I.P. Gurskii

Elementary Physics

PROBLEMS AND SOLUTIONS

Rs 60

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И. П. Гурский

ЭЛЕМЕНТАРНАЯ ФИЗИКА

С ПРИМЕРАМИ РЕШЕНИЯ ЗАДАЧ

Под редакцией И. В. Савельева

Издательство «Наука»
Москва

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Mir Publishers
Moscow

Translated from Russian
by Natalia Wadhwa

First published 1987
Revised from the 1984 Russian edition

На английском языке

Printed in the Union of Soviet Socialist Republics

Foreword

This is a translation of the third revised edition of *Elementary Physics: Problems and Solutions*. It was prepared by the editorial staff of Nauka Publishers, Moscow, after the death of Dr. Isaak Gurskii (1903-1980). The book is intended for high-school students preparing for the entrance examinations to universities and colleges. It can also be used independently by a student.

The theoretical parts of the book have been significantly revised for the new edition. The formulation and solution of problems have been refined. Detailed arithmetic sections have been eliminated in view of the higher standards in high schools.

The terminology and the system of units employed in the book have been updated for the revised edition.

The editorial staff is deeply indebted to Prof. I. V. Savelyev, Head of the General Physics Department at the Moscow Institute of Physical Engineering for his invaluable assistance in the preparation of the revised manuscript of the book.

From the Preface to the Second Russian Edition

The book is intended for those preparing for university entrance examinations in physics. The contents and sequence of topics are in keeping with the requirements for such examinations. The few sections beyond the entrance examination programme are marked by circles. In view of the introduction of the elements of higher mathematics to the high-school curriculum, some problems have also been illustrated using differential calculus. The author has endeavoured to present the basic principles of school physics in a compact form to help the candidates revise the entire course in the shortest possible time. All sections have been illustrated with problems to give a better understanding of the subject. Each problem and its solution is followed by one or more exercises on the same topic, the exercises corresponding to problems that have been solved in the text are assigned the same number.
Those intending to use this book independently are advised to attempt the exercises after going through the theoretical part. The relevant solved problems should be consulted if difficulties are encountered while solving the exercises. After this, the exercise should be tried again, and if there is more than one exercise bearing the same number, another exercise (preferably the last one) should be tackled. In most cases, the last exercises in a series are the most difficult.

Even if the exercises do not pose any difficulty, it is advisable to look at the relevant solved problem to ensure that the approach to the problem is correct and rational.

The author

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INTRODUCTION

I.1. SI System of Units

The analysis of physical phenomena and processes is associated with measuring physical quantities. To measure a physical quantity is to compare it with a similar quantity which is conventionally adopted as a unit.

Such a unit could be chosen for every quantity independently of other quantities. It is expedient, however, to proceed in a different way, i.e. to establish the units of several quantities (which are called **base units**) independently and then express the remaining units in terms of the base units by using physical regularities. For example, velocity is expressed in terms of two independent quantities, viz. length and time. The units established not independently but with the help of formulas relating them to base units are called **derived units.** The set of base and derived units is called a **system of units**.

In 1960, the XI General Conference on Weights and Measures in Paris adopted the International System of Units (SI) which was subsequently supplemented and refined. It includes seven base units: metre (m) as the unit of length, kilogram (kg) as the unit of mass, second (s) as the unit of time, ampere (A) as the unit of electric current, kelvin (K) as the unit of absolute (thermodynamic) temperature, candela (cd) as the unit of luminous intensity, and mole (mol) as the unit of the amount of substance. Supplementary SI units are radian (rad) as the unit of plane angle and steradian (sr) as the unit of solid angle.

Using mathematical expressions for physical quantities in terms of other quantities, all the remaining units of physical quantities can be expressed in terms of the base units. For example, the SI unit of force is a derived unit: a newton (N) is the force that imparts an acceleration of 1 m/s^2 to a body whose mass is 1 kg.

Special **standards** are manufactured to keep the base units unchanged. These are measures and measuring instruments intended for the storage and reproduction of units of physical quantities to the highest accuracy that can be attained at a given level of science and engineering. The selection and definition of units, as well as the storage and reproduction of their standards, form a special branch of science called *metrology*.

Definitions and standards of the SI base units

1. Prior to the XI General Conference on Weights and Measures, the international standard metre was a platinum-iridium bar kept in Paris. The distance between the marks engraved on this bar at 0 °C was taken as a metre. This distance was close to one ten-millionth of a quarter of the Paris meridian measured at the end of the 18th century. The accuracy of this standard was limited by the width of the marks, which is insufficient at the present time. Besides, such an artificial standard would be difficult to reproduce if it were lost or damaged. For this reason, the XI General Conference on Weights and Measures held in 1960 established that a **metre** will be defined as the distance equal to 1 650 763.73 wavelengths of the radiation corresponding to the translation between orange-red energy levels of the crypton-86 atom in vacuum. This conference specified the construction of the source of these waves, viz. a crypton-discharge lamp.

2. The SI unit of mass, viz. a **kilogram**, is equal to the mass of the international prototype of kilogram. The standard kilogram is a platinum-iridium prototype adopted in 1901 at the III General Conference on Weights and Measures and kept in Paris.

3. In order to define the unit of time, viz. a second, more accurately, the XIII General Conference on Weights and Measures associated it with the period of the radiation corresponding to the transition between certain levels of the caesium-133 atom (earlier, it was associated with the motion of the Earth around its axis or

with the motion of the Earth around the Sun). A **second** is equal to 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom. The standard of the unit of time is reproduced with the help of a special caesium atomic oscillator.

4. The definition of the unit of temperature, viz. a **kelvin**, is given in Sec. 3.8. Gas thermometers are used as standard instruments for reproducing the unit of temperature.

5. The previously used definition of the SI unit of current in terms of the mass of the substance deposited on an electrode in electrolysis could not ensure the sufficient accuracy of the standard. For this reason, the present-day SI unit of current is established on the basis of the law of interaction of thin parallel rectilinear current-carrying conductors of infinite length (see Sec. 4.26). In spite of the fact that it is impossible to manufacture thin conductors of infinite length, the force of interaction of thin conductors of finite length can be calculated to a sufficiently high degree of accuracy by the law describing the interaction of conductors of infinite length. The instrument for reproducing the standard unit of current, viz. an **ampere**, is based on this law and is called the current (or ampere) balance.

6. The SI unit of luminous intensity, viz. a **candela**, is equal to the luminous intensity of light emitted in a given direction by a source of monochromatic radiation of frequency 540×10^{12} Hz. The radiant intensity of this source in this direction is 1/683 W/sr. Incandescent lamps of various designs and various colour temperatures serve as the standard candela.

7. The unit of the amount of substance, viz. a **mole**, is the amount of substance in a system which contains as many structural elements as there are atoms in the carbon-12 sample whose mass is 0.012 kg. When a mole is used, structural elements must be specified. They can be atoms, molecules, ions, electrons, and other particles or structural groups of particles.

I.2. Vectors. Some Mathematical Operations on Vectors

A **vector** is defined by its magnitude and direction (Fig. 1).

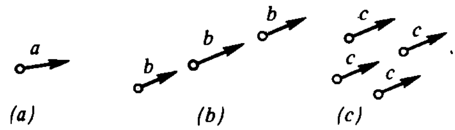

Fig. 1

Depending on the point of application, vectors can be of different types, i.e. bound, sliding, and free vectors.

A vector whose point of application is defined in space and cannot be displaced is called a **bound vector** (Fig. 1*a*). A vector whose point of application can be displaced along the line of its action[1] is called a **sliding vector** (Fig. 1*b*). Finally, a vector whose point of application can be displaced to any point in space is called a **free vector** (Fig. 1*c*).

Vectors are denoted by bold-face letters (**v**, **a**) or by an arrow above the letters ($\overrightarrow{AB}$, where A and B are the tail and head of the vector). The magnitude of a vector is denoted as $|\mathbf{a}|$, $|\overrightarrow{AB}|$ or by the same letters as the vector itself but in light face (v, a, AB).

The **sum of two vectors** applied at the same point of a body is represented (in magnitude and direction) by the diagonal of the parallelogram plotted with the component vectors as its sides (Fig. 2). Instead of constructing a parallelogram, we can plot a triangle (Fig. 3). From the head of one vector, say, **a**, we draw a

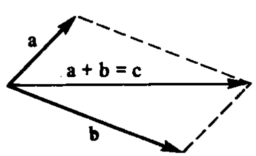

Fig. 2

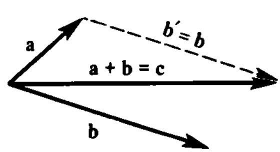

Fig. 3

[1] The line of action is a straight line containing the vector. The vector has one of the two possible directions, along this line of action or against it.

vector $\mathbf{b}'$ which is equal in magnitude and parallel to the vector $\mathbf{b}$. Connecting the tail of vector $\mathbf{a}$ with the head of vector $\mathbf{b}'$, we obtain a vector equal to their sum. Comparing this figure with Fig. 2, we see that instead of a parallelogram, we have plotted one of the triangles constituting it. This method of addition of vectors is called the **triangle rule**.

The **addition of several vectors** lying in the same plane and applied at the same point at an angle to each other can be carried out consecutively. First the vector sum of two vectors is found, then this vector is added to a third vector, and so on. Alternately, the vectors can be added pairwise, the resultant vectors are also added in pairs, and so on.

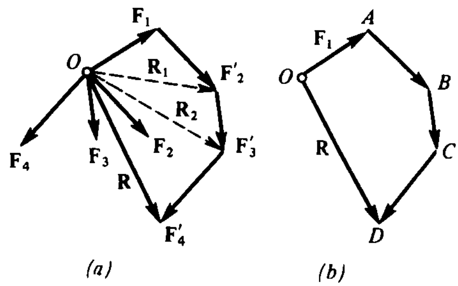

Fig. 4

Figure 4*a* shows a consecutive addition of four vectors with the help of the triangle rule. The addition of vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ gives their vector sum $\mathbf{R}_1$. Adding this vector to vector $\mathbf{F}_3$ by the triangle rule, we obtain their sum $\mathbf{R}_2$. Finally, adding $\mathbf{R}_2$ and $\mathbf{F}_4$, we get the vector sum $\mathbf{R}$ of all the given four vectors.

Instead of all these intermediate constructions, we can proceed in a simpler way shown in Fig. 4*b*. From the head A of vector $\mathbf{F}_1$, we draw vector $\overrightarrow{AB}$ equal in magnitude and parallel to vector $\mathbf{F}_2$. From the head of this vector, we plot vector $\overrightarrow{BC}$ equal in magnitude and parallel to vector $\mathbf{F}_3$, and proceed in the same way until we have used all the given vectors. We thus obtain an unclosed polygon $OABCD$. Vector $\overrightarrow{OD}$ closing this polygon and opposite to the direction of circumvention (see Fig. 4*b*) of the unclosed polygon constructed from the given vectors represents the sum of these vectors. This rule of addition of several vectors is called the **polygon rule**.

This method can be used for the addition of several free vectors as well as sliding vectors whose *lines of action intersect at one point*. Since sliding vectors can be displaced along the lines of their action to the point of intersection of these lines, their resultant is applied at the same point or lies on a straight line passing through this point and coinciding with the resultant. Figure 5*a* illustrates the addition of three vectors $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, whose lines of action intersect at point O. The vector polygon is constructed separately in Fig. 5*b*. The vector sum is applied at point O or at any other point lying on the straight line AOB, say, at point C.

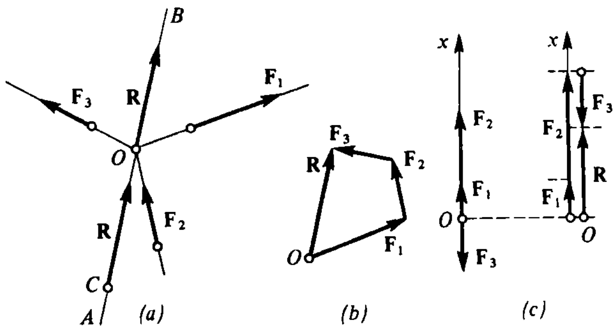

Fig. 5

Free vectors whose *lines of action do not intersect at one point* can be added pairwise (according to the parallelogram or triangle rule).

In the particular case of the addition of vectors directed along a straight line, the polygon degenerates into a straight line on which the tail of the second vector coincides with the head of the first, the tail of the third vector with the head of the second, and so on. Figure 5*c* shows the addition of three vectors $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ applied at point O. One of the components, $\mathbf{F}_3$, and the resultant $\mathbf{R}$ are somewhat displaced for the sake of clarity.

It follows from what has been said above that *vectors are add-*

ed geometrically unlike scalar quantities which are added algebraically.

The **subtraction of one vector from another** is the same as the addition of the "minuend" vector and a vector equal in magnitude and antiparallel to the "subtrahend" vector.

The addition and subtraction of vectors directed along the same straight line can be simplified by replacing the geometrical addition and subtraction by the algebraic addition of projections of these vectors onto an axis having the same direction. In this case, as usual, the projections of vectors having the same direction as the chosen axis are assumed to be positive and the projections of vectors having the opposite directions are assumed to be negative. For example, vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ in Fig. 5*c* have the same direction as the x-axis, and their projections are positive. On the other hand, the projection of vector $\mathbf{F}_3$ onto this axis is negative. Instead of the vector equality

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

we can write

$$R = F_1 + F_2 - F_3.$$

If $\mathbf{F}_3 > \mathbf{F}_1 + \mathbf{F}_2$, the sum of the three vectors is negative, i.e. vector $\mathbf{R}$ has the same direction as vector $\mathbf{F}_3$ (see Fig. 5*c*).

When a vector $\mathbf{F}$ is multiplied by a scalar quantity α, we obtain a vector $\alpha\mathbf{F}$ applied at the same point. Its magnitude is $|\alpha|$ times larger than the magnitude F of the vector being multiplied. The direction of the resultant vector coincides with that of original vector if $\alpha > 0$ and is opposite to it if $\alpha < 0$.

Multiplication of two vectors. There are two kinds of vector products: the dot (scalar) product and the cross (vector) product.

As a result of **scalar multiplication** of two vectors (which is denoted by $\mathbf{F}_1\mathbf{F}_2$ or $\mathbf{F}_1 \cdot \mathbf{F}_2$), we obtain a **scalar quantity** equal to the product of the magnitudes of these vectors by the cosine of the angle formed by these vectors:

$$\mathbf{F}_1\mathbf{F}_2 = F_1F_2\cos(\widehat{\mathbf{F}_1, \mathbf{F}_2}).$$

As a result of **vector multiplication** of two vectors (which is denoted by $[\mathbf{F}_1\mathbf{F}_2]$ or $\mathbf{F}_1 \times \mathbf{F}_2$), we obtain a **vector** whose magnitude is equal to the area of the parallelogram constructed on these two vectors:

$$|\mathbf{F}_1 \times \mathbf{F}_2| = F_1 F_2 \sin(\widehat{\mathbf{F}_1, \mathbf{F}_2}).$$

The line of action of this vector is normal to the plane in which the two vectors being multiplied lie. Its direction is such that as we look from its head, the shortest rotation about this vector from the first to the second vector being multiplied is in the anticlockwise direction (Fig. 6).[2] Hence it follows that vector product *is not commutative*, i.e. $\mathbf{F}_1 \times \mathbf{F}_2 \neq \mathbf{F}_2 \times \mathbf{F}_1$. Instead, we have $\mathbf{F}_1 \times \mathbf{F}_2 = -\mathbf{F}_2 \times \mathbf{F}_1$. In other words, a change in the order of vectors in a vector (cross) product leads to a vector having the same magnitude but the opposite direction.

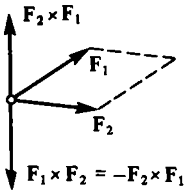

Fig. 6

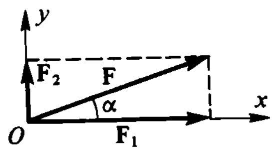

Fig. 7

The **decomposition of a vector into components** in two directions is carried out in accordance with the parallelogram rule. The vector being decomposed is the diagonal and the component vectors are the sides of the parallelogram. In the particular case of vector decomposition in two *mutually perpendicular directions*, the parallelogram becomes a rectangle. Figure 7 illustrates the decomposition of vector $\mathbf{F}$ into vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ along the coordinate axes. The magnitudes of the component vectors are $\mathbf{F}_1 = F\cos\alpha$ and $\mathbf{F}_2 = F\sin\alpha$.

[2] In practice, the "corkscrew rule" is conveniently used: the direction of the resultant vector in a vector product coincides with the direction of translation of a corkscrew whose handle is rotated from the first vector to the second along the shortest path.

I.3. Projections of Points and Vectors onto an Axis

The **projection of a point** A onto a straight line (axis) OO' is the base a of the perpendicular dropped from this point onto the straight line (Fig. 8).

The **projection of a vector** $\vec{AB}$ (and also $\overrightarrow{A'B'}$, $\vec{CD}$, and $\overrightarrow{C'D'}$) onto the axis OO' is the segment ab (and also $a'b'$, cd, and $c'd'$) bounded by the projections of the tail and head of the vector onto this axis (Fig. 9). If the direction from the projection

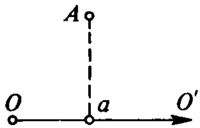

Fig. 8

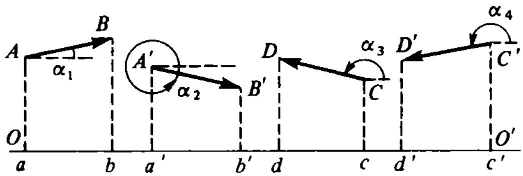

Fig. 9

of the vector tail to the projection of its head coincides with the direction chosen for the axis, the projection of the vector is assumed to be positive (ab, $a'b'$). Otherwise, the projection is negative (cd, $c'd'$).

It can be seen from Fig. 9 that the projection of a vector onto an axis is equal to the product of the vector length and the cosine of the angle between the positive direction of the axis and the vector:

$$ab = AB\cos\alpha_1, \quad a'b' = A'B'\cos\alpha_2,$$
$$cd = CD\cos\alpha_3, \quad c'd' = C'D'\cos\alpha_4.$$

As is usually done in trigonometry, angles are measured in the anticlockwise direction. An angle in the clockwise direction is assumed to be negative. (In actual practice, the cosine of the acute angle between the axis and vector is taken, and the sign of the projection is determined from the drawing.)

It follows from what has been said above that the projection of a vector onto an axis is a scalar quantity. *All mathematical operations on projections are carried out algebraically.*

1.4. General Methodical Hints to the Solution of Problems

The main peculiarity of a physical problem is that a *physical process* is considered in it. Although the solution of the problem is reduced to a number of mathematical operations, the correct solution of the problem in physics is possible only if the physical process involved is understood correctly. Therefore, the following hints for solving physical problems are appropriate here.

1. The formulation of the problem should be read very attentively and more than once until it becomes clear which physical process or phenomenon is actually considered in it.

2. After this, the given and required physical quantities should be written down. The quantities that are given should be written thoroughly, without missing anything. The values of the quantities which are not given explicitly but which can be immediately obtained from the conditions of the problem should also be written. For example, if a braking process leads to a complete halt, we must write that the final velocity $v_f = 0$. If the conditions of a problem imply that some quantity x can be neglected, it is necessary to write that $x \simeq 0$ or $x = 0$, and so on.

3. Next, the physical laws governing the given process should be recollected, as well as the mathematical formulas describing these laws. If there are several formulas describing these laws, we must compare the quantities appearing in the formulas with those given in the problem and choose the formulas containing the given and required quantities.

4. As a rule, a problem in physics is solved in the general form, i.e. we derive the formula in which the required quantity is expressed in terms of the given quantities. Finally, the numerical values of the given quantities are substituted together with their dimensions into the obtained formula. Thus, the numerical value and dimensions of the required quantity are determined. Solving the problem in this way, we do not accumulate errors as in the case when the approximate values of intermediate quantities are calculated and are then substituted into the formula for obtaining the value of the required quantity. The exceptions to this rule (which are quite rare) are of two types: (a) the formula for

calculating an intermediate quantity is so cumbersome that the calculation of this quantity simplifies the subsequent form of the solution; (b) the numerical solution of the problem is much simpler than the derivation of the final formula and does not deteriorate the accuracy of the result.

5. Before substituting the numerical values of given quantities into the computational formulas obtained for required quantities, all the values of the given quantities should be recalculated to express them in the same system of units, preferably in SI. An exception to this rule is the case when formulas contain a certain quantity as a multiplier in the numerator and denominator. Such quantities can be expressed in any (naturally, the same) units.

6. The correctness of the obtained result can be judged to a certain extent from dimensional analysis. If the computational formula is a sum of several terms, the dimensions of the addends should be the same.

7. The correctness of the result can also be controlled by the order of magnitude of the obtained numerical values. The order must be commensurate with the physical meaning of the required quantity.

8. The answer should be obtained to a certain degree of accuracy corresponding to the accuracy of given quantities. However, insufficient and excessive accuracy are equally harmful. For example, if the initial quantities are measured or given to within 1 cm, and as a result of calculations we obtain 287 mm, it is expedient to write the answer either in the form of 29 cm, or 0.29 m rather than 28.7 cm, or 0.287 m. On the other hand, if the initial quantities are given to within 1 mm, and we get 29 cm, the answer should be written in the form of 29.0 cm, or 290 mm, or 0.290 m rather than 0.29 m, or 29 cm.

In this book, wherever possible, the problems following each topic are arranged in the increasing order of difficulty. More complicated problems are marked by asterisks (*).

1. MECHANICS

1.1. Basic Concepts

Mechanics is the branch of physics for studying the motion of bodies, i.e. the change in their position in space and time.

In most cases, the deformation of bodies during their motion is not taken into account. In other words, perfectly rigid bodies are considered. A **perfectly rigid body** is a body in which the mutual arrangement of particles remains unchanged during its motion. While considering the motion of a body, we can often neglect its size and shape. In such cases, instead of studying the motion of a perfectly rigid body, we can analyze the motion of a material point. A **material point (particle)** is a body whose size can be neglected in a given problem.

The replacement of a rigid body by a material point is a sort of abstraction which may turn out to be admissible for analyzing some types of motion of a body but inadmissible for studying some other types of motion. For example, while analyzing the motion of the Earth around the Sun, the Earth and the Sun can be considered to be material points. On the other hand, while studying the motion of the Earth around its axis, it cannot be treated as a material point.

The position of a body in space can be specified only relative to some other body or bodies. Therefore, when we speak about motion, we mean **relative motion**, i.e. the motion of a body relative to another body which is conditionally assumed to be fixed. If we mentally attach a coordinate system to the body taken as fixed and called the **reference body**, this system, together with the chosen method of measuring time, forms the **reference system.** Normally, the *Cartesian coordinate system*, which is a set of three

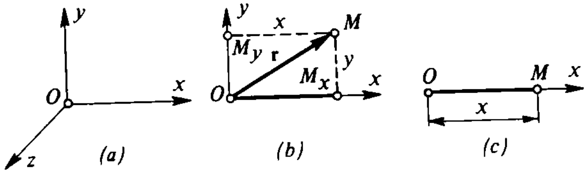

Fig. 10

mutually perpendicular axes Ox, Oy, and Oz, is used (Fig. 10*a*).

While studying the motion of a material point M, its position at each instant relative to a reference system is determined either by its coordinates or by the radius vector $\mathbf{r}$, i.e. the vector drawn from the origin to this point (Fig. 10*b*).

In an analysis of terrestrial motions, i.e. in all practical cases, we can generally take a system of coordinate axes rigidly fixed to the Earth surface as the reference system. When the rotation of the Earth has to be taken into account (as, for example, in an analysis of the motion of Earth's satellites), a *geocentric reference system* is used. This is a system whose origin coincides with the centre of the Earth and whose axes are directed towards three chosen "stationary" stars. When the motion of the Earth and other planets of the solar system relative to the Sun has to be taken into consideration (as, for example, in an analysis of the motion of spacecrafts launched to the planets of the solar system), a *heliocentric reference system* is used. This is a system whose origin coincides with the centre of the Sun and whose axes are directed towards three chosen "stationary" stars.

Two coordinate axes, Ox and Oy (Fig. 10*b*), are sufficient to describe the motion of a body (material point) in one plane, since the position of any point on a plane, say, point M, is determined by two coordinates: $x = M_x$ and $y = M_y$. These coordinates are the distances from the origin to the points M_x and M_y, which are the bases of the perpendiculars dropped from the given point onto the respective coordinate axes.

If a material point is moving in a straight line, we can make one of the coordinate axes coincide with this line and specify the position of the point at any instant by a single coordinate, viz. the distance from this point to the point chosen as the origin

(Fig. 10c). The motion of the body in this case can be described by using just one coordinate axis.

We shall be mainly interested in two cases when a body can be treated as a material point: (1) the size of the body is small in comparison with the distance covered by it during its motion or the distances from this body to other bodies; (2) all points of the body perform the same motion, i.e. have equal velocities at any instant and describe similar *trajectories* (Fig. 11). Such a motion of a body is called **translatory motion**.

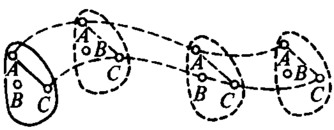

Fig. 11

Depending on the trajectory, the translatory motion of bodies can be of two types: *rectilinear motion* (if the trajectory is a straight line) and *curvilinear motion* (if the trajectory is an arbitrary curve).

A. KINEMATICS

1.2. Kinematics of Translatory Motion

In translatory motion, all points of a body move identically. Therefore, instead of considering the motion of each point of the body, we can analyze the motion of only one of its points. Hence, the study of the translatory motion of a body is reduced to an analysis of the motion of a material point.

The main characteristics of motion of a material point are its trajectory, displacement, path length (the distance covered by it), its coordinates, velocity, and acceleration.

The **displacement** of a material point over a certain time interval is the *vector* $\Delta\mathbf{r}$ directed from its position at the initial instant to the position at the final instant (Fig. 12). The figure shows that $\mathbf{r}_B = \mathbf{r}_A + \Delta\mathbf{r}$, whence the displacement vector

$$\Delta\mathbf{r} = \mathbf{r}_B - \mathbf{r}_A.$$

The **path length** s of a material point is a *scalar quantity* equal to the distance covered by it along its trajectory (see Fig. 12). *For the motion of a body in a straight line in one direction, the path length and the magnitude of the displacement vector coincide:* $s = |\Delta \mathbf{r}|$. In all other cases, the magnitude of displacement is *smaller* than the path length. For example, when a body vibrates about a

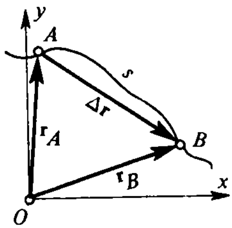

Fig. 12

point, its displacement can be zero at certain instants. The path length, however, is a positive scalar quantity which does not decrease with time.

The **velocity** of a material point is a *vector* characterizing the variation of the displacement of the material point relative to a reference body with time. Similarly, **acceleration** characterizes the variation of the velocity of a material point relative to a reference body with time.

In kinematics, the **law of independent motion** holds: *if a material point simultaneously takes part in several motions, the resultant displacement of the point is the vector sum of the displacements of the point in all its motions.*

1.3. Uniform Rectilinear Motion. Velocity. Graphs of Velocity and Path Length in Uniform Motion

Uniform rectilinear motion is a motion in which a material point moves in a straight line and covers equal distances in equal intervals of time.

The velocity in a uniform rectilinear motion is defined as the ratio of the displacement and the time over which this displacement occurs:

$$\mathbf{v} = \Delta\mathbf{r}/t.$$

For the x-axis, we take the line along which the material point is moving. We choose the direction of motion of the point as the positive direction of the axis. Projecting vectors $\mathbf{r}$ and $\mathbf{v}$ onto this axis, we obtain

$$v = \Delta r/t,$$

where v and Δr are the magnitudes of the corresponding vectors. This gives

$$\Delta r = vt.$$

The path length s of a body in a uniform rectilinear motion is equal to the magnitude of the displacement. Consequently, the path length in the motion is equal to the magnitude of the velocity multiplied by the time:

$$s = vt.$$

Thus we obtain the following expression for the x-coordinate of the body at instant t:

$$x = x_0 + s = x_0 + vt,$$

where x_0 is the coordinate of the body at the initial instant $t = 0$.

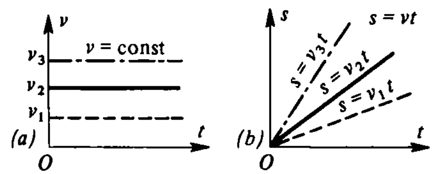

Fig. 13

Figure 13 represents the time graphs of velocity and path length for a uniform rectilinear motion. The slope of the path length vs time curve is the larger, the higher the velocity. When $x_0 = 0$, the time graphs of the coordinate and the path length in a uniform rectilinear motion coincide (Fig. 13*b*).

The SI unit of velocity is metre per second (m/s). Besides, other units such as kilometre per hour (km/h) and kilometre per second (km/s) are also used.

1.4. Nonuniform Motion. Average and Instantaneous Velocities. Acceleration

Nonuniform motion is a motion in which the velocity varies with time.

For a nonuniform motion, the ratio of the displacement $\Delta\mathbf{r}$ of a point to the time interval Δt over which this displacement occurs defines the **average velocity** over time Δt:

$$\mathbf{v}_{av} = \Delta\mathbf{r}/\Delta t.$$

Sometimes, average velocity is defined as a scalar quantity v_{av} equal to the ratio of the distance Δs covered by a body over the time interval Δt to this interval:

$$v_{av} = \Delta s/\Delta t.$$

It is this velocity that is meant when we speak about the average velocity of motion of a motorcar or the average velocity of a train.

