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AN UNUSUAL ALGEBRA

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НЕОБЫКНОВЕННАЯ АЛГЕБРА

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Preface

The present book is based on the lecture given by the author to senior pupils in Moscow on the 20th of April of 1966. The distinction between the material of the lecture and that of the book is that the latter includes exercises at the end of each section (the most difficult problems in the exercises are marked by an asterisk). At the end of the book are placed answers and hints to some of the problems. The reader is advised to solve most of the problems, if not all, because only after the problems have been solved can the reader be sure that he understands the subject matter of the book. The book contains some optional material (in particular, Sec. 7 and Appendix which are starred in the table of contents) that can be omitted in the first reading of the book. The corresponding parts of the text of the book are marked by one star at the beginning and by two stars at the end. However, in the second reading of the book it is advisable to study Sec. 7 since it contains some material important for practical applications of the theory of Boolean algebras.

The bibliography given at the end of the book lists some books which can be of use to the readers who want to study the theory of Boolean algebras more thoroughly.

The author is grateful to S. G. Gindikin for valuable advice and to F. I. Kizner for the thoroughness and initiative in editing the book.[1]

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[1] The present translation incorporates suggestions made by the author. — *Tr.*

Contents

[1]) The starred items indicate those sections which may be omitted in the first reading of the book.

1. Algebra of Numbers and Algebra of Sets

When studying arithmetic and algebra at school pupils deal with numbers of various nature. At the beginning the pupils study whole numbers whose understanding causes no difficulties because most pupils are to some extent familiar with these numbers before they start going to school. However, in the further course of the study of mathematics the pupils come across new and still new "numbers" (such as fractional numbers, irrational numbers etc.). When the pupils get used to a new class of numbers they are no longer puzzled by them, but yet at every stage of the extension of the notion of a number they lose some of their illusions. A whole number gives information on how many objects there are in a given collection, for instance, how many apples there are in a basket or how many pages there are in a book or how many boys there are in a class. As to the fractions, there cannot of course be $33^1/_2$ boys in a class or $3^1/_4$ plates on the table, but at the same time, there can be $4^1/_2$ apples on the table, a film can last for $1^3/_4$ hours, and there can even be $6^1/_2$ books on a book shelf. (Of course, in this case we cannot say that the owner of the books handles them well!) The moment we get used to the fact that a fractional number of objects in a collection can make sense we pass to negative numbers. Of course, there cannot be -3 books on a book shelf. (This would be quite unnatural!) But a thermometer can read $-5°$ and it even makes sense to say that a person has -50 copecks (the latter situation may worry the person but this is of no importance to mathematics!). Senior pupils study still more "frightening" numbers: first the so-called irrational numbers such as $\sqrt{2}$ and then the imaginary numbers such as $1 + 2i$[1]) (the terms "irrational" and "imaginary" clearly indicate how strange these numbers seemed to the people until they got used to them). By the way, if the reader is not yet familiar with irrational and imaginary numbers, this will not prevent

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[1]) In modern mathematics the numbers of the type of $1 + 2i$ are called *complex numbers* while by *imaginary* (or *pure imaginary*) *numbers* are meant such numbers as $2i$ or $-\sqrt{2}i$ (in contrast to them, such numbers as 1 or $-^3/_2$ or $\sqrt{2}$ are termed *real numbers*).
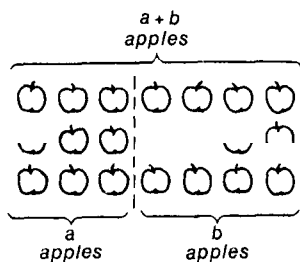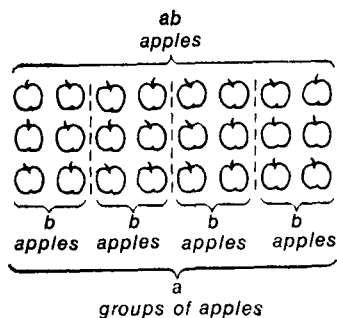
Fig. 1

Fig. 2

him from reading this book[1]). Although the concepts of irra
tional and imaginary numbers have little in common with
the primary idea of a whole number as a quantitative charac-
teristic of a collection of objects we nevertheless speak of
them as "numbers".

So it is natural to ask what is the common feature of
all these kinds of numbers which allows us to apply the
term "number" to all of them. It can easily be noticed that
the main common feature of all these kinds of numbers
is that all the numbers can be added together and multi-
plied by each other[2]). However, this similarity between the
various kinds of numbers is conditional: the matter is that
although we can perform addition and multiplication of all
kinds of numbers these operations themselves have a different
sense in different cases. For instance, *when we add together
two positive integers a and b we find the number of objects
in the union of two collections the first of which contains a objects
and the second b objects.* If there are 35 pupils in a class
and 39 in another class then there are $35 + 39 = 74$ pupils
in both classes (also see Fig. 1). Similarly, *when we multiply
two positive integers a and b we find the number of objects
in a union of a collections each of which contains b objects.*

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[1]) An elementary representation of various numbers can be found
in the book: I. Niven, *Numbers: Rational and Irrational*, New York,
Random House, 1961.

[2]) But not subtracted or divided: if, for instance, we are familiar
only with positive numbers, we cannot subtract the number 5 from
the number 3 and if we know only the whole numbers we cannot divide
the number 7 by the number 4.

If, for instance, there are 3 classes in each of which there are 36 pupils then there are $3 \cdot 36 = 108$ pupils in all these classes (also see Fig. 2). It is however evident that this interpretation of addition and multiplication applies neither to the operations on fractions nor to the operations on negative numbers. For instance, the sum and the product of *rational numbers (fractions)* are defined by the following rules:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

and

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

(here a, b, c and d are whole numbers). We also know that, for instance, for the *signed numbers* there exists the rule

$$(-a) \cdot (-b) = ab$$

etc.[1]).

Thus, we can draw the following conclusion: the term "number" is applied to numbers of different kinds because they can be added together and multiplied by each other but the operations of addition and multiplication themselves are completely different for different kinds of numbers. However, here we have gone too far: it turns out that there is in fact a great similarity between the operation of addition of the whole numbers and the operation of addition of the fractions. More precisely, the *definitions* of these operations are different but the *general properties* of the operations are completely similar. For instance, for the numbers of *any* nature we always have the identities

$$a + b = b + a$$

the commutative law for addition of numbers

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[1]) We do not discuss in detail irrational and imaginary numbers here and only note that the *complex numbers* are added together and multiplied by each other according to the rules

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

These rules may seem strange to the reader who is not yet familiar with complex numbers but they are much simpler than the definitions of the sum and of the product of *irrational numbers.*

and

$$ab = ba$$

and also the identities

$$(a + b) + c = a + (b + c)$$

and

$$(ab) c = a (bc)$$

Also, in all the cases there exist two "special" numbers 0
and 1 such that the addition of the first of them to any num-
ber and the multiplication of any number by the second
one do not change the original number: for any number a
we have

$$a + 0 = a \quad \text{and} \quad a \cdot 1 = a$$

What has been said accounts for the point of view of modern
mathematics according to which the aim of algebra is to
study some (different) systems of numbers (and other objects)
for which the operations of addition and multiplication are
defined so that the above laws and some other laws which
will be stated later are fulfilled. For instance, for any num-
bers a, b and c the identity

$$(a + b) c = ac + bc$$

must hold.

There is a certain kind of analogy between the operations
of addition and multiplication which is particularly notice-
able because the properties of addition are in many respects
similar to the properties of multiplication. For instance,
if we set the unusual "proportion"

$$\frac{\text{addition}}{\text{subtraction}} = \frac{\text{multiplication}}{?}$$

then everybody will substitute the word "division" for the
interrogation sign even without analysing the meaning of
the "proportion". It is due to this analogy that so many
people often confuse the notion of the (*additive*) *inverse*
of a number a (which is the number $-a$ whose addition to
the given number a results in 0) and the notion of the *recip-*

$_{rocal}$ (the *multiplicative inverse*) of a number a (which is the number $\frac{1}{a}$ whose product by the given number a is equal to 1). By the same reason, we see much similarity between the properties of an arithmetic progression (which is a sequence of numbers for which the difference between any member and the preceding member is one and the same) and a geometric progression (which is a sequence of numbers for which the ratio of any member to the preceding one is one and the same).

However, this analogy is not complete. For instance, the number 0 plays a special role not only in the addition but also in the multiplication because we have

$$a \cdot 0 = 0$$

for any number a (in particular, from the last identity it follows that a number different from 0 cannot be divided by 0). But if we replace in this identity multiplication by addition and zero by unity we arrive at the meaningless "equality"

$$a + 1 = 1$$

which can hold only when $a = 0^1$). Further, if we take the distributive law $(a + b) c = ac + bc$ and interchange addition and multiplication we get the "equality"

$$ab + c = (a + c)(b + c)$$

with which nobody of course can agree. (Since we obviously have

$$(a + c)(b + c) = ab + ac + bc + c^2 =$$
$$= ab + c(a + b + c)$$

it follows that $(a + c)(b + c) = ab + c$ only when $c = 0$ or when $a + b + c = 1$.)

There are however some other algebraic systems whose elements are not numbers such that it is also possible to define the operations of addition and multiplication for them and the similarity between the addition and the multiplication of these elements is even closer than the similarity

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[1]) If the equality $a + 1 = 1$ were fulfilled for any a then it would be impossible to subtract 1 from any number different from 1. In reality this is not the case: for instance, $3 - 1 = 2$.
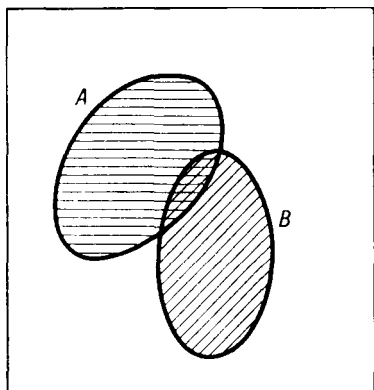
Fig. 3

b t····· th · ··· p···· tions on numbers. For instance, let us consider a very important example of an *algebra of sets.* By a *set* is meant any collection of arbitrary objects which are called the *elements* of the set. For instance, we can consider the set of the pupils in a given class, the set of points bounded by a circle, the set of points in q ···, ····· ······ ments in the periodic system, the set of even numbers, the set of elephants in India, the set of grammar mistakes in your composition etc. It is quite evident that the addition of two sets can be defined in the following way: *by the sum A + B of a set A and a set B we shall simply mean the union of these sets.* For instance, if *A* is the set of the boys in a class and *B* is the set of the girls in that class then *A + B* is the set of all the pupils in the class; similarly, if *A* is the set of all even positive integers and *B* is the set of the positive integers divisible by 3 then

{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, ...}

is the set *A + B* which consists of both groups of numbers. If the set *A* consists of the points belonging to the area shaded by horizontal lines in Fig. 3 and the set *B* consists of the points in the area shaded by inclined lines then the set *A + B* is the whole area shaded in Fig. 3.

The fact that we have defined a completely new operation and have called it "addition" must not seem strange: it should be remembered that earlier when we passed from numbers of one kind to numbers of another kind we defined the operation of addition in a different way. It is clear that the addition of positive numbers and the addition of negative numbers are completely different operations: for instance, the sum of the numbers 5 and (-8) is equal to the difference between the (positive) numbers 5 and 8. Similarly, the addition of fractions performed according to the rule $a/b + c/d = (ad + bc)/bd$ differs from the addition of whole

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numbers: the definition of the addition of the positive integers (see Fig. 1) given on page 8 is inapplicable to the description of the addition of fractions. The usage of one and the same term "addition" in all the cases was accounted for by the fact that the general laws for the operation of addition of whole numbers remained valid when we passed, for instance, to fractions: in both cases the operation of addition turned out to be commutative and associative.

Now let us check whether these laws remain valid for the new operation of "addition", that is for the addition of sets. To facilitate the analysis it is convenient to consider special diagrams demonstrating operations on sets. Let us make the following convention: the set of all the elements under consideration (for instance, the set of all whole numbers or the set of all pupils in a school) will be represented as a square; within this square we can mark different points representing some concrete elements of the set (for instance, the numbers 3 and 5 or the pupils Peter and Mary). In this representation the sets consisting of some of the elements of the given set (for instance the set of even numbers or the set of excellent pupils) are represented by some parts of that square. Such diagrams are often called $Venn's\ diagrams$ after the English mathematician John Venn (1834-1923) who used these diagrams in his study of mathematical logic. It would be more precise to call them $Euler's\ diagrams$ because L. Euler[1]) used such diagrams much earlier than J. Venn[2]).

As is clearly seen from Fig. 3, we have

$$A + B = B + A$$

for any two sets A and B, which means that *the commutative law holds for the addition of sets.* Further, it is obvious that for any sets A, B and C there always holds the identity

$$(A + B) + C = A + (B + C)$$

This means that *the addition of sets obeys the associative law.* It follows that the set $(A + B) + C$ (or, which is the same,

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[1]) Leonard Euler (1707-1783), a famous Swiss mathematician who spent most of his life in Russia and died in St. Petersburg.

[2]) In his studies in mathematical logic L. Euler represented different sets of objects by *circles* in the plane; therefore, the corresponding diagrams (which, in principle, do not differ from Venn's diagrams) are often referred to as $Euler's\ circles.$
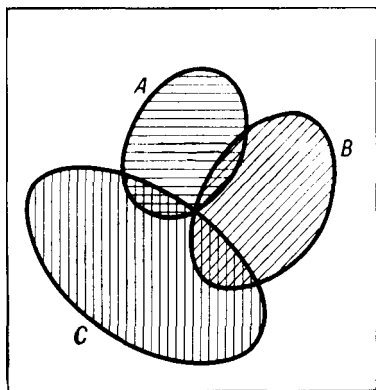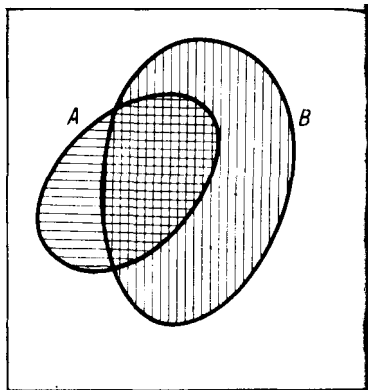
Fig. 4   Fig. 5

the set $A + (B + C)$) can simply be denoted as $A + B + C$ without using parentheses; the set $A + B + C$ is nothing but the union of all the three sets A, B and C (see Fig. 4 where the set $A + B + C$ coincides with the whole area shaded in this figure).

Now let us agree that *by the product AB of two sets A and B will be meant their common part, that is the intersection of these sets* (also called the *meet* of A and B). For instance, if A is the set of the chess-players in your class and B is the set of the swimmers in your class then AB is the set of those chess-players who can also swim; if A is the set of the positive integers and B is the set of numbers divisible by 3 then the set AB is

$$\{6,\ 12,\ 18,\ 24,\ \dots\}$$

This set consists of all positive integers divisible by 6. If the set A consists of the points lying in the area shaded in Fig. 5 by horizontal lines and the set B consists of the points in the area shaded by vertical lines then the set AB is the area in the figure which is cross-hatched. It is quite clear that *the multiplication of sets defined in this way obeys the commutative law*, that is for any two sets A and B we have

$$AB = BA$$

(see the same Fig. 5, it is also obvious that "the set of the chess-players who can swim" and "the set of the swimmers

who can play c ess" coin-
cide: this is simply one
and the same set). Further,
it is quite evident that *the
associative law also holds
for the multiplication of sets*,
that is for any three sets
A, B and C we have

$$(AB)\,C = A\,(BC)$$

The associative law allows
us to denote the set $(AB)C$
or, which is the same, the
set $A(BC)$ simply as ABC
w.t..out using parenth....,
the set ABC is the com-

Fig. 6

mon part (the intersection) of the three sets A, B and
C (in Fig. 6 the set ABC is covered by the network of
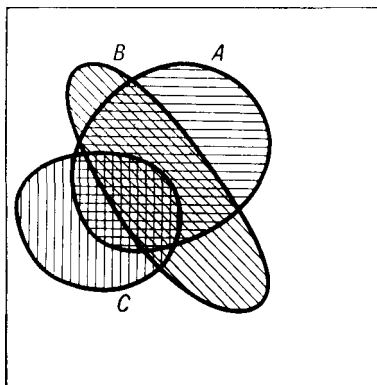the three families of hatching lines[1]).

A remarkable fact is that *for any three sets A, B and C
there also holds the distributive law*

$$(A + B)\,C = AC + BC$$

which means that *intersection distributes over union*. Indeed,
if, for instance, A is the set of the chess-players in your
class and B is the set of the pupils who can play draughts
while C is the set of the swimmers then the set $A + B$
is the union of the set of the chess-players and the set of
the pupils who can play draughts, that is the set of those
pupils who can play chess or draughts or both. The set
$(A + B)\,C$ can be obtained from the set $A + B$ if we choose
from the latter only those pupils who can swim. But it is
quite clear that exactly the same set can be obtained if we
form the union $AC + BC$ of the set AC of the chess-players

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[1] Here is one more example demonstrating the associative law
for the product of sets. Let A be the set of integers divisible by 2,
B the set of integers divisible by 3 and C the set of integers divisible
by 5. Then AB is the set of integers divisible by 6 and $(AB)C$ is the
set of integers divisible both by 6 and 5, that is divisible by 30. On
the other hand, BC is the set of integers divisible by 15 and $A(BC)$
is the set of even integers divisible by 15; hence, we see that $A(BC)$
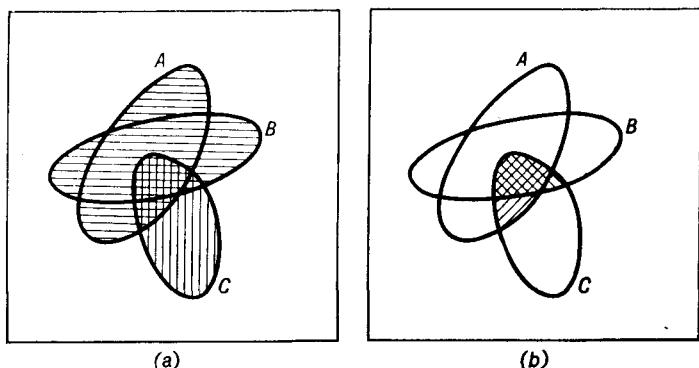coincides with the set $(AB)C$ of all integers divisible by 30.

Fig. 7

who can swim and the set *BC* of the pupils who can play draughts and can swim.

The verbal explanation of the distributive law is rather lengthy. To explain this law it is also possible to use the graphical demonstration. In Fig. 7,*a* the set $A + B$ is shaded by horizontal lines and the set *C* by vertical lines so that the set $(A + B) C$ is covered by a network of hatching lines. In Fig. 7,*b* the sets *AC* and *BC* are shaded by lines with different inclination; the set $AC + BC$ in the figure coincides with the whole shaded area. But it is clearly seen that the area $AC + BC$ in Fig. 7,*b* does not differ from the area $(A + B) C$ covered by cross-hatching in Fig. 7,*a*.

It is easy to understand which "set" plays the role of "zero" in the "algebra of sets". Indeed, if this set (we shall denote it *O*) is added to an arbitrary set the latter must not change and therefore the set *O* must contain no elements at all. Thus, *O* is an *empty set* (it is also referred to as a *void* or *null* set). One may think that since the set *O* is empty and contains no elements there is no need to take it into account. But it would be in fact quite unreasonable to exclude the empty set from the consideration. If we did so this would resemble the exclusion of the number 0 from the number system: a "collection" containing zero elements is also "empty" and it may seem senseless to speak of the "number" of elements contained in such a collection. But it is in fact by far not senseless, moreover, it is quite meaningful. If we did not introduce the number 0 it would be impossible

to subtract any number from any other number (because, for instance, the difference $3 - 3$ would be equal to no number in this case). Without having the number 0 at our disposal it would be very difficult to write the number 108 in the decimal number system because this number contains one hundred, eight ones and no tens! There are many other important things which would be impossible without the num-
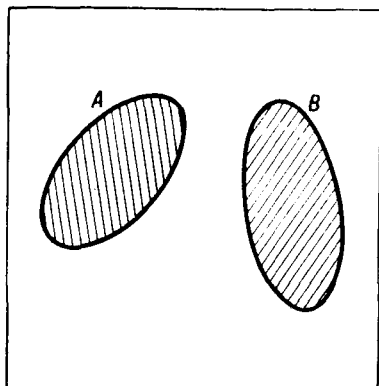
Fig. 8

ber zero. That is why the introduction of the number zero is considered one of the most remarkable events in the history of the development of arithmetic. Similarly, if we did not introduce the notion of an empty set O it would be impossible to speak of the product (the intersection) of any two sets: for instance, the intersection of the sets A and B shown in Fig. 8 is empty. Analogously, the intersection of the set of excellent pupils in your class and the set of elephants is also empty. If we did not have the notion of an "empty" set it would be even impossible to mention some sets: for instance, it would be impossible to speak of "the set of the pupils in a class whose name is Peter" because this set may not exist at all, that is it may turn out to be an empty set.

It is quite clear that if O is an empty set then we have

$$A + O = A$$

for any set A. It is also evident that for any set A we always have

$$AO = O$$

because the intersection of an arbitrary set A and the set O (which contains no elements) must be empty. (For instance, the intersection of the set of the girls in your class and the set of all those pupils whose height exceeds 2.5 m is empty.) The last identity is known as one of the *intersection laws* (the other intersection law will be presented later).

Fig. 9

Now we proceed to a more complicated question concerning a set playing the role analogous to that of the number 1 in the system of numbers. This set (we shall den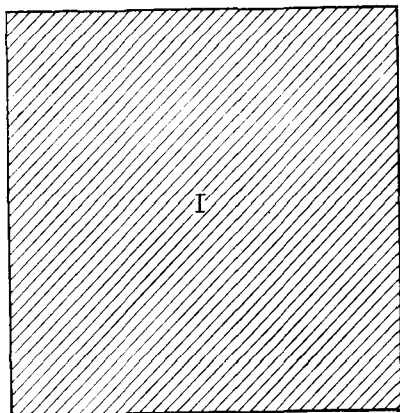ote it as I) must be such that the product (that is the intersection) of the set I and any set A coincides with A. From this requirement it follows that the set I must contain all the elements of all the sets A! It is clear however that such a set can only exist if we limit ourselves to those sets whose elements are taken from a definite store of "objects": for instance, if we limit ourselves to the sets of pupils of one definite school or of one definite class (such a set A may be the set of excellent pupils and another set B may be the set of the chess-players). Similarly, we can limit ourselves to the positive integers (then A can be, for instance, the set of the even positive integers and B can be the set of prime numbers which have no divisors except themselves and unity). We can also consider the sets of points forming various geometrical figures lying within a definite square such as those represented in Figs. 3-8. (We remind the reader that when we introduced Venn's diagrams on page 13 we stipulated the existence of such a "set of all the elements under consideration".) Thus, by I we shall always mean some underlying basic set containing all the objects admissible in a particular problem or discussion. For instance, as the set I we can take the set of all the pupils of a given school or class or the set of all positive integers or the set of all the points of

a square (Fig. 9). In "algebra of sets" the set I is referred to as the *universal set*. It is evident that for any "smaller" set A (and even for the set A coinciding with I) we have the second *intersection law*

$$AI = A$$

The last identity resembles the well-known arithmetic equality defining the number unity.

Thus, we see that the operation laws for the "algebra of sets" we have constructed are in many respects similar to the laws of elementary algebra dealing with numbers but at the same time the former laws do not completely coincide with the latter laws. As has been shown, almost all basic laws which are known for numbers also hold for the algebra of sets but the algebra of sets also has some other completely different laws which may seem strange when they are first encountered. For instance, as was already mentioned, in the general case the rule obtained from the equality $a \cdot 0 = 0$ by replacing multiplication by addition and zero by unity does not hold for numbers: for almost all numbers a we have $a + 1 \neq 1$. As to the algebra of sets, the situation is quite different: in this case we always have

$$A + I = I$$

Indeed, by definition, the set I contains all the objects under consideration and therefore it cannot be enlarged: when we add an arbitrary set A (this set A must of course belong to the class of sets we deal with) to the universal set I we always obtain the same set I.

Further, if we take the distributive law $(a + b)\,c = ac + bc$ which holds for numbers and interchange addition and multiplication in it we arrive at the meaningless "equality" $ab + c = (a + c)(b + c)$ which turns out to be wrong for numbers in almost all cases. But in the algebra of sets the situation is reverse: in this case we always have (for any sets A, B and C) the equality

$$AB + C = (A + C)(B + C)$$

expressing the *second distributive law of set theory* (this is the *distributive law for addition over multiplication*). Indeed, let, for instance, A be again the set of the chess-players, B the set of the pupils who play draughts and C the set of the

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swimmers in your class. Then, obviously, the intersection AB of the sets A and B consists of all the pupils who can play both chess and draughts and the union $AB + C$ of the sets AB and C consists of all the pupils who can play both chess and draughts or can swim (or, perhaps, can play chess, draughts and can swim). On the other hand, the unions $A + C$ and $B + C$ of the sets A and C and of the sets B and C respectively consist of the pupils who can play chess or can swim or both and of the pupils who can play draughts or can swim or both. It is clear that the intersection $(A + C)(B + C)$ of these two unions includes all the pupils who can swim and also those pupils who cannot swim but can play both chess and draughts, which means that this intersection coincides with the set $AB + C$.

This verbal explanation may seem too lengthy and therefore we shall also present the graphical demonstration of the second distributive law of the set theory. In Fig. 10,a the intersection AB (of the sets A and B) and the set C are shaded by hatching lines with different inclinations; the whole shaded area in the figure represents the set $AB + C$. In Fig. 10,b the union $A + C$ of the sets A and C is shaded by horizontal lines and the union $B + C$ of the sets B and C is shaded by vertical lines; the intersection $(A + C)(B + C)$ of these two unions is covered by the "network" of hatching lines. But it is readily seen that the area covered by the network of horizontal and vertical lines in Fig. 10,b exactly coincides with the whole area shaded in Fig. 10,a, which proves the second distributive law.
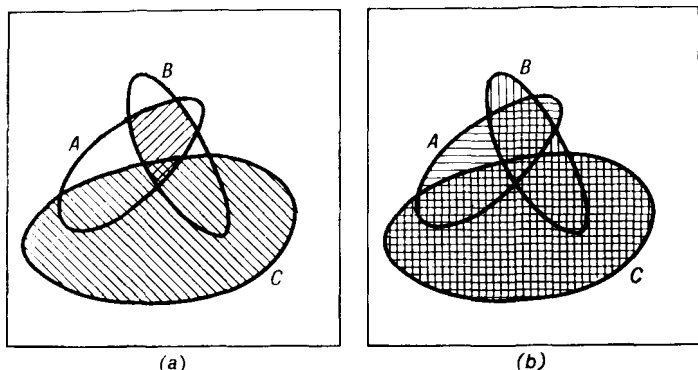

(a)   (b)

Fig. 10

In conclusion we present two more laws of algebra of sets which essentially differ from what is known from elementary algebra dealing with numbers. It is easy to understand that for any set A the union of this set and another replica of the same set and also the intersection of the set A with itself coincide with the original set A:

$$A + A = A \quad \text{and} \quad AA = A$$

These two identities are called *idempotent laws* (the first of them is the *idempotent law for addition* and the second is the *idempotent law for multiplication*).

The fact that the general laws of algebra retain one and the same form for all kinds of numbers is very important: this makes it possible to use our experience acquired in studying numbers of one kind when we pass to another kind of numbers (for instance, when we pass from whole numbers to fractions or to signed numbers supplied with the signs "plus" or "minus"). In other words, in all such cases related to algebra of numbers we have to learn some new facts and laws but the facts learned previously remain valid. However, when we pass from numbers to sets we encounter a completely different situation: it turns out that there are a number of laws of algebra of numbers which are inapplicable to the algebra of sets[1]).

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[1]) It is this distinction between the laws of algebra of sets and the laws of algebra of numbers that accounts for the fact that in many books the addition and the multiplication of sets (that is the operations of forming the union and the intersection of sets) are denoted not by the usual symbols $+$ and $\cdot$, but in a completely different way: the union of sets A and B is denoted as $A \cup B$ and the intersection of these sets as $A \cap B$. In the present book we shall deal not only with algebra of sets but also with some other algebraic systems in which the operations of "addition" and "multiplication" obey the same laws as in algebra of sets; therefore for our aims it is natural to use the ordinary symbols $+$ and $\cdot$, instead of the symbols $\cup$ and $\cap$ used in set theory. The usage of the symbols $+$ and $\cdot$ makes it possible to indicate in a visual way the similarity between elementary algebra and the new algebraic systems. But it is nevertheless expedient to write down here the basic laws of set algebra using the standard set-theoretic notation:

$$A \cup B = B \cup A \quad \text{and} \quad A \cap B = B \cap A$$
the commutative laws

$$(A \cup B) \cup C = A \cup (B \cup C) \quad \text{and} \quad (A \cap B) \cap C = A \cap (B \cap C)$$
the associative laws

Let us enumerate these new laws. We first of all mention the relation

$$A + I = I$$

which indicates a significant distinction between the universal set I and the number 1. Another peculiarity of algebra of sets is the way in which the "parentheses are opened" using the second distributive law

$$(A + C)(B + C) = AB + C$$

For instance, in algebra of sets we have

$$(A + D)(B + D)(C + D) = [(A + D)(B + D)](C + D) =$$
$$= (AB + D)(C + D) = (AB)C + D = ABC + D$$

Finally, the idempotent laws

$$A + A = A \quad \text{and} \quad AA = A$$

are completely new to us; we can express the meaning of these laws verbally by saying that *the algebra of sets involves neither exponents nor coefficients.* For in the algebra of sets we have

$$\underbrace{A + A + \ldots + A}_{n \text{ times}} = A$$

and

$$\underbrace{A \cdot A \cdot \ldots \cdot A}_{n \text{ times}} = A$$

for any A and n; that is why, for instance, we have

$$(A + B)(B + C)(C + A) =$$
$$= ABC + AAB + ACC + AAC +$$
$$+ BBC + ABB + BCC + ABC =$$
$$= (ABC + ABC) + (AB + AB) +$$

$$A \cup O = A \quad \text{and} \quad A \cap I = A$$

$$A \cup I = I \quad \text{and} \quad A \cap O = O$$

the properties of the empty set O and of the universal set I

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C) \quad \text{and} \quad (A \cap B) \cup C =$$
$$= (A \cup C) \cap (B \cup C)$$

the distributive laws

$$A \cup A = A \quad \text{and} \quad A \cap A = A$$

the idempotent laws

$$+ (AC + AC) + (BC + BC) =$$
$$= ABC + AB + AC + BC$$

(cf. Exercise 6 below).

