rigid body to be made up of a discreet set of masses. But now, if you notice, as you go from 1 to 3, m_1 and m_2 and m_3 are three different numbers. We don't know what the masses are. r_1 and r_2 and r_3 are three different numbers. We don't know; each one could be at a different distance from the origin. But the ω doesn't have a subscript i; there's nothing called ω for this one and ω for that one. There's only one ω for everything.

So we can write $K = \frac{1}{2}\left(\sum_{i=1}^{n} m_i r_i^2\right)\omega^2$, and lets say $I = \sum_{i=1}^{n} m_i r_i^2$

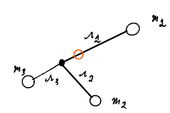
We will then write $K = \frac{1}{2} I \omega^2$, w/ '$I$' being the moment of inertia

So let's now construct a new analog.

$K = \frac{1}{2}mv^2$	$K = \frac{1}{2}I\omega^2$
...	...

I is called the moment of inertia. So, the moment of inertia is determined not only by the masses that make up the body but how far they are from the center. If all the masses just fell on top of the center, the body would have no moment of inertia. It'll weigh the same; the moment of inertia would vanish. And likewise, if the mass is spread out the moment of inertia is more.

And here's another important thing. If I decide to rotate the body about a new location...



The moment of inertia will change, because all the distances r will change. The moment of inertia -- If someone says, "Here are the masses, here's where they are, please find me the moment of inertia," you will say, "I cannot do it." You will say, "I cannot do it until you tell me the point around which you plan to rotate the body. Unless that is given I cannot tell you what the moment of inertia is." Therefore, the moment of inertia is with respect to a point, and there's nothing called the mass with the respect to the point. The mass is just the mass. The moment of inertia depends on the point around which you're computing the moment of inertia; it's a variable.

I = moment of inertia

It's also worth knowing that if the body's being rotated about mass m_3, then m_3 doesn't contribute to the moment of inertia; it's out. Anything sitting on the axis doesn't contribute. Things which are further away contribute more in proportion to their mass, in proportion to the square of the distance from the axis.

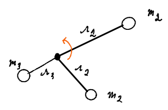
For our analog, in rotational kinetic energy mass is played by the moment of inertia.

$K = \frac{1}{2}mv^2$	$K = \frac{1}{2}I\omega^2$
m	I (units are $kg\,m^2$ which doesn't have a name)

So, this is going to be the analog of mass in our world of rotations. You remember, in the world of translation, we define something called momentum, which is mass times velocity.

$K = \frac{1}{2}mv^2$	$K = \frac{1}{2}I\omega^2$
m	I
$p = mv$	$L = I\omega$ w/ L being angular momentum

One extremely important point that you guys should notice is that all the concepts I'm using, like mass and energy, they are the same old concepts that we learned earlier on. The fact that it's a rotating body doesn't



Three massless rigid rods with masses attached to the end of it. Imagine the center point is pinned to the paper therefore the masses can only spin around the center point.

$p = mv \quad | \quad L = I\omega$ w/ L being angular momentum

One extremely important point that you guys should notice is that all the concepts I'm using, like mass and energy, they are the same old concepts that we learned earlier on. The fact that it's a rotating body doesn't change anything, okay? Kinetic energy still means the same; this K, though it looks so different, is in fact the ½ mv^(2) of every part of the body, summed over the bodies.

So, now we are looking for the most important equation in mechanics, which is F = ma.

$$K = \frac{1}{2}mv^2 \quad | \quad K = \frac{1}{2}I\omega^2$$

$$m \quad | \quad I$$

$$p = mv \quad | \quad L = I\omega$$

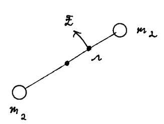
$$F = \frac{dp}{dt} = ma \quad | \quad ? = \frac{dL}{dt} = I\alpha$$

m is not changing, neither is I because distance from point doesn't change

What is the analog for force itself?

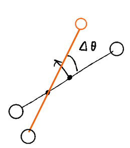
First of all, it's pretty clear that if you want the angular momentum of the body to change, ω has to change. You can see if something's spinning around at some fixed ω, you want to change it, you've got to push or pull or do something. So obviously, you're going to apply a force, but the thing that comes on the left-hand side is not simply the force but something else and we'll find out what the something is. And that's done by the following device.

$\Delta W = F \cdot d = \Delta K_E$ Let's apply this to a simplified rigid body.



Force F applied at r away from point of rotation.

Let's say the body moves an amount $\Delta\theta$.



What is the distance moved?
$r\Delta\theta$

Therefore $\Delta W = Fr\Delta\theta = \Delta K$, $\Delta K = \frac{1}{2}I(\omega - \omega_0)^2$

By the way, these are two very neighboring instants of time. If in the neighboring instants of time, during that brief period we may take the body to have a constant acceleration; therefore, α is constant. So, this formula you can also calculate when α is constant, period. In a real body, when it's rotating, α need not be constant, but for a small enough interval, as small as you wish, we may apply the formula that works when α is a constant. $(\Delta\theta \to 0)$

So we can write $\Delta K = \frac{1}{2}I(\omega - \omega_0)^2 = \frac{1}{2}I\,2\alpha\Delta\theta$

We get $2\alpha\Delta\theta$ from $\omega^2 = \omega_0^2 + 2\alpha(\Delta\theta)$.

So now, $Fr\Delta\theta = \frac{1}{2}I\,2\alpha\Delta\theta$ gives $Fr = I\alpha$

So, this fellow here, Fr, is the analog of force, and it's called the torque.

Fr = torque $= \tau$

It's very important to notice that the torque due to this force is the value of the force times the distance from the point of rotation, and it's only the external force that I'm talking about. There are also internal forces in a rod. Every part of the rod is being dragged along by another part of the rod, but that analysis in terms of forces can be done but it's very tricky and complicated. It's much easier to look at the energy where the energy changes only due to external forces. If you only look at forces you've got to be careful. This pivot point is also applying a force on the rod, but it's not moving and not doing any work. So, the only external force is the one I'm applying and is responsible for the change in the kinetic energy. There are other forces inside the rod. In other words, this mass is going to pick up speed, not because I'm directly pushing it.

I'm saying, whatever is happening internally, I don't care; the external force times distance is a change in the kinetic energy of an object.

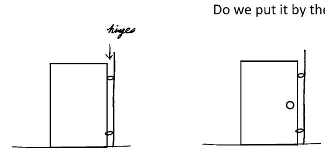
Now, I think it's very clear that if I had several forces acting on the body, then the torque will be…

$$\tau = \sum F_i r_i$$

Now, you've got to realize that torque is the ability to change the rotational state of a body, and what we are realizing is that it depends on the force you apply and depends on how far away the force is from the point of rotation.

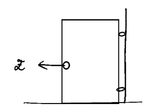
So if we make a door, where should we put the doorknob?

Do we put it by the hinges?

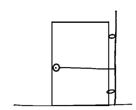


Even if you apply a lot of force you won't get anywhere. force was everything in linear motion, force is not everything in rotational motion. If you want to get your money's worth, you've got to take the doorknob as far as you can and put it near the end.

Then you come along and you say, "Okay, I've put the doorknob at the right place, open the door," then you're applying enormous force this way.

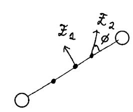


And again frustrated, again you're getting no results, you say, "I'm going as far as I can from the hinge," and it dawns on you that if you really want to get something going, you should really take the line of separation between where it's rotating and where you're applying the force, and apply a force perpendicular to it.



So in reality, we would have to pull the doorknob straight towards us.

Any part parallel to it is not doing anything, unless you kind of rip it off the hinges. You don't want to do that.



So here, F_2 has a useless part, and a useful part.

The useless part is trying to pull the rod off the hinges, it is going to be balanced by a force from the hinges, because the rod is a rigid rod, it's not going to move. But the one perpendicular to the rod can, in fact, turn the rod without affecting the rigidity. So, we need an extra factor here…

$$\tau = \sum F_i r_i \sin\theta_i$$

θ is the angle between the line joining the point of application of the force, and the direction of the force.

$\theta = Ø$ in this case.

$\tau = \sum F_i r_i \sin\theta_i$ is now a complete definition.

Therefore our analog will be…

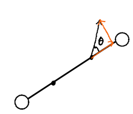
$$K = \frac{1}{2}mv^2 \quad | \quad K = \frac{1}{2}I\omega^2$$

$$m \quad | \quad I$$

$$p = mv \quad | \quad L = I\omega$$

$$F = \frac{dp}{dt} = ma \quad | \quad \tau = \frac{dL}{dt} = I\alpha \quad \text{w/} \quad \tau = \sum F_i r_i \sin\theta_i$$

F sin θ is just a component of the force perpendicular to the separation r.



There is no parallel displacement, only perpendicular.

Anyway, I hope we understand intuitively why this is the guy that's responsible for rotating objects. You want to rotate something around some axis, you've got a certain amount of force, you're best off applying the force as far as you can from the point of rotation.

$$K = \frac{1}{2}mv^2 \quad | \quad K = \frac{1}{2}I\omega^2$$

$$m \quad | \quad I$$

$$p = mv \quad | \quad L = I\omega$$

$$F = \frac{dp}{dt} = ma \quad | \quad \tau = \frac{dL}{dt} = I\alpha$$

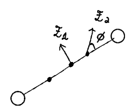
$$\Delta W = F\,dx \quad | \quad \Delta W = \tau\,\Delta\theta$$

kinetic energy of an object.

Now, I think it's very clear that if I had several forces acting on the body, then the torque will be...

$$\tau = \sum F_i r_i$$

There's just one final caveat to this thing, one final qualification, which is that, in practice, the forces that are used to rotate a body may not always be perpendicular. You can apply a force in some other direction, like that.



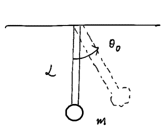
$$p = mv \qquad L = I\omega$$

$$F = \frac{dp}{dt} = ma \qquad \tau = \frac{dL}{dt} = I\alpha$$

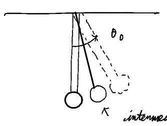
$$\Delta W = F\,dx \qquad \Delta W = \tau\,\Delta\theta$$

When a torque rotates a body by an angle Δθ, then the work done is τ times Δθ.

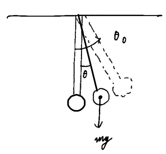
massless rod hanging from ceiling w/ length L.

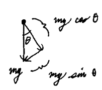


I want to know how much work I do; that's what we're going to calculate.

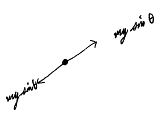


intermediate position, what is the actual force?





To hold the pendulum from falling when the angle is θ, I have to apply an opposite tangential force in of size mg sin θ.