The motion of a material point at each given instant of time, i.e. at each point of its trajectory, is characterized by the so-called **instantaneous velocity**, or simply **velocity**. *Instantaneous velocity* is equal to the *limit* of the average velocity as the time interval Δt over which it is determined tends to zero:

$$\mathbf{v} = \lim_{\Delta t \to 0} (\Delta\mathbf{r}/\Delta t).$$

Instantaneous velocity is directed along the tangent to the trajectory of motion (Fig. 14). Its magnitude is equal to the limit of the ratio of the distance Δs covered by the body over the time interval Δt to this interval ($\Delta s = |\Delta\mathbf{r}|$ as $\Delta t \to 0$):

$$v = \lim_{\Delta t \to 0} (\Delta s/\Delta t). \tag{1.4.1}$$

If a material point takes part in several motions, the resultant

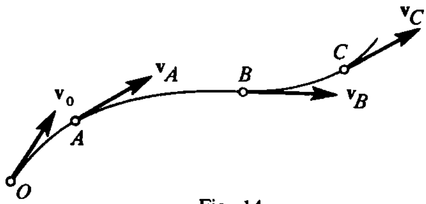

Fig. 14

velocity is equal to the vector sum of the velocities of these motions.

The difference between the final and initial values of a quantity f is called the **increment** of this quantity and is denoted by Δf:

$$\Delta f = f_{\mathrm{f}} - f_{\mathrm{in}}.$$

The change in the velocity of a material point in nonuniform motion is characterized by acceleration. The average acceleration over a time interval Δt is the quantity defined as the ratio of the velocity increment $\Delta \mathbf{v}$ to Δt:

$$\mathbf{a}_{\mathrm{av}} = \Delta \mathbf{v}/\Delta t.$$

We obtain the acceleration of a material point at a given instant of time by taking the limit of the average acceleration as $\Delta t \to 0$:

$$\mathbf{a} = \lim_{\Delta t \to 0} (\Delta \mathbf{v}/\Delta t). \tag{1.4.2}$$

Since the SI unit of velocity is metre per second, the unit of acceleration in this system is metre per second per second.

Velocity and acceleration are vectors *related* to the material point whose motion they characterize. The velocities of a material point taking part in several motions, as well as its accelerations, are added in accordance with the rules of addition of vectors (see Sec. 1.2).

Let us consider the motion of a material point along the x-axis. Since

$$\lim_{\Delta t \to 0} (\Delta x/\Delta t) = dx/dt = x'(t),$$

$$\lim_{\Delta t \to 0} (\Delta v/\Delta t) = dv/dt = v'(t),$$

the velocity of the material point is the time derivative of the coordinate, while its acceleration is the time derivative of the velocity, or the second time derivative of the coordinate:

$$v = dx/dt = x'(t),$$
$$a = dv/dt = v'(t), \quad a = d^2x/dt^2 = x''(t). \tag{1.4.3}$$

1.5. Uniformly Variable Motion. Graphs of Velocity and Path Length in Uniformly Variable Motion

Uniformly variable motion is a motion with a constant acceleration. In such a motion, the acceleration is defined as

$$\mathbf{a} = (\mathbf{v} - \mathbf{v}_0)/t,$$

where $\mathbf{v}_0$ is the velocity of a body at the initial instant of time, t is the time that has elapsed from the beginning of the motion, and $\mathbf{v}$ is the velocity of the body at instant t.

Uniformly variable motion can be curvilinear. In this case, the velocity $\mathbf{v}$ changes only in direction (this case is considered in Sec. 1.25).

If a uniformly variable motion is rectilinear, i.e. the velocity $\mathbf{v}$ changes only in magnitude, it is convenient to take the straight line in which a material point moves as one of the coordinate axes (say, the x-axis). The positive direction of this axis coincides with the direction of the initial velocity $\mathbf{v}_0$. Then the acceleration can be calculated as a scalar quantity, viz. the projection of the acceleration vector, which is equal to $+a$ if the direction of the acceleration coincides with the direction of $\mathbf{v}_0$, and to $-a$ if $\mathbf{a}$ and $\mathbf{v}_0$ have opposite directions. In this case, the acceleration of a body can be calculated from the formula[1]

$$a = (v - v_0)/t.$$

Hence we obtain the following formula for the velocity $\mathbf{v}$ of uniformly variable motion at instant t:

$$v = v_0 + at. \tag{1.5.1}$$

[1] It would be more correct to write this formula as follows: $a_x = (v_x - v_{0x})/t$, considering that the formula contains the x-projections of velocities and accelerations, which can be either positive or negative. In this and subsequent formulas, the subscripts "x" are omitted to simplify the notation.

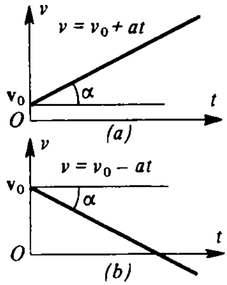

Fig. 15

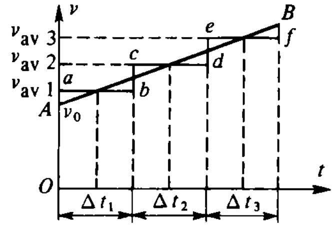

Fig. 16

While using this formula, it should be borne in mind that the x-axis is directed along vector $\mathbf{v}_0$ of the initial velocity. If the acceleration $\mathbf{a}$ is antiparallel to vector $\mathbf{v}_0$, i.e. $a < 0$, the value of the velocity $\mathbf{v}$ at an instant may turn out to be negative. This means that the velocity $\mathbf{v}$ at this instant is antiparallel to $\mathbf{v}_0$.

In the particular case of motion from the state of rest, i.e. when $v_0 = 0$, we have

$$v = at.$$

In this case, the directions of $\mathbf{v}$ and $\mathbf{a}$ always coincide.

The time graph of the magnitude of velocity in a uniformly variable motion of a material point is a straight line (Fig. 15). The slope α of this straight line is equal to the acceleration a of a body. If the acceleration is positive (the directions of $\mathbf{a}$ and $\mathbf{v}_0$ coincide), $\tan\alpha > 0$ (Fig. 15*a*). If the acceleration is negative, $\tan\alpha < 0$ (Fig. 15*b*).

Let us consider a **uniformly accelerated motion** ($a > 0$). We divide the time of motion into a set of intervals (for the sake of simplicity, three intervals are taken in Fig. 16). The distance covered by a body (material point) can then be approximately calculated as follows:

$$s \simeq v_{\text{av}\,1}\Delta t_1 + v_{\text{av}\,2}\Delta t_2 + v_{\text{av}\,3}\Delta t_3.$$

The right-hand side of this equality is numerically equal to the area of the figure bounded by the broken line *abcdef* from above. As the number of intervals increases, the broken line obviously approaches the straight line AB more closely. Consequently, the distance covered by the body is equal in magnitude to the area of the trapezoid bounded by the straight line AB from above and by the t-axis from below:

$$s = [(v_0 + v)/2]t, \qquad (1.5.2)$$

where $(v_0 + v)/2$ is the average velocity of motion of the material point over time t.[2] It has been shown earlier that

$$v = v_0 + at. \qquad (1.5.3)$$

Substituting this expression for v into (1.5.2), we get

$$s = [v_0 + (v_0 + at)]t/2, \qquad (1.5.4)$$

or

$$s = v_0t + at^2/2. \qquad (1.5.5)$$

Thus we obtain the following expression for the coordinate of the body at any instant of time:

$$x = x_0 + s = x_0 + v_0t + at^2/2, \qquad (1.5.6)$$

where x_0 is the coordinate of the body at the initial instant.

The derivation of formulas (1.5.3) and (1.5.5) is simplified if we introduce the concepts of a differential and an integral. From formula (1.4.3), we get

$$dv = a\,dt, \qquad v = \int a\,dt.$$

For a uniformly variable motion (a = const), we obtain

$$v = a\int dt = at + C.$$

The value of the arbitrary constant C can be found from the initial condition: for $t = 0$, $v = v_0$, i.e. $v_0 = a \cdot 0 + C$. Consequently,

[2] For a uniformly variable motion, when the time dependence of velocity is linear ($v = v_0 + at$), the average velocity is equal to the arithmetic mean of the initial and final velocities of a body. Let us find, for example, the average velocity of a motorcar moving from town A to town B with a velocity v_1 and from B to A with a velocity v_2. By definition of the average velocity, $v_{av} = 2s/t$, where s is the distance covered by the car and t is the time of motion. But $t = t_1 + t_2 = s/v_1 + s/v_2$. Hence $v_{av} = 2v_1v_2/(v_1 + v_2)$ and not $(v_1 + v_2)/2$!

$C = v_0$, and

$$v = v_0 + at.$$

Using expression (1.4.3), we obtain

$$dx = v\,dt.$$

For a uniformly variable motion, we obtain

$$dx = (v_0 + at)\,dt,$$

whence

$$x = \int (v_0 + at)\,dt = v_0\int dt + a\int t\,dt = v_0t + at^2/2 + C.$$

We find the value of the arbitrary constant $C = x_0$ from the initial condition that the coordinate $x = x_0$ for $t = 0$. Consequently,

$$x = x_0 + v_0t + at^2/2,$$

and for $x_0 = 0$,

$$x = v_0t + at^2/2.$$

If we eliminate time t from Eqs. (1.5.3) and (1.5.5), we obtain the following relation between the velocity of a material point in a uniformly variable motion and the distance covered by it:

$$v = \sqrt{v_0^2 + 2as}. \qquad (1.5.7)$$

In the particular case of a uniformly accelerated motion starting from the state of rest, we have

$$v = at, \qquad s = at^2/2, \qquad v = \sqrt{2as}. \qquad (1.5.8)$$

If the motion is not uniformly variable, i.e. $a \neq$ const, the velocity graph of such a motion will differ from a straight line and will be a curve. However, in this case also the distance covered by a body is equal to the area bounded by this curve and the t-axis even if the velocity changes its sign. For example, Fig. 17 shows the velocity graph of a rectilinear motion with an acceleration antiparallel to the velocity. It can be seen that up to the instant t_1 when the velocity becomes zero ($v_1 = 0$), the average velocity of motion, as before, is equal to the half-sum of the velocities at the beginning and end of the motion:

$$v_{av} = (v_0 + v_1)/2 = v_0/2.$$

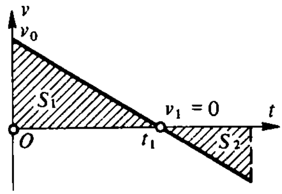

Fig. 17

The distance covered in this motion is numerically equal to the area S_1 and is calculated by the same formula (1.5.5) as for $a > 0$. In spite of the fact that now $a < 0$, the motion of the body for $t < t_1$ takes place in the direction of the velocity, so that the displacement of the body is parallel to its velocity, and the magnitude of the displacement is equal to the path length.

If the body continues its motion with the same acceleration after its velocity becomes zero (but now in the opposite direction), the magnitude of the displacement decreases since it is equal to the *difference* in areas S_1 and S_2. It can be easily shown that in this case, the coordinate of the body and its displacement at an instant $t > t_1$ should be calculated by the same formulas (1.5.6) and (1.5.5), bearing in mind that the acceleration is negative. If the motion continues further, the projection of the displacement onto the direction of vector $\mathbf{v}_0$ may become negative ($s < 0$) in accordance with formula (1.5.5). On the other hand, the distance covered by the body continues to increase after the instant t_1 also and cannot be calculated by formula (1.5.5). This distance is equal to the *sum* and not the difference of areas S_1 and S_2. Starting from instant t_1, the distance covered should be calculated as in the case of motion when the directions of velocity and acceleration coincide, taking the instant when the velocity of the body is zero as the reference point.

Figure 18 represents the time dependences of acceleration, velocity, and displacement of a body (i.e. its coordinate for $x_0 = 0$) for a uniformly variable motion.

The solid curve in Fig. 19 is the time dependence of the coordinate of the body for $x_0 = 0$, while the dashed line shows the time dependence of the path length when the body has a negative ac-

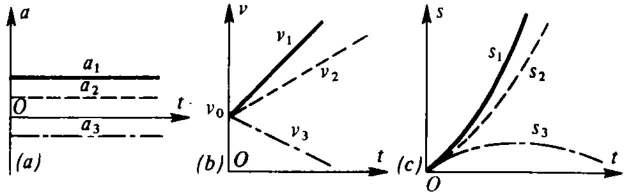

Fig. 18

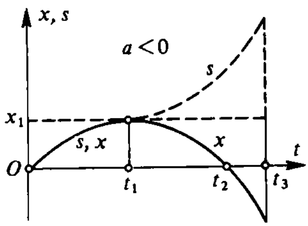

Fig. 19

celeration. These graphs coincide only at $t = t_1$, when the velocity of the body vanishes. Obviously, for $t > t_1$, the time dependence of the path length is just the graph of the coordinate, reflected from the horizontal line passing through the point (t_1, x_1).

The equation of motion with a negative acceleration is often written in the form

$$x = v_0 t - at^2/2.$$

In this case, the acceleration a on the right-hand side is a positive quantity, viz. the magnitude of acceleration. The minus sign in front of a indicates that the acceleration is directed against the coordinate axis. The motion with a constant negative acceleration is called a **uniformly decelerated (retarded) motion**.

It should be noted that the graph of the coordinate has the following property: the slope of the tangent to the coordinate graph of a body is equal to the velocity of the body at a given instant. Since the velocity of the body cannot change jumpwise, the graphs of coordinate and velocity of the body do not have breakpoints (otherwise, the tangents to the curve to the right and to the left of a breakpoint would have different slopes which would mean that the velocity changed jumpwise at the moment of time corresponding to the breakpoint on the graph of the coordinate or the path length).

Problems with Solutions

1. A motorboat covers the distance between two sports on the river in $t_1 = 8$ h and $t_2 = 12$ h downstream and upstream respectively. What is the time required for the boat to cover this distance in still water?

Solution. Let us denote the distance between the spots by L, the velocity of the motorboat in still water by v, and the velocity of water flow in the river by u. Then the equations of motion of the boat downstream and upstream can be written as $L = (v + u)t_1$ and $L = (v - u)t_2$. From these equations, we must find $t = L/v$. For this purpose, we write these equations so as to eliminate u:

$$L/t_1 = v + u, \quad L/t_2 = v - u.$$

Adding these equations, we obtain $L/t_1 + L/t_2 = 2v$. This gives

$$t = L/v = 2t_1t_2/(t_1 + t_2) = 9.6\,\text{h}.$$

2. A passenger sitting by the window of a train moving with a velocity $v_1 =$ 72 km/h sees for 10 s a train moving with a velocity $v_2 = 32.4$ km/h in the opposite direction. Find the length of the second train.

Solution. We take the passenger as the reference body and direct the coordinate axis along the motion of the second train. This train moves with a velocity $v = v_1 + v_2$ relative to the passenger. The length of the train is $l = (v_1 + v_2)t$. In SI units, we have $v_1 = 20$ m/s and $v_2 = 9$ m/s. Thus $l = 290$ m.

3. A car moves southwards with a velocity $v_1 = 80$ km/h. Another car moves eastwards with a velocity $v_2 = 60$ km/h. Find the velocity (magnitude and direction) of the second car relative to the first.

Solution. Suppose that at a certain instant of time, the first car is at point A, while the second car is at point B (Fig. 20). We take the first car as the reference

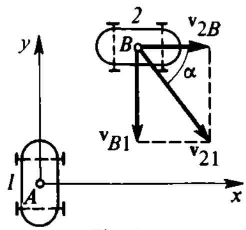

Fig. 20

body and attach to it the coordinate system xOy, which we assume to be fixed. In this system, point B on the surface of the Earth has a velocity which is equal to v_1 and is directed southwards: $v_{B1} = v_1 = 80$ km/h. Relative to point B, the second car has a velocity which is equal to v_2 and is directed eastwards: $v_{2B} = v_2 =$ 60 km/h. Adding vectors $\mathbf{v}_{B1}$ and $\mathbf{v}_{2B}$ in accordance with the parallelogram rule, we obtain $v_{21} = 100$ km/h. This velocity is directed along the diagonal of a rectangle having vectors $\mathbf{v}_{2B}$ and $\mathbf{v}_{B1}$ as its sides. Vector $\mathbf{v}_{21}$ forms an angle α with the x-axis, such that

$$\cos\alpha = v_{2B}/v_{21} = 0.60, \quad \alpha \simeq -53°.$$

4. An aeroplane flies from point A to point B at a distance $L = 1000$ km northwards of A with a velocity $v_1 = 500$ km/h relative to air. A north-westerly wind is blowing with a velocity $v_2 = 100$ km/h all the time during the flight. What is the time of flight if the plane flies in a straight line? At what compass must the pilot direct his plane?

Solution. We take the surface of the Earth as the reference body. The velocity of the plane relative to the Earth (Fig. 21) is the vector sum of the velocity $\mathbf{v}_1$

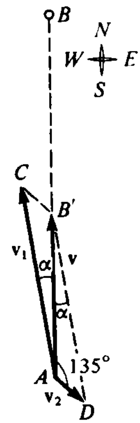

Fig. 21

relative to air and the velocity $\mathbf{v}_2$ of air relative to the Earth. It is equal to the diagonal of the parallelogram whose sides are v_1 and v_2, vector $\mathbf{v}_2$ pointing southeastwards, and vector $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ pointing northwards.

The magnitude of the velocity vector $\mathbf{v}$ can be found from $\triangle AB'D$ (or from $\triangle ACB'$) in which the two sides AD and $B'D$ and $\angle B'AD = 135°$ are known. From the law of sines, we have $\sin\alpha/\sin 135° = v_2/v_1$, whence

$$\sin\alpha = (v_2/v_1)\sin 135° \simeq 0.14, \quad \alpha \simeq 0.14\,\text{rad} \simeq 8°.$$

The required point is NW 8°. Using the law of cosines and taking into account that $B'D = v_1$, we obtain $v_1^2 = v^2 + v_2^2 - 2vv_2\cos 135°$. This gives

$$v^2 + 100\sqrt{2}v - 24 \times 100^2 = 0,$$

whence $v = 300\sqrt{2} \approx 424$ km/h. The required time of flight is

$$t = L/v = 2.36\,\text{h}.$$

5. The velocity of a train between two points is $v_1 = 80$ km/h, the average velocity over the entire path is $v_2 = 60$ km/h. The time $t_h = 1$ h is spent on halts. Find the distance L between the points.

Solution. We denote the duration of the journey by t, and write two equations: $L = v_1 t$ and $L = v_2(t + t_h)$. Solving them together, we find $L = v_1 v_2 t_h/(v_1 - v_2) = 240$ km.

6. A train having a velocity $v_0 = 70$ km/h starts moving with a uniform deceleration and its velocity decreases to $v = 52$ km/h over time $t = 10$ s. What is its acceleration on this segment of the path? What distance does it cover?

Solution. Having expressed the velocities in SI units, we find the acceleration by definition:

$$a = (v - v_0)/t = -0.5\,\text{m/s}^2.$$

The acceleration of the train is negative, since it is directed against the train velocity. In this case, the distance covered by the train is equal to its displacement, since the train moves in a straight line in the same direction:

$$s = x = v_{av} t = \frac{v_0 + v}{2} t \approx 170\,\text{m}.$$

7. A train leaves a station with an acceleration $a = 0.4$ m/s^2 and covers a distance $s = 0.5$ km. What is its velocity at the end of the path and the time which the train takes to gather speed?

Solution. For determining the velocity of the train, we use the formula $v = \sqrt{2as}$, since all the quantities appearing on the right-hand side of this formula are given: $v = 20$ m/s $= 72$ km/h. The time which the train takes to gather speed is $t = v/a = 50$ s.

8. A body moving with a uniform acceleration from the state of rest covers a distance $s = 450$ m during time $t = 6$ s. At what distance from the initial position is the body four seconds after the beginning of motion?

Solution. The body covers a distance $s = at^2/2$ during time t, while it covers a distance $s' = at'^2/2$ during time t'. Dividing the first equation by the second, we get $s'/s = t'^2/t^2$, whence $s' = 200$ m.

Note. Sometimes, such problems are solved as follows. The acceleration is found from the first equation and is substituted into the second equation. However, there is no need to do this since acceleration is easily eliminated by dividing the equations termwise.

9. A body with an initial velocity $v_0 = 4$ m/s covers a distance $\Delta s = 2.9$ m during the sixth second of its motion. Find the acceleration of the body.

Solution. The distance covered by the body during the sixth second of its motion is

$$\Delta s = s_6 - s_5 = (v_0 t_6 + a t_6^2/2) - (v_0 t_5 + a t_5^2/2),$$

whence

$$a = 2(v_0 \cdot 1\,\text{s} - \Delta s)/(t_5^2 - t_6^2) = -0.2\,\text{m/s}^2.$$

The body slows down with an acceleration $a = -0.2$ m/s^2 directed against the velocity.

10. A body moving in a straight line with a constant acceleration a loses 3/4 of its initial velocity v_0. What is the time of motion and the distance covered by the body during this time?

Solution. Since the body loses 3/4 of its initial velocity, its velocity at the end of the required time interval is $v = v_0/4$. We direct the coordinate axis along the velocity of motion. Then $v = v_0/4 = v_0 + at$, whence $t = -3v_0/4a$, $a < 0$. The distance covered by the body during this time is $s = v_{av}t = -15v_0^2/32a$, $a < 0$.

11. A train passes an observer standing on a platform. The first carriage of the train passes the observer during time $t_1 = 1$ s and the second, during time $t_2 = 1.5$ s. Find the velocity of the train at the beginning and end of observation and its acceleration, assuming that the motion of the train is uniformly variable. The length of each carriage is $l = 12$ m.

Solution. Considering that $v_{av} = l/t$, we can write three equations:

$$v_0 + v_1 = 2l/t_1 \qquad \text{for the first carriage,} \qquad (1)$$

$$v_1 + v_2 = 2l/t_2 \qquad \text{for the second carriage,} \qquad (2)$$

$$v_0 + v_2 = 4l/(t_1 + t_2) \qquad \text{for two carriages.} \qquad (3)$$

Solving these equations, we obtain

$$v_0 = l\left(\frac{1}{t_1} - \frac{1}{t_2} + \frac{2}{t_1 + t_2}\right) = 13.6\,\text{m/s}.$$

Equation (3) gives

$$v_2 = 4l/(t_1 + t_2) - v_0 = 5.6\,\text{m/s}.$$

The acceleration $a = (v_2 - v_0)/(t_1 + t_2) = -3.2$ m/s^2. The minus sign indicates that the motion of the train is decelerated.

12. Two motorcars start moving one after the other with an interval $\Delta t = 1$ min and move with an acceleration $a = 0.4$ m/s^2 each. How long after the start of the first car is the distance between them equal to $s_1 - s_2 = 4.2$ km?

Solution. Let us find the distances covered by each car: $s_1 = at^2/2$ and $s_2 = a(t - \Delta t)^2/2$. Subtracting s_2 from s_1, we get $s_1 - s_2 = a\,\Delta t\,(t - \Delta t/2)$, whence

$$t = (s_1 - s_2)/(a\,\Delta t) + \Delta t/2 = 205\text{ s}.$$

13. A material point starts moving eastwards with a uniform acceleration $a = 0.5$ m/s^2 directed against its initial velocity $v_0 = 20$ m/s. Find the distance covered by the point and its displacement in $t_1 = 0.5$ min and $t_2 = 2$ min after the beginning of motion.

Solution. Let us first find the displacement of the point over time t_1. We direct the coordinate axis along vector $\mathbf{v}_0$ and take the initial position of the point as the origin. Then the displacement coincides with its coordinate:

$$x_1 = v_0t_1 + at_1^2/2 = 375\text{ m}.$$

Similarly,

$$x_2 = v_0t_2 + at_2^2/2 = -1200\text{ m}.$$

Half a minute after the beginning of motion, the material point is at a distance of 375 m to the east of the initial position, and two minutes after, at 1200 m to the west.

In the general case, the distance covered by the body is the sum of two distances: from the origin to the point at which the velocity $v = 0$ (this is the maximum displacement of the body in the direction of vector $\mathbf{v}_0$) and the distance covered by it in the opposite direction. In order to find these distances, we must know the instant t_0 when the velocity becomes equal to zero, i.e. when the direction of its motion is reversed. If we substitute the value $t = t_0$ into the formula $v_0 + at$, we obtain $0 = v_0 + at_0$. Hence $t_0 = -v_0/a = 40$ s. Since $t_1 < 40$ s, the distance s_1 covered over time t_1 is equal to the displacement over the same time: $s_1 = x_1 = 375$ m. The distance s_2 covered over time t_2 can be found as the sum $s_2 = s_2' + s_2''$, where s_2' is the distance covered over time $t_0 = 40$ s and s_2'' is the distance covered in the opposite direction. The distance s_2' is equal to the displacement x of the body over time t_0:

$$s_2' = x = v_0 t_0 + a t_0^2/2 = 400 \text{ m}.$$

The distance s_2'' can be found by taking the point at which the body is at instant t_0

$t_0 = 80$ s. Obviously, the distance s_2'' is equal to the displacement of the body in this coordinate system:

$$s_2'' = a' t_3^2/2 = 1600 \text{ m}.$$

Consequently, $s_2 = s_2' + s_2'' = 2000$ m.

14. Two cyclists move towards each other. The first cyclist, whose initial velocity $v_{01} = 5.4$ km/h, descends the hill, gathering speed with an acceleration $a_1 = 0.2$ m/s^2. The second cyclist, whose initial velocity $v_{02} = 18$ km/h, climbs the hill with an acceleration $a_2 = -0.2$ m/s^2. How long does it take for the cyclists to meet if the distance x_0 separating them at the initial instant of time is 195 m?

Solution. We place the origin at the initial position of the first cyclist and choose the positive direction of the x-axis so that it coincides with the direction of its initial velocity. Then the equations for the coordinates of the first and second cyclists are

$$x_1 = v_{01} t + a_1 t^2/2, \quad x_2 = x_0 + v_{02} t + a_2 t^2/2.$$

From the condition of the problem, we have $|a_1| = |a_2| = 0.2$ m/s^2, $v_{01} = 1.5$ m/s, and $v_{02} = -5$ m/s. At the moment of their meeting, $x_1 = x_2$. Equating the right-hand sides of these equations, we find $t = x_0/(v_{01} - v_{02}) = 30$ s.

At this stage, we could terminate the solution of the problem. However, in our case we must convince ourselves that the result obtained has a physical meaning. For this purpose, we find the velocities of the cyclists 30 seconds after the beginning of motion:

$$v_1 = v_{01} + a_1 t = 7.5 \text{ m/s}, \quad v_2 = v_{02} + a_2 t = 1 \text{ m/s}.$$

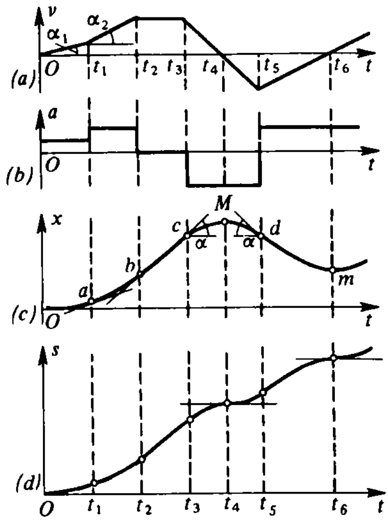

Fig. 22

It turns out that by this time the second cyclist will descend the hill rather than climb it. Obviously, the problem is formulated not quite correctly. It should be indicated how the second cyclist moves while descending the hill or a distance smaller than x_0 should be given.

15. Given the time dependence of the velocity of a material point (Fig. 22*a*). Plot the time graphs for the acceleration, displacement, and the distance covered by this point.

Solution. It follows from the given velocity graph that in the first time interval $(0, t_1)$ the material point moves with a constant acceleration a_1 = const. In the second interval (t_1, t_2), it moves with a constant acceleration $a_2 > a_1$, since the slope of the velocity vs time curve is larger in this interval ($\tan \alpha_2 > \tan \alpha_1$). In the third interval (t_2, t_3), the motion is uniform ($a_3 = 0$). In the fourth and fifth intervals (t_3, t_5), the motion is uniformly decelerated, and then it is again uniformly accelerated with an acceleration a_2, since the slope of the velocity vs time curve is the same here as on the second interval. The acceleration graph shown in Fig. 22*b* is constructed on the basis of these considerations.

Going over to the displacement curve (Fig. 22*c*), we find that on the first and second segments, the graph consists of two concave parabolas, the parabola on the second segment being steeper since the velocity graph is also steeper here. It should be noted that at a common point *a* the parabolas have a common tangent since the velocity varies smoothly at this point. On the third segment (t_2, t_3), where the motion is uniform, the graph is a straight line coinciding with the tangent to the second parabola at the point b and on the segment (t_3, t_5), with the tangent to the convex parabola at the point c. At the point t_4, where the velocity is zero, the material point changes the direction of its motion (its displacement is maximum at this point). On the segment (t_5, t_6), the displacement of the material point decreases, and at the point t_6 it starts to increase again.

On the segment $(0, t_4)$ and beyond the point t_6, the distance graph covered by the point (Fig. 22*d*) coincides with the displacement curve. On the segment (t_4, t_6), it is the mirror reflection of the displacement curve from the tangent at point M.

Exercises

1. (a) A motor ship covers a distance of 300 km between two localities on a river in 10 h downstream and in 12 h upstream. Find the flow velocity of the river and the cruising velocity of the motor ship in still water, assuming that these velocities are constant.
Answer. 2.5 km/h, 27.5 km/h.

(b) A motorboat covers the distance between two localities on a river and returns in 14 h. What is this distance if the velocity of the boat in still water is 35 km/h, and the flow velocity of the river is 5 km/h?
Answer. 240 km.

(c*) A motor ship covers the distance between two localities on a river in 60 h downstream and in 80 h upstream. How many days does it take for a raft to float between these localities?
Answer. 20 days.

2. (a) A passenger sits by the window of a train moving with a velocity of 90 km/h. A 700-m long train moves from the opposite direction with a velocity of 36 km/h. How long does the second train pass the passenger?
Answer. 20 s.

(b) A motorboat moves on a lake with a velocity of 72 km/h and overtakes a motor ship holding a parallel course with a velocity of 54 km/h in 1 min. Find the length of the motor ship.
Answer. 300 m.

