$$y_i = \beta_1 + \beta_2 \cdot X_{2i} + \beta_3 \cdot X_{3i} + u_i$$

$$\hat{\beta} = (X'X)^{-1} \cdot X'y$$

$$Y = \hat{\beta}_1 + \hat{\beta}_2 \cdot X_2 + \hat{\beta}_3 \cdot X_3 + \hat{u}$$

$$(1) \quad \hat{\beta}_1 = \bar{y} - \hat{\beta}_2 \cdot \bar{X}_2 - \hat{\beta}_3 \bar{X}_3$$

$\mathrm{COV}(X_2, Y)$, $\mathrm{Cov}(X_3, Y)$ - ?

$$(2) \quad \hat{\beta}_2 \mathrm{Var}(X_2) + \hat{\beta}_3 \cdot \mathrm{Cov}(X_2, X_3) = \mathrm{Cov}(X_2, Y)$$

$$(3) \quad \hat{\beta}_2 \mathrm{Cov}(X_2, X_3) + \hat{\beta}_3 \cdot \mathrm{Var}(X_3) = \mathrm{Cov}(X_3, Y)$$

$$\left(\begin{array}{cc|c} \mathrm{Var}(X_2) & \mathrm{Cov}(X_2, X_3) & \mathrm{Cov}(X_2, Y) \\ \mathrm{Cov}(X_2, X_3) & \mathrm{Var}(X_3) & \mathrm{Cov}(X_3, Y) \end{array} \right)$$

$$\Delta = \mathrm{Var}(X_2)\mathrm{Var}(X_3) - \mathrm{Cov}^2(X_2, X_3)$$

$$\Delta_1 = \mathrm{Cov}(X_2, Y)\mathrm{Var}(X_3) - \mathrm{Cov}(X_2, X_3)\mathrm{Cov}(X_3, Y)$$

$$\Delta_2 = \mathrm{Var}(X_2)\mathrm{Cov}(X_3, Y) - \mathrm{Cov}(X_2, Y)\mathrm{Cov}(X_2, X_3)$$

$$\hat{\beta}_2 = \frac{\Delta_1}{\Delta} \qquad \hat{\beta}_3 = \frac{\Delta_2}{\Delta}$$

$$R^2 = 1 - \frac{RSS}{TSS}$$

$$R^2_{adj} = 1 - \frac{RSS/(n-k)}{TSS/(n-1)}$$

F-test for significance

$$F = t^2$$

UR: $y_i = \beta_0 + \beta_1 X_{1i} + \dots + \beta_k X_{ki} + u_i$

R: $y_i = \beta_0 + u_i$

$H_0: \beta_1 = \dots = \beta_k = 0$

$H_a: \exists i \ \beta_i \neq 0$

$$F = \frac{R^2/(k-1)}{(1-R^2)/n-k} = \frac{ESS/k-1}{RSS/n-k} \overset{H_0}{\sim} F(k-1, n-k)$$

F test for linear restrictions

UR: $y_i = \beta_0 + \beta_1 X_{1i} + \dots + \beta_k x_{ki} + u_i$

R: $\begin{bmatrix} \beta_1 = 0 \\ \beta_2 = 0 \end{bmatrix}$ $y_i = \beta_0 + \beta_3 X_{3i} + \dots + \beta_k x_{ki} + u_i$

$H_0: \beta_1 = 0, \ \beta_2 = 0$

$H_a: \beta_1 \neq 0$ or $\beta_2 \neq 0$

$$F = \frac{ESS/k-1}{RSS/n-k}$$

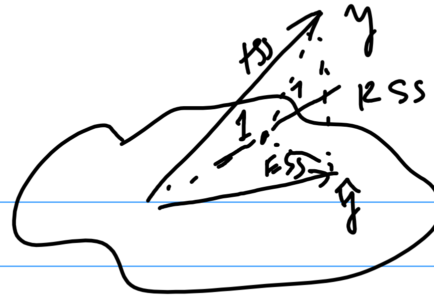
$$F = \frac{(RSS_R - RSS_{UR})/q}{RSS_{UR}/(n-k-1)} = \frac{(R^2_{UR} - R^2_R)/q}{(1-R^2_{UR})/(n-k-1)}$$