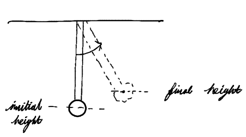


Therefore $\tau = (mg\sin\theta)\,L$

$$W = \int_0^{\theta_0} mgL\sin\theta\, d\theta = -mgL\cos\theta \Big|_{\theta=0}^{\theta=\theta_0} = mgL(1-\cos\theta_0)$$

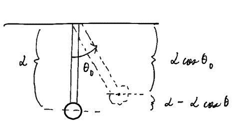
So, that's the work done to take a pendulum which is horizontal, which is vertical and turn it to an angle θ_0.

I'm going to do a cross-check on this to tell you that it works out from another point of view.



So, you can ask, "What happened to the work I did?" Going back to the Work Energy Theorem, let's observe the initial mass. Forces were acting on it. One was the rod, but the force of the rod was always perpendicular to the motion of the movement, so it couldn't do any work. The only work was done by me. What happened to the work I did? The work I did has to be changing the total energy of the body, but the bob was not moving before, and it's not moving after, because I give just the right force to cancel gravity; I didn't give it any speed. So, it must be that the potential energy of this bob when it's here should explain the work I did.

Lets $mgL(1-\cos\theta_0) = mgL - (mgL\cos\theta_0) = mg(L - L\cos\theta_0)$



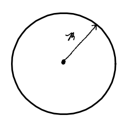
so $mg(L - L\cos\theta) = mgh$

That's where your work went in.

So, what's the point of this exercise? This exercise is to get you used to the notion of calculating work done, not as force times distance, but as torque times angle. And here's the problem where the torque

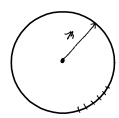
Okay, now there is one technical obstacle you've got to overcome if you want to do rigid body dynamics, and that is to know how to compute the moment of inertia for all kinds of objects. If I give you 37 masses, each with a distance r from the point of rotation, it's a trivial thing. Do mr^(2) for each one and add it. But here is the kind of objects you may be given, not a discreet set of masses but a continuous blob. Just like in the case of center of mass, when you had a bunch ofmasses, you took m_1x_1 + m_2x_2 + m_3x_3 and so on. But if you had a rod which is continuous, you have to do an integral. So you will have to do an integral here also.

So, let's take certain rigid bodies and try to find the moment of inertia, rigid bodies which are continuous.



Ring, not disk, of mass M with a radius of R.

$I = ?$



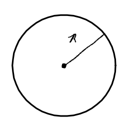
Partition the ring into partitions so small that they can be thought of as a point.

$$I = \left(\sum_{i=1}^{\infty} \delta m_i\right) r^2, \quad r^2 \text{ is constant in summation.}$$

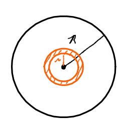
$$\left(\sum_{i=1}^{\infty} \delta m_i\right) r^2 = M r^2$$

And so we have the moment of inertia for a ring.

But let us now find the moment of inertia for a disk of radius r.



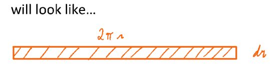
So, here is where you have to organize your thinking. I plan to do the moment of inertia through the center. First of all, if I didn't tell you the moment of inertia through where you cannot even start; I tell you I want it through the center. To want it through the center we have the ability then to think of this as made up of a whole bunch of concentric rings



Each ring will be at a radius of r and a thickness of dr.

If you can find the moment of inertia of this tiny shaded region, I sum over all the little washers, the annuli that I have, to fill up the whole disk. So, what's the moment of inertia of this annulus? Well, I already told you. (Mr^2)

So, the question is, "What is the mass of the shaded region?" If we take it out the ring and make it a rectangle, it will look like...

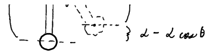


mass per unit area

$$\frac{M}{\pi r^2}\, 2\pi r\, r^2\, dr$$

Thus we will have $\displaystyle\int_0^R \frac{M}{\pi r^2}\, 2\pi r\, r^2\, dr = \frac{MR^2}{2}$

The correct answer is if it is MR^(2) it means the entire mass is at a distance R^(2). But we know some of the mass is a lot closer to the center. In fact, some of it is right at the center, so you cannot get the same R^(2) as a contribution from all the pieces. Some will be at 0 distance; some will be at the full distance. So, whenever you do the moment of inertia for a disk, it's got to look like MR^(2) times a number which is definitely less than 1. It turns out it's half; if we got a third or a fourth you wouldn't know it's obviously wrong. But if I got 1 in front, or worse, if I got 2 times MR^(2), then you know I've made a mistake.

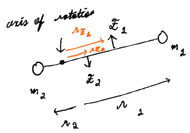


That's where your work went in.

So, what's the point of this exercise? This exercise is to get you used to the notion of calculating work done, not as force times distance, but as torque times angle. And here's the problem where the torque itself is changing with the angle. The torque is itself a function of θ, so you have to do an integral and that's how you get the work.

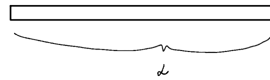
Important note: torque being done counter clockwise is positive, clockwise is negative

So, it's possible that in a given body, there could be torques trying to move it in different directions. So, for example, in the simplest rigid body made up of, say, two masses...



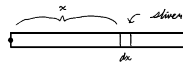
$$\frac{\alpha = \tau_1(r_{\tau_1}) - \tau_2(r_{\tau_2})}{I = m_1 r_1^2 + m_2 r_2^2}$$

the full distance. So, whenever you do the moment of inertia for a disk, it's got to look like MR^(2) times a number which is definitely less than 1. It turns out it's half; if we got a third or a fourth you wouldn't know it's obviously wrong. But if I got 1 in front, or worse, if I got 2 times MR^(2), then you know I've made a mistake.



Rod with a length of L and a mass of M.

To find the moment of inertia the first thing we have to do is pick the point of rotation (axis of rotation). Let's put this at the leftmost point of the rod.

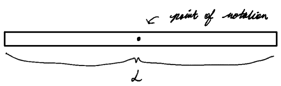


That sliver is small enough for me to consider there's a point mass. Therefore, the moment of inertia of that tiny portion will look like...

mass per unit length

$x^2 \cdot \frac{M}{L} \cdot dx$, so we have $\int_0^L x^2 \frac{M}{L}\, dx = \frac{ML^2}{3}$ (1 L gets canceled out)

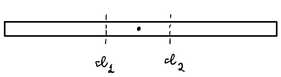
The final thing that I will now calculate is the moment of inertia around the center. Let's see what's going to happen.



We will use the same integral but some values change...

$$\frac{M}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \cdot dx = \frac{2M}{L}\int_0^{\frac{L}{2}} x^2\, dx = \frac{2M}{L} \cdot \frac{1}{3} \cdot \frac{L^3}{8} = \frac{ML^2}{12}$$

So, the moment of inertia is not a fixed number. It got a lot smaller when I took it around the center of mass. When I took it around the left edge, I got ML^(2)/3; I went to the midpoint, I got ML^(2)/12. I think it's pretty clear that I'm going from here I reduce the answer. So, when I keep going, it will get even smaller. That's where your common sense will tell you that

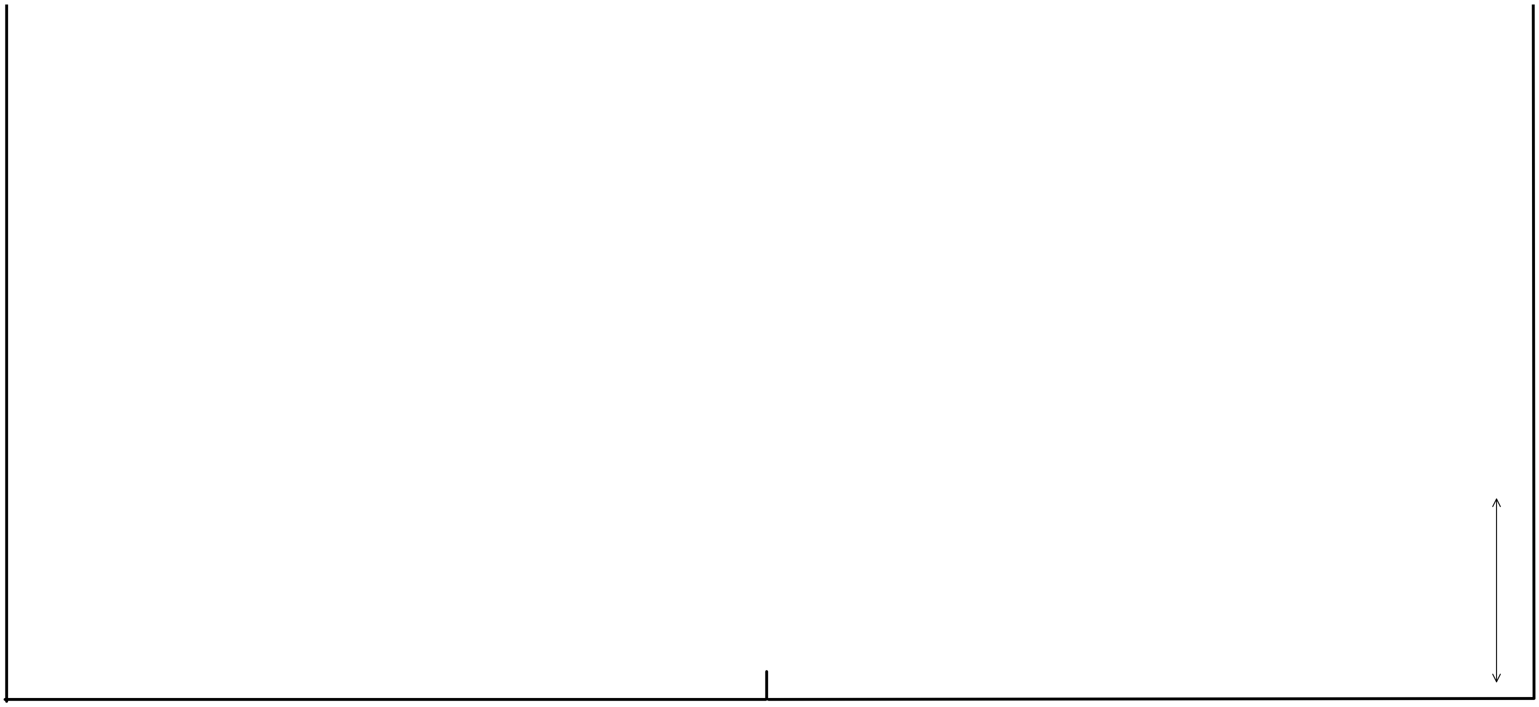


$$I_1 = I_2$$

Now the thing to check is $I_{end} = I_{cm} + M\left(\frac{L}{2}\right)^2$

Therefore $\frac{ML^2}{12} + M\left(\frac{L}{2}\right)^2 = \frac{ML^3}{3}$

And this happens to be a very general theorem I'll talk about next time.



Rotations, Part 2: Parallel Axis Theorem

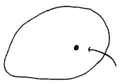
Saturday, October 14, 2023 10:54 AM

Lets us analyze a result of using vectors in order to prove the parallel axis theorem.

