$$Y_i = \beta_0 + \beta_1 \cdot X_i + \varepsilon_i$$

$$\hat{\beta}_1 = \frac{Cov(X,Y)}{Var(X)} = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2}$$

$$= \frac{\sum (X_i - \bar{X})(\cancel{\beta_0} + \beta_1 \cdot X_i + \varepsilon_i - \cancel{\beta_0} - \beta_1 \bar{X} - \bar{\varepsilon})}{\sum (X_i - \bar{X})^2}$$

$$= \frac{\sum (X_i - \bar{X})\, \beta_1 (X_i - \bar{X})}{\sum (X_i - \bar{X})^2} + \frac{\sum (X_i - \bar{X})(\varepsilon_i - \bar{\varepsilon})}{\sum (X_i - \bar{X})^2}$$

$$= \beta_1 + \frac{\sum (X_i - \bar{X})\, \varepsilon_i}{\sum (X_i - \bar{X})^2} - \underbrace{\frac{\bar{\varepsilon} \sum (X_i - \bar{X})}{\sum (X_i - \bar{X})^2}}_{} {}^{= 0}$$

$$\sum (X_i - \bar{X}) \overset{?}{=} 0$$

$$n \cdot \frac{\sum X_i}{n} - n \bar{X} = 0$$

$$\hat{\beta}_1 = \beta_1 + \frac{\sum (X_i - \bar{X})\, \varepsilon_i}{\sum (X_i - \bar{X})^2} = \beta_1 + \sum a_i \varepsilon_i$$

$$a_i = \frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}$$

1) Equation: $y_i = \beta_1 + \beta_2 \cdot X_i + \varepsilon_i$ ($= u_i$)

2) Assumption: $E(\varepsilon_i) = 0$, $E(\varepsilon_i^2) = \sigma^2$

$E(\varepsilon_i, \varepsilon_j) = 0$

3) Method: OLS $\hat{y}_i = \hat{\beta}_1 + \hat{\beta}_2 \cdot X_i$

$y_i = \hat{\beta}_1 + \hat{\beta}_2 \cdot X_i + \hat{\varepsilon}_i$ ($= e_i$)

4) Properties

Assumptions of Linear Regression

1) $E(\varepsilon_i) = 0$ $\left(E(\varepsilon_i | X) = 0\right)$
2) (X_i, Y_i) i.i.d.
3) $\underbrace{E(X_i^4) < \infty}_{\text{stoch.}}$, $E(\varepsilon_i^4) < \infty$
4) $\text{rank}(X) = K$
5) $Var(\varepsilon_i) = \sigma^2_\varepsilon$
6) $\varepsilon_i \sim N(0, \sigma^2_\varepsilon)$

Assumption for Gauss Markov THM:

1) $E(\varepsilon) = 0$
2) $E(\varepsilon\varepsilon') = \sigma^2_\varepsilon \cdot I$
3) $\text{rank}(X) = K$

$\Rightarrow \hat{\beta}_{OLS}$ - BLUE

$$a_i = \frac{x_i - \bar{x}}{\sum (x_i - \bar{x})^2}$$

1) $\sum a_i = 0$

$$\bar{X} \cdot \sum (X_i - \bar{X}) = 0$$

2) $$\sum a_i^2 = \sum \left(\frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2} \right)^2 = \frac{\sum (X_i - \bar{X})^2}{\left(\sum (X_i - \bar{X})^2\right)^2} =$$

$$= \frac{1}{\sum (X_i - \bar{X})^2} = \frac{1}{\sum x_i^2} \qquad x_i = X_i - \bar{X}$$

3) $$\sum a_i X_i = \frac{\sum (X_i - \bar{X}) X_i - \sum (X_i - \bar{X}) \cdot \bar{X}}{\sum (X_i - \bar{X})^2} = 1$$

$$\sum (X_i - \bar{X})(X_i - \bar{X})$$

$$E(\hat{\beta}_1) = E(\beta_1 + \sum a_i \varepsilon_i) =$$

$$= \beta_1 + \sum a_i \underbrace{E(\varepsilon_i)}_{0} = \beta_1$$

$$\hat{\beta}_1 = \frac{Cov(X,Y)}{Var(X)} = \frac{Cov(X, \beta_0 + \beta_1 X + \varepsilon)}{Var(X)} =$$

$$= 0 + \beta_1 \cdot \frac{\cancel{Var}(X)}{\cancel{Var}(X)} + \frac{Cov(X, \varepsilon)}{Var(X)}$$

$$E(\hat{\beta}_1) = \beta_1 + E\left(\frac{Cov(X, \varepsilon)}{Var(X)}\right)$$