Exercises

Prove the following equalities in which capital letters denote sets (the letter O always denotes the empty set and the letter I denotes the universal set):

1. $(A + B) (A + C) (B + D) (C + D) = AD + BC$
2. $A (A + B) = A$
3. $AB + A = A$
4. $A (A + C) (B + C) = AB + AC$
5. $A (A + I) (B + O) = AB$
6. $(A + B) (B + C) (C + A) = AB + BC + CA$
7. $(A + B) (B + C) (C + D) = AC + BC + BD$
8. $(A + B) (A + I) + (A + B) (B + O) = A + B$
9. $(A + B) (B + I) (A + O) = A$
10. $(A + B + C) (B + C + D) (C + D + A) =$
    $= AB + AD + BD + C$

Example: $A (A + C) (B + C)$ $=$ $A [(A + C) \times$

the associativity
of multiplication

$\times (B + C)]$ $=$ $A (AB + C)$ $=$ $(AB + C) A =$

the second
distributive law

the commutativity
of multiplication

$=$ $(AB) A + CA$ $=$ $(AA) B + AC =$

the first distributive
law

the commutativity and
the associativity of
multiplication

$=$ $AB + AC$

the idempotent
law for multiplication

2. Boolean Algebra

Let us write down all the general laws of algebra of sets we have established:

$$A + B = B + A \quad \text{and} \quad AB = BA$$

the commutative laws

$$(A + B) + C = A + (B + C) \quad \text{and} \quad (AB)C = A(BC)$$

the associative laws

$$(A + B)C = AC + BC \text{ and } AB + C = (A + C)(B + C)$$

the distributive laws

$$A + A = A \text{ and } AA = A$$

the idempotent laws

Besides, the set algebra contains two "special" elements (sets) O and I such that

$$A + O = A \quad \text{and} \quad AI = A$$
$$A + I = I \quad \text{and} \quad AO = O$$

These laws (identities) are similar to the ordinary laws of the algebra of numbers but they do not coincide completely with the latter; the algebra of sets is, of course, also an "algebra" but it is new to us and rather unusual.

Now it should be noted that we have in fact not one ordinary algebra of numbers but many such "algebras": indeed, we can consider the "algebra of positive integers", the "algebra of rational numbers" (by the rational numbers are meant both the integers and the fractions), the "algebra of signed numbers" (that is of the positive and non-positive numbers) and also the "algebra of real numbers" (that is of the rational and irrational numbers), the "algebra of complex numbers" (that is of the real and the imaginary numbers) and so on. All these "algebras" differ from one another in the numbers on which the operations are performed and in the definitions of the operations of addition and multiplication but the general properties of the operations remain the same in all the cases. In this connection it now appears natural to ask, what is the situation in the unusual algebra of sets? In other words, is there only one realization of such an algebra or is there also a number of these "algebras" which differ from one another in the elements on which the operations are performed and in the definitions of the operations (as before we shall call these operations addition and multiplication) but at the same time are similar in the basic properties of the operations?

The reader can undoubtedly anticipate the answer to the question: there are in fact many algebras similar to the algebra of sets (in these algebras the same general operation rules hold). First of all, there are a variety of algebras

of sets themselves: for instance, we can consider the "algebra of sets of pupils in your class", the "algebra of sets of animals in a zoo" (this is, of course, a completely different algebra!), the "algebra of sets of numbers" (these numbers can be of different kinds), the "algebra of sets of points lying within a square" (see Figs. 3-10), the "algebra of sets of books in a library" and the "algebra of sets of stars in the sky". But there are also completely different examples of algebras having similar properties; below we shall discuss some examples of this kind.

Before proceeding to these examples the reader should realize that *to define the operations of addition and multiplication in a set of objects (elements) a, b, . . . means to indicate some rules according to which we attribute to any two objects a and b two other objects c and d called, respectively, the sum and the product of a and b*:

$$c = a + b \quad \text{and} \quad d = ab$$

These rules must be chosen so that all the laws characteristic of the algebra of sets are fulfilled. But after these rules have been chosen we have no right to ask, why the sum of a and b is equal to c? Indeed, we *define* the sum $a + b$ as being the element c and, as is known, the definitions must not be discussed when they are consistent, that is when they satisfy definite general logical requirements. Some definitions given below may seem strange because they are new, and new things always seem strange when they are first encountered. For instance, pupils are first told that the sum of two numbers a and b is the number of objects in the union of a collection containing a objects and a collection of b objects (see Fig. 1 on page 8) and that the product ab is equal to the number of objects in the union of a collections each of which contains b objects (Fig. 2 on page 8). Later the pupils are taught fractions and are told that the sum and the product of fractions a/b and c/d are defined according to the rules given on page 9; these new rules and definitions may of course seem strange to the pupils until they get used to them.

Now we proceed to the examples.

Example 1. *Algebra of two numbers* (elements). Let us assume that the algebra consists of only two elements; for the sake of simplicity, we shall call these elements

"numbers" and denote them by the familiar symbols 0 and **1** (but in the case under consideration these symbols have a completely new meaning). We shall define the *multiplication* of these numbers in exactly the same way as in ordinary arithmetic, that is by means of the following "multiplication table":

·	0	1
0	0	0
1	0	1

As to the *addition*, we shall define it in an "almost ordinary way" with the only distinction from the ordinary arithmetic that now the sum $1 + 1$ is not equal to 2 (this "algebra of two numbers" does not contain the number 2 at all!) but is equal to 1. Thus, the "addition table" in this new algebra has the form

+	0	1
0	0	1
1	1	1

It is obvious that in the algebra thus defined both commutative laws hold:

$$a + b = b + a \quad \text{and} \quad ab = ba \quad \text{for any } a \text{ and } b$$

It can readily be verified that the associative laws also hold for this algebra:

$$(a + b) + c = a + (b + c) \quad \text{and} \quad (ab) c = a (bc)$$
$$\text{for any } a, b \text{ and } c$$

There is no need to verify the associative law for multiplication because the "new" multiplication completely coincides with the multiplication of numbers for which, as we know, the associative law holds. It is also clearly seen that the idempotent laws also hold for this algebra:

$$a + a = a \quad \text{and} \quad aa = a \text{ for any } a$$

that is for $a = 0$ and for $a = 1$ (now we see why it was necessary to put $1 + 1 = 1$!). It is a little more difficult

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to verify the distributive laws:

$$(a + b)\, c = ac + bc \quad \text{and} \quad ab + c = (a + c)\, (b + c)$$
$$\text{for any } a,\, b \text{ and } c$$

For instance, in this algebra of two elements we have

$$(1 + 1) \cdot 1 = 1 \cdot 1 = 1 \quad \text{and} \quad (1 \cdot 1) + (1 \cdot 1) = 1 + 1 = 1$$
$$(1 \cdot 1) + 1 = 1 + 1 = 1 \quad \text{and} \quad (1 + 1) \cdot (1 + 1) = 1 \cdot 1 = 1$$

Finally, if we agree that the number 0 plays the role of the element O and the number 1 plays the role of the element I, the rules concerning the "special" elements O and I will also hold, that is we shall always have (for $a = 0$ and for $a = 1$)

$$a + 0 = a \quad \text{and} \quad a \cdot 1 = a; \qquad a + 1 = 1 \quad \text{and} \quad a \cdot 0 = 0$$

Example 2. *Algebra of four numbers (elements).* This is a slightly more complicated example of the same kind. Suppose that the elements of the algebra are four "numbers" which we shall denote as the digits 0 and 1 and the letters p and q. The addition and the multiplication in this algebra will be defined with the aid of the following tables:

$+$	0	p	q	1
0	0	p	q	1
p	p	p	1	1
q	q	1	q	1
1	1	1	1	1

$\cdot$	0	p	q	1
0	0	0	0	0
p	0	p	0	p
q	0	0	q	q
1	0	p	q	1

As can readily be checked by means of the direct computation, in this case we also have

$$a + b = b + a \quad \text{and} \quad ab = ba \text{ for any } a \text{ and } b$$
$$(a + b) + c = a + (b + c) \quad \text{and} \quad (ab)\, c = a\, (bc)$$
$$\text{for any } a,\, b \text{ and } c$$
$$(a + b)\, c = ac + bc \quad \text{and} \quad ab + c = (a + c)\, (b + c)$$
$$\text{for any } a,\, b \text{ and } c$$
$$a + a = a \quad \text{and} \quad aa = a \quad \text{for any } a$$
$$\text{(that is for } a = 0,\ a = p,\ a = q \text{ and } a = 1)$$

Besides, the numbers 0 and 1 play the roles of the elements O and I of the algebra of sets respectively because for any a we have

$$a + 0 = a \quad \text{and} \quad a \cdot 1 = a; \ a + 1 = 1 \text{ and } a \cdot 0 = 0$$

Example 3. *Algebra of maxima and minima.* As the elements of the algebra let us take the numbers contained in an arbitrary (bounded) number set; for instance, let us agree that the elements of the algebra are some (or, perhaps, all) numbers x satisfying the condition $0 \leqslant x \leqslant 1$, that is numbers lying between 0 and 1 and the numbers 0 and 1 themselves. As to the operations of addition and multiplication, we shall define them in a completely new manner. To avoid the confusion between the ordinary addition and multiplication and the new operations we shall even denote the latter by the new symbols $\oplus$ (addition) and $\otimes$ (multiplication). Namely, let us assume that the *sum* $x \oplus y$ of two numbers x and y is equal to the *greatest* (the *maximum*) of these numbers in case $x \neq y$ and to any of them in case $x = y$. By the product $x \otimes y$ of two numbers x and y we shall mean the *least* (the *minimum*) of these numbers in case $x \neq y$ and any of them in case $x = y$. For instance, if the elements of the algebra are the numbers 0, 1/3, 1/2, 2/3 and 1 then the "addition table" and the "multiplication table" for these numbers have the form

$\oplus$	0	$\dfrac{1}{3}$	$\dfrac{1}{2}$	$\dfrac{2}{3}$	1
0	0	$\dfrac{1}{3}$	$\dfrac{1}{2}$	$\dfrac{2}{3}$	1
$\dfrac{1}{3}$	$\dfrac{1}{3}$	$\dfrac{1}{3}$	$\dfrac{1}{2}$	$\dfrac{2}{3}$	1
$\dfrac{1}{2}$	$\dfrac{1}{2}$	$\dfrac{1}{2}$	$\dfrac{1}{2}$	$\dfrac{2}{3}$	1
$\dfrac{2}{3}$	$\dfrac{2}{3}$	$\dfrac{2}{3}$	$\dfrac{2}{3}$	$\dfrac{2}{3}$	1
1	1	1	1	1	1

$\otimes$	0	$\dfrac{1}{3}$	$\dfrac{1}{2}$	$\dfrac{2}{3}$	1
0	0	0	0	0	0
$\dfrac{1}{3}$	0	$\dfrac{1}{3}$	$\dfrac{1}{3}$	$\dfrac{1}{3}$	$\dfrac{1}{3}$
$\dfrac{1}{2}$	0	$\dfrac{1}{3}$	$\dfrac{1}{2}$	$\dfrac{1}{2}$	$\dfrac{1}{2}$
$\dfrac{2}{3}$	0	$\dfrac{1}{3}$	$\dfrac{1}{2}$	$\dfrac{2}{3}$	$\dfrac{2}{3}$
1	0	$\dfrac{1}{3}$	$\dfrac{1}{2}$	$\dfrac{2}{3}$	1

The maximum number among two or several numbers $u, v, \ldots, z$ is often denoted in mathematics as max $|u, v, \ldots, z|$ and the minimum number among these

numbers as min $[u, v, \ldots, z]$. Thus, in this "algebra of maxima and minima", we have, by definition,

$$x \oplus y = \max [x, y] \quad \text{and} \quad x \otimes y = \min [x, y]$$

We can also agree to represent numbers as points on the number line. Then the numbers x satisfying the condition $0 \leqslant x \leqslant 1$ are represented by the points of a horizontal line segment of length 1, the sum $x \oplus y$ of two numbers x and y is represented by the rightmost of the points x and y and their product $x \otimes y$ is represented by the leftmost point (Fig. 11).

It is clear that the new operations of addition and multiplication we have defined satisfy the commutative laws:

$$x \oplus y = y \oplus x \quad \text{and} \quad x \otimes y = y \otimes x$$

The associative laws

$(x \oplus y) \oplus z = x \oplus (y \oplus z)$ and

$$(x \otimes y) \otimes z = x \otimes (y \otimes z)$$

also obviously hold because the number $(x \oplus y) \oplus z$ (or, which is the same, the number $x \oplus (y \oplus z)$) which can simply be denoted as $x \oplus y \oplus z$ without the parentheses is nothing but $\max [x, y, z]$ (Fig. 12) while the number $(x \otimes y) \otimes z$ (or, which is the same, $x \otimes (y \otimes z)$) which can simply be written as $x \otimes y \otimes z$ without the parentheses is nothing but $\min [x, y, z]$ (see the same Fig. 12). It is also quite clear that the idempotent laws also hold here:

$x \oplus x = \max [x, x] = x \quad \text{and} \quad x \otimes x = \min [x, x] = x$

Finally, let us check the validity of the distributive laws

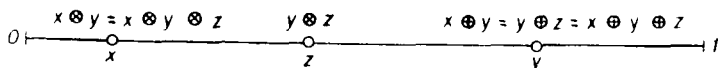$$(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$$

Fig. 12

$(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$

$y \otimes z \quad x \otimes z \qquad x \oplus y$

$0 \vdash$ — — — $\dashv 1$

$y \quad z \qquad x$

(a)

$(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z)$

$y \otimes z \quad x \oplus y = x \otimes z$

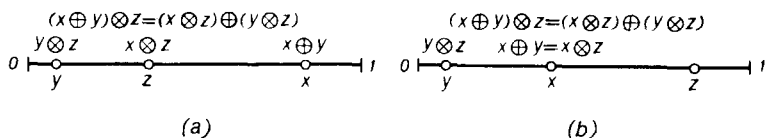$0 \vdash$ — — — $\dashv 1$

$y \quad x \qquad z$

(b)

Fig. 13

and
$$(x \otimes y) \oplus z = (x \oplus z) \otimes (y \oplus z)$$

It is evident that the number

$$(x \oplus y) \otimes z = \min \{\max [x, y], z\}$$

is equal to z if *at least one of the numbers x and y is greater than z* and is equal to the greatest of these numbers *if both x and y are less than z* (Fig. 13,a and b). It is also clear that the number

$$(x \otimes z) \oplus (y \otimes z) = \max \{\min [x, z], \min [y, z]\}$$

is equal to the same value (see again Fig. 13). Analogously, the number

$$(x \otimes y) \oplus z = \max \{\min [x, y], z\}$$

is equal to z if *at least one of the numbers x and y is less than z* and is equal to the minimum of the numbers x and y *if both x and y exceed z* (Fig. 14,a and b). As is seen from the same Fig. 14, the number

$(x \oplus z) \otimes (y \oplus z) = \min \{\max [x, z], \max [y, z]\}$

is also equal to the same value.

Now to make sure that all the laws of the algebra of sets hold for the new unusual algebra of maxima and minima it is sufficient to note that the role of the elements O and I of the algebra of sets is played by the smallest number 0 among all the numbers under consideration and by the greatest number 1 respectively. Indeed, for any number x satis-

$(x \otimes y) \oplus z = (x \oplus z) \otimes (y \oplus z)$

$x \otimes y \qquad x \oplus z = y \oplus z$

$0 \vdash$ — — — $\dashv 1$

$y \quad x \qquad z$

(a)

$(x \otimes y) \oplus z = (x \oplus z) \otimes (y \oplus z)$

$x \otimes y = x \oplus z \quad y \oplus z$

$0 \vdash$ — — — $\dashv 1$
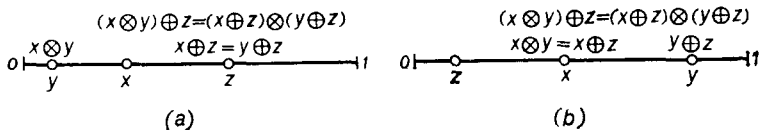
$z \quad x \qquad y$

(b)

Fig. 14

fying the condition $0 \leqslant x \leqslant 1$ we always have

$$x \oplus 0 = \max [x, 0] = x \quad \text{and} \quad x \otimes 1 = \min [x, 1] = x$$
$$x \oplus 1 = \max [x, 1] = 1 \quad \text{and} \quad x \otimes 0 = \min [x, 0] = 0$$

Example 4. *Algebra of least common multiples and greatest common divisors.* Let N be an arbitrary integer. As the elements of the new algebra we shall take all the possible divisors of the number N. For instance, if $N = 210 = 2 \cdot 3 \cdot 5 \cdot 7$ then the elements of the algebra in question are the numbers 1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 35, 42, 70, 105 and 210. In this example we shall define the addition and the multiplication of the numbers in a completely new way: by the *sum* $m \oplus n$ of two numbers m and n we shall mean their *least common multiple* (that is the smallest positive integer which is divisible by both numbers m and n) and as the *product* $m \otimes n$ of the numbers m and n we shall take their *greatest common divisor* (that is the greatest integer by which both m and n are divisible). For instance, if $N = 6$ (in this case the algebra contains only the four numbers 1, 2, 3 and 6) the addition and the multiplication of the elements of the algebra are specified by the following tables:

$\oplus$	1	2	3	6		$\otimes$	1	2	3	6
1	1	2	3	6		1	1	1	1	1
2	2	2	6	6	and	2	1	2	1	2
3	3	6	3	6		3	1	1	3	3
6	6	6	6	6		6	1	2	3	6

In "higher arithmetic" (number theory) the least common multiple of two or several numbers $m, n, \ldots, s$ is often denoted as $[m, n, \ldots, s]$ and the greatest common divisor of the same numbers is denoted as $(m, n, \ldots, s)$. Thus, for this algebra we have, by definition,

$$m \oplus n = [m, n] \quad \text{and} \quad m \otimes n = (m, n)$$

For instance, if the algebra contains the numbers 10 and 15 then

$$10 \oplus 15 = [10, 15] = 30 \quad \text{and} \quad 10 \otimes 15 = (10, 15) = 5$$

It is evident that in this algebra we always have

$$m \oplus n = n \oplus m \quad \text{and} \quad m \otimes n = n \otimes m$$

Further, we have

$$(m \oplus n) \oplus p = m \oplus (n \oplus p) = [m, \ n, \ p]$$

(we can denote this number simply as $m \oplus n \oplus p$ without the parentheses) and also

$$(m \otimes n) \otimes p = m \otimes (n \otimes p) = (m, \ n, \ p)$$

(the latter number can simply be denoted as $m \otimes n \otimes p$). The idempotent laws

$$m \oplus m = [m, \ m] = m \text{ and } m \otimes m = (m, \ m) = m$$

are also quite evident.

The verification of the distributive laws is a little more lengthy. The number

$$(m \oplus n) \otimes p = ([m, \ n], \ p)$$

is nothing but *the greatest common divisor of the number p and the least common multiple of the numbers m and n* (think carefully about this expression!). This number contains those and only those prime factors which are contained in p and are simultaneously contained in at least one of the numbers m and n. But it is evident that these (and only these) prime factors are also contained in the number

$$(m \otimes p) \oplus (n \otimes p) = [(m, \ p), \ (n, \ p)]$$

and therefore we always have

$$(m \oplus n) \otimes p = (m \otimes p) + (n \otimes p)$$

For instance, if we limit ourselves to the divisors of the number 210 then we have

$$(10 \oplus 14) \otimes 105 = ([10, \ 14], \ 105) = (70, \ 105) = 35$$

and

$$(10 \otimes 105) \oplus (14 \otimes 105) = [(10, \ 105), \ (14, \ 105)] =$$
$$= [5, \ 7] = 35$$

Analogously, the number

$$(m \otimes n) \oplus p = [(m, \ n), \ p]$$

is *the least common multiple of the number p and the greatest common divisor of the numbers m and n*; it contains those and only those prime factors which are contained in p or in both numbers m and n or in all the three numbers p, m

and n. But exactly the same factors are contained in the number

$$(m \oplus p) \otimes (n \oplus p) = ([m, \, p], \, [n, \, p])$$

and therefore we always have

$$(m \otimes n) \oplus p = (m \oplus p) \otimes (n \oplus p)$$

For instance,

$$(10 \otimes 14) \oplus 105 = [(10, \, 14), \, 105] = [2, \, 105] = 210$$

and

$$(10 \oplus 105) \otimes (14 \oplus 105) = ([10, \, 105], \, [14, \, 105]) =$$
$$= (210, \, 210) = 210$$

Finally, in this case the roles of the elements O and I of the algebra of sets are played by the smallest number 1 among the collection of numbers we deal with and by the greatest number N respectively. Indeed, this algebra only contains the divisors of the number N and we obviously have

$$m \oplus 1 = [m, \, 1] = m \quad \text{and} \quad m \otimes N = (m, \, N) = m$$
$$m \oplus N = [m, \, N] = N \quad \text{and} \quad m \otimes 1 = (m, \, 1) = 1$$

We thus see that for this algebra all the laws of the algebra of sets are fulfilled.

Hence, there are different systems of "objects" (of *elements* of the algebra in question) for which it is possible to define the operations of *addition* and *multiplication* satisfying all the known rules fulfilled in the algebra of sets: the two commutative laws, the two associative laws, the two distributive laws, the two idempotent laws and the four rules specifying the properties of the "special" elements whose role in these algebras is close to that of zero and unity. Later on we shall consider two more important and interesting examples of such algebras.

Now we proceed to the study of the general properties of all such algebras and our immediate aim is to give a general name to all of these algebras. Since algebras with such strange properties were first considered by the distinguished English mathematician George Boole who lived in the 19th century, all the algebras of this kind are called *Boolean algebras*[1]). For the basic operations on the elements of a Boo-

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[1]) A rigorous definition of a Boolean algebra is stated in Appendix on page 117.
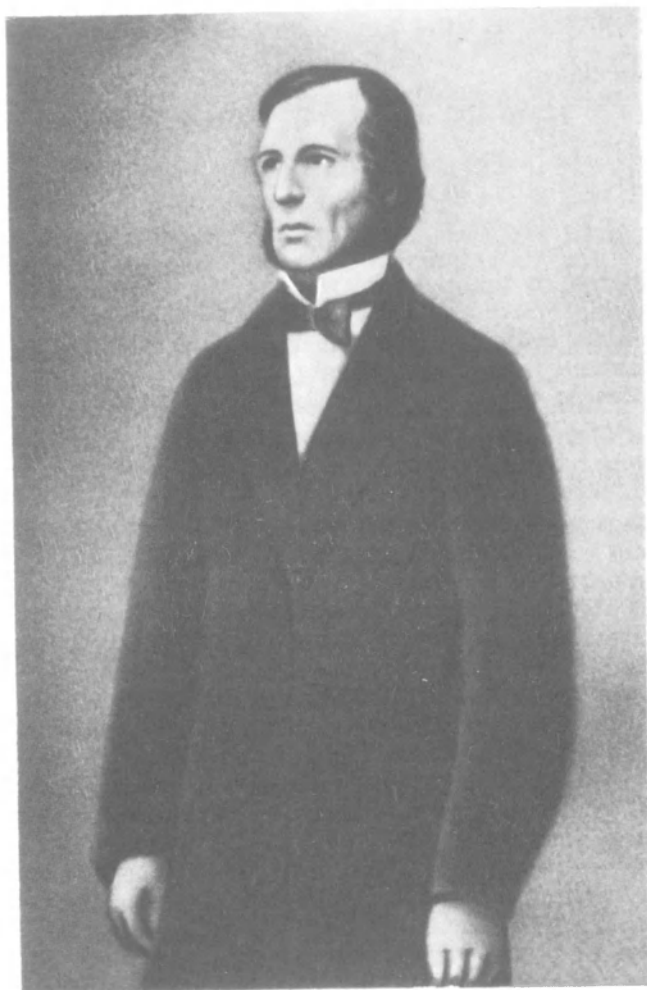
GEORGE BOOLE
(1815-1864)

lean algebra we shall retain the terms "addition" and "multiplication" (but the reader should bear in mind that in the general case these operations differ from the ordinary addition and multiplication of numbers!). We shall also sometimes refer to these operations as the *Boolean addition* and the *Boolean multiplication*.

In his "Laws of Thought" which first appeared in 1854, that is more than a hundred years ago, G. Boole investigated in detail this unusual algebra. The title of G. Boole's work may first seem strange; however, after the reader has studied this book it will become clear what is the relationship between unusual algebras considered in the book and the laws of human thought. At present we only note that it is this relationship between the Boolean algebras and the "laws of thought" that accounts for the fact that the work of G. Boole to which his contemporaries paid little attention is of such great interest nowadays. In recent years the book by G. Boole has been many times republished and translated into various languages.

Exercises

1. Verify directly that for all triples of elements of the "Boolean algebra containing two numbers" (see Example 1 on page 25) there hold both distributive laws.

2. Check the validity of both distributive laws for several triples of elements of the "Boolean algebra with four elements" (Example 2 page 27).

3. (a) Let there be a family in which there is only one schoolboy. Then all the "sets of the schoolboys in the family" are the following: the set I containing one schoolboy and the set O containing no schoolboys (the empty set). Compile the "addition table" and the "multiplication table" for the "algebra of sets of schoolboys in the family" (this algebra consists of only two elements O and I) and compare these tables with the tables on page 26. Proceeding from this comparison show that in the "algebra of two numbers" considered in Example 1 of this section all the laws of the Boolean algebra hold.

(b) Let there be a family in which there are two schoolchildren Peter and Mary who go to school. Then the "algebra of sets of schoolchildren in the family" consists of four elements: the set I containing both schoolchildren, the two

3 *

sets P (Peter) and M (Mary) each of which contains one of the two schoolchildren and the empty set O. Compile the "addition table" and the "multiplication table" for this algebra of sets and compare them with the tables on page 27. Proceeding from this comparison show that in the "algebra of four elements" considered in Example 2 of this section there hold all the laws of the Boolean algebra.

4. Check that

(a) $\min\left\{\max\left[\dfrac{1}{2},\ \dfrac{1}{3}\right],\ \dfrac{1}{4}\right\} =$
$$= \max\left\{\min\left[\dfrac{1}{2},\ \dfrac{1}{5}\right],\ \min\left[\dfrac{1}{3},\ \dfrac{1}{4}\right]\right\}$$

and

$$\max\left\{\min\left[\dfrac{1}{2},\ \dfrac{1}{3}\right],\ \dfrac{1}{4}\right\} =$$
$$= \min\left\{\max\left[\dfrac{1}{2},\ \dfrac{1}{4}\right],\ \max\left[\dfrac{1}{3},\ \dfrac{1}{4}\right]\right\}$$

(b) $([12, 30], 8) = [(12, 8), (30, 8)]$
and
   $[(12, 30), 8] = ([12, 8],\quad [30, 8])$

5. (a) Compile the "addition table" and the "multiplication table" for the Boolean algebra consisting of the three numbers 0, $\dfrac{1}{2}$ and 1 in which $x \oplus y = \max [x,\ y]$ and $x \otimes y = \min [x,\ y]$. Verify the validity of the laws of the Boolean algebra for this algebra of three elements.

(b) Compile the "addition table" and the "multiplication table" for the algebra of the divisors of the number 12 in which $m \oplus n = [m,\ n]$ and $m \otimes n = (m,\ n)$. Show that some of the laws of the Boolean algebra are fulfilled for this algebra of the divisors.