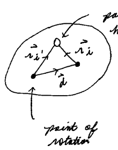
Given $\vec{A}$, $A^2 = \vec{A}\cdot\vec{A}$

$(\vec{A}+\vec{B})\cdot(\vec{A}+\vec{B}) = (\vec{A})^2 + (\vec{B})^2 + 2\vec{A}\cdot\vec{B}$, we will use this result

Let's take some arbitrary object...



Mentally, divide this body into a hundred thousand million tiny little squares, which when tiled together form this object because it's easier for me to work with a sum than with an integral.



r_i' → distance from point of rotation (arbitrary)

r_i → distance from center of mass (given)

So what is the moment of inertia of i?

$$I = \sum_i m_i (\vec{r}_i')^2, \quad \text{but } (\vec{r}_i')^2 = (\vec{r}_i + \vec{d})\cdot(\vec{r}_i + \vec{d})$$

$$\text{So } I = \sum_i m_i (\vec{r}_i + \vec{d})\cdot(\vec{r}_i + \vec{d}) = \underbrace{\sum_i m_i \vec{r}_i^{\,2}}_{\text{inertia for center of mass } (I_{cm})} + \sum_i \overset{\text{constant}}{d^2} \underbrace{m_i}_{M} + \underbrace{\dots}_{\text{more stuff}}$$

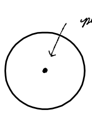
$$I = I_{cm} + M\vec{d}^{\,2} + 2\vec{d}\cdot\underbrace{\sum_i m_i \vec{r}_i}_{\text{every mass multiplied by its position}}$$

$$\frac{\sum_i m_i \vec{r}_i}{M}\cdot M$$

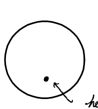
center of mass in a coordinate system w/ center of mass at the origin. $\Rightarrow 0$ because $\vec{r}_i = \vec{0}$

$$\text{Since } \cancel{\frac{\sum_i m_i \vec{r}_i}{M}}_{0}\cdot M, \quad I = I_{cm} + M\vec{d}^{\,2}$$

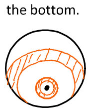
One place where this result is useful is in the moment of inertia for a disk.



If I run the moment of inertia through the center, I truly have to calculate it. You divide it into concentric rings. The answer for each ring is simple, you do an integral, you get MR^(2)/2. But suppose I say I want the moment of inertia through....



If we try to divide the disk into concentric rings, the rings will eventually be non-rings. They will get cut off at the bottom.

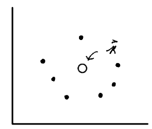


So we can use $I = \frac{MR^2}{2} + M\vec{d}^{\,2}$, where d is the distance your new point of rotation is at from the center.

So now, we have proved the parallel axis theorem is…

$I = I_{cm} + M\vec{d}^{\,2}$, again, d being the distance between the new point of rotation and the center of mass.

Suppose you have a bunch of whole random masses in a two-dimensional plane.



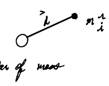
M located at $\vec{R}$

As the bodies move, this is not a fixed situation. The whole--It could be a bunch of planets traveling as seen by another galaxy. The whole Solar System is drifting. As the whole thing is drifting, the center of mass, which is the weighted average of the location, will also drift with that. Now, I want to write a formula for the kinetic energy of this complex

$$K = \frac{1}{2}\sum_i m_i v_i^2$$

But now, I want to look at the velocity of this object, of every object v_i;

So if,



& $\vec{v}$, $\hat{\vec{v}}_i$, if $\vec{v}_i = \vec{v} + \hat{\vec{v}}_i$, $\vec{v}_i = \vec{v} + \hat{\vec{v}}_i$

Therefore, $\vec{v}_i = \underset{\substack{\uparrow \\ \text{velocity of center of mass}}}{\vec{v}} + \underset{\substack{\uparrow \\ \text{velocity relative to center of mass}}}{\hat{\vec{v}}_i}$,

Every little speck is moving. But let's ride in a frame moving with the center of mass. Wherever it goes, we go with that. We latch onto it. As seen by us, the body will have a velocity v_i relative to us, but from the point of view of the ground, we should add to that velocity, the velocity of the center of mass to get the velocity with respect to the ground, or some fixed coordinate system. So let's do that same trick with

$$K = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i\left(\vec{v}^{\,2} + \hat{v}_i^{\,2} + (2\vec{v}\,\hat{\vec{v}}_i)\right) \quad \text{use } (\vec{A}\cdot\vec{B})^2 \text{ for } v \text{ \& } \hat{v}_i$$

$$K = \frac{1}{2}Mv^2 + \underbrace{\frac{1}{2}\sum_i m_i |\hat{v}_i|^2}_{\text{kinetic energy relative to C.O.M}} + \vec{v}\underbrace{\sum_i m_i \hat{\vec{v}}_i}_{\sum_i \hat{P}_i = \vec{P}}$$

↳ sum of momenta for all particles

But it is the momentum of the center of mass as measured by a person co-moving with the center of mass. So, what will that person get? $\Rightarrow 0$

If there're two masses moving at different velocities, there're only two of them. If you go to the center of mass, and ride with the center of mass, whatever momentum one guy has, the other will have the opposite momentum. So, the momentum of a system will appear to be zero as seen by the center of mass.

Therefore $K = \frac{1}{2}mv^2 + K_{relative}$

Here is the result. The kinetic energy of a collection of masses is the kinetic energy that you can associate with the center of mass motion, plus kinetic energy relative to center of mass. Let's just call it K relative, but it's understood as relative to center of mass.

Now, apply it to the case where these particles form a rigid body. Think about it now. It's not a swarm of bees, they're all connected by rigid rods, so they form a rigid complex. In that case, what motion is allowed, if you ride with the center of mass?

If you are riding with the rigid body and you are sitting wherever the center of mass is sitting, then the only motion allowed for you is the rotation around the center of mass. Because if it is not around the center of mass, then the center of mass will be moving relative to the axis of rotation. But that's not allowed, because according to you the center of mass is not moving; therefore, any rotation that takes place must be around an axis passing through the center of mass. In that case, we know what the kinetic energy is.

$$K = \frac{1}{2}Mv^2 + \frac{1}{2}I_{cm}\,\omega^2$$

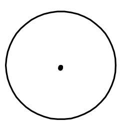
Meaning $K = K_{Translational} + K_{Rotational}$

$K = K_{Translational} + K_{Rotational}$

This is the statement that the kinetic energy is the sum of two terms, the translational part, and the rotational part, provided you separate it as the kinetic energy of translation of the center of mass, and rotation relative to center of mass. So far what we did is in the early part of the course, we took bodies which just translate. They are point particles; they have a mass; they have a velocity. We got used to ½ MV^(2) as the answer. Last lecture, we took a rigid body which has a size and a shape, we nailed one part of it so it couldn't move, so all it could do was rotate around a point, and we learned that energy of rotation written somewhere there is ½ Iω^(2). But in reality, a body can translate and rotate. Throw anything you want in the air, it will spin or move and also rotate. Then the kinetic energy of that complicated motion remarkably is the sum of two terms. One, you ignore all the wobbling. Just zero in on the center of mass. Treat it like a point, and associate an energy with its motion. Then, think of the rotation around the center of mass, which is all that is allowed to that [co moving] observer, and add that energy.

For the more mathematically-minded people, what is interesting is if you take A + B whole squared, there's an A^(2) term, and a B^(2) term, and they told you, long back, not to forget the 2AB term, the cross term. The fact is, the cross term always vanishes both in that derivation and in this derivation. It has to do with the fact that the point that I'm talking about is not any old point, but the center of mass.

Let's apply this to a simple problem...

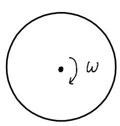


Disk / cylinder

A disk is like a coin. The cylinder is a stack of coins. But the moment of inertia for a disk or a cylinder is the same. Do you guys understand that? If I took one coin, it has got MR^(2)/2. Then, we put a second coin on top of it and rotate them through the same axis, then it's MR^(2)/2 for this guy plus MR^(2)/2 for that guy. It then is equal to the total MR^(2). For a cylinder or a disk, the moment of inertia is just MR^(2)/2 through the center of mass.

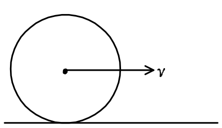
$I_{cm} = \frac{MR^2}{2}$ ← for cylinder / disk

Now, let's take this disk, which is sitting now and give it some energy of motion. Think of it as a tire in a car, one of the four tires. They're spinning. So, there are different things you could imagine. Lift the car off the ground in a service station, and you let the tires spin; that is rotational energy.



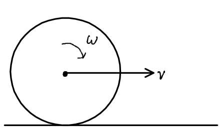
only ω

Let's do something else. Let the tire hit the ground. And the car is going at say 50-miles-an hour; then you slam the brakes, so you prevent the tire from turning. If you prevent the tire from turning, you lock the brakes, the tire can still move with a certain velocity v, which is just the translational motion of the center of mass. So, this is a car which is skidding



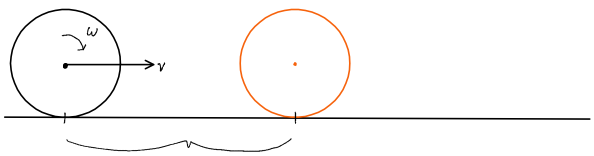
only v.

In a real car both things are happening. In the real car, the tire is rolling along the ground and also rotating around the center and also translating. It's got both motions. Now, the two motions, namely the linear velocity of the center and the angular velocity around the center in general don't have to be connected in any way.



ω & v.

Suppose you start your car on a slippery road. The tire's spinning, the car's not moving. So, that's ω equal to not zero, but v = 0. Or when the car is moving along and you slam the brake, then you kill the ω but not the v. I'm just trying to tell you, in general, there are two velocities. The linear motion of the center and the angular motion around the center are independent. But one is a skidding car, and one is a car that's spinning its wheels. But I want to talk about the car the way we like it. In a car that's moving the way we like it, there's a correlation between the angular velocity and the linear velocity. And that is called "rolling without slipping." That's the term people use. And when it's rolling without slipping, I think you have an intuitive feeling. If this car is to roll without slipping on the ground, can you see that by the time it finishes one full revolution, when it has moved a distance, the center will have moved the distance equal to circumstance of that tire. So, every part of the tire touches the ground; then, that comes down and that comes down, and when it goes all the way to this end, it has also finished one full revolution.



So, what does it mean to roll without slipping? What's the velocity?

$v = \frac{2\pi r}{T}$, T = time to make one full revolution

$v = 2\pi f r$, $f = \frac{1}{T}$

$v = \omega r$, $\omega = 2\pi f$

Here is the connection between linear velocity and angular velocity, if you're rolling without slipping.

