A COLLECTION OF PROBLEMS ON THE EQUATIONS OF MATHEMATICAL PHYSICS

Bitsadze and D.F. Kalinichenko

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PREFACE

The present book is a collection of problems on the equations of mathematical physics studied in colleges with comprehensive mathematical programme. It consists of two parts, the first of which contains the conditions of the problems and the second the answers to the problems and detailed explanations of the solutions of the most difficult problems. The material of the first part is divided into five chapters in which the problems are grouped according to the type of the partial differential equations. Much emphasis is placed on practical methods most frequently used in the solution of partial differential equations. Each chapter begins with the necessary prerequisites taken from the corresponding division of the theory of the equations of mathematical physics which facilitates the understanding of the subject.

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§ 1. Partial Differential Equation and Its Solution. Systems of Partial Differential Equations

We shall denote by D a domain in the n-dimensional Euclidean space E_n of points $x = (x_1, x_2, \ldots, x_n)$ with orthogonal Cartesian coordinates $x_1, x_2, \ldots, x_n$ $(n \geqslant 2)$.

Let us denote $F \equiv F(x, \ldots, p_{i_1 \ldots i_n}, \ldots)$ a given real function dependent on the points x of the domain D and on real variables $p_{i_1 \ldots i_n}$ where $i_1, \ldots, i_n$ $\left(\sum\limits_{j=1}^{n} i_j = k,\ k = 0, \ldots, m,\ m \geqslant 1\right)$ are nonnegative integral indices, and let at least one of the partial derivatives

$$\frac{\partial F}{\partial p_{i_1 \ldots i_n}}, \quad \sum_{j=1}^{n} i_j = m$$

of the function F be different from zero.

An equation of the form

$$F\left(x, \ldots, \frac{\partial^k u}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}, \ldots\right) = 0, \quad x \in D \qquad (1)$$

is called a *partial differential equation of the mth order* with respect to the unknown function $u(x) = u(x_1, \ldots, x_n)$. The expression F on the left-hand side of equality (1), symbolizing certain operations on the function u, is called a *partial differential operator of the mth order.*

Every real function $u(x)$ defined in the domain D and continuous together with its partial derivatives involved in equation (1) which turns the equation into an identity is called a *regular solution* of partial differential equation (1).

Besides regular solutions, an important role is also played in the theory of partial differential equations by the so-called *fundamental* (*elementary*) solutions which are not everywhere regular.

In the case when F is an N-dimensional vector $F = (F_1, \ldots, F_N)$ with components $F_i(x, \ldots, p_{i_1 \ldots i_n}, \ldots)$ $(i = 1, \ldots, N)$ dependent on $x \in D$ and on the M-dimensional vectors $p_{i_1 \ldots i_n} = (p^1_{i_1 \ldots i_n}, \ldots, p^M_{i_1 \ldots i_n})$ the vector equality (1) is called a *system of partial differential equations* with respect to the unknown functions $u_1, \ldots, u_M$ (or, which is the same, with respect to the unknown vector $u = (u_1, \ldots, u_M)$).

An equation of form (1) is said to be *linear* when F is a linear function of all the derivatives $\frac{\partial^k u}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}$, $0 \leqslant k \leqslant m$.

A linear partial differential equation can be written in the form

$$Lu = f(x), \quad x \in D$$

where L is a *linear partial differential operator* involving only the first powers of the derivatives $\frac{\partial^k u}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}$, $0 \leqslant k \leqslant m$. A linear partial differential equation may be called *homogeneous* or *non-homogeneous* depending on whether $f(x) \equiv 0$ or $f(x) \not\equiv 0$.

Equation (1) is said to be *quasi-linear* when F depends linearly only on the highest derivatives

$$\frac{\partial^m u}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}, \quad \sum_{j=1}^{n} i_j = m.$$

Find whether the equalities below are partial differential equations:

1. $\cos(u_x + u_y) - \cos u_x \cos u_y + \sin u_x \sin u_y = 0.$

2. $u_{xx}^2 + u_{yy}^2 - (u_{xx} - u_{yy})^2 = 0.$
3. $\sin^2(u_{xx} + u_{xy}) + \cos^2(u_{xx} + u_{xy}) - u = 1.$
4. $\sin(u_{xy} + u_x) - \sin u_{xy} \cos u_x - \cos u_{xy} \sin u_x + 2u = 0.$
5. $\frac{\partial}{\partial x} \tan u - u_x \sec^2 u - 3u + 2 = 0.$
6. $\ln|u_x u_y| - \ln|u_x| - \ln|u_y| + 5u - 6 = 0.$

Determine the orders of the following partial differential equations:

7. $\ln|u_{xx}u_{yy}| - \ln|u_{xx}| - \ln|u_{yy}| + u_x + u_y = 0.$
8. $u_x u_{xy}^2 + (u_{xx}^2 - 2u_{xy}^2 + u_y)^2 - 2_{xy} = 0.$
9. $\cos^2 u_{xy} + \sin^2 u_{xy} - 2u_x^2 - 3u_y + u = 0.$
10. $2(u_x - 2u)u_{xy} - \frac{\partial}{\partial y}(u_x - 2u)^2 - xy = 0.$
11. $\frac{\partial}{\partial x}(u_{yy}^2 - u_y) - 2u_{yy}\frac{\partial}{\partial y}(u_{xy} - u_x) - 2u_x + 2 = 0.$
12. $2u_{xx}u_{xxy} - \frac{\partial}{\partial y}(u_{xx} - u_y)^2 - 2u_y u_{xxy} + u_x = 0.$

Find which of the following partial differential equations are linear (homogeneous or non-homogeneous) and which are non-linear (quasi-linear):

13. $u_x u_{xy}^2 + 2xuu_{yy} - 3xyu_y - u = 0.$
14. $u_y u_{xx} - 3x^2 uu_{xy} + 2u_x - f(x, y)u = 0.$
15. $2\sin(x + y)u_{xx} - x\cos yu_{xy} + xyu_x - 3u + 1 = 0.$
16. $x^2yu_{xxy} + 2e^x y^2 u_{xy} - (x^2y^2 + 1)u_{xx} - 2u = 0.$
17. $3u_{xy} - 6u_{xx} + 7u_y - u_x + 8x = 0.$
18. $u_{xy}u_{xx} - 3u_{yy} - 6xu_y + xyu = 0.$
19. $a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + h(x, y) = 0.$
20. $a(x, y, u_x, u_{xy})u_{xyy} + b(x, y, u_{yy})u_{yyy} + 2uu_{xy}^2 - f(x, y) = 0.$
21. $u_{xy} + u_y + u^2 - xy = 0.$
22. $u_{xy} + 2\frac{\partial}{\partial x}(u_x^2 + u) - 6x\sin y = 0.$
23. $2xu_{xy} - 6\frac{\partial}{\partial x}(u^2 - xy) + u_{yy} = 0.$
24. $\frac{\partial}{\partial y}(yu_y + u_x^2) - 2u_x u_{xy} + u_x - 6u = 0.$

§ 2. Classification of Partial Differential Equations and Systems of Partial Differential Equations

A form

$$K(\lambda_1, \ldots, \lambda_n) \equiv \sum_{i_1 \ldots i_n} \frac{\partial F}{\partial p_{i_1 \ldots i_n}} \lambda_1^{i_1} \ldots \lambda_n^{i_n}, \quad \sum_{j=1}^{n} i_j = m \quad (2)$$

of the mth order with respect to the real parameters $\lambda_1, \ldots, \lambda_n$ is called the *characteristic form* corresponding to partial differential equation (1).

In the case of a linear partial differential equation of the second order

$$\sum_{i, j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} B_i \frac{\partial u}{\partial x_i} + Cu = f \quad (3)$$

characteristic form (2) is a *quadratic form*:

$$Q(\lambda_1, \ldots, \lambda_n) = \sum_{i, j=1}^{n} A_{ij}(x) \lambda_i \lambda_j$$

For every fixed point $x \in D$ the quadratic form Q can be reduced to its *standard form*

$$Q = \sum_{i=1}^{n} \alpha_i \xi_i^2 \quad (4)$$

by means of a *non-singular* (*non-degenerate*) *affine transformation* $\lambda_i = \lambda_i(\xi_1, \ldots, \xi_n)$ $(i = 1, \ldots, n)$ of the variables where the coefficients α_i assume the values 1, -1 and 0. As is known, the numbers of the negative and of the zero coefficients of the form Q in expression (4) are independent of the way in which the form has been brought to the standard form, and this underlies the classification of linear equations (3).

A linear equation (3) is said to be *elliptic*, *hyperbolic* or *parabolic in the domain D* if for each point $x \in D$ the coefficients α_i of form (4) are all different from zero and have one sign, are all different from zero and are not all of one sign or at least one of them (but not all) is equal to zero, respectively.

An equation (3) which is elliptic in the domain D (or, as we say, *is of elliptic type* in the domain D) is said to be *uniformly elliptic* in that domain if there exist real numbers $k_0 \neq 0$ and $k_1 \neq 0$ of one sign such that

$$k_0 \sum_{i=1}^{n} \lambda_i^2 \leqslant Q(\lambda_1, \ldots, \lambda_n) \leqslant k_1 \sum_{i=1}^{n} \lambda_i^2$$

for all $x \in D$.

In the case of a linear partial differential equation of the mth order of the form

$$\sum_{i_1 \ldots i_n} a_{i_1 \ldots i_n}(x) \frac{\partial^m u}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}} + L_1 u = f(x), \qquad \sum_{j=1}^{n} i_j = m \quad (5)$$

where L_1 is a linear partial differential operator whose order is less than m characteristic form (2) is written as

$$K(\lambda_1, \ldots, \lambda_n) = \sum_{i_1 \ldots i_n} a_{i_1 \ldots i_n}(x) \lambda_1^{i_1} \ldots \lambda_n^{i_n}, \quad \sum_{j=1}^{n} i_j = m \quad (6)$$

If for a fixed point $x \in D$ there is an affine transformation $\lambda_i = \lambda_i(\mu_1, \ldots, \mu_n)$ $(i = 1, \ldots, n)$ which brings expression (6) to a form containing only l $(0 < l < n)$ variables μ_l we speak of the *parabolic degeneration* of equation (5) *at the point* $x \in D$.

If there is no parabolic degeneration and if the equation

$$K(\lambda_1, \ldots, \lambda_n) = 0 \quad (7)$$

possesses no real solutions other than $\lambda_1 = 0, \ldots, \lambda_n = 0$, partial differential equation (5) is said to be *elliptic at the point* $x \in D$.

A partial differential equation of form (5) is said to be *hyperbolic at a point* $x \in D$ if in the n-dimensional space of the variables $\lambda_1, \ldots, \lambda_n$ there is a straight line possessing the following property: if one of the new variables $\mu_1, \ldots, \mu_n$ obtained from $\lambda_1, \ldots, \lambda_n$ by means of an affine transformation is reckoned along that straight line then the transformed equation resulting from (7), when resolved with respect to that variable reckoned along the straight line, has exactly m real roots (simple or multiple) for any choice of the values of the other variables.

The non-linear partial differential equations of the mth order are classified in an analogous manner depending on the character of form (2). However, for a non-linear equation the coefficients of form (2) depend not only on the point $x \in D$ but also on the sought-for solution and its derivatives, and therefore in this case the classification by type can only be carried out for a given particular solution.

In case equality (1) is a system of N partial differential equations with respect to N unknown functions (that is in case $M = N$ and the order of each of the equations is equal to m) we consider the square matrices

$$\left\| \frac{\partial F_i}{\partial p^j_{i_1 \ldots i_n}} \right\|; \quad i, j = 1, \ldots, N; \quad \sum_{k=1}^{n} i_k = m$$

and construct, with the aid of these matrices, the form

$$K(\lambda_1, \ldots, \lambda_n) \equiv$$
$$\equiv \det \sum_{i_1 \ldots i_n} \left\| \frac{\partial F_i}{\partial p^j_{i_1 \ldots i_n}} \right\| \lambda_1^{i_1} \ldots \lambda_n^{i_n}, \quad \sum_{k=1}^{n} i_k = m \quad (8)$$

of the Nmth order involving the n real scalar parameters $\lambda_1, \ldots, \lambda_n$. The classification of system (1) by type is carried out depending on the character of form (8) by a complete analogy with the case of one partial differential equation of the mth order considered above.

Determine the type of each of the following partial differential equations:

25. $u_{xx} + 4u_{xy} + u_{yy} + u_x + u_y + 2u - x^2y = 0.$

26. $2u_{xx} + 2u_{xy} + u_{yy} + 2u_x + 2u_y - u = 0.$

27. $u_{xx} + 2u_{xy} + u_{yy} + u_x + u_y + 3u - xy^2 = 0.$

28. $4u_{xx} + 2u_{yy} - 6u_{zz} + 6u_{xy} + 10u_{xz} + 4u_{yz} + 2u = 0.$

29. $2u_{xy} - 2u_{xz} + 2u_{yz} + 3u_x - u = 0.$

30. $u_{xx} + 2u_{xy} + 2u_{yy} + 4u_{yz} + 5u_{zz} - xu_x + yu_z = 0.$

31. $u_{xx} - 4u_{xy} + 2u_{xz} + 4u_{yy} + u_{zz} - 2xyu_x + 3xu = 0.$

32. $u_{xy} + u_{yz} + u_{xz} - 3x^2u_y + y \sin xu + xe^{-y} = 0.$

33. $5u_{xx} + u_{yy} + 5u_{zz} + 4u_{xy} - 8u_{xz} - 4u_{yz} - u + yz^2 \sin x = 0.$

34. $u_{xx} + 2u_{xy} + 2u_{yy} - 2u_{yz} + 3u_z - u = 0.$
35. $3u_{xx} + 4u_{yy} + 5u_{zz} + 4u_{xy} - 4u_{yz} + 2u_x - u_y +$
$+ xye^z = 0.$ 3
36. $y^{2m+1}u_{xx} + u_{yy} - u_x = 0$ where m is a nonnegative integer.
37. $xu_{xx} + yu_{yy} - u = 0.$

For the given solutions $u\ (x,\ y)$ determine the type of the following partial differential equations:

38. $u_{xx}^2 + (u_{xx} - 2)\ u_{xy} - u_{yy}^2 = 0,\ u = x^2 + y^2.$
39. $u_{xy}^2 + u_{xx}u_{yy} + u_{yy}^2 = 8,\ u = x^2 + y^2$ and $u = 2\sqrt{2}xy.$
40. $u_{xx}^2 - 4u_{xy} + u_{yy}^2 = 0,\quad u = (x+y)^2,\ u = x$ and $u = x^2 + \frac{y^2}{4} + \frac{17}{16}xy.$
41. $u_{xx} + u_{xy}u_{yy} + u_{yy}^2 - 4u_{yy} = 0,\quad u = 2y^2,\quad u = 5xy,$ and $u = x.$
42. $3u_{xx}^3 - 6u_{xy} + u_{yy} - 4 = 0,\quad u = \frac{1}{2}(x^2 + y^2)$ and $u = 2y^2.$
43. $u_{xx}^2u_{xy} - 5u_{yy} + u_x - 2\ (x + y) - 8 = 0,$ $u = x^2 + 2xy.$
44. $u_{xx}^4 + 2u_{xy}^2 - 3u_{yy} + u_y - 2x = 0,\ \ u = 2xy - 8y.$
45. $2u_{xx}^3 + 2u_{xy}^5 + 3u_{yy} - 2u_y + 2x = 0,\ u = xy - \frac{1}{2}x^2.$
46. $5u_{xx}^5 - 7u_{xy} + 25u_{yy} - 150y = 0,\quad u = \frac{x^2}{2} + y^3 + \frac{5}{7}xy.$
47. $u_{xx}^2 + 5u_{xy}^2 + 6u_{yy}^2 = 12,\quad u = \frac{1}{2}(x+y)^2,\quad u = \sqrt{3}\,x^2.$
48. $u_{xx}^3 - 4u_{xy}^2 + 7u_{yy} - 4u_x + u_y + 3x + 4y + 3 = 0,$ $u = \frac{1}{2}x^2 + xy.$
49. $u_{xx}^2 - 2u_{xy}^2 + u_{yy}^2 + 2u_x - 2\,(x+y) = 0,\quad u = \frac{1}{2}(x+y)^2.$
50. $u_{xy}^2 + u_{xx}u_{yy} + u_{yy}^2 + 2u_{xx} + 2u_{yy} = 0,$ $u = x^2 - y^2$ and $u = x.$
51. Let the function F in the equation

$$F\ (x,\ y,\ u,\ u_x,\ u_y,\ u_{xx},\ u_{xy},\ u_{yy}) = 0$$

be continuously differentiable with respect to the last three variables and let at least one of the derivatives of F with respect to these three variables be different from zero. Find the conditions for the ellipticity, parabolicity and hyperbolicity of this partial differential equation.

Determine the type of the following systems of partial differential equations:

52. $2u_x + 3u_y - 3v_y + u = 0,$
$-u_x + u_y + v_x + xy = 0.$

53. $2u_x + 3v_y + 3u_y - 6u = 0,$
$u_x + u_y + v_x + x^2u = 0.$

54. $2u_x + 3v_y + 3u_y - 2u = 0,$
$u_x + v_x - u + xy^2 = 0.$

55. $2u_x - 4v_x + 3u_y + 8v_y - u = 0,$
$3u_x - 2v_x + 6u_y + 3v_y + 2u = 0.$

56. $2u_x + v_x + 12u_y - 2u = 0,$
$v_x + 4u_y + v_y + xy = 0.$

57. $2u_x + v_x + 7u_y - 2u = 0,$
$3u_x + 3v_x + 31u_y + v_y - e^y \sin x = 0.$

58. $5u_x + 22.5v_x + 2u_y + v_y - 6u = 0,$
$5v_x + 2u_y + 3v_y - 2xu = 0.$

59. $v_x + 12u_y + v_y + 3u - 32xe^y = 0,$
$-5u_x + \frac{5}{6}v_x + u_y + v_y - e^x u = 0.$

60. $15u_x + 9v_x + 12u_y + 17v_y - 3x \cos y = 0,$
$3u_x + 2v_x + v_y - 6u = 0.$

61. $3u_x + 3v_x + 3u_y + 4v_y = 0,$
$2u_x + 3v_x - v_y - 3u = 0.$

62. $u_x - v_y + 2u_z - 3v_z - u = 0,$
$u_y + 2v_x - 2u_z + v_y + 2u = 0.$

63. $u_x - u_y + 2v_y - 3v_z + 2u = 0,$
$u_x + 2u_z - v_x + v_z - u = 0.$

64. $u_x + u_y + v_y + v_z - xyu = 0,$
$v_x - u_y - v_y + u_z + 2u = 0.$

Depending on the values of the parameter k, determine the type of the following systems of partial differential equations:

65. $u_x - kv_y = 0,$
$u_y + v_x = 0.$

66. $u_y - kv_x + v_y = 0,$
$u_x + kv_y - u = 0.$

67. $u_y - kv_x + kv_y = 0,$
$u_x + v_y + 2v = 0.$

§ 3. Reduction to Normal Form of Linear Partial Differential Equations of the Second Order in Two Independent Variables

The general form of a linear partial differential equation of the second order in two independent variables is

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu + g = 0 \qquad (9)$$

where a, b, c, d, e, f, and g are given functions of the independent variables x and y.

Let us denote by Δ the discriminant $b^2 - ac$ of the quadratic form

$$Q(\lambda_1, \lambda_2) = a\lambda_1^2 + 2b\lambda_1\lambda_2 + c\lambda_2^2 \qquad (10)$$

corresponding to equation (9).

The curves determined by the equations of the form $\Omega(x, y) = \text{const}$ where Ω is an arbitrary solution of the non-linear partial differential equation of the first order

$$a\Omega_x^2 + 2b\Omega_x\Omega_y + c\Omega_y^2 = 0$$

are called *characteristic curves* (or, simply, *characteristics*) of equation (9). The components of the tangent vector (dx, dy) to a characteristic curve satisfy the equality

$$a\,dy^2 - 2b\,dy\,dx + c\,dx^2 = 0 \qquad (11)$$

for each point (x, y) of the curve.

According to the classification of partial differential equations discussed in § 2, equation (9) is elliptic, hyperbolic or parabolic depending on whether form (10) (or form (11)) is definite, indefinite or semi-definite, that is on whether the discriminant $\Delta = b^2 - ac$ of the form is negative, positive or equal to zero respectively.

In the elliptic case equation (9) can be reduced to the *normal form*

$$v_{\xi\xi} + v_{\eta\eta} + d_1v_\xi + e_1v_\eta + f_1v + g_1 = 0 \qquad (12)$$

with the aid of a transformation of independent variables of the form

$$\xi = \varphi(x, y), \quad \eta = \psi(x, y) \qquad (13)$$

where $\varphi(x, y)$ and $\psi(x, y)$ are solutions of the system of linear partial differential equations of the first order

$$a\varphi_x + b\varphi_y + \sqrt{-\Delta}\,\psi_y = 0, \quad a\psi_x + b\psi_y - \sqrt{-\Delta}\,\varphi_y = 0$$

for which the Jacobian $\frac{\partial(\varphi, \psi)}{\partial(x, y)}$ is different from zero.

In the hyperbolic case transformation (13) with functions $\varphi(x, y)$ and $\psi(x, y)$ satisfying the partial differential equations

$$a\varphi_x + (b + \sqrt{\Delta})\,\varphi_y = 0, \quad a\psi_x + (b - \sqrt{\Delta})\,\psi_y = 0$$

brings equation (9) to the form

$$v_{\xi\eta} + d_1 v_\xi + e_1 v_\eta + f_1 v + g_1 = 0 \tag{14}$$

The new change of variables $\xi = \alpha + \beta$, $\eta = \alpha - \beta$ reduces equation (14) to the normal form

$$w_{\alpha\alpha} - w_{\beta\beta} + d_2 w_\alpha + e_2 w_\beta + f_2 w + g_2 = 0 \tag{15}$$

Finally, in the case when equation (9) is parabolic a transformation of variables of form (13) where $\varphi(x, y)$ is a non-constant solution of the equation

$$a\varphi_x + b\varphi_y = 0$$

and $\psi(x, y)$ is an arbitrary smooth function satisfying the condition

$$a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2 \neq 0$$

reduces the equation to the normal form

$$v_{\eta\eta} + d_1 v_\xi + e_1 v_\eta + f_1 v + g_1 = 0 \tag{16}$$

In equations (12), (14) and (16) we have $v(\xi, \eta) = u[x(\xi, \eta), y(\xi, \eta)]$ where $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$ are solutions of system (13). The condition

$$\frac{\partial(\varphi, \psi)}{\partial(x, y)} \neq 0$$

guarantees that system (13) is solvable at least "in the small".

As is known from the theory of linear partial differential equations of the first order, in the case when $\Delta > 0$ we can take as functions $\varphi(x, y)$ and $\psi(x, y)$ determining transformation (13) the left-hand sides of general integrals $\varphi(x, y) = \text{const}$ and $\psi(x, y) = \text{const}$ of the ordinary differential equations

$$\frac{dx}{a} = \frac{dy}{b + \sqrt{\Delta}} \quad \text{and} \quad \frac{dx}{a} = \frac{dy}{b - \sqrt{\Delta}}$$

respectively; in the case when $\Delta = 0$ we can take as a function $\varphi(x, y)$ the left-hand side of a general integral $\varphi(x, y) = \text{const}$ of the differential equation

$$\frac{dx}{a} = \frac{dy}{b}$$

If $\Delta < 0$, then the function

$$\Omega(x, y) = \varphi(x, y) + i\psi(x, y)$$

is a solution of the differential equation

$$a\Omega_x + (b + i\sqrt{-\Delta})\,\Omega_y = 0$$

and therefore in this case as well transformation (13) can easily be found in an analogous manner.

A quasi-linear partial differential equation of the form

$$au_{xx} + 2bu_{xy} + cu_{yy} + F(x, y, u, u_x, u_y) = 0$$

whose coefficients a, b and c are given functions dependent solely on the arguments x and y can also be reduced to the normal form using the same scheme.

Since the functions $\varphi(x, y)$ and $\psi(x, y)$ in transformation (13) are solutions of linear partial differential equations of the first order whose coefficients are expressed in terms of the functions $a(x, y)$, $b(x, y)$ and $c(x, y)$ these functions should satisfy the condition requiring that they should not turn into zero simultaneously and should also possess some differentiability properties.

It should be noted that if the coefficients of equation (9) are constant then, after the equation has been reduced to one of the normal forms (12), (15), (16), some further simplifications of the equation can be performed. For instance, by introducing a new unknown function $w(\xi, \eta)$ specified by the formula

$$v(\xi, \eta) = e^{\lambda\xi + \mu\eta} w(\xi, \eta)$$

and choosing in a proper manner the constants λ and μ, we can make the coefficients in the first derivatives of the function w turn into zero in the elliptic and hyperbolic cases. In the parabolic case we can make one of the coefficients in the first derivatives of w and the coefficient in the function w itself turn into zero.

Reduce to normal form the following partial differential equations in each of the domains where the type of the equations is preserved:

68. $u_{xx} + 2u_{xy} + 5u_{yy} - 32u = 0.$

69. $u_{xx} - 2u_{xy} + u_{yy} + 9u_x + 9u_y - 9u = 0.$

70. $2u_{xx} + 3u_{xy} + u_{yy} + 7u_x + 4u_y - 2u = 0.$

71. $u_{xx} + u_{xy} - 2u_{yy} - 3u_x - 15u_y + 27x = 0.$

72. $9u_{xx} - 6u_{xy} + u_{yy} + 10u_x - 15u_y - 50u + x - 2y = 0.$

73. $u_{xx} + 4u_{xy} + 10u_{yy} - 24u_x + 42u_y + 2(x + y) = 0.$

74. $u_{xx} + 4u_{xy} + 13u_{yy} + 3u_x + 24u_y - 9u + 9(x + y) = 0.$

75. $(1 + x^2)^2 u_{xx} + u_{yy} + 2x(1 + x^2) u_x = 0.$

76. $y^2u_{xx} + 2xyu_{xy} + x^2u_{yy} = 0.$

77. $u_{xx} - (1 + y^2)^2 u_{yy} - 2y(1 + y^2) u_y = 0.$

78. $(1 + x^2) u_{xx} + (1 + y^2) u_{yy} + xu_x + yu_y - 2u = 0.$

79. $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} - 2yu_x + ye^{y/x} = 0.$

80. $xy^2u_{xx} - 2x^2yu_{xy} + x^3u_{yy} - y^2u_x = 0.$

81. $u_{xx} - 2\sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0.$

82. $e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} - xu = 0.$

83. $u_{xx} - 2xu_{xy} = 0.$

84. $xu_{xx} + 2xu_{xy} + (x - 1) u_{yy} = 0.$

85. $yu_{xx} + u_{yy} = 0.$

86. $xu_{xx} + yu_{yy} + 2u_x + 2u_y = 0.$

87. $u_{xx} + 2\sin x u_{xy} - (\cos^2 x - \sin^2 x) u_{yy} + \cos x u_y = 0.$

88. $u_{xx} + xyu_{yy} = 0.$

Reduce the following partial differential equations to normal form and perform further possible simplification of the equations:

89. $u_{xx} - 4u_{xy} + 5u_{yy} - 3u_x + u_y + u = 0.$

90. $u_{xx} - 6u_{xy} + 9u_{yy} - u_x + 2u_y = 0.$

91. $2u_{xy} - 4u_{yy} + u_x - 2u_y + u + x = 0.$
92. $u_{xy} + 2u_{yy} - u_x + 4u_y + u = 0.$
93. $2u_{xx} + 2u_{xy} + u_{yy} + 4u_x + 4u_y + u = 0.$
94. $u_{xx} + 2u_{xy} + u_{yy} + 3u_x - 5u_y + 4u = 0.$
95. $u_{xx} - u_{yy} + u_x + u_y - 4u = 0.$
96. $u_{xy} + u_{xx} - u_y - 10u + 4x = 0.$
97. $3u_{xx} + u_{xy} + 3u_x + u_y - u + y = 0.$
98. $u_{xx} + 4u_{xy} + 5u_{yy} - 2u_x - 2u_y + u = 0.$
99. $5u_{xx} + 16u_{xy} + 16u_{yy} + 24u_x + 32u_y + 64u = 0.$
100. $u_{xx} - 2u_{xy} + u_{yy} - 3u_x + 12u_y + 27u = 0.$

Reduce to normal form the following partial differential equations:

101. $u_{xx} + 2u_{xy} + 2u_{yy} + 4u_{yz} + 5u_{zz} = 0.$
102. $u_{xx} - 4u_{xy} + 2u_{xz} + 4u_{yy} + u_{zz} + 3u_x = 0.$
103. $u_{xy} + u_{xz} + u_{yz} - u_x + u_y = 0.$
104. $3u_{xy} - 2u_{xz} - u_{yz} - u = 0.$
105. $u_{xx} + 3u_{yy} + 3u_{zz} - 2u_{xy} - 2u_{xz} - 2u_{yz} - 8u = 0.$
106. $u_{xx} + u_{yy} + u_{zz} + 6u_{xy} + 2u_{xz} + 2u_{yz} + 2u_x + 2u_y + 2u_z + 4u = 0.$
107. $2u_{xx} + 5u_{yy} + 2u_{zz} - 6u_{xy} - 4u_{xz} + 6u_{yz} - 3u + y - 2z = 0.$
108. $3u_{yy} - 2u_{xy} - 2u_{yz} + 4u = 0.$
109. $u_{xx} + 4u_{yy} + u_{zz} + 4u_{xy} + 2u_{xz} + 4u_{yz} + 2u = 0.$
110. $u_{xx} + 4u_{yy} + 9u_{zz} + 4u_{xy} + 6u_{xz} + 12u_{yz} - 2u_x - 4u_y - 6u_z = 0.$

§ 4. Mathematical Models for Some Phenomena Studied in Mathematical Physics

There are many cases when the investigation of various natural phenomena can be reduced to the determination of solutions of partial differential equations referred to as *equations of mathematical physics.* To make use of the methods of mathematical physics one should first of all establish what quantities determine the character of the phenomenon in question. Then, taking into account the physical laws (principles) expressing the relationship between these quantities, one should derive a partial differential equation (or a system of equations) and state some additional conditions (most often *boundary* and *initial* conditions) from which the unknown quantities completely characterizing the phenomenon can be uniquely determined. It is important to take into account that a given problem of mathematical physics can serve as a model for various completely different natural phenomena.

Some problems leading to partial differential equations of hyperbolic type. Hyperbolic equations (or systems of equations) result from mathematical modelling of oscillation processes. When equations of oscillation of mechanical systems are derived the well-known *variational principle of stationary action* (also known as *Hamilton's principle of least action*) proves to be very useful. As an example, let us consider the process of *plane transverse oscillation of a string*; the mathematical description of this process and the derivation of the partial differential equation of oscillation of a string is based on Hamilton's principle.

By a *string* is meant a flexible elastic thread (in other words, a one-dimensional elastic medium). It is supposed that in the state of rest the string is taut (along the x-axis) and that the potential energy gained in the oscillation process by an infinitesimal element of the string is directly proportional to the increment of the length of the element. The corresponding proportionality coefficient is spoken of as the *tension* of the string. The basic quantity characterizing the oscillation of the string is the deviation $u = u(x, t)$ of the string in the xu-plane at the point x at time t from the equilibrium position. On denoting by K the kinetic energy of the string and by U its potential energy which are expressed in terms of the function $u(x, t)$ and its derivatives we can form the integral

$$\int_{t_1}^{t_2} (K-U)\,dt \tag{17}$$

called the (*Hamiltonian*) *action*. This integral taken over the interval $t_1 \leqslant t \leqslant t_2$ during which the oscillation phenomenon is observed must assume a stationary value, that is there must exist a definite function $u(x, t)$ for which the first variation of functional (17) turns into zero. Euler's equation for this functional is nothing other than the sought-for partial differential equation which is called the *equation of oscillation of a string*. When computing the variation of functional (17) we impose certain conditions on the function $u(x, t)$ which must be fulfilled at the end points of the string; these additional conditions are the *boundary conditions* characterizing the state of the end points of the string (in particular, the ways of fixing the end points) in the oscillation process.

Physical considerations imply that in order to describe an oscillation process uniquely, we must also know, besides the differential equations and boundary conditions, the initial position of the string (that is the shape of the string at the initial instant) and the initial velocity of the string. The equation of oscillation of the string can be considerably simplified in the case when the amplitude of the oscillation is small, that is when it is allowable to neglect the powers of u_x exceeding the second contained in the expression of the potential energy U. We also note that the same equation can serve as a mathematical model for the description of some other phenomena, for instance, the oscillation of a gas in a tube, electric oscillation in wires etc. Following this scheme and using Hamilton's principle we can state mathematically the problems on longitudinal oscillation of a bar, transverse oscillation of a membrane or a plate and the like.

111. A string $(0 \leqslant x \leqslant l)$ with line density $\rho = \rho(x)$ oscillates in the xu-plane, the oscillation being described by a function $u = u(x, t)$. Find the kinetic energy K of the string for the following cases:

(a) the string has no localized masses;

(b) there are localized masses m_i $(i = 1, \ldots, n)$ at points x_i.

112. For the following cases find the potential energy of a string $(0 \leqslant x \leqslant l)$ whose transverse oscillation in the xu-plane is described by a function $u(x, t)$:

(a) the ends of the string are rigidly fixed;

(b) the ends of the string are rigidly fixed and the powers of u_x higher than the second can be dropped;

(c) some forces $\nu_1(t)$ and $\nu_2(t)$ directed perpendicularly to the x-axis are applied to the ends of the string;

(d) the ends of the string have elastic fixing, that is they are under the action of forces which are proportional to the deviation of the ends and are directed opposite to the deviations.

By a *membrane* is meant a flexible elastic film (that is a two-dimensional elastic medium) which occupies a region in the plane when it is at rest; it is meant that the work expended on the deformation of an infinitesimal element of the membrane is proportional to the increment of the area of the element (the corresponding proportionality coefficient is referred to as the *tension per unit length*).

113. A membrane coincides with a domain D in the plane of the variables x and y when it is at rest. The surface density of the membrane is $\rho = \rho(x, y)$; the membrane is in a process of transverse oscillation described by a function $u = u(x, y, t)$. For the cases enumerated below find the kinetic energy K of the membrane:

(a) the membrane has no localized masses;

(b) there are localized masses m_i $(i = 1, \ldots, n)$ at points (x_i, y_i).

114. For the cases enumerated below find the potential energy of a membrane D which undergoes transverse oscillation described by a function $u = u(x, y, t)$:

(a) the edge of the membrane is rigidly fixed;

(b) the edge of the membrane is rigidly fixed and the powers of u_x and u_y higher than the second can be neglected;

(c) the edge of the membrane has elastic fixing, that is the points (x, y) belonging to the edge of the membrane are subject to resistance forces proportional to the deviation $u = u(x, y, t)$ of these points;

(d) the edge of the membrane is rigidly fixed and the membrane is acted upon by a transverse force $F(x, y, t)$, the powers of u_x and u_y higher than the second can be dropped.

115. A string $(0 \leqslant x \leqslant l)$ with line density $\rho = \rho(x)$ and tension T undergoes small transverse oscillation described by a function $u(x, t)$. Let the functions $\varphi(x)$ and $\psi(x)$ describe, respectively, the initial deviations and velocities of the points of the string (for $t = 0$). Neglecting the powers of u_x higher than the second in the expression for the potential energy of the string and also the action of the force of gravity, use Hamilton's principle to state the problem of the determination of the deviation $u(x, t)$ $(t > 0)$ of the points of the string from the equilibrium position for the following cases:

(a) the ends of the string are rigidly fixed;

(b) the ends of the string are free;

(c) starting from the initial instant $t = 0$, the ends $x = 0$ and $x = l$ are acted upon by given transverse forces $F(t)$ and $\Phi(t)$, respectively;

(d) the ends of the string have elastic fixing, that is each end is subject to a resistance force proportional to the deviation of the end;

(e) the end $x = 0$ is rigidly fixed, the end $x = l$ has elastic fixing (that is, it undergoes resistance proportional to its deviation) and, beginning with the initial instant $t = 0$, the string is acted upon by a transverse force $F(x, t)$;

(f) beginning with the initial instant $t = 0$, a transverse force $F(t)$ is applied to a point x_0 $(0 < x_0 < l)$ of the string, the ends of the string being rigidly fixed;

(g) the ends of the string have elastic fixing and at points x_i $(0 < x_i < l;\ i = 1, \ldots, n)$ of the string there are localized masses m_i.

116. A homogeneous membrane occupies a domain D with boundary L in the plane (x, y) when it is at rest (in the equilibrium position). Let ρ be the surface density of the membrane and T, the tension (per unit length) of the membrane. Let the initial deviations and the initial velocities (at $t = 0$) of the points (x, y) of the membrane be described by functions $\varphi(x, y)$ and $\psi(x, y)$ respectively. Use Hamilton's principle to state the problem on the determination of the transverse deviation $u(x, y, t)$ $(t > 0)$ of the points of the membrane from the equilibrium position, neglecting the force of gravity and the powers of u_x and u_y higher than the second, for the following cases:

(a) the edge of the membrane is rigidly fixed;

(b) the edge of the membrane is free;

(c) beginning with the initial instant $t = 0$, a transverse force $F(x, y, t)$ $((x, y) \in L)$ is applied to the edge of the membrane;

(d) the edge of the membrane has elastic fixing, that is the points belonging to the edge are under the action of resistance forces proportional to their deviations;

(e) starting with the initial instant $t = 0$, the membrane is acted upon by a transverse force $F(x, y, t)$, the edge of the membrane being rigidly fixed;

(f) the oscillating membrane is placed in a medium exerting resistance proportional to the deviation, the edge of the membrane being rigidly fixed;

(g) at a point $(x_0, y_0) \in D$ of the membrane there is a localized mass m, the edge of the membrane being rigidly fixed.

The equation of oscillation of a string also describes the *longitudinal oscillation of an elastic bar*. Indeed, let the coordinate axis x coincide with the direction of the longitudinal axis of an elastic bar of length l. Let us suppose that the transverse sections of the bar (the sections orthogonal to the x-axis) receive displacements along the x-axis (that is they can oscillate longitudinally). We shall assume that the transverse sections $S = S(x)$ of the bar remain plane and orthogonal to the x-axis when they are displaced. This assumption is quite reasonable when the width of the bar is sufficiently small in comparison with its length.

Let us denote by $u = u(x, t)$ the deviation at time t of the section of the bar which has an abscissa x when the bar is at rest. Let $\rho = \rho(x)$, $F = F(x, t)$, $E = E(x)$ and $T = T(x, t)$ be the density of the bar, the volume density of the external forces acting on the bar and directed along the x-axis, Young's modulus of the bar and the tension of the bar respectively. Let us separate mentally a sufficiently small part W inside the bar which lies between two transverse sections with coordinates x and $x + \Delta x$, respectively, when the bar is at rest. We shall derive the equation of motion of that part of the bar on the basis of *D'Alembert's principle*. In accordance with this principle, *the sum of all forces acting on W in the direction of possible displacements* (along the x-axis), *including the inertia forces, must be equal to zero*, that is

$$T(x + \Delta x, t) + T(x, t) + S(\bar{x}) F(\bar{x}, t) \Delta x -$$

$$- S(\tilde{x}) \rho(\tilde{x}) u_{tt}(\tilde{x}, t) \Delta x = 0$$

$$\bar{x}, \tilde{x} \in (x, x + \Delta x)$$

Now, taking into account that, according to *Hook's law*, the tension $T(x, t)$ is proportional to *unit elongation*, that is

$$T(x, t) = E(x)\, u_x(x, t), \quad (0 < x < l)$$

we obtain

$$(ESu_x)(x + \Delta x, t) - (ESu_x)(x, t) + \\ + (SF)(\bar{x}, t)\,\Delta x = (S\rho u_{tt})(\tilde{x}, t)\,\Delta x \qquad (18)$$

Next we use *Lagrange's theorem on finite increments* which makes it possible to rewrite equality (18) in the form

$$\frac{\partial}{\partial x}(ESu_x)(x^*, t)\,\Delta x + (SF)(\bar{x}, t)\,\Delta x = (\rho S u_{tt})(\tilde{x}, t)\,\Delta x \quad (19)$$

$$x^* \in (x, x + \Delta x)$$

On cancelling equality (19) by Δx and then making Δx tend to zero we obtain the (partial differential) *equation of the longitudinal oscillation of the bar*:

$$\frac{\partial}{\partial x}[S(x)\, E(x)\, u_x(x, t)] + S(x)\, F(x, t) = \\ = \rho(x)\, S(x)\, u_{tt}(x, t), \quad (0 < x < l) \quad (20)$$

When the bar is homogeneous, that is S, ρ and E are constant, equation (20) takes the form

$$u_{tt}(x, t) = a^2 u_{xx}(x, t) + f(x, t) \quad (0 < x < l) \quad (20')$$

where $a^2 = E/\rho$ and $f(x, t) = F(x, t)/\rho$.

For the sought-for function $u(x, t)$ in equation (20) or (20′) to be determined uniquely we must prescribe the initial deviation $u(x, 0)$ and the initial velocity $u_t(x, 0)$ of the points of the bar and also set some boundary conditions. To derive the boundary conditions we must consider two parts W_0 and W_l of the bar having sufficiently small length Δx which adjoin the end point sections of the bar, write the equation of motion for each of these parts and then pass to the limit for $\Delta x \to 0$.

117. State the problem on longitudinal oscillation of a homogeneous elastic bar of constant cross-sectional area S and length l for arbitrary initial deviation and velocity for the following cases:

(a) the ends of the bar are free;

(b) beginning with the initial instant $t = 0$, some forces $F(t)$ and $\Phi(t)$ acting along the x-axis are applied to the ends $x = 0$ and $x = l$ of the bar;

(c) the ends of the bar have elastic fixing, that is they are subject to resistance forces proportional to their displacements;

(d) the end $x = 0$ is subject to a resistance force proportional to the velocity and the end $x = l$ is rigidly fixed;

(e) starting with the initial instant $t = 0$, the bar is acted upon by a force directed along the x-axis and distributed with volume density $F(x, t)$ over the bar (for instance, such a force can be exerted by a magnetic field), and the ends of the bar are rigidly fixed;

(f) the bar (its every unit mass) is under the action of a force resisting the displacement, the magnitude of that force being proportional to the velocity; the ends $x = 0$ and $x = l$ oscillate according to some given laws $\mu(t)$ and $\nu(t)$ respectively;

(g) the end $x = 0$ of the bar is fixed while the end $x = l$ is free and carries a localized mass m.

118. For the cases enumerated below state the problem on longitudinal oscillation of a homogeneous elastic bar of variable cross-section $S = S(x)$ and length l for arbitrary initial conditions:

(a) the bar has the shape of a frustum of a cone with radii of the bases r and R $(r < R)$, the end point sections being rigidly fixed;

(b) the end $x = 0$ of the bar is rigidly fixed while, beginning with the initial instant $t = 0$, the end $x = l$ is acted upon by a longitudinal force, the magnitude of the force per unit area of the cross-section being $F(t)$.

119. The end point sections of two semi-bounded homogeneous elastic bars having the same (constant) cross-sectional areas S are joined together to form one unbounded bar. Let ρ_1 and E_1 be, respectively, the density and the modulus of elasticity of one of the bars, and let ρ_2 and E_2 be the same quantities characterizing the other bar. State the boundary-value problem on the determination of the deviations of the sections of the unbounded bar (for $t > 0$) from their positions at rest for the given initial deviation

$\varphi(x)$ and initial velocity $\psi(x)$ (at $t = 0$). Consider the following cases:

(a) the end-point sections of the constituent bars are butt joined;

(b) between the ends of the constituent bars joined together there is a negligibly thick rigid gasket of mass m.

The problem on electric oscillation in a wire also leads to hyperbolic partial differential equations.

Let a wire be located along the coordinate axis x. We shall denote by $i = i(x, t)$, $v = v(x, t)$ the current intensity and the voltage, and by R, L, C and G the ohmic resistance, the self-inductance, the capacitance and the insulation leakage per unit length of the wire respectively. We shall derive the equations describing the variation of the electric current and of the voltage along the wire neglecting the electromagnetic oscillation in the medium surrounding the wire and assuming that the leakage (due to imperfect insulation) is proportional to the voltage. For definiteness, we shall suppose that the direction of the electric current coincides with that of the x-axis. According to *Ohm's law*, for a sufficiently small inner part $(x, x + \Delta x)$ of the wire we can write

$$v(x, t) - v(x + \Delta x, t) = i(x', t) R \Delta x + i_t(x'', t) L \Delta x$$
$$x', x'' \in (x, x + \Delta x)$$

Now, using Lagrange's theorem on finite increments, we obtain, cancelling by Δx and then passing to the limit for $\Delta x \to 0$, the equation

$$v_x(x, t) + Ri(x, t) + Li_t(x, t) = 0 \qquad (21)$$

Further, equating the electric charge flowing into the element Δx of the wire during time Δt (from t to $t + \Delta t$)

$$[i(x, \bar{t}) - i(x + \Delta x, \bar{t})] \Delta t = -i_x(\bar{x}, \bar{t}) \Delta x \Delta t,$$
$$\bar{x} \in (x, x + \Delta x), \quad \bar{t} \in (t, t + \Delta t)$$

to the charge which is expended on charging the element Δx and on the leakage through the imperfect insulation of that element we obtain

$$C\,[v\,(\tilde{x},\ t+\Delta t)-v\,(\tilde{x},\ t)]\,\Delta x+Gv\,(\tilde{x},\ \tilde{t})\,\Delta x\,\Delta t=$$
$$=[Cv_t\,(\tilde{x},\ t')+Gv\,(\tilde{x},\ \tilde{t})]\,\Delta x\,\Delta t,$$
$$\tilde{x}\in(x,\ x+\Delta x);\quad \tilde{t},\ t'\in(t,\ t+\Delta t)$$

On equating the last two expressions for the electric charge we find, by analogy with the derivation of equality (21), another equation

$$i_x\,(x,\ t)+Cv_t\,(x,\ t)+Gv\,(x,\ t)=0 \qquad (22)$$

Equations (21) and (22) form the so-called *system of telegraphy equations*. On eliminating $v\,(x,\ t)$ or $i\,(x,\ t)$ from this system of partial differential equations we obtain, respectively,

$$i_{xx}=ai_{tt}+bi_t+ci \quad \text{or} \quad v_{xx}=av_{tt}+bv_t+cv$$

where $a=CL$, $b=CR+GL$ and $c=GR$.

To derive the boundary conditions (for instance, in the case of a finite wire $0\leqslant x\leqslant l$) it is necessary to consider the voltage drop and the inflow of the electric charge for the parts $(0,\ \Delta x)$ and $(l-\Delta x,\ l)$ of the wire adjoining its ends. Here it should be taken into account that if a circuit contains a part involving a series connection of concentrated ohmic resistance R_0, self-inductance L_0 and capacitance C_0 then the voltage drop corresponding to that part is expressed by the formula

$$\Delta v=R_0i+L_0i_t+\frac{1}{C_0}\int i\,dt$$

120. Let $\varphi\,(x)$ and $\psi\,(x)$ be the initial current intensity and the initial voltage along the wire $(0\leqslant x\leqslant l)$ at the initial instant $t=0$. For the cases enumerated below, state the boundary-value problem for the determination of the current intensity and the voltage in the process of electric oscillation in that wire for $t>0$ neglecting the ohmic resistance and the electric leakage:

(a) beginning with the initial instant $t=0$, an electromotive force $E\,(t)$ is applied to the end $x=0$, while the end $x=l$ is grounded;

(b) the end $x = 0$ is grounded with the aid of a concentrated capacitance C_0 and beginning with the initial time $t = 0$ an electromotive force $E(t)$ is applied to the end $x = l$ through a concentrated ohmic resistance R_0;

(c) beginning with the initial instant $t = 0$, an electromotive force $E(t)$ is applied to the end $x = 0$ through a concentrated self-inductance L_0 while the end $x = l$ is grounded with the aid of a concentrated self-inductance L_l.

121. Let $\varphi(x)$ and $\psi(x)$ be the initial (for $t = 0$) current intensity and voltage in a wire ($0 \leqslant x \leqslant l$). State the boundary-value problem for the determination of the current intensity and voltage for $t > 0$ in electric oscillation in that wire for the following cases:

(a) the end $x = 0$ of the wire is grounded with the aid of a concentrated ohmic resistance R_0, and beginning with the initial instant $t = 0$ an electromotive force $E(t)$ is applied to the end $x = l$ through a concentrated ohmic resistance R_l;

(b) the end $x = 0$ is grounded with the aid of a concentrated ohmic resistance R_0 and self-inductance L_0 and beginning with the initial instant $t = 0$ an electromotive force $E(t)$ is applied to the end $x = l$ through a concentrated self-inductance L_l.

Some problems leading to partial differential equations of parabolic type. We shall begin with the problem of determining the distribution of temperature along a bar. Let the axis of the bar go along the coordinate axis x. We shall suppose that the temperature at any cross-section of the bar orthogonal to its axis is independent of the position of the points of the section. Let us denote by $\rho = \rho(x)$, $k = k(x)$, $\varkappa = \varkappa(x)$, $c = c(x)$, $S = S(x)$, $\sigma = \sigma(x)$, $q = q(x, t)$, $u = u(x, t)$ and $u_0 = u_0(t)$ the *line density* of the bar, the *coefficient of thermal conductivity*, the *heat transfer coefficient*, the *specific heat*, the area of the cross-section, the perimeter of the cross-section, the *volume density of heat sources*, the temperature at the section x at time t and the temperature of the surrounding medium respectively. To derive the partial differential equation for the function $u = u(x, t)$ let us separate mentally, within the bar, an arbitrary part W of sufficiently small size lying between two sections of the bar orthogonal to the x-axis and passing through points x and $x + \Delta x$. The amount of heat flowing into the element W during time Δt can be written as

$$Q = Q_1 + Q_2 + Q_3$$

Here the quantity Q_1 is the inflow of heat through the sections x and $x + \Delta x$. According to *Fourier's law of heat conduction*, Q_1 is expressed by the formula

$$Q_1 = [(kSu_x)(x + \Delta x,\ t') - (kSu_x)(x,\ t')]\,\Delta t = \\ = \left[\frac{\partial}{\partial x}(kSu_x)\right](x',\ t')\,\Delta x\,\Delta t$$

$$x' \in (x,\ x + \Delta x), \quad t' \in (t,\ t + \Delta t)$$

The quantity Q_2 is the inflow of heat through the lateral surface of the bar. By *Newton's law*, Q_2 is proportional to the temperature difference:

$$Q_2 = [\varkappa\sigma(u_0 - u)](\bar{x},\ \bar{t})\,\Delta x\,\Delta t, \quad \bar{x} \in (x,\ x + \Delta x), \\ \bar{t} \in (t,\ t + \Delta t)$$

Finally, the amount of heat Q_3 appears due to the action of the heat sources, and we have

$$Q_3 = (qS)(\tilde{x},\ \tilde{t})\,\Delta x\,\Delta t, \quad \tilde{x} \in (x,\ x + \Delta x), \quad \tilde{t} \in (t,\ t + \Delta t)$$

Consequently,

$$Q = \left\{\left[\frac{\partial}{\partial x}(kSu_x)\right](x',\ t') + [\varkappa\sigma(u_0 - u)](\bar{x},\ \bar{t}) + \\ + (Sq)(\tilde{x},\ \tilde{t})\right\}\Delta x\,\Delta t \qquad (23)$$

This amount of heat is expended on heating the element W from the temperature $u(x, t)$ to the temperature $u(x, t + \Delta t)$ and therefore we can write for Q another expression:

$$Q = (c\rho S)(x'')\,[u(x'',\ t + \Delta t) - u(x'',\ t)]\,\Delta x = \\ = (c\rho Su_t)(x'',\ t'')\,\Delta x\,\Delta t \qquad (24)$$

$$x'' \in (x,\ x + \Delta x), \quad t'' \in (t,\ t + \Delta t)$$

On equating (23) to (24), cancelling the resultant equality by $\Delta x\,\Delta t$ and then passing to the limit for $\Delta x \to 0$ and $\Delta t \to 0$ we receive the following partial differential equation for the unknown function $u(x, t)$:

$$(c\rho S u_t)(x, t) = \frac{\partial}{\partial x}[(kSu_x)(x, t)] -$$

$$- [\varkappa\sigma(u - u_0)](x, t) + (Sq)(x, t) \quad (25)$$

Equation (25) is called the *heat conduction equation.* In particular, if c, ρ, k, $\varkappa$, σ and S are constant quantities equation (25) takes the form

$$u_t = a^2 u_{xx} - bu + f(x, t)$$

where

$$a^2 = \frac{k}{c\rho}, \quad b = \frac{\varkappa\sigma}{c\rho S}, \quad f(x, t) = \frac{Sq(x, t) + \varkappa\sigma u_0(t)}{c\rho S}$$

To solve the problem on the determination of the temperature $u(x, t)$ in the bar at time instant $t > 0$ we must know, besides the partial differential equation, the initial temperature of the bar at $t = 0$ and also the boundary conditions specifying the thermal conditions at the ends of the bar. The boundary conditions can be derived if, like in the derivation of the heat conduction equation, we consider the balance of heat for two infinitesimal elements W_0 and W_l of the bar adjoining the corresponding ends.

Similarly, proceeding from the *Nernst law* for the flux of a substance, for instance, of a gas, through a surface, we can state the diffusion problems (on the determination of the concentration of the substance in a tube). Let us suppose that there is a gas in an arbitrary volume Ω in a porous medium. Let us denote by $u = u(x, t)$, $D = D(x)$, $c = c(x)$, $F = F(x, t)$ and ν the concentration of the gas at point $x = (x_1, x_2, x_3)$ at time moment t, the *diffusion coefficient*, the *porosity coefficient* which is equal to the ratio of the volume of the pores to the whole volume in question, the volume density of gas sources and the unit vector of outer normal to the surface bounding the volume Ω respectively. To derive the *diffusion equation* let us separate mentally an arbitrary volume V within Ω with a sufficiently smooth boundary S and write the *gas balance equation* for that volume for an arbitrary sufficiently small time interval $(t, t + \Delta t)$. By the Nernst law, the amount of gas passing through the element dS of the surface S in the direction of ν (here ν is the normal to dS) during unit time is equal to

$$dQ = -D(x)\frac{\partial u}{\partial \nu}\,dS$$

In the derivation of the diffusion equation we must take into account the following facts:

(1) the amount of gas flowing into the volume V through the surface S during time Δt is equal to

$$Q_1 = \int\limits_t^{t+\Delta t} dt \int\limits_S D\frac{\partial u}{\partial \nu}\,dS$$

Using the *Gauss-Ostrogradsky formula* we can write

$$\int\limits_S D\frac{\partial u}{\partial \nu}\,dS = \int\limits_V \operatorname{div}(D\operatorname{grad} u)\,dV$$

whence

$$Q_1 = \int\limits_t^{t+\Delta t} dt \int\limits_V \operatorname{div}(D\operatorname{grad} u)\,dV$$

(2) the amount of gas produced by the gas sources within the volume V during time Δt (for instance, when there is a chemical reaction accompanied by liberation of gas) is equal to

$$Q_2 = \int\limits_t^{t+\Delta t} dt \int\limits_V F(x,\ t)\,dV$$

(3) the total increment of the amount of gas in the volume V appearing due to the increment

$$u(x,\ t+\Delta t) - u(x,\ t) \approx u_t(x,\ t)\,\Delta t$$

gained by the function $u(x,\ t)$ during time Δt is expressed by the formula

$$Q_3 = \int\limits_t^{t+\Delta t} dt \int\limits_V cu_t\,dV$$

Consequently, we have

$$Q_1+Q_2-Q_3=\int\limits_t^{t+\Delta t} dt \int\limits_V [\operatorname{div}(D \operatorname{grad} u)+F-cu_t]\, dV=0$$

Now let us suppose that the integrand in the last formula is a continuous function. Since the volume V and the time interval $(t,\ t+\Delta t)$ are quite arbitrary, it follows that at each point of the domain Ω and for any t $(t>0)$ the integrand is equal to zero, that is

$$cu_t = \operatorname{div}(D \operatorname{grad} u) + F \tag{26}$$

Equation (26) is the *diffusion equation* we intended to derive. When the medium is homogeneous the quantities c and D are constant, and equation (26) takes the form

$$u_t=a^2\Delta u+f, \quad a^2=\frac{D}{c}, \quad f=\frac{F}{c}$$

When investigating a diffusion phenomenon we must consider, besides the diffusion equation, the initial distribution of the concentration of the substance (for instance, at $t=0$) and the boundary conditions specifying the diffusion on the boundary of the volume under consideration.

122. The lateral surface of a homogeneous bar $(0 \leqslant x \leqslant l)$ is heat insulated, and the initial temperature (at $t=0$) of the bar is described by a function $\varphi(x)$. For the cases enumerated below state the problem of the determination of the temperature u in the bar for $t>0$:

(a) the ends of the bar are heat insulated;

(b) beginning with the instant $t=0$, heat fluxes $q(t)$ and $Q(t)$ are maintained at the ends $x=0$ and $x=l$ respectively;

(c) convective heat exchange with the media adjoining the ends goes on at the ends $x=0$ and $x=l$ of the bar, the temperature of the media being $\tau(t)$ and $\theta(t)$ respectively (the convective heat exchange obeys Newton's law);

(d) at the end $x=l$ of the bar there is a concentrated mass m of the same material as the bar itself, and this end of the bar is heat insulated; at the end $x=0$, a given temperature $\mu(t)$ is maintained beginning with the instant $t=0$;

(e) at both ends of the bar there are equal concentrated masses m of the same material as the bar itself, the end $x = 0$ of the bar is heat insulated, and at the end $x = l$, beginning with the initial time $t = 0$, a heat flux $q(t)$ is maintained.

123. A tube of length l with constant cross-sectional area S is filled with a homogeneous porous substance. A gas diffusion takes place within the tube, the initial concentration of the gas (at $t = 0$) being equal to $\varphi(x)$. For the cases enumerated below state the problem of determining the concentration u of the gas in the tube for $t > 0$ under the assumption that the lateral surface of the tube is gas-proof:

(a) beginning with the initial time $t = 0$, a concentration of the gas equal to $\mu(t)$ is maintained at the end $x = 0$ while the end $x = l$ is gas-proof;

(b) at the end $x = 0$, beginning with the initial instant $t = 0$, a gas flux $q(t)$ is maintained while at the end $x = l$ there is a porous diaphragm, that is, at the end $x = l$ there is gas exchange with the surrounding medium obeying a law analogous to Newton's law for the convective heat exchange; the concentration of the gas in the surrounding medium is permanently equal to zero.

124. The initial temperature (at $t = 0$) of a homogeneous bar ($0 \leqslant x \leqslant l$) having a constant cross-section S is equal to $\varphi(x)$. Convective heat exchange (obeying Newton's law) with the surrounding medium whose temperature is $v(t)$ occurs on the surface of the bar whose ends $x = 0$ and $x = l$ are held in massive clamps with given heat capacities C and Q respectively and with sufficiently high thermal conductivity. State the problem on the determination of the temperature u for $t > 0$ in that bar for the following cases:

(a) the bar is being heated by a direct electric current I;

(b) heat sources with volume density $F(x, t)$ begin acting within the bar at time moment $t = 0$;

(c) heat is absorbed proportionally to u_t at each point of the bar.

125. A tube ($0 \leqslant x \leqslant l$) of constant cross-section S is filled with a gas whose initial concentration (at $t = 0$) is $\varphi(x)$. The surface and the end faces of the tube are porous so that concentration exchange goes on through them with the surrounding medium (obeying a law analogous to Newton's law for convective heat exchange). The concentration of gas in the surrounding medium is equal to $v(t)$. For the cases enumerated below state the boundary-value problem for the determination of the concentration u of the gas for $t > 0$:

(a) the particles of the gas disintegrate (this is the case when the gas is unstable), the rate of the disintegration at each point inside the tube being proportional to the square root of the concentration of the gas;

(b) the particles of the gas multiply, the rate of multiplication at each point inside the tube being proportional to the product uu_t.

126. A homogeneous ball of radius R with centre at the origin has temperature T. State the boundary-value problem on the cooling of the ball for the following cases:

(a) a chemical reaction goes on within the ball resulting in heat absorption at each point of the ball proportional to the temperature u at that point, the surface S of the ball being heat insulated;

(b) there are heat sources of constant intensity Q within the ball and on the surface S of the ball there is convective heat exchange with the surrounding medium whose temperature is equal to zero.

Some problems leading to partial differential equations of elliptic type. Partial differential equations of elliptic type arise in the study of *stationary* (*steady-state*) processes. For instance, if we assume that the sought-for quantities in the problems considered above are independent of time, then all the differential equations derived for the determination of these quantities become elliptic (provided that the number of the spatial variables exceeds unity).

127. For the cases enumerated below state the boundary-value problem for the determination of a stationary (steady-state) concentration of an unstable gas in a cylinder of radius r_0 and altitude h on condition that, due to a chemical reaction, within the cylinder there are gas sources of constant intensity Q and the rate of the disintegration of the particles of the gas is proportional to its concentration u:

(a) on the bases $z = 0$ and $z = h$ of the cylinder a zero concentration of the gas is maintained, the lateral surface of the cylinder being gas-proof;

(b) the bases $z = 0$ and $z = h$ of the cylinder are porous and allow the gas to diffuse obeying a law analogous to Newton's law for convective heat exchange, while on the lateral surface of the cylinder a zero concentration of the gas is maintained, the concentration of that gas in the surrounding medium is permanently equal to zero.

128. Let $u(x, y)$ and $v(x, y)$ be the components of the velocity of a plane steady-state flow of an incompressible fluid at the point (x, y). We shall denote by D an arbitrary simply connected domain lying in the plane of the flow whose boundary is a smooth curve S with normal ν and tangent s. The integrals

$$\int_S (u \cos \widehat{\nu x} + v \cos \widehat{\nu y})\, dS$$

and

$$\int_S (u \cos \widehat{sx} + v \cos \widehat{sy})\, dS$$

express the flux of the fluid through the contour S and the circulation of the fluid along S respectively. Using these integrals and assuming that there are no sources and no circulation show that the functions u and v satisfy the system of partial differential equations

$$u_x + v_y = 0, \quad u_y - v_x = 0$$

and that each of the functions satisfies Laplace's equation.

129. Show that the velocity potential $\varphi(x, y)$ and the stream function $\psi(x, y)$ specified by the equalities $\varphi_x = u$, $\varphi_y = v$, $\psi_x = -v$ and $\psi_y = u$ (where u and v are the same as in Problem 128) are solutions of the *Cauchy-Riemann system of partial differential equations*

$$\varphi_x - \psi_y = 0, \quad \varphi_y + \psi_x = 0$$

and that each of the functions $\varphi(x, y)$ and $\psi(x, y)$ satisfies Laplace's equation.

130. Explain the physical meaning of an equation of the form $\psi = \text{const}$ where $\psi(x, y)$ is a stream function (see Problems 128 and 129).

131. Let a membrane be deformed and let it be in equilibrium in the state of deformation; this means that for that state the function u describing the deflection of the membrane is independent of time. In this case in expression (17) only the potential energy U remains nonzero. Therefore, if the powers of u_x and u_y higher than the second are neglected, then, by virtue of Hamilton's principle, the function $u(x, y)$ must minimize the *Dirichlet integral*

$$D(u) = \iint_G (u_x^2 + u_y^2)\, dx\, dy$$

where G is the domain occupied by the membrane when it is at rest. Under these assumptions solve the following problems:

(a) show that the function $u(x, y)$ describing the deflection of the membrane is a solution of Laplace's equation

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in G$$

(b) explain the physical meaning of the Dirichlet problem with a condition of the form

$$u(x, y) = f(x, y), \quad (x, y) \in S$$

and of the Neumann problem with a condition of the form

$$\frac{\partial u(x, y)}{\partial \nu} = f(x, y), \quad (x, y) \in S$$

where S is the boundary of the domain G, ν is the normal to S and $f(x, y)$ is a given function defined on S.

Chapter 2

ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

§ 1. Basic Properties of Harmonic Functions

The simplest example of an elliptic partial differential equation is *Laplace's equation*

$$\Delta u = 0$$

where $\Delta \equiv \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is *Laplace's operator* (also called the *Laplacian*).

Regular solutions of Laplace's equation are called *harmonic functions*.

132. Find the expression of Laplace's operator

(a) in general *curvilinear coordinates*

$$x = \varphi(\xi, \eta), \quad y = \psi(\xi, \eta)$$

(b) in the *polar coordinates*

$$x = r \cos \varphi, \quad y = r \sin \varphi$$

(c) in the *cylindrical coordinates*

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad z = z$$

(d) in the *spherical coordinates*

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta$$

(e) in the *oblate spheroidal coordinates*

$$x = \xi\eta \sin \varphi, \quad y = \sqrt{(\xi^2 - 1)(1 - \eta^2)} \quad z = \xi\eta \cos \varphi$$

133. Let $u = u(x_1, \ldots, x_n)$ be a harmonic function. Find which of the functions written below are harmonic and which are not:

(a) $u(x + h)$ where $h = (h_1, \ldots, h_n)$ is a constant vector;
(b) $u(\lambda x)$ where λ is a constant scalar;

(c) $u(Cx)$ where C is a constant orthogonal matrix;

(d) $\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}$, $n=2$;

(e) $\frac{\partial u}{\partial x_1}\frac{\partial u}{\partial x_2}$, $n>2$;

(f) $x_1\frac{\partial u}{\partial x_1}+x_2\frac{\partial u}{\partial x_2}+x_3\frac{\partial u}{\partial x_3}$, $n=3$;

(g) $x_1\frac{\partial u}{\partial x_1}-x_2\frac{\partial u}{\partial x_2}$, $n=2$;

(h) $x_2\frac{\partial u}{\partial x_1}-x_1\frac{\partial u}{\partial x_2}$, $n=2$;

(i) $\dfrac{\frac{\partial u}{\partial x_1}}{\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2}$, $n=2$;

(j) $\left(\frac{\partial u}{\partial x_1}\right)^2-\left(\frac{\partial u}{\partial x_2}\right)^2$, $n=2$;

(k) $\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2$, $n=2$.

134. Find the value of the constant k for which the functions written below are harmonic:

(a) $x_1^3+kx_1x_2^2$;
(b) $x_1^2+x_2^2+kx_3^2$;
(c) $e^{2x_1}\cosh kx_2$;
(d) $\sin 3x_1\cosh kx_2$;
(e) $\frac{1}{|x|^k}$, $|x|^2=\sum_{i=1}^{n}x_i^2$, $|x|\neq 0$.

In the theory of harmonic functions an important role is played by the following *extremum principle*: *a function $u(x)$ harmonic in a domain D and different from a constant can achieve its extremum at no interior point x of the domain D.*

135. Show that if $u(x)$ is a harmonic function then so is the function $v(x)=(1/|x|^{n-2})\,u(x/|x|^2)$ at any point where it is defined.

136. Using the extremum principle find whether the level lines of a harmonic function can intersect in the domain of harmonicity of the function.

137. Find a monotone increasing function $y=f(x)$ describing a level line of the function $u(x, y)=x^2-y^2$ which passes through the point $(0, 0)$.

138. Find the level line of the harmonic function $u = \sin x\cosh y$ passing through the point $(-\pi/2, 0)$ and possessing the property that when the moving point recedes to infinity along that line the function $u = \cos x \sinh y$ tends to minus infinity.

139. Find the points of extremum of the harmonic function $u = xy$ in the closed region $\overline{D}$: $x^2 + y^2 \leqslant 1$.

140. Find the points of extremum of the harmonic function $u = x^2 - y^2$ in the closed region $\overline{D}$: $x^2/4 + y^2/9 \leqslant 1$.

141. Let $w(x)$ be a continuous function in a domain D possessing in this domain continuous partial derivatives up to the second order and satisfying the condition $\Delta w < 0$ ($\Delta w > 0$). Show that the function $w(x)$ cannot have a negative relative minimum (a positive relative maximum) in the domain D.

142. Compute the directional derivatives $\frac{\partial u}{\partial \nu}$ along the outer normal ν to the boundary S of the domain D at the points of extremum of the functions u spoken of in Problems 139 and 140.

143. Let u be a harmonic function in a domain D with a sufficiently smooth boundary S and let it be continuous, including the boundary S, together with its partial derivatives of the first order. Show that at the point $x_0 \in S$ where u achieves its extremum in $D \cup S$ the normal derivative $\frac{\partial u}{\partial \nu}$ is different from zero and that if ν is the outer normal then we have $\frac{\partial u}{\partial \nu} < 0$ at the point of minimum and $\frac{\partial u}{\partial \nu} > 0$ at the point of maximum (this fact is known as *Zaremba's principle*).

The real and the imaginary parts of an analytic function $f(z) = u(x, y) + iv(x, y)$ of the complex variable $z = x + iy$ are harmonic (they are called *conjugate harmonic functions*). Due to this fact, there is a close connection between the theory of harmonic functions of two independent variables and the theory of analytic functions of one complex variable.

144. Show that if $u(x, y)$ is a harmonic function then $\varphi(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$ is an analytic function.

For the cases enumerated below use complex integration along a line in order to reconstruct the function $f(z)$ analytic in a simply connected domain D from the given real part $u(x, y) = \operatorname{Re} f(z)$:

145. $u = x^3 - 3xy^2$.

146. $u = e^x \sin y$.

147. $u = \sin x \cosh y$.

Find the harmonic functions u satisfying the following conditions:

148. $\dfrac{\partial u}{\partial x} = 3x^2y - y^3$.

149. $\dfrac{\partial u}{\partial z} = e^x (x \cos y - y \sin x) + 2z$.

150. For a function $f(z)$ analytic in a simply connected domain D prove the *Goursat formula*

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0, 0) + iC, \qquad z = x + iy, \qquad (0, 0) \in D \tag{1}$$

which makes it possible to reconstruct the function $f(z)$ from its given real part $u(x, y)$ to within an arbitrary pure imaginary constant iC without using integration.

151. Solve Problems 145-147 using Goursat's formula (1) and compare the results with those obtained earlier.

152. Prove the harmonicity of the function

$$u(x) = \sum_{k=0}^{\infty} (-1)^k \left[\frac{x_n^{2k}}{(2k)!} \Delta^k \tau(x_1, \ldots, x_{n-1}) + \frac{x_n^{2k+1}}{(2k+1)!} \Delta^k \nu(x_1, \ldots, x_{n-1})\right] \tag{2}$$

where τ and ν are arbitrary infinitely differentiable functions under the assumption that the series on the right-hand side of formula (2) can be differentiated termwise the required number of times.

153. Consider the elliptic equation

$$\sum_{k=1}^{n} a_k \frac{\partial^2 u}{\partial x_k^2} = 0$$

with real constant coefficients a_k ($k = 1, \ldots, n$) which are all of one sign. Prove that all regular solutions of the equation can be represented in the form

$$u(x_1, \ldots, x_n) = v\left(\frac{x_1}{\sqrt{|a_1|}}, \ldots, \frac{x_n}{\sqrt{|a_n|}}\right)$$

where $v(x_1, \ldots, x_n)$ is an arbitrary harmonic function.

154. Prove that the formula

$$u(x, y) = e^{\lambda x + \mu y} v(x, y)$$

where λ and μ are constants and where $v(x, y)$ is an arbitrary harmonic function, expresses the general solution of the elliptic equation

$$u_{xx} + u_{yy} - 2\lambda u_x - 2\mu u_y + (\lambda^2 + \mu^2) u = 0$$

155. Verify directly that the function $E(x, y)$ dependent on the two points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ which is expressed by the formula

$$E(x, y) = \begin{cases} \dfrac{1}{n-2}|x-y|^{2-n} & \text{for } n > 2 \\ -\ln|x-y| & \text{for } n = 2 \end{cases} \tag{3}$$

where $|x - y|$ is the distance between the points x and y, satisfies Laplace's equation for $x \neq y$ both as a function of x and as a function of y.

The function $E(x, y)$ determined by formula (3) is called the *fundamental* (*elementary*) *solution* of Laplace's equation.

156. Show that all the solutions of Laplace's equation which are different from a constant and depend solely on the distance $|x - y|$ are of the form $CE(x, y)$ where C is an arbitrary constant and $E(x, y)$ is the fundamental solution of that equation.

The elementary solution

$$E(M, M_0) = \frac{1}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}}$$

of Laplace's equation in the three variables x, y and z can be given a simple physical interpretation. Namely, a concentrated electric charge μ placed at the point $M_0(x_0, y_0, z_0)$ generates an electrostatic field whose potential $u(x, y, z) = u(M)$ at any point $M(x, y, z)$ distinct from M_0 is expressed by the formula

$$u(M) = \mu E(M, M_0)$$

157. Let $M'(x', y', z')$ and $M''(x'', y'', z'')$ be two points lying on a straight line with direction vector ν and symmetric about a third point $M_0(x_0, y_0, z_0)$ on that line. Let there be two electric charges $-\mu_0$ and μ_0 concentrated at the points M' and M'', respectively, such that in the limit when $|M' - M''| \to 0$ we have

$$\mu_0 \, |M' - M''| = \mu(M_0)$$

The potential of the field generated by these charges at a point $M(x, y, z)$ different from M_0, M' and M'' is expressed as

$$\frac{\mu_0}{|M''-M|} - \frac{\mu_0}{|M'-M|}$$

The limiting configuration of the charges $-\mu_0$ and μ_0 for $|M' - M''| \to 0$ is called a *dipole*, and μ_0 and ν are referred to as the *polarization* (the *dipole moment*) and the *axis* of the dipole respectively. Compute the value of the potential of the dipole at the point $M(x, y, z)$.

158. Let concentrated electric charges μ_k be located at points $M_k(x_k, y_k, z_k)$ $(k = 1, \ldots, m)$. Write down the expression of the potential generated by these charges at an arbitrary point $M(x, y, z)$ distinct from the points $M_k(x_k, y_k, z_k)$.

159. An electric charge is distributed over the sphere

$$(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 = R^2$$

with a constant density of C. Compute the value of the potential of the field generated by this charged sphere at its centre $M(x, y, z)$.

160. Derive the formula for the potential generated by a charge distributed along a spatial curve L with continuous line density $\mu(\xi, \eta, \zeta)$ where

$$\xi = \xi(t), \quad \eta = \eta(t), \quad \zeta = \zeta(t), \quad t_0 \leqslant t \leqslant t_1$$

are parametric equations of the curve L.

§ 2. The Dirichlet and the Neumann Bound-ary-Value Problems for Harmonic Functions

Let D be a domain in an n-dimensional space E_n with boundary S. By a *function of class* C^m $(D \cup S)$ is meant a one-valued function continuous in $D \cup S$ together with its partial derivatives of order m. In case $m = 0$ this class reduces to the collection of all continuous functions defined in $D \cup S$.

In the theory of harmonic functions an extremely important role is played by the so-called *Dirichlet* and the *Neumann boundary-value problems* which are also referred to as the *first* and the *second boundary-value problems* respectively.

The Dirichlet problem: it is required to find a function $u(x)$ *of class* C^0 $(D \cup S)$ *which is harmonic in the domain* D *and satisfies the boundary condition*

$$u(x) = \varphi(x), \quad x \in S \tag{4}$$

where $\varphi(x)$ *is an arbitrary continuous function defined on* S.

The Neumann problem: under the assumption that the boundary S *of the domain* D *is smooth it is required to determine a function* $u(x)$ *of class* C^1 $(D \cup S)$ *which is harmonic in the domain* D *and satisfies the boundary condition*

$$\frac{\partial u}{\partial \nu} = \varphi(x), \quad x \in S \tag{5}$$

where ν *is the outer normal to* S *and* $\varphi(x)$ *is an arbitrary continuous function defined on* S.

Besides the Dirichlet and the Neumann problems, an important role is played in applications by the so-called *mixed boundary-value problems*; in these problems the values of the sought-for harmonic function are prescribed on one part of the boundary S of the domain D while on the other part of the boundary the values of its normal derivative are prescribed.

In the theory of boundary-value problems an important role is played by *Green's function. Green's function of the Dirichlet problem for harmonic functions* is defined as a function $G(x, y)$ dependent on two points x and y and possessing the following properties:

(1) this function has the form

$$G(x, y) = E(x, y) + g(x, y)$$

where $E(x, y)$ is the fundamental (elementary) solution of Laplace's equation specified by formula (3) and $g(x, y)$ is a function harmonic both with respect to $x \in D$ and with respect to $y \in D$;

(2) $G(x, y) = 0$ when at least one of the points x and y lies on S.

We shall begin with the case when D is a bounded domain with a smooth boundary S.

161. Let $u(x)$ and $v(x)$ be two functions of class $C^1(D \cup S)$ harmonic in a domain D. Derive the identity

$$\int_S \left[v(y) \frac{\partial u(y)}{\partial \nu_y} - u(y) \frac{\partial v(y)}{\partial \nu_y} \right] dS_y = 0 \tag{6}$$

where ν_y is the outer normal to S at the point $y \in S$ and dS_y is the element of area of the surface S whose position on S is specified by the coordinate y.

162. Let $u(x)$ be a function of class $C^1(D \cup S)$ harmonic in a domain D. Derive the integral representation

$$u(x) = \frac{1}{\omega_n} \int_S \left[E(x, y) \frac{\partial u(y)}{\partial \nu_y} - u(y) \frac{\partial E(x, y)}{\partial \nu_y} \right] dS_y \tag{7}$$

where ω_n is the area of unit sphere in E_n:

$$\omega_n = \frac{2}{\Gamma\left(\frac{n}{2}\right)} \pi^{\frac{n}{2}}$$

(and Γ is Euler's gamma function).

163. Prove the symmetry of Green's function $G(x, y)$, that is the identity

$$G(y, x) \equiv G(x, y)$$

164. Prove that for any function $u(x)$ of class $C^1(D \cup S)$ harmonic in a domain D the equality

$$\int_S \frac{\partial u}{\partial \nu} dS = 0$$

holds where ν is the normal to the boundary S of the domain D.

165. Let a ball $|y-x| \leqslant R$ lie within the domain of harmonicity of a function $u(x)$. Prove the *mean-value theorem* asserting that

$$\text{(a)} \quad u(x) = \frac{1}{\omega_n R^{n-1}} \int\limits_{|y-x|=R} u(y)\, dS_y$$

for the sphere $|y-x| = R$;

and

$$\text{(b)} \quad u(x) = \frac{n}{\omega_n R^n} \int\limits_{|y-x|<R} u(y)\, d\tau_y$$

for the ball $|y-x| < R$.

166. From the formula expressing the mean-value theorem derive the *extremum principle* for harmonic functions.

167. Using the extremum principle prove the uniqueness of the solution of the Dirichlet problem for harmonic functions with boundary condition (4).

168. Prove that, given Green's function $G(x, y)$, the solution $u(x)$ of the Dirichlet problem with boundary condition (4) where $\varphi(x)$ belongs to the class $C^1(D \cup S)$ can be expressed in quadratures:

$$u(x) = -\frac{1}{\omega_n} \int\limits_S \frac{\partial G(x, y)}{\partial \nu_y} \varphi(y)\, dS_y \qquad (8)$$

169. Show that the formula

$$G(x, y) = E(x, y) - E\left(|x|\, y, \frac{x}{|x|}\right)$$

specifies Green's function of the Dirichlet problem for the ball $|x| < 1$.

170. Using Green's function, derive *Poisson's formula*

$$u(x) = \frac{1}{\omega_n} \int\limits_{|y|=1} \frac{1-|x|^2}{|y-x|^n} \varphi(y)\, dS_y$$

expressing the solution of the Dirichlet problem for harmonic functions with boundary condition (4) in the ball $|x| < 1$.

171. Construct the solution of the Dirichlet problem with boundary condition (4) for the ball $|x - x_0| < R$.

172. From Poisson's formula derive the formula expressing the *mean-value theorem for a sphere.*

173. Prove the identity

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |x|^2}{|y - x|^2} \, d\psi = 1$$

where the point $x = (x_1, x_2)$ belongs to the interior of the circle $|x| < 1$ and the point $y = (\cos\psi, \sin\psi)$ belongs to the circumference $|y| = 1$ of the circle.

174. Verify directly that the function $u(x)$ represented by Poisson's formula in the ball $|x| < 1$ is harmonic and that

$$\lim_{\substack{x \to x_0 \\ |x_0| = 1}} u(x) = \varphi(x_0)$$

175. Show that a nonnegative function $u(x)$ harmonic in a ball $|x| < R$ satisfies the inequalities

$$R^{n-2} \frac{R - |x|}{(R + |x|)^{n-1}} u(0) \leqslant u(x) \leqslant R^{n-2} \frac{R + |x|}{(R - |x|)^{n-1}} u(0)$$

176. Can a function harmonic throughout E_n and different from a constant retain sign?

177. Can a function harmonic throughout E_n and bounded above be different from a constant?

178. Prove that if for a continuous function defined in a domain D there holds the mean-value theorem in a neighbourhood of each point of the domain D then this function is harmonic in D.

179. Prove that if $u(x)$ is a function of class $C^1(D)$ such that the integral of the normal derivative $\frac{\partial u}{\partial \nu}$ over any sphere lying in D is equal to zero then this function is harmonic in D.

180. Find the solution $u(x, y)$ of the Dirichlet problem for harmonic functions in the circle $x^2 + y^2 < 1$ satisfying the boundary condition

$$u = \sin 2\varphi \quad (0 \leqslant \varphi \leqslant 2\pi)$$

on the circumference $x^2 + y^2 = 1$ of the circle.

181. Solve the Dirichlet problem

$$\Delta u(x, y) = 0, \quad (x, y) \in D;$$
$$u(x, y) = 4x^3 + 6x - 1, \quad (x, y) \in \partial D$$

where the domain D is the circle $x^2 + y^2 + 2x < 0$ and ∂D is its boundary (circumference).

The Dirichlet problem can be stated not only for a bounded domain. When this problem is considered for an unbounded domain it is required that the sought-for harmonic function should be bounded for $|x| \to \infty$ in the case $n = 2$ and should tend to zero not slower than $1/|x|^{n-2}$ in the case $n > 2$.

182. Prove the validity of formula (7) (see Problem 162) for a harmonic function $u(x)$ in the half-space $x_n > 0$.

183. Prove that the expression

$$G(x, y) = E(x, y) - E(x, y')$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ and y' is the point belonging to E_n which is symmetric to the point y about the plane $y_n = 0$, satisfies all the requirements enumerated in the definition of Green's function; also, from formula (8) derive *Poisson's formula*

$$u(x) = \Gamma\left(\frac{n}{2}\right) \pi^{-\frac{n}{2}} x_n \int\limits_{y_n=0} \frac{\varphi(y_1, \ldots, y_{n-1})}{\left[\sum\limits_{i=1}^{n-1} (y_i - x_i)^2 + x_n^2\right]^{n/2}} \, dy_1 \ldots dy_{n-1}$$

expressing the solution of the Dirichlet problem for Laplace's equation with boundary condition (4) in the half-space $x_n > 0$.