6*. Let the decomposition of a (positive integral) number N into prime factors be of the form

$$N = p_1^{A_1} p_2^{A_2} \ldots p_k^{A_k}$$

Then any two divisors m and n of this number can be written as

$$m = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$$

where $0 \leqslant a_1 \leqslant A_1,\ 0 \leqslant a_2 \leqslant A_2,\ \ldots,\ 0 \leqslant a_k \leqslant A_k$

and

$$n = p_1^{b_1} p_2^{b_2} \ldots p_k^{b_k}$$

where

$$0 \leqslant b_1 \leqslant A_1, \ 0 \leqslant b_2 \leqslant A_2, \ldots, 0 \leqslant b_k \leqslant A_k$$

(some of the numbers $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ can be equal to zero). What form have in this case the decompositions into prime factors of the numbers $[m, n]$ (the least common multiple of the numbers m and n) and of the numbers (m, n) (the greatest common divisor of the numbers m and n)? Using these decompositions prove that the set of all the divisors of the number N with the operations $m \oplus n = [m, n]$ and $m \otimes n = (m, n)$ is a Boolean algebra.

3. Further Properties of Boolean Algebras. Principle of Duality. Boolean Equalities and Inequalities

Let us continue to study Boolean algebras. We first of all see a complete parallelism between the properties of the Boolean addition and the Boolean multiplication; the similarity between the operations is so close that *in every* (correct) *formula of a Boolean algebra we can interchange the addition and the multiplication*: the equality resulting from the interchange remains valid. For instance, in a Boolean algebra there holds the equality

$$A (A + C) (B + C) = AB + AC$$

(which was proved earlier for the algebra of sets; see the exercises to Sec. 1 on page 23). On interchanging the addition and the multiplication in this equality we obtain

$$A + AC + BC = (A + B)(A + C)$$

The latter equality is also valid (see page 39). It should also be taken into account that *when an equality fulfilled for a Boolean algebra involves the "special" elements O and I then the interchange of the Boolean addition and the Boolean multiplication in this equality must be followed by the interchange of the elements O and I*. For instance, the validity

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of the equality

$$(A + B)(A + I) + (A + B)(B + O) = A + B$$

(see Exercise 8 on page 23) implies that the equality

$$(AB + AO)(AB + BI) = AB$$

must also hold.

The just stated property of the Boolean algebras which allows us to obtain automatically (that is without proof) from any equality a new one[1]) is called the *Principle of Duality* and the formulas which are obtained from each other with the aid of this principle are called *dual formulas*. The principle of duality follows from the fact that the list of the basic laws of a Boolean algebra (when proving various Boolean relations we can only proceed from these laws) is completely "symmetric", i.e. together with every law it also includes another law dual to the former, that is the law which is obtained from the former law by interchanging the addition and the multiplication and by interchanging simultaneously the elements O and I. Examples of dual pairs of laws are the commutative law for addition and the commutative law for multiplication, the associative law for addition and the associative law for multiplication, the idempotent law for addition and the idempotent law for

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[1]) The "new" equality obtained from a formula of a Boolean algebra by means of the interchange of the addition and the multiplication and of the elements O and I may sometimes coincide with the original relation. In this case we deal with a *self-dual* relation and the application of the duality principle does not give us a new formula. For instance, if we take the correct equality

$$(A + B)(B + C)(C + A) = AB + BC + CA$$

(see Exercise 6 on page 23) and interchange the addition and the multiplication in it we obtain the equality

$$AB + BC + CA = (A + B)(B + C)(C + A)$$

coinciding with the original equality. Similarly, the application of the principle of duality to the correct equality

$$(A + B)(B + C)(C + D) = AC + BC + BD$$

(see Exercise 7 on page 23) results in the equality

$$AB + BC + CD = (A + C)(B + C)(B + D)$$

which only slightly differs from the original relation (it simply turns into the original relation if we interchange the letters B and C).

multiplication. Similarly, the first and the second distributive laws are dual and, finally, the equalities $A + O = A$ and $A + I = I$ are dual to the equalities $AI = A$ and $AO = O$ respectively. That is why when we prove an equality by using some basic laws of a Boolean algebra we can similarly prove the dual equality by using the corresponding dual laws.

Example. Let us prove the equality

$$A + AC + BC = (A + B)(A + C)$$

It can readily be seen that this equality is the dual of the relation

$$A(A + C)(B + C) = AB + AC$$

Indeed,

$$A + AC + BC \underset{\substack{\text{the associativity}\\\text{of addition}}}{=} A + (AC + BC) \underset{\substack{\text{the first}\\\text{distributive law}}}{=}$$

$$= A + (A + B)C \underset{\substack{\text{the commutativity}\\\text{of addition}}}{=} (A + B)C + A \underset{\substack{\text{the second}\\\text{distributive law}}}{=}$$

$$= [(A + B) + A](C + A) \underset{\substack{\text{the commutativity}\\\text{and the associati-}\\\text{vity of addition}}}{=}$$

$$= [(A + A) + B](A + C) \underset{\substack{\text{the idempotent law}\\\text{of addition}}}{=} (A + B)(A + C)$$

(cf. the proof of the equality $A(A + C)(B + C) = AB + AC$ on page 23).

An alternative proof of the principle of duality is connected with a special operation defined in Boolean algebras which transforms every element A of a Boolean algebra into a new element $\overline{A}$ and under which *addition and multiplication are interchanged.* In other words, this operation (we shall refer to it as the "*bar*" operation) is such that

$$\overline{A + B} = \overline{A}\overline{B} \quad \text{and} \quad \overline{AB} = \overline{A} + \overline{B}$$

Further, this operation possesses the properties

$$\overline{O} = I \quad \text{and} \quad \overline{I} = O$$

Finally, under the "bar" operation the element $\overline{A}$ goes into the original element A, that is for every element A
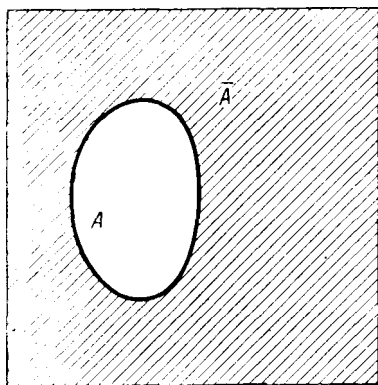
Fig. 15

of a Boolean algebra we have

$$\overline{\overline{A}} = (\overline{\overline{A}}) = A$$

In the algebra of sets the "bar" operation (this operation makes it possible to form a new element of a Boolean algebra from one given element of the algebra and not from two given elements as in the case of addition or multiplication) has the following meaning.

By $\overline{A}$ we mean the so-called *complement* of the set A which is, by definition, *the set containing those and only those elements of the universal set I which are not contained in the set A* (see Fig. 15). For instance, if we take the set of all the pupils in your class as the universal set and if A is the set of those pupils who got at least one bad mark then $\overline{A}$ is the set of those pupils who got no bad marks.

The definition of the complement $\overline{A}$ of the set A directly implies that

$$\overline{\overline{A}} = (\overline{\overline{A}}) = A$$

It also follows from the definition that

$$A + \overline{A} = I \quad \text{and} \quad A\overline{A} = O$$

(see Fig. 15; the last two equalities expressing the so-called *complementation laws* can even be taken as the definition of the set $\overline{A}$). It is also evident that

$$\overline{O} = I \quad \text{and} \quad \overline{I} = O$$

Finally, let us prove that in the set algebra there hold the following highly important properties of the "bar" operation:

$$\overline{A + B} = \overline{A}\,\overline{B} \quad \text{and} \quad \overline{AB} = \overline{A} + \overline{B}$$

The last two relations express the so-called *laws of dualization*; they are also referred to as the *De Morgan formulas*
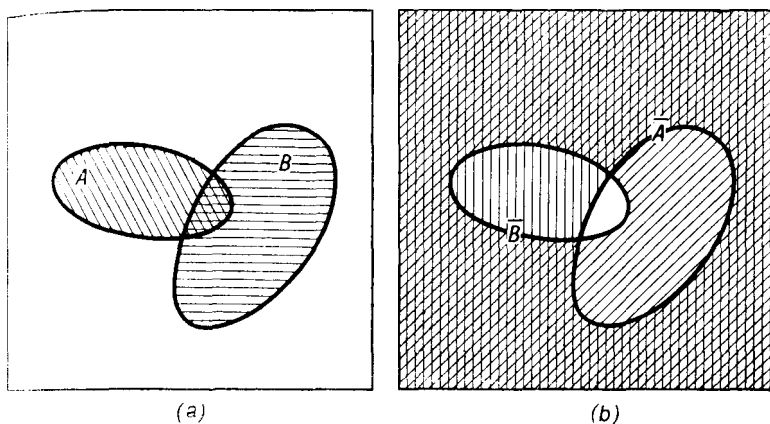
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(a)   (b)

Fig. 16

(or *rules*) after the English mathematician Augustus De
Morgan (1806-1871), a contemporary and associate of George
Boole. The first of these relations is also called the *De
Morgan theorem for union-complement* and the second is called
the *De Morgan theorem for intersection-complement*. In
Fig. 16a the area representing the set A is shaded by hatch-
ing lines inclined to the left; in Fig. 16b the complement $\overline{A}$
of the set A (with respect to the whole square I) is shaded
with hatching lines inclined to the right. The horizontal
lines in Fig. 16a cover the area representing the set B;
the vertical lines in Fig. 16b show the complement $\overline{B}$ of the
set B. The whole shaded area in Fig. 16a represents the
set $A + B$ while the cross-hatched area in Fig. 16b repre-
sents the set $\overline{A}\overline{B}$. The comparison of Fig. 16a and Fig. 16b
indicates that the cross-hatched area in Fig. 16b is the
complement of the set represented by the whole shaded
area in Fig. 16a, which proves the first of the De Morgan
formulas:

$$\overline{A + B} = \overline{A}\overline{B}$$

On the other hand, the cross-hatched area in Fig. 16a repre-
sents the set AB; the whole shaded area in Fig. 16b repre-
sents the set $\overline{A} + \overline{B}$. It is evident, that these two areas
(sets) are also the complements of each other, that is

$$\overline{AB} = \overline{A} + \overline{B}$$

which proves the second of the De Morgan formulas.

*

Now let us discuss the meaning of the "bar" operation for the other examples of Boolean algebras considered earlier. For the algebra of two elements (Example 1 on page 25) we put

$$0 = 1 \quad \text{and} \quad 1 = 0$$

It is quite evident that for any element a of this algebra (that is for $a = 0$ and for $a = 1$) we have $\overline{\overline{a}} = a$. Further, the comparison of the "addition table" and the "multiplication table" compiled for the numbers $0 = 1$ and $1 = 0$ which have the form

+	0	1
0	0	1
1	1	1

and

·	$\overline{0}=1$	$\overline{1}=0$
$\overline{0}=1$	1	0
$\overline{1}=0$	0	0

shows that $\overline{a + b} = \overline{a}\,\overline{b}$ in all the cases. De Morgan's second rule is verified in an analogous way: $\overline{ab} = \overline{a} + \overline{b}$.

For the algebra of four elements (Example 2 on page 27) we put

$$\overline{0} = 1, \quad \overline{p} = q, \quad \overline{q} = p \quad \text{and} \quad \overline{1} = 0$$

In this case it is also quite clear that $\overline{\overline{a}} = a$ for any element a of this algebra. As above, to prove the relation $\overline{a + b} = \overline{a}\,\overline{b}$ it suffices to compare the following two tables:

+	0	p	q	1
0	0	p	q	1
p	p	p	1	1
q	q	1	q	1
1	1	1	1	1

·	$\overline{0}=1$	$\overline{p}=q$	$\overline{q}=p$	$\overline{1}=0$
$\overline{0}=1$	1	q	p	0
$\overline{p}=q$	q	q	0	0
$\overline{q}=p$	p	0	p	0
$\overline{1}=0$	0	0	0	0

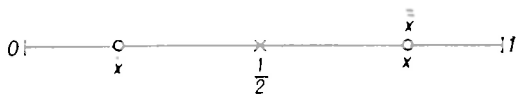The relation $\overline{ab} = \overline{a} + \overline{b}$ can be checked in a similar manner.

Fig. 17

Now let us consider the *algebra of maxima and minima* whose elements are numbers x such that $0 \leqslant x \leqslant 1$ for which the Boolean addition $\oplus$ and the Boolean multiplication $\otimes$ are defined as

$$x \oplus y = \max [x, y] \quad \text{and} \quad x \otimes y = \min [x, y]$$

For the De Morgan rules to hold in this algebra we must have

$$\overline{x \oplus y} = \overline{x} \otimes \overline{y} \quad \text{and} \quad \overline{x \otimes y} = \overline{x} \oplus \overline{y}$$

This means that there must be

$$\overline{\max [x, y]} = \min [\overline{x}, \overline{y}] \quad \text{and} \quad \overline{\min [x, y]} = \max [\overline{x}, \overline{y}]$$

and it is also necessary that the "bar" operation should reverse the order of elements, that is it is necessary that the condition $x \leqslant y$ should imply $\overline{x} \geqslant \overline{y}$ (why?). Therefore, when the elements of the algebra are *all* the numbers x satisfying the condition $0 \leqslant x \leqslant 1$ then, for instance, we can put

$$\overline{x} = 1 - x$$

In other words, we can assume that *the points $\overline{x}$ and x are symmetric about the midpoint* $1/2$ *of the closed interval* $[0, 1]$ (Fig. 17). Then, obviously,

$$\overline{0} = 1, \quad \overline{1} = 0$$

and

$$\overline{\overline{x}} = x$$

In this case the De Morgan rules also obviously hold:

$$\overline{x \oplus y} = \overline{x} \otimes \overline{y} \quad \text{and} \quad \overline{x \otimes y} = \overline{x} \oplus \overline{y}$$

(see Figs. 18a and b). However, unfortunately, the rules $x + \overline{x} = 1$ and $x\overline{x} = 0$ do not hold here (cf. what is said in this connection on page 117).

Finally, let us consider *the algebra of least common multiples and greatest common divisors* whose elements are all
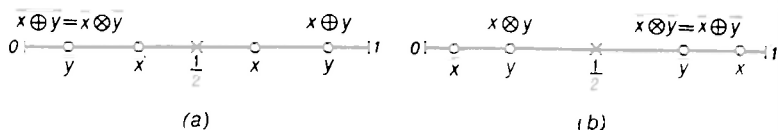
Fig. 18

the possible divisors of a positive integer N for which the Boolean addition $\oplus$ and the Boolean multiplication $\otimes$ are defined as

$$m \oplus n = [m, n] \text{ and } m \otimes n = (m, n)$$

where $[m, n]$ is the least common multiple of the numbers m and n and (m, n) is their greatest common divisor. Let us put

$$\overline{m} = \frac{N}{m}$$

for this algebra. For instance, in the case $N = 210$ considered earlier we have

$$\overline{1} = 210, \ \overline{2} = 105, \ \overline{3} = 70, \ \overline{5} = 42, \ \overline{6} = 35, \ \overline{7} = 30,$$

$$\overline{10} = 21, \ \overline{14} = 15, \ \overline{15} = 14, \ \overline{21} = 10, \ \overline{30} = 7, \ \overline{35} = 6,$$

$$\overline{42} = 5, \ \overline{70} = 3, \ \overline{105} = 2, \ \overline{210} = 1$$

It is clear that in the general case of an arbitrary number N we have

$$\overline{1} = N \quad \text{and} \quad \overline{N} = 1$$

Besides, it is evident that

$$\overline{\overline{m}} = \frac{N}{N/m} = m$$

The De Morgan rules also hold here:

$$\overline{m \oplus n} = \overline{m} \otimes \overline{n} \quad \text{and} \quad \overline{m \otimes n} = \overline{m} \oplus \overline{n}$$

For instance, in the case $N = 210$ we have

$$6 \oplus 21 = [6, 21] = 42, \ \overline{6} \otimes \overline{21} = 35 \otimes 10 =$$

$$= (35, 10) = 5 \text{ and } \overline{42} = 5$$

and also

$$6 \otimes 21 = (6, 21) = 3, \ \overline{6} \oplus \overline{21} = 35 \oplus 10 =$$

$$= [35, 10] = 70 \text{ and } \overline{3} = 70$$

Let the reader prove the De Morgan rules for the general case of an arbitrary N (in this connection also see Exercise 5* on page 52).

<p style="text-align:center">*　　*</p>

Now suppose that we have an arbitrary relation holding in any Boolean algebra, for instance, the equality

$$A (A + C) (B + C) = AB + AC$$

which we have already mentioned. The application of the "bar" operation to both members of this equality results in

$$\overline{A (A+C) (B+C)} = \overline{AB + AC}$$

However, by virtue of the De Morgan rules, we have

$$\overline{A (A + C) (B + C)} = \overline{[A (A+C)] (B+C)} =$$

$$= \overline{A (A+C)} + \overline{B+C} = \overline{A} + \overline{A+C} + \overline{B}\overline{C} = \overline{A} + \overline{A}\overline{C} + \overline{B}\overline{C}$$

and

$$\overline{AB + AC} = \overline{AB}\ \overline{AC} = (\overline{A} + \overline{B}) (\overline{A} + \overline{C})$$

Thus, we finally obtain

$$\overline{A} + \overline{A}\overline{C} + \overline{B}\overline{C} = (\overline{A} + \overline{B}) (\overline{A} + \overline{C})$$

Since the last equality is fulfilled for any $\overline{A}, \overline{B}$ and $\overline{C}$ it remains valid if we simply denote the elements $\overline{A}$, $\overline{B}$ and $\overline{C}$ of the Boolean algebra by the letters A, B and C; this yields the equality

$$A + AC + BC = (A + B)(A + C)$$

which is the dual of the original equality.

We see that the principle of duality is a consequence of the properties of the "bar" operation (and first of all, of the De Morgan rules). It should be however borne in mind that if the original equality involves the "special" elements O and I then, by virtue of the equalities

$$\overline{O} = I \quad \text{and} \quad \overline{I} = O$$

the transformed (dual) equality involves I instead of O and O instead of I; in other words, in the passage to the dual equality we must interchange O and I.

For instance, on applying the "bar" operation to both sides of the equality

$$A \, (A \, + \, I) \, (B \, + \, O) = AB$$

(see Exercise 5 on page 23) we obtain

$$\overline{A \, (A + I) \, (B + O)} = \overline{AB}$$

Now, since

$$\overline{A \, (A + I) \, (B + O)} = \overline{A \, (A + I)} + \overline{B + O} = \overline{A} + \overline{A + I} +$$
$$+ \, \overline{B + O} = \overline{A} + \overline{A} \overline{I} + \overline{B} \overline{O} = \overline{A} + \overline{A} O + \overline{B} I$$

and

$$\overline{AB} = \overline{A} + \overline{B}$$

we can also write the relation

$$\overline{A} = \overline{A} O + \overline{B} I = \overline{A} + \overline{B}$$

The last relation (note that $\overline{A}$ and $\overline{B}$ are arbitrary in it) is equivalent to the relation

$$A \, + \, AO \, + \, BI \, = A \, + \, B$$

which can be obtained from the original equality by *interchanging addition and multiplication and interchanging simultaneously O and I*.

Now we also note that the proof of the principle of duality we have presented allows us to extend immediately its statement. Namely, up till now we have spoken of those "Boolean equalities" which only involve the operations of addition and multiplication. It turns out that the relation $\overline{\overline{A}} = A$ makes it possible to extend the duality principle to equalities involving the "bar" operation as well. It is obvious that if, for instance, a given formula involves an element $\overline{A}$ then the application of the "bar" operation to both members of the formula results in the transformation of $\overline{A}$ into the element $\overline{\overline{A}} = A$. Finally, if we replace in the resultant equality the elements $\overline{A}$, B, C etc. by their

complements $\bar{\bar{A}} = A$, $\bar{\bar{B}} = B$, $\bar{\bar{C}} = C$ etc. then instead of $\bar{A}$ we must again write A. It follows that when we pass *from a given formula to its dual the "bar" operation goes into itself.* For instance, the De Morgan formulas

$$\overline{A+B} = \bar{A}\bar{B} \quad \text{and} \quad \overline{AB} = \bar{A} + \bar{B}$$

are the duals of each other and, similarly, the dual of the equality $A + \bar{A}B = A + B$ (cf. Exercise 2 (d) below) is the evident relation $A(\bar{A} + B) = AB$.

It turns out that the principle of duality has even a wider range of application because it applies not only to Boolean equalities but also to "Boolean inequalities". However, to explain this fact we must present one more notion playing an extremely important role in the theory of Boolean algebras.

Every Boolean algebra involves the equality relation between elements of the algebra (an equality $A = B$ simply means that A and B are one and the same element of the Boolean algebra) and it also involves one more important relation (the *inclusion relation*) between elements whose role is analogous to that of the relation "greater than" (or "less than") in the algebra of numbers. The inclusion relation is denoted by the symbol $\supset$ (or $\subset$); for two elements A and B there may exist the inclusion relation

$$A \supset B$$

or, which is the same,

$$B \subset A$$

The last two relations have one and the same meaning (note that the form of these relations resembles that of the relations $a > b$ and $b < a$ in the algebra of numbers). In the algebra of sets the relation $A \supset B$ means that the *set A contains the set B as its part* (see Fig. 19). For instance, if A_2 is the set of the even numbers and A_6 is the set of the integers divisible by 6 then obviously $A_2 \supset A_6$. Similarly, if A is the set of the pupils in your class who have no bad marks and B is the set of the excellent pupils then, of course, $A \supset B$. It should also be taken into account that when two sets A and B coincide it is also correct to write $A \supset B$ because in this case as well the set B is entirely contained in the set A. We see, that the relation $\supset$ for ele-
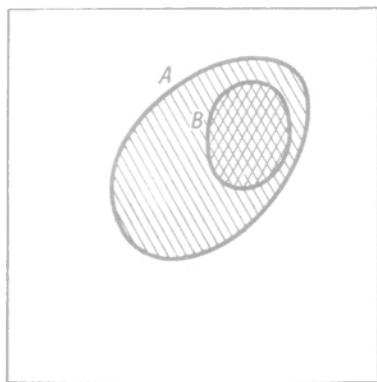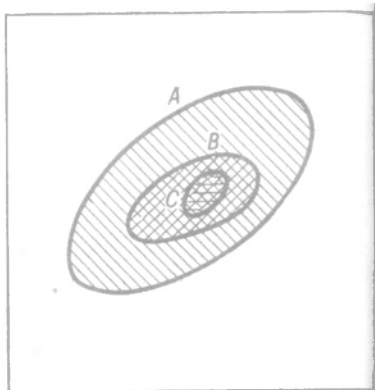
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Fig. 19   Fig. 20

ments of a Boolean algebra is closer to the relation $\geqslant$ ("greater than or equal to") than to the relation $>$ ("greater than") used in the algebra of numbers.

It is clear that

$$\text{if } A \supset B \quad \text{and} \quad B \supset C, \quad \text{then} \quad A \supset C$$

(see Fig. 20). Similarly, when we deal with numbers the relations $a \geqslant b$ and $b \geqslant c$ imply that $a \geqslant c$. Further,

$$\text{if } A \supset B \quad \text{and} \quad B \supset A, \quad \text{then} \quad A = B$$

For numbers we know the similar fact that the relations $a \geqslant b$ and $b \geqslant a$ imply $a = b$. Finally (this fact is particularly important),

$$\text{if } A \supset B \quad \text{then} \quad \bar{A} \subset \bar{B}$$

(see Fig. 21). For instance, since the set of the pupils in your class is wider than the set of the excellent pupils it follows that the set of the pupils having at least one bad mark is contained in the set of the pupils who are not excellent pupils.

Up till now we have compared the properties of the relation $\supset$ for sets with the properties of the relation $\geqslant$ for numbers to stress the similarity between the relations. Now we indicate an essential distinction between these relations. Any two (real) numbers a and b are comparable in the sense that at least one of the relations $a \geqslant b$ and

$b \geqslant a$ must be fulfilled for them[1]). In contrast to it, in the general case, for two arbitrary sets A and B neither of the relations $A \supset B$ and $B \supset A$ holds (see Fig. 22).

We also note that for any element A of an algebra of sets we have

$$I \supset A, \quad A \supset O$$

and that there always hold (for any A and B) the inclusion relations

Fig. 21

$A + B \supset A$ and $AB \subset A$

(Fig. 23). Finally, it is evident that

if $A \supset B$ then $A + B = A$ and $AB = B$

(see Fig. 19). Since $A \supset A$ for any A the last two equalities can be considered a generalization of the idempotent laws $A + A = A$ and $AA = A$.
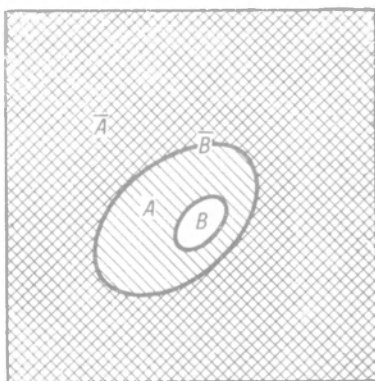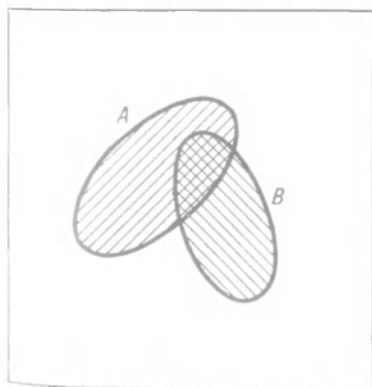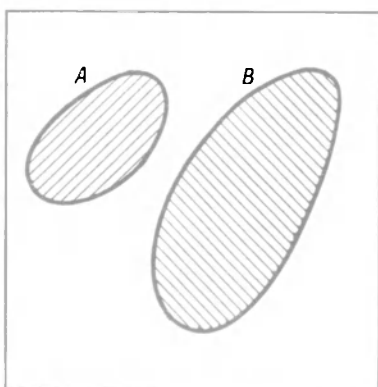

(a)   (b)

Fig. 22

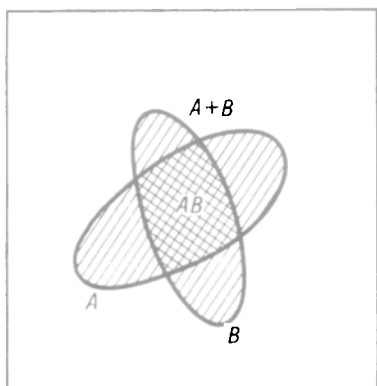[1] In case both relations hold simultaneously the numbers a and b are simply equal.

Fig. 23

Let us discuss the meaning of the relation $\supset$ for the other Boolean algebras known to us. For the "algebra of two numbers" (Example 1 on page 25) this relation is specified by the condition

$$1 \supset 0$$

For the "algebra of four numbers" (Example 2 on page 27) the relation $\supset$ is specified by the conditions

$$1 \supset 0, \; 1 \supset p, \; 1 \supset q, \; p \supset 0 \quad \text{and} \quad q \supset 0$$

(the elements p and q of this algebra are *incomparable*, that is neither of the relations $p \supset q$ and $q \supset p$ holds for them). For the "algebra of maxima and minima" (Example 3 on page 28) the relation $\supset$ coincides with the relation $>$: we assume that two elements x and y of this algebra are connected by the relation $x \supset y$ if the number x is not less than the number y (for instance, we have $1/2 \supset 1/3$ and $1 \supset 1$ in this case)[1]. Finally, in the "algebra of least common multiples and greatest common divisors" (Example 4 on page 31) the relation $m \supset n$ means that the number n is a divisor of the number m; for instance, in this case we have $42 \supset 6$ while the numbers 42 and 35 are incomparable in this algebra (that is neither of the relations $42 \supset 35$ and $42 \subset 35$ takes place). Let the reader check that the relation $\supset$ defined in the above indicated way in each of the algebras we have considered possesses all the enumerated properties of the relation $\supset$ in the algebra of sets.

* *

Now it appears natural to apply the term *"Boolean inequality"* to any formula whose left-hand and right-hand members are connected by the relation $\supset$ (or $\subset$). We shall

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[1] In this Boolean algebra for any two elements x and y at least one of the relations $x \supset y$ and $y \supset x$ holds.

speak of only those inequalities which hold for all the possible values of the elements A, B, C, ... of a Boolean algebra entering into the inequalities in question. For instance, such are the inequalities $I \supset A$, $A \supset O$, $A + B \supset \supset A$ and $A \supset AB$ considered above.

The principle of duality states that *if addition and multiplication are interchanged in such an inequality and if the elements O and I are also interchanged* (provided that O or I or both enter into the enequality) *then, on changing the sign of inequality to the opposite* (that is on replacing the relation $\supset$ by the relation $\subset$ or vice versa) *we again arrive at a correct inequality* (that is at an inequality which is fulfilled for all the values of the elements of the Boolean algebra which enter into it). For instance, from the relation

$$(A + B)(A + C)(A + I) \supset ABC$$

(see Exercise 8 (b) on page 53) it follows that we always have

$$AB + AC + AO \subset A + B + C$$

*

To prove the general principle of duality it suffices to apply the "bar" operation to both members of the original inequality. For instance, since the inequality $(A + B) \times \times (A + C)(A + I) \supset ABC$ holds and since we have the rule "if $A \supset B$ then $\bar{A} \subset \bar{B}$, it follows that the inequality

$$\overline{(A + B)(A + C)(A + I)} \subset \overline{ABC}$$

is also valid. By the De Morgan rules, taking into account that $\bar{I} = O$, we obtain

$$\overline{(A + B)(A + C)(A + I)} = \overline{(A + B)} \ \overline{(A + C)} + \overline{A + I} =$$
$$= \overline{A + B} + \overline{A + C} + \bar{A} + \bar{I} = \bar{A}\bar{B} + \bar{A}\bar{C} + \bar{A}O$$

Similarly,

$$\overline{ABC} = \bar{A} + \bar{B} + \bar{C}$$

Thus, we conclude that for any A, B and C there holds the inequality

$$\bar{A}\bar{B} + \bar{A}\bar{C} + \bar{A}O \subset \bar{A} + \bar{B} + \bar{C}$$

Now, since $\overline{A}$, B and $\overline{C}$ are arbitrary here we can simply denote them as A. B and C respectively. We thus arrive at the inequality

$$AB + AC + AO \subset A + B + C$$

which is dual to the original inequality in the sense described above.