(c*) A motorcar passes a parking lot 10 min after a lorry moving in the same direction. At what distance from the parking lot does the motorcar overtake the lorry if it travels with a constant velocity of 60 km/h, while the velocity of the lorry is 40 km/h?
Answer. 20 km.

(d*) Two trains move in the same direction along parallel tracks. One of them is a 200-m long passenger train travelling with a velocity of 72 km/h. The second one is a 800-m long goods train travelling with a velocity of 27 km/h. How long will it take for the passenger train to overtake the goods train?
Answer. 80 s.

(e*) The person bringing up the rear of a 2.5-km long column of troops moving with 5 km/h sends a motorcyclist with a dispatch to the commander at the head of the column. The commander receives the dispatch and writes down his reply for 3 min, standing at the road side. Find the average velocity of the motorcyclist if he returns to the rear in 9 min 45 s.
Answer. 45 km/h.

3. (a) A swimmer crosses a 200-m wide channel with straight banks and returns in 10 min at a point 300 m below the starting point (downstream). Find the magnitude and direction of the velocity of the swimmer relative to the bank if he heads towards the bank of the channel all the time at right angles.
Answer. 3 km/h, at an angle of arctan (4/3) = 53° to the bank.

(b) The velocity of a ship in still water is 20 km/h. What is the velocity of a motorboat approaching the ship at right angles to its course if it appears to people on board the ship that the motorboat heads towards the ship at 60°?
Answer. 34.6 km/h.

(c*) Raindrops fall vertically at a speed of 20 m/s. At what angle do they fall on the windscreen of a motorcar moving with a velocity of 54 km/h if the windscreen is inclined at an angle of 10° to the vertical?
Answer. arctan (3/4) + 10° ≈ 47°.

4. (a) A boatman crossing a 120-m wide channel heads all the time at 60° to the bank (against the flow). After two minutes he reaches the opposite bank 10 m below the starting point (downstream). Find the flow velocity of water in the channel (assuming that it is constant over the entire channel width) and the velocity of the boat relative to water.
Answer. 2.4 km/h, 4.2 km/h.

(b*) A motorboat heads upstream towards the opposite bank of a 200-m wide channel in a direction which forms an angle of 60° with the bank. At what angle to the bank must the motorboat head in order to reach its destination if the velocity of the boat relative to water is 18 km/h and the flow velocity of water in the channel is 0.5 m/s? How long does it take for the boat to reach its destination? For small angles, it can be assumed that $\sin\alpha \approx \tan\alpha \approx \alpha$ rad.
Answer. 55°49′, 1.5 min.

5. (a) A bus covers a distance of 10 km in 25 min, spending 5 min for all its halts. Find the velocity of the bus between the halts and its average velocity over the entire distance.
Answer. 30 km/h, 24 km/h.

(b) A train covers a distance of 330 km with a velocity of 60 km/h. What is the average velocity of the train over this distance if it spends 30 min for all its halts at intermediate stations?
Answer. 55 km/h.

6. How long does it take for a train to increase its velocity in a uniformly accelerated motion from 12 km/h to 60 km/h over a distance of 800 m? What is its acceleration?
Answer. 1 min, 2.2. m/s^2.

7. (a) A train starts from a station with an acceleration of 0.5 m/s^2 and attains the velocity of 60 km/h. What is the distance covered by it?
Answer. 278 m.

(b) As a result of braking, a train moving with a velocity of 90 km/h is subjected to a uniform acceleration of -0.3 m/s^2. Find its velocity at a distance of 1 km from the point where the brakes are applied.
Answer. 18 km/h.

(c) A train having a velocity of 32.4 km/h starts gaining speed with a uniform acceleration of 0.2 m/s^2. What is its velocity when it covers 0.8 km?
Answer. 71.8 km/h.

(d) The engine driver of a train applies the brakes when the train approaches a station with a velocity of 50 km/h and stops the train at the station. At what distance from the station are the brakes applied if the train covers this distance with a uniform acceleration of -0.5 m/s^2?
Answer. 192 m.
8. (a) Two motorcars start off with a gap of 2 min. How long after the departure of the second car is the distance covered by it equal to 1/9 of the distance covered by the first car?
Answer. 1 min.
(b) Two motorcars start off with a certain time gap and move with the same acceleration. Two minutes after the start of the second car, the distance covered by it is 4/9 of the distance covered by the first car by this moment. How long after the departure of the first car does the second car start?
Answer. 1 min.
(c) A body starts moving with a uniform acceleration from the state of rest and covers the first hundred metres in half a second. How long does it take for the body to cover the first metre of its path? How long does it take for the body to cover 10 km from the starting point?
Answer. 0.05 s, 5 s.
(d) Moving with a uniform acceleration, a body covers a certain distance in 12 s. How long does it take for the body to cover the last one third of the distance?
Answer. 2.2 s.
(e*) Moving with a uniform acceleration from the state of rest, a body covers 22 m during the sixth second of its motion. What is the distance covered by it during the first six seconds? During the next six seconds?
Answer. 72 m, 216 m.
9. (a) Moving with a uniform acceleration, a body covers 150 m during 10 s so that it covers 24 m during the tenth second. Find the initial velocity and acceleration of the body.
Answer. 5.0 m/s, 2.0 m/s^2.
(b) Moving with a constant acceleration of -1.2 m/s^2, a body covers 4.2 m during the fourth second. Find its initial velocity, the distance covered by it, and its displacement over 10 s.
Answer. 8.4 m/s, 34.8 m, 24.0 m.
10. A body moving with a constant acceleration a loses 2/3 of its initial velocity v_0. Find the time during which this occurs and the distance covered by the body during this time.
Answer. $2v_0/3a$, $4v_0^2/9a$.
11. (a) A material point covers 24 m and 64 m during two consecutively equal time intervals of 4 s each. Find its velocity at the beginning and end of each segment of the path and the acceleration if the motion of the point is uniformly accelerated.
Answer. 1 m/s, 11 m/s, 21 m/s, 2.5 m/s^2.
(b) Moving with a constant acceleration, a material point covers 24 m in 2 s and the next 24 m, in 4 s. Find the velocity of the point at the beginning and end of the distance and the acceleration of the point.
Answer. 14.0 m/s, 2.0 m/s, -2.0 m/s^2.
12. (a) Two motorcars leave with a 1-min gap and move with an acceleration of

0.2 m/s^2. How long after the departure of the second car does the distance between them become equal to three times its initial value?
Answer. 1.0 min.

(b) Two motorcars leave one after the other and move with an acceleration of 0.4 m/s^2. Two minutes after the departure of the first car, the distance between them becomes 1.90 km. What is the time interval between their departures?
Answer. 50 s.

(c) Two motorcars leave one after the other and move with an acceleration of 0.5 m/s^2. Two minutes after the departure of the second car, the distance between them becomes 8.5 km. What is the time interval between their departures?
Answer. 100 s. However, the velocity of the first car at this instant would be 396 km/h. Consequently, the problem is formulated incorrectly and does not have a real solution.

13. A material point starts a uniformly variable motion northwards with an initial velocity of 54 km/h and an acceleration of 0.2 m/s^2 directed against the initial velocity. Find its displacement and the distance covered in 1 min and 3 min after the beginning of motion.
Answer. x_1 = 540 m northwards from the initial position, x_2 = 540 m southwards from the initial position, s_1 = 540 m, s_2 = 1665 m.

14*. Two cyclists approach each other from opposite directions. One of them descends the hill with an initial velocity of 7.2 km/h and an acceleration of 0.30 m/s^2. The other climbs the hill with an initial velocity of 36 km/h and an acceleration of 0.20 m/s^2 directed against this velocity. What is the distance between the cyclists at the initial instant if they meet half a minute after the beginning of motion? What is the maximum length of the side of the hill for which the problem has a solution?
Answer. 405 m, 725 m.

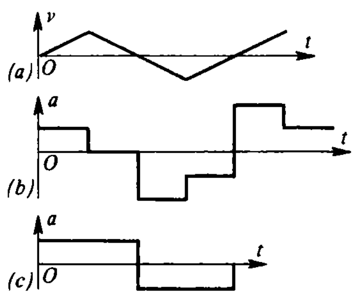

Fig. 23

15. (a) Given the graph of the velocity of a material point as a function of time (Fig. 23*a*). Plot the acceleration, displacement, and the distance covered by the point as functions of time.[3]
Answer. See Fig. I.

[3] The answers to problems in the form of graphs are given at the end of the book.

(b) Given the graph of the acceleration of a material point as a function of time (Fig. 23*b*). Plot the velocity, displacement, and the distance covered by the point as functions of time, assuming that the motion starts from the state of rest.
Answer. See Fig. II.

(c) Given the graph of acceleration as a function of time (Fig. 23*c*). Plot the displacement as a function of velocity.
Answer. See Fig. III.

B. DYNAMICS OF TRANSLATORY MOTION

1.6. Force

A change in the motion of a body, i.e. a change in its velocity (the acceleration of the body), is caused by the action of other bodies on it. If a body acts on another body and causes its acceleration, the measure of this action is a *vector quantity* which is called **force**.

In an analysis of motion of a body, the action of other bodies on it is represented by vectors—forces—applied to this body. This allows us, after analyzing the interaction of bodies, to digress the very process of interaction and assume that an acceleration is imparted to a body whose motion is being studied by a force applied to it. If a force or system of forces applied to a body can be replaced by another force or system of forces without altering the motion of the body, such forces or systems of forces are called **equivalent**. In particular, when a system of forces is replaced by a single force, this force is called the **resultant.** Experiments show that the resultant of a system of several forces acting along the straight lines intersecting at a point is applied at the same point and is equal to the geometrical (vector) sum of all the forces of the system.

The SI unit of force (see Sec. I.1) is a **newton** (N).

1.7. Newton's First Law. Inertial and Noninertial Reference Systems

Newton's first law can be formulated as follows: *a material point preserves its state of rest or uniform motion in a straight line if the actions of other bodies on it are mutually balanced*, or: *a material* *point preserves its state of rest or uniform motion in a straight line if the resultant of all the forces applied at it is zero.*

The property of a body to preserve its velocity in the absence of external actions on it by other bodies is called **inertia**. The quantitative characteristic of inertia is a physical quantity called the **mass** of the body.

The SI unit of mass is a **kilogram** (kg).

The motion of a body is *relative*. It depends on the choice of the reference system. We can consider the following simple example: the driver of a motorcar moving with an acceleration is at rest relative to the reference frame fixed to the car. On the other hand, he moves with an acceleration equal to the acceleration of the car relative to a man standing on the side of the road. For this reason, formulating the laws of motion, we should obviously indicate the reference system in which the motion is considered. Newton's first law formulated above, viz. the law of inertia, is valid not in all reference systems.

A reference system in which Newton's first law is valid is called an **inertial reference system.** A reference system in which the law of inertia does not hold is called a **noninertial reference system**.

Let us elaborate the above statement. Suppose that a smooth ball lies on a perfectly smooth table in the coach of a train. As long as the train is at rest or moves uniformly in a straight line, the ball is stationary. An observer standing on the platform sees the ball at rest if the train is stationary, and moving with a constant velocity equal to the velocity of the train if the latter moves uniformly in a straight line. An observer in the train sees the ball at rest in both cases. In both reference systems, the motion of the ball obeys Newton's first law. The two reference frames, viz. the one connected to the platform and the one fixed to the train, are inertial systems.

But as soon as the coach acquires an acceleration, the observer in the train sees that the ball rolls backwards although the action of other bodies on it remains unchanged. As a matter of fact, the reference frame fixed to the train moving with an acceleration is a noninertial system. At the same time, for the observer on the platform, the ball preserves its state of rest or uniform motion in a straight line with an accuracy admissible to the observer's eye. With this accuracy, the reference frame fixed to the surface of the Earth is an inertial system.

On the other hand, the Earth moves in a curvilinear trajectory relative to the Sun and the stars. Besides, the Earth rotates about its axis. Consequently, the velocity of any point on the Earth's surface continuously changes both in magnitude and direction relative to the Sun. In other words, any point on the Earth moves with an *acceleration* relative to the Sun. Consequently, a geocentric reference system, and a reference frame fixed to the surface of the Earth, are not *exactly* inertial systems. However, the accuracy required for analyzing the motion of motorcars, trains, aeroplanes, and other terrestrial bodies allows us to treat these reference systems as inertial.

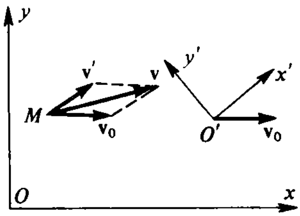

Fig. 24

Any reference system moving relative to a certain inertial reference system with a constant velocity, i.e. uniformly in a straight line, is also an inertial reference system. Indeed, let us consider the motion of a material point M in two reference systems xOy and $x'Oy'$ (Fig. 24). It is known that the second system is inertial and moves with a constant velocity $\mathbf{v}_0$ relative to the first system. Suppose that the velocity of point M relative to the system $x'Oy'$ is constant and equal to $\mathbf{v}'$. Then the velocity $\mathbf{v}$ of this point relative to the system xOy is equal to the geometrical sum of the velocities $\mathbf{v}_0$ and $\mathbf{v}'$ and is represented by the diagonal of the parallelogram constructed on the vectors $\mathbf{v}_0$ and $\mathbf{v}'$. Hence it is also a constant quantity. This proves that the reference system xOy is also inertial.

1.8. Newton's Second Law. Momentum of a Body

Newton's second law states that the *acceleration of a body is directly proportional to the force* $\mathbf{F}$ *acting on the body and inversely proportional to its mass*:

$$\mathbf{a} = k\mathbf{F}/m.$$

The SI unit of force is chosen so that the proportionality factor k is equal to unity. Consequently, the equation for Newton's second law becomes

$$\mathbf{a} = \mathbf{F}/m. \qquad (1.8.1)$$

If several forces are applied to a body, the acceleration of the body is the geometrical sum of the accelerations imparted by each of the forces. In other words, the force $\mathbf{F}$ should be treated as the resultant of all the forces applied to the body. The magnitude of acceleration is calculated by the formula

$$a = F/m,$$

where F is the magnitude of force.

The **momentum** $\mathbf{p}$ of a body is a vector equal to the product of the mass of the body and its velocity

$$\mathbf{p} = m\mathbf{v}. \qquad (1.8.2)$$

Assuming that the force is constant, we multiply both sides of (1.8.1) by the mass of the body and the time of action of the force. This gives

$$m\mathbf{a}t = \mathbf{F}t.$$

Since the acceleration $\mathbf{a}$ is also constant when the force $\mathbf{F}$ is constant, the product $\mathbf{a}t$ gives the velocity increment over time t. Consequently,

$$m(\mathbf{v}_2 - \mathbf{v}_1) = \mathbf{F}t. \qquad (1.8.3)$$

Taking into account (1.8.2), we obtain the following relation:

$$\mathbf{p}_2 - \mathbf{p}_1 = \mathbf{F}t.$$

Thus, Newton's second law can be formulated as follows: *the* *increment of the momentum of a body is equal to the force acting on it multiplied by the time of its action.*

After the formulation of the special theory of relativity, it turned out that formula (1.8.3) is more general than (1.8.1).

While considering a motion caused by a varying force, we must divide the time t of action of the force in infinitely small intervals dt. Formula (1.8.1) leads to the following relation for each time interval:

$$m\mathbf{a}\,dt = \mathbf{F}\,dt.$$

Since $\mathbf{a}\,dt = d\mathbf{v}$ and m is constant, we have $m\mathbf{a}\,dt = m\,d\mathbf{v} = d(m\mathbf{v}) = d\mathbf{p}$. Thus, we arrive at the equality

$$d\mathbf{p} = \mathbf{F}\,dt. \tag{1.8.4}$$

Finally, integrating over time between t_1 and t_2, we obtain

$$\mathbf{p}_2 - \mathbf{p}_1 = \int_{t_1}^{t_2} \mathbf{F}\,dt. \tag{1.8.5}$$

Let us consider two examples involving Newton's second law.

1. Anvils are made as heavy as possible, since the acceleration imparted to an anvil by the impact of hammer is the lower, the larger the mass of the anvil.

2. A ball attracted by a magnet moves with an increasing acceleration, since the force of attraction grows as the ball approaches the pole of the magnet.

Newton's second law holds only in inertial reference systems. For example, a ball lying on the floor of a train coach moving uniformly in a straight line is at rest relative to the reference frame fixed to the train and moves uniformly in a straight line relative to the reference frame fixed to the rails. As soon as the train receives an acceleration $\mathbf{a}$, the ball starts to move relative to the reference frame fixed to the train with an acceleration having the opposite direction with respect to the acceleration of the train, although the action of other bodies on the ball remains unchanged. Relative to the noninertial reference frame moving with the acceleration $\mathbf{a}$, the ball will move as if the force $-m\mathbf{a}$ were applied to it. This force is called the **inertial force**. It is due to the properties of a noninertial reference frame and not to interaction with other bodies.

Remark. If we put $\mathbf{F} = 0$ in Eq. (1.8.1) expressing Newton's second law, we obtain $\mathbf{a} = 0$. This means that a body not experiencing the action of other bodies moves without an acceleration. It may appear that Newton's first law is a corollary of Newton's second law. Actually, this is not so, and Newton's first law is important on its own. The physical meaning of Newton's first law lies in that it defines inertial reference systems.

Newton's first law states that inertial reference systems do exist. These are the systems in which a body which does not experience the action of other bodies (the actions of other bodies can be mutually balanced) moves uniformly in a straight line. (The state of rest is a particular case of uniform rectilinear motion.)

1.9. Newton's Third Law

The action of one body on another is an interaction. If body *1* acts on body *2* with a force $\mathbf{F}_1$, body *2*, in turn, acts on body *1* with a force $\mathbf{F}_2$.

Newton's third law states that *the forces with which two bodies act on each other are equal in magnitude and have opposite directions* (Fig. 25):

$$\mathbf{F}_1 = -\mathbf{F}_2.$$

It follows from Newton's third law that forces appear in pairs: any force applied to a body can be juxtaposed to a force of equal magnitude and opposite direction, applied to another body interacting with the given body.

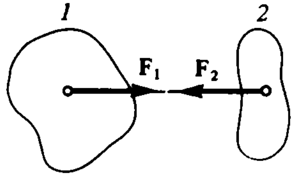

Fig. 25

Newton's third law is valid not in all cases. It strictly holds for interactions occurring during direct contact of bodies as well as for interacting bodies at rest separated by a certain distance.

An example illustrating the violation of Newton's third law is a system of two charged particles moving apart so that their velocities are at right angles, or a system of two electrically neutral particles having masses m_1 and m_2, separated by a distance r, and moving with velocities close to the speed of light.

1.10. Principle of Independence of Action

The **principle of independence of action** consists in that *each force acting on a body imparts to it an acceleration which is independent of the motion of the body as well as of the action of other forces on it.*

The manifestation of this principle can be illustrated by the following examples.

1. A passenger drops a load from the window of a train. The time taken by the falling load to reach the rails and the velocity of the fall (i.e. the vertical component of the velocity of motion) at this moment do not depend on the velocity of the train, since the action of the force of gravity due to which the vertical displacement of the load occurs is independent of its horizontal motion by inertia.

2. The time of fall and the vertical component of the velocity with which a shell fired horizontally at a certain height above the ground reaches the Earth's surface are the same as the time of fall and the velocity of a body dropped from the same height.

1.11. Addition of Forces Acting at an Angle

To **add** the forces means to find their resultant. The **resultant** of two or more forces is the *force* whose action is equivalent to the action of these forces.

The following guiding principle confirmed by experiments should be used while adding forces. Two forces of equal magnitude and opposite directions, applied to a rigid body so that they lie on the same straight line connecting the points of their application, do not alter the motion of the body (Fig. 26). This statement leads to the following important corollary: the point of application of a force can be translated along the line of action of the force to any point of a rigid body, without changing the action of this force on the body.

Indeed, suppose that a force **F** is applied to a body at point A (Fig. 27). We take some point B on the line of action of the force

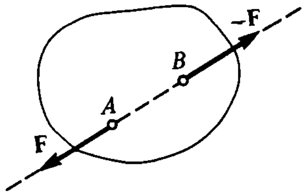

Fig. 26

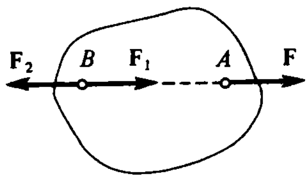

Fig. 27

F and apply two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ equal in magnitude to the force **F** and having opposite directions along the line of action of the force **F**, i.e. along the straight line AB. Since the sum of the forces **F** and $\mathbf{F}_2$ is zero, these forces do not alter the motion of the body. We are left with the force $\mathbf{F}_1$, equal to the given force **F** and applied at point B. Consequently, the force acting on a rigid body is a sliding vector.

Forces are added geometrically, as was described in Sec. 1.2.[4]

1.12. Resolution of a Force into Two Components at an Angle to Each Other

The **resolution** of a given force is its replacement by two or more other forces whose action is equivalent to the action of the given force. We shall consider here only the problem of resolving a force into two components.

The resolution of a force into two components at an angle to each other is based on the same parallelogram rule that is used for the addition of forces. The force being resolved is the diagonal of a parallelogram, while the components are its sides emerging from the point of application of the original force.

[4] The addition of forces whose lines of action do not lie in the same plane is beyond the scope of this book.

The problem has a unique solution if besides the force to be resolved, the lines of action of the two components are specified. Then this problem is reduced to the geometrical problem of constructing a parallelogram (or a rectangle in a particular case) from the three known parameters.

1. Resolution for a given diagonal (the force being resolved) and two adjacent angles (the directions of the components) (Fig. 28). Given a force **R** that is to be resolved along two directions *AB* and *CD*. We draw straight lines parallel to *AB* and *CD* through the point of application *O* and the tip *K* of the force until they intersect at points *M* and *N*. The components $\mathbf{F}_1$ and $\mathbf{F}_2$ coincide with the sides of the obtained parallelogram intersecting at the point of application *O* of the force being resolved. These forces are also applied at point *O*.

2. Resolution for a given diagonal (the force being resolved), a side of the parallelogram, and the angle between the diagonal and this side (one of the components) (Fig. 29). Given a force **R** being

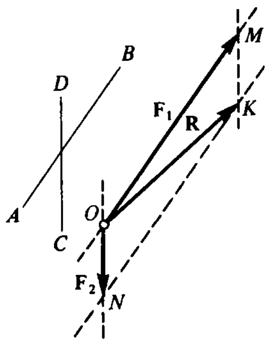

Fig. 28

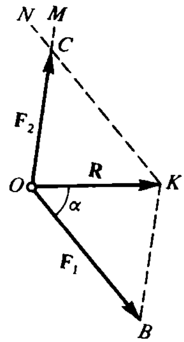

Fig. 29

resolved and a component $\mathbf{F}_1$, i.e. its magnitude (*OB*) and direction (angle α). We connect the tips of the two given forces **R** and $\mathbf{F}_1$ and draw a straight line $OM \parallel BK$ through the point of application *O* and a straight line $KN \parallel OB$ through the point *K* until they intersect at point *C*. Vector $\overrightarrow{OC}$ is the required component $\mathbf{F}_2$.

1.13. Law of Momentum Conservation

A **mechanical system** is a set of bodies singled out for an analysis. The forces acting on the bodies constituting the system can be classified as internal and external forces. **Internal forces** are those exerted on a given body by other bodies of the system. **External forces** are due to the action of bodies which do not belong to the system. If there are no external forces, a system is called **closed**.

For a closed system, the **law of momentum conservation** is valid. According to this law, the sum of the momenta of bodies constituting a closed system remains constant (is conserved):

$$\sum \mathbf{p}_i = \sum m_i \mathbf{v}_i = \text{const.} \tag{1.13.1}$$

The law of momentum conservation follows from Newton's laws. Let us demonstrate this by using a two-body system as an example. In accordance with Newton's third law, the forces with which two bodies act on each other are connected by the relation $\mathbf{F}_1 = -\mathbf{F}_2$. Multiplying this equality by the time during which the interaction takes place, we obtain $\mathbf{F}_1 \Delta t = -\mathbf{F}_2 \Delta t$. But $\mathbf{F}_1 \Delta t$ is the increment $\mathbf{p}_1' - \mathbf{p}_1$ of the momentum of the first body, while $\mathbf{F}_2 \Delta t$ is the increment $\mathbf{p}_2' - \mathbf{p}_2$ of the momentum of the second body. Thus,

$$\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1' + \mathbf{p}_2'. \tag{1.13.2}$$

Replacing $\mathbf{p}$ by $m\mathbf{v}$, we obtain

$$m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = m_1 \mathbf{v}_1' + m_2 \mathbf{v}_2'. \tag{1.13.3}$$

The left-hand sides of these equalities contain the sum of the momenta of the bodies before the interaction, while the right-hand sides contain the sum of these quantities after the interaction. Thus, the sum of the momenta of the bodies remains unchanged upon their interaction.

Since the projections of equal vectors onto coordinate axes are equal, from Eqs. (1.13.1) and (1.13.2) we obtain

$$\sum m v_x = \text{const}, \quad \sum m v_y = \text{const}, \quad \sum m v_z = \text{const},$$

or

$$\begin{aligned} m_1 v_{1x} + m_2 v_{2x} &= m_1 v'_{1x} + m_2 v'_{2x}, \\ m_1 v_{1y} + m_2 v_{2y} &= m_1 v'_{1y} + m_2 v'_{2y}, \\ m_1 v_{1z} + m_2 v_{2z} &= m_1 v'_{1z} + m_2 v'_{2z}. \end{aligned} \tag{1.13.4}$$

Thus, for any change in the mechanical state of a system, *internal forces do not alter its total momentum*.

If a system is not closed or external forces act on the bodies of the system, the total momentum of the system generally does not remain constant. In this case, its increment is equal to the product of the sum of external forces and the time of their action:

$$\Delta \mathbf{p} = \sum \mathbf{F}_{\text{ext}} \Delta t.$$

When the sum of external forces is equal to zero, the momentum of the system will be conserved.

All the relations obtained above are also valid for systems consisting of more than two bodies.

The following examples illustrate Newton's third law and the law of momentum conservation.

1. A man walks on the ground, pushing himself off the Earth by his feet. The force with which the Earth acts on the man is equal in magnitude to the force with which the man acts on the Earth.

2. A shell has exploded in air into several fragments. Here, the vector sum of the momenta of all the fragments is equal to the momentum of the shell before the explosion, because the duration of the explosion is so small that the action of external forces can be neglected.

1.14. Idea of Reaction Propulsion

The principle of reaction propulsion can be illustrated by the following simple example (Fig. 30). A vessel with a tap filled with a fluid under pressure is mounted on a cart. When the tap is closed, the vessel is at rest. Let us open the tap so that a fluid jet flows out of it with a velocity $\mathbf{v}_1$. Then the vessel starts to move with a velocity $\mathbf{v}_2$ in the direction opposite to that of the outflowing jet.

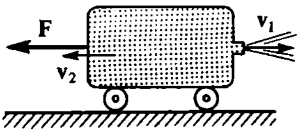

Fig. 30

Since no forces act in the horizontal direction on the vessel containing the fluid, the horizontal component of the momentum of the vessel-fluid system remains unchanged in accordance with the law of momentum conservation:

$$m\mathbf{v}_1 + M\mathbf{v}_2 = 0,$$

where m is the mass of the outflowing fluid jet, M is the mass of the vessel with the remaining fluid, $\mathbf{v}_1$ is the velocity of the particles in the jet, and $\mathbf{v}_2$ is the velocity of the vessel containing the fluid. From this equation, we obtain

$$M\mathbf{v}_2 = -m\mathbf{v}_1, \quad \text{or} \quad \mathbf{v}_2 = -(m/M)\mathbf{v}_1.$$

The velocity $\mathbf{v}_2$ of the vessel containing the fluid is opposite to the velocity $\mathbf{v}_1$ of the outflowing jet. The magnitude of the velocity of the vessel is given by the formula $v_2 = (m/M)v_1$. (Actually, the velocity of the vessel is always somewhat lower than the value obtained by this formula since it does not take into account the external forces of resistance to motion, viz. air drag and friction.)

Since the vessel started to move, i.e. its velocity has changed, this means that the force acted on it when the fluid jet flowed out of it. This force is called the **reactive force** of the outflowing jet and can be found from the equality $M\mathbf{v}_2 = \mathbf{F}t$:

$$\mathbf{F} = M\mathbf{v}_2/t = -m\mathbf{v}_1/t.$$

The reactive force of the jet is directed along the motion of the vessel, i.e. against the velocity of the jet. A force having the same magnitude but opposite direction acted on the fluid. The mass of the fluid flowing out of the vessel per unit time is $m_t = m/t$. Hence the reactive force of the jet is $F = m_t v_1$.

Depending on the type of fuel, jet engines are divided into solid-propellant engines (flares and jet missiles) and liquid-propellant engines (using kerosene, petrol, and special jet fuels).

Figure 31 is a schematic diagram of a *liquid-propellant engine* used in rockets and missiles, as well as in high-speed aircraft. The fuel and oxidizer are pumped from the tanks to the spray injectors of the combustion chamber, where they are atomized and ignited. Expanding gas (combustion products) is ejected from the nozzle with a high velocity and thus has a large momentum. Due to the reaction of the jet on the rear wall of the combustion chamber, the engine and the vehicle on which it is mounted acquire an acceleration in a direction opposite to the direction of the outflowing jet of combustion products.