184. Solve the Dirichlet problem with boundary condition (4) for Laplace's equation in the half-space $x_n < 0$.

185. Find the function $u(x, y)$ harmonic in the half-plane $y > 0$ and satisfying the condition

$$u(x, 0) = \frac{x}{x^2 + 1}$$

186. Find the function $u(x, y, z)$ harmonic in the half-space $z < 0$ and satisfying the boundary condition

$$u(x, y, 0) = \frac{1}{(1 + x^2 + y^2)^{3/2}}$$

Let D^+ be a bounded domain with boundary S and let D^- be the *complement* $C(D^+ \cup S)$ of $D^+ \cup S$ with respect to the whole space E_n. The Dirichlet problems for the domains D^+ and D^- are usually referred to as the *interior* and the *exterior Dirichlet problems* respectively.

187. Show that the *inversion transformation*

$$\xi = \frac{x}{|x|^2}$$

reduces the exterior Dirichlet problem to the interior problem.

188. Construct the solution $u(x, y)$ of the Dirichlet problem with boundary condition

$$u(x, y) = \varphi(x, y), \quad x^2 + y^2 = 1$$

for the exterior of the circle $x^2 + y^2 \leqslant 1$.

189. Find the necessary condition for the existence of the solution of the Neumann problem with boundary condition (5).

190. Prove the uniqueness, to within an arbitrary additive constant, of the solution of the interior Neumann problem.

191. Show that the function

$$u(x, y) = -\frac{1}{\pi} \int_S \ln \sqrt{(\xi - x)^2 + (\eta - y)^2}\, g(\xi, \eta)\, dS + C$$

is the solution of the Neumann problem for the circle $x^2 + y^2 < R^2$ with the boundary condition

$$\frac{\partial u(x, y)}{\partial \nu} = g(x, y), \quad x^2 + y^2 = R^2$$

provided that the function g satisfies the condition

$$\int_S g\, dS = 0$$

where S is the boundary $x^2 + y^2 = 1$ of the circle.

192. Derive the relation

$$f(z) = u(x, y) + iv(x, y) = \frac{1}{2\pi i} \int_{|t|=1} \frac{t+z}{t-z} \frac{u(t)\, dt}{t} + iC$$

known as *Schwarz' formula* (C is a real constant) from Poisson's formula

$$u(x, y) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-x^2-y^2}{1-2x\xi-2y\eta+x^2+y^2} u(e^{i\varphi})\, d\varphi$$

$$\xi = \cos\varphi, \quad \eta = \sin\varphi$$

using Goursat's formula (1).

193. Find the function $u(x, y)$ harmonic in the open semi-circle D: $|z| < 1$, $\operatorname{Im} z > 0$, and continuous in the closed semi-circle $\overline{D}$: $|z| \leqslant 1$, $\operatorname{Im} z \geqslant 0$, whose derivatives are continuous within the semi-closed region $D \cup d$ where d is the diameter $-1 \leqslant x \leqslant 1$, $y = 0$, of the semi-circle, satisfying the boundary conditions

$$u = (x, y) = \varphi(x, y), \quad x^2 + y^2 = 1, \quad y > 0;$$

$$\left.\frac{\partial u(x, y)}{\partial y}\right|_{y=0} = 0, \quad -1 < x < 1$$

194. Find the function $u(x, y)$ harmonic in the semi-circle $|z| < 1$, $\operatorname{Im} z > 0$ and satisfying the boundary conditions

$$u(x, y) = 0, \quad |z| = 1, \quad \operatorname{Im} z > 0;$$

$$u(x, 0) = \varphi(x), \quad -1 \leqslant x \leqslant 1$$

195. Prove that the formula

$$u(x, y) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{1-2r\cos(\theta-\varphi)+r^2} - \frac{1}{1-2r\cos(\theta+\varphi)+r^2} \right) \times (1-r^2) f(\theta)\, d\theta$$

$$r^2 = x^2 + y^2, \quad x = r\cos\varphi, \quad y = r\sin\varphi$$

specifies a harmonic function in the semi-circle $|z| < 1$, $\operatorname{Im} z > 0$, satisfying the boundary conditions

$$u(\xi, \eta) = f(\theta), \quad \xi = \cos\theta, \quad \eta = \sin\theta, \quad 0 \leqslant \theta \leqslant \pi$$

$$u(x, 0) = 0, \quad -1 \leqslant x \leqslant 1$$

where f is a given continuous function such that

$$f(0) = u(1, 0) = f(\pi) = u(0, -1) = 0$$

§ 3. Potential Functions

By a *volume potential with density* μ in a domain D of the space E_n (or, more precisely, a *potential function for a volume distribution of mass with density* μ) is meant the function

$$u(x) = \int_D E(x, y)\,\mu(y)\,d\tau_y \tag{9}$$

where $E(x, y)$ is the fundamental (elementary) solution of Laplace's equation and $d\tau_y$ is the element of volume whose position within D is specified by the coordinate y. The potential function $u(x)$ is harmonic in the exterior of the closed region $D \cup S$ where $S = \partial D$ is the boundary of D. In case the function μ is continuous and bounded in D the volume potential is continuous together with its partial derivatives of the first order throughout the space E_n. In case μ possesses partial derivatives of the first order continuous and bounded in D the volume potential has partial derivatives of the second order in D, and

$$\Delta u = -\omega_n \mu(x), \quad x \in D \tag{10}$$

where ω_n is the area of unit sphere in E_n.

196. Investigate the behaviour of the potential function for a volume distribution of mass as $|x| \to \infty$.

197. For a bounded domain D, find a sufficient condition for the potential function of volume distribution of mass to tend to zero as $|x| \to \infty$ in the case when $n = 2$.

198. Show that the expression

$$u(x) = -\frac{1}{\omega_n} \int_D G(x, y)\, f(y)\, d\tau_y$$

where $G(x, y)$ is Green's function of the Dirichlet problem in the domain D is the solution of Poisson's equation

$$\Delta u = f(x), \quad x \in D \tag{11}$$

satisfying the boundary condition

$$\lim_{x \to x_0} u(x) = 0, \quad x_0 \in S$$

199. Under the assumption that the Dirichlet problem for harmonic functions with boundary condition

$$\lim_{x \to x_0} v(x) = \varphi(x_0), \quad x_0 \in S$$

is solvable, prove, using the result of Problem 198, the existence of the solution $u(x)$ of Poisson's equation (11) satisfying the non-homogeneous boundary condition

$$\lim_{x \to x_0} u(x) = \varphi(x_0), \quad x_0 \in S \tag{12}$$

200. Is the solution of Problem (11), (12) unique?

201. Prove the formula

$$\int_{\sigma} \frac{\partial E(x, y)}{\partial \nu_x} ds_x = \begin{cases} -\omega_n & \text{for } y \in d \\ -\omega_n/2 & \text{for } y \in \sigma \\ 0 & \text{for } x \in C(d \cup \sigma) \end{cases}$$

where $E(x, y)$ is the elementary solution of Laplace's equation, d is an arbitrary bounded domain in the space E_n with a smooth boundary σ and $C(d \cup \sigma)$ is the *complement* of $d \cup \sigma$ with respect to the whole space E_n.

202. Prove *Gauss' formula*

$$\int_{\sigma} \frac{\partial u(x)}{\partial \nu_x} d\sigma_x = -\omega_n \int_{D \cap d} \mu(y)\, d\tau_y$$

where d is an arbitrary bounded domain in the space E_n with a smooth boundary σ, for the volume potential $u(x)$ of a mass distributed over a domain $D \subset E_n$ with density $\mu(x)$.

203. Can a function harmonic in a domain D be a potential function of mass distribution over D with a non-zero density?

204. Find the density μ of distribution of mass over a domain D on condition that the volume potential of the mass is

$$u = (x^2 + y^2 + z^2)^2 - 1$$

205. For the conditions of Problem 204 find the total mass M filling the interior of a ball $x^2 + y^2 + z^2 < r^2$ lying in the domain D.

206. Find a particular solution of Poisson's equation $\Delta u = f\left(\sum_{k=1}^{n} a_k x_k\right)$ where a_k $(k=1, \ldots, n)$ are real constants such that $\sum_{k=1}^{n} a_k^2 = A^2 \neq 0$.

207. The volume potential of a mass distributed over a domain D is specified by the function

$$u(x, y) = x^2y^2$$

Determine the mass M filling the square $-1 \leqslant x \leqslant 1$, $-1 \leqslant y \leqslant 1$ on condition that this square lies within D.

208. Show that the potential $u(x, y)$ of a mass distributed over the circle $x^2 + y^2 < 1$ with density $\mu = 1$ is expressed by the formula

$$u(x, y) = \begin{cases} -\pi \ln r & \text{for} \quad r \geqslant 1 \\ \frac{\pi}{2}(1-r^2) & \text{for} \quad r \leqslant 1 \end{cases}$$

where $r^2 = x^2 + y^2$.

209. Show that the function $u(x, y, z)$ expressed by the formula

$$u(x, y, z) = \begin{cases} 4\pi/3r & \text{for} \quad r \geqslant 1 \\ 2\pi\left(1-\frac{1}{3}r^2\right) & \text{for} \quad r \leqslant 1 \end{cases}$$

(where $r^2 = x^2 + y^2 + z^2$) is the potential function of the volume distribution of mass over the ball $r^2 < 1$ with density $\mu = 1$.

210. The potential $u(x, y)$ of a mass distributed over the circle $r^2 = x^2 + y^2 < 1$ is expressed, for the interior points of the circle, by the formula

$$u(x, y) = \frac{\pi x}{4}(3-r^2)$$

Determine the mass density μ and the values of the potential $u(x, y)$ outside the closed circle $r^2 \leqslant 1$.

211. The potential $u(x, y)$ of a mass distributed over the circle $r^2 = x^2 + y^2 < 1$ is expressed by the formula

$$u(x, y) = \frac{\pi}{8}(1-r^4)$$

for the interior points of the circle. Determine the mass M lying within the annulus

$$\frac{1}{4}<x^2+y^2<\frac{1}{2}$$

212. Compute the integral $I=\int\limits_{x^2+y^2<1}\frac{\partial u(x)}{\partial \nu_x}\,ds_x$ where $u(x)$ is the potential of a mass distributed over the square $-1\leqslant x\leqslant 1$, $-1\leqslant y\leqslant 1$ with density $\mu=xy$.

Let S be a smooth or piecewise-smooth surface in the space E_n (that is, a smooth or a piecewise-smooth $(n-1)$-dimensional manifold in E_n); let μ be a continuous function defined on S.

The expressions

$$u(x)=\frac{1}{\omega_n}\int\limits_S \mu(y)\frac{\partial E(x,\ y)}{\partial \nu_y}\,dS_y \tag{13}$$

and

$$u(x)=\frac{1}{\omega_n}\int\limits_S \mu(y)\,E(x,\ y)\,dS_y \tag{14}$$

where $E(x,\ y)$ is the fundamental (elementary) solution of Laplace's equation, ν_y is the outer normal to S at the point y and ω_n is the area of unit sphere in E_n, are called, respectively, the *double-layer potential* (or, more precisely, the *potential function for a double layer of distribution of dipoles over the surface S with density* μ) and the *single-layer potential* (or a *potential function for a surface distribution of mass over S with density* μ).

At each point x in the space E_n not lying on S the double-layer potential and the single-layer potential are harmonic functions. Expressions (13) and (14) also make sense when the point x varies over the surface S, and they are continuous functions on S.

Let S be a sufficiently smooth closed $(n-1)$-dimensional surface (in the case $n=2$ this is a curve with continuous curvature) and let D^+ and D^- be, respectively, the bounded and the unbounded domains with boundary S.

The double-layer potential expressed by formula (13) possesses the following two important properties: (1) when the point x passes from the domain D^+ to the domain D^- the double-layer potential suffers a jump so that there hold the equalities

$$u^+(x_0) = -\frac{1}{2}\mu(x_0) + u(x_0) \tag{15}$$

and

$$u^-(x_0) = \frac{1}{2}\mu(x_0) + u(x_0) \tag{16}$$

where

$$u(x_0) = \frac{1}{\omega_n}\int_S \mu(y)\,\frac{\partial E(x_0, y)}{\partial \nu_y}\,dS_y$$

and

$$u^+(x_0) = \lim_{\substack{x\to x_0\\ x\in D^+}} u(x), \quad u^-(x_0) = \lim_{\substack{x\to x_0\\ x\in D^-}} u(x), \quad x_0 \in S$$

(2) for $x \to x_0 \in S$ there exist the limits

$$\lim_{\substack{x\to x_0\\ x\in D^+}} \frac{\partial u(x)}{\partial \nu_x} \quad \text{and} \quad \lim_{\substack{x\to x_0\\ x\in D^+}} \frac{\partial u(x)}{\partial \nu_x}$$

(they are commonly denoted as

$$\left(\frac{\partial u(x_0)}{\partial \nu_{x_0}}\right)^+ \quad \text{and} \quad \left(\frac{\partial u(x_0)}{\partial \nu_{x_0}}\right)^-$$

respectively) and the equality

$$\left(\frac{\partial u(x_0)}{\partial \nu_{x_0}}\right)^+ = \left(\frac{\partial u(x_0)}{\partial \nu_{x_0}}\right)^-$$

holds at each point $x_0 \in S$.

Accordingly, the single-layer potential expressed by formula (14) possesses the following properties: (1) it remains continuous when the point x passes from the domain D^+ to the domain D^- and (2) there exist the limits

$$\lim_{\substack{x\to x_0\\ x\in D^+}} \frac{\partial u(x)}{\partial \nu_x} = \left(\frac{\partial u(x_0)}{\partial \nu_{x_0}}\right)^+ \quad \text{and} \quad \lim_{\substack{x\to x_0\\ x\in D^-}} \frac{\partial u(x)}{\partial \nu_x} = \left(\frac{\partial u(x_0)}{\partial \nu_{x_0}}\right)^-$$

for which the relations

$$\left(\frac{\partial u(x_0)}{\partial \nu_{x_0}}\right)^+ = \frac{1}{2}\mu(x_0) + \frac{\partial u(x_0)}{\partial \nu_{x_0}} \tag{15$'$}$$

and

$$\left(\frac{\partial u(x_0)}{\partial \nu_{x_0}}\right)^- = -\frac{1}{2}\mu(x_0) + \frac{\partial u(x_0)}{\partial \nu_{x_0}} \qquad (16')$$

hold. Here the expression $\frac{\partial u(x_0)}{\partial \nu_{x_0}}$ designates the normal derivative of single-layer potential (14) for $x = x_0 \in S$. This expression has quite a definite meaning.

213. Investigate the behaviour of the double-layer potential and single-layer potential for $|x| \to \infty$.

214. For the case $n = 2$ state a sufficient condition for the single-layer potential to tend to zero as $|x| \to \infty$.

215. Write down *Fredholm's integral equations of the second kind* to which the Dirichlet and the Neumann problems (both the interior and the exterior ones) reduce for harmonic functions.

216. Find the solution of the Neumann problem for harmonic functions $u(x, y)$ in the half-plane $y > 0$, the boundary condition being

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = \varphi(x), \quad -\infty < x < \infty$$

To this end, reduce the Neumann problem to the Dirichlet problem in the same half-plane for the harmonic function $v(x, y)$ conjugate to $u(x, y)$.

217. Find the single-layer potential $u(x, y)$ of a mass distributed along the circle $x^2 + y^2 = R^2$ with density $\mu = 1$.

218. Find the single-layer potential $u(x, y, z)$ of a mass distributed over the sphere $x^2 + y^2 + z^2 = 1$ with density $\mu = 1$.

219. Find the double-layer potential $u(x, y)$ of dipoles distributed along the circle $x^2 + y^2 = 1$ with density $\mu = x$.

220. Find the function $u(x, y)$ harmonic in the exterior of the closed circle $x^2 + y^2 \leqslant 1$ and satisfying the boundary condition

$$u(x, y) = \cos^2\varphi - \sin^2\varphi - 1$$

$$(x = \cos\varphi,\ y = \sin\varphi,\quad 0 \leqslant \varphi < 2\pi)$$

221. Find the solution $u(x, y, z)$ of the Dirichlet problem for harmonic functions in the exterior of the ball $x^2 + y^2 +$

$+ z^2 \leqslant 1$ satisfying the boundary condition

$$u(x, y, z) = x^2 - y^2 - 1, \quad x^2 + y^2 + z^2 = 1$$

222. Find the solution $u(x, y, z)$ of the Dirichlet problem for harmonic functions in the domain $x^2 + y^2 + z^2 > 1$, the boundary condition being

$$u(x, y, z) = z, \quad x^2 + y^2 + z^2 = 1$$

223. Let the formula

$$u(x, y) = -\frac{y}{2r^2} \quad (r^2 = x^2 + y^2)$$

express the values of a single-layer potential of a mass distributed along the circle $r^2 = 1$, in the exterior of that circle. Determine the values of this potential function inside the circle $r^2 < 1$.

224. Find the double-layer potential of dipoles distributed over the circle $x^2 + y^2 = 1$ with density $\mu = 1$.

225. The values of a single-layer potential of a mass distributed over the circumference $x^2 + y^2 = 1$ of the closed circle $x^2 + y^2 \leqslant 1$ are expressed in the exterior of that circle by the formula

$$u(x, y) = \frac{x}{r^2}\left(1 + \frac{2y}{r^2}\right) \quad (r^2 = x^2 + y^2)$$

Determine the density μ of the mass distribution.

§ 4. Some Other Classes of Elliptic Partial Differential Equations

Among the partial differential equations of elliptic type an important role is played in various applications by the *Helmholtz equation*

$$\Delta u + \lambda u = 0 \tag{17}$$

where Δ is Laplace's operator and λ is a real constant, and by the *biharmonic equation*

$$\Delta \Delta u = 0 \tag{18}$$

226. Verify directly that the expression

$$u(x, y) = J_0\left(\mu \sqrt{(z-t)\,z}\right)$$

(where $J_0(r)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{r}{2}\right)^{2n}$ is *Bessel's function of order zero*, $\mu^2=\lambda$, $z=x+iy$, $\bar{z}=x-iy$ and $t=\xi+i\eta$), considered as a function of the variables x and y, satisfies equation (17).

227. Using the result of Problem 226 show that the formula

$$u(x, y)=\operatorname{Re}\int_0^z J_0\left(\mu\sqrt{(z-t)\bar{z}}\right) f(t)\,dt$$

where f is an arbitrary analytic function of the complex variable t and $\mu^2=\lambda$, expresses the regular solutions of equation (17).

228. For an arbitrary domain D, prove the following *extremum principle*: if $\lambda<0$ then a regular solution of equation (17) in D cannot attain at any interior point of the domain D either a positive maximum or a negative minimum.

229. Does the solution of Dirichlet problem (17), (4) possess the uniqueness property in a bounded domain for $\lambda<0$?

230. Show that the function

$$E(r)=\int_{-\infty}^{-1}\frac{e^{\mu rt}\,dt}{\sqrt{t^2-1}}$$

where $\mu^2=-\lambda$ and $r^2=(x-\xi)^2+(y-\eta)^2$ is a solution of equation (17) for $n=2$ and $r\neq 0$.

231. Using the expression of Laplace's operator in spherical coordinates prove that for $n=3$ one of the solutions of equation (17) which are dependent solely on the distance $r=|x-y|$ has the form

$$E(r)=\frac{e^{-\mu r}}{r} \tag{19}$$

where $\lambda=-\mu^2$.

232. For the solutions $u(x)$ of equation (17) regular in a domain $D\subset E_3$ and belonging to the class $C^1(D\cup S)$

prove that if $\lambda = -\mu^2$ in that equation then the identity

$$u(x) = \frac{1}{4\pi} \int_S \left[E(x, y) \frac{\partial u(y)}{\partial \nu_y} - u(y) \frac{\partial E(x, y)}{\partial \nu_y} \right] dS_y$$

holds where the function $E(x, y) = E(r)$ is specified by formula (19).

233. For the case $n = 2$ write equation (18) in the form

$$\frac{\partial^4 u}{\partial z^2\, \partial \bar{z}^2} = 0$$

to prove that all the solutions of this equation in a simply connected domain can be represented in the form

$$u = \operatorname{Re} [\bar{z}\varphi(z) + \psi(z)]$$

where φ and ψ are arbitrary analytic functions of the complex variable $z = x_1 + ix_2$.

234. Prove that in the case $n = 2$ the function

$$E(r) = r^2 \ln r \quad (r = |x - y|)$$

satisfies equation (18) for $r \neq 0$.

235. Verify directly that the function

$$u(x) = v_0(x) + |x|^2 v_1(x)$$

where v_0 and v_1 are harmonic functions and $|x|^2 = \sum_{i=1}^{n} x_i^2$ satisfies equation (18).

236. Consider the function

$$u(x) = \sum_{k=1}^{m} u_k(x)$$

where $u_k(x)$ are solutions of the equations $\Delta u_k - \lambda_k u_k = 0$; $k = 1, \ldots, m$, and the numbers λ_k are the roots of the polynomial $\sum_{k=0}^{m} a_k \lambda^{m-k}$. Prove that every function of this type is a solution of the elliptic partial differential equation of the $2m$-th order with constant coefficients of the form

$$\sum_{k=0}^{m} a_k \Delta^{m-k} u = 0$$

237. Prove that the functions of the form

$$u(x, y) = \operatorname{Re}[\varphi(z_1) + \psi(z_2)]$$

where φ and ψ are analytic functions of the complex variables $z_1 = x - \sqrt{2}(1 + i)y/2$ and $z_2 = x + \sqrt{2}(1 + i)y/2$, respectively, are solutions of the elliptic equation

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4} = 0$$

238. Consider the elliptic equation with constant coefficients

$$au_{xx} + 2bu_{xy} + cu_{yy} = 0$$

where $b^2 - ac < 0$. Prove that all regular solutions of the equation are expressed by the general formula

$$u(x, y) = \operatorname{Re} f\left[x - \frac{1}{c}\left(b + i\sqrt{ac - b^2}\right)y\right]$$

where $f(t)$ is an arbitrary analytic function of the complex variable $t = x - (b + i\sqrt{ac - b^2})\,y/c$.

239. For the elliptic partial differential equation

$$a^2 u_{xx} + b^2 u_{yy} = 0 \quad (a = \text{const}, \quad b = \text{const})$$

with boundary condition (4) write down the solution $u(x, y)$ of the interior Dirichlet problem for the ellipse $x^2/a^2 + y^2/b^2 < 1$ using Poisson's formula for the circle.

240. Prove that for an arbitrary constant a the general solution $u(x, y)$, $v(x, y)$ of the system of partial differential equations

$$au_x - v_y = 0, \quad av_x + u_y = 0$$

is specified by the relation

$$u(x, y) + iv(x, y) = f(z)$$

where $f(z)$ is an arbitrary analytic function of the complex variable $z = x + aiy$.

241. Prove that the Cauchy problem for the system of partial differential equations considered in Problem 240 with data

$$u = f_1, \quad v = f_2$$

prescribed on any arc S cannot possess more than one solution.

242. Can the Cauchy problem

$$a^2 u_{xx} + b^2 u_{yy} = 0, \quad u(x, 0) = 0, \quad u_y(x, 0) = 0,$$
$$0 < x < \varepsilon, \quad \varepsilon = \text{const}$$

possess a nonzero solution?

243. Prove that all regular solutions of the elliptic system of partial differential equations

$$u_{xx} - u_{yy} - 2v_{xy} = 0, \quad v_{xx} - v_{yy} + 2u_{xy} = 0$$

in a simply connected domain can be obtained from the formula

$$u(x, y) + iv(x, y) = \bar{z}\varphi(z) + \psi(z) \tag{20}$$

where φ and ψ are arbitrary analytic functions of the complex variable $z = x + iy$.

244. Using general expression (20) for the solutions of the elliptic system of partial differential equations considered in Problem 243, show that the homogeneous Dirichlet problem with the boundary conditions

$$u(t) = 0, \quad v(t) = 0, \quad |t| = 1$$

for this system in the circle $|z| < 1$ possesses infinitely many solutions $u(x, y)$, $v(x, y)$ satisfying the equality

$$u(x, y) + iv(x, y) = (1 - z\bar{z})\psi(z)$$

where $\psi(z)$ is an arbitrary analytic function in the circle $|z| < 1$; also prove, for the same system, that the nonhomogeneous Dirichlet problem with the boundary conditions

$$u(x) = f_1(t), \quad v(t) = f_2(t), \quad |t| = 1$$

has no solutions at all.

245. Verify that the system of partial differential equations

$$u_{xx} - u_{yy} + \sqrt{2}v_{xy} = 0, \quad v_{xx} - v_{yy} - \sqrt{2}u_{xy} = 0$$

is elliptic, show that in the circle $x^2 + y^2 < 1$ for this system the homogeneous Dirichlet problem with boundary conditions

$$u(x, y) = 0, \quad v(x, y) = 0, \quad x^2 + y^2 = 1$$

possesses non-trivial solutions $u^k(x, y)$, $v^k(x, y)$ $(k = 1, 2, \ldots)$ satisfying the relations

$$u^k(x, y) + iv^k(x, y) = [(\mu z + \bar{z})^2 - 4\mu^2]^k - (\mu z - \bar{z})^{2k}$$

where $z = x + iy$ and $\mu = i/(1 + \sqrt{2})$.

246. Verify directly that the function

$$u(x, y, z) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{c^{2n}} \frac{z^{2n}}{(2n)!} \left(a^2 \frac{\partial^2}{\partial x^2} + b^2 \frac{\partial^2}{\partial y^2}\right)^n \tau\left(\frac{x}{|a|}, \frac{y}{|b|}\right) + \sum_{n=0}^{\infty} (-1)^n \frac{1}{c^{2n+1}} \frac{z^{2n+1}}{(2n+1)!} \left(a^2 \frac{\partial^2}{\partial x^2} + b^2 \frac{\partial^2}{\partial y^2}\right)^n \nu\left(\frac{x}{|a|}, \frac{y}{|b|}\right)$$

where τ and ν are arbitrary polynomials in two variables, satisfies the elliptic partial differential equation with constant coefficients

$$a^2 u_{xx} + b^2 u_{yy} + c^2 u_{zz} = 0$$

247. Show that the system of partial differential equations

$$\begin{Vmatrix} 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & 0 & -\frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & 0 & -\frac{\partial}{\partial x} \\ \frac{\partial}{\partial z} & -\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{Vmatrix} (u, v, w, \varphi) = 0$$

is elliptic and prove that each component of its solution (u, v, w, φ) is a harmonic function of the variables x, y and z.

We remind the reader that if $A = \| A_{ij} \|$ is a *square matrix of order* n and $x = (x_1, \ldots, x_n)$ is a vector, then $y = Ax$ *is understood as a new vector* $y = (y_1, \ldots, y_n)$ (the vector y results from the *linear transformation* of x, A being the *transformation matrix*) whose components are

$$y_i = \sum_{k=1}^{n} A_{ik} x_k \quad (i = 1, \ldots, n)$$

Chapter 3

HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

§ 1. Wave Equation

Here and henceforth we shall denote by the symbol x the collection of the spatial variables $x_1, \ldots, x_n$ corresponding to the (variable) point (x, t) belonging to the $(n+1)$-dimensional space E_{n+1} where t is time.

As was mentioned in § 4, Chapter 1, under certain assumptions some oscillation (wave) processes are described by the equation

$$\sum_{i=1}^{n} u_{x_i x_i} - u_{tt} = 0 \tag{1}$$

That is why the solutions of this equation are usually referred to as *waves*, and the equation itself is called the *wave equation*.

Since the characteristic form $Q(\lambda)$ corresponding to equation (1) is

$$Q(\lambda) = \sum_{i=1}^{n} \lambda_i^2 - \lambda_{n+1}^2$$

the wave equation is a partial differential equation of hyperbolic type.

By a *characteristic hypersurface* of equation (1) is meant an n-dimensional manifold

$$\varphi(x, t) = 0$$

in the space E_{n+1} on which the quadratic form $Q(\text{grad}\,\varphi)$ turns into zero:

$$Q(\text{grad}\,\varphi) \equiv \sum_{i=1}^{n} \varphi_{x_i}^2 - \varphi_t^2 = 0$$

One of the most important problems in the theory of propagation of waves is the *Cauchy problem.* In the present section we shall consider the following statement of this problem: *it is required to find the solution $u(x, t)$ of equation (1) satisfying the initial conditions*

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x) \tag{2}$$

where φ and ψ are given functions of the variables $x_1, \ldots, x_n$.

248. Write down the general expression for all characteristic curves of the *equation of oscillation of a string*

$$u_{xx} - u_{tt} = 0 \tag{3}$$

249. Find the characteristic surfaces of the second order for the *equation of oscillation of a membrane*

$$u_{x_1x_1} + u_{x_2x_2} - u_{tt} = 0 \tag{4}$$

250. Find all the characteristic planes of the *equation of propagation of sound*

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - u_{tt} = 0 \tag{5}$$

251. Show that the expression

$$u(x_1, x_2, x_3, t) = tM(\mu)$$

where

$$M(\mu) = \int\limits_{|y|=1} \mu(x_1 + ty_1, x_2 + ty_2, x_3 + ty_3)\, dS_y$$

and $\mu(x_1, x_2, x_3)$ is a given function defined in the space E_3 of the variables x_1, x_2, x_3 and possessing continuous partial derivatives of the second order, is a solution of equation (5).

252. Prove that *Kirchhoff's formula*

$$u(x_1, x_2, x_3, t) = \frac{1}{4\pi} tM(\psi) + \frac{1}{4\pi} \frac{\partial}{\partial t} [tM(\varphi)] \tag{6}$$

where φ and ψ are real functions defined in the space E_3 and possessing continuous partial derivatives of the third and of the second order, respectively, and $M(\mu)$ is the expression defined in Problem 251, represents the solution of the Cauchy problem with initial conditions (2).

253. Consider the function

$$u(x, t) = \sum_{k=0}^{\infty} \left[\frac{t^{2k}}{(2k)!} \Delta^k \tau(x_1, \ldots, x_n) + \right.$$
$$\left. + \frac{t^{2k+1}}{(2k+1)!} \Delta^k \nu(x_1, \ldots, x_n) \right] \quad (7)$$

where Δ is Laplace's operator in the variables $x_1, \ldots, x_n$ and τ and ν are infinitely differentiable functions. Verify directly that expression (7) is the solution of equation (1) satisfying the initial conditions

$$u(x, 0) = \tau(x), \quad u_t(x, 0) = \nu(x)$$

under the assumption that the series on the right-hand side of formula (7) and the series resulting from two-fold termwise differentiation of that series with respect to $x_1, \ldots, x_n$ and t are all uniformly convergent.

254. On the basis of formula (6) derive the *Huygens principle*: at any point (x_1, x_2, x_3, t) of the space E_4 the value of the solution of the Cauchy problem (5) for wave equation with conditions (2) is completely determined by the values assumed by the functions φ, $\frac{\partial \varphi}{\partial \nu}$ and ψ on the sphere

$$(z_1 - x_1)^2 + (z_2 - x_2)^2 + (z_3 - x_3)^2 = t^2$$

of radius $|t|$ with centre at the point (x_1, x_2, x_3).

255. From Kirchhoff's formula (6), under the assumption that the functions φ and ψ depend solely on the two spatial variables x_1 and x_2, derive *Poisson's formula*

$$u(x_1, x_2, t) = \frac{1}{2\pi} \int_d \frac{\psi(y_1, y_2)\, dy_1 dy_2}{\sqrt{t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} +$$
$$+ \frac{1}{2\pi} \frac{\partial}{\partial t} \int_d \frac{\varphi(y_1, y_2)\, dy_1 dy_2}{\sqrt{t^2 - (y_1 - x_1)^2 - (y_2 - x_2)^2}} \quad (8)$$

where d is the circle $(y_1 - x_1)^2 + (y_2 - x_2)^2 \leqslant t^2$.

256. Show that Poisson's formula (8) expresses the solution of the Cauchy problem (4), (2).

257. Does the Huygens principle hold for the solutions of the Cauchy problem (4), (2)?

258. From Poisson's formula (8), under the assumption that the functions φ and ψ depend solely on one spatial variable $x = x_1$, derive D'Alembert's formula

$$u(x, t) = \frac{1}{2}\left[\varphi(x+t) + \varphi(x-t) + \int_{x-t}^{x+t} \psi(\tau)\, d\tau\right] \qquad (9)$$

expressing the solution of the Cauchy problem with conditions (2) for equation (3).

259. In the equation of oscillation of a string (3) pass to the characteristic variables $\zeta = x + t$ and $\eta = x - t$ and show, using the transformed equation, that the general solution of original equation (3) has the form

$$u(x, t) = f(x + t) + \varphi(x - t) \qquad (10)$$

where f and φ are arbitrary twice continuously differentiable functions.

Find the general solution for each of the following equations:

260. $2u_{xx} - 5u_{xy} + 3u_{yy} = 0$.

261. $2u_{xx} + 6u_{xy} + 4u_{yy} + u_x + u_y = 0$.

262. $3u_{xx} - 10u_{xy} + 3u_{yy} - 2u_x + 4u_y + \frac{5}{16} u = 0$.

263. $3u_{xx} + 10u_{xy} + 3u_{yy} + u_x + u_y + \frac{1}{16} u - 16\, xe^{-\frac{x+y}{16}} = 0$.

264. $u_{yy} - 2u_{xy} + 2u_x - u_y = 4e^x$.

265. $u_{xx} - 6u_{xy} + 8u_{yy} + u_x - 2u_y + 4e^{5x + \frac{3}{2}y} = 0$.

266. $u_{xx} - 2\cos x u_{xy} - (3 + \sin^2 x)\, u_{yy} + u_x + (\sin x - \cos x - 2)\, u_y = 0$.

267. $e^{-2x}u_{xx} - e^{-2y}u_{yy} - e^{-2x}u_x + e^{-2y}u_y + 8e^y = 0$.

268. $u_{xy} + yu_y - u = 0$.

269. $u_{xy} + xu_x - u + \cos y = 0$.

270. $\cosh x\, u_{xy} + (\sinh x + y \cosh x)\, u_y - \cosh x\, u = 0$.

271. $\frac{\partial}{\partial y}(u_x + u) + 2x^2y\,(u_x + u) = 0$.

272. $\frac{\partial}{\partial y}(u_x + u) + x\,(u_x + u) + x^2y = 0$.

Solve the following Cauchy problems:

273. $4y^2u_{xx} + 2(1 - y^2)\, u_{xy} - u_{yy} - \frac{2y}{1+y^2}(2u_x - u_y) = 0$,

$$u(x, y)\big|_{y=0} = \varphi(x), \quad u_y(x, y)\big|_{y=0} = \psi(x).$$

274. $u_{xx} - 2u_{xy} + 4e^y = 0,$
$u(x, y)|_{x=0} = \varphi(y), \quad u_x(x, y)|_{x=0} = \psi(y).$

275. $u_{xx} + 2\cos x\, u_{xy} - \sin^2 x\, u_{yy} - \sin x\, u_y = 0,$
$u(x, y)|_{y=\sin x} = x + \cos x, u_y(x, y)|_{y=\sin x} = \sin x.$

276. $3u_{xx} - 4u_{xy} + u_{yy} - 3u_x + u_y = 0,$
$u(x, y)|_{y=0} = \varphi(x), \quad u_y(x, y)|_{y=0} = \psi(x).$

277. $e^y u_{xy} - u_{yy} + u_y = 0,$
$u(x, y)|_{y=0} = -x^2/2, \ u_y(x, y)|_{y=0} = -\sin x.$

278. $u_{xx} - 2\sin x\, u_{xy} - (3 + \cos^2 x)\, u_{yy} - \cos x\, u_y = 0,$
$u(x, y)|_{y=\cos x} = \sin x, \quad u_y(x, y)|_{y=\cos x} = e^x/2.$

279. $u_{xx} - 2\sin x\, u_{xy} - (3 + \cos^2 x)\, u_{yy} + u_x +$
$+ (2 - \sin x - \cos x)\, u_y = 0,$
$u(x, y)|_{y=\cos x} = 0, \quad u_y(x, y)|_{y=\cos x} = e^{-x/2} \cos x.$

280. $u_{xx} + 2\sin x\, u_{xy} - \cos^2 x\, u_{yy} + u_x +$
$+ (\sin x + \cos x + 1)\, u_y = 0,$
$u(x, y)|_{y=-\cos x} = 1 + 2\sin x, \ u_y(x, y)|_{y=-\cos x} =$
$= \sin x.$

281. Find the *domain of dependence* for Problem (1), (2) in the cases $n = 1$, $n = 2$ and $n = 3$.

282. Prove that for each solution $u(x, t)$ of equation (3) there holds the following *mean-value formula*:

$$u(x_1, t_1) + u(x_3, t_3) = u(x_2, t_2) + u(x_4, t_4)$$

where (x_1, t_1), (x_2, t_2), (x_3, t_3) and (x_4, t_4) are the consecutive vertices of a characteristic rectangle, that is of a rectangle bounded by characteristic straight lines of equation (3).

283. Construct the solution $v(x_1, x_2, x_3, t, \tau)$ of equation (5) dependent on the variables x_1, x_2, x_3, t and on the parameter τ which satisfies the initial conditions

$$v(x_1, x_2, x_3, \tau, \tau) = 0, \quad \left.\frac{\partial v}{\partial t}\right|_{t=\tau} = g(x_1, x_2, x_3, \tau)$$

284. Let $v(x_1, x_2, x_3, t, \tau)$ be the solution of Problem 283. Show that the function

$$u(x_1, x_2, x_3, t) = \int_0^t v(x_1, x_2, x_3, t, \tau)\, d\tau$$

is the solution of the non-homogeneous equation

$$u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3} - u_{tt} = -g(x_1, x_2, x_3, t)$$

satisfying the homogeneous initial conditions

$$u(x_1, x_2, x_3, 0) = 0, \quad u_t(x_1, x_2, x_3, 0) = 0$$

285. Represent the function $u(x_1, x_2, x_3, t)$ mentioned in Problem 284 in the form

$$u(x_1, x_2, x_3, t) = \frac{1}{4\pi} \int\limits_{r^2 \leqslant t^2} \frac{g(y_1, y_2, y_3, t-r)}{r} \, d\tau_y, \quad r = |y - x|$$

and explain why it is called a *retarded potential*.

286. Find the solution of equation (4) satisfying the initial conditions

$$u(x_1, x_2, 0) = x_1^3 x_2^2, \quad \left. \frac{\partial u(x_1, x_2, t)}{\partial t} \right|_{t=0} = x_1^2 x_2^4 - 3x_1^3$$

287. Construct the solution of the non-homogeneous equation

$$u_{xx} - u_{tt} = g(x, t)$$

satisfying non-homogeneous conditions of form (2).

288. Show that if $u(x, t)$ is a solution of equation (3) then the function

$$v(x, t) = u\left(\frac{x}{x^2 - t^2}, \frac{t}{x^2 - t^2}\right)$$

is also a solution of that equation at each point where it is defined.

289. Using D'Alembert's formula for the solution $u(x, t)$ of the Cauchy problem

$$u_{tt} = a^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad -\infty < x < \infty$$

show that if the functions $\varphi(x)$ and $\psi(x)$ are simultaneously odd then $u(x, t)|_{x=0} = 0$ and if they are simultaneously even then $u_x(x, t)|_{x=0} = 0$.

290. Show that if the function $f(x, t)$ in the Cauchy problem

$$u_{tt} = a^2 u_{xx} + f(x, t), \quad -\infty < x < \infty, \quad t > 0$$
$$u(x, 0) = u_t(x, 0) = 0, \quad -\infty < x < \infty$$

is odd with respect to x then $u(x, t)|_{x=0} = 0$ and if it is even then $u_x(x, t)|_{x=0} = 0$.

Using the results established in Problems 289 and 290 and extending in a proper manner the Cauchy data to the whole number line $-\infty < x < \infty$ solve the following problems for the half-line $x > 0$:

291. $u_{tt} = a^2 u_{xx}, \quad x > 0, \quad t > 0,$
$u(0, t) = 0, \quad t > 0; \quad u(x, 0) = \varphi(x),$
$u_t(x, 0) = \psi(x), \quad x > 0.$

292. $u_{tt} = a^2 u_{xx}, \quad x > 0, \quad t > 0,$
$u_x(0, t) = 0, \quad t > 0, \quad u(x, 0) = \varphi(x),$
$u_t(x, 0) = \psi(x), \quad x > 0.$

293. $u_{tt} = a^2 u_{xx} + f(x, t), \ x > 0, \ t > 0, \quad u(0, t) = 0,$
$t > 0; \quad u(x, 0) = u_t(x, 0) = 0, \quad x > 0.$

294. $u_{tt} = a^2 u_{xx} + f(x, t), \ x > 0, \ t > 0,$
$u_x(0, t) = 0, \quad t > 0;$
$u(x, 0) = u_t(x, 0) = 0, \quad x > 0.$

295. $u_{tt} = a^2 u_{xx} + f(x, t), \quad x > 0, \quad t > 0,$
$u(0, t) = 0, \quad t > 0, \quad u(x, 0) = \varphi(x),$
$u_t(x, 0) = \psi(x), \quad x > 0.$

296. $u_{tt} = a^2 u_{xx} + f(x, t), \quad x > 0, \quad t > 0,$
$u_x(0, t) = 0, \ t > 0, \ u(x, 0) = \varphi(x),$
$u_t(x, 0) = \psi(x), \ x > 0.$

Every function $f(x - at)$ dependent on the argument $x - at$ is referred to as a *direct wave*.

Find the solutions of the following problems in the form of a direct wave generated by the propagation of the end point perturbation:

297. $u_{tt} = a^2 u_{xx}, \quad x > 0, \ t > 0,$
$u(0, t) = \mu(t), \quad t > 0, \quad u(x, 0) = u_t(x, 0) = 0,$
$x > 0.$

298. $u_{tt} = a^2 u_{xx}, \quad x > 0, \quad t > 0,$
$u_x(0, t) = \nu(t), \quad t > 0; \quad u(x, 0) = u_t(x, 0) = 0,$
$x > 0.$

299. $u_{tt} = a^2 u_{xx}, \quad x > 0, \quad t > 0,$
$u_x(0, t) - hu(0, t) = \varkappa(t), \quad t > 0, \quad h > 0,$
$u(x, 0) = u_t(x, 0) = 0, \quad x > 0.$

Solve the following problems:

300. $u_{tt} = a^2 u_{xx} + f(x, t), \quad x > 0, \quad t > 0,$
$u(0, t) = \mu(t), \ t > 0, \quad u(x, 0) = \varphi(x),$
$u_t(x, 0) = \psi(x), \quad x > 0.$

301. $u_{tt} = a^2 u_{xx} + f(x, t), \quad x > 0, \quad t > 0,$
$u_x(0, t) = \nu(t), \quad t > 0, \quad u(x, 0) = \varphi(x),$
$u_t(x, 0) = \psi(x), \quad x > 0.$

302. Find the solution $u(x, y, t)$ of the equation

$$u_{xx} + u_{yy} - u_{tt} = xyt$$

satisfying the initial conditions

$$u(x, y, 0) = 0, \quad u_t(x, y, 0) = xy$$

303. Consider the formula

$$u(x, y, t) = \sum_{k \geqslant 0} \frac{(-1)^k \rho^{2k+2} \square^k \Phi}{[2 \cdot 4 \dots (2k+2)] \{(2n-1)(2n-3) \dots [2n-(2k-1)]\}}$$

where $\rho^2 = x^2 + y^2 - t^2$, $\square = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial t^2}$ and Φ is a homogeneous polynomial of degree $n-2$ in the variables x, y and t. Prove that the function $u(x, y, t)$ satisfies the non-homogeneous equation

$$u_{xx} + u_{yy} - u_{tt} = \Phi(x, y, t).$$

304. Show by means of the direct verification that if $u(x, t)$ $(x = (x_1, \dots, x_n))$ is a solution of equation (1) then the function

$$v(x, t) = \frac{1}{(|x|^2 - t^2)^{\frac{n-2}{2}}} u\left(\frac{x}{|x|^2 - t^2}, \frac{t}{|x|^2 - t^2}\right)$$

where

$$|x|^2 = \sum_{i=1}^{n} x_i^2, \quad |x|^2 \neq t^2$$

is also a solution of that equation.

305. Find all linearly independent homogeneous polynomials of the 3rd degree in the variables x_1, x_2 and t satisfying equation (4).

306. What is the number of all linearly independent homogeneous polynomials of the kth degree in the variables $x_1, \dots, x_n$ and t satisfying equation (1)?

307. Let $u(x, t)$ be a function possessing continuous partial derivatives of the third order and satisfying equation

(3). Show that the function

$$v(x, t) = \frac{\partial u}{\partial x} \frac{\partial u}{\partial t}$$

also satisfies the same equation.

308. Show that if $u(x, t)$ is a function satisfying equation (3) then the following functions also satisfy that equation:

(a) $xu_x + tu_t$,

(b) $u_x^2 + u_t^2$,

(c) $\dfrac{u_t}{u_x^2 - u_t^2}$, $\quad u_x^2 \neq u_t^2$.

309. Determine the value of the exponent k = const for which equation (1) possesses a solution of the form

$$u(x, t) = \frac{1}{(|x|^2 - t^2)^k} \quad \left(|x|^2 = \sum_{i=1}^{n} x_i^2\right)$$

310. Show that if $u(x, t)$ is a solution of equation (1) and a_i $(i = 1, \ldots, n + 1)$ are constants of one sign then the function

$$v(x, t) = u\left(\frac{x_1}{\sqrt{|a_1|}}, \ldots, \frac{x_n}{\sqrt{|a_n|}}, \frac{t}{\sqrt{|a_{n+1}|}}\right)$$

satisfies the hyperbolic partial differential equation

$$\sum_{i=1}^{n} a_i u_{x_i x_i} - a_{n+1} u_{tt} = 0$$

311. Find the relationship between the constants m_i $(i = 1, \ldots, n + 1)$ for which equation (1) possesses a solution $u(x, t)$ of the form of a *plane wave*:

$$u(x, t) = \Phi(m_1 x_1 + \ldots + m_n x_n + m_{n+1} t)$$

312. Prove that the most general expression for the solutions of equation (5) depending solely on r and t is of the form

$$u(r, t) = \frac{f_1(r+t)}{r} + \frac{f_2(r-t)}{r}, \quad r \neq 0$$

where $r^2 = x_1^2 + x_2^2 + x_3^2$, f_1 and f_2 being arbitrary twice continuously differentiable functions.

Such solutions are referred to as *spherical waves.*

313. Verify directly that if a function $u(x, t)$ possessing partial derivatives of the third order is a solution of equation (1) then so is the function

$$v(x, t)=\sum_{i=1}^{n} x_i u_{x_i}+tu_t$$

314. Show that the expression

$$u(x_1, x_2, x_3, t)=\Box\,[\varphi(r+t)+\psi(r-t)]$$

where $\Box=\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}-\frac{\partial^2}{\partial t^2}$, $r^2=x_1^2+x_2^2+x_3^2$, and φ and ψ are arbitrary functions possessing continuous partial derivatives up to the third order, satisfies equation (5).

315. Find the solution of the Cauchy problem for equation (5) with the initial conditions

$$u(x, 0)=\varphi(r), \quad u_t(x, 0)=\psi(r) \quad (r^2=x_1^2+x_2^2+x_3^2)$$

where φ and ψ are given twice continuously differentiable functions.

Find the solutions of equation (5) satisfying the following initial conditions:

316. $u(x, 0)=x_1x_2x_3, \quad u_t(x, 0)=x_1^2x_2^2x_3^2$.

317. $u(x, 0)=r^2, \quad u_t(x, 0)=x_1x_2$.

318. $u(x, 0)=e^{x_1}\cos x_2, \quad u_t(x, 0)=x_1^2-x_2^2$.

319. $u(x, 0)=x_1^2+x_2^2, \quad u_t(x, 0)=1$.

320. $u(x, 0)=e^{x_1}, \quad u_t(x, 0)=e^{-x_1}$.

321. $u(x, 0)=1/x_1, \quad u_t(x, 0)=0, \quad x_1\neq 0, \quad x^2\neq t^2$.

322. Prove the uniqueness of the solution of the Cauchy problem with initial conditions (2) for the equation

$$\sum_{i=1}^{2}\frac{\partial^2 u}{\partial x_i^2}-\frac{\partial^2 u}{\partial t^2}=g(x, t)$$

323. Find the velocity of propagation of the plane wave

$$u(x_1, x_2, x_3, t)=\varphi(m_1x_1+m_2x_2+m_3x_3+mt)$$

324. Can a function of the form

$$u(x_1, x_2, x_3, t)=x_1^2+x_2^2+x_3^2-x_1t^2$$

describe the process of propagation of a wave?

325. Show that the function

$$u(x_1, x_2, x_3, t) = x_1^2 + x_2^2 + x_3^2 + a^2t^2$$

describes a process of propagation of a wave and find the velocity of the wave.

Find the domain of propagation of the waves for the following cases:

326. The velocity of the wave is $a = 5$, $n = 1$ and the initial data $u(x, 0)$ and $u_t(x, 0)$ are prescribed on the segment $l_1 \leqslant x_1 \leqslant l_2$ of the straight line $t = 0$.

327. The velocity of the wave is $a = 1$, $n = 2$ and the initial data $u(x, 0)$ and $u_t(x, 0)$ are prescribed in the annular domain $1 \leqslant x_1^2 + x_2^2 \leqslant 4$ lying in the plane $t = 0$.

328. The velocity of the wave is $a = 2$, $n = 3$ and the initial data $u(x, 0)$ and $u_t(x, 0)$ are prescribed on the ball $x_1^2 + x_2^2 + x_3^2 \leqslant 1$, $t = 0$.

329. The velocity of the wave is $a = 1$, $n = 1$ and the initial data $u(x, 0)$ and $u_t(x, 0)$ are prescribed on the segments $-2 \leqslant x_1 \leqslant -1$ and $1 \leqslant x_1 \leqslant 2$ of the straight line $t = 0$. Determine the set of points in the plane E_2 of the variables x_1 and t which is the common domain of influence of these two line segments.

§ 2. Well-Posed Problems for Hyperbolic Partial Differential Equations

In the foregoing section we considered the Cauchy problem for the wave equation with the initial data $u(x, t_0)$ and $u_t(x, t_0)$ prescribed on the whole plane $t = t_0$ or on its definite part. However, in applications an important role is also played by some more general problems for hyperbolic partial differential equations with the Cauchy data prescribed on manifolds distinct from a plane $t = t_0$ or from its part. In this connection, it should be noted that by far not every manifold (however smooth) can serve for this purpose.

A problem stated for a hyperbolic partial differential equation is said to be *well-posed* (or *correctly set*) if its solution exists, is unique and stable. Here the stability of the solution is understood in the sense that to small variations of the initial data of the problem there correspond small variations of its solution.

In § 1 we defined a *characteristic hypersurface* as a manifold $\varphi(x, t) = 0$ at whose every point the equality

$$Q(\text{grad}\,\varphi) \equiv \sum_{i=1}^{n} \varphi_{x_i}^2 - \varphi_t^2 = 0$$

holds. A hypersurface specified by an equation $\psi(x, t) = 0$ in the space E_{n+1} is said to be *non-characteristic* if at its every point the expression $Q(\text{grad}\,\psi)$ is different from zero. A non-characteristic hypersurface will be called a *space-like hypersurface* if for its every point the inequality

$$Q(\text{grad}\,\psi) \equiv \sum_{i=1}^{n} \psi_{x_i}^2 - \psi_t^2 < 0$$

is fulfilled. Let us denote by S a part of a sufficiently smooth space-like hypersurface. The general statement of the Cauchy problem reads: *it is required to find the solution of equation* (1) *satisfying on S the conditions*

$$u(x, t) = F(M) \quad \text{and} \quad \frac{\partial u}{\partial N}(x, t) = \Phi(M) \tag{11}$$

where $F(M)$ and $\Phi(M)$ are given sufficiently smooth functions of the variable point M running over the hypersurface S and N is a direction which is at no point tangent to S. It can be proved that this Cauchy problem is well-posed.

It should be noted that in the special case of one spatial variable $x_1 = x$ the hypersurface S on which the Cauchy data of type (11) are prescribed is a curve $\psi(x, t) = 0$; in this case not the condition $\psi_x^2 - \psi_t^2 < 0$ but the requirement that $\psi_x^2 - \psi_t^2 \neq 0$ is important.

All that has been said does not mean that when stating the Cauchy problem for hyperbolic partial differential equations we cannot prescribe the data on characteristic hypersurfaces. For instance, in the case when a characteristic hypersurface $\psi(x, t) = 0$ is a cone described by the equation

$$\sum_{i=1}^{n} (x_i - x_i^0)^2 - (t - t_0)^2 = 0 \tag{12}$$

the so-called *Cauchy characteristic problem* can be stated: *it is required to find the solution* $u(x, t)$ *of equation* (1) *regular inside cone* (12) *and assuming on cone* (12) *some prescribed values.*

In the case of one spatial variable $x_1 = x$ a cone of type (12) is nothing other than a pair of straight lines $x - x_0 = t - t_0$ and $x - x_0 = t_0 - t$ passing through the point (x_0, t_0). These straight lines split the plane E_2 of the variables x and t into four angular domains. Let D be one of them. A characteristic problem stated for such a domain D is usually referred to as the *Goursat problem*; its statement reads: *it is required to find the solution* $u(x, t)$ *of equation* (3) *regular in* D *satisfying the conditions*

$$\begin{gathered} u = \varphi \quad \text{for} \quad x - x_0 = t - t_0, \\ u = \psi \quad \text{for} \quad x - x_0 = t_0 - t \\ \varphi(x_0, t_0) = \psi(x_0, t_0) \end{gathered} \tag{13}$$

330. Show that the problem of determining the regular solution $u(x, t)$ of equation (3) corresponding to given values of the function $u(x, t)$ and of its normal derivative $\frac{\partial u}{\partial \nu} = \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right) \Big/ \sqrt{2}$ prescribed on the characteristic $x - t = 0$ is *improperly posed* (*not well posed*). (This problem has no solution at all in those cases when the uniqueness property does not take place.)

331. Find for what values of the constant k Cauchy data (11) can be prescribed for equation (3) on the straight line $x = kt$. Also solve the following problems:

(a) find the solution of this problem in the case when the direction N is specified by the vector with components $(1/\sqrt{2}, 1/\sqrt{2})$ and the Cauchy data are prescribed on the line segment joining the points $A(0, 0)$ and $B(1, 1/k)$ of the straight line N;

(b) determine the domain of dependence, the domain of influence and the domain of propagation;

(c) prove the stability of the solution.

332. Find for what values of the positive constants φ_0 and φ_1 the Cauchy data in Problem (11) for equation (3) can be prescribed on the arc $\varphi_0 \leqslant \varphi \leqslant \varphi_1$ of the circle

$x = \cos\varphi,\ t = \sin\varphi\ (0 \leqslant \varphi \leqslant 2\pi)$ and determine the solution of this problem for the case when N coincides with the normal to the circle.

333. Let S be the arc of a curve $x = f(t)$ joining two points $A(x_0, t_0)$ and $B(x_1, t_1)$. Let the curve $x = f(t)$ possess continuous curvature and have no points at which it is tangent to characteristics of equation (3). Let N be the normal to the arc AB. Construct the solution $u(x, t)$ of Problem (3), (11) for this case.

334. Find the domain of propagation of the wave constructed in Problem 333 and prove the uniqueness of that solution.

335. Find the condition of the constants a, b and c under which the Cauchy data of type (11) for equation (4) can be prescribed on the plane Π specified by the equation $ax_1 + bx_2 + ct = 0$ and construct the solution of the Cauchy problem with the data

$$u = -\frac{a}{c}x_1 - \frac{b}{c}x_2, \qquad \frac{\partial u}{\partial N} = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

on that plane where N is the normal to Π.

336. Find the solution of the Goursat problem for equation (3) with the data

$$u(x, x) = \varphi(x), \quad 0 \leqslant x \leqslant a$$
$$u(x, -x) = \psi(x), \quad 0 \leqslant x \leqslant b$$
$$\varphi(0) = \psi(0)$$

prescribed on the characteristics $x - t = 0$ and $x + t = 0$.

337. Find the domain of propagation of the wave determined in Problem 336 and prove the uniqueness of that solution.

338. Prove the uniqueness of the solution $u(x, t)$ of the Cauchy characteristic problem for equation (4) with data prescribed on the lower part of the characteristic cone

$$x_1^2 + x_2^2 - (t-1)^2 = 0$$

339. Let us denote by S the part of the characteristic cone $x^2 + y^2 - t^2 = 0$ lying between the origin $(0, 0, 0)$ and the plane $t = -h\ (h > 0)$. Find the solution $u(x, y, t)$ of the following Cauchy characteristic problem:

$$u_{xx} + u_{yy} - u_{tt} = xyt, \quad u\,|_S = 0$$

340. Determine the domain of propagation for the wave found in Problem 339 and prove the uniqueness of that solution.

341. Consider the Dirichlet problem for equation (3) in a characteristic rectangle, the data for $u(x, t)$ being prescribed on all the sides of the rectangle. Is this problem well-posed?

The problem of determining the solution of equation (1) from given values of $u(x, t)$ is well-posed not only in the case when these data are prescribed on characteristic hypersurfaces of the equation but in some other cases as well. To demonstrate what has been said we shall limit ourselves to the consideration of equation (3).

Let D be a domain lying in the characteristic angle between the straight lines $x - x_0 = t - t_0$ and $x - x_0 = t_0 - t$ $(x \geqslant x_0)$ and bounded by some curves

$$S_1\colon t = s_1(x) \quad \text{and} \quad S_2\colon t = s_2(x) \quad (x \geqslant x_0, \quad s_1(x_0) = s_2(x_0))$$

which possess continuous curvature and satisfy the conditions

$$-1 \leqslant \frac{ds_1}{dx} < \frac{ds_2}{dx} \leqslant 1$$

It can be proved that the following problem is well-posed:

The *Darboux problem: it is required to find the solution* $u(x, t)$ *of equation* (3) *regular in the domain* D *and satisfying the conditions*

$$u|_{S_1} = \varphi(x), \quad u|_{S_2} = \psi(x), \quad x \geqslant x_0$$

where φ *and* ψ *are given twice continuously differentiable functions satisfying the condition*

$$\varphi(x_0) = \psi(x_0)$$

342. Consider the following problem: it is required to determine the solution $u(x, t)$ of equation (3) regular in the first quadrant of the xt-plane and satisfying the conditions

$$u(x, 0) = \varphi(x), \quad 0 \leqslant x < \infty$$
$$u(0, t) = \psi(t), \quad 0 \leqslant t < \infty$$
$$\varphi(0) = \varphi(0), \quad \varphi''(0) = \psi''(0)$$

Is this problem well-posed?

343. Let us denote by D the angle bounded by the straight lines $x = 0$, $t = x/2$, $t \geqslant 0$, $x \geqslant 0$. Find whether the problem of determining the solution of equation (3) in the domain D satisfying the conditions

$$u(0, t) = \varphi(t), \quad u(x, x/2) = \psi(x), \quad t \geqslant 0, \quad x \geqslant 0$$
$$\varphi(0) = \psi(0), \quad \varphi''(0) = \psi''(0)$$

is well-posed.

Problems 344, 345, 347, 350, 353, 354, 355 and 374 reduce to a functional equation of the form

$$P(x) + \mu P[\lambda(x)] = f(x) \tag{14}$$

whose solution, for instance, under the condition $|\mu^m f[\lambda^m(x)]| < M^m$, can be constructed by the iteration method in the form

$$P(x) = \sum_{m=0}^{\infty} (-1)^m \mu^m f[\lambda^m(x)] \tag{15}$$

Here M is a constant such that $0 < M < 1$, by μ^m is meant the ordinary mth power of μ while $\lambda^m(x)$ is understood in the sense that

$$\lambda^m(x) = \lambda^{m-1}[\lambda(x)], \quad \lambda^0(x) = x$$

344. Let D be the angular domain lying between the straight lines $t = k_1 x$ and $t = k_2 x$ $(x \geqslant 0)$ where $-1 \leqslant k_1 < k_2 \leqslant 1$. Find the regular solution of equation (3) in the domain D satisfying the conditions

$$u(x, k_1 x) = \varphi(x), \quad u(x, k_2 x) = \psi(x), \quad k_1 = 0,$$
$$k_2 = k > 0$$

where φ and ψ are given twice continuously differentiable functions such that $\varphi(0) = \psi(0)$.

345. Put $k_1 = -1/4$, $k_2 = 1/4$, $0 \leqslant x \leqslant a$, $\varphi(x) = x$ and $\psi(x) = x$ in Problem 344 and prove the existence and the uniqueness of the solution $u(x, t)$.

346. Determine the domain of propagation of the wave corresponding to the solution $u(x, t)$ of Problem 345.

347. Let us denote by D the angular domain between the straight lines $t = x/4$ and $t = 0$ $(x \geqslant 0)$. Find the regular solution $u(x, t)$ of equation (3) in the domain D satisfying the conditions

$$u(x, x/4) = x \quad \text{and} \quad u(x, 0) = \sin x$$

348. Determine the domain of propagation of the wave found in Problem 347 on condition that $0 \leqslant x \leqslant 1$.

Find the solutions of equation (3) and the corresponding domains of propagation for the following conditions:

349. $u(x, 0) = \varphi(x)$, $u(x, x) = \psi(x)$, $0 \leqslant x \leqslant a$, $\varphi(0) = \psi(0)$.

350. $u(x, 0) = \varphi(x)$, $u(x, x/2) = \psi(x)$, $0 \leqslant x \leqslant 2/3$, $\varphi(0) = \psi(0)$.

351. $u(0, t) = t^2$, $u(t, t) = t^3$, $0 \leqslant t \leqslant 2$.

352. $u(0, t) = \sin t$, $0 \leqslant t \leqslant 1$; $u(t, t) = 0$, $0 \leqslant t \leqslant 2$.

353. $u(x, 0) = \varphi(x)$, $u[x, \tau(x)] = \psi(x)$, $0 \leqslant x \leqslant 1$, $\varphi(0) = \psi(0)$,

where τ is a given twice continuously differentiable function satisfying the condition

$$0 < \frac{d\tau}{dx} < 1$$

354. Consider the solution $u(x, t)$ of equation (3) with the data prescribed on the following two arcs of the curves:

$$t = \sin x, \qquad 0 \leqslant x \leqslant \pi/4$$

and

$$t = -\sin x, \quad 0 \leqslant x \leqslant \pi/4$$

Let the data be

$$u(x, \sin x) = x \quad \text{and} \quad u(x, -\sin x) = x$$

Determine the wave $u(x, t)$ and its domain of propagation.

355. Let the data for the solution $u(x, t)$ of equation (3) be prescribed on the arc of the parabola $t = x^2/4$, $0 \leqslant x \leqslant 1$, and on the segment $0 \leqslant x \leqslant 2$ of the straight line $t = 0$. Let the data be

$$u(x, x^2/4) = x^3 \quad \text{and} \quad u(x, 0) = 0$$

Find this solution $u(x, t)$ of equation (3) and its domain of propagation.

356. Find the solution $u(x, t)$ of equation (3) satisfying the conditions

$$u(x, x) = \varphi(x), \quad \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t}\right)\Big|_{t=-x} = \psi(x), \quad 0 \leqslant x < \infty$$

and prove the uniqueness of the solution.

357. Determine the solution $u(x, t)$ of equation (3) satisfying the conditions

$$\frac{\partial u}{\partial t}\Big|_{t=0} = \varphi(x), \quad 0 \leqslant x \leqslant a$$

and

$$u(x, x) = \psi(x), \quad 0 \leqslant x \leqslant b$$

and determine the domain of propagation for this solution.

358. Find whether the problem of determining the solution of equation (3) satisfying the conditions

$$u(x, x) = \varphi(x), \quad 0 \leqslant x < \infty$$

and

$$\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right)\Big|_{t=x} = \psi(x), \quad 0 \leqslant x < \infty$$

is well-posed.

§ 3. Some Other Classes of Hyperbolic Partial Differential Equations. The Cauchy Problem for Laplace's Equation

The problems considered in the foregoing section can also be stated for the general hyperbolic partial differential equation of the form

$$\sum_{i, j=1}^{n} A_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - \frac{\partial^2 u}{\partial t^2} + \sum_{i=1}^{n} B_i \frac{\partial u}{\partial x_i} + Cu = F \quad (16)$$

As a manifold on which Cauchy data (11) are prescribed for equation (16) we can take the one specified by an equation of the form $\varphi(x, t) = 0$ where the function $\varphi(x, t)$ satisfies the condition

$$\sum_{i, j=1}^{n} A_{ij} \varphi_{x_i} \varphi_{x_j} - \varphi_t^2 < 0$$

As in the foregoing section, in the Cauchy characteristic problem for equation (16) the data are prescribed on a characteristic hypersurface $\varphi(x, t) = 0$ where, by definition, the function $\varphi(x, t)$ satisfies the condition

$$\sum_{i, j=1}^{n} A_{ij}\varphi_{x_i}\varphi_{x_j} - \varphi_t^2 = 0$$

In the case of one spatial variable $x = x_1$ it is most convenient to write equation (16) in the form

$$Lu = \frac{\partial^2 u}{\partial\xi\,\partial\eta} + a(\xi, \eta)\frac{\partial u}{\partial\xi} + b(\xi, \eta)\frac{\partial u}{\partial\eta} + c(\xi, \eta)u = F(\xi, \eta) \tag{17}$$

In the theoretical study of equation (17) an important role is played by the so-called *Riemann function* $R(\xi, \eta; \xi_1, \eta_1)$ dependent on two points (ξ, η) and (ξ_1, η_1) and possessing the following properties:

(a) the expression $R(\xi, \eta; \xi_1, \eta_1)$ considered as a function of the variables ξ and η is a solution of the equation

$$L^*R = \frac{\partial^2 R}{\partial\xi\,\partial\eta} - \frac{\partial}{\partial\xi}(aR) - \frac{\partial}{\partial\eta}(bR) + cR = 0$$

adjoint to (17); as a function of the variables ξ_1 and η_1 this expression is a solution of the equation $LR = 0$ in which instead of ξ and η are meant the variables ξ_1 and η_1;

(b) $$\frac{\partial R(\xi_1, \eta; \xi_1, \eta_1)}{\partial\eta} - a(\xi_1, \eta)R(\xi_1, \eta; \xi_1, \eta_1) = 0,$$

$$\frac{\partial R(\xi, \eta_1; \xi_1, \eta_1)}{\partial\xi} - b(\xi, \eta_1)R(\xi, \eta_1; \xi_1, \eta_1) = 0$$

and

$$R(\xi_1, \eta_1; \xi_1, \eta_1) = 1;$$

(c) $$\frac{\partial R(\xi, \eta; \xi, \eta_1)}{\partial\eta_1} + a(\xi, \eta_1)R(\xi, \eta; \xi, \eta_1) = 0,$$

$$\frac{\partial R(\xi, \eta; \xi_1, \eta)}{\partial\xi_1} + b(\xi_1, \eta)R(\xi, \eta; \xi_1, \eta) = 0$$

and

$$R(\xi, \eta; \xi, \eta) = 1.$$

These conditions uniquely specify the function $R(\xi, \eta; \xi_1, \eta_1)$ in case the coefficients a and b are functions of class C^1 and the coefficient c belongs to the class C^0.

If the Riemann function is known it is possible to express in quadratures the solutions of equation (17) both for the Cauchy problem and for the Goursat problem.

The solution of the Goursat problem for equation (17) with the conditions

$$u(\xi, \eta_0) = \varphi(\xi), \quad u(\xi_0, \eta) = \psi(\eta), \quad \varphi(\xi_0) = \psi(\eta_0)$$

where φ and ψ are given continuously differentiable functions, is given by the formula

$$\begin{aligned} u(\xi, \eta) = {} & R(\xi, \eta_0; \xi, \eta)\,\varphi(\xi) + R(\xi_0, \eta; \xi, \eta)\,\psi(\eta) - \\ & - R(\xi_0, \eta_0; \xi, \eta)\,\varphi(\xi_0) + \\ & + \int_{\xi_0}^{\xi} \left[b(t, \eta_0) R(t, \eta_0; \xi, \eta) - \frac{\partial}{\partial t} R(t, \eta_0; \xi, \eta) \right] \varphi(t)\,dt + \\ & + \int_{\eta_0}^{\eta} \left[a(\xi_0, \tau) R(\xi_0, \tau; \xi, \eta) - \frac{\partial}{\partial \tau} R(\xi_0, \tau; \xi, \eta) \right] \psi(\tau)\,d\tau + \\ & + \int_{\xi_0}^{\xi} dt \int_{\eta_0}^{\eta} R(t, \tau; \xi, \eta) F(t, \tau)\,d\tau. \end{aligned} \tag{18}$$

Let σ be a non-closed Jordan curve possessing continuous curvature which is not tangent to the characteristics of equation (17) at any of its points. The solution of the Cauchy problem for equation (17) with prescribed values of u and $\frac{\partial u}{\partial N} = \frac{\partial \xi}{\partial \nu}\frac{\partial u}{\partial \eta} + \frac{\partial \eta}{\partial \nu}\frac{\partial u}{\partial \xi}$ on σ where ν is the outer normal to σ at the point (ξ, η) has the form

$$\begin{aligned} u(P) = {} & \frac{1}{2} u(Q) R(Q, P) + \frac{1}{2} u(Q') R(Q', P) + \\ & + \int_G F(P') R(P', P)\, d\xi_1\, d\eta_1 - \\ & - \frac{1}{2} \int_{QQ'} \left[\frac{\partial u(P')}{\partial N} R(P', P) - u(P') \frac{\partial R(P', P)}{\partial N} \right] d\sigma_{P'} - \\ & - \int_{QQ'} \left[a(P') \frac{\partial \xi_1}{\partial N} + b(P') \frac{\partial \eta_1}{\partial N} \right] R(P', P)\, u(P')\, d\sigma_{P'} \end{aligned} \tag{19}$$

where Q' and Q are the points of intersection of the arc σ with the characteristics $\xi_1 = \xi$ and $\eta_1 = \eta$ issued from the point $P(\xi, \eta)$ and G is the finite domain in the plane of the variables ξ, η bounded by the part QQ' of the arc σ and by the segments PQ and PQ' of the characteristics.

The expression

$$\int_G F(P')\, R(P', P)\, d\xi_1\, d\eta_1$$

is a particular solution of non-homogeneous equation (17).

359. Show that the Riemann function $R(\xi, \eta; \xi_1, \eta_1)$ for equation (3) expressed in terms of the characteristic variables is identically equal to unity.

360. Using Riemann's function mentioned in Problem 359 write down the solutions of the Cauchy problem and of the Goursat problem for equation (3).

361. Verify directly that the formula

$$R(\xi, \eta; \xi_1, \eta_1) = J_0\left(\mu \sqrt{(\xi - \xi_1)(\eta - \eta_1)}\right)$$

where $\mu^2 = -\lambda$ specifies the Riemann function for the equation

$$u_{xx} - u_{tt} + \lambda u = 0 \tag{20}$$

expressed in terms of the variables $\xi = x + t$ and $\eta = x - t$.

Using Riemann's function mentioned in Problem 361 express in quadratures the solutions of equation (20) satisfying the following conditions:

362. $u(x, 0) = \varphi(x)$, $u_t(x, 0) = \psi(x)$.

363. $u(x, x) = \varphi(x)$, $u(x, -x) = \psi(x)$, $0 \leqslant x < \infty$, $\varphi(0) = \psi(0)$.

364. Construct the solution of the equation

$$u_{xx} - u_{tt} + \lambda u = 1$$

satisfying the conditions $u(x, x) = u(x, -x) = 0$.

Find the solutions of the Cauchy problem with the data

$$u(0, t) = \varphi(t), \quad u_x(0, t) = \psi(t)$$

and of the Goursat problem with the data

$$u(x, x) = \varphi(x), \quad u(x, -x) = \psi(x), \quad x \geqslant 0, \quad \varphi(0) = \psi(0)$$

for the following partial differential equations:

365. $u_{xx}-u_{tt}+au_x+\frac{a^2}{4}u=0,\ a=\text{const}.$

366. $u_{xx}-u_{tt}+bu_t-\frac{b^2}{4}u=0,\ b=\text{const}.$

367. $u_{xx}-u_{tt}+au_x+bu_t+\frac{a^2}{4}u-\frac{b^2}{4}u=0,$
$a=\text{const},\quad b=\text{const}.$

368. For the equations indicated in Problems 365-367 find the solutions satisfying the conditions

$$u(x, 0)=\varphi(x),\quad u(x, x)=\psi(x),\quad \varphi(0)=\psi(0)$$

369. Show that the general solution of the system of partial differential equations

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}=0,\quad \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0$$

has the form

$$u(x, y)=f(x+y)+f_1(x-y),\quad v(x, y)=$$
$$=f(x+y)-f_1(x-y)$$

where f and f_1 are arbitrary continuously differentiable functions.

For the system of partial differential equations indicated in Problem 369 construct the solutions satisfying the following conditions:

370. $u(x, 0)=\varphi(x),\quad v(x, 0)=\psi(x).$

371. $u(x, x)=\varphi(x),\quad v(x, -x)=\psi(x),\quad x\geqslant 0.$

372. $u(x, 0)=\varphi(x),\quad v(x, -x)=\psi(x),\quad x\geqslant 0.$

373. $u(x, 0)=\varphi(x),\quad v(x, x)=\psi(x),\quad x\geqslant 0.$

374. $u(x, 0)=\varphi(x),\quad v(x, -x/2)=\psi(x),\quad x\geqslant 0,$
$\varphi(0)=0,\quad \psi(0)=0,$

where φ and ψ are given continuously differentiable functions.

375. Prove that the system of partial differential equations

$$a\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\quad \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0$$

is hyperbolic if and only if $a > 0$; also show that for $a = \text{const} > 0$ its general solution has the form

$$u(x, y) = \frac{1}{\sqrt{a}} f(x + \sqrt{a}y) + \frac{1}{\sqrt{a}} f_1(x - \sqrt{a}y)$$

$$v(x, y) = -f(x + \sqrt{a}y) + f_1(x - \sqrt{a}y)$$

where f and f_1 are arbitrary continuously differentiable functions.

376. For the system of partial differential equations indicated in Problem 375 find the solution satisfying the conditions

$$u\left(x, \frac{1}{\sqrt{a}} x\right) = \varphi(x), \quad v\left(x, -\frac{1}{\sqrt{a}} x\right) = \psi(x), \quad x \geqslant 0$$

where φ and ψ are given real continuously differentiable functions.

377. Find the relationship connecting the real constants a, b and c for which the hyperbolic partial differential equation

$$a^2 u_{xx} + b^2 u_{yy} - c^2 u_{zz} = 0$$

possesses solutions of the form

$$u(x, y, z) = f(\alpha x + \beta y + \gamma z)$$

where α, β and γ are real constants and f is an arbitrary twice continuously differentiable function.

378. Show that the equation indicated in Problem 377 possesses a solution of the form

$$u(x, y, z) = \sum_{n=0}^{\infty} \frac{1}{c^{2n}} \frac{z^{2n}}{(2n)!} \left(a^2 \frac{\partial^2}{\partial x^2} + b^2 \frac{\partial^2}{\partial y^2}\right)^n \tau\left(\frac{x}{a}, \frac{y}{b}\right) +$$

$$+ \sum_{n=0}^{\infty} \frac{1}{c^{2n+1}} \frac{z^{2n+1}}{(2n+1)!} \left(a^2 \frac{\partial^2}{\partial x^2} + b^2 \frac{\partial^2}{\partial y^2}\right)^n \nu\left(\frac{x}{a}, \frac{y}{b}\right)$$

where τ and ν are polynomials.

379. For the equation indicated in Problem 377 find the solution of the Cauchy problem with the data

$$u(x, y, 0) = x^2 - y^2, \quad u_z(x, y, 0) = xy$$

380. Verify directly that the function

$$u(x, y) = \frac{\pi \sqrt[3]{4}}{3\Gamma^3\left(\frac{1}{3}\right)} \int_0^1 \tau\left[x + \frac{2}{3}(-y)^{\frac{3}{2}}(2t - 1)\right] t^{-\frac{5}{6}}(1-t)^{-\frac{5}{6}}\, dt + \frac{\sqrt[3]{6}\,\Gamma^3\left(\frac{1}{3}\right)}{4\pi^2}\left(\frac{4}{3}\right)^{\frac{2}{3}} y \int_0^1 \nu\left[x + \frac{2}{3}(-y)^{\frac{3}{2}}(2t-1)\right] t^{-\frac{1}{6}}(1-t)^{-\frac{1}{6}}\, dt$$

is the solution of the Cauchy problem with the data

$$u(x, 0) = \tau(x), \quad \frac{\partial u(x, 0)}{\partial y} = \nu(x), \quad 0 \leqslant x \leqslant 1$$

for the *Tricomi equation*

$$yu_{xx} + u_{yy} = 0$$

for $y < 0$.

381. Show that the function

$$u(x, t) = f(t + ax) + \varphi(t + bx) + \psi(t + cx)$$

where f, φ and ψ are arbitrary functions possessing continuous partial derivatives up to the third order, is a solution of the hyperbolic partial differential equation

$$\frac{\partial^3 u}{\partial x^3} - (a+b+c)\frac{\partial^3 u}{\partial x^2\, \partial t} + (ab+ac+bc)\frac{\partial^3 u}{\partial x\, \partial t^2} - abc\frac{\partial^3 u}{\partial t^3} = 0$$

382. For the equation considered in Problem 381 solve the Cauchy problem with the data

$$u(x, 0) = \varphi_1(x), \quad u_t(x, 0) = \varphi_2(x), \quad u_{tt}(x, 0) = \varphi_3(x)$$

383. Determine the type of the system of partial differential equations

$$\left.\begin{aligned} u_{xx} + u_{yy} - 2v_{xy} = 0 \\ v_{xx} + v_{yy} - 2u_{xy} = 0 \end{aligned}\right\}$$

and show that the functions

$$u(x, y) = (x - y)\varphi(x + y) + (x + y)\varphi_1(x - y) + \psi(x + y) + \psi_1(x - y)$$

and

$$v(x, y) = (x - y)\varphi(x + y) - (x + y)\varphi_1(x - y) + \\ + \psi(x + y) - \psi_1(x - y)$$

where φ, φ_1, ψ and ψ_1 are arbitrary twice continuously differentiable functions, form a solution of that system.

384. For the angular region bounded by the straight lines $y = x/2$ and $y = -x/2$ $(x > 0)$ find the solution of the system of partial differential equations considered in Problem 383 satisfying the conditions

$$u(x, x/2) = \tau(x), \quad u(x, -x/2) = \nu(x)$$
$$v(x, x/2) = \tau_1(x), \quad v(x, -x/2) = \nu_1(x), \quad x \geqslant 0$$
$$\tau(0) = \nu(0), \quad \tau_1(0) = \nu_1(0)$$

where τ, τ_1, ν and ν_1 are given twice continuously differentiable functions.

385. For the system of partial differential equations indicated in Problem 383 construct the solution of the Cauchy problem with the data

$$u(x, 0) = \tau_1(x), \quad v(x, 0) = \tau_2(x)$$
$$u_y(x, 0) = \nu_1(x), \quad v_y(x, 0) = \nu_2(x)$$

386. Find for what values of the real constants a, b, c and k the system of partial differential equations

$$\left.\begin{aligned} au_x + bu_y + kcv_x = 0 \\ av_x + bv_y + \frac{c}{k} u_x = 0 \end{aligned}\right\}$$

is hyperbolic and construct the general solution of the system.

387. Find for what values of the constants a, b, c and k guaranteeing the hyperbolicity of the system of partial differential equations considered in Problem 386 the Cauchy data can be prescribed on the straight line $y = 0$.

388. Construct the solution $u(x, t)$ of the Cauchy problem for Laplace's equation $u_{xx} + u_{yy} = 0$ with the conditions

$$u(x, 0) = p_n(x), \quad u_y(x, 0) = q_m(x)$$

under the assumption that $p_n(x)$ and $q_m(x)$ are polynomials of degrees n and m respectively.

389. For Laplace's equation $u_{xx} + u_{yy} = 0$ construct the solution $u(x, y)$ of the Cauchy problem with the data

$$u(x, 0) = 0, \quad u_y(x, 0) = \frac{\sin nx}{n}$$

and prove the instability of that solution.

390. Let D be the domain in the plane of the variables x, t bounded by the line segment with end points $A(0, 0)$ and $B(1, 0)$ lying on the straight line $t = 0$ and by the characteristics $x + t = 0$ and $x - t - 1 = 0$ of equation (3). Show that the regular solution $u(x, t)$ of equation (3) in the domain D which is continuous in $\bar{D}$ and is equal to zero on the characteristic $x + t = 0$ attains its extremum in D on the line segment AB.

391. Show that the Cauchy problem for the equation

$$y^2 u_{xx} + y u_{yy} + \frac{1}{2} u_y = 0$$

with the data

$$u(x, 0) = \varphi(x), \quad u_y(x, 0) = \psi(x), \quad 0 \leq x < 1$$

is improperly posed (not well posed) for $y < 0$.

392. For the equation considered in Problem 391 find, for $y < 0$, the solution $u(x, y)$ satisfying the conditions

$$u(x, 0) = \tau(x), \quad \lim_{y \to -0} (-y)^{-1/2} \frac{\partial u(x, y)}{\partial y} = \nu(x), \quad 0 < x < 1$$

393. Show that the general solution of the partial differential equation

$$u_{xx} - y u_{yy} - \frac{1}{2} u_y = 0, \qquad y > 0$$

has the form

$$u(x, y) = f_1(x + 2y^{1/2}) + f_2(x - 2y^{1/2})$$

where f_1 and f_2 are arbitrary twice continuously differentiable functions.

394. For the equation considered in Problem 393 find the solution $u(x, y)$ satisfying the conditions

$$u(x, 0) = \tau(x), \quad 0 < x < 1; \quad |\lim_{y \to +0} u_y| < \infty.$$

395. For $y > 0$ find the solution of the equation considered in Problem 393 satisfying the conditions

$$u(x, 0) = \tau(x), \quad \lim_{y \to +0} y^{1/2} \frac{\partial u}{\partial y} = \nu(x)$$

396. Show that the general solution of the hyperbolic partial differential equation

$$\frac{\partial^4 u}{\partial x^4} - 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0 \tag{21}$$

has the form

$$u(x, y) = (x + y)\varphi(x - y) + (x - y)\psi(x + y) + \\ + \rho_1(x - y) + \psi_1(x + y)$$

where φ, φ_1, ψ and ψ_1 are arbitrary functions possessing partial derivatives up to the fourth order.