In general, V and ω are independent, but rolling without slipping refers to a particular situation. Now, in that case, let's find the kinetic energy of an automobile tire that's rolling without slipping.

$K = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$

$K = \frac{1}{2} m v^2 + \frac{1}{2} \cdot \frac{M r^2}{2} \cdot \omega^2$, we now know $v = \omega r$

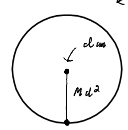
so $K = \frac{1}{2} m v^2 + \frac{1}{4} M v^2$, $K = \frac{1}{2}\left(\frac{3M}{2}\right) v^2$ = total kinetic energy

So, the kinetic energy of a rotating tire is not just ½ MV^(2). If it's rolling without slipping, it's incumbent on the tire to have some rotational energy that's correlated with its translational energy. So, the total energy will be this. But I'm going to write this in a different way.

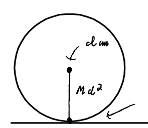
$K = \frac{1}{2}\left(\frac{3}{2} M\right) v^2 = \frac{1}{2}\left(M + \frac{M}{2}\right) r^2 \omega^2 = \frac{1}{2}\left(\underset{\uparrow}{Mr^2} + \underset{\uparrow}{\frac{Mr^2}{2}}\right)\omega^2$

looks like parallel axis theorem

looks like moment of inertia for disk



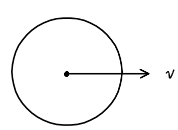
Therefore $K = \frac{1}{2}\left(Mr^2 + \frac{Mr^2}{2}\right)\omega^2 = \frac{1}{2}\, "I_{road}"\, \omega^2$



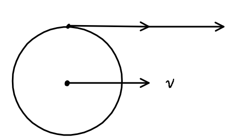
moment of inertia w/ respect to contact on the road

Now, you have to think about this result. It's a very interesting result. It is telling you the entire energy of this tire is as though it is rotating around the point of contact with the road with angular velocity ω. Forget translation, it's a just a pure rotation around this point. Now, when it's rotating around the point, it means that point cannot be moving. And I will now convince you that that is actually true.

If you look at your car tire, how fast are the different parts of the tire moving?

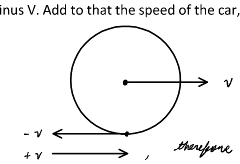


On top of this, the tire is spinning so...



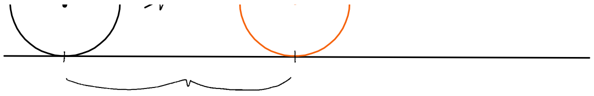
extra tangential velocity $v + \omega r = 2v$

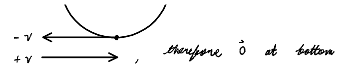
If you come to the bottom part of it, as seen by the car, it's moving backwards at this velocity minus V. Add to that the speed of the car, you get 0.



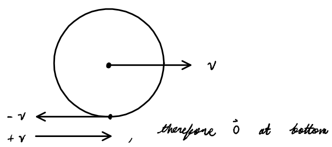
therefore 0 at bottom

In other words, the top of the tire is moving along with the car, with the same velocity as the center. The



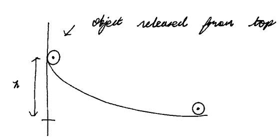


In other words, the top of the tire is moving along with the car, with the same velocity as the center. The bottom of the tire is moving opposite to the car with the same velocity as the center, so its real velocity is seen by the ground as zero. So, this is a very surprising result.



A car going past you at 200 mph. There is a part of the car that has zero velocity relative to the ground. It's not obvious that a zooming car has one part of it that's not moving at all. It's not the same part, otherwise the car is really not moving. But at every instant, the part of the tire that touches the ground has zero velocity. I hope you understand why it's a cancellation of the central motion, of the overall motion with the rotational motion. You can see the rotational motion on the top adds to the speed of the car. Rotational motion on the bottom subtracts from the speed of the car. And if you write V = ωR, the two precisely cancel here and double there. So, if the bottom point is at rest, then the only thing the car can do is to rotate around that point. And that's the reason why the energy comes out to be purely rotational energy around the center. So, this is a simple problem that illustrates all the things you have learned today: how to use the Parallel Axis Theorem, how to do the energy calculation, how to interpret the energy calculation. This is useful for the following example.

Suppose we have



For a point mass $mgh = \frac{1}{2} mV^2$, but this is a rolling cylinder so

$mgh \neq \frac{1}{2} mV^2$, $mgh = \frac{1}{2}\left(\frac{3}{2} M\right) v^2$ (see previous work)

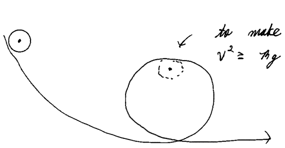
So we will get $v^2 = \frac{4}{3} gh$ which is much slower than a particle

So, what did I do? I used the Law of Conservation of Energy. That's all I did. But in finding the kinetic energy at the bottom, I was careful to not just take ½ MV^(2). This is the energy of a point particle, but take the kinetic, the energy of translation and rotation, which I've done for you previously. That's the kinetic energy. The translation, that's the kinetic energy of rotation. By the way, don't make the mistake of thinking it is 3/4V^(2) for everything. It is 3/4V^(2) for a cylinder. If it's a sphere or something, you got to add a different thing. But every time you will have to find a moment of inertia, you do rotation and add to the energy of the rotation, the energy of translation. You follow that? For each body, you'll have a different moment of inertia.

For a sphere $I = \frac{2}{5} Mr^2$ (this was just given).

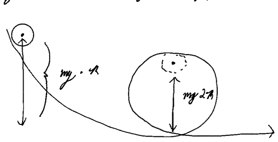
So, you will find instead of just the mass, you'll get some multiple of the mass entering here in the kinetic energy. And to the extent that's bigger than the mass, the velocity will be that much slower.

So, there're all kinds of variants you can have. Here's the variation of this theme.



to make it here as a point $v^2 \geq rg$ so it doesnt fall

For a disk, $v^2 \geq rg$ aswell but we need to use the law of conservation of energy.

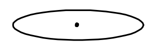


$k_E \neq \frac{1}{2} m v^2$ is the main point

To summarize in one sentence what I have told you all of this time is that, when rigid bodies move, they have a translational and rotational energy. In general, they're independent numbers, but when you have rolling without slipping, the angular velocity and the linear velocity are connected in V = ωR. So, it's not surprising that the total energy has got a contribution from both, which you can write either in terms of the angular velocity or the linear velocity. In other words, if I know how fast the wheel is spinning, I can tell you how fast

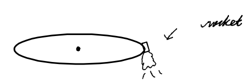
And let me remind you where we stand. When I did F = ma after telling you a little bit about how mass is measured, and everybody knew how acceleration is measured, what did we do? We first took a bunch of problems where some forces are acting on a body. Maybe something's pulling, something's pushing, then we learned how to find the acceleration. Then, I took other cases where there are many bodies exerting a mutual force on each other. Then, if there are no external forces, we saw the total momentum of the system is concerned. We're going to do analogous things with our newfound equation of τ = Iα. (I believe professor Shankar is talking about his lectures for newton's laws of motion and then the dynamics of multi body systems).

Lets use $\tau = I\alpha$



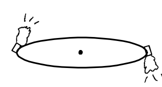
disk like a merry-go-round

I want to apply a torque. So, one way to apply the torque is the way kids do. You know, you grab one end and you run with it on the side; as you run with it, it picks up speed, but what you're trying to do is apply a tangential force as long as you can. And the force times the distance will turn into a torque. And that'll produce angular acceleration. Another way to do that if you're really high-tech is to buy yourself a rocket

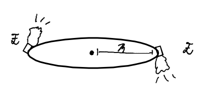


There will be translational motion left to itself, because that is the force. And F = ma has not gone away just because we're in chapter 11; that's still true. So, what'll happen is, this pivot point is trying to hold the merry-go-round in place. They're putting a lot of force on it, because it'll try to move. It has to counter the force at every instant. As long as the merry-go-round doesn't fly off, what'll happen is, that the rocket's emitting gas one way, and applying a force that way, the pivot will apply an equal and opposite force. And as the rocket goes there, it'll apply an opposite force in the other direction.

So to counteract the force on the pivot we will add another rocket.



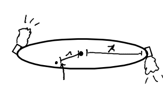
So, when you buy a rocket, they will tell you, "This rocket has a thrust of so many Newtons." That's the force it will exert by emitting the gas.



Therefore $\tau = 2Fr$, & $\alpha = \frac{\tau}{I_{disk}} = \frac{2Fr}{Mr^2/2}$

(r = radius of disk)

What if I put some passengers on this? Suppose I put one lone passenger here on some mass; let me call it little m. Then what happens?



r = distance between C.O.M & m

$\alpha = \frac{2Fr}{Mr^2/2 + mr^2}$

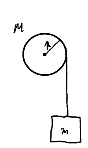
Once the rockets start burning, you got this α. I can ask you things like, "How many degrees has it rotated after 19 seconds?" And you should know what to do.

Use $\theta = \theta_0 + \omega t + \frac{1}{2}\alpha t^2$

"How many revolutions has it made?"

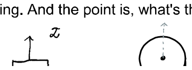
Divide by 2π, one full revolution.

Let us analyze a slightly more complicated problem.



M = mass of pulley
m = mass of block

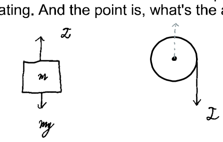
There's a combination of rotational problems and translation. The mass is translating, the pulley is rotating. And the point is, what's the acceleration of the block?



Grey force isnt relevant ..

To summarize in one sentence what I have told you all of this time is that, when rigid bodies move, they have a translational and rotational energy. In general, they're independent numbers, but when you have rolling without slipping, the angular velocity and the linear velocity are connected in V = ωR. So, it's not surprising that the total energy has got a contribution from both, which you can write either in terms of the angular velocity or the linear velocity. In other words, if I know how fast the wheel is spinning, I can tell you how fast the car is moving. If I tell you how fast the car is moving, I know the rate at which the wheels are spinning, provided it's not skidding. If it's skidding, these bets are off.

There's a combination of rotational problems and translation. The mass is translating, the pulley is rotating. And the point is, what's the acceleration of the block?

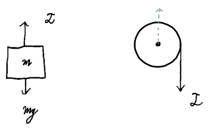


Grey force isn't relevant...

If you take a torque through a certain point, the forces acting through that point, they don't get to contribute because in the formula for torque, Fr sin θ, r vanishes, and the forces don't do anything. That's why I don't pay any attention to what the pivot is doing.

$\tau = I r$, so $I r = \frac{Mr^2}{2}\alpha$

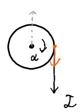
So we have…



$I r = \frac{Mr}{2}\alpha$ ← acceleration for wheel

$I - mg = ma$ ← acceleration for block

Notice that



$a_I = r\alpha$

The accelerations after all are not independent of each other.

