Simultaneous Equation Models

$$\begin{cases} C = \alpha + \beta Y + u & (1) \\ Y = C + I & (2) \end{cases}$$

Structural form model

$$\begin{cases} Y = \frac{\alpha}{1-\beta} + \frac{1}{1-\beta} I + \frac{u}{1-\beta} & (3) \\ C = \frac{\alpha}{1-\beta} + \frac{\beta}{1-\beta} I + \frac{u}{1-\beta} & (4) \end{cases}$$

Reduced form model

a) $\hat{\beta}_{OLS}$ is inconsistent?

$$\hat{\beta} = \frac{\hat{Cov}(Y,C)}{\hat{Var}(Y)} = \beta + \frac{\hat{Cov}(Y,u)}{\hat{Var}(Y)} \xrightarrow{plim} \beta + \frac{\sigma_{Y,u}}{\sigma_Y^2}$$

$$\sigma_{Y,u} = Cov\left(\frac{\alpha}{1-\beta} + \frac{1}{1-\beta} I + \frac{u}{1-\beta}, u\right) = \frac{\sigma_u^2}{1-\beta}$$

$$\sigma_Y^2 = Var\left(\frac{\alpha}{1-\beta} + \frac{1}{1-\beta} I + \frac{u}{1-\beta}\right) =$$

$$\frac{1}{(1-\beta)^2} Var(I + u) = \frac{\sigma_I^2 + \sigma_u^2}{(1-\beta)^2}$$

$$plim\,\hat{\beta} = \beta + \frac{\sigma_u^2/1-\beta}{\sigma_I^2 + \sigma_u^2/(1-\beta)^2} = \beta + \frac{1}{1-\beta} \cdot \frac{\sigma_u^2}{\sigma_I^2 + \sigma_u^2}$$

b) Obtain consistent estimator of β from

$$C = \underbrace{\frac{\alpha}{1-\beta}}_{\hat{\alpha}} + \underbrace{\frac{\beta}{1-\beta}}_{\hat{\beta}} I + \frac{u}{1-\beta} \qquad (4)$$

$$\hat{\beta} = \frac{\widehat{Cov}(I,C)}{\widehat{Var}(I)} \qquad \text{plim } \hat{\beta} = \frac{\beta}{1-\beta}$$

$$\hat{\beta} = \frac{\hat{\beta}_{ILS}}{1-\hat{\beta}_{ILS}} \Rightarrow \hat{\beta}_{ILS} = \frac{\hat{\beta}}{1+\hat{\beta}}$$

$$\text{plim } \hat{\beta}_{ILS} = \text{plim} \frac{\hat{\beta}}{1+\hat{\beta}} = \frac{\beta/_{1-\beta}}{1+\beta/_{1-\beta}} = \beta$$

c) Obtain consistent est. α from (4)

$$\text{plim } \hat{\alpha} = \frac{\alpha}{1-\beta}$$

$$\hat{\alpha} = \frac{\hat{\alpha}_{ILS}}{1-\hat{\beta}_{ILS}} \Rightarrow \hat{\alpha}_{ILS} = \hat{\alpha} \cdot (1-\hat{\beta}_{ILS})$$

$$\text{plim } \hat{\alpha}_{ILS} = \text{plim } \hat{\alpha} \cdot (1-\hat{\beta}_{ILS}) =$$

$$= \frac{\alpha}{1-\beta}(1-\beta) = \alpha$$

d) $\hat{\beta}_{ILS} = \hat{\beta}_{IV}$

$$\hat{\beta}_{ILS} = \frac{\hat{\beta}}{1+\hat{\beta}} = \frac{\frac{\widehat{Cov}(I,C)}{\widehat{Var}(I)}}{1+\frac{\widehat{Cov}(I,C)}{\widehat{Var}(I)}} = \frac{\widehat{Cov}(I,C)}{\widehat{Cov}(I,I)+\widehat{Cov}(I,C)} =$$

$$= \frac{\widehat{Cov}(I,C)}{\widehat{Cov}(I,I+C)} = \frac{\widehat{Cov}(I,C)}{\widehat{Cov}(I,Y)} = \hat{\beta}_{IV}$$