397. For equation (21) find the solution of the Cauchy problem with the conditions

$$u(x, 0) = \tau(x), \quad u_y(x, 0) = 0$$
$$u_{yy}(x, 0) = 0, \quad u_{yyy}(x, 0) = 0$$

398. Determine the solution $u(x, y)$ of equation (21) satisfying the conditions

$$u(x, x) = \tau_1(x), \quad u(x, -x) = \tau_2(x)$$
$$\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right)\Big|_{y=-x} = \tau_3(x), \quad \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)\Big|_{y=x} = \tau_4(x), \quad x \geqslant 0$$
$$\tau_1(0) = \tau_2(0), \quad \tau_1'(0) = \tau_2'(0)$$
$$\tau_2'(0) = \tau_3(0) = \tau_4(0), \quad \tau_3'(0) = \tau_4'(0)$$

399. Show that the general solution of the partial differential equation

$$\frac{\partial^3 u}{\partial x^3} - \frac{\partial^3 u}{\partial x \partial y^2} = 0 \tag{22}$$

has the form

$$u(x, y) = f_1(x + y) + f_2(x - y) + f_3(y)$$

where f_1, f_2 and f_3 are arbitrary sufficiently smooth functions.

400. Is the problem of determining the solution of equation (22) with the data

$$u(x, 0) = \varphi_1(x), \; u_y(x, 0) = \varphi_2(x), \quad u_{yy}(x, 0) = \varphi_3(x)$$

well-posed?

401. Determine the solution $u(x, y)$ of equation (22) for the data

$$u(0, y) = \varphi_1(y), \quad u_x(0, y) = \varphi_2(y), \quad u_{xx}(0, y) = \varphi_3(y)$$

Chapter 4

PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

§ 1. Heat Conduction Equation

As was mentioned in § 4 of Chapter 1, under certain assumptions the study of such phenomena as conduction of heat, diffusion etc. leads to the equation

$$\sum_{i=1}^{n} u_{x_i x_i} - u_t = 0 \tag{1}$$

known as the *heat conduction equation*; this is a typical example of a parabolic partial differential equation.

Let D be a domain in the space of the variables (x, t) possessing the property that its intersection with every plane $t = T$ where $T_0 \leqslant T \leqslant T_1$ is a simply connected n-dimensional domain in the space of the variables $x_1, \ldots, x_n$. Let us denote by S the union of the lateral part of the surface bounding the domain D and its lower base $t = T_0$.

The so-called *first boundary-value problem* or the *Dirichlet problem* for equation (1) is stated as follows: *it is required to find the solution $u(x, t)$ of equation* (1) *regular in the domain D including its upper base $t = T_1$ and assuming prescribed values on S:*

$$u\,|_S = \varphi \tag{2}$$

where φ is a given function.

The so-called *second boundary-value problem* (the *Cauchy-Dirichlet problem*) can also be stated for equation (1): *it is required to determine the solution $u(x, t)$ of equation* (1) *regular in the half-space $t > 0$ and satisfying the condition*

$$u(x, 0) = \varphi(x) \tag{3}$$

where $\varphi(x_1, \ldots, x_n)$ is a given function.

402. Derive a partial differential equation for the function $v(\xi, \eta) = u(\eta, \xi - a\eta)$ where a is a constant and $u(x, t)$ is a solution of equation (1).

403. Show that the function $u(x, t)$ defined as the sum of the series

$$u(x, t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta^k \tau(x_1, \ldots, x_n) \tag{4}$$

such that it can be differentiated termwise the required number of times is a solution of equation (1).

404. Verify directly that the function

$$E(x, t) = \frac{1}{(t-t_0)^{n/2}} \exp\left[-\frac{\sum_{i=1}^{n} (x_i - y_i)^2}{4(t-t_0)}\right]$$

where $y_1, \ldots, y_n$ are real parameters, is a solution of equation (1) for $t > t_0$. (This function is called the *fundamental* (*elementary*) *solution* of equation (1).)

405. Show that if $u(x, t)$ is a solution of equation (1) then so is the function $u(\lambda x, \lambda^2 t)$ ($\lambda = \text{const}$) in the whole domain of its definition.

406. Prove that for equation (1) there holds the following *extremum principle*: *a solution of equation* (1) *regular in a domain D with boundary S and continuous in $D \cup S$ attains its extremum on the boundary S of D.*

407. Prove the uniqueness property for the solution of Problem (1), (2).

408. Prove that for $n = 2$, in the prismatic domain D: $0 < t < T$, $0 < x_1 < l_1$, $0 < x_2 < l_2$, the function

$$u(x_1, x_2, t) = \exp\left[-\pi^2\left(\frac{i^2}{l_1^2} + \frac{j^2}{l_2^2}\right) t\right] \sin\frac{ix_1\pi}{l_1} \sin\frac{jx_2\pi}{l_2}$$

where i and j are natural numbers, is the solution of equation (1) satisfying the conditions

$$u(x_1, x_2, 0) = \sin\frac{ix_1\pi}{l_1} \sin\frac{jx_2\pi}{l_2}, \quad u|_\sigma = 0$$

where σ is the lateral part of the surface bounding of the domain D.

409. For the rectangle $0 < t < T_0$, $0 < x < \pi$ construct the solution $u(x, t)$ of the equation

$$u_{xx} - u_t = 0 \tag{1'}$$

regular in that rectangle and satisfying the boundary conditions

$$u(0, t) = u(\pi, t) = 0, \quad 0 \leqslant t \leqslant T_0$$
$$u(x, 0) = \varphi(x), \qquad 0 \leqslant x \leqslant \pi$$

where φ is a given sufficiently smooth function.

410. Show that the function

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(y) e^{-\frac{(x-y)^2}{4t}} dy, \quad t > 0$$

where $\varphi(y)$ $(-\infty < y < \infty)$ is a given bounded continuous function, is the solution of equation (1′) satisfying the condition

$$u(x, 0) = \varphi(x), \quad -\infty < x < \infty \tag{3'}$$

411. Show that for a solution $u(x, t)$ of equation (1′) regular in the half-space $t > 0$ there hold the inequalities

$$m \leqslant u(x, t) \leqslant M$$

where

$$m = \inf u(x, 0), \quad M = \sup u(x, 0), \quad -\infty < x < \infty$$

412. Prove the uniqueness of the solution $u(x, t)$ of Cauchy-Dirichlet problem (1′), (3′).

413. Verify directly that the function

$$u(x, t) = \int_0^t v(x, t, \tau)\, d\tau$$

where

$$v(x, t, \tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4(t-\tau)}} g(y, \tau)\, dy, \quad t > \tau$$

and $g(x, \tau)$ is a given bounded continuous function defined for $-\infty < x < \infty$, $-\infty < \tau < \infty$, satisfies the equation

$$u_{xx} - u_t = -g(x, t)$$

414. For the rectangle $0 < t < T_0$, $0 < x < 1$ reduce the first boundary-value problem for the equation

$$u_{xx} - u_t = f(x, t) \tag{5}$$

with non-homogeneous conditions

$$u(0, t) = \alpha(t), \; u(1, t) = \beta(t), \quad 0 \leqslant t \leqslant T_0$$

on the sides $x = 0$ and $x = 1$ of the rectangle to the first boundary-value problem with new homogeneous boundary conditions on these sides.

415. Construct a particular solution of equation (5) for the case when

$$f(x, t) = \sin nx \, f_n(t)$$

where $f_n(t)$ is a given continuous function.

416. For $t > T$ construct the solution of the Cauchy-Dirichlet problem for equation (1) with the condition

$$u(x, T) = e^{x_1} \cosh x_2$$

Extend the given data to the whole x-axis in a proper manner and solve the following problems:

417. $u_t = a^2 u_{xx}$, $0 < x < \infty$, $t > 0$,
$u(0, t) = 0$, $t > 0$; $u(x, 0) = \varphi(x)$, $0 < x < \infty$.

418. $u_t = a^2 u_{xx}$, $0 < x < \infty$, $t > 0$,
$u_x(0, t) = 0$, $t > 0$; $u(x, 0) = \varphi(x)$, $0 < x < \infty$.

419. $u_t = a^2 u_{xx} - hu$, $0 < x < \infty$, $t > 0$,
$u(0, t) = 0$, $t > 0$; $u(x, 0) = \varphi(x)$, $0 < x < \infty$.

420. $u_t = a^2 u_{xx} - hu$, $0 < x < \infty$, $t > 0$,
$u_x(0, t) = 0$, $t > 0$; $u(x, 0) = \varphi(x)$, $0 < x < \infty$.

421. $u_t = a^2 u_{xx} + f(x, t)$; $0 < x < \infty$, $t > 0$,
$u(0, t) = 0$, $t > 0$; $u(x, 0) = 0$, $0 < x < \infty$.

422. $u_t = a^2 u_{xx} + f(x, t)$, $0 < x < \infty$, $t > 0$,
$u_x(0, t) = 0$, $t > 0$; $u(x, 0) = 0$, $0 < x < \infty$.

423. $u_t = a^2 u_{xx} - hu + f(x, t)$, $0 < x < \infty$, $t > 0$,
$u(0, t) = 0$, $t > 0$; $u(x, 0) = 0$, $0 < x < \infty$.

424. $u_t = a^2 u_{xx} - hu + f(x, t)$, $0 < x < \infty$, $t > 0$,
$u_x(0, t) = 0$, $t > 0$; $u(x, 0) = 0$, $0 < x < \infty$.

425. $u_t = a^2 u_{xx} - hu + f(x, t)$, $0 < x < \infty$, $t > 0$,
$u(0, t) = 0$, $t > 0$; $u(x, 0) = \varphi(x)$, $0 < x < \infty$.

426. $u_t = a^2 u_{xx} - hu + f(x, t)$, $0 < x < \infty$, $t > 0$,
$u_x(0, t) = 0$, $t > 0$; $u(x, 0) = \varphi(x)$, $0 < x < \infty$.

Let D be the domain in the space of the variables x, y, t bounded by the planes $t = 0$ and $t = T > 0$ and by the circular cylinder S: $x^2 + y^2 = 1$. Determine the solutions $u(x, y, t)$ of the equation

$$u_{xx} + u_{yy} - u_t = 0$$

regular in D and satisfying the following conditions:

427. $u|_S = -4t, \quad u(x, y, 0) = 1 - x^2 - y^2$.

428. $u|_S = -32t^2 - 16t, \quad u(x, y, 0) = 1 - (x^2 + y^2)^2$.

429. $u|_S = 1 + 4t, \quad u(x, y, 0) = x^2 + y^2$.

430. $u|_S = e^{2t+\cos\varphi+\sin\varphi}, \quad 0 \leqslant \varphi \leqslant 2\pi$;
$u(x, y, 0) = e^{x+y}$.

431. $u|_S = e^t I_0(1), \quad u(x, y, 0) = I_0(r), \quad r^2 = x^2 + y^2$, where $I_0(r) = J_0(ir)$ and J_0 is Bessel's function of order zero.

Construct the solutions of the Cauchy-Dirichlet problem for equation (1) satisfying the following conditions:

432. $u(x, 0) = \sin lx_1$.

433. $u(x, 0) = \cos lx_1$.

434. $u(x, 0) = \cosh lx_1$.

435. $u(x, 0) = \sinh lx_1$.

436. $u(x, 0) = \sin l_1x_1 \sin l_2x_2$.

437. $u(x, 0) = \sin l_1x_1 \cos l_2x_2$.

438. $u(x, 0) = \cos l_1x_1 \cos l_nx_n$.

439. $u(x, 0) = \cos l_1x_1 \sin l_2x_2$.

440. $u(x, 0) = \sin l_1x_1 \sin l_2x_2 \ldots \sin l_nx_n$.

441. $u(x, 0) = \sin l_1x_1 + \cos l_nx_n$.

§ 2. Some Other Examples of Parabolic Partial Differential Equations

442. Find the general solution of the equation

$$a^2u_{xx} + 2au_{xy} + u_{yy} = 0, \quad a = \text{const}$$

443. Show that the function

$$u(x, y, t) = \sum_{k=0}^{\infty} \frac{t^k}{p^k k!} \Delta^k \tau(x, y) \tag{4'}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $\tau(x, y)$ is an arbitrary polynomial in the variables x and y, satisfies the equation

$$u_{xx} + u_{yy} - pu_t = 0, \quad p = \text{const}$$

444. For $t > 1$ solve the Cauchy-Dirichlet problem

$$u_{xx} + u_{yy} - u_t = 0$$
$$u(x, y, 1) = 1 - (x^2 + y^2)^2$$

445. Express in quadratures the solution $u(x, t)$ of the Cauchy-Dirichlet problem

$$u_{xx} - pu_t = 0, \quad p = \text{const} > 0$$
$$u(x, 0) = \varphi(x), \quad -\infty < x < \infty$$

in the half-plane $t > 0$.

446. For the half-plane $x < by$ find the solution of the Cauchy-Dirichlet problem

$$b^2u_{xx} + 2bu_{xy} + u_{yy} + bu_x = 0$$

$$u\left(x, \frac{x}{b}\right) = \varphi(x), \quad -\infty < x < \infty$$

where φ is a given bounded continuous function.

447. Consider the equation indicated in Problem 446 in the parallelogram bounded by the straight lines $y = x/b$, $y = x/b + 1$, $y = 0$ and $y = 1$. For this parallelogram find the solution $u(x, y)$ of the equation satisfying the boundary conditions

$$u\left(x, \frac{x}{b}\right) = \varphi(x), \quad 0 \leqslant x \leqslant b$$

$$u(x, 0) = 0, \quad -b \leqslant x < 0; \quad u(x, 1) = 0, \quad 0 \leqslant x \leqslant b$$

where φ is a given sufficiently smooth function.

448. For the rectangle bounded by the straight lines $x = 0$, $x = \pi$, $y = 0$ and $y = T > 0$ find the solution $u(x, y)$ of the equation

$$u_{xx} + pu_x - u_y + \frac{p^2}{4} u = 0, \quad p = \text{const}$$

satisfying the conditions

$$u(0, y) = u(\pi, y) = 0, \quad 0 \leqslant y \leqslant T$$

$$u(x, 0) = \sin x \cdot e^{-\frac{p}{2}x}, \quad 0 \leqslant x \leqslant \pi$$

449. For the equation considered in Problem 448 express in quadratures the solution of the Cauchy-Dirichlet problem

$$u(x, 0) = \varphi(x), \quad -\infty < x < \infty$$

and find the conditions on $\varphi(x)$ guaranteeing the existence of the integral in the expression of the solution obtained in a formal manner.

450. Show that the functions

$$e^{-\lambda t} J_k(\lambda r) \cos k\varphi$$

and

$$e^{-\lambda t} J_k(\lambda r) \sin k\varphi \qquad (k = 0, 1, \ldots)$$

satisfy the equation

$$u_{xx} + u_{yy} - \lambda u_t = 0, \quad \lambda = \text{const}$$

where $x = r \cos \varphi$, $y = r \sin \varphi$ and J_k is Bessel's function of an integral order k.

451. For the domain D lying in the space of the variables x, y, t and bounded by the planes $t = 0$ and $t = T > 0$ and by the circular cylinder $x^2 + y^2 = (\lambda_1/\lambda)^2$ find the solution $u(x, y, t)$ of the equation considered in Problem 450 which satisfies the conditions

$$u(x, y, 0) = J_0(\lambda r)$$

$$u|_{x^2+y^2=(\lambda_1/\lambda)^2} = 0$$

where $J_0(z)$ is Bessel's function of order zero and λ_1 is its root.

452. Determine the type of the equation

$$\Delta\Delta u - \frac{\partial u}{\partial t} = 0 \tag{6}$$

and show that the function

$$u(x, t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta^{2k} \tau(x)$$

where $\tau(x)$ is an arbitrary infinitely differentiable function and the series on the right-hand side can be differentiated termwise the required number of times, is a solution of equation (6).

Construct the solutions of equation (6) satisfying the following boundary conditions:

453. $u(x, 0) = P_n(x)$ where $P_n(x)$ is a polynomial of the nth degree in the variables $x_1, \ldots, x_n$.

454. $u(x, 0) = \sin l_1 x_1 \cos l_n x_n$.

455. Determine the type of the equation

$$\Delta\Delta u - \frac{\partial^2 u}{\partial t^2} = 0 \tag{7}$$

and show that the function

$$u(x, t) = \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} \Delta^{2k}\tau(x) + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \Delta^{2k}\nu(x)$$

where τ and ν are arbitrary infinitely differentiable functions and the series on the right-hand side can be differentiated termwise the required number of times, is a solution of equation (7).

Find the solutions of equation (7) satisfying the following conditions:

456. $u(x, 0) = P_n(x)$, $u_t(x, 0) = 0$, where $P_n(x)$ is a polynomial of the nth degree.

457. $u(x, 0) = \sin x_1$, $u_t(x, 0) = \cos x_1$.

Chapter 5

BASIC PRACTICAL METHODS FOR THE SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

§ 1. The Method of Separation of Variables (The Fourier Method)

The *Fourier method* of separation of variables is used for the construction of the solutions of the so-called *mixed problems* for a wide class of partial differential equations.

Let us denote by D a domain in the space of the variables $x_1, \ldots, x_n, t$ bounded by the plane $t = 0$ and by a cylindrical surface S, whose generators are parallel to the t-axis, which lies in the region where the equation

$$\sum_{i,j=1}^{n} A_{ij}(x)\, u_{x_i x_j} + \sum_{i=1}^{n} B_i(x)\, u_{x_i} + C(x)\, u - \alpha(t)\, u_{tt} - \beta(t)\, u_t - \gamma(t)\, u = 0 \quad (1)$$

is defined.

Let us suppose that the quadratic form $\sum_{i,j=1}^{n} A_{ij}(x)\,\lambda_i\lambda_j$ is positive definite and that the coefficient $\alpha(t)$ is either greater than zero or identically equal to zero; in the latter case we shall assume that the inequality $\beta(t) > 0$ is fulfilled. Under these assumptions equation (1) is either hyperbolic or parabolic respectively.

The general mixed problem (*the boundary-initial-value problem*) for equation (1) is stated as follows: *it is required to determine in the domain D the regular solution $u(x, t)$ of the equation satisfying the boundary condition*

$$\sum_{i=1}^{n} a_i(x)\, u_{x_i} + b(x)\, u = 0, \quad x \in S, \quad t \geqslant 0 \quad (2)$$

and the initial conditions

$$u(x, 0) = \varphi(x), \; u_t(x, 0) = \psi(x) \quad (3)$$

in the case of a hyperbolic equation and *the initial condition*

$$u(x, 0) = \varphi^*(x) \tag{4}$$

in the case of a parabolic equation.

For the sought-for solution to be continuous in D including its boundary, the data in conditions (2), (3) and (4) should be coherent in a certain sense.

The basic idea of the method of separation of variables is the following.

A non-trivial solution $u(x, t)$ of equation (1) satisfying boundary condition (2) is constructed in the form of a product of two functions $T(t)$ and $X(x) = X(x_1, \ldots, x_n)$:

$$u(x, t) = T(x) X(x) \tag{5}$$

On substituting expression (5) for $u(x, t)$ into equation (1) and into boundary condition (2), we obtain

$$\frac{1}{X(x)}\left[\sum_{i,j=1}^{n} A_{ij}(x) X_{x_i x_j} + \sum_{i=1}^{n} B_i(x) X_{x_i} + C(x) X\right] =$$

$$= \frac{1}{T(t)}[\alpha(t) T'' + \beta(t) T' + \gamma(t) T] = -\lambda = \text{const}, \quad (x, t) \in D \tag{6}$$

and

$$\left[\sum_{i=1}^{n} a_i(x) X_{x_i} + b(x) X\right] T(t) = 0, \quad x \in S, \quad t \geqslant 0 \tag{7}$$

Since $X(x)$ and $T(t)$ are ot identically equal to zero, equalities (6) and (7) yield

$$\alpha(t) T'' + \beta(t) T' + [\gamma(t) + \lambda] T = 0, \quad t > 0 \tag{8}$$

$$\sum_{i,j=1}^{n} A_{ij}(x) X_{x_i x_j} + \sum_{i=1}^{n} B_i(x) X_{x_i} + [C(x) + \lambda] X = 0, \quad x \in d \tag{9}$$

and

$$\sum_{i=1}^{n} a_i(x) X_{x_i} + b(x) X = 0, \quad x \in s \tag{10}$$

where d and s are the projections of the domain D and of the surface S on the plane $t = 0$ respectively.

A value of λ for which boundary-value problem (9), (10) possesses a non-trivial solution $X(x)$ is called an *eigenvalue* and the solution $X(x)$ itself is called an *eigenfunction*.

The set of all eigenvalues of problem (9), (10) is called the *spectrum*, and the problem of the determination of the spectrum and of the system of eigenfunctions corresponding to the spectrum is referred to as an *eigenvalue problem*.

In many cases the spectrum of problem (9), (10) is countable:

$$\lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots, \quad \lim_{k\to\infty} \lambda_k = \infty$$

and the corresponding linearly independent system of eigenfunctions

$$X_1(x), \quad X_2(x), \ \ldots \tag{11}$$

is complete. Only these cases will be considered in what follows.

Let us denote by $T_k(t)$ the general solution of ordinary differential equation (8) corresponding to $\lambda = \lambda_k$ in the case when $\alpha(t) > 0$:

$$T_k(t) = \alpha_k T_{k1}(t) + \beta_k T_{k2}(t) \tag{12}$$

where α_k and β_k are arbitrary real constants and $T_{k1}(t)$ and $T_{k2}(t)$ are the solutions of equation (8) satisfying the conditions

$$T_{k1}(0) = 1, \quad T'_{k1}(0) = 0; \quad T_{k2}(0) = 0, \quad T'_{k2}(0) = 1 \tag{13}$$

For $\alpha(t) \equiv 0$, $\beta(t) > 0$ the general solution $T_k(x)$ of equation (8) has the form

$$T_k(t) = \alpha_k^* T_{k1}(t) \tag{12'}$$

where

$$T_{k1}(0) = 1 \tag{13'}$$

and α_k^* is an arbitrary constant.

It is evident that a function $u(x, t)$ of the form

$$u(x, t) = \sum_{k=1}^{\infty} T_k(t) X_k(x) \tag{14}$$

is a solution of equation (1) satisfying boundary condition (2) provided that the series on the right-hand side of equality (14) and the series obtained by means of the termwise differentiation of series (14) the required number of times are all uniformly convergent. Let us impose on formula (14) the condition that the function $u(x, t)$ should also satisfy initial conditions (3) or (4); then we obtain

$$\sum_{k=1}^{\infty} \alpha_k X_k(x) = \varphi(x), \quad \sum_{k=1}^{\infty} \beta_k X_k(x) = \psi(x) \tag{15}$$

or

$$\sum_{k=1}^{\infty} \alpha_k^* X_k(x) = \varphi^*(x) \tag{15'}$$

respectively.

In the case when system of eigenfunctions (11) is complete and orthonormal we obtain the following formulas for the coefficients α_k, β_k and α_k^* from (15) and (15′) respectively:

$$\alpha_k = \int_d \varphi(x) X_k(x)\, d\tau_x, \quad \beta_k = \int_d \psi_k X_k(x)\, d\tau_x \tag{16}$$

and

$$\alpha_k^* = \int_d \varphi^*(x) X_k(x)\, d\tau_x \tag{16'}$$

On substituting the values of α_k, β_k and α_k^* determined by the formulas (16) and (16′) into (12) and (12′), respectively, we find $T_k(t)$. Consequently, formula (14) expresses the solution of the mixed problem stated above.

In the case $n = 1$ equation (9) is an ordinary linear differential equation of the form

$$\begin{gathered} A(x) X'' + B(x) X' + [C(x) + \lambda] X = 0 \\ A(x) = A_{11}(x_1), \quad x_1 = x \end{gathered} \tag{17}$$

and the domain D coincides with the half-strip $0 < x < l$, $t > 0$ while boundary condition (10) assumes the form

$$a_1 X'(0) + b_1 X(0) = 0, \quad a_2 X'(l) + b_2 X(l) = 0 \tag{18}$$

where a_k and b_k $(k = 1, 2)$ are constants because in this case boundary condition (2) has the form

$$\begin{gathered} a_1 u_x(0, t) + b_1 u(0, t) = 0, \\ a_2 u_x(l, t) + b_2 u(l, t) = 0 \end{gathered} \tag{19}$$

The spectral problem determined by (17), (18) is referred to as the *Sturm-Liouville problem*. In the general case the investigation of Sturm-Liouville problem (17), (18) is rather intricate. It becomes still more difficult in the case when the coefficient $A(x)$ turns into zero at separate points of the interval of variation of the variable x. In such cases it is necessary to introduce the so-called *special functions*.

If $n = 1$ and the coefficients of equation (1) are constant the solution of Sturm-Liouville problem (17), (18) can be expressed explicitly. For instance, in the case of the equation of oscillation of a string

$$u_{xx} - \frac{1}{a^2} u_{tt} = 0, \quad a = \text{const}$$

equations (8) and (17) have the form

$$T''(t) + a^2\lambda T(t) = 0 \tag{8'}$$

and

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < l \tag{17'}$$

respectively. For the sake of simplicity, we shall assume that the coefficients in boundary conditions (18) are $a_1 = a_2 = 0$, $b_1 = b_2 = 1$ and that $l = \pi$, that is

$$X(0) = 0, \quad X(\pi) = 0 \tag{18'}$$

The spectrum of Problem (17'), (18') coincides with the sequence of all natural numbers, and the corresponding system of linearly independent eigenfunctions $X_k(x) = \sin kx$ $(k = 1, 2, \ldots)$ is complete in the interval $(0, \pi)$. As to the solution $T_k(t)$ of equation (8') corresponding to $\lambda = k^2$, it is given by the formula

$$T_k(t) = \alpha_k \cos akt + \beta_k \sin akt$$

Under the same assumptions, in the case of the heat conduction equation

$$a^2 u_{xx} - u_t = 0$$

the eigenfunctions are again equal to $X_k(x) = \sin kx$ $(k = 1, 2, \ldots)$ and

$$T_k(t) = \alpha_k^* e^{-k^2a^2t}, \quad k = 1, 2, \ldots$$

since in this case equation (8) has the form $T' + a^2k^2T = 0$.

The method of separation of variables also makes it possible to construct the solutions of mixed problems in those cases when the given partial differential equation and the boundary conditions are non-homogeneous.

We shall limit ourselves to the study of the following non-homogeneous mixed problem. Let us consider the partial differential equation

$$u_{xx}-\frac{1}{a^2}u_{tt}=f(x,t) \tag{20}$$

with the boundary conditions

$$\begin{aligned} &a_1u_x(0,t)+b_1u(0,t)=\mu(t)\\ &a_2u_x(l,t)+b_2u(l,t)=\nu(t)\\ &a_k^2+b_k^2\neq 0,\quad k=1,2 \end{aligned} \tag{21}$$

and the initial conditions

$$u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x) \tag{22}$$

It should be noted that under some additional assumptions concerning a_1, b_1, a_2 and b_2 there exist constants γ_1, γ_2, γ_3, δ_1, δ_2 and δ_3 such that the transformation of the sought-for function expressed by the formula

$$u(x,t)=v(x,t)+w(x,t)$$

where

$$w(x,t)=(\gamma_1x^2+\gamma_2x+\gamma_3)\mu(t)+(\delta_1x^2+\delta_2x+\delta_3)\nu(t)$$

reduces problem (20), (21), (22) to the mixed problem for the equation

$$v_{xx}-\frac{1}{a^2}v_{tt}=F(x,t) \tag{20$'$}$$

with the homogeneous boundary conditions

$$\begin{aligned} &a_1v_x(0,t)+b_1v(0,t)=0,\\ &a_2v_x(l,t)+b_2v(l,t)=0 \end{aligned} \tag{21$'$}$$

and the initial conditions

$$v(x,0)=\varphi_1(x),\quad v_t(x,0)=\psi_1(x) \tag{22$'$}$$

where

$$F(x,t)=f(x,t)-w_{xx}+\frac{1}{a^2}w_{tt}$$

and

$$\varphi_1(x) = \varphi(x) - w(x, 0), \quad \psi_1(x) = \psi(x) - w_t(x, 0)$$

We shall assume that there exists a complete orthonormal linearly independent system of eigenfunctions $X_k(x)$ ($k = 1, 2, \ldots$) of the Sturm-Liouville problem

$$X'' + \lambda X = 0 \tag{23}$$

$$a_1 X'(0) + b_1 X(0) = 0, \quad a_2 X'(l) + b_2 X(l) = 0 \tag{24}$$

Let us represent the functions $F(x, t)$, $\varphi_1(x)$ and $\psi_1(x)$ as the sums of the series

$$F(x, t) = \sum_{k=1}^{\infty} c_k(t) X_k(x) \tag{25}$$

and

$$\varphi_1(x) = \sum_{k=1}^{\infty} d_k X_k(x), \quad \psi_1(x) = \sum_{k=1}^{\infty} e_k X_k(x) \tag{26}$$

We shall construct the solution of Problem (20′), (21′), (22′) having the form

$$v(x, t) = \sum_{k=1}^{\infty} T_k(t) X_k(x) \tag{27}$$

On substituting the expressions of $F(x, t)$, $\varphi_1(x)$, $\psi_1(x)$ and $v(x, t)$ given by the formulas (25), (26) and (27) into equation (20′) and conditions (22′), we obtain

$$\sum_{k=1}^{\infty} \left[T_k(t) X_k''(x) - \frac{1}{a^2} T_k''(t) X_k(x) \right] = \sum_{k=1}^{\infty} c_k(t) X_k(x) \tag{28}$$

and

$$\begin{aligned} \sum_{k=1}^{\infty} T_k(0) X_k(x) &= \sum_{k=1}^{\infty} d_k X_k(x), \\ \sum_{k=1}^{\infty} T_k'(0) X_k(x) &= \sum_{k=1}^{\infty} e_k X_k(x) \end{aligned} \tag{29}$$

On the basis of (23) we can rewrite equality (28) in the form

$$\sum_{k=1}^{\infty} [T_k''(t) + a^2 \lambda_k T_k(t)] X_k(x) = -a^2 \sum_{k=1}^{\infty} c_k(t) X_k(x) \tag{28'}$$

By virtue of the linear independence of the system of the functions $X_k(x)$ $(k = 1, 2, \ldots)$ we obtain from (28′) and (29) the following problem for the determination of the function $T_k(t)$:

$$T_k''(t) + a^2\lambda_k T_k(t) = -a^2 c_k(t)$$

$$T_k(0) = d_k, \quad T_k'(0) = e_k$$

The solution of the last problem can easily be found and expressed in quadratures.

On condition that the functions F, φ_1 and ψ_1 are such that the series $\sum\limits_{k=1}^{\infty} T_k(t)\, X_k(x)$ and the series obtained by means of the termwise differentiation of $\sum\limits_{k=1}^{\infty} T_k(t)\, X_k(x)$ the required number of times are uniformly convergent, the substitution of the values of $T_k(x)$ found above into the right-hand side of (27) results in the sought-for solution of Problem (20′), (21′), (22′).

Let us consider the special case when the right-hand member of equation (20) is a function dependent solely on one variable x:

$$f(x, t) \equiv f(x)$$

We shall also suppose that the right-hand members in conditions (21) are constant: $\mu = \mu_0 =$ const and $\nu = \nu_0 =$ = const, and that the condition

$$a_1 b_2 - a_2 b_1 - b_1 b_2 l \neq 0 \tag{30}$$

holds. Then the change of the sought-for function expressed by the formula

$$u(x, t) = v(x, t) + w(x)$$

where

$$w''(x) = f(x) \tag{31}$$

$$a_1 w'(0) + b_1 w(0) = \mu_0, \quad a_2 w'(l) + b_2 w(l) = \nu_0$$

reduces Problem (20), (21), (22) to the mixed problem for the homogeneous equation

$$v_{xx} - \frac{1}{a^2} v_{tt} = 0$$

with the homogeneous boundary conditions

$$a_1 v_x(0, t) + b_1 v(0, t) = 0, \quad a_2 v_x(l, t) + b_2 v(l, t) = 0$$

and the initial conditions

$$v(x, 0) = \varphi(x) - w(x), \quad v_t(x, 0) = \psi(x)$$

Finally, in the case when condition (30) is fulfilled the new problem (31) is always solvable.

The method of separation of variables is also used for constructing solutions of certain classes of elliptic partial differential equations.

1°. *Problems for Wave Equation*

458. Construct the class of the solutions $u(x, t)$ of the form $u(x, t) = v(x)w(t)$ for the equation of oscillation of a string $u_{xx} = u_{tt}$.

459. For the half-strip $a < x < b$, $t > 0$ construct the solution of the boundary-value problem

$$u_{xx} = u_{tt}, \; u(a, t) = u(b, t) = 0$$

Is the solution of this problem unique?

460. For the half-strip $0 < x < \pi$, $t > 0$ solve the problem

$$u_{xx} = u_{tt}, \quad u(0, t) = u(\pi, t) = 0$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x)$$

where $\varphi(x)$ $(\varphi(0) = \varphi(\pi) = 0, \; \varphi''(0) = \varphi''(\pi) = 0)$ and $\psi(x)$ $(\psi(0) = \psi(\pi) = 0)$ are sufficiently smooth functions (this is the so-called *basic mixed problem* for the equation $u_{xx} = u_{tt}$).

461. Does the solution of Problem 460 possess the uniqueness property?

462. For the strip $0 < x < \pi$, $-\infty < t < \infty$ find the natural vibrations (harmonics) corresponding to boundary-value problem

$$u_{xx} = u_{tt}, \quad u(0, t) = u_x(\pi, t) = 0$$

For the equation $u_{tt} = a^2 u_{xx}$ find the solutions of the mixed problems in the half-strip $0 < x < l$, $t > 0$ satisfying the following conditions:

463. $u(0, t) = u(l, t) = 0,$

$$u(x, 0) = 0, \quad u_t(x, 0) = \sin\frac{2\pi}{l}x.$$

464. $u(0, t) = u(l, t) = 0,$
$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x).$

465. $u(0, t) = u_x(l, t) = 0,$
$u(x, 0) = \sin \frac{5\pi}{2l} x, \quad u_t(x, 0) = \cos \frac{\pi}{2l} x.$

466. $u(0, t) = u_x(l, t) = 0,$
$u(x, 0) = x, \quad u_t(x, 0) = \sin \frac{\pi}{2l} x + \sin \frac{3\pi}{2l} x.$

467. $u_x(0, t) = u(l, t) = 0,$
$u(x, 0) = \cos \frac{\pi}{2l} x, u_t(x, 0) = \cos \frac{3\pi}{2l} x + \cos \frac{5\pi}{2l} x.$

468. $u_x(0, t) = u(l, t) = 0, u(x, 0) = \varphi(x),$
$u_t(x, 0) = \psi(x).$

469. $u_x(0, t) = u_x(l, t) = 0, \quad u(x, 0) = x,$
$u_t(x, 0) = 1.$

470. $u_x(0, t) = u_x(l, t) = 0, \quad u(x, 0) = \varphi(x),$
$u_t(x, 0) = \psi(x).$

471. $u(0, t) = u_x(l, t) + hu(l, t) = 0,$
$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad h > 0.$

472. $u_x(0, t) = u_x(l, t) + hu(l, t) = 0,$
$u(x, 0) = 0, \quad u_t(x, 0) = 1, \ h > 0.$

473. $u_x(0, t) - hu(0, t) = u_x(l, t) = 0,$
$u(x, 0) = \varphi(x), \ u_t(x, 0) = \psi(x), \ h > 0.$

474. $u_x(0, t) - hu(0, t) = u_x(l, t) + hu(l, t) = 0,$
$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad h > 0.$

For the half-strip $0 < x < l$, $t > 0$ solve the following mixed problems:

475. $u_{tt} = a^2 u_{xx} + f(x), \quad u(0, t) = \alpha, \quad u(l, t) = \beta,$
$u(x, 0) = u_t(x, 0) = 0.$

476. $u_{tt} = a^2 u_{xx} + f(x), \quad u_x(0, t) = \alpha, \quad u_x(l, t) = \beta,$
$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x).$

477. $u_{tt} = a^2 u_{xx} + f(x),$
$u_x(0, t) - hu(0, t) = \alpha,$
$u(l, t) = \beta, \quad u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x),$
$h > 0.$

478. $u_{tt} = a^2 u_{xx} + f(x),$
$u_x(0, t) = \alpha, \quad u_x(l, t) + hu(l, t) = \beta,$
$u(x, 0) = u_t(x, 0) = 0, \quad h > 0.$

479. $u_{tt} = u_{xx}$,
$u_x(0, t) - hu(0, t) = \alpha$,
$u_x(l, t) + hu(l, t) = -\alpha$,
$u(x, 0) = 0, \quad u_t(x, 0) = 0$.

Using the substitution $u(x, t) = v(x, t) + w(x, t)$ choose the function $w(x, t)$ in such a way that the problems below reduce to the corresponding problems for a non-homogeneous equation of the form $v_{xx} - v_{tt} = F(x, t)$ with homogeneous boundary conditions and with initial conditions changed in the adequate manner:

480. $u_{xx} = u_{tt}$,
$u(0, t) = \mu(t), \quad u(l, t) = \nu(t)$,
$u(x, 0) = \varphi(x), \; u_t(x, 0) = \psi(x)$.

481. $u_{xx} = u_{tt}, \quad u_x(0, t) = \mu(t), \quad u(l, t) = \nu(t)$,
$u(x, 0) = \varphi(x), \quad u_t(x, 0) = 0$.

482. $u_{xx} = u_{tt} + f(x, t)$,
$u(0, t) = \mu(t), \quad u_x(l, t) + hu(l, t) = \nu(t), \quad h > 0$,
$u(x, 0) = 0, \quad u_t(x, 0) = \psi(x)$.

483. $u_{xx} = u_{tt} + f(x, t)$,
$u_x(0, t) - hu(0, t) = \mu(t), \quad h > 0, \quad u_x(l, t) = \nu(t)$,
$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x)$.

484. $u_{xx} = u_{tt}$,
$u_x(0, t) - hu(0, t) = \mu(t), \quad u_x(l, t) + gu(l, t) = \nu(t), \quad h > 0, \quad g > 0$,
$u(x, 0) = 0, \quad u_t(x, 0) = 0$.

For the half-strip $0 < x < l$, $t > 0$ solve the mixed problems for the equation $u_{tt} = a^2 u_{xx} + f(x, t)$ with the initial conditions $u(x, 0) = 0$, $u_t(x, 0) = 0$ and with the following boundary conditions:

485. $u(0, t) = u(l, t) = 0, \quad f(x, t) = Ae^{-t} \sin \frac{\pi}{l} x$.

486. $u(0, t) = u(l, t) = 0, \quad f(x, t) = Axe^{-t}$.

487. $u(0, t) = u_x(l, t) = 0, \quad f(x, t) = A \sin t$.

488. $u(0, t) = u_x(l, t) = 0$.

489. $u_x(0, t) = u(l, t) = 0, \quad f(x, t) = Ae^{-t} \cos \frac{\pi}{2l} x$.

490. $u_x(0, t) = u_x(l, t) = 0$.

Solve the following mixed problems:

491. $u_{xx} = u_{tt}$, $u(0, t) = t^2$, $u(\pi, t) = t^3$,
$u(x, 0) = \sin x$, $u_t(x, 0) = 0$, $0 < x < \pi$, $t > 0$.

492. $u_{xx} = u_{tt}$, $u(0, t) = e^{-t}$, $u(\pi, t) = t$,
$u(x, 0) = \sin x \cos x$,
$u_t(x, 0) = 1$, $0 < x < \pi$, $t > 0$.

493. $u_{xx} = u_{tt}$, $u(0, t) = t$, $u_x(\pi, t) = 1$,
$u(x, 0) = \sin \frac{1}{2} x$, $u_t(x, 0) = 1$, $0 < x < \pi$,
$t > 0$.

494. $u_{tt} = a^2 u_{xx}$, $u_x(0, t) = 0$, $u_x(l, t) = Ae^{-t}$,

$$u(x, 0) = \frac{Aa \cosh \frac{x}{a}}{\sinh \frac{l}{a}}, \quad u_t(x, 0) = -\frac{Aa \cosh \frac{x}{a}}{\sinh \frac{l}{a}},$$

$$0 < x < l, \quad t > 0.$$

495. $u_{tt} = a^2 u_{xx} + \sin 2t$,

$u_x(0, t) = 0$, $u_x(l, t) = \frac{2}{a} \sin \frac{2l}{a} \sin 2t$,

$u(x, 0) = 0$, $u_t(x, 0) = -2 \cos \frac{2x}{a}$,

$0 < x < l$, $t > 0$.

496. State the problems to which the separation of variables of the form $u(x, y, t) = v(x, y) w(t)$ reduces the boundary-initial-value problem for the equation

$$u_{xx} + u_{yy} - u_{tt} = 0 \tag{32}$$

with the conditions

$$\begin{gathered} u(x, y, t) = 0, \quad t > 0, \quad (x, y) \in C \\ u(x, y, 0) = \varphi(x, y), \quad u_t(x, y, 0) = \psi(x, y), \quad (x, y) \in G \end{gathered} \tag{33}$$

where G is a domain in the xy-plane with boundary G, and $\varphi(x, y)$ and $\psi(x, y)$ are given continuous functions.

497. Prove the uniqueness of the solution of the mixed problem indicated in Problem 496 (see (32), (33)).

Consider the equation

$$v_{xx} + v_{yy} + \lambda v = 0, \quad (x, y) \in G \tag{34}$$

with the condition

$$v(x, y) = 0, \quad (x, y) \in C \tag{35}$$

where G is a domain in the xy-plane with boundary C and λ is a constant parameter. For Problem (34), (35) prove the following facts:

498. The eigenvalues of the problem are positive.

499. Any two eigenfunctions $v_k(x, y)$ and $v_m(x, y)$ of the problem corresponding to two different eigenvalues λ_k and λ_m are orthogonal, that is

$$\int_G v_k(x, y)\, v_m(x, y)\, dx\, dy = 0 \quad \text{for} \quad \lambda_k \neq \lambda_m$$

500. For the cases enumerated below, neglecting the resistance of the surrounding medium, determine the transverse oscillation of the rectangular membrane $0 \leqslant x \leqslant s$, $0 \leqslant y \leqslant p$ with a rigidly fixed edge:

(a) the initial deflection of the points of the membrane is described by the function $\sin \pi x/s \cdot \sin \pi y/p$ and the initial velocity is equal to zero;

(b) at the initial instant $t = 0$ the membrane receives a concentrated transverse impulse I at a point (x_0, y_0) $(0 < x_0 < s,\ 0 < y_0 < p)$ and at that initial moment the membrane is at rest;

(c) the oscillation of the membrane is produced by a transverse force distributed over the membrane with the density

$$f(x, y, t) = e^{-t} x \sin \frac{2\pi}{p} y$$

501. Consider a rectangular membrane $0 \leqslant x \leqslant s$, $0 \leqslant y \leqslant p$ the parts $x = s$, $0 \leqslant y < p$ and $y = p$, $0 \leqslant x < s$ of whose boundary are free while the other part of the boundary is rigidly fixed. Neglecting the resistance of the surrounding medium, determine the transverse oscillation of the membrane for the following cases:

(a) the oscillation is produced by the initial deviation Axy;

(b) the oscillation is produced by a concentrated impulse I received by the membrane at the initial instant $t = 0$ at a point (x_0, y_0), $0 < x_0 < s$, $0 < y_0 < p$.

2°. *Problems for Parabolic Partial Differential Equations*

For the half-strip $0 < x < l$, $l > 0$ solve the mixed problems for the equation $u_t = a^2 u_{xx}$ with the following conditions:

502. $u(0, t) = u(l, t) = 0, \quad u(x, 0) = Ax.$

503. $u(0, t) = u_x(l, t) = 0, \ u(x, 0) = \varphi(x).$

504. $u_x(0, t) = u(l, t) = 0, \quad u(x, 0) = A(l - x).$

505. $u_x(0, t) = u_x(l, t) = 0, \quad u(x, 0) = U.$

506. $u_x(0, t) = u_x(l, t) + hu(l, t) = 0,$
$u(x, 0) = \varphi(x), \ h > 0.$

507. $u_x(0, t) - hu(0, t) = u(l, t) = 0, \quad u(x, 0) = U,$
$h > 0.$

508. $u_x(0, t) - hu(0, t) = u_x(l, t) + hu(l, t) = 0,$
$u(x, 0) = U, \quad h > 0.$

For the half-strip $0 < x < l$, $t > 0$ solve the following mixed problems:

509. $u_t = a^2 u_{xx} - \beta u,$
$u(0, t) = u(l, t) = 0, \quad u(x, 0) = \varphi(x).$

510. $u_t = a^2 u_{xx} - \beta u,$
$u(0, t) = u_x(l, t) = 0, \quad u(x, 0) = \sin \frac{\pi x}{2l}.$

511. $u_t = a^2 u_{xx} - \beta u,$
$u_x(0, t) = u_x(l, t) = 0, \quad u(x, 0) = \varphi(x).$

512. $u_t = a^2 u_{xx} - \beta u,$
$u_x(0, t) - hu(0, t) = u_x(l, t) = 0, \quad u(x, 0) = U,$
$h > 0.$

513. $u_t = a^2 u_{xx},$
$u(0, t) = T, \quad u(l, t) = U, \quad u(x, 0) = 0.$

514. $u_t = a^2 u_{xx} + f(x),$
$u(0, t) = 0, \quad u_x(l, t) = q, \quad u(x, 0) = \varphi(x).$

515. $u_t = a^2 u_{xx},$
$u_x(0, t) = u_x(l, t) = q, \quad u(x, 0) = Ax.$

516. $u_t = a^2 u_{xx},$
$u(0, t) = T, \ u_x(l, t) + hu(l, t) = U, \quad u(x, 0) =$
$= 0, \quad h > 0.$

517. $u_t = a^2 u_{xx} - \beta u + \sin \frac{\pi x}{l},$
$u(0, t) = u(l, t) = 0, \quad u(x, 0) = 0.$

518. $u_t = a^2 u_{xx}$,
$u(0, t) = 0, \quad u_x(l, t) = Ae^{-t}, \quad u(x, 0) = T.$

519. $u_t = a^2 u_{xx}$,
$u_x(0, t) = At, \quad u_x(l, t) = T, \quad u(x, 0) = 0.$

520. The initial temperature of a homogeneous ball $0 \leqslant r < R$ of radius R with centre at the origin is equal to T. Determine the temperature of the ball for the following cases:

(a) on the surface of the ball a zero temperature is permanently maintained;

(b) there is a constant heat influx of density q into the ball through its surface.

521. The initial temperature of an infinite bar $0 \leqslant x \leqslant p$, $0 \leqslant y \leqslant s$, $-\infty < z < \infty$ with a rectangular transverse cross-section is described by an arbitrary function $f(x, y)$. Determine the temperature in the bar for the following cases:

(a) the part $x = 0$, $0 < y < s$ of the lateral surface of the bar is heat insulated while on the other part of the boundary a zero temperature is constantly maintained;

(b) on the part $x = p$, $0 < y < s$ of the surface of the bar there is convective heat exchange with the surrounding medium having zero temperature, the part $y = 0$, $0 < x < p$ of the surface is heat insulated and on the remaining part of the surface of the bar a zero temperature is maintained.

522. In a cube $0 \leqslant x, y, z \leqslant l$ there is diffusion of a substance whose particles disintegrate at a rate proportional to the concentration of the substance. Determine the concentration of the substance within the cube on condition that the initial concentration of the substance in that cube is of a constant magnitude U while on the boundary of the cube the concentration of the substance is permanently equal to zero.

3°. *Problems for Elliptic Partial Differential Equations*

523. Find the solutions $u(x, y)$ of Laplace's equation in the rectangle $0 < x < p$, $0 < y < s$ satisfying the following boundary conditions:

(a) $u(0, y) = u_x(p, y) = 0, \quad u(x, 0) = 0, \quad u(x, s) = f(x)$;

(b) $u_x(0, y) = u_x(p, y) = 0, \quad u(x, 0) = A, \quad u(x, s) = Bx$;

(c) $u(0, y) = U, \quad u_x(p, y) = 0, \quad u_y(x, 0) = T \sin \frac{\pi x}{2p}$, $u(x, s) = 0$.

524. Find the solutions of Laplace's equation in the half-strip $0 < x < \infty$, $0 < y < l$ for the following boundary conditions:

(a) $u(x, 0) = u_y(x, l) = 0, \quad u(0, y) = f(y)$, $u(\infty, y) = 0$;

(b) $u_y(x, 0) = u_y(x, l) + hu(x, l) = 0, \quad u(0, y) = f(y)$; $u(\infty, y) = 0, \quad h > 0$.

525. Find the functions $u(r, \varphi)$ harmonic within the ring $a < r < b$ and satisfying the following boundary conditions:

(a) $u(a, \varphi) = 0, \quad u(b, \varphi) = \cos \varphi$;

(b) $u(a, \varphi) = A, \quad u(b, \varphi) = B \sin 2\varphi$;

(c) $u_r(a, \varphi) = q \cos \varphi, \quad u(b, \varphi) = Q + T \sin 2\varphi$.

526. Find the functions $u(r, \varphi)$ harmonic in the circular sector $0 < r < R$, $0 < \varphi < \alpha$ and satisfying the following boundary conditions:

(a) $u(r, 0) = u(r, \alpha) = 0, \quad u(R, \varphi) = A\varphi$;

(b) $u_\varphi(r, 0) = u(r, \alpha) = 0, \quad u(R, \varphi) = f(\varphi)$.

§ 2. Special Functions. Asymptotic Expansions

1°. *Some Problems Involving Special Functions*

As was already mentioned earlier, a number of mixed problems with one spatial variable often reduce to ordinary linear differential equations of the second order whose coefficients in the highest derivatives turn into zero at some separate points. The solutions of such equations are usually referred to as *special functions.*

For instance, the following ordinary differential equations belong to the class of equations whose solutions are special functions:

1. *Bessel's equation*:

$$x^2v'' + xv' + (x^2 - \mu^2)\, v = 0, \quad \mu = \text{const}$$

The solutions of this equation are called *Bessel's* (or *cylindrical*) *functions of order* μ.

2. *Chebyshev's equation*:

$$(1 - x^2)\, v'' - xv' + n^2v = 0, \quad n = \text{const}$$

The solutions of this equation are referred to as *Chebyshev's functions*.

3. *Laguerre's equation*:

$$xv'' + (1 - x)\, v' + \lambda v = 0, \quad \lambda = \text{const}$$

The solutions of this equation are called *Laguerre's functions*.

4. *Legendre's equation*:

$$(1 - x^2)\, v'' - 2xv' + m\,(m + 1)\, v = 0, \quad m = \text{const}$$

The solutions of this equation are called *Legendre's functions*.

5. The equation for the *associated Legendre's functions*:

$$(1 - x^2)\, v'' - 2xv' + \left[m\,(m+1) - \frac{n^2}{1-x^2}\right] v = 0$$

$$m = \text{const}, \quad n = \text{const}$$

527. Prove that Bessel's functions

$$J_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{n+2k}}{2^{n+2k} k!\,(n+k)!}, \qquad n = 1, 2, \ldots$$

satisfy the following identities:

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$$

and

$$xJ_n'(x) = -xJ_{n+1}(x) + nJ_n(x)$$

528. Verify the validity of the following integral representation for Bessel's function $J_0(x)$:

$$J_0(x) = \frac{2}{\pi} \int_0^1 \frac{\cos tx}{\sqrt{1-t^2}}\, dt$$

529. Prove that

$$J_0'(x) = -J_1(x)$$

and

$$J_0''(x) = \frac{1}{2}[J_2(x) - J_0(x)]$$

530. Verify the validity of the identities

$$(\alpha^2 - \beta^2)\, xJ_n(\alpha x)\, J_n(\beta x) = \\ = \frac{d}{dx}\left[xJ_n(\alpha x)\frac{d}{dx}J_n(\beta x) - xJ_n(\beta x)\frac{d}{dx}J_n(\alpha x)\right]$$

and

$$2\alpha^2 x J_n^2(\alpha x) = \frac{d}{dx}\left\{(\alpha^2 x^2 - n^2)\, J_n^2(\alpha x) + \left[x\frac{d}{dx}J_n(\alpha x)\right]^2\right\}$$

where α and β are constants and $n > -1$.

531. For $n > -1$ prove the following assertions: if $J_n(\alpha) = J_n(\beta) = 0$ then

$$\int_0^1 xJ_n(\alpha x)\, J_n(\beta x)\, dx = 0, \quad \alpha \neq \beta$$

and

$$\int_0^1 xJ_n^2(\alpha x)\, dx = \frac{1}{2} J_{n+1}^2(\alpha)$$

and if $J_{n+1}(\alpha) = 0$ then

$$\int_0^1 xJ_n^2(\alpha x)\, dx = \frac{1}{2} J_n^2(\alpha)$$

532. Using the results established in Problem 531, show that the roots of the equation $J_n(x) = 0$ $(n = 0, 1, \ldots)$ can only be real; also prove that the equations $J_n(x) = 0$ and $J_m(x) = 0$ $(n, m = 0, 1, \ldots)$ cannot have common roots different from zero for $n > 0$, $m > 0$, $n \neq m$.

533. Prove that the functions

$$u_n(r, \theta) = I_n(\mu r)\cos n\theta$$

and

$$v_n(r, \theta) = I_n(\mu r)\sin n\theta$$

($n = 0, 1, \ldots$) where $I_n(x)$ is Bessel's function of a pure imaginary argument, that is $I_n(x) \equiv i^{-n} J_n(ix)$ (the function $I_n(x)$ is called a *modified Bessel function of the first kind*), satisfy the equation

$$\Delta u - \mu^2 u = 0 \quad (x^2 + y^2 = r^2, \quad x = r\cos\theta, \quad y = r\sin\theta)$$

534. For an arbitrary index n, express Bessel's function $J_n(x)$ as the sum of the series

$$J_n(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{n+2k}}{2^{n+2k} k!\, \Gamma(n+k+1)}$$

and derive the formulas

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad \text{and} \quad J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

535. Show that the change of the variable $x = \cos\theta$ brings Chebyshev's equation

$$(1 - x^2) v'' - x v' + n^2 v = 0$$

to the form

$$\frac{d^2 u}{d\theta^2} + n^2 u = 0$$

where $u(\theta) = v(\cos\theta)$.

536. Using the formula

$$\cos n\theta = \cos^n \theta - \binom{n}{2} \cos^{n-2}\theta \sin^2\theta + \ldots$$

show that Chebyshev's function

$$T_n(x) = \frac{1}{2}\left[\left(x + i\sqrt{1 - x^2}\right)^n + \left(x - i\sqrt{1 - x^2}\right)^n\right]$$

is a polynomial of the nth degree.

537. Construct Chebyshev's polynomials

$$T_0(x), \quad T_1(x), \quad T_2(x) \quad \text{and} \quad T_3(x)$$

538. Prove that Chebyshev's polynomials are orthogonal with weight function $1/\sqrt{1 - x^2}$ in the interval $(-1, 1)$, that is

$$\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,dx=0, \quad n\neq m$$

539. Compute the norm

$$\|T_n\|=\sqrt{\int_{-1}^{1} \frac{T_n^2(x)\,dx}{\sqrt{1-x^2}}}$$

of Chebyshev's polynomial $T_n(x)$.

540. Prove that the functions

$$L_n(x)=\frac{1}{n!}e^x\frac{d^n}{dx^n}(x^ne^{-x}); \quad n=0,\ 1,\ \ldots$$

are solutions of Laguerre's equation

$$xv''+(1-x)\,v'+nv=0$$

541. Compute the coefficients of Laguerre's polynomials

$$L_0(x), \quad L_1(x), \quad L_2(x) \quad \text{and} \quad L_3(x)$$

542. Show that

$$\int_0^\infty e^{-x}L_n(x)\,L_m(x)\,dx=0, \quad n\neq m$$

543. Prove that

$$\|L_n\|^2=\int_0^\infty e^{-x}L_n^2(x)\,dx=1$$

544. Using the formulas

$$u_\alpha^m(x,\ y,\ z)=\sum_{n\geqslant 0}(-1)^n\frac{z^{2n}}{(2n)!}\Delta^n(x^\alpha y^{m-\alpha}); \quad \alpha=0,\ \ldots,\ m$$

and

$$u_{m+\beta+1}^m(x,\ y,\ z)=\sum_{n\geqslant 0}(-1)^n\frac{z^{2n+1}}{(2n+1)!}\Delta^n(x^\beta y^{m-\beta-1});$$

$$\beta=0,\ \ldots,\ m-1$$

find all linearly independent *solid spherical harmonics* of degree 3 dependent on the variables x, y, and z.

545. Using the formula

$$Y_m^k(\varphi, \theta) = r^m u_k^m\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right); \quad k = 0, \ldots, 2m$$

and the results established in Problem 544, find *Laplace's (surface) spherical harmonics* $Y_3^k(\varphi, \theta)$, $k = 0, \ldots, 6$.

546. Show that Laplace's surface spherical harmonics $Y_3^k(\varphi, \theta)$ (for instance, $Y_3^5(\varphi, \theta)$) satisfy the equation

$$\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + 12Y = 0$$

547. Check that the expressions

$$P_1(t) = t \text{ and } Q_1(t) = \frac{1}{2}t\ln\frac{t+1}{t-1} - 1$$

are *Legendre's functions of the first* and *of the second kind* respectively, that is they satisfy Legendre's equation

$$(1 - t^2)v'' - 2tv' + m(m+1)v = 0 \tag{36}$$

for $m = 1$.

548. Verify directly that the expressions

$$P_1^1(t) = \sqrt{1-t^2} \text{ and } Q_1^1(t) = \frac{1}{2}\sqrt{1-t^2}\ln\frac{t+1}{t-1} + \frac{t}{\sqrt{1-t^2}}$$

$$-1 < t < 1$$

are *Legendre's associated functions*, that is they are solutions of the equation

$$(1-t^2)v'' - 2tv' + \left[m(m+1) - \frac{n^2}{1-t^2}\right]v = 0 \tag{37}$$

for $m = 1$, $n = 1$.

549. Show that the functions

$$P_m(t) = \frac{1}{2^m m!}\frac{d^m}{dt^m}(t^2-1)^m; \quad m = 1, 2, \ldots$$

are *Legendre's polynomials*, that is they are solutions of equation (36).

550. For Legendre's polynomials derive the recurrence relations

$$(n+1)\,P_{n+1}(t)-(2n+1)\,tP_n(t)+nP_{n-1}(t)=0$$

and

$$P_n(t)=\frac{1}{2n+1}\,[P'_{n+1}(t)-P'_{n-1}(t)]$$

551. Prove that for $m=0, 1, \ldots$ the functions

$$P_m(t)=\sum_{k=0}^{m}\frac{(m+k)!\,(-1)^k}{(m-k)!\,(k!)^2\,2^{k+1}}\,[(1-t)^k+(-1)^m\,(1+t)^k] \qquad (38)$$

are Legendre's polynomials.

552. Using the expression for $P_m(t)$ indicated in Problem 549 check the orthogonality of Legendre's polynomials, that is prove the equality

$$\int_{-1}^{1}P_m(t)\,P_n(t)\,dt=0, \qquad m\neq n$$

553. Verify that the norm of $P_m(t)$ (see Problem 549) is expressed by the equality

$$\|P_m\|^2=\frac{2}{2m+1}$$

554. Show that the product by 15 of the factor dependent on θ which is contained in the expression of Laplace's spherical harmonic $Y_3^5(\varphi, \theta)$ (see Problem 545) is equal to *Legendre's associated function of the first kind* $P_3^2(\cos\theta)=P_3^2(t)$, that is this product is a solution of equation (37) for $m=3$, $n=2$.

555. Show that if $v(t)$ is a solution of Legendre's equation (36) then the function $y=\frac{d^nv}{dt^n}$ is a solution of the equation

$$(1-t^2)\,y''-2\,(n+1)\,ty'+(m-n)\,(m+n+1)\,y=0$$

556. Verify directly that for *Legendre's functions of the second kind* there holds the representation

$$Q_m(t)=\frac{1}{2^m m!}\,\frac{d^m}{dt^m}\left[(t^2-1)^m\ln\frac{t+1}{t-1}\right]-\frac{1}{2}\,P_m(t)\ln\frac{t+1}{t-1},$$
$$-1<t<1$$

557. Check directly by computation that the function $P_2^1(t) = 3t\sqrt{1-t^2}$, $-1 < t < 1$, is *Legendre's associated function of the first kind.*

558. Using the result established in Problem 555, show that the functions

$$P_m^n(t) = (1-t^2)^{n/2} \frac{d^n}{dt^n} P_m(t), \quad -1 < t < 1$$

where m is a nonnegative integral number, are Legendre's associated functions, that is they are solutions of equation (37).

559. Show that Legendre's associated functions of the second kind $Q_m^n(t)$ can be represented in the form

$$Q_m^n(t) = (1-t^2)^{n/2} \frac{d^n}{dt^n} Q_m(t), \quad -1 < t < 1$$

560. Check directly that

$$P_m(\cos\theta) = \frac{1}{\pi} \int_0^\pi (\cos\theta + i\sin\theta\cos t)^m \, dt \qquad (39)$$

561. Using representation (39), show that

$$|P_m(t)| \leqslant 1, \quad -1 < t < 1$$

for any integral $m \geqslant 0$.

562. Show that Legendre's polynomial $P_m(x)$ is orthogonal to any polynomial of degree less than m in the interval $(-1, 1)$.

563. Show that $P_m(1) = 1$, $P_m(-1) = (-1)^m$; $m = 0, 1, \ldots$

564. Using the result established in Problem 560, compute $P_m(0)$.

565. Verify directly that the functions of the form

$$u(x, y, z) = \int_{-\pi}^{\pi} f(z + ix\cos t + iy\sin t,\ t)\, dt$$

where $f(\tau, t)$ is an arbitrary function analytic with respect to τ and continuous with respect to t, are harmonic.

566. Prove that there holds the equality

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}(z+ix\cos t+iy\sin t)^m\,dt=r^mP_m(\cos\theta)$$

where $x = r\cos\varphi\sin\theta$, $y = r\sin\varphi\sin\theta$, $z = r\cos\theta$.

A solution $y(z)$ of an ordinary differential equation

$$L(y)\equiv p(z)\,y''+q(z)\,y'+r(z)\,y=0$$

can sometimes be conveniently found in the form of an integral

$$y(z)=\int\limits_{C}K(z,\,t)\,v(t)\,dt \tag{40}$$

where C is a piecewise smooth contour, $K(z,\ t)$ is an analytic function of the variables z and t satisfying a partial differential equation of the form

$$p(z)\,K_{zz}+q(z)\,K_z+r(z)\,K=$$
$$=a(t)\,K_{tt}+b(t)\,K_t+c(t)\,K$$

and $v(t)$ is a solution of the equation

$$(av)_{tt}-(bv)_t+cv=0 \tag{41}$$

567. Show that Bessel's equation

$$z^2y''+zy'+(z^2-n^2)\,y=0 \tag{42}$$

possesses a solution expressed by formula (40) in which

$$K(z,\,t)=\mp\frac{1}{\pi}e^{-iz\sin t}$$

For this case write down equation (41) and its solutions.

568. Let the contour C (see formula (40)) lying in the complex plane of the variable $t=\xi+i\eta$ be expressed by the relations

$$\xi=0,\quad -\infty<\eta\leqslant 0;\quad -\pi\leqslant\xi\leqslant 0,\quad \eta=0;$$
$$\xi=-\pi,\quad 0\leqslant\eta<\infty$$

or

$$\xi=0,\quad -\infty<\eta\leqslant 0;\quad 0\leqslant\xi\leqslant\pi,\quad \eta=0;$$
$$\xi=\pi,\quad 0\leqslant\eta<\infty$$

and let $K(z, t) = -e^{-iz \sin t}/\pi$ or $K(z, t) = e^{-iz \sin t}/\pi$ respectively. Find the solutions of equation (42) in the form of integrals. These solutions are called *Hankel's functions* (or *Bessel's functions of the third kind*) and are denoted $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ respectively.

569. Using the fact that Bessel's function $J_n(z)$ is expressed in terms of Hankel's functions in the form

$$J_n(z) = \frac{1}{2}[H_n^{(1)}(z) + H_n^2(z)]$$

and taking into account the result established in Problem 568, derive the integral representation

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z \sin \xi - n\xi)\, d\xi$$

for Bessel's function $J_n(z)$ with integral index n.

570. Prove that for integral indices n we have

$$J_{-n}(z) = (-1)^n J_n(z)$$

571. Using the representation

$$J_n(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z \sin \xi - n\xi)\, d\xi$$

show that Bessel's functions with integral indices are uniformly bounded for the real values of z.

572. Prove the harmonicity of the function

$$u(x, y, z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda(z+ix \sin t+iy \cos t)} e^{imt}\, dt$$

and the validity of the equality

$$u(x, y, z) = e^{\lambda z} e^{im\varphi} J_{-m}(\lambda\rho)$$

where λ is a real constant, m is an integral number, $x = \rho \cos \varphi$ and $y = \rho \sin \varphi$.

Using the method of separation of variables (and special functions) solve the following problems:

573. Neglecting the resistance of the surrounding medium, consider a process of transverse oscillation of a homogeneous circular membrane of radius R with a rigidly fixed edge and centre at the origin. Determine the oscillation of the membrane for the following cases:

(a) the oscillation is produced by the initial deflection $f(r) = A(R^2 - r^2)$ of the membrane;

(b) the oscillation is produced by a constant initial velocity U of the points of the membrane.

574. For the cases enumerated below find the distribution of temperature in an infinite homogeneous circular cylinder of radius R whose initial temperature is equal to Ur^2:

(a) the surface of the cylinder is heat insulated;

(b) on the surface of the cylinder there is convective heat exchange with the surrounding medium having a zero temperature;

(c) on the surface of the cylinder a constant temperature of T is permanently maintained.

575. Let the initial temperature in the homogeneous bounded cylinder $0 \leqslant r \leqslant R$, $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant z \leqslant l$ be equal to $A(R^2 - r^2)z$. Determine the distribution of temperature in that cylinder at an arbitrary time instant $t > 0$ for the following cases:

(a) a zero temperature is maintained on the lateral surface and on the lower base of the cylinder while the upper base is heat insulated;

(b) a zero temperature is maintained on the upper base of the cylinder, the lower base is heat insulated and on the lateral surface there is heat exchange with the surrounding medium having a zero temperature.

576. Determine the distribution of temperature in a homogeneous ball of radius R and centre at the origin on condition that a zero temperature is maintained on the surface of the ball and the initial temperature of the ball is equal to $f(r, \theta)$.

577. Find the stationary distribution of temperature in a homogeneous cylinder ($0 \leqslant r \leqslant R$, $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant z \leqslant l$) for the following cases:

(a) the lower base of the cylinder has a temperature of T while the temperature of the other part of the surface of the cylinder is equal to zero;

(b) the lower base of the cylinder has zero temperature, the upper base is heat insulated and the temperature of the lateral surface of the cylinder is equal to $f(z)$;

(c) the cylinder contains heat sources with volume density Q while the temperature of the surface of the cylinder is equal to zero.

578. A cylindrical hole of radius R is drilled in the unbounded homogeneous plate $0 \leqslant r < \infty$, $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant z \leqslant l$, the axis of the hole coinciding with the coordinate axis z. Determine the stationary distribution of temperature in that plate on condition that the temperature of the wall of the cylindrical hole is equal to T and the faces of the plate have zero temperature.

579. The concentration of a gas on the boundary of a spherical vessel of radius R with centre at the origin is equal to $f(\theta)$. Determine the stationary distribution of the concentration of the gas (a) inside the vessel and (b) outside the vessel.

2°. *Asymptotic Expansions*

In applications it is sometimes very important to obtain a precise, in a certain sense, description of the behaviour of functions in the neighbourhood of some points we are interested in (for instance, the description of the behaviour of special functions in the neighbourhood of their singular points). For this aim the so-called *asymptotic expansions* of functions are used.

We shall denote by E a set of points in the complex plane of the variable z for which the point at infinity is a limit point. Let there be a function $f(z)$ defined on E, and let us consider finite sums of the form

$$S_n(z) = \sum_{k=0}^{n} \frac{a_k}{z^k}$$

where a_k are some given numbers.

If for any fixed n there holds the relation

$$\lim_{z\to\infty} z^n \left[f(z) - S_n(z)\right] = 0, \quad z \in E \tag{43}$$

we say that the series

$$a_0 + \frac{a_1}{z} + \ldots + \frac{a_n}{z^n} + \ldots \tag{44}$$

provides an *asymptotic expansion* of $f(z)$ on E, irrespective of whether or not series (44) is convergent, and write

$$f(z) \sim \sum_{k=0}^{\infty} \frac{a_k}{z^k}$$

In this case expression (44) is called an *asymptotic series.*

If the behaviour of the function $S_n(z)$ is known then relation (43) provides important information about the behaviour of the function $f(z)$ on the set E in the neighbourhood of the point at infinity. Equalities (43) imply the following formulas for the determination of the coefficients of asymptotic expansion (44):

$$a_0 = \lim_{z\to\infty} f(z)$$

and

$$a_n = \lim_{z\to\infty} z^n [f(z) - S_{n-1}(z)]; \quad n = 1, 2, \ldots$$

580. For the set $E = \{0 < z < \infty\}$ find the asymptotic expansion of the function e^{-z}.

581. Demonstrate by examples that one and the same series can serve as asymptotic expansion for various different functions.

Using integration by parts, derive the following asymptotic expansions:

582. $$\int_z^{\infty} e^{z^2-t^2}\, dt \sim \frac{1}{2z} + \sum_{k=1}^{\infty} (-1)^k \frac{1\cdot 3 \ldots (2k-1)}{2^{k+1} z^{2k+1}},$$

$z < t < \infty$, $E = \{0 < z < \infty\}$, $z \to \infty$.

583. $$\int_z^{\infty} e^{z-t} \frac{t\, dt}{z^2} \sim \frac{1}{z} + \frac{1}{z^2}, \quad z < t < \infty,$$

$E = \{0 < z < \infty\}$, $z \to \infty$.

584. $$\int_0^{\infty} \frac{e^{-t}}{t+z}\, dt \sim \sum_{k=1}^{\infty} \frac{(-1)^{k-1} (k-1)!}{z^k},$$

$z \to \infty$, $|\arg z| \leqslant \pi - \delta < \pi$.

585. $$\int_z^{\infty} e^{-t} t^{a-1}\, dt \sim e^{z} \sum_{k=1}^{\infty} \frac{\Gamma(a)\, z^{a-k}}{\Gamma(a-k+1)},$$

$0 < z < \infty$, $z \to +\infty$, a is a real number.

586. $\int\limits_{z}^{\infty} t^{-a}e^{it}\,dt \sim \frac{ie^{iz}}{z^a}\sum\limits_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(a)\,(iz)^k}$,

$0<z<\infty,\quad z\to+\infty,\quad a>0.$

587. Using the result established in Problem 582, derive the following asymptotic expansion for the *error function**:

$$\frac{2}{\sqrt{\pi}}\int\limits_{z}^{\infty} e^{-\tau^2}\,d\tau \sim e^{-z^2}\sum_{k=1}^{\infty}\frac{z^{1-2k}}{\Gamma\left(\frac{3}{2}-k\right)},\quad 0<z<\infty,\quad z\to+\infty.$$

588. Separate the real and the imaginary parts in the result obtained in Problem 586 to find the asymptotic expansions of *Fresnel's integrals*

(a) $\int\limits_{u}^{\infty} \cos\theta^2\,d\theta$

and

(b) $\int\limits_{u}^{\infty} \sin\theta^2\,d\theta$

for $u\to+\infty$.

With the aid of integration by parts find the asymptotic expansions for the following functions:

589. $\mathrm{Ei}(z)=\int\limits_{-\infty}^{z}\frac{e^{\xi}}{\xi}\,d\xi,\quad -\infty<z<0,\quad z\to-\infty$

(the *exponential integral*).

590. $\mathrm{Ci}(z)=\int\limits_{\infty}^{z}\frac{\cos\xi}{\xi}\,d\xi,\quad 0<z<\infty,\quad z\to\infty$

(the *cosine integral*).

---

* Any of the functions $Erf\,(z)\equiv\int\limits_{0}^{z}e^{-\tau^2}\,d\tau$, $Erfc(z)\equiv\int\limits_{z}^{\infty}e^{-\tau^2}\,d\tau$ and $Erfi\,(z)\equiv\int\limits_{0}^{z}e^{\tau^2}dt$ is also called the error function.— *Tr.*

591. $\operatorname{Si}(z)=\int\limits_{\infty}^{z}\frac{\sin\xi}{\xi}\,d\xi,\quad -\infty<z<\infty,\ |z|\to\infty$

(the *sine integral*).

In order to obtain asymptotic expansions for various classes of functions the so-called *saddle-point method* and the *Watson method* are most frequently used.

Below we present the basic idea of Watson's method.

Let $\varphi(t)$ be a continuous function defined in an interval $0\leqslant t\leqslant N$, $0<N\leqslant\infty$. Then the function $F(z)$ represented by the integral

$$F(z)=\int\limits_{0}^{N}t^{m}\varphi(t)\,e^{-zt^{\alpha}}\,dt,\quad \alpha>0,\quad m>-1$$

is analytic.

If the function $\varphi(t)$ is equal to the sum of a power series in some interval $0\leqslant t\leqslant h_1\leqslant N$, that is

$$\varphi(t)=\sum_{k=0}^{\infty}c_kt^k,\quad c_0\neq 0$$

and if for a fixed value $z=z_0>0$ there holds the inequality

$$\int\limits_{0}^{N}t^{m}|\varphi(t)|e^{-z_0t^{\alpha}}\,dt<M=\text{const}$$

then on the set $E=\{0<z<\infty\}$ the asymptotic expansion of the function $F(z)$ for $z\to\infty$ is given by the formula

$$F(z)\sim\sum_{k=0}^{\infty}\frac{c_k}{\alpha}\,\Gamma\left(\frac{m+k+1}{\alpha}\right)z^{-\frac{m+k+1}{\alpha}}\tag{45}$$

where Γ is Euler's gamma function.

To obtain the asymptotic expansion of the function

$$F(z)=\int\limits_{-A}^{N}\varphi(t)\,e^{-\frac{1}{2}zt^{2}}\,dt,\quad A=\text{const}>0$$

on condition that

$$\varphi(t)=\sum_{k=0}^{\infty}c_kt^k,\quad c_0\neq 0,\quad 0\leqslant t\leqslant h\leqslant\min(A,\ N)$$

it is sufficient to represent $F(2z)$ in the form

$$F(2z)=\int_0^N \varphi(t)\,e^{-zt^2}\,dt+\int_0^A \varphi(-t)\,e^{-zt^2}\,dt$$

and use the following *Watson formula*:

$$F(2z)\sim\sum_{k=0}^{\infty} c_{2k}\Gamma\left(\frac{2k+1}{2}\right)z^{-\frac{2k+1}{2}}=$$

$$=\sqrt{\pi}\sum_{k=0}^{\infty} c_{2k}\,\frac{1\cdot 3\ldots(2k-1)}{2^k}\,z^{-\frac{2k+1}{2}} \qquad (45')$$

Using Watson's method derive the following asymptotic expansions:

592. $\displaystyle\int_0^\infty e^{-zt}\frac{dt}{1+t^{2n}}\sim\sum_{k=0}^{\infty}(-1)^k\frac{(2nk)!}{z^{2nk+1}},\quad 0<t<\infty,$

$0<z<\infty,\quad z\to\infty,\quad n>0\quad (n \text{ is an integer}).$

593. $\displaystyle\int_0^1 t^{p-1}e^{-zt}\,dt\sim\frac{\Gamma(p)}{z^p},$

$0<t<1,\quad 0<z<\infty,\quad z\to\infty,\quad p>0.$

594. $\displaystyle\int_{-A}^{N}\sin te^{-zt^2}\,dt\sim 0,\quad A>0,\quad N>0,$

$0<z<\infty,\quad z\to\infty.$

595. $\displaystyle\int_{-1}^{2}\cos t\,e^{-\frac{1}{2}zt^2}\,dt\sim$

$$\sim\sqrt{2\pi}\sum_{k=0}^{\infty}(-1)^k\frac{1\cdot 3\ldots(2k-1)}{(2k)!}\,z^{-\frac{2k+1}{2}},$$

$-1<t<2,\quad 0<z<\infty,\quad z\to\infty.$

596. Find the asymptotic expansion of the function

$$F(z)=\int_0^\infty e^{-z^2t^2}\,dt$$

for $z\to\infty$, $0<z<\infty$.

597. Prove that

$$e^{-z^2}\int_0^z e^{\xi^2}\,d\xi=\frac{1}{2z}\,[1+o(1)]\quad \text{for}\quad z\to\infty.$$

§ 3. Method of Integral Transformations

We shall begin with the definition of the transform (or image) of a given function $f(t)$ under an *integral transformation.*

By the *transform* (or the *image*) of a function $f(t)$ is meant a function $F(z)$ determined by the formula

$$F(z)=\int_a^b K(z,\ t)\,f(t)\,dt$$

where $K(z,\ t)$ is a given function called a *kernel.* The transformation from $f(t)$ to $F(z)$ is called an *integral transformation.* The given function $f(t)$ is called the *original* (or the *preimage* or the *inverse transform*) of its *transform* (*image*) $F(t)$ corresponding to the *integral transformation* expressed by the above formula. The integral transformation of functions $f(t)$ belonging to a certain class of functions is specified by the choice of the kernel $K(z,\ t)$ of the transformation and of the interval of integration $(a,\ b)$.

Let a given real or complex function $f(t)$ dependent on the real variable t, $0\leqslant t<\infty$, satisfy the following conditions:

(1) $f(t)$ is continuous everywhere except, possibly, a finite number of points at which it can have discontinuities of the first kind;

(2) there exist some constants $M>0$ and $\xi_0>0$ such that

$$|f(t)|<Me^{\xi_0 t}\quad \text{for all}\ t$$

Given a function $f(t)$ of this kind, the integral

$$F(\zeta)=\int_0^\infty e^{-\zeta t}f(t)\,dt$$

exists for all values of ζ whose real parts satisfy the inequality $\operatorname{Re}\zeta > \xi_0$; besides, this integral is an analytic function of the complex variable $\zeta = \xi + i\eta$ in the half-plane $\operatorname{Re}\zeta > \xi_0$.

The function $F(\zeta)$ thus defined is called the *Laplace transform* (or *image*) of the function $f(t)$ and the function $f(t)$ is the *original* (corresponding to the transformation expressed by the last integral; the transformation defined in this way is called the *Laplace transformation*; the original $f(t)$ is also called the *inverse image* or the *preimage* or the *inverse transform of $F(\zeta)$ under the Laplace transformation*).

Under certain conditions, the original $f(t)$ corresponding to a known Laplace transform (image) $F(\zeta)$ can be found from $F(\zeta)$ by means of the *Laplace inverse transformation* expressed by the formula

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{(a+i\eta)t} F(a+i\eta)\, d\eta \tag{46}$$

where the constant a satisfies the inequality $a > \xi_0$.

In the case when the function $f(t)$ is defined for all real values of t the so-called *Fourier transformation* can be defined by means of the formula

$$F(\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\eta t} f(t)\, dt \tag{47}$$

In this case $F(\eta)$ is called the *Fourier transform* of $f(t)$ and $F(\eta)$ is the *inverse Fourier transform* (or the *original* or the *preimage*) of the function $F(\eta)$ corresponding to the Fourier transformation.

If Condition (1) is fulfilled then for the existence of the Fourier transform it is sufficient that the integral

$$\int_{-\infty}^{\infty} f(t)\, dt$$

should be absolutely convergent.

Fourier's inverse transformation (that is Fourier's *inversion formula*) corresponding to (47) is expressed as

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\eta t} F(\eta)\, d\eta \tag{48}$$

It should be noted that if $f(t)$ is an even function then Fourier transformation (47) and Fourier inverse transformation (48) go into the so-called *Fourier cosine transformation*

$$F(\eta) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos \eta t f(t)\, dt$$

and the *Fourier inverse cosine transformation*

$$f(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \cos \eta t F(\eta)\, d\eta$$

respectively. Similarly, if the function $f(t)$ is odd then (47) and (48) go into *Fourier's sine transformation*

$$F(\eta) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin \eta t f(t)\, dt$$

and *Fourier's inverse sine transformation*

$$f(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \sin \eta t F(\eta)\, d\eta$$

respectively.

Among other integral transformations we mention here the *Hankel transformation* (also referred to as the *Fourier-Bessel transformation*)

$$G_n(\eta) = \int_{0}^{\infty} t J_n(\eta t)\, g(t)\, dt$$

and the corresponding inverse transformation

$$g(t) = \int_{0}^{\infty} \eta J_n(\eta t)\, G_n(\eta)\, d\eta$$

and also the *Mellin transformation*

$$G(z)=\int_0^\infty t^{z-1}g(t)\,dt, \quad \operatorname{Re} z=b$$

and the corresponding inverse transformation

$$g(t)=\frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} t^{-z}G(z)\,dz, \quad t>0$$

Accordingly, the function $G_n(\eta)$ is *Hankel's transform* of $g(t)$ and the function $g(t)$ is *Hankel's inverse transform* of $G_n(\eta)$. Analogously, the function $G(z)$ is *Mellin's transform* of $g(t)$, the function $g(t)$ being *Mellin's inverse transform* of $G(z)$.

Integral transformations make it possible to find the solutions of a number of problems of mathematical physics. As an example, let us use the Laplace transformation in order to find the solution $u(x, t)$ of the mixed problem

$$\begin{gathered} a(x)\,u_{xx}+b(x)\,u_{tt}+c(x)\,u_x+d(x)\,u_t+e(x)\,u=0 \\ u(x, 0)=\varphi(x), \quad u_t(x, 0)=\psi(x) \\ u(0, t)=f_1(t), \quad u(l, t)=f_2(t) \end{gathered} \tag{49}$$

in the half-strip $t>0$, $0<x<l$.

Let the parameter ζ and the class of functions among which the solution $u(x, t)$ of this problem is sought be such that the integrals

$$\begin{gathered} v(x, \zeta)=\int_0^\infty e^{-\zeta t}u(x, t)\,dt \\ F_1(\zeta)=\int_0^\infty e^{-\zeta t}u(0, t)\,dt, \quad F_2(\zeta)=\int_0^\infty e^{-\zeta t}u(l, t)\,dt \end{gathered} \tag{50}$$

exist and the following operations on them are legitimate:

$$v_x(x,\ \zeta)=\int_0^\infty e^{-\zeta t}u_x(x,\ t)\,dt,\quad v_{xx}(x,\ \zeta)=\int_0^\infty e^{-\zeta t}u_{xx}(x,\ t)\,dt$$

$$\int_0^\infty e^{-\zeta t}u_t(x,\ t)\,dt=e^{-\zeta t}u(x,\ t)\Big|_0^\infty+\zeta\int_0^\infty e^{-\zeta t}u(x,\ t)\,dt=$$
$$=\zeta v(x,\ \zeta)-u(x,\ 0)\quad (51)$$

$$\int_0^\infty e^{-\zeta t}u_{tt}(x,\ t)\,dt=\zeta^2 v(x,\ \zeta)-\zeta u(x,\ 0)-u_t(x,\ 0)$$

On multiplying both sides of the given equation and the last two conditions of Problem (49) by $e^{-\zeta t}$ and integrating over the interval $(0, \infty)$ of variation of t, we arrive, by virtue of (49), (50) and (51), at the equation

$$a(x)\,v_{xx}+c(x)\,v_x+[e(x)+\zeta d(x)+\zeta^2 b(x)]\,v=$$
$$=\zeta b(x)\,\varphi(x)+b(x)\,\psi(x)+d(x)\,\varphi(x)\quad (52)$$

with the conditions

$$v(0,\ \zeta)=F_1(\zeta),\quad v(l,\ \zeta)=F_2(\zeta)\quad (53)$$

Thus, the solution of mixed problem (49) has been reduced to the determination of the solution $v(x,\ \zeta)$ (which depends on the parameter ζ) of the boundary-value problem (52), (53) for ordinary differential equation (52). After the solution $v(x,\ \zeta)$ of Problem (52), (53) has been constructed, the sought-for solution of Problem (49) can be found by means of the Laplace inverse transformation:

$$u(x,\ t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} v(x,\ a+i\eta)\,e^{(a+i\eta)t}\,d\eta,\quad a>\xi_0$$

In applications, when solving some concrete problems for partial differential equations, it is preferable to use the Fourier transformation because the conditions guaranteeing the existence of the inverse Fourier transform are in many cases naturally fulfilled.