* *

Exercises

1. Write down the dual equalities for all the equalities whose proof is discussed in Exercises 1-10 on page 23.

2. Prove the following identities of algebra of the sets:

(a) $(A + B)(A + \overline{B}) = A$

(b) $AB + (A + B)(\overline{A} + \overline{B}) = A + B$

(c) $\overline{\overline{A}\overline{B}C}\ \overline{\overline{A}B}\ \overline{AC} = O$

(d*) $A + \overline{A}B = A + B$

3. What equalities are obtained from the equalities in Exercises 2 (a), (b) and (c) by means of the principles of duality?

4. Check that in the "algebra of four numbers" (Example 2 on page 27) there holds De Morgan's second rule $\overline{ab} = \overline{a} + \overline{b}$.

5*. (a) Let $N = p_1 p_2 \ldots p_k$ where all prime numbers p_1, p_2, ..., p_k are pairwise different. Prove that in this case the "algebra of least common multiples and greatest common divisors" whose elements are the divisors of the number N (see Example 4 on page 31) reduces to the "algebra of the subsets of the universal set $I = \{p_1, p_2, \ldots, p_k\}$". Proceeding from this fact show that in this "algebra of least common multiples and greatest common divisors" all the laws of a Boolean algebra hold including the De Morgan rules.

(b) Let $N = p^A$ where p is a prime number and A is a positive integer. Prove that in this case the "algebra of least common multiples and greatest common divisors" whose elements are the divisors of the number N reduces to the "algebra of maxima and minima" defined in the set consisting of the numbers $0, 1, 2, \ldots, A$. Show that in this "algebra

of least common multiples and greatest common divisors" all the laws of a Boolean algebra hold including the De Morgan rules.

(c) Let $N = p_1^{A_1} p_2^{A_2} \ldots p_k^{A_k}$ and $m = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$ where $0 \leqslant a_1 \leqslant A_1$, $0 \leqslant a_2 \leqslant A_2$, $\ldots 0 \leqslant a_k \leqslant A_k$ (cf. Exercise 6 on page 36). What is the decomposition of the number $\bar{m} = N/m$ into prime factors? Use the formula obtained for this decomposition to prove the De Morgan rules in the general case of an arbitrary "algebra of least common multiples and greatest common divisors".

6*. For which of the Boolean algebras known to you do the equalities

$$A + \bar{A} = I \quad \text{and} \quad A\bar{A} = O$$

hold and for which do they not hold?

7. Check that all the properties of the relation $\supset$ hold for the following algebras:

(a) the algebra of two elements (see Example 1 on page 25);

(b) the algebra of four elements;

(c) the "algebra of maxima and minima";

(d) the "algebra of least common multiples and greatest common divisors".

8. Prove the following inequalities of the set algebra:

(a) $A + B + C \supset (A + B)(A + C)$

(b) $(A + B)(A + C)(A + I) \supset ABC$

(c) $(A + B)(B + C)(C + A) \supset ABC$

(d) $A + B \supset \bar{A}B + AB$

9. Write down the inequalities obtained from the inequalities in Exercises 8 (a)-(c) using the duality principle; also prove these inequalities directly without resorting to the duality principle.

10. Prove that if a Boolean inequality involves the "bar" operation then is also valid the inequality obtained from the original one by interchanging the Boolean addition and the Boolean multiplication and by interchanging simultaneously the element O and the element I while the "bar" operation is retained at each place it occupies in the original inequality and the sign of the inequality is changed to the opposite. Use this principle to form a new inequality from the inequality in Exercise 8 (d).

11. Verify all the properties of the relation $\supset$ for

(a) the "algebra of maxima and minima";

(b) the "algebra of least common multiples and greatest common divisors".

12*. Let some sets A and B be such that $A \supset B$. Simplify the following expressions:

(a) $A + B$; (b) AB; (c) $A + \bar{B}$; (d) AB

4. Sets and Propositions. Propositional Algebra

Let us come back to the Boolean algebra of sets which plays the most important role in the present book. Let us discuss the methods for specifying the sets which are the elements of this algebra. It is obvious that the simplest method is to specify a given set by *tabulation*, that is by enumerating all the elements of the set. For instance, we can consider the "set of the pupils: Peter, John, Tom and Mary" or the "set of the numbers: 1, 2, 3, 4" or the "set of the four operations of arithmetic: addition, subtraction, multiplication and division". In mathematics the elements of a set which is defined by tabulation are usually written in curly brackets; for instance, the sets we have mentioned can be written as

$$A = \{\text{Peter, John, Tom, Mary}\}$$
$$B = \{1, 2, 3, 4, 5\}$$

and

$$C = \{+, -, \times, :\}$$

(in the last expression the signs of the operations symbolize the operations themselves)[1]).

However, this method of representing a set is highly inconvenient in case there are very many elements in the set; it becomes completely inapplicable when the set in question is infinite (we cannot enumerate an infinite number of the elements of the set!). Besides, even in those cases when a set can be defined by tabulation and the tabulation is quite simple it may nevertheless happen that the enumeration itself does not indicate why these elements are collected to form the set.

[1]) Also see Exercise 5 (a) on page 52.

Therefore another method which specifies the sets implicitly by *description* is more widely used. When a set is defined by description we indicate a property characterizing all the elements of the set. For instance, we can consider the "set of all excellent pupils in your class" (it may turn out that the set A mentioned above coincides with this set of excellent pupils) or the "set of all integers x such that $0 \leqslant x \leqslant 5$" (this set exactly coincides with the set B mentioned above) or the "set of all animals in a zoo". The descriptive method for specifying sets is quite applicable for the definition of infinite sets such as the "set of all integers" or the "set of all triangles with area equal to 1"; moreover, as has been mentioned, infinite sets can be defined by description only.

The descriptive method of representing sets connects the sets with *propositions* which are studied in mathematical logic. Namely, the essence of the method is that we fix a collection of the objects we are interested in (for instance, the collection of the pupils in your class or the collection of the integral numbers) and then state a proposition which is true for all the elements of a set under consideration and only for these elements. For instance, if we are interested in the sets whose elements are some (or all) pupils in your class then such propositions can be "he is an excellent pupil", "he is a chess-player", "his name is George" and the like. The set A of all those elements of the universal set I in question (for instance, the set of the pupils, the set of the numbers, etc.) which satisfy the condition mentioned as the characteristic property in a given proposition a is called the *truth set* of this proposition[1]) (for instance, see Fig. 24)[2]).

Thus, there is a "two-way connection" between sets and propositions: every set is described by a proposition (in particular, such a proposition may simply reduce to the enumeration of the elements of a given set, for instance, "the name of the pupil is Peter or John, or Tom, or Mary")

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[1]) According to the terminology of modern mathematical logic, it would be more precise to use the term *propositional function* or *open sentence* (or *open statement*) and to speak of the truth set of the given propositional function but we shall simply speak of propositions throughout the present translation. — *Tr.*

[2]) We shall denote propositions by small letters and the truth sets corresponding to the propositions by the same capital letters.

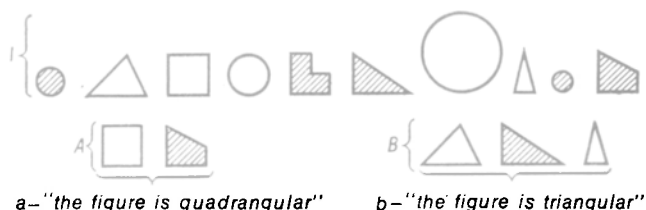a–"the figure is quadrangular"   b–"the figure is triangular"

Fig. 24

and to every proposition there corresponds a definite set which is the truth set of this proposition. It is also important that for any collection of propositions (even for propositions concerning objects of different kinds) it is always possible to indicate a certain universal set I corresponding to all the propositions in question and containing all the objects mentioned in these propositions. Another highly important condition is that *by a proposition we shall only mean a statement about which it makes sense to say that it is either true or false* when it is applied to a definite element of the given universal set. This means, for instance, that such statements as "the person has two heads and sixteen arms" or "$2 \times 3 = 6$" are propositions (the second of these sentences is even completely independent of the choice of the universal set I) while such sentences as the exclamation "Be careful!" or "Oh!" are not considered propositions. Finally, it should also be borne in mind that such sentences like "two hours is a long time" or "the examination in mathematics is a highly unpleasant procedure" are not considered propositions either because they are quite subjective and their truth or falsity depends on a number of circumstances and on the character of the person who states these sentences.

When considering propositions we are only interested in the sets they describe. Therefore any two propositions a and b to which one and the same truth set corresponds will be identified and will be considered *equivalent* ("equal"). When two given propositions a and b (for instance, "he is an excellent pupil" and "he has only the highest marks" or "the number x is odd" and "the division of the number x by 2 gives 1 in the remainder") are equivalent we shall write

$$a = b$$

All the *necessarily true* propositions, that is the propositions which are always true irrespective of which element of the set I is considered, are also regarded as equivalent to one another. Examples of necessarily true propositions are "$2 \times 3 = 6$", "this pupil is a boy or a girl", "the height of the pupil does not exceed 3m" and the like. Let us agree to denote all necessarily true propositions by the letter i. Similarly, all the *necessarily false* (that is *contradictory*) propositions which are never true, that is the propositions whose truth sets are empty, will also be regarded as equivalent. We shall denote such propositions by the letter o; examples of necessarily false propositions are "$2 \times 2 = 6$", "this pupil can fly like a bird", "the height of the pupil exceeds 4m" and "the number x is greater than 3 and less than 2".

The connection between the sets and the propositions makes it possible to define some algebraic operations on propositions similar to those introduced earlier for the algebra of sets. Namely, *by the sum of two propositions a and b we shall mean a proposition whose truth set coincides with the sum of the truth set A of the proposition a and the truth set B of the proposition b.* Let us agree to denote this new proposition by the symbol $a + b^1)$. Since the sum of two sets is nothing but the union of all the elements contained in both sets, the sum $a + b$ of two propositions a and b is simply the proposition "a or b" where the word "or" means that at least one of the propositions a and b (or both propositions) is true. For instance, if the proposition a states "the pupil is a chess-player" and if among the pupils in your class the truth set corresponding to this proposition is

$$A = \{\text{Peter, John, Tom, George, Mary, Ann, Helen}\}$$

while the proposition b asserts that "the pupil can play draughts" and its truth set is

$$B = \{\text{Peter, Tom, Bob, Harry, Mary, Alice}\}$$

then $a + b$ is the proposition "the pupil can play chess *or* the pupil can play draughts" (or, briefly, "the pupil can

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[1]) In mathematical logic the sum of two propositions a and b is usually called the *disjunction* of these propositions and is denoted by the symbol $a \vee b$ (cf. the notation $A \cup B$ for the sum of two sets A and B).
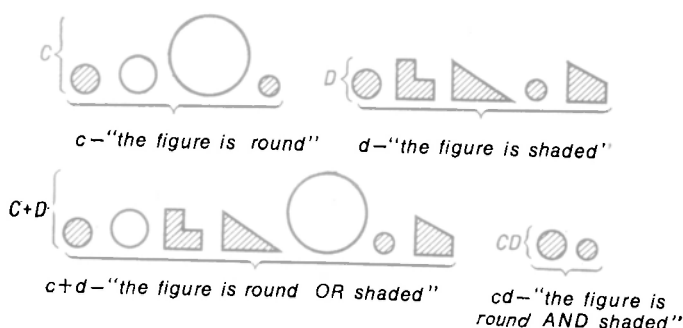
c — "the figure is round"   d — "the figure is shaded"

c+d — "the figure is round OR shaded"   cd — "the figure is round AND shaded"

Fig. 25

play chess *or* draughts"). The truth set corresponding to this proposition $a + b$ is

$$A + B = \{\text{Peter, John, Tom, George, Bob, Harry,}$$
$$\text{Mary, Ann, Helen, Alice}\}$$

If the universal set is the set of the geometrical figures shown in Fig. 24 and if the propositions c and d assert that "the figure is round" and "the figure is shaded" respectively then the proposition $c + d$ asserts that "the figure is round *or* shaded" (see Fig. 25).

Similarly, *by the product ab of two propositions a and b with truth sets A and B we shall mean a proposition whose truth set coincides with the product AB of the sets A* and B[1]. Since the product of two sets A and B is nothing but their intersection (that is their common part) containing those and only those elements of the universal set I which are contained in both sets A and B, the product ab of the propositions a and b is the proposition "a **and** b" where the word "and" means that both propositions a and b are true. For instance, if the propositions a and b concerning the pupils in your class are the same as above, then the proposition ab asserts that "the pupil can play chess *and* the pupil can play draughts" (or, briefly, "the pupil can play chess *and* draughts"); the truth set corresponding to this

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[1] In mathematical logic the product of two propositions a and b is more often called the *conjunction* of these propositions and is denoted by the symbol $a \wedge b$ (cf. the notation $A \cap B$ for the product of two sets A and B).

proposition is

$$AB = \{\text{Peter, Tom, Mary}\}$$

If two propositions c and d concerning the set of the geometrical figures shown in Fig. 24 mean that "the figure is round" and "the figure is shaded" respectively, then the proposition cd asserts that "the figure is round *and* shaded" (see Fig. 25).

The connection between the sets and the propositions makes it possible to extend to the propositions all the rules of the algebra of sets:

$$a + b = b + a \quad \text{and} \quad ab = ba$$

the commutative laws of algebra of propositions

$$(a + b) + c = a + (b + c) \quad \text{and} \quad (ab) c = a (bc)$$

the associative laws of algebra of propositions

$$(a + b) c = ac + bc \quad \text{and} \quad ab + c = (a + c) (b + c)$$

the distributive laws of algebra of propositions

$$a + a = a \quad \text{and} \quad aa = a$$

the idempotent laws of algebra of propositions

Besides, if i is a necessarily true proposition and o is a necessarily false proposition then we always have (that is for any proposition a) the relations

$$a + o = a, \qquad ai = a$$

and

$$a + i = i, \qquad ao = o$$

For instance, the proposition "the pupil has only the highest marks or the pupil has two heads" is equivalent to the proposition "the pupil has only highest marks" while the proposition "the pupil can swim and the pupil is not yet 200 years old" is equivalent to the proposition "the pupil can swim"[1]).

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[1]) We shall also write the rules we have enumerated in the form in which they are usually given in mathematical logic:

$$a \vee b = b \vee a \qquad\qquad a \wedge b = b \wedge a$$
$$(a \vee b) \vee c = a \vee (b \vee c) \qquad (a \wedge b) \wedge c = a \wedge (b \wedge c)$$
$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) \qquad (a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$$
$$a \vee a = a \qquad\qquad a \wedge a = a$$
$$a \vee o = a \qquad\qquad a \wedge i = a$$
$$a \vee i = i \qquad\qquad a \wedge o = o$$

To demonstrate how the rules of the algebra of propositions are derived from the rules of the algebra of sets let us consider, for instance, the derivation of the second distributive law. Since the truth set of the sum of two propositions is the union of the truth sets of these propositions and since the truth set of the product of two propositions is the intersection of the truth sets of the given propositions, it is evident that the truth set of the compound proposition $ab + c$ which means "the proposition "*a and b*" *or* the proposition c is true" is the set $AB + C$ where A, B and C are the truth sets of the propositions a, b and c respectively. Similarly, the truth set of the (compound) proposition $(a + c) (b + c)$ is the set $(A + C) (B + C)$. By virtue of the second distributive law of the set algebra, we have

$$AB + C = (A + C)(B + C)$$

Thus, the truth sets of the propositions $ab + c$ and $(a + c) \times (b + c)$ coincide, which means that the propositions $ab + c$ and $(a + c) (b+c)$ are equivalent. (Also see page 20 where we indicated that the propositions "the pupil can play chess and draughts or can swim" and "the pupil can play chess or can swim and also can play draughts or can swim" have one and the same sense, that is

$$ab + c - (a + c)(b + c)$$

where the propositions a, b and c are "the pupil can play chess", "the pupil can play draughts" and "the pupil can swim" respectively.)

* *

Like the operations of addition and multiplication of sets, the "bar" operation of the algebra of sets can also be extended to the algebra of propositions. Namely, *by $\bar{a}$ should be meant the proposition whose truth set is the set $\overline{A}$ where A is the truth set of the proposition a.* In other words, the truth set of the proposition $\bar{a}$ contains those and only those elements of the universal set I which are not contained in the set A, that is the elements which are not contained in the truth set of the proposition a. For instance, if the

b- "IT IS FALSE THAT the figure is triangular"

Fig. 26

proposition a asserts that "the pupil has bad marks" then the proposition $\bar{a}$ means "the pupil has no bad marks". If the universal set I consists of the geometrical figures shown in Fig. 24 and the proposition b asserts that "the figure is triangular" then the proposition $\bar{b}$ means "it is false that the figure is triangular" (that is, simply, "the figure is not triangular"; see Fig. 26). Generally, the proposition $\bar{a}$ has the sense "**not** a"; hence, the "bar" operation of the propositional algebra is the operation of forming the *negation* (*denial*) $\bar{a}$ of the proposition a. The proposition $\bar{a}$ can be formed from a by prefixing "it is false that".

Now let us enumerate the rules of the algebra of propositions related to the operation of forming negation:

$$\bar{\bar{a}} = a$$

$$a + \bar{a} = i \quad \text{and} \quad a\bar{a} = o$$

$$\bar{o} = i \quad \text{and} \quad \bar{i} = o$$

$$\overline{a + b} = \bar{a}\bar{b} \quad \text{and} \quad \overline{ab} = \bar{a} + \bar{b}$$

Indeed, the negation of a necessarily false proposition (for instance, "*it is false that* 2×2 is equal to 5" or "*it is false that* the pupil has two heads") is always a necessarily true proposition while the negation of a necessarily true proposition (for instance, "*it is false that* the pupil is not yet 120 years old") is always necessarily false. All the other laws can also be readily checked (let the reader check them). By the way, there is no need to verify them since they simply follow from the corresponding rules of the set algebra[1]).

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[1]) For instance, since the truth sets of the propositions $\overline{a + b}$ and $\overline{ab}$ are $\overline{A + B}$ and $\overline{AB}$ where A and B are the truth sets of the propositions a and b respectively and since $\overline{A + B} = \bar{A}\bar{B}$, we have, according to the definition of the equivalence (equality) of propositions, the equality $\overline{a + b} = \bar{a}\bar{b}$.

Exercises

1. Give three examples of necessarily true propositions and two examples of necessarily false propositions.

2. Let the proposition a assert that:

(a) "$2 \times 2 = 4$";

(b) "the pupil is a boy";

(c) "an elephant is an insect";

(d) "he can fly".

What is the meaning of the proposition $\bar{a}$ in all these cases? Is the proposition $\bar{a}$ necessarily true? Is it necessarily false?

3. Let the proposition a mean "the pupil can play chess" and let the proposition b be "the pupil can play draughts". Explain the meaning of the following propositions:

(a) $a + b$; (b) ab; (c) $\bar{a} + b$; (d) $a + \bar{b}$;

(e) $\bar{a} + \bar{b}$; (f) $\bar{a}b$; (g) $a\bar{b}$; (h) $\overline{ab}$

4. Let a be the proposition "he is an excellent pupil", b "he is dark" and let c mean "he can swim". Explain the meaning of the propositions

$$\text{(a)} \quad (a + b)\,c \quad \text{and} \quad ac + bc$$

and

$$\text{(b)} \quad ab + c \quad \text{and} \quad (a + c)(b + c)$$

5. Let the propositions a and b mean "the given positive integer is even" and "the given positive integer is a prime number" respectively. State the following propositions:

(a) ab; (b) $\bar{a} + b$; (c) $\bar{a}b$; (d) $a\bar{b}$; (e) $\bar{a} + \bar{b}$

What are the truth sets of these propositions?

6. Let a and b be the propositions "this pupil is interested in mathematics" and "this pupil sings well" respectively. State the propositions

(a) $\overline{a + b}$ and $\bar{a}\bar{b}$

and

(b) $\overline{ab}$ and $\bar{a} + \bar{b}$

5. "Laws of Thought".
Rules for Deduction

Now we can explain why George Boole called his work
(where the "unusual algebra" considered in the present book
was constructed) "Laws of Thought". The matter is that
the algebra of propositions is closely related to the rules
of the process of thinking because the sum and the product
of propositions defined in Sec. 4 reduce to nothing but the
logical (propositional) connectives "or" and "and" respec-
tively, the "bar" operation has the sense of the negation and
the laws of the propositional algebra describe the basic rules
for the logical operations which all the people follow in the
process of thinking. Of course, in everyday life few people
think of these rules as mathematical laws of thought but
even children freely use them. Indeed, nobody doubts that
to say "he is a good runner and a good jumper" is just the
same as to say "he is a good jumper and a good runner",
that is all the people know (although they may not be aware
of it) that propositions ab and ba have the same meaning,
or, which is the same, are equivalent.

We can also explain why nowadays George Boole's
approach to the mathematical interpretation of the laws
of logic as certain "algebraic rules" has become an object of
intense interest. As long as logical operations were per-
formed only by people who used them in the process of
thinking quite intuitively there was no need to formulate
the logical rules rigorously. In recent decades the situation
has changed and nowadays we want to make electronic
computers perform the functions which in the past could
only be performed by people, for instance, such functions
as production process control, traffic schedulling, solving
mathematical problems, translating books from one language
into another, economic planning and finding necessary
data in scientific literature; as is known, modern electronic
computers can even play chess! It is obvious that in order
to compile the necessary programs for the computers it
is necessary to state rigorously the "rules of the game", that
is the "laws of thought", which must be followed by the
"thinking" machines constructed by the man. People can
use the rules of logic intuitively but for a computer these
rules must be stated in a clear manner using the only

"language" which a mathematical machine can "understand", that is the language of mathematics[1]).

Now let us come back to the "laws of thought" themselves. The most interesting logical rules are connected with the logical operation of negation; some of them have special names in logic. For instance, the rule

$$a + \bar{a} = i$$

expresses the so-called *law (principle) of excluded middle*: it means that either a is true (here a is an arbitrary proposition) or the proposition $\bar{a}$ is true and therefore the proposition $a + \bar{a}$, that is "a or not a", is always true. For instance, even without having any information on the tallest pupil in your school we can definitely assert that this pupil is "either an excellent pupil or not an excellent pupil" or that this pupil "either can play chess or cannot play chess" etc. The rule expressed by the relation

$$a\bar{a} = o$$

is called the *law (principle) of contradiction*; this law asserts that the propositions a and $\bar{a}$, that is a and "not a", can never be true simultaneously and therefore the product of these propositions is always false. (The law of excluded middle and the law of contradiction are referred to as the *laws of complement* or the *complementation laws*.) For instance, if a pupil has no bad marks then the proposition "the pupil has bad marks" is of course false when applied to this pupil; if a whole number n is even then the proposition "the number n is odd" is of course false for this number. The rule expressed by the formula

$$\bar{\bar{a}} = a$$

is called the *law of double negation* (or the *law of double denial*). It asserts that the double negation of a propo-

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[1]) We must warn the reader that the elementary algebra of propositions to which the present book is devoted does not provide sufficient means for constructing modern electronic computers and for posing complex mathematical problems in the form in which they can be "inserted" into computers. For this purpose a more intricate mathematical and logical apparatus must be developed which is not considered here.

sition is equivalent to the original proposition itself. For instance, the negation of the proposition "the given integer is even" is the proposition "the given integer is odd"; as to the negation of the latter proposition, it asserts that "the given integer is not odd" and hence it is equivalent to the original proposition asserting that the integer is even. Similarly, the negation of the proposition "the pupil has no bad marks" means that "the pupil has bad marks" and the double negation of the former proposition states that "it is false that the pupil has bad marks" and is therefore equivalent to the original proposition asserting that the pupil has no bad marks.

The De Morgan rules

$$\overline{a+b} = \overline{a}\overline{b} \quad \text{and} \quad \overline{ab} = \overline{a} + \overline{b}$$

for the propositions are also very important; the verbal statement of these rules is a little more complicated (in this connection see Exercise 1 below). All the other rules of the propositional algebra such as the distributive laws

$$(a + b) c = ac + bc \quad \text{and} \quad ab + c = (a + c)(b + c)$$

or the idempotent laws

$$a + a = a \quad \text{and} \quad aa = a$$

are definite "laws of thought", that is logical rules which the people follow when deducing new inferences from those which are already known to be true.

A particularly important role is played by the relation $\supset$ which can be extended from the algebra of sets to the mathematical logic (propositional calculus). Up till now we have not considered this relation in connection with propositions and only discussed it in connection with the algebra of sets. However, the "two-way connection" between sets and propositions allows us to extend easily the relation $\supset$ of the algebra of sets (the inclusion relation) to the algebra of propositions. Namely, let us agree that the relation

$$a \supset b$$

written for two propositions a and b should be understood in the sense that *the proposition a follows from the proposition b* or, which is the same, *the proposition a is a consequence of the proposition b*: the meaning of what has been said is that *the truth set A of the proposition a contains the*
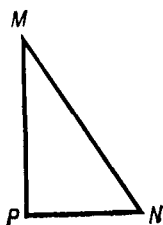
M

P _____ N

truth set B of the proposition b. In other
words, the above relation $a \supset b$ means that

$$A \supset B$$

For instance, since the set B of excellent pupils
in your class is obviously contained in the
set A of all pupils having no bad marks,
the proposition a stating that "the pupil has
no bad marks" is a consequence of the proposi-
tion b asserting that "the pupil has only the
highest marks". Similarly, the set

$$A_6 = \{6, 12, 18, \ldots\}$$

of the whole numbers divisible by 6 is contained in the set

$$A_2 = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, \ldots\}$$

of the even whole numbers, therefore the proposition "the
number is even" (it is meant that the universal set I we
deal with consists of all the positive integers) follows from
the proposition "the number is divisible by 6"[1]).

The process of establishing the fact that two propositions
a and b are connected by the relation $a \supset b$ is called *deduc-
tion.* When showing that the relation $a \supset b$ takes place we
deduce the conclusion a from the condition b. In everyday
life and in science we often deal with deduction; for instance,
as a rule, the proof of a mathematical theorem reduces to
deduction. In a mathematical proof it is usually required
to show that from a condition b of the theorem (for instance,
from the condition that "the angle P of a triangle MNP
is a right angle"; see Fig. 27) follows a conclusion a of the
theorem (for instance, the conclusion that "$MP^2 + NP^2 =
= MN^2$"; in this case the relation $a \supset b$ is equivalent to the
Pythagoras theorem). In the deduction processes (for instance,
in the proof of a theorem) we always use the basic properties
of the relation $\supset$ (sometimes without being aware of it).
These properties can be stated as the following rules for

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[1]) If $a \supset b$ we also say that the proposition b is a *sufficient con-
dition* for the proposition a (for instance, for a pupil to have no bad
marks it is of course sufficient that this pupil should have only the
highest marks) while the proposition a is said to be a *necessary condition*
for the proposition b (for instance, for a pupil to have only the highest
marks it is of course necessary that this pupil should have no bad
marks).

deduction[1]):

$$a \supset a$$

$$\text{if } a \supset b \quad \text{and} \quad b \supset a \quad \text{then } a = b$$

$$\text{if } a \supset b \quad \text{and} \quad b \supset c \quad \text{then } a \supset c$$

$$i \supset a \quad \text{and} \quad a \supset o \quad \text{for any} \quad a$$

$$a + b \supset a \quad \text{and} \quad a \supset ab \quad \text{for any } a \quad \text{and} \quad b$$

$$\text{if } a \supset b \quad \text{then } \quad \overline{b} \supset \overline{a}$$

For instance, we know that *if the diagonals of a quadrilateral bisect each other* (the proposition *b*) *the quadrilateral is a parallelogram* (the proposition *a*). On the other hand, we know that "in a parallelogram *the opposite angles are congruent*" (the proposition *c*). Thus, we have

$$a \supset b^2) \quad \text{and} \quad c \supset a$$

and therefore

$$c \supset b$$

In other words, *if the diagonals of a quadrilateral bisect each other then its opposite angles are congruent.*

Let us dwell in more detail on the application of the rule asserting that *if $a \supset b$ then $\overline{b} \supset \overline{a}$.* This rule serves as the basis for the so-called *proofs by contradiction* (Latin *reductio ad absurdum* proofs). Let it be required to prove that the relation $a \supset b$ holds, that is to prove that the proposition *a* follows from the proposition *b*. It often turns out that it is easier to prove the fact that if *a* is false then *b* cannot be true, that is to show that the proposition "not *b*" (that is the proposition $\overline{b}$) follows from the proposition "not *a*" (that is from the proposition $\bar{a}$).

Let us consider an example of a proof by contradiction. Let it be required to prove that *if an integer n which is greater than 3 is a prime number* (the proposition *b*) *then n has the form $6k \pm 1$ (where k is an integer),* that is when

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[1]) The second of these rules asserting that "*if $a \supset b$ and $b \supset a$ then $a = b$*" is sometimes stated as "*if b is a necessary and sufficient condition for a then the propositions a and b are equivalent*" (from this point of view we have assumed the propositions *a* and *b* are considered equal in this case).

