Simultaneous Equation Models

P1.
$$\begin{cases} C = \alpha + \beta Y + u & (1) \\ Y = C + I & (2) \end{cases}$$
Structural Form

$$\begin{cases} Y = \dfrac{\alpha}{1-\beta} + \dfrac{1}{1-\beta} \cdot I + \dfrac{u}{1-\beta} & (3) \\ C = \dfrac{\alpha}{1-\beta} + \dfrac{\beta}{1-\beta} \cdot I + \dfrac{u}{1-\beta} & (4) \end{cases}$$
Reduced Form

a) $\hat{\beta}_{OLS}$ in (1) is inconsistent?

$$C = \alpha + \beta Y + u \qquad (1)$$

$$\hat{\beta} = \frac{\widehat{Cov}(Y,C)}{\widehat{Var}(Y)} = \beta + \frac{\widehat{Cov}(Y,u)}{\widehat{Var}(Y)} \xrightarrow{plim} \beta + \frac{\sigma_{Y,u}}{\sigma_Y^2}$$

$$\sigma_{Y,u} = Cov\left\{ \frac{\alpha}{1-\beta} + \frac{1}{1-\beta} \cdot I + \frac{u}{1-\beta}, u \right\} = \frac{1}{1-\beta} \cdot \sigma_u^2$$

$$\sigma_Y^2 = Var\left\{ \frac{\alpha}{1-\beta} + \frac{1}{1-\beta} \cdot I + \frac{u}{1-\beta} \right\} =$$

$$\frac{1}{(1-\beta)^2} Var\{I + u\} = \frac{\sigma_I^2 + \sigma_u^2}{(1-\beta)^2}$$

$$\operatorname{plim} \hat{\beta} = \beta + \frac{\frac{1}{1-\beta} \cdot \sigma_u^2}{\frac{\sigma_I^2 + \sigma_u^2}{(1-\beta)^2}} = \beta + (1-\beta) \cdot \frac{\sigma_u^2}{\sigma_I^2 + \sigma_u^2}$$

b) Obtain consisten est β from (4)

$$C = \underbrace{\frac{\alpha}{1-\beta}}_{\hat{\alpha}} + \underbrace{\frac{\beta}{1-\beta}}_{\hat{\beta}} \cdot I + \frac{u}{1-\beta}$$

$$\hat{\beta} = \frac{\widehat{Cov}(I, C)}{\widehat{Var}(I)} \qquad \operatorname{plim} \hat{\beta} = \frac{\beta}{1-\beta}$$

$$\hat{\beta} = \frac{\hat{\beta}_{ILS}}{1-\hat{\beta}_{ILS}} \Rightarrow \hat{\beta}_{ILS} = \frac{\hat{\beta}}{1+\hat{\beta}}$$

$$\operatorname{plim} \hat{\beta}_{ILS} = \operatorname{plim} \frac{\hat{\beta}}{1+\hat{\beta}} = \frac{\beta / 1-\beta}{1 + \beta / 1-\beta} = \beta$$

c) Obtain consisten est of α from (4)

$$\operatorname{plim} \hat{\alpha} = \frac{\alpha}{1-\beta} \qquad \hat{\alpha} = \frac{\hat{\alpha}_{ILS}}{1-\hat{\beta}_{ILS}}$$

$$\hat{\alpha}_{ILS} = \hat{\alpha}(1-\hat{\beta}_{ILS})$$

$$\text{plim } \hat{\alpha}_{ILS} = \frac{\alpha}{1-\beta}(1-\beta) = \alpha$$

d) ILS is equivalent to IV :

$$\hat{\beta}_{ILS} = \frac{\hat{\beta}}{1+\hat{\beta}} = \frac{\frac{Cov(I,C)}{Var(I)}}{1+\frac{Cov(I,C)}{Var(I)}} =$$

$$= \frac{Cov(I,C)}{Cov(I,I)+Cov(I,C)} = \frac{Cov(I,C)}{Cov(I,I+C)} =$$

$$= \frac{Cov(I,C)}{Cov(I,Y)} = \hat{\beta}_{IV}$$

$$\begin{cases} \underline{C} = \alpha + \beta \underline{Y} + u & (1) \\ \underline{Y} = \underline{C} + I & (2) \end{cases}$$

Identification

\# endog. regr.

\# instruments

Exactly Identified	$r = k$	Consistent estimators
Over identified	$r < k$	can obtain different but consistent est.
underidentified	$r > k$	can't obtain consist. estimators

Example 1 Recursive SEM

$$\begin{cases} y_1 = \alpha + \beta y_2 + \gamma \cdot x + u_1 & (1) \\ y_2 = \delta + \varepsilon \cdot x + u_2 & (2) \end{cases}$$

1 step (2) using OLS $\Rightarrow \hat{y}_2$

2 step plug $\hat{y}_2$ in (1) + use OLS

$\Rightarrow$ Both eq. are identified

$\Rightarrow$ system is identified

Example 2: (IV rule)

$$\begin{cases} y_1 = \alpha + \beta y_2 + \gamma x + u_1 & (1) \\ y_2 = \delta + \varepsilon y_1 + u_2 & (2) \end{cases}$$

Eq (2) 1 end. regr. y_1 = 1 ex. reg from (1)

$\Rightarrow$ exactly identified

Eq(1) 1 end. regr y_2 > 0 ex. reg from (2)
=> underidentified

=> System is partially identified

Order Condition

G - # equations / endog. var. in SEM

j - # endog. regr. missing from equation of SEM

$(G-j-1)$ - # regr. available on the right side
- at least # of instruments is needed

$(j+(G-1-j)) = G-1$

- minimum # of variables missing from the equation

Necessary Condition for Identification

Order Condition Rule:

- equation is likely over identified if $> (G-1)$ are missing from it
- equation is likely exactly identified if $(G-1)$ are missing from it

Problem 2

$$\begin{cases} W_t = \alpha_0 + \alpha_1 P_t + \alpha_2 U_t + \alpha_3 Z_t + \varepsilon_{1t} & (1) \\ P_t = \beta_0 + \beta_1 W_t + \beta_2 U_t + \beta_3 Z_t + \varepsilon_{2t} & (2) \end{cases}$$

a) Structural form => Reduced Form

b) both (1) and (2) are underidentified.

(i) $\alpha_2 = \alpha_3 = 0$

(ii) $\alpha_2 = \beta_3 = 0$

(iii) $\alpha_2 = \alpha_3 = \beta_3 = 0$

$y \mid x_1, \dots, x_k, w_1, \dots, w_m$

$x_1 \mid z_1, \dots z_p, w_1, \dots, w$

(i) $\alpha_2 = \alpha_3 = 0$

$$\begin{cases} w_t = \alpha_0 + \alpha_1 p_t + \varepsilon_{1t} & (1) \\ p_t = \beta_0 + \beta_1 w_t + \beta_2 u_t + \beta_3 z_t + \varepsilon_{2t} & (2) \end{cases}$$

Eq (1) overidentified

Eq (2) underidentified

$\Rightarrow$ Partially identified