Figure 32 represents a *turbojet engine* used in aeroplanes. It operates as follows. Air and fuel are atomized in spray injector

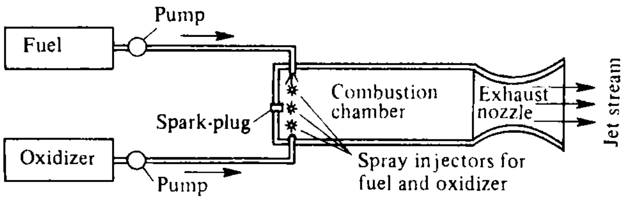

Fig. 31

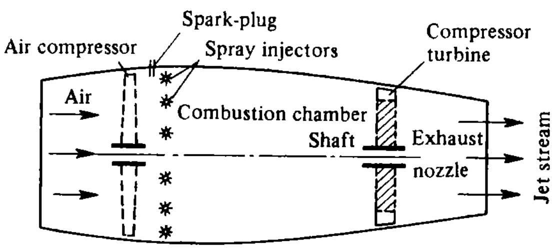

Fig. 32

and fed to the combustion chamber of the engine. Here this mixture is ignited and burnt. The combustion products are ejected from the engine nozzle with a high velocity and rotate the turbine blades past which they are flowing. The turbine rotates the compressor which supplies air to the combustion chamber. Thus, the gas jet flowing out of the nozzle exerts a reactive force on the engine and rotates the turbine running the engine.

1.15. Friction. Coefficient of Friction

Friction is a force emerging in the relative motion of two bodies in contact.

Depending on the nature of the relative displacement of solid bodies in contact, two types of friction are possible: (a) **sliding friction**, appearing when a body is sliding over the surface of another body; (b) **rolling friction**, caused by a body rolling over the surface of another body.

We shall consider only sliding friction. This force is directed along the contact surface of bodies against the displacement. For the same solid bodies, sliding friction is nearly proportional to the force with which one body is pressed against the other, i.e. to the force of pressure of one body on the other. This force is normal to the contact surface between the bodies:

$$F_{fr} = fN.$$

The proportionality factor f is called the **coefficient of sliding friction**: $f = F_{fr}/N$.

Friction appears not only when one surface slides over another surface but also when attempts are made to cause such a slip by applying force $\mathbf{R}$ to the bodies. In this case, we deal with **static friction**. As long as the magnitude of the external force $\mathbf{R}$ is less than fN, there is no slip between the bodies, since the force $\mathbf{R}$ is balanced by static friction which automatically assumes a value equal to R. As soon as the force $\mathbf{R}$ attains a value equal to fN, the body starts to slide over the surface of the other body, and static friction becomes sliding friction. Thus, static friction may assume various values between zero and fN. Figure 33 shows the dependence of friction on the magnitude of the force $\mathbf{R}$ applied to the bodies.

The coefficient of friction depends on the force N and the velocity of slippage. For solving many practical problems, the coefficient of friction can be assumed to be constant to an admissible degree of accuracy.

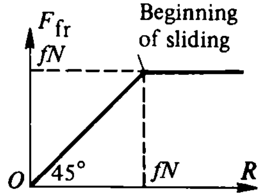

Fig. 33

When a body moves in a fluid, there also appears a force which hampers the motion of the body. This force is exerted on the body by the fluid particles. In this case, however, the resistance of the medium differs from the friction between two solid surfaces in that there is no static friction. A body floating in a liquid can be displaced by a force as weak as desired.

The friction acting on a body in a fluid, as well as the friction between two solid surfaces, is always directed against the motion of the body and depends on its velocity. For sufficiently low velocities, we can assume that friction is proportional to the velocity of a body: $F_l = \alpha v$, while for high velocities, it is proportional to the square of the velocity: $F_h = \beta v^2$ (α and β are the proportionality factors which depend on the property of a fluid and on the shape and size of a moving body).

1.16. Elastic Force. Hooke's Law

While solving some problems in mechanics, it is often necessary to take into consideration the deformations of a body caused by forces applied to it. In such problems, the model of a perfectly rigid body is inapplicable, since elastic bodies are considered.

The deformation of a body is characterized by its **strain (relative elongation)** ε, viz. by the ratio of the change in the length of the body to its initial length:

$$\varepsilon = \Delta l / l, \qquad (1.16.1)$$

where Δl is the elongation (contraction) of the body and l is its initial length.

The elongation of a body caused by a force acting on it is due to a stress in the body. **Tensile** or **compressive stress** is the ratio of the (tensile or compressive) force to the cross-sectional area of the body, which is normal to the direction of the force:

$$\sigma = F/S, \tag{1.16.2}$$

where F is the applied force, S is the cross-sectional area which is normal to the direction of the force, and σ is the stress.

Experiments show that if the deformation (relative elongation) of a body under the action of applied forces does not exceed a certain value specified for each material, the body restores its original shape and size after the deforming forces cease to act on it.

The property of a deformed body to be restored to its original shape and size after being subject to a deforming force is called **elasticity**. In a deformed body, there emerge forces which ensure the restoration of its shape and size. These are **elastic forces**. The limiting deformation under which a body still preserves elastic properties is called the **elastic limit**.

The elastic limit is specified either in the form of the limiting strain $\varepsilon_{\mathrm{lim}}$ for which the material still possesses elastic properties or (which is more often) as the limiting elastic stress σ_{lim}. Each material has its own elastic limit in a given physical state, in particular, at a given temperature.

If the strain exceeds the elastic limit for the material of a body undergoing deformation, the body no longer restores its original shape after the removal of deforming forces. In this case, a **residual deformation** is observed.

Unlike **elastic deformation** which takes place within the elastic limit, the deformation beyond this limit is called **plastic deformation**. Bodies with a low elastic limit (e.g. bodies made of lead, soft clay or wax) are called **plastic bodies** in contrast to **elastic bodies** which have a high elastic limit.

The relation between the elastic strain and the deforming force is expressed by **Hooke's law**: *strain is proportional to the force responsible for this deformation, i.e. to stress*:

$$\sigma = E\varepsilon, \tag{1.16.3}$$

where E is the **elasticity modulus**, or **Young's modulus**.

Plastic deformations increase more rapidly than the forces causing them. Figure 34 represents the dependence of stress on strain, viz. the relative elongation of a stretched body or contraction of a compressed body. The figure shows that within the elastic limit, strain increases in proportion to stress (Hooke's law).

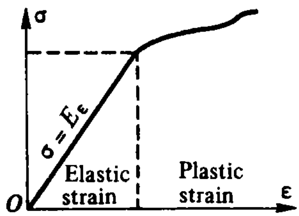

Fig. 34

Let us substitute expressions (1.16.1) and (1.16.2) into (1.16.3). This gives

$$F/S = E(\Delta l/l), \quad \text{or} \quad F = (ES/l)\Delta l, \quad \text{i.e.} \quad F \propto \Delta l.$$

This is true for all types of elastic deformations: $F = k\,\Delta l$. The proportionality factor k is determined by elastic properties of the material, its initial length, and cross-sectional area. For an elastic body, $k = ES/l$. When a spring or a rubber cord are considered, k is refered to as *rigidity*.

1.17. Law of Universal Gravitation

The **law of universal gravitation** was established in 1687 by I. Newton in the following form: *two material points attract each other with the forces directly proportional to their masses and inversely proportional to the square of the distance between them*:

$$F = G(m_1 m_2/r^2), \tag{1.17.1}$$

where G is the **gravitational constant** whose value is the same for all bodies.

The gravitational constant is determined experimentally. It was obtained for the first time by Cavendish who measured the forces of attraction between two bodies of a known mass with the help of a torsion balance. The experimental set-up used by Caven-

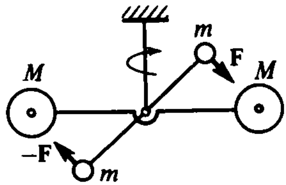

Fig. 35

dish is shown schematically in Fig. 35. Two lead spheres of a known mass m were suspended from a horizonal beam attached to a vertical straight torsion wire in the vicinity of two fixed spheres of mass M. The torque created by the forces of attraction of the spheres m exerted by the spheres M was determined from the torsion angle of the wire.

At the present time, the gravitational constant for most calculations is taken as

$$G \simeq 6.67 \times 10^{-11}\,\mathrm{N \cdot m^2/kg^2}.$$

The physical meaning of the gravitational constant is that it is equal to the force of interaction (in newtons) between two masses of 1 kg separated by a distance of 1 m.

It can be easily seen that gravitational forces are very weak. For example, the force of attraction between two cars weighing 30 t each and separated by a distance of 5 m is 2.4×10^{-3} N.

The gravitational interaction between bodies takes place through the gravitational field generated by the bodies and is one of the forms in which matter exists.

Formula (1.17.1) for the force of gravitational interaction contains the quantities m_1 and m_2, viz. the *masses* of the interacting bodies. These masses characterize the gravitational interaction between bodies. On the other hand, it has been established earlier that mass is a measure of inertia in phenomena described by Newton's second law. For this reason, it is called *gravitational mass* in the former case and *inertial mass* in the latter case. Since the equivalence of the two masses can be assumed to have been established at present, we shall not distinguish between them henceforth. However, it should be borne in mind that the same quantity—mass—characterizes two different properties of a body.

1.18. Force of Gravity. Free Fall of Bodies

One of the manifestations of the force of universal gravitation is the **force of gravity**, i.e. the force of attraction of bodies to the Earth. If the force of gravity alone acts on a body, it falls freely to the Earth. Consequently, **free fall** is the motion of a body in empty space (*vacuum*) under the action of the force of gravity.

Free fall is a uniformly accelerated motion, since near the Earth's surface it is caused by the force of gravity whose magnitude and direction remain unchanged.

The free-fall acceleration (denoted by g) at the surface of the Earth at a given latitude is the same for all bodies. At the pole, $g = 9.83\ \mathrm{m/s^2}$, on the equator $g = 9.78\ \mathrm{m/s^2}$, while at the latitude $\varphi = 45°$, $g = 9.80\ \mathrm{m/s^2}$. When solving problems, we shall assume that $g = 9.8\ \mathrm{m/s^2}$.

The relations between kinematic characteristics of free fall are obtained from the formulas for uniformly accelerated motion by putting $a = g$ in them. For $v_0 = 0$, we have

$$v = gt, \quad H = gt^2/2, \quad v = \sqrt{2gH}.$$

A falling body always experiences the air resistance. For a given body, the air resistance is the larger, the higher the velocity of fall. Consequently, as the velocity of a falling body increases, the air resistance increases, its acceleration decreases, and the free-fall acceleration vanishes at the moment when the air resistance becomes equal to the force of gravity. After this, the motion of the body becomes uniform. For example, a parachutist performing a free-fall jump at first is falling with an acceleration. The air resistance increases with the velocity, and becomes equal to the force of gravity acting on the parachutist at a certain instant after the opening of the parachute. From this moment, the parachutist is falling uniformly.

1.19. Weight of a Body. Weighing

The **weight** of a body is the force with which the body acts on a horizontal support or a vertical suspender due to the gravitational attraction to the Earth. This definition is valid for the reference frame fixed to the support or suspender and corresponds to practical determination of weight in terrestrial conditions with the help of a spring balance (dynamometer) (Fig. 36).

Henceforth, we shall consider only this method of weighing. Essentially weighing by a beam balance is a comparison of the weight of a body with the weight of standard loads which are known beforehand rather than its direct measurement. If we bring the beam balance to another place on the Earth, this will not change anything in this comparison, although the weights of the body as well as of standard loads will change. What remains unchanged is their masses. If we weigh a body moving uniformly in a straight line relative to the Earth's surface (say, if the weighing is made in the car of a train moving uniformly in a straight line), the result will be the same as when weighing is done on the surface of the Earth.

Generally, the weight of a body is the same in all inertial reference systems. It is equal in magnitude to the force of gravity.[5] However, these two forces should not be considered identical. The weight of a body is the force applied to the support or suspender by the body, while the force of gravity is exerted on the body by the Earth. In noninertial reference systems, the magnitudes of the weight and force of gravity are different.

Figure 37 illustrates the weighing of a body in a cart moving over a horizontal plane. As long as the cart is at rest or moving uniformly in a straight line, the suspender (spring balance) with

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[5] To be more precise, according to Newton's third law, the weight is equal and opposite to the force of normal reaction of the horizontal support (or to the tension of the vertical suspender). In turn, in accordance with Newton's second law, the force of normal reaction of the support is equal and opposite to the force of gravity since under the combined action of these forces, a body is at rest or moves uniformly in a straight line. Thus, the weight of a body is indeed equal to the force of gravity in an inertial reference system.

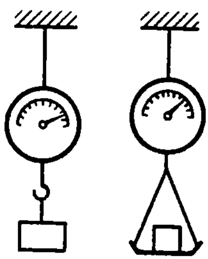

Fig. 36

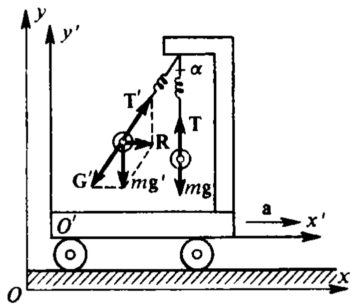

Fig. 37

the load is in a vertical position, and the pointer of the spring balance indicates the weight $G = mg$ (mg is the force of gravity) in the $x'O'y'$ system fixed to the cart, which is the same as the weight in the xOy system fixed to the surface of the Earth.

If the cart starts to move to the right with an acceleration **a** relative to the Earth, the suspender with the load will deviate to the left at an angle α and will move with the same acceleration along with the cart. The body under investigation is acted upon by two forces, viz. the force of gravity $m\mathbf{g}' = m\mathbf{g}$ and the tension $\mathbf{T}'$ of the suspender. The resultant **R** of these forces has the same direction as the acceleration **a** of the cart. Figure 37 shows that in the $x'O'y'$ reference system, the tension $\mathbf{T}'$ of the suspender is larger than the force of gravity $m\mathbf{g}$, and hence larger than the weight of the body in the xOy reference system:

$$T' = G' > mg = T.$$

1.20. Weightlessness

Let us consider the weighing of a body suspended in a lift. As long as the lift is in the state of rest or uniform motion, the reading of the spring balance (and hence the weight of the body) corresponds to the force of gravity (Fig. 38*a*):

$$G = T = mg.$$

When the lift acquires an acceleration **a** directed upwards (Fig. 38*b*), the spring of the balance is stretched, and as a result

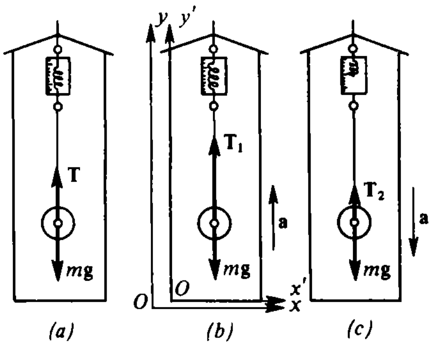

Fig. 38

the body will move relative to the reference frame fixed to the Earth's surface with the same acceleration **a** as the lift. This acceleration is imparted to the body by the resultant **R** of the two forces applied to the body being weighed, viz. the force of gravity $m\mathbf{g}$ (due to the action of the Earth) and the force $\mathbf{T}_1$ exerted by the stretched spring of the balance. The magnitude of the resultant is $R = T_1 - mg$. According to Newton's second law, $R = ma$. Consequently, $T_1 - mg = ma$. Hence the tension of the suspender is $T_1 = mg + ma$. This means that in the reference frame fixed to the lift, the weight of the body is larger than the force of gravity:

$$G_1 = T_1 > mg.$$

Let us consider a lift moving with a downward acceleration (Fig. 38c). The acceleration is now imparted to the body by the force $R = mg - T_2$, or $ma = mg - T_2$. Consequently, $T_2 = mg - ma$, i.e. the weight of the body is less than the force of gravity:

$$G_2 = T_2 < mg.$$

If the lift moves with the free-fall acceleration, the reading of the spring balance will be

$$G = T = mg - mg = 0.$$

The weight of the body (and hence the reaction of the support or suspender) in the freely falling lift turns out to be equal to zero. In particular, the position of the body in the falling lift will not change if its suspension to the spring balance is cut. The behaviour of a body on a horizontal support fixed to a freely falling lift will be similar: it will not exert a pressure on the support and will not change its position relative to the lift if the support is removed.

It is important to note that during a free fall not only the forces exerted by a body on the support or suspender vanish but also the forces of pressure and tension exerted by one part of the body on another part inside it. In a free fall, if the resistance of the medium is vanishingly small, a man experiences weightlessness physiologically (see also Sec. 1.27). On the contrary, when the acceleration is directed against the force of gravity, a state of overload is felt.

The state of a body observed when its weight is zero is called **weightlessness**, or **zero-gravity state**. It follows from what has been said above that any freely falling body (i.e. a body moving only under the action of the force of gravity) is in the weightless state.

Problems with Solutions

16. A body slides along an inclined plane forming an angle $\alpha = 30°$ with the horizontal. Find its acceleration if the coefficient of friction $f = 0.3$. What is the magnitude of the angle α at which the body starts to slide if the coefficient of static friction is equal to the coefficient of sliding friction?

Solution. The body is acted upon (Fig. 39) by the force of gravity $m\mathbf{g}$ applied to its centre of mass, the force $\mathbf{N}$ of normal reaction of the inclined plane applied

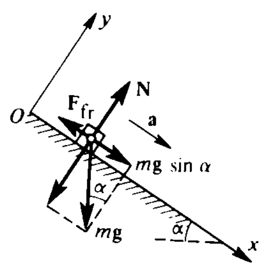

Fig. 39

to the surface of the body in contact with the support (this force is normal to the inclined plane), and the sliding friction force $\mathbf{F}_{fr}$ which is parallel to the inclined plane and is directed against the motion of the body. We direct the x-axis of the xOy reference system fixed to the inclined plane so that its positive direction coincides with the direction of acceleration $\mathbf{a}$.

Let us find the x-projections of all the forces. The force $\mathbf{N}$ is perpendicular to this axis, and hence its x-projection is zero. The force $\mathbf{F}_{\text{fr}}$ is parallel to the x-axis and has the opposite direction. Hence the x-projection of the force $\mathbf{F}_{\text{fr}}$ is $-\mathbf{F}_{\text{fr}}$. It can be seen from Fig. 39 that the x-projection of the force $m\mathbf{g}$ is equal to $mg \sin \alpha$. In accordance with Newton's second law, we can now write the equation for the x-projections of all the forces acting on the body:

$$mg \sin \alpha - F_{\text{fr}} = ma. \tag{1}$$

Since the body moves, the friction has the largest possible value $F_{\text{fr}} = fN$. Consequently,

$$mg \sin \alpha - fN = ma. \tag{2}$$

The acceleration of the body along the y-axis (the y-projection of vector $\mathbf{a}$) is zero. Hence we can write the equation for the y-projections of all the forces acting on the body:

$$mg \cos \alpha - N = 0, \quad \text{or} \quad N = mg \cos \alpha. \tag{3}$$

Substituting this expression into Eq. (2), we get

$$mg \sin \alpha - fmg \cos \alpha = ma,$$

whence

$$a = g(\sin \alpha - f \cos \alpha). \tag{4}$$

Let us analyze this result before substituting the numerical values of the quantities into it. When $\sin \alpha = f \cos \alpha$, i.e. when $\tan \alpha = f$, we have $a = 0$, which means that the body is either at rest or slides uniformly. If it is at rest, an increase in the value of α will make it slide. Obviously, $a \geqslant 0$ since the acceleration of the body cannot be directed to the top of the inclined plane. This means that our solution is valid only for $\sin \alpha \geqslant f \cos \alpha$, i.e. for $\tan \alpha \geqslant f$, or $\alpha \geqslant \arctan f$. For $\alpha < \arctan f$, the body cannot slide down the inclined plane, and $F_{\text{fr}} \neq fN$. In this case, $mg \sin \alpha - F_{\text{fr}} = 0$, i.e. $F_{\text{fr}} = mg \sin \alpha$. Substituting now the numerical values of g, f, and α into formula (4), we get $a = 2.35\ \text{m/s}^2$. It follows from the above arguments that the body start sliding when $\alpha \geqslant \arctan f = \arctan 0.3$.

17. A body slides along an inclined plane with an angle $\alpha = 30°$ to the horizontal. Find its velocity at the end of the third second from the beginning of sliding if the coefficient of friction $f = 0.25$.

Solution. Since the acceleration of the body is not specified in the conditions of the problem, it is reasonable to avoid operations involving its determination and to use Newton's second law for a uniformly accelerated motion in the form $mv = Ft$. Hence we can immediately determine the required velocity: $v = Ft/m$. The tangential force (viz. the force directed parallel to the plane) was found in the preceding problem:

$$F = mg(\sin \alpha - f \cos \alpha).$$

Consequently,

$$v = gt(\sin\alpha - f\cos\alpha) = 8.3 \text{ m/s}.$$

18. A shell with a mass $m = 10$ kg is fired from a gun with a velocity $v = 500$ m/s. Find the force of pressure of the gunpowder gas, assuming it to be constant for the entire period $t = 0.01$ s during which the shell moves inside the barrel of the gun.

Solution. It follows from the equation $Ft = mv$ for a uniformly accelerated motion that $F = mv/t = 500$ kN. In this problem, we use the projections of the force **F** acting on the shell and of its velocity **v** at the moment it leaves the barrel onto the axis of the barrel although it has not been specially indicated.

Remark. Such problems are often solved as follows. First, the force of pressure is expressed in terms of the acceleration of the shell: $F = ma$. Then the acceleration is written as $a = v/t$ and this expression is substituted into the previous formula, which gives $F = mv/t$, i.e. the expression following from $Ft = mv$. This, however, is another form of Newton's second law which can be used directly, without deriving the formula anew.

19. An automatic-coupling carriage of mass $m_1 = 10$ t moving with a velocity $v_1 = 12$ m/s catches up with another carriage of mass $m_2 = 20$ t moving with a velocity $v_2 = 6$ m/s, and is coupled with it. Moving together, the two carriages collide with a third carriage having a mass $m_3 = 7.5$ t, which stands on the rails. Find the velocities of the carriages on different segments of the track. Friction should be neglected.

Solution. From the law of momentum conservation, we have

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v_4 = (m_1 + m_2 + m_3)v_5,$$

where v_4 is the common velocity of the two carriages and v_5 is the velocity of the three carriages moving together. Solving the equation $m_1v_1 + m_2v_2 = (m_1 + m_2)v_4$, we obtain

$$v_4 = (m_1v_1 + m_2v_2)/(m_1 + m_2) = 8 \text{ m/s}.$$

From the equation $(m_1 + m_2)v_4 = (m_1 + m_2 + m_3)v_5$, we get

$$v_5 = (m_1 + m_2)v_4/(m_1 + m_2 + m_3) = 6.4 \text{ m/s}.$$

20. A bullet leaves a rifle with a velocity $v_b = 900$ m/s. Find the recoil velocity of the rifle if the mass m_r of the rifle is 500 times as large as the mass m_b of the bullet.

Solution. The momentum of the rifle before the bullet is fired is zero. Since we can assume the rifle-bullet system as isolated (the external forces acting on the system are not equal to zero but they mutually cancel out), its momentum remains unchanged. Having projected all the momenta onto an axis parallel to the velocity of the bullet and coinciding with it in direction, we can write

$$m_bv_b + m_rv_r = 0,$$

whence

$$v_r = -(m_b/m_r)v_b = -1.8 \text{ m/s}.$$

The minus sign indicates that the direction of the recoil velocity of the rifle is opposite to the direction of the velocity of the bullet.

21. A grenade flying with a velocity $v = 15$ m/s explodes into two fragments having masses $m_1 = 6$ kg and $m_2 = 14$ kg. The larger fragment has a velocity $v_2 = 24$ m/s in the same direction as the velocity of the grenade before the explosion. Find the magnitude and direction of the velocity of the smaller fragment.

Solution. During the explosion of the grenade, its momentum changes insignificantly due to the action of the force of gravity, so that the change of its momentum is $mg\,\Delta t$, and the time Δt of explosion is very short. Hence we can assume that the grenade and its fragments form an isolated system during the explosion. According to the law of momentum conservation, we have

$$(m_1 + m_2)\mathbf{v} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2. \quad (1)$$

Since the directions of velocities $\mathbf{v}$ and $\mathbf{v}_2$ coincide, the direction of $\mathbf{v}_1$ will be either the same or the opposite. We make the coordinate axis coincide with this direction, assuming that the direction of vectors $\mathbf{v}$ and $\mathbf{v}_2$ is the positive direction of the axis. Then we project Eq. (1) onto the chosen coordinate axis. We obtain the scalar equation

$$(m_1 + m_2)v = m_1v_1 + m_2v_2, \quad \text{or}$$
$$v_1 = [(m_1 + m_2)v - m_2v_2]/m_1 = -6 \text{ m/s}.$$

The minus sign indicates that the velocity $\mathbf{v}_1$ is directed against the velocity of the grenade.

22. A shell having a mass $m = 50$ kg and flying at an angle of 60° to the horizontal with a velocity $v_1 = 400$ m/s hits a cart filled with sand having a mass $M = 1000$ kg and gets stuck in the sand. What is the velocity acquired by the cart?

Solution. Applying the law of momentum conservation to the x-projections (the first equation of system (4) in Sec. 1.13), we can write

$$(M + m)v_2 = mv_{1x}, \quad \text{or} \quad (M + m)v_2 = mv_1\cos\alpha.$$

Hence $v_2 = mv_1\cos\alpha/(M + m) = 9.5$ m/s.

23. A body with a mass $m = 5$ kg slides along the surface of a trihedral prism with a mass $M = 35$ kg, whose upper plane is inclined at an angle $\alpha = 30°$ to the horizontal. The prism rests on a horizontal plane having a vertical ledge at the rear edge of the prism to keep it at rest. Find the force of pressure exerted by the prism on the ledge during the motion of the body over the surface of the prism. What is the pressure exerted by the prism on the base of the plane? Friction should be neglected.

Solution. During the motion of the body over the inclined plane of the prism, the momentum of the prism-body system relative to the xOy reference frame fixed to the plane is $m\mathbf{v}$. The forces exerted by the plane (Fig. 40*a*) are $\mathbf{N}_1$ and $\mathbf{N}_2$. Consequently, we can write

$$(\mathbf{N}_1 + \mathbf{N}_2 + M\mathbf{g} + m\mathbf{g})t = m\mathbf{v}.$$

In the projections onto the coordinate axes, we have

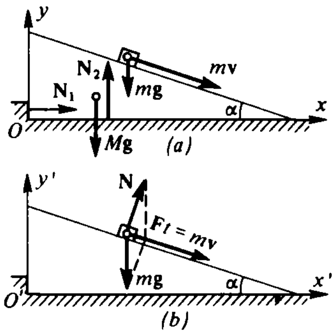

Fig. 40

$$N_1 t = mv\cos\alpha, \quad (N_2 - Mg - mg)t = -mv\sin\alpha. \tag{1}$$

Let us now consider the motion of the body relative to the prism in the $x'O'y'$ reference frame fixed to the prism (Fig. 40*b*). The body is acted upon by two forces, viz. the force of gravity $m\mathbf{g}$ and the reaction $\mathbf{N}$ of the prism. Their resultant $\mathbf{F}$ is directed along the inclined plane of the prism: $F = mg\sin\alpha$. The impulse of this force is $Ft = mgt\sin\alpha$. Consequently, according to Newton's second law, the momentum of the body is

$$mv = mgt\sin\alpha. \tag{2}$$

Taking this relation into account, we can write the system of equations (1) as follows:

$$\begin{aligned} N_1 t &= mgt\sin\alpha\cos\alpha, \\ N_2 - Mg - mg &= -mg\sin^2\alpha. \end{aligned} \tag{3}$$

The first of these equations gives the force exerted on the prism by the ledge:

$$N_1 = (mg/2)\sin 2\alpha = 21.2\text{ N}.$$

The second equation in system (3) gives the force exerted on the prism by the plane:

$$N_2 = Mg + mg - mg\sin^2\alpha = g(M + m\cos^2\alpha) = 380\text{ N}.$$

In accordance with Newton's third law, the forces of pressure exerted by the prism on the plane are equal in magnitude to the obtained forces and have the opposite directions. The prism presses against the ledge with a force of 21.2 N directed to the left. The normal pressure of the prism on the horizontal plane is 380 N.

24*. A hydraulic machine with a mass $m = 2000$ kg moves on wheels over a horizontal floor by ejecting a water jet with a relative velocity $v = 12$ m/s through a hole of diameter $d = 4$ cm. At a certain moment, the velocity of the machine is $v_m = 5$ m/s. What will be the velocity increment in a time interval Δt if this interval is sufficiently small? How can we calculate the velocity increment in 1 s, 2 s, etc.? The friction and reduction of the mass of the machine due to water ejection should be neglected.

Solution. The volumetric flow rate[6] of water is $Q = (\pi d^2/4)v$. The mass of water flowing out over the time Δt is $m_1 = \rho Q \Delta t = \rho(\pi d^2/4)v \Delta t$ (the density of water $\rho = 1 \times 10^3$ kg/m^3). The velocity of the jet relative to an immobile reference system at the beginning of the time interval Δt under consideration is $v - v_m$. By the end of this interval Δt, the velocity v_m of the machine will increase relative to the fixed reference system, while the velocity $v - v_m$ of the jet will decrease. If this decrease is insignificant, it can be neglected. Then, according to the law of momentum conservation, we have

$$m_1(v - v_m) = m \Delta v_m \quad \text{or} \quad \rho(\pi d^2/4)v(v - v_m) \Delta t = m \Delta v_m,$$

whence

$$\Delta v_m = \rho(\pi d^2/4m)v(v - v_m) \Delta t = 0.05\Delta t \text{ m/s}. \tag{1}$$

The velocity of the machine will increase by the end of the second second by 0.10 m/s. Its average velocity over two seconds is (5 + 0.05) m/s, and for $v - v_m = 7$ m/s we can substitute a more exact value of 6.95 m/s into Eq. (1). This, however, would change the result only by 0.7%, i.e. within the accuracy of calculations. Consequently, over a few seconds, Δv_m can be calculated by formula (1).

25. Two loads having a total mass $m_1 + m_2 = 30$ kg are suspended from the ends of a rope around a pulley attached to the ceiling. The loads move with an acceleration $a = 0.3g$, the right load moving downwards. Find the masses of the loads, neglecting the masses of the pulley and rope, as well as the friction at the pulley axis.