When using Fourier's transformation, it is very convenient to resort to the notion of a convolution. By the *convolution* $f*\varphi$ (or German *faltung*) of two functions $f(x)$ and $\varphi(x)$ defined in the interval $-\infty < x < \infty$ is meant the function of x equal to the integral $\int_{-\infty}^{\infty} f(t)\,\varphi(x-t)\,dt$:

$$f*\varphi = \int_{-\infty}^{\infty} f(t)\,\varphi(x-t)\,dt \tag{54}$$

If Fourier's transforms

$$F(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\zeta} f(t)\,dt$$

and

$$\Phi(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it\zeta} \varphi(t)\,dt$$

of the functions $f(t)$ and $\varphi(t)$ exist and if Fourier's inverse transformations

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\zeta} F(\zeta)\,d\zeta$$

and

$$\varphi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{it\zeta} \Phi(\zeta)\,d\zeta$$

make sense, convolution (54) can be written in the form

$$f*\varphi = \int_{-\infty}^{\infty} F(\zeta)\,\Phi(\zeta)\,e^{i\zeta x}\,d\zeta \tag{55}$$

Lengthy calculations encountered when the Fourier transformation is applied can be considerably simplified by using the so-called *Dirac delta function* $\delta(x)$. The δ-function can be defined formally as the Fourier transform of the constant $1/\sqrt{2\pi}$:

$$\delta(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix\xi}\,d\xi \tag{56}$$

The transformation inverse to (56) is expressed (also formally) by the formula

$$\frac{1}{\sqrt{2\pi}}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ix\xi}\delta(\xi)\,d\xi$$

In the ordinary sense, transformation (56) does not exist, and therefore the definition of the δ-function stated above is purely formal; in modern mathematical analysis a rigorous definition of the δ-function is stated in the theory of the so-called *generalized functions.*

598. Let $f(x)$ be a function defined for $-\infty < x < \infty$. Suppose that $f(x)$ satisfies the conditions under which the Fourier transformation and its inversion can be applied. Prove the following basic property of Dirac's δ-function:

$$f*\delta = f(x)$$

599. Prove the equality $\int_{-\infty}^{\infty}\delta(t)\,dt=1$.

It should be mentioned that in physics Dirac's function is sometimes formally defined as a function which is equal to zero for all real values of x different from zero, turns into infinity at $x=0$ and satisfies the condition

$$\int_{-\infty}^{\infty}\delta(t)\,dt=1$$

Using the Fourier integral transformation solve the problems stated below.

For the half-plane $-\infty < x < \infty$, $t>0$ find the solutions of the following problems:

600. $u_{tt}=a^2u_{xx}$, $u(x, 0)=\varphi(x)$, $u_t(x, 0)=\psi(x)$.
601. $u_{tt}=a^2u_{xx}+f(x, t)$, $u(x, 0)=u_t(x, 0)=0$.
602. $u_t=a^2u_{xx}$, $u(x, 0)=\varphi(x)$.
603. $u_t=a^2u_{xx}+f(x, t)$, $u(x, 0)=0$.

For the quadrant $0 < x < \infty$, $t > 0$ find the solutions of the following problems:

604. $u_t = a^2 u_{xx}$, $u(0, t) = \mu(t)$, $u(x, 0) = 0$.

605. $u_t = a^2 u_{xx}$, $u_x(0, t) = \nu(t)$, $u(x, 0) = 0$.

606. $u_t = a^2 u_{xx} + f(x, t)$, $u(0, t) = u(x, 0) = 0$.

For the half-space $-\infty < x, y < \infty$, $t > 0$ find the solutions of the following problems:

607. $u_t = a^2 (u_{xx} + u_{yy})$, $u(x, y, 0) = \varphi(x, y)$.

608. $u_t = a^2 (u_{xx} + u_{yy}) + f(x, y, t)$, $u(x, y, 0) = 0$.

For the part $-\infty < x < \infty$, $0 < y < \infty$, $t > 0$ *of the space of the variables* x, y, t find the solutions of the following problems:

609. $u_t = a^2 (u_{xx} + u_{yy})$, $u(x, 0, t) = 0$,
$u(x, y, 0) = f(x, y)$.

610. $u_t = a^2 (u_{xx} + u_{yy})$, $u(x, 0, t) = f(x, t)$,
$u(x, y, 0) = 0$.

611. $u_t = a^2 (u_{xx} + u_{yy})$, $u_y(x, 0, t) = 0$,
$u(x, y, 0) = f(x, y)$.

Using the Laplace integral transformation solve the following problems:

612. $u_y = u_{xx} + a^2 u + f(x)$, $u(0, y) = u_x(0, y) = 0$,
$0 < x < \infty$, $0 < y < \infty$.

613. $u_y = u_{xx} + u + B \cos x$, $u(0, y) = Ae^{-3y}$,
$u_x(0, y) = 0$,
$0 < x < \infty$, $0 < y < \infty$.

614. The initial temperature (at $t = 0$) of a thin homogeneous bar is equal to zero. Determine the temperature $u(x, t)$ in the bar for $t > 0$ for the following cases:

(a) the bar is of a finite length $(0 < x < l)$ and

$$u(+0, t) = \delta(t), \quad u(l - 0, t) = 0;$$

(b) the bar is semi-infinite $(0 < x < \infty)$ and

$$u(0, t) = \delta(t), \quad u(\infty, t) = 0;$$

(c) the bar is semi-infinite $(0 < x < \infty)$ and

$$u(0, t) = \mu(t), \quad u(\infty, t) = 0;$$

where $\delta(t)$ is the Dirac delta-function and $\mu(t)$ is a given (ordinary) function.

615. Starting with the initial instant $t = 0$, an electromotive force $E(t)$ is applied to the end $x = 0$ of a semi-infinite insulated wire ($0 \leqslant x < \infty$). For $t > 0$, find the voltage $u(x, t)$ in the wire on condition that the initial voltage and the initial current in the wire are equal to zero for the cases enumerated below:

(a) there are no losses in the wire, that is $R = G = 0$;

(b) the condition $RC = LG$ is fulfilled (the absence of "distortion").

616. Solve the following problem:

$$u_{tt} - a^2 u_{xx} = 0, \quad 0 < x < \infty, \quad 0 < t < \infty$$

$$u_x(0, t) - hu(0, t) = \varphi(t), \quad u(\infty, t) = 0, \quad 0 < t < \infty$$

$$u(x, 0) = u_t(x, 0) = 0, \quad 0 < x < \infty$$

Using Hankel's integral transformation solve the problems stated below.

617. It is required to find the stationary distribution of temperature in the half-space $0 \leqslant r < \infty$, $0 \leqslant \varphi \leqslant 2\pi$, $z > 0$ for the following cases:

(a) the temperature of the boundary $z = 0$ is equal to $f(r)$;

(b) the temperature of the boundary $z = 0$ is equal to T for $r < R$ and to zero for $r > R$;

(c) the half-space is heated by a heat flux of constant density q incident on the part $r \leqslant R$, $z = 0$, $0 \leqslant \varphi \leqslant 2\pi$ of the boundary. On the whole boundary there is heat exchange (obeying Newton's law) with the surrounding medium having zero temperature.

§ 4. Method of Finite Differences

Let the variables x and y be the orthogonal Cartesian coordinates of a point in the plane. Let us cover this plane by the grid $x = mh$, $y = nh$ ($m, n = 0, \pm 1, \ldots$) of points where h is a given positive number. The vertices of each square cell of that grid are referred to as *grid-points* and the number h is called the *grid-size*.

On condition that the six grid-points (x, y), $(x - h, y)$, $(x + h, y)$, $(x, y - h)$, $(x, y + h)$ and $(x + h, y + h)$ belong to the domain D where a function $u(x, y)$ of class $C^{(2)}(D)$ is defined, we can approximate the values of the derivatives of $u(x, y)$ at the grid-point (x, y) according to the formulas

$$
\begin{aligned}
&u_x \approx \frac{u(x, y)-u(x-h, y)}{h}, \quad u_y \approx \frac{u(x, y)-u(x, y-h)}{h} \\
&u_{xx} \approx \frac{u(x+h, y)+u(x-h, y)-2u(x, y)}{h^2} \\
&u_{xy} \approx \frac{u(x+h, y+h)-u(x+h, y)-u(x, y+h)+u(x, y)}{h^2} \\
&u_{yy} \approx \frac{u(x, y+h)+u(x, y-h)-2u(x, y)}{h^2}
\end{aligned}
\tag{57}
$$

Proceeding from formulas (57), for each grid-point (x, y), we can approximate a partial differential equation of the form

$$
a(x, y)\, u_{xx} + 2b(x, y)\, u_{xy} + c(x, y)\, u_{yy} + d(x, y)\, u_x + \\
+ e(x, y)\, u_y + f(x, y)\, u = g(x, y)
$$

defined in a domain D by the finite difference equation

$$
\begin{aligned}
&a(x, y)\,[u(x+h, y) + u(x-h, y) - 2u(x, y)] + \\
&+ 2b(x, y)\,[u(x+h, y+h) - u(x+h, y) - \\
&\qquad - u(x, y+h) + u(x, y)] + \\
&+ c(x, y)\,[u(x, y+h) + u(x, y-h) - 2u(x, y)] + \\
&\qquad + hd(x, y)\,[u(x, y) - u(x-h, y)] + \\
&\qquad + he(x, y)\,[u(x, y) - u(x, y-h)] + \\
&\qquad\qquad + h^2 f(x, y)\, u(x, y) = h^2 g(x, y)
\end{aligned}
\tag{58}
$$

Let the point (x, y) range over the grid-points belonging to the domain D. Then, using formula (58), we obtain a system of algebraic linear equations with respect to the values of the function $u(x, y)$ at these points. Some of these values can be found directly, independently of system (58), from the initial and boundary conditions, or these conditions generate some additional algebraic linear equations, and then system (58) together with the additional equations form a finite-difference approximation to the whole original problem.

The solution of the system of algebraic linear equations thus derived is taken as an approximate solution of the problem in question.

As an instance, let us consider the finite-difference analogue of the Dirichlet problem for harmonic functions. In this case the boundary conditions are taken into account as follows. Let us denote by Q_δ the collection of all those square cells of the grid lying within the domain D at least one of whose vertices lies at a distance not greater than δ from the boundary S of the domain D where $\delta > h$ is a fixed number and h is the grid-size. For each grid-point (x, y) coinciding with a vertex of a square belonging to the collection Q_δ we take as the value of $u(x, y)$ the value of the function $\varphi(x, y)$ (which describes the values of the sought-for harmonic function on S) assumed at the point of the boundary S lying at the shortest distance from (x, y). In the case when there are several such points on S we choose arbitrarily one of the values of the given function $\varphi(x, y)$ assumed at these points and take this value of φ as an approximation to the corresponding boundary value of $u(x, y)$.

618. Construct the finite-difference approximation to the Laplace's equation $u_{xx} + u_{yy} = 0$ for a domain D with boundary S.

619. For the circle $x^2 + y^2 < 16$, putting $h = 1$ and $\delta = h + 1/8$, find the approximate solution of the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in \{x^2 + y^2 < 16\}$$
$$u(x, y) = \varphi(x, y), \quad (x, y) \in \{x^2 + y^2 = 16\}$$

for each of the following cases:

(a) $\varphi(x, y) \equiv 0$;
(b) $\varphi(x, y) = 1$;
(c) $\varphi(x, y) = x$.

For the problems stated above compare the approximate solutions obtained with the aid of the finite-difference method with the exact solutions which can readily be found directly.

620. In the rectangle Q with vertices at the points $A(-3, 4)$, $B(3, 4)$, $C(3, -4)$ and $D(-3, -4)$ find the approximate solution of the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \quad (x, y) \in Q$$
$$u(x, y) = \varphi(x, y), \quad (x, y) \in S$$

where S is the boundary of the rectangle Q; for this problem put $h = 1$, $\delta = h + 1/8$ and consider the following cases:

(a) $\varphi(x, y) = 1$;
(b) $\varphi(x, y) = y$;
(c) $\varphi(x, y) = x + y$.

Compare the approximate solutions found for these cases with the corresponding exact solutions.

Let D be a domain in the xt-plane bounded by segments OA and MN of the straight lines $t = 0$ and $t = H$ $(H > 0)$ and by smooth curves OM and AN each of which intersects every straight line $t = \text{const}$ at not more than one point. We shall denote by S the part of the boundary of the domain D consisting of OM, OA and AN. Let us consider the approximate solution of the first boundary-value problem for the heat conduction equation:

$$u_{xx} - u_t = 0, \qquad (x, t) \in D \tag{59}$$
$$u(x, t) = \varphi(x, t), \qquad (x, t) \in S \tag{60}$$

In order to take into account boundary condition (60), let us denote by Q_h the collection of all square cells of the grid with grid-size h which do not fall outside the closed region $\bar{D}$ and by ∂Q_h the boundary of Q_h.

Let q_h be the collection of all squares belonging to Q_h at least one of whose vertices lies on ∂Q_h with the exception of the internal squares belonging to the uppermost row of square cells adjoining the upper part of the boundary of the domain D. For the values of $u(x, t)$ at the grid-points which are the vertices of the squares belonging to q_h we shall take the values of $\varphi(x, t)$ at the points of the boundary S lying at the shortest distance from the corresponding grid-points. The unknown values of $u(x, t)$ at the other grid-points lying within D can be found by solving the algebraic linear system of equations resulting from the finite-difference approximation of equation (59).

621. Construct the finite-difference analogue of the heat conduction equation $u_{xx} - u_t = 0$ in the domain where the solution of the first boundary-value problem (59), (60) is sought.

622. Putting $h = 1$, find the approximate solution of the first boundary-value problem

$$u_{xx} - u_t = 0, \qquad (x, t) \in Q$$
$$u(x, t) = \varphi(x, t), \qquad (x, t) \in S$$

for the rectangle Q with vertices at the points $A(0, 0)$, $B(0, 5)$, $C(4, 5)$ and $D(4, 0)$ (S denotes the boundary of Q) where the function $\varphi(x, t)$ is such that

$$\varphi(x, 0) = x, \quad \varphi(0, t) = 0, \quad \varphi(4, t) = 4$$

623. For the rectangle Q with vertices at the points $A(0, 0)$, $B(0, 3)$, $C(5, 3)$ and $D(5, 0)$ find the approximate finite-difference solution of the problem

$$u_{xx} - u_t = 0, \quad u(0, t) = t, \quad u(5, t) = t + 25/2,$$
$$u(x, 0) = x^2/2$$

for $h = 1$.

Compare this approximate solution of the problem with its exact solution $u(x, t) = t + x^2/2$.

624. Use the finite-difference scheme with grid-size $h = 1$ to find the values of the approximate solution $u(x, y)$ of the Goursat problem

$$u_{xy} = 0, \quad 0 < x < \infty, \quad 0 < y < \infty$$
$$u(0, y) = \varphi(y), \; 0 < y < \infty; \; u(x, 0) = \psi(x), \; 0 < x < \infty$$

at the grid-points (2, 2), (2, 3) and (2, 4) for the following cases:

(a) $\varphi(y) = 0, \quad \psi(x) = x$;
(b) $\varphi(y) = y, \quad \psi(x) = 0$;
(c) $\varphi(y) = y, \quad \psi(x) = x$.

§ 5. Variational Methods

Partial differential equations which are encountered in applications can often be regarded as Euler's equations for the corresponding variational problems.

As is known, Laplace's equation $\Delta u = 0$ can serve as Euler's equation of the minimum problem for the *Dirichlet* *integral*

$$D(u) = \int_D (u_x^2 + u_y^2)\, dx\, dy \tag{61}$$

taken over a domain D with boundary S.

Let us consider the class of continuous functions defined in $D \cup S$ and possessing piecewise continuous partial derivatives of the first order in the domain D for which Dirichlet's integral (61) is finite and which assume on the boundary S of D prescribed values equal to a given continuous function $\varphi(x, y)$. The functions of this class will be referred to as *admitted functions*. The problem of determining the function belonging to the class of the admitted functions for which Dirichlet's integral (61) attains its minimum is called the *first variational problem*.

If d is the minimum value of Dirichlet's integral or, generally, of an arbitrary functional $\Phi(u)$, a sequence $\{u_n\}$; $n = 1, 2, \ldots$, of admitted functions possessing the property

$$\lim_{n \to \infty} \Phi(u_n) = d$$

is referred to as a *minimizing sequence*.

The main role in the theory of variational methods is played by the construction of a minimizing sequence. One of the methods for the construction of such a sequence was suggested by Ritz. The essence of *Ritz' method* is the following.

Let $\{\varphi_n\}$; $n = 1, 2, \ldots$, be a complete system of functions belonging to the class of admitted functions for a functional $\Phi(u)$. Such a sequence $\{\varphi_n\}$ is referred to as a *system of coordinate functions*. Let us construct a new sequence of the form

$$u_n = \sum_{k=1}^{n} c_k \varphi_k; \quad n = 1, 2, \ldots$$

where c_k are indeterminate coefficients; let us choose these coefficients in such a way that the expression $\Phi_n = \Phi(u_n)$ considered as function of the variables $c_1, \ldots, c_n$ assumes the minimum value. For some classes of functionals it is possible to prove that such sequences $\{u_n\}$ are minimizing sequences and that the limits of these sequences are the solutions of the variational problems under consideration.

625. Let a function $\varphi(x, y)$ defined on the boundary S of a domain D be such that the class of admitted functions assuming on S the values equal to $\varphi(x, y)$ is not void. Prove that in this case the Dirichlet problem

$$\Delta u(x, y) = 0, \qquad (x, y) \in D$$
$$u(x, y) = \varphi(x, y), \qquad (x, y) \in S$$

and the first variational problem are equivalent.

626. Show that among the admitted functions $y(x)$, $0 \leqslant x \leqslant 1$ satisfying the conditions $y(0) = 0$ and $y(1) = a$ the function $y(x) = ax^n$ minimizes the functional

$$I_n(y) = \int_0^1 \left[\left(\frac{dy}{dx}\right)^2 + \frac{n^2}{x^2} y^2\right] x\, dx$$

where n is a positive integer. Compute $\min I_n(y)$.

627. Consider the class of admitted functions $u(x, y)$ defined in the square Q: $0 \leqslant x \leqslant \pi$, $0 \leqslant y \leqslant \pi$, which turn into zero on the boundary of that square. The function $u(x, y) = \frac{2}{\pi} \sin x \sin y$ belonging to this class minimizes the functional

$$I(u) = \frac{D(u)}{H(u)} \tag{62}$$

where $D(u) = \int\limits_Q (u_x^2 + u_y^2)\, dx\, dy$ and $H(u) = \int\limits_Q u^2\, dx\, dy$.

Using this fact prove that the inequality

$$H(u) \leqslant \frac{1}{2} D(u)$$

holds for all the admitted functions.

628. Among the continuously differentiable functions $y(x)$ defined on the interval $0 \leqslant x \leqslant \pi$ and satisfying the conditions

$$y(0) = y(\pi) = 0 \quad \text{and} \quad H(y) = \int_0^\pi y^2(x)\, dx = 1$$

find the one which minimizes the functional

$$D(y) = \int_0^\pi y'^2(x)\, dx$$

629. Prove that the admitted functions $y(x)$ indicated in Problem 628 satisfy the inequality

$$H(y) \leqslant D(y)$$

630. Find the first approximation to the solution of the minimum problem for the functional

$$D(y) = \int_0^1 (y'^2 + y^2 + 2xy)\, dx, \quad y(0) = y(1) = 0$$

in the case when the coordinate functions are of the form $\varphi_0(x) = 0$, $\varphi_n(x) = x^n(x-1)$; $n = 1, 2, \ldots$.

631. Reduce the Dirichlet problem

$$\Delta u(x, y) = -1, \quad (x, y) \in D; \quad u(x, y) = 0, \quad (x, y) \in S$$

to the minimum problem for the functional

$$D(u) = \int_D (u_x^2 + u_y^2 - 2u)\, dx\, dy, \quad u|_S = 0$$

where the domain D is the square $-1 < x < 1$, $-1 < y < 1$, and find the first approximation $u_1(x, y)$ to the solution of the latter problem for the case of the coordinate functions

$$v_1(x, y) = (x^2 - 1)(y^2 - 1),$$
$$v_2(x, y) = (x^2 - 1)(y^3 - 1)(x^2 + y^2), \ldots$$

(according to Ritz' scheme in this case $u_1(x, y)$ is constructed in the form $u_1(x, y) = cv_1(x, y)$).

632. Reduce the Dirichlet problem

$$\Delta u(x, y) = xy, \quad (x, y) \in D; \quad u(x, y) = 0, \quad (x, y) \in S$$

to the corresponding variational problem and find the approximate solution $u_1(x, y) = cxy(x-1)(y-1)$ for the case when the domain D is the square $0 < x < 1$, $0 < y < 1$.

633. Let us consider the admitted functions for functional (62) which are defined in the circle Q: $x^2 + y^2 < 1$ and turn into zero on the boundary of the circle. Using Ritz' method find the function which minimizes functional (62).

634. Using the result established in Problem 633, derive the inequality

$$H(u) \leqslant CD(u)$$

and determine the exact value of the constant C for the case when the domain Q is the circle $x^2 + y^2 < 1$.

ANSWERS, HINTS AND SOLUTIONS

Chapter 1

1. No. **2.** Yes. **3.** No. **4.** No. **5.** No. **6.** No. **7.** The first. **8.** The second. **9.** The first. **10.** The first. **11.** The second. **12.** The second. **13.** Non-linear. **14.** Quasi-linear. **15.** Non-homogeneous linear. **16.** Homogeneous linear. **17.** Non-homogeneous linear. **18.** Non-linear. **19.** Linear (and non-homogeneous for $h(x, y) \not\equiv 0$). **20.** Quasi-linear. **21.** Quasi-linear. **22.** Quasi-linear. **23.** Quasi-linear (linear with respect to the highest derivatives). **24.** Homogeneous linear. **25.** Hyperbolic. **26.** Elliptic. **27.** Parabolic.

28. The equation is of parabolic type. Indeed, the form

$$Q(\lambda_1, \lambda_2, \lambda_3) = 4\lambda_1^2 + 2\lambda_2^2 - 6\lambda_3^2 + 6\lambda_1\lambda_2 + 10\lambda_1\lambda_3 + 4\lambda_2\lambda_3 =$$
$$= \frac{1}{4}(4\lambda_1 + 3\lambda_2 + 5\lambda_3)^2 - \frac{1}{4}(\lambda_2 + 7\lambda_3)^2$$

corresponding to this partial differential equation reduces to its standard form $K(\xi_1, \xi_2, \xi_3) = \xi_1^2 - \xi_2^2$ under the non-singular linear transformation of variables

$$\lambda_1 = \frac{1}{2}\xi_1 - \frac{3}{2}\xi_2 + 4\xi_3, \quad \lambda_2 = 2\xi_2 - 7\xi_3, \quad \lambda_3 = \xi_3$$

whence it follows that the equation is parabolic. **29.** Hyperbolic. **30.** The equation is elliptic because the corresponding characteristic form

$$Q(\lambda_1, \lambda_2, \lambda_3) = \lambda_1^2 + 2\lambda_1\lambda_2 + 2\lambda_2^2 + 4\lambda_2\lambda_3 + 5\lambda_3^2$$

is positive definite. In this problem and in Problems 33 and 35 it is advisable to use *Sylvester's theorem* which implies a *necessary and sufficient condition for a symmetric quadratic form*

$$Q = a_{11}\lambda_1^2 + 2a_{12}\lambda_1\lambda_2 + 2a_{13}\lambda_1\lambda_3 + a_{22}\lambda_2^2 + 2a_{23}\lambda_2\lambda_3 + a_{33}\lambda_3^2$$

to be positive definite; this condition states that it is necessary and sufficient that all the *principal minors*

$$A_{11}=a_{11},\quad A_{22}=\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix},\quad A_{33}=\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$$

of the matrix $\begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}$ should be positive.

31. The equation is of hyperbolic type. Indeed, the corresponding characteristic form

$$Q(\lambda_1, \lambda_2, \lambda_3) = \\ = \lambda_1^2 - 4\lambda_1\lambda_2 + 2\lambda_1\lambda_3 + 4\lambda_2^2 + \lambda_3^2 = \\ = (\lambda_1 - 2\lambda_2 + \lambda_3)^2 + (\lambda_2 + \lambda_3)^2 - (\lambda_2 - \lambda_3)^2$$

reduces to its standard form $K(\mu_1, \mu_2, \mu_3) = \mu_1^2 + \mu_2^2 - \mu_3^2$ under the non-singular linear transformation of variables

$$\lambda_1 = \mu_1 + \frac{1}{2}\mu_2 + \frac{3}{2}\mu_3,\quad \lambda_2 = \frac{1}{2}(\mu_2 + \mu_3),\quad \lambda_3 = \frac{1}{2}(\mu_2 - \mu_3)$$

32. The equation is of hyperbolic type because the corresponding characteristic form

$$Q(\lambda_1, \lambda_2, \lambda_3) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \\ = \frac{1}{4}(\lambda_1 + \lambda_2 + 2\lambda_3)^2 - \frac{1}{4}(\lambda_1 - \lambda_2)^2 - \lambda_3^2$$

reduces to its standard form $K(\mu_1, \mu_2, \mu_3) = \mu_1^2 - \mu_2^2 - \mu_3^2$ under the transformation of variables

$$\lambda_1 = \mu_1 + \mu_2 - \mu_3,\quad \lambda_2 = \mu_1 - \mu_2 - \mu_3,\quad \lambda_3 = \mu_3$$

33. Elliptic. **34.** The equation is hyperbolic. Indeed, the non-singular linear transformation of variables $\lambda_1 = \mu_1 - \mu_2 - \mu_3$, $\lambda_2 = \mu_2 + \mu_3$, $\lambda_3 = \mu_3$ brings the corresponding characteristic form

$$Q(\lambda_1, \lambda_2, \lambda_3) = \lambda_1^2 + 2\lambda_1\lambda_2 + 2\lambda_2^2 - 2\lambda_2\lambda_3 = \\ = (\lambda_1 + \lambda_2)^2 + (\lambda_2 - \lambda_3)^2 - \lambda_3^2$$

to the standard form $K(\mu_1, \mu_2, \mu_3) = \mu_1^2 + \mu_2^2 - \mu_3^2$.

35. Elliptic. **36.** The equation is of parabolic type for $y = 0$, of hyperbolic type for $y < 0$ and of elliptic type for $y > 0$. **37.** The equation is of parabolic type for $x = 0$, $y \neq 0$ and for $y = 0$, $x \neq 0$; it is of hyperbolic type when $\operatorname{sgn} x \neq \operatorname{sgn} y$ and of elliptic type when $\operatorname{sgn} x = \operatorname{sgn} y$. **38.** The equation is of hyperbolic type. **39.** The equation is of elliptic type for $u = x^2 + y^2$ and of hyperbolic type for $u = 2\sqrt{2}xy$. **40.** The equation is of elliptic type for $u = (x + y)^2$, of hyperbolic type for $u = x$ and of parabolic type for $u = x^2 + y^2/4 + 17xy/16$. **41.** The equation is of parabolic type for $u = 2y^2$, of elliptic type for $u = 5xy$ and of hyperbolic type for $u = x$. **42.** The equation is of parabolic type for $u = (x^2 + y^2)/2$ and of hyperbolic type for $u = 2y^2$. **43.** The equation is of hyperbolic type. **44.** The equation is of hyperbolic type. **45.** The equation is of hyperbolic type. **46.** The equation is of elliptic type. **47.** The equation is of hyperbolic type for $u = (x + y)^2/2$ and of parabolic type for $u = \sqrt{3}x^2$. **48.** The equation is of elliptic type. **49.** The equation is of parabolic type. **50.** For the solution $u = x^2 - y^2$ the equation belongs to none of the three types because $K(\lambda_1, \lambda_2) = 0$; for the solution $u = x$ the equation is of elliptic type.

51. The equation is hyperbolic, elliptic or parabolic if the expression $\frac{\partial F}{\partial u_{xx}} \frac{\partial F}{\partial u_{yy}} - \frac{1}{4}\left(\frac{\partial F}{\partial u_{xy}}\right)^2$ is less than, greater than or equal to zero respectively.

52. Elliptic. **53.** Hyperbolic. **54.** Parabolic. **55.** Hyperbolic. **56.** Hyperbolic. **57.** Elliptic. **58.** Parabolic. **59.** Elliptic. **60.** Parabolic. **61.** Hyperbolic. **62.** Hyperbolic. **63.** Hyperbolic. **64.** Parabolic.

65. The equation is of hyperbolic type for $k < 0$, of parabolic type for $k = 0$ and of elliptic type for $k > 0$. **66.** The equation is of hyperbolic type for $-0.5 < k < 0.5$, of parabolic type for $k = \pm 0.5$ and of elliptic type for $|k| > 0.5$. **67.** The equation is of parabolic type for $k = 0$ and for $k = 4$, of elliptic type for $0 < k < 4$ and of hyperbolic type for $k < 0$ and for $k > 4$.

68. The equation is elliptic everywhere; the normal form of the equation is

$$v_{\xi\xi} + v_{\eta\eta} - 8v = 0; \quad \xi = y - x, \quad \eta = 2x$$

69. The equation is parabolic everywhere; the normal form of the equation is

$$v_{\eta\eta}+18v_{\xi}+9v_{\eta}-9v=0;\quad \xi=x+y,\quad \eta=x$$

70. The equation is hyperbolic everywhere; the normal form of the equation is

$$v_{\xi\eta}+3v_{\xi}-v_{\eta}+2v=0;\quad \xi=y-x,\quad \eta=2y-x$$

71. The equation is hyperbolic everywhere; the normal form of the equation is

$$v_{\xi\eta}+v_{\xi}-2v_{\eta}+\xi+\eta=0;\quad \xi=2x-y,\quad \eta=x+y$$

72. The equation is parabolic everywhere; the normal form of the equation is

$$27v_{\eta\eta}-105v_{\xi}+30v_{\eta}-150v-2\xi+5\eta=0;$$
$$\xi=x+3y,\quad \eta=x$$

73. The equation is elliptic everywhere; the normal form of the equation is

$$v_{\xi\xi}+v_{\eta\eta}+15v_{\xi}-4\sqrt{6}\,v_{\eta}+\frac{1}{3}\xi+$$
$$+\frac{1}{\sqrt{6}}\eta=0;\quad \xi=y-2x,\quad \eta=\sqrt{6}\,x$$

74. The equation is elliptic everywhere; the normal form of the equation is

$$v_{\xi\xi}+v_{\eta\eta}-2v_{\xi}+v_{\eta}-v+\eta-\xi=0;$$
$$\xi=2x-y,\quad \eta=3x$$

75. The equation is elliptic everywhere; the normal form of the equation is

$$v_{\xi\xi}+v_{\eta\eta}=0;\quad \xi=y,\quad \eta=\arctan x$$

76. The equation is parabolic everywhere except the origin (at the origin the equation degenerates); the normal form of the equation is

$$v_{\eta\eta}-\frac{\xi}{2\eta(\xi+\eta)}v_{\xi}+\frac{1}{2\eta}v_{\eta}=0;\quad \xi=y^2-x^2,\quad \eta=x^2$$

77. The equation is hyperbolic everywhere; the normal form of the equation is

$$v_{\xi\eta} = 0; \quad \xi = x + \arctan y, \quad \eta = x - \arctan y$$

78. The equation is elliptic everywhere; the normal form of the equation is

$$v_{\xi\xi} + v_{\eta\eta} - 2v = 0; \quad \xi = \ln\left(x + \sqrt{1+x^2}\right),$$
$$\eta = \ln\left(y + \sqrt{1+y^2}\right)$$

79. The equation is parabolic everywhere except the origin (at the origin the equation degenerates); the normal form of the equation is

$$v_{\eta\eta} + 2\frac{\xi^2}{\eta^2} v_\xi + \frac{1}{\eta} e^{\xi} = 0; \quad \xi = \frac{y}{x}, \quad \eta = y$$

80. The equation is parabolic everywhere except the coordinate axis $x = 0$ (on the axis $x = 0$ the equation degenerates); the normal form of the equation is

$$v_{\eta\eta} + \frac{2\eta^2}{\xi - \eta^2} v_\xi - \frac{1}{\eta} v_\eta = 0; \quad \xi = x^2 + y^2, \quad \eta = x$$

81. The equation is hyperbolic everywhere; the normal form of the equation is

$$v_{\xi\eta} = 0; \quad \xi = x + y - \cos x, \quad \eta = -x + y - \cos x$$

82. The equation is parabolic everywhere; the normal form of the equation is

$$v_{\eta\eta} - \frac{\xi}{1 + \xi e^{\eta}} v_\xi - \eta e^{-2\eta} v = 0; \quad \xi = e^{-y} - e^{-x}, \quad \eta = x$$

83. The equation is parabolic for $x = 0$, the normal form being $u_{xx} = 0$; the equation is hyperbolic for $x \neq 0$, the normal form being

$$v_{\xi\eta} - \frac{1}{2(\xi - \eta)} v_\xi = 0; \quad \xi = x^2 + y, \quad \eta = y$$

84. The equation is parabolic for $x = 0$, the normal form being $u_{yy} = 0$; the equation is hyperbolic for $x > 0$, the normal form being

$$v_{\xi\eta}+\frac{1}{2(\xi-\eta)}(v_\xi-v_\eta)=0;\quad \xi=y-x+$$
$$+2\sqrt{x},\quad \eta=y-x-2\sqrt{x}$$

The equation is elliptic for $x<0$; in this case its normal form is

$$v_{\xi\xi}+v_{\eta\eta}-\frac{1}{\eta}v_\eta=0;\quad \xi=y-x,\quad \eta=2\sqrt{-x}$$

85. The equation is parabolic for $y=0$, the normal form being $u_{yy}=0$; the equation is hyperbolic for $y<0$, the corresponding normal form being

$$v_{\xi\eta}+\frac{1}{6(\xi+\eta)}(v_\xi+v_\eta)=0;\quad \xi=\frac{2}{3}(-y)^{3/2}+x,$$
$$\eta=\frac{2}{3}(-y)^{3/2}-x$$

The equation is elliptic for $y>0$; in this case the normal form of the equation is

$$v_{\xi\xi}+v_{\eta\eta}+\frac{1}{3\xi}v_\xi=0;\quad \xi=\frac{2}{3}y^{3/2},\quad \eta=x$$

86. The equation is parabolic for $x=0$, $y\neq 0$ and for $x\neq 0$, $y=0$, the normal forms being $u_{yy}+\frac{2}{y}(u_x+u_y)=0$ and $u_{xx}+\frac{2}{x}(u_x+u_y)=0$ respectively (at the origin the equation degenerates). The equation is hyperbolic for $x>0$, $y<0$ and for $x<0$, $y>0$, the normal form being $v_{\xi\eta}-\frac{3}{\xi^2-\eta^2}(\eta v_\xi-\xi v_\eta)=0$ (the corresponding transformations of variables are: $\xi=\sqrt{-y}+\sqrt{x}$, $\eta=\sqrt{-y}-\sqrt{x}$ for $x>0$, $y<0$ and $\xi=\sqrt{y}+\sqrt{-x}$, $\eta=\sqrt{y}-\sqrt{-x}$ for $x<0$, $y>0$). The equation is elliptic for $x>0$, $y>0$ and for $x<0$, $y<0$, the normal form being $v_{\xi\xi}+v_{\eta\eta}+$ $+3\left(\frac{1}{\xi}v_\xi+\frac{1}{\eta}v_\eta\right)=0$ (the corresponding transformations of variables are: $\xi=\sqrt{y}$, $\eta=\sqrt{x}$ for $x>0$, $y>0$ and $\xi=\sqrt{-y}$, $\eta=\sqrt{-x}$ for $x<0$, $y<0$). **87.** The equation is parabolic on the straight lines $x=(2k+1)\pi/2$; $k=0$,

$\pm 1, \ldots$. The equation is hyperbolic everywhere except the straight lines $x=(2k+1)\pi/2$, $k=0, \pm 1, \ldots$, the normal form being

$$v_{\xi\eta}+\frac{\xi-\eta}{2[4-(\xi-\eta)^2]}(v_\xi-v_\eta)=0$$

$$\xi=y+\cos x+\sin x, \quad \eta=y+\cos x-\sin x$$

88. The equation is parabolic on the coordinate axes $x=0$ and $y=0$ where it reduces to $u_{xx}=0$. The equation is hyperbolic for $x>0$, $y<0$ and for $x<0$, $y>0$, its normal form being

$$v_{\xi\eta}-\frac{1}{3(\xi^2-\eta^2)}[(2\xi-\eta)v_\xi-(2\eta-\xi)v_\eta]=0$$

(the corresponding transformations of variables are: $\xi=-2(-y)^{1/2}+2x^{3/2}/3$, $\eta=-2(-y)^{1/2}-2x^{3/2}/3$ for $x>0$, $y<0$ and $\xi=2y^{1/2}+2(-x)^{3/2}/3$, $\eta=2y^{1/2}-2(-x)^{3/2}/3$ for $x<0$, $y>0$). The equation is elliptic for $x>0$, $y>0$ and for $x<0$, $y<0$, the normal form being $v_{\xi\xi}+v_{\eta\eta}-\frac{1}{\xi}v_\xi+\frac{1}{3\eta}v_\eta=0$ (the corresponding transformations of variables are: $\xi=2y^{1/2}$, $\eta=2x^{3/2}/3$ for $x>0$, $y>0$ and $\xi=2(-y)^{1/2}$, $\eta=2(-x)^{3/2}/3$ for $x<0$, $y<0$).

89. $w_{\xi\xi}+w_{\eta\eta}-\frac{15}{2}w=0,$

$$\xi=2x+y, \quad \eta=x, \quad v(\xi,\eta)=u(\eta,\xi-2\eta)=$$
$$=e^{\frac{5\xi+3\eta}{2}}w(\xi,\eta).$$

90. $w_{\eta\eta}-w_\xi=0,$

$$\xi=3x+y, \quad \eta=x, \quad v(\xi,\eta)=u(\eta,\xi-3\eta)=$$
$$=e^{\frac{-\xi+2\eta}{4}}w(\xi,\eta).$$

91. $w_{\xi\eta}+\frac{1}{2}w+\frac{\eta}{2}e^{\frac{\xi}{2}}=0,$

$$\xi=2x+y, \quad \eta=x, \quad v(\xi,\eta)=u(\eta,\xi-2\eta)=$$
$$=e^{-\frac{\xi}{2}}w(\xi,\eta).$$

92. $w_{\xi\eta}-7w=0,$

$\xi=2x-y,\quad \eta=x,\quad v(\xi,\eta)=u(\eta,\ 2\eta-\xi)=$
$=e^{-\xi-6\eta}w(\xi,\eta).$

93. $w_{\xi\xi}+w_{\eta\eta}-\frac{3}{2}w=0,$

$\xi=2y-x,\quad \eta=x,\quad v(\xi,\ \eta)=u\left(\eta,\ \frac{\xi+\eta}{2}\right)=$
$=e^{-\xi-\eta}w(\xi,\ \eta).$

94. $w_{\eta\eta}-2w_{\xi}=0,$

$\xi=y-x,\quad \eta=y+x,$

$v(\xi,\ \eta)=u\left(\frac{\eta-\xi}{2},\ \frac{\eta+\xi}{2}\right)=e^{\frac{15\xi+8\eta}{32}}w(\xi,\ \eta).$

95. $w_{\xi\eta}-w=0,$

$\xi=x-y,\quad \eta=x+y,$

$v(\xi,\ \eta)=u\left(\frac{\eta+\xi}{2},\ \frac{\eta-\xi}{2}\right)=e^{-\frac{1}{2}\xi}w(\xi,\ \eta).$

96. $w_{\xi\eta}+9w+4(\xi-\eta)e^{\xi+\eta}=0,$
$\xi=y-x,\quad \eta=y,\quad v(\xi,\ \eta)=u(\eta-\xi,\ \eta)=$
$=e^{-\xi-\eta}w(\xi,\ \eta).$

97. $w_{\xi\eta}-w+\xi e^{\eta}=0,$
$\xi=y,\quad \eta=x-3y,\quad v(\xi,\ \eta)=u(\eta+3\xi,\ \xi)=$
$=e^{-\eta}w(\xi,\ \eta).$

98. $w_{\xi\xi}+w_{\eta\eta}-w=0,$
$\xi=2x-y,\quad \eta=x,\quad v(\xi,\ \eta)=u(\eta,\ 2\eta-\xi)=$
$=e^{\xi+\eta}w(\xi,\ \eta).$

99. $w_{\xi\xi}+w_{\eta\eta}+2w=0,$

$\xi=y,\quad \eta=4x-2y,$

$v(\xi,\ \eta)=u\left(\frac{\eta+2\xi}{4},\ \xi\right)=e^{\xi-\eta}w(\xi,\eta).$

100. $w_{\xi\xi}+w_{\eta}=0,$
$\xi=2x-y,\quad \eta=x+y,$

$v(\xi,\ \eta)=u\left(\frac{\xi+\eta}{3},\ \frac{2\eta-\xi}{3}\right)=e^{\xi-2\eta}w(\xi,\ \eta).$

101. $v_{\xi\xi}+v_{\eta\eta}+v_{\zeta\zeta}=0,$

$\xi=x,\quad \eta=-x+y,\quad \zeta=2x-2y+z.$

The characteristic quadratic form corresponding to the original equation is $Q = \lambda_1^2 + 2\lambda_1\lambda_2 + 2\lambda_2^2 + 4\lambda_2\lambda_3 + 5\lambda_3^2$. This form can be brought to the form $Q = (\lambda_1 + \lambda_2)^2 + (\lambda_2 + 2\lambda_3)^2 + \lambda_3^2$ (for instance, by using *Lagrange's method*). Let us denote $\mu_1 = \lambda_1 + \lambda_2$, $\mu_2 = \lambda_2 + 2\lambda_3$ and $\mu_3 = \lambda_3$; then Q assumes the standard form $Q = \mu_1^2 + \mu_2^2 + \mu_3^2$. Hence, the non-singular (non-degenerate) affine transformation $\lambda_1 = \mu_1 - \mu_2 + 2\mu_3$, $\lambda_2 = \mu_2 - 2\mu_3$, $\lambda_3 = \mu_3$ whose matrix is

$$M = \begin{Vmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{Vmatrix}$$

reduces Q to its standard form $Q = \mu_1^2 + \mu_2^2 + \mu_3^2$.

The matrix of the non-degenerate affine transformation under which the original partial differential equation reduces to its normal form is the transpose of the matrix M, that is

$$M^* = \begin{Vmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{Vmatrix}$$

and the transformation itself is written as

$$\xi = x, \quad \eta = -x + y, \quad \zeta = 2x - 2y + z$$

Using this transformation and denoting $u(x, y, z) = v(\xi, \eta, \zeta)$ we find:

$$u_{xx} = v_{\xi\xi} + v_{\eta\eta} + 4v_{\zeta\zeta} - 2v_{\xi\eta} + 4v_{\xi\zeta} - 4v_{\eta\zeta}$$
$$u_{yy} = v_{\eta\eta} + 4v_{\zeta\zeta} - 4v_{\eta\zeta}, \quad u_{zz} = v_{\zeta\zeta}$$
$$u_{xy} = -v_{\eta\eta} - 4v_{\zeta\zeta} + v_{\xi\eta} - 2v_{\xi\zeta} + 4v_{\eta\zeta}, \quad u_{yz} = -2v_{\zeta\zeta} + v_{\eta\zeta}$$

On substituting the expressions of the derivatives thus found into the original equation we obtain $v_{\xi\xi} + v_{\eta\eta} + v_{\zeta\zeta} = 0$.

102. $v_{\xi\xi} + v_{\eta\eta} - v_{\zeta\zeta} + 3v_\xi + \frac{3}{2} v_\eta - \frac{9}{2} v_\zeta = 0,$

$$\xi = x, \quad \eta = \frac{1}{2}(x + y + z), \quad \zeta = -\frac{1}{2}(3x + y - z).$$

103. $v_{\xi\xi} - v_{\eta\eta} - v_{\zeta\zeta} + 2v_\eta = 0,$

$$\xi = x + y, \quad \eta = -x + y, \quad \zeta = -x - y + z.$$

104. $v_{\xi\xi}-v_{\eta\eta}+v_{\zeta\zeta}+v=0$,

$$\xi=y+z,\quad \eta=-y+z,\quad \zeta=\frac{1}{\sqrt{6}}x-\frac{2}{\sqrt{6}}y+\frac{\sqrt{6}}{2}z.$$

105. $v_{\eta\eta}+v_{\zeta\zeta}-8v=0$,

$$\xi=x+\frac{1}{2}y+\frac{1}{2}z,\quad \eta=-\frac{1}{2}(y+z),$$

$$\zeta=\frac{1}{2\sqrt{2}}(y-z).$$

106. $v_{\xi\xi}-v_{\eta\eta}+v_{\zeta\zeta}+2v_{\xi}-\sqrt{2}\,v_{\eta}+\sqrt{2}\,v_{\zeta}+4v=0$,

$$\xi=x,\ \ \eta=-\frac{1}{2\sqrt{2}}(3x-y),\ \ \zeta=-\frac{1}{2\sqrt{2}}(x+y-4z).$$

107. $v_{\xi\xi}+v_{\eta\eta}-3v+\frac{1}{\sqrt{2}}(\xi-\eta)-2\zeta=0$,

$$\xi=\frac{1}{\sqrt{2}}x,\quad \eta=\frac{3}{\sqrt{2}}x+\sqrt{2}\,y,\quad \zeta=x+z.$$

108. $v_{\xi\xi}-v_{\eta\eta}+4v=0$,

$$\xi=y+z,\quad \eta=-y-2z,\quad \zeta=x-z.$$

109. $v_{\xi\xi}+2v=0$,

$$\xi=x,\quad \eta=-2x+y,\quad \zeta=-x+z.$$

110. $v_{\xi\xi}-2v_{\xi}=0$,

$$\xi=x,\quad \eta=-2x+y,\quad \zeta=-3x+z.$$

111. (a) $K=\frac{1}{2}\int\limits_0^l \rho(x)\,u_t^2(x,\ t)\,dx$;

(b) $K=\frac{1}{2}\int\limits_0^l \rho(x)\,u_t^2(x,\ t)\,dx+\frac{1}{2}\sum\limits_{i=1}^{n} m_i u_t^2(x_i,\ t)$.

112. (a) $U=T\int\limits_0^l \left(\sqrt{1+u_x^2(x,\ t)}-1\right)dx$;

(b) $U=\frac{T}{2}\int\limits_0^l u_x^2(x,\ t)\,dx$;

(c) $U=\frac{T}{2}\int\limits_0^l u_x^2(x,t)\,dx-\nu_1(t)\,u(0,t)-\nu_2(t)\,u(l,t)$;

(d) $U = \frac{T}{2} \int_0^l u_x^2(x, t)\,dx + \frac{\sigma_1}{2} u^2(0, t) + \frac{\sigma_2}{2} u^2(l, t)$;

where T is the *tension* of the string and σ_1 and σ_2 are the stiffness factors of the elastic fixing.

113. (a) $K = \frac{1}{2} \int_D \rho(x, y)\, u_t^2(x, y, t)\,dx\,dy$;

(b) $$K = \frac{1}{2} \int_D \rho(x, y)\, u_t^2(x, y, t)\,dx\,dy + \\ + \frac{1}{2} \sum_{i=1}^{n} m_i u_t^2(x_i, y_i, t).$$

114. (a) $U = T \int_D [\sqrt{1 + u_x^2(x, y, t) + u_y^2(x, y, t)} - 1]\,dx\,dy$;

(b) $U = \frac{T}{2} \int_D [u_x^2(x, y, t) + u_y^2(x, y, t)]\,dx\,dy$;

(c) $$U = T \int_D [\sqrt{1 + u_x^2(x, y, t) + u_y^2(x, y, t)} - 1] \times \\ \times dx\,dy + \int_L \sigma(s)\, u^2(s, t)\,ds;$$

(d) $$U = \frac{T}{2} \int_D [u_x^2(x, y, t) + u_y^2(x, y, t)]\,dx\,dy + \\ + \int_D F(x, y, t)\, u(x, y, t)\,dx\,dy;$$

where T is the *tension per unit length* of the membrane, L is the boundary of the domain D, s is the variable point on the curve L, ds is the element of length of the curve L and $\sigma(s)$ is the stiffness factor of the elastic fixing.

115. (a) $\rho u_{tt} = T u_{xx}, \quad 0 < x < l, \quad t > 0,$
$u(0, t) = u(l, t) = 0, \quad t > 0,$
$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 < x < l;$

(b) $\rho u_{tt} = T u_{xx}, \quad 0 < x < l, \quad t > 0,$
$u_x(0, t) = u_x(l, t) = 0, \quad t > 0,$
$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 < x < l;$

(c) $\rho u_{tt} = Tu_{xx}, \quad 0 < x < l, \quad t > 0,$
$Tu_x(0, t) = -F(t), \quad Tu_x(l, t) = \Phi(t), \quad t > 0,$
$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 < x < l;$

(d) $\rho u_{tt} = Tu_{xx}, \quad 0 < x < l, \quad t > 0$
$Tu_x(0, t) - \sigma_1 u(0, t) = 0,$
$Tu_x(l, t) + \sigma_2 u(l, t) = 0, \quad t > 0$
$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 < x < l$

where σ_1 and σ_2 are the stiffness factors of the elastic fixing of the ends of the string;

(e) $\rho u_{tt} = Tu_{xx} + F(x, t), \quad 0 < x < l, \quad t > 0$
$u(0, t) = 0, \quad Tu_x(l, t) + \sigma u(l, t) = 0, \quad t > 0$
$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 < x < l$

where σ is the stiffness factor of the elastic fixing;

(f) $\rho u_{tt} = Tu_{xx} + F(t)\,\delta(x - x_0), \; 0 < x < l, \; t > 0$
$u(0, t) = u(l, t) = 0, \quad t > 0$
$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 < x < l$

Here and below $\delta(x - \xi)$ denotes the Dirac delta function (see Chapter 5 § 3);

(g) $$[\rho(x) + \sum_{i=1}^{n} m_i \delta(x - x_i)]\, u_{tt} = Tu_{xx},$$
$$0 < x < l, \quad t > 0$$
$$Tu_x(0, t) - \sigma_1 u(0, t) = 0, \quad Tu_x(l, t) + \sigma_2 u(l, t) = 0, \quad t > 0$$
$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 < x < l$$

where σ_1 and σ_2 are the stiffness factors of the elastic fixing at the ends of the string.

116. (a) $u_{tt} = a^2 \Delta u, \quad (x, y) \in D, \quad t > 0, \quad a^2 = \frac{T}{\rho}$
$u(x, y, t) = 0, \quad (x, y) \in L, \quad t > 0$
$u(x, y, 0) = \varphi(x, y), \quad u_t(x, y, 0) = \psi(x, y), \quad (x, y) \in D$

(b) $u_{tt} = a^2 \Delta u, \quad (x, y) \in D, \quad t > 0, \quad a^2 = \frac{T}{\rho}$

$\frac{\partial u(x, y, t)}{\partial \nu} = 0, \; (x, y) \in L, \; t > 0$ (ν is the outer normal to L)

$$u(x, y, 0) = \varphi(x, y), \quad u_t(x, y, 0) = \psi(x, y),$$
$$(x, y) \in D$$

(c) $u_{tt} = a^2 \Delta u, \quad (x, y) \in D, \quad t > 0, \quad a^2 = \frac{T}{\rho}$

$$\frac{\partial u(x, y, t)}{\partial \nu} = \frac{1}{T} F(x, y, t), \quad (x, y) \in L, \quad t > 0$$

(ν is the outer normal to L)

$$u(x, y, 0) = \varphi(x, y), \quad u_t(x, y, 0) = \psi(x, y), \quad (x, y) \in D$$

(d) $u_{tt} = a^2 \Delta u, \quad (x, y) \in D, \quad t > 0, \quad a^2 = \frac{T}{\rho}$

$$T \frac{\partial u(x, y, t)}{\partial \nu} + \sigma u(x, y, t) = 0, \quad (x, y) \in L, \quad t > 0$$
$$u(x, y, 0) = \varphi(x, y), \quad u_t(x, y, 0) = \psi(x, y), \quad (x, y) \in D$$

where ν is the outer normal to L and σ is the stiffness factor of the elastic fixing of the edge of the membrane;

(e) $u_{tt} + a^2 \Delta u + \frac{1}{\rho} F(x, y, t), \ (x, y) \in D, \ t > 0, \ a^2 = \frac{T}{\rho}$

$$u(x, y, t) = 0, \quad (x, y) \in L, \quad t > 0$$
$$u(x, y, 0) = \varphi(x, y), \quad u_t(x, y, 0) = \psi(x, y), \quad (x, y) \in D$$

(f) $u_{tt} = a^2 \Delta u - \alpha u, \ (x, y) \in D, \ t > 0, \ a^2 = \frac{T}{\rho}, \ \alpha = \frac{\beta}{\rho}$

$$u(x, y, t) = 0, \quad (x, y) \in L, \quad t > 0$$
$$u(x, y, 0) = \varphi(x, y), \quad u_t(x, y, 0) = \psi(x, y), \quad (x, y) \in D$$

where β is the proportionality factor in the expression $-\beta u$ of the resistance force of the medium;

(g) $[\rho + m\delta(x - x_0, y - y_0)]\, u_{tt} = T\, \Delta u,$

$$(x, y) \in D, \quad t > 0$$
$$u(x, y, t) = 0, \quad (x, y) \in L$$
$$u(x, y, 0) = \varphi(x, y), \quad u_t(x, y, 0) = \psi(x, y),$$
$$(x, y) \in D$$

117. (a) $u_{tt}=a^2u_{xx}, \quad 0<x<l, \quad t>0, \quad a^2=\frac{E}{\rho}$

$$u_x(0,\ t)=u_x(l,\ t)=0, \quad t>0$$

$$u(x,\ 0)=\varphi(x), \quad u_t(x,\ 0)=\psi(x), \quad 0<x<l$$

(b) $u_{tt}=a^2u_{xx}, \quad 0<x<l, \quad t>0, \quad a^2=\frac{E}{\rho}$

$$u_x(0,\ t)=-\frac{1}{SE}F(t), \quad u_x(l,\ t)=\frac{1}{SE}\Phi(t), \quad t>0$$

$$u(x,\ 0)=\varphi(x), \quad u_t(x,\ 0)=\psi(x), \quad 0<x<l$$

(c) $u_{tt}=a^2u_{xx}, \quad 0<x<l, \quad t>0, \quad a^2=\frac{E}{\rho}$

$$SEu_x(0,\ t)-\sigma_1\ u(0,\ t)=0, \ SEu_x(l,\ t)+\sigma_2u(l,\ t)=0, \ t>0$$

$$u(x,\ 0)=\varphi(x), \quad u_t(x,\ 0)=\psi(x), \quad 0<x<l$$

where σ_1 and σ_2 are the stiffness factors of the elastic fixing at the ends:

(d) $u_{tt}=a^2u_{xx}, \quad 0<x<l, \quad t>0, \quad a^2=\frac{E}{\rho}$

$$\alpha u_t(0,\ t)+SEu_x(0,\ t)=0, \quad u(l,\ t)=0, \quad t>0$$

$$u(x,\ 0)=\varphi(x), \quad u_t(x,\ 0)=\psi(x), \quad 0<x<l$$

where α is the proportionality factor in the expression $-\alpha u_t(0,\ t)$ of the resistance force acting on the end $x=0$;

(e) $u_{tt}=a^2u_{xx}+\frac{1}{\rho}F(x,\ t), \ 0<x<l, \ t>0, \ a^2=\frac{E}{\rho}$

$$u(0,\ t)=u(l,\ t)=0, \quad t>0$$

$$u(x,\ 0)=\varphi(x), \quad u_t(x,\ 0)=\psi(x), \quad 0<x<l$$

(f) $u_{tt}=a^2u_{xx}-\alpha u_t, \quad 0<x<l, \quad t>0, \quad a^2=\frac{E}{\rho}$

$$u(0,\ t)=\mu(t), \quad u(l,\ t)=\nu(t), \quad t>0$$

$$u(x,\ 0)=\varphi(x), \quad u_t(x,\ 0)=\psi(x), \quad 0<x<l$$

where α is the proportionality factor in the expression $-\alpha u_t$ of the force (resisting the deviation) which acts on unit mass;

(g) $u_{tt} = a^2 u_{xx}, \quad 0 < x < l, \quad t > 0, \quad a^2 = \frac{E}{\rho}$

$$u(0, t) = 0, \quad -SEu_x(l, t) = mu_{tt}(l, t), \quad t > 0$$
$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 < x < l$$

118. (a) $\left[r + \frac{(R-r)}{l} x\right]^2 u_{tt} = \frac{E}{\rho} \frac{\partial}{\partial x} \left\{\left[r + \frac{(R-r)}{l} x\right]^2 u_x\right\}$

$$0 < x < l, \quad t > 0$$
$$u(0, t) = u(l, t) = 0, \quad t > 0$$
$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 < x < l$$

(b) $\rho S u_{tt} = E \frac{\partial}{\partial x}(S u_x), \quad 0 < x < l, \quad t > 0$

$$S(0) E u_x(0, t) - \sigma u(0, t) = 0, \quad E u_x(l, t) = F(t), \quad t > 0$$
$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 < x < l$$

where σ is the stiffness factor of the elastic fixing.

119. On denoting

$$u(x, t) = \begin{cases} u_1(x, t) & \text{for } -\infty < x < 0 \\ u_2(x, t) & \text{for } \quad 0 < x < \infty \end{cases}$$

we arrive at the following problems:

(a) $\rho_1 u_{1tt} = E_1 u_{1xx}, \quad -\infty < x < 0, \quad t > 0$

$\rho_2 u_{2tt} = E_2 u_{2xx}, \quad 0 < x < \infty, \quad t > 0$

$u_1(0, t) = u_2(0, t), \; E_1 u_{1x}(0, t) = E_2 u_{2x}(0, t),$

$$t > 0$$
$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x),$$
$$-\infty < x < \infty$$

(b) $\rho_1 u_{1tt} = E_1 u_{1xx}, \quad -\infty < x < 0, \quad t > 0$

$\rho_2 u_{2tt} = E_2 u_{2xx}, \quad 0 < x < \infty, \quad t > 0$

$$u_1(0, t) = u_2(0, t), \quad t > 0$$

$SE_2 u_{2x}(0 + 0, t) - SE_1 u_{1x}(0 - 0, t) =$

$$= m u_{1tt}(0, t) = m u_{2tt}(0, t), \quad t > 0$$
$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad 0 < x < l$$

120. (a) $v_x + Li_t = 0,\ i_x + Cv_t = 0,\ 0 < x < l,\ t > 0$

$$v(0, t) = E(t), \quad v(l, t) = 0, \quad t > 0$$

$$v(x, 0) = \psi(x), \quad i(x, 0) = \varphi(x), \quad 0 < x < l$$

whence

$$v_{xx} = CLv_{tt}, \quad 0 < x < l, \quad t > 0$$

$$v(0, t) = E(t), \quad v(l, t) = 0, \quad t > 0$$

$$v(x, 0) = \psi(x), \quad Cv_t(x, 0) = -\varphi'(x),$$

$$0 < x < l$$

and

$$i_{xx} = CLi_{tt}, \quad 0 < x < l, \quad t > 0$$

$$i_x(0, t) = -CE'(t), \quad i_x(l, t) = 0, \quad t > 0$$

$$i(x, 0) = \varphi(x),\ Li_t(x, 0) = -\psi'(x),\ 0 < x < l$$

(b) $v_x + Li_t = 0,\ i_x + Cv_t = 0,\ 0 < x < l,\ t > 0$

$$C_0 v_t(0, t) + i(0, t) = 0,$$

$$v(l, t) - R_0 i(l, t) = E(t), \quad t > 0$$

$$v(x, 0) = \psi(x), \quad i(x, 0) = \varphi(x), \quad 0 < x < l$$

whence

$$v_{xx} = CLv_{tt}, \quad 0 < x < l, \quad t > 0$$

$$LC_0 v_{tt}(0, t) - v_x(0, t) = 0,$$

$$Lv_t(l, t) + R_0 v_x(l, t) = E'(t), \quad t > 0$$

$$v(x, 0) = \psi(x), \quad v_t(x, 0) = -\frac{1}{C}\varphi'(x), \quad 0 < x < l$$

and

$$i_{xx} = CLi_{tt}, \quad 0 < x < l, \quad t > 0$$

$$C_0 i_x(0, t) - Ci(0, t) = 0,$$

$$i_x(l, t) + CR_0 i_t(l, t) = E'(t), \quad t > 0$$

$$i(x, 0) = \varphi(x), \quad i_t(x, 0) = -\frac{1}{L}\psi'(x), \quad 0 < x < l$$

(c) $v_x + Li_t = 0,\ i_x + Cv_t = 0,\ 0 < x < l,\ t > 0$

$$L_0 i_t(0, t) + v(0, t) = E(t),$$

$$L_l i_t(l, t) - v(l, t) = 0, \quad t > 0$$
$$i(x, 0) = \varphi(x), \quad v(x, 0) = \psi(x), \quad 0 < x < l$$

whence

$$v_{xx} = CLv_{tt}, \quad 0 < x < l, \quad t > 0$$
$$L_0 v_x(0, t) - Lv(0, t) = -LE(t),$$
$$L_l v_x(l, t) + Lv(l, t) = 0, \quad t > 0$$
$$v(x, 0) = \varphi(x), \quad v_t(x, 0) = -\frac{1}{C}\varphi'(x), \quad 0 < x < l$$

and

$$i_{xx} = CLi_{tt}, \quad 0 < x < l, \quad t > 0$$
$$L_0 C i_{tt}(0, t) - i_x(0, t) = CE'(t),$$
$$CL_l i_{tt}(l, t) + i_x(l, t) = 0, \quad t > 0$$
$$i(x, 0) = \varphi(x), \quad i_t(x, 0) = -\frac{1}{L}\psi'(x), \quad 0 < x < l$$

121. (a) $v_x + Li_t + Ri = 0, \quad i_x + Cv_t + Gv = 0,$

$$0 < x < l, \quad t > 0$$
$$v(0, t) + R_0 i(0, t) = 0,$$
$$v(l, t) - R_l i(l, t) = E(t), \quad t > 0$$
$$i(x, 0) = \varphi(x), \quad v(x, 0) = \psi(x), \quad 0 < x < l$$

whence

$$v_{xx} = CLv_{tt} + (CR + GL)v_t + GRv,$$
$$0 < x < l, \quad t > 0$$
$$R_0 v_x(0, t) - Lv_t(0, t) - Rv(0, t) = 0$$
$$R_l v_x(l, t) + Lv_t(l, t) + Rv(l, t) =$$
$$= LE'(t) + RE(t), \quad t > 0$$
$$v(x, 0) = \psi(x), \quad v_t(x, 0) = -\frac{1}{C}\varphi'(x) - \frac{G}{C}\psi(x), \quad 0 < x < l$$

and

$$i_{xx} + CLi_{tt} + (CR + GL)i_t + CRi,$$
$$0 < x < l, \quad t > 0$$

$$i_x(0,\ t) - CR_0 i_t(0,\ t) - GR_0 i(0,\ t) = 0$$

$$i_x(l,\ t) + CR_l i_t(l,\ t) + GR_l i(l,\ t) = \\ = -CE'(t) - GE(t), \quad t > 0$$

$$i(x,\ 0) = \varphi(x), \quad i_t(x,\ 0) = -\frac{1}{L}\psi'(x) - \frac{R}{L}\varphi(x), \ 0 < x < l$$

(b) $v_x + Li_t + Ri = 0, \quad i_x + Cv_t + Gv = 0,$

$$0 < x < l, \quad t > 0$$

$$v(0,\ t) + L_0 i_t(0,\ t) + R_0 i(0,\ t) = 0,$$

$$v(l,\ t) - L_l i_t(l,\ t) = E(t), \quad t > 0$$

$$i(x,\ 0) = \varphi(x), \quad v(x,\ 0) = \psi(x), \quad 0 < x < l$$

To determine the current intensity $i(x,\ t)$ we can set the following problem:

$$i_{xx} = CLi_{tt} + (CR + GL)\,i_t + GRi,$$

$$0 < x < l, \quad t > 0$$

$$CL_0 i_{tt}(0,\ t) + (CR_0 + GL_0)\,i_t(0,\ t) - \\ - i_x(0,\ t) + GR_0 i(0,\ t) = 0, \quad t > 0$$

$$CL_l i_{tt}(l,\ t) + GL_l i_t(l,\ t) + i_x(l,\ t) + \\ + CE'(t) + GE(t) = 0, \quad t > 0$$

$$i(x,\ 0) = \varphi(x), \quad i_t(x,\ 0) = -\frac{1}{L}[\varphi'(x) + R\varphi(x)], \ 0 < x < l$$

122. (a) $S\dfrac{\partial u}{\partial t} = a^2\dfrac{\partial}{\partial x}\left(S\dfrac{\partial u}{\partial x}\right), \ 0 < x < l, \ t > 0, \ a^2 = \dfrac{k}{c\rho}$

$$u_x(0,\ t) = u_x(l,\ t) = 0, \quad t > 0$$

$$u(x,\ 0) = \varphi(x), \quad 0 < x < l$$

(b) $S\dfrac{\partial u}{\partial t} = a^2\dfrac{\partial}{\partial x}\left(S\dfrac{\partial u}{\partial x}\right), \ 0 < x < l, \quad t > 0, \ a^2 = \dfrac{k}{c\rho}$

$$u_x(0,\ t) = -\frac{1}{kS(0)}q(t), \quad u_x(l,\ t) = \frac{1}{kS(l)}Q(t), \quad t > 0$$

$$u(x,\ 0) = \varphi(x), \quad 0 < x < l$$

(c) $S\dfrac{\partial u}{\partial t} = a^2\dfrac{\partial}{\partial x}\left(S\dfrac{\partial u}{\partial x}\right), \quad 0 < x < l, \quad t > 0, \ a^2 = \dfrac{k}{c\rho}$

$$u_x(0,\ t) - h_1[u(0,\ t) - \tau(t)] = 0$$

$$u_x(l,\ t)+h_2[u(l,\ t)-\theta(t)]=0, \quad t>0$$

$$h_i=\frac{\varkappa_i}{k}, \quad i=1,\ 2,\ \ldots$$

$$u(x,\ 0)=\varphi(x), \quad 0<x<l$$

where $\varkappa_i$ is the *heat transfer coefficient* characterizing heat exchange at the ends;

$$\text{(d)}\ \ S\frac{\partial u}{\partial t}=a^2\frac{\partial}{\partial x}\left[S\frac{\partial u}{\partial x}\right], \quad 0<x<l, \ \ t>0, \ \ a^2=\frac{k}{c\rho}$$

$$u(0,\ t)=\mu(t), \quad kS(l)\,u_x(l,\ t)+cmu_t(l,\ t)=0, \quad t>0$$

$$u(x,\ 0)=\varphi(x), \quad 0<x<l$$

$$\text{(e)}\ \ S\frac{\partial u}{\partial t}=a^2\frac{\partial}{\partial x}\left(S\frac{\partial u}{\partial x}\right), \quad 0<x<l, \ \ t>0, \ \ a^2=\frac{k}{c\rho}$$

$$kS(0)\,u_x(0,\ t)-cmu_t(0,\ t)=0, \quad t>0$$

$$kS(l)\,u_x(l,\ t)+cmu_t(l,\ t)=q(t), \quad t>0$$

$$u(x,\ 0)=\varphi(x), \quad 0<x<l$$

123. (a) $u_t=a^2u_{xx}, \quad 0<x<l, \quad t>0, \quad a^2=\dfrac{\alpha D}{c}$

$$u(0,\ t)=\mu(t), \quad u_x(l,\ t)=0, \quad t>0$$

$$u(x,\ 0)=\varphi(x), \quad 0<x<l$$

where α is the *porosity coefficient* of the cross-section of the tube which is equal to the ratio of the area of the pores in the given section to the area of that cross-section;

$$\text{(b)}\ \ u_t=a^2u_{xx}, \quad 0<x<l, \quad t>0, \quad a^2=\frac{\alpha D}{c}$$

$$u_x(0,\ t)=-\frac{1}{\alpha SD}q(t), \quad u_x(l,\ t)+\frac{d}{D}u(l,\ t)=0, \quad t>0$$

$$u(x,\ 0)=\varphi(x), \quad 0<x<l$$

where α is the porosity coefficient of the cross-section equal to the ratio of the area of the pores in that section to the area of the cross-section and d is the coefficient characterizing the diffusion through the porous diaphragm.

124. (a) $u_t = \frac{k}{c\rho} u_{xx} - \frac{\varkappa\sigma}{c\rho S} u + \frac{\varkappa\sigma}{c\rho S} v(t) + \frac{\beta I^2 R}{c\rho S}$,

$$0 < x < l, \quad t > 0$$

$$kSu_x(0, t) = cu_t(0, t), \; kSu_x(l, t) = = Qu_t(l, t), \quad t > 0$$

$$u(x, 0) = \varphi(x), \quad 0 < x < l$$

where β is the proportionality factor in the formula $q = \beta I^2 R \, \Delta x$ expressing the amount of heat generated by the electric current in unit time within the element $(x, x + \Delta x)$ of the wire;

(b) $u_t = \frac{k}{c\rho} u_{xx} - \frac{\varkappa\sigma}{c\rho S} u + \frac{\varkappa\sigma}{c\rho S} v(t) + \frac{1}{c\rho S} F(x, t)$,

$$0 < x < l, \quad t > 0$$

$$kSu_x(0, t) = cu_t(0, t), \; kSu_x(l, t) = = Qu_t(l, t), \quad t > 0$$

$$u(x, 0) = \varphi(x), \quad 0 < x < l$$

(c) $u_t = \frac{k}{c\rho} u_{xx} - \frac{\alpha}{c\rho S} u_t - \frac{\varkappa\sigma}{c\rho S} u + \frac{\varkappa\sigma}{c\rho S} v(t)$,

$$0 < x < l, \quad t > 0$$

$$kSu_x(0, t) = cu_t(0, t), \; kSu_x(l, t) = = Qu_t(l, t), \quad t > 0$$

$$u(x, 0) = \varphi(x), \quad 0 < x < l$$

where α is the proportionality factor in the formula $q = \alpha u_t S \, \Delta x$ expressing the amount of heat absorbed within the volume $S \, \Delta x$ of the element $(x, x + \Delta x)$ of the bar.

125. (a) $u_t = Du_{xx} - \gamma u^{1/2} - \frac{\sigma d}{S} [u - v(t)], \; 0 < x < l, \; t > 0$

$$u_x(0, t) - \frac{d}{D} [u(0, t) - v(t)] = 0,$$

$$u_x(l, t) + \frac{d}{D} [u(l, t) - v(t)] = 0, \quad t > 0$$

$$u(x, 0) = \varphi(x), \quad 0 < x < l$$

where γ is the proportionality factor involved in the expression for the law of the disintegration and d is the coefficient characterizing the diffusion through the porous diaphragm;

$$\text{(b)}\quad u_t = Du_{xx} + \gamma u u_t - \frac{\sigma d}{S}[u - v(t)], \quad 0 < x < l,$$

$$t > 0$$

$$u_x(0, t) - \frac{d}{D}[u(0, t) - v(t)] = 0,$$

$$u_x(l, t) + \frac{d}{D}[u(l, t) - v(t)] = 0, \quad t > 0$$

$$u(x, 0) = \varphi(x), \quad 0 < x < l$$

where γ is the proportionality factor characterizing the multiplication and d is the coefficient of diffusion through the porous diaphragm.

126. (a) $u_t = a^2\Delta_r u - \beta u, \quad 0 \leqslant r < R, \ t > 0,$

$$a^2 = \frac{k}{c\rho}, \quad \beta = \frac{\alpha}{c\rho}$$

$$\frac{\partial u(R, t)}{\partial r} = 0, \quad t > 0$$

$$u(r, 0) = T, \quad 0 \leqslant r < R$$

where $\Delta_r u \equiv \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)$ is the part of the expression of Laplace's operator in spherical coordinates dependent solely on r and α is the coefficient characterizing the absorption of heat;

$$\text{(b)}\quad u_t = a^2\Delta_r u + \frac{Q}{c\rho}, \quad 0 \leqslant r < R, \quad a^2 = \frac{k}{c\rho}$$

$$k\frac{\partial u(R, t)}{\partial r} + \alpha u(R, t) = 0, \quad t > 0$$

$$u(r, 0) = T, \quad 0 \leqslant r < R$$

where $\Delta_r u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)$ is the part of the expression of Laplace's operator in spherical coordinates depending solely on the radius r and α is the heat-exchange coefficient.

127. (a) $k\Delta u-\gamma u+Q=0, \quad 0\leqslant r<r_0,\ 0<z<h,\ t>0$

$$u(r,\ 0)=u(r,\ h)=0, \quad 0\leqslant r<r_0,$$

$$\frac{\partial u(r_0,\ z)}{\partial r}=0, \quad 0<z<h$$

where γ is the coefficient characterizing the rate of the disintegration of the particles of the gas;

(b) $k\Delta u-\gamma u+Q=0, \quad 0\leqslant r<r_0, \quad 0<z<h, \quad t>0$

$$D\frac{\partial u(r,\ 0)}{\partial z}-du(r,\ 0)=0, \quad D\frac{\partial u(r,\ h)}{\partial z}+du(r,\ h)=0$$

$$0\leqslant r<r_0, \quad u(r_0,\ z)=0, \quad 0<z<h$$

where d is the coefficient of diffusion through the porous diaphragm and γ is the disintegration coefficient of the gas.

130. The formula $\psi(x,\ y)$ = const describes the set of the stream lines.

131. (b) The function $f(x,\ y)$ is proportional to the force acting at the points $(x,\ y)\in S$ in the direction orthogonal to the plane in which the membrane lies when it is at rest.

Chapter 2

132. As is known from the course of mathematical analysis, the expression

$$\Delta u=\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}$$

goes into

$$\Delta u=\frac{1}{\sqrt{g}}\sum_{j,\ k=1}^{n}\frac{\partial}{\partial y_j}\left(\sqrt{g}\,g^{jk}\frac{\partial u}{\partial y_k}\right)$$

under the transformation from the Cartesian orthogonal coordinates $x_1,\ \ldots,\ x_n$ to arbitrary curvilinear coordinates $y_1,\ \ldots,\ y_n$ where $g=\det\|g_{jk}\|$, $g^{jk}=G^{jk}/g$, $G^{jk}=G^{kj}$ is the cofactor of the element g_{ik} (or g_{ki}) of the determinant $\det\|g_{jk}\|$ and

$$g_{jk}(y_1,\ \ldots,\ y_n)=\sum_{i=1}^{n}\frac{\partial x_i}{\partial y_j}\,\frac{\partial x_i}{\partial y_k}$$

In the case when $y_1, \ldots, y_n$ are orthogonal curvilinear coordinates we have $g_{jk} = 0$ for $j \neq k$.

(a) the expression of Laplace's operator is

$$\Delta u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi}\left(\sqrt{g}\, g^{11} \frac{\partial u}{\partial \xi}\right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi}\left(\sqrt{g}\, g^{12} \frac{\partial u}{\partial \eta}\right) + \\ + \frac{1}{\sqrt{g}} \frac{\partial}{\partial \eta}\left(\sqrt{g}\, g^{21} \frac{\partial u}{\partial \xi}\right) + \frac{1}{\sqrt{g}} \frac{\partial}{\partial \eta}\left(\sqrt{g}\, g^{22} \frac{\partial u}{\partial \eta}\right)$$

where $g = (x_\xi y_\eta - x_\eta y_\xi)^2$, $g^{11} = g^{-1}(x_\eta^2 + y_\eta^2)$, $g^{12} = g^{21} = -g^{-1}(x_\xi x_\eta + y_\xi y_\eta)$ and $g^{22} = g^{-1}(x_\xi^2 + y_\xi^2)$;

(b) $\Delta u = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2}$;

(c) $\Delta u = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2}$;

(d) $$\Delta u = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \\ + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 u}{\partial \varphi^2};$$

(e) $$\Delta u = \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{\xi\eta(\xi^2-\eta^2)} \left\{ \frac{\partial}{\partial \xi}\left[\sqrt{\frac{\xi^2-1}{1-\eta^2}}\, \xi\eta \frac{\partial u}{\partial \xi}\right] + \right. \\ + \frac{\partial}{\partial \eta}\left[\sqrt{\frac{1-\eta^2}{\xi^2-1}}\, \xi\eta \frac{\partial u}{\partial \eta}\right] + \\ \left. + \frac{\partial}{\partial \varphi}\left[\frac{\xi^2-\eta^2}{\xi\eta} \frac{1}{\sqrt{(\xi^2-1)(1-\eta^2)}} \frac{\partial u}{\partial \varphi}\right]\right\}$$

133. (a) Harmonic; (b) harmonic; (c) harmonic; (d) harmonic; (e) not harmonic; (f) harmonic; (g) not harmonic; (h) harmonic; (i) harmonic. In this problem the direct computations are rather lengthy. Here it is advisable to take into account that, given a harmonic function $u = u(x_1, x_2)$, we can take it as $\operatorname{Re} f(z)$ $(z = x_1 + ix_2)$ where $f(z) = u + iv$ is an analytic function, after which the function $v(x_1, x_2) = \operatorname{Im} f(z)$ can be constructed. The Cauchy-Riemann system of partial differential equations corresponding to the function $f(z)$ has the form $\frac{\partial u}{\partial x_1} = \frac{\partial v}{\partial x_2}$, $\frac{\partial u}{\partial x_2} = -\frac{\partial v}{\partial x_1}$. It is evident that the function $w(z) = \frac{\partial u}{\partial x_1} + i\frac{\partial v}{\partial x_1}$ is also analytic, and, according to the above Cauchy-Riemann system, it can be written in the form $w(z) = \frac{\partial u}{\partial x_1} - i\frac{\partial u}{\partial x_2}$.

The function

$$\frac{1}{w(z)}=\frac{1}{\frac{\partial u}{\partial x_1}-i\frac{\partial u}{\partial x_2}}=\frac{\frac{\partial u}{\partial x_1}+i\frac{\partial u}{\partial x_2}}{\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2}$$

is also analytic and its real part $\frac{\partial u}{\partial x_1}\left[\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2\right]^{-1}$ is a harmonic function; (j) harmonic; (k) not harmonic.

134. (a) $k=-3$; (b) $k=-2$; (c) $k=\pm 2i$; in this case $\cosh kx_2=\cos 2x_2$; (d) $k=\pm 3$; (e) $k=0$ and $k=n-2$ for $n>2$.

135. Since $\Delta\,|x|^{2-n}=0$ for $x\neq 0$, we have

$$\Delta v=2\sum_{i=1}^{n}\frac{\partial}{\partial x_i}|\xi|^{n-2}\frac{\partial u(\xi)}{\partial x_i}+|\xi|^{n-2}\Delta u(\xi)$$

where $\Delta=\sum\limits_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$, $\xi=x/|x|^2$, $|x|=1/|\xi|$ and $\xi_i=x_i/|x|^2$.
Now, taking into account the harmonicity of the function $u(\xi)$ and the equalities

$$\sum_{i=1}^{n}\left(\frac{\partial\xi_l}{\partial x_i}\right)^2=\left(\frac{\partial\xi_l}{\partial x_l}\right)^2+\sum_{\substack{i=1\\ i\neq l}}^{n}\left(\frac{\partial\xi_l}{\partial x_i}\right)^2=$$

$$=(|\xi|^2-2\xi_l)^2+4\xi_l^2\sum_{\substack{i=1\\ i\neq l}}^{n}\xi_i^2=$$

$$=|\xi|^4-4|\xi|^2\xi_l^2+4\xi_l^2\sum_{i=1}^{n}\xi_i^2=|\xi|^4$$

and

$$\sum_{\substack{i=1\\ l\neq j}}^{n}\frac{\partial\xi_l}{\partial x_i}\frac{\partial\xi_j}{\partial x_i}=\frac{\partial\xi_l}{\partial x_l}\frac{\partial\xi_j}{\partial x_l}+\frac{\partial\xi_l}{\partial x_j}\frac{\partial\xi_j}{\partial x_j}+\sum_{\substack{i=1\\ i\neq l,\,j}}^{n}\frac{\partial\xi_l}{\partial x_i}\frac{\partial\xi_j}{\partial x_i}=$$

$$=-2(|\xi|^2-2\xi_l^2)\,\xi_j\xi_l-2(|\xi|^2-2\xi_j^2)\,\xi_j\xi_l+$$

$$+4\xi_l\xi_j\sum_{\substack{i=1\\ \neq l,\,j}}^{n}\xi_i^2=-4\xi_j\xi_l|\xi|^2+4\xi_j\xi_l\sum_{i=1}^{n}\xi_i^2=0$$

we obtain

$$2\sum_{i=1}^{n} \frac{\partial}{\partial x_i} |\xi|^{n-2} \frac{\partial u(\xi)}{\partial x_i} =$$

$$= 2 \sum_{i,\,j,\,l=1}^{n} \frac{\partial}{\partial \xi_j} |\xi|^{n-2} \frac{\partial u(\xi)}{\partial \xi_l} \frac{\partial \xi_j}{\partial x_i} \frac{\partial \xi_l}{\partial x_i} =$$

$$= 2|\xi|^4 \sum_{j=1}^{n} \frac{\partial}{\partial \xi_j} |\xi|^{n-2} \frac{\partial u(\xi)}{\partial \xi_j} =$$

$$= 2(n-2)|\xi|^n \sum_{j=1}^{n} \xi_j \frac{\partial u(\xi)}{\partial \xi_j}$$

and

$$\Delta u(\xi) = \sum_{i=1}^{n} \frac{\partial^2 u(\xi)}{\partial x_i^2} = \sum_{i,\,j,\,l=1}^{n} \frac{\partial}{\partial \xi_l}\left(\frac{\partial u}{\partial \xi_j}\frac{\partial \xi_j}{\partial x_i}\right)\frac{\partial \xi_l}{\partial x_i} =$$

$$= |\xi|^4 \sum_{j=1}^{n} \frac{\partial^2 u}{\partial \xi_j^2} + \sum_{i,\,j=1}^{n} \frac{\partial u}{\partial \xi_j} \frac{\partial^2}{\partial x_i^2}\left(\frac{x_j}{|x|^2}\right) =$$

$$= \sum_{j=1}^{n} \frac{\partial u}{\partial \xi_j}\left[\frac{\partial^2}{\partial x_j^2}\left(\frac{x_j}{|x|^2}\right) + \sum_{\substack{i=1\\ i\neq j}}^{n} \frac{\partial^2}{\partial x_i^2}\left(\frac{x_j}{|x|^2}\right)\right] =$$

$$= \sum_{j=1}^{n} \frac{\partial u}{\partial \xi_j}\left[-6|\xi|^2\xi_j + 8\xi_j^3 + \sum_{\substack{i=1\\ i\neq j}}^{n} (8\xi_j\xi_i^2 - 2|\xi|^2\xi_j)\right] =$$

$$= 2(2-n)|\xi|^2 \sum_{j=1}^{n} \xi_j \frac{\partial u}{\partial \xi_j}$$

Consequently, $\Delta v(x) = 0$.