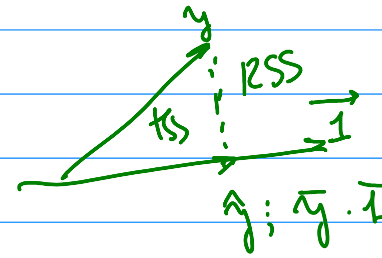
$$RSS_R \geq RSS_{UR}$$

$$TSS_R = TSS_{UR}$$

$$ESS_R \leq ESS_{UR}$$

$$R^2_R \leq R^2_{UR}$$





(a) $H_0: \beta_2 = 1$

$H_a: \beta_2 \neq 1$

$$y_i = \beta_1 + \beta_2 \cdot X_{2i} + \beta_3 \cdot Y_{3i} + u_i$$

$k = 3$

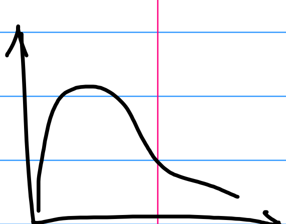
$$t = \frac{\hat{\beta}_2 - 1}{se(\hat{\beta}_2)} \overset{H_0}{\sim} t_{n-3}$$

$$|t| > t^{crit}_{1-\alpha/2,\, n-3}$$

(b) $H_0: \beta_2 = \beta_3 = 0$

$H_a: \beta_2 \neq 0$ or $\beta_3 \neq 0$

$$y_i = \beta_1 + \beta_2 \cdot X_{2i} + \beta_3 \cdot Y_{3i} + u_i$$



$$F^{obs} = \frac{R^2 / k-1}{(1-R^2)/n-k} \overset{H_0}{\sim} F(\ldots)$$

$F^{obs} > F^{crit}_{1-\alpha,\, k-1,\, n-k} \Rightarrow H_0$ is rej

(c) $H_0: \beta_2 = \beta_3$ UR: $y_i = \beta_1 + \beta_2 \cdot X_{2i} + \beta_3 \cdot X_{3i} + u_i$

$H_a: \beta_2 \neq \beta_3$ R: $y_i = \beta_1 + \beta_2 \underbrace{(X_{2i} + X_{3i})}_{z_i} + u_i$

$$F = \frac{(RSS_R - RSS_{UR})/q}{RSS_{UR}/(n-k-1)} = \frac{(R^2_{UR} - R^2_R)/q}{(1 - R^2_{UR})/(n-k-1)}$$

(d) $H_0: \beta_3 + \beta_2 = 1$ UR: $y_i = \beta_1 + \beta_2 \cdot X_{2i} + \beta_3 \cdot X_{3i} + u_i$

$H_a: \beta_3 + \beta_2 \neq 1$ R: $y_i = \beta_1 + \beta_2 X_{2i} + (1 - \beta_2) X_{3i} + u_i$

$y_i = \beta_1 + X_{3i} + \beta_2 \underbrace{(X_{2i} - X_{3i})}_{z_i} + u_i$

$\underbrace{y_i - X_{3i}}_{W_i} = \beta_1 + \beta_2 \cdot Z_i + u_i$

$$F = \frac{(RSS_R - RSS_{UR})/q}{RSS_{UR}/(n-k-1)}$$

$q = 1$

$k = 2$

(e) $H_0: \beta_2 + \beta_3 - 1 = 0$ UR: $y_i = \beta_1 + \beta_2 \cdot X_{2i} + \beta_3 \cdot X_{3i} + u_i$

$H_a: \beta_2 + \beta_3 - 1 < 0$

R: $\underbrace{y_i - X_{2i}}_{w_i} = \beta_1 + (\beta_2 + \beta_3 - 1) X_{2i} +$