Solution. Let us consider the motion of the left load (Fig. 41). It is acted upon by a force of gravity $m_1\mathbf{g}$ and by the tension $\mathbf{T}$ of the rope. For the x-projections of these forces, we can write: $T - m_1 g = m_1 a$. Similarly, for the right load, we can write: $T - m_2 g = -m_2 a$. Subtracting the second equation from the first, we obtain $(m_2 - m_1)g = (m_2 + m_1)a$, which gives $m_2 - m_1 = (m_2 - m_1)a/g$. Substituting the given values into this equation, we obtain $m_2 - m_1 = 9$ kg. Thus, $m_1 = 10.5$ kg and $m_2 = 19.5$ kg.

26. The pulley with loads described in the previous problem is suspended from a spring balance attached to the ceiling of a lift moving upwards with an acceleration $a_l = 0.1g$. Determine the reading $F_{s.b}$ on the spring balance.

Solution. The acceleration of the left load (Fig. 42) relative to the Earth is directed upwards: $a_1 = a + a_l = 0.4g$. We can write the following equation for the

[6] The **volumetric flow rate** Q of a fluid is its volume passing through a given cross section (of a tube, channel) per unit time.

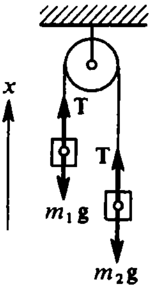

Fig. 41

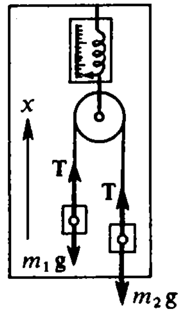

Fig. 42

x-projections of forces acting on this load:

$$T - m_1 g = m_1 a_1. \qquad (1)$$

The acceleration of the right load is directed downwards: $a_2 = a - a_1 = 0.2g$. For the x-projections of forces acting on this load, we have

$$T - m_2 g = -m_2 a_2. \qquad (2)$$

Adding Eqs. (1) and (2), we get

$$2T - (m_1 + m_2)g = m_1 a_1 - m_2 a_2.$$

Since the ropes act on the spring balance with two identical forces, each equal to the tension T of the rope, the reading on the spring balance is

$$F_{s.b} = 2T = (m_1 + m_2)g + (m_1 a_1 - m_2 a_2)$$
$$= (m_1 + m_2)(g + a_1) + (m_1 - m_2)a. \qquad (3)$$

For solving the problem, we must find the difference $m_2 - m_1$, since the other quantities appearing in Eq. (3) are known. For this purpose, we subtract (2) from (1):

$$(m_2 - m_1)g = m_1 a_1 + m_2 a_2 = m_1(a + a_1) + m_2(a - a_1),$$

whence $m_2 - m_1 = (m_1 + m_2)a/(a_1 + g)$. Substituting this expression into (3), we obtain

$$F_{s.b} = (m_1 + m_2)\,[g + a_1 - a^2/(g + a_1)] = 300 \text{ N}.$$

27. A pulley is attached to the upper edge of an inclined plane having a slope $\alpha = 60°$, and a rope is passed through it as shown in Fig. 43. A load with a mass $m_1 = 240$ g is fixed to the vertical end of the rope. This load is used to slide two loads, each having a mass $m_2 = m_3 = 100$ g up the inclined plane with an acceleration $a = 0.1g$. The loads are connected to each other and fixed to the other end of the rope. Find the coefficient of friction between the surface of the inclined plane and

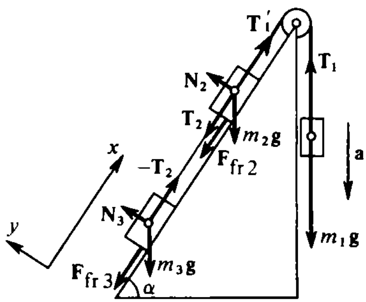

Fig. 43

the loads, and the tension of the rope connecting the loads being lifted. The masses of the pulley and rope, as well as the friction at the pulley axis, can be neglected.

Solution. Let us consider the motion of the load m_1. It is acted upon by the force of gravity $m_1\mathbf{g}$ and the tension $\mathbf{T}_1$ of the rope (see Fig. 43). The resultant of these forces imparts a downward acceleration $\mathbf{a}$ to the load. We write the equation of motion for this load, having projected all the vectors onto the axis coinciding with the acceleraton $\mathbf{a}$:

$$m_1 g - T_1 = m_1 a, \tag{1}$$

where $a = 0.1g$. Next, we consider the motion of the loads m_2 and m_3 along the inclined plane. The following forces are applied to the system formed by the two loads connected by the rope: the forces of gravity $m_2\mathbf{g}$ and $m_3\mathbf{g}$, the tension $\mathbf{T}_1'$ of the rope ($T_1' = T_1$ in magnitude), the normal reactions $\mathbf{N}_2$ and $\mathbf{N}_3$ of the plane, and the frictions $F_{fr\,2}$ and $F_{fr\,3}$. Since the loads are moving along the plane, $F_{fr\,2} = fN_2$ and $F_{fr\,3} = fN_3$. According to Newton's second law, we can write the following equations for the x- and y-projections of these forces:

$$\begin{gathered} T_1 - m_3 g \sin\alpha - m_2 g \sin\alpha - F_{fr\,3} - F_{fr\,2} = (m_2 + m_3)a, \\ N_2 - m_2 g \cos\alpha = 0, \quad N_3 - m_3 g \cos\alpha = 0. \end{gathered} \tag{2}$$

Hence

$$N_2 = m_2 g \cos\alpha, \quad N_3 = m_3 g \cos\alpha,$$

$$F_{fr\,2} = fN_2 = fm_2 g \cos\alpha, \quad F_{fr\,3} = fN_3 = fm_3 g \cos\alpha.$$

Substituting the expressions for $F_{fr\,2}$ and $F_{fr\,3}$ into the first equation from (2), we obtain

$$T_1 - g(m_2 + m_3)\sin\alpha - fg(m_2 + m_3)\cos\alpha = (m_2 + m_3)a. \tag{3}$$

But according to Eq. (1) $T_1 = m_1 g - m_1 a = m_1(g - a)$. Therefore, Eq. (3) can be written in the form

$$m_1(g - a) - g(m_2 + m_3)\sin\alpha - a(m_2 + m_3) = fg(m_2 + m_3)\cos\alpha.$$

This gives

$$f = \frac{m_1(g - a)}{(m_3 + m_2)g \cos\alpha} - \tan\alpha - \frac{a}{g\cos\alpha} \simeq 0.23.$$

The tension T_2 of the rope can be found from the equation of motion for the third load:

$$T_2 - m_3 g \sin\alpha - f m_3 g \cos\alpha = m_3 a,$$

whence

$$T_2 = m_3 a + m_3 g(\sin\alpha + f\cos\alpha) = m_3 g(a/g + \sin\alpha + f\cos\alpha) = 1.06 \text{ N}.$$

28. A load of mass $m = 100$ kg is lifted with the help of a double block (consisting of two pulleys having different radii and rigidly fixed to each other on the same axis) by a force $F = 500$ N, which is directed at an arbitrary angle to the vertical (Fig. 44). The block axis rotates in a bearing fixed to an arm which is attached to a

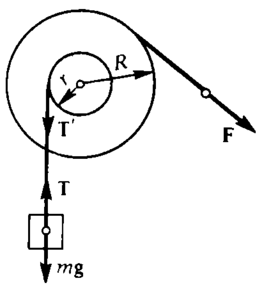

Fig. 44

wall. Find the acceleration of the load if the radii of the pulleys are $r = 10$ cm and $R = 25$ cm. (The rope attached to the load is wound on the smaller pulley.) The mass of the block and ropes, as well as the friction at the block axis, should be neglected.

Solution. Let us first consider the motion of the load. It is acted upon by the force of gravity $m\mathbf{g}$ and the tension $\mathbf{T}$ of the rope. The equation of motion for the load along the vertical is

$$T - mg = ma. \tag{1}$$

Next, we consider the motion of the block. The force $\mathbf{F}$ is applied to it on the right, and the tension $\mathbf{T}' = -\mathbf{T}$ acts on the left. Since the block is assumed to be weightless by the conditions of the problem, the moment of force $\mathbf{T}'$ which rotates the block counterclockwise must be equal in magnitude to the moment of force $\mathbf{F}$ which rotates the block clockwise: $T'r = FR$, whence $T' = T = FR/r$. Substituting this quantity into Eq. (1), we obtain $FR/r - mg = ma$. Now, we can find the acceleration of the load:

$$a = FR/mr - g = 2.7 \text{ m/s}^2.$$

29. A body falls from a height $h = 1000$ m with zero initial velocity. Another body falls simultaneously from a height $H = 1100$ m with a certain initial velocity. The two bodies reach the surface of the Earth at the same instant. Find the initial velocity of the second body.[7]

Solution. We denote the time of fall of the bodies by t. Then

$$h = gt^2/2, \quad H = v_{02}t + gt^2/2.$$

Substituting $gt^2/2 = h$ and $t = \sqrt{2h/g}$ from the first equation into the second, we obtain

$$H = v_{02}\sqrt{2h/g} + h, \quad \text{or} \quad v_{02} = \frac{H-h}{\sqrt{2h/g}} = \frac{H-h}{2h}\sqrt{2gh} = 7.0 \text{ m/s}.$$

30. A body is thrown vertically upwards with an initial velocity v_0. At the moment it reached the point at a height $H = 100$ m, a second body is thrown from the same starting point and with the same initial velocity. At what height h will the bodies meet? Which velocities will they have at this moment? What is the initial velocity of the bodies?

Solution. Obviously, the time of ascent of the second body to height h is equal to the time of descent of the first body from H to h. Hence

$$h = v_0t - gt^2/2, \quad H - h = gt^2/2.$$

Since the velocity of the body at height H is zero, $v_0 = \sqrt{2gH} = 44.2$ m/s. Solving these equations together, we obtain $h = \sqrt{2gH}t - gt^2/2 = (3/4)H = 75$ m. The velocities of the bodies at the moment of their meeting are equal in magnitude and opposite in direction: $v = \sqrt{2g(H-h)} = 22.1$ m/s.

31. An aerostat ascends vertically with a certain constant acceleration. When the velocity of ascent is $v_1 = 10$ m/s, a body is dropped from the aerostat. At what height does this happen if the body falls to the Earth in $t_0 = 5$ s? Find the velocity of the body at the Earth's surface.

Solution. The solution of the problem can be split into two stages if we make use of the formulas expressing the time dependence of the distance: (a) determining the maximum height to which the body is raised and the time of ascent to this height, and (b) calculating the motion of the body freely falling from this height. The problem is solved in a simpler way if we use the formulas describing the time dependence of the displacement of the body.

Let us choose a reference system fixed to the Earth and such that its origin coincides with the point at which the body falls while the positive direction of the y-axis coincides with the direction of the initial velocity $\mathbf{v}_1$ of the falling body (Fig. 45). Then the time variation of the y-coordinate is described by the following formula:

$$y = v_1t - gt^2/2. \tag{1}$$

---

[7] Here and below, the air resistance should be neglected unless otherwise specified in the conditions of a problem.

At the moment t_0 when the body reaches the ground, $y = -h$. Substituting this value of the coordinate into Eq. (1), we get

$$h = gt_0^2/2 - v_1 t_0 = 72.5 \text{ m}.$$

Let us now find the velocity of the body at the moment it touches the ground: $v = v_1 - gt_0 = -39$ m/s. The velocity v (to be more precise, the y-projection of the velocity **v**) is negative. This means that the final velocity is directed downwards.

32. A bullet fired from a pistol fixed in the horizontal position at a height $H = 80$ cm from the floor has the initial velocity $v = 5$ m/s. Find the range l of the bullet flight and the magnitude and direction of its final velocity $\mathbf{v}_f$.

Solution. Let us write the equations of motion for the bullet in the x- and y-projections in the xOy coordinate system (Fig. 46): $x = vt$ and $y = gt^2/2$. At the

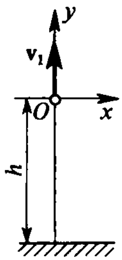

Fig. 45

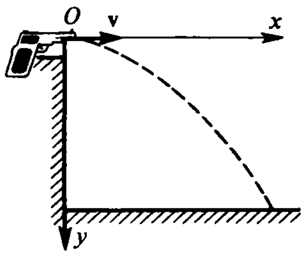

Fig. 46

moment t_0 when the bullet reaches the ground, $x = l$ and besides $y = H$. Substituting these values of the coordinates and time t_0 into the equations of motion, we obtain $l = vt_0$ and $H = gt_0^2/2$. We express from the second equation the time of motion t_0 and substitute it into the first equation:

$$t_0 = \sqrt{2H/g}, \quad l = v\sqrt{2H/g} \simeq 2.02 \text{ m}.$$

The horizontal component of the bullet velocity does not change with time: $v_x = v = \text{const}$, while the vertical component varies as $v_y = gt$. At the moment the bullet hits the ground, $v_x = v$ and $v_y = gt_0 = g\sqrt{2H/g} = \sqrt{2gH}$. Consequently, the final velocity of the bullet at this moment is

$$v_f = \sqrt{v_x^2 + v_y^2} = \sqrt{v^2 + 2gH} \simeq 6.4 \text{ m/s}.$$

The velocity vector $\mathbf{v}_f$ forms an angle α with the horizontal, such that $\cos\alpha = v/v_f = 0.8$; hence $\alpha \doteq 37°$.

33. A spring pistol is mounted on a horizontal surface so that its barrel forms an angle α with the horizontal. At what value of α is the range of a bullet the maximum? Find the maximum range of the bullet for the initial velocity $v_0 = 7$ m/s.

Solution. The motion of the bullet along the x-axis is uniform (Fig. 47), and the velocity $v_x = v_{0x} = \quad \cos\alpha$. Therefore, its x-coordinate varies with time ac-

Fig. 47

cording to the formula

$$x = (v_0 \cos\alpha)t. \tag{1}$$

In the vertical direction, the bullet moves with a constant acceleration $a = -g$ (directed downwards). The initial velocity in this direction is $v_{0y} = v_0 \sin\alpha$. The y-coordinate of the bullet at the moment t can be found by the formula

$$y = v_{0y}t - gt^2/2 = (v_0 \sin\alpha)t - gt^2/2.$$

Let us find the time t_0 of flight of the bullet. At the moment t_0 it falls to the ground, $y = 0$. Consequently, we can write

$$0 = (v_0 \sin\alpha)t_0 - gt_0^2/2, \quad \text{whence} \quad t_0 = (2v_0 \sin\alpha)/g.$$

Substituting this expression for the time of flight into Eq. (1) describing the motion along the x-axis, we can find the range of the bullet:

$$l = x_{max} = (v_0 \cos\alpha)t_0 = (2v_0^2/g)\sin\alpha\cos\alpha = (v_0^2/g)\sin 2\alpha.$$

The range is at a maximum when $\sin 2\alpha = 1$, i.e. $2\alpha = 90°$, $\alpha = 45°$. Consequently, the maximum range of the bullet is

$$l_{max} = v_0^2/g = 5.0 \text{ m}.$$

34. A shell is fired from a gun from the bottom of a hill along its slope. The slope of the hill is $\alpha = 30°$, and the angle of the barrel to the horizontal is $\beta = 60°$. The initial velocity v of the shell is 21 m/s. Find the distance from the gun to the point at which the shell falls.

Solution. The shell moves along the x-axis (Fig. 48) uniformly with a velocity $v_x = v\cos\beta$. In the vertical direction, its motion is uniformly accelerated with an

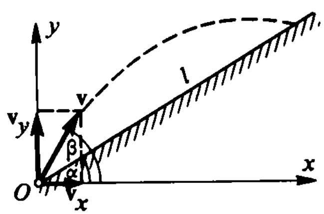

Fig. 48

initial velocity directed upwards and equal to $v_{0y} = v \sin\beta$, and acceleration $a = -g$. We can write two equations of motion:

$$x = v_x t, \qquad \text{or} \quad x = vt\cos\beta, \tag{1}$$

$$y = v_{0y}t - gt^2/2, \qquad \text{or} \quad y = vt\sin\beta - gt^2/2. \tag{2}$$

At the moment t_0 the shell falls to the ground, $x = l\cos\alpha$ and $y = l\sin\alpha$. Thus,

$$l\cos\alpha = vt_0\cos\beta, \quad l\sin\alpha = vt_0\sin\beta - gt_0^2/2. \tag{3}$$

We express the time t_0 of flight of the shell from the first equation in (3) and substitute it into the second equation:

$$t_0 = \frac{l\cos\alpha}{v\cos\beta}, \quad l\sin\alpha = \frac{l\cos\alpha\sin\beta}{\cos\beta} - \frac{gl^2\cos^2\alpha}{2v^2\cos^2\beta},$$

whence

$$l = \frac{2v^2}{g}\,\frac{\sin\beta\cos\alpha - \cos\beta\sin\alpha}{\cos^2\alpha}\cos\beta = \frac{2v^2}{g}\,\frac{\sin(\beta-\alpha)\cos\beta}{\cos^2\alpha} = 30.0\text{ m}.$$

35. A ball rolls down a curvilinear trough fixed to a wall at a height $h = 3.0$ m from the floor (Fig. 49). The initial velocity $v_0 = 7.0$ m/s of the ball forms an

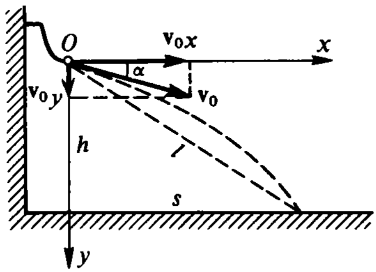

Fig. 49

angle $\alpha = 30°$ with the horizontal. At what distance l from the end of the trough does the ball fall on the floor?

Solution. The projections of the ball velocity onto the coordinate axes (see Fig. 49) are

$$x = v_{0x}t, \qquad \text{or} \quad x = v_0 t\cos\alpha,$$

$$y = v_{0y}t + gt^2/2, \qquad \text{or} \quad y = v_0 t\sin\alpha + gt^2/2.$$

At the moment t_0 the ball falls on the floor, $x = s$ and $y = h$. Therefore,

$$s = v_0 t_0\cos\alpha, \quad h = v_0 t_0 \sin\alpha + gt_0^2/2.$$

From the first equation, we obtain $t_0 = s/(v_0\cos\alpha)$. Substituting this expression for t_0 into the second equation, we get

$$h = s\tan\alpha + \frac{gs^2}{2v_0^2\cos^2\alpha},$$

or

$$\frac{g}{2v_0^2\cos^2\alpha}s^2 + (\tan\alpha)s - h = 0.$$

It is more convenient to solve this equation numerically for s:

$$\frac{9.8}{2\times 7^2\cos^2 30°}s^2 + (\tan 30°)s - 3.0 = 0,\quad s^2 + \frac{5\sqrt{3}}{2}s - \frac{45}{2} = 0,$$

which gives $s = 3.0$ m. The required distance $l = \sqrt{s^2 + h^2} = 4.2$ m.

36. At what height above the Earth's surface does the force of gravity decrease by 10%? The radius R of the Earth is 6370 km.

Solution. The force of gravity at the surface of the Earth is

$$F_1 = GmM/R^2. \tag{1}$$

The force of gravity at a height H is given by

$$F_2 = GmM/(R + H)^2, \tag{2}$$

where M is the mass of the Earth and m is the mass of a body. Dividing Eq. (1) by (2), we obtain $F_1/F_2 = (R + H)^2/R^2$, whence

$$(R + H)^2 = (F_1/F_2)R^2,\quad R + H = R\sqrt{F_1/F_2},$$
$$H = R(\sqrt{F_1/F_2} - 1) = 350 \text{ km}.$$

37. Find the mass and average density of the Earth. The radius R of the Earth is 6400 km.

Solution. The force of gravity for any body of mass m at the surface of the Earth is $mg = GmM/R^2$. Hence the mass of the Earth is $M = gR^2/G = 6\times 10^{24}$ kg. The average density of the Earth is $\rho = 3M/4\pi R^3 \simeq 5.5\times 10^3$ kg/m^3.

38. What is the maximum height H_{max} of a massive brick column of a constant cross-sectional area s for which the column deformation due to the force of gravity is within the elastic limit? What is the maximum relative compression of the brickwork in this case? Young's modulus of brick is $E = 3000$ MPa, the density of brick is $\rho = 1.8\times 10^3$ kg/m^3, and the limiting compressive stress $\sigma_{lim} = 3$ MPa.

Solution. The maximum stress and maximum relative compression will be observed at the base of the column: $\sigma = Mg/s$, where the mass of the column is $M = \rho sH$. Consequently, $\sigma = \rho gsH/s = \rho gH$, whence $H = \sigma/\rho g$. The maximum height of the column is $H_{max} = \sigma_{lim}/\rho g = 160$ m. The maximum relative compression $\varepsilon = \sigma_{lim}/E = 0.001$.

39. A rod made of a material with Young's modulus $E = 200$ GPa has a cross-sectional area $s_1 = 2$ cm^2 over a length $l_1 = 1$ m and $s_2 = 3$ cm^2 over a length $l_2 = 2$ m. The rod is fixed in a vertical plane and is acted upon by two tensile forces $F_1 = 600$ kN and $F_2 = 800$ kN (Fig. 50). Find the maximum tensile stress in the

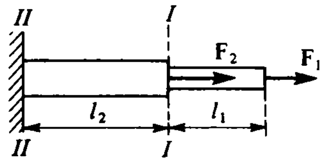

Fig. 50

section *I-I* and the stress in the section *II-II* where the rod is fixed, appearing under the action of these forces, and the rod elongation caused by these forces. The mass of the rod should be neglected.

Solution. Let us consider the right-hand part of the rod whose length $l_1 =$ 1 m. This part is stretched by the force $F_1 = 600$ kN. The tensile stress is $\sigma_1 = F_1/s_1 = 3000$ MPa. From formulas $\sigma = E\varepsilon$ and $\varepsilon = \Delta l/l$ (see Sec. 1.16), we obtain $\Delta l = l\varepsilon = l\sigma/E$. Consequently, the elongation $\Delta l_1 = l_1\sigma_1/E = 1.5$ cm. The left-hand part of the rod is stretched by the force $F = F_1 + F_2 = 1400$ kN. The corresponding tensile stress is $\sigma_2 = F/s_2 = 3500$ MPa. The rod elongation on this segment is $\Delta l_2 = l_2\sigma_2/E = 3.5$ cm. The total elongation of the rod $\Delta l = \Delta l_1 + \Delta l_2 = 5$ cm.

Exercises

16. A body slides with an acceleration of 3.2 m/s^2 down an inclined plane with a slope of 30°. Find the coefficient of friction.
Answer. 0.2.
17. A body slides down an inclined plane. The coefficient of sliding friction is 0.3. Find the slope of the plane if the body is known to acquire a velocity of 5.6 m/s by the end of the second second.
Answer. 30°.
18. (a) A shell whose mass is 8 kg is fired with a velocity of 600 m/s from a gun whose barrel diameter is 100 mm. The average pressure of the gunpowder gas in the barrel is 1 MPa. Find the time for which the shell moves in the barrel.
Answer. 0.61 s.

(b) Brakes are applied to a car moving with a disengaged engine, bringing it to a halt after 2 s. What is its velocity at the moment when the brakes are applied if the coefficient of friction between the road and the tyres is 0.4?
Answer. 27.5 km/h.
19. (a) A man having a mass of 75 kg and standing by a railroad steps on the footboard of a trolley moving with a velocity of 3 m/s. What is the new velocity of the trolley if its mass is 300 kg?
Answer. 2.4 m/s.

(b) An automatic-coupling carriage whose mass is 5000 kg moves with a velocity of 2.0 m/s and catches up with another carriage having a mass of 7500 kg and a velocity of 1.0 m/s, and is coupled with it. Moving together, the two carriages collide with a third carriage moving in the opposite direction with a velocity

of 1.6 m/s, after which the three carriages start to move in the direction of the third carriage with a velocity of 0.1 m/s. Find the velocity of the two carriages after coupling and the mass of the third carriage. Friction should be neglected.
Answer. 1.4 m/s, 12.5 t.

20. (a) A bullet is fired from a gun with a velocity of 1000 m/s. The recoil velocity of the gun is 2 m/s. What is the ratio of the masses of the gun and the bullet?
Answer. 500.

(b) A skater having a mass of 70 kg and standing on ice throws a thin plate whose mass is 2 kg so that it slides over the ice surface and stops in 4 s, having covered a distance of 28 m. What is the magnitude and direction of the velocity of the skater after the throw? How long does it take for him to stop if the coefficient of friction between the skates and the surface of ice is 0.04?
Answer. 0.4 m/s, 10 s.

21. A grenade having a mass of 10 kg and flying with a velocity of 10 m/s explodes into two fragments. The larger fragment has a velocity of 25 m/s and is directed as the initial velocity of the grenade, while the smaller fragment has a velocity of 12.5 m/s in the opposite direction. Find the masses of the fragments.
Answer. 6 kg, 4 kg.

22. (a) A grenade flying horizontally with a velocity of 12 m/s explodes into two fragments with masses of 10 kg and 5 kg. The velocity of the larger fragment is 25 m/s and forms an angle of 330° with the horizontal. Find the magnitude and direction of the velocity of the smaller fragment.
Answer. 26 m/s, 164° to the horizontal.

(b) A 50-kg shell is fired from a self-propelled cannon having a mass of 20×10^3 kg and moving with a velocity of 9 km/h. The shell has a velocity of 1000 m/s directed at 60° to the horizontal. Find the velocity of the cannon after the shot if it is fired in a direction opposite to the direction of its motion (in the same direction).
Answer. 13.5 km/h (45 km/h).

23. (a*) A body whose mass is 6 kg slides over the smooth surface of a trihedral prism having a mass of 24 kg and a slope of 15°. The prism lies on a smooth horizontal plane. Find the velocity of motion and the acceleration of the prism by the end of the fourth second after the beginning of motion of the body. Friction should be neglected.
Answer. 3.9 m/s, 0.98 m/s^2.

(b) A body having a mass of 4 kg slides over the smooth surface of a trihedral prism with a slope of 30°. The prism lies on a smooth horizontal plane and acquires an acceleration of 1.7 m/s^2 during the motion of the body. Find the mass of the prism and the horizontal projection of the acceleration of the body relative to a fixed reference system. Friction should be neglected.
Answer. 16 kg, 6.8 m/s^2.

(c) A body whose mass is 5 kg slides over the smooth surface of an oblique prism lying on a horizontal plane. The plane has a vertical ledge at the rear wall, which keeps the prism at rest. What is the slope of the prism if the force of pressure exerted by it on the ledge during the motion of the body is 12.25 N? Friction should be neglected.
Answer. 15°.

24*. A jet plane increases its velocity in a horizontal flight from 200 m/s to

220 m/s over 0.2 s. The combustion products, whose density is 1.5 kg/m^3, are ejected through a nozzle having a diameter of 60 cm with a velocity of 800 m/s relative to the plane. Find the mass of the plane. While calculating the velocity of combustion products relative to the Earth, assume that the velocity of the plane is 210 m/s. The increase in the resistance of air flow to the motion of the plane over the time interval under consideration should be neglected.
Answer. 1.96×10^3 kg.
25. Two loads of equal mass hang at the ends of a rope passing through a pulley suspended from the ceiling. After an additional load of a 4-kg mass has been placed on the right load, the loads start to move with an acceleration of $0.25g$. Find the initial mass of the loads and the tension of the rope during the motion. The masses of the pulley and rope, as well as the friction at the pulley axis, should be neglected.
Answer. 6 kg, 73.5 N.
26*. A pulley is suspended from a spring balance attached to the ceiling of a lift moving with a downward acceleration of $0.2g$. Two loads with a total mass of 48 kg are attached to the rope of the pulley. The loads move with an acceleration of $0.3g$ relative to the lift. Find the masses of the loads and the reading of the spring balance. The masses of the pulley and rope, as well as the friction at the pulley axis, should be neglected.
Answer. 15 kg, 33 kg, 323 N.
27*. A pulley with a rope is fixed at the upper edge of an inclined plane with a slope of 30°. A 200-kg load attached to the vertical part of the rope lifts two loads with a mass of 100 kg each, connected by a rope, up the inclined plane. Find the acceleration of the loads and the tension of the rope if the coefficient of friction between the loads and the inclined plane is 0.2. The masses of the pulley and rope, as well as the friction at the pulley axis, should be neglected.
Answer. 1.6 m/s^2, 818 N.
28*. A 50-kg load is suspended from a rope attached to the smaller pulley of a double block (consisting of two pulleys of different radii, rigidly fixed to each

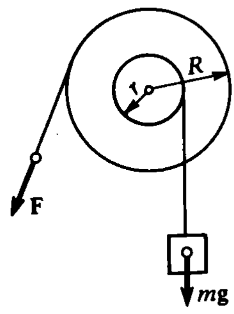

Fig. 51

other and having a common axis), rotating in a bearing arm on an axle fixed to the ceiling (Fig. 51). The load is lifted with an acceleration of 2.0 m/s^2 by a force ap-

plied to the rope fixed to the larger pulley. Find this force if the radii of the pulleys are 12 cm and 30 cm respectively. The rope attached to the load is wound on the smaller pulley. The masses of the block and rope, as well as the friction at the block axis, should be neglected.
Answer. 236 N.
29. (a) A body falls from a height of 1000 m with zero initial velocity. Simultaneously, another body falls from a certain height with an initial velocity of 8.0 m/s. The two bodies reach the ground at the same time. What is the initial height of the second body?[8]
Answer. 1114 m.

(b) A body falling freely from a certain height covers the distance from 1100 m to 120 m above the sea level during 10 s. What is the initial height of the body?
Answer. 1222.5 m.

(c) A body falls from a certain height. Two seconds later, another body falls from the same height. How long after the beginning of motion of the first body is the distance between the bodies twice the distance at the moment the second body starts to fall?
Answer. 3 s.