136. They can.

137. $y = x$.

138. The function $\cos x \sinh y$ tends to $-\infty$ when the point (x, y) recedes to infinity along the part of the level line $\sin x \cosh y = -1$ of the function $u = \sin x \cosh y$

the tangent line at whose point $(-\pi/2, 0)$ forms an angle of $3\pi/4$ with the x-axis. The coordinate y of the variable point (x, y) on that part of the level line $\sin x \cosh y = -1$ decreases from $+\infty$ to $-\infty$ when the coordinate x decreases from $-\pi$ to 0.

139. $u_{\max} = 1/2$ at the points $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$; $u_{\min} = -1/2$ at the points $(-1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$.

140. $u_{\max} = 4$ at the points $(-2, 0)$ and $(2, 0)$; $u_{\min} = -9$ at the points $(0, -3)$ and $(0, 3)$.

141. Let $x \in D$ be a point at which the function $w(x)$ has a negative relative minimum. Then for this point we have $w_{x_i} = 0$ $(i = 1, \ldots, n)$ and $\sum\limits_{i, k=1}^{n} w_{x_i x_k} \lambda_i \lambda_k \geqslant 0$. Since the quadratic form $\sum\limits_{i, k=1}^{n} w_{x_i x_k} \lambda_i \lambda_k$ can be represented in the form

$$\sum_{i, k=1}^{n} w_{x_i x_k} \lambda_i \lambda_k = \sum_{i, s=1}^{n} (g_{is} \lambda_s)^2$$

at the point x, we have $w_{x_i x_i} = \sum\limits_{j=1}^{n} g_{ji} g_{jk}$, and, consequently, $\Delta w = \sum\limits_{i=1}^{n} w_{x_i x_i} = \sum\limits_{i, j=1}^{n} g_{ji}^2 \geqslant 0$, which contradicts the condition $\Delta w < 0$. The second part of the assertion stated in the problem can be proved analogously.

142. Under the conditions of Problem 139 we have $\frac{\partial u}{\partial \nu} = 1$ at the points of maximum $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$, and $\frac{\partial u}{\partial \nu} = -1$ at the points of minimum $(-1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$. For Problem 140 we have $\frac{\partial u}{\partial \nu} = 4$ at the points of maximum $(2, 0)$ and $(-2, 0)$, and $\frac{\partial u}{\partial \nu} = -6$ at the points of minimum $(0, 3)$ and $(0, -3)$.

143. Let us choose a point $x^* \in D$ on the inner normal to S passing through the point $y_0 \in S$ at which the function

$u(x)$ harmonic in the domain D attains its minimum so that the closed ball d_1: $|x - x^*| \leqslant |x^* - y_0|$ has the only common point y_0 with the boundary S. Let d_2: $|x - y_0| \leqslant \rho < |x^* - y_0|$ be a closed ball not containing x^*. We shall denote by d the intersection of the two closed balls d_1 and d_2 and consider the function

$$v(x) = e^{-\gamma|x^*-y_0|^2} - e^{-\gamma|x-x^*|^2}$$

where γ is a positive indeterminate constant. By the extremum principle, there must be $u(x) - u(y_0) > 0$ everywhere in D. Now let us choose the constant $\lambda > 0$ so that the inequalities $-\lambda v(x) \leqslant u(x) - u(y_0)$ hold on the boundary of d. Since

$$\Delta[u(x) - u(y_0) + \lambda v(x)] = 2\lambda\gamma\{n - 2\gamma|x - x^*|^2\} e^{-\gamma|x-x^*|^2}$$

the constant γ can always be chosen in such a way that $\Delta[u(x) - u(y_0) + \lambda v(x)] < 0$. Therefore (see Problem 141) the inequality $u(x) - u(y_0) \geqslant -\lambda v(x)$ also holds for the closed domain $\overline{d}$. It follows that the directional derivative of $u(x)$ along the outer normal ν to S at the point $y_0 \in S$ satisfies the inequality

$$\frac{\partial u}{\partial \nu} \leqslant -2\lambda\gamma|x^* - y_0| e^{-\gamma|x^*-y_0|^2} < 0$$

The second part of the assertion stated in the problem can be proved analogously.

144. The function $\varphi(z)$ is analytic because its real part $U(x, y) = u_x$ and its imaginary part $V(x, y) = -u_y$ are continuous together with their partial derivatives of the first order and satisfy the Cauchy-Riemann system of partial differential equations

$$U_x - V_y = u_{xx} + u_{yy} = 0, \quad U_y + V_x = u_{xy} - u_{xy} = 0$$

145. The real part $u(x, y)$ and the imaginary part $v(x, y)$ of the analytic function $f(z) = u(x, y) + iv(x, y)$ satisfy the Cauchy-Riemann system of partial differential equations $u_x - v_y = 0$, $u_y + v_x = 0$. Therefore the expression $dv = v_x\,dx + v_y\,dy = -u_y\,dx + u_x\,dy$ is a total differential because $(u_x)_x + (u_y)_y = \Delta u = 0$. Consequently, the line integral $\int dv = \int -u_y\,dx + u_x\,dy$ taken from a fixed point (x_0, y_0) to the variable point (x, y) within the simply connected domain D is independent of the path of integration. As the path of integration we can take, for instance, the one consisting of the line segments joining the points (x_0, y_0), (x, y_0) and (x, y_0), (x, y) (provided this path lies inside the domain D) or a step-like broken line with a finite number of segments connecting the points (x_0, y_0) and (x, y). In the case under consideration we have

$$f(z) = x^3 - 3xy^2 + i\left[\int_{x_0}^{x} 6xy_0\,dx + \int_{y_0}^{y} 3(x^2 - y^2)\,dy\right] + iC =$$

$$= x^3 - 3xy^2 + i(3x^2y - y^3) + i(-3x_0^2y_0 + y_0^3 + C)$$

where $-3x_0^2y_0 + y_0^3 + C$ plays the role of an arbitrary real constant.

146. $f(z) = e^x \sin y - ie^x \cos y + i(e^{x_0} \cos y_0 + C)$.

147. $f(z) = \sin x \cosh y + i \cos x \sinh y + i(-\cos x_0 \sinh y_0 + C)$.

148. The sought-for harmonic function can be written as $u(x, y) = x^3y - xy^3 + Cy + C_0$ where C and C_0 are arbitrary real constants.

149. The sought-for harmonic function has the form $u(x, y, z) = xze^x \cos y - zye^x \sin y + z^2 - x^2 + \varphi(x, y)$ where $\varphi(x, y)$ is an arbitrary real harmonic function.

150. A function $u(x, y)$ harmonic in a simply connected domain D is analytic in that domain in the sense that for each point $(x_0, y_0) \in D$ there exists its neighbourhood (lying inside D) within which the function u can be expanded into a power series (involving the powers of $x - x_0$ and $y - y_0$). Therefore we can assume that the function $u(x, y)$ can be continued analytically to the complex values of x and y. For the real values of x and y we have $f(z) = u(x, y) + iv(x, y)$ and $\overline{f}(\overline{z}) = u(x, y) - iv(x, y)$ whence

$$f(z) = 2u(x, y) + \overline{f}(\overline{z})$$

If x and y contained in the last equality are considered complex then the expressions $z = x + iy$ and $\overline{z} = x - iy$ are no longer mutually complex conjugate; since $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/2i$, we have

$$f(z) = 2u\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) - \overline{f}(\overline{z})$$

On putting $\bar{z} = \bar{z}_0$ in the last equality we obtain the Goursat formula

$$f(z) = 2u\left(\frac{z+\bar{z}_0}{2},\ \frac{z-\bar{z}_0}{2i}\right) - u(x_0, y_0) + iC$$

where $C = \operatorname{Im} f(z_0)$ is an arbitrary real constant. On putting $\bar{z}_0 = 0$ we obtain the required equality.

151. For Problem 145 we have $f(z) = z^3 + iC$; for Problem 146 we have $f(z) = -ie^z + i(1 + C)$; for Problem 147 we have $f(z) = \sin z + iC$.

152. Since

$$\sum_{i=1}^{n-1} u_{x_i x_i} = \sum_{k=0}^{\infty} (-1)^k \left[\frac{x_n^{2k}}{(2k)!}\Delta^{k+1}\tau + \frac{x_n^{2k+1}}{(2k+1)!}\Delta^{k+1}\nu\right]$$

and 70

$$u_{x_n x_n} = \sum_{k=1}^{\infty} (-1)^k \left[\frac{x_n^{2k-2}}{(2k-2)!}\Delta^k\tau + \frac{x_n^{2k-1}}{(2k-1)!}\Delta^k\nu\right] =$$

$$= -\sum_{k=0}^{\infty} (-1)^k \left[\frac{x_n^{2k}}{(2k)!}\Delta^{k+1}\tau + \frac{x_n^{2k+1}}{(2k+1)!}\Delta^{k+1}\nu\right]$$

there must be $\Delta u = 0$.

153. On performing the change of variables $y_k = x_k/\sqrt{|a_k|}$ $(k = 1, \ldots, n)$ we obtain $\sum_{k=1}^{n} a_k u_{x_k x_k} = \pm\sum_{k=1}^{n} v_{y_k y_k} = 0$ whence it follows that $u(x_1, \ldots, x_n) = v\left(x_1/\sqrt{|a_1|}, \ldots, x_n/\sqrt{|a_n|}\right)$.

154. The assertion stated in the problem follows from the fact that the transformation $u = e^{\lambda x + \mu y} v(x, y)$ of the sought-for function u brings the given equation to the form $\Delta v = 0$.

155. For $x \neq y$ we have

$$E_{x_i x_i} = -|x-y|^{-n} - n|x-y|^{-n-2}(x_i - y_i)^2; \quad i = 1, \ldots, n$$

Consequently,

$$\Delta E = -n|x-y|^{-n} - n|x-y|^{-n-2}\sum_{i=1}^{n}(x_i-y_i)^2 =$$

$$= -n|x-y|^{-n} - n|x-y|^{-n} = 0$$

156. Since the function $E(x, y)$ depends solely on the distance $|x - y| = r$, the expression of Laplace's operator in the spherical coordinates with origin at the point $x = y$ shows that for $r \neq 0$ the function $E(r)$ is a solution of the ordinary differential equation $\frac{d}{dr}\left(r^{n-1}\frac{dE}{dr}\right) = 0$ whence it follows that $E = C/r^{n-2} + C_1$ for $n > 2$ and $E = C \ln r + C_1$ for $n = 2$ where C and C_1 are arbitrary constants.

157. The value of the potential at M is equal to $\mu(M_0) \times \times \frac{\partial}{\partial \nu}\frac{1}{|M-M_0|}$. Indeed, according to the definition of a dipole, we can write, for the potential of the dipole at the point M distinct from M', M'' and M_0, the relation

$$\lim_{|M'-M''|\to 0}\left(\frac{\mu_0}{|M''-M|} - \frac{\mu_0}{|M'-M|}\right) =$$

$$= \mu(M_0)\lim_{|M'-M''|\to 0}\frac{1}{|M'-M''|}\left(\frac{1}{|M''-M|} - \frac{1}{|M'-M|}\right) =$$

$$= \mu(M_0)\frac{\partial}{\partial \nu}\frac{1}{|M-M_0|}$$

158. The potential is expressed by the formula $u(M) = \sum_{k=1}^{m}\frac{\mu_k}{|M_k - M|}$ where $|M_k - M|$ is the distance between the points M_k and M. **159.** The sought-for value of the potential is equal to RC.

160. $u(x, y, z) =$

$$= \frac{1}{4\pi}\int_{t_0}^{t_1}\frac{\mu[\xi(t), \eta(t), \zeta(t)]\sqrt{[\xi'(t)]^2 + [\eta'(t)]^2 + [\zeta'(t)]^2}}{\sqrt{[\xi(t)-x]^2 + [\eta(t)-y]^2 + [\zeta(t)-z]^2}}\,dt.$$

161. Apply the Gauss-Ostrogradsky formula

$$\int_D \sum_{i=1}^{n}\frac{\partial F_i}{\partial x_i}\,d\tau_x = \int_S \sum_{i=1}^{n} F_i \cos\widehat{\nu y_i}\,ds_y$$

to the identity

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(v\frac{\partial u}{\partial x_i}-u\frac{\partial v}{\partial x_i}\right)=v\,\Delta u-u\,\Delta v=0$$

162. Let x be a point belonging to D. We shall denote by D_ε the part of the domain D exterior to the closed ball $|y-x|\leqslant\varepsilon$ of a sufficiently small radius ε with centre at the point x. Since the function $E(x, y)$ is harmonic in D_ε, we can apply formula (6) to the boundary of the domain D_ε; putting $v=E(x, y)$ in that formula we obtain

$$\int\limits_S \left[E(x, y)\frac{\partial u(y)}{\partial \nu_y}-u(y)\frac{\partial E(x, y)}{\partial \nu_y}\right]dS_y=$$

$$=\int\limits_{|y-x|=\varepsilon}\left[E(x, y)\frac{\partial u(y)}{\partial \nu_y}-u(y)\frac{\partial E(x, y)}{\partial \nu_y}\right]dS_y$$

For the points of the sphere $|y-x|=\varepsilon$ we have

$$E(x, y)=\begin{cases}\dfrac{1}{(n-2)\,\varepsilon^{n-2}} & \text{for}\quad n>2\\ -\ln\varepsilon & \text{for}\quad n=2\end{cases}$$

and

$$\frac{\partial E(x, y)}{\partial \nu_y}=\begin{cases}-\dfrac{1}{\varepsilon^{n-1}} & \text{for}\quad n>2\\[2mm] -\dfrac{1}{\varepsilon} & \text{for}\quad n=2\end{cases}$$

Besides, we also have the relations

$$\lim_{\varepsilon\to 0}\int\limits_{|y-x|=\varepsilon}[u(y)-u(x)]\frac{\partial E(x, y)}{\partial \nu_y}dS_y=0$$

and

$$\int\limits_{|y-x|=\varepsilon}\frac{dS_y}{\varepsilon^{n-1}}=\omega_n$$

Therefore, on passing to the limit for $\varepsilon\to 0$, we arrive at formula (7).

163. Let x and y $(x\neq y)$ be two points belonging to D. We shall denote by D_ε the part of the domain D lying outside the two closed balls $|z-x|\leqslant\varepsilon$ and $|z-y|\leqslant\varepsilon$ of sufficiently small radius ε with centres at the points x and y respectively, $z\in D_\varepsilon$ being the variable point. On applying formula (6), Problem 161, to the domain D_ε for $u(z)=G(z, x)$ and $v(z)=G(z, y)$ we obtain

$$\int\limits_{|z-x|=\varepsilon}\left[G(z, x)\frac{\partial G(z, y)}{\partial \nu_z}-G(z, y)\frac{\partial G(z, x)}{\partial \nu_z}\right]ds_z=$$

$$=\int\limits_{|z-y|=\varepsilon}\left[G(z, y)\frac{\partial G(z, x)}{\partial \nu_z}-G(z, x)\frac{\partial G(z, y)}{\partial \nu_z}\right]ds_z$$

Now, taking into account that $G(z, x)=E(z, x)+g(z, x)$ and $G(z, y)=E(z, y)+g(z, y)$ where $g(z, x)$ and $g(z, y)$ are harmonic functions and using the same argument as in the solution of the foregoing problem we obtain, after passing to the limit for $\varepsilon\to 0$, the equality $-G(x, y)=$ $=-G(y, x)$.

164. Integrate over the domain D both members of the identity

$$\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(v\frac{\partial u}{\partial x_i}\right)=\sum_{i=1}^{n}\frac{\partial v}{\partial x_i}\frac{\partial u}{\partial x_i}$$

which holds for the harmonic function u, apply it to the left-hand side of the equality the Gauss-Ostrogradsky formula and then put $v\equiv 1$ in the final result. The required equality also follows from formula (6), Problem 161, if we put $v\equiv 1$ in it.

165. (a) The formula expressing the mean-value theorem for a sphere follows from formula (7), Problem 162, if we assume that the surface S involved in the formula is the sphere $|y-x|=R$ with centre at the point x; (b) for the ball $|y-x|<R$ we can obtain the formula expressing the mean-value theorem if we write the formula expressing that theorem for the sphere $|y-x|=\rho$ in the form

$$\rho^{n-1}u(x)=\frac{1}{\omega_n}\int\limits_{|y-x|=\rho}u(y)\,dS_y$$

and then integrate both members of the last equality with respect to ρ from $\rho=0$ to $\rho=R$.

166. The assumption that a function $u(x)$ harmonic in a domain D and different from a constant attains its maximum at a point $x_0 \in D$ leads to a contradiction. Indeed, let us use the formula expressing the mean-value theorem:

$$u(x_0) = \frac{n}{\omega_n R^n} \int\limits_{|y-x_0| \leqslant R} u(y)\, d\tau_y$$

It follows that the function $u(x)$ is equal to $u(x_0)$ everywhere within the ball $|y - x_0| < R$ lying in the domain D. Indeed, if at a point y_0, $|y_0 - x_0| < R$, there holds the inequality $u(y_0) < u(x_0)$ (the inequality $u(y_0) > u(x_0)$ is impossible) then the same inequality must hold in a neighbourhood $|y - y_0| < \varepsilon$ of the point y_0, and, consequently, $u(x_0) < u(x_0)$. The contradiction we have obtained implies that $u(x) = u(x_0)$ everywhere in the ball $|y - x_0| < R$. Now let us denote by x an arbitrary point belonging to the domain D and join the points x and x_0 by a continuous curve L whose distance from the boundary of the domain D is equal to $\delta > 0$. If the centre y^* of the ball $|y - y^*| < \delta$ is moved from the point x_0 to the point x along the curve L, then for every y^* there must be $u(y^*) = u(x_0)$, whence it follows that the equality $u(x) = u(x_0)$ holds, which is impossible. The case of a minimum can be considered in like manner.

167. Apply the extremum principle to the difference $u_1 - u_2$ of two arbitrary solutions u_1 and u_2 of the Dirichlet problem: $\Delta u(x) = 0$ for $x \in D$, $u(x) = f(x)$ for $x \in S$.

168. Let $G(x, y) = E(x, y) + g(x, y)$ be Green's function and let $u(x)$ be the solution of the Dirichlet problem. The required formula (8) can be obtained if we take formula (7), Problem 162, written for the solution $u(x)$ and subtract from it termwise formula (6), Problem 161, written for the functions $u(y)$, $g(x, y)$ and multiplied by ω_n^{-1}.

169. Verify directly the validity of the equalities

$$\left| |x|\, y - \frac{x}{|x|} \right| = [|x|^2 |y|^2 - 2(x, y) + 1]^{1/2} =$$

$$= \left| |y|\, x - \frac{y}{|y|} \right| = |y| \left| x - \frac{y}{|y|^2} \right| = |x| \left| y - \frac{x}{|x|^2} \right|$$

which imply that the function $g(x, y) = E(|x|y, x/|x|)$ is harmonic within the unit sphere both with respect to x and with respect to y, and $g(x, y) = E(x, y)$ when $|x| = 1$ or $|y| = 1$. Here (x, y) denotes the scalar product of the vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Thus, the function $G(x, y)$ satisfies all the conditions enumerated in the definition of Green's function.

170. We have

$$\frac{\partial G(x, y)}{\partial \nu_y} = -\sum_{i=1}^{n} \left\{ \frac{y_i(y_i - x_i)}{|y-x|^n} - |x| \frac{y_i\left(|x| y_i - \frac{x_i}{|x|}\right)}{\left| |x| y - \frac{x}{|x|} \right|^n} \right\} = \frac{1-|x|^2}{|y-x|^n}$$

(see Problem 169). Therefore formula (8), Problem 168, implies the required Poisson's formula.

171. The solution is

$$u(x) = \frac{1}{\omega_n R} \int\limits_{|y-x_0|=R} \frac{R^2 - |x-x_0|^2}{|y-x|^n} \varphi(y)\, ds_y$$

Indeed, the change of variables $x = Rz + x_0$ reduces the given problem to the Dirichlet problem for the harmonic function $v(z) = u(Rz + x_0)$ in the ball $|z| < 1$:

$$\Delta v(z) = 0 \quad \text{for} \quad |z| < 1,$$

$$v(z) = \varphi(Rz + x_0) \quad \text{for} \quad |z| = 1$$

The solution of the last problem has the form

$$v(z) = \frac{1}{\omega_n} \int\limits_{|y|=1} \frac{1-|z|^2}{|y-z|^n} \varphi(Ry + x_0)\, ds_y$$

(see Problem 170). The inverse transformation $z = (x - x_0)/R$ from the variable z to the variable x results in the required answer to the problem.

172. Put $x = x_0$ in Poisson's formula (see the answer to Problem 171).

173. For the function $u(x) \equiv 1$ which is harmonic in the circle $|x| < 1$ we obtain, using Poisson's formula (see

Problem 170), the equality

$$1=\frac{1}{2\pi}\int_{0}^{2\pi}\frac{1-|x|^2}{|y-x|^2}\,d\psi$$

174. Since the kernel in Poisson's formula coincides with the normal derivative $\frac{\partial G(x, y)}{\partial \nu_y}$ of Green's function, this kernel is a harmonic function for $|x|<1$. Since the integral in the formula is uniformly convergent in a sufficiently small neighbourhood of any point x lying within the ball $|x|<1$, Laplace's operator can be written under the integral sign. This shows that the function $u(x)$ is harmonic.

For the second part of the problem we shall limit ourselves to the investigation of the case $n=2$. We have

$$u(x)=\frac{1}{2\pi}\int_{0}^{2\pi}\frac{1-|x|^2}{|y-x|^2}\,\varphi(y)\,dy$$

whence, using the identity of Problem 173, we obtain

$$u(x)-\varphi(x_0)=\frac{1}{2\pi}\int_{0}^{2\pi}\frac{1-|x|^2}{|y-x|^2}\,[\varphi(y)-\varphi(x_0)]\,d\psi,$$

$$|x|<1, \quad |x_0|=1$$

Since the function $\varphi(x)$ is uniformly continuous on the circumference $|x|=1$ of the circle $|x|\leqslant 1$, for any $\varepsilon>0$ there exists a number $\delta=\delta(\varepsilon)>0$ such that $|\varphi(y)-\varphi(x_0)|<\varepsilon$ for all ψ and ψ_0 ($y_1=\cos\psi$, $y_2=\sin\psi$, $x_{10}=\cos\psi_0$, $x_{20}=\sin\psi_0$) satisfying the condition $|\psi-\psi_0|<\delta$. The expression $u(x)-\varphi(x_0)$ can be represented in the form $u(x)-\varphi(x_0)=I_1+I_2$ where

$$I_1=\frac{1}{2\pi}\int_{\psi_0-\delta}^{\psi_0+\delta}\frac{1-|x|^2}{|y-x|^2}\,[\varphi(y)-\varphi(x_0)]\,d\psi$$

and

$$I_2=\frac{1}{2\pi}\left(\int\limits_0^{\psi_0-\delta}\frac{1-|x|^2}{|y-x|^2}[\varphi(y)-\varphi(x_0)]\,d\psi+\right.$$
$$\left.+\int\limits_{\psi_0+\delta}^{2\pi}\frac{1-|x|^2}{|y-x|^2}[\varphi(y)-\varphi(x_0)]\,d\psi\right)$$

whence we conclude that $|I_1|<\varepsilon$. On choosing $\delta(\varepsilon)$ and making x tend to x_0 we obtain

$$\left(\int\limits_0^{\psi_0-\delta}\frac{1-|x|^2}{|y-x|^2}\,d\psi+\int\limits_{\psi_0+\delta}^{2\pi}\frac{1-|x|^2}{|y-x|^2}\,d\psi\right)<\frac{\pi\varepsilon}{M},$$
$$M=\max_{0\leqslant\psi\leqslant 2\pi}|\varphi(y)|,\quad |y|=1$$

whence $|I_2|<\varepsilon$. Consequently, $|u(x)-\varphi(x_0)|<2\varepsilon$, that is $\lim\limits_{x\to x_0}u(x)=\varphi(x_0)$ for $|x|<1$ and $|x_0|=1$.

175. The validity of the inequalities is implied by Poisson's formula (see Problem 171)

$$u(x)=\frac{1}{\omega_n R}\int\limits_{|y|=R}\frac{R^2-|x|^2}{|y-x|^n}\varphi(y)\,dy$$

To show this we must take into account the inequalities $R-|x|<|y-x|<R+|x|$ which hold for $|x|<R$ and $|y|=R$ and then make use of the formula expressing the mean-value theorem (see Problem 165).

176. It cannot. This follows from the inequalities of Problem 175. Indeed, without loss of generality, we can assume that $u(x)>0$, and then, after we pass to the limit for $R\to\infty$, the indicated inequalities imply $u(x)=u(0)=\text{const}$.

177. It cannot. Indeed, if $M=\sup u(x)$ then the harmonic function $M-u(x)$ must retain sign, and, consequently, there must be $M-u(x_0)=M-u(x)$, that is the equality $u(x)=u(x_0)$ must hold throughout E_n. We thus see that the answer to the problem is negative.

178. To prove the assertion stated in the problem let us take an arbitrary point x_0 in the domain D and consider a ball $|y-x_0|\leqslant\varepsilon$ lying inside D. Let $u(x)$ be a function continuous in D for which the formula expressing the mean-value theorem holds in the neighbourhood of each point of the domain D. Let us denote by $v(x)$ a harmonic function in the ball $|y - x_0| < \varepsilon$ assuming on the sphere $|y - x_0| = \varepsilon$ the same values as $u(x)$. Then the formula expressing the mean-value theorem also holds for the difference $u(x) - v(x) = w(x)$, whence it follows that the extremum principle is valid for $w(x)$ (see Problem 166). Further, since $w(x) = 0$ on the sphere $|y - x_0| = \varepsilon$, we must have $w(x) \equiv 0$ throughout the ball $|y - x_0| \leqslant \varepsilon$, which proves the harmonicity of the function $u(x)$ in the neighbourhood of each point $x_0 \in D$.

179. According to the Gauss-Ostrogradsky formula and the condition of the problem, we have

$$\int\limits_{|x-x_0|<\varepsilon} \Delta u \, d\tau = \int\limits_{|x-x_0|=\varepsilon} \frac{\partial u}{\partial \nu} ds = 0$$

for any ball $|x - x_0| \leqslant \varepsilon$ lying within the domain D. Since the point x_0 is quite arbitrary and the number ε is positive, it follows that $\Delta u = 0$.

180. $u(x, y) = 2xy$. **181.** $u(x, y) = x^3 - 3x^2 - 3xy^2 + 3y^2 + 12x - 1$.

182. Write formula (7) for the hemisphere $|y| = R$, $y_n \geqslant 0$ and make R tend to infinity.

184. The solution of the problem is obtained by substituting $-x_n$ for x_n in the formula established in Problem 183.

185. $u(x, y) = x/[x^2 + (y + 1)^2]$.

186. $u(x, y, z) = -(z - 1)/[x^2 + y^2 + (z - 1)^2]^{3/2}$.

187. Let D^+ be a bounded domain in the Euclidean space E_n (consisting of the points $\xi = (\xi_1, \ldots, \xi_n)$) with a sufficiently smooth boundary S; the points belonging to the boundary S will be denoted as $\eta = (\eta_1, \ldots, \eta_n)$ and the complement of $D^+ \cup S$ with respect to E_n will be denoted as D^-. Without loss of generality, we can assume that the unit ball $|\xi| < 1$ belongs to D^+. To find the solution $u(\xi)$ of the exterior Dirichlet problem

$$\Delta u(\xi) = 0, \quad \lim_{\xi \to \eta} u(\xi) = \varphi(\eta), \quad \xi \in D^-, \quad \eta \in S \qquad (*)$$

let us perform the inversion transformation $\xi = x/|x|^2$ of the space $E_n(\xi)$ (with respect to the unit sphere $|\xi| = 1$), Under the inversion the unbounded domain D^- with boundary S is mapped onto a domain d of points $x = (x_1, \ldots, x_n)$ with boundary σ (the points belonging to the boundary σ will be denoted as $y = (y_1, \ldots, y_n)$. Next we construct the function

$$v(x) = \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right) \qquad (**)$$

It can be verified directly that the function $v(x)$ is harmonic if the function $u(\xi)$ is harmonic too (see Problem 135). Besides, putting $\eta = y/|y|^{n-2}$, we can write

$$\lim_{x\to y} v(x) = \lim_{x\to y} \frac{1}{|x|^{n-2}} u\left(\frac{x}{|x|^2}\right) =$$

$$= \frac{1}{|y|^{n-2}} \varphi\left(\frac{y}{|y|^2}\right), \quad x \in d, \quad y \in \sigma$$

Thus, for the function $v(x)$ we have the (interior) Dirichlet problem:

$$\Delta v(x) = 0, \ \lim_{x\to y} v(x) = \frac{1}{|y|^{n-2}} \varphi\left(\frac{y}{|y|^2}\right), \quad x \in d, \quad y \in \sigma$$

On solving this problem we find $v(x)$. Further, knowing the function $v(x)$, we can use formula (**) and the inverse transformation $\xi = x/|x|^2$ to obtain the solution $u(\xi) = v(\xi/|\xi|^2)/|\xi|^{n-2}$ of exterior Dirichlet problem (*). It can easily be checked that the function $u(\xi)$ we have found satisfies the boundary condition of Problem (*).

188. $u(x, y) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} \frac{x^2+y^2-1}{(x-x_1)^2+(y-y_1)^2} \varphi(x_1, y_1)\, ds$

189. $\displaystyle\int_S \varphi\, ds = 0$ (see Problem 164).

190. Let u_1 and u_2 be any two solutions of Neumann problem (5) for harmonic functions. Then their difference $v = u_1 - u_2$ satisfies the condition $\left.\dfrac{dv}{d\nu}\right|_S = 0$, whence, taking into account the obvious identity

$$\int_D \sum_{i=1}^{n} \left(\frac{\partial v}{\partial x_i}\right)^2 d\tau = \int_S v \frac{\partial v}{\partial \nu}\, ds = 0$$

we conclude that $v = C = \text{const}$, that is $u_1 = u_2 + C$,

191. Let us denote by $v(x, y)$ the harmonic conjugate function to $u(x, y)$. Then we have

$$\frac{dv}{ds}=\frac{\partial v}{\partial x}\frac{dx}{ds}+\frac{\partial v}{\partial y}\frac{dy}{ds}=\frac{\partial u}{\partial y}\frac{dy}{\partial \nu}+\frac{\partial u}{\partial x}\frac{dx}{d\nu}=\frac{du}{d\nu}=g(s)$$

and therefore $v(s)=\int\limits_0^s g(t)\,dt+C$, $0\leqslant s\leqslant 2\pi R$, where C is an arbitrary constant. The necessary condition $\int\limits_0^{2\pi R} g(t)\,dt=0$ for the solvability of the Neumann problem guarantees the continuity of the function $v(s)$ at the point $s=0$ (which coincides with the point $s=2\pi R$).

The function $v(x, t)$ is harmonic in the circle $x^2+y^2<R^2$ and is specified by Poisson's formula (see Problem 171):

$$v(x, y)=\frac{1}{2\pi R}\int\limits_S \frac{R^2-(x^2+y^2)}{(z-t)(\bar{z}-\bar{t})}\left(\int\limits_0^s g(\tau)\,d\tau\right)ds+C$$

where $z=x+iy$ and $t=\xi+i\eta$.

The analytic function $\varphi(z)=v(x, y)+iu(x, y)$ is determined by the formula

$$\varphi(z)=2v\left(\frac{z}{2}, \frac{z}{2i}\right)-v(0, 0)+iC_1=$$

$$=\frac{R}{\pi}\int\limits_S \frac{ds}{\bar{t}(t-z)}\int\limits_0^s g(\tau)\,d\tau-v(0, 0)+2C+iC_1$$

where $t=Re^{i\varphi}$ and $\bar{t}=Re^{-i\varphi}$. Since $R/\bar{t}=e^{i\varphi}$ and $ds=dt/ie^{i\varphi}$, we have

$$\varphi(z)=\frac{1}{\pi i}\int\limits_S \frac{dt}{t-z}\int\limits_0^s g(\tau)\,d\tau-v(0, 0)+2C+iC_1$$

The integration by parts yields

$$\varphi(z)=-\frac{1}{\pi i}\int\limits_S [\ln|t-z|+i\arg(t-z)]\,g(s)\,ds-$$

$$-v(0, 0)+2C+iC_1$$

On separating the imaginary part we finally obtain

$$u(x, y) = \frac{1}{\pi} \int_S g(s) \ln |t - z| \, ds + C_1$$

192. Let us write down Poisson's formula (see Problem 191) in the form

$$u(x, y) = \frac{1}{2\pi} \int_{|t|=1} \frac{1 - z\bar{z}}{(t - z)(\bar{t} - \bar{z})} u(t) \, d\varphi$$

and use Goursat's formula (1); then we obtain the relation

$$f(z) = u(x, y) + iv(x, y) = \frac{1}{\pi} \int_{|t|=1} \frac{u(t)}{(t - z)\bar{t}} \, d\varphi - u(0, 0) + iC$$

Now, since $t = e^{i\varphi}$, $\bar{t} = e^{-i\varphi}$, $d\varphi = -i\bar{t}\,dt$ and $u(0, 0) = \frac{1}{2\pi i} \int_{|t|=1} \frac{u(t)}{t} \, dt$, we conclude that

$$f(z) = \frac{1}{2\pi i} \int_{|t|=1} \frac{t + z}{t - z} \frac{u(t)}{t} \, dt + iC$$

193. The function $f(z) = u(x, y) + iv(x, y)$ is analytic in the semi-circle $|z| < 1$, $\operatorname{Im} z > 0$. By virtue of the condition $\left.\frac{\partial v}{\partial x}\right|_{y=0} = -\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0$ which implies $v(x, 0) = \text{const}$, we have $\operatorname{Im}\left.[f(z) - \text{const}]\right|_{y=0} = 0$. This means that the function $u(x, y)$ can be continued harmonically from the upper semi-circle $|z| < 1$, $\operatorname{Im} z > 0$ to the lower semi-circle $|z| < 1$, $\operatorname{Im} z < 1$, and we have $u(x, y) = u(x, -y)$ for $y < 0$. Consequently, the function $u(x, y)$ is harmonic in the circle $|z| < 1$ and satisfies the boundary conditions $u(x, y)|_{\sigma_1} = \varphi(x, y)$ and $u(x, y)|_{\sigma_2} = \varphi(x, -y)$ where σ_1 and σ_2 are the arcs $x^2 + y^2 = 1$, $y > 0$ and $x^2 + y^2 = 1$, $y < 0$ of the circumference of the circle $x^2 + y^2 \leqslant 1$ respectively. Now, using Poisson's formula, we find

$$u(x,\ y)=\int\limits_{\sigma_1}\frac{1-x^2-y^2}{(\xi-x)^2+(\eta-y)^2}\,\varphi(\xi,\ \eta)\,ds+$$

$$+\int\limits_{\sigma_2}\frac{1-x^2-y^2}{(\xi-x)^2+(\eta-y)^2}\,\varphi(\xi,\ -\eta)\,ds=$$

$$=\int\limits_{\sigma_1}(1-x^2-y^2)\left[\frac{1}{(\xi-x)^2+(\eta-y)^2}+\right.$$

$$\left.+\frac{1}{(\xi-x)^2+(\eta+y)^2}\right]\varphi(\xi,\ \eta)\,ds$$

194. The sought-for function is

$$u(x,\ y)=$$

$$=\frac{y}{\pi}\int\limits_{-1}^{1}\left[\frac{1}{(t-x)^2+y^2}-\frac{1}{(1-tx)^2+t^2y^2}\right]\varphi(t)\,dt.$$

Indeed, the function $u(x,\ y)$ can be continued harmonically from the semi-circle $|z|<1$, $\operatorname{Im} z>0$ to the domain $|z|>1$, $\operatorname{Im} z>0$, and for $|z|>1$, $\operatorname{Im} z>0$ we have

$$u(x,\ y)=-u\left(\frac{x}{x^2+y^2},\ \frac{y}{x^2+y^2}\right)$$

Hence, the function $u(x,\ y)$ is harmonic in the upper half-plane $y>0$ and satisfies the boundary condition

$$u(x,\ 0)=\begin{cases}-\varphi(1/x) & \text{for}\quad -\infty<x\leqslant -1\\ \varphi(x) & \text{for}\quad -1\leqslant x\leqslant 1\\ -\varphi(1/x) & \text{for}\quad 1\leqslant x<\infty\end{cases}$$

Therefore, by virtue of Poisson's formula (see Problem 183), we have

$$u(x,\ y)=\frac{y}{\pi}\left\{-\int\limits_{-\infty}^{-1}\frac{1}{(t-x)^2+y^2}\,\varphi\left(\frac{1}{t}\right)dt+\right.$$

$$\left.+\int\limits_{-1}^{1}\frac{1}{(t-x)^2+y^2}\,\varphi(t)\,dt-\int\limits_{1}^{\infty}\frac{1}{(1-x)^2+y^2}\,\varphi\left(\frac{1}{t}\right)dt\right\}=$$

$$=\frac{y}{\pi}\int\limits_{-1}^{1}\left[\frac{1}{(t-x)^2+y^2}-\frac{1}{(1-tx)^2+t^2y^2}\right]\varphi(t)\,dt$$

195. To obtain the formula it is sufficient to show that the function $u(x, y)$ can be continued harmonically from the upper semi-circle $|z| < 1$, $\operatorname{Im} z > 0$ to the lower semi-circle $|z| < 1$, $\operatorname{Im} z < 0$ and that $u(x, y) = -u(x, -y)$ for $y < 0$. The concluding part of the proof reduces to the usual application of Poisson's formula (see Problem 194).

196. In case D is a bounded domain, formula (9) shows that for $n > 2$ the potential function of a volume distribution of mass tends to zero as $x \to \infty$. When $n = 2$ we represent the function $\ln |x - y|$ in the form $\ln |x - y| = \ln \frac{|x-y|}{|x|} + \ln |x|$ and then show that for $|x| \to \infty$ in this case the volume potential behaves like the function $\ln |x| \int\limits_D \mu(y)\, d\tau_y$.

197. $\int\limits_D \mu(y)\, d\tau_y = 0$ (see Problem 196).

198. Let the function $f(x)$ be continuous and bounded in D together with its partial derivatives of the first order. The function $u(x)$ can be represented in the form

$$u(x) = -\frac{1}{\omega_n} \int\limits_D E(x, y) f(y)\, d\tau_y - \frac{1}{\omega_n} \int\limits_D g(x, y) f(y)\, d\tau_y$$

Now we can use the relations

$$\Delta \int\limits_D E(x, y) f(y)\, d\tau_y = -\omega_n f(x)$$

and

$$\Delta \int\limits_D g(x, y) f(y)\, d\tau_y = 0$$

to show that equality (11) holds. As to the condition $\lim\limits_{x \to x_0} u(x) = 0$, $x_0 \in S$, its validity cannot be established by means of the direct passage to the limit under the integral sign in the expression

$$u(x) = -\frac{1}{\omega_n} \int\limits_D G(x, y) f(y)\, d\tau_y$$

because Green's function $G(x, y)$ tends to zero, as $x \to x_0$, not uniformly with respect to $y \in D$. Let us write the function $u(x)$ in the form

$$u(x) = -\frac{1}{\omega_n}\int_{D_\delta} G(x, y) f(y)\, d\tau_y - \frac{1}{\omega_n}\int_{d_\delta} G(x, y) f(y)\, d\tau_y$$

where $d_\delta = D \cap \{|y - x_0| < \delta\}$ and D is the part of the domain D lying outside the ball $|y - x_0| \leqslant \delta$. It is evident that

$$\lim_{x \to x_0} \int_{D_\delta} G(x, y) f(y)\, d\tau_y = \int_{d_\delta} \lim_{x \to x_0} G(x, y) f(y)\, d\tau_y = 0 \qquad (*)$$

Let Q_R be a ball $|x - y| < R$ with centre at the point $y \in D$ such that $D \cup S \subset Q_R$ for any $y \in D$. If z is a point belonging to the sphere $|z - y| = R$ then for the function $\Omega(x, y) \equiv E(x, y) - E(z, y)$ we have $\Omega(x, y) \geqslant 0$, and on the boundary S of the domain D the inequalities $G(x, y) - \Omega(x, y) \leqslant 0$ hold. Since the function $G(x, y) - \Omega(x, y)$ is harmonic in D, the theorem on the maximum and on the minimum of a harmonic function implies that $G(x, y) - \Omega(x, y) \leqslant 0$ everywhere in $\overline{D}$. Therefore, taking $x \in d_\delta$ and denoting $M = \sup |f(y)|$, $y \in D$, we can write

$$\left|\int_{d_\delta} G(x, y) f(y)\, d\tau_y\right| \leqslant \int_{d_\delta} G(x, y)\, |f(y)|\, d\tau_y \leqslant$$

$$\leqslant M \int_{d_\delta} G(x, y)\, d\tau_y \leqslant M \int_{d_\delta} \Omega(x, y)\, d\tau_y =$$

$$= M \int_{d_\delta} [E(x, y) - E(z, y)]\, d\tau_y \leqslant M \int_{d_\delta} E(x, y)\, d\tau_y \leqslant$$

$$\leqslant M \int_{|y - x_0| < \delta} E(x, y)\, d\tau_y = \frac{M}{n-2} \int_{|y - x_0| < \delta} \frac{1}{|y - x|^{n-2}}\, d\tau_y \leqslant$$

$$\leqslant \frac{M}{n-2} \int_{|y - x| < 2\delta} \frac{1}{|y - x|^{n-2}}\, d\tau_y =$$

$$= \frac{M}{n-2} \int_{|y - x| < 2\delta} \frac{1}{r^{n-2}}\, r^{n-1}\, dr\, d\sigma = \frac{\omega_n M}{n-2} \int_0^{2\delta} r\, dr = \frac{2M\omega_n \delta^2}{n-2}$$

where $d\sigma$ is the element of area of unit sphere.

From the inequalities we have established it follows that

$$\lim_{\delta \to 0} \int_{d_\delta} G(x, y) f(y)\, d\tau_y = 0 \qquad (**)$$

Further, let us fix an arbitrary number $\varepsilon > 0$. From (*) and (**) it follows that there exist numbers $\delta_1 = \delta_1(\varepsilon) > 0$ and $\delta_2 = \delta_2(\varepsilon) > 0$ such that

$$\left| \frac{1}{\omega_n} \int_{d_{\bar\delta}} G(x, y) f(y)\, d\tau_y \right| < \frac{\varepsilon}{2}$$

for any $\bar\delta < \delta_1$ and for all x such that $|x - x_0| < \delta_2$ and inequality

$$\left| \frac{1}{\omega_n} \int_{D_{\bar\delta}} G(x, y) f(y)\, d\tau_y \right| < \frac{\varepsilon}{2}$$

is fulfilled. Now we see that for $\delta = \min(\delta_1, \delta_2)$ and for $|x - x_0| < \delta$ the inequalities

$$\left| \frac{1}{\omega_n} \int_{d_\delta} G(x, y) f(y)\, d\tau_y \right| < \frac{\varepsilon}{2}$$

$$\text{and} \quad \left| \frac{1}{\omega_n} \int_{D_\delta} G(x, y) f(y)\, d\tau_y \right| < \frac{\varepsilon}{2}$$

hold.

The last two inequalities imply that $|u(x)| < \varepsilon$, which means that $\lim\limits_{x \to x_0} u(x) = 0$, $x \in D$, $x_0 \in S$; this is what we intended to prove.

199. The difference $u(x) - v(x) = w(x)$ is the solution of the problem

$$\Delta w(x) = f(x) \quad \text{for} \quad x \in D, \quad w(y) = 0 \quad \text{for} \quad y \in S$$

Consequently (see Problem 198),

$$u(x) = v(x) + \frac{1}{\omega_n} \int_D G(x, y) f(y)\, d\tau_y$$

200. Yes (see the solution of Problem 167).

201. For $y \in C\ (d \cup \sigma)$ the validity of the third of the equalities we had to prove is quite evident (see Problem 164). Now let $y \in d$; we shall denote as d_ε the part of the domain d lying outside a sufficiently small closed ball $|x-y| \leqslant \varepsilon$. Using the result established in Problem 164 we can write for the domain d_ε the equality

$$\int\limits_{\sigma} \frac{\partial E(x, y)}{\partial \nu_x} ds_x = \int\limits_{|x-y|=\varepsilon} \frac{\partial E(x, y)}{\partial \nu_x} ds_x$$

whence, passing to the limit for $\varepsilon \to 0$ and taking into account the equality $\left.\frac{\partial E(x, y)}{\partial \nu_x}\right|_{|x-y|=\varepsilon} = -1/\varepsilon^{n-1}$, we obtain $\int\limits_{\sigma} \frac{\partial E(x, y)}{\partial \nu_x} ds_x = -\omega_n$. It now remains to consider the case $y \in \sigma$. Let us again denote by d_ε the part of the domain d lying outside a sufficiently small ball $|x-y| \leqslant \varepsilon$. Let σ_1 be the part of σ exterior to that ball and let σ_2 be the part of the sphere $|x-y| = \varepsilon$ lying within d. In this case we can also use the result of Problem 164 to write

$$\int\limits_{\sigma_1} \frac{\partial E(x, y)}{\partial \nu_x} ds_x = \int\limits_{\sigma_2} \frac{\partial E(x, y)}{\partial \nu_x} ds_x = -\int\limits_{\sigma_2} \frac{1}{\varepsilon^{n-1}} ds_x$$

whence, after passing to the limit for $\varepsilon \to 0$, we obtain $\int\limits_{\sigma} \frac{\partial E(x, y)}{\partial \nu_x} ds_x = -\omega_n/2,\ y \in \sigma$.

202. The required formula is a direct consequence of the equality

$$\int\limits_{\sigma} \frac{\partial}{\partial \nu_x} E(x, y)\, ds_x = \begin{cases} -\omega_n & \text{for} \quad y \in d \\ 0 & \text{for} \quad y \in C(d \cup \sigma) \end{cases}$$

(see Problem 201) where d is a bounded domain with boundary σ. Indeed, we have

$$\int_\sigma \frac{\partial u(x)}{\partial \nu_x} ds_x = \int_\sigma ds_x \int_D \mu(y) \frac{\partial E(x, y)}{\partial \nu_x} d\tau_y =$$

$$= \int_D \mu(y)\, d\tau_y \int_\sigma \frac{\partial E(x, y)}{\partial \nu_x} ds_x = \int_{D\cap d} \mu(y)\, d\tau_y \int_\sigma \frac{\partial E(x, y)}{\partial \nu_x} ds_x +$$

$$+ \int_{d_1} \mu(y)\, d\tau_y \int_\sigma \frac{\partial E(x, y)}{\partial \nu_x} ds_x$$

where d_1 is the part of D lying outside $d \cup \sigma$.

203. It cannot. **204.** $\mu = -5(x^2+y^2+z^2)/\pi$.

205. $M = -4r^5$. **206.** $u(x) = \frac{1}{A^2} \int_0^\omega dt \int_0^t f(\tau)\, d\tau$, $\omega = \sum_{k=1}^n a_k x_k$. **207.** $M = -\frac{8}{3\pi}$.

208. Solve the following problem:

$$\Delta u(r) = \begin{cases} -2\pi & \text{for} \quad 0 \leqslant r < 1 \\ 0 & \text{for} \quad r > 1 \end{cases}$$

$$u(1+0) = u(1-0), \quad u_r(1+0) = u_r(1-0)$$

209. Solve the following problem:

$$\Delta u(r) = \begin{cases} -4\pi & \text{for} \quad 0 \leqslant r < 1 \\ 0 & \text{for} \quad r > 1 \end{cases}$$

$$u(1-0) = u(1+0), \quad u_r(1-0) = u_r(1+0), \quad |u(r)| < \infty$$

$$\lim_{r\to\infty} u(r) = 0$$

210. $\mu = x$ and $u = \pi(x + x/r^2)/4$. **211.** $M = \frac{3\pi}{32}$.

212. The result is $I = 0$. To find the solution of the problem it suffices to make use of Gauss' formula (see Problem 202).

213. In case the surface over which the dipoles are distributed lies inside a bounded domain in the space E_n, the double-layer potential tends to zero for $|x| \to \infty$. In the case $n > 2$ the single-layer potential also possesses this property.

214. This condition has the form $\int\limits_S \mu(y)\, ds_y = 0$.

215. Let us consider the case $n = 2$. We shall look for the solution of the (interior or exterior) Dirichlet problem with a boundary condition $u\,|_S = g$ in the form of a double-layer potential with density μ. Then, using formulas (15) and (16) for the determination of μ, we obtain the following integral equations to which the interior and the exterior Dirichlet problems reduce respectively:

$$\mu(s) + \int\limits_S K(s, t)\,\mu(t)\,dt = -2g(s)$$

and

$$\mu(s) - \int\limits_S K(s, t)\,\mu(t)\,dt = 2g(s)$$

where $K(s, t) = \frac{1}{\pi}\frac{\partial}{\partial \nu_y} \ln|y - x|, \quad x = x(s), \quad y = y(t)$.

The solution of the (interior or exterior) Neumann problem with the boundary condition $\left.\frac{\partial u}{\partial \nu}\right|_S = g$ can be sought in the form of a single-layer potential with density μ. Then, using formulas (15′) and (16′) respectively, we reduce the interior and the exterior Neumann problems to the following integral equations:

$$\mu(s) + \int\limits_S K^*(s, t)\,\mu(t)\,dt = 2g(s)$$

and

$$\mu(s) - \int\limits_S K^*(s, t)\,\mu(t)\,dt = -2g(s)$$

where

$$K^*(s, t) = -\frac{1}{\pi}\frac{\partial}{\partial \nu_x} \ln|y - x| = -\frac{1}{\pi}\frac{d}{ds}\arctan\frac{y_2(t) - x_2(s)}{y_1(t) - x_1(s)}$$

216. The solution is

$$u(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \ln [(t-x)^2 + y^2] \varphi(t)\, dt + C$$

Indeed, denoting by $v(x, y)$ the harmonic function conjugate to $u(x, y)$, we arrive at the Dirichlet problem

$$\Delta v(x, y) = 0 \text{ for } y > 0, \quad v(x, 0) = \int_{0}^{x} \varphi(t)\, dt + C = \psi(x) \quad (*)$$

for the function $v(x, y)$. Assuming that for sufficiently large values of $|x|$ there holds the inequality $|\psi(x)| < A|x|^{-\delta}$, $\delta > 0$, and using Poisson's formula (see Problem 183) we find the solution of Problem (*) in the form

$$v(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\psi(t)\, dt}{(t-x)^2 + y^2}$$

Knowing $v(x, y)$ we can in an ordinary way reconstruct $u(x, y)$.

217. The sought-for potential is

$$u(x, y) = \begin{cases} -R \ln R & \text{for } x^2 + y^2 \leqslant R \\ -R \ln \sqrt{x^2+y^2} & \text{for } x^2 + y^2 \geqslant R \end{cases}$$

When solving this problem one should take into account the angular symmetry of the density μ of the distribution.

218.

$$u(x, y, z) = \begin{cases} 1 & \text{for } x^2 + y^2 + z^2 \leqslant 1 \\ (x^2+y^2+z^2)^{-1/2} & \text{for } x^2 + y^2 + z^2 \geqslant 1 \end{cases}$$

219. $u(x, y) = \begin{cases} -x/2 & \text{for } x^2 + y^2 < 1 \\ x/2r^2 & \text{for } x^2 + y^2 > 1 \end{cases}$

where $r = \sqrt{x^2 + y^2}$.

220. $u(x, y) = \dfrac{x^2 - y^2}{(x^2+y^2)^2} - 1$

221. $u(x, y, z) = \dfrac{x^2 - y^2}{(x^2+y^2+z^2)^{5/2}} - \dfrac{1}{(x^2+y^2+z^2)^{1/2}}$

222. $u(x, y, z) = \frac{z}{(x^2+y^2+z^2)^{3/2}}$

223. $u(x, y) = -\frac{y}{2}$

224. $u(x, y) = \begin{cases} -1 & \text{for} \quad x^2+y^2<1 \\ 0 & \text{for} \quad x^2+y^2>1 \end{cases}$

225. $\mu = 2x + 8xy$.

226. The transformation of the variables $z = x + iy$, $\bar{z} = x - iy$ reduces equation (17) to the form

$$\frac{\partial^2 u}{\partial z\, \partial \bar{z}} + \frac{\lambda}{4} u = 0 \qquad (17')$$

On representing the function $J_0\left(\mu \sqrt{(z-t)\bar{z}}\right)$ in the form of the sum of the series

$$u(x, y) = J_0\left(\mu \sqrt{(z-t)\bar{z}}\right) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\mu}{2}\right)^{2n} \frac{(z-t)^n \bar{z}^n}{(n!)^2} =$$

$$= \sum_{n=0}^{\infty} \left(-\frac{\lambda}{4}\right)^n \frac{(z-t)^n \bar{z}^n}{(n!)^2}$$

we find

$$\frac{\partial^2 u(x, y)}{\partial z\, \partial \bar{z}} = \sum_{n=0}^{\infty} \left(-\frac{\lambda}{4}\right)^n \frac{(z-t)^{n-1} \bar{z}^{n-1}}{[(n-1)!]^2} =$$

$$= -\frac{\lambda}{4} \sum_{n=0}^{\infty} \left(\frac{\lambda}{4}\right)^n \frac{(z-t)^n \bar{z}^n}{(n!)^2}$$

The substitution of these expressions for $u(x, y)$ and $\frac{\partial^2 u(x, y)}{\partial z \partial \bar{z}}$ into the left-hand member of (17′) shows that $u(x, y)$ is a solution of equation (17′).

227. We have

$$u(x, y) = \frac{1}{2}\left\{\int_0^z J_0\left(\mu\sqrt{(z-t)\bar{z}}\right) f(t)\,dt + \right.$$

$$\left. + \int_0^{\bar{z}} J_0\left(\mu\sqrt{(\bar{z}-\bar{t})\,z}\right) \bar{f}(\bar{t})\,d\bar{t}\right\}$$

and

$$\frac{\partial^2 u}{\partial z\,\partial \bar{z}} = \frac{1}{2}\left\{\int_0^z \frac{\partial^2 J_0\left(\mu\sqrt{(z-t)\bar{z}}\right)}{\partial z\,\partial \bar{z}} f(t)\,dt + \right.$$

$$\left. + \int_0^{\bar{z}} \frac{\partial^2 J_0\left(\mu\sqrt{(\bar{z}-\bar{t})\,z}\right)}{\partial z\,\partial \bar{z}} \bar{f}(\bar{t})\,d\bar{t}\right\}$$

and the function $J_0\left(\mu\sqrt{(\bar{z}-\bar{t})\,z}\right)$, like the function $J_0\left(\mu\sqrt{(z-t)\,\bar{z}}\right)$, is a solution of equation (17). Therefore the assertion becomes obvious.

228. If we assume that $u(x, y)$ attains a positive maximum at an interior point $(x, y) \in D$, this will lead to a contradiction. Indeed, at the point (x, y) where $u(x, y)$ attains its maximum there must be $u_{xx} + u_{yy} < 0$. Since the maximum is positive and $\lambda < 0$, the equality $u_{xx} + u_{yy} + \lambda u = 0$ is impossible. The second part of the assertion can be proved quite similarly.

229. Yes. (This follows from the extremum principle stated in Problem 228).

230. Since in the polar coordinates $x - \xi = r\cos\varphi$, $y - \eta = r\sin\varphi$ equation (17) assumes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} - \mu^2 u = 0$$

and since $\dfrac{\partial^2 E}{\partial \varphi^2} = 0$, we must have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E}{\partial r}\right) - \mu^2 E = 0$$

As to the last equality, it follows from the equalities

$$r\frac{\partial E}{\partial r}=\int_{-\infty}^{-1}\frac{r\mu te^{r\mu t}\,dt}{\sqrt{t^2-1}}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E}{\partial r}\right)=\frac{1}{r}\int_{-\infty}^{-1}\frac{\mu te^{r\mu t}\,dt}{\sqrt{t^2-1}}+\int_{-\infty}^{-1}\frac{\mu^2t^2e^{r\mu t}\,dt}{\sqrt{t^2-1}}=$$

$$=-\mu^2\int_{-\infty}^{-1}\sqrt{t^2-1}\,e^{r\mu t}\,dt+\mu^2\int_{-\infty}^{-1}\sqrt{t^2-1}\,e^{r\mu t}\,dt+$$

$$+\mu^2\int_{-\infty}^{-1}\frac{e^{r\mu t}\,dt}{\sqrt{t^2-1}}$$

231. The function $E(r)$ can be found as a solution of the differential equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)-\mu^2u=0$$

As can easily be seen, for $E(r)=e^{-\mu r}/r$ we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial E}{\partial r}\right)=-\frac{1}{r^2}\frac{\partial}{\partial r}\left(\mu re^{-\mu r}+e^{-\mu r}\right)=\frac{\mu^2}{r}e^{-\mu r}$$

and, consequently, $\Delta E-\mu^2E=0$.

232. Let x be a point belonging to D. We shall denote by D_ε the part of the domain D exterior to the ball $|y-x|\leqslant\varepsilon$ and lying inside the domain D, the radius ε of the ball being sufficiently small. Since $\Delta u=\mu^2u$ and $\Delta E=\mu^2E$, we obtain, using the Gauss-Ostrogradsky formula, the equality

$$\int\limits_{|y-x|=\varepsilon}\left[u(y)\frac{\partial E(x,y)}{\partial\nu_y}-E(x,y)\frac{\partial u(y)}{\partial\nu_y}\right]d\sigma_y=$$

$$=\int\limits_S\left[u(y)\frac{\partial E(x,y)}{\partial\nu_y}-E(x,y)\frac{\partial u(y)}{\partial\nu_y}\right]ds_y$$

Now, since

$$\lim_{\varepsilon\to 0} \varepsilon^2 E(\varepsilon) = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} \varepsilon^2 \left[\frac{\partial}{\partial\varepsilon} \frac{e^{-\mu\varepsilon}}{\varepsilon}\right] = -1$$

we obtain, after passing to the limit for $\varepsilon \to 0$, the relation

$$-4\pi u(x) = \int_S \left[u(y) \frac{\partial E(x, y)}{\partial \nu_y} - E(x, y) \frac{\partial u(y)}{\partial \nu_y}\right] ds_y$$

233. On integrating twice the equation $\frac{\partial^4 u}{\partial z^2\, \partial \bar{z}^2} = 0$ we obtain

$$\frac{\partial^3 u}{\partial z\, \partial \bar{z}^2} = \bar{\varphi}_2(\bar{z}) \quad \text{and} \quad \frac{\partial^2 u}{\partial \bar{z}^2} = z\bar{\varphi}_2(\bar{z}) + \bar{\psi}_2(\bar{z})$$

where $\bar{\varphi}_2(\bar{z})$ and $\bar{\psi}_2(\bar{z})$ are arbitrary analytic functions of the complex variable $\bar{z} = x_1 - ix_2$. Further, we have

$$\frac{\partial u}{\partial \bar{z}} = z\bar{\varphi}_1(\bar{z}) + \bar{\psi}_1(\bar{z}) + \lambda_1(z)$$

where $\bar{\varphi}_1(\bar{z}) = \int_0^{\bar{z}} \bar{\varphi}_2(\bar{t})\, d\bar{t}$ and $\bar{\psi}_1(\bar{z}) = \int_0^{\bar{z}} \bar{\psi}_2(\bar{t})\, d\bar{t}$. Consequently, $u = z\bar{\varphi}_*(\bar{z}) + \bar{\psi}_*(\bar{z}) + \bar{z}\chi(z) + \omega(z)$ where $\bar{\varphi}_*(\bar{z}) = \int_0^{\bar{z}} \varphi_1(\bar{t})\, d\bar{t}$, $\bar{\psi}_*(\bar{z}) = \int_0^{\bar{z}} \bar{\psi}_1(\bar{t})\, d\bar{t}$, $\chi(z)$ and $\omega(z)$ are arbitrary analytic functions of the complex variable $z = x_1 + ix_2$. Since $u(x, y)$ is a real function, there must be $\chi(z) = \varphi_*(z)$ and $\omega(z) = \psi_*(z)$, and therefore $u = z\bar{\varphi}_*(\bar{z}) + \bar{\psi}_*(\bar{z}) + \bar{z}\varphi_*(z) + \psi_*(z)$. On putting $\varphi_*(z) = \varphi(z)/2$ and $\psi_*(z) = \psi(z)/2$ we obtain $u(x_1, x_2) = \operatorname{Re}[\bar{z}\varphi(z) + \psi(z)]$.

234. The function $E(r) = r^2 \ln r$ is obtained from the formula

$$u = \operatorname{Re}[\bar{z}\varphi(z) + \psi(z)], \quad z = x_1 - y_1 + i(x_2 - y_2)$$

(see Problem 233), if we put $\psi(z) = 0$ and $\varphi(z) = z \ln z = z(\ln z + i \arg z)$. Therefore the function $E(r)$ satisfies equation (18) for $r \neq 0$.

236. Let $\lambda_1, \ldots, \lambda_\mu$ be the zeros of the polynomial $\sum_{k=0}^{m} a_k\lambda^{m-k}$. Let their multiplicities be $\nu_1, \ldots, \nu_\mu$ respectively. The differential operator under consideration can be written in the form

$$\sum_{k=0}^{m} a_k\Delta^{m-k} = a_0 \prod_{k=1}^{\mu} (\Delta-\lambda_k)^{\nu_k}$$

whence follows the assertion stated in the problem.

237. On writing the differential operator in the form

$$\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}=\left(\frac{\partial^2}{\partial x^2}+i\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2}-i\frac{\partial^2}{\partial y^2}\right)$$

we easily see that both $\varphi(z_1)$ and $\psi(z_2)$ are solutions of the given equation. Indeed, we have

$$\left\{1+i\left[\frac{\sqrt{2}}{2}(1+i)\right]^2\right\}\operatorname{Re}\varphi''(z_1)=0$$

and

$$\left\{1+i\left[\frac{\sqrt{2}}{2}(1+i)\right]^2\right\}\operatorname{Re}\psi''(z_2)=0$$

238. The roots λ and $\bar{\lambda}$ of the quadratic equation $c\lambda^2 + 2b\lambda + a = 0$ are

$$\lambda=-\frac{b}{c}-\frac{i}{c}\sqrt{ac-b^2} \quad\text{and}\quad \bar{\lambda}=-\frac{b}{c}+\frac{i}{c}\sqrt{ac-b^2}$$

In terms of the variables $z = x + \lambda y$ and $\bar{z} = x + \bar{\lambda} y$ the given partial differential equation is expressed as $u_{z\bar{z}} = 0$. Consequently,

$$u(x, y)=\frac{1}{2}f(z)+\frac{1}{2}\bar{f}(\bar{z})$$

239. The solution is

$$u(x, y)=\frac{1}{2\pi}\int\limits_{\frac{t^2}{a^2}+\frac{\tau^2}{b^2}=1}\frac{1-\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}{\left(\frac{t-x}{a}\right)^2+\left(\frac{\tau-y}{b}\right)^2}\varphi(t, \tau)\,ds$$

Indeed, under the transformation of variables $x = a\xi$, $y = b\eta$, $u(x, y) = u(a\xi, b\eta) = v(\xi, \eta)$ the ellipse $x^2/a^2 + y^2/b^2 = 1$ goes into the circle $\xi^2 + \eta^2 = 1$ and the given equation is transformed into Laplace's equation $v_{\xi\xi} + v_{\eta\eta} = 0$. On denoting by $v(\xi, \eta)$ the solution of the Dirichlet problem for Laplace's equation in the circle $\xi^2 + \eta^2 < 1$, we derive the following formula for the sought-for solution $u(x, y)$:

$$u(x, y) = v\left(\frac{x}{a}, \frac{y}{b}\right)$$

240. In terms of the variables $z = x + iay$ and $\bar{z} = x - iay$ the given system can be expressed in the form $w_{\bar{z}} = 0$, where $w = u + iv$, whence it follows that

$$u(x, y) + iv(x, y) = f(x + iay)$$

241. For the corresponding homogeneous Cauchy problem with the data

$$u_0(x, y) = 0, \quad v_0(x, y) = 0, \quad (x, y) \in S$$

the analytic function $f(z) = u_0 + iv_0$ (see Problem 240) of the complex variable $z = x + iay$ turns into zero at all points belonging to S. Therefore, by virtue of the uniqueness theorem for analytic functions, the function $f(z)$ is identically equal to zero, whence follows the uniqueness of the solution of the Cauchy problem.

242. It cannot. Indeed, let us denote $\xi = \frac{1}{a}x$, $\eta = \frac{1}{b}y$ $(a > 0,\ b > 0)$, $u_\xi = v$ and $u_\eta = w$. The given equation reduces to the Cauchy-Riemann system of partial differential equations $w_\xi - v_\eta = 0$, $w_\eta + v_\xi = 0$, and we have $v(\xi, 0) = 0$, and $w(\xi, 0) = 0$ for $\eta = 0$ and $0 \leqslant \xi \leqslant a\varepsilon$. Therefore (see the answer to Problem 241) both $v(\xi, \eta)$ and $w(\xi, \eta)$ are identically equal to zero.

243. On denoting $z = x + iy$, $\bar{z} = x - iy$ and $w = u + iv$ we can write the given system in the form $w_{\bar{z}\bar{z}} = 0$, whence follows representation (20).

244. Formula (20) implies that

$$\varphi(t) + t\psi(t) = t[f_1(t) + if_2(t)]$$

on the circle $|t| = 1$. It follows that (a) the Dirichlet problem can possess a solution only under the condition that the function $t[f_1(t) + if_2(t)]$ represents the limiting values on the circumference $|t| = 1$ of the circle $|z| < 1$ of a function analytic within that circle; (b) in case $f_1(t) = f_2(t) = 0$ there must be $\varphi(t) = -t\psi(t)$, $|t| = 1$. Therefore, by virtue of the uniqueness theorem for analytic functions, we must have $\varphi(z) = -z\psi(z)$ everywhere within the circle $|z| \leqslant 1$. Consequently, the homogeneous problem possesses infinitely many linearly independent solutions

$$u(x, y) + iv(x, y) = (1 - z\bar{z})\psi(z)$$

247. According to formula (8), Chapter 1, the characteristic determinant of the given system is

$$D(\lambda_1, \lambda_2, \lambda_3) = \begin{vmatrix} 0 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & 0 & -\lambda_3 & \lambda_2 \\ \lambda_2 & \lambda_3 & 0 & -\lambda_1 \\ \lambda_3 & -\lambda_2 & \lambda_1 & 0 \end{vmatrix} = -(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)^2$$

Consequently, the system is elliptic.

If u, v, w and φ are twice continuously differentiable functions, the matrix differential operator

$$\begin{Vmatrix} 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial z} & -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & -\frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 0 \end{Vmatrix}$$

transforms the given equation into

$$\begin{Vmatrix} \Delta & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 \\ 0 & 0 & \Delta & 0 \\ 0 & 0 & 0 & \Delta \end{Vmatrix} (u, v, w, \varphi) = 0$$

whence follows the harmonicity of the functions u, v, w and φ.

Chapter 3

248. $x - t = \text{const}, \ \ x + t = \text{const}.$

249. The sought-for surfaces are described by the equation $(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2 - (t - t^0)^2 = 0$ where (x_1^0, x_2^0, t^0) is an arbitrary fixed point in the space E_3 of the variables x_1, x_2, t.

250. These planes are described by the equation $a_1x_1 + a_2x_2 + a_3x_3 + at = \text{const}$ where a_1, a_2, a_3 and a are arbitrary real constants satisfying the relation $a_1^2 + a_2^2 + a_3^2 = a^2$.

251. From the conditions of the problem it follows that

$$\sum_{i=1}^{3} u_{x_ix_i} = t \int\limits_{|y|=1} \sum_{i=1}^{3} \mu_{z_iz_i}\, ds_y$$

and

$$u_{tt} = \frac{\partial}{\partial t} \left\{ \int\limits_{|y|=1} \mu(x_1 + ty_1, x_2 + ty_2, x_3 + ty_3)\, ds_y + \right.$$

$$\left. + t \int\limits_{|y|=1} \sum_{i=1}^{3} \mu_{z_i} y_i\, ds_y \right\} = \frac{\partial}{\partial t}\left(\frac{u}{t} + \frac{1}{t} I\right) = \frac{1}{t} I_t$$

where $\mu(z_1, z_2, z_3) = \mu(x_1 + ty_1, x_2 + ty_2, x_3 + ty_3)$, $I = \int\limits_{|z-x|^2=t^2} \sum_{i=1}^{3} \mu_{z_i}\nu_i\, ds_z$ and $\nu = (\nu_1, \nu_2, \nu_3)$ is the outer normal to the sphere $|z - x|^2 = t^2$ at the point z. The Gauss-Ostrogradsky formula

$$\int\limits_D \sum_{i=1}^{3} A_{z_i}\, d\tau = \int\limits_{\partial D} \sum_{i=1}^{3} A_i\nu_i\, ds_z$$

makes it possible to write the expression of I in the form

$$I = \int\limits_{|z-x|^2 \leqslant t^2} \sum_{i=1}^{3} \mu_{z_iz_i}\, d\tau = \int_0^t \rho^2\, d\rho \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} \Delta\mu\, d\varphi$$

where

$$z_1 - x_1 = \rho \sin\theta \cos\varphi, \ \ z_2 - x_2 = \rho \sin\theta \sin\varphi, \ \ z_3 - x_3 = \rho\cos\theta$$

and

$$\Delta = \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} + \frac{\partial^2}{\partial z_3^2}$$

It follows that

$$I_t = t^2 \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} \Delta\mu \, d\varphi = t^2 \int_{|y|=1} \sum_{i=1}^{3} \mu_{z_i z_i} \, ds_y$$

and

$$u_{tt} = t \int_{|y|=1} \sum_{i=1}^{3} \mu_{z_i z_i} \, ds_y$$

Consequently, $\sum\limits_{i=1}^{3} u_{x_i x_i} - u_{tt} = 0$.

252. Since the function ψ is continuous together with its partial derivatives of the second order, the first summand on the right-hand side of formula (6) satisfies equation (5). Further, the continuity of the partial derivative of the third order of the function φ is sufficient for the existence of the third-order derivatives $\frac{\partial^3}{\partial x_i^2 \partial t} [tM(\varphi)]$ and $\frac{\partial^3}{\partial t^3} [tM(\varphi)]$, and hence

$$\sum_{i=1}^{3} \frac{\partial^3}{\partial x_i^2 \, \partial t} [tM(\varphi)] - \frac{\partial^3}{\partial t^3} [tM(\varphi)] =$$

$$= \frac{\partial}{\partial t} \{\Delta [tM(\varphi)]\} - \frac{\partial}{\partial t} \{\Delta [tM(\varphi)]\} = 0$$

Consequently, function (6) satisfies equation (5). Besides, from (6) we find

$$u(x_1, x_2, x_3, 0) = \frac{1}{4\pi} \int_{|y|=1} \varphi(x_1, x_2, x_3) \, ds_y = \varphi(x_1, x_2, x_3)$$

and

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \frac{1}{4\pi} \int_{|y|=1} \psi(x_1, x_2, x_3) \, ds_y + \frac{1}{4\pi} \frac{\partial^2}{\partial t^2} [tM(\varphi)]_{t=0} =$$

$$= \psi(x_1, x_2, x_3) + \frac{1}{4\pi} [tM(\Delta\varphi)]_{t=0} = \psi(x_1, x_2, x_3)$$

254. On rewriting formula (6) in the form

$$u(x_1, x_2, x_3, t) = \frac{1}{4\pi t} \int\limits_{|z-x|^2=t^2} \psi(z_1, z_2, z_3)\, ds_z +$$

$$+ \frac{1}{4\pi t^2} \int\limits_{|z-x|^2=t^2} \varphi(z_1, z_2, z_3)\, ds_z + \frac{1}{4\pi t} \int\limits_{|z-x|^2=t^2} \frac{\partial\varphi}{\partial\nu}\, ds_z$$

(see Problem 252), we readily prove that the value of the function determined by formula (6) at the point (x_1, x_2, x_3, t) is specified by the values of φ, $\frac{\partial\varphi}{\partial\nu}$ and ψ on the sphere $(z_1 - x_1)^2 + (z_2 - x_2)^2 + (z_3 - x_3)^2 = t^2$.

255. In case $\varphi = \varphi(x_1, x_2)$, and $\psi = \psi(x_1, x_2)$, formula (6) specifies a function $u(x_1, x_2, t)$ dependent on two spatial variables; it can be written in the form

$$u(x_1, x_2, t) = \frac{1}{4\pi t} \int\limits_{|z|^2=t^2} \psi(x_1 + z_1, x_2 + z_2)\, ds_z +$$

$$+ \frac{1}{4\pi} \frac{\partial}{\partial t} \left\{ \frac{1}{t} \int\limits_{|z|^2=t^2} \varphi(x_1 + z_1, x_2 + z_2)\, ds_z \right\}$$

To compute the integrals of the right-hand side of the last formula we must project the upper and the lower hemispheres of the sphere $z_1^2 + z_2^2 + z_3^2 = t^2$ on the circle d: $z_1^2 + z_2^2 \leqslant t^2$, $z_3 = 0$. The area $dz_1 dz_2$ of the projection of the surface element ds_z of the sphere $|z|^2 = t^2$ on the circle d is expressed in terms of ds_z in the form

$$dz_1\, dz_2 = ds_z \cos(\widehat{i_3, \nu}) = \frac{z_3}{\sqrt{z_1^2 + z_2^2 + z_3^2}}\, ds_z$$

where i_3 is unit vector along the x_3-axis, ν is the normal to the sphere $|z|^2 = t^2$ at the point (z_1, z_2, z_3) and $z_3 = \pm\sqrt{t^2 - z_1^2 - z_2^2}$. The computations result in

$$u(x_1, x_2, t) = \frac{1}{2\pi} \int\limits_d \frac{\psi(x_1 + z_1, x_2 + z_2)}{\sqrt{t^2 - z_1^2 - z_2^2}}\, dz_1\, dz_2 +$$

$$+ \frac{1}{2\pi} \frac{\partial}{\partial t} \int\limits_d \frac{\varphi(x_1 + z_1, x_2 + z_2)}{\sqrt{t^2 - z_1^2 - z_2^2}}\, dz_1\, dz_2 \qquad (8')$$

whence, performing the change of variables $x_1 + z_1 = y_1$, $x_2 + z_2 = y_2$, we obtain formula (8).

256. Reversing the course of the argument given in the solution of Problem 255 we can bring formula (8′) to form (6), whence it follows that the function $u(x_1, x_2, t)$ is the solution of Problem (4), (2).

257. No, it does not, because the value of the function $u(x_1, x_2, t)$ at the point (x_1, x_2, t) (see formula (8)) is specified not only by the values of the functions φ and ψ assumed on the circumference $(y_1 - x_1)^2 + (y_2 - x_2)^2 = t^2$ of the circle $(y_1 - x_1)^2 + (y_2 - x_2)^2 \leqslant t^2$ but also by their values within the whole circle.

258. In case φ and ψ depend solely on one variable $x_1 = x$ formula (8) (see the answer to Problem 255) yields

$$u(x, t) = \frac{1}{2\pi} \int_{-t}^{t} \psi(x + \eta_1)\, d\eta_1 \int_{-\sqrt{t^2-\eta_1^2}}^{\sqrt{t^2-\eta_1^2}} \frac{d\eta_2}{\sqrt{t^2 - \eta_1^2 - \eta_2^2}} +$$

$$+ \frac{1}{2\pi} \frac{\partial}{\partial t} \int_{-t}^{t} \varphi(x + \eta_1)\, d\eta_1 + \int_{-\sqrt{t^2-\eta_1^2}}^{\sqrt{t^2-\eta_1^2}} \frac{d\eta_2}{\sqrt{t^2 - \eta_1^2 - \eta_2^2}} =$$

$$= \frac{1}{2} \int_{-t}^{t} \psi(x + \eta_1)\, d\eta_1 + \frac{1}{2} \frac{\partial}{\partial t} \int_{-t}^{t} \varphi(x + \eta_1)\, d\eta_1 =$$

$$= \frac{1}{2} \varphi(x + t) + \frac{1}{2} \varphi(x - t) + \frac{1}{2} \int_{x-t}^{x+t} \psi(\tau)\, d\tau$$

259. Make the change of variables $\xi = x + t$, $\eta = x - t$ and the transformation $v(\xi, \eta) = u\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right) = u(x, y)$ of the unknown function in equation (3) and integrate the resultant equation.

260. In terms of the characteristic variables $\xi = x + y$ and $\eta = 3x + 2y$ the given equation is expressed in the form $v_{\xi\eta}(\xi, \eta) = 0$. On integrating the last equation we find $u = f(x + y) + \varphi(3x + 2y)$ where f and φ are arbitrary twice continuously differentiable functions.

261. $u=\varphi(y-x)+e^{\frac{x-y}{2}}\psi(y-2x)$.

262. $u=[\varphi(x+3y)+\psi(3x+y)]\,e^{-\frac{7x+y}{16}}$.

263. The general solution of the equation is

$$u=\left[\varphi(y-3x)+\psi(3y-x)-\frac{1}{8}x(y-3x)(3y-x)\right]e^{-\frac{x+y}{16}}$$

Indeed, the original equation is expressed in terms of the characteristic variables $\xi=y-3x$ and $\eta=3y-x$ in the form

$$32v_{\xi\eta}+v_\xi-v_\eta-\frac{1}{32}v-(3\xi-\eta)\,e^{-\frac{\xi-\eta}{32}}=0$$

where $v(\xi,\ \eta)=u\left(\frac{\eta-3\xi}{8},\ \frac{3\eta-\xi}{8}\right)$. On performing the substitution $v(\xi,\eta)=e^{\frac{\xi-\eta}{32}}w(\xi,\eta)$ in the last equation we arrive at the new equation $32w_{\xi\eta}-3\xi+\eta=0$. Finally, we integrate the last equation and return to the original variables x, y, which results in the above answer to the problem.

264. $u=2e^x+e^{\frac{1}{2}(x+2y)}[\varphi(x)+\psi(x+2y)]$.

265. $u=e^{x+y/2}\,[(2x+y)\,e^{4x+y}+\varphi(2x+y)+$
$+\psi(4x+y)]$.

266. $u=\varphi(y+2x+\sin x)+e^{-\frac{1}{4}(y+2x+\sin x)}\psi(y-$
$-2x+\sin x)$.

267. The general solution of the equation is

$$u=e^y(e^{2y}-e^{2x})+\varphi(e^y+e^x)+\psi(e^y-e^x)$$

268. The general solution of the equation is

$$u=y\varphi(x)+\varphi'(x)+\int_0^y(y-\eta)\,e^{-x\eta}f(\eta)\,d\eta$$

To solve the problem transform the original equation to the form $v_{xy}+yv_y=0$ using the notation $v=u_y$.

269. The general solution of the equation is

$$u = \cos y + x\varphi(y) + \varphi'(y) + \int_0^x (x-\xi)\, e^{-y\xi} f(\xi)\, d\xi$$

To solve the problem construct the solution in the form $u = v + \cos y$ using the hint to Problem 268.

270. The general solution of the equation is

$$u = \frac{1}{\cosh x}\left\{ y\varphi(x) + \varphi'(x) + \int_0^y (y-\eta)\, e^{-x\eta}\psi(\eta)\, d\eta \right\}$$

To solve the problem use the notation $v = \cosh x u_y$ in order to transform the original equation to the form $v_{xy} + yv_y = 0$.

271. The general solution of the equation is

$$u = e^{-x}\left\{ \varphi(y) + \int_0^x e^{\xi - \xi^2 y^2}\psi(\xi)\, d\xi \right\}$$

272. The general solution of the equation is

$$u = (1+y)(1-e^{-x}) - xy + e^{-x}\left\{ \varphi(y) + \int_0^x e^{\xi(1-y)}\psi(\xi)\, d\xi \right\}$$

To solve the problem we use the notation $u_x + u = e^{-xy}v$ and transform the original equation to the equation $v_y = -x^2ye^{xy}$ from which v is found. Further, we substitute the expression of v thus found into the equality $u_x + u = e^{-xy}v$ and thus arrive at the equation $u_x + u = 1 - xy + e^{-xy}\psi(x)$ whose integration gives the sought-for answer to the problem.

273. The solution is

$$u(x, y) = \varphi\left(x - \frac{2}{3}y^3\right) + \frac{1}{2}\int_{x-\frac{2}{3}y^3}^{x+2y} \psi(\alpha)\, d\alpha$$

The problem is solved thus: the original equation reduces to the form $v_{\xi\eta} = 0$ in terms of the characteristic variables

$\xi = x - \frac{2}{3} y^3$ and $\eta = x + 2y$. On integrating the new equation and using the initial data we arrive at the answer to the problem.

274. The solution is

$$u(x, y) = (1 + 2x - e^{2x}) e^y + \varphi(y) + \frac{1}{2} \int_y^{2x+y} \psi(z)\, dz$$

To solve the problem make the change of variables $\xi = y$, $\eta = y + 2x$ in the given equation.

275. The solution is

$$u(x, y) = x + \cos(x - y + \sin x)$$

To solve the problem perform the change of variables $\xi = y - x - \sin x$, $\eta = y + x - \sin x$ in the given equation.

276. The solution is

$$u(x, y) = \frac{3}{2} e^{-y} \varphi(x + y) - \frac{1}{2} \varphi(x + 3y) + \\ + \frac{1}{4} e^{-\frac{1}{2}(x+3y)} \int_{x+y}^{x+3y} e^{\frac{z}{2}} [3\varphi(z) + 2\psi(z)]\, dz$$

To solve the problem one should first use the change of variables $\xi = x + 3y$, $\eta = x + y$ to bring the original equation to the normal form $v_{\xi\eta} = -\frac{1}{2} v_\eta$ and then integrate the resultant equation in order to find its general solution.

