Simultaneous Equations Model

$$\begin{cases} C = \alpha + \beta Y + u & (1) \\ Y = C + I & (2) \end{cases}$$ Structural Equation form

$$\begin{cases} Y = \dfrac{\alpha}{1-\beta} + \dfrac{1}{1-\beta} \cdot I + \dfrac{u}{1-\beta} & (3) \\ C = \dfrac{\alpha}{1-\beta} + \dfrac{\beta}{1-\beta} \cdot I + \dfrac{u}{1-\beta} & (4) \end{cases}$$ Reduced form

a) Show $\hat{\beta}_{OLS}$ is inconsistent:

$$\hat{\beta} = \frac{\hat{Cov}(Y,C)}{\hat{Var}(Y)} = \beta + \frac{\hat{Cov}(Y,u)}{\hat{Var}(Y)} \xrightarrow{plim} \beta + \frac{\sigma_{Y,u}}{\sigma_Y^2}$$

$$\sigma_{Y,u} = Cov\left(\frac{\alpha}{1-\beta} + \frac{1}{1-\beta} \cdot I + \frac{u}{1-\beta}, u\right) = \frac{\sigma_u^2}{1-\beta}$$

$$\sigma_Y^2 = Var\left(\frac{\alpha}{1-\beta} + \frac{1}{1-\beta} I + \frac{u}{1-\beta}\right) =$$

$$\frac{1}{(1-\beta)^2} Var(I + u) = \frac{\sigma_I^2 + \sigma_u^2}{(1-\beta)^2}$$

$$plim\ \hat{\beta} = \beta + \frac{\sigma_u^2/1-\beta}{\sigma_u^2 + \sigma_I^2/(1-\beta)^2} = \beta + (1-\beta) \cdot \frac{\sigma_u^2}{\sigma_u^2 + \sigma_I^2}$$

b) Obtain consist. est. of β from

$$C = \underbrace{\frac{\alpha}{1-\beta}}_{\hat{\alpha}} + \underbrace{\frac{\beta}{1-\beta}}_{\hat{\beta}} \cdot I + \frac{u}{1-\beta} \quad (4)$$

$$\text{plim } \hat{\beta} = \frac{\beta}{1-\beta} \qquad \hat{\beta} = \frac{\widehat{Cov}(I, C)}{\widehat{Var}(I)}$$

$$\hat{\beta} = \frac{\hat{\beta}_{ILS}}{1 - \hat{\beta}_{ILS}} \Rightarrow \hat{\beta}_{ILS} = \frac{\hat{\beta}}{1+\hat{\beta}}$$

$$\text{plim } \hat{\beta}_{ILS} = \text{plim } \frac{\hat{\beta}}{1+\hat{\beta}} = \frac{\beta/_{1-\beta}}{1 + \beta/_{1-\beta}} = \beta$$

c) Obtain consistent est α from (4)

$$\text{plim } \hat{\alpha} = \frac{\alpha}{1-\beta}$$

$$\hat{\alpha} = \frac{\hat{\alpha}_{ILS}}{1-\hat{\beta}_{ILS}} \Rightarrow \hat{\alpha}_{ILS} = \hat{\alpha}(1 - \hat{\beta}_{ILS})$$

$$\text{plim } \hat{\alpha}_{ILS} = \text{plim } \hat{\alpha}(1-\hat{\beta}_{ILS}) = \frac{\alpha}{1-\beta} \cdot (1 - \beta) = \alpha$$

d) Show that $\hat{\beta}_{ILS} = \hat{\beta}_{IV}$

$$\hat{\beta}_{ILS} = \frac{\hat{\beta}}{1+\hat{\beta}} = \frac{\frac{\widehat{Cov}(I,C)}{\widehat{Var}(I)}}{1+\frac{\widehat{Cov}(I,C)}{\widehat{Var}(I)}} = \frac{\widehat{Cov}(I,C)}{\widehat{Cov}(I,I)+\widehat{Cov}(I,C)} =$$

$$= \frac{\widehat{Cov}(I,C)}{\widehat{Cov}(I,I+C)} = \frac{\widehat{Cov}(I,C)}{\widehat{Cov}(I,Y)} = \hat{\beta}_{IV}$$

Identification of SEM

	# end. regressors / # instruments	
Exactly identified	$m = p$	Consistent est.
Overidentified	$m < p$	can obtain diff. estimates (consist.)
underidentified	$m > p$	can't obtain consistent est.