(d) A 5-kg body falls vertically with an acceleration of 14.7 m/s^2. What force acts on the body in addition to the force of gravity?
Answer. 24.5 N.
30*. (a) A body falls freely from a height of 50 m. Simultaneously, another body is thrown from the surface of the Earth with a certain initial velocity. The two bodies meet at a height of 10 m. What is the initial velocity of the second body?
Answer. 17.5 m/s.

(b) A body thrown vertically upwards passes twice by a point at a height of 10 m over a time interval of 4 s. Find the initial velocity of the body.
Answer. 24.1 m/s.
31*. (a) A helicopter takes off from the Earth's surface vertically with an acceleration of 2.45 m/s^2. Eight seconds after the beginning of motion, a body falls from it. In what time and with what velocity will this body hit the ground?
Answer. 6.5 s, 44 m/s.

(b) A helicopter ascends from the ground vertically with a velocity of 10 m/s. At a height of 100 m, a body is dropped from it with a downward velocity of 2 m/s relative to the helicopter. Find the maximum height reached by the body, the time of its motion, and the velocity with which it hits the ground.
Answer. 103.4 m, 5.4 s, 45 m/s.

(c) A helicopter ascends uniformly with a velocity of 5 m/s. At a height of 100 m, a bullet, whose mass is 10 g, is shot downwards from a toy pistol in whose barrel a force of 20 N acts on the bullet during 0.01 s. In what time and with what velocity will the bullet hit the ground?
Answer. 3.2 s, 46.7 m/s.

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[8] It should be recalled that here and below the air resistance should be neglected if the opposite is not stipulated. (*Editor's note.*)

(d) A small ball is suspended from the ceiling of a lift ascending with an acceleration of 1 m/s^2 at 2.0 m from the floor. When the lift is at 3 m from the bottom of the shaft, the thread from which the ball is suspended breaks, and the ball falls to the bottom of the shaft through the hole in the floor of the lift. Find the maximum height to which the ball is lifted, the time required for the ball to hit the bottom of the shaft, and its final velocity.
Answer. 5.31 m, 1.27 s, 10.0 m/s.
32. (a) A bomb is dropped from an aeroplane flying horizontally at a height of 500 m. It falls at a distance of 1 km along the horizontal from the place where it has been dropped. Find the velocity of the aeroplane at the moment the bomb is dropped and the angle at which the bomb hits the ground.
Answer. 99.0 m/s, 45°.
(b) A stone thrown horizontally from a certain height falls to the ground in 3 s at 60° to the vertical. What is the initial velocity of the stone?
Answer. 25.5 m/s.
(c) Two stones are thrown horizontally with velocities of 5 m/s and 7.5 m/s from two points at the same height. The stones fall into water simultaneously and the first stone has a horizontal range of 10 m. Find the height from which the stones are thrown, the duration of their flights, and the horizontal range of the second stone.
Answer. 19.6 m, 2 s, 15 m.
(d*) A skater moving with a velocity of 10 m/s throws a 2-kg stone in the forward direction, as a result of which his velocity decreases to 9.5 m/s. Find the horizontal distance covered by the stone between the point from which it is thrown and the point at which it falls if the mass of the skater (without the stone) is 80 kg, and the stone is thrown from a height of 160 cm in the horizontal direction. What is the distance between the skater and the point at which the stone falls at this moment?
Answer. 17.1 m, 11.7 m.
(e*) A grenade is thrown from the high bank of a river with a horizontal velocity of 4.9 m/s. Half a second later, the grenade explodes into two fragments of equal mass, one of which falls vertically and reaches the surface of water with a velocity of 9.8 m/s, while the other starts to fly in the horizontal direction. Find the minimum distance between the point from which the grenade is thrown and the point at which the second fragment falls.
Answer. 8.8 m.
33. (a) Find the horizontal ranges of bodies thrown at angles of 15°, 45°, and 75° to the horizontal with a velocity of 10 m/s. Show that the maximum range corresponds to the angle of 45° to the horizontal. Find the maximum height for this case.
Answer. 5.1 m, 10.2 m, 5.1 m, 3.6 m.
(b) A body is thrown at an angle of 15° to the horizontal so that it falls at a distance of 30 m along the horizontal. What is its initial velocity? What is the maximum height reached by it? In what time and at what horizontal distance from the initial point will the body be when its height is 1 m?
Answer. 24.2 m/s, 2 m, 0.19 s, 4.5 m, 1.09 s, 25.5 m.

(c*) A grenade is thrown at an angle of 30° to the horizontal with an initial velocity of 10 m/s. At the maximum height, it explodes into two fragments of equal mass, one of which falls vertically with an initial velocity of 5 m/s. At what distance from the point where the grenade has been thrown does the second fragment fall?
Answer. 26 m.
34*. A shell is fired from a gun mounted at the bottom of a hill. The shell falls at a distance of 500 m. The slope of the hill is 30°, while the angle between the barrel of the gun and the horizontal is 60°. What is the initial velocity of the shell leaving the barrel? How long does it take for the shell to traverse the trajectory between two points corresponding to the height at which it falls?
Answer. 86 m/s, 7 s.
35*. (a) A stone is thrown from the high bank of a river with a velocity of 10 m/s at an angle of 330° to the horizontal. Find the height from which the stone is thrown if its horizontal range is 20 m.
Answer. 37.7 m.
(b) A grenade is thrown from the high bank of a river with a velocity of 10 m/s directed at 330° to the horizontal. At the point where the velocity vector of the grenade forms an angle of 315° with the horizontal, it explodes into two fragments of equal mass, one of which falls vertically and the other moves in the horizontal direction. The magnitudes of the velocities of the fragments are equal. Half a second after the explosion, the fragments fall into water. Find the height of the bank, the horizontal range, and the final velocity of the second fragment. The air resistance should be neglected.
Answer. 12.4 m, 15.2 m, 28 m/s.
36. A body is lifted to 1600 km above the surface of the Earth. What is the decrease (in percent) in the force of gravity acting on the body? The Earth's radius should be taken as 6400 km.
Answer. 36%.
37. (a) Find the mass and the average density of the Moon, assuming that its radius is 1740 km and the free-fall acceleration on the Moon is 1.6 m/s^2.
Answer. 7.4×10^{22} kg, 3400 kg/m^3.
(b) Find the free-fall acceleration on Mars, assuming that its mass is 0.65×10^{24} kg and its diameter is 6800 km.
Answer. 0.9 m/s^2.
38. What is the maximum height of a brick column for which the stress due to the force of gravity does not exceed 3 MPa? The column cross section has a ledge at one third of its height such that the cross-sectional area of the upper part of the column is equal to half the cross-sectional area of the lower part. Assume that the density of the building material is 2000 kg/m^3. The possibility of toppling the column should be disregarded.
Answer. 230 m.
39. Compressive forces of 300 kN and 900 kN are applied to a horizontal rod (Fig. 52) having a cross-sectional area of 3.0 cm^2 over a length of 2.0 m and 5.0 cm^2 over a length of 3.0 m. Find the maximum compressive stress in section *I-I* and the stress in section *II-II* where the rod is fixed, as well as the change in the

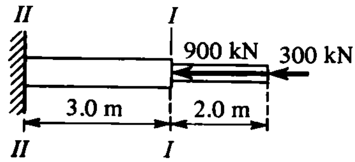

Fig. 52

rod length under the action of the compressive forces, assuming that Young's modulus is 10^5 MPa. The mass of the rod should be neglected.
Answer. 1 GPa, 2.4 GPa, 9.2 cm.

1.21. Work and Power

Work is a physical quantity equal to the product of the magnitudes of force and displacement of a body under the action of this force and the cosine of the angle between the vectors of force and displacement (or velocity):

$$A = Fs \cos \alpha$$

(Fig. 53*a*). If the angle between the directions of force and velocity varies over different segments of the path (Fig. 53*b*), the total

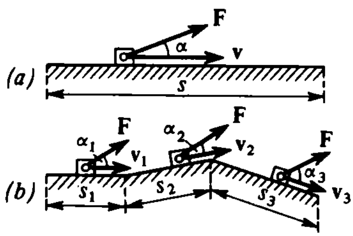

Fig. 53

work is calculated as the sum of the works done on different segments.

Figure 53*a* shows that the product $F \cos \alpha$ is the projection of force onto the direction of displacement. Consequently, we can say that work is equal to the projection of force onto the direction of displacement, multiplied by the length of the path traversed by the point of application of the force: $A = F_s s$.

Particular cases.

1. If the direction of force coincides with the direction of displacement, i.e. if $\alpha = 0$, we have $A = Fs$.

2. If the force is directed at right angles to the displacement, i.e. if $\alpha = 90°$, then $A = 0$. Consequently, a force does not perform any work if the body is displaced in a direction normal to the direction of the force. In particular, the work of the force of gravity is equal to the product of this force and the vertical displacement of the body irrespective of the path along which it moves:

$$A = mg(H - h).$$

3. If the angle between the directions of force and displacement is obtuse, i.e. if $\alpha > 90°$, then $A < 0$ since in this case $\cos\alpha < 0$.

4. If a body is displaced in a direction opposite to the direction of force, i.e. if $\alpha = 180°$, then $\cos\alpha = -1$, and $A = -Fs$. This means that the work of a resistance force is negative.

When the directions of force and displacement coincide, the work is numerically equal to the area bounded by the curve representing the magnitude of force as a function of distance (Fig. 54*a*-*c*). If the direction of force is opposite to the direction of

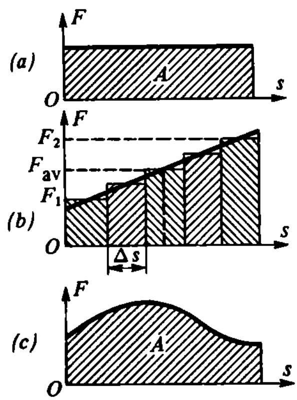

Fig. 54

displacement, the work is equal to the same area but has minus sign.

If the magnitude of force varies linearly during the time of motion (see Fig. 54*b*), we can divide the displacement into small segments Δs and, while calculating work, assume that the force is constant over such a small segment and is equal to a certain average force. Then the total work is numerically equal to the area bounded by the s-axis and the step-like line. The smaller the value of Δs, the closer the step-like line approaches the straight line describing force as a function of displacement. In the limit as $s \to 0$, these lines coincide. In this case, the work is equal to the product of the average force $F_{av} = (F_1 + F_2)/2$ and the displacement. In a similar way, we can show that for an arbitrary dependence of force on displacement (Fig. 54*c*), the work is numerically equal to the area under the curve representing force as a function of displacement.

An important characteristic of operation of machines and mechanisms is power. **Power** is the work done per unit time:

$$N = \Delta A / \Delta t.$$

Here Δt is the time interval which is so small that the magnitude F of the force and the angle α between the force and displacement can be considered constant. Substituting the expression $F \Delta s \cos \alpha$ for ΔA and considering that $\Delta s/\Delta t$ is the magnitude of velocity, we obtain the following expression for power:

$$N = Fv \cos \alpha,$$

where α is the angle between the vectors $\mathbf{F}$ and $\mathbf{v}$.

The SI unit of work is a **joule** (J): 1 J = 1 N·m. The corresponding unit of power is a **watt** (W): 1 W = 1 J/s.

The work of machines and mechanisms is characterized by the **efficiency** η. The efficiency is the ratio of the useful work done by a machine to the total work. Useful work is always less than total work due to losses associated with friction and other phenomena.

1.22. Energy. Kinetic and Potential Energies

Depending on the physical nature of processes, various kinds of energy are encountered, e.g. mechanical, internal, electromagnetic energy.

Mechanical energy is a physical quantity that characterizes the ability of a body to do work.

The unit of energy is the same as the unit of work.

There are two kinds of mechanical energy, viz. kinetic energy and potential energy. **Kinetic energy** is due to the motion of a body, while **potential energy** is due to the interaction between bodies. The sum of kinetic and potential energies is called the **total mechanical energy** of a body.

Let us consider a body (material point) of mass m in the state of rest. Obviously, its kinetic energy is equal to zero. Let us apply a constant force $\mathbf{F}$ to the body for time t. During this time, the force does a work A on the body which, by definition, should be equal to the increment of the mechanical energy of the body.

During time t, the body moves with a constant acceleration $a = v/t$, where v is the velocity acquired by the body by the instant t. Consequently, the magnitude of force $\mathbf{F}$ can be represented in the form

$$ma = mv/t.$$

Suppose that the distance s covered in this case is $vt/2$, and the projection of force on the direction of displacement is equal to the magnitude of force. Multiplying the magnitude of force by the distance, we obtain the work A which, as has been shown above, is equal to the increment of the kinetic energy W_k of the body:

$$A = Fs = (mv/t)(vt/2) = mv^2/2 = \Delta W_k.$$

Since the initial value of W_k is zero, ΔW_k is equal to the value of the kinetic energy of the body at the moment t when its velocity is equal to v. Thus, the kinetic energy of a body of mass m moving with a velocity v is defined by the following formula:

$$W_k = mv^2/2.$$

The potential energy depends on the mutual position of interacting bodies. For example, a body of mass m lifted above the surface of the Earth to a height h has the potential energy

$$W_p = mgh.$$

Descending from the height h, this body does a work equal to mgh.

Potential energy is determined to within an arbitrary additive constant. If in the above example we measure height not from the surface of the Earth but from a certain level located at a height h_0, the potential energy will be $mg(h - h_0)$. Thus, potential energy is counted from an arbitrary chosen position of a body to which we ascribe the zero value of the potential energy W_p.

A stretched (or compressed) spring also possesses a potential energy equal to the work which must be done to change the length of the spring by Δl. The force with which the spring should be stretched varies according to the law $F = kx$, where k is the rigidity of the spring, $x = 0$, and $x = \Delta l$ at the beginning and end of the deformation. The average value of the force is $k\,\Delta l/2$. The distance covered by the point of application of the force is Δl. Consequently,

$$W_p = A = \frac{k\,\Delta l}{2}\,\Delta l = \frac{k\,(\Delta l)^2}{2}.$$

Thus, the potential energy of a stretched or compressed spring is

$$W_p = \frac{k\,(\Delta l)^2}{2}.$$

It should be noted that the expression obtained above for the potential energy of a spring is valid under the condition that the potential energy of the undeformed spring is zero.

1.23. Law of Energy Conservation

For certain forces existing in nature the work done on a body is determined only by the initial and final positions of the body and not by the path along which the body moves from the initial to the final point. Such forces are called *conservative*. Figure 55 shows two trajectories along which a body (material point) passes from position *1* to position *2*. If the forces acting on the body in this case are conservative, their work over path *I* is the same as over path *II* or over any other possible path starting at point *1* and terminating at point *2*.

Conservative forces include gravitational forces, as well as Coulomb electric forces (i.e. the forces obeying Coulomb's law). Forces of friction are *nonconservative*.

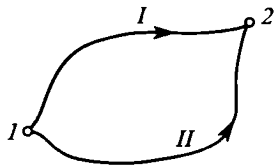

Fig. 55

The law of energy conservation is one of the fundamental laws of nature. In mechanics, this law is formulated as follows: *the total mechanical energy of a closed system of bodies, whose interaction is described by conservative forces, remains unchanged (is conserved).*

When the total mechanical energy is constant, the kinetic and potential energy may change only in such a way that their sum remains unchanged. For example, a body dropped from a height h has initial values $W_k = 0$ and $W_p = mgh$ of its kinetic and potential energy. As the body is falling, its kinetic energy W_k increases, while its potential energy W_p decreases. However, at each instant of time their sum is mgh. At the ground, the potential energy W_p vanishes, while $W_k = mv^2/2$ becomes equal to mgh. Hence we obtain the familiar formula for the velocity of a body dropped from a height h:

$$v = \sqrt{2gh}.$$

Problems with Solutions

40. A load whose mass $m = 3$ t is lifted by a winch with an acceleration $a = 2$ m/s^2. Find the work done during the first one and a half seconds from the beginning of motion.

Solution. The height to which the body is lifted during the first t seconds is $h = at^2/2$. We can find the tension of the rope from the equation of motion for the load. Two forces act on the load: the force of gravity $m\mathbf{g}$ and the tension $\mathbf{T}$ of the rope. According to Newton's second law, $T - mg = ma$, whence $T = m(a + g)$. The work done by the winch is given by

$$A = Th = m(g + a)at^2/2 = 79.65 \text{ J}.$$

41. A box is moved over a horizontal path by applying to a rope a force $F = 60$ N at an angle $\alpha = 30°$ to the horizontal. What is the work done during the displacement of the box over a distance of 0.5 km?

Solution. The work $A = Fs \cos\alpha = 26.0$ kJ.

42. Find the power of the engine of a crane lifting a load of mass $m = 3$ t with a constant velocity $v = 6$ m/min if the crane efficiency $\eta = 0.8$.

Solution. The power N_{cr} of the crane is mgv. The power N of its engine is $mgv/\eta = 3.675$ kW.

43. A hydroelectric power station is designed for the volumetric flow rate $Q = 0.8 \times 10^3$ m^3/s of water at a head[9] $H = 15.0$ m. The efficiency η of the station is 0.8. Find the rated power of the station.

Solution. The mass of water flowing per unit time is $m_t = m/t = Q\rho$, where $\rho = 10^3$ kg/m^3 is the density of water. The required power is $N = Q\rho gH\eta = 93$ MW.

44. Find the power N of a water jet flowing through an orifice of diameter $d = 20$ cm with a velocity $v = 5$ m/s.

Solution. The power of the jet is equal to the ratio of the kinetic energy of water having a mass m to the time during which this mass flows out: $N = mv^2/2t$, $m = Q\rho t$, where $\rho = 10^3$ kg/m^3 and Q is the volumetric flow rate of water. Since the volumetric flow rate of water is equal to the area of the jet cross section multiplied by the flow velocity, $Q = \pi d^2 v/4$, the water mass $m = \pi d^2 v\rho t/4$ and the jet power $N = Q\rho v^2/2 = \pi d^2 v^3 \rho/8 = 1.96$ kW.

45. The volumetric flow rate of water through a hydraulic turbine is $Q = 3.0$ m^3/s. Water enters the turbine with a velocity $v_1 = 6.0$ m/s and leaves it with a velocity $v_2 = 2.0$ m/s at a level which is below the inlet level by $H = 1.5$ m. The efficiency η of the turbine with an engine is 0.8. Find the power N_t of the turbine and the yearly energy output corresponding to this power, W_{year}.

Solution. For the zero level of the potential energy, we take the level of water leaving the turbine. Then the power of the water flow through the turbine exit is $N_1 = N_{p1} + N_{k1}$, where $N_{p1} = mgH/t = Q\rho gH$ and $N_{k1} = mv_1^2/2t = Q\rho v_1^2/2$. Consequently,

$$N_1 = Q\rho gH + Q\rho v_1^2/2 = Q\rho(gH + v_1^2/2).$$

The power of water flow at the turbine exist is equal to its kinetic energy per unit time:

$$N_2 = N_{k2} = Q\rho v_2^2/2.$$

The power of water flow lost in the turbine is

$$N = N_1 - N_2 = Q\rho[gH + (v_1^2 - v_2^2)/2].$$

Hence the turbine power is

$$N_t = \eta N = \eta Q\rho[gH + (v_1^2 - v_2^2)/2] = 73.7 \text{ kW}.$$

The possible yearly energy output is

$$W_{year} = 73.7 \text{ kW} \times 24 \text{ h/day} \times 365 \text{ days} = 645.6 \text{ MWh}.$$

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[9] The head is the difference in water levels in the upper and lower reaches of a hydroelectric power station. (*Editor's note.*)

46. An elastic spring whose length $l_0 = 30$ cm is compressed to $l = 22$ cm. Find the potential energy of the compressed spring if the force $F_l = 0.5$ MN/cm is required to reduce its length by unity.

Solution. The energy of a compressed spring is equal to the work that is spent for its compression. During the compression, the force acting on the spring increases from zero to the maximum value corresponding to the complete compression (see Sec. 1.16): $F_{max} = F_l(l - l_0)$. Since the spring is elastic, $F = k\,\Delta l$, where k is the rigidity of the spring. The physical meaning of k is the force required to compress the spring by the unit of length: $k = F/\Delta l = 0.5$ MN/cm. The force deforming the spring linearly depends on the change in its length (Fig. 56). Therefore, the

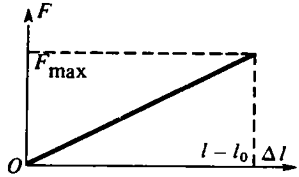


Fig. 56

work done for compressing the spring is

$$A = F_{max}(l - l_0)/2, \quad \text{hence} \quad W_s = A = k(l - l_0)^2/2 \approx 160 \text{ kJ}.$$

47. What work is done to lift from the ground the materials required for constructing a uniform inclined bar whose one end is supported by the foundation at the ground level, while the other end leans upon a wall of height $H = 4.0$ m? The length l of the bar is 15.0 m, its cross-sectional area $s = 20 \times 40$ cm^2. The density of the material is $\rho = 2.5 \times 10^3$ kg/m^3.

Solution. The cross-sectional area $s = ab$, the volume $u = ls = lab$, and the mass $m = u\rho = lab\rho = 300$ kg. We shall consider the ground level to be the zero potential energy level. The work done in lifting the materials from the Earth's surface is equal to the potential energy of the bar which, in turn, is equal to the product of the force of gravity acting on the bar and the height of its centre of mass above the ground level.

Let us prove this statement for an arbitrary body. We divide the body into small parts that can be regarded as material points with masses $m_1, m_2, \ldots, m_n$, located at heights $y_1, y_2, \ldots, y_n$ respectively above the zero level. The potential energy of the body is then given by

$$W_p = W_{p1} + W_{p2} + \ldots + W_{pn} = m_1 g y_1 + m_2 g y_2 + \ldots + m_n g y_n$$
$$= (m_1 y_1 + m_2 y_2 + \ldots + m_n y_n) g. \quad (1)$$

In Sec. 1.35 we shall obtain formula (1.35.2) for the coordinate of the centre of mass of a body:

$$y_{c.m} = (m_1 y_1 + m_2 y_2 + \ldots + m_n y_n)/m.$$

Hence it follows that $m_1y_1 + m_2y_2 + \ldots + m_ny_n = my_{c.m}$. Substituting this expression into Eq. (1), we obtain $W_p = mgy_{c.m}$. Various methods of determining the centres of mass of different bodies are also considered in Sec. 1.35. On the basis of above arguments, we find that the potential energy of the bar is $W_p = mgH/2$, where $H/2$ is the height of the centre of mass of the bar. Thus, the work $A = W_p = mgH/2 = 68.8$ kJ.

48. Having developed the maximum power, a diesel locomotive pulls a train with a mass of 2000 t up the hill with a slope $\alpha_1 = 0.005$ and a velocity $v_1 = 60$ km/h. Working at 60% of the maximum power, the locomotive can pull the same train up the hill with a slope $\alpha_2 = 0.003$ and a velocity $v_2 = 50$ km/h. Find the maximum power of the locomotive and the coefficient of friction.

Solution. The slope is the ratio of the height to the length of the path: $h/l = \sin\alpha$, where α is the angle between the displacement and the horizontal. If $\sin\alpha \ll 1$, $h/l \simeq \alpha$. The friction $F_{fr} = fmg$, since at a small slope the force of normal pressure $F_p = mg$. In the first case, the increase in the potential energy of the train over a certain distance is $\Delta W_{p1} = mgv_1t\alpha_1$. The energy loss measured by the work done by the friction is $W_{loss} = A_{fr} = F_{fr}v_1t = fmgv_1t$. Consequently,

$$N_{max} = (\Delta W_{p1} + W_{loss})/t = mgv_1\alpha_1 + mgv_1f = mgv_1(\alpha_1 + f). \quad (1)$$

Similarly, for the second case, we can write

$$0.6N_{max} = mgv_2(\alpha_2 + f). \quad (2)$$

From Eqs. (1) and (2) we find N_{max} and f. Dividing the second equation by the first, we obtain $0.6 = v_2(\alpha_2 + f)/v_1(\alpha_1 + f)$, or $0.6v_1\alpha_1 - v_2\alpha_2 = v_2f - 0.6v_1f$, whence

$$f = (0.6v_1\alpha_1 - v_2\alpha_2)/(v_2 - 0.6v_1) \simeq 0.002.$$

Substituting this value of the coefficient of friction into Eq. (1), we get $N_{max} = 2.29$ MW.

49. A body falls from a large height with zero initial velocity. Find the ratio of the kinetic energy acquired by it by the end of the first three seconds after the beginning of motion to the increment in the kinetic energy during the next three seconds.

Solution. The kinetic energy is proportional to the square of velocity. It is known that the velocity of free fall from the state of rest increases in proportion to the time of fall. Consequently, the kinetic energy is proportional to the square of the time of fall. Therefore, the kinetic energy in six seconds after the beginning of fall is four times higher than the kinetic energy in three seconds after the start. Thus, the required ratio is

$$W_{k1}/(W_{k2} - W_{k1}) = W_{k1}/(4W_{k1} - W_{k1}) = 1/3.$$

50. A motorcar moves over a horizontal road with a velocity $v_0 = 54$ km/h. After its engine is disengaged and the brakes are applied, it stops having covered a distance $l = 50$ m. Find the coefficient of friction between the wheels of the car and the road.

Solution. The kinetic energy of the car is converted into the internal energy of the wheels and the road due to friction. The measure of energy conversion is the work of friction:

$$mv_0^2/2 = F_{fr}l = fmgl, \quad \text{whence} \quad f = v_0^2/2gl = 0.23.$$

51. A rope whose length $l = 80$ cm and mass $m = 2$ kg is hanging from the end of a plane so that the length l_0 of the vertical segment is 50 cm. The other end of the rope is fixed by a nail. At a certain instant, the nail is pulled out. What is the velocity of the rope at the moment it completely slides off the plane? Friction should be neglected.

Solution. We take for the zero potential energy level the lower end of the rope when the entire rope slides off the plane, i.e. $l = 80$ cm below the level of the plane. As the rope slides off the plane, its potential energy decreases, being converted into kinetic energy. Since the losses in mechanical energy due to friction are disregarded in the problem, we have

$$W_{p1} = W_{k2} + W_{p2}. \tag{1}$$

The potential energy of the horizontal segment of the fixed rope of the length $l - l_0 = 30$ cm is

$$W'_{p1} = \frac{m(l - l_0)}{l} gl = mg(l - l_0).$$

The potential energy of the hanging segment of the rope is (see solution to Problem 47)

$$W'_{p1} = \frac{ml_0}{l} g \left(l - \frac{l_0}{2}\right),$$

where $l - l_0/2$ is the height of the middle of the segment of the rope hanging above the zero potential energy level. When the entire rope slides off the plane, its potential energy $W_{p2} = mgl/2$, while its kinetic energy $W_{k2} = mv^2/2$, since all particles of the rope material have the same velocity v. Substituting these expressions into Eq. (1), we obtain

$$mg(l - l_0) + mg\frac{l_0}{l}\left(l - \frac{l_0}{2}\right) = mg\frac{l}{2} + \frac{mv^2}{2}.$$

Hence the velocity of the rope is $v = \sqrt{g(l^2 - l_0^2)/l} = 2.18$ m/s.

52. A sledge slides down an ice-covered hill of length $l = 2.5$ m and height $h = 1.5$ m, and then moves over a horizontal ice surface until it comes to a halt. During what time does the sledge move over the horizontal path if the coefficient of friction $f = 0.04$? At what distance L from the foot of the hill will the sledge stop?

Solution. It is easier to solve first the second part of the problem, i.e. to find the length of the horizontal path traversed by the sledge. Since the kinetic energy at the beginning and end of motion is zero, the work of friction is equal to the change in the potential energy of the sledge. Assuming the level of the horizontal ice surface as the zero potential energy level, we obtain $W_p = A_{fr}$, where $A_{fr} = A_1 + A_2$. The work of friction over the inclined region is $A_1 = fmgl\cos\alpha = fmg\sqrt{l^2 - h^2}$, where α is the angle formed by the inclined plane with the horizontal. The work on the horizontal path is $A_2 = fmgL$. Thus,

$$mgh = fmg\sqrt{l^2 - h^2} + fmgL, \quad \text{or} \quad L = h/f - \sqrt{l^2 - h^2} = 35.5 \text{ m}.$$

To determine the time for which the sledge slides over the horizontal path, we first find the velocity of the sledge at the foot of the ice-covered hill. For this we equate the potential energy of the sledge at the top of the hill to the sum of the kinetic energy at the foot of the hill plus the work of friction over the inclined path:

$$W_p = mv^2/2 + A_1, \quad mgh = mv^2/2 + fmg\sqrt{l^2 - h^2},$$

whence $v = \sqrt{2g(h - f\sqrt{l^2 - h^2})}$. It is convenient to calculate the value of this quantity at this stage to avoid further complications: $v = 5.27$ m/s. The time for which the sledge slides over the horizontal surface is $t = L/v_{av} = 2L/v = 13.5$ s.

53. Two loads with the total mass $m_1 + m_2 = 30$ kg are attached to a nonstretchable rope passing through a locked pulley suspended from the ceiling. When the lock is removed, the loads start to move and, having covered the distance $h = 120$ cm, acquire a velocity $v = 2.0$ m/s. Find the masses of the loads. The masses of the rope and pulley, as well as the friction at the pulley axis, should be neglected.

Solution. During the motion of the loads, the potential energy of the larger load decreases, while that of the smaller load increases. Since the distances covered by the loads are equal, the total potential energy of the system decreases and the total kinetic energy increases by the same amount. The measure of energy conversion is the work done by the forces of gravity acting on the two loads, the work of the force of gravity of the larger load is positive and the work of the force of gravity of the smaller load is negative. The total work $A = m_2gh - m_1gh = (m_2 - m_1)gh$. Equating this work to the kinetic energy of the loads, we obtain

$$(m_2 - m_1)gh = (m_1 + m_2)v^2/2, \quad \text{or} \quad m_2 - m_1 = (m_1 + m_2)v^2/2gh = 5.1 \text{ kg}.$$

Since $m_2 + m_1 = 30$ kg, $m_2 = 17.55$ kg, and $m_1 = 12.45$ kg.