[2]) In this case we even have $a \supset b$ and $b \supset a$, that is $\boldsymbol{a} = b$.

n is divided by 6 the remainder is $+1$ or -1 (the proposition a). It is rather difficult to prove this fact directly without using the rule *if $a \supset b$ then $\bar{b} \supset \bar{a}$*; therefore we shall try to resort to a proof by contradiction. To this end let us suppose that the proposition $\bar{a}$ is true, that is the number n (which is an integer greater than 3) cannot be represented in the form $6k \pm 1$. Since the remainder obtained when an arbitrary integer n is divided by 6 can be equal to 0 (in this case n is divisible by 6) or to 1, or to 2, or to 3, or to 4, or to 5 (the last case is equivalent to the one when the remainder is equal to -1), it follows that the assumption that $\bar{a}$ is true means that *when n is divided by 6 we obtain* 0 *in the remainder* (that is n is divisible by 6) *or* 2, *or* 3, *or* 4. An integer divisible by 6 cannot be a prime number; if an integer $n > 3$ gives 2 or 4 in the remainder when it is divided by 6, then this integer is even and therefore it cannot be a prime number; if the division of n by 6 gives 3 in the remainder, then n is divisible by 3 and cannot be a prime number either. Thus, $\bar{b}$ follows from $\bar{a}$ (or, symbolically, $\bar{b} \supset \bar{a}$) whence we conclude that

$$a \supset b$$

which is what we intended to prove[1]).

Exercises

1. Give the verbal statement of the De Morgan rules of the algebra of propositions: $\overline{a + b} = \bar{a}\bar{b}$ and $\overline{ab} = \bar{a} + \bar{b}$.

2. Give examples demonstrating the following rules:

(a) the law of excluded middle;

(b) the law of contradiction;

(c) the law of double negation.

3. Give one example to demonstrate each of the properties of the relation $\supset$ for propositions enumerated on page 67.

4. Give an example of a proof by contradiction and write it in the symbolic form.

5. Let $a \supset b$. Simplify the expressions of the sum $a + b$ and of the product ab of the two propositions a and b.

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[1]) The following argument is more precise: from the relation $\bar{b} \supset a$ we have proved it follows that $\overline{(a)} \supset \overline{(\bar{b})}$, that is $\bar{\bar{a}} \supset \bar{\bar{b}}$; now, since by virtue of the law of double negation we have $\bar{\bar{a}} = a$ and $\bar{\bar{b}} = b$, there must be $a \supset b$.

6. Further Examples of Application of Rules for Deduction. Implication

The rules of propositional algebra can be applied to the solution of logical problems whose conditions form a collection of propositions using which we must establish the truth or the falsity of some other propositions. Below is an example of this kind.

A family consisting of a father (F), a mother (M), a son (S) and two daughters $(D_1$ and $D_2)$ spends its vacation at the sea shore. They often swim early in the morning and it is known that when the father goes swimming then the mother and the son always go swimming together with him; it is also known that if the son goes swimming his sister D_1 goes with him. The second daughter D_2 goes swimming then and only then when her mother does and it is known that at least one of the parents goes swimming every morning. Finally, it is known that last Sunday only one of the daughters went swimming. The question is: who of the members of the family went swimming last Sunday morning?

Let us denote the propositions "the father went swimming on Sunday morning", "the mother went swimming on Sunday morning", "the son went swimming on Sunday morning", "the first daughter went swimming on Sunday morning" and "the second daughter went swimming on Sunday morning" by the same symbols F, M, S, D_1 and D_2 respectively. As usual, the negations of these propositions will be denoted by the same symbols supplied with the bar. In the symbolic form the conditions of the problem are written thus:

(1) $FMS + \bar{F} = i$

(2) $SD_1 + \bar{S} = i$

(3) $MD_2 + \overline{MD_2} = i$

(4) $F + M = i$

(5) $D_1\overline{D_2} + \overline{D_1}D_2 = i$

where the letter i denotes a necessarily true proposition. On multiplying all these equalities we obtain the relation

$$FMS + \overline{F})(SD_1 + \bar{S})(MD_2 + \overline{MD_2})(F + M)(D_1\overline{D_2} + \overline{D_1}D_2) = i$$

which is equivalent to the system of equalities (1)-(5) because the product of propositions is true if and only if all the multiplicand propositions are true.

Let us open the parentheses in the expression on the left-hand side using the first distributive law and also the commutative, the associative, the idempotent laws and the law of contradiction $A\overline{A} = O$ together with the relations $A + O = A$ and $AO = O$. To simplify the transformation we change the order of the factors:

$$(FMS + \overline{F})(F + M) = FMS + \overline{F}M$$

$$(FMS + \overline{F}M)(MD_2 + \overline{M}\overline{D}_2) = FMSD_2 + \overline{F}MD_2$$

$$(FMSD_2 + \overline{F}MD_2)(D_1\overline{D}_2 + \overline{D}_1D_2) = FMS\overline{D}_1D_2 + \overline{F}M\overline{D}_1D_2$$

$$(FMS\overline{D}_1D_2 + \overline{F}M\overline{D}_1D_2)(SD_1 + \overline{S}) = \overline{F}M\overline{S}\overline{D}_1D_2$$

Thus, we finally obtain

$$\overline{F}M\overline{S}\overline{D}_1D_2 = i$$

which means that only the mother M and the second daughter D_2 went swimming on Sunday morning.

*

The solution of the problem we have presented is based on the algebraic transformations by means of which we have simplified a rather complex expression

$$(FMS + \overline{F})(SD_1 + \overline{S})(MD_2 + \overline{M}\overline{D}_2)(F + M)(D_1\overline{D}_2 + \overline{D}_1D_2)$$

and have brought it to the form $\overline{F}M\overline{S}\overline{D}_1D_2$. This procedure turns out to be useful in many other cases and therefore we shall dwell on it in more detail.

Suppose that we are given an arbitrary algebraic expression $f = f(p_1, p_2, \ldots, p_n)$ composed of propositions $p_1, p_2, \ldots, p_n$ with the aid of the basic operations $+$, $\cdot$ and the "bar" operation of the algebra of propositions. We shall prove that *if the compound (composite) proposition f is not necessarily false* (that is f is not equivalent to o) *then it can be reduced to the form*

$$f = \sum p_1' p_2' \ldots p_n' \tag{*}$$

where the symbol p'_j *(j* = 1, 2, . . ., *n) in each term of the sum denotes either* p_j *or* $\bar{p}_j$ *and all the terms of the sum are pairwise different. Besides, if two composite propositions* f_1 *and* f_2 *are equal (equivalent) they have the same forms of type* (*) *and if they are different (not equal to each other) their forms of type* (*) *are also different.* Form (*) of a composite proposition *f* is called its *additive normal form*[1]).

The proof of the assertion we have stated is quite simple. First of all, using the De Morgan formulas we can transform the composite proposition $f(p_1, p_2, \ldots, p_n)$ so that the sign of negation (the bar) stands only above some (or all) constituent (prime) propositions $p_1, p_2, \ldots, p_n$ but not above their combinations (their sums or products). Further, using the first distributive law we can open the parentheses in all those cases when they mean that to obtain the expression *f* it is necessary to multiply by one another sums of prime propositions p_j and their negations $\bar{p}_j$ or some more complex combinations of the propositions. On opening all the parentheses of this kind we reduce the compound proposition *f* to an "additive" form written as a sum of a number of terms each of which is a product of prime propositions p_j and their negations.

Further, if a term *A* of the sum we have obtained contains, for instance, neither the proposition p_1 nor its negation $\bar{p}_1$ we can replace it by the equivalent expression

$$A(p_1 + \bar{p}_1) = Ap_1 + A\bar{p}_1$$

which is a sum of two terms each of which contains the factor p_1 or $\bar{p}_1$. In this way we can bring the sum *f* to the form in which all the summands contain as factors all the propositions $p_1, p_2, \ldots, p_n$ or their negations. If a term in this sum contains both factors p_j and $\bar{p}_j$ it can be simply dropped (because $p\bar{p} = o$ for any *p*); if a term in the sum contains one and the same proposition p_h (or $\bar{p}_h$) several times as a factor then we can retain only one such factor and if the sum *f* contains several identical terms we can also retain only one of them (we remind the reader that the Boolean algebra is an "algebra without exponents and coefficients"; see page 22). If all these operations result

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[1]) In mathematical logic form (*) of a composite proposition is more often referred to as its *disjunctive normal form*.

in the disappearance of all the terms in the sum f we shall have

$$f = o$$

If otherwise, the given composite proposition f is reduced to its additive (disjunctive) normal form (*).

Finally, it is evident that if two compound propositions f_1 and f_2 reduce to one and the same form (*) they must be equal. On the other hand, if forms (*) of two propositions are different these propositions cannot be equal (equivalent). For if, for instance, $n = 4$ and form (*) of a proposition f_1 involves the summand

$$p_1\overline{p_2}p_3p_4$$

while form (*) of a proposition f_2 does not involve such term the proposition f_1 is true if the propositions p_1, p_3 and p_4 are true while the proposition p_2 is false; as to the proposition f_2, it cannot be true in the latter case. We have thus proved that to every proposition f there corresponds its *uniquely determined* additive normal form.

The last property can be used for verifying whether two composite propositions f_1 and f_2 are equivalent or different. Obviously, we can always assume that any two given composite propositions f_1 and f_2 contain *the same* prime propositions p, q, r, etc. because if, for instance, a proposition p is contained in the expression of f_1 but is not contained in the expression of f_2 we can write f_2 in the form

$$f_2(p + \overline{p})$$

which involves the prime proposition p. It also turns out that the additive (disjunctive) normal form of a compound proposition is very useful in many other cases.

One example of this kind was considered above. Below is one more problem whose mathematical content is close to that of the "swimming problem" solved above. Let us consider a simplified curriculum in which there are only three educational days a week, namely, Monday, Wednesday and Friday, and let there be not more than three lessons every educational day. It is required that during a week the pupils should have three lessons in mathematics, two lessons in physics, one lesson in chemistry, one lesson in history and one lesson in English. It is also required that the time-table should satisfy the following conditions.

(1) The mathematics teacher insists that his lessons should never be the last and that at least twice a week they should be the first;

(2) The physics teacher does not want his lessons to be the last either; at least once a week he wants his lesson to be the first; besides, on Wednesday his lesson must not be the first while, on the contrary, he wants to have the first lesson on Friday;

(3) The history teacher can only teach on Monday and on Wednesday; he wants to have the first or the second lesson on Monday or the second lesson on Wednesday; besides, he does not want his lesson to precede the English lesson;

(4) The chemistry teacher insists that his lessons should not be on Friday and that the day he has his lesson the pupils should have no lesson in physics;

(5) The English teacher insists that his lesson should be the last and, besides, he cannot teach on Friday;

(6) It is naturally required that every educational day the pupils should have not more than one lesson in every subject;

(7) In this curriculum there are only $3 + 2 + 1 + 1 + 1 = 8$ lessons a week while the total number of possible lessons is $3 \times 3 = 9$ and hence once a week the pupils have only two lessons. The last requirement to be satisfied is that the pupils should have time of leisure either instead of the last lesson on Friday or instead of the first lesson on Monday.

How can the time-table satisfying all these conditions be worked out?

Let us index the 9 possible lessons in succession by the numbers from 1 to 9; then to solve the problem we must establish the truth or the falsity of the 54 propositions M_j, Ph_j, Ch_j, H_j, E_j and L_j (where $j = 1, 2, \ldots, 9$) meaning, respectively, that the jth lesson is devoted to *mathematics*, *physics*, *chemistry*, *history*, *English* or is the time of *leisure*. The conditions of the problem can now be written as the following system of relations:

(1) $f_1 = \overline{M}_3 \overline{M}_6 \overline{M}_9 (M_1 M_4 + M_1 M_7 + M_4 M_7) = i$

(2) $f_2 = \overline{Ph}_3 \overline{Ph}_6 \overline{Ph}_9 (Ph_1 + Ph_4 + Ph_7) \overline{Ph}_4 \overline{Ph}_8 \overline{Ph}_9 = i$

(3) $f_3 = (H_1 + H_2 + H_5) (\overline{H_1 E_2 + H_2 E_3 + H_4 E_5}$
$\overline{+ H_5 E_6 + H_7 E_8 + H_8 E_9}) = i$

(4) $f_4 = \overline{Ch_7}\overline{Ch_8}\overline{Ch_9}\,(\overline{Ph_1Ch_2} + \overline{Ph_1Ch_3} + \overline{Ph_2Ch_4} + \overline{Ph_2Ch_3} +$
$\qquad + \ldots + \overline{Ph_9Ch_7} + \overline{Ph_9Ch_8}) = i$

(5) $f_5 = \overline{E_3 + E_6 + E_9 + E_2L_3 + E_5L_6 + E_8L_9}\;\overline{E_7}\overline{E_8}\overline{E_9} = i$

(6) $f_6 = (\overline{M_1M_2 + M_1M_3 + M_2M_3 + M_4M_5 + \ldots + M_8M_9}) \times$
$\qquad \times (\overline{Ph_1Ph_2 + Ph_1Ph_3 + \ldots + Ph_8Ph_9}) = i$

(7) $f_7 = \overline{L_1 + L_9} = i$

This system of relations is equivalent to one equality

$$f = f_1 f_2 f_3 f_4 f_5 f_6 f_7 = i$$

The additive normal form of the proposition f is

$$f = M_1 H_2 Ch_3 M_4 Ph_5 E_6 Ph_7 M_8 L_9 \ldots +$$
$$+ M_1 Ph_2 E_3 M_4 H_5 Ch_6\, Ph_7 M_8 L_9 \ldots$$

where the dots symbolize $54 - 9 = 45$ multiplicands entering into each of the two terms of the sum f with the sign of negation (the bar). Hence, *there are only two ways of working out a time-table satisfying all the requirements stated*:

(1) *Monday*: *mathematics, history, chemistry*; *Wednesday*: *mathematics, physics, English*; *Friday*: *physics, mathematics, time of leisure*; *or*

(2) *Monday*: *mathematics, physics, English*; *Wednesday*: *mathematics, history, chemistry*; *Friday*: *physics, mathematics, time of leisure*.

Likewise, it should be noted that every compound proposition $f = f\,(p_1,\ p_2,\ \ldots,\ p_n)$ can also be brought to its *multiplicative normal form*[1]

$$f = \prod (p_1' + p_2' + \ldots + p_n') \qquad (**)$$

where the symbols p_j' have the same sense as in formula $(*)$ and all the terms in the product are pairwise different; this form is also uniquely determined for a given proposition and characterizes completely the whole class of the propositions equal (equivalent) to that proposition. The proof of this assertion is quite analogous to the argument which we used in proving that every compound proposition f

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[1] In mathematical logic form $(**)$ of a composite proposition is usually referred to as its *conjunctive normal form*.

can be brought to form (*); the difference between the proofs is very slight, for instance, instead of the first distributive law we must use the second one.

* *

Let us come back to the "swimming problem". Let us discuss the instructive comparison of the solution of the problem presented above with a solution which can be given by a pupil who is not familiar with elements of mathematical logic.

Such a pupil would replace equalities (1)-(5) and their formal transformations presented above by a "non-formalized" argument (that is an argument which is based on "common sense" instead of the laws of logic stated in a rigorous way) such as the following: "if the father went swimming on Sunday morning then the mother and the son would go with him; but the first daughter would follow the son and the second daughter would go together with her mother; however, since only one of the daughters went swimming that morning the father could not go swimming" and so on. It is however quite evident that an argument of this kind is in fact also based on the rigorous laws of propositional algebra and that the so-called "common sense" exactly follows these laws. For instance, the above argument can be stated thus: "by the conditions of the problem, we have

$$M \supset F \quad \text{and} \quad S \supset F$$

Besides,

$$D_1 \supset S, \quad D_2 \supset M \quad \text{and} \quad M \supset D_2$$

Therefore $M = D_2$ and, consequently,

$$D_2 \supset F$$

while from $D_1 \supset S$ and $S \supset F$ it follows that

$$D_1 \supset F$$

Thus, from proposition F follow propositions D_1 and D_2. Since only one of these propositions is true we conclude that the proposition $\overline{F}$ is true" and so on. Thus, the "formalization" of ordinary inferences which was demonstrated in the solution of the problem we presented above reduces simply to the exact enumeration of all the conditions used

in the argument and to the introduction of mathematical symbols making it possible to write in a concise form both the given conditions and the course of the solution.

The solution of the "swimming problem" can easily be obtained by using an electronic computer because the rules of propositional algebra on which the solution given in this book is based can easily be inserted in the "memory" of the computer and the further course of the solution becomes automatical.

Problems of this kind are rather often encountered in practice. For instance, the problem of working out a real time-table for an educational institution has a similar character because it is necessary to take into account many interrelated conditions such as the wishes and the possibilities of teachers and pupils or students, the necessary alternation of subjects of different character and different difficulty, lectures, lessons, laboratory work, etc. (cf. the problem mentioned on pages 72-75). A traffic controller deals with a similar problem when introducing a rational dispatching system and the like. At present many problems of this type are often solved on electronic computers; the programming of the work of a computer is based on the laws of mathematical logic and, in particular, on "propositional calculus" to which Secs. 5 and 6 of the present book are devoted.

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The relation $p \supset q$ between two propositions p and q plays an important role and that is why it is advisable to consider one more binary operation of algebra of propositions which is connected with this relation. This binary operation forms a new proposition called the *implication* of propositions p and q; we shall write this operation as $q \Rightarrow p$. The proposition $q \Rightarrow p$ is formed of the propositions q and p by connecting them with the expression "*if* . . ., *then* . . ." or with the word "*implies*"; thus, the proposition $q \Rightarrow p$ reads: "*if q, then p*" or, which is the same, "*q implies p*". For instance, if the propositions q and p are "*Peter is an excellent pupil*" and "*an elephant is an insect*", respectively, then the proposition $q \Rightarrow p$ means: "*if Peter is an excellent pupil, then an elephant is an insect*" or, which is the same, "*from the fact that Peter is an excellent pupil it follows that an elephant is an insect*" or "*the fact that Peter is an excellent*

pupil implies that an elephant is an insect". By definition, we shall consider an implication $q \Rightarrow p$ to be true when and only when $p \supset q$; thus, the composite proposition $q \Rightarrow p$ is equivalent to the proposition "the relation $p \supset q$ takes place". Therefore, in case the proposition q is false the proposition $q \Rightarrow p$ is true for any proposition p (because a false proposition q implies any proposition p). For instance, if a pupil whose name is Peter has bad marks we shall consider the above proposition $q \Rightarrow p$ which means "*if Peter is an excellent pupil then an elephant is an insect*" to be true.

It should be stressed that there is a great difference between the *operation* of forming an implication $q \Rightarrow p$ (this is one of the operations of propositional algebra) and the relation $p \supset q$. The composite proposition $q \Rightarrow p$ can be formed of *any* constituent (prime) propositions p and q; as any other proposition, the proposition $q \Rightarrow p$ may turn out to be true or false. As to the relation $p \supset q$, it connects only some pairs of propositions; the fact that the relation $p \supset q$ holds is not a proposition but is a fact concerning these two propositions p and q.

We should stress a peculiarity of the implication $q \Rightarrow p$ of two given propositions q and p: in contradistinction to the operations of forming the sum ("disjunction") $p + q$ and the product ("conjunction") pq of propositions, the operation $q \Rightarrow p$ is *non-commutative*, that is, in the general case, the proposition $p \Rightarrow q$ differs from the proposition $q \Rightarrow p$. The implication $p \Rightarrow q$ is called the *converse* of the implication $q \Rightarrow p$. The relationship between the implication $q \Rightarrow p$ and its converse $p \Rightarrow q$ is quite similar to the relationship between the direct theorem "*if q, then p*" (for instance, "*if all the sides of a quadrilateral are equal, then its diagonals are mutually perpendicular*") and the converse theorem "*if p, then q*" ("*if the diagonals of a quadrilateral are mutually perpendicular, then all its sides are equal*"). As is well-known, generally, the statements of a direct theorem and the corresponding converse theorem are not necessarily equivalent: one of them may turn out to be true while the other can be false. At the same time, the implication $\bar{p} \Rightarrow \bar{q}$ called the *contrapositive* of the implication $q \Rightarrow p$ is equivalent to the latter for the relation $p \supset q$ holds when and only when the relation $\bar{q} \supset \bar{p}$ holds The relationship between the proposition $q \Rightarrow p$ (which

means "*if q, then p*") and its contrapositive $\bar{p} \Rightarrow \bar{q}$ (that is the proposition "*if p is false then q is also false*") is quite similar to the relationship between a direct theorem (for instance, "*if all the sides of a quadrilateral are equal, then its diagonals are mutually perpendicular*") and the converse of the theorem which is the *inverse* of the original theorem, "if the diagonals of a quadrilateral are not mutually perpendicular, then all its sides are not equal". By the *inverse* of a theorem (implication) $q \Rightarrow p$ is meant the implication $\bar{q} \Rightarrow \bar{p}$, and the converse of the latter is the implication $\bar{p} \Rightarrow \bar{q}$ (to the example of the direct theorem we have given there corresponds the converse of the inverse asserting that "*if the diagonals of a quadrilateral are not mutually perpendicular, then it is false that all its sides are equal*"). In other words, the theorem $\bar{p} \Rightarrow \bar{q}$ is equal to the direct theorem $q \Rightarrow p$. As to the implication $\bar{q} \Rightarrow \bar{p}$ expressing the *inverse* of the theorem (implication) $q \Rightarrow p$ (in the case of the above example, $\bar{q} \Rightarrow \bar{p}$ is the theorem which reads: "*if it is false that all the sides of a quadrilateral are equal, then its diagonals are not mutually perpendicular*"), it is not equivalent to the direct theorem $(q \Rightarrow p)$ but is equivalent to the converse theorem (implication) $p \Rightarrow q$ which means "*if p, then q*" (because the property "if $a \supset b$, then $\bar{b} \supset \bar{a}$" which is one of the basic properties of the relation $\supset$ implies that the relations $q \supset p$ and $\bar{p} \supset \bar{q}$ hold or do not hold simultaneously).

Together with the implication $q \Rightarrow p$, in mathematical logic is sometimes considered the so-called *biconditional proposition* formed of two propositions q and p; we shall denote it by the symbol $q \Longleftrightarrow p$. The biconditional proposition $q \Longleftrightarrow p$ reads: "*p, if and only if q*"[1]). For instance, for the propositions p and q given as examples on page 76 the proposition $q \Longleftrightarrow p$ reads: "*Peter is an excellent pupil if and only if an elephant is an insect*". The last statement is a new proposition (although rather funny!): the operation of forming the biconditional proposition from two given propositions q and p is also a binary operation of propositional algebra which assigns to every pair q and p

---

[1]) Instead of "*if and only if*" in English mathematical literature is often used the expression "*iff*" which is the abbreviation for the former. — *Tr.*

of propositions a new proposition which we denote $q \Leftrightarrow p$. It is quite clear that the relationship between the biconditional proposition $q \Leftrightarrow p$ and the equivalence relation $p = q$ is similar to the relationship between an implication $q \Rightarrow p$ and the relation $q \supset p$: *the proposition $q \Leftrightarrow p$ is true when and only when the equivalence $p = q$ takes place.* In contrast to the implication $q \Rightarrow p$ of two propositions q and p, the biconditional proposition is *commutative*: the propositions $q \Leftrightarrow p$ and $p \Leftrightarrow q$ are equivalent for any two propositions q and p, that is we always have

$$(p \Leftrightarrow q) = (q \Leftrightarrow p)$$

The notion of the "truth set" of a proposition (see page 55) makes it possible to extend the new operations $q \Rightarrow p$ and $q \Leftrightarrow p$ of algebra of propositions to algebra of sets. Let q and p be two arbitrary propositions and let Q and P be their truth sets respectively. We shall denote the truth set of the proposition $q \Rightarrow p$ as $Q \Rightarrow P$ and the truth set of the proposition $q \Leftrightarrow p$ as $Q \Leftrightarrow P$. It is obvious that the implication $q \Rightarrow p$ is true if and only if either the proposition q is false (a false hypothesis implies any conclusion) or the propositions q and p are simultaneously true (a true hypothesis implies any other true statement). This means that *the set $Q \Rightarrow P$ is the union of the complement of the set Q and the intersection of the sets Q and P* (see Fig. 28,a). It follows that

$$Q \Rightarrow P = \overline{Q} + QP$$

and consequently

$$q \Rightarrow p = \overline{q} + qp$$

Thus, the implication $q \Rightarrow p$ formed of two propositions q and p can be defined in terms of the basic operations of propositional algebra, that is in terms of the operations of addition of propositions, multiplication of propositions and the operation of forming the negation. For instance, according to the last relation, the above proposition "*if Peter is an excellent pupil, then an elephant is an insect*" is equivalent to the proposition "*Peter is not an excellent pupil or Peter is an excellent pupil and an elephant is an insect*".

Similarly, a biconditional proposition $q \Leftrightarrow p$ is true if and only if either both propositions q and p are true or

$P \Rightarrow Q$
(a)

$P \Leftrightarrow Q$
(b)

$P \Rightarrow Q$
(c)
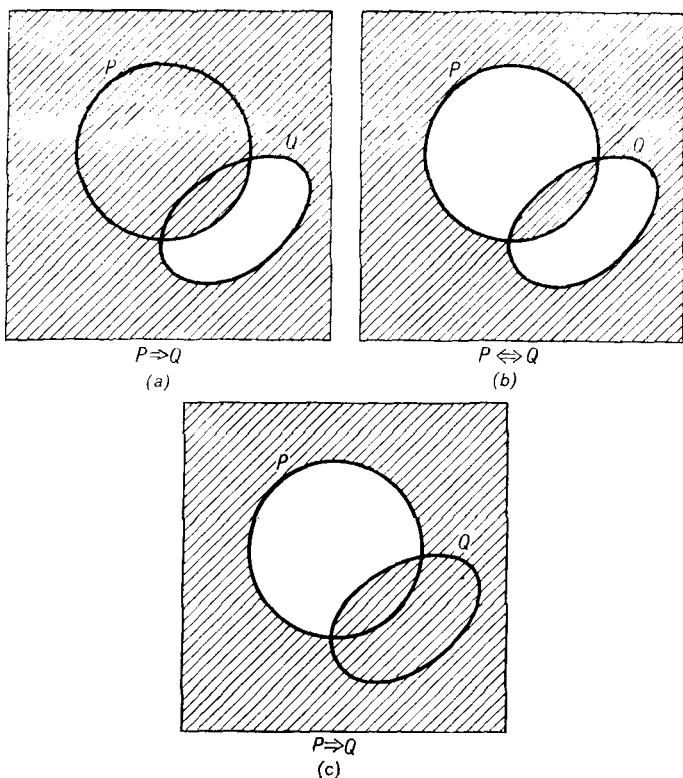
Fig. 28

both propositions q and p are false. Hence, *the set $Q \Leftrightarrow P$ is the union of the intersection of the sets Q and P and the intersection of the sets $\overline{Q}$ and $\overline{P}$* (see Fig. 28,b):

$$Q \Leftrightarrow P = QP + \overline{Q}\overline{P}$$

It follows that the biconditional proposition $q \Leftrightarrow p$ formed of two propositions q and p can also be expressed in terms of the operations of propositional algebra studied earlier, namely:

$$q \Leftrightarrow p = qp + \overline{q}\overline{p}$$

The formulas we have written readily show that the operation $\Leftrightarrow$ of forming a biconditional proposition is a

commutative operation of algebra of propositions while an implication $q \Rightarrow p$ is non-commutative:

$$q \Leftrightarrow p = qp + \bar{q}\bar{p} = p \Leftrightarrow q$$

but

$$q \Rightarrow p = \bar{q} + qp \neq \bar{p} + pq = p \Rightarrow q$$

(see Fig. 28c, where the truth set $P \Rightarrow Q$ of the converse $p \Rightarrow q$ of the implication $q \Rightarrow p$ is shown). On the other hand, the contrapositive $\bar{p} \Rightarrow \bar{q}$ of the implication $q \Rightarrow p$ is equivalent to the latter:

$$\bar{p} \Rightarrow \bar{q} = \overline{(\bar{p})} + \bar{p}\bar{q} = p + \bar{p}\bar{q} = \bar{q} + pq = q \Rightarrow p$$

because from Fig. 28,a it is seen that the sets $Q \Rightarrow P = \bar{Q} + QP$ and $\bar{P} \Rightarrow \bar{Q} = P + \bar{P}\bar{Q}$ coincide. The equivalence between the implication $q \Rightarrow p$ of propositions q and p and the contrapositive $\bar{p} \Rightarrow \bar{q}$ of the implication can also be proved without resorting to the Venn diagram:

$$\bar{p} \Rightarrow \bar{q} = p + \bar{p}\bar{q} = p\,(q + \bar{q}) + \bar{p}\bar{q} =$$
$$= pq + p\bar{q} + \bar{p}\bar{q} = pq + (p + \bar{p})\,\bar{q} = pq + \bar{q} = \bar{q} + qp = q \Rightarrow p$$

(here we have used the commutative, the associative and the distributive laws and also the identities $pi = p$ and $p + \bar{p} = i$). Similarly, the proposition $\bar{q} \Rightarrow \bar{p}$ is equivalent to the converse $p \Rightarrow q$ of the implication $q \Rightarrow p$:

$$\bar{q} \Rightarrow \bar{p} = q + \bar{q}\bar{p} = \bar{p} + pq = p \Rightarrow q$$

*

The formulas $q \Rightarrow p = \bar{q} + qp$ and $q \Leftrightarrow p = qp + \bar{q}\bar{p}$ express the operations $\Rightarrow$ and $\Leftrightarrow$ in terms of the basic operations of addition and multiplication and the "bar" operation of the Boolean algebra. That is why, although, for instance, the operation of forming an implication is highly important we do not include it into the list of the basic operations which form the foundation of the definition of a Boolean algebra. However, it also turns out that the three original operations $+$, $\cdot$ and $-$ are not independent: using the De Morgan formulas we can express each of the two operations $+$ and $\cdot$ in terms of the other and the "bar" operation. For instance, the "Boolean multiplication" of
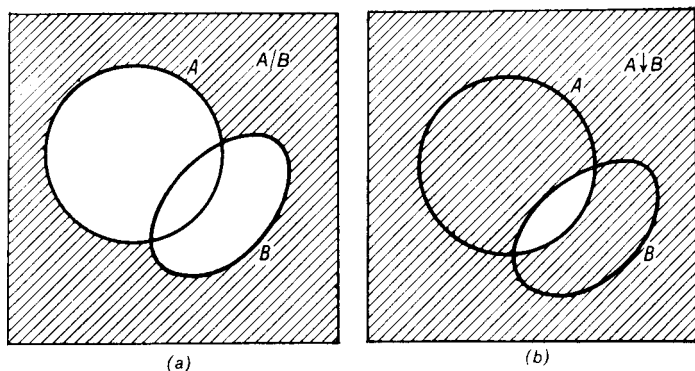
Fig. 29

propositions can be *defined* thus:

$$pq = \overline{\overline{p} + \overline{q}}$$

Moreover, it turns out that there is an operation defined in Boolean algebra in terms of which all the three operations $+$, $\cdot$ and $-$ can be expressed. This makes it possible to reduce all the variety of operations used in Boolean algebras to a single operation and its various combinations. One of the well-known operations of this kind is the so-called (*Sheffer*[1])) *stroke operation* $\alpha \mid \beta$ (here α and β are elements of an arbitrary Boolean algebra) which is expressed in terms of the "Boolean multiplication" and the "bar" operation as

$$\alpha \mid \beta = \overline{\alpha}\overline{\beta}$$

In the case when the Boolean algebra in question is an algebra of sets whose elements A, B, C, ... are some sets for which the operation of addition $A + B$, the operation of multiplication AB and the operation of forming the complement $\overline{A}$ of any set A are defined as was done in Secs. 1 and 2, the Sheffer operation $A \mid B$ reduces to forming *the intersection of the complements of the sets A and B* (see Fig. 29,*a*).

