$$y_i = \beta_1 + \beta_2 \cdot X_{2i} + \beta_3 X_{3i} + u_i$$

$$\hat{\beta} = (X'X)^{-1} X'y \qquad \hat{y} = \hat{\beta}_1 + \hat{\beta}_2 \cdot X_2 + \hat{\beta}_3 \cdot X_3$$

$$(1) \quad \hat{\beta}_1 = \bar{y} - \hat{\beta}_2 \cdot \bar{X}_2 - \hat{\beta}_3 \cdot \bar{X}_3$$

$$Cov(X_2, Y), \quad Cov(X_3, Y) = ?$$

$$(2) \quad \hat{\beta}_2 \cdot Var(X_2) + \hat{\beta}_3 Cov(X_2, X_3) = Cov(X_2, Y)$$

$$(3) \quad \hat{\beta}_2 \cdot Cov(X_2, X_3) + \hat{\beta}_3 Var(X_3) = Cov(X_3, Y)$$

$$\left(\begin{array}{cc|c} Var(X_2) & Cov(X_2, X_3) & Cov(X_2, Y) \\ Cov(X_2, X_3) & Var(X_3) & Cov(X_3, Y) \end{array}\right)$$

$$\Delta = Var(X_2) \cdot Var(X_3) - Cov^2(X_2, X_3)$$

$$\Delta_1 = Cov(X_2, Y) \cdot Var(X_3) - Cov(X_2, X_3) \cdot Cov(X_3, Y)$$

$$\Delta_2 = Var(X_2) \cdot Cov(X_3, Y) - Cov(X_2, X_3) \cdot Cov(X_2, Y)$$

$$\hat{\beta}_2 = \frac{\Delta_1}{\Delta} \qquad \hat{\beta}_3 = \frac{\Delta_2}{\Delta}$$

$$R^2 = 1 - \frac{RSS}{TSS}$$

$$R^2_{adj} = 1 - \frac{RSS/n-K}{TSS/n-1}$$

$y_i = \beta_0 + \beta_1 X_{1i} + \dots + \beta_k x_{ki} + u_i$

F for regression significance

$H_0: \beta_1 = \dots = \beta_k = 0$

$H_a: \exists i \quad \beta_i \neq 0$

$$F = \frac{R^2/(k-1)}{(1-R^2)/(n-k)} = \frac{ESS/k-1}{RSS/n-k} \overset{H_0}{\sim} F(k-1, n-k)$$

For pair linear regression

$$F = \frac{ESS}{RSS/_{n-2}} = \frac{\sum(\hat{Y}_i - \bar{Y})^2}{\sum \hat{u}_i^2 / _{n-2}} =$$

$$= \frac{\sum(\cancel{\hat{\beta}_0} - \hat{\beta}_1 x_i - \cancel{\hat{\beta}_0} + \hat{\beta}_1 \cdot \bar{x}_i)^2}{\hat{\sigma}^2_u} = \frac{\hat{\beta}_1^2}{\hat{\sigma}_u^2} \sum (x_i - \bar{x})^2$$

$$= \frac{\hat{\beta}_1^2}{se^2(\hat{\beta}_1)} = t^2$$

F-test for linear restrictions

UR: $y_i = \beta_0 + \beta_1 x_{1i} + \dots \beta_k x_{ki} + u_i$

R $\begin{bmatrix} \beta_1 = 0 \\ \beta_2 = 0 \end{bmatrix}$: $y_i = \beta_0 + \beta_3 x_{3i} + \dots + \beta_k X_{ki} + u_i$

R: $y_i = \beta_0 + u_i$

H_0: $\beta_1 = 0$ and $\beta_2 = 0$

H_1: $\beta_1 \neq 0$ or $\beta_2 \neq 0$

$$F = \frac{ESS / k-1}{RSS / n-k-1}$$

$$F = \frac{(RSS_R - RSS_{UR}) / q}{RSS_{UR} / n-k-1} = \frac{(R^2_{UR} - R^2_R) / q}{(1 - R^2_{UR}) / n-k-1}$$

$$RSS_R \geq RSS_{UR}$$

$$TSS_R = TSS_{UR}$$

$$ESS_R \leq ESS_{UR}$$

$$R^2_R \leq R^2_{UR}$$

(a) $H_0: \beta_2 = 1$ $\qquad y_i = \beta_1 + \beta_2 \cdot X_{2i} + \beta_3 X_{3i} + u_i$

$H_a: \beta_2 \neq 1$

$$t = \frac{\hat{\beta}_2 - 1}{se(\hat{\beta}_2)} \overset{H_0}{\sim} t_{n-k} \qquad k = 3$$

$$|t^{obs}| > t^{crit}_{1-\alpha/2,\, n-k} \Rightarrow H_0 \text{ is rejected}$$

(b) $H_0: \beta_2 = \beta_3 = 0$ $\qquad y_i = \beta_1 + \beta_2 \cdot X_{2i} + \beta_3 X_{3i} + u_i$