54. A pulley with a nonstretchable rope is fixed to the upper end of an inclined plane with a slope $\alpha = 30°$. A load of mass $m_2 = 100$ kg attached to the hanging vertical part of the rope makes a load of mass $m_1 = 150$ kg move up the inclined plane (Fig. 57). Having covered a distance $h = 80$ cm with zero initial velocity, the

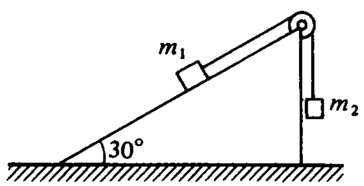


Fig. 57

ıds acquire a velocity $v = 0.5$ m/s. Find the coefficient of friction between the ıane and the load moving over it. The masses of the pulley and rope, as well as the friction at the pulley axis, should be neglected.

Solution (cf. solution to Problem 27). During the motion of the loads, their ɔtal potential energy decreases, while their kinetic energy increases. A part of the potential energy is converted into the internal energy of the loads m_1 and m_2 and of the plane due to friction. The total decrease in the potential energy of the loads is measured by the work of the forces of gravity acting on them. The part of energy coverted into the internal energy is equal to the work against friction. Consequently,

$$A_1 + A_2 = m_1 v^2/2 + m_2 v^2/2 + |A_{\text{fr}}| = (m_1 + m_2)v^2/2 + |A_{\text{fr}}|,$$

where $A_2 = m_2 gh$, $A_1 = -m_1 gs \sin\alpha = -m_1 gh \sin\alpha$, and $A_{\text{fr}} = -fm_1 gs \cos\alpha = -fm_1 gh \cos\alpha$. Thus

$$(m_2 - m_1 \sin\alpha)gh = (m_2 + m_1)v^2/2 + fm_1 gh \cos\alpha,$$

whence

$$f = \frac{(m_2 - m_1 \sin 30^\circ)gh - (m_2 + m_1)v^2/2}{m_1 gh \cos 30^\circ} = 0.16.$$

55. A helicopter ascends with a velocity $v_0 = 10$ m/s. At a height $H = 50$ m, a heavy body is dropped from it. With what velocity does this body reach the ground?

Solution (cf. solution to Problem 31). The total mechanical energy of the body at the height H has the form

$$W_H = W_{\text{p}} + W_{\text{k}} = mgH + mv_0^2/2. \tag{1}$$

At the ground level,

$$W = W_{\text{k}} = mv^2/2. \tag{2}$$

Equating Eqs. (1) and (2), we get

$$mgH + mv_0^2/2 = mv^2/2, \quad \text{whence} \quad v = \sqrt{2gH + v_0^2} = 33 \text{ m/s}.$$

56. (a) A stone is thrown at an angle α to the horizontal from a height $H = 6.0$ m above a horizontal surface with an initial velocity $v_0 = 10.0$ m/s. What is the velocity with which the stone reaches the surface? Find the relation between the direction of this velocity and the direction of the initial velocity of the stone.

(b) A body is thrown at an angle $\alpha = 30^\circ$ to the horizontal with a velocity $v_0 = 15$ m/s. Find the maximum height attained by the body and the magnitude and direction of its velocity at a height $h = 1.2$ m.

Solution. (a) (cf. solution to Problem 32). A body thrown with a velocity v_0 has the kinetic energy $W_{\text{k}} = mv_0^2/2$ and the potential energy $W_{\text{p}} = mgH$. Consequently, the total mechanical energy of the stone is

$$W = W_{\text{k}} + W_{\text{p}} = mv_0^2/2 + mgH.$$

If we neglect the air resistance, the potential energy of the falling stone is completely converted into its kinetic energy. At the ground, the potential energy vanishes. According to the law of energy conservation, we can write

$$mv^2/2 = mv_0^2/2 + mgH,$$

where v is the velocity with which the stone reaches the ground. Hence $v = \sqrt{v_0^2 + 2gH} = 14.75$ m/s.

The angle β between the direction of the velocity and the horizontal can be found from the expression $\cos\beta = v_{hor}/v$. The horizontal velocity can be found as the horizontal projection of the initial velocity of the stone: $v_{hor} = v_0 \cos\alpha$. Consequently.

$$\cos\beta = (v_0 \cos\alpha)/v, \quad \text{whence} \quad \cos\beta/\cos\alpha = v_0/v.$$

It can be seen that the ratio of the angles of the initial and final velocities to the horizontal is the same for bodies thrown up and down at the same angle.

(b) (cf. solution to Problem 33). When a body thrown at an angle to the horizontal ascends, its kinetic energy is partially converted into its potential energy and when it descends, the converse process takes place. The body thrown at an angle α to the horizontal with an initial velocity v_0 has the kinetic energy $W_{k0} = mv_0^2/2$. Its potential energy relative to the level from which it has been thrown is zero. At the upper point, the potential and kinetic energies of the body are mgH and $mv_{hor}^2/2$, where v_{hor} is the horizontal velocity of the body at the upper point. The total mechanical energy of the body at the upper point is

$$W_H = mgH + mv_{hor}^2/2, \quad W_H = W_0.$$

Since $W_0 = mv_0^2/2$ and $v_{hor} = v_0 \cos\alpha$, we have

$$mgH + m(v_0 \cos\alpha)^2/2 = mv_0^2/2.$$

Hence the maximum height reached by the body is

$$H = (v_0^2 - v_0^2 \cos^2\alpha)/2g = (v_0 \sin\alpha)^2/2g = 2.87 \text{ m}.$$

The total mechanical energy of the body at a height h is

$$W_h = mgh + mv_h^2/2, \quad W_h = W_0,$$

and hence

$$mgh + mv_h^2/2 = mv_0^2/2, \quad \text{whence} \quad v_h = \sqrt{v_0^2 - 2gh} = 14.2 \text{ m/s}.$$

If the velocity v_h forms an angle β with the horizontal, we obtain

$$\cos\beta = v_{hor}/v_h = (v_0 \cos\alpha)/v_h = 0.915, \quad \beta = \arccos 0.915.$$

57. A stone is thrown upwards from the high bank of a river at an angle $\alpha = 30°$ to the horizontal and with a velocity $v_0 = 20.0$ m/s. It fell into water at an angle $\beta = 60°$ to the horizontal. Find the maximum height H reached by the stone (above the river level) and the magnitude and direction of the stone velocity at a height $h = 2.0$ m and the height H_0 of the point from which the stone has been thrown.

Solution (cf. solutions to Problems 33 and 35). A body thrown at an angle α to the horizontal with an initial velocity v_0 has the kinetic energy $W_{k0} = mv_0^2/2$. Its potential energy relative to the level from which it has been thrown is zero.

At the upper point, the potential and kinetic energies of the body are mgH and $mv_{hor}^2/2$ respectively, where v_{hor} is the horizontal velocity of the body at the upper point. The total mechanical energy of the body at the upper point is

$$W_H = mgH + mv_{hor}^2/2, \quad W_H = W_0.$$

Since $W_0 = W_{k0}$ and $v_{hor} = v_0 \cos\alpha$, we have

$$mgH + m(v_0^2 \cos^2\alpha)/2 = mv_0^2/2.$$

Hence the maximum height attained by the body is

$$H = (v_0^2 - v_0^2 \cos^2\alpha)/2g = (v_0 \sin\alpha)^2/2g = 5.1 \text{ m}.$$

The total mechanical energy of the stone at the intermediate height h is

$$W_h = mgh + mv_h^2/2, \quad W_h = W_0.$$

Therefore,

$$mgh + mv_h^2/2 = mv_0^2/2, \quad \text{whence} \quad v_h = \sqrt{v_0^2 - 2gh} = 19.0 \text{ m/s}.$$

The direction of the velocity of the stone at the height h is given by

$$\cos\varphi = v_{hor}/v_h = (v_0 \cos\alpha)/v_h = 0.918, \quad \varphi = \arccos 0.918.$$

Since $\cos\beta/\cos\alpha = v_0/v$ (cf. solution to Problem 56), the velocity with which the stone falls into water is $v = (v_0 \cos\alpha)/\cos\beta$. The kinetic energy of the stone at this moment is

$$W_k = mv^2/2 = (mv_0^2/2)(\cos\alpha/\cos\beta)^2.$$

Taking the water level for the zero potential energy level, we find the total mechanical energy of the stone at the moment it has been thrown:

$$W = mv_0^2/2 + mgH_0, \quad W = W_k.$$

Thus,

$$mgH_0 + mv_0^2/2 = (mv_0^2/2)(\cos\alpha/\cos\beta)^2,$$

whence the height of the bank is

$$H_0 = (v_0^2/2g)[(\cos\alpha/\cos\beta)^2 - 1] = 40.8 \text{ m}.$$

58. A grenade is thrown up with a velocity $v_0 = 10.0$ m/s at an angle $\alpha = 30°$ to the horizontal. At the upper point of the flight the grenade explodes into two fragments of equal mass. One of them falls vertically downwards with an initial velocity $v_1 = 10.0$ m/s. Find the maximum height of the flight of the second fragment and the velocity with which it reaches the ground. The air resistance should be neglected. What additional energy is imparted to the fragments during the explosion if the mass of the grenade is $m = 1$ kg?

Solution. The momentum of the grenade at the point of explosion is directed along the horizontal: $p_0 = mv_0 \cos\alpha$ (Fig. 58). The time of explosion is very short, hence it can be assumed that the total momentum of the grenade and its fragments

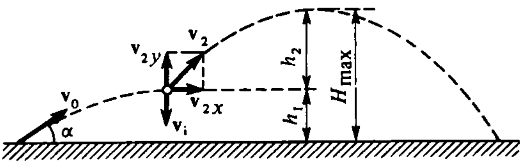


Fig. 58

is conserved during the explosion. The impulse of external forces over the explosion time is neglected (see Sec. 1.13). Since the momentum of the first fragment is directed vertically, the horizontal component of the momentum of the second fragment must be equal to p_0, i.e. $mv_{2x}/2 = mv_0 \cos\alpha$, whence the horizontal component of the initial velocity of the second fragment is $v_{2x} = 2v_0 \cos\alpha = 10\sqrt{3}$ m/s. The vertical component of the momentum of the second fragment is equal in magnitude to the momentum of the first fragment and has the opposite direction, i.e. is directed upwards: $v_{2y} = v_1 = 10.0$ m/s.

The maximum height reached by the second fragment can be found as the sum of the height to which the grenade ascends before the explosion and the height attained by the fragment above this point. Equating the initial kinetic energy of the grenade to its total energy at the moment before the explosion, we obtain

$$mv_0^2/2 = m(v_0 \cos\alpha)^2/2 + mgh_1,$$

whence $h_1 = (v_0 \sin\alpha)^2/2g$.

Equating further the kinetic energy of the second fragment at the point of explosion to its total mechanical energy at the maximum height (we take this point for the zero potential energy level), we obtain

$$\frac{mv_2^2}{2\times 2} = \frac{mv_{2x}^2}{2\times 2} + \frac{m}{2}gh_2, \quad \text{i.e.} \quad \frac{m(v_{2x}^2 + v_{2y}^2)}{4} = \frac{mv_{2x}^2}{4} + \frac{m}{2}gh_2,$$

whence $h_2 = v_{2y}^2/2g$. Hence,

$$H_{max} = h_1 + h_2 = (v_0 \sin\alpha)^2/2g + v_{2y}^2/2g = 6.4 \text{ m}.$$

We can find the velocity with which the second fragment falls to the ground by equating its total mechanical energy at the maximum height to the kinetic energy at the ground (we take this point for the zero potential energy level):

$$mv_{2x}^2/4 + mgH_{max}/2 = mv_k^2/4, \quad \text{or} \quad v_k = \sqrt{v_{2x}^2 + 2gH_{max}} = 20.6 \text{ m/s}.$$

The additional energy acquired by the grenade fragments during the explosion is equal to the difference between the kinetic energies of the grenade fragments after the explosion and of the grenade before the explosion:

$$\Delta W = \left[\frac{mv_1^2}{2 \times 2} + \frac{m(v_{2x}^2 + v_{2y}^2)}{2 \times 2}\right] - \frac{m(v_0 \cos\alpha)^2}{2}$$

$$= \frac{m}{4}(v_1^2 + v_{2x}^2 + v_{2y}^2 - 2v_0^2\cos^2\alpha) = 87.5\ \text{J}.$$

59. A bullet of mass m flying horizontally with a velocity v_1 hits a pendulum of mass M and stucks in it. To what height h does the pendulum rise? What fraction of the mechanical energy of the flying bullet is converted into the mechanical energy of the pendulum with the bullet?

Solution. When the bullet hits the pendulum, a fraction of the mechanical energy of the bullet is converted into the internal energy due to friction and deformation of the bodies associated with the penetration of the bullet into the pendulum, and the remaining fraction is converted into the mechanical (kinetic) energy of the pendulum with the bullet. This fraction of the entire mechanical energy of the bullet is then converted into the potential energy of the pendulum with the bullet as the pendulum swings to the maximum height h (Fig. 59). The law of momentum conservation makes it possible to find the fraction of the entire mechanical (kinetic) energy of the bullet converted into the mechanical energy of the pendulum with the bullet.

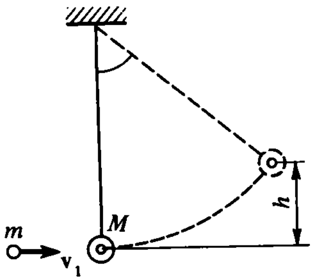


Fig. 59

Let us find the velocity with which the pendulum starts to move together with the bullet at the moment after the bullet hits the pendulum. For this purpose, we write the law of momentum conservation for the projections onto the horizontal axis: $mv_1 = (m + M)v_2$, whence the initial horizontal velocity of the pendulum with the bullet is $v_2 = mv_1/(m + M)$. The kinetic energy of the bullet is $W_1 = mv_1^2/2$. The kinetic energy of the pendulum with the bullet at the beginning of their motion is given by

$$W_2 = \frac{(m + M)v_2^2}{2} = \frac{m + M}{2}\left(\frac{mv_1}{m + M}\right)^2 = \frac{m^2v_1^2}{2(m + M)}.$$

The ratio of these energies is

$$\alpha = \frac{W_2}{W_1} = \frac{m^2 v_1^2}{2(m + M)} : \frac{m v_1^2}{2} = \frac{m}{m + M}.$$

We now equate the kinetic energy of the pendulum with the bullet at the beginning of their motion to the potential energy in an extreme position when their total velocity vanishes: $(m + M)v_2^2/2 = (m + M)gh$. Hence the maximum height to which the pendulum rises is

$$h = \frac{v_2^2}{2g} = \left(\frac{m}{m + M}\right)^2 \frac{v_1^2}{2g}.$$

60. A bullet of mass $m = 10$ g hits a 1.5-kg ball of mass suspended from a nonstretchable thread of length $l = 55$ cm and sticks in it. The bullet flies down at an angle $\alpha = 30°$ to the horizontal (Fig. 60) with a velocity $v = 400$ m/s. Through what angle φ is the ball with the bullet deflected from the vertical?

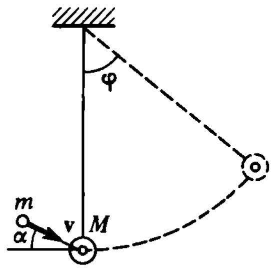


Fig. 60

Solution. The horizontal component of the momentum of the bullet before it hits the ball is equal to the momentum of the ball with the bullet after the impact. From the law of momentum conservation for the projections onto the horizontal axis, we have $mv \cos \alpha = (M + m)u$, where u is the initial velocity of the ball with the bullet stuck in it. Hence the initial velocity of the ball is $u = m(v \cos \alpha)/(M + m)$. Equating the kinetic energy of the ball with the bullet after the impact to the increment of their potential energy in the uppermost position, we obtain $(M + m)u^2/2 = (M + m)gh$, whence

$$h = \frac{u^2}{2g} = \left(\frac{m}{M + m}\right)^2 \frac{v^2 \cos^2 \alpha}{2g} = 0.27 \text{ m}.$$

We can now find the angle φ: $\cos \varphi = (l - h)/l \simeq 0.5$; $\varphi \simeq 60°$.

Exercises

40. Find the work done by the force of gravity acting on a freely falling body of 10-kg mass over the first, second, and third seconds of motion.
Answer. 480 J, 1440 J, 2400 J.

41. (a) A tugboat pulls a barge. The angle between the tug rope and the direction of motion of the tug is 30°. The tensile force of the rope is 30 kN, the force of resistance to the motion of the tug is 8 kN. Find the work done to move the tug with the barge over 2.0 km.
Answer. 68 MJ.

(b) A 100-kg load is uniformly lifted along an inclined plane with the help of a weightless rope. The slope of the plane is 30°. Find the work required for displacing the load over a distance of 80 cm. Friction should be neglected.
Answer. 392 J.

42. Determine the efficiency of an engine having a power of 12 kW and mounted on a crane which lifts a 5-t load with a velocity of 12 m/min.
Answer. 81.7%.

43. The efficiency of a 1-MW hydroelectric power plant is 80% at a head of 25 m. Find the volumetric flow rate of water at this plant.
Answer. 3.3 m^3/s.

44. A water jet flowing through a hole of diameter 10 cm has a power of 1 kW. What is the velocity of the jet?
Answer. 6.3 m/s.

45. The volumetric flow rate of water through a 1-MW hydraulic turbine is 20 m^3/s. The water enters the turbine with a velocity of 9 m/s and leaves it with a velocity of 3.0 m/s at a level of 2.0 m lower than the level of the inlet. Find the efficiency of the turbine and the maximum possible yearly output of energy if the efficiency of the electric motor connected to the turbine is 90%.
Answer. 90%, 7.9 GWh.

46*. (a) A rubber cord having a length of 0.5 m is stretched to twice its length. Find the potential energy of the stretched cord if the rigidity of rubber is 100 N/m.
Answer. 12.5 J.

(b) Find the rigidity of a spring if the work done by the force of compression of the spring is known to be 200 kJ when the length of the spring is reduced by 10 cm.
Answer. 40 MN/m.

(c) The spring of a pistol, having a length of 30 cm in the undeformed state and a rigidity of 200 N/m, is compressed to 20 cm and then released. What is the velocity of a 9-g bullet shot from this pistol if its efficiency is 90%?
Answer. 4.5 m/s.

47. (a) What work should be done to lift from the ground the material required for constructing a 20-m column with a cross-sectional area of 1.5 m^2? The density of the material is 2.6×10^3 kg/m^3.
Answer. 7.78 MJ.

(b) A 4.0-m column is lifted at one end so that its longitudinal axis forms an angle of 30° with the horizontal. The work done for this is 99.2 kJ. Find the cross-sectional area of the column if the density of its material is 2.5×10^3 kg/m^3.
Answer. 0.4 m^2.

(c*) A 10-m column has a stepped cross section: 40×50 cm^2 from the base to a height of 4 m, 30×40 cm^2 from 4 m to 7 m, and 20×30 cm^2 from 7 m to the top. What work should be done for lifting from the ground the material required

for constructing the column if its average density is 2.5×10^3 kg/m^3?
Answer. 127.5 kJ.

(d*) A load whose mass is 900 kg is lifted from a 100-m deep shaft with the help of a rope. The work done for this is 980 kJ. Find the mass per unit length of the rope and the efficiency of the mechanism.
Answer. 2.0 kg/m, 90%.

48*. (a) Developing a constant power, a locomotive can pull a 1000-t train up the hill with a slope of 0.005 and a velocity of 30 km/h. If the slope is 0.003, the velocity of the train is 40 km/h. Find the friction, assuming that it is the same in both cases, and the maximum power of the locomotive.
Answer. ~ 50 kN, ~ 800 kW.

(b) Using 80% of its maximum power, a diesel locomotive pulls a 3000-t train up the hill with a slope of 0.006 and a velocity of 75 km/h. Using 40% of its maximum power, it can pull the same train up the hill with a slope of 0.003 and a velocity of 60 km/h. Find the maximum power of the locomotive and the coefficient of friction.
Answer. ~ 6125 kW, 0.002.

49. (a) A motorcar starts off with a constant acceleration during 10 s. What is the ratio of the increment of the kinetic energy of the car over the first and second 5-second intervals of its motion?
Answer. 1/3.

(b*) A 1500-kg motorcar starts off with a constant acceleration of 2.0 m/s^2. The coefficient of resistance to its motion is 0.05. Find the work of the car engine over the first five seconds of motion. What is the ratio of this work to the work done during the next five seconds? During the next ten seconds?
Answer. 93.4 kJ, 1/3, 1/8.

50. A motorcar moves over a horizontal road with a velocity of 72 km/h. Find its braking distance if the coefficient of friction between the road and the wheels is 0.3.
Answer. 68 m.

51*. (a) A 120-cm rope lies on the surface of a table so that its one end is at the edge. As a result of a slight impact, whose strength can be neglected, it starts to slide off the table. What is the velocity of the rope at the moment it completely leaves the surface of the table? Friction should be neglected.
Answer. 3.4 m/s.

(b) A 1-m rope whose mass is 0.5 kg hangs by 20 cm from the edge of a plane. A 0.8-kg load is attached to the hanging end of the rope, while its other end is fixed to the plane. The fixed end is rapidly released. What is the velocity of the load and rope at the moment the entire rope slides off the plane? The size of the load and friction should be neglected.
Answer. ~ 3.6 m/s.

52*. A skater who has gathered speed up to 10 m/s passes a horizontal 50-m segment and then starts to move over an ice hill having a slope of 30°. The coefficient of friction between the skates and ice is 0.06. What distance is covered by the skater before he comes to halt? Find the time required to cover the entire distance. The mechanical energy lost when he moves up the hill should be neglected.
Answer. ~ 3.8 m, 7.3 s.

53. (a) Two loads with masses of 4 kg and 6 kg are attached to the ends of a nonstretchable rope passing through a locked pulley suspended from the ceiling. The larger load is 1.5 m above the smaller load. When the lock is removed, the loads start to move. Find the velocity of the loads at the moment they are at the same level. The masses of the pulley and rope, as well as the friction at the pulley axis, should be neglected.
Answer. 1.7 m/s.

(b) Two loads with masses of 4 kg and 6 kg are attached to a nonstretchable rope passing through a pulley suspended from the ceiling. The mass of a 1-m piece of the rope is 1 kg. The pulley is locked in the position when both loads are 2.0 m below the pulley axis. When the lock is removed, the loads start to move, and the heavier load reaches the floor with a velocity of 2.05 m/s. At what height are the loads at the beginning of their motion? The masses of the pulley and rope, as well as the friction at the pulley axis, should be neglected.
Answer. 1.5 m.

54*. A pulley with a nonstretchable rope is fixed at the upper end of an inclined plane having a slope of 60°. A 100-kg load attached to the vertical end of the rope pulls a 90-kg load upwards along the inclined plane. What is the velocity of the loads after they have covered 1.0 m from the state of rest if the coefficient of friction between the plane and the load moving over it is 0.15? The masses of the pulley and rope, as well as the friction at the pulley axis, should be neglected.
Answer. 0.8 m/s.

55*. An aerostat ascends from the ground with an acceleration of 2.45 m/s^2. Eight seconds after the beginning of motion, a body drops from its car. What is the velocity of the body when it hits the ground if the work of air resistance amounts to 10% of the mechanical energy of the body at the moment it is dropped?
Answer. 43.4 m/s.

56. (a) A stone thrown from the bank of a river, which is 4.0 m above the water level, reaches the surface of water with a velocity of 10.0 m/s. At what angle to the horizontal is it thrown and what is its initial velocity if it falls into water at an angle $\beta = \arccos 0.23$?
Answer. 60°, ~4.6 m/s.

(b*) A stone is thrown from the high bank of a river at an angle of 30° to the horizontal with a velocity of 10.0 m/s and falls into water at 60° to the horizontal. What is the height from which it is thrown?
Answer. 10.2 m.

57*. (a) A stone is thrown upwards from the high bank of a river at an angle of 30° to the horizontal. It falls into the river at an angle of 60° to the horizontal with a velocity of 30.0 m/s. Find the magnitudes and directions of its velocity at the levels of 2.0 m below and above the point from which it is thrown.
Answer. 16.1 m/s, arccos 0.93, 18.4 m/s, arccos 0.815.

(b) A stone is thrown upwards from the high bank of a river at an angle of 60° to the horizontal with a velocity of 10.0 m/s. It falls into the river at an angle of 80° to the horizontal. Find the maximum height reached by the stone and the height of the point from which it is thrown above the water level. Determine the

same quantities for the case when the work of the air resistance to the stone ascent amounts to 10% of the kinetic energy of the stone at the point from which it is thrown and, in the further flight, to 10% of the kinetic energy the stone would have at the moment it reached the water surface if the air resistance were absent.
Answer. 3.8 m, 3.4 m, 34 m.

58*. (a) A grenade is thrown at an angle of 60° to the horizontal with a velocity of 10.0 m/s. At the upper point of the flight, the grenade explodes into two fragments of equal mass. One of them starts to move vertically downwards with an initial velocity of 15.0 m/s. Find the maximum height above the ground level reached by the second fragment and the velocity with which it hits the ground. What additional energy is imparted to the fragments during the explosion if the mass of the grenade is 1 kg?
Answer. 15.3 m, 20.0 m/s, 125 J.

(b) A grenade is thrown upwards from the bank of a river which is 2.0 m above the water level at an angle of 30° to the horizontal with a velocity of 10.0 m/s. It explodes into two fragments on the same level from which it is thrown. The larger fragment whose mass amounts to 3/4 of the grenade mass starts to move at an angle of 330° to the horizontal with a velocity of 15.0 m/s. Find the maximum height reached by the second fragment, as well as the magnitude and direction of the velocity with which it falls into water.
Answer. 0.32 m, 8.0 m/s, the fragment falls at 0.312 rad to the horizontal in the direction opposite to the initial velocity of the grenade.

59*. A 10-g bullet flying horizontally with a velocity of 500 m/s hits a 1.0-kg ball which is suspended from a long nonstretchable thread and sticks in the ball. To what height does the ball rise and what part of the mechanical energy of the bullet is converted into the mechanical energy of the ball with the bullet? The mass of the thread should be neglected.
Answer. 1.25 m, 1/101.

60*. A 10-g bullet flying at an angle of 60° to the horizontal hits a 1.6-kg ball which is suspended from a nonstretchable thread whose length is 80 cm and sticks in the ball. As a result, the ball with the bullet is deflected by 30° to the horizontal. What is the velocity of the bullet? The mass of the thread should be neglected.
Answer. 464 m/s.

C. KINEMATICS AND DYNAMICS OF ROTATIONAL MOTION OF A RIGID BODY

1.24. Uniform Rotational Motion. Angular Velocity. Linear Velocity

Figure 61 illustrates two types of rotational motion of a body, viz. the rotation of a body about an axis passing through this body

(Fig. 61*a*) and the rotation of a body about an axis lying at a certain distance from it (Fig. 61*b* shows such a motion in the plane normal to the rotational axis). When the distance from the body

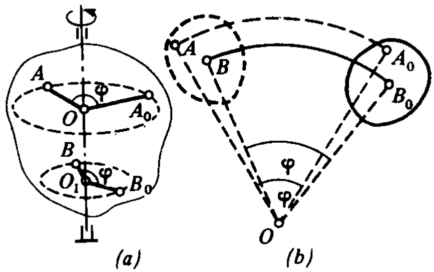


Fig. 61

to the rotational axis is large in comparison with the size of the body, the rotating body can be regarded as a material point. The motion of this point in a circle of the corresponding radius should be regarded as translatory motion.

The measure of rotation of a body is the angle φ by which a straight line connected to the body, which is normal to the rotational axis and passes through it, turns. Straight lines drawn in such a way through any point of a rotating rigid body turn by the same angle during the same interval of time. This angle φ is called the **angle of rotation** of the body. It is usually measured in radians (rad).

Henceforth, we shall consider only *uniform rotation* in which a body turns through the same angles during equal intervals of time. The main characteristic of a uniform rotational motion of a rigid body is its **angular velocity** which is defined as the angle of rotation of the body per unit time:

$$\omega = \varphi/t.$$

(Strictly speaking, this is the magnitude of the angular velocity which is a vector.) The unit of angular velocity is a radian per second (rad/s). The angular velocity ω is connected with the rotational frequency n of body (i.e. the number of revolutions completed by the body per unit time) through the relation $\omega = 2\pi n$.

In a rotational motion of a rigid body, all its points describe circles (see Fig. 61). The radius of the circular path of each point of the body is equal to the distance from this point to the rotational axis.

The magnitude of the velocity $\mathbf{v}$ of a point of a body moving in a circular path is given by

$$v = 2\pi r/T,$$

where r is the distance from this point to the rotational axis and T is the time of one revolution of the point about this axis. As distinct from the angular velocity ω, $\mathbf{v}$ is called the **linear velocity**.

Let us establish the relation between the linear velocity of points of a rotating body and its angular velocity. Figure 61 shows that the distance l covered by any point along the circular path can be expressed in terms of the angle φ of rotation of the body:

$$l = \varphi r,$$

where r is the distance from this point to the rotational axis. The linear velocity of a point in a uniform rotational motion is $v = l/t = \varphi r/t = (\varphi/t)r$. But $\varphi/t = \omega$, and hence

$$v = \omega r. \qquad (1.24.1)$$

The linear velocity of any point of a rotating body is equal to the product of the angular velocity of the body and the radius of the circle in which the point moves.

1.25. Centripetal Acceleration

Let us find the acceleration of points of a uniformly rotating rigid body.