277. The solution is

$$u(x, y) = -\frac{x^2}{2} + \cos(x - 1 + e^y) - \cos x$$

To facilitate the computation of the general solution of the original equation one should reduce it to the normal form with the aid of the change of variables $\xi = x$, $\eta = x + e^y$.

278. The solution is

$$u(x, y) = e^x \sinh\left(\frac{y - \cos x}{2}\right) + \sin x \cos\left(\frac{y - \cos x}{2}\right)$$

To reduce the equation indicated in the condition of the problem to its normal form use the change of variables $\xi = 2x - y + \cos x$, $\eta = 2x + y - \cos x$.

279. The solution is

$$u(x, y) = 2e^{-\frac{1}{4}(2x-y+\cos x)} \cos x \sin \frac{1}{2}(y - \cos x)$$

To solve the problem one should make the change of variables $\xi = 2x - y + \cos x$, $\eta = 2x + y - \cos x$ in order to reduce the original equation indicated in the condition of the problem to its normal form

$$4v_{\xi\eta} + v_\eta = 0$$

where $v(\xi, \eta) = u((\xi + \eta)/4, (\eta - \xi)/2 + \cos(\xi + \eta)/4) = u(x, y)$. The general solution of the last equation has the form $v(\xi, \eta) = f(\xi) + e^{-\xi/4}F(\eta)$ where f and F are arbitrary twice continuously differentiable functions. Returning to the original variables x and y, we find the general solution of the original equation:

$$u = f(2x - y + \cos x) + e^{-\frac{1}{4}(2x-y+\cos x)} F(2x + y - \cos x)$$

Further, using the initial data, one should determine the functions f and F.

280. The solution is

$$u(x, y) = 1 - \sin(y - x + \cos x) + e^{y+\cos x} \sin(x + y + \cos x)$$

To solve the problem one should start with performing the change of variables $\xi = -x + y + \cos x$, $\eta = x + y + \cos x$ in order to reduce the equation indicated in the condition of the problem to its normal form. The further course of the solution follows the procedure indicated in the hint to the solution of Problem 279.

281. For the cases $n=3$, $n=2$ and $n=1$ the domains of dependence, on the manifold $t = 0$, corresponding to the point $(y, \tau) \in E_{n+1}$ are the sphere $|x - y|^2 = \tau^2$, the circle $|x - y|^2 \leqslant \tau^2$ and the line segment $|x - y|^2 \leqslant \tau^2$ respectively.

282. Since the sides of the characteristic rectangle with vertices at the points (x_1, t_1), (x_2, t_2), (x_3, t_3) and (x_4, t_4) are the straight lines $x - x_1 = t - t_1$, $x - x_2 = t_2 - t$, $x - x_3 = t - t_3$ and $x - x_4 = t_4 - t$, we have $x_2 - x_1 = t_2 - t_1$, $x_3 - x_2 = t_2 - t_3$, $x_4 - x_3 = t_4 - t_3$ and $x_1 - x_4 = t_4 - t_1$. Therefore, by virtue of formula (10), there must be

$$u(x_1, t_1) + u(x_3, t_3) = f(x_1 + t_1) + \varphi(x_1 - t_1) + f(x_3 + t_3) + \varphi(x_3 - t_3)$$

and

$$u(x_2, t_2) + u(x_4, t_4) = f(x_2 + t_2) + \varphi(x_2 - t_2) + f(x_4 + t_4) + \varphi(x_4 - t_4) = f(x_3 + t_3) + \varphi(x_1 - t_1) + f(x_1 + t_1) + \varphi(x_3 - t_3)$$

whence follows the assertion stated in the problem.

283. The sought-for solution is

$$v(x_1, x_2, x_3, t, \tau) = \frac{t-\tau}{4\pi} \int\limits_{|\xi|=1} g\, d\sigma_\xi$$

where $g = g[x_1 + (t-\tau)\xi_1,\ x_2 + (t-\tau)\xi_2 + x_3 + (t-\tau)\xi_3,\ \tau]$.

286. $$u(x_1, x_2, x_3, t) = x_1^3 x_2^2 + (3x_1 x_2^2 + x_1^3)\, t^2 + x_1 t^4 + (x_1^2 x_2^4 - 3x_1^3)\, t + \frac{1}{3}(x_2^4 - 9x_1 + 6x_1^2 x_2^2)\, t^3 + \frac{1}{5}(2x_2^2 + x_1^2)\, t^5 + \frac{1}{35} t^7$$

287. $$u(x, t) = \frac{1}{2}\varphi(x+t) + \frac{1}{2}\varphi(x-t) + \frac{1}{2}\int\limits_{x-t}^{x+t} \psi(\tau)\, d\tau - \frac{1}{2}\int\limits_0^t d\tau \int\limits_{x-t+\tau}^{x+t-\tau} g(\tau_1, \tau)\, d\tau_1$$

289. D'Alembert's formula

$$u(x, t) = \frac{\varphi(x-at) + \varphi(x+at)}{2} + \frac{1}{2a}\int\limits_{x-at}^{x+at} \psi(z)\, dz$$

shows directly that

(a) in case both functions $\varphi(x)$ and $\psi(x)$ are odd we have

$$u(0, t) = \frac{\varphi(-at)+\varphi(at)}{2} + \frac{1}{2a}\int_{-at}^{at} \psi(z)\, dz = 0$$

(b) in case both functions $\varphi(x)$ and $\psi(x)$ are even we have

$$u_x(0, t) = \frac{\varphi'(-at)+\varphi'(at)}{2} + \frac{\psi(at)-\psi(-at)}{2a} = 0$$

290. The solution of the given Cauchy problem is expressed by the formula

$$u(x, t) = \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(z, \tau)\, dz\, d\tau$$

whence we find directly that

(a) if the function $f(x, t)$ is odd with respect to the variable x then

$$u(0, t) = \frac{1}{2a}\int_0^t d\tau \int_{-a(t-\tau)}^{a(t-\tau)} f(z, \tau)\, dz = 0$$

(b) if the function $f(x, t)$ is even with respect to the variable x then

$$u_x(0, t) = \frac{1}{2a}\int_0^t \{f[a(t-\tau), \tau] - f[-a(t-\tau), \tau]\}\, d\tau = 0$$

291. The solution is

$$u(x, t) = \begin{cases} \dfrac{\varphi(x+at)+\varphi(x-at)}{2} + \dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at} \psi(z)\, dz \\ \qquad\qquad \text{for } \quad x>0, \quad t<\dfrac{x}{a} \\ \dfrac{\varphi(x+at)-\varphi(at-x)}{2} + \dfrac{1}{2a}\displaystyle\int_{at-x}^{x+at} \psi(z)\, dz \\ \qquad\qquad \text{for } \quad x>0, \quad t>\dfrac{x}{a} \end{cases}$$

To obtain the solution we reduce the given problem to the Cauchy problem for the infinite x-axis. To this end let us continue the initial data $\varphi(x)$ and $\psi(x)$ to the whole x-axis in an odd manner, that is let us construct the functions

$$\Phi(x)=\begin{cases}\varphi(x) & \text{for } x>0\\ -\varphi(-x) & \text{for } x<0\end{cases}$$

$$\text{and}\quad \Psi(x)=\begin{cases}\psi(x) & \text{for } x>0\\ -\psi(-x) & \text{for } x<0\end{cases}$$

and consider the following Cauchy problem:

$$\begin{gathered} U_{tt}=a^2U_{xx},\quad -\infty<x<\infty,\quad t>0\\ U(x,\ 0)=\Phi(x),\quad U_t(x,\ 0)=\Psi(x),\quad -\infty<x<\infty \end{gathered} \tag{*}$$

As is known, the solution of Problem (*) is given by D'Alembert's formula

$$U(x,\ t)=\frac{\Phi(x+at)+\Phi(x-at)}{2}+\frac{1}{2a}\int_{x-at}^{x+at}\Psi(z)\,dz$$

Since the functions $\Phi(x)$ and $\Psi(x)$ are odd, we have $U(0,\ t)=$ $=0$ (see Problem 289), and for $x>0$ the relations

$$U(x,\ 0)=\Phi(x)=\varphi(x),\quad U_t(x,\ 0)=\Psi(x)=\psi(x)$$

hold. Hence, the function $U(x,\ t)$ thus constructed satisfies all the conditions of Problem 291 for $x\geqslant 0$, $t\geqslant 0$, and, consequently, $u(x,\ t)=U(x,\ t)$ $(x\geqslant 0,\ t\geqslant 0)$ is the solution of the problem. On expressing the function $U(x,\ t)$ in terms of the initial data $\varphi(x)$ and $\psi(x)$ of the original problem for $x\geqslant 0$ and $t\geqslant 0$, we arrive at the expression of $u(x,\ t)$ indicated above.

292. The solution is

$$u(x,\ t)=\begin{cases} \dfrac{\varphi(x+at)+\varphi(x-at)}{2}+\dfrac{1}{2a}\displaystyle\int\limits_{x-at}^{x+at}\psi(z)\,dz \\ \qquad\qquad\qquad \text{for}\quad x>0,\quad t<\dfrac{x}{a} \\ \dfrac{\varphi(x+at)+\varphi(at-x)}{2}+ \\ \qquad +\dfrac{1}{2a}\left\{\displaystyle\int\limits_{0}^{x+at}\psi(z)\,dz+\int\limits_{0}^{at-x}\psi(z)\,dz\right\} \\ \qquad\qquad\qquad \text{for}\quad x>0,\quad t>\dfrac{x}{a} \end{cases}$$

To obtain this solution we consider the following auxiliary Cauchy problem:

$$U_{tt}=a^2U_{xx},\quad -\infty<x<\infty,\quad t>0$$
$$U(x,\ 0)=\Phi(x),\quad U_t(x,\ 0)=\Psi(x),\quad -\infty<x<\infty$$

where

$$\Phi(x)=\begin{cases}\varphi(x) & \text{for}\quad x>0\\ \varphi(-x) & \text{for}\quad x<0\end{cases}$$

$$\text{and}\quad \Psi(x)=\begin{cases}\psi(x) & \text{for}\quad x>0\\ \psi(-x) & \text{for}\quad x<0\end{cases}$$

The solution of the latter problem is given by D'Alembert's formula:

$$U(x,\ t)=\frac{\Phi(x+at)+\Phi(x-at)}{2}+\frac{1}{2a}\int\limits_{x-at}^{x+at}\Psi(z)\,dz$$

Since the functions $\Phi(x)$ and $\Psi(x)$ are even, we have $U_x(0,\ t)=0$ (see Problem 289), and for $x>0$ the relations

$$U(x,\ 0)=\Phi(x)=\varphi(x)\quad\text{and}\quad U_t(x,\ 0)=\Psi(x)=\psi(x)$$

hold.

Consequently, the function $U(x, t)$ is the sought-for solution for $x \geqslant 0$, $t \geqslant 0$, that is $u(x, t) = U(x, t)$ $(x \geqslant 0, t \geqslant 0)$. On expressing the function $U(x, t)$ in terms of the data $\varphi(x)$ and $\psi(x)$ of the original problem for $x \geqslant 0$ and $t \geqslant 0$, we arrive at the solution $u(x, t)$ indicated above.

293.
$$u(x, t) = \begin{cases} \dfrac{1}{2a} \displaystyle\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(z, \tau)\, dz\, d\tau & \text{for} \quad x > 0, \quad t < \dfrac{x}{a} \\ \dfrac{1}{2a} \displaystyle\int_0^{t-\frac{x}{a}} \int_{a(t-\tau)-x}^{x+a(t-\tau)} f(z, \tau)\, dz\, d\tau + & \\ + \dfrac{1}{2a} \displaystyle\int_{t-\frac{x}{a}}^{t} \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(z, \tau)\, dz\, d\tau & \text{for } x > 0, \ t > \dfrac{x}{a} \end{cases}$$

To obtain the solution of this form, we extend the function $f(x, t)$ to the whole x-axis so that the resultant function $F(x, t)$ is odd with respect to the variable x:

$$F(x, t) = \begin{cases} f(x, t) & \text{for} \quad x > 0 \\ -f(-x, t) & \text{for} \quad x < 0 \end{cases}$$

Now let us consider the following Cauchy problem:

$$U_{tt} = a^2 U_{xx} + F(x, t), \quad -\infty < x < \infty, \quad t > 0$$
$$U(x, 0) = U_t(x, 0) = 0, \quad -\infty < x < \infty$$

The solution of the last problem is the function

$$U(x, t) = \frac{1}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} F(z, \tau)\, dz\, d\tau \qquad (*)$$

Since the function $F(x, t)$ is odd with respect to x, we have $U(0, t) = 0$ (see Problem 290), and for $x > 0$, the equalities $U(x, 0) = U_t(x, 0) = 0$ hold. Consequently, the function $U(x, t)$ coincides with the sought-for solution for $x \geqslant 0$, $t \geqslant 0$, that is $u(x, t) = U(x, t)$ $(x \geqslant 0, t \geqslant 0)$. To transform the solution we have found to the form indicated above we shall consider separately the following cases:

(1) let $x > 0$ and $x - at > 0$ (that is $t < x/a$). Then

$$x - a(t - \tau) = x - at + a\tau > 0$$

Therefore

$$u(x, t) = U(x, t) = \frac{1}{2a} \int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(z, \tau)\, dz\, d\tau$$

(2) let $x > 0$ and $x - at < 0$ (that is $t > x/a$). Then

$$x - a(t - \tau) = x - at + a\tau < 0 \text{ for } 0 < \tau < t - x/a$$

and

$$x - a(t - \tau) = x - at + a\tau > 0 \text{ for } \tau > t - x/a$$

Consequently

$$u(x, t) = U(x, t) = \frac{1}{2a} \int_0^{t-\frac{x}{a}} \int_{x-a(t-\tau)}^{x+a(t-\tau)} F(z, \tau)\, dz\, d\tau +$$

$$+ \frac{1}{2a} \int_{t-\frac{x}{a}}^{t} \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(z, \tau)\, dz\, d\tau =$$

$$= \frac{1}{2a} \int_0^{t-\frac{x}{a}} \int_{x-a(t-\tau)}^{0} -[f(-z, \tau)]\, dz\, d\tau +$$

$$+ \frac{1}{2a} \int_0^{t-\frac{x}{a}} \int_0^{x+a(t-\tau)} f(z, \tau)\, dz\, d\tau +$$

$$+ \frac{1}{2a} \int_{t-\frac{x}{a}}^{t} \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(z, \tau)\, dz\, d\tau$$

On replacing $-z$ by z in the first integral we finally obtain

$$u(x,\ t)=\frac{1}{2a}\int\limits_{0}^{t-\frac{x}{a}}\int\limits_{a(t-\tau)-x}^{x+a(t-\tau)} f(z,\ \tau)\,dz\,d\tau+$$

$$+\frac{1}{2a}\int\limits_{t-\frac{x}{a}}^{t}\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} f(z,\ \tau)\,dz\,d\tau$$

294.

$$u(x,\ t)=\begin{cases}\dfrac{1}{2a}\displaystyle\int\limits_{0}^{t}\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} f(z,\ \tau)\,dz\,d\tau \quad \text{for} \quad x>0, \quad t<\dfrac{x}{a}\\[2ex] \dfrac{1}{2a}\displaystyle\int\limits_{0}^{t-\frac{x}{a}}\left[\int\limits_{0}^{a(t-\tau)-x}+\int\limits_{0}^{x+a(t-\tau)}\right] f(z,\ \tau)\,dz\,d\tau+\\[2ex] +\dfrac{1}{2a}\displaystyle\int\limits_{t-\frac{x}{a}}^{t}\int\limits_{x-a(t-\tau)}^{x+a(t-\tau)} f(z,\ \tau)\,dz\,d\tau \quad \text{for} \quad x>0, \quad t>\dfrac{x}{a}\end{cases}$$

To obtain the solution of this form let us extend the function $f(x,\ t)$ (with respect to the variable x) to the whole x-axis so that the resultant function $F(x,\ t)$ is even:

$$F(x,\ t)=\begin{cases} f(x,\ t) & \text{for} \quad x>0\\ f(-x,\ t) & \text{for} \quad x<0\end{cases}$$

Next we consider the following Cauchy problem:

$$U_{tt}=a^2U_{xx}+F(x,\ t), \quad -\infty<x<\infty, \quad t>0$$
$$U(x,\ 0)=U_t(x,\ 0)=0, \quad -\infty<x<\infty$$

In the further course of the solution use the procedure presented in the answer to Problem 293, taking into account however that in the case under consideration the extended function $F(x,\ t)$ is even with respect to the variable x.

295. The solution should be sought in the form $u(x,\ t)=v(x,\ t)+w(x,\ t)$ where $v(x,\ t)$ and $w(x,\ t)$ are the solutions of Problems 291 and 293 respectively.

296. The solution should be sought in the form $u(x, t) = v(x, t) + w(x, t)$ where $v(x, t)$ and $w(x, t)$ are the solutions of Problems 292 and 294 respectively.

297. The perturbation at the end $x = 0$ generates a wave propagating from the point $x = 0$ along the x-axis, and therefore seek the solution of the problem in the form of a direct wave $u(x, t) = f(x - at)$. From the initial condition we obtain $u(x, 0) = f(x) = 0$ for $x > 0$, whence it follows directly that for $x > 0$ the condition $u_t(x, 0) = -af'(x) = 0$ must hold. From the boundary condition we find $u(0, t) = f(-at) = \mu(t)$, $t > 0$. Hence, $f(z) = 0$ for $z \geqslant 0$ and $f(z) = \mu(-z/a)$ for $z \leqslant 0$, and, consequently,

$$u(x, t) = \begin{cases} 0 \quad \text{for} \quad 0 < t \leqslant x/a \\ \mu(t - x/a) \quad \text{for} \quad t \geqslant x/a \end{cases}$$

298. The solution is

$$u(x, t) = \begin{cases} 0 \quad \text{for} \quad 0 < t \leqslant x/a \\ -a \int\limits_0^{t - x/a} \nu(s)\, ds \quad \text{for} \quad t \geqslant x/a \end{cases}$$

As in the foregoing problem, the solution should be sought for in the form of a direct wave $u(x, t) = f(x - at)$.

299. The solution is

$$u(x, t) = \begin{cases} 0 \quad \text{for} \quad 0 < t \leqslant x/a \\ -ae^{h(x - at)} \int\limits_0^{t - x/a} e^{ahs} \varkappa(s)\, ds \quad \text{for} \quad t \geqslant x/a \end{cases}$$

Since the oscillation is generated by the perturbation at the end point $x = 0$, we seek the solution of the form of a direct wave $u(x, t) = f(x - at)$. From the initial condition we find $u(x, 0) = f(x) = 0$, $x > 0$, whence it follows immediately that $u_t(x, 0) = 0$ because $u_t(x, 0) = -af'(x) = 0$ for $x > 0$. From the boundary condition we find $u_x(0, t) - hu(0, t) = f'(-at) - hf(-at) = \varkappa(t)$, $t \geqslant 0$, that is $f'(z) - hf(z) = \varkappa(-z/a)$ for $z \leqslant 0$.

Integrating the last relation we obtain

$$f(z) = -ae^{hz} \int_0^{-z/a} e^{ahs} \varkappa(s)\, ds, \quad z \leqslant 0$$

Thus,

$$f(z) = \begin{cases} 0 & \text{for} \quad z > 0 \\ -ae^{hz} \displaystyle\int_0^{-z/a} e^{ahs} \varkappa(s)\, ds & \text{for} \quad z \leqslant 0 \end{cases}$$

Finally, the answer to the problem indicated above is obtained by putting $z = x - at$ in the last equality.

300. The solution should be sought in the form $u(x, t) = v(x, t) + w(x, t) + z(x, t)$ where $v(x, t)$, $w(x, t)$ and $z(x, t)$ are the solutions of Problems 291, 293 and 297 respectively.

301. The solution should be sought in the form $u(x, t) = v(x, t) + w(x, t) + z(x, t)$ where $v(x, t)$, $w(x, t)$ and $z(x, t)$ are the solutions of Problems 292, 294 and 298 respectively.

302. $u(x, t) = xyt - \frac{1}{6} xyt^3$.

303. Indeed, if $w(x, y, t)$ is a homogeneous polynomial of degree $n - 2m \geqslant 0$ then, according to the property of homogeneous functions, we have $xw_x + yw_y + tw_t = (n - 2m)\, w$; therefore

$$\square\, w\rho^{2m} = 2m\,(2n - 2m + 1)\, w\rho^{2m-2} + \rho^{2m} \square\, w \quad (*)$$

Let us consider the function $u_1(x, y, t) = v + \sum_{k=1}^{\infty} A_k \rho^{2k} \square^k v$

where A_k are constants and v is a homogeneous polynomial of degree n. Using relation $(*)$ we can write

$$\square\, u_1 = \square\, v + \sum_{k=1}^{\infty} A_k \left[2k\,(2n - 2k + 1)\, \rho^{2k-2} \square^k v + \rho^{2k} \square^{k+1} v\right]$$

Under the assumption that $2k\,(2n - 2k + 1)\, A_k = -A_{k-1}$ for $k \geqslant 2$ and $2\,(2n - 1)\, A_1 = -1$ we obtain $\square\, u_1 = 0$. Now we put $\square v = \Phi$ and $u = u_1 + v$ and thus obtain $\square u = \Phi$, which is what we intended to prove.

304. See Problem 135.

305. There are altogether seven polynomials of this kind: x^3+3xt^2, x^2y+yt^2, xy^2+xt^2, y^3+3yt^2, $x^2t+\frac{1}{3}t^3$, $y^2t+\frac{1}{3}t^3$ and xyt.

306. For $n=1$ the number of the polynomials is equal to two. In case $n\geqslant 2$ the sought-for polynomials can be obtained from formula (7); they have the form

$$u(x_1, \ldots, x_n, t)=\sum_{m=0}^{\infty}\frac{t^{2m}}{(2m)!}\Delta^m x_1^{\alpha_1}\ldots x_n^{\alpha_n}$$

and

$$u(x_1, \ldots, x_n, t)=\sum_{m=1}^{\infty}\frac{t^{2m+1}}{(2m+1)!}\Delta^m x_1^{\beta_1}\ldots x_n^{\beta_n}$$

where $x_1^{\alpha_1}\ldots x_n^{\alpha_n}$ and $x_1^{\beta_1}\ldots x_n^{\beta_n}$ are linearly independent monomials of degrees k and $k-1$ respectively. Since there are altogether $\binom{k+n-1}{n-1}$ and $\binom{k+n-2}{n-1}$ monomials of these kinds respectively, the number l of the sought-for polynomials is

$$l=\binom{k+n-1}{n-1}+\binom{k+n-2}{n-1}$$

309. $k=\frac{n-2}{2}$. **311.** $\sum_{i=1}^{n}m_i^2=m_{n+1}^2$.

312. For a solution u of equation (5) dependent solely on r and t the equation reduces to the form

$$\frac{1}{r}\left[\frac{\partial^2(ru)}{\partial^2 r}-\frac{\partial^2(ru)}{\partial t^2}\right]=0$$

(see Problem 132 d). From the last equation we find

$$ru(r, t)=f_1(r+t)+f_2(r-t)$$

(see Problem 259).

314. Using the expression of Laplace's operator $\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ in spherical coordinates we obtain the function

$$u(x_1, x_2, x_3, t) = \square [\varphi(r+t) + \psi(r-t)] =$$

$$= \frac{1}{r} \left\{ \frac{\partial^2}{\partial r^2} [r\varphi(r+t) + r\psi(r-t)] - \right.$$

$$\left. - \frac{\partial^2}{\partial t^2} [r\varphi(r+t) + r\psi(r-t)] \right\} = 2r^{-1} [\varphi'(r+t) + \psi'(r-t)]$$

which satisfies equation (5) (see Problem 312).

315. Using the representation for the solutions of equation (5) of the form $u(r, t)$ found in Problem 312 we obtain

$$u(r, t) = \frac{(r+t)\varphi(r+t) + (r-t)\varphi(r-t)}{2r} + \frac{1}{2r} \int_{r-t}^{r+t} \tau\psi(\tau)\, d\tau$$

When solving Problems 316-321, it is advisable to use formula (7) in which one should put $\tau(x) = u(x, 0)$ and $\nu(x) = u_t(x, 0)$.

316. $u = x_1x_2x_3 + x_1^2x_2^2x_3^2 t + \frac{1}{3}(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) t^3 +$

$$+ \frac{1}{15}(x_1^2 + x_2^2 + x_3^2) t^5 + \frac{1}{105} t^7.$$

317. $u = x_1^2 + x_2^2 + x_3^2 + 3t^2 + x_1x_2t$.

318. $u = e^{x_1}\cos x_2 + t(x_1^2 - x_2^2)$. **319.** $u = x_1^2 + x_2^2 + t + 2t^2$.

320. $u = e^{x_1}\cosh t + e^{-x_1}\sinh t$. **321.** $u = \frac{x_1}{x_1^2 - t^2}$.

322. It is sufficient to show that the solution $u(x, t)$ of the homogeneous problem

$$u_{x_1x_1} + u_{x_2x_2} - u_{tt} = 0, \quad u(x, 0) = u_t(x, 0) = 0$$

is identically equal to zero.

Indeed, let (x_1^0, x_2^0, t^0) $(t^0 > 0)$ be an arbitrary point and let K be the cone $\sqrt{(x_1 - x_1^0)^2 + (x_2 - x_2^0)^2} = t_0 - t$. By D we shall denote the domain in the space of the variables x_1, x_2, t bounded by the cone K and by the plane $t = 0$. Next, using the Gauss-Ostrogradsky formula and the equalities $u(x_1, x_2, 0) = ut(x_1, x_2, 0) = 0$, we integrate over the domain D the obvious identity

$$(u_{x_1}^2)_t + (u_{x_2}^2)_t + (u_t^2)_t - 2(u_t u_{x_1})_{x_1} - 2(u_t u_{x_2})_{x_2} = 0$$

This yields

$$\int_K \frac{1}{\nu_3} [(u_{x_1}\nu_3 - u_t\nu_1)^2 + (u_{x_2}\nu_3 - u_t\nu_2)^2]\, ds = 0$$

where $\nu_3 = 1/\sqrt{2}$, $\nu_1^2 + \nu_2^2 = \nu_3^2$. Consequently, the expressions $u_{x_1}\nu_3 - u_t\nu_1$ and $u_{x_2}\nu_3 - u_t\nu_2$ (which are proportional to directional derivatives of u along the inner normal to the cone K) are equal to zero on K, which means that $u(x_1, x_2, t) = 0$ on K, that is $u = 0$ at all the points belonging to the cone K. Since the point (x_1^0, x_2^0, t) is quite arbitrary, we conclude that $u(x_1, x_2, t) = 0$ everywhere in the domain of propagation of the wave.

323. $|m|(m_1^2 + m_2^2 + m_3^2)^{-1/2}$.

324. It cannot (because the function under consideration does not satisfy the wave equation).

325. The velocity of the wave is equal to $a/\sqrt{3}$.

326. The domain of propagation is the parallelogram bounded by the straight lines

$$x - 5t = l_1, \quad x + 5t = l_2, \quad x + 5t = l_1$$
$$\text{and } x - 5t = l_2$$

327. The domain of propagation is the torus obtained by the rotation about the t-axis of the square bounded by the straight lines $x_1 - t = 1$, $x_1 + t = 1$, $x_1 - t = 2$ and $x_1 + t = 2$ lying in the x_1t-plane.

328. The domain of propagation is bounded by the two cones

$$\sqrt{\sum_{i=1}^{3} x_i^2} = \frac{1}{2}(1-t), \quad 0 \leqslant t \leqslant 1, \quad \text{and} \quad \sqrt{\sum_{i=1}^{3} x_i^2} = \frac{1}{2}(1+t), \quad -1 \leqslant t \leqslant 0$$

329. The common domain of influence consists of the two domains bounded by the straight lines

$$x - t = -1, \quad x + t = 1, \quad t > 1$$

and

$$x - t = 1, \quad x + t = -1, \quad t < -1$$

respectively.

330. Since the data

$$u(x, x) = f_1(x), \quad \frac{\partial u}{\partial \nu} = \varphi_1(x)$$

are prescribed on the straight line $x - t = 0$ we conclude, by virtue of (10), that $f(2x) + \varphi(0) = f_1(x)$ and $\sqrt{2}\, f'(2x) = \varphi_1(x)$, whence $f(x) = f_1(x/2) - \varphi(0)$ and $f'(x) = \frac{1}{\sqrt{2}} \varphi_1(x/2)$. Consequently, the problem is solvable only when the condition

$$f_1'(x) = \sqrt{2}\, \varphi_1(x)$$

holds. Under this condition the solution of the problem is given by the formula

$$u(x, t) = f_1\left(\frac{x+t}{2}\right) - \varphi(0) + \varphi(x - t)$$

where φ is an arbitrary twice continuously differentiable function; hence the solution is not unique.

331. The Cauchy data can be prescribed on the straight lines $t = x/k$ only when $|k| \neq 1$.

(a) Let us assume that $k > 0$ ($k \neq 1$) and $\nu = (1/\sqrt{2}, 1/\sqrt{2})$, and that the data are prescribed on the line segment AB of the straight line $t = x/k$ where $A = A(0, 0)$ and $B = B(1, 1/k)$; let

$$u|_{AB} = f_1(x), \quad \left.\frac{\partial u}{\partial \nu}\right|_{AB} = \varphi_1(x), \quad 0 \leqslant x \leqslant 1$$

Then from formula (10) we obtain

$$f\left(\frac{k+1}{k} x\right) + \varphi\left(\frac{k-1}{k} x\right) = f_1(x)$$

and

$$\sqrt{2}\, f'\left(\frac{k+1}{k} x\right) = \varphi_1(x), \quad 0 \leqslant x \leqslant 1$$

Consequently,

$$f(x)=\frac{1}{\sqrt{2}}\int_0^x \varphi_1\left(\frac{k}{k+1}\tau\right)d\tau+f(0)$$

and

$$\varphi(x)=f_1\left(\frac{k}{k-1}x\right)-\frac{k+1}{k\sqrt{2}}\int_0^{\frac{k}{k-1}x}\varphi_1(\tau)\,d\tau-f(0)$$

and hence the sought-for solution can be written in the form

$$u(x,\ t)=f_1\left[\frac{k}{k-1}(x-t)\right]+\frac{k+1}{k\sqrt{2}}\int_{\frac{k}{k-1}(x-t)}^{\frac{k}{k+1}(x+t)}\varphi_1(\tau)\,d\tau$$

(b) let $C=C\left[\frac{k}{k-1}(x-t),\ \frac{1}{k-1}(x-t)\right]$ and $D=D\left[\frac{k}{k+1}(x+t),\ \frac{1}{k+1}(x+t)\right]$, for a point $(x,\ t)$ the domain of dependence is the intersection of the line segments AB and CD with the straight line $x=kt$. The domains of influence are bounded by the straight lines $x+t=0$, $x=kt$, $x-1=t-1/k$ and $x-t=0$, $x=kt$, $x-1=1/k-t$ respectively. The domain of propagation is the rectangle bounded by the straight lines $x-t=0$, $x-1=1/k-t$, $x+t=0$ and $x-1=t-1/k$.

(c) the stability of the solution follows directly from the formula expressing that solution.

332. The data can be prescribed on any arc S of the circle in question located within its arcs with end points

$A(1/\sqrt{2},\ 1/\sqrt{2})$ and $B(-1/\sqrt{2},\ 1/\sqrt{2})$

$B(-1/\sqrt{2},\ 1/\sqrt{2})$ and $C(-1/\sqrt{2},\ -1/\sqrt{2})$

$C(-1/\sqrt{2},\ -1/\sqrt{2})$ and $D(1/\sqrt{2},\ -1/\sqrt{2})$

$D(1/\sqrt{2},\ -1/\sqrt{2})$ and $A(1/\sqrt{2},\ 1/\sqrt{2})$

Let Q and Q' be the points of intersection of the arc S with the characteristics L_1: $\xi - x = t - \tau$ and L_2: $\xi - x = \tau - t$ of equation (3) issued from the point $P(x, y)$. Integrating the identity $(u_\xi)_\xi - (u_\tau)_\tau = 0$ over the domain bounded by the segments PQ and $Q'P$ of the characteristics L_1 and L_2 and by the part QQ' of the arc S and applying the Gauss-Ostrogradsky formula we find

$$u(P) = \frac{1}{2} F(Q) + \frac{1}{2} F(Q') + \\ + \frac{1}{2} \int_Q^{Q'} [\cos 2\theta \Phi(\xi, \eta) - \sin 2\theta F'_\theta(\xi, \eta)]\, d\theta$$

where $\xi = \cos\theta$ and $\eta = \sin\theta$.

333. The solution is

$$u(P) = \frac{1}{2} F(Q) + \frac{1}{2} F(Q') + \\ + \frac{1}{2} \int_Q^{Q'} [(\tau_s^2 - \xi_s^2)\Phi(\xi, \tau) - (\tau_s \tau_N - \xi_s \xi_N) F'_s] \frac{ds}{\tau_s \xi_N - \xi_s \tau_N}$$

where Q and Q' are the points of intersection of the characteristics $\xi - x = t - \tau$ and $\xi - x = \tau - t$ issued from the point $P(x, t)$ with the arc S of the curve $\xi = f(\tau)$ (see Problem 332).

334. The domain of propagation of the wave is the rectangle bounded by the characteristics $x - x_0 = t - t_0$, $x - x_0 = t_0 - t$, $x - x_1 = t - t_1$ and $x - x_1 = t_1 - t$. The uniqueness can be proved by means of the usual argument based on the integration of the identity $(u_\xi^2)_\tau + (u_\tau^2)_\tau - 2(u_\tau u_\xi)_\xi = 0$ over the domain bounded by the straight lines $\xi - x = \tau - t$ and $\xi - x = t - \tau$ and by the arc S.

335. $a^2 + b^2 - c^2 < 0$; $u(x_1, x_2, t) = t$.

336. $u(x, t) = \varphi\left(\frac{x+t}{2}\right) + \psi\left(\frac{x-t}{2}\right) - \varphi(0)$.

337. The domain of propagation of the wave is bounded by the straight lines $x - t = 0$, $x + t = 0$, $x - a = a - t$ and $x - b = t + b$.

338. Integrating the identity

$$(u_{x_1}^2)_t + (u_{x_2}^2)_t + (u_t^2)_t - 2(u_{x_1} u_t)_{x_1} - 2(u_{x_2} u_t)_{x_2} = 0$$

over the domain bounded by the cone $\sqrt{x_1^2 + x_2^2} = 1 - t$ and by the plane $t = h$, $h < 1$, where h is an arbitrary constant we conclude that $u = u_t = 0$ for $t = h$. Now, by virtue of the uniqueness of the solution of the Cauchy problem (see Problem 322), we conclude that the assertion stated in the problem is true.

339. By virtue of the formula established in Problem 303, we have

$$u(x, y, t) = \frac{1}{18}(x^2 + y^2 - t^2)\,xyt$$

340. The domain of propagation of the wave is bounded by the cones

$$t = -\sqrt{x^2 + y^2}, \quad -h \leqslant t \leqslant 0$$

and

$$t = -2h + \sqrt{x^2 + y^2}, \quad -2h \leqslant t \leqslant -h$$

The proof of the uniqueness of the solution can be carried out by means of the same argument as the one used in the solution of Problem 322 with the corresponding replacement of the initial conditions by the conditions stated in the problem in question.

341. No, it is not, because the solution of the Goursat problem with the data prescribed on the adjoining sides of the characteristic rectangle is determined uniquely (see Problem 336 or 337).

342. No, it is not, because the corresponding homogeneous problem possesses non-trivial solutions of the form

$$u(x, t) = \begin{cases} \omega\left(\frac{x+t}{2}\right) - \omega\left(\frac{x-t}{2}\right) & \text{for} \quad x - t \geqslant 0 \\ \omega\left(\frac{x+t}{2}\right) - \omega\left(\frac{t-x}{2}\right) & \text{for} \quad x - t \leqslant 0 \end{cases}$$

where ω is an arbitrary twice continuously differentiable function satisfying the conditions $\omega'(0) = \omega''(0) = 0$.

343. The problem is not well-posed because the corresponding homogeneous problem possesses non-trivial solutions of the form

$$u(x, t)=\begin{cases}\omega\left(\frac{x+t}{2}\right)-\omega\left[\frac{3}{2}(x-t)\right] & \text{for} \quad \frac{x}{2}\leqslant t\leqslant x\\ \omega\left(\frac{x+t}{2}\right)-\omega\left(\frac{t-x}{2}\right) & \text{for} \quad t\geqslant x\end{cases}$$

where ω is an arbitrary twice continuously differentiable function satisfying the conditions $\omega'(0)=\omega''(0)=0$.

344. The general solution of equation (3) has the form

$$u(x, t)=f_1(x+t)+f_2(x-t)$$

(see formula (10)). Using the data prescribed on the boundary of the domain D we find

$$u(x, 0)=f_1(x)+f_2(x)=\varphi(x)$$

$$u(x, kx)=f_1(x+kx)+f_2(x-kx)=\psi(x) \qquad (*)$$

On eliminating f_1 from the last two equations we obtain a functional equation of form (14):

$$f_2(x)-f_2\left(\frac{1-k}{1+k}x\right)=\varphi(x)-\psi\left(\frac{x}{1+k}\right)$$

Using formula (15) we can write the solution of this equation in the form

$$f_2(x)=\sum_{m=0}^{\infty}\left\{\varphi(\alpha^m x)-\psi\left(\frac{\alpha^m}{1+k}x\right)\right\}$$

where $\alpha=(1-k)/(1+k)$. The substitution of the expression of f_2 we have found into the first equality (*) results in

$$f_1(x)=\varphi(x)-\sum_{m=0}^{\infty}\left\{\varphi(\alpha^m x)-\psi\left(\frac{\alpha^m}{1+k}x\right)\right\}$$

Consequently

$$u(x, t)=\varphi(x+t)-\sum_{m=0}^{\infty}\left\{\varphi[\alpha^m(x+t)]-\psi\left[\frac{\alpha^m}{1+k}(x+t)\right]\right\}+$$

$$+\sum_{m=0}^{\infty}\left\{\varphi[\alpha^m(x-t)]-\psi\left[\frac{\alpha^m}{1+k}(x-t)\right]\right\}$$

345. From general solution (10) of equation (3) written in the form

$$u(x, t) = f_1(x+t) + f_2(x-t)$$

we obtain

$$f_1\left(\frac{3}{4}x\right) + f_2\left(\frac{5}{4}x\right) = x \text{ and } f_1\left(\frac{5}{4}x\right) + f_2\left(\frac{3}{4}x\right) = x$$

and therefore $f_1(x) = x/2$ and $f_2(x) = x/2$, whence we obtain the formula

$$u(x, t) = \frac{1}{2}(x+t) + \frac{1}{2}(x-t) = x$$

The uniqueness of the solution follows from the same formula.

346. The domain of propagation of the wave is bounded by the straight lines $x = -a/4$, $x = a/4$, $x - t = 5a/4$ and $x + t = 5a/4$.

347. $$u(x, t) = \sin(x+t) - \sum_{m=0}^{\infty} \sin\left(\frac{3}{5}\right)^m (x+t) + \sum_{m=0}^{\infty} \sin\left(\frac{3}{5}\right)^m (x-t) + 4t.$$

348. The domain of propagation of the wave is bounded by the straight lines $t = 0$, $t = x/4$, $x - t = 1$ and $x + t = 5/4$.

349. The solution is

$$u(x, t) = \varphi(x-t) + \psi\left(\frac{x+t}{2}\right) - \psi\left(\frac{x-t}{2}\right)$$

The domain of propagation of the wave is bounded by the straight lines $t = 0$, $x - t = 0$, $x - t = a$ and $x + t = 2a$.

350. The solution is

$$u(x,\ t)=\varphi(x+t)-$$
$$-\sum_{m=0}^{\infty}\left\{\varphi\left[\frac{1}{3^m}(x+t)\right]-\psi\left[\frac{2}{3^{m+1}}(x+t)\right]\right\}+$$
$$+\sum_{m=0}^{\infty}\left\{\varphi\left[\frac{1}{3^m}(x-t)\right]-\psi\left[\frac{2}{3^{m+1}}(x-t)\right]\right\}$$

The domain of propagation of the wave is bounded by the straight lines $t = 0$, $t = x/2$, $x - t = 2/3$ and $x + t = 1$.

351. The solution is

$$u(x,\ t)=\frac{1}{8}(x+t)^3+(x-t)^2+\frac{1}{8}(x-t)^3$$

The domain of propagation of the wave is bounded by the straight lines $x = 0$, $x - t = 0$, $t - x = 2$ and $x + t = 4$.

352. The solution is $u(x,\ t) = \sin(t - x)$. The domain of propagation of the wave is bounded by the straight lines $x = 0$, $x = t$, $t - x = 1$ and $t + x = 4$.

353. The solution is

$$u(x,\ t)=\varphi(x+t)-\sum_{k=0}^{\infty}\{\varphi[\theta^k(x+t)]-$$
$$-\varphi[\theta^k(x-t)]-\psi[\omega(\theta^k(x+t))]+\psi[\omega(\theta^k(x-t))]\}$$

where $x=\omega(\xi)$ is the solution of the equation $x+\tau(x) = \xi$, $\theta(\xi) = \omega(\xi) - \tau[\omega(\xi)]$ and $\theta^k(x) = \theta^{k-1}(x)\,\theta(x)$, $\theta^0(\xi) = \xi$. The domain of propagation of the wave is bounded by the straight lines $t = 0$, $t = \tau(x)$, $x + t = 1 + \tau(1)$ and $x - t = 1$.

354. The sought-for wave is $u(x,\ t) = x$. The domain of propagation of the wave is bounded by the lines $t = \sin x$, $t = -\sin x$, $x - t = \pi/4 + 1/\sqrt{2}$ and $x + t = \pi/4 + 1/\sqrt{2}$.

355. The solution is

$$u(x,\ t)=8\sum_{k=0}^{\infty}\{[1-\sqrt{1+\theta^k(x+t)}]^3-$$
$$-[1-\sqrt{1+\theta^k(x-t)}]^3\}$$

where $\theta(\xi) = 4(1+\xi) - \xi - 4$, $\theta^0(\xi) = \xi$ and $\theta^k = \theta^{k-1}\theta$. The domain of propagation of the wave is bounded by the parabola $t = x^2/4$ and by the straight lines $t = 0$, $x + t = 2$ and $x - t = 3/4$.

356. From general solution (10) of equation (3) written in the form $u(x, t) = f_1(x+t) + f_2(x-t)$ we find $f_1(2x) = \varphi(x) - f_2(0)$ and $2f_2'(2x) = \psi(x)$, whence

$$f_1(x) = \varphi\left(\frac{x}{2}\right) - f_2(0) \text{ and } f_2(x) = \frac{1}{2}\int_0^x \psi\left(\frac{\tau}{2}\right) d\tau + f_2(0)$$

Consequently, $u(x, t) = \varphi\left(\frac{x+t}{2}\right) + \frac{1}{2}\int\limits_0^{x-t} \psi\left(\frac{\tau}{2}\right) d\tau$. The uniqueness of the solution follows from the formula we have derived.

357. The solution is

$$u(x, t) = \psi\left(\frac{x+t}{2}\right) + \psi\left(\frac{x-t}{2}\right) - \psi(0) - \int_0^{x-t} \varphi(\tau)\, d\tau$$

The domain of propagation of the wave is bounded by the straight lines $t = x$, $t = 0$, $x - t = a$ and $x + t = 2b$.

358. The problem is not well-posed. Indeed, the problem is solvable only when the condition $\varphi'(x) = \psi(x)$ is fulfilled. Under this condition the solution of the problem has the form

$$u(x, t) = \varphi\left(\frac{x+t}{2}\right) - f(x-t)$$

where f is an arbitrary twice continuously differentiable function satisfying the relation $f(0) = \varphi(0)$.

359. If we pass to the characteristic variables $\xi = x + t$, $\eta = x - t$, equation (3) takes the form $v_{\xi\eta} = 0$ where $v(\xi, \eta) = u[(\xi+\eta)/2, (\xi-\eta)/2]$. For the last equation the Riemann function $R(\xi, \eta; \xi_1, \eta_1)$ is uniquely determined by the conditions

$$R(\xi_1, \eta_1; \xi_1, \eta_1) = 1$$

and

$$\frac{\partial R(\xi_1, \eta; \xi_1, \eta_1)}{\partial \eta} = \frac{\partial R(\xi, \eta_1; \xi_1, \eta_1)}{\partial \xi} = \frac{\partial^2 R}{\partial \xi\, \partial \eta} = 0$$

It is evident that the function $R(\xi, \eta; \xi_1, \eta_1) \equiv 1$ satisfies these conditions.

360. The conditions of the Cauchy problem with the given data prescribed on the arc σ can be written in the form

$$v(\xi, \eta)|_\sigma = \varphi(P'), \quad \frac{\partial v}{\partial N}\bigg|_\sigma = \frac{\partial \xi}{\partial \nu}\frac{\partial v}{\partial \eta} + \frac{\partial \eta}{\partial \nu}\frac{\partial v}{\partial \xi} = \psi(P')$$

and, consequently, by virtue of formula (19), we obtain

$$v(\xi, \eta) = \frac{1}{2}\varphi(Q) + \frac{1}{2}\varphi(Q') - \frac{1}{2}\int_{QQ'} \psi(P')\, ds_{P'}$$

where Q and Q' are the points of intersection of the straight lines $\xi_1 = \xi$ and $\eta_1 = \eta$ with the arc σ: $\xi_1 = \xi_1(s)$, $\eta_1 = \eta_1(s)$.

As to the Goursat problem, its conditions, for instance, with the data $u(x, x) = \varphi(x)$, $u(x, -x) = \psi(x)$, $\varphi(0) = \psi(0)$, can be written for the function v in the form

$$v(\xi, 0) = u\left(\frac{\xi}{2}, \frac{\xi}{2}\right) = \varphi\left(\frac{\xi}{2}\right), \quad v(0, \eta) = u\left(\frac{\eta}{2}, -\frac{\eta}{2}\right) = \psi\left(\frac{\eta}{2}\right)$$

Therefore from formula (18) we obtain

$$u(x, t) = v(\xi, \eta) = \varphi\left(\frac{\xi}{2}\right) + \psi\left(\frac{\eta}{2}\right) - \varphi(0) = \varphi\left(\frac{x+t}{2}\right) + \psi\left(\frac{x-t}{2}\right) - \varphi(0)$$

361. Let us transform the given equation to the form

$$v_{\xi\eta} + \frac{\lambda}{4} v = 0$$

where $v(\xi, \eta) = u\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2}\right)$ and $\xi = x + t$ and $\eta = x - t$ are the characteristic variables. Further, we write Bessel's function $J_0\left(\mu\sqrt{(\xi-\xi_1)(\eta-\eta_1)}\right)$ in the form of the sum of the power series:

$$J_0\left(\mu\sqrt{(\xi-\xi_1)(\eta-\eta_1)}\right) = \sum_{k=0}^{\infty}\left(\frac{\lambda}{4}\right)^k \frac{(\xi-\xi_1)^k(\eta-\eta_1)^k}{(k!)^2}$$

From this series we find

$$\frac{\partial^2 J_0}{\partial\xi\,\partial\eta} = -\frac{\lambda}{4}\sum_{k=0}^{\infty}\left(\frac{\lambda}{4}\right)^k \frac{(\xi-\xi_1)^k(\eta-\eta_1)^k}{(k!)^2} = -\frac{\lambda}{4}J_0$$

whence it follows that $\frac{\partial^2 J_0}{\partial\xi\,\partial\eta}+\frac{\lambda}{4}J_0=0$. Besides,

$$\frac{\partial J_0(\xi_1,\ \eta;\ \xi_1,\ \eta_1)}{\partial\eta}=0, \qquad \frac{\partial J_0(\xi,\ \eta_1;\ \xi_1,\ \eta_1)}{\partial\xi}=0$$

and $J_0(\xi_1,\ \eta_1;\ \xi_1,\eta_1)=1$. Consequently, the function $J_0\left(\mu\sqrt{(\xi-\xi_1)(\eta-\eta_1)}\right)$ satisfies all requirements imposed on the Riemann function and is specified by them uniquely.

362. The conditions of the Cauchy problem for the function u imply the corresponding conditions for the function v:

$$v(\xi,\ \xi)=\varphi(\xi), \quad \frac{\partial v}{\partial N}=\frac{1}{\sqrt{2}}\psi(\xi)$$

Besides, $ds_{P'}=\sqrt{2}\,d\xi_1$, $P=P(\xi,\ \eta)$, $Q=Q(\xi,\ \xi)$, $Q'=Q'(\eta,\ \eta)$, $v(\xi,\ \xi)=\varphi(x+t)$, $v(\eta,\ \eta)=\varphi(x-t)$, $R(Q,\ P)=1$ and $R(Q',\ P)=1$; therefore we obtain from (18) the expression

$$u(x,\ t)=v(\xi,\ \eta)=\frac{1}{2}\varphi(x+t)+\frac{1}{2}\varphi(x-t)+$$

$$+\frac{1}{2}\int_{x-t}^{x+t} J_0\left(\mu\sqrt{(x+t-\xi_1)(x-t-\xi_1)}\right)\psi(\xi_1)\,d\xi_1-$$

$$-\frac{1}{2}\int_{x-t}^{x+t}\left(\frac{\partial}{\partial\xi_1}+\frac{\partial}{\partial\eta_1}\right)J_0\left(\mu\sqrt{(\xi-\xi_1)(\eta-\eta_1)}\right)\Big|_{\eta_1=\xi_1}\varphi(\xi_1)\,d\xi_1$$

363. The conditions of the Goursat problem for the function $u(x,\ t)$ generate the corresponding conditions for the function $v(\xi,\ \eta)$:

$$v(\xi,\ \xi)=u\left(\frac{\xi}{2},\ \frac{\xi}{2}\right)=\varphi\left(\frac{\xi}{2}\right), \quad v(\eta,\ \eta)=$$

$$=u\left(\frac{\eta}{2},\ -\frac{\eta}{2}\right)=\psi\left(\frac{\eta}{2}\right)$$

Besides, $R(\xi, 0; \xi, \eta) = 1$, $R(0, \eta; \xi, \eta) = 1$ and $R(0, 0; \xi, \eta) = J_0(\mu\sqrt{x^2 - t^2})$. Therefore, by virtue of formula (18), the sought-for solution has the form

$$u(x, t) = \varphi\left(\frac{x+t}{2}\right) + \psi\left(\frac{x-t}{2}\right) - J_0(\mu\sqrt{x^2-t^2})\,\varphi(0) -$$

$$-\int_0^{x+t} \frac{\partial}{\partial\tau} J_0\left(\mu\sqrt{(\tau-x-t)(t-x)}\right)\varphi\left(\frac{\tau}{2}\right)d\tau -$$

$$-\int_0^{x-t} \frac{\partial}{\partial\tau} J_0\left(\mu\sqrt{(x+t)(x-t-\tau)}\right)\psi\left(\frac{\tau}{2}\right)d\tau$$

364. From formula (18) it follows that the sought-for solution is

$$u(x, t) = \frac{1}{4}\int_0^{\xi} d\xi_1 \int_0^{\eta} \sum_{k=0}^{\infty} (-1)^k \left(\frac{\lambda}{4}\right)^k \frac{(\xi-\xi_1)^k(\eta-\eta_1)^k}{(k!)^2}\,d\eta_1 =$$

$$= \frac{1}{4}\sum_{k=0}^{\infty} (-1)^k \left(\frac{\lambda}{4}\right)^k \frac{\xi^{k+1}\eta^{k+1}}{[(k+1)!]^2} = \frac{1}{4}\sum_{k=0}^{\infty} (-1)^k \left(\frac{\lambda}{4}\right)^k \frac{(x^2-t^2)^{k+1}}{[(k+1)!]^2}$$

365. *The solution of the Cauchy problem.* The transformation $u(x, t) = e^{-ax/2}v(x, t)$ of the sought-for function results in the following Cauchy problem for the equation of oscillation of a string with the unknown function $v(x, t)$:

$$v_{xx} - v_{tt} = 0,\ v(0, t) = \varphi(t),\quad v_x(0, t) = \frac{a}{2}\varphi(t) + \psi(t)$$

Using the general solution $v(x, t) = f_1(x+t) + f_2(x-t)$ of the equation $v_{xx} - v_{tt} = 0$ we conclude that

$$f_1(t) + f_2(-t) = \varphi(t),\ f_1'(t) + f_2'(-t) = \frac{a}{2}\varphi(t) + \psi(t)$$

whence

$$f_1(t) = \frac{1}{2}\varphi(t) + \frac{1}{2}\int_0^t \left[\frac{a}{2}\varphi(\tau) + \psi(\tau)\right]d\tau + C$$

and

$$f_2(-t)=\frac{1}{2}\varphi(t)-\frac{1}{2}\int_0^t\left[\frac{a}{2}\varphi(\tau)+\psi(\tau)\right]d\tau-C$$

where C is an arbitrary constant. Consequently,

$$u(x, t)=e^{-\frac{a}{2}x}v(x, t)=\frac{1}{2}e^{-\frac{a}{2}x}\left\{\varphi(x+t)+\varphi(t-x)+\right.$$
$$\left.+\int_{t-x}^{t+x}\left[\frac{a}{2}\varphi(\tau)+\psi(\tau)\right]d\tau\right\}$$

The solution of the Goursat problem. On performing the transformation $u(x, t)=e^{-ax/2}v(x, t)$ we arrive at the following Goursat problem for the function $v(x, t)$:

$$v_{xx}-v_{tt}=0,\quad v(x, x)=e^{\frac{a}{2}x}\varphi(x),\quad v(x, -x)=e^{\frac{a}{2}x}\psi(x)$$

Consequently,

$$u(x, t)=e^{-\frac{a}{2}x}v(x, t)=e^{-\frac{a}{2}x}\left\{e^{\frac{a}{4}(x+t)}\varphi\left(\frac{x+t}{2}\right)+\right.$$
$$\left.+e^{\frac{a}{4}(x-t)}\psi\left(\frac{x-t}{2}\right)-\varphi(0)\right\}$$

366 *The solution of the Cauchy problem:*

$$u(x, t)=\frac{1}{2}\left[e^{-\frac{b}{2}x}\varphi(x+t)+e^{\frac{b}{2}x}\varphi(t-x)\right]+$$
$$+\frac{1}{2}\int_{t-x}^{t+x}e^{\frac{b}{2}(t-\tau)}\psi(\tau)\,d\tau$$

The solution of the Goursat problem:

$$u(x, t)=e^{\frac{b}{2}t}\left[e^{-\frac{b}{4}(x+t)}\varphi\left(\frac{x+t}{2}\right)+\right.$$
$$\left.+e^{\frac{b}{4}(x-t)}\psi\left(\frac{x-t}{2}\right)-\varphi(0)\right]$$

367. *The solution of the Cauchy problem:*

$$u(x, t) = \frac{1}{2} e^{-\frac{a}{2}x + \frac{b}{2}t} \left\{ e^{-\frac{b}{2}(x+t)} \varphi(x+t) + \right.$$

$$\left. + e^{\frac{b}{2}(x+t)} \varphi(t-x) + \int\limits_{t-x}^{t+x} e^{-\frac{b}{2}\tau} \left[\frac{a}{2}\varphi(\tau) + \psi(\tau) \right] d\tau \right\}$$

The solution of the Goursat problem:

$$u(x, t) = e^{-\frac{a}{2}x + \frac{b}{2}t} \left[e^{\frac{(a-b)(x+t)}{4}} \varphi\left(\frac{x+t}{2}\right) + \right.$$

$$\left. + e^{\frac{(a+b)(x-t)}{4}} \psi\left(\frac{x-t}{2}\right) - \varphi(0) \right]$$

368. For the equation indicated in Problem 365 the solution is

$$u(x, t) = e^{-\frac{a}{4}(x-t)} \psi\left(\frac{x+t}{2}\right) +$$

$$+ e^{-\frac{a}{2}t} \varphi(x-t) - e^{-\frac{a}{4}(x+t)} \psi\left(\frac{x-t}{2}\right)$$

For the equation indicated in Problem 366 the solution is

$$u(x, t) = e^{-\frac{b}{4}(x-t)} \psi\left(\frac{x+t}{2}\right) + e^{\frac{b}{2}t} \varphi(x-t) -$$

$$- e^{-\frac{b}{4}(x-3t)} \psi\left(\frac{x-t}{2}\right)$$

For the equation indicated in Problem 367 the solution is

$$u(x, t) = e^{-\frac{a}{2}x + \frac{b}{2}t} \left[e^{\frac{(a-b)(x+t)}{4}} \psi\left(\frac{x+t}{2}\right) + \right.$$

$$\left. + e^{\frac{a(x-t)}{2}} \varphi(x-t) - e^{\frac{(a-b)(x-t)}{4}} \psi(x-t) \right]$$

369. The introduction of the new variables $\xi = x + y$ and $\eta = x - y$ reduces the given system to the form $u_\xi + u_\eta - v_\xi + v_\eta = 0$, $u_\xi - u_\eta - v_\xi - v_\eta = 0$, that is $(u-v)_\xi = 0$, $(u+v)_\eta = 0$. Therefore $u - v = 2f_1(\eta)$ and $u + v = 2f(\xi)$, whence

$$u(x, y) = f(x+y) + f_1(x-y),$$
$$v(x, y) = f(x+y) - f_1(x-y)$$

370. $u(x, y) = \frac{1}{2}[\varphi(x+y) + \psi(x+y) + \varphi(x-y) - \psi(x-y)]$

$$v(x, y) = \frac{1}{2}[\varphi(x+y) + \psi(x+y) - \varphi(x-y) + \psi(x-y)]$$

371. $u(x, y) = \varphi\left(\frac{x+y}{2}\right) - \psi\left(\frac{x-y}{2}\right) + \psi(0),$

$$v(x, y) = \varphi\left(\frac{x+y}{2}\right) + \psi\left(\frac{x-y}{2}\right) - \varphi(0)$$

372. $u(x, y) = \varphi(x+y) + \psi\left(\frac{x+y}{2}\right) - \psi\left(\frac{x-y}{2}\right)$

$$v(x, y) = \varphi(x+y) + \psi\left(\frac{x+y}{2}\right) + \psi\left(\frac{x-y}{2}\right) - \varphi(0) - \psi(0)$$

373. $u(x, y) = \psi\left(\frac{x+y}{2}\right) + \varphi(x-y) - \psi\left(\frac{x-y}{2}\right)$

$$v(x, y) = \psi\left(\frac{x+y}{2}\right) - \varphi(x-y) + \psi\left(\frac{x-y}{2}\right) + \varphi(0) - \psi(0)$$

374. The solution is

$$u(x, y) = f_1(x+y) + f_2(x-y)$$
$$v(x, y) = f_1(x+y) - f_2(x-y)$$

where $f_1(\tau) = \sum_{k=0}^{\infty}(-1)^k[\varphi(\tau/3^k) + \psi(2\tau/3^{k+1})]$ and $f_2(\tau) = \varphi(\tau) - f_1(\tau)$.

375. The characteristic determinant of the given system has the form

$$\begin{vmatrix} a\lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_1 \end{vmatrix} = a\lambda_1^2 - \lambda_2^2$$

that is

$$(u+v)_{\eta\eta}=0, \quad (u-v)_{\xi\xi}=0$$

It follows that

$$u+v=2\eta\varphi(\xi)+2\psi(\xi) \quad \text{and} \quad u-v=2\xi\varphi_1(\eta)+2\psi_1(\eta)$$

Therefore

$$u(x, y)=(x-y)\varphi(x+y)+(x+y)\varphi_1(x-y)+ \\ +\psi(x+y)+\psi_1(x-y)$$

$$v(x, y)=(x-y)\varphi(x+y)-(x+y)\varphi_1(x-y)+ \\ +\psi(x+y)-\psi_1(x-y)$$

384. $$u(x, y)=\frac{3(x-y)}{16(x+y)}\left\{\nu[2(x+y)]+\right. \\ \left.+\nu_1[2(x+y)]-\tau\left[\frac{2}{3}(x+y)\right]-\tau_1\left[\frac{2}{3}(x+y)\right]\right\}+ \\ +\frac{3(x+y)}{16(x-y)}\left\{\tau[2(x-y)]-\tau_1[2(x-y)]-\nu\left[\frac{2}{3}(x-y)\right]+\right. \\ \left.+\nu_1\left[\frac{2}{3}(x-y)\right]\right\}+\frac{1}{16}\left\{9\tau\left[\frac{2}{3}(x+y)\right]+\right. \\ \left.+9\tau_1\left[\frac{2}{3}(x+y)\right]-\nu[2(x+y)]-\nu_1[2(x+y)]\right\}+ \\ +\frac{1}{16}\left\{9\nu\left[\frac{2}{3}(x-y)\right]-9\nu_1\left[\frac{2}{3}(x-y)\right]-\right. \\ \left.-\tau[2(x-y)]+\tau_1[2(x-y)]\right\}$$

$$v(x, y)=\frac{3(x-y)}{16(x+y)}\left\{\nu[2(x+y)]+\nu_1[2(x+y)]-\right. \\ \left.-\tau\left[\frac{2}{3}(x+y)\right]-\tau_1\left[\frac{2}{3}(x+y)\right]\right\}- \\ -\frac{3(x+y)}{16(x-y)}\left\{\tau[2(x-y)]-\tau_1[2(x-y)]-\nu\left[\frac{2}{3}(x-y)\right]+\right. \\ \left.+\nu_1\left[\frac{2}{3}(x-y)\right]\right\}+\frac{1}{16}\left\{9\tau\left[\frac{2}{3}(x+y)\right]+\right. \\ \left.+9\tau_1\left[\frac{2}{3}(x+y)\right]-\nu[2(x+y)]-\nu_1[2(x+y)]\right\}- \\ -\frac{1}{16}\left\{9\nu\left[\frac{2}{3}(x-y)\right]-9\nu_1\left[\frac{2}{3}(x-y)\right]-\right. \\ \left.-\tau[2(x-y)]+\tau_1[2(x-y)]\right\}$$

385. $u(x, y) =$

$$= \frac{(x-y)}{4}[\tau_1'(x+y) + \tau_2'(x+y) - \nu_1(x+y) - \nu_2(x+y)] +$$

$$+ \frac{(x+y)}{4}[\tau_1'(x-y) - \tau_2'(x-y) + \nu_1(x-y) - \nu_2(x-y)] +$$

$$+ \frac{1}{2}[\tau_1(x+y) + \tau_2(x+y) + \tau_1(x-y) - \tau_2(x-y)] -$$

$$- \frac{(x+y)}{4}[\tau_1'(x+y) + \tau_2'(x+y) - \nu_1(x+y) - \nu_2(x+y)] -$$

$$- \frac{(x-y)}{4}[\tau_1'(x-y) - \tau_2'(x-y) + \nu_1(x-y) - \nu_2(x-y)]$$

$$v(x, y) = \frac{(x-y)}{4}[\tau_1'(x+y) + \tau_2'(x+y) -$$

$$- \nu_1(x+y) - \nu_2(x+y)] -$$

$$- \frac{(x+y)}{4}[\tau_1'(x-y) - \tau_2'(x-y) + \nu_1(x-y) - \nu_2(x-y)] +$$

$$+ \frac{1}{2}[\tau_1(x+y) + \tau_2(x+y) - \tau_1(x-y) + \tau_2(x-y)] -$$

$$- \frac{(x+y)}{4}[\tau_1'(x+y) + \tau_2'(x+y) - \nu_1(x+y) - \nu_2(x+y)] +$$

$$+ \frac{(x-y)}{4}[\tau_1'(x-y) - \tau_2'(x-y) + \nu_1(x-y) - \nu_2(x-y)]$$

386. If $a^2 - c^2$ and b do not turn into zero simultaneously the given system of partial differential equations is hyperbolic for any real values of a, b, c and k because in this case the roots of the characteristic determinant

$$\begin{vmatrix} a\lambda - b & kc\lambda \\ \frac{c}{k}\lambda & a\lambda - b \end{vmatrix} = (a^2 - c^2)\lambda^2 - 2ab\lambda + b^2$$

are real:

$$\lambda_1 = \frac{dy}{dx} = \frac{b}{a-c}, \quad \lambda_2 = \frac{dy}{dx} = \frac{b}{a+c}$$

After the transformation from the variables x and y to the characteristic variables

$$\xi = (a-c)y - bx, \quad \eta = (a+c)y - bx$$

has been performed, the given system assumes the form

$$u_\xi - u_\eta + kv_\xi + kv_\eta = 0, \quad u_\xi + u_\eta + kv_\xi - kv_\eta = 0$$

that is

$$(u + kv)_\xi = 0, \quad (u - kv)_\eta = 0$$

Consequently, $u + kv = 2f_1(\eta)$ and $u - kv = 2f(\xi)$. Therefore the general solution of the original system is expressed by the formulas

$$u(x, y) = f[(a-c)y - bx] + f_1[(a+c)y - bx]$$
$$v(x, y) = -\frac{1}{k} f[(a-c)y - bx] + \frac{1}{k} f_1[(a+c)y - bx]$$

387. The Cauchy data $u(x, 0) = \tau(x)$ and $v(x, 0) = \tau_1(x)$ can be prescribed on the straight line $y = 0$ for all real values of a, b, c and k on condition that $k \neq 0$ and $k \neq \infty$.

388. The solution can be constructed using formula (2), Chapter 2. It has the form of a sum of two (finite) series:

$$u(x, y) = \sum_{k\geqslant 0} (-1)^k \frac{y^{2k}}{(2k)!} p_n^{(2k)}(x) + \sum_{k\geqslant 0} (-1)^k \frac{y^{2k+1}}{(2k+1)!} q_m^{(2k)}(x)$$

Beginning with the values of k satisfying the conditions $2k > n$ and $2k > m$, respectively, the terms of the two series on the right-hand side of this formula become equal to zero.

389. The solution is given by the formula $u(x, y) = \frac{\sinh ny \cdot \sin nx}{n^2}$ which follows from formula (2), Chapter 2, on condition that

$$u(x, 0) = \tau(x) = 0, \quad \left.\frac{\partial u(x, y)}{\partial y}\right|_{y=0} = \nu(x) = \frac{\sin nx}{n}$$

The instability of this solution follows from the fact that for sufficiently large values of n the function $\nu(x)$ assumes arbitrarily small values whereas the function $u(x, y)$ is unbounded for $n \to \infty$.

390. By virtue of formula (10), any solution $u(x, t)$ of equation (3), which turns into zero on the characteristic $x + t = 0$, has the form $u(x, t) = f(x + t) - f(0)$. It follows that the value $u(x_1, t_1) = f(x_1 + t_1) - f(0)$, which the function $u(x, t)$ assumes at a point (x_1, t_1) belonging to the domain D (including the extremal value), is also assumed by this function at the point $(x_1 + t_1, 0)$ belonging to the line segment AB.

391. The transformation of variables $\xi = x + 2(-y)^{3/2}/3$, $\eta = x - 2(-y)^{3/2}/3$ (which is non-singular for $y < 0$) reduces the given equation to the form $u_{\xi\eta} = 0$, whence it follows that the general solution of the original equation is

$$u(x, y) = f_1\left[x + \frac{2}{3}(-y)^{3/2}\right] + f_2\left[x - \frac{2}{3}(-y)^{3/2}\right]$$

where $f_1(t)$ and $f_2(t)$ are arbitrary twice continuously differentiable functions. For the function $u(x, y)$ to satisfy the Cauchy initial conditions it is necessary and sufficient that the relations

$$f_1(x) + f_2(x) = \varphi(x)$$

and

$$\lim_{y \to -0} (-y)^{1/2}\left\{-f_1'\left[x + \frac{2}{3}(-y)^{3/2}\right] + f_2'\left[x - \frac{2}{3}(-y)^{3/2}\right]\right\} = \psi(x)$$

should be fulfilled for $0 < x < 1$. The second of these relations cannot hold when $\psi(x) \not\equiv 0$. If $\psi(x) \equiv 0$ $(0 < x < 1)$ the solution exists but is not unique because in this case

$$u(x, y) = \varphi\left[x + \frac{2}{3}(-y)^{3/2}\right] - f_2\left[x + \frac{2}{3}(-y)^{3/2}\right] + f_2\left[x - \frac{2}{3}(-y)^{3/2}\right]$$

where $f_2(t)$ is an arbitrary twice continuously differentiable function.

392. $$u(x, y) = \frac{1}{2}\tau\left[x + \frac{2}{3}(-y)^{3/2}\right] + \frac{1}{2}\tau\left[x - \frac{2}{3}(-y)^{3/2}\right] - \frac{1}{2}\int\limits_{x-\frac{2}{3}(-y)^{3/2}}^{x+\frac{2}{3}(-y)^{3/2}} \nu(\xi)\,d\xi.$$

393. The change of variables $\xi = x + 2y^{1/2}$, $\eta = x - 2y^{1/2}$ reduces the given equation to the form $u_{\xi\eta} = 0$; integrating the last equation we find

$$u(x, y) = f_1(x + 2y^{1/2}) + f_2(x - 2y^{1/2})$$

394. $$u(x, y) = \frac{1}{2}\tau(x + 2y^{1/2}) + \frac{1}{2}\tau(x - 2y^{1/2}).$$

395. $$u(x, y) = \frac{1}{2}\tau(x + 2y^{1/2}) + \frac{1}{2}\tau(x - 2y^{1/2}) + \frac{1}{2}\int\limits_{x-2y^{1/2}}^{x+2y^{1/2}} \nu(\xi)\,d\xi.$$

396. The change of variables $\xi = x + y$, $\eta = x - y$ reduces equation (21) to the equivalent equation

$$\frac{\partial^4 u}{\partial\xi^2\,\partial\eta^2} = 0$$

whose integration makes it possible to find the general solution of the original equation:

$$u(x, y) = \xi\varphi(\eta) + \varphi_1(\eta) + \eta\psi(\xi) + \psi_1(\xi) = (x + y)\varphi(x - y) + \varphi_1(x - y) + (x - y)\psi(x + y) + \psi_1(x + y)$$

397. Using the general solution of equation (21) (see Problem 396) we obtain the system of equalities

$$\left.\begin{aligned} x\varphi(x)+\varphi_1(x)+x\psi(x)+\psi_1(x)=\tau(x)\\ \varphi(x)-x\varphi'(x)-\varphi_1'(x)-\psi(x)+x\psi'(x)+\psi_1'(x)=0\\ -2\varphi'(x)-2\psi'(x)+x\varphi''(x)+x\psi''(x)+\\ +\varphi_1''(x)+\psi_1''(x)=0\\ 3\varphi''(x)-3\psi''(x)-x\varphi'''(x)+x\psi'''(x)-\\ -\varphi_1'''(x)+\psi_1'''(x)=0 \end{aligned}\right\}$$

On determining the functions $\varphi(x)$, $\psi(x)$, $\varphi_1(x)$ and $\psi_1(x)$ from this system we find

$$u(x, y)=\frac{1}{2}\tau(x+y)+\frac{1}{2}\tau(x-y)+$$
$$+\frac{1}{4}y\tau'(x-y)-\frac{1}{4}y\tau'(x+y)$$

398. $$u(x, y)=\frac{1}{2}(x+y)\tau_3\left(\frac{x-y}{2}\right)+$$
$$+\frac{1}{2}(x-y)\tau_4\left(\frac{x+y}{2}\right)+\tau_2\left(\frac{x-y}{2}\right)+\tau_1\left(\frac{x+y}{2}\right)-$$
$$-\frac{1}{4}(x^2-y^2)\tau_4'(0)-\frac{(x+y)}{2}\tau_2'(0)-\frac{(x-y)}{2}\tau_4(0)-\tau_2(0).$$

399. Let us write equation (22) in the form
$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\right)=0.$$
It follows that

$$u_{xx}-u_{yy}=-f_3''(y) \qquad (*)$$

where $f_3(y)$ is an arbitrary twice continuously differentiable function. Since $f_3(y)$ is a particular solution of equation (*) and since the expression $f_1(x+y)+f_2(x-y)$ where f_1 and f_2 are arbitrary functions possessing continuous partial derivatives up to the third order is, according to formula (10), the general solution of the homogeneous equation corresponding to (*), we conclude that

$$u(x, y)=f_1(x+y)+f_2(x-y)+f_3(y)$$

400. No, it is not, because using the formula expressing the general solution of equation (22) (see Problem 399) we obtain the system of equalities

$$\left.\begin{array}{l} f_1(x)+f_2(x)+f_3(0)=\varphi_1(x) \\ f_1'(x)-f_2'(x)+f_3'(0)=\varphi_2(x) \\ f_1''(x)+f_2''(x)+f_3''(0)=\varphi_3(x) \end{array}\right\}$$

This system shows that the problem in question cannot have a solution in case $\varphi_1''(x) \not\equiv \varphi_3(x) - f_3''(0)$. If the condition $\varphi_1''(x) \equiv \varphi_3(x) - f_3''(0)$ is fulfilled then

$$f_1(x) = \frac{1}{2}\varphi_1(x) + \frac{1}{2}\int_0^x \varphi_2(t)\,dt - \frac{1}{2}f_3'(0)\,x - \frac{1}{2}f_3(0) + C$$

and

$$f_2(x) = \frac{1}{2}\varphi_1(x) - \frac{1}{2}\int_0^x \varphi_2(t)\,dt + \frac{1}{2}f_3'(0)\,x - \frac{1}{2}f_3(0) - C$$

where C is an arbitrary constant. Consequently, the sought-for solution is given by the formula

$$u(x, y) = \frac{1}{2}\varphi_1(x+y) + \frac{1}{2}\varphi_1(x-y) + \\ + \frac{1}{2}\int_{x-y}^{x+y} \varphi_1(t)\,dt + f_3(y) - f_3'(0)\,y - f_3(0)$$

and is not unique.

401. The solution is

$$u(x, y) = \varphi_1(y) + \frac{1}{2}\int_{y-x}^{y+x} \varphi_2(t)\,dt + \frac{1}{2}\int_0^{x+y} d\tau \int_0^{\tau} \varphi_3(t)\,dt + \\ + \frac{1}{2}\int_0^{y-x} d\tau \int_0^{\tau} \varphi_3(t)\,dt - \int_0^{y} d\tau \int_0^{\tau} \varphi_3(t)\,dt$$

(see Problem 399).

Chapter 4

402. The function v satisfies the equation $a^2v_{\xi\xi} + 2av_{\xi\eta} + v_{\eta\eta} - v_{\xi} = 0$. To derive this equation put $n = 1$ and $x = x_1$ in equation (1) and then perform the transformation

$$x = \eta, \quad t = \xi - a\eta, \quad v(\xi, \eta) = u(\eta, \xi - a\eta)$$

403. If the conditions guaranteeing the uniform convergence of series (4) and of the series obtained from series (4) by differentiating it termwise once with respect to t and twice with respect to x are fulfilled, the sum $u(x, t)$ of series (4) satisfies the relation

$$\sum_{i=1}^{n} u_{x_i x_i} - u_t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta^{k+1}\tau - \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!} \Delta^{k}\tau =$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta^{k+1}\tau - \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta^{k+1}\tau = 0$$

404. For $t > t_0$ we have

$$\sum_{i=1}^{n} E_{x_i x_i} - E_t = -\frac{n}{2} \frac{E}{t-t_0} + \frac{E}{4(t-t_0)^2} \sum_{i=1}^{n} (x_i - y_i)^2 +$$

$$+ \frac{n}{2} \frac{E}{t-t_0} - \frac{E}{4(t-t_0)^2} \sum_{i=1}^{n} (x_i - y_i)^2 = 0$$

406. We shalll consider the case of a maximum. Let $M = \max u(x, t)$, $(x, t) \in D \cup \partial D$ and $m = \max u \times (x, t)$, $(x, t) \in S$ where $u(x, t)$ is a solution of equation (1) regular in D and continuous in $D \cup \partial D$. Let us suppose that $m < M$. Then the value M is attained by the function $u(x, t)$ at a point $(x_0, t_0) \in D$ where $0 < t_0 \leqslant T_1$, $M = u(x_0, t_0)$. Let us consider the function

$$v(x, t) = u(x, t) + \frac{M-m}{2nd^2} \sum_{i=}^{n} (x_i - x_{0i})^2$$

where d is the diameter of the domain D. Since $\sum_{i=1}^{n}(x_i - x_{0i})^2 \leqslant d^2$ and $m < M$, we obviously have

$$(1)\quad v(x, t) \leqslant m + \frac{M-m}{2n} = \left(1 - \frac{1}{2n}\right) m + \frac{M}{2n} < M \text{ for } (x, t) \in S,$$

and

(2) $v(x_0, t_0) = M$.

From (1) and (2) it follows that the function $v(x, t)$, like the function $u(x, t)$, attains its maximum value not on S but at a point $(x^*, t^*) \in D$ where $0 < t^* \leqslant T_1$. For that point we have $v_{x_ix_i} \leqslant 0$ and $v_t \geqslant 0$ ($v_t = 0$ if $t^* < T_1$ and $v_t \geqslant 0$ if $t^* = T_1$), and therefore at the point (x^*, t^*) the relation

$$\sum_{i=1}^{n} v_{x_ix_i} - v_t \leqslant 0 \qquad (*)$$

must be fulfilled.

On the other hand, taking into account the expression of $v(x, t)$, we find that

$$\sum_{i=1}^{n} v_{x_ix_i} - v_t = \sum_{i=1}^{n} u_{x_ix_i} + \frac{M-m}{d^2} - u_t = \frac{M-m}{d^2} > 0$$

at the point (x^*, t^*) which contradicts $(*)$. This contradiction implies the equality $m = M$, which is what we had to prove. The case of a minimum is considered analogously.

407. Let $u_1(x, t)$ and $u_2(x, t)$ be two solutions of Problem (1), (2) regular in D and continuous in $D \cup \partial D$. Then the function $u(x, t) = u_1(x, t) - u_2(x, t)$ is also a solution of equation (1) regular in D and continuous in $D \cup \partial D$; this solution satisfies the condition $u|_S = 0$. By the extremum principle, there must be $u(x, t) = 0$ throughout $D \cup \partial D$, that is

$$u_1(x, t) = u_2(x, t)$$

408. On putting $\tau(x_1, x_2) = \sin i\pi x_1/l_1 \sin j\pi x_2/l_2$ in formula (4), Problem 403, and taking into account that

$$\Delta^k \tau(x_1, x_2) = (-1)^k \left[\left(\frac{i\pi}{l_1} \right)^{2k} + \left(\frac{j\pi}{l_2} \right)^{2k} \right] \sin \frac{i\pi}{l_1} x_1 \sin \frac{j\pi}{l_2} x_2$$

we obtain the function

$$u(x_1, x_2, t) = \sin \frac{i\pi}{l_1} x_1 \sin \frac{j\pi}{l_2} x_2 \exp \left[-\pi^2 \left(\frac{i^2}{l_1^2} + \frac{j^2}{l_2^2} \right) t \right]$$

satisfying all the requirements of the problem in question.

409. The solution is $u(x, t) = \sum\limits_{k=1}^{\infty} \sin kx e^{-k^2 t}$.

To derive this formula let us suppose that the function $\varphi(x)$ is continuously differentiable in the interval $0 \leqslant x \leqslant \pi$; then it can be represented as the sum of its Fourier's series

$$\varphi(x) = \sum_{k=1}^{\infty} a_k \sin kx, \quad 0 \leqslant x \leqslant \pi$$

where

$$a_k = \frac{2}{\pi} \int_0^{\pi} \varphi(x) \sin kx \, dx$$

and the series is absolutely and uniformly convergent. Taking into account that the function $u_k(x, t) = \sin kx \, e^{-k^2 t}$ is the solution of equation (1′) in the rectangle $0 < x < \pi$, $0 < t < T_0$, $T_0 > 0$ (see Problem 408) satisfying the conditions $u(x, 0) = \sin kx$ and $u(0, t) = u(\pi, t) = 0$, we conclude that the solution of the problem under consideration is

$$u(x, t) = \sum_{k=1}^{\infty} \sin kx e^{-k^2 t}$$

Since

$$\lim_{k \to \infty} k^m e^{-k^2 t} = 0$$

in a neighbourhood of every point (x, t) of the rectangle $0 < x < \pi$, $0 < t < T_0$, the series whose sum is equal to $u(x, t)$ can be differentiated termwise any number of times.

410. The integral

$$u(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \varphi(y)\, dy, \quad t > 0 \qquad (*)$$

is convergent. Indeed, denoting $M = \max_{-\infty<y<\infty} |\varphi(y)|$ we can write the inequality

$$|u(x, t)| \leqslant \frac{M}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \frac{dy}{2\sqrt{t}} = \frac{M}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2}\, d\eta = M$$

It is also easy to show that the integrals obtained from integral (*) by differentiating it with respect to x and with respect to t under the integral sign any number of times are also convergent. Besides, all these integrals are uniformly convergent in a neighbourhood of any point (x, t) provided that $t > 0$. It follows that for $t > 0$ the function $u(x, t)$ possesses partial derivatives of all orders which can be computed using the formula

$$\frac{\partial^{m+n} u(x, t)}{\partial x^m \partial t^n} = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(y) \frac{\partial^{m+n}}{\partial x^m \partial t^n} \left[\frac{1}{\sqrt{t}} e^{-\frac{(x-y)^2}{4t}} \right] dy$$

To prove the condition

$$\lim_{t \to 0} u(x, t) = u(x, 0) = \varphi(x), \quad -\infty < x < \infty$$

it is sufficient to take into account that the integral on the right-hand side of (*) is uniformly convergent in a neighbourhood of any point (x, t) for $t > 0$. On making the change of the variable y using the formula $y = x + 2\eta\sqrt{t}$ we obtain

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \varphi(x + 2\eta\sqrt{t})\, e^{-\eta^2}\, d\eta$$

Since the last integral is uniformly convergent and the function φ is continuous, it follows that

$$\lim_{t \to 0} u(x, t) = \varphi(x) \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\eta^2}\, d\eta = \varphi(x)$$

411. Let $u(x, t)$ be a solution of equation (1′) continuous and bounded for $t \geqslant 0$. We shall present the proof of the inequality $u(x, t) \leqslant M$ (the proof of the inequality $u(x, t) \geqslant m$ can be reduced to the former by changing the sign of the function u). Let us fix an arbitrary number $\varepsilon > 0$. We shall show that $u(x_0, t_0) \leqslant M + \varepsilon$ for any point (x_0, t_0) belonging to the half-space $t \geqslant 0$. To this end let us consider the function $v(x, t) = x^2 + 2t$ which satisfies equation (1′). Let $N = \sup_{t \geqslant 0} |u(x, t)|$. The function $\varepsilon v(x, t)/v(x_0, t_0) + M - u(x, t)$ satisfying equation (1′) for $t > 0$ is nonnegative for $t = 0$ and for $|x| = [(N - M) v(x_0, t_0)/\varepsilon + |x_0|]^{1/2}$. According to the extremum principle for a bounded domain (see the answer to Problem 406), this function must be nonnegative throughout the rectangle $\{0 \leqslant t < T, \ |x| \leqslant [(N - M) \times v(x_0, t_0)/\varepsilon]^{1/2}\}$, to which the point (x_0, t_0) belongs. Consequently, $u(x, t) \leqslant M + \varepsilon v(x, t)/v(x_0, t_0)$ within that rectangle whence it follows that $u(x_0, t_0) \leqslant M + \varepsilon$. Finally, since the point (x_0, t_0) and the number ε are quite arbitrary, there must be $u(x, t) \leqslant M$ for $t \geqslant 0$.