Identification in SEM

	# endogenous regr / # instruments	
Exactly identified	$p = m$	consistent est.
Over identified	$p < m$	can obtain dif. est. (consistent)
Under identified	$p > m$	can't obtain consistent est

IV - rule

Example 1 : Recursive SEM

$$\begin{cases} y_1 = \alpha + \beta y_2 + \gamma_1 x + u_1 & (1) \\ y_2 = \delta + \gamma_2 x + u_2 & (2) \end{cases}$$

Eq (2) is identified

1 step use OLS for (2) $\Rightarrow \hat{y}_2$

2 step plug $\hat{y}_2$ in (1) and

use OLS $\Rightarrow \hat{\beta}$ consistent

Both (1) and (2) are identified

$\Rightarrow$ system is identified

Example 2

$y_1 \mid x_1, \dots, x_p, w_1, \dots, w_k$

$x_1 \mid z_1, \dots, z_m, w_1, \dots, w_k$

$$\begin{cases} y_1 = \alpha + \beta y_2 + \gamma_1 x_1 + u_1 & (1) \\ y_2 = \delta + \gamma_2 y_1 + u_2 & (2) \end{cases}$$

Eq (2) $m = p = 1$ exactly identified

Eq (1) $p = 1$ $m = 0$ under identified

$\Rightarrow$ system is partially identified

Order Condition

(Necessary condition for identification)

G – # end. variables / # equation is SEM

j – # end. variables missing from eq.

$(G-1-j)$ – # end. variables on the right

– min # of instruments needed

$$j + (G-1-j) = G-1$$

if $(G-1)$ variable is missing $\Rightarrow$
is likely to be identified

if $> (G-1)$ variable is missing $\Rightarrow$
is likely to be overidentified

Problem 1. $W_t = \alpha_0 + \alpha_1 P_t + \alpha_2 U_t + \alpha_3 Z_t + \varepsilon_{1t}$ (1)

$P_t = \beta_0 + \beta_1 W_t + \beta_2 U_t + \beta_2 Z_t + \varepsilon_{2t}$ (2)

(a) Eq (1) underidentified

Eq (2) underidentified

=> system is unidentified

(b) $\alpha_2 = \alpha_3 = 0$

$W_t = \alpha_0 + \alpha_1 P_t + \varepsilon_{1t}$ (1)

$P_t = \beta_0 + \beta_1 W_t + \beta_2 U_t + \beta_2 Z_t + \varepsilon_{2t}$ (2)

Eq (1) overidentified (2SLS)

Eq (2) underidentified

=> system is partially identified

c) $\alpha_2 = \beta_3 = 0$

$W_t = \alpha_0 + \alpha_1 P_t + \alpha_3 Z_t + \varepsilon_{1t}$ (1)

$P_t = \beta_0 + \beta_1 W_t + \beta_2 U_t + \varepsilon_{2t}$ (2)

(1), (2) exactly ident. => system is ident.

(IV or ILS)

d) $\alpha_2 = \alpha_3 = \beta_3 = 0$

$$w_t = \alpha_0 + \alpha_1 p_t + \varepsilon_{1t} \quad (1)$$

$$p_t = \beta_0 + \beta_1 w_t + \beta_2 u_t + \varepsilon_{2t} \quad (2)$$

(1) exactly
(2) underidentified $\Rightarrow$ system is partially identified

(e) $\alpha_2 = \alpha_3 = 0$

$$w_t = \alpha_0 + \alpha_1 p_t + \varepsilon_{1t} \quad (1)$$

$$p_t = \beta_0 + \beta_1 w_{t-1} + \beta_2 u_t + \beta_2 z_t + \varepsilon_{2t} \quad (2)$$

Eq (1) over identified

w_{t-1} predetermined variable

for t is exogenous