$H_a: \beta_2 \neq 0$ or $\beta_3 \neq 0$

$$F = \frac{R^2 / 2}{1-R^2 / n-3} \overset{H_0}{\sim} F(2, n-3)$$

(c) $H_0: \beta_2 = \beta_3$ $\qquad UR: y_i = \beta_1 + \beta_2 \cdot X_{2i} + \beta_3 X_{3i} + u_i$

$H_a: \beta_2 \neq \beta_3$ $\qquad R: y_i = \beta_1 + \beta_2 \underbrace{(X_{2i} + X_{3i})} + u_i$

$$F = \frac{(RSS_R - RSS_{UR})/q}{RSS_{UR} / n-k-1} = \frac{(R^2_{UR} - R^2_R)/q}{(1 - R^2_{UR}) / n-k-1}$$

(d) $H_0: \beta_2 + \beta_3 = 1$ $\qquad UR: y_i = \beta_1 + \beta_2 \cdot X_{2i} + \beta_3 X_{3i} + u_i$

$H_a: \beta_2 + \beta_3 = 1$ $\qquad R: y_i = \beta_1 + \beta_2 X_{2i} + (1-\beta_2) X_{3i} + u_i$

$$y_i = \beta_1 + \beta_2 \underbrace{(X_{2i} - X_{3i})}_{z_i} + X_{3i} + u_i$$

$$y_i - X_{3i} = W_i$$

$$W_i = \beta_1 + \beta_2 \cdot Z_i + u_i$$

$$F = \frac{(RSS_R - RSS_{UR})/q}{RSS_{UR} / n-k-1}$$

(e) $H_0: \beta_2 + \beta_3 - 1 = 0$ $\quad$ UR: $y_i = \beta_1 + \beta_2 \cdot X_{2i} + \beta_3 X_{3i} + u_i$

$H_a: \beta_2 + \beta_3 - 1 < 0$ $\quad$ R: $y_i - X_{2i} = \beta_1 + (\beta_2 + \beta_3 - 1) X_{2i}$

$\gamma = \beta_2 + \beta_3 - 1$ $\quad\quad + \beta_3 (X_{3i} - X_{2i}) + u_i$

$H_0: \gamma = 0$

$H_a: \gamma < 0$ $\quad \Rightarrow$ t-test

Predictions $\quad y_i = \beta_0 + \beta_1 \cdot x_i + u_i$

$n+1$: $\quad y_{n+1} = \hat{\beta}_0 + \hat{\beta}_1 x_{n+1} + \hat{u}_{n+1} \qquad y_{n+1} = \beta_0 + \beta_1 x_{n+1} + u_{n+1}$

$$E(\hat{y}_{n+1}) = \beta_0 + \beta_1 \cdot x_{n+1}$$

$$E(y_{n+1}) = \beta_0 + \beta_1 \cdot x_{n+1}$$

$$E(y_{n+1} - \hat{y}_{n+1})^2 = E\left((\beta_0 - \hat{\beta}_0) + (\beta_1 - \hat{\beta}_1)x_{n+1} + u_{n+1}\right)^2 =$$

$$Var(\hat{\beta}_0) + x_{n+1}^2 Var(\hat{\beta}_1) + \sigma_u^2 + 2\,Cov(\hat{\beta}_0, \hat{\beta}_1)\, x_{n+1} =$$

$$\sigma^2(\hat{\beta}) = \sigma_u^2 (X'X)^{-1} = \begin{bmatrix} \frac{\sigma^2 \sum x_i^2}{n\sum(x_i - \bar{x})^2} & \frac{-\bar{x}\sigma^2}{\sum(x_i - \bar{x})^2} \\ \frac{-\bar{x}\sigma^2}{\sum(x_i - \bar{x})^2} & \frac{\sigma^2}{\sum(x_i - \bar{x})^2} \end{bmatrix}$$

$$= \sigma_u^2 \left(1 + \frac{1}{n} + \frac{(x_{n+1} - \bar{x})^2}{\sum(x_i - \bar{x})^2}\right)$$

$$se(\hat{u}_{n+1}) = \sqrt{s^2\left(1 + \frac{1}{n} + \frac{(x_{n+1} - \bar{x})^2}{\sum(x_i - \bar{x})^2}\right)}$$

$$y_{n+1}: \left\{ \hat{y}_{n+1} \pm t_{1-\frac{\alpha}{2}, n-k} \cdot se(\hat{u}_{n+1}) \right\}$$