The Sheffer operation is obviously commutative, that is

$$\alpha \mid \beta = \beta \mid \alpha$$

---

[1]) H. M. Sheffer, an American logician of the beginning of the 20th century.

for any α and β. Further, from the basic properties of the operations of a Boolean algebra it follows that

$$(\alpha \mid \beta) \mid (\alpha \mid \beta) = \overline{(\overline{\alpha}\overline{\beta})} \, (\overline{\overline{\alpha}\overline{\beta}}) = \overline{[(\overline{\alpha}) + (\overline{\beta})]} \, \overline{[(\overline{\alpha}) + (\overline{\beta})]} = \alpha + \beta$$

$$(\alpha \mid \alpha) \mid (\beta \mid \beta) = \overline{(\overline{\alpha}\,\overline{\alpha})} \cdot \overline{(\overline{\beta}\,\overline{\beta})} = \overline{[(\overline{\alpha}) + (\overline{\alpha})]} \, \overline{[(\overline{\beta}) + (\overline{\beta})]} = \alpha\beta$$

and

$$(\alpha \mid \alpha) = \overline{\alpha}\,\overline{\alpha} = \overline{\alpha}$$

Thus, if we take the Sheffer operation $\alpha \mid \beta$ as the basic one it is possible to *define* $\alpha + \beta$, $\alpha\beta$ and $\overline{\alpha}$ as $(\alpha \mid \beta) \mid (\alpha \mid \beta)$, $(\alpha \mid \alpha) \mid (\beta \mid \beta)$ and $\alpha \mid \alpha$ respectively.

The role analogous to that of the Sheffer operation can also be played by another binary operation $\alpha \downarrow \beta$ defined as

$$\alpha \downarrow \beta = \overline{\alpha} + \overline{\beta}$$

(for algebra of sets the operation $A \downarrow B$ reduces to forming *the union of the complements of the sets A and B*; see Fig. 29b). It can readily be seen that

$$(\alpha \downarrow \alpha) \downarrow (\beta \downarrow \beta) = \overline{(\overline{\alpha + \overline{\alpha}})} + \overline{(\overline{\beta + \overline{\beta}})} = \overline{\overline{\overline{\alpha}}} + \overline{\overline{\overline{\beta}}} = \alpha + \beta$$

$$(\alpha \downarrow \beta) \downarrow (\alpha \downarrow \beta) = \overline{(\overline{\alpha + \overline{\beta}})} + \overline{(\overline{\alpha + \overline{\beta}})} = \overline{\overline{\overline{\alpha}\overline{\beta}}} + \overline{\overline{\overline{\alpha}\overline{\beta}}} = \alpha\beta$$

and

$$(\alpha \downarrow \alpha) = \overline{\alpha} + \overline{\alpha} = \overline{\alpha}$$

Therefore the operations $\alpha + \beta$, $\alpha\beta$ and $\overline{\alpha}$ can also be *defined* in terms of the operation $\alpha \downarrow \beta$.

The definition of a Boolean algebra is sometimes stated using only one "*ternary*" operation $\{\alpha, \beta, \gamma\}$ defined as

$$\{\alpha\beta\gamma\} = \alpha\beta + \beta\gamma + \gamma\alpha = (\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)$$

This operation assigns a new element $\delta = \{\alpha, \beta, \gamma\}$ to every *triple* of elements α, β, γ of a Boolean algebra (cf. Exercise 6 on page 23). (The operations $+$ and $\cdot$ which assign new elements to any pair of elements of a Boolean algebra are *binary* operations; the "bar" operation assigns a new element $\overline{\alpha}$ to one element α and is an example of a "*unary*" operation.) For an algebra of sets the element $\{ABC\} = AB + BC + CA$ is the set coinciding with *the union of the pairwise intersections of the sets A, B and C* (see Fig. 30)
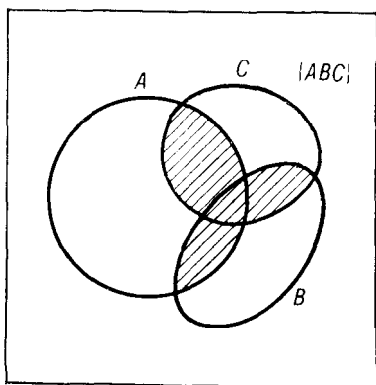
Fig. 30

or, which is the same, with ⸻ ⸺ ⸻ ⸻ ⸺ ⸻ ⸺ ⸻ pair-wise unions of these sets.

The ternary operation $\{\ \}$ is obviously commutative with respect to the interchange of any two of the elements entering into it:

$$\{\alpha\beta\gamma\} = \{\alpha\gamma\beta\} = \{\beta\alpha\gamma\} =$$
$$= \{\beta\gamma\alpha\} = \{\gamma\alpha\beta\} = \{\gamma\beta\alpha\}$$

Further, this operation possesses a certain kind of distributivity:

$$\{\alpha\beta\{\gamma\delta\varepsilon\}\} = \{\{\alpha\beta\gamma\}\,\delta\,\{\alpha\beta\varepsilon\}\}$$

It also possesses a (weakened) associativity:

$$\{\alpha\beta\{\alpha\beta\delta\}\} = \{\{\alpha\beta\gamma\}\,\beta\delta\}$$

Finally, for this operation there holds a law analogous to the idempotent laws for addition and multiplication:

$$\{\alpha\alpha\beta\} = \alpha$$

The operation $\bar{\alpha}$ can be *defined* with the help of the ternary operation $\{\alpha\beta\gamma\}$ by means of the following condition similar to the idempotent law we have written:

$$\{\alpha\bar{\alpha}\beta\} = \beta \tag{A}$$

Since this condition is symmetric with respect to the elements α and $\bar{\alpha}$ it obviously implies that

$$\bar{\bar{\alpha}} = \alpha$$

Further, if we fix a "special element" ι among the elements α, β, γ, ... forming a set for which the ternary operation $\{\alpha\beta\gamma\}$ is defined and put $\bar{\iota} = o$ then it is also possible to define the basic (binary) operations of the Boolean algebra in terms of the ternary operation $\{\alpha\beta\gamma\}$:

$$\alpha + \beta = \{\alpha\beta\iota\} \quad \text{and} \quad \alpha\beta = \{\alpha\beta o\} \tag{B}$$

By virtue of (A) and the idempotent law, we shall also have

$$\alpha + o = \{\alpha o \iota\} = \{\alpha\bar{\iota}\iota\} = \alpha \cdot \quad \text{and} \quad \alpha\iota = \{\alpha\iota o\} = \{\alpha\iota\bar{\iota}\} = \alpha$$
$$\alpha + \iota = \{\alpha\iota\iota\} = \iota \quad \text{and} \quad \alpha o = \{\alpha o o\} = o$$

Definitions (A) and (B) make it possible to state all the properties of the operations of the Boolean algebra so that the corresponding expressions involve the ternary operation $\{\alpha\beta\gamma\}$ solely.

In propositional algebra the Sheffer operation $p \mid q$ and the ternary operation $\{pqr\}$ have the following meaning: $p \mid q$ reads: "*neither p nor q is true*" (that is why in logic the Sheffer operation is sometimes referred to as the *joint negation*) and the proposition $\{pqr\}$ reads: "*at least two of the three propositions p, q and r are true*". Further, we have

$$p + q = (p \mid q) \mid (p \mid q)$$
$$pq = (p \mid p) \mid (q \mid q)$$

and

$$\overline{p} = p \mid p$$

and, consequently,

$$q \Rightarrow p = \overline{q} + qp = [(q \mid q) \mid ((p \mid p) \mid (q \mid q))] \mid [(q \mid q) \mid ((p \mid p) \mid (q \mid q))]$$

and

$$q \Longleftrightarrow p = pq + \overline{p}\,\overline{q} = \overline{r} \mid \overline{r}$$

where

$$r = [(p \mid p) \mid (q \mid q)] \mid [((p \mid p) \mid (p \mid p)) \mid ((q \mid q) \mid (q \mid q))]$$

With the aid of the ternary operation $\{pqr\}$ the sum ("disjunction") and the product ("conjunction") of two propositions p and q are expressed thus:

$$p + q = \{pq\iota\} \quad \text{and} \quad pq = \{pqo\}$$

* *

We have already discussed the relationship between the operations $q \Rightarrow p$ and the relation $q \supset p$. Using algebraic symbols we can express this relationship as

$$p \supset (q \Rightarrow p)\, q$$

In other words, *if the implication $q \Rightarrow p$ is true and the proposition q is true then the proposition p is also true*. This obviously follows from the expression of the implication in terms of the other operations of a Boolean algebra: we have $q \Rightarrow p = \overline{q} + qp$ and consequently

$$(q \Rightarrow p)\, q = (\overline{q} + qp)\, q = o + qp = qp$$

whence it follows that

$$p \supset qp - (q \Rightarrow p) \, q$$

The relation $p \supset (q \Rightarrow p) \, q$ expresses the form of a logical statement known as the classical *syllogism*; for instance, a typical syllogism is:

"*All men are mortal* (that is if N is a man then N is mortal; this can be written as an implication $q \Rightarrow p$);

Peter is a man (q);

Consequently, Peter is mortal (p)".

The logical statement expressed by the relation

$$\bar{q} \supset (q \Rightarrow p) \, \bar{p}$$

is also true; it means that *if the implication $q \Rightarrow p$ is true and the proposition p is false then the proposition q is also false*. The last relation also readily follows from the formula for the implication: we have

$$(q \Rightarrow p) \, \bar{p} = (\bar{q} + pq) \, \bar{p} = \bar{q} \, \bar{p} + (p\bar{p}) \, q = \bar{q} \, \bar{p} + o = \bar{q} \, \bar{p}$$

and therefore $\bar{q} \supset (q \Rightarrow p) \, \bar{p} = \overline{qp}$.

Here is an example demonstrating the application of the logical rule $\bar{q} \supset (q \Rightarrow p) \, \bar{p}$:

"*All mathematicians reason logically* (that is if N is a mathematician then he reasons logically; this can be regarded as an implication $q \Rightarrow p$);

Paul reasons illogically $(\bar{p})$;

Consequently, Paul is not a mathematician $(\bar{q})$".

Similarly, we have

$$\bar{q} \supset (q \Rightarrow p) \, (p \Leftrightarrow r) \, \bar{r}$$

which means that *if q implies a proposition p, p is equivalent to r and r is false then q is also false*. Indeed, by virtue of the formulas expressing the implication and the biconditional proposition, we have

$$(q \Rightarrow p) \, (p \Leftrightarrow r) \, \bar{r} = (\bar{q} + pq) \, (pr + \bar{p}\bar{r}) \, \bar{r} =$$
$$= \bar{q} p \, (r\bar{r}) + \bar{q} \, \bar{p} \, \bar{r} + pq \, (r\bar{r}) + (p\bar{p}) \, q\bar{r} = o + \bar{q} \, \bar{p} \, \bar{r} + o + o = \bar{q} \, \bar{p} \, \bar{r}$$

and, consequently,

$$\bar{q} \supset \bar{q} \, \bar{p} \, \bar{r} = (q \Rightarrow p) \, (p \Leftrightarrow r) \, \bar{r}$$

Here is an example of an argument following this rule:

"*If the sides of a quadrilateral are equal then the quadrilateral is a parallelogram* (or, more precisely, a rhombus; this proposition is an implication $q \Rightarrow p$);

A quadrilateral is a parallelogram if and only if its diagonals bisect each other ($p \Leftrightarrow r$)";

The diagonals of the given quadrilateral ABCD do not bisect each other ($\bar{r}$);

Therefore it is false that all the sides of the quadrilateral ABCD are equal ($\bar{q}$)".

In contrast to the above, the following two relations may turn out to be false:

$$q \supset (q \Rightarrow p)\, p$$

and

$$\bar{p} \supset (q \Rightarrow p)\, \bar{q}$$

Indeed, we have

$$(q \Rightarrow p)\, p = (\bar{q} + qp)\, p = \bar{q}p + qp = (q + \bar{q})\, p = ip = p$$

and

$$(q \Rightarrow p)\, \bar{q} = (\bar{q} + qp)\, \bar{q} = \bar{q} + (q\bar{q})\, p = \bar{q} + op = \bar{q}$$

and the relation

$$q \supset p$$

and the (equivalent) relation

$$\bar{p} \supset \bar{q}$$

do not, of course, follow from the rules of propositional algebra and may not take place. That is why the following two statements (which, unfortunately, are rather frequently used, particularly, by non-mathematicians) do not follow from the rules for deduction and are therefore incorrect (it should be noted that an electronic computer which was "taught" the theory of Boolean algebras can never make such a mistake!):

"*q implies p; p is true; therefore the proposition q is also true*" (for instance, "*opposite sides of a parallelogram are equal; the opposite sides AB and CD of the given quadrilateral ABCD are equal; consequently ABCD is a parallelogram*");
and

"*q implies p; the proposition q is false; therefore the proposition p is also false*" (for instance, "*lawyers speak well; N is not a lawyer; consequently N does not speak well*").

The examples we have considered (their number can easily be increased) demonstrate the role which mathematical rules of propositional algebra play even in everyday life.

Exercises

1. Reduce the composite propositions

(a) $pq + \bar{p} + \bar{q}$;

(b) $pqr + p + q + r$;

(c) $(p + q)(q + r)(r + p)$

to

(1) form (*) (see page 70);

and to

(2) form (**) (see page 74).

2. Rewrite the following proposition in the form involving only the addition (disjunction) and the "bar" operation (negation):

(a) $q \Rightarrow (p + q)$;

(b) $pq \Rightarrow q$;

(c) $(p + q) \Rightarrow [(p + r) \Rightarrow (q + r)]$;

(d) $pq \Leftrightarrow rs$.

3. Find which of the following propositions are true and which are false:

(a) $p + q \Rightarrow p$;

(b) $pq \Rightarrow q$;

(c) $\overline{(p \Rightarrow p)} \Leftrightarrow p$;

(d) $(p \Rightarrow \bar{p}) \Leftrightarrow p$;

(e) $pq \Leftrightarrow qp$.

4. Write the negations of the following propositions in the form which only involves the sign of negation (the bar) over the propositions p, q and r themselves but not over their combinations:

(a) $\bar{p} \Rightarrow q$;

(b) $(p + q)\,r$;

(c) $(p + q) \Rightarrow r$;

(d) $\bar{p} \Leftrightarrow (p + \bar{q})$.

5. Express the implication (⇒) and the biconditional proposition (⇔) in terms of

(a) the operation ↓ (see page 83);

(b) the operation {}.

7. Propositions and Switching Circuits

Here we shall discuss one more example of a Boolean algebra which may seem rather unexpected. As the elements of this algebra we shall consider various switching circuits, that is electric circuits with a number of switches each of which can be open or closed. Separate sections of such a circuit (for instance, see a section shown in Fig. 31) will be denoted by capital letters; it is these sections that are elements of the peculiar algebra under consideration (earlier we used capital letters for denoting sets).

Since a section of an electric circuit is only meant for conducting an electric current, we shall consider any two sections which are similar in this sense to be identical ("equal"). In other words, any two sections containing the same switches and simultaneously permitting or not permitting passage of current for the same states of all the switches (every switch can be in one of the two states: "open" or "closed") will be considered "equal" to each other.

Fig. 31

(a)

(b)

Fig. 32

89
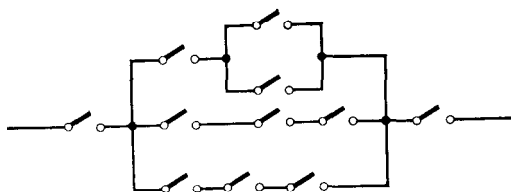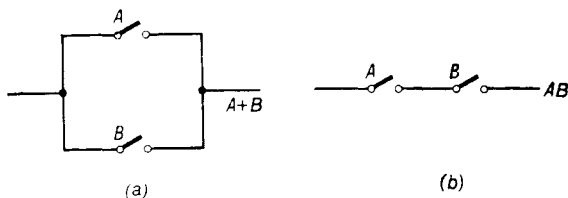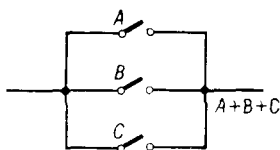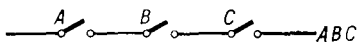
(a)   (b)

Fig. 33
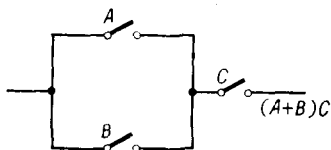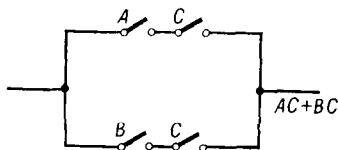
(a)   (b)

Fig. 34
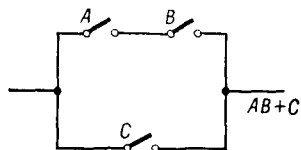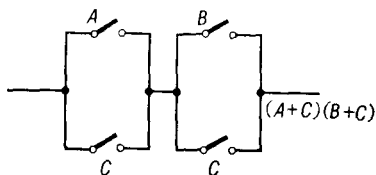
(a)   (b)

Fig. 35

Further, let us agree that by the sum $A + B$ of two sections A and B will be meant a circuit with these two given sections A and B in parallel connection and that by the product AB will be meant a circuit section with the sections A and B in series connection. For instance, see Fig. 32,a and b where each of the sections A and B of the circuit contains only one switch. It is clear that the addition and the multiplication of the sections of an electric circuit are *commutative*:

$$A + B = B + A \quad \text{and} \quad AB = BA$$

These operations are also *associative*:

$$(A + B) + C = A + (B + C) = A + B + C$$

and

$$AB\,(C) = A\,(BC) = ABC$$

see Fig. 33a and b where the "triple sum" $A + B + C$ of three switches and their "triple product" ABC are shown). The *idempotent* laws

$$A + A = A \quad \text{and} \quad AA = A$$

also hold for these operations because when two switches which are in one and the same state (that is when they are open or closed simultaneously) are in series connection or in parallel connection the resultant circuit section gives the same result as a single switch in that state. The verification of the *distributive* laws

$$(A + B) C = AB + BC \quad \text{and} \quad AB + C =$$
$$= (A + C)(B + C)$$

in this "algebra of switching circuits" is a little more complicated. However, as can be seen from Figs. 34 and 35, these laws also hold here (it can readily be checked that the switching circuit shown in Fig. 34,a is "equal to" the circuit in Fig. 34,b while the circuit in Fig. 35,a is equal to that in Fig. 35,b).

Finally, let us agree that I denotes an *always-closed* switch (Fig. 36,a) and let O denote an *always-open* switch (Fig. 36,b). It is evident that

$$A + O = A \quad \text{and} \quad AI = A$$

(see Fig. 37) and that

$$A + I = I \quad \text{and} \quad AO = O$$

(a)   (b)

Fig. 36

(a)   (b)
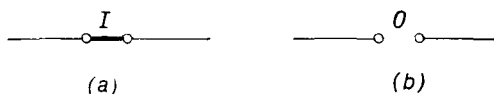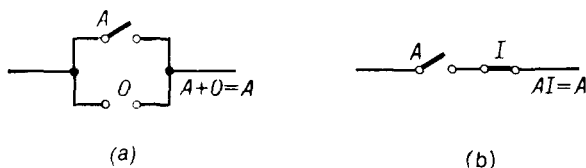
Fig. 37
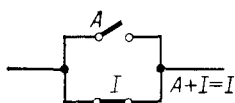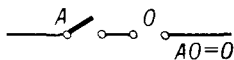
(a)   (b)
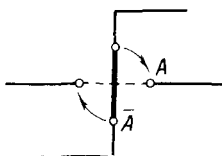
Fig. 38

Fig. 39
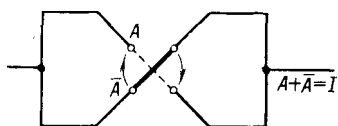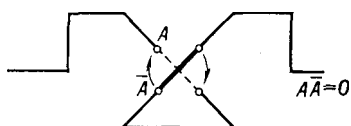
(a)   (b)

Fig. 40

(see Fig. 38). Thus, the roles of the "special" elements I and O of this Boolean algebra are played by the circuit sections equal to an always-closed and an always-open switches respectively.

Let us also agree to denote as A and $\overline{A}$ a pair of switches such that *when the switch A is closed the switch $\overline{A}$ is necessarily open and vice versa*; such a pair of switches can easily be constructed (see Fig. 39). It is evident that

$$\overline{\overline{A}} = A, \ \overline{I} = O \quad \text{and} \quad \overline{O} = I$$

and also

$$A + \overline{A} = I \quad \text{and} \quad A\overline{A} = O$$

(see Fig. 40,a and b). The De Morgan rules

$$\overline{A + B} = \overline{A}\,\overline{B} \quad \text{and} \quad \overline{AB} = \overline{A} + \overline{B}$$

are proved in a more intricate manner but they also hold in this algebra (for instance, see Fig. 41,a and b where the sections $A + B$ and $\overline{A + B}$ of the circuit satisfy the condi-
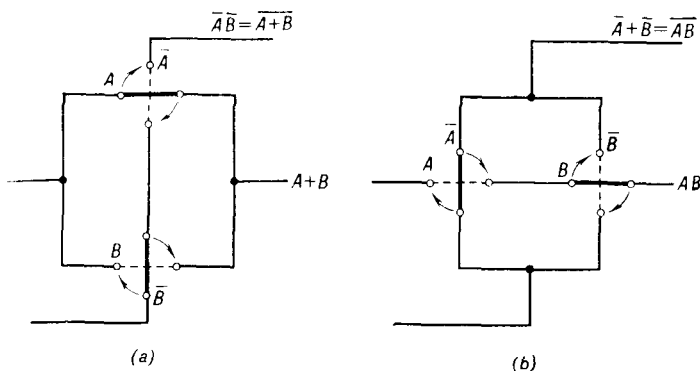
Fig. 41
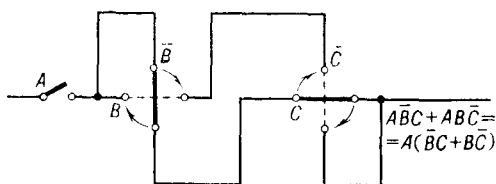
Fig. 42

tion that when the section $A + B$ permits passage of current the section $\overline{A + B}$ does not and vice versa).

The similarity between the "algebra of switching circuits" and the "algebra of propositions" is extremely valuable. In the first place, this similarity makes it possible to model composite propositions by means of electric circuits. For instance, let us consider the composite proposition

$$d = a\overline{b}c + ab\overline{c}$$

where a, b and c are some "prime" propositions and the addition, the multiplication of propositions and the "bar" operations are understood in the ordinary sense as the logical connectives "or", "and" and the negation of a proposition respectively. Let us associate some switches A, B and C with the given propositions a, b and c; then the composite proposition d can be represented by the circuit in Fig. 42 which corresponds to the combination

$$D = A\overline{B}C + AB\overline{C}$$
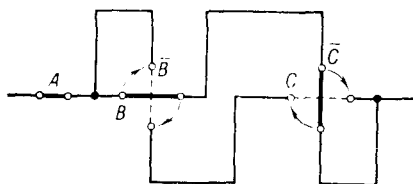
Fig. 43

of the switches A, B and C. To verify whether the proposition d is true when, for instance, the propositions a and b are true while the proposition c is false it suffices to close the switches A and B in the circuit D and to open the switch C (see Fig. 43). If the circuit D with the switches A, B and C in these states permits the electric current to flow then D corresponds to the true proposition i (that is to the circuit I conducting electric current). In other words, in this case the proposition d is true. In case the circuit D does not permit passage of current (that is it is "equal" to the circuit O) for the given states of the switches then the proposition d is equivalent to the false proposition o when a and b are true while c is false.

In the second place, the similarity between the algebra of switching circuits and the algebra of propositions allows us to use the rules of logic for constructing switching circuits satisfying some given conditions (which can be rather complex). Here we shall give two examples to demonstrate what has been said.

*

Example 1. *It is required to design an electric circuit for a bedroom with one electric lamp and with two switches one of which is by the door and the other by the bed-side. The condition which must be satisfied is that when each of the switches is operated on the circuit must become open if it is closed before the operation and must become closed if it is open before the operation irrespective of the state of the other switch.*

Solution. Let us denote as A and B the switches in the circuit. The problem reduces to designing a combination C of the switches A and B (and perhaps $\overline{A}$ and $\overline{B}$) such that the *change of the state of any of the two switches changes the state of the whole circuit C* to the opposite, that is transforms

the circuit permitting the passage of current into the one not permitting it and vice versa. In other words, we have to find a combination c of two propositions a and b such that the replacement of the true proposition a by the false proposition $\bar{a}$ or vice versa changes to the opposite the sense (the "truth" or the "falsity") of the whole proposition c, and the same requirement refers to the proposition b. The condition stated is satisfied by a proposition c which is true when both propositions a and b are simultaneously true or simultaneously false and which is false in all the other cases (that is when one of the two propositions a and b is true while the other is false). This description of the circuit involves the connective "or", which hints that it is possible to represent the proposition c as a *sum* of two propositions one of which is true when a and b are true while the other is true when $\bar{a}$ and $\bar{b}$ are true (that is when a and b are false). Further, since the descriptions of the summands of the sought-for sum involve the connective "and" we conclude that these summands are

$$ab \quad \text{and} \quad \bar{a}\,\bar{b}$$

Thus, we finally obtain

$$c = ab + \bar{a}\,\bar{b}$$

It can readily be seen that this proposition c satisfies all the requirements stated above.

Now, passing back from the propositions to the switching circuits we see that the electric circuit C we are interested in can be expressed by the formula

$$C = AB + \bar{A}\,\bar{B}$$

The construction of such a circuit clearly involves no difficulties (Fig. 44).
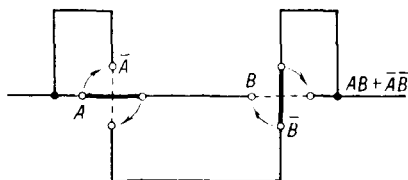

Fig. 44

Example 2. *It is required to design an electric circuit for controlling a lift. For the sake of simplicity, we shall assume that there are only two floors; we shall also confine ourselves to the circuit controlling the downward motion of the lift[1]).* This circuit must involve two switches (push-buttons) one of which is in the car (the descent button) and the other is placed by the lift shaft door on the first floor (the call button). The circuit also involves the following additional switches: a switch which is closed only when the car is on the second floor, two switches connected with the outer (lift shaft) doors on the first and on the second floors which are closed when the doors are closed, a switch connected with the door of the car (the inner door) which is closed when the inner door is closed and a switch connected with the floor of the car which is closed when a person is in the car and the weight of the person exerts pressure on the floor. The electric circuit controlling the downward motion of the lift must be closed only when the car is on the second floor and, besides, when one of the two following conditions is fulfilled:

(1) the outer doors on the first and on the second floors and the inner door (in the car) are closed; a person is in the car and pushes the descent button;

(2) both outer doors (of the lift shaft) are closed while the door of the car is closed or open; there is no person in the car; a person on the first floor presses the call button.