IV - rule

Example 1

Recursive SEM

$$\begin{cases} y_1 = \alpha + \beta \cdot y_2 + \gamma_1 x + u_1 & (1) \\ y_2 = \delta + \gamma_2 \cdot x + u_2 & (2) \end{cases}$$

Eq (2) identified =>

1 step. Estimate (2) using OLS => $\hat{y}_2$

2 step. Plug $\hat{y}_2$ in (1) and use OLS for (1)

Example 2

$$\begin{cases} y_1 = \alpha + \beta y_2 + \gamma_1 x + u_1 & (1) \\ y_2 = \delta + \gamma_2 \cdot y_1 + u_2 & (2) \end{cases}$$

Eq (2) : $m = 1$ $p = 1$ => exactly identified

Eq (1) : $m = 1$ $p = 0$ => underidentified

=> System is partially identified

Order Condition

(Necessary cond. for identification)

G – # end. variables / # of equations in SEM

j – # end. variables missing from the equation

$(G-1-j)$ – # end variables in right part of eq.

– min # instruments needed

$$j + (G-1-j) = G-1$$

– min # missing variables

OCR:

if $G-1$ var. is missing $\Rightarrow$ equation is likely to be exactly identified

if $>(G-1)$ var. is missing $\Rightarrow$ equation is likely to be overidentified

Problem 1.

$$\begin{cases} W_t = \alpha_0 + \alpha_1 P_t + \alpha_2 u_t + \alpha_3 z_t + \varepsilon_{1t} & (1) \\ P_t = \beta_0 + \beta_1 W_t + \beta_2 u_t + \beta_3 z_t + \varepsilon_{2t} & (2) \end{cases}$$

(a) both eq. are underidentified $\Rightarrow$

system is underidentified

(b) $\alpha_2 = \alpha_3 = 0$

$$\begin{cases} W_t = \alpha_0 + \alpha_1 P_t + \varepsilon_{1t} & (1) \\ P_t = \beta_0 + \beta_1 W_t + \beta_2 u_t + \beta_3 z_t + \varepsilon_{2t} & (2) \end{cases}$$

Eq (1) overidentified TSLS

Eq (2) underidentified

$\Rightarrow$ system is partially identified

(c) $\alpha_2 = \beta_3 = 0$

$$\begin{cases} W_t = \alpha_0 + \alpha_1 P_t + \alpha_3 z_t + \varepsilon_{1t} & (1) \\ P_t = \beta_0 + \beta_1 W_t + \beta_2 u_t + \varepsilon_{2t} & (2) \end{cases}$$

both eq. are exactly ident. $\Rightarrow$ syst. ident.

IV or ILS

(d) $\alpha_2 = \alpha_3 = \beta_3 = 0$

$$\begin{cases} W_t = \alpha_0 + \alpha_1 P_t + \varepsilon_{1t} & (1) \\ P_t = \beta_0 + \beta_1 W_t + \beta_2 U_t + \varepsilon_{2t} & (2) \end{cases}$$

eq (1) exactly ident.

eq (2) underidentified

$\Rightarrow$ syst. partially identified

(e) $\alpha_2 = \alpha_3 = 0$

$$\begin{cases} W_t = \alpha_0 + \alpha_1 P_t + \varepsilon_{1t} & (1) \\ P_t = \beta_0 + \beta_1 W_{t-1} + \beta_2 U_t + \beta_3 Z_t + \varepsilon_{2t} & (2) \end{cases}$$

W_{t-1} – predetermined

for period t W_{t-1} is exog.

$\Rightarrow$ estimate recursively