So, the mass can go one inch only if the pulley turns by a corresponding amount to release one more inch.

We will then say $a = r\alpha$

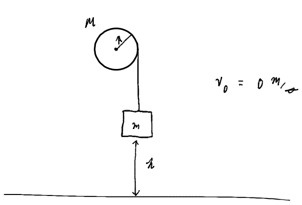
So $I r = \frac{Mr}{2} a$ & $mg - I = ma$

$$\begin{array}{r} I r = \frac{Mr}{2} a \\ + \; mg - I = ma \\ \hline mg = a\left(m + \frac{M}{2}\right) \end{array}$$

, therefore $a = \frac{gm}{m + M/2}$

With this result we can test how different values of mass influence the acceleration, so we can verify whether or not the acceleration is reasonable (try plugging in 0 for M).

Let's do another analyzation.



$v_0 = 0\ m/s$

Just before the block hits the ground, what is its velocity?

Given a = constant

$v^2 = 2ah$

$v^2 = 2\left(\frac{gm}{m + M/2}\right)h$

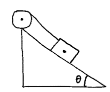
$v^2 = \frac{2}{m + M/2}\, mgh$ Let's see if energy is conserved.

Does $mgh = k$?

$\frac{m}{2}v^2 + \frac{1}{2}\left(\frac{M}{2}\right)v^2 = mgh$

$\underbrace{\frac{1}{2}mv^2}\; + \;\underbrace{\frac{1}{2}(Mr^2)\omega^2} = mgh$

One variation of this problem is to have a block connected to a pulley on a ramp.



There is no friction, so what he's saying is you can use the Law of Conservation of Energy. For example, if there's dropped a certain height h, from that height to the ground, your answer should satisfy the condition that mgh was equal to the kinetic energy of the mass and the pulley at the end of the day, and your answer will satisfy it. Another thing you can do is if this angle θ goes to 90 degrees, look what happens. If θ goes to 90, forget about the block, it is just falling freely, and that's the previous problem. You can check if the answer you get agrees with the previous problem. So, I'm just teaching you that it's good to have constant checks on the problem. One way to check the calculation is to go back and do it again. Most of the time, if you screw it up the first time, you'll probably screw up the second time because one of the basic rules of nature is that if you repeat a certain experiment under the same conditions, you'll get the same answer. That's why when I proofread anything, I write, I don't find any mistakes. But my readers are gleefully pointing out this and that typo, because we cannot detect those errors. That's true even for a scientific calculation. Because for some reason you forgot something the first time, you'll keep making the same mistake. That is known in the business as "chasing after π."

So, how do you beat those things is by all these extreme limits. That's another thing I want to tell you with this. All of us are partial to working these things with symbols and not with numbers. I think it's a big issue in Introductory Physics. There's a resistance to working with symbols. People want to know what's the mass M. What's the number R? Let me put that in. If you put numbers in too early, you cannot test your answer for anything. In the end you just got a number. You cannot take the limit of one particular number. But if you got a function that depends on all the parameters, you can vary them and in some extreme limit, you would know what the answer is.

Now, I'm going to take a subset of problems, in nature, where the torque, the total torque on a collection of bodies is zero. So, I'm going to take a system. There are many torques, but the external torque, due to forces outside the things forming the system is zero.

$\tau_{ext} = 0$

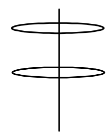
Then, you remember when the external forces vanished, then the momentum of the system was the same. $(p_1 + p_2 = p_1' + p_2')$

Therefore $L_1 + L_2 = L_1' + L_2'$

In other words, angular momentum is conserved so long as the external torques sum up to 0.

Let's take an example.

Two disks on a spindle, bottom disk spinning



I_2 (at rest)

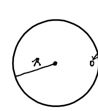
$I_1\ \omega_1$

If the top disk falls onto the bottom disk, what would happen?

The two masses would stick together and spin at the same angular velocity.

Therefore $I_1\omega_1 + \overset{\text{top disk}}{0} = (I_1 + I_2)\,\omega$

Let's take a merry-go-around.

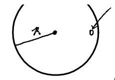


piece of chewing gum falls on merry-go-round

$I_1\ \omega_1$

$$\underbrace{\frac{1}{2} m v^2}_{\text{kinetic energy of block}} + \underbrace{\frac{1}{2}(MR^2)\omega^2}_{\text{kinetic energy of pulley}} = mgh$$

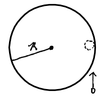
So, you got to understand the linear velocity of the mass and the angular velocity of the pulley are connected. And the connection factor's always R. R is the number that connects linear to angular quantities. So indeed, this answer satisfies the Law of Conservation of Energy. That's a very good test.



piece of chewing gum falls on merry-go-round

$I_1 \ \omega_1$

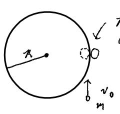
Obviously the merry-go-round will slow down.



$I_1 \ \omega_1$ But what if the chewing gum is shot onto the merry go round like this?

So, let me repeat what he said. This moving chewing gum has to be brought to rest [in the frame of the disk]. It'll be brought to rest by the disk that'll push it backwards, and the chewing gum will push it forward, so it'll apply a torque. That means it'll change the angular momentum.

Instead of going through that route of finding the torque, if I only knew the angle of momentum of this putty, then that will add it to the angle of momentum of the disk. Then, I say, that's the total angular momentum that should not change. So, the question is, "What's the angular momentum of a mass that's not really moving in a circle, but that's moving in a straight line?" And the answer is…



pretend its a pad here moving at a certain velocity

In other words, if you are the moving putty, and you look to your left, the disk is spinning, and you are going at a certain speed, at that instant you are not distinguishable from another body that's actually orbiting the center. But what will be the relation of your velocity to your angular velocity?

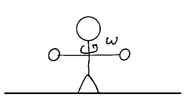
$\omega = \frac{v_0}{R}$, therefore $L = m R^2 \omega$

$L = m R^2 \frac{v_0}{R} = m v_0 R = pR$

So, when a body comes with a tangential momentum p, and it plants itself on the disk, it actually is an angular momentum p times R. This is not very surprising. Just like R times F gives the force, R times p gives the angular momentum. So, the surprising thing for you guys is, things don't have to be going around in a circle to have an angular momentum. They can be going in a straight line, but at every instant, I can find the angular momentum by asking, "What's the moment of inertia, namely MR^(2), and what's the angular velocity?" The angular velocity is easiest to find when this guy is right next to the disk. You know from what we did earlier, angular velocity is linear velocity divided by R.

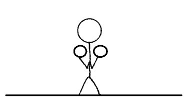
Say we have a single particle in one dimension.

$p = mv \quad \frac{dp}{dt} = F = 0$ means $v =$ constant



Weightless ice dancer holding two dumbbells, spinning on ice.

They drop their arms and bring the dumbbells very close.



You really reduce the moment of inertia. So, what'll happen?

I is changing. Therefore we now have a $I_2 \ \omega_2$

If you go back and find thekinetic energy, you'll find the kinetic energy's not the same…

$K_0 = \frac{1}{2} I_1 \omega_1^2$

$K = \frac{1}{2} I_2 \left(\frac{I_1}{\ }\right) \omega_1^2$

If you go back and find thekinetic energy, you'll find the kinetic energy's not the same…

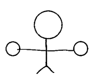
$$K_0 = \frac{1}{2} I_1 \omega_1^2$$

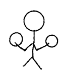
$$K = \frac{1}{2} I_2 \left(\frac{I_1}{I_2}\right)\omega_1^2$$

And you can easily compare the two numbers in the geometry I've given you to show the final kinetic energy's actually higher.

When you start spinning faster, in fact, the bigger the kinetic energy. Because if your ω doubles because your moment of inertia went down by a factor of half, that keeps Iω constant. But you're supposed to square the ω, so in the product Iω^(2), when you reduce one by half and double the other, you gain a factor of 2. So, take the simple case where I_2 is half of I_1 and you will see it. Okay, so you can ask, "Where does that energy come from? How does this ice skater pick up speed, pick up angular momentum or kinetic energy?" The angular momentum doesn't change, but the rotational energy changes.

When the dumbbells are spinning, they're going in a circle. And we know that they have an acceleration v^(2)/R, and we know somebody's got to provide mv^(2)/R, and the person is providing that force. But as long as she's providing the force, but not moving, you can move tangentially. But the force and velocity are perpendicular, so you're not doing any work. But the minute you pull in, you're applying a force and a displacement which are both parallel. Then F times d is no longer 0, you will find that these things are going to fly off, as he said, and you're trying to not only hold on to them, but trying to bring them in. During that time, it'll be just like lifting weights. If you want to lift weights, and you go to a planet where there is no gravity, tell somebody to spin you, then when you pull the stuff in, it'll be like lifting weights, okay? That's what the person's doing, and that's where the energy's coming from.





As you bring in the dumbbells closer the dumbbells will want to fly back out to where they are, but us being the external work prevents that.

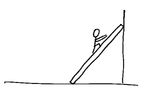
Torque

Tuesday, October 17, 2023 4:42 PM

Today, I'm going to consider an extreme case where there is no torque at all. If there's no torque, we know α is zero and the angular velocity is constant. But I'm going to take a case where even angular velocity is zero. There is no motion; there is no torque. So you can say, "You know, what's there to study?" Well, sometimes it's of great interest to us that the object has no angular velocity.

$\tau = I\alpha = 0$

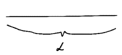
Consider the following example...



We are on a ladder. We do not want this ladder to have an angular velocity or we would probably fall and get hurt.

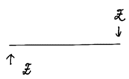
So you want to ask, "What does it take to keep the ladder from falling over?"

Let's first start with a simple problem.



Horizontal rod

"What does is take to keep this rod from moving around?" We should know that the sum of all of the forces should add up to 0. Otherwise F = ma would tell us that the rod is flying off somewhere.



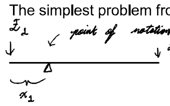
This is insufficient because the rod will spin around the center of mass. Although the summation of forces adds up to 0, the rod is spinning.

We then have to ask what does it take to keep this rod from rotating. Lets write the conditions of this planar body that moves in the plane of the paper that keep this object completely still.

$\sum F_{x_i} = 0, \ \sum F_{y_i} = 0$ most importantly, $\sum \tau_i = 0$

We will work on problems that strictly fall under these conditions. If an object meets these equations we say the object is in equilibrium.

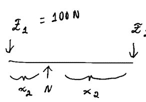
The simplest problem from this family of problems is the seesaw.



$F_2 = ?$ to balance F_1

Lets say force 1 and force 2 are kids sitting on the seesaw.

Lets rewrite our image.



These three forces should conspire to keep the seesaw in equilibrium.

Note that F_1, x_1, x_2 are all known numbers.

Horizontally, $0 = 0$.

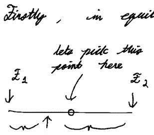
Vertically, $F_1 + F_2 = N$, which has two unknowns F_2 & N.