Solution. Let us denote the switches in the circuit as follows: S—the switch which is closed only when the car is on the second floor, D_1 and D_2—the switches which are closed when the outer doors on the first and on the second floors respectively are closed, D—an analogous switch connected with the door of the car, F—the switch connected with the floor of the car, B_d and B_c—the switches connected with the descent button in the car and with the call button by the lift shaft door on the first floor respectively. According to the conditions of the problem the sought-for circuit C_d controlling the descent of the lift must be closed (must conduct electric current) only if:

(1) the switch S is closed *and* the switch D_1 is closed *and* the switch D_2 is closed *and* the switch D is closed *and* the

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[1]) The circuit controlling the upward motion of the lift can be designed in just the same way (see Exercise 6 on page 100).
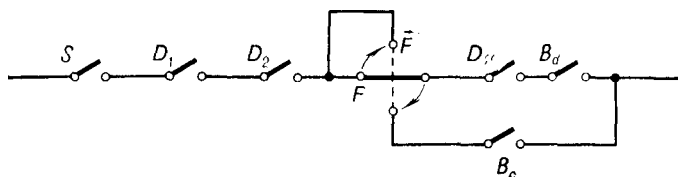
Fig. 45

switch F is closed *and* the switch B_d is closed

or

(2) the switch S is closed *and* the switch D_1 is closed *and* the switch D_2 is closed *and* the switch D is closed *or* open and the switch B_c is closed *and* the switch F is open.

Taking into account that the logical operation "and" corresponds to the product of propositions (of switches) and the logical operation "or" corresponds to their sum we readily find

$$C_d = SD_1D_2DFB_d + SD_1D_2(D + \overline{D})\,B_c\overline{F}$$

Using the equality

$$D + \overline{D} = I$$

and the property of the switch I ($AI = A$ for any switch A) and also the commutative law for multiplication and the distributive law we can simplify the expression we have derived:

$$C_d = SD_1D_2(FDB_d + \overline{F}B_c)$$

Such a circuit can easily be constructed (see Fig. 45).

✱

We also note that the possibility of expressing all the operations of a Boolean algebra in terms of only one Sheffer operation (see Sec. 6) is equivalent to the possibility of designing any electric switching circuit using only one special component (we denote it Σ) with two inputs and one output such that *the output electric current can flow if and only if neither of the inputs is supplied with electric current.* Such an element can easily be constructed (see Fig. 46). For instance, to this end we can agree that to
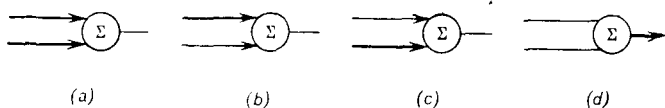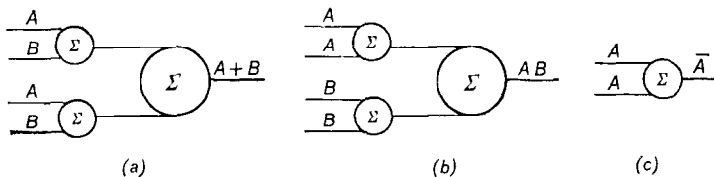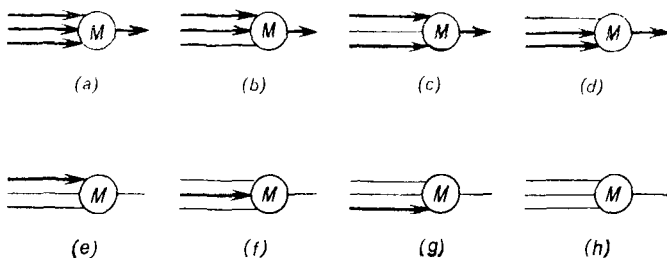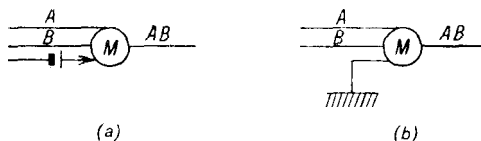
Fig. 46

Fig. 47

Fig. 48

Fig. 49

every section of the circuit there correspond two conductors through one of which the current flows permanently. In Fig. 47,*a-c* the sum $A + B$ and the product AB of two circuits A and B are shown and also the scheme of the circuit $\overline{A}$ corresponding to the circuit A constructed with the aid of the "Sheffer component" Σ (cf. pages 82-85).

An analogous role can also be played by a component M with three inputs and one output such that *the output current can flow only when at least two of the three inputs of the element M are supplied with electric current* (see the scheme in

98

Fig. 48; the element M corresponds to the operation

$$\{ABC\} = AB + BC + CA = (A + B)(B + C)(C + A)$$

of the Boolean algebra, cf. page 83). In Fig. 49a and b we see how the "addition" and the "multiplication" of two circuits A and B can be realized by means of the element M.

Exercises

1. Draw switching circuits corresponding to the following composite propositions:

(a) $(a + b)(c + d)$

(b) $abc + a\overline{b} + \overline{a}$

(c) $ab\overline{c} + a\overline{b}c + \overline{a}bc$

(d) $(a + b)(\overline{a} + \overline{b}) + ab + \overline{a}\overline{b}$

2. Sketch switching circuits corresponding to the propositions

$$(a + c)(b + c)(a + d)(b + d) \quad \text{and} \quad ab + cd$$

and check the "equality" of these circuits.

3*. Design an electric circuit E containing switches A, B, C and D (and also perhaps the switches $\overline{A}$, $\overline{B}$, $\overline{C}$ and $\overline{D}$) such that

(a) the circuit E is closed only when *all* the switches A, B, C and D are closed or *none* of the switches is closed;

(b) the circuit E is closed only in the case when *some but not all* of the switches A, B, C and D are closed.

4. (a) A committee consists of three members. Design an electric circuit showing the results of voting: every member of the committee presses a button when he agrees with the proposal under discussion; the bulb in the circuit must only be switched when the majority votes for the proposal.

(b) Design an analogous circuit for a committee consisting of a chairman and five members. It is required that the lamp should give light only in the case when the majority votes for the proposal or when the numbers of the votes for the proposal and against it are equal and the chairman votes for the proposal.

5*. Design an electric circuit with a lamp making it possible to switch on and to switch off the light by means of

(a) three independent switches (cf. Example 1 on page 94);

(b) n independent switches.

6. For the conditions of Example 2 on page 96 design an electric circuit controlling the upward motion of the lift.

<p style="text-align:center">*   *</p>

8. Normed Boolean Algebras

The material of Secs. 1-6 of this small book does not exhaust the extensive theory of Boolean algebras. The notion of a Boolean algebra can be generalized in various ways. In this section we shall discuss one notion which is directly related to the notion of a Boolean algebra and has very many applications[1]).

A Boolean algebra consisting of elements α, β, γ, . . . etc. and containing a "zero" element o and a "unit" element ι is called a *normed Boolean algebra* if to every element α is assigned its "norm" ("absolute value") $|\alpha|$ which is a non-negative number satisfying the following two conditions[2]):

(1) $0 \leqslant |\alpha| \leqslant 1$; $|o| = 0$; $|\iota| = 1$;

and

(2)   if $\alpha\beta = o$, then $|\alpha + \beta| = |\alpha| + |\beta|$.

Examples.

1°. The "algebra of two elements" (see page 25) consists of two "numbers" 0 and 1. These numbers can be taken as the norms of the corresponding elements:

$$|0| = 0, \quad |1| = 1$$

---

[1]) One of the most important applications lies in the foundation of the so-called *probability theory* on which, unfortunately, we cannot dwell in the present small book.

[2]) From the fact that $\alpha o = o$ and $\alpha + o = \alpha$ for any element α of the Boolean algebra and from condition (2) it follows that

$$|\alpha| = |\alpha + o| = |\alpha| + |o|$$

whence

$$|o| = 0$$

Thus, using property (2) we can prove that $|o| = 0$ and, consequently, the equality $|o| = 0$ must not necessarily be included into the list of conditions defining a norm. (Similarly, the equality $|\iota| = 1$ specifying the "unit norm" is not very important either; however, the conditions $|\alpha| \geqslant 0$ and $|\iota| > 0$ are essential.)

Then condition (1) of the definition of the norm of an element is obviously fulfilled. Further, since

$$0 \cdot 0 = 0 \quad \text{and} \quad |\,0 + 0\,| = 0 = |\,0\,| + |\,0\,|$$
$$0 \cdot 1 = 0 \quad \text{and} \quad |\,0 + 1\,| = 1 = |\,0\,| + |\,1\,|$$

condition (2) also holds. (Generally, for any Boolean algebra in which condition (1) holds condition (2) also always holds if at least one of the elements α and β coincides with o because in this case $\alpha o = o$ and $|\,\alpha + o\,| = |\,\alpha\,| = = |\,\alpha\,| + 0 = |\,\alpha\,| + |\,o\,|$.) Hence, with this definition of the norm of an element the Boolean algebra of two elements 0 and 1 becomes a normed Boolean algebra.

2°. For the "algebra of four elements" considered in Example 2 on page 27 we have $pq = 0$ and $p + q = 1$; therefore, in order to satisfy conditions (1) and (2), we must put

$$|\,0\,| = 0, \quad |\,1\,| = 1 \quad \text{and} \quad |\,p\,| + |\,q\,| = |\,1\,| = 1$$

Let the "numbers" (elements) p and q entering into the definition of this Boolean algebra be two arbitrary positive numbers *whose sum is equal to unity* and let

$$|\,1\,| = 1, \quad |\,0\,| = 0, \quad |\,p\,| = p \quad \text{and} \quad |\,q\,| = q$$

Then condition (1) will be fulfilled. Condition (2) will also be fulfilled because the only pair of nonzero elements of this Boolean algebra whose product is equal to zero is the pair of the elements p and q, and we have

$$|\,p + q\,| = |\,1\,| = 1 = p + q = |\,p\,| + |\,q\,|$$

Thus, this Boolean algebra of four elements with the absolute values (norms) of the elements we have defined becomes a normed Boolean algebra.

3°. Now we shall consider an example which elucidates the essence of the notion of a normed Boolean algebra itself. Let the Boolean algebra under consideration be the algebra of sets which we considered in Sec. 1; we shall also assume that the universal set (we denote it J here) is finite, for instance, let J contain N elements. Let us define the norm of any subset A of the universal set J as a number *proportional to the numbers k of elements contained in A*; for the condition $|\,J\,| = 1$ to be fulfilled the proportionality factor must obviously be $1/N$ so that $|\,A\,|$ is the ratio of the number of elements contained in A to the number of

elements in the universal set J:

$$|A| = \frac{k}{N}$$

Then condition (1) is fulfilled. Condition (2) is also fulfilled and its meaning is quite clear: if two sets A and B do not intersect (that is $AB = O$) then the number of elements contained in their sum can simply be obtained by adding together the number k of elements in the set A and the number l of elements in the set B whence it follows that

$$|A+B| = \frac{k+l}{N} = \frac{k}{N} + \frac{l}{N} = |A| + |B| \ (AB = O)$$

Thus, the Boolean algebra in question with the norm of the elements we have defined is a normed Boolean algebra.

The above definition of the norm $|A|$ of a subset A of the universal set J admits of further generalization. Suppose that different elements $a_1, a_2, \ldots, a_N$ of the set J are assumed to have different "weights" (different "prices"). For instance, if J is the set of all chessmen it often turns out that it is natural to consider different chessmen as having different "prices". When we try to "teach" an electronic computer to play chess we usually assume that a bishop or a knight "costs" approximately 3 times as much as a pawn, the rook "costs" 4 or 5 times as much as a pawn, the "price" of the queen is 8 or 9 times that of a pawn while the king "costs" much more than a pawn, say, its "price" is 1000 times that of a pawn. Let the "weights" ("prices") of different elements $a_1, a_2, \ldots, a_N$ of the set J be equal to some nonnegative numbers $t_1, t_2, \ldots, t_N$ respectively; it is also convenient to choose "unit price" so that

$$t_1 + t_2 + \ldots + t_N = 1$$

Now let us put

$$|A| = |\{a_{i_1}, a_{i_2}, \ldots, a_{i_n}\}| = t_{i_1} + t_{i_2} + \ldots + t_{i_n}$$

where $i_1, i_2, \ldots, i_N$ are some of the numbers $1, 2, \ldots, N$ chosen quite arbitrarily ($i_1, i_2, \ldots, i_n$ are of course pairwise different); then the algebra of the subsets of the set J becomes a normed Boolean algebra.

(If we put

$$t_1 = t_2 = \ldots = t_N = \frac{1}{N}$$

then the new generalized definition of the norm | A | of a subset A reduces to the definition in the example above.)

4°. The next example is in many respects analogous to the previous one. As before, let us assume that the Boolean algebra under consideration is the algebra of subsets of a set J. But now we choose, for instance, as the universal set J a unit square so that the various subsets A of the set J are some geometrical figures lying within the square J. By the norm (absolute value) of a set A we shall mean the *area* of the figure A. It is quite clear that, under this definition, condition (1) of the general definition of the norm will be satisfied. Condition (2) will also hold: in this case it simply means that if a figure C is split into non-intersecting parts A and B (that is $AB = O$) then its area is equal to the sum of the areas of the figures A and B. We see that the conditions imposed on the norm have simple meaning in this case and they are similar to the conditions which *define* the notion of the area of a geometrical figure. The algebra of subsets of the square J supplied with the norm thus defined becomes a normed Boolean algebra. Of course, almost nothing changes if the role of the universal set J is played not by a unit square but by some other geometrical figure with area S. In the latter case the norm of a figure A should be defined as its area divided by the number S, that is as the "relative input" of the figure A into the whole area of J. Similarly, if we take as J a *three-dimensional* solid then it is natural to define the norm of its subset A as the volume of A (divided, when necessary, by the volume of the whole solid J).

*

This example also admits of an important generalization. Suppose that the solid J is a thin plate of uniform thickness made of an arbitrary nonhomogeneous material. Let the specific weight of the material at a point $M = (x, y)$ (that is the weight per unit area) be determined by a (nonnegative!) function $f(M) = f(x, y)$. As the norm | A | of a subset A let us take the *weight* of the part A of the plate J; this weight is computed by means of the integral

$$|A| = \int_A f(M)\, d\sigma = \int\int_A f(x, y)\, dx\, dy$$

where $d\sigma$ is the (infinitesimal) element of area of the plate adjoining (or containing) the point M. The "unit weight" should be chosen so that the weight of the whole plate J is equal to unity, that is

$$\int\limits_J f(M)\, d\sigma = \int\int\limits_J f(x,\ y)\, dx\, dy = 1$$

It is easy to understand that the introduction of the norm defined in this way (with the aid of an arbitrary nonnegative function $f(x,\ y)$ satisfying only one "normalization condition" written above) transforms the Boolean algebra of figures A into a normed Boolean algebra.

In the same manner we can also construct a normed Boolean algebra whose elements are arbitrary domains contained in a given three-dimensional solid J assuming that the solid is made of a non-homogeneous material and that the norm of a domain lying within J is equal to its weight.

5°. Let us consider the Boolean algebra whose elements are the various divisors of a positive integer N for which the "sum" and the "product" of numbers are defined, respectively, as their least common multiple and their greatest common divisor (see Example 4 on page 31). In this case we can define the norm $|\,a\,|$ of a number a as the *logarithm* of that number or, more precisely, as the ratio $\log a/\log N$ because it is required that the norm of the number N (which plays the role of the element ι of the Boolean algebra) should be equal to unity[1]). Indeed, it is obvious that condition (1) is fulfilled in this case. Further, if the condition $a \otimes b = (a,\ b) = 1$ is fulfilled for some numbers a and b (the role of the element o of the Boolean algebra in question is played by the number 1!) means that the two given numbers a and b are mutually prime; in this case we have

$$a \oplus b = [a,\ b] = ab$$

---

[1]) The choice of the base of logarithms is not important here because the ratio $\log a/\log N$ is independent of the base. Indeed, the change of the base of logarithms from b to c simply reduces to the multiplication of all the logarithms by the constant factor $\log_c b$ (by the modulus of the former system of logarithms to base b with respect to the latter system of logarithms to base c):

$$\log_c m = \log_c b \cdot \log_b m$$

Instead of the equality $|\,a\,| = \log a/\log N$ we can also write $|\,a\,| = \log_N a$ (because $\log_N a = \log_n a/\log_n N$ for any n).

and, consequently,

$$\log (a \oplus b) = \log (ab) = \log a + \log b$$

that is

$$| a \oplus b | = | a | + | b |$$

We have thus proved that condition (2) entering into the definition of a normed Boolean algebra is also fulfilled here. Hence, we have a normed Boolean algebra in this example.

* *

6°. Let us assume that the elements of the Boolean algebra are all real numbers x such that $0 \leqslant x \leqslant 1$; let the operations be defined as $a \oplus b = \max [a, b]$, $a \otimes b = \min [a, b]$ and $\bar{a} = 1 - a$ (besides, $o = 0$, $\iota = 1$ and let $a \supset b$ when $a \geqslant b$; see Example 3 on page 28). As can readily be seen, this Boolean algebra also becomes a normed algebra if we put $| a | = a$. Then condition (1) of the definition of the norm of an element of a Boolean algebra is obviously fulfilled. As to condition (2), it follows from the fact that here we have $a \otimes b = 0$ only when one of the elements a and b of the Boolean algebra coincides with 0; if, for instance, $b = 0$ then obviously $a \oplus b = a$ and $| a \oplus b | = | a | = = | a | + 0 = | a | + | b |$.

7°. Another interesting example of a normed Boolean algebra can be obtained if we introduce a norm into the algebra of electric switching circuits (see Sec. 7). By definition, let the norm $| A |$ of a circuit section A be equal to 1 when this section permits passage of current for the given states of all switches it contains and let $| A | = 0$ when the current does not flow through the section A. Then, naturally, condition (1) of the definition of a normed Boolean algebra is fulfilled because all the possible values of the norm are equal to 0 or 1, the norm of the section O which plays the role of the element o is equal to 0 (this circuit section never permits passage of current) and the norm of the section I which plays the role of the element ι is equal to 1. Further, the equality $AB = O$ means that at least one of the sections A and B, say A, does not permit passage of current; therefore when $AB = O$ the circuit $A + B$ permits passage of current when the other circuit (B) conducts current and does not conduct current if otherwise,

whence it follows that for such circuits A and B we have $|A + B| = |A| + |B|$.

8°. Let us consider an example which is very similar to the example of the normed Boolean algebra considered above. Let us introduce the norm $|p|$ of a proposition p in the algebra of propositions by putting $|p| = 1$ when the proposition p is true and $|p| = 0$ when p is false. Here we also have $0 \leqslant |p| \leqslant 1$ (or, more precisely, $|p| = 0$ or $|p| = 1$) and $|o| = 0$, $|\iota| = 1$. Further, the relation $pq = 0$ means that the proposition "p and q" is false, that is at least one of the two propositions p and q, say p, is false. Therefore, in case $pq = O$ the proposition $p + q$ (that is the proposition "p or q") is true if and only if the other proposition (q) is true. Whence it readily follows that in this case $|p + q| = |p| + |q|$.

The most important normed Boolean algebras we have considered are of course those in Examples 7° and 8°. The normalization condition for the elements of the algebra of propositions assigns to every proposition one of the two numbers 0 and 1 which is the *truth value* of the proposition. All the operations of propositional algebra can be characterized by the indication of the truth values of the composite propositions obtained from the constituent (prime) propositions by means of these operations depending on the truth values of the constituent propositions. The sum (disjunction) $p + q$ of two propositions p and q is characterized by the condition that $p + q$ is true if and only if at least one of the propositions p and q is true while the product (conjunction) pq of these same propositions is true if and only if both propositions p and q are true. Hence, the operations $p + q$ and pq can be described by means of the following "truth tables":

$|p|$	$|q|$	$|p+q|$	$|pq|$
1	1	1	1
1	0	1	0
0	1	1	0
0	0	0	0

In just the same way we can compile the truth table corresponding to any other composite proposition (cf. Exercises 1-2 below); the truth table corresponding to the ope-

ration of forming negation is particularly simple:

$|p|$	$\cdot|\bar{p}|$
1	0
0	1

It is evident that such truth tables completely characterize the propositions to which they correspond. Analogously, the norm of an electric circuit (see Example 7°) characterizes the *conductivity* of the circuit which is its only important characteristic: this measure of conductivity is equal to 1 or 0 depending on whether the current flows or does not flow through this circuit.

Here we shall not dwell in more detail on the truth tables for propositions and the values characterizing the conductivity of switching circuits (for these questions see bibliography at the end of the book). Let us consider some other examples of the application of normed Boolean algebras to elementary mathematical problems.

From properties (1) and (2) of the norm (the absolute value) of an element of a Boolean algebra we can derive some further properties of the norm. First of all it follows that if $\alpha\bar{\alpha} = o$ and $\alpha + \bar{\alpha} = \iota$ then

$$|\alpha| + |\bar{\alpha}| = |\alpha + \bar{\alpha}| = |\iota| = 1$$

Consequently, we see that

$$|\bar{\alpha}| = 1 - |\alpha| \quad \text{for all } \alpha$$

Further, for any two elements α and β of a Boolean algebra for which the relation $\alpha \supset \beta$ holds there exists an element ξ (the "difference" between the elements α and β) such that

$$\alpha = \beta + \xi \quad \text{and} \quad \beta\xi = o$$

(see Exercise 3 below). It follows that

$$|\alpha| = |\beta + \xi| = |\beta| + |\xi| \geqslant |\beta|$$

or, in other words,

$$\text{if} \quad \alpha \supset \beta, \quad \text{then} \quad |\alpha| \geqslant |\beta|$$

The existence of the "difference" of two elements α and β for which $\alpha \supset \beta$ also implies that for any two elements
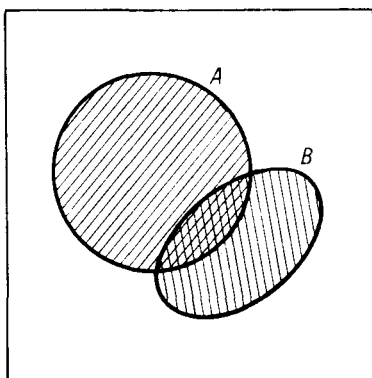
Fig. 50

α and β of a Boolean algebra we have

$$\alpha + \beta = \alpha + (\beta - \alpha\beta)$$

where the element $\beta - \alpha\beta$ (the "difference" between β and $\alpha\beta$) possesses the property that $\alpha\cdot(\beta - \alpha\beta) = o$ and, besides,

$$\beta = \alpha\beta + (\beta - \alpha\beta)$$

where $\alpha\beta\cdot(\beta - \alpha\beta) = o$ whence, by virtue of condition (2), we obtain

$$|\alpha + \beta| = |\alpha| + |\beta - \alpha\beta| \quad \text{and}$$

$$|\beta| = |\alpha\beta| + |\beta - \alpha\beta|$$

On subtracting the second of the last equalities from the first one (these are number equalities) and transposing the term $|\beta|$ to the right, we obtain the relation

$$|\alpha + \beta| = |\alpha| + |\beta| - |\alpha\beta| \qquad \text{(A)}$$

For instance, let $|A|$ and $|B|$ be the areas of two geometrical figures A and B (see Example 4 above); then the sum $|A| + |B|$ involves twice the area of the intersection AB (see Fig. 50) and therefore

$$|A + B| = |A| + |B| - |AB|$$

Example. *Let there be a group of 22 students among whom 10 students are chess-players, 8 students can play draughts and 3 can play both chess and draughts. How many students can play neither chess nor draughts?*

Let us denote the set of the students who can play chess by the symbol Ch and the set of the students who can play draughts by the symbol Dr. Let us also define the norm in the algebra of the sets of the students in the group as it was done for a finite universal set in the Example $3°$. We have to determine the number of students in the set

$$\overline{Ch \cdot Dr} = \overline{Ch + Dr}$$

(see the corresponding De Morgan formula). We have (cf. page 102)

$$|Ch| = \frac{10}{22}, \quad |Dr| = \frac{8}{22} \text{ and } |Ch \cdot Dr| = \frac{3}{22}$$

whence, by virtue of formula (A), we obtain

$$|Ch + Dr| = |Ch| + |Dr| - |Ch \cdot Dr| = \frac{10}{22} + \frac{8}{22} - \frac{3}{22} = \frac{15}{22}$$

Consequently,

$$|\overline{Ch \cdot Dr}| = |\overline{Ch + Dr}| = 1 - |Ch + Dr| = 1 - \frac{15}{22} = \frac{7}{22}$$

Thus, *there are 7 students in the group who cannot play either chess or draughts.*

It is clear that equality (A) is a generalization of property (2) of the norm. For $\alpha\beta = o$ it goes into property (2). From (A) it also follows that *for any two elements α and β of a normed Boolean algebra we have*

$$|\alpha + \beta| \leqslant |\alpha| + |\beta| \qquad \text{(B)}$$

This property of the absolute values (norms) of the elements of a normed Boolean algebra is analogous to the well-known property of the absolute values of numbers.

<p style="text-align:center">*</p>

We also note that equality (A) admits of a further generalization. Let us consider three arbitrary elements α_1, α_2 and α_3 of a normed Boolean algebra. By virtue of (A), we have

$$|\alpha_1 + \alpha_2 + \alpha_3| = |\alpha_1 + (\alpha_2 + \alpha_3)| = |\alpha_1| + |\alpha_2 + \alpha_3| - |\alpha_1(\alpha_2 + \alpha_3)| =$$
$$= |\alpha_1| + (|\alpha_2| + |\alpha_3| - |\alpha_2\alpha_3|) - |\alpha_1\alpha_2 + \alpha_1\alpha_3| =$$
$$= |\alpha_1| + |\alpha_2| + |\alpha_3| - |\alpha_2\alpha_3| - (|\alpha_1\alpha_2| + |\alpha_1\alpha_3| - |\alpha_1\alpha_2\alpha_3|) =$$
$$= |\alpha_1| + |\alpha_2| + |\alpha_3| - |\alpha_1\alpha_2| - |\alpha_1\alpha_3| - |\alpha_2\alpha_3| + |\alpha_1\alpha_2\alpha_3|$$

because, obviously $(\alpha_1\alpha_2)\cdot(\alpha_1\alpha_3) = \alpha_1\alpha_2\alpha_3$. We can similarly express the norm of a sum of four elements:

$$|\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4| = |\alpha_1 + (\alpha_2 + \alpha_3 + \alpha_4)| =$$
$$= |\alpha_1| + |\alpha_2 + \alpha_3 + \alpha_4| - |\alpha_1(\alpha_2 + \alpha_3 + \alpha_4)| =$$
$$= |\alpha_1| + (|\alpha_2| + |\alpha_3| + |\alpha_4| - |\alpha_2\alpha_3| - |\alpha_2\alpha_4| - |\alpha_3\alpha_4| +$$
$$+ |\alpha_2\alpha_3\alpha_4|) - |\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4| =$$
$$= |\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4| - |\alpha_2\alpha_3| - |\alpha_2\alpha_4| - |\alpha_3\alpha_4| +$$
$$+ |\alpha_2\alpha_3\alpha_4| - (|\alpha_1\alpha_2| + |\alpha_1\alpha_3| + |\alpha_1\alpha_4| - |\alpha_1\alpha_2\alpha_3| -$$
$$- |\alpha_1\alpha_2\alpha_4| - |\alpha_1\alpha_3\alpha_4| + |\alpha_1\alpha_2\alpha_3\alpha_4|) =$$
$$= |\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4| - |\alpha_1\alpha_2| - |\alpha_1\alpha_3| - |\alpha_1\alpha_4| -$$
$$- |\alpha_2\alpha_3| - |\alpha_2\alpha_4| - |\alpha_3\alpha_4| + |\alpha_1\alpha_2\alpha_3| + |\alpha_1\alpha_2\alpha_4| +$$
$$+ |\alpha_1\alpha_3\alpha_4| + |\alpha_2\alpha_3\alpha_4| - |\alpha_1\alpha_2\alpha_3\alpha_4|$$

Generally,

$$|\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n| = \sum_i |\alpha_i| - \sum_{(i_1, i_2)} |\alpha_{i_1}\alpha_{i_2}| +$$
$$+ \sum_{(i_1, i_2, i_3)} |\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}| - \ldots +$$
$$+ (-1)^{n-2} \sum_{(i_1, i_2, \ldots, i_{n-1})} |\alpha_{i_1}\alpha_{i_2} \ldots \alpha_{i_{n-1}}| +$$
$$+ (-1)^{n-1} |\alpha_1\alpha_2\alpha_3 \ldots \alpha_n| \qquad (A')$$

where the symbol $(i_1, i_2, \ldots, i_k)$ under the summation sign indicates that the summation extends over all the possible combinations of the pairwise different indices $i_1, i_2, \ldots$ $\ldots, i_k$ each of which can be equal to 1, 2, 3, $\ldots, n$; in the case under consideration we have $k = 1, 2, \ldots, n$ and the sum $\sum_{(i_1, i_2, \ldots, i_n)}$ contains only one term (because of which the last term in the expression on the right-hand side does not involve the summation sign $\sum$). Let the reader prove formula (A') by induction.

