$$\hat{\beta}_1 = \frac{Cov(X,Y)}{Var(X)} = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})^2} =$$

$$= \frac{\sum (X_i - \bar{X})(\cancel{\beta_0} + \beta_1 \cdot X_i + \varepsilon_i - \cancel{\beta_0} - \beta_1 \bar{X} \cdot \bar{\varepsilon})}{\sum (X_i - \bar{X})^2}$$

$$= \beta_1 \frac{\sum (X_i - \bar{X})^2}{\sum (X_i - \bar{X})^2} + \frac{\sum (X_i - \bar{X})(\varepsilon_i - \bar{\varepsilon})}{\sum (X_i - \bar{X})^2} =$$

$$= \beta_1 + \frac{\sum (X_i - \bar{X})\varepsilon_i}{\sum (X_i - \bar{X})^2} + \frac{\bar{\varepsilon} \sum (X_i - \bar{X})}{\sum (X_i - \bar{X})^2} = 0$$

$$\sum (X_i - \bar{X}) = 0$$

$$\sum X_i = \sum \bar{X}$$

$$\hat{\beta}_1 = \beta_1 + \sum a_i \varepsilon_i$$

$$a_i = \frac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}$$

1) Equation : $y_i = \beta_0 + \beta_1 \cdot x_i + \varepsilon_i$ errors (u_i)

2) Assumptions : $E(\varepsilon_i) = 0$, $E(\varepsilon_i^2) = \sigma^2$, $E(\varepsilon_i \varepsilon_j) = 0$ $i \neq j$

3) Method : OLS $y_i = \hat{\beta}_0 + \hat{\beta}_1 x_i + \hat{\varepsilon}_i$ residuals (e_i)

4) Properties

Assumptions of linear model $\hat{\beta}_{OLS} = (X'X)^{-1}X'y$

Assumptions of GMT

- stochastic regressors

1) $E(\varepsilon_i | X) = 0$ $E(\varepsilon_i) = 0$
2) (x_i, y_i) i.i.d.
3) $E(X_i^4) < \infty$, $E(\varepsilon_i^4) < \infty$
4) $\mathrm{rank}(X) = K$

$\Leftrightarrow$ no perfect m.c.

5) $\mathrm{Var}(\varepsilon_i | X_i) = \sigma_\varepsilon^2$ $\mathrm{Var}(\varepsilon_i) = \sigma_\varepsilon^2$

6) $\varepsilon_i | X_i \sim N(0, \sigma_\varepsilon^2)$ $\varepsilon_i \sim N(0, \sigma_\varepsilon^2)$

GMT:

1) $E(\varepsilon) = 0$

$E(\varepsilon | X)$

2) $E(\varepsilon\varepsilon') = \sigma_\varepsilon^2 \cdot I$

$E(\varepsilon\varepsilon' | X)$

3) $\mathrm{rank}(X) = K$

$\Rightarrow \hat{\beta}_{OLS}$ - BLUE

$$a_i = \frac{x_i - \bar{x}}{\sum (x_i - \bar{x})^2}$$

$$\bar{x} \cdot \sum (x_i - \bar{x}) = 0$$

1) $\sum a_i = 0$

2) $$\sum a_i^2 = \sum \left(\frac{x_i - \bar{x}}{\sum (x_i - x)^2} \right)^2 = \frac{\cancel{\sum (x_i - x)^2}}{\left(\sum (x_i - \bar{x})^2\right)^{\cancel{2}}} = \frac{1}{\sum (x_i - \bar{x})^2}$$

3) $$\sum a_i x_i = \frac{\sum (x_i - \bar{x}) x_i - \overbrace{\sum (x_i - \bar{x})\bar{x}}^{=0}}{\sum (x_i - \bar{x})^2} = 1$$

$$E(\hat{\beta}_1) = \beta_1 + E\left(\sum a_i \varepsilon_i\right) \overset{E(\varepsilon_i) = 0}{=} \beta_1$$

$$E(\hat{\beta}) = \frac{\mathrm{Cov}(X, Ey)}{\mathrm{Var}(X)} = \frac{\mathrm{Cov}(X, \beta_0 + \beta_1 \cdot X + E\varepsilon)}{\mathrm{Var}(X)} =$$

$$\sigma^2_{\hat\beta_1} = \frac{\sigma^2_\varepsilon}{\sum (x_i - \bar{x})^2} \qquad \hat\beta_1 = \beta_1 + \sum a_i \varepsilon_i$$

$$\sigma^2_{\hat\beta_1} = E\left[\left(\hat\beta_1 - E(\hat\beta_1)\right)^2 \right] = E\left((\hat\beta_1 - \beta_1)^2 \right) =$$