This is where torque is going to play a role. Question is, how do we find the torque due to any force?

Understand that the location in which torque is applied changes the strength of the torque.

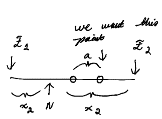
But we understand that although there are an infinite amount of points to apply torque to, we are interested in force 2 and the normal 2.

Firstly, in equilibrium, $\sum F_i = 0$.



so we have $\sum F_i x_i = 0$

If we then say

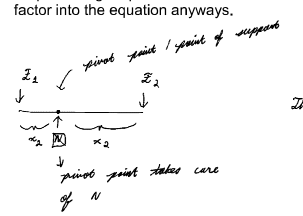


then we would have $\sum F_i (x_i + a)$

$$\sum F_i (x_i + a) = \underbrace{\sum F_i x_i}_{0} + a \underbrace{\sum F_i}_{0}$$

In other words, if you find any one point and make sure the torques around that point add up to zero, then torques at any other point will also add up to zero. This is provided that the sum of the forces also adds up to zero.

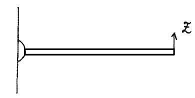
So if we want to find the value of force 2, we want to choose a point of pivot that is most optimal. The most optimal pivot point being one that prevents the normal force from being an acting force. Take the torque through a point where an unknown force is acting since the unknown force isn't going to play a factor into the equation anyways.



Therefore $F_1 x_1 = F_2 x_2$

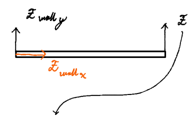
So if $x_1 = 1m$, $x_2 = 6m$, $F_1 = 10N$, we get $10 = 6F_2$ so $F_2 = \frac{10}{6}$ newtons

Once we find F_2, you can plug it into $F_1 + F_2 = N$ to find the normal force.



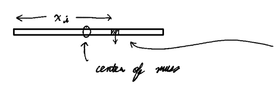
There is a rod on a wall with a length of L and a mass of m supported by a pivot on the wall. We want to be able to hold this rod up with a force of F.

free-body diagram



We pick our pivot point to be a point at which the force of the wall is disregarded.

$\tau = FL$



Torque acting on small shaded area.

$m_i x_i g$

Torque around pivot point for different small segments of rod.

Lets $\sum m_i x_i g$, and add an extra $\left(\frac{\sum m_i x_i}{M}\right) g M$ to make it easy.

$\left(\frac{\sum m_i x_i}{M}\right) g M = Mg x$ center of mass for rod

The rod doesn't have to have a uniformly distributed mass, but gravity must act with a constant value throughout the rod. For a laboratory size rod, gravity will be constant, but for a rod that as big as the size of the Earth gravity will not be constant.

Thats why we can say Mgx.

So we can say $FL = Mg\left(\frac{L}{2}\right)$ meaning $F = \frac{Mg}{2}$

It's also worth noting that there could possibly by a required horizontal force necessary to keep the rod in equilibrium, namely H prime, but we can actually disregard the horizontal force.
However we now know that...

$F = \frac{Mg}{2}$ is the force required to keep the rod in equilibrium

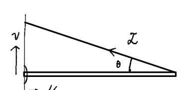
Now that we have this force, lets find the vertical force at the pivot.

$V + F = Mg$

$V + \frac{Mg}{2} = Mg$

$V = \frac{Mg}{2}$

Let's go to a problem that has a support with horizontal and vertical force.

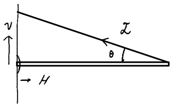


Rod of mass M, length of L

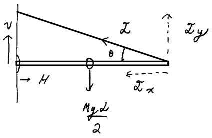
$H \rightarrow$ horizontal pivot force

$V \rightarrow$ vertical pivot force.

So we are going to now find what the tension in the rope is. Assume we are concerned about the tension in the rope as we need to make sure it doesn't snap.



Instead of using newtons second law, we will use the analog of the torque equation. For the usage of the torque equation to be efficient we will choose our pivot point to be at the pivot point so we can eliminate V and H.



$T = d \sin\theta = \frac{Mgd}{2}$

$T = \frac{Mg}{2\sin\theta}$, we dont want theta to be too small.

So if asked "what is the minimum angle so that the rope doesn't snap given that the rope can support a maximum of 1000 newtons?" We will set T equal to 1000 newtons and the mass of the rod should be given, gravity is given.

We can be asked to find more than tension however. What is occurring at the pivot point?

Horizontal direction: $H = T\cos\theta$

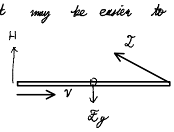
$H = \frac{Mg}{2}\cot\theta$

Vertical direction: $V + T\sin\theta = Mg$

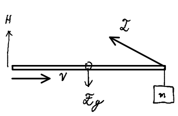
$V + \frac{Mg}{2} = Mg$

$V = \frac{Mg}{2}$

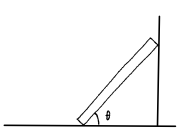
It may be easier to isolate the rod and draw the forces...



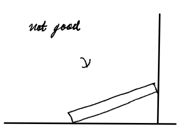
Even if we add an additional mass at the end it does not change much.



So let's now return to the ladder.



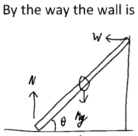
Is there a limit of θ?
lower limit or upper?



For equilibrium there will be a lower limit on the ladder.

We also dont want to go over 90°.
Our limit is from 0° to 90°.

By the way the wall is frictionless, but the floor has friction. We will look at the ladder as a rod.

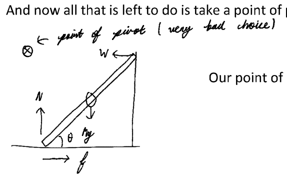


W = wall exerting a force

$\rightarrow f$, friction force inward since ladder is trying to push out

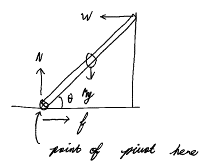
$\sum T_x = 0 \qquad W = f$

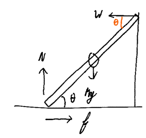
And now all that is left to do is take a point of pivot for our torque.



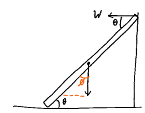
Our point of pivot can be at any point.

Pick the dang point of pivot at a key point to where we can disregard forces.





This choice gives $W \cdot d \cdot \sin\theta = Mg\frac{d}{2}(\cos\theta)$ → This isnt a mistake, lets focus on Mg



We can say that the angle in which Mg is acting is phi, but we notice that the sin of phi is equal to the cosine of theta.

$\sin\phi = \cos\theta$

$W \cdot d \cdot \sin\theta = Mg\frac{d}{2}\cos\theta$

$W = \frac{Mg}{2}\cot\theta$, so torque equation tells us what we need.

W must $= f$. But f cannot be arbitrarily big.

Remember, $f \leq \mu_s N$.

We know $N = Mg$ so $f \leq \mu_s \cdot Mg$

$\frac{Mg}{2}\cot\theta \leq \mu_s Mg$

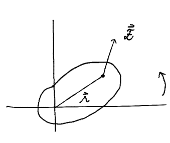
$\cot\theta \leq 2\mu_s$, but its easier to write $\tan\theta \leq \frac{1}{2\mu_s}$ ($3 < 10$, take reciprocal and we get $\frac{1}{3} > \frac{1}{10}$)

This way we can understand tan θ has to be bigger than some number. That means θ has to be bigger than some number, because tan θ increases with θ. So, you tell me what your coefficient of friction is. Maybe it's .5, then this is 1. Tan θ should be bigger than 1, θ should be bigger than π /4.

These problems are in the xy plane, the plane of the paper. This makes rigid dynamics pretty easy. The mass is concentrated on the plane, the vectors separating an applied torque from the pivot point is in the plane. But real physics isn't like this. A potato is not uniform, a couch isn't uniform. Rigid body dynamics in three dimensions is actually pretty challenging. In fact it's one of the most complicated fields in physics. This is why we are going to simplify rigid dynamics in three dynamics; we just aren't ready yet.

We will limit our context to a single point mass.

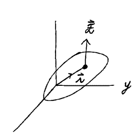
What does the analogue of τ = Iα when the mass is not moving in the xy plane but just running around all over three dimensions. How do you define torque? How do you define angular momentum in 3D? You will find this a--it's not a straightforward extension. You have to think a little harder.



Sheet of metal in x-y plane.

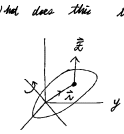
Torque rotates this way

But lets move to 3D

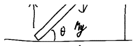


The sheet can still be flat, but torque is acting on this sheet of metal as it is floating in space.

What does this torque do?



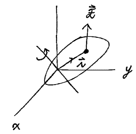
Torque will be around an axis of rotation.



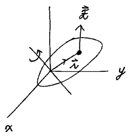
$\to f$, friction force inward since ladder is trying to push out

$\sum F_x = 0 \quad W = f$

$\sum F_y \to N = mg$



Torque will be around an axis of rotation.



We will use a trick that indicates the effect of the torque will produce a rotation around a specific axis of rotation. To do this we introduce a new vector...

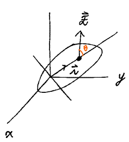
$\vec{\tau}$, now a vector, no longer a scalar

$\vec{\tau} = \vec{r} \times \vec{F}$

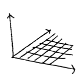
↑ cross product

Vectors need a magnitude and direction

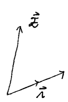
$\tau = rF\sin\theta$



Now having a vector in a plane gives a perpendicular plane.



But we can draw an up or down perpendicular plane, we must make a choice to which direction we want associated with torque. This is where we use the right hand rule.



$\vec{r} \times \vec{F} = ?$

We must pick a convention to where the perpendicular plane is either coming towards us out of the paper or away from us into the paper.

If you make a thumbs up with your right hand and align it with the perpendicular plane of these vectors, the direction your thumb points determines the direction of the cross product, the directions of your fingers must align with the rotation r needs to make to get to F.

So the right hand rule gives $\vec{r} \times \vec{F} = -\vec{F} \times \vec{r}$.

This is an accidental property of working in three dimensions.

$\vec{C} = \vec{A} \times \vec{B}$

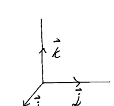
$C = AB\sin\theta$

And the direction of C is found by taking the plane in which A and B lie and drawing a perpendicular to the plane in the sense that you rotate a screwdriver from A to B. It's a beautiful property of three dimensions that one is able to take two vectors and manufacture from them a third vector.

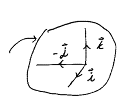
Some cross product rules: $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

$\vec{A} \times \vec{A} = -\vec{A} \times \vec{A} = 0$

Some simple cross products



$\vec{i} \times \vec{j} = \vec{k} = -\vec{j} \times \vec{i}$



If you went from J to I, you turned a screwdriver from J to I, the screw will go down the z axis. So, I cross J and J cross I are opposite of each other, and I cross I is zero.