412. To solve the problem apply the inequalities obtained in the solution of Problem 411 to the difference $u(x, t) = u_1(x, t) - u_2(x, t)$ of two solutions of Problem (1′), (3′).

414. The transformation $u(x, t) = v(x, t) + \alpha(t) + x[\beta(t) - \alpha(t)]$ of the sought-for function leads to the following problem:

$$v_{xx} - v_t = f(x, t) + \alpha'(t) + x[\beta'(t) - \alpha'(t)],$$
$$v(0, t) = 0, \quad v(1, t) = 0$$

415. $u(x, t) = \sin nx \int\limits_0^t e^{-n^2(t-\tau)} f_n(\tau)\, d\tau.$

416. The solution is $u(x, t) = e^{x_1} \cosh x_2 e^{2t}$ (it does not belong to the class of uniquely determined solutions).

417. The solution is

$$u(x, t) = \frac{1}{2a\sqrt{\pi t}} \int\limits_0^\infty \left[e^{-\frac{(x-\xi)^2}{4a^2t}} - e^{-\frac{(x+\xi)^2}{4a^2t}}\right] \varphi(\xi)\, d\xi$$

To derive this formula let us extend the function $\varphi(x)$ to the negative semi-axis $-\infty < x < 0$ in an odd manner; the extended function will be denoted as $\Phi(x)$:

$$\Phi(x)=\begin{cases} \varphi(x) & \text{for} \quad x>0 \\ -\varphi(-x) & \text{for} \quad x<0 \end{cases} \qquad (*)$$

Now let us consider the Cauchy-Dirichlet problem

$$U_t - a^2 U_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0;$$
$$U(x, 0) = \Phi(x), \quad -\infty < x < \infty$$

As is known, the solution of this problem is given by the formula

$$U(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4a^2t}} \Phi(\xi)\, d\xi \qquad (**)$$

It is evident that $U(x, 0) = \varphi(x)$, $0 \leqslant x < \infty$. Further from (**) and (*) we obtain

$$U(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_{0}^{\infty} e^{-\frac{(x-\xi)^2}{4a^2t}} \varphi(\xi)\, d\xi -$$
$$- \frac{1}{2a\sqrt{\pi t}} \int_{0}^{\infty} e^{-\frac{(x+\xi)^2}{4a^2t}} \varphi(\xi)\, d\xi$$

whence we find that $U(0, t) = 0$, and therefore $U(x, t) = u(x, t)$ for $x \geqslant 0$.

418. The solution is

$$u(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_{0}^{\infty} \left[e^{-\frac{(x-\xi)^2}{4a^2t}} + e^{-\frac{(x+\xi)^2}{4a^2t}}\right] \varphi(\xi)\, d\xi$$

To obtain this formula one should solve the following auxiliary problem:

$$U_t - a^2 U_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0;$$
$$U(x, 0) = \Phi(x), \quad -\infty < x < \infty$$

where

$$\Phi(x)=\begin{cases} \varphi(x) & \text{for} \quad x>0 \\ \varphi(-x) & \text{for} \quad x<0 \end{cases}$$

419. The solution is

$$u(x,\ t)=\frac{e^{-ht}}{2a\sqrt{\pi t}}\int_0^\infty [e^{-\frac{(x-\xi)^2}{4a^2t}}-e^{-\frac{(x+\xi)^2}{4a^2t}}]\,\varphi(\xi)\,d\xi$$

To solve the problem we make the transformation $u(x, t) = e^{-ht}v(x,\ t)$ of the sought-for function and arrive at Problem 417 for the new unknown function $v(x,\ t)$.

420. The solution is

$$u(x,\ t)=\frac{e^{-ht}}{2a\sqrt{\pi t}}\int_0^\infty [e^{-\frac{(x-\xi)^2}{4a^2t}}+e^{-\frac{(x+\xi)^2}{4a^2t}}]\,\varphi(\xi)\,d\xi$$

421. The solution is

$$u(x,\ t)=\frac{1}{2a\sqrt{\pi}}\int_0^t\int_0^\infty \frac{1}{\sqrt{t-\tau}}[e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}-e^{-\frac{(x+\xi)^2}{4a^2(t-\tau)}}]\,f(\xi,\ \tau)\,d\xi\,d\tau$$

To derive this formula we consider the following auxiliary problem:

$$U_t=a^2U_{xx}+F(x,\ t),\quad U(x,\ 0)=0,\qquad -\infty<x<\infty,\quad t>0$$

where

$$F(x,\ t)=\begin{cases} f(x,\ t) & \text{for}\quad x>0\\ -f(-x,\ t) & \text{for}\quad x<0\end{cases}\qquad (*)$$

The solution of the last problem is given by the formula

$$U(x,\ t)=\frac{1}{2a\sqrt{\pi}}\int_0^t\int_{-\infty}^\infty \frac{1}{\sqrt{t-\tau}}e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}F(\xi,\ \tau)\,d\xi\,d\tau\qquad (**)$$

By virtue of (*), this formula can be written in the form

$$U(x,\ t)=\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{1}{\sqrt{t-\tau}}\left\{\int_0^\infty e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}f(\xi,\ \tau)\,d\xi-\int_0^\infty e^{-\frac{(x+\xi)^2}{4a^2(t-\tau)}}f(\xi,\ \tau)\,d\xi\right\}d\tau$$

Consequently, $U_t = a^2 U_{xx} + f(x, t)$, $x > 0$, $t > 0$; $U(0, t) = 0$, $t \geqslant 0$; $U(x, 0) = 0$, $x \geqslant 0$, and therefore $U(x, t) = u(x, t)$ for $x \geqslant 0$, $t \geqslant 0$.

422. The solution is

$$u(x, t) = \frac{1}{2a\sqrt{\pi}} \int_0^t \int_0^\infty \frac{1}{\sqrt{t-\tau}} \left[e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} + e^{-\frac{(x+\xi)^2}{4a^2(t-\tau)}}\right] f(\xi, \tau)\, d\xi\, d\tau$$

423. The solution is

$$u(x, t) = \frac{1}{2a\sqrt{\pi}} \int_0^t \int_0^\infty \frac{e^{-h(t-\tau)}}{\sqrt{t-\tau}} \left[e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} - e^{-\frac{(x+\xi)^2}{4a^2(t-\tau)}}\right] f(\xi, \tau)\, d\xi\, d\tau$$

To solve the problem we make the transformation $u(x, t) = e^{-ht} v(x, t)$ of the unknown function then, for the new unknown function $v(x, t)$, we arrive at the problem considered in the solution of Problem 421.

424. The solution is

$$u(x, t) = \frac{1}{2a\sqrt{\pi}} \int_0^t \int_0^\infty \frac{e^{-h(t-\tau)}}{\sqrt{t-\tau}} \left[e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} + e^{-\frac{(x+\xi)^2}{4a^2(t-\tau)}}\right] f(\xi, \tau)\, d\xi\, d\tau$$

425. The solution is

$$u(x, t) = \frac{e^{-ht}}{2a\sqrt{\pi t}} \int_0^\infty \left[e^{-\frac{(x-\xi)^2}{4a^2 t}} - e^{-\frac{(x+\xi)^2}{4a^2 t}}\right] \varphi(\xi)\, d\xi + \frac{1}{2a\sqrt{\pi}} \int_0^t \int_0^\infty \frac{e^{-h(t-\tau)}}{\sqrt{t-\tau}} \left[e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} - e^{-\frac{(x+\xi)^2}{4a^2(t-\tau)}}\right] f(\xi, \tau)\, d\xi\, d\tau$$

426. The solution is

$$u(x, t) = \frac{e^{-ht}}{2a\sqrt{\pi t}} \int_0^\infty \left[e^{-\frac{(x-\xi)^2}{4a^2t}} + e^{-\frac{(x+\xi)^2}{4a^2t}}\right] \varphi(\xi)\, d\xi +$$

$$+ \frac{1}{2a\sqrt{\pi}} \int_0^t \int_0^\infty \frac{e^{-h(t-\tau)}}{\sqrt{t-\tau}} \left[e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}} + e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}\right] f(\xi, \tau)\, d\xi\, d\tau$$

To obtain the solutions of Problems 427-441 it is convenient to use formula (4) (see Problem 403) putting $\tau = u(x, 0)$ $(x = x_1, \ldots, x_n)$ in it.
427. $u = 1 - x^2 - y^2 - 4t$.
428. $u = 1 - (x^2 + y^2)^2 - 16(x^2 + y^2)t - 32t^2$.
429. $u = x^2 + y^2 + 4t$.
430. $u = e^{x+y+2t}$.
431. $u = I_0(r) e^t$.
432. $u = e^{-l^2t} \sin lx_1$.
433. $u = e^{-l^2t} \cos lx_1$.
434. $u = e^{l^2t} \cosh lx_1$.
435. $u = e^{l^2t} \sinh lx_1$.
436. $u = e^{-(l_1^2+l_2^2)t} \sin l_1x_1 \sin l_2x_2$.
437. $u = e^{-(l_1^2+l_2^2)t} \sin l_1x_1 \cos l_2x_2$.
438. $u = e^{-(l_1^2+l_n^2)t} \cos l_1x_1 \cos l_nx_n$.
439. $u = e^{-(l_1^2+l_2^2)t} \cos l_1x_1 \sin l_2x_2$.
440. $u = e^{-\sum\limits_{i=1}^{n} l_i^2 t} \sin l_1x_1 \sin l_2x_2 \ldots \sin l_nx_n$.
441. $u = e^{-l_1^2t} \sin l_1x_1 + e^{-l_n^2t} \cos l_nx_n$.
442. The solution is $u(x, y) = yf_1(ay - x) + f_2(ay - x)$ where f_1 and f_2 are arbitrary twice continuously differentiable functions.

To integrate the equation we make the change of variables $\xi = ay - x$, $\eta = y$ which reduces the given equation to the form $u_{\eta\eta} = 0$.

443. Using the variables x, y and $z = t/p$ we form the function $v(x, y, z) \equiv u(x, y, pz)$; for the function $v(x, y, z)$ the original equation implies the new equation $v_{xx} +$

$+ v_{yy} - v_z = 0$. Therefore formula (4′) is a direct consequence of formula (4) established in Problem 403.

444. The solution is

$$u(x, y, t) = 1 - (x^2 + y^2)^2 - 16(x^2 + y^2)(t-1) - 32(t-1)^2$$

To solve the problem we use the variables x, y, $z = t - 1$ and consider the function $v(x, y, z) \equiv u(x, y, z + 1)$; the original problem then implies the following problem for the new function $v(x, y, z)$: $v_{xx} + v_{yy} - v_z = 0$, $z > 0$; $v(x, y, 0) = u(x, y, 1) = 1 - (x^2 + y^2)^2$. Using formula (4) (see Problem 403) in which we put $\tau = u(x, y, 1)$, we obtain $v(x, y, z) = 1 - (x^2 + y^2)^2 - 16(x^2 + y^2)z - 32z^2$, whence the sought-for solution can readily be found.

445. The expression of the solution in quadratures is

$$u(x, t) = \frac{1}{2}\sqrt{\frac{p}{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{p(x-y)^2}{4t}} \varphi(y)\, dy$$

446. The expression of the solution in quadratures is

$$u(x, y) = \frac{1}{2\sqrt{\pi b(by-x)}} \int_{-\infty}^{\infty} e^{-\frac{(by-z)^2}{4b(by-x)}} \varphi(z)\, dz, \quad x < by$$

To solve the problem we use the change of variables $\xi = y$, $t = y - x/b$ and the transformation $u(x, y) = u(b\xi - bt, \xi) = v(\xi, t)$ of the unknown function, which reduces the original problem to the following problem:

$$v_{\xi\xi} - v_t = 0, \quad -\infty < \xi < \infty, \quad t > 0$$
$$v(\xi, 0) = u(b\xi, \xi) = \varphi(b\xi), \quad -\infty < \xi < \infty, \quad b > 0$$

447. The solution is

$$u(x, y) = \sum_{k=1}^{\infty} a_k e^{-\frac{k^2\pi^2(by-x)}{b}} \sin k\pi y$$

where

$$a_k = 2\int_0^1 \varphi(b\xi) \sin k\pi\xi\, d\xi; \qquad k = 1, 2, \ldots$$

To solve the problem we use the change of variables $\xi = y$, $t = y - x/b$ and the transformation $u(x, y) = u(b\xi - bt, \xi) = v(\xi, t)$ of the unknown function, which reduces the original problem to the new problem:

$$v_{\xi\xi} - v_t = 0, \quad 0 < \xi < 1, \quad 0 < t < 1$$
$$v(\xi, 0) = u(b\xi, \xi) = \varphi(b\xi), \quad 0 \leqslant \xi \leqslant 1$$
$$v(0, t) = u(-bt, 0) = 0; \quad v(1, t) = u(b - bt, 1) = 0, \quad 0 \leqslant t \leqslant 1$$

The solution of the last problem has the form

$$v(\xi, t) = \sum_{k=1}^{\infty} a_k \sin k\pi\xi \, e^{-k^2\pi^2 t}$$

where

$$a_k = 2\int_0^1 \varphi(b\xi) \sin k\pi\xi \, d\xi; \qquad k = 1, 2, \ldots$$

(see Problem 409).

448. The solution is $u(x, y) = e^{-\left(\frac{p}{2}x + y\right)} \sin x$.

To derive this formula make the transformation $u(x, y) = e^{-px/2} v(x, y)$ of the sought-for function, which simplifies the problem.

449. The expression of the solution in quadratures is

$$u(x, y) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\left[\frac{(x-\xi)^2}{4y} + \frac{p(x-\xi)}{2}\right]} \varphi(\xi) \, d\xi$$

For the integral on the right-hand side to exist it is sufficient that the function $\varphi(x)$ should be continuous and the expression $e^{px/2}\varphi(x)$ should be bounded for $-\infty < x < \infty$.

450. It is advisable to construct the solution $u(x, t)$ in the form $u(x, t) = e^{-\lambda t} v(x, t)$; using this transformation we obtain for $v(x, t)$ Helmholtz' equation $v_{xx} + v_{yy} + \lambda^2 v = 0$. According to the solution of Problem 227, the functions $J_k(\lambda r) \cos k\varphi$ and $J_k(\lambda r) \sin k\varphi$ satisfy the last equation.

451. $u(x, y, t) = e^{-\lambda t} J_0(\lambda r)$.

452. The equation is parabolic. Indeed, the characteristic form $K(\lambda_1, \ldots, \lambda_n, \lambda_{n+1}) = \left(\sum\limits_{i=1}^{n}\lambda_i^2\right)^2$ corresponding to equation (6) does not involve the parameter λ_{n+1}.

453. $u(x, t) = \sum\limits_{k=0}^{[n/4]} \frac{t^k}{k!} \Delta^{2k} P_n(x)$.

454. $u(x, t) = e^{-(l_1^2+l_n^2)^2 t} \sin l_1 x_1 \cos l_n x_n$.

455. The equation is parabolic (see the answer to Problem 452).

456. $u(x, t) = \sum\limits_{k=0}^{[n/4]} \frac{t^{2k}}{(2k)!} \Delta^{2k} P_n(x)$.

457. $u(x, t) = \sin x_1 \cosh t + \cos x_1 \sinh t$.

Chapter 5

458. The required class is described by the formula $u_\lambda(x, t) = v_\lambda(x) w_\lambda(t)$ where $v_\lambda(x)$ and $w_\lambda(t)$ are solutions of ordinary differential equations $v''(x) + \lambda v(x) = 0$ and $w''(t) + \lambda w(t) = 0$ respectively.

459. The problem possesses infinitely many solutions of the form

$$u_n(x, t) = \left(a_n \cos \frac{\pi n}{b-a} t + b_n \sin \frac{\pi n}{b-a} t\right) \sin \frac{\pi n}{b-a}(x-b);$$
$$n = 1, 2, \ldots,$$

where a_n and b_n are arbitrary real constants.

460. The solution is

$$u(x, t) = \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \sin nx$$

where

$$a_n = \frac{2}{\pi} \int_0^\pi \varphi(x) \sin nx \, dx \quad \text{and} \quad b_n = \frac{2}{\pi n} \int_0^\pi \psi(x) \sin nx \, dx$$

461. Yes, it does. To prove this assertion it is sufficient to show that for $\varphi(x) = \psi(x) = 0$, $0 \leqslant x \leqslant \pi$, the problem has only a trivial solution. As is known, the Cauchy problem $u_{xx} = u_{tt}$, $u(x, 0) = u_t(x, 0) = 0$, $0 \leqslant x \leqslant \pi$ for the triangle with vertices at the points $A(0, 0)$, $B(\pi, 0)$ and $C(\pi/2, \pi/2)$ has the single solution $u(x, t) \equiv 0$. The solution $u(x, t)$ of the problem under consideration is also equal to zero in the triangle with vertices at the points $A(0, 0)$, $C(\pi/2, \pi/2)$ and $D(0, \pi/2)$. Indeed, the integration of the obvious identity $-2(u_x u_t)_x + (u_x^2)_t + (u_t^2)_t = 0$ over a triangular domain with vertices at the points $A(0, 0)$, $C_\tau(\tau, \tau)$ and $D_\tau(0, \tau)$ for any fixed τ, $0 < \tau < \pi/2$, results in

$$\int_{C_\tau D_\tau} (u_x^2 + u_t^2)\, dx = 0$$

because $u(x, t) = 0$ on the line segments AC_τ and AD_τ. Consequently, $u_x = u_t = 0$ on $D_\tau C_\tau$, and therefore $u(x, t) = 0$ throughout the triangle ACD. It can similarly be proved that for the triangle BCD_1 where $D_1 = D_1(\pi, \pi/2)$ we also have $u(x, t) = 0$. Thus,

$$u(x, \pi/2) = u_t(x, \pi/2) = 0, \quad 0 \leqslant x \leqslant \pi$$

On repeating the above argument, we conclude that $u(x, t) = 0$ throughout the half-strip $0 \leqslant x \leqslant \pi$, $t \geqslant 0$.

462. The harmonics are

$$u_n(x, t) = \\ = [a_n \cos(n + 1/2)t + b_n \sin(n + 1/2)t] \sin(n + 1/2)x$$

where a_n and b_n are arbitrary constants and $n = 0, 1, \ldots$

463. $u(x, t) = \frac{l}{2\pi a} \sin \frac{2\pi a}{l} t \sin \frac{2\pi}{l} x$.

464. $u(x, t) = \sum_{k=1}^{\infty} \left(a_k \cos \frac{ak\pi}{l} t + b_k \sin \frac{ak\pi}{l} t\right) \sin \frac{k\pi}{l} x$,

where

$$a_k = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{k\pi}{l} x\, dx$$

and

$$b_k = \frac{2}{ak\pi} \int_0^l \psi(x) \sin \frac{k\pi}{l} x \, dx.$$

465. $u(x, t) = \frac{2l}{a\pi} \sin \frac{a\pi}{2l} t \sin \frac{\pi}{2l} x + \cos \frac{5a\pi}{2l} t \sin \frac{5\pi}{2l} x.$

466. $u(x, t) = \frac{2l}{a\pi} \sin \frac{a\pi}{2l} t \sin \frac{\pi}{2l} x +$

$$+ \frac{2l}{3a\pi} \sin \frac{3a\pi}{2l} t \sin \frac{3\pi}{2l} x +$$

$$+ \frac{8l}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \cos \frac{(2k+1)a\pi}{2l} t \sin \frac{(2k+1)\pi}{2l} x.$$

467. $u(x, t) = \cos \frac{a\pi}{2l} t \cos \frac{\pi}{2l} x + \frac{2l}{3a\pi} \sin \frac{3a\pi}{2l} t \cos \frac{3\pi}{2l} x +$

$$+ \frac{2l}{5a\pi} \sin \frac{5a\pi}{2l} t \cos \frac{5\pi}{2l} x.$$

468. $u(x, t) = \sum_{k=0}^{\infty} \left[a_k \cos \frac{(2k+1)a\pi}{2l} t + \right.$

$$\left. + b_k \sin \frac{(2k+1)a\pi}{2l} t \right] \cos \frac{(2k+1)\pi}{2l} x,$$

where

$$a_k = \frac{2}{l} \int_0^l \varphi(x) \cos \frac{(2k+1)\pi}{2l} x \, dx,$$

$$b_k = \frac{4}{(2k+1)a\pi} \int_0^l \psi(x) \cos \frac{(2k+1)\pi}{2l} x \, dx.$$

469. $u(x, t) = t + \frac{l}{2} -$

$$- \frac{4l}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} \cos \frac{(2k+1)a\pi}{l} t \cos \frac{(2k+1)\pi}{l} x.$$

470. $u(x, t) = a_0 + b_0 t + \sum_{k=1}^{\infty} \left(a_k \cos \frac{ka\pi}{l} t + \right.$

$$\left. + b_k \sin \frac{ka\pi}{l} t \right) \cos \frac{k\pi}{l} x,$$

where

$$a_0 = \frac{1}{l}\int_0^l \varphi(x)\,dx, \quad b_0 = \frac{1}{l}\int_0^l \psi(x)\,dx$$

and

$$a_k = \frac{2}{l}\int_0^l \varphi(x)\cos\frac{k\pi}{l}x\,dx, \quad b_k = \frac{2}{ka\pi}\int_0^l \psi(x)\cos\frac{k\pi}{l}x\,dx.$$

471. The solution is

$$u(x, t) = \sum_{k=1}^{\infty}(a_k\cos a\lambda_k t + b_k\sin a\lambda_k t)\sin\lambda_k x$$

where

$$a_k = \frac{1}{\|\sin\lambda_k x\|^2}\int_0^l \varphi(x)\sin\lambda_k x\,dx$$

$$b_k = \frac{1}{a\lambda_k\|\sin\lambda_k x\|^2}\int_0^l \psi(x)\sin\lambda_k x\,dx$$

$$\|\sin\lambda_k x\|^2 = \int_0^l \sin^2\lambda_k x\,dx = \frac{l(h^2+\lambda_k^2)+h}{2(h^2+\lambda_k^2)}$$

and λ_k are the positive roots of the equation $h\tan\lambda l = -\lambda$.

472. The solution is

$$u(x, t) = \frac{2h}{a}\sum_{k=1}^{\infty}\frac{\sqrt{h^2+\lambda_k^2}}{\lambda_k^2[l(h^2+\lambda_k^2)+h]}\sin a\lambda_k t\cos\lambda_k x$$

where λ_k are the positive roots of the equation $\lambda\tan\lambda l = h$.

473. The solution is

$$u(x, t) = \sum_{k=1}^{\infty}(a_k\cos a\lambda_k t + b_k\sin a\lambda_k t)(\lambda_k\cos\lambda_k x + h\sin\lambda_k x)$$

where

$$a_k = \frac{1}{\|\Phi_k(x)\|^2} \int_0^l (\lambda_k \cos \lambda_k x + h \sin \lambda_k x)\, \varphi(x)\, dx$$

$$b_k = \frac{1}{a\lambda_k \|\Phi_k(x)\|^2} \int_0^l (\lambda_k \cos \lambda_k x + h \sin \lambda_k x)\, \psi(x)\, dx$$

$$\|\Phi_k(x)\|^2 = \int_0^l (\lambda_k \cos \lambda_k x + h \sin \lambda_k x)^2\, dx = \frac{l(h^2+\lambda_k^2)+h}{2}$$

and λ_k are the positive roots of the equation $h \cot \lambda l = \lambda$.

474. The solution is

$$u(x, t) = \sum_{k=1}^{\infty} (a_k \cos a\lambda_k t + b_k \sin a\lambda_k t)(\lambda_k \cos \lambda_k x + h \sin \lambda_k x)$$

where

$$a_k = \frac{1}{\|\Phi_k(x)\|^2} \int_0^l (\lambda_k \cos \lambda_k x + h \sin \lambda_k x)\, \varphi(x)\, dx$$

$$b_k = \frac{1}{a\lambda_k \|\Phi_k(x)\|^2} \int_0^l (\lambda_k \cos \lambda_k x + h \sin \lambda_k x)\, \psi(x)\, dx$$

$$\|\Phi_k(x)\|^2 = \int_0^l (\lambda_k \cos \lambda_k x + h \sin \lambda_k x)^2\, dx = \frac{l(h^2+\lambda_k^2)+2h}{2}$$

and λ_k are the nonnegative roots of the equation $\cot \lambda l = \frac{1}{2}(\lambda/h - h/\lambda)$.

475. The solution is $u(x, t) = v(x, t) + w(x)$ where

$$v(x, t) = \sum_{k=1}^{\infty} a_k \cos \frac{ka\pi}{l} t \sin \frac{k\pi}{l} x,$$

$$a_k = -\frac{2}{l} \int_0^l w(x) \sin \frac{k\pi}{l} x\, dx$$

and

$$w(x) = -\frac{1}{a^2}\int_0^x \left[\int_0^y f(\xi)\,d\xi\right] dy +$$

$$+\frac{x}{la^2}\int_0^l \left[\int_0^y f(\xi)\,d\xi\right] dy + \frac{\beta-\alpha}{l}x + \alpha$$

476. The solution is

$$u(x,\ t) = \frac{\beta-\alpha}{2l}x^2 + \alpha x + \Phi_0 + \psi_0 t + \frac{F_0}{2}t^2 +$$

$$+\sum_{k=1}^{\infty}\left\{\left(\frac{l}{ak\pi}\right)^2 F_k + \left[\Phi_k - \left(\frac{l}{ak\pi}\right)^2 F_k\right]\cos\frac{ak\pi t}{l} +\right.$$

$$\left.+\frac{l\psi_k}{ak\pi}\sin\frac{ak\pi t}{l}\right\}\cos\frac{k\pi x}{l}$$

where

$$F_k = \frac{\varepsilon_k}{l}\int_0^l \left[f(x) + \frac{(\beta-\alpha)\,a^2}{l}\right]\cos\frac{k\pi x}{l}\,dx$$

$$\Phi_k = \frac{\varepsilon_k}{l}\int_0^l \left[\varphi(x) - \frac{(\beta-\alpha)\,x^2}{2l} - \alpha x\right]\cos\frac{k\pi x}{l}\,dx$$

and

$$\psi_k = \frac{\varepsilon_k}{l}\int_0^l \psi(x)\cos\frac{k\pi x}{l}\,dx;$$

$$k = 0,\ 1,\ \ldots;\quad \varepsilon_0 = 1;\quad \varepsilon_k = 2;\quad k = 1,\ 2,\ \ldots$$

To solve the problem construct the solution in the form $u(x,\ t) = w(x) + v(x,\ t)$ where $w(x) = (\alpha_1 x^2 + \beta_1 x)\,\alpha + (\alpha_2 x^2 + \beta_2 x)\,\beta$; the constants α_1, β_1, α_2 and β_2 should be chosen so that the function $w(x)$ satisfies the boundary conditions of the original problem, that is $w_x(0) = \alpha$, $w_x(l) = \beta$.

477. The solution is

$$u(x,\ t) = w(x) + \sum_{k=1}^{\infty}(a_k\cos a\lambda_k t +$$

$$+ b_k\sin a\lambda_k t)(\lambda_k\cos\lambda_k x + h\sin\lambda_k x)$$

where

$$w(x) = -\frac{1}{a^2}\int_0^x \left[\int_0^y f(\xi)\,d\xi\right] dy +$$

$$+\left\{\beta - \alpha l + \frac{1}{a^2}\int_0^l \left[\int_0^y f(\xi)\,d\xi\right] dy\right\}\frac{1+hx}{1+hl} + \alpha x$$

$$a_k = \frac{2}{h + l(h^2+\lambda_k^2)}\int_0^l [\varphi(x) - w(x)]\,(\lambda_k \cos\lambda_k x + h\sin\lambda_k x)\,dx$$

$$b_k = \frac{2}{a\lambda_k\,[h + l(h^2+\lambda_k^2)]}\int_0^l \psi(x)\,(\lambda_k\cos\lambda_k x + h\sin\lambda_k x)\,dx$$

and λ_k are the positive roots of the equation $h \tan \lambda l = -\lambda$.

478. The solution is

$$u(x, t) =$$

$$= w(x) - 2\sum_{k=1}^{\infty}\left[\frac{h^2+\lambda_k^2}{h+l(h^2+\lambda_k^2)}\int_0^l w(\xi)\cos\lambda_k\xi\,d\xi\right]\times$$

$$\times \cos a\lambda_k t\cos\lambda_k x$$

where

$$w(x) = -\frac{1}{a^2}\int_0^x\left[\int_0^y f(\xi)\,d\xi\right]dy + \frac{\beta-\alpha}{h} - \alpha(l-x) +$$

$$+\frac{1}{a^2}\int_0^l\left[\int_0^y f(\xi)\,d\xi\right]dy + \frac{1}{a^2 h}\int_0^l f(\xi)\,d\xi$$

and λ_k are the positive roots of the equation $\lambda \tan \lambda l = h$.

479. The solution is

$$u(x, t) = -\frac{\alpha}{h} +$$

$$+ 4\alpha\sum_{n=0}^{\infty}\frac{1}{\lambda_{2n+1}\,[2h + l(h^2+\lambda_{2n+1}^2)]}(\lambda_{2n+1}\cos\lambda_{2n+1}x +$$

$$+ h\sin\lambda_{2n+1}x)\cos\lambda_{2n+1}t$$

where λ_{2n+1} are the positive roots of the equation $\cot\lambda l = \frac{1}{2}(\lambda/h - h/\lambda)$.

480. The sought-for function is $w(x, t) = (1 - x/l)\,\mu(t) + x\nu(t)/l$.

To solve the problem construct $w(x, t)$ in the form $w(x, t) = (\alpha_1 x + \beta_1)\,\mu(t) + (\alpha_2 x + \beta_2)\,\nu(t)$. The coefficients α_1, β_1, α_2 and β_2 should be chosen so that the function $w(x, t)$ satisfies the (non-homogeneous) boundary conditions of the given problem.

481. The sought-for function is $w(x, t) = (x - l) \times \mu(t) + \nu(t)$. See the hint to the solution of Problem 480.

482. $w(x, t) = \left(1 - \frac{hx}{1+lh}\right)\mu(t) + \frac{x}{1+lh}\nu(t)$.

483. $w(x, t) = -\frac{1}{h}\mu(t) + \left(x + \frac{1}{h}\right)\nu(t)$.

484. The sought-for function is

$$w(x, t) = \frac{[g(x-l)-1]\,\mu(t) + (1+hx)\,\nu(t)}{g + h(1+lg)}$$

See the hint to the solution of Problem 480.

485. The solution is

$$u(x, t) = \frac{A}{1+\left(\frac{a\pi}{l}\right)^2}\left(e^{-t} - \cos\frac{a\pi}{l}t + \frac{l}{a\pi}\sin\frac{a\pi}{l}t\right)\sin\frac{\pi}{l}x$$

To solve the problem we construct the solution as a series of the form

$$u(x, t) = \sum_{k=1}^{\infty} T_k(t)\,\Phi_k(x) \tag{a}$$

where $\Phi_k(x) = \sin\frac{k\pi}{l}x$ are the eigenfunctions of the Sturm-Liouville problem $\Phi'' + \lambda^2\Phi = 0$, $0 < x < l$, $\Phi(0) = \Phi(l) = 0$ corresponding to the eigenvalues $\lambda_k = k\pi/l$; $k = 1, 2, \ldots$. To determine the coefficients $T_k(t)$ of series (a) we require that the function $u(x, t)$ specified by series (a) should satisfy the original equation. This yields

$$\sum_{k=1}^{\infty}\left[T_k''(t) + \left(\frac{ak\pi}{l}\right)^2 T_k(t)\right]\sin\frac{k\pi}{l}x = Ae^{-t}\sin\frac{\pi}{l}x \tag{b}$$

From (b) it follows that

$$T_1''(t)+\left(\frac{a\pi}{l}\right)^2 T_1(t)=Ae^{-t} \tag{c}$$

and

$$T_k''(t)+\left(\frac{ka\pi}{l}\right)^2 T_k(t)=0; \qquad k=2,\ 3,\ \dots \tag{d}$$

From formula (a) and from the initial conditions of the problem we find

$$T_k(0)=T_k'(0)=0; \qquad k=1,\ 2,\ \dots \tag{e}$$

On solving equations (c) and (d) and using conditions (e) we obtain

$$T_1(t)=\frac{A}{1+\left(\frac{a\pi}{l}\right)^2}\left(e^{-t}-\cos\frac{a\pi}{l}t+\frac{l}{a\pi}\sin\frac{a\pi}{l}t\right)$$

and

$$T_k(t)\equiv 0; \quad k=2,\ 3,\ \dots$$

486. $u(x,\ t)=$

$$=\frac{2lA}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k\left[1+\left(\frac{ka\pi}{l}\right)^2\right]}\left(e^{-t}-\cos\frac{ka\pi}{l}t+\right.$$

$$\left.+\frac{l}{ka\pi}\sin\frac{ka\pi}{l}t\right)\sin\frac{k\pi}{l}x.$$

487. $$u(x,\ t)=\frac{4A}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)\left\{\left[\frac{a\pi(2k+1)}{2l}\right]^2-1\right\}}\times$$

$$\times\left[\sin t-\frac{2l}{a\pi(2k+1)}\sin\frac{a\pi(2k+1)}{2l}t\right]\sin\frac{(2k+1)\pi}{2l}x.$$

488. We construct the solution of the problem as a series of the form

$$u(x,\ t)=\sum_k T_k(t)\,\Phi_k(x) \tag{a}$$

where $\Phi_k(x)$ are the eigenfunctions of the corresponding Sturm-Liouville problem. To determine the eigenfunctions $\Phi_k(x)$ we look for the non-trivial solutions of the auxiliary problem

$$v_{tt} = a^2 v_{xx}, \quad 0 < x < l, \quad t > 0; \quad v(0, t) = v_x(l, t) = 0, \quad t > 0 \tag{b}$$

having the form

$$v(x, t) = P(t)\,\Phi(x) \not\equiv 0 \tag{c}$$

(in other words, we use the method of separation of variables). The substitution of (c) into (b) and the separation of the variables lead to the following Sturm-Liouville problem for the function $\Phi(x)$:

$$\Phi''(x) + \lambda^2 \Phi(x) = 0, \quad 0 < x < l, \quad \Phi(0) = \Phi'(l) = (0)$$

On solving the last problem, we find the eigenvalues $\lambda_k = (2k+1)\pi/2l$ and the corresponding eigenfunctions

$$\Phi_k(x) = \sin\frac{(2k+1)\pi}{2l}x; \quad k = 0, 1, \ldots$$

Further, let us expand the function $f(x, t)$ into a series with respect to the eigenfunctions $\Phi_k(x)$ we have found:

$$f(x, t) = \sum_{k=0}^{\infty} \tau_k(t)\,\Phi_k(x) \tag{d}$$

where

$$\tau_k(t) = \frac{2}{l}\int_0^l f(x, t)\sin\frac{(2k+1)\pi}{2l}x\,dx; \quad k = 0, 1, \ldots$$

Next we start to determine the coefficients $T_k(t)$ of series (a). The substitution of (a) and (d) into the equation of the original problem yields

$$\sum_{k=0}^{\infty}\left\{T_k''(t) + \left[\frac{(2k+1)a\pi}{2l}\right]^2 T_k(t) - \tau_k(t)\right\}\sin\frac{(2k+1)\pi}{2l}x = 0$$

whence

$$T_k''(t) + \left[\frac{(2k+1)a\pi}{2l}\right]^2 T_k(t) = \tau_k(t), \quad k = 0, 1, \ldots \tag{e}$$

Similarly, on substituting (a) into the initial conditions of the original problem, we find

$$T_k(0) = T_k'(0) = 0; \quad k = 0, 1, \ldots \tag{f}$$

The solution of Problem (e), (f) gives

$$T_k(t)=\frac{2l}{(2k+1)a\pi}\int_0^t \tau_k(\xi)\sin\frac{(2k+1)a\pi}{2l}(t-\xi)\,d\xi;$$

$$k=0,\ 1,\ \dots$$

Finally, the substitution of the expressions of $\Phi_k(x)$ and $T_k(t)$ we have found into (a) results in the solution of the original problem:

$$u(x,t)=\frac{2l}{a\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left[\int_0^t \tau_k(\xi)\sin\frac{(2k+1)a\pi}{2l}(t-\xi)\,d\xi\right]\times$$
$$\times\sin\frac{(2k+1)\pi}{2l}x$$

489. $$u(x,t)=\frac{A}{1+\left(\frac{a\pi}{2l}\right)^2}\left(e^{-t}-\cos\frac{a\pi}{2l}t+\right.$$
$$\left.+\frac{2l}{a\pi}\sin\frac{a\pi}{2l}t\right)\cos\frac{\pi}{2l}x.$$

490. $$u(x,t)=\int_0^t\left[\int_0^\tau f_0(\xi)\,d\xi\right]d\tau+$$
$$+\frac{l}{a\pi}\sum_{k=1}^{\infty}\left[\frac{1}{k}\int_0^t f_k(\xi)\sin\frac{ka\pi}{l}(t-\xi)\,d\xi\right]\cos\frac{k\pi}{l}x,$$

where

$$f_0(\xi)=\frac{1}{l}\int_0^l f(x,\xi)\,dx \text{ and } f_k(\xi)=$$
$$=\frac{2}{l}\int_0^l f(x,\xi)\cos\frac{k\pi}{l}x\,dx;\quad k=1,\ 2,\ \dots$$

491. The solution is

$$u(x,t)=\left(1-\frac{x}{\pi}\right)t^2+\frac{x}{\pi}t^3+\sin x\cos t+$$
$$+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^3}\left[(-1)^k 3t-1+\cos kt-\frac{(-1)^k 3}{k}\sin kt\right]\sin kx$$

To find the solution we construct it in the form $u(x, t) = w(x, t) + v(x, t)$ where the function $w(x, t)$ should be chosen so that it satisfies the boundary conditions of the problem (see the answer to Problem 480).

492. The solution is

$$u(x,t) = \left(1 - \frac{x}{\pi}\right) e^{-t} + \frac{xt}{\pi} + \frac{1}{2}\cos 2t \sin 2x -$$

$$- \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k(1+k^2)} \left[e^{-t} + k^2 \cos kt - \left(2k + \frac{1}{k}\right) \sin kt\right] \sin kx$$

See the hint to the solution of Problem 491.

493. The solution is

$$u(x,t) = x + t + \cos\frac{t}{2}\sin\frac{x}{2} -$$

$$- \frac{8}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \cos\frac{(2k+1)}{2} t \sin\frac{(2k+1)}{2} x$$

See the hint to the solution of Problem 491.

494. The solution is $u(x, t) = Aae^{-t} \cosh x/a (\sinh l/a)^{-1}$. To find the solution construct it in the form $u(x, t) = v(x, t) + e^{-t} f(x)$.

495. The solution is $u(x, t) = t/2 - (1/4 + \cos 2x/a) \times \sin 2t$. To obtain the solution construct it in the form $u(x, t) = v(x, t) + f(x) \sin 2t$.

496. The function $v(x, y)$ is the solution of the problem

$$v_{xx} + v_{yy} + \lambda v = 0, \quad (x, y) \in G; \quad v(x, y) = 0, \quad (x, y) \in C$$

and the function $w(t)$ satisfies the equation $w''(t) + \lambda w(t) = 0$. The existence of the sequence of the solutions

$$u_n(x, y, t) = (a_n \cos \mu_n t + b_n \sin \mu_n t)\, v_n(x, y)$$

where $v_n(x, y)$ are the non-trivial solutions of Problem (34), (35) for $\lambda = \mu_n$ and a_n, b_n are arbitrary real constants, makes it possible to construct the solution $u(x, y, t)$ of the original problem satisfying the initial conditions as well.

497. By virtue of the uniqueness of the solution of the Cauchy problem

$$u_{xx}+u_{yy}-u_{tt}=0, \quad u(x, y, 0)=u_t(x, y, 0)=0$$

we conclude that $u(x, y, t)\equiv 0$ in the domain bounded by the cone $\sqrt{x^2+y^2}=1-t$ and the plane $t=0$. Integrating the identity

$$-2\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right)-2\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial t}\right)+ \\ +\frac{\partial}{\partial t}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]+\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2=0$$

over the domain D_τ bounded by the cylindrical surface $x^2+y^2=1$, the cone $\sqrt{x^2+y^2}=1-t$ and the plane $t=\tau$ $(\tau>0)$, and taking into account that $u(x, y, t)=0$ for $x^2+y^2=1$, $t\geqslant 0$ and for $t=1-\sqrt{x^2+y^2}$, we derive the relation

$$\int\limits_{1-\tau\leqslant\sqrt{x^2+y^2}\leqslant 1}\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial t}\right)^2\right]_{t=\tau} dx\,dy=0$$

whence $u(x, y, \tau)=\left.\frac{\partial u(x, y, t)}{\partial t}\right|_{t=\tau}=0$. The repetition of this argument for $t>1$ leads to the conclusion that $u(x, y, t)=0$ in the semi-cylinder $0\leqslant x^2+y^2\leqslant 1$, $t\geqslant 0$.

498. Let $v(x, y)$ be the solution of Problem (34), (35). The integration of the identity

$$\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2=\frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(v\frac{\partial v}{\partial y}\right)-v\,\Delta v$$

over the domain G results in

$$\int\limits_G(v_x^2+v_y^2)\,dx\,dy=\int\limits_{\partial G} v\frac{\partial v}{\partial \nu}\,ds-\int\limits_G v\,\Delta v\,dx\,dy=\lambda\int\limits_G v^2\,dx\,dy$$

whence follows the required assertion.

499. Let $v_k(x, y)$ and $v_m(x, y)$ be two eigenfunctions of Problem (34), (35) corresponding to two eigenvalues λ_k and λ_m $(\lambda_k\neq\lambda_m)$ respectively. The integration of the identity

$$\frac{\partial}{\partial x}\left(v_k \frac{\partial v_m}{\partial x} - v_m \frac{\partial v_k}{\partial x}\right) + \\ + \frac{\partial}{\partial y}\left(v_k \frac{\partial v_m}{\partial y} - v_m \frac{\partial v_k}{\partial y}\right) = v_k \Delta v_m - v_m \Delta v_k$$

results in

$$\int_G (v_k \Delta v_m - v_m \Delta v_k)\, dx\, dy = (\lambda_m - \lambda_k) \int_G v_k v_m \, dx\, dy = 0$$

500. (a) The solution of the problem

$$u_{tt} = a^2 (u_{xx} + u_{yy}), \quad 0 < x < s, \quad 0 < y < p, \quad t > 0$$
$$u(0, y, t) = u(s, y, t) = u(x, 0, t) = \\ = u(x, p, t) = 0, \quad t > 0$$
$$u(x, y, 0) = \sin \frac{\pi}{s} x \sin \frac{\pi}{p} y, \\ u_t(x, y, 0) = 0, 0 < x < s, \quad 0 < y < p$$

is the function

$$u(x, y, t) = \cos \frac{\sqrt{s^2 + p^2}\, a\pi t}{sp} \sin \frac{\pi x}{s} \sin \frac{\pi y}{p}$$

(b) The solution of the problem

$$u_{tt} = a^2 (u_{xx} + u_{yy}), \quad 0 < x < s, \quad 0 < y < p, \quad t > 0$$
$$u(0, y, t) = u(s, y, t) = u(x, 0, t) = \\ = u(x, p, t) = 0, \quad t > 0$$
$$u(x, y, 0) = 0, \quad u_t(x, y, 0) = \\ = \frac{I}{\rho} \delta(x - x_0)\, \delta(y - y_0), \quad 0 < x < s, \quad 0 < y < p$$

is the function

$$u(x, y, t) = \frac{4I}{a\pi\rho} \sum_{k, n=1}^{\infty} \frac{\sin \frac{k\pi x_0}{s} \sin \frac{n\pi y_0}{p}}{\sqrt{k^2 p^2 + n^2 s^2}} \times \\ \times \sin\left(\sqrt{\frac{k^2}{s^2} + \frac{n^2}{p^2}}\, a\pi t\right) \sin \frac{k\pi}{s} x \sin \frac{n\pi}{p} y$$

where ρ is the area density of the distribution of the mass over the membrane.

(c) The solution of the problem

$$u_{tt}=a^2(u_{xx}+u_{yy})+\frac{1}{\rho}e^{-t}x\sin\frac{2\pi}{p}y,$$
$$0<x<s,\quad 0<y<p,\quad t>0$$
$$u(0,\ y,\ t)=u(s,\ y,\ t)=u(x,\ 0,\ t)=$$
$$=u(x,\ p,\ t)=0,\quad t>0$$
$$(x,\ y,\ 0)=u_t(x,\ y,\ 0)=0,\quad 0<x<s,\quad 0<y<p$$

is the function

$$u(x,y,t)=\sin\frac{2\pi}{p}y\sum_{k=1}^{\infty}a_k\Big(e^{-t}-$$
$$-\cos a\,\pi\omega_k t+\frac{1}{a\pi\omega_k}\sin a\pi\omega_k t\Big)\sin\frac{k\pi}{s}x$$

where $a_k=(-1)^{k+1}2s/\pi\rho k\,(1+a^2\pi^2\omega_k^2)$, $\omega_k=\sqrt{k^2/s^2+4/p^2}$ and ρ is the area density of the distribution of the mass over the membrane.

501. (a) The solution of the problem

$$u_{tt}=a^2(u_{xx}+u_{yy}),\quad 0<x<s,\quad 0<y<p,\quad t>0$$
$$u(0,\ y,\ t)=u_x(s,\ y,\ t)=u(x,\ 0,\ t)=$$
$$=u_y(x,\ p,\ t)=0,\quad t>0$$
$$u(x,\ y,\ 0)=Axy,\quad u_t(x,\ y,\ 0)=0,$$
$$0<x<s,\quad 0<y<p$$

is the function

$$u(x,\ y,\ t)=\sum_{k,\ n=0}^{\infty}a_{kn}\cos\left(a\pi\sqrt{\frac{(2k+1)^2}{4s^2}+\frac{(2n+1)^2}{4p^2}}\,t\right)\times$$
$$\times\sin\frac{(2k+1)\pi}{2s}x\sin\frac{(2n+1)\pi}{2p}y$$

where $a_{kn}=(-1)^{k+n}64spA/\pi^4(2k+1)^2(2n+1)^2$.

(b) The solution of the problem

$$u_{tt} = a^2 (u_{xx} + u_{yy}), \quad 0 < x < s, \quad 0 < y < p, \quad t > 0$$

$$u(0, y, t) = u_x(s, y, t) = u(x, 0, t) =$$
$$= u_y(x, p, t) = 0, \quad t > 0$$

$$u(x, y, 0) = 0, \quad u_t(x, y, 0) = \frac{I}{\rho} \delta(x - x_0) \delta(y - y_0),$$
$$0 < x < s, \quad 0 < y < p$$

is the function

$$u(x, y, t) = \sum_{k, n=0}^{\infty} a_{kn} \sin\left(\frac{a\pi}{2} \sqrt{\frac{(2k+1)^2}{s^2} + \frac{(2n+1)^2}{p^2}}\, t\right) \times$$
$$\times \sin \frac{(2k+1)\pi}{2s} x \sin \frac{(2n+1)\pi}{2p} y$$

where

$$a_{kn} = \frac{8I}{a\pi\rho s p} \frac{\sin \frac{(2k+1)\pi x_0}{2s} \sin \frac{(2n+1)\pi y_0}{2p}}{\sqrt{\frac{(2k+1)^2}{s^2} + \frac{(2n+1)^2}{p^2}}}$$

and ρ is the area density of the distribution of the mass over the membrane.

502. $u(x, t) = \frac{2lA}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} e^{\left(-\frac{ak\pi}{l}\right)^2 t} \sin \frac{k\pi}{l} x.$

503. The solution is

$$u(x, t) = \sum_{k=0}^{\infty} a_k e^{-\left[\frac{(2k+1)a\pi}{2l}\right]^2 t} \sin \frac{(2k+1)\pi}{2l} x$$

where $a_k = \frac{2}{l} \int_0^l \varphi(x) \sin \frac{(2k+1)\pi}{2l} x \, dx.$

504. $u(x, t) =$

$$= \frac{8lA}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} e^{-\left[\frac{(2k+1)a\pi}{2l}\right]^2 t} \cos \frac{(2k+1)\pi}{2l} x.$$

505. $u(x, t) = U$.

506. The solution is

$$u(x, t) = 2\sum_{k=1}^{\infty} \left\{ \frac{h^2+\lambda_k^2}{l(h^2+\lambda_k^2)+h} \int_0^l \varphi(\xi) \cos \lambda_k \xi \, d\xi \right\} e^{-a^2\lambda_k^2 t} \cos \lambda_k x$$

where λ_k are the positive roots of the equation $\lambda \tan \lambda l = h$.

507. The solution is

$$u(x, t) = 2U \sum_{k=1}^{\infty} \frac{h-(-1)^k \sqrt{h^2+\lambda_k^2}}{\lambda_k [l(h^2+\lambda_k^2)+h]} e^{-a^2\lambda_k^2 t} \Phi_k(x)$$

where $\Phi_k(x) = \lambda_k \cos \lambda_k x + h \sin \lambda_k x$ and λ_k are the positive roots of the equation $h \tan \lambda l = -\lambda$.

508. The solution is

$$u(x, t) = \sum_{k=1}^{\infty} a_k e^{-a^2\lambda_k^2 t} (\lambda_k \cos \lambda_k x + h \sin \lambda_k x)$$

where $a_k = 2U [h/\lambda_k + (h^2 + \lambda_k^2) \sin \lambda_k l/2\lambda_k^2] [l(h^2 + \lambda_k^2) + 2h]^{-1}$ and λ_k are the positive roots of the equation $\cot \lambda l = (\lambda/h - h/\lambda)/2$.

509. $u(x, t) =$

$$= \frac{2}{l} \sum_{k=1}^{\infty} \left(\int_0^l \varphi(\xi) \sin \frac{k\pi}{l} \xi \, d\xi \right) e^{-\left[\left(\frac{ak\pi}{l}\right)^2+\beta\right] t} \sin \frac{k\pi}{l} x.$$

510. $u(x, t) = e^{-\left(\frac{a^2\pi^2}{4l^2}+\beta\right) t} \sin \frac{\pi}{2l} x$.

511. $u(x, t) = \sum\limits_{k=0}^{\infty} a_k e^{-\left[\left(\frac{ak\pi}{l}\right)^2+\beta\right] t} \cos \frac{k\pi}{l} x$, where

$$a_0 = \frac{1}{l} \int_0^l \varphi(x)\, dx, \quad a_k = \frac{2}{l} \int_0^l \varphi(x) \cos \frac{k\pi}{l} x \, dx; \quad k = 1, 2, \ldots$$

512. The solution is

$$u(x, t) = 2hU \sum_{k=1}^{\infty} \frac{1}{\lambda_k [l(h^2+\lambda_k^2)+h]} e^{-(a^2\lambda_k^2+\beta)t} \Phi_k(x)$$

where $\Phi_{h_k}(x) = \lambda_k \cos \lambda_k x + h \sin \lambda_k x$ and λ_k are the positive roots of the equation $h \cot \lambda l = \lambda$.

513. $u(x, t) = \frac{(U-T)}{l} x + T +$

$$+\frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} [(-1)^k U - T] e^{-\left(\frac{ak\pi}{l}\right)^2 t} \sin \frac{k\pi}{l} x.$$

514. The solution is

$$u(x, t) = w(x) + \sum_{k=0}^{\infty} a_k e^{-\frac{(2k+1)^2 a^2 \pi^2}{4l^2} t} \sin \frac{(2k+1)\pi}{2l} x$$

where

$$w(x) = -\frac{1}{a^2} \int_0^x \left[\int_0^y f(\xi)\, d\xi \right] dy + \frac{x}{a^2} \int_0^l f(\xi)\, d\xi + qx$$

and

$$a_k = \frac{2}{l} \int_0^l [\varphi(x) - w(x)] \sin \frac{(2k+1)\pi}{2l} x\, dx$$

515. $u(x, t) = qx + \frac{(A-q)l}{2} -$

$$-\frac{4l(A-q)}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} e^{-\frac{(2k+1)^2 a^2 \pi^2}{l^2} t} \cos \frac{(2k+1)\pi}{l} x.$$

516. $u(x, t) = \frac{U - hT}{1 + lh} x + T -$

$$-2 \sum_{k=1}^{\infty} \frac{h^2 + \lambda_k^2}{\lambda_k [l(h^2 + \lambda_k^2) + h]} \left[T - \frac{(-1)^k U}{\sqrt{h^2 + \lambda_k^2}} \right] e^{-a^2 \lambda_k^2 t} \sin \lambda_k x$$

where λ_k are the positive roots of the equation $h \tan \lambda l = -\lambda$.

517. $u(x, t) = \frac{1}{\beta + \left(\frac{a\pi}{l}\right)^2} \left[1 - e^{-\left[\beta + \left(\frac{a\pi}{l}\right)^2\right] t} \right] \sin \frac{\pi}{l} x.$

518. The solution is

$$u(x, t) = \frac{aA}{\cos\frac{l}{a}} e^{-t} \sin\frac{x}{a} +$$

$$+\frac{2}{l}\sum_{k=0}^{\infty}\left[\frac{T}{\omega_k}+\frac{(-1)^k Aa^2}{1-a^2\omega_k^2}\right] e^{-a^2\omega_k^2 t}\sin\omega_k x$$

where $\omega_k = (2k+1)\pi/2l$, $\omega_k \neq 1/a$; $k = 0, 1, \ldots$

To find the solution construct it in the form $u(x, t) = f(x) e^{-t} + v(x, t)$ where the function $v(x, t)$ must satisfy the homogeneous equation and the homogeneous boundary conditions.

519. The solution is

$$u(x, t) = -\frac{a^2A}{2l}t^2 - \left(\frac{A}{2l}x^2 - Ax + \frac{Al}{3} - \frac{a^2T}{l}\right)t +$$

$$+\frac{T}{2l}x^2 - \frac{lT}{6} + \frac{2l}{a^2\pi^4}\sum_{k=1}^{\infty}\frac{1}{k^4}\{Al^2 -$$

$$-[Al^2 + (-1)^k T(ak\pi)^2] e^{-\left(\frac{ak\pi}{l}\right)^2 t}\}\cos\frac{k\pi x}{l}$$

To find the solution construct it in the form $u(x, t) = w(x, t) + v(x, t)$ where $w(x, t)$ has the form $w(x, t) = (\alpha_1 x^2 + \beta_1 x) At + (\alpha_2 x^2 + \beta_2 x) T$, the constants α_1, β_1, α_2 and β_2 being chosen so that $w(x, t)$ satisfies the boundary conditions of the problem.

520. (a) The solution of the problem

$$u_t = a^2\Delta u, \quad 0 \leqslant r < R \quad \text{where} \quad \Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)$$

$$u(R, t) = 0; \quad |u(0, t)| < \infty, \quad t > 0; \quad u(r, 0) = T,$$

$$0 \leqslant r < R$$

is the function

$$u(r, t) = \frac{2RT}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} e^{-\left(\frac{ka\pi}{R}\right)^2 t}\frac{\sin\frac{k\pi r}{R}}{r}$$

To obtain this solution we pass to the new unknown function $v(r, t) = ru(r, t)$; this reduces the original problem to the new problem

$$v_t = a^2 v_{rr}, \quad 0 < r < R, \quad t > 0$$

$$v(0, t) = v(R, t) = 0, \quad t > 0; \quad v(r, 0) = Tr, \; 0 < r < R$$

(b) The solution of the problem

$$u_t = a^2 \Delta u, \quad 0 \leqslant r < R, \quad t > 0$$

$$\text{where} \quad \Delta u = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right)$$

$$k u_r(R, t) = q, \quad t > 0; \quad u(r, 0) = T, \quad 0 \leqslant r < R$$

is the function

$$u(r, t) = T + \frac{qR}{k}\left(\frac{3a^2 t}{R^2} - \frac{3R^2 - 5r^2}{10R^2}\right) -$$

$$- \frac{2qR^2}{k} \sum_{n=1}^{\infty} \frac{1}{\mu_n^3 \cos \mu_n} e^{-\left(\frac{a\mu_n}{R}\right)^2 t} \frac{\sin \frac{\mu_n r}{R}}{r}$$

where μ_n are the positive roots of the equation $\tan \mu = \mu$.

521. (a) The solution of the problem

$$u_t = a^2 (u_{xx} + u_{yy}), \quad 0 < x < p, \quad 0 < y < s, \quad t > 0$$

$$u_x(0, y, t) = u(p, y, t) = 0, \quad 0 < y < s, \quad t > 0$$

$$u(x, 0, t) = u(x, s, t) = 0, \quad 0 < x < p, \quad t > 0$$

$$u(x, y, 0) = f(x, y), \quad 0 < x < p, \quad 0 < y < s$$

is the function

$$u(x, y, t) = \sum_{k=1,\, n=0}^{\infty} a_{kn} e^{-a^2 \omega_{kn}^2 t} \sin \frac{k\pi}{s} y \cos \frac{(2n+1)\pi}{2p} x$$

where

$$a_{kn} = \frac{4}{ps} \int_0^p \int_0^s f(x, y) \sin \frac{k\pi}{s} y \cos \frac{(2n+1)\pi}{2p} x \, dx \, dy$$

$$\text{and} \quad \omega_{kn}^2 = \frac{k^2 \pi^2}{s^2} + \frac{(2n+1)^2 \pi^2}{4p^2}$$

(b) The solution of the problem

$$u_t = a^2 (u_{xx} + u_{yy}), \quad 0 < x < p, \quad 0 < y < s, \quad t > 0$$

$$u(0, y, t) = 0, \quad u_x(p, y, t) + hu(p, y, t) = 0,$$

$$0 < y < s, \quad t > 0$$

$$u_y(x, 0, t) = u(x, s, t) = 0, \quad 0 < x < p, \quad t > 0$$

$$u(x, y, 0) = f(x, y), \quad 0 < x < p, \quad 0 < y < s$$

is the function

$$u(x, y, t) = \sum_{k=1,\ n=0}^{\infty} a_{kn} e^{-a^2\omega_{kn}^2 t} \sin \mu_k x \cos \frac{(2n+1)\pi}{2s} y$$

where

$$a_{kn} = \frac{4(h^2+\mu_k^2)}{s[p(h^2+\mu_k^2)+h]} \int_0^p \int_0^s f(x, y) \sin \mu_k x \cos \frac{(2n+1)\pi}{2s} y\, dx\, dy$$

$$\omega_{kn}^2 = \mu_k^2 + \frac{(2n+1)^2 \pi^2}{4s^2}$$

and μ_k are the positive roots of the equation $h \tan p\mu = -\mu$.

522. The solution of the problem

$$u_t = a^2 (u_{xx} + u_{yy} + u_{zz}) - \beta u, \quad 0 < x, y, z < l, \quad t > 0$$

$$u(0, y, z, t) = u(l, y, z, t) = 0, \quad 0 < y, z < l, \ t > 0$$

$$u(x, 0, z, t) = u(x, l, z, t) = 0, \quad 0 < x, \quad z < l, \quad t > 0$$

$$u(x, y, 0, t) = u(x, y, l, t) = 0, \quad 0 < x, y < l, \quad t > 0$$

$$u(x, y, z, 0) = U, \quad 0 < x, y, z < l$$

is the function

$$u(x, y, z, t) = \frac{64U}{\pi^3} \sum_{k,\ m,\ n=0}^{\infty} A_{kmn} e^{-\omega_{kmn} t} \sin \frac{(2k+1)\pi x}{l} \times$$

$$\times \sin \frac{(2m+1)\pi y}{l} \sin \frac{(2n+1)\pi z}{l}$$

where

$$A_{kmn} = [(2k+1)(2m+1)(2n+1)]^{-1}$$

$$\omega_{kmn} = \beta + \frac{a^2\pi^2}{l^2} [(2k+1)^2 + (2m+1)^2 + (2n+1)^2]$$

and β is the coefficient characterizing the rate of the disintegration.

523. (a) $u(x, y) = \sum_{k=0}^{\infty} a_k \sin \frac{(2k+1)\pi}{2p} x \sinh \frac{(2k+1)\pi}{2p} y,$

where

$$a_k = \frac{2}{p}\left[\sinh \frac{(2k+1)\pi s}{2p}\right]^{-1} \int_0^p f(x) \sin \frac{(2k+1)\pi}{2p} x\, dx$$

(b) $$u(x, y) = \frac{(pB-2A)\,y}{2s} + A -$$

$$-\frac{4pB}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2 \sinh \frac{(2k+1)\pi s}{p}} \times$$

$$\times \cos \frac{(2k+1)\pi}{p} x \sinh \frac{(2k+1)\pi}{p} y.$$

(c) $$u(x, y) = U + \frac{2p}{\pi}\left[T \sinh \frac{\pi}{2p} y -\right.$$

$$\left. -\left(\cosh \frac{\pi s}{2p}\right)^{-1}\left(\frac{2U}{p} + T \sinh \frac{\pi s}{2p}\right) \cosh \frac{\pi}{2p} y\right] \sin \frac{\pi}{2p} x -$$

$$-\frac{4U}{\pi} \sum_{k=1}^{\infty} \frac{\left[\cosh \frac{(2k+1)\pi s}{2p}\right]^{-1}}{2k+1} \cosh \frac{(2k+1)\pi}{2p} y \sin \frac{(2k+1)\pi}{2p} x.$$

524. (a) The solution is

$$u(x, y) = \sum_{k=0}^{\infty} a_k e^{-\frac{(2k+1)\pi}{2l} x} \sin \frac{(2k+1)\pi}{2l} y$$

where

$$a_k = \frac{2}{l} \int_0^l f(y) \sin \frac{(2k+1)\pi}{2l} y\, dy$$

(b) The solution is

$$u(x, y) = \sum_{k=1}^{\infty} \left\{ \frac{2(h^2+\lambda_k^2)}{l(h^2+\lambda_k^2)+h} \int_0^l f(\xi) \cos \lambda_k \xi \, d\xi \right\} e^{-\lambda_k x} \cos \lambda_k y$$

where λ_k are the positive roots of the equation $\lambda \tan \lambda l = h$.

525. (a) $u(r,\varphi)=\frac{b}{b^2-a^2}\left(r-\frac{a^2}{r}\right)\cos\varphi.$

(b) $u(r,\varphi)=A\frac{\ln\frac{r}{b}}{\ln\frac{a}{b}}+\frac{Bb^2}{b^4-a^4}\left(r^2-\frac{a^4}{r^2}\right)\sin 2\varphi.$

(c) $$u(r,\varphi)=Q+\frac{a^2q}{a^2+b^2}\left(r-\frac{b^2}{r}\right)\cos\varphi+$$
$$+\frac{b^2T}{a^4+b^4}\left(r^2+\frac{a^4}{r^2}\right)\sin 2\varphi$$

526. (a) $u(r,\varphi)=\frac{2A\alpha}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\left(\frac{r}{R}\right)^{\frac{k\pi}{\alpha}}\sin\frac{k\pi\varphi}{\alpha}.$

(b) $u(r,\varphi)=\sum_{k=0}^{\infty}a_k r^{\frac{(2k+1)\pi}{2\alpha}}\cos\frac{(2k+1)\pi}{2\alpha}\varphi,$

where

$$a_k=\frac{2}{\alpha}R^{-\frac{(2k+1)\pi}{2\alpha}}\int_0^{\alpha}f(\varphi)\cos\frac{(2k+1)\pi}{2\alpha}\varphi\,d\varphi$$

527. Indeed,

$$J_{n-1}(x)+J_{n+1}(x)=\frac{x^{n-1}}{2^{n-1}(n-1)!}+$$
$$+\sum_{k=1}^{\infty}(-1)^k\frac{x^{n-1+2k}}{2^{n-1+2k}(k-1)!\,(n-1+k)!}\left(\frac{1}{k}-\frac{1}{n+k}\right)=$$
$$=\frac{2n}{x}\left[\frac{x^n}{2^n n!}+\sum_{k=1}^{\infty}(-1)^k\frac{x^{n+2k}}{2^{n+2k}k!\,(n+k)!}\right]=\frac{2n}{x}J_n(x)$$

The other two identities are verified in like manner.

528. Since

$$\int_0^1\frac{t^{2k}}{\sqrt{1-t^2}}\,dt=\frac{1}{2^k}\binom{2k}{k}\frac{\pi}{2}$$

we have

$$\frac{2}{\pi}\int_0^1 \frac{\cos tx}{\sqrt{1-t^2}}\,dt=\frac{2}{\pi}\sum_{k=}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}\int_0^1\frac{t^{2k}}{\sqrt{1-t^2}}\,dt=$$

$$=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{2^{2k}(k!)^2}=J_0(x)$$

529. The first identity can be verified in the same way as the identities indicated in Problem 527. Further, we have

$$J_0''(x)=-J_1'(x)=-\frac{1}{2}[J_0(x)-J_2(x)]$$

The last equality follows fro the second identity established in Problem 527.

530. Both identities can be established directly on the basis of the identities indicated in Problem 527. Indeed, taking into account that

$$J_n'(\alpha x)=J_{n-1}(\alpha x)-\frac{n}{\alpha x}J_n(\alpha x)$$

and

$$J_n'(\alpha x)=\frac{n-1}{\alpha x}J_{n-1}(\alpha x)-J_n(\alpha x)$$

we obtain

$$\frac{d}{dx}\{x[\beta J_n(\alpha x)J_n'(\beta x)-\alpha J_n(\beta x)J_n'(\alpha x)]\}=$$

$$=\frac{d}{dx}\left\{\beta xJ_n(\alpha x)\left[\frac{n}{\beta x}J_n(\beta x)-J_{n+1}(\beta x)\right]-\right.$$
$$\left.-\alpha xJ_n(\beta x)\left[\frac{n}{\alpha x}J_n(\alpha x)-J_{n+1}(\alpha x)\right]\right\}=$$
$$=\alpha J_n(\beta x)J_{n+1}(\alpha x)-\beta J_n(\alpha x)J_{n+1}(\beta x)+$$
$$+x[\alpha^2J_{n+1}'(\alpha x)J_n(\beta x)+\alpha\beta J_n'(\beta x)J_{n+1}(\alpha x)-$$
$$-\beta^2J_{n+1}'(\beta x)J_n(\alpha x)-\alpha\beta J_n'(\alpha x)J_{n+1}(\beta x)]=$$
$$=\alpha J_n(\beta x)J_{n+1}(\alpha x)-\beta J_n(\alpha x)J_{n+1}(\beta x)+$$
$$+\alpha^2xJ_n(\beta x)\left[J_n(\alpha x)-\frac{n+1}{\alpha x}J_{n+1}(\alpha x)\right]+$$
$$+\alpha\beta xJ_{n+1}(\alpha x)\left[\frac{n}{\beta x}J_n(\beta x)-J_{n+1}(\beta x)\right]-$$

$$-\alpha\beta x J_{n+1}(\beta x)\left[\frac{n}{\alpha x} J_n(\alpha x) - J_{n+1}(\alpha x)\right] -$$
$$-\beta^2 x J_n(\alpha x)\left[J_n(\beta x) - \frac{n+1}{\beta x} J_{n+1}(\beta x)\right] =$$
$$= (\alpha^2 - \beta^2) x J_n(\alpha x) J_n(\beta x)$$

Similarly,

$$\frac{d}{dx}\left\{(\alpha^2 x^2 - n^2) J_n^2(\alpha x) + \left[x \frac{d}{dx} J_n(\alpha x)\right]^2\right\} =$$
$$= 2\alpha^2 x J_n^2(\alpha x) + 2\alpha^3 x^2 J_n(\alpha x) J_n'(\alpha x) +$$
$$+ 2\alpha^2 x J_{n-1}^2(\alpha x) + 2\alpha^3 x^2 J_{n-1}(\alpha x) J_{n-1}'(\alpha x) -$$
$$- 2n\alpha J_{n-1}(\alpha x) J_n(\alpha x) - 2n\alpha^2 x J_{n-1}'(\alpha x) J_n(\alpha x) -$$
$$- 2n\alpha^2 x J_{n-1}(\alpha x) J_n'(\alpha x) = 2\alpha^2 x J_n^2(\alpha x)$$

531. According to Problem 530, we have the equalities $(\alpha^2 - \beta^2) x J_n(\alpha x) J_n(\beta x) =$

$$= \frac{d}{dx}\{\alpha x J_n(\beta x) J_{n+1}(\alpha x) - \beta x J_n(\alpha x) J_{n+1}(\beta x)\}$$

and

$$2\alpha^2 x J_n^2(\alpha x) = \frac{d}{dx}\{(\alpha^2 x^2 - n^2) J_n^2(\alpha x) + [n J_n(\alpha x) - \alpha x J_{n+1}(\alpha x)]^2\}$$

The integration of these equalities results in

$$(\alpha^2 - \beta^2)\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = 0$$

and

$$2\alpha^2 \int_0^1 x J_n^2(\alpha x)\, dx = \begin{cases} \alpha^2 J_{n+1}^2(\alpha) & \text{for} \quad J_n(\alpha) = 0 \\ \alpha^2 J_n^2(\alpha) & \text{for} \quad J_{n+1}(\alpha) = 0 \end{cases}$$

532. Indeed, if α is a complex root of the function $J_n(x)$ then $\bar{\alpha}$ is also its root. Therefore (see Problem 531) we obtain

$$\int_0^1 x J_n(\alpha x) J_n(\bar{\alpha} x)\, dx = \int_0^1 x\, |J_n(\alpha x)|^2\, dx = 0$$

that is $J_n(\alpha x)$ is identically equal to zero for $0 \leqslant x \leqslant 1$. By the analyticity of $J_n(\alpha x)$, it follows that $J_n(\alpha x) = 0$

for all the values of x (both real and complex), which is impossible. Similarly, assuming that $J_n(\alpha) = J_{n+1}(\alpha) = 0$ for $\alpha \neq 0$ we arrive at the contradiction:

$$\int_0^1 x\,|J_n(\alpha x)|^2\,dx = 0$$

Consequently, $J_n(x)$ and $J_{n+1}(x)$ cannot have common zeros (roots). By virtue of the first identity established in Problem 527, it follows that for any two nonnegative integral indices m and n the functions $J_m(x)$ and $J_n(x)$ cannot have common zeros (roots).

533. Let us write down the expression of Laplace's operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in the polar coordinates $r = \sqrt{x^2+y^2}$, $\theta = \operatorname{arc\,tan} y/x$:

$$\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Now, taking, for instance, $u_n(r, \theta)$ we obtain

$$\Delta u_n(r, \theta) - \mu^2 u_n(r, \theta) = \frac{\partial^2 I_n(\mu r)}{\partial r^2} + \\ + \frac{1}{r}\frac{\partial I_n(\mu r)}{\partial r} - \left(\frac{n^2}{r^2} + \mu^2\right) I_n(\mu r) = 0$$

because the function $I_n(x) = i^{-n} J_n(ix)$ satisfies the equation

$$I_n''(x) + \frac{1}{x} I_n'(x) - \left(1 + \frac{n^2}{x^2}\right) I_n(x) = 0$$

534. Since $\Gamma(k+1+1/2) = \sqrt{\pi}\,(2k+1)!/2^{2k+1}k!$, we obtain

$$J_{1/2}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{1/2+2k}}{2^{1/2+2k}k!\,\Gamma(k+1+1/2)} = \\ = \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = \sqrt{\frac{2}{\pi x}} \sin x$$

The other identity is derived in a similar manner.

535. The assertion stated in the problem follows from the fact that the change of the variable x according to the formula $x = \cos\theta$ $(-1 < x < 1)$ results in

$$\frac{\partial}{\partial x} = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} \quad \text{and} \quad \frac{\partial^2}{\partial x^2} = -\frac{\cos\theta}{\sin^3\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\theta^2}$$

536. For $x = \cos\theta$ we have

$$T_n(x) = T_n(\cos\theta) = \frac{1}{2}[(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n] =$$

$$= \frac{1}{2}[e^{in\theta} + e^{-in\theta}] = \cos n\theta = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \ldots$$

Therefore

$$T_n(x) = x^n - \binom{n}{2}x^{n-2}(1-x^2) + \ldots$$

537. Using the formula for $T_n(x)$ (see the solution of Problem 536) we obtain

$T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$ and

$T_3(x) = 4x^3 - 3x$

538. Using the formula $T_n(x) = \cos n\theta$ (here x $\cos\theta$; see the solution of Problem 536) we obtain

$$\int_{-1}^{1}\frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\,dx = \int_0^{\pi}\cos n\theta\cos m\theta\,d\theta = 0, \quad n \neq m$$

539. $\|T_0\| = \sqrt{\pi}$ and $\|T_n\| = \sqrt{\pi/2}$; $n = 1, 2, \ldots$.

540. Using Leibniz' formula we find

$$L_n(x) = \frac{e^x}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{d^k e^{-x}}{dx^k}\frac{d^{n-k}x^n}{dx^{n-k}} = \sum_{k=0}^{n}\binom{n}{n-k}\frac{(-1)^k}{k!}x^k$$

whence it follows that

$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) =$$

$$= \sum_{k=1}^{n-1}(-1)^k\left[\binom{n}{n-k}\frac{n}{k!} - \binom{n}{n-k}\frac{1}{(k-1)!} - \right.$$

$$\left. - \binom{n}{n-k+1}\frac{1}{(k-1)!} - \binom{n}{n-k-1}\frac{1}{k!}\right]x^k =$$

$$= \sum_{k=1}^{n-1}(-1)^k\frac{x^k}{k!}\left[\binom{n}{n-k}(n-k) - \binom{n}{n-k+1}(k+1)\right] = 0$$

541. $L_0 = 1, \quad L_1 = 1 - x, \quad L_2 = 1 - 2x + x^2/2!$ and $L_3 = 1 - 3x + 3x^2/2 - x^3/3!$.

542. Let $n < m$. Integrating by parts $n+1$ times we obtain

$$\int_0^\infty e^{-x} L_n(x) L_m(x)\, dx = \frac{1}{m!} \int_0^\infty L_n(x) \frac{d^m}{dx^m}(x^m e^{-x})\, dx =$$

$$= (-1)^{n+1} \frac{1}{(m+1)!} \int_0^\infty \frac{d^{n+1}}{dx^{n+1}} L_n(x) \frac{d^{m-n-1}}{dx^{m-n-1}} (x^m e^{-x})\, dx = 0$$

543. Integrating by parts n times we obtain

$$\int_0^\infty e^{-x} L_n^2(x)\, dx = \frac{1}{n!} \int_0^\infty L_n(x) \frac{d^n}{dx^n}(x^n e^{-x})\, dx =$$

$$= (-1)^{2n} \frac{1}{n!} \int_0^\infty x^n e^{-x}\, dx = \frac{\Gamma(n+1)}{n!} = 1$$

because $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$.

544. $u_0^3 = y^3 - 3yz^2, \quad u_1^3 = xy^2 - xz^2, \quad u_2^3 = yx^2 - yz^2,$

$u_3^3 = x^3 - 3xz^2, \quad u_4^3 = zy^2 - \frac{1}{3} z^3, \quad u_5^3 = xyz,$

$u_6^3 = x^2 z - \frac{1}{3} z^3.$

545. $Y_3^0 = \sin\varphi \sin\theta (\sin^2\varphi \sin^2\theta - 3\cos^2\theta),$

$Y_3^1 = \cos\varphi \sin\theta (\sin^2\varphi \sin^2\theta - \cos^2\theta),$

$Y_3^2 = \sin\varphi \sin\theta (\cos^2\varphi \sin^2\theta - \cos^2\theta),$

$Y_3^3 = \cos\varphi \sin\theta (\cos^2\varphi \sin^2\theta - 3\cos^2\theta),$

$Y_3^4 = \cos\theta \left(\sin^2\varphi \sin^2\theta - \frac{1}{3}\cos^2\theta\right),$

$Y_3^5 = \sin\varphi \cos\varphi \sin^2\theta \cos\theta,$

$Y_3^6 = \cos\theta \left(\cos^2\varphi \sin^2\theta - \frac{1}{3}\cos^2\theta\right).$

546. Since the function

$$\frac{1}{r} u_5^3\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right) = \frac{1}{r^4} Y_3^5(\varphi, 0)$$

is harmonic, the factors

$$w(r)=\frac{1}{r^4} \quad \text{and} \quad Y(\varphi, \theta)=Y_3^5(\varphi, \theta)$$

are solutions of the equations

$$\frac{d}{dr}\left(r^2 \frac{dw}{dr}\right)-12w=0$$

and

$$\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\varphi^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+12Y=0$$

respectively, which can easily be shown by using the expression for Laplace's operator in the spherical coordinates.

549. The assertion stated in the problem can easily be proved because it follows from the identity

$$(t^2-1)\frac{d^{m+2}}{dt^{m+2}}(t^2-1)^m+ \\ +2t\frac{d^{m+1}}{dt^{m+1}}(t^2-1)^m-m(m+1)\frac{d^m}{dt^m}(t^2-1)^m=0$$

which is obtained by differentiating $m+1$ times the obvious identity

$$(t^2-1)\frac{d}{dt}(t^2-1)^m=2mt(t^2-1)^m$$

550. These relations can easily be verified by using the representation of $P_n(t)$ derived in Problem 549. For instance, by virtue of this representation, we have

$$P'_{n+1}(t)-P'_{n-1}(t)= \\ =\frac{1}{2^{n+1}(n+1)!}\frac{d^n}{dt^n}[2(n+1)(t^2-1)^n+4n(n+1)t^2(t^2-1)^{n-1}]- \\ -\frac{1}{2^{n-1}(n-1)!}\frac{d^n}{dt^n}(t^2-1)^{n-1}= \\ =P_n(t)+\frac{1}{2^{n-1}(n-1)!}\frac{d^n}{dt^n}t^2(t^2-1)^{n-1}- \\ -\frac{1}{2^{n-1}(n-1)!}\frac{d^n}{dt^n}(t^2-1)^{n-1}=P_n(t)+2nP_n(t)= \\ =(1+2n)P_n(t)$$

551. The easiest way to prove the assertion is to verify it directly, that is to substitute expression (38) of $P_m(t)$ into the left-hand side of equation (36).

552. Let $m>n$. In this case, integrating by parts n times, we find

$$\int_{-1}^{1} P_m(t)\,P_n(t)\,dt=\frac{1}{2^{m+n}m!\,n!}\int_{-1}^{1}\frac{d^m}{dt^m}(t^2-1)^m\cdot\frac{d^n}{dt^n}(t^2-1)^n\,dt=$$

$$=\frac{(-1)^n}{2^{m+n}m!\,m!}\int_{-1}^{1}\frac{d^{m-n}}{dt^{m-n}}(t^2-1)^m\,\frac{d^{2n}}{dt^{2n}}(t^2-1)^n\,dt=$$

$$=\frac{(-1)^n\,(2n)!}{2^{m+n}m!\,n!}\int_{-1}^{1}\frac{d^{m-n}}{dt^{m-n}}(t^2-1)^m\,dt=0$$

553. For $n=m$ we have

$$\int_{-1}^{1} P_m^2(x)\,dx=\frac{(2m)!}{2^{2m}(m!)^2}\int_{-1}^{1}(1-t^2)^m\,dt=\frac{2}{2m+1}$$

(see the answer to Problem 552).

554. Since

$$Y_3^5(\varphi,\ \theta)=\sin\varphi\cos\varphi\sin^2\theta\cos\theta$$

(see the answer to Problem 545), we obtain for $P_3^2(\cos\theta)=$ $=P_3^2(t)$ $(t=\cos\theta)$ the expression

$$P_3^2(\cos\theta)=15\sin^2\theta\cos\theta=15\,(1-\cos^2\theta)\cos\theta=$$
$$=15t\,(1-t^2)$$

which obviously satisfies equation (37) for $m=3$, $n=2$.

555. The required result can readily be obtained if we differentiate equation (36) n times and then put

$$y(t)=\frac{d^n v(t)}{dt^n}$$

558. The assertion can readily be proved by substituting the expression

$$y(t)=(1-t^2)^{-n/2}P_m^n(t)$$

into the equation indicated in Problem 555.

559. Put

$$y(t)=(1-t^2)^{-n/2}Q_m^n(t)$$

in the equation indicated in Problem 555.

561. The assertion readily follows from formula (39) if we take into account the inequality

$$|\cos\theta + i\sin\theta\cos t| = \sqrt{\cos^2\theta + \sin^2\theta\cos^2 t} \leqslant 1$$

562. Let $r_n(t) = a_n t^n + \ldots + a_0$ be an arbitrary polynomial of degree less than m. Integrating by parts n times we obtain

$$\int_{-1}^{1} P_m(t)\,r_n(t)\,dt = \frac{1}{2^m m!}\int_{-1}^{1} r_n(t)\,\frac{d^m}{dt^m}(t^2-1)^m\,dt =$$

$$= \frac{(-1)^n a_n n!}{2^m m!}\int_{-1}^{1}\frac{d^{m-n}}{dt^{m-n}}(t^2-1)^m\,dt = 0$$

563. On putting $\theta = 0$ and $\theta = \pi$ in formula (39) we readily prove the assertion stated in the problem.

564. $P_{2m+1}(0) = 0, \quad P_{2m}(0) = (-1)^m \dfrac{(2m)!}{2^{2m}(m!)^2}$.

565. Since Laplace's operator can be written under the integral sign, the assertion follows from the relation

$$\Delta f(z + ix\cos t + iy\sin t,\ t) = (1 - \cos^2 t - \sin^2 t)\,\frac{\partial^2 f}{\partial \xi^2} = 0$$

where

$$\xi = z + ix\cos t + iy\sin t$$

566. According to the assertion proved in Problem 560, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(z + ix\cos t + iy\sin t)^m\,dt =$$

$$= \frac{r^m}{2\pi}\int_{-\pi}^{\pi}[\cos\theta + i\sin\theta\cos(t-\varphi)]^m\,dt =$$

$$= \frac{r^m}{2\pi}\int_{-\pi}^{\pi}[\cos\theta + i\sin\theta\cos\tau]^m\,d\tau = r^m P_m(\cos\theta)$$

567. Let us construct the solution of equation (42) by using formula (40) in which we put

$$K(z, t) = \mp \frac{1}{\pi} e^{-iz \sin t}$$

Then equation (41) assumes the form

$$v_{tt} + n^2 v = 0$$

and the function $v(t) = e^{\pm it}$ is a solution of the last equation.