$$= \beta_1 + \frac{Cov(X, \overset{=0}{E(\varepsilon)})}{Var(X)} = \beta_1$$

2) $$\sigma^2_{\hat{\beta}_1} = \frac{\sigma^2_\varepsilon}{\sum (x_i - \bar{x})^2}$$

$$\sigma^2_{\hat{\beta}_1} = E(\hat{\beta}_1 - E(\hat{\beta}_1))^2 = E(\overset{\beta_1 + \sum a_i \varepsilon_i}{\hat{\beta}_1} - \beta_1)^2$$

$$= E(\sum a_i \cdot \varepsilon_i)^2 =$$

$$= E\left(\sum_{i=1}^{n} a_i^2 \cdot \varepsilon_i^2 + \sum_{i=1}^{n} \sum_{i \neq j} a_i a_j \, \varepsilon_i \cdot \varepsilon_j\right)$$

$$= \sum a_i^2 \cdot \underbrace{E(\varepsilon_i^2)}_{\sigma^2_\varepsilon} + \sum\sum a_i a_j \underbrace{E(\varepsilon_i \cdot \varepsilon_j)}_{0} =$$

$$= \sigma^2_\varepsilon \cdot \sum a_i^2 = \frac{\sigma^2_\varepsilon}{\sum (x_i - \bar{x})^2}$$

$$\sigma^2_{\hat{\beta}_0} = \sigma^2_\varepsilon \cdot \left(\frac{1}{n} + \frac{\bar{X}^2}{\sum (X_i - \bar{X})^2} \right)$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \cdot \bar{X} = \beta_0 + \beta_1 \bar{X} + \bar{\varepsilon} - \hat{\beta}_1 \cdot \bar{X}$$

$$= \beta_0 + \bar{X}(\beta_1 - \hat{\beta}_1) + \bar{\varepsilon} =$$

$$= \beta_0 - \bar{X} \cdot \sum a_i \varepsilon_i + \bar{\varepsilon} =$$

$$= \beta_0 + \sum c_i \varepsilon_i \qquad c_i = \frac{1}{n} - a_i \bar{X}$$

$$\sigma^2_{\hat{\beta}_0} = E\left(\sum c_i \varepsilon_i\right)^2 = \sigma^2_\varepsilon \cdot \sum c_i^2 =$$

$$\sigma^2_\varepsilon \sum \left(\frac{1}{n} - a_i \bar{X}\right)^2 = \sigma^2_\varepsilon \Big(n \cdot \frac{1}{n^2} - \underbrace{2 \cdot \frac{\bar{X}}{n} \sum a_i}_{\sum a_i = 0} + \bar{X}^2 \underbrace{\sum a_i^2}_{\sum a_i^2 = \frac{1}{\sum (x_i - \bar{x})^2}}\Big)$$

$$= \sigma^2_\varepsilon \cdot \left(\frac{1}{n} + \frac{\bar{X}}{\sum (x_i - \bar{X})^2} \right)$$

$$\operatorname{Var}(\hat{\beta}_j) = \frac{\sigma^2_\varepsilon}{TSS_j \cdot (1 - R_j^2)}$$

$$TSS_j = \sum_{i=1}^{n} (x_{ij} - \bar{x}_j)^2$$

(*) Var. $\hat{\beta}_j$ increases

1) level of noise in the data $\sigma^2_\varepsilon \nearrow$

2) if X_j is close to const.

3) if $R_j^2 \approx 1$ $\Rightarrow$ M.C.

R_j^2 - coef. of determination

$$X_j \mid X_1, \dots X_{j-1}, X_{j+1} \dots X_n$$

$$X_j \mid X_{-j}$$

is regressed

$$\operatorname{Var}(\hat\beta) = \sigma_\varepsilon^2 \cdot (X'X)^{-1}$$

$$(X'X) = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} =$$

$$\begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}$$

$$D = n\sum x_i^2 - \left(\sum x_i\right)^2$$

$$\sigma_\varepsilon^2 (X'X)^{-1} = \begin{bmatrix} \dfrac{\sigma^2 \sum x_i^2}{n\sum (x_i - \bar x)^2} & \dfrac{-\bar x \sigma^2}{\sum (x_i - \bar x)^2} \\ \dfrac{-\bar x \sigma^2}{\sum (x_i - \bar x)^2} & \dfrac{\sigma^2}{\sum (x_i - \bar x)^2} \end{bmatrix}$$

$$\operatorname{Var}(\hat\beta) = \operatorname{Var}\left(\begin{pmatrix} \hat\beta_0 \\ \hat\beta_1 \end{pmatrix}\right) \qquad \begin{bmatrix} \operatorname{Var}(\hat\beta_0) & \operatorname{Cov}(\hat\beta_0, \hat\beta_1) \\ \operatorname{Cov}(\hat\beta_0, \hat\beta_1) & \operatorname{Var}(\hat\beta_1) \end{bmatrix}$$