For cross products, $(\vec{i} A_x + \vec{j} A_y + \vec{k} A_z) \times (\vec{i} B_x + \vec{j} B_y + \vec{k} B_z)$

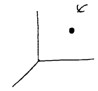
There are, in principle, nine terms you can get in the cross product. Three of them will vanish, because when you take I cross I you will get zero. When you take I cross J, you will get something.

We get $(A_y B_z - A_z B_y) + \ldots$ more stuff we don't need.

So what's the point of understanding cross products?

$\vec{\tau} = \vec{r} \times \vec{F}$

If a force is acting at the point with vector r measured from some origin, the torque due to that force is r cross F.



↙ single point mass

$\vec{L} = \vec{r} \times \vec{F}$

↑ momentum of mass

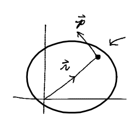
What's the first thing we would want this new momentum definition to satisfy?

$\frac{d\vec{L}}{dt}$ should $= \tau$.

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \underbrace{\vec{v} \times \vec{p}}_{\text{0, velocity and momentum are parallel}} + \underbrace{\vec{r} \times \vec{F}}_{\text{torque}}$$

$\frac{dL}{dt} = \vec{\tau} = \vec{r} \times \vec{F}$

But these three dimensional definition is comparable to our two dimensional definitions.



↙ piece of gum embedded

$L = |r \times p|, \quad p = mv$

$= rmv$

$= rm\omega r$

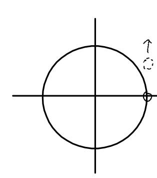
$= mr^2\omega, \quad mr^2 = I$

$= I\omega$

This is only applicable to bodies circling a point since they are stuck/embedded on a rigid body.

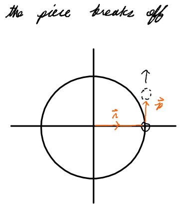
Think about the following fact. Whenever you take a rigid body that's made up of many bodies, each one of them has angular momentum Iω. You added them all up. But as a vector, each one has an angular momentum coming out of the blackboard or going into the blackboard. Because if r and p lie in the xy plane, then r cross p lies along the z axis, either out or in.

Consider

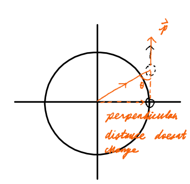


Speck breaks off and flies, does speck have angular momentum?

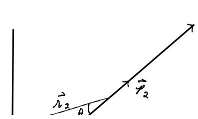
W/ L now being $\vec{L} = \vec{r} \times \vec{p}$, there will be angular momentum even after the piece breaks off



, $L = \vec{r} \times \vec{p}$



, $L = rp\sin\theta$

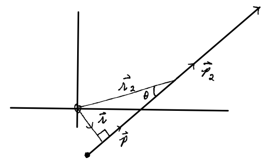


Body moving at constant momentum.

Notice $r = \vec{r}_2 \sin\theta$

you take i cross i you will get zero. When you take i cross j, you will get something.

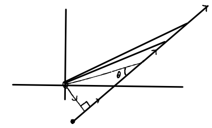
We get $(A_y B_y - A_z B_y) + \dots$ more stuff we dont need.



Body moving at constant momentum.

Notice $r = \vec{r}_2 \sin\theta$

No matter what new r is, momentum doesnt change.



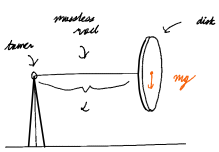
Now for many bodies, you can use...

$$\sum \frac{d\vec{L}_i}{dt} = \sum \vec{\tau}_i$$

$$\frac{d\vec{L}}{dt} = \sum \vec{\tau}_i \, , \quad \sum \vec{\tau}_i = \underset{\text{external torques}}{\vec{\tau}_{ext}} + \underset{\text{internal torques}}{\vec{\tau}_{int}}$$

With translational momentum it was easy to say that internal forces were cancelled, but it is not the same for torque. It's not so obvious you can cancel them because even though the force that I exert on you and you exert on me are equal and opposite, to find the torque, you multiply the force you exert on me cross product with the distance of me from the origin; whereas, with you it's the distance of you from the origin. So, they don't necessarily cancel. And you can show that only if the force the one body exerted on the other was in the line joining them; then that would cancel and angular momentum would be conserved. So, angular momentum conservation requires more than just usual momentum conservation.

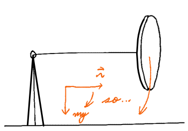
Let's now work with the gyroscope.



If the disk isn't spinning and we let the disk go, the disk will drop. Let's make sure that agrees with...

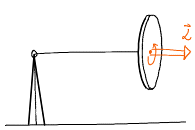
$$\tau = I \alpha$$
$$\downarrow$$
$$mg\ell = I\alpha$$

What is the vector torque?



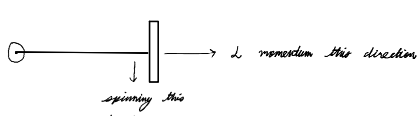
$\hat{r} \times \hat{z}$, right hand rule shows angular momentum goes into paper

But let's say the gyro is spinning.



If the disk is spinning like this, towards the counter clockwise rotation from our observation point, using the right hand rule reveals that the disk will have a momentum pointing away from the tower.

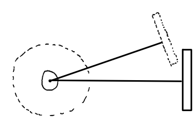
A top down view shows...



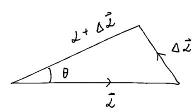
We know that from the right hand rule the wheel would be going into the paper from our first image, meaning after a certain time period we would observe the following motion.

$$\Delta L = \tau \, \Delta t$$

So from a top down view the motion of the wheel would run parallel to the dotted circle.



The last thing to calculate is the rate at which it goes around the tower.



$$\Delta L = L\,\Delta\theta$$

$$\frac{\Delta L}{\Delta t} = L\frac{\Delta\theta}{\Delta t}$$

$$\tau = L\,\omega_p$$

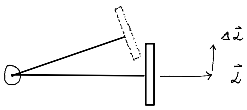
$$\omega_p = \frac{\tau}{L}$$

ω_p: precessional frequency; τ: torque; L: angular momentum

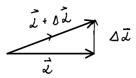
$$\omega_p = \frac{mg\ell}{I\omega}$$

We know that from the right hand rule the wheel would be going into the paper from our first image, meaning after a certain time period we would observe the following motion.

$\Delta L = \tau \Delta t$



Expressing this in a vector illustration...

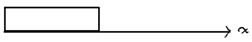


Introduction to Relativity

Friday, October 20, 2023 11:39 PM

The theory of relativity came before Einstein.

The standard technique of relativity is to take the following situation.



Train traveling along one dimension plane.

All of the blinds on this train are closed. You pour a drink, juggle some ping pong balls, etc. You are living your life like usual. Then when you sleep, the train speeds up by 200 mph. When you wake up, can you tell if you're moving or not?

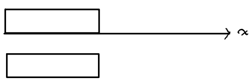
The claim is that you will not be able to tell. If you change speed while you are awake you will be able to tell since the train would need to accelerate or decelerate.

At the time of Galileo and Newton, the claim would be that you will be unable to detect a constant velocity no matter how high.

We said that in order for the laws of Newton to work, you need to be an inertial observer.

We say that everything looks the same, or that we are unable to tell we are moving at a high velocity, which means the same thing as Newton's laws are able to still work.

Now if there is a train next to us that is not moving, we lift a blind and observe the other train not moving.



If we sleep then wake up at a velocity of 200 mph, can we tell that we are moving?

The train appears to simply be moving the other direction at 200 miles per hour. The claim is that we cannot tell if it is us that is moving or if it is the other train.

This argument is only valid to uniform relative motion.

We cannot say the same for acceleration as if we were the ones to accelerate then we would fly backwards on the train and get slammed at the back wall (if we accelerate very fast to 200mph).

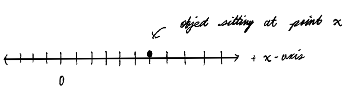
If you are in a rocket, you can't claim that the earth is accelerating downwards, since you are the only one sensing the acceleration.

Uniform velocity produces no detectable sensations so you cannot ascertain who is actually moving.

Now if we are still on the train, and we open the blinds, and we see that the landscape is going the opposite direction, then we usually claim that we are the ones moving, as it's unreasonable to claim that the ground is moving at a uniform velocity 200mph opposite to us. This is a sociological assumption, not a physical one. It's not actually unreasonable, it's reasonable to have the entire landscape on wheels and move it.

But nobody is doing all that, this is why we focus on the train.

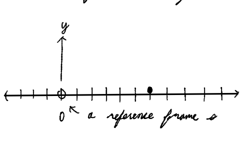
Let's prove that after we wake up and we are at a uniform velocity of 200mph, that Newton's laws still apply.



An event is (x, t)

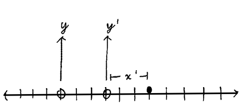
If a firecracker goes off at some time, x is where the firecracker went off and t is when it happened.

So this object will give a frame of reference s.



Now we are going to be sliding to the right in a reference frame of s'. Our velocity will be denoted by u.

As we go to the right, at some instant we will pass the reference frame of s and have a new y coordinate.

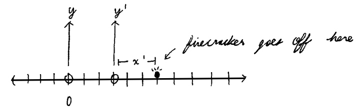


We have arranged it so that as we pass the reference frame of s, we will track our time as 0.

For s, $(x = 0, t = 0)$

For s', $(x' = 0, t = 0)$

Then we will have a second event.

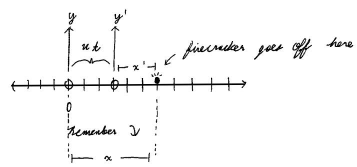


Let's say that s is a reference frame in which multiple people are moving with us. We have agents at each point in the x axis moving at the same speed over the same time. Let's also say this is true for s'. This way both reference frames know where the firecracker went off.

For s, $(x = 0, t = 0)$ \
For s', $(x' = 0, t = 0)$ } event 1, crossing

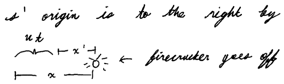
Firecracker event $\begin{cases} \text{me} = (x, t) \\ \text{you} = (x', t) \end{cases}$

What is the relation between x and x'?



The distance s origin and s' origin is to the right by ut.

$x' = x - ut$ since



↑ distance of s' origin

This is the law of transformation for coordinates in Newtonian mechanics.

A formal definition can say...

$t' = t$, since we are measuring the same time.

This is called the Galilean transformation.

Let's discuss the consequences of the Galilean transformation.

$x' = x - ut$

Let's say the firecracker isn't a firecracker event, but actually a moving bullet. Let's say it is moving to the right.

$V = \frac{dx}{dt}$ for reference frame s

$W = \frac{dx'}{dt}$ for reference frame s'

So previously we defined a singular event at a point and transformed the coordinates for two different reference frames. But now this point is going to be a moving bullet.