568. We have

$$H_n^{(1)}(z) = \frac{1}{i\pi} \int_{-\infty}^{0} \exp(z \sinh \eta - n\eta)\, d\eta +$$

$$+ \frac{1}{\pi} \int_{-\pi}^{0} \exp(-iz \sin \xi + in\xi)\, d\xi +$$

$$+ \frac{1}{i\pi} \int_{0}^{\infty} \exp(-z \sinh \eta - n\eta - in\pi)\, d\eta$$

and

$$H_n^{2}(z) = -\frac{1}{i\pi} \int_{-\infty}^{0} \exp(z \sinh \eta - n\eta)\, d\eta +$$

$$+ \frac{1}{\pi} \int_{0}^{\pi} \exp(-iz \sin \xi + in\xi)\, d\xi -$$

$$- \frac{1}{i\pi} \int_{0}^{\infty} \exp(-z \sinh \eta - n\eta + in\pi)\, d\eta$$

569. $$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(-iz \sin \xi + in\xi)\, d\xi =$$

$$= \frac{1}{\pi} \int_{0}^{\pi} \cos(z \sin \xi - n\xi)\, d\xi.$$

570. Using the expression of $J_n(z)$ (see Problem 569) we obtain

$$J_{-n}(z)=\frac{1}{\pi}\int_0^{\pi}\cos(z\sin\xi+n\xi)\,d\xi=$$

$$=\frac{1}{\pi}\int_0^{\pi}\cos[z\sin(\pi-t)+n(\pi-t)]\,dt=$$

$$=(-1)^n\frac{1}{\pi}\int_0^{\pi}\cos(z\sin t-nt)\,dt=(-1)^n J_n(z)$$

571. The assertion follows from the inequality $|\cos t|\leqslant\leqslant 1$ which is valid for all real values of t.

572. Using the expression for $J_n(z)$ indicated in Problem 571 we obtain

$$u(x,\,y,\,z)=\frac{1}{2\pi}e^{\lambda z}\int_{-\pi}^{\pi}e^{i\lambda\rho\sin(t-\varphi)}e^{imt}\,dt=$$

$$=\frac{1}{2\pi}e^{\lambda z}\int_{-\pi}^{\pi}e^{i\lambda\rho\sin\psi}e^{im(\varphi+\psi)}\,d\psi=$$

$$=\frac{1}{\pi}e^{\lambda z}e^{im\varphi}\int_0^{\pi}\cos(\lambda\rho\sin\psi+m\psi)\,d\psi=e^{\lambda z}e^{im\varphi}J_{-m}(\lambda\rho)$$

573. (a) The solution of the problem

$$u_{tt}=a^2\Delta u,\quad 0\leqslant r<R,\quad t>0\quad\text{where}\quad \Delta u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

$$u(R,\,t)=0,\quad |u(0,\,t)|<\infty,\quad t>0$$

$$u(r,\,0)=A(R^2-r^2),\quad u_t(r,\,0)=0,\quad 0\leqslant r\leqslant R$$

is the function

$$u(r,\,t)=8AR^2\sum_{k=1}^{\infty}\frac{J_0\left(\frac{\mu_k r}{R}\right)}{\mu_k^3 J_1(\mu_k)}\cos\frac{a\mu_k t}{R}$$

where μ_k are the positive roots of the equation $J_0(\mu)=0$.

(b) The solution of the problem

$$u_{tt}=a^2\Delta u,\quad 0\leqslant r<R,\quad t>0\quad \text{where}\quad \Delta u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

$$u(R,\ t)=0,\quad |u(0,\ t)|<\infty,\quad t>0;$$
$$u(r,\ 0)=0,\quad u_t(r,\ 0)=U,\quad 0\leqslant r\leqslant R$$

is the function

$$u(r,\ t)=\frac{2RU}{a}\sum_{k=1}^{\infty}\frac{1}{\mu_k^2 J_1(\mu_k)}\sin\left(\frac{a\mu_k}{R}t\right)J_0\left(\frac{\mu_k r}{R}\right)$$

μ_k are the positive roots of the equation $J_0(\mu)=0$.

574. (a) The solution of the problem

$$u_t=a^2\Delta u,\quad 0\leqslant r<R,\quad t>0\quad \text{where}\quad \Delta u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

$$u_r(R,\ t)=0,\quad |u(0,\ t)|<\infty,\quad t>0;$$
$$u(r,\ 0)=Ur^2,\quad 0\leqslant r\leqslant R$$

is the function

$$u(r,\ t)=\frac{1}{2}UR^2+4UR^2\sum_{k=1}^{\infty}\frac{1}{\mu_k^2 J_0(\mu_k)}e^{-\left(\frac{a\mu_k}{R}\right)^2 t}J_0\left(\frac{\mu_k}{R}r\right)$$

where μ_k are the positive roots of the equation $J_0'(\mu)=0$.

(b) The solution of the problem

$$u_t=a^2\Delta u,\quad 0\leqslant r<R,\quad t>0\quad \text{where}\quad \Delta u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

$$u_r(R,\ t)+hu(R,\ t)=0,\quad |u(0,\ t)|<\infty,\quad t>0$$
$$u(r,\ 0)=Ur^2,\quad 0\leqslant r\leqslant R$$

is the function

$$u(r,\ t)=2R^2U\sum_{k=1}^{\infty}\frac{(2+hR)\mu_k^2-4hR}{\mu_k^2(\mu_k^2+h^2R^2)J_0(\mu_k)}e^{-\left(\frac{a\mu_k}{R}\right)^2 t}J_0\left(\frac{\mu_k}{R}r\right)$$

where μ_k are the positive roots of the equation $\mu J_0'(\mu)+$ $+hRJ_0(\mu)=0$.

(c) The solution of the problem

$$u_t = a^2\Delta u, \quad 0 \leqslant r < R, \quad t > 0 \quad \text{where} \quad \Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

$$u(R, t) = T, \quad |u(0, t)| < \infty, \quad t > 0;$$

$$u(r, 0) = Ur^2, \quad 0 \leqslant r < R$$

is the function

$$u(r, t) = T + 2\sum_{k=1}^{\infty} \frac{(UR^2 - T)\mu_k^2 - 4UR^2}{\mu_k^3 J_1(\mu_k)} e^{-\left(\frac{a\mu_k}{R}\right)^2 t} J_0\left(\frac{\mu_k}{R} r\right)$$

where μ_k are the positive roots of the equation $J_0(\mu) = 0$.

575. (a) The solution of the problem

$$u_t = a^2\Delta u, \quad \Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}$$

$$|u(r, z, t)| < \infty, \quad u(R, z, t) = u(r, 0, t) =$$

$$= u_z(r, l, t) = 0$$

$$u(r, z, 0) = A(R^2 - r^2)z, \quad 0 \leqslant r < R,$$

$$0 < z < l, \quad t > 0$$

is the function

$$u(r, z, t) = \sum_{\substack{k=1 \\ n=0}}^{\infty} a_{kn} e^{-a^2(\lambda_k^2 + \eta_n^2)t} J_0\left(\frac{\mu_k}{R} r\right) \sin\frac{(2n+1)\pi}{2l} z$$

where $a_{kn} = (-1)^n 32AlR^2 J_2(\mu_k)/(2n+1)^2\pi^2\mu_k^2 J_1^2(\mu_k)$, $\lambda_k = \mu_k/R$, $\eta_n = (2n+1)\pi/2l$ and μ_k are the positive roots of the equation $J_0(\mu) = 0$.

(b) The solution of the problem

$$u_t = a^2\Delta u \quad \text{where} \quad \Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2},$$

$$|u(r, z, t)| < \infty,$$

$$u_z(r, 0, t) = u(r, l, t) = u_r(R, z, t) + hu(R, z, t) = 0$$

$$u(r, z, 0) = A(R^2 - r^2)z, \quad 0 \leqslant r < R, \quad 0 < z < l,$$

$$t > 0$$

is the function

$$u(r, z, t)=\sum_{\substack{k=1\\n=0}}^{\infty} a_{kn}\, e^{-a^2(\lambda_k^2+\eta_n^2)t} J_0\left(\frac{\mu_k}{R} r\right) \cos \frac{(2n+1)\pi}{2l} z$$

where $a_{kn} = 16AlR^2 [(-1)^n (2n + 1) \pi -$

$$-2]\, J_2(\mu_k)[(2n+1)^2 \pi^2 (\mu_k^2+h^2R^2)\, J_0^2(\mu_k)]^{-1}, \quad \lambda_k=\mu_k/R$$

$\eta_n = (2n + 1)\pi/2l$ and μ_k are the positive roots of the equation $\mu J_0'(\mu) + hRJ_0(\mu) = 0$.

576. The solution of the problem

$$u_t=a^2\Delta u \quad \text{where} \quad \Delta u=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+ +\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)$$

$$|u(r, \theta, t)| < \infty, \quad u(R, \theta, t) = 0, \quad u(r, \theta, 0) = f(r, \theta)$$

$$0 \leqslant r < R, \quad 0 < \theta < \pi, \quad t > 0$$

is the function

$$u(r, \theta, t)=\sum_{\substack{n=0\\m=1}}^{\infty} a_{nm} e^{-\left(\frac{a\mu_{nm}}{R}\right)^2 t} P_n(\cos\theta)\, \frac{J_{n+1/2}\left(\frac{\mu_{nm}}{R} r\right)}{\sqrt{r}}$$

where

$$a_{nm}=\frac{2n+1}{R^2[J'_{n+1/2}(\mu_{nm})]^2}\int_0^R\int_0^\pi r^{3/2} f(r, \theta)\, J_{n+1/2} \times \times\left(\frac{\mu_{nm}}{R} r\right) P_n(\cos\theta)\sin\theta\, dr\, d\theta$$

and μ_{nm} are the positive roots of the equation $J_{n+1/2}(\mu) = 0$.

577. (a) The solution of the problem

$$\Delta u=0 \quad \text{where} \quad \Delta u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial z^2}$$

$$|u(r, z)| < \infty, \quad u(R, z) = u(r, l) = 0, \quad u(r, 0) = T$$

$$0 \leqslant r < R, \quad 0 < z < l$$

is the function

$$u(r, z)=2T\sum_{k=1}^{\infty}\frac{1}{\mu_k J_1(\mu_k)}\times$$

$$\times\left(\cosh\frac{\mu_k}{R}z-\coth\frac{\mu_k}{R}l\sinh\frac{\mu_k}{R}z\right)J_0\left(\frac{\mu_k}{R}r\right)$$

where μ_k are the positive roots of the equation $J_0\mu=0$.

(b) The solution of the problem

$$\Delta u=0 \text{ where } \Delta u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial z^2}$$

$$|u(r, z)|<\infty, \quad u(r, 0)=u_z(r, l)=0, \quad u(R, z)=f(z)$$

$$0\leqslant r<R, \quad 0<z<l$$

is the function

$$u(r, z)=\frac{2}{l}\sum_{k=1}^{\infty}\frac{\int\limits_0^l f(\xi)\sin\frac{(2k+1)\pi}{2l}\xi\,d\xi}{I_0\left[\frac{(2k+1)\pi R}{2l}\right]}\times$$

$$\times I_0\left[\frac{(2k+1)\pi}{2l}r\right]\sin\frac{(2k+1)\pi}{2l}z$$

(c) The solution of the problem

$$\Delta u=-\frac{Q}{k} \text{ where } \Delta u=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial z^2}$$

$$|u(r, z)|<\infty, \quad u(r, 0)=u(r, l)=u(R, z)=0$$

$$0\leqslant r<R, \quad 0<z<l$$

is the function

$$u(r, z)=\frac{Q}{4k}(R^2-r^2)+$$

$$+\frac{QR^2}{k}\sum_{n=1}^{\infty}\frac{J_2(\mu_n)}{\mu_n^2 J_1^2(\mu_n)\sinh\frac{\mu_n}{R}l}\left\{\left(\cosh\frac{\mu_n}{R}l-1\right)\sinh\frac{\mu_n}{R}z-\right.$$

$$\left.-\sinh\frac{\mu_n}{R}l\cosh\frac{\mu_n}{R}z\right\}J_0\left(\frac{\mu_n}{R}r\right)$$

where μ_k are the positive roots of the equation $J_0(\mu)=0$.

Hint: construct the solution in the form $u(r, z) = w(r) + v(r, z)$ where w satisfies the equation $\Delta w = -Q/k$.

578. The solution of the problem

$$\Delta u = 0 \quad \text{where} \quad \Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2}$$

$$|u(r, z)| < \infty, \quad u(r, 0) = u(r, l) = 0, \quad u(R, z) = T$$

$$R < r < \infty, \quad 0 < z < l$$

is the function

$$u(r, z) = \frac{4T}{\pi}\sum_{n=0}^{\infty}\frac{K_0\left[\frac{(2n+1)\pi}{l}r\right]\sin\frac{(2n+1)\pi}{l}z}{(2n+1)K_0\left[\frac{(2n+1)\pi}{l}R\right]}$$

where $K_0(\xi)$ is the *Macdonald function* (also called the *modified Hankel function* or the *modified Bessel function of the second kind*).

579. The solution of the problem

$$\Delta u = 0 \text{ where } \Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)$$

$$|u(r, \theta)| < \infty, \quad u(R, \theta) = f(\theta)$$

is the function $u(r, \theta)$ expressed by the formula

(a) $u(r, \theta) =$

$$= \sum_{n=0}^{\infty}\frac{(2n+1)}{2}\left\{\int_0^{\pi} f(\xi)P_n(\cos\xi)\sin\xi\, d\xi\right\}\left(\frac{r}{R}\right)^n P_n(\cos\theta)$$

for $0 \leqslant r < R$, $0 \leqslant \theta \leqslant \pi$, and by the formula

(b) $u(r, \theta) =$

$$= \sum_{n=0}^{\infty}\frac{(2n+1)}{2}\left\{\int_0^{\pi} f(\xi)P_n(\cos\xi)\sin\xi\, d\xi\right\}\left(\frac{R}{r}\right)^{n+1} P_n(\cos\theta)$$

for $R \leqslant r < \infty$, $0 \leqslant \theta \leqslant \pi$.

580. $e^{-z} \sim 0 + \frac{0}{z} + \ldots + \frac{0}{z^n} + \ldots = 0.$

581. The series indicated in the answer to Problem 580 provides the asymptotic expansion as $z \to 0$ for all functions of the form $f(z) = e^{-\omega z}$ where ω is an arbitrary positive number.

582. Since $0 < z < t < \infty$, the consecutive repetition of integration by parts results in

$$\int_z^\infty e^{(z^2-t^2)}\,dt = -\frac{1}{2}\int_z^\infty \frac{1}{t}\,de^{z^2-t^2} =$$

$$= \frac{1}{2z} - \frac{1}{2}\int_z^\infty e^{z^2-t^2}\frac{dt}{t^2} =$$

$$= \frac{1}{2z} - \frac{1}{2^2z^3} + \frac{1\cdot 3}{2^3z^5} - \frac{1\cdot 3\cdot 5}{2^4z^7} + \ldots$$

$$\ldots + (-1)^k\frac{1\cdot 3\cdot 5\ldots(2k-1)}{2^{k+1}z^{2k+1}} +$$

$$+ (-1)^{k+1}\frac{1\cdot 3\cdot 5\ldots(2k+1)}{2^{k+1}}\int_z^\infty \frac{e^{z^2-t^2}}{t^{2k+2}}\,dt$$

The integration by parts in the last integral on the right-hand side of this formula shows that the remainder satisfies inequality

$$\frac{1\cdot 3\cdot 5\ldots(2k+1)}{2^{k+1}}\int_z^\infty \frac{e^{z^2-t^2}}{t^{2k+2}}\,dt < \frac{1\cdot 3\cdot 5\ldots(2k+1)}{2^{k+2}z^{2k+3}}$$

whence follows the required asymptotic expansion.

583. $\displaystyle\int_z^\infty e^{z-t}\frac{t\,dt}{z^2} = -e^{z-t}\frac{t}{z^2}\Big|_z^\infty + \frac{1}{z^2}\int_z^\infty e^{z-t}\,dt = \frac{1}{z} + \frac{1}{z^2}.$

584. The condition $|\arg z| \leqslant \pi - \delta < \pi$ guarantees that the consecutive repetition of integration by parts is legitimate. Therefore we can write

$$\int_0^\infty \frac{e^{-t}}{t+z}\,dt = -\frac{e^{-t}}{t+z}\Big|_0^\infty - \int_0^\infty \frac{e^{-t}}{(t+z)^2}\,dt =$$

$$= \frac{1}{z} - \int_0^\infty \frac{e^{-t}}{(t+z)^2}\,dt = \frac{1}{z} + \frac{e^{-t}}{(t+z)^2}\Big|_0^\infty + 2\int_0^\infty \frac{e^{-t}}{(t+z)^3}\,dt =$$

$$= \frac{1}{z} - \frac{1}{z^2} + \ldots + \frac{(-1)^{n-1}(n-1)!}{z^n} +$$

$$+ (-1)^n n! \int_0^\infty \frac{e^{-t}}{(t+z)^{n+1}}\,dt$$

585. To derive the asymptotic expansion we write

$$\int_z^\infty e^{-t}t^{a-1}\,dt = -e^{-t}t^{a-1}\Big|_z^\infty + (a-1)\int_z^\infty e^{-t}t^{a-2}\,dt =$$

$$= e^{-z}z^{a-1} - (a-1)\,e^{-t}\,t^{a-2}\Big|_z^\infty + (a-1)(a-2)\int_z^\infty e^{-t}t^{a-3}\,dt =$$

$$= e^{-z}\,[z^{a-1} + (a-1)\,z^{a-2} + (a-1)(a-2)\,z^{a-3} + \ldots$$

$$\ldots + (a-1)(a-2)\ldots(a-k+1)\,z^{a-k}] +$$

$$+ (a-1)(a-2)\ldots(a-k+1)(a-k)\int_z^\infty e^{-t}e^{a-k-1}\,dt$$

Now, taking into account the identity $\Gamma(a+k) = (a+k-1)\,\Gamma(a+k-1)$ and the inequality

$$\left|\frac{\Gamma(a)}{\Gamma(a-k)}\int_z^\infty e^{-t}t^{a-k-1}\,dt\right| < \left|\frac{\Gamma(a)}{\Gamma(a-k)}\right| z^{a-k-1}\int_z^\infty e^{-t}\,dt =$$

$$= \left|\frac{\Gamma(a)}{\Gamma(a-k)}\right| e^{-z}z^{a-k-1}$$

which holds for $k > a-1$ we obtain the required asymptotic expansion.

586. The condition $a > 0$ guarantees that the consecutive integration by parts is legitimate. Therefore we have

$$\int_z^\infty t^{-a}e^{it}\,dt = \frac{ie^{iz}}{z^a} - ia\int_z^\infty t^{-a-1}e^{it}\,dt$$

The repetition of integration by parts k times yields

$$\int_z^\infty t^{-a}e^{it}\,dt = \frac{ie^{iz}}{z^a}\left[1 + \frac{a}{iz} + \frac{a(a+1)}{(iz)^2} + \dots\right.$$
$$\left.\dots + \frac{a(a+1)\dots(a+k-1)}{(iz)^k}\right] +$$
$$+ \frac{a(a+1)\dots(a+k)}{i^{k+1}}\int_z^\infty \frac{e^{it}}{t^{a+k+1}}\,dt$$

Finally, taking into account the identity $\Gamma(a+k) = (a+k-1)\Gamma(a+k-1)$ and the inequality

$$\frac{\Gamma(a+k+1)}{\Gamma(a)}\left|\int_z^\infty \frac{e^{it}}{t^{a+k+1}}\,dt\right| \leqslant$$
$$\leqslant \frac{\Gamma(a+k+1)}{\Gamma(a)}\int_z^\infty \frac{dt}{t^{a+k+1}} = \frac{\Gamma(a+k)}{\Gamma(a)\,z^{a+k}}$$

we obtain the required asymptotic expansion.

587. Using the result established in Problem 582 we obtain

$$\frac{2}{\sqrt{\pi}}\int_z^\infty e^{-\tau^2}\,d\tau = \frac{2}{\sqrt{\pi}}e^{-z^2}\int_z^\infty e^{z^2-\tau^2}\,d\tau \sim$$
$$\sim \frac{2}{\sqrt{\pi}}e^{-z^2}\left[\frac{1}{2z} + \sum_{k=1}^\infty (-1)^k\frac{1\cdot 3\dots(2k-1)}{2^{k+1}z^{2k+1}}\right] =$$
$$= e^{-z^2}\sum_{k=1}^\infty \frac{z^{1-2k}}{\Gamma\left(\frac{3}{2}-k\right)}$$

588. Putting $a = 1/2$, $t = \theta^2$ and $u = z$ in the expression indicated in Problem 586 we obtain

$$\int_u^\infty e^{i\theta^2}\,d\theta \sim \frac{ie^{iu}}{2\sqrt{\pi u}} \sum_{k=0}^{\infty} \frac{(-1)^k\, i^k \Gamma\left(k+\frac{1}{2}\right)}{u^k} =$$

$$= \frac{ie^{iu}}{2\sqrt{\pi u}} \sum_{k=0}^{\infty} \frac{(-1)^k\, \Gamma\left(2k+\frac{1}{2}\right)}{u^{2k}} \left[1 - \frac{i\left(2k+\frac{1}{2}\right)}{u}\right]$$

whence

$$\int_u^\infty \cos\theta^2\,d\theta \sim$$

$$\sim \frac{1}{2\sqrt{\pi u}} \sum_{k=0}^{\infty} \frac{(-1)^k\, \Gamma\left(2k+\frac{1}{2}\right)}{u^{2k}} \left[\frac{\left(2k+\frac{1}{2}\right)}{u}\cos u - \sin u\right]$$

and

$$\int_u^\infty \sin\theta^2\,d\theta \sim$$

$$\sim \frac{1}{2\sqrt{\pi u}} \sum_{k=0}^{\infty} \frac{(-1)^k\, \Gamma\left(2k+\frac{1}{2}\right)}{u^{2k}} \left[\frac{\left(2k+\frac{1}{2}\right)}{u}\sin u + \cos u\right]$$

589. $\mathrm{Ei}(z) \sim \dfrac{e^z}{z} \displaystyle\sum_{k=0}^{\infty} \frac{k!}{z^k}$.

590. $\mathrm{Ci}(z) \sim \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{z^{2k+1}} \left(\sin z - \frac{2k+1}{z}\cos z\right)$.

591. $\mathrm{Si}(z) \sim -\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{z^{2k+1}} \left(\cos z + \frac{2k+1}{z}\sin z\right)$.

592. In the case under consideration we have $N = \infty$, $m = 0$, $\alpha = 1$ and $\varphi(t) = \sum\limits_{k=0}^{\infty} (-1)^k t^{2nk}$; for $z_0 > 0$ the

integral $\int\limits_0^\infty \frac{e^{-zt}}{1+t^{2n}}\,dt$ is absolutely convergent. Therefore, by virtue of Watson's formula (45), the sought-for asymptotic series has the form

$$\sum_{k=0}^{\infty} \frac{(-1)^k\,\Gamma(2nk+1)}{z^{2nk+1}}$$

593. Since $m=p-1>-1$, $\alpha=1$ and $\varphi(t)\equiv 1$, Watson's formula (45) implies the required assertion.

594. The assertion stated in the problem follows from Watson's formula (45′) and from the identity $\sin t + \sin(-t)\equiv 0$.

595. Take into account that in the case under consideration $A=-1$, $N=2$, $\varphi(t)+\varphi(-t)=2\cos t$ and use Watson's formula (45′).

596. $F(z)=\int\limits_0^\infty e^{-z^2t^2}\,dt \sim \frac{1}{2}\Gamma\left(\frac{1}{2}\right)z^{-1}=\frac{\sqrt{\pi}}{2z}$.

597. Let us denote $F(z)=e^{-z^2}\int\limits_0^z e^{\xi^2}\,d\xi$. On using twice L'Hospital's rule we find $\lim\limits_{z\to\infty} 2zF(z)=1$, whence $2zF(z)=1+o(1)$, that is $F(z)=\frac{1}{2z}[1+o(1)]$ for $z\to+\infty$, which is what we had to prove.

600. Let us use the Fourier transformation with respect to the variable x:

$$U(\xi,t)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty} e^{-i\xi x}u(x,t)\,dx$$

This reduces the equation of the original problem to the form

$$U_{tt}+a^2\xi^2U=0$$

whence $U(\xi,t)=A(\xi)\,e^{-i\xi at}+B(\xi)\,e^{i\xi at}$ where $A(\xi)$ and $B(\xi)$ are arbitrary functions of the parameter ξ. Further, resorting to the inverse Fourier transformation, we obtain

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} U(\xi, t)\, d\xi =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [A(\xi)\, e^{i\xi(x-at)} + B(\xi)\, e^{i\xi(x+at)}]\, d\xi =$$

$$= A(x-at) + B(x+at)$$

Finally, taking into account the initial conditions of the problem we find its solution in the form

$$u(x, t) = \frac{1}{2}\varphi(x-at) + \frac{1}{2}\varphi(x+at) + \frac{1}{2a}\int_{x-at}^{x+at} \psi(z)\, dz$$

601. Using the Fourier integral transformation

$$U(\xi, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} u(x, t)\, dx$$

with respect to the variable x we reduce the original problem to the new problem

$$U_{tt} + a^2\xi^2 U = F(\xi, t), \quad U(\xi, 0) = U_t(\xi, 0) = 0$$

where

$$F(\xi, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} f(x, t)\, dx$$

The solution of the last problem yields

$$U(\xi, t) = \frac{1}{a\xi} \int_{0}^{t} F(\xi, \tau) \sin a\xi(t-\tau)\, d\tau \qquad (*)$$

The application of Fourier's inverse transformation gives

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} U(\xi, t)\, d\xi \qquad (**)$$

where $U(\xi, t)$ is expressed by formula (*). Now, taking into account that $\sin a\xi(t-\tau) = (2i)^{-1}[e^{ia\xi(t-\tau)} - e^{-ia\xi(t-\tau)}]$ we derive from (*) and (**) the expression

$$u(x, t) = \frac{1}{2a\sqrt{2\pi}} \int_0^t d\tau \int_{-\infty}^{\infty} \frac{1}{i\xi} \{e^{i\xi[x+a(t-\tau)]} - e^{i\xi[x-a(t-\tau)]}\} F(\xi, \tau)\, d\xi$$

Since

$$\frac{1}{i\xi}\{e^{i\xi[x+a(t-\tau)]} - e^{i\xi[x-a(t-\tau)]}\} = \int_{x-a(t-\tau)}^{x+a(t-\tau)} e^{i\xi\eta}\, d\eta$$

we obtain

$$u(x, t) = \frac{1}{2a} \int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} \left\{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi\eta} F(\xi, \tau)\, d\xi\right\} d\eta$$

that is

$$u(x, t) = \frac{1}{2a} \int_0^t d\tau \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\eta, \tau)\, d\eta$$

602. Using Fourier's transformation

$$U(\xi, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x} u(x, t)\, dx$$

(with respect to the variable x) we reduce the original problem to the new problem

$$U_t + a^2\xi^2 U = 0, \quad U(\xi, 0) = \Phi(\xi)$$

where $\Phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\xi x}\varphi(x)\, dx$. The solution of the last problem has the form

$$U(\xi, t) = \Phi(\xi)\, e^{-a^2\xi^2 t}$$

On applying the inverse transformation we obtain

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\xi x} U(\xi, t)\, d\xi =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\eta)\, d\eta \int_{-\infty}^{\infty} e^{-a^2\xi^2 t} e^{-i\xi(\eta - x)}\, d\xi =$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(\eta)\, d\eta \int_{0}^{\infty} e^{-a^2\xi^2 t} \cos \xi (\eta - x)\, d\xi$$

whence, taking into account the equality

$$\int_{0}^{\infty} e^{-a^2\xi^2 t} \cos \xi (\eta - x)\, d\xi = \frac{1}{2a} \sqrt{\frac{\pi}{t}}\, e^{-\frac{(x-\eta)^2}{4a^2 t}} \qquad (*)$$

we find

$$u(x, t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \varphi(\eta)\, e^{-\frac{(x-\eta)^2}{4a^2 t}}\, d\eta$$

603. The solution is

$$u(x, t) = \frac{1}{2a\sqrt{\pi}} \int_{0}^{t} d\tau \int_{-\infty}^{\infty} f(\eta, \tau)\, \frac{e^{-\frac{(x-\eta)^2}{4a^2(t-\tau)}}}{\sqrt{t-\tau}}\, d\eta$$

See the solution of Problem 602.

604. To solve the problem we shall use *Fourier's sine-transform*

$$U(\xi, t) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} u(x, t) \sin \xi x\, dx$$

of the function $u(x, t)$. Taking into account the boundary condition $u(0, t) = \mu(t)$ and assuming that the function u and its partial derivative with respect to x tend to zero sufficiently fast for $x \to \infty$ we obtain

$$U_t(\xi, t) = a^2 \sqrt{\frac{2}{\pi}} \int_0^\infty u_{xx} \sin \xi x \, dx = a^2 \sqrt{\frac{2}{\pi}} u_x \sin \xi x \Big|_0^\infty -$$

$$-a^2\xi \sqrt{\frac{2}{\pi}} \int_0^\infty u_x \cos \xi x \, dx =$$

$$= -a^2\xi \sqrt{\frac{2}{\pi}} \int_0^\infty u_x \cos \xi x \, dx =$$

$$= -a^2\xi \sqrt{\frac{2}{\pi}} \left\{ u \cos \xi x \Big|_0^\infty + \xi \int_0^\infty u(x, t) \sin \xi x \, dx \right\} =$$

$$= a^2\xi \sqrt{\frac{2}{\pi}} \mu(t) - a^2\xi^2 U(\xi, t)$$

Thus, the original problem reduces to the new problem

$$U_t + a^2\xi^2 U = a^2\xi \sqrt{\frac{2}{\pi}} \mu(t), \quad U(\xi, 0) = 0$$

The solution of the last problem is

$$U(\xi, t) = a^2\xi \sqrt{\frac{2}{\pi}} \int_0^t e^{-a^2\xi^2(t-\tau)} \mu(\tau) \, d\tau$$

On performing the inverse transformation, we arrive at the expression

$$u(x, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty U(\xi, t) \sin \xi x \, d\xi =$$

$$= \frac{2a^2}{\pi} \int_0^t \mu(\tau) \, d\tau \int_0^\infty \xi e^{-a^2\xi^2(t-\tau)} \sin \xi x \, d\xi =$$

$$= -\int_0^t \frac{\mu(\tau) \, d\tau}{\pi(t-\tau)} \left[e^{-a^2\xi^2(t-\tau)} \sin \xi x \Big|_{\xi=0}^{\xi=\infty} - \right.$$

$$-x\int_0^\infty e^{-a^2\xi^2(t-\tau)}\cos\xi x\,d\xi\Big]=$$

$$=\frac{x}{\pi}\int_0^t\frac{\mu(\tau)\,d\tau}{(t-\tau)}\int_0^\infty e^{-a^2\xi^2(t-\tau)}\cos\xi x\,d\xi$$

whence, taking into account equality (*) established in the solution of Problem 602, we obtain the sought-for answer to the problem under consideration:

$$u(x,\ t)=\frac{x}{2a\sqrt{\pi}}\int_0^t\frac{\mu(\tau)}{(t-\tau)^{3/2}}e^{-\frac{x^2}{4a^2(t-\tau)}}\,d\tau$$

605. The solution is

$$u(x,\ t)=-\frac{a}{\sqrt{\pi}}\int_0^t\frac{\nu(\tau)}{\sqrt{t-\tau}}e^{-\frac{x^2}{4a^2(t-\tau)}}\,d\tau$$

To solve the problem use *Fourier's cosine-transform* of the function $u(x,\ t)$; also see the solution of Problem 604.

606. $$u(x,\ t)=\frac{1}{2a\sqrt{\pi}}\int_0^t\frac{d\tau}{\sqrt{t-\tau}}\int_0^\infty\Big[e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)}}-$$
$$-e^{-\frac{(x+\xi)^2}{4a^2(t-\tau)}}\Big]f(\xi,\ \tau)\,d\xi$$

607. The solution is

$$u(x,\ y,\ t)=\frac{1}{(2a\sqrt{\pi t})^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}}\varphi(\xi,\ \eta)\,d\xi\,d\eta$$

To solve the problem use the *(two-dimensional) Fourier's transformation* and the corresponding *inverse transformation* which are specified by the formulas

$$F(\xi,\ \eta)=\frac{1}{(\sqrt{2\pi})^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i(\xi x+\eta y)}f(x,\ y)\,dx\,dy$$

and

$$f(x, y) = \frac{1}{(\sqrt{2\pi})^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(\xi x + \eta y)} F(\xi, \eta) \, d\xi \, d\eta$$

respectively.

608. The solution is

$$u(x, y, t) =$$

$$= \frac{1}{(2a\sqrt{\pi})^2} \int_0^t \frac{d\tau}{(\sqrt{t-\tau})^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2(t-\tau)}} f(\xi, \eta, \tau) \, d\xi \, d\eta$$

See the hint to the solution of Problem 607.

609. The solution is

$$u(x, y, t) =$$

$$= \frac{1}{(2a\sqrt{\pi t})^2} \int_{-\infty}^{\infty} d\xi \int_0^{\infty} [e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}} - e^{-\frac{(x-\xi)^2+(y+\eta)^2}{4a^2t}}] f(\xi, \eta) \, d\eta$$

Hint: use the *Fourier integral transformation* with the kernel

$$K(x, y; \xi, \eta) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} e^{-ix\xi} \sin y\eta$$

for $-\infty < x < \infty$, $0 < y < \infty$.

610. The solution is

$$u(x, y, t) = \frac{y}{(2a\sqrt{\pi})^2} \int_0^t \frac{d\tau}{(t-\tau)^2} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2+y^2}{4a^2(t-\tau)}} f(\xi, \tau) \, d\xi$$

See the hint to the solution of Problem 609.

611. The solution is

$$u(x, y, t) =$$

$$= \frac{1}{(2a\sqrt{\pi t})^2} \int_{-\infty}^{\infty} d\xi \int_0^{\infty} [e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4a^2t}} + e^{-\frac{(x-\xi)^2+(y+\eta)^2}{4a^2t}}] f(\xi, \eta) \, d\eta$$

To solve the problem make use of Fourier's integral transformation with the kernel

$$K(x, y; \xi, \eta) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} e^{-i\xi x} \cos y\eta$$

for $-\infty < x < \infty$, $0 < y < \infty$.

612. Let us pass to the *Laplace transforms* (*images*) $U(\zeta, y)$ and $F(\zeta)$ of the functions $u(x, y)$ and $f(x)$ with respect to the variable x. Then the original problem reduces to the equation

$$U_y - (\zeta^2 + a^2) U = F$$

whence

$$U(\zeta, y) = Ce^{(\zeta^2+a^2)y} - \frac{F(\zeta)}{\zeta^2+a^2}$$

Since $y > 0$ and $U(\zeta, y) \to 0$ for $\zeta \to \infty$, there must be $C = 0$, that is $U(\zeta, y) = -F(\zeta)/(\zeta^2 + a^2)$. Consequently,

$$U(x, y) = -\frac{1}{a} \int_0^x f(x-\xi) \sin a\xi \, d\xi$$

613. The solution is $u(x, y) = Ae^{-3y} \cos 2x - Bx \sin x/2$. See the solution of Problem 612.

614. (a) The mathematical statement of the problem reads thus:

$$u_t = a^2 u_{xx}, \quad 0 < x < l, \quad t > 0$$

$$u(+0, t) = \delta(t), \quad u(l-0, t) = 0, \quad t > 0;$$

$$u(x, +0) = 0, \quad 0 < x < l$$

Let us pass to the Laplace transform of the function $u(x, t)$ with respect to the variable t:

$$U(x, \zeta) = \int_0^\infty e^{-\zeta t} u(x, t) \, dt$$

Then the problem we have stated reduces to the new problem

$$U_{xx} - \frac{\zeta}{a^2} U = 0, \quad 0 < x < l,$$

$$U(+0, \zeta) = 1, \quad U(l-0, \zeta) = 0$$

whose solution is

$$U(x,\ \zeta)=\frac{\sinh\frac{l-x}{a}\sqrt{\zeta}}{\sinh\frac{l}{a}\sqrt{\zeta}}$$

To obtain the sought-for solution $u(x,\ t)$ (which is the *original* of the function $U(x,\ \zeta)$) we first transform the expression of $U(x,\ \zeta)$:

$$U(x,\ \zeta)=\frac{e^{-\frac{x}{a}\sqrt{\zeta}}-e^{-\frac{(2l-x)}{a}\sqrt{\zeta}}}{1-e^{-\frac{2l}{a}\sqrt{\zeta}}}=$$

$$=\left(e^{-\frac{x}{a}\sqrt{\zeta}}-e^{-\frac{(2l-x)}{a}\sqrt{\zeta}}\right)\sum_{n=0}^{\infty}e^{-\frac{2ln}{a}\sqrt{\zeta}}=$$

$$=\sum_{n=0}^{\infty}e^{-\frac{(2nl+x)}{a}\sqrt{\zeta}}-\sum_{n=1}^{\infty}e^{-\frac{(2nl-x)}{a}\sqrt{\zeta}} \qquad (*)$$

For $\xi>0$ the Laplace transform of the function $\psi(\xi,\ t)=$ $=\xi e^{-\xi^2/4t}/2\sqrt{\pi}\ t^{3/2}$ is equal to $e^{-\xi\sqrt{\zeta}}$ (see the table of the originals and their Laplace transforms at the end of the book); therefore from (*) we readily find the original $u(x,\ t)$ of the transform $U(x,\ \zeta)$:

$$u(x,\ t)=\sum_{n=0}^{\infty}\psi\left(\frac{2nl+x}{a},\ t\right)-\sum_{n=1}^{\infty}\psi\left(\frac{2nl-x}{a},\ t\right)$$

Finally, taking into account that the function $\psi(x,\ t)$ is odd with respect to the variable x, we obtain

$$u(x,\ t)=\sum_{n=-\infty}^{\infty}\psi\left(\frac{2nl+x}{a},\ t\right)=$$

$$=\frac{1}{2a\sqrt{\pi}\,t^{3/2}}\sum_{n=-\infty}^{\infty}(2nl+x)\,e^{-\frac{(2nl+x)^2}{4a^2t}}$$

(b) The solution of the problem

$$u_t = a^2 u_{xx}, \quad 0 < x < \infty, \quad t > 0$$

$$u(+0, t) = \delta(t), \quad u(\infty - 0, t) = 0, \quad t > 0;$$

$$u(x, +0) = 0, \quad 0 < x < \infty$$

is the function

$$u(x, t) = \frac{x}{2a\sqrt{\pi}\, t^{3/2}} e^{-\frac{x^2}{4a^2 t}}$$

(see also Case (a)).

(c) The solution of the problem

$$u_t = a^2 u_{xx}, \quad 0 < x < \infty, \quad t > 0$$

$$u(+0, t) = \mu(t), \quad u(\infty - 0, t) = 0, \quad t > 0;$$

$$u(x, +0) = 0, \quad 0 < x < \infty$$

is the function

$$u(x, t) = \frac{x}{2a\sqrt{\pi}} \int_0^t \mu(\tau) \frac{e^{-\frac{x^2}{4a^2(t-\tau)}}}{(t-\tau)^{3/2}}\, d\tau$$

(see also Case (a)). Compare the result with the solution of Problem 604.

615. (a) The mathematical statement of the problem is the following:

$$u_{tt} = a^2 u_{xx}, \quad 0 < x < \infty, \quad t > 0, \quad a = \frac{1}{\sqrt{LC}}$$

$u(0, t) = E(t)$, $t > 0$ and $u(x, t)$ is bounded for $x \to \infty$

$$u(x, 0) = u_t(x, 0) = 0, \quad 0 < x < \infty$$

Using the Laplace transformation with respect to the variable t we find the solution of this problem in the form

$$u(x, t) = \begin{cases} 0 & \text{for} \quad t < \frac{x}{a} = x\sqrt{LC} \\ E(t - x\sqrt{LC}) & \text{for} \quad t > x\sqrt{LC} \end{cases}$$

(b) The solution of the problem

$$u_{xx} = a^2 u_{tt} + 2bu_t + c^2 u, \quad 0 < x < \infty, \quad t > 0$$

$$u(0, t) = E(t), \quad t > 0; \quad u(x, t) \text{ is bounded for } x \to \infty$$

$$u(x, 0) = u_t(x, 0) = 0, \quad 0 < x < \infty$$

$$a^2 = LC, \quad b = \frac{1}{2}(CR + LG), \quad c^2 = RG$$

is the function

$$u(x, t) = \begin{cases} 0 & \text{for } t < ax \\ e^{-amx} E(t - ax) & \text{for } t > ax \end{cases}$$

where $m = b/a^2$.

616. $$u(x, t) = \begin{cases} 0 & \text{for } x > at, \\ -ae^{h(x-at)} \int\limits_0^{t-\frac{x}{a}} e^{ah\tau} \varphi(\tau)\, d\tau & \text{for } x < at. \end{cases}$$

617. (a) The mathematical statement of the problem of the determination of the temperature $u \equiv u(r, z)$ is:

$$\Delta u \equiv \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 \leqslant r < \infty, \quad z > 0$$

$$u(r, 0) = f(r), \quad u(r, \infty) = 0, \quad 0 \leqslant r < \infty$$

$$u(\infty, z) = u_r(\infty, z) = 0, \quad z > 0$$

Let us multiply both members of the equation

$$\frac{\partial^2 u}{\partial z^2} = -\frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right)$$

by $rJ_0(\eta r)$ and integrate the resultant relation with respect to r from 0 to ∞. Integrating by parts and using the boundary conditions $u(\infty, z) = u_r(\infty, z) = 0$ we derive the relation

$$\int_0^\infty rJ_0(\eta r)\frac{\partial^2 u}{\partial z^2}\,dr = -\int_0^\infty J_0(\eta r)\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)dr =$$

$$= -rJ_0(\eta r)\frac{\partial u}{\partial r}\Big|_{r=0}^{r=\infty} + \eta\int_0^\infty rJ_0'(\eta r)\frac{\partial u}{\partial r}\,dr =$$

$$= \eta\int_0^\infty rJ_0'(\eta r)\frac{\partial u}{\partial r}\,dr = \eta\left\{ruJ_0'(\eta r)\Big|_{r=0}^{r=\infty} -\right.$$

$$\left.-\int_0^\infty u\frac{\partial}{\partial r}[rJ_0'(\eta r)]\,dr\right\} = -\eta\int_0^\infty u\frac{\partial}{\partial r}[rJ_0'(\eta r)]\,dr =$$

$$= -\eta\int_0^\infty uJ_0'(\eta r)\,dr - \eta^2\int_0^\infty ruJ_0''(\eta r)\,dr$$

On finding $J_0'(\eta r)$ from the equation

$$\frac{d^2}{dr^2}J_0(\eta r) + \frac{1}{r}\frac{d}{dr}J_0(\eta r) + \eta^2 J_0(\eta r) = 0$$

we obtain

$$\int_0^\infty rJ_0(\eta r)\frac{\partial^2 u}{\partial z^2}\,dr = \eta^2\int_0^\infty rJ_0(\eta r)u\,dr$$

whence

$$U_{zz} = \eta^2 U$$

where $U(\eta, z) = \int_0^\infty rJ_0(\eta r)\,u(r, z)\,dr$ is the *Hankel transform* of the function $u(r, z)$.

Thus, with the aid of *Hankel's transformation* the problem under consideration reduces to the new problem

$$U_{zz} - \eta^2 U = 0, \quad 0 < z < \infty; \quad U(\eta, 0) = F(\eta),$$
$$U(\eta, \infty) = 0$$

where $F(\eta) = \int_0^\infty rJ_0(\eta r)\,f(r)\,dr$. The solution of the last problem is

$$U(\eta, z) = F(\eta)\, e^{-\eta z}$$

Finally, performing *Hankel's inverse transformation* we find the solution of the original problem:

$$u(r, z) = \int_0^\infty \eta J_0(\eta r) F(\eta)\, e^{-\eta z}\, d\eta =$$

$$= \int_0^\infty \eta J_0(\eta r)\, e^{-\eta z} \left[\int_0^\infty \rho J_0(\eta\rho) f(\rho)\, d\rho \right] d\eta$$

(b) The solution is

$$u(r, z) = TR \int_0^\infty J_0(\eta r) J_1(R\eta)\, e^{-\eta z}\, d\eta$$

See the solution of the problem for Case (a).

(c) The solution of the problem

$$\Delta u \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = 0, \quad 0 \leqslant r < \infty, \quad z > 0$$

$$u_z(r, 0) = \begin{cases} -\frac{q}{k} + hu(r, 0) & \text{for} \quad 0 \leqslant r < R \\ hu(r, 0) & \text{for} \quad R \leqslant r < \infty \end{cases}$$

$$u(r, \infty) = 0, \quad 0 \leqslant r < \infty; \quad u(\infty, z) = u_r(\infty, z) = 0, \quad z > 0$$

is the function

$$u(r, z) = \frac{qR}{k} \int_0^\infty \frac{e^{-\eta z}}{\eta + h} J_0(\eta r) J_1(\eta R)\, d\eta$$

See the solution of the problem for Case (a).

618. $u(x+h, y) + u(x-h, y) + u(x, y+h) + u(x, y-h) - 4u(x, y) = 0.$

619. In the case under consideration the finite-difference approximation of Laplace's equation has the form

$$u(x+1, y) + u(x-1, y) + u(x, y+1) + u(x, y-1) - 4u(x, y) = 0.$$

In accordance with the finite-difference scheme, we must prescribe for the vertices of the squares belonging to Q_δ the corresponding boundary values of $u(x, y)$; for the points (0, 0), (1, 0), (—1, 0), (0, 1) and (0, —1) the values of $u(x, y)$ must be found from the algebraic linear system of equations

$$
\begin{aligned}
&u(1, 0)+u(-1, 0)+u(0, 1)+u(0, -1)-4u(0, 0)=0\\
&4u(1, 0) - u(0, 0)=u(2, 0)+u(1, 1)+u(1, -1)\\
&4u(0, 1 - u(0, 0)=u(1, 1)+u(-1, 1)+u(0, 2)\\
&4u(-1, 0) - u(0, 0)=u(-2, 0)+u(-1, 1)+u(-1, -1)\\
&4u(0, -1)- u(0, 0)=u(1, -1)+u(-1, -1)+u(0, -2)
\end{aligned}
$$

whose determinant is different from zero. On solving this system for each of the cases under consideration we obtain the following results:

(a) $u(1, 0) = u(-1, 0) = u(0, 1) = u(0, -1) = u(0, 0) = 0$; the corresponding exact solution is $u(x, y) = 0$;

(b) $u(1, 0) = u(-1, 0) = u(0, 1) = u(0, -1) = u(0, 0) = 1$; the corresponding exact solution is $u(x, y) = 1$;

(c) $u(1, 0) = -u(-1, 0) = 1 + \sqrt{2}$; $u(0, 1) = u(0, -1) = u(0, 0) = 0$; the exact solution is $u(x, y) = x$.

620. The values of $u(x, y)$ at the vertices of the squares belonging to Q_δ are found according to the above finite-difference scheme; the values $u(0, 0)$, $u(0, 1)$ and $u(0, -1)$ are found from the linear system

$$
\begin{aligned}
&4u(0, 0)- u(0, 1) - u(0, -1)=u(1, 0)+u(-1, 0)\\
&u(0, 0)-4u(0, 1) = -u(1, 1)-u(-1, 1)-u(0, 2)\\
&u(0, 0) -4u(0, -1)= -u(1, -1)-u(-1, -1)-u(0, -2)
\end{aligned}
$$

The solution of the system yields the following results:

(a) $u(0, 0) = u(0, 1) = u(0, -1) = 1$; the exact solution is $u(x, y) = 1$;

(b) $u(0, 0) = 0$, $u(0, 1) = \frac{3}{2}$, $u(0, -1) = -\frac{3}{2}$; the exact solution is $u(x, y) = y$;

(c) $u(0, 0) = 0$, $u(0, 1) = \frac{3}{2}$, $u(0, -1) = -\frac{3}{2}$; the exact solution is $u(x, y) = x + y$.

621. $u(x+h, t) + u(x-h, t) - 2u(x, t) - hu(x, t) + hu(x, t-h) = 0.$

622. The values of $u(x, t)$ at the points (1, 5), (1, 4), (1, 3), (1, 2), (1, 1), (2, 1), (3, 1), (3, 2), (3, 3), (3, 4) and (3, 5) are expressed in terms of the boundary values according to the above finite-difference scheme; to determine the values $u(2, 2)$, $u(2, 3)$ and $u(2, 4)$ we must solve the algebraic linear system of equations

$$\left.\begin{aligned} u(2, 2) - 3u(2, 3) &= -u(1, 3) - u(3, 3) \\ u(2, 3) - 3u(2, 4) &= -u(1, 4) - u(3, 4) \\ 3u(2, 2) &= u(1, 2) + u(2, 1) + \\ &\quad + u(3, 2) \end{aligned}\right\}$$

whose determinant is different from zero.

In the case under consideration we obtain $u(2, 2) = u(2, 3) = u(2, 4) = 2$; the exact solution is $u(x, y) = x$.

623. $u(2, 2) = 31/8$, $u(3, 2) = 61/8$.

624. The finite-difference approximation of the given equation is written thus:

$$u(x+h, y+h) - u(x+h, y) - u(x, y+h) + u(x, y) = 0$$

As the value of $u(x, y)$ at each grid-point which is a vertex of a square cell adjoining the coordinate axis we take the given value of $u(x, y)$ at the grid-point lying on the coordinate axis which is the nearest to that grid-point. To determine $u(2, 2)$, $u(2, 3)$ and $u(2, 4)$ we must solve the system of linear equations

$$\left.\begin{aligned} u(2,2)-u(2,3) &= u(1,2)-u(1,3) \\ u(2,3)-u(2,4) &= u(1,3)-u(1,4) \\ u(2,2) &= u(2,1)+u(1,2)-u(1,1) \end{aligned}\right\}$$

For each of the cases under consideration the solutions of that linear system are:

(a) $u(2, 2) = u(2, 3) = u(2, 4) = 2$ or $u(2, 2) = u(2, 3) = u(2, 4) = 1$;

(b) $u(2, 2) = 2$, $u(2, 3) = 3$, $u(2, 4) = 4$ or $u(2, 2) = 1$, $u(2, 3) = 2$, $u(2, 4) = 3$;

(c) $u(2, 2) = 3$, $u(2, 3) = 4$, $u(2, 4) = 5$.

The two solutions in each of the cases (a) and (b) appear due to the two possible values of $u(1, 1)$ at the grid-point $(1, 1)$ lying at the same distance from the coordinate axes on which different data are prescribed.

625. Let us assume additionally that the boundary S of the domain D and the functions $u(x, y)$ and $h(x, y)$ considered below are such that the identities

$$u_x h_x + u_y h_y = (u_x h)_x + (u_y h)_y - h\Delta u, \quad (x, y) \in D$$

$$D(u, h) = \int_D (h_x u_x + h_y u_y)\, dx\, dy = \int_S h \frac{\partial u}{\partial \nu}\, ds - \int_D h\Delta u\, dx\, dy \quad (*)$$

hold. Then, if $u(x, y)$ is the solution of the Dirichlet problem

$$\Delta u(x, y) = 0, \quad (x, y) \in D; \quad u(x, y) = \varphi(x, y), \quad (x, y) \in S$$

we can take as the class of the admitted functions the collection of the functions of the form $u(x, y) + \varepsilon h(x, y)$ where ε is an arbitrary constant and $h(x, y)$ is an arbitrary function belonging to the class of the admitted functions and satisfying the boundary condition $h(x, y) = 0$, $(x, y) \in S$. In this case from the identity

$$D(u + \varepsilon h) = D(u) + 2\varepsilon D(u, h) + \varepsilon^2 D(h) \quad (**)$$

it follows that $D(u) \leqslant D(u + \varepsilon h)$, which means that $u(x, y)$ is the minimizing function.

Now let us suppose that $u(x, y)$ is the minimizing function. Then identity (**) implies $D(u, h) = 0$. Indeed, if otherwise, we can choose the constant ε so that the expression $\varepsilon D(u, h)$ becomes negative and therefore identity (**) results in the inconsistent inequality $D(u) > D(u + \varepsilon h)$. On the basis of the equalities $D(u, h) = 0$ and $h(x, y) = 0$ holding for $(x, y) \in S$ we conclude from (*) that

$$\int_D h\Delta u\, dx\, dy = 0$$

whence, since h is arbitrary, it follows that $\Delta u = 0$. This means that $u(x, y)$ is the solution of the Dirichlet problem.

626. The functional I_n exists for the class of continuously differentiable functions $y(x)$, $0 \leqslant x \leqslant 1$ satisfying the condition $y(0) = 0$. Since $y(0) = 0$ and $y(1) = a$ we can write

$$I_n(y) = \int_0^1 \left(\frac{dy}{dx} - \frac{n}{x} y\right)^2 x\, dx + 2n \int_0^1 y \frac{dy}{dx}\, dx = \int_0^1 \left(\frac{dy}{dx} - \frac{n}{x} y\right)^2 x\, dx + na^2$$

It follows that the minimizing function must satisfy the ordinary differential equation

$$\frac{dy}{dx} - \frac{n}{x} y = 0$$

whence

$$y = ax^n \quad \text{and} \quad \min I_n = na^2$$

627. Since

$$D\left(\frac{2}{\pi} \sin x \sin y\right) = 2 \quad \text{and} \quad H\left(\frac{2}{\pi} \sin x \sin y\right) = 1$$

we have

$$\frac{D(u)}{H(u)} \geqslant \frac{D\left(\frac{2}{\pi} \sin x \sin y\right)}{H\left(\frac{2}{\pi} \sin x \sin y\right)} = 2, \quad \text{that is } H(u) \leqslant \frac{1}{2} D(u)$$

for any admitted function $u(x, y)$.

628. Let us take as the coordinate functions the system of functions $\{\sin kx\}$ $(k=1, 2, \ldots)$.

According to Ritz' method, we write $y_n=\sum_{k=1}^{n} c_k \sin kx$.

Under the condition that $H(y_n)=\frac{\pi}{2}\sum_{k=1}^{n} c_k^2=1$ the minimum of the expressions

$$D(y_n)=\frac{\pi}{2}\sum_{k=1}^{n} c_k^2 k^2; \qquad n=1, 2, \ldots$$

is attained for $c_1^2=\frac{2}{\pi}$ and $c_k=0$; $k=2, 3, \ldots$ Consequently,

$$y_n=\sqrt{\frac{2}{\pi}}\sin x, \quad \lim_{n\to\infty} D(y_n)=D(y)=1$$

629. According to the solution of Problem 628, we have $\min\frac{D(y)}{H(y)}=1$. Therefore, for any function $y(x)$ continuously differentiable on the interval $0\leqslant x\leqslant \pi$ and satisfying the conditions $y(0)=y(\pi)=0$ the inequality $H(y)\leqslant D(y)$ is fulfilled.

630. $y_1(x)=\frac{5}{2}x(x-1)$.

631. $u_1(x, y)=\frac{5}{16}(x^2-1)(y^2-1)$.

632. The given equation is nothing other than Euler's equation for the functional

$$D(u)=\int\limits_D (u_x^2+u_y^2-2xyu)\,dx\,dy$$

On finding the minimum of the expression $D(u_1)=c^2/45-c/72$ we obtain $c=5/16$. Consequently, $u_1(x, y)=(5/16)\,xy(x-1)(y-1)$.

633. As the system of coordinate functions we shall take

$$v_{kl}=J_k(\rho_{kl}r)\cos k\theta, \quad v_{kl}^*=J_{kl}(\rho_{kl}r)\sin k\theta;$$
$$k, l=0, 1, \ldots$$

where $x=r\cos\theta$, $y=r\sin\theta$ and ρ_{kl} are the positive roots of Bessel's function $J_k(z)$ (these roots are indexed with the letter l in the increasing order). The functions v_{kl} and v_{kl}^* are the eigenfunctions of *Helmholtz' equation* $\Delta v + \rho_{kl}^2 v = 0$ in the circle Q. Let

$$u_{mn} = \sum_{k=0}^{m} \sum_{l=0}^{n} (\alpha_{kl} v_{kl} + \beta_{kl} v_{kl}^*); \qquad m, n = 0, 1, \ldots$$

where α_{kl} and β_{kl} are arbitrary real constants. From the obvious equalities

$$D(u, v) = \lambda^2 H(u, v) = \mu^2 H(u, v)$$

which hold for any pair of eigenfunctions u and v corresponding to two eigenvalues λ and μ, respectively, we conclude that

$$d_{mn} = D(u_{mn}) = \sum_{k=0}^{m} \sum_{l=0}^{n} \rho_{kl}^2 \int_Q (\alpha_{kl}^2 v_{kl}^2 + \beta_{kl}^2 v_{kl}^{*2})\, dx\, dy$$

It is obvious that for any m and n the minimum of this functional is obtained when, except α_{00}, all the numbers α_{kl} and β_{kl} are equal to zero and

$$2\pi\alpha_{00}^2 \int_0^1 J_0^2(\rho_{00} r)\, r\, dr = 1$$

In this case

$$\min d_{mn} = d_{00} = 2\pi\alpha_{00}^2 \rho_{00}^2 \int_0^1 r J_0^2(\rho_{00} r)\, dr = \rho_{00}^2$$

and the minimizing function is

$$u_{00} = \frac{J_0(\rho_{00} r)}{\sqrt{\pi}\, J_1(\rho_{00})}$$

634. Since

$$\min \frac{D(u)}{H(u)} = \rho_{00}^2$$

we have the inequality

$$\rho_{00}^2 H(u) \leqslant D(u)$$

for any admitted function indicated in Problem 633; hence, $C = 1/\rho_{00}^2$ where ρ_{00} is the smallest positive root of Bessel's function $J_0(r)$.

APPENDIX

1. Square Matrices and Quadratic Forms

A collection of scalar quantities a_{ik}; $i, k = 1, \ldots, n$, belonging to a commutative field P and arranged in the form of a table

$$a = \left\| \begin{matrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \cdot & \cdot\cdot\cdot & \cdot \\ a_{n1} & \ldots & a_{nn} \end{matrix} \right\| \tag{1}$$

is referred to as a *square matrix of order* n (or an $(n \times n)$ matrix); the quantities a_{ik} themselves are called the *elements* of the matrix a.

The group of the elements a_{ii}; $i = 1, \ldots, n$, of the matrix a is called the *principal diagonal* of a. A matrix a is called *triangular* when all its elements a_{ik} are equal to zero for $i > k$. A triangular matrix a is called *diagonal* when all its elements a_{ik} are equal to zero for $i \neq k$. A diagonal matrix is called the *unit matrix* in case $a_{ii} = 1$; $i = 1, \ldots, n$. The unit matrix is usually denoted by the letter E or J.

The expression

$$\det a = \begin{vmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \cdot & \cdot\cdot\cdot & \cdot \\ a_{n1} & \ldots & a_{nn} \end{vmatrix} \tag{2}$$

is called the *determinant of the matrix* a. A matrix a is said to be *non-singular* (*non-degenerate*) when $\det a \neq 0$. For a non-singular matrix a (see expression (1)) the notion of the *inverse matrix* a^{-1} can be defined; the elements of the matrix a^{-1} are the quantities

$$\frac{A_{ki}}{\det a}; \quad i, k = 1, \ldots, n \tag{3}$$

where A_{ik} are the *cofactors* of the elements a_{ik} of determinant (2) of the matrix a.

The matrix a' with elements $a'_{ik} = a_{ki}$ $(i, k = 1, \ldots, n)$ is called the *transpose* of the matrix a. A matrix a whose elements are real numbers is said to be *symmetric* when $a_{ik} = a_{ki}$ $(i, k = 1, \ldots, n)$. By the *sum* of two $(n \times n)$ matrices $a = \| a_{ik} \|$ and $b = \| b_{ik} \|$ is meant an $(n \times n)$ matrix c whose elements are the quantities

$$c_{ik} = a_{ik} + b_{ik}$$

The sum c of a and b is denoted $c = a + b$.

By the *product* of two matrices a and b is meant the matrix c whose elements are

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}; \quad i, k = 1, \ldots, n \tag{4}$$

The product c of a and b is denoted $c = ab$. It can easily be checked that $aa^{-1} = E$.

From (4) it is readily seen that for any $(n \times n)$ matrix a and the $n \times n$ matrix E the relations

$$aE = Ea = a$$

always hold.

The *product* of a scalar quantity λ belonging to the field P *by an* $(n \times n)$ *matrix* a is defined as a new $(n \times n)$ matrix c with elements

$$c_{ik} = \lambda a_{ik}; \quad i, k = 1, \ldots, n$$

The product c of λ by a is denoted as $c = \lambda a$.

By virtue of equalities (4) and the definition of the product of determinants, we conclude that if a and b are two matrices of one order then

$$\det ab = \det a \cdot \det b \tag{5}$$

By virtue of (3), from equality (5) it follows that if a is a non-singular matrix then

$$\det aa^{-1} = 1$$

A matrix a is said to be *orthogonal* when the conditions

$$\sum_{j=1}^{n} a_{ji}a_{jk} = \delta_{ik}; \quad i, k = 1, \ldots, n$$

hold where

$$\delta_{ik} = \begin{cases} 1 & \text{for} \quad i = k \\ 0 & \text{for} \quad i \neq k \end{cases}$$

An n-dimensional vector p with components $p_1, \ldots, p_n$ belonging to a field P will be denoted as $p = (p_1, \ldots, p_n)$.

The *product* of a scalar λ belonging to the field P by an *n-dimensional vector* p is understood as the n-dimensional vector

$$r = \lambda p = (\lambda p_1, \ldots, \lambda p_n)$$

The *sum* of two *n-dimensional vectors* $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ is defined as the n-dimensional vector

$$r = p + q = (p_1 + q_1, \ldots, p_n + q_n)$$

By the *scalar* (or *inner*) *product* of two *n-dimensional vectors* $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n)$ is meant the scalar quantity

$$p \cdot q = \sum_{i=1}^{n} p_i q_i$$

By definition, the product of an $(n \times n)$ matrix a by an n-dimensional vector $p = (p_1, \ldots, p_n)$ is the vector $q = ap$ with components

$$q_i = \sum_{k=1}^{n} a_{ik} p_k; \quad i = 1, \ldots, n \tag{6}$$

When saying that a matrix a or a vector p is real or continuous or differentiable or of class $C^{(m, h)}$ or has a certain singularity etc. we shall always mean that each element of the matrix a or each component of the vector p possesses the corresponding property.

A point $x = (x_1, \ldots, x_n)$ belonging to an n-dimensional Euclidean space E_n with orthogonal Cartesian coordinates $x_1, \ldots, x_n$ is identified with an n-dimensional vector called the *radius vector*.

According to the above definition of the product of an $(n \times n)$ matrix a by an n-dimensional vector (see formulas (6)), a *linear transformation* in the space E_n expressed by the formulas

$$y_i = \sum_{k=1}^{n} a_{ik} x_k; \qquad i = 1, \ldots, n \tag{7}$$

can be written in the form

$$y = ax \tag{8}$$

Each of the expressions y_i defined by formula (7) is a *linear form* in the n variables $x_1, \ldots, x_n$.

Transformation (7) is said to be *non-singular* (*non-degenerate*) if so is the matrix a. For a non-degenerate linear transformation there exists a uniquely determined *inverse transformation*.

In case the matrix a of linear transformation (8) is symmetric or orthogonal, transformation (7) (or, which is the same, (8)) is said to be *symmetric* or *orthogonal* respectively.

The form

$$A(x, y) = \sum_{i, k=1}^{n} a_{ik} x_i y_k \tag{9}$$

of the second degree in the variables $x_1, \ldots, x_n$; $y_1, \ldots, y_n$ is called *bilinear*. Using the definitions of the product of an $(n \times n)$ matrix a by an n-dimensional vector x and of the *inner* (*scalar*) *product* of two n-dimensional vectors, we can represent bilinear form (9) as

$$A(x, y) = (ay) \cdot x$$

Bilinear form (9) is called a *quadratic form* in case the radius vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ coincide. A quadratic form $A(x, x)$ is usually denoted as $Q(x)$:

$$Q(x) = A(x, x) = \sum_{i, k=1}^{n} a_{ik} x_i x_k = (ax) \cdot x \tag{10}$$

The matrix $a = \| a_{ik} \|$ is referred to as the *matrix of the quadratic form* $Q(x)$.

For any quadratic form Q there exists a non-degenerate linear transformation

$$x = by \tag{11}$$

with an $(n \times n)$ matrix b under which quadratic form (10) *is reduced to its standard form*:

$$Q(x) = Q(by) = Q^*(y) = \sum_{i=1}^{n} \alpha_i y_i^2 \tag{12}$$

where the coefficients α_i assume the values 1, -1 and 0.

There holds the following very important property known as the *inertia law for a quadratic form*: *the total number of the nonzero coefficients* (the *rank* of the quadratic form), *the number of the positive coefficients* (the *index* of the form) *and the difference between the number of the positive coefficients and the number of the negative coefficients* (the *signature* of the form) *in standard form* (12) *are invariant under all non-degenerate linear transformations* (11).

2. Hamilton's Principle

In the derivation of partial differential equations of mathematical physics *Hamilton's variational principle* is most often used.

Let us consider a material system whose state is completely determined by a finite number of spatial parameters $q_1, \ldots, q_n$. The law of motion of the system is considered to be known when the values of all these parameters as functions of time t $(t_0 \leqslant t \leqslant t_1)$ are known.

The *kinetic* and the *potential* energy of the system will be denoted as

$$T = T(t, q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n)$$

and

$$U = U(t, q_1, \ldots, q_n)$$

respectively, where $\dot{q}_i$ is the derivative of the first order of q_i with respect to time t. As is known, the kinetic energy T is a *positive definite* quadratic form in the variables $\dot{q}_1, \ldots$

$\ldots, \dot{q}_n$ with coefficients dependent on $q_1, \ldots, q_n$ and t:

$$T = \sum_{i,\,k=1}^{n} T_{ik}(t, q_1, \ldots, q_n)\, \dot{q}_i \dot{q}_k \tag{1}$$

By *admitted functions* (describing the *admitted motions*) we shall mean the functions

$$q_i^*(t) = q_i(t) + \delta q_i(t); \quad i = 1, \ldots, n \tag{2}$$

where $\delta q_i(t)$ $(i = 1, \ldots, n)$ are arbitrary sufficiently small quantities satisfying the conditions

$$\delta q_i(t_0) = \delta q_i(t_1) = 0; \quad i = 1, \ldots, n \tag{3}$$

Hamilton's principle: the real motion of the system is such that for the functions $q_1, \ldots, q_n$ the integral (the *functional*)

$$J = \int_{t_0}^{t_1} (T - U)\, dt \tag{4}$$

assumes a stationary value in comparison with the values assumed for all admitted functions (2). *Consequently, for the functions $q_i = q_i(t)$ $(i = 1, \ldots, n)$ to describe a real motion of the system it is necessary and sufficient that the first variation of functional* (4) *should be equal to zero:*

$$\delta \int_{t_0}^{t_1} (T - U)\, dt = 0 \tag{5}$$

By virtue of (2), using the theorem on finite increments we derive from (5) the equality

$$\int_{t_0}^{t_1} \sum_{i=1}^{n} \left[\left(\frac{\partial T}{\partial q_i} - \frac{\partial U}{\partial q_i} \right) \delta q_i + \frac{\partial T}{\partial \dot{q}_i}\, \delta \dot{q}_i \right] dt = 0 \tag{6}$$

Proceeding from (3) and integrating equality (6) by parts we can rewrite it in the form

$$\int_{t_0}^{t_1} \sum_{k=1}^{n} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} \right) \delta q_i\, dt = 0 \tag{7}$$

because

$$\delta\dot{q}_i = \frac{d}{dt}\,\delta q_i; \quad i = 1, \ldots, n$$

Since the quantities δq_i are arbitrary, equality (7) implies

$$\frac{d}{dt}\,\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0; \quad i = 1, \ldots, n \tag{8}$$

Equality (8) is the system of differential equations describing the motions of the given material system.

In case the functions T and U do not depend explicitly on time t and the system is in equilibrium, formulas (1) and (8) yield

$$\frac{\partial U}{\partial q_i} = 0; \quad i = 1, \ldots, n \tag{9}$$

As is known, equalities (9) express a sufficient condition for the function U to have an extremum. An equilibrium state specified by the values of $q_1, \ldots, q_n$ satisfying the finite system of equations of form (9) is *stable* when the function U attains its minimum for these values of the arguments $q_1, \ldots, q_n$.

Now let us again suppose that T and U do not depend on time explicitly. On multiplying each of equalities (8) by the quantity $\dot{q}_i dt = dq_i$ and adding together the results we obtain

$$\sum_{i=1}^{n} \left(\frac{\partial U}{\partial q_i} - \frac{\partial T}{\partial q_i} \right) dq_i + d \sum_{i=1}^{n} \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} - \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} d\dot{q}_i = 0 \tag{10}$$

Taking into account that expression (1) for T is a homogeneous function of the second degree with respect to the variables $\dot{q}_i$ and using the well-known *Euler theorem* expressed by the formula

$$\sum_{i=1}^{n} \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T$$

we can rewrite equality (10) in the form

$$\sum_{i=1}^{n}\left(\frac{\partial U}{\partial q_i}-\frac{\partial T}{\partial q_i}\right)dq_i-\sum_{i=1}^{n}\frac{\partial T}{\partial \dot{q}_i}d\dot{q}_i+2\,dT=$$
$$=d(U-T)+2\,dT=d(U+T)=0$$

whence

$$U+T=\text{const} \tag{11}$$

Since the expression $U+T$ is the *total energy* of the mechanical system in question, equality (11) expresses nothing other than the *law of conservation of energy.*

Under the hypotheses we have assumed, the function U can be found from equality (11); the substitution of the expression of U thus found into (5) results in

$$\delta\int_{t_0}^{t_1} T\,dt=0 \tag{12}$$

Hamilton's principle written in the form of equality (12) is referred to as *Lagrange's principle of least action.*

3. Expressions for the Laplace Operator in Various Coordinate Systems

(a) for the orthogonal Cartesian coordinates x, y and z we have

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

(b) for the cylindrical coordinates r, φ and z we have

$$\Delta=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}$$
$$x=r\cos\varphi,\quad y=r\sin\varphi,\quad z=z$$

(c) for the spherical coordinates r, φ and θ we have

$$\Delta=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$
$$x=r\cos\varphi\sin\theta,\quad y=r\sin\varphi\sin\theta,\quad z=r\cos\theta$$

4. Some Special Functions

1. Euler's gamma functions $\Gamma(z)$ and some of its properties:

(a) Euler's gamma function is represented in the form of the integral

$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\,dt, \quad \operatorname{Re} z>0$$

(b) it is analytic in the half-plane $\operatorname{Re} z>0$;

(c) $\Gamma(z+1)=z\Gamma(z)$, $\Gamma(1)=1$, $\Gamma(1/2)=\sqrt{\pi}$;

(d) it can be continued analytically across the axis $\operatorname{Re} z=0$ to the whole complex plane of the variable z; at the points $z=0, -1, \ldots, -n, \ldots$ it has poles of the first order; the residues of $\Gamma(z)$ at the poles are

$$\operatorname{Res}\Gamma(-n)=\frac{(-1)^n}{n!}, \quad n\geqslant 0$$

(e) it has no zeros.

2. Cylindrical functions. The equation for the cylindrical functions $y=y(\xi)$ has the form

$$y''+\frac{1}{\xi}y'+\left(k^2-\frac{\nu^2}{\xi^2}\right)y=0, \quad k=\text{const}, \quad \nu=\text{const}$$

The change of the variable ξ according to the formula $x=k\xi$ transforms this equation into *Bessel's equation*

$$z''+\frac{1}{x}z'+\left(1-\frac{\nu^2}{x^2}\right)z=0, \quad z=z(x)=y\left(\frac{x}{k}\right)$$

The general solutions of these equations have the form

$$y_\nu(\xi)=C_1J_\nu(k\xi)+C_2N_\nu(k\xi)$$

and

$$z_\nu(x)=C_1J_\nu(x)+C_2N_\nu(x)$$

respectively, where C_1, C_2 are arbitrary constants,

$$J_\nu(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}$$

is *Bessel's function of order* ν and

$$N_\nu(x) = \begin{cases} \dfrac{J_\nu(x)\cos\pi\nu - J_{-\nu}(x)}{\sin\pi\nu} & \text{for a non-integral } \nu \\ \dfrac{1}{\pi}\left[\dfrac{\partial J_\nu(x)}{\partial\nu} - (-1)^\nu \dfrac{\partial J_{-\nu}(x)}{\partial\nu}\right] & \\ & \text{for an integral } \nu \end{cases}$$

The expression $N_\nu(x)$ is called *Neumann's function of order* ν (or *Bessel's function of the second kind*).

Some properties of Bessel's functions:

(a) for an integral n we have $J_{-n}(x) = (-1)^n J_n(x)$; we also have the formulas

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x \quad \text{and} \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

(b) the *orthogonality* is stated thus: if α and β are real roots of the equation

$$\mu J_\nu(\gamma) + \eta\gamma J'_\nu(\gamma) = 0, \quad \mu \geqslant 0, \quad \eta \geqslant 0, \quad \mu + \eta > 0$$

then, when $\nu > -1$

$$\int_0^l x J_\nu\left(\frac{\alpha}{l}x\right) J_\nu\left(\frac{\beta}{l}x\right) dx = 0$$

for $\alpha \neq \beta$;

(c) $$\int_0^l x J_\nu^2\left(\frac{\alpha}{l}x\right) dx = \frac{l^2}{2}\left\{[J'_\nu(\alpha)]^2 + \left(1 - \frac{\nu^2}{\alpha^2}\right) J_\nu^2(\alpha)\right\};$$

(d) the *recurrence relations for Bessel's functions*:

$$\frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x)$$

$$\frac{d}{dx}[x^{-\nu} J_\nu(x)] = -x^{-\nu} J_{\nu+1}(x)$$

$$J_{\nu+1}(x) + J_{\nu-1}(x) = \frac{2\nu}{x} J_\nu(x)$$

$$J_{\nu+1}(x) - J_{\nu-1}(x) = -2J'_\nu(x)$$

$$J_{n+1/2}(x) = (-1)^n \sqrt{\frac{2}{\pi}}\, x^{n+1/2} \left(\frac{d}{x\,dx}\right)^n \frac{\sin x}{x}$$

$$J_{n-1/2}(x) = \sqrt{\frac{2}{\pi}}\, x^{n+1/2} \left(\frac{d}{x\,dx}\right)^n \frac{\cos x}{x},$$

$n \geqslant 0$ is an integer

3. Modified Bessel's functions. Under the change of the variable x according to the formula $x = it$ Bessel's equation goes into the equation

$$v'' + \frac{1}{t} v' - \left(1 + \frac{\nu^2}{t^2}\right) v = 0, \quad v = v(t) = z(it)$$

whose general solution has the form

$$v(t) = C_1 I_\nu(t) + C_2 K_\nu(t)$$

where C_1, C_2 are arbitrary constants,

$$I_\nu(t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\,\Gamma(k+\nu+1)} \left(\frac{t}{2}\right)^{2k+\nu}$$

is *modified Bessel's function of the first kind of order* ν (or *Bessel's function of a pure imaginary argument*) and

$$K_\nu(t) = \begin{cases} \dfrac{\pi}{2\sin \pi\nu} [I_{-\nu}(t) - I_\nu(t)] & \text{for a non-integral } \nu \\ \dfrac{(-1)^\nu}{2} \left[\dfrac{\partial J_{-\nu}(t)}{\partial \nu} - \dfrac{\partial J_\nu(t)}{\partial \nu}\right] & \text{for an integral } \nu \end{cases} \qquad (1)$$

is *Macdonald's function* (or *modified Hankel's function* or *modified Bessel's function of the second kind*)* of order ν.

4. Asymptotic formulas:

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + O(x^{-3/2}), \quad x \to +\infty$$

$$N_\nu(x) = \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right) + O(x^{-3/2}), \quad x \to +\infty$$

* The definition of K_ν is sometimes taken as the product of $\cos \nu\pi$ by the value expressed by formula (1). The asymptotic expansion for K_ν written below corresponds to definition (1).—*Tr.*

$$I_\nu(x) = \sqrt{\frac{1}{2\pi x}}\, e^x [1 + O(1/x)], \quad x \to +\infty$$

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} [1 + O(1/x)], \quad x \to +\infty$$

5. Legendre's polynomials $P_n(x)$ $(n = 0, 1, \ldots)$ possess the following properties:

(a) they are solutions of *Legendre's equation*

$$\frac{d}{dx}\left[(1 - x^2)\frac{dy}{dx}\right] + n(n+1)\,y = 0;$$

(b) they are represented in the form

$$P_n(x) = \frac{1}{2^n n!}\,\frac{d^n}{dx^n}[(x^2 - 1)^n];$$

(c) they satisfy the recurrence relations

$$(n+1)\,P_{n+1}(x) - (2n+1)\,xP_n(x) + nP_{n-1}(x) = 0$$

and

$$P_n(x) = \frac{1}{2n+1}[P'_{n+1}(x) - P'_{n-1}(x)]$$

(d) they are orthogonal in the interval $(-1, 1)$:

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = 0, \quad m \neq n$$

and, besides,

$$\int_{-1}^{1} q_m(x)\,P_n(x)\,dx = 0$$

where $q_m(x)$ is an arbitrary polynomial of degree $m < n$;

(e) $\displaystyle\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$;

(f) $P_n(1) = 1$, $P_n(-1) = (-1)^n$; $n = 0, 1, \ldots$

6. Legendre's associated functions $P_n^{(m)}(x)$ possess the following properties:

(a) they satisfy the equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+\left[n(n+1)-\frac{m^2}{1-x^2}\right]y=0$$

(b) they are represented in the form

$$P_n^{(m)}(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)$$

(c) $\int\limits_{-1}^{1}[P_n^{(m)}(x)]^2\,dx=\frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!}$.

5. The Laplace Transformation

The *integral transformation*

$$F(\zeta)=\int\limits_0^\infty e^{-\zeta t}f(t)\,dt,\quad \zeta=\xi+i\eta$$

is called the *Laplace transformation* and is symbolized as $f(t)\div F(\zeta)$ where f is the *original* (or the *inverse transform* or the *inverse image* or the *preimage* of $F(\zeta)$ and F is the *Laplace transform* (*image*) of f. The *Laplace inverse transformation* from $F(\zeta)$ to $f(t)$ is symbolized as $F(\zeta)\div f(t)$.

Let $f(t)$ and $g(t)$ be two originals satisfying the conditions $|f(t)|\leqslant Ae^{at}$ and $|g(t)|\leqslant Be^{bt}$ and let $F(\zeta)$ and $G(\zeta)$ be their Laplace transforms respectively. Then the following properties hold:

(a) $\lim\limits_{\zeta\to\infty}F(\zeta)=0$ (here $\zeta\to\infty$ so that $\operatorname{Re}\zeta\to+\infty$);

(b) $\alpha f(t)+\beta g(t)\div\alpha F(\zeta)+\beta G(\zeta)$ where α and β are arbitrary (generally speaking, complex) constants;

(c) $f(\alpha t)\div\frac{1}{\alpha}F\left(\frac{\zeta}{\alpha}\right)$, $\alpha>0$;

(d) $f^{(n)}(t)\div\zeta^nF(\zeta)-\zeta^{n-1}f(0)-\zeta^{n-2}f'(0)-\ldots-f^{n-1}(0)$;

(e) $F^{(n)}(\zeta)\div(-1)^n t^n f(t)$;

(f) $\int\limits_0^t f(t)\,dt \div \frac{F(\zeta)}{\zeta}$;

(g) $\frac{f(t)}{t} \div \int\limits_\zeta^\infty F(\zeta)\,d\zeta$;

(h) $f(t-\tau) \div e^{-\zeta\tau}F(\zeta)$;

(i) $e^{\zeta_0 t}f(t) \div F(\zeta-\zeta_0)$;

(j) $F(\zeta)\,G(\zeta) \div \int\limits_0^t f(\tau)\,g(t-\tau)\,d\tau$;

(k) $f(t)\cdot g(t) \div \frac{1}{2\pi i}\int\limits_{c-i\infty}^{c+i\infty} F(\mu)\,G(\zeta-\mu)\,d\mu,$

$$c > a,\quad \operatorname{Re}\zeta > b+c;$$

(l) for

$$F(\zeta) = \sum_{k=1}^{\infty} \frac{c_k}{\zeta^k},\quad |\zeta| \geqslant R > 0$$

we have

$$F(\zeta) \div f(t) = \sum_{k=1}^{\infty} \frac{c_k}{(k-1)!}\,t^{k-1}$$

(m) for

$$F(\zeta) = \frac{q_m(\zeta)}{q_n(\zeta)},\quad m < n$$

we have

$$F(\zeta) \div f(t) = \sum_{(\zeta_k)} \operatorname{Res}\,[F(\zeta)\,e^{\zeta t}]$$

where $q_i(\zeta)$ denotes a polynomial of the ith degree and ζ_k are the poles of $F(\zeta)$.

6. Table of Some Originals and Their Laplace Transforms

	Original	Laplace transform
1	1	$1/\zeta$
2	$e^{\alpha t}$	$\dfrac{1}{\zeta-\alpha}$
3	$\sin\omega t$	$\dfrac{\omega}{\zeta^2+\omega^2}$
4	$\cos\omega t$	$\dfrac{\zeta}{\zeta^2+\omega^2}$
5	$\sinh\omega t$	$\dfrac{\omega}{\zeta^2-\omega^2}$
6	$\cosh\omega t$	$\dfrac{\zeta}{\zeta^2-\omega^2}$
7	$t^{\alpha}\ (\alpha>-1)$	$\dfrac{\Gamma(\alpha+1)}{\zeta^{\alpha+1}}$
8	$e^{-\beta t}t^{\alpha}$	$\dfrac{\Gamma(\alpha+1)}{(\zeta+\beta)^{\alpha+1}}$
9	$\delta(t)$	1
10	$\dfrac{e^{bt}-e^{at}}{t}$	$\ln\dfrac{\zeta-a}{\zeta-b}$
11	$\dfrac{e^{-\alpha t}}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{\zeta+\alpha}}$
12	$\dfrac{1}{\sqrt{\pi t}}e^{-\alpha^2/4t}$	$\dfrac{e^{-\alpha\sqrt{\zeta}}}{\sqrt{\zeta}}$
13	$\dfrac{\xi}{2\sqrt{\pi}\,t^{3/2}}\ e^{-\xi^2/4t}$	$e^{-\xi\sqrt{\zeta}}$
14	$\dfrac{1}{\sqrt{\pi\alpha}}\sin 2\sqrt{\alpha t}$	$\dfrac{1}{\zeta\sqrt{\zeta}}e^{-\alpha/\zeta}$

(*continued*)

	Original	Laplace transform
15	$\frac{1}{\sqrt{\pi\alpha}} \cos 2\sqrt{\alpha t}$	$\frac{1}{\sqrt{\zeta}} e^{-\alpha/\zeta}$
16	$J_n(t)\ (n > -1)$	$\frac{(\sqrt{\zeta^2+1}-\zeta)^n}{\sqrt{\zeta^2+1}}$
17	$t^{n/2} J_n(2\sqrt{t})\ (n > -1)$	$\zeta^{-(n+1)} e^{-1/2\zeta}$

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