$E(\varepsilon_i \varepsilon_j) = 0$

$$= E\left(\left(\sum a_i \varepsilon_i\right)^2 \right) = E\left(\sum_{i=1}^{n} a_i^2 \varepsilon_i^2 + \sum_{i=1}^{n} \sum_{j \neq i} a_i a_j \varepsilon_i \varepsilon_j \right) =$$

$$= \sum a_i^2 \cdot \sigma^2_\varepsilon + 0 = \frac{\sigma^2_\varepsilon}{\sum (x_i - \bar{x})^2}$$

$$\sigma^2_{\hat\beta_0} = \sigma^2_\varepsilon \cdot \left(\frac{1}{n} + \frac{\bar{x}}{\sum (x_i - \bar{x})^2} \right)$$

$$\hat\beta_0 = \bar{y} - \hat\beta_1 \bar{x} = \beta_0 + \beta_1 \bar{x} + \bar{\varepsilon} - \hat\beta_1 \bar{x} =$$

$$= \beta_0 + \bar{x}(\beta_1 - \hat\beta_1) + \bar{\varepsilon} =$$

$$= \beta_0 - \bar{x} \sum a_i \varepsilon_i + \frac{1}{n} \cdot \sum \varepsilon_i = \qquad \underbrace{c_i = \frac{1}{n} - a_i \bar{x}}$$

$$= \beta_0 + \sum c_i \varepsilon_i$$

$$\sigma^2_{\hat\beta_0} = E\left(\left(\hat\beta_0 - \underset{\substack{\| \\ \beta_0}}{E(\hat\beta_0)}\right)^2 \right) = E\left(\left(\sum c_i \varepsilon_i\right)^2 \right) =$$

$$= \sigma^2_\varepsilon \cdot \sum c_i^2 = \sigma^2_\varepsilon \sum \left(\frac{1}{n} - a_i \bar{x} \right)^2 =$$

$$= \sigma_\varepsilon^2 \left(\frac{1}{n} - 2 \cdot \frac{\bar{X}}{n} \underbrace{\sum a_i}_{=0} + \bar{X}^2 \cdot \sum a_i^2 \right) =$$

$$= \sigma_\varepsilon^2 \left(\frac{1}{n} + \frac{\bar{X}^2}{\sum (x_i - \bar{X})^2} \right)$$

$$Var(\hat{\beta}_j) = \frac{\sigma_\varepsilon^2}{TSS_j \cdot (1 - R_j^2)}$$

$Var(\hat{\beta}_j) \nearrow$

1) the level noise in the data $\uparrow$

2) regressor close to a const

3) $R_j^2 \approx 1$ multicollinearity

$$TSS = \sum_{i=1}^{n} (x_{ij} - \bar{x}_j)^2$$

R_j^2 - coef. of determination

$X_j \,/\, X_{-j}$

$X_j \,|\, X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_k$

$$Var(\hat{\beta}) = \sigma^2 (X'X)^{-1}$$

$$(X'X) = \begin{bmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix}$$

$$\Delta = n \Sigma x_i^2 - (\Sigma x_i)^2 = n \Sigma x_i^2 - n^2 \bar{x}^2$$

$$\sigma^2 \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \begin{bmatrix} Var(\hat{\beta}_0) & Cov(\hat{\beta}_0, \hat{\beta}_1) \\ Cov(\hat{\beta}_0, \hat{\beta}_1) & Var(\hat{\beta}_1) \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\sigma_\varepsilon^2 \Sigma x_i^2}{\Sigma (x_i - \bar{x})^2} & \frac{-\bar{x} \sigma_\varepsilon^2}{\Sigma (x_i - \bar{x})^2} \\ \frac{-\bar{x} \sigma_\varepsilon^2}{\Sigma (x_i - \bar{x})^2} & \frac{\sigma_\varepsilon^2}{\Sigma (x_i - \bar{x})^2} \end{bmatrix}$$

$N = 37$

$Y_i = A \cdot K_i^{\alpha} \cdot L_i^{\beta} \varepsilon_i$

$\log Y_i = \underbrace{\log A}_{\beta_0} + \underbrace{\alpha}_{\beta_1} \log K_i + \underbrace{\beta}_{\beta_2} \log L + \varepsilon_i$

$cloth_i$ - log exp. on cloth.

yd_i - log. income

pc_i - log price (all) log - log

ps_i - log price (shoes)

$x_i \uparrow 1\%$ $y_i \uparrow \hat{\beta}_1 \%$

$N = 37$

$$cloth_i = -3{,}26 + \underbrace{1{,}02}_{\hat{\beta}_1} yd_i - \underbrace{0{,}24}_{\hat{\beta}_2} pc_i - \underbrace{0{,}43}_{\hat{\beta}_3} ps_i + \hat{\varepsilon}_i$$

$$(1{,}53) \quad (0{,}12) \quad (0{,}13) \quad (0{,}19)$$

$R^2 = 0{,}992$

a) $H_0: \beta_1 = 1$

$$t = \frac{0{,}02}{0{,}12} = \frac{1}{6} \approx 0{,}177\ldots \qquad t_{0{,}975,\,33} \approx 2{,}035$$

b) 95%- conf. int. for pc_i

$$\{-0{,}24 \pm 0{,}13 \cdot 2{,}035\}$$

$$-0{,}509 < \beta_2 < 0{,}03$$