$W = V - u$

So if the bullet were to be traveling at 300 m/s to the right, and we are already travelling 100 m/s to the right, reference frame s' would see the bullets velocity as 200 m/s.

Now for acceleration...

$\frac{dW}{dt} = \frac{dV}{dt} = 0$ because u is a constant.

This means that both reference s and s' will agree on the acceleration of the bullet. This is because we only differ by a constant velocity, so if the bullet's velocity changes then the velocity change will be the same for both reference frames.

$a' = \frac{dW}{dt} = \frac{dV}{dt} = a$

Let's now look at Newton's second law.

$$m \frac{d^2x}{dt^2} = \mathcal{F}$$

$$m \frac{d_2 x'}{dt^2} = \mathcal{F}'$$

We claim $\mathcal{F} = \mathcal{F}'$

Let us consider two bodies that are feeling a certain force due to gravitation. This is a fictitious force (real gravity acts in three dimensions), let us say...

$$\mathcal{F}_{12} = \frac{1}{x_1 - x_2} \qquad \mathcal{F}_{21} = -\frac{1}{x_2 - x_1}$$

$$m_1 \frac{d^2 x_1}{dt^2} = \frac{1}{x_1 - x_2}$$

$$m_2 \frac{d^2 x_2}{dt^2} = -\frac{1}{x_1 - x_2}$$

Note, we are ignoring the constants $G\ m_1\ m_2$.

So we have discovered two bodies that feel a force for each other with this fashion. It only depends on x_1 and x_2 for reference frame. Another reference frame s' studies this force and finds...

same as $d^2 x_1$

$$m_1 \frac{d^2 x_1'}{dt} = \frac{1}{x_1' - x_2'}$$

same $x_1 - x_2$

If there are two bodies feeling a force and we see it while moving, the distance between the two particles will be the same, and the acceleration will be the same.

That is why the Newtonian laws will be the same. To finish our law we write...

$$m_2 \frac{d^2 x_2'}{dt} = -\frac{1}{x_1' - x_2'}$$

This is the way we prove with Newtonian mechanics, the principle of relativity.

For reference frame s, and reference frame s' if we see a force acting we will both be able to observe it. If we see an acceleration we will both be able to see it.

300 years later after electricity and magnetism is discovered, light was discovered to be measurable electrical and magnetic waves moving in space. These waves had a certain calculable velocity. Maxwell discovered this velocity to be...

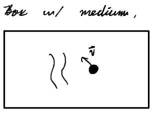
$$C = 3 \times 10^8\ m/s$$

People wanted to know the medium that is responsible for carrying electromagnetic waves (light).

We know it's everywhere since, we can see the sun, and the sun is very far from us and there's empty space between us. But how dense is this medium? Because the more dense a medium is the faster waves are able to travel.

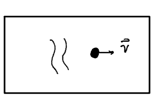
Planets move through this medium, and they have been for millions of year and they've never slowed down. So how fast do planets move with respect to this medium (let's call it ether)? This was the big question.

Box w/ medium, waves and earth in box. Earth is going around the Sun.



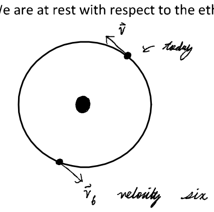
Velocity of light : $C - v$

Suppose the earth is traveling to the right.



Velocity of light is still $c - v$.

In the 1900s, Michaelson and his assistant measured that the speed relative to the ether is 0.
We are at rest with respect to the ether.



If we think about speed of sound, no matter where earth is, it will still be the same speed. This is why we always get the answer of C for the speed of electromagnetic waves.

This is because earth is carrying the air. Even if the earth moves the air will move with us so the relative velocity of air to the Earth, or Earth to the air will always be 0. This is why in the 1900s we claimed that the ether was moving with Earth as well.

This was the impasse the 1900s had to face.

It's as if there is a car going to the right of C, and another car going to the right at C/2, the expected outcome is the other car to have a velocity of C/2. But the other car keeps measuring at C.

This is very contrary to the Galilean Transformation.

Newtonian physics didn't work.

$$v \neq C - u$$

Einstein came along and said that light is behaving in this way since if it didn't if we wake up in a moving train, we can just measure the speed of light and the difference would be the speed of train. Then it would have been possible to detect the velocity of the train without looking outside.

Even though mechanical laws involving F = ma, the laws of electricity and magnetism betray your understanding of velocity. Because this would mean that uniform velocity would cause a change.

Electricity and magnetism appear to be in a conspiracy to hide your velocity, but Einstein knew that nature wouldn't have a system where mechanical laws didn't agree with electrical laws.

Einstein postulated that all phenomena will be unaffected if we are traveling at a constant velocity.

Physicists put faith in that the natural systems of the universe have elegance.

Einstein claimed that all laws of physics should follow the principles of relativity. This claim turned out to be right.

So the two great postulates of relativity are the following.

Postulate 1: All inertial observers are equivalent, with respect to all natural phenomena. The laws of nature are not an accidental observation depending on our motion. There is complete symmetry between two uniformed motion observers.

Postulate 2: Velocity of light is independent of the state of motion of source/observer. All people will get the same answer for light.

These postulates cannot be derived, they are simply claims that make our understanding of the universe work harmoniously.

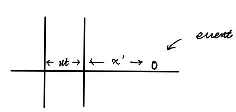
Basically we are saying light behaves this way because it plays a part in hiding uniform motion.

You will see that these two postulates restore the relativity principle to all phenomena, but in exchange you must give up all cherished notions of Newtonian physics.

This is because us claiming light has a constant speed is analogous to us following a car traveling at 200mph, while we are going at 50mph, is not moving relative to us at 150mph, but at 200mph.

So we must find a new rule that is better than the Galilean transformation.

First of all for these assumptions to be true, a meter stick now appears to not be constant, and time appears to not be constant. If we thought we traveled for 4 seconds but we only travelled for 1 second then our clock has to be slower. Something has to give in our understanding of the universe.



We used to say $x' = x - ut$, conversely, $x = x' + ut'$

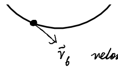
We will now say that whatever length we used to get, we will multiply it by a factor of gamma.

$$x' = (x - ut)\ \gamma \qquad `\gamma' = \text{gamma}$$
$$x = (x' + ut')\ \gamma$$

Essentially, we are saying that the units for the lengths for reference frame s, and reference frame s' must be multiplied by a factor of gamma to fit their own respective units. It is the absolute difference in measurements for both reference frames.

This allows symmetry between both reference frames.

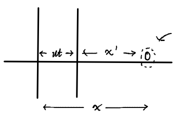
Now let's apply this to the event itself.



$\vec{v}_6$ velocity six months later

When $x = t = 0,\quad x' = t' = 0$

Let's say we emit a light pulse at the event.



light signal detected at

$(x, t) = s \qquad (x', t') = s'$

We are saying x may not $= x'$

t may not $= t'$

We must write down a very important condition that relates x' and t' (and also x and t).

$$x = ct$$
$$x' = ct'$$

We will combine these relations with...

$$x' = (x - ut)\,\gamma$$
$$x = (x' + ut')\,\gamma$$

First multiply these equations with each other then simplify.

$xx' = \gamma(xx' + uxt' - ux't - u^2tt')$, now divide everything by xx'

$$1 = \gamma^2\left(1 + \frac{ut'}{x'} - \frac{ut}{x} - u^2\left(\frac{t}{x}\right)\left(\frac{t'}{x'}\right)\right)$$

$$1 = \gamma\left(1 - u^2\left(\frac{t}{x}\right)\left(\frac{t'}{x'}\right)\right)$$

Now we know

$x = ct$ therefore $c = \frac{x}{t}$ so $\frac{1}{c} = \frac{t}{x}$

$x' = ct'$ therefore $c = \frac{x'}{t'}$ so $\frac{1}{c} = \frac{t'}{x'}$

Thus we can write

$$1 = \gamma^2\left(1 - u^2\left(\frac{1}{c}\right)\left(\frac{1}{c}\right)\right)$$

$$1 = \gamma^2\left(1 - \frac{u^2}{c^2}\right)$$

$$\gamma = \sqrt{\frac{1}{(1 - u^2/c^2)}}$$

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}}$$

Now isolating t' in $x' = \gamma(x - ut')$ and plugging in x' gives

$$t' = \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}}$$

These equations are the GREATEST result from the theory of relativity. These were derived with simple algebraic skills! THE LORENTZ TRANSFORMATION!

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}} \qquad t' = \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}}$$

Remember, u is simply a velocity. And c is the velocity of light. This means that any normal human velocity (a car traveling at 60mph) is incomparable to the speed of light, therefore our equations end up reducing to the Galilean transformation. u/c << 1

c is the greatest assumed constant.

But what are the formulas connecting/relating?

velocity (a car traveling at 60mph) is incomparable to the speed of light, therefore our equations end up reducing to the Galilean transformation. $u/c << 1$

c is the greatest assumed constant.

But what are the formulas connecting/relating?

These formulas claim that an event in space has a specific x coordinate and t coordinate. But if there is a person moving to the right at a speed of u, then that person will have special coordinates x' and t' with the relation being the Lorentz transformation.

Note: the laws of newton will still apply, since these are simply relations between a position and time.

Einstein got credit for turning our world into four dimensions, because T is now a variable coordinate. Space and time mix with each other when you change your frame of reference.

Lorentz Transformation

Thursday, November 2, 2023 11:48 PM

Previously we have derived

$$x' = \frac{x - ut}{\sqrt{1 - u^2/c^2}} \quad \& \quad t' = \frac{t - \frac{ux}{c^2}}{\sqrt{1 - \frac{u^2}{c^2}}}$$

These two equations allow us to make some of the most ambitious famous theories such as e = mc^2 and the twin paradox.

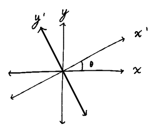
The mathematics are very simple, compared to modern physics which requires 1000 times more math to calculate the stress and strain for a beam under a weight.

Results from these equations simply result from analysis of these equations.

It's more of an issue of courage than intelligence, you allow the equations to take you places.

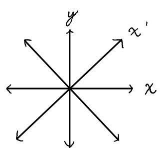
Remember that these equations give you a relation between x and y and x' and t'.

You can think of the relation as an analog to…



$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\cos\theta + y\sin\theta$$

Lets consider when $\theta = 45°$.

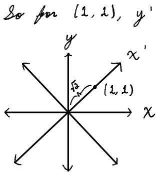


In this case…

$$x' = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$$

$$y' = -\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$$

So for $(1, 1)$, $y' = 0$, $x' = \sqrt{2}$



Think of the Lorentz transformation as analogous to the rotation of axes. You can also write the Lorentz transformation as…

$$x' = \frac{x}{\sqrt{\ldots}} - \frac{ut}{\sqrt{\ldots}}, \text{ notice the similarity to } x' = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\,!$$