In the same manner, we can derive from formula (B) the relation

$$|\alpha_1 + \alpha_2 + \ldots + \alpha_n| \leqslant |\alpha_1| + |\alpha_2| + \ldots + |\alpha_n| \tag{B'}$$

which can readily be proved by induction.

Below are some examples demonstrating various applications of formula (A′).

Example 1. *It is known that in a group of people*
60% *of the people speak English;*
30% *of the people speak French;*
20% *of the people speak German;*
15% *of the people speak English and French;*
5% *of the people speak English and German;*
2% *of the people speak French and German;*
1% *of the people speak all the three languages.*

What is the percentage of the people who can speak none of the three languages?

Solution. Let us denote the sets of the people speaking English, speaking French and speaking German as e, f and g respectively. Then, for instance, ef is the set of the people speaking both English and French. Let us consider the algebra of the sets of the people in the group as a normed Boolean algebra in which the norm is introduced as in the first Example 3°. By virtue of formula (A′), we have

$$|\,e + f + g\,| = |\,e\,| + |\,f\,| + |\,g\,| - |\,ef\,| - |\,eg\,| -$$
$$- |\,fg\,| + |\,efg\,| = 0.6 + 0.3 + 0.2 -$$
$$- 0.15 - 0.05 - 0.02 + 0.01 = 0.89$$

and consequently the percentage of the people who speak none of the three languages is equal to[1])

$$|\,\overline{efg}\,| = |\,\overline{e + f + g}\,| = 1 - |\,e + f + g\,| = 1 - 0.89 = 0.11 = 11\%$$

Thus, 11% *of the people speak none of the three languages.*

Example 2. [2]). *There are 25 pupils in a class among whom 17 pupils are cyclists, 13 pupils are swimmers and 8 pupils are skiers. None of the pupils is good at all the three kinds*

---

[1]) In this solution we use the following fact which can easily be proved by induction: for any number k of elements $\alpha_1, \alpha_2, \ldots, \alpha_k$ of a Boolean algebra there holds the relation

$$\overline{\alpha_1} \cdot \overline{\alpha_2} \cdot \ldots \cdot \overline{\alpha_k} = \overline{\alpha_1 + \alpha_2 + \ldots + \alpha_k}$$

For $k = 2$ this relation goes into the De Morgan formula $\overline{\alpha_1 \alpha_2} = \overline{\alpha_1} + \overline{\alpha_2}$. In particular, we use the fact that $\overline{efg} = \overline{e + f + g}$.

[2]) The idea and the solution of this problem are adopted from the book: H. Steinhaus, *One Hundred Problems in Elementary Mathematics*, Pergamon Press, Oxford, 1963.

of sport. The cyclists, the swimmers and the skiers have satisfactory marks in mathematics (let us agree that the progress of the pupils is appraised with the three marks "good", "satisfactory" and "bad") and it is known that 6 *pupils in the class have bad marks in mathematics. How many pupils in the class have good marks in mathematics and how many swimmers can ski?*

Solution. Let us denote the sets of the cyclists, swimmers and skiers by the symbols Cy, Sw and Sk respectively. Further, let G denote the set of the pupils who have only good marks in mathematics, S denote the set of the pupils having satisfactory marks and, finally, let B be the set of the pupils having bad marks in mathematics. Then the conditions of the problem can be written in the form

(1) $|Cy| = \dfrac{17}{25}$, $|Sw| = \dfrac{13}{25}$, $|Sk| = \dfrac{8}{25}$

(2) $|Cy\ Sw\ Sk| = 0$

(3) $|S\ Cy| = |Cy|$, $|S\ Sw| = |Sw|$, $|S\ Sk| = |Sk|$

and

(4) $|B| = \dfrac{6}{25}$

and it is required to determine the values of norms $|G|$ and $|Sw\ Sk|$.

It is obvious that

$$|G| + |S| + |B| = |G + S + B| = |J| = 1$$

and, consequently,

$$|G| + |S| = 1 - |B| = 1 - \frac{6}{25} = \frac{19}{25}$$

Let us denote by S_0, S_1 and S_2 the following sets of the pupils having satisfactory marks in mathematics: the set of the pupils who cannot cycle, swim and ski, the set of the pupils who are good at one of these kinds of sport and the set of the pupils who are good at two of the kinds of sport (remember that none of the pupils is good at all the three kinds of sport!). Then we can write

$$|S| = |S_0| + |S_1| + |S_2|$$

Consequently

$$|G| + |S_0| + |S_1| + |S_2| = \frac{19}{25} \tag{a}$$

But we have

$$|S_1| + |S_2| = |S\ Cy + S\ Sw + S\ Sk| =$$
$$= |S\ Cy| + |S\ Sw| + |S\ Sk| - |S\ Cy\ Sw| -$$
$$- |S\ Cy\ Sk| - |S\ Sw\ Sk| =$$
$$= |Cy| + |Sw| + |Sk| - |S\ Cy\ Sw| -$$
$$- |S\ Cy\ Sk| - |S\ Sw\ Sk| =$$
$$= \frac{17}{25} + \frac{13}{25} + \frac{8}{25} - |S_2| = \frac{38}{25} - |S_2|$$

(here we have again used the fact that none of the pupils is good at all the three kinds of sport). It follows that

$$|S_1| + 2|S_2| = \frac{38}{25} \tag{b}$$

Now let us duplicate equality (a) and subtract equality (b) from the result. This yields

$$2|G| + 2|S_0| + |S_1| = 0$$

Since *the sum of three nonnegative numbers can only be equal to zero when each of the numbers is equal to zero* we thus obtain

$$|G| = 0, \quad |S_0| = 0 \quad \text{and} \quad |S_1| = 0$$

Further, note that the condition $|S_1| = 0$ means that each of the pupils who is good at least at one of the kinds of sport is also good at one more kind of sport. Taking this into account we arrive at the following system of equations:

$$\left.\begin{array}{l} \frac{17}{25} = |Cy| = |Cy\ Sw + Cy\ Sk| = |Cy\ Sw| + |Cy\ Sk| \\[2mm] \frac{13}{25} = |Sw| = |Cy\ Sw + Sw\ Sk| = |Cy\ Sw| + |Sw\ Sk| \\[2mm] \frac{8}{25} = |Sk| = |Cy\ Sk + Sw\ Sk| = |Cy\ Sk| + |Sw\ Sk| \end{array}\right\}$$

from which we find

$$|Cy\ Sw| = \frac{11}{25}, \quad |Cy\ Sk| = \frac{6}{25} \quad \text{and} \quad |Sw\ Sk| = \frac{2}{25}$$

Thus, *the number of the pupils having good marks in mathematics is equal to zero and the number of swimmers who can ski is equal to 2.*

Example 3. [1]). *A coat of area* 1 *has* 5 *patches the area of each of which is not less than* 1/2. *Prove that there are at least two patches which overlap so that their common part has an area not less than* 1/5.

Solution. Let us denote the patches (which are regarded as subsets of this coat J having unit area) as A_1, A_2, A_3, A_4 and A_5; their pairwise intersections will be denoted A_{12}, A_{13} etc. We know that $|A_i| \geqslant \frac{1}{2}$ where $i = 1, 2, 3, 4, 5$ (see Example 4° on page 103); it is necessary to estimate the quantities $|A_{ij}|$. According to formula (A′) we can write

$$1 = |J| \geqslant |A_1 + A_2 + A_3 + A_4 + A_5| =$$

$$= \sum_{i=1}^{5} |A_i| - \sum_{(i,j)} |A_{ij}| + \sum_{(i,j,k)} |A_{ijk}| -$$

$$- \sum_{(i,j,k,l)} |A_{ijkl}| + |A_{12345}|$$

whence

$$1 - \sum_i |A_i| + \sum_{(i,j)} |A_{ij}| - \sum_{(i,j,k)} |A_{ijk}| +$$

$$+ \sum_{(i,j,k,l)} |A_{ijkl}| - |A_{12345}| \geqslant 0 \qquad (1)$$

From the inequality we have obtained we now eliminate the term $\sum |A_{ijk}|$ and the following terms. To this end we use the same formula (A′) to obtain

$$|A_1| \geqslant |A_{12} + A_{13} + A_{14} + A_{15}| =$$

$$= \sum_{i=2}^{5} |A_{1i}| - \sum_{(i,j)} |A_{1ij}| + \sum_{(i,j,k)} |A_{1ijk}| - |A_{12345}|$$

where the summation indices i, j, k run over the values 2, 3, 4, 5. In just the same way we can write analogous inequalities for $|A_2|$, $|A_3|$, $|A_4|$ and $|A_5|$. On adding together all the inequalities thus obtained we get

$$|\sum_{i=1}^{5} |A_i| \geqslant 2 \sum_{(i,j)} |A_{ij}| - 3 \sum_{(i,j,k)} |A_{ijk}| +$$

$$+ 4 \sum_{(i,j,k,l)} |A_{ijkl}| - 5|A_{12345}$$

---

[1]) This example admits of an extensive generalization (see problems 59 and 60 in the book by D. O. Shklyarsky, I. M. Yaglom and N. N. Chentsov, *Geometrical Estimates and Problems in Combinatorial Geometry*, Moscow, "Nauka", 1974; in Russian).

whence

$$\sum_{i=1}^{5} |A_i| - 2 \sum_{(i,\, j)} |A_{ij}| + 3 \sum_{(i,\, j,\, k)} |A_{ijk}| - $$
$$- 4 \sum_{(i,\, j,\, k,\, l)} |A_{ijkl}| + 5 |A_{12345}| \geqslant 0 \qquad (2)$$

Let us multiply inequality (2) by $1/3$ and add the result to inequality (1). The inequality obtained in this way does not contain the term with $\sum |A_{ijk}|$:

$$1 - \frac{2}{3} \sum_{i} |A_i| + \frac{1}{3} \sum_{(i,\, j)} |A_{ij}| - \frac{1}{3} \sum_{(i,\, j,\, k,\, l)} |A_{ijkl}| + $$
$$+ \frac{2}{3} |A_{12345}| \geqslant 0 \qquad (3)$$

It is clear that each of the five quantities $|A_{1234}|$, $|A_{1235}|$, $|A_{1245}|$, $|A_{1345}|$ and $|A_{2345}|$ is not less than $|A_{12345}|$ and therefore

$$\sum_{(i,\, j,\, k,\, l)} |A_{ijkl}| \geqslant 5 |A_{12345}|$$

and

$$\frac{1}{3} \sum_{(i,\, j,\, k,\, l)} |A_{ijkl}| - \frac{2}{3} |A_{12345}| \geqslant |A_{12345}| \geqslant 0$$

Consequently, inequality (3) can also be rewritten as

$$1 - \frac{2}{3} \sum_{i} |A_i| + \frac{1}{3} \sum_{(i,\, j)} |A_{ij}| \geqslant 0$$

whence we derive the inequality

$$\sum_{(i,\, j)} |A_{ij}| \geqslant 2 \sum_{i} |A_i| - 3 \geqslant 2 \left(5 \cdot \frac{1}{2} \right) - 3 = 2$$

Now, since the number of the terms entering in the sum $\sum_{(i,j)} |A_{ij}|$ is equal to $\binom{5}{2} = 10$, we conclude that at least one of these terms is not less than

$$2 : 10 = \frac{1}{5}$$

which is what we intended to prove.

Exercises

1. Compile the truth tables for the following operations:

(a) the implication $\Rightarrow$;

(b) the biconditional proposition $\Leftrightarrow$;

(c) the Sheffer operation $|$; the operation $\downarrow$ (see page 83);

(d) the operation $\{\ \ \}$ (see page 83).

2. Compile the truth tables for the following compound propositions:

(a) $p\bar{q} + \bar{p} + q$;

(b) $(p + q)\,(\bar{p} + \bar{q})$;

(c) $pq + \bar{r}$

3*. Prove that if α and β are elements of a Boolean algebra such that $\alpha \supset \beta$ then there exists the "difference" ξ of the elements α and β, that is an element of the Boolean algebra such that $\beta + \xi = \alpha$ and $\beta\xi = 0$.

4*. Let $N = p_1^{k_1} p_2^{k_2} \ldots p_n^{k_n}$ be a positive integer where $p_1, p_2, \ldots, p_n$ are the prime factors of the integer. How can we determine the number $\varphi\,(N)$ of the positive integers which are less than N and relatively prime to N? (The function $\varphi\,(N)$ of the positive integral argument N is called *Euler's function*.)

5*. Let $a_1, a_2, \ldots, a_k$ be arbitrary positive integers. Let the numbers themselves and the greatest common divisors of any combinations of these numbers be known. It is required to find the least common multiple $[a_1 a_2 \ldots a_k]$ of the numbers $a_1, a_2, \ldots, a_k$. Apply the formula you obtain to the case $k = 4$, $a_1 = 10$, $a_2 = 12$, $a_3 = 30$, $a_4 = 45$.

6. Let a, b, c and d be 4 arbitrary numbers. Prove that

$$\max\,[a,\ b,\ c,\ d] = a + b + c + d - \min\,[a,\ b] -$$

$$- \min\,[a,\ c] - \ldots - \min\,[c,\ d] +$$

$$+ \min\,[a,\ b,\ c] + \min\,[a,\ b,\ d] + \min\,[a,\ c,\ d] +$$

$$+ \min\,[b,\ c,\ d] - \min\,[a,\ b,\ c,\ d]$$

Appendix

Definition of a Boolean Algebra

A **Boolean algebra** is an arbitrary set of elements α, β, γ, . . ., for which two operations called *addition* and *multiplication* are defined which associate with any two elements α and β their *sum* $\alpha + \beta$ and their *product* $\alpha\beta$[1]) and for which the "bar" operation is defined which associates with any element α a new element $\bar{\alpha}$[2]). It is also required that two "special" elements o and ι should exist and that the following rules should hold:

Rules for addition	Rules for multiplication
(1) $\alpha + \beta = \beta + \alpha$	(1a) $\alpha\beta = \beta\alpha$

<div align="center">the commutative laws</div>

(2) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$	(2a) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$

<div align="center">the associative laws</div>

(3) $\alpha + \alpha = \alpha$	(3a) $\alpha\alpha = \alpha$

<div align="center">the idempotent laws</div>

<div align="center">Rules connecting addition and multiplication</div>

(4) $(\alpha + \beta)\gamma = \alpha\gamma + \beta\gamma$	(4a) $\alpha\beta + \gamma = (\alpha + \gamma)(\beta + \gamma)$

<div align="center">the distributive laws</div>

<div align="center">Rules concerning the elements o and ι</div>

(5) $\alpha + o = \alpha$	(5a) $\alpha\iota = \alpha$
(6) $\alpha + \iota = \iota$	(6a) $\alpha o = o$

<div align="center">Rules concerning the "bar" operation</div>

<div align="center">(7) $\bar{\bar{\alpha}} = \alpha$</div>

(8) $\bar{o} = \iota$	(8a) $\bar{\iota} = o$

[1]) Cf. page 25.
[2]) As was mentioned, addition (multiplication) is a *binary* operation; it associates a new element $\alpha + \beta$ ($\alpha\beta$) with every two elements α and β of the Boolean algebra. In a Boolean algebra a *unary* operation (the "bar" operation) is also defined which associates a new element $\bar{\alpha}$ with one element α of the Boolean algebra.

Rules connecting the "bar" operation
with addition and multiplication

(9) $\overline{\alpha + \beta} = \overline{\alpha}\,\overline{\beta}$ (9a) $\overline{\alpha\beta} = \overline{\alpha} + \overline{\beta}$

<div align="center">the De Morgan formulas</div>

It is unnecessary to include into the definition of a Boolean algebra the requirement that the inclusion relation $\supset$ should exist for some pairs of elements α and β; the matter is that the inclusion relation $\alpha \supset \beta$ can be *defined* by means of any of the two conditions $\alpha + \beta = \alpha$ and $\alpha\beta = \beta$ from each of which all the properties of the relation $\supset$ can be derived, namely:

$$\alpha \supset \alpha$$
$$\text{if } \alpha \supset \beta \quad \text{and} \quad \beta \supset \alpha \quad \text{then} \quad \alpha = \beta$$
$$\text{if } \alpha \supset \beta \quad \text{and} \quad \beta \supset \gamma \quad \text{then} \quad \alpha \supset \gamma$$
$$\iota \supset \alpha \quad \text{and} \quad \alpha \supset o$$
$$\alpha + \beta \supset \alpha \quad \text{and} \quad \alpha \supset \alpha\beta$$
$$\text{if } \alpha \supset \beta \quad \text{then} \quad \overline{\beta} \supset \overline{\alpha}$$

(let the reader derive them). Moreover, it is even unnecessary to require in the definition of a Boolean algebra that one of the two operations of addition and multiplication should exist, that is it suffices to require that only one of these operations and also the "bar" operation should exist. For instance, if the operation of addition and the "bar" operation exist, we can *define* the multiplication using the corresponding De Morgan rule:

$$\alpha\beta = \overline{\overline{\alpha} + \overline{\beta}}$$

However, if only the operations of addition and multiplication exist (but the "bar" operation is not defined) they do not specify a Boolean algebra.

The above definition of a Boolean algebra is highly "non-economical" in the sense that many of the properties we have enumerated can be derived from some other properties and it is therefore unnecessary to require that the former should be fulfilled. On this question see, for instance, books [1], [2] and [5] (see the bibliography). We also note that, in contradistinction to the definition of a Boolean algebra we use in this book, most of the books and scientific papers

add two more conditions connecting the "bar" operation with addition and multiplication, namely

(10) $\alpha + \bar{\alpha} = \iota$ and (10a) $\alpha\bar{\alpha} = o$

(cf. page 53). If we include these conditions in the definition then the "algebra of maxima and minima" (see Example 3 on page 28) is a Boolean algebra in the single case when the set of numbers in question contains only two elements: 1 and 0; then this algebra coincides with the "algebra of two numbers" (page 25). With these two additional conditions included the "algebra of least common multiples and greatest common divisors" (Example 4 on page 31) is a Boolean algebra when the number N decomposes only into pairwise different prime factors. However, for this approach to the notion of a "Boolean algebra" Examples 3 and 4 in Sec. 2 can naturally be regarded as examples of "incomplete" (or "generalized") Boolean algebras because properties (10) and (10a) in which they differ from the "complete" Boolean algebras (for which these conditions hold) are not so very important.

Answers and Hints

Section 1

1. $(A + B)(A + C)(B + D)(C + D) = [(B + A) \times (C + A)] \cdot [(B + D)(C + D)] = (BC + A)(BC + D) = (A + BC)(D + BC) = AD + BC$ (here we use the second distributive law).

2. $A(A + B) = AA + AB = A + AB = AI + AB = A(I + B) = AI = A$.

5. $A(A + I)(B + O) = A \cdot I \cdot B = AB$.

6. $(A + B)(B + C)(C + A) = ABC + AB + AC + BC = ABC + ABI + AC + BC = AB(C + I) + AC + BC = ABI + AC + BC = AB + BC + CA$ (see the identity proved on page 23).

7. $[(A + B)(B + C)](C + D) = (AC + B)(C + D) = AC + ACD + BC + BD = AC + BC + BD$

10. $[(A + B + C)(B + C + D)](C + D + A) = [AD + (B + C)](C + D + A) = [(AD + B) + C] \times [(A + D) + C] = (AD + B)(A + D) + C = AD + AD + AB + BD + C = AB + AD + BD + C$

Section 2

3. (a)

+	O	I
O	O	I
I	I	I

and

·	O	I
O	O	O
I	O	I

(b)

+	O	P	Q	I
O	O	P	Q	I
P	P	P	I	I
Q	Q	I	Q	I
I	I	I	I	I

and

·	O	P	Q	I
O	O	O	O	O
P	O	P	O	P
Q	O	O	Q	Q
I	O	P	Q	I

6. $[m, n] = p_1^{\max [a_1, b_1]} p_2^{\max [a_2, b_2]} \ldots p_k^{\max [a_k, b_k]}$

$(m, n) = p^{\min [a_1, b_1]} p_2^{\min [a_2, b_2]} \ldots p_k^{\min [a_k, b_k]}$

Section 3

1. $AB + AC + BD + CD = (A + D)(B + C)$ (see Exercise 1, Sec. 1); $A + AB = A$ (see Exercise 2, Sec. 1); $AB + BO + AI = A$ (see Exercise 9, Sec. 1); $ABC + BCD + CDA = (A + B)(A + D)(B + D)C$ (see Exercise 10, Sec. 1).

2. (a) $(A + B)(A + \bar{B}) = AA + A\bar{B} + BA + B\bar{B} = A + A\bar{B} + BA + O = A + BA + \bar{B}A = A + (B + \bar{B})A = A + IA = A + A = A.$

(b) $AB + (A + B)(\bar{A} + \bar{B}) = AB + A\bar{A} + A\bar{B} + \bar{B}A + B\bar{B} = AB + O + A\bar{B} + B\bar{A} + O = AB + A\bar{B} + B\bar{A} = (AB + A\bar{B}) + (AB + \bar{A}B) = A(B+\bar{B}) + (A + \bar{A})B = AI + IB = A + B.$

(c) $\overline{\overline{ABC}}\ \overline{AB}\ \bar{A}C = (\bar{\bar{A}} + \bar{\bar{B}} + \bar{C})(\bar{\bar{A}} + \bar{B})\bar{A}C = [(A + B) + \bar{C}](A + \bar{B})\bar{A}C = A(A + \bar{B})\bar{A}C + B(A + \bar{B})\bar{A}C + (A + \bar{B})\bar{A}(C\bar{C}) = (A\bar{A})(A + \bar{B})C + [B(A\bar{A})C + (B\bar{B})\bar{A}C] + (A + \bar{B})\bar{A}O = O(A + \bar{B})C + [BOC + O\bar{A}C] + O = O.$

(d) $A + B = A + IB = A + (A + \bar{A})B = A + AB + \bar{A}B = (AI + AB) + \bar{A}B = A(I + B) + \bar{A}B = AI + \bar{A}B = A + \bar{A}B.$

3. Apply the "bar" operation to both members of the equality in question and use the fact that $\bar{\bar{A}} = A.$

4. $AB + A\bar{B} = A$ (see Exercise 2 (a)); $(A + B)(AB + \bar{A}\bar{B}) = AB$ (see Exercise 2 (b)); $A(\bar{A} + B) = AB$ (see Exercise 2 (d)).

6. (a) To every divisor m of the number N there corresponds   subset of the set $I = \{p_1, p_2, \ldots, p_k\}$ of the prime factors of the number N consisting of those prime factors of N which are simultaneously prime factors of m. If A and B are the subsets of the set I corresponding to two numbers m and n then to the numbers $m \oplus n = [m, n]$, $m \otimes n = (m, n)$ and $\bar{m} = N/m$ there correspond the sets $A + B$, AB and $\bar{A}$ respectively.

(b) If $m = p^a$ and $n = p^b$ then $m \oplus n = [m, n] = p^{\max [a, b]}$, $m \otimes n = (m, n) = p^{\min [a, b]}$ and $\bar{m} = N/m = p^{A-a}.$

(c) $\overline{m} = N/m = p_1^{A_1-a_1} p_2^{A_2-a_2} \ldots p_k^{A_k-a_k}$.

7. These equalities do not hold for the "algebra of maxima and minima" (except for the case when the algebra consists of only two numbers) and for the "algebra of least common multiples and greatest common divisors" (except for the case when the number N decomposes into pairwise different prime factors; cf. Exercise 6 (a)).

8. (a) $(A + B)(A + C) = A + AC + AB + BC =$
$= AI + AC + AB + BC = A(I + C + B) + BC =$
$= AI + BC = A + BC \subset A + B \subset A + B + C$.

(b) $(A + B)(A + C)(A + I) = (A + B)(A + C)I =$
$= (A + B)(A + C) = A + BC \supset A \supset ABC$ (cf. Exercise (a)).

(c) $(A + B)(B + C)(C + A) = AB + BC + CA \supset$
$\supset AB \supset ABC$ (see Exercise 6, Sec. 1).

(d) Since $A \supset A\bar{B}$ and $B \supset \bar{A}B$ we have $A + B \supset$
$\supset A\bar{B} + \bar{A}B$.

9. $ABC \subset AB + AC$ (see Exercise 8(a); $AB + AC + AO \subset A + B + C$ (see Exercise 8(b)); $AB + BC + CA \subset A + B + C$ (see Exercise 8(c)).

10. $AB \subset (\bar{A} + B)(A + \bar{B})$.

12. (a) A; (b) B; (c) I; (d) O.

Section 4

5. (a) "the number is even *and* prime"; the truth set is $\{2\}$; (b) "the number is odd *or* prime"; the truth set is $\{1, 2, 3, 5, 7, 9, 11, 13, 15, 17, \ldots\}$; it differs from the set of all odd numbers in the fact that it includes the number 2; (c) "the number is odd *and* prime"; the truth set is $\{3, 5, 7, 11, 13, 17, 19, \ldots\}$ and it differs from the set of all prime numbers in the fact that it excludes the number 2; (d) "the number is even *and* not prime"; the truth set is $\{4, 6, 8, 10, 12, 14, 16, \ldots\}$ and it differs from the set of all even numbers in the fact that it excludes the number 2; (e) "the number is odd *or* not prime"; the truth set is $\{1, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\}$ and coincides with the set of all positive integers with the exception of the number 2.

Section 5

5. $a + b = a$; $ab = b$.

Section 6

1. (a) $\overline{pq} + pq + \overline{pq} + p\overline{q}$; (b) $pqr + p\overline{q}r + p\overline{q}\overline{r} + \overline{p}qr$.
2. (d) $pqrs + \overline{pr} + \overline{ps} + \overline{qr} + \overline{qs}$.
3. Propositions (b), (d) and (e) are true (for all p and q).

Section 7

1. (a) See Fig. 51; (b) see Fig. 52.
3. (a) See Fig. 53a, (b) see Fig. 53,b.
4. (a) $D = AB + AC + BC$ (A, B and C are the push buttons which the members of the committee press).

(b) $G = A (BC + BD + BE + BF + CD + CE + CF + DE + DF + EF) + BCDE + BCDF + BCEF + BDEF + CDEF$ (A is the push button which the chairman presses and B, C, D, E and F are the push buttons for the members of the committee).

5. (a) $D = ABC + A\overline{B}\overline{C} + \overline{A}B\overline{C} + A\overline{B}\overline{C}$.

Fig. 51

Fig. 52

$E = ABCD + \overline{A}\overline{B}\overline{C}\overline{D}$

(a)

$E = (A+B+C+D)(\overline{A}+\overline{B}+\overline{C}+\overline{D})$
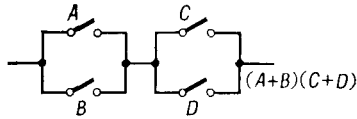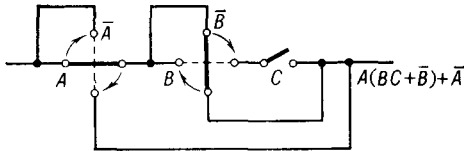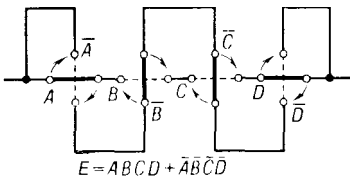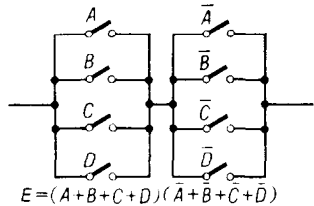
(b)

Fig. 53

Section 8

3. $\xi = \alpha\bar{\beta}$.

4. $\varphi(N) = N\left(1 - \dfrac{1}{p_1}\right)\left(1 - \dfrac{1}{p_2}\right) \cdot \cdot \cdot \left(1 - \dfrac{1}{p_n}\right)$.

This formula can be obtained by applying (A') to Example $3°$ in which the roles of the elements of a Boolean algebra are played by the sets of natural numbers not exceeding N and divisible by p_i (where $i = 1, 2, \ldots, n$).

5. Apply formula (A') to the Boolean algebra in Example $5°$.

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   This book is in many respects similar to [3] but Boolean algebras are considered in it in less detail; however much attention is paid to problems concerning computers.

2. Courant, R., Robbins, H., *What is Mathematics?*, Oxford University Press, New York, 1948.
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   This book does not require any prerequisites except the knowledge of elementary mathematics but the reader of the book is supposed to have some experience in studying mathematical literature; all the questions considered in the present book are discussed and there are many problems for self-training.

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Name Index

Subject Index

TO THE READER

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This book is devoted to fundamentals of the theory of
Boolean algebras which play an important role in
mathematical logic and in the development of electro-
nic computers and cybernetics.
The book includes the definition of the notion of a
Boolean algebra and many examples of such algebras; in
particular, it deals with algebra of propositions.
Some applications of this algebra to the automation of
mathematical proofs are discussed. There are also many
exercises with answers and hints to some of them
placed at the end of the book. The solution of the
problems in the exercises facilitates the
understanding of the material of the book.
The book is intended for senior pupils and can also
be of use for adult readers interested in mathematics.