$\gamma = \beta_2 + \beta_3 - 1$

$\beta_3 \underbrace{(X_{3i} - X_{2i})}_{z_i} + u_i$

$H_0: \gamma = 0$

$H_a: \gamma < 0$

t-test for γ

$$y_i = \beta_0 + \beta_1 x_i + u_i$$

$$\hat{y}_{n+1} = \hat{\beta}_0 + \hat{\beta}_1 x_{n+1}$$

$$E(\hat{y}_{n+1}) = \beta_0 + \beta_1 \cdot x_{n+1}$$

$$E(y_{n+1}) = \beta_0 + \beta_1 \cdot x_{n+1}$$

$$E(y_{n+1} - \hat{y}_{n+1})^2 = E(\beta_0 + \beta_1 x_{n+1} + u_{n+1} -$$

$$\hat{\beta}_0 - \hat{\beta}_1 x_{n+1})^2 = E((\beta_0 - \hat{\beta}_0) + (\beta_1 - \hat{\beta}_1) x_{n+1} +$$

$$u_{n+1})^2 = Var(\hat{\beta}_0) + Var(\hat{\beta}_1) \cdot x_{n+1}^2 +$$

$$\sigma_u^2 + Cov(\hat{\beta}_0, \hat{\beta}_1) x_{n+1} \; =$$

$$Var(\hat{\beta}) = \sigma^2 (X'X)^{-1} = \begin{bmatrix} \frac{\sigma^2 \sum X_i^2}{n \sum (X_i - \bar{X})^2} & \frac{-\bar{X}\sigma^2}{\sum (X_i - \bar{X})^2} \\ \frac{-\bar{X}\sigma^2}{\sum (X_i - \bar{X})^2} & \frac{\sigma^2}{\sum (X_i - \bar{X})^2} \end{bmatrix}$$

$$= \sigma_u^2 \left(1 + \frac{1}{n} + \frac{(x_{n+1} - \bar{x})^2}{\sum (X_i - \bar{X})^2} \right)$$

$$\updownarrow \; s^2$$

$$\hat{\sigma}^2 = \frac{1}{n} \cdot \sum (X_i - \bar{X})^2$$

$$s^2 = \frac{1}{n-1} \cdot \sum (X_i - \bar{X})^2$$

$$se(\hat{u}_{n+1}) = \sqrt{s^2\left(1 + \frac{1}{n} + \frac{(x_{n+1} - \bar{x})^2}{\sum (x_i - \bar{x})^2}\right)}$$

$$y_{n+1} : \quad \{ \hat{y}_{n+1} \pm t_{0.975, n-k} \cdot se(\hat{u}_{n+1}) \}$$

Example: $n = 10$ $\quad y_i = \beta_1 + \beta_2 x_i + \varepsilon_i$

$Cov(X,Y) = E(XY) - E(X)E(Y)$

$Var(X) = E(X^2) - E^2(X)$

$x_{11} = 5$

$\sum x_i = 20 \quad \sum x_i^2 = 50 \quad \sum x_i y_i = 10$

$\sum y_i = 8 \quad \sum y_i^2 = 26$

$$\hat{\beta}_2 = \frac{\widehat{Cov}(X,Y)}{\widehat{Var}(X)} = \frac{\overline{XY} - \bar{X} \cdot \bar{Y}}{\overline{X^2} - \bar{X}^2} = \frac{1 - 2 \cdot 0.8}{5 - 4} = -0.6$$

$$\hat{\beta}_1 = \bar{y} - \hat{\beta}_2 \cdot \bar{x} = 0.8 + 0.6 \cdot 2 = 2$$

$$\hat{y}_{11} = 2 - 0.6 \cdot 5 = -1$$

$$RSS = \sum \hat{\varepsilon}_i^2 = \sum \hat{\varepsilon}_i (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) =$$

$$= \sum \hat{\varepsilon}_i y_i = \sum (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) y_i =$$

$$= 26 - 2 \cdot 8 + 0.6 \cdot 10 = 16 = s^2$$

$$se(\hat{\varepsilon}_{11}) = 2$$

TSS = RSS + ESS

$$TSS = \sum(y_i - \bar{y}_i)^2$$

$$ESS = \sum(\hat{y}_i - \bar{y})^2$$