Figure 62 shows that when a body rotates through an angle $\Delta\varphi$ during the time Δt, the velocity $\mathbf{v}$ of any point of this body, remaining constant in magnitude, turns through the same angle $\Delta\varphi$. The right part of the figure shows the triangle formed by vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\Delta\mathbf{v}$. For a very small $\Delta\varphi$, the length of the base of this triangle, equal to $|\Delta\mathbf{v}|$, practically coincides with the length of the arc of the circle of radius $v (v_1 = v_2 = v)$. Hence we can write $\Delta\varphi = (|\Delta\mathbf{v}|)/v$. This gives $|\Delta\mathbf{v}| = v\,\Delta\varphi$. For a very small $\Delta\varphi$, the angle between vectors $\mathbf{v}$ and $\Delta\mathbf{v}$ can be treated as right angle.

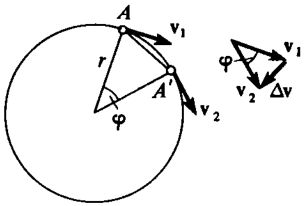


Fig. 62

By definition, the acceleration $\mathbf{a} = \Delta\mathbf{v}/\Delta t$. Substituting the obtained value of $|\Delta\mathbf{v}|$, we get

$$a = v\,\Delta\varphi/\Delta t = v\omega.$$

Using relation (1.24.1), we obtain the final expression for the magnitude of the acceleration of a point moving uniformly with a velocity v in a circle of radius r:

$$a = v^2/r = \omega^2 r.$$

This acceleration has the same direction as vector $\Delta\mathbf{v}$, viz. along the normal to vector $\mathbf{v}$. A normal to a circle is directed towards its centre. In view of what has been said above, the acceleration of a point moving uniformly in a circle is called **normal** or **centripetal acceleration**.

It can be shown that for a nonuniform motion along an arbitrary curved path, the acceleration is equal to the sum of the normal (centripetal) acceleration (in this case, r is treated as the radius of curvature of the path at the corresponding point) and the tangential acceleration directed along the tangent to the path, whose magnitude is determined by the time derivative dv/dt of the magnitude of velocity.

A material point moves uniformly in a circle under the action of a force $\mathbf{F}$ which is a constant magnitude and whose direction at each instant is normal to the velocity vector $\mathbf{v}$.[10]

[10] The resultant force directed to the centre of rotation is sometimes called the **centripetal force.** It should be remembered, however, that this force does not emerge by its own. It is the resultant of the forces of interaction between a given body and other bodies, which actually exist.

Let us consider two examples.

1. A body lies on a horizontal disc rotating uniformly about a vertical axis (Fig. 63). The body is kept on the circular path (shown by the dashed line in the figure) by friction which is a result of action of the disc on the given body. Since the rotation is uniform and there is no acceleration along the tangent to the path, the friction is directed to the centre of rotation and imparts only a centripetal acceleration to the body under consideration.

2. A heavy ball suspended from a thread is uniformly rotated so that it describes a circle in a horizontal plane (Fig. 64). In this

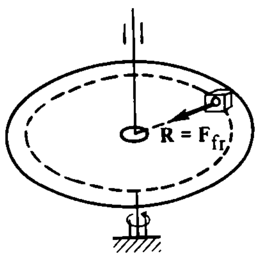


Fig. 63

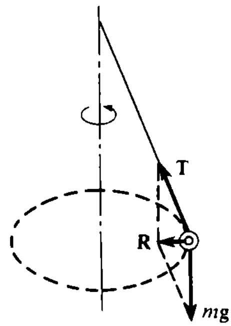


Fig. 64

case, the resultant is the sum of two forces: the force of gravity $m\mathbf{g}$ due to the action of the Earth and the tension **T** of the thread. The resultant **R** of these two forces is directed along the horizontal to the centre of the circle in which the ball moves. This force imparts a centripetal acceleration to the ball.

Some examples from engineering. In Examples 1 to 5, the dimensions of a rotating body are small in comparison with the radius of rotation. Therefore, the rotating body is treated as a material point, and its motion is regarded as the motion of a material point in a circle.

1. *A motorcar moving uniformly over a convex bridge* (Fig. 65). When the car is at the middle of the bridge, it is acted upon by the force of gravity $m\mathbf{g}$ and the normal reaction **N** of the bridge. The resultant of these forces, $\mathbf{R} = m\mathbf{g} + \mathbf{N}$, is directed

downwards and imparts a centripetal acceleration to the car. The magnitude of this force is $R = mg - N$.

2. *A cyclist on the bend of a road* (Fig. 66). When the cyclist is inclined to the centre of the rounding of its path, the resultant of the force of gravity $m\mathbf{g}$ and of the force $\mathbf{N}$ exerted on the bicycle at the points of contact between the wheels and the ground is directed horizontally to the centre of the circular path of the cycle. This resultant force imparts a centripetal acceleration to the cyclist.

3. *The motion of a motorcyclist over a vertical cylindrical wall* (Fig. 67). A centripetal acceleration is imparted to the motor-

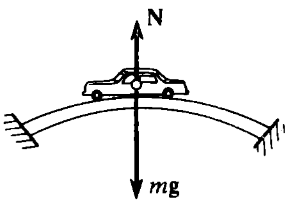


Fig. 65

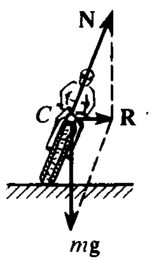


Fig. 66

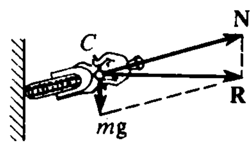


Fig. 67

cyclist by the resultant of the force of gravity $m\mathbf{g}$ and the normal reaction $\mathbf{N}$ of the wall.

4. *A train carriage on the rounding* (Fig. 68). The outer rail on the rounding of a railway is higher than the inner rail. The angle of inclination of the railroad is chosen depending on the radius of curvature of the road and the rated velocity of the train so that the resultant of the normal reaction $\mathbf{N}$ of the rails on the wheels of the carriage and of the force of gravity $m\mathbf{g}$ is directed along the horizontal. It imparts to the carriage a centripetal acceleration required for the motion of the carriage in the circle of a given radius with a given velocity.

5. *A steep turn of an aeroplane* (Fig. 69). The turn is the motion of an aeroplane along a curve of a certain radius in a horizontal plane. A centripetal acceleration is imparted to the aeroplane by the resultant of the lifting force $\mathbf{N}$ of its wing and of the force of gravity $m\mathbf{g}$ acting on the aeroplane.

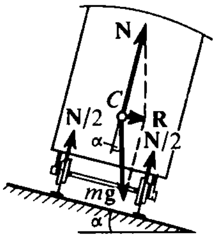


Fig. 68

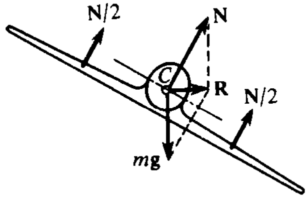


Fig. 69

1.26. Weight of a Body Considering the Rotation of the Earth

A body located on the surface of the Earth at a certain latitude φ (Fig. 70) experiences the action of the force of attraction to the Earth, or the force of gravity $m\mathbf{g}$, and the reaction $\mathbf{N}$ of the Earth's surface. Their resultant

$$\mathbf{F}_\varphi = m\mathbf{g} + \mathbf{N} \tag{1.26.1}$$

is directed to the centre O_1 of the parallel corresponding to this latitude. The magnitude of this force is

$$F_\varphi = m\omega^2_{\text{Earth}} r, \quad \text{where} \quad r = R_{\text{Earth}} \cos\varphi. \tag{1.26.2}$$

Here m is the mass of the body, ω_{Earth} is the angular velocity of the daily rotation of the Earth, r is the radius of the parallel at the latitude φ, and R_{Earth} is the average radius of the Earth. This force imparts a centripetal acceleration to the body and keeps the body on the surface of the Earth. The force of gravity is directed to the centre of the Earth at all points of its surface and is practically constant: $mg = mg_0$, where $g_0 = GM_{\text{Earth}}/R^2_{\text{Earth}} \approx 9.8\ \text{m/s}^2$.[11]

[11] Actually, the force of gravity is not the same at different latitudes due to the fact that the globe has the form of a geoid rather than a sphere, i.e. it is slightly flattened at the poles. Therefore, the distance from the centre to the surface of the Earth increases from the pole to the equator. However, this difference is very small and its effect on the free-fall acceleration is even weaker than the effect of rotation of the Earth.

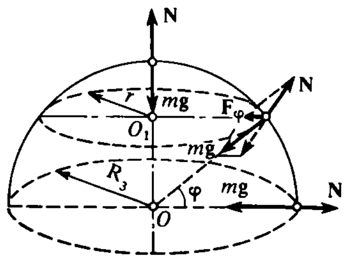


Fig. 70

Therefore, to satisfy condition (1.26.1), the magnitude and direction of the force **N** should depend on the latitude φ.

According to the generally accepted definition (see Sec. 1.19), the **weight** of a body is the force **P** with which the body acts on the surface of the Earth (on a horizontal support which is at rest relative to it) or on a vertical suspender. In view of Newton's third law, the weight **P** of the body must be equal and opposite to the normal reaction **N** of the Earth's surface. Consequently, if we take into account the rotation of the Earth, the weight of the body should depend on the latitude φ.

For an arbitrary φ, the weight **P** of a body differs from the force of gravity $m\mathbf{g}$ both in magnitude and direction (see Fig. 70). These two forces (applied to different bodies!) may coincide in direction and magnitude only at a pole where $\varphi = 90°$, $F_\varphi = 0$, and $\mathbf{P} = \mathbf{N} = m\mathbf{g} = m\mathbf{g}_0$.

The maximum difference in the magnitudes of the force of gravity and the weight of a body is observed on the equator (where they have the same direction). In this case, $\varphi = 0$, $F_\varphi = ma_c$ (a_c is the centripetal acceleration of the body on the equator). The forces $m\mathbf{g}$ and **N** are now antiparallel, so that condition (1.26.1) yields

$$P = N = mg - F_\varphi = mg_0 - ma_c = m(g_0 - a_c).$$

The centripetal acceleration of the body on the equator is

$$a_c = \omega^2_{\text{Earth}} R_{\text{Earth}} = \frac{4\pi^2 \times 6.37 \times 10^6}{(24 \times 3600)^2} \simeq 0.03\ \text{m/s}^2.$$

Here we take into account the fact that $\omega_{\text{Earth}} = 2\pi/T$, where the period T of the daily rotation of the Earth is 24×3600 s and $R_{\text{Earth}} = 6370$ km $= 6.37 \times 10^6$ m. The magnitude of a_c amounts to about 0.3% of g_0. For this reason, in most problems the change in the free-fall acceleration as a result of the rotation of the Earth can be neglected.

1.27. Reasons Behind the Emergence of Weightlessness in Artificial Satellites. Orbital Velocity

It was shown in Sec. 1.19 that the weight of a body is equal to the force of gravity acting on it not only for a body at rest relative to the Earth's surface but also for a body moving uniformly in a straight line relative to the Earth. The reference system fixed to the surface of the Earth, as well as to any reference body moving uniformly in a straight line relative to the Earth's surface, can be treated as an inertial system (with an accuracy sufficiently high for analysis of circumterrestrial motions). Experiments show that the weight of a body is equal to its force of gravity in any inertial system.

In a noninertial system, however, the weight of a body differs from the force of gravity acting on it. It was shown in Sec. 1.20, for example, that for a body suspended in a lift moving with a certain acceleration $\mathbf{a}$ directed to the centre of the Earth, the tension of the support is $T = mg - ma = m(g - a)$. This force is the weight of the body in the noninertial reference system fixed to the lift. If the acceleration of the lift is equal to the free-fall acceleration, $\mathbf{a} = \mathbf{g}$, it follows from what has been said above that the tension of the support is equal to zero, i.e. the weight of the body relative to the lift vanishes. Similarly, a body placed on a horizontal support in such a lift does not exert a pressure on the support.

When a spacecraft is orbiting the Earth, its acceleration is equal to the free-fall acceleration ($\mathbf{a}_s = \mathbf{g}$), i.e. the satellite is in the state of a free fall. Therefore, the force with which the cosmonaut acts on the support is zero, as well as the forces of pressure acting within the cosmonaut's body between its different parts. This causes a physiological sensation of weightlessness.

Let us find the velocity v_s of the satellite, assuming, for simplicity, that its orbit is a circle of radius equal to the radius of the Earth. To a sufficiently high degree of accuracy (neglecting the effect of rotation of the Earth), we can equate the acceleration of the satellite to the free-fall acceleration on the surface of the Earth: $a_s = g_0 = GM_{\text{Earth}}/R^2_{\text{Earth}}$. On the other hand, the centripetal acceleration of the satellite is $a_s = v_s^2/R_{\text{Earth}}$. Therefore,

$$v_s^2/R_{\text{Earth}} = g_0, \quad \text{whence} \quad v_s = \sqrt{R_{\text{Earth}} g_0} \simeq 7.9 \text{ km/s}.$$

This velocity is called the **orbital velocity** of a body. A body of any mass, having acquired such a velocity, becomes an Earth's satellite.

Problems with Solutions

61. A flywheel of a diameter $D = 1.5$ m rotates with a speed $n = 600$ rpm. The mass of the flywheel is $m = 0.5$ t. Find the angular velocity ω of the flywheel, the linear velocity of points on its rim, and the kinetic energy of the flywheel, assuming that its entire mass is concentrated on the rim. Express the kinetic energy in terms of the angular velocity of the flywheel.

Solution. The number of revolutions per second is $n = 600/60$ s $= 10$ s^{-1}. The angular velocity of the flywheel is $\omega = 2\pi n = 62.8$ rad/s. The linear velocity of the points on the rim is $v = \omega R = 47.1$ m/s. The kinetic energy of the flywheel is $W_k = mv^2/2 = 556$ kJ. Since $v = \omega R$, we have $W_k = mv^2/2 = m(\omega R)^2/2 = mR^2\omega^2/2$.

62. A hoop rolls over a horizontal plane with a constant velocity **v** without slippage. Find the velocity of point A of the hoop (Fig. 71).

Solution. We shall solve this problem by two methods.

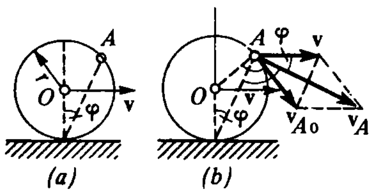


Fig. 71

Method I. Let us first find the velocity of point A in the coordinate system fixed to the centre of the hoop. Since the centre of the hoop moves with a velocity **v** relative to the Earth (Fig. 71*a*), in the coordinate system fixed to the centre of the hoop, the point of the hoop in contact with the ground moves with a velocity $-$**v**, i.e. with a velocity of the same magnitude but of opposite direction relative to the centre of the hoop. But in this system, all the points of the hoop have the same velocity (in magnitude). Consequently, the velocity of point A is $v_{A0} = v$ and is directed along the tangent to the circle.

Let us now go over to the coordinate system fixed to the plane (Fig. 71*b*). For this purpose, we must add the velocity $\mathbf{v}$ of the centre of the hoop relative to the Earth to the velocity $\mathbf{v}_{A0}$ of point A in the old coordinate system (we recall that $v = v_{A0}$). Thus, the velocity $\mathbf{v}_A$ of point A in the stationary coordinate system is the diagonal of the rhombus with sides equal to v. Consequently, $v_A = 2v \cos \varphi$.

Method II. At each instant of time the motion of the hoop can be considered as a rotation about point O', viz. the instantaneous centre of rotation[12] (Fig. 72). Let us find the angular velocity of this rotation. The velocity of point O is v. But $v = \omega r$, whence $\omega = v/r$. We can now easily find the velocity of point A. It is normal to the segment $O'A$, i.e. $v_A = \omega \cdot O'A = (v/r) \cdot O'A$. Considering $\triangle O'AB$, we see that $O'A = 2r \cos \varphi$. Consequently, $v_A = (v/r)2r \cos \varphi = 2v \cos \varphi$.

63. A hoop of mass $m = 5$ kg rolls with a velocity $v = 4$ m/s without slippage. Find its kinetic energy.

Solution (see solution to Problem 62). The velocity of a small segment of the hoop of mass Δm near point A (Fig. 73) is $2v \cos \alpha$, while the velocity of the same segment near point B which is diametrically opposite to point A is $2v \cos (90° - \alpha) = 2v \sin \alpha$. The total kinetic energy of these segments,

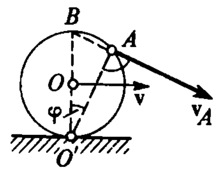


Fig. 72

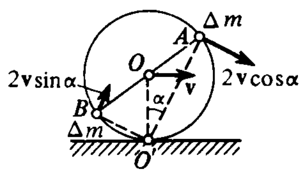


Fig. 73

$$\frac{\Delta m(2v \cos \alpha)^2}{2} + \frac{\Delta m(2v \sin \alpha)^2}{2} = \frac{4\Delta m v^2(\cos^2 \alpha + \sin^2 \alpha)}{2} = 2\Delta m v^2,$$

is independent of α. Consequently, the kinetic energy of any pair of segments of the hoop lying on the two ends of a diameter is equal to their total mass $2\Delta m$ multiplied by v^2. Let us divide the hoop into such pairs. Then the kinetic energy of the hoop is equal to the sum of the kinetic energies of these segments:

$$W_k = \sum (2\Delta m)v^2 = v^2 \sum (2\Delta m) = mv^2 = 80 \text{ J}.$$

---

[12] The instantaneous centre of rotation is a point whose velocity at a given moment is equal to zero.

64. A stone attached to a rope of length $l = 80$ cm is rotated in a vertical plane with a speed $n = 240$ rpm. At the moment when the velocity of the stone is directed vertically upwards, the rope ruptures. To what height does the stone rise? The air resistance should be neglected.

Solution. The number of revolutions per second is $n = 240/60$ s $= 4\ s^{-1}$. At the moment of rupture the velocity of the stone is $v = 2\pi nl$. At this moment the kinetic energy of the stone is $W_k = mv^2/2 = 2m\pi^2n^2l^2$. We equate this value to the potential energy of the stone at the maximum height:

$$mgh = 2m\pi^2n^2l^2, \quad \text{whence} \quad h = 2\pi^2n^2l^2/g = 20.6 \text{ m}.$$

65. A shaft of diameter $d = 40$ cm and a pulley of diameter 2.0 m fit tight on it rotate uniformly. What are the ratios of the linear velocities and centripetal accelerations on the rim of the pulley and on the outer surface of the shaft?

Solution. The linear velocities are $v_1 = \omega r$ and $v_2 = \omega R$. Consequently, $v_2/v_1 = \omega R/\omega r = R/r = D/d = 5$. The centripetal accelerations are $a_1 = \omega^2 r$ and $a_2 = \omega^2 R$. Their ratios are $a_2/a_1 = \omega^2 R/\omega^2 r = R/r = D/d = 5$.

66. A stone of mass $m_1 = 2$ kg moves uniformly in a circle of radius $r_1 = 0.5$ m with a velocity $v_1 = 10$ m/s. Another stone whose mass $m_2 = 550$ g moves in a circle of radius $r_2 = 30$ cm with a velocity $v_2 = 4$ m/s. Find the ratio of the resultant forces applied to the stones.

Solution. The resultant forces are $R_c^{I} = m_1v_1^2/r_1$ and $R_c^{II} = m_2v_2^2/r_2$. Their ratio is

$$K = \frac{R_c^{I}}{R_c^{II}} = \frac{m_1v_1^2r_2}{m_2v_2^2r_1} = \frac{m_1}{m_2}\frac{r_2}{r_1}\left(\frac{v_1}{v_2}\right)^2 = 15.$$

67. When the angular velocity of a uniformly rotating body has increased thrice, the resultant of forces applied to it increases by $\Delta R = 60$ N. Find the accelerations of the body in the two cases if the mass of the body is $m = 3$ kg.

Solution. $R_1 = m\omega_1^2 r$ and $R_2 = m\omega_2^2 r$, where r is the radius of the circular path of the body. By the conditions of the problem,

$$m\omega_2^2 r - m\omega_1^2 r = \Delta R, \quad \text{or} \quad mr(\omega_2^2 - \omega_1^2) = \Delta R.$$

Substituting $\omega_2 = 3\omega_1$ into this formula, we obtain $mr(9\omega_1^2 - \omega_1^2) = \Delta R$, or $8mr\omega_1^2 = \Delta R$. This gives

$$a_1 = \omega_1^2 r = \Delta R/8m = 2.5 \text{ m/s}^2, \quad a_2 = \omega_2^2 r = 9\omega_1^2 r = 22.5 \text{ m/s}^2.$$

68. A load lies on horizontal rotating platform at a distance $R = 75$ cm from the rotational axis. What must the coefficient of friction be so that the load does not slide with the rotational speed $n = 12$ rpm?

Solution. The number of revolutions per second is $n = 12/60$ s $= 0.2\ s^{-1}$. For the load to remain on the platform without slip with a given speed of rotation, the resultant of all the forces acting on it must be equal to $F_c = 4\pi^2n^2Rm$ and directed to the rotational axis. Since the only horizontal force applied to the body is the static friction, it is this force that imparts a centripetal acceleration to it, i.e.

$fP = F_c$, or $fP = 4\pi^2 n^2 Rm$, whence

$$f = 4\pi^2 n^2 R/g = 0.12.$$

The coefficient of friction must be greater than 0.12.

69. A cyclist moves with a velocity $v = 36$ km/h. What must be the maximum angle to the vertical when he makes a turn with this velocity if the coefficient of sliding friction of the bicycle wheel in the transverse direction is $f = 0.3$? What is the radius of curvature of the rounding?

Solution. When the cyclist inclines at the turn (Fig. 74), he experiences the action of the force of gravity mg applied at the centre of mass of the cyclist (with the

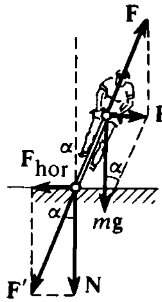


Fig. 74

bicycle). Besides, at the points of contact of the two wheels with the road, two forces exerted by the road act in the plane of the bicycle. Their resultant **F** also lies in the plane of the bicycle and passes through the centre of mass of the cyclist (with the bicycle). The resultant **R** of all these forces has the horizontal direction and imparts a centripetal acceleration $a = v^2/r$ to the cyclist. It can be seen from the figure that $R = mg \tan\alpha$. The force **F**′ exerted by the bicycle on the road is, by Newton's third law, $\mathbf{F}' = -\mathbf{F}$, i.e. $F' = F$.

The vertical component of the force **F**′ is equal in magnitude to the force of gravity $N = mg$, while the horizontal component of **F**′ is $F_{hor} = N\tan\alpha = mg\tan\alpha = R$.

On the other hand, the force F_{hor} cannot exceed the maximum possible friction: $F_{hor} \leqslant fmg = fN$. Consequently, $mg\tan\alpha \leqslant fmg$, whence $\tan\alpha \leqslant f$ and $\tan\alpha \leqslant 0.3$. For small angles, $\tan\alpha \simeq \alpha$. Thus, $\alpha \leqslant 0.3$, and the maximum angle of inclination $\alpha_{max} = 0.3$ rad $\simeq 17°$. In this case, the resultant of all the forces applied to the cyclist is $R = mg\tan\alpha = fmg$.

Let us write the equation of motion of the cyclist:

$$R = ma_c = mv^2/r, \quad \text{or} \quad fmg = mv^2/r.$$

From this expression, we can obtain the radius of the rounding: $r = mv^2/fmg = v^2/fg \simeq 34$ m. This is the smallest permissible radius of the rounding of the cyclist's trajectory under the given conditions.

70. A motorcyclist moves over a cylindrical wall whose radius $r = 5$ m. The coefficient of friction between the wall and the wheels for transverse motion is $f = 0.3$. Find the minimum admissible velocity of the motorcyclist with which he would not slip down the wall.

Solution. The vertical component of the force **F** exerted on the motorcyclist by the wall (Fig. 67) balances the force of gravity $m\mathbf{g}$. The horizontal component **R** of the force **F** imparts a centripetal acceleration to the motorcyclist. This means that the horizontal component **N** of the force $-\mathbf{F}$ with which the motorcyclist acts on the wall (the force of normal pressure) is equal in magnitude to $N = mv^2/r$, where v is the velocity of the motorcyclist and m is the total mass of the motorcyclist and the motorcycle.

The vertical component $\mathbf{F}_{ver}$ of the force $-\mathbf{F}$ due to friction is equal in magnitude to the force of gravity: $F_{ver} = mg$. Obviously, the magnitude of the force $\mathbf{F}_{ver}$ cannot exceed the maximum static friction $F_{fr} = fN = fmv^2/r$. Consequently, in order that the motorcyclist do not slip off the wall, the following condition must be satisfied: $F_{ver} \leqslant F_{fr}$, or $mg \leqslant fmv^2/r$. Hence $v^2 \geqslant gr/f$. The minimum velocity of motion of the motorcyclist is $v_{min} = \sqrt{gr/f} = 46$ km/h.

71. A 3-t motorcar moves uniformly with a velocity $v = 36$ km/h over a convex bridge whose radius of curvature $r = 50$ m. At what point of the bridge is the pressure of the car maximum? Find the force of pressure. (The position of points on the bridge surface should be determined by the angle between the vertical and the straight line connecting the centre of curvature and a given point.) What is the relation between the velocity of the car and its force of pressure on the bridge?

Solution. The car moving over the bridge experiences the action of the following forces (Fig. 75): the force of gravity $m\mathbf{g}$, the tractive force $\mathbf{F}_{tr}$, the resistance

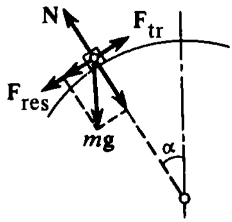


Fig. 75

$\mathbf{F}_{res}$, and the normal reaction **N** of the bridge. Since we have a uniform curvilinear motion, the resultant of these forces imparts a centripetal acceleration to the car and is thus directed to the centre of the bridge curvature. We resolve the force of gravity $m\mathbf{g}$ into two components, i.e. along the radius of curvature and normal to it. It is obvious that the forces normal to the radius of curvature balance each other, while the resultant of the forces creating a centripetal acceleration is $F_c = mg\cos\alpha - N$. The force of pressure of the car on the bridge is $\mathbf{F}_{pr} = -\mathbf{N}$. Its magnitude is given by

$$F_{pr} = N = mg\cos\alpha - ma_c = mg\cos\alpha - mv^2/r = m(g\cos\alpha - v^2/r). \quad (1)$$

This force attains its maximum at $\cos\alpha = 1$, i.e. for $\alpha = 0$:

$$F_{\mathrm{pr\,max}} = m(g - v^2/r) \doteq 23.4\ \mathrm{kN}. \quad (2)$$

It follows from Eqs. (1) and (2) that as the velocity of the car increases, its force of pressure on the bridge becomes smaller. If its velocity attains the value at which $v^2/r = g\cos\alpha_0$, where α_0 is the angle corresponding to the point of the trajectory at the bridge entrance, i.e. if the velocity of the car entering the bridge is $v = \sqrt{gr\cos\alpha_0}$, the force of pressure of the car on the bridge at this point is zero. Over the remaining path on the bridge, $F_{\mathrm{pr}} > 0$ since $g\cos\alpha - g\cos\alpha_0 > 0$.

If the velocity of the car at the bridge entrance increases to $v = \sqrt{gr\cos\alpha_1}$, where $\alpha_1 < \alpha_0$ (i.e. $\cos\alpha_1 > \cos\alpha_0$), then for $\alpha_1 < \alpha < \alpha_0$ Eq. (1) is not valid since, according to this equation, we would have $F_{\mathrm{pr}} = m(g\cos\alpha - g\cos\alpha_1) < 0$, which is impossible. Here we should have $F_{\mathrm{pr}} = 0$, i.e. $g\cos\alpha - v^2/r_1 = 0$, or $g\cos\alpha - r(g\cos\alpha_1)/r_1 = 0$, whence $r_1/r = \cos\alpha_1/\cos\alpha$, i.e. $r_1 > r$. Consequently, on a segment corresponding to $\alpha_1 < \alpha < \alpha_0$, the car will be detached from the road and move along the trajectory of a varying (increasing) radius (the spring-board effect).

72. An open vessel filled with water to a height $h = 10$ cm is rotated with a constant angular velocity in a vertical plane by a rope such that the distance l from the centre of rotation to the bottom of the vessel is 80 cm. What must be the minimum angular velocity of rotation of the vessel for water not to flow out of it? What is the ratio of the tension of the rope to the force of gravity acting on the vessel with water at the moment it passes through the lowest point?

Solution. Water does not flow out of the vessel if the centripetal acceleration $a = v^2/r = \omega^2 r$ of any portion of water of mass Δm at the uppermost point (here ω is the angular velocity of rotation of the vessel and r is the radius of the circle in which the portion of water moves) is greater than or equal to the free-fall acceleration g:

$$\omega^2 r \geqslant g. \quad (1)$$

The rope is stretched in this case, and the resultant of the force of gravity and the tension of the rope imparts the centripetal acceleration to the entire vessel, while the resultant of the force of gravity and the forces exerted on this portion of water by other portions imparts the acceleration to this portion. Inequality (1) yields $\omega \geqslant \sqrt{g/r}$, whence $\omega_{\min} = \sqrt{g/r}$. It follows from this expression that the angular velocity must be the higher, the smaller r, viz. the distance from the small portion to the rotational axis. This distance is the smallest for a water portion on the surface, where $r = l - h$. Then $\omega_{\min} = \sqrt{g/(l-h)} = 3.7\ \mathrm{s}^{-1}$.

At the lowest point of the trajectory, the centripetal acceleration $a' = \omega^2 r'$ is imparted to the vessel by the resultant $\mathbf{R}$ of the force of gravity $M\mathbf{g}$ and the tension $\mathbf{T}$ of the rope (here M is the mass of the vessel with water and r' is the distance from the rotational axis of the vessel to its centre, $r' = l - h/2$). The force of gravity $M\mathbf{g}$ is directed downwards, while the tension $\mathbf{T}$ of the rope acts upwards. In accordance with Newton's second law, $T - Mg = M\omega^2 r'$. Consequently,

$$T = Mg + M\omega^2 r' = Mg(1 + \omega^2 r'/g).$$

This means that $k = T/Mg = 1 + \omega^2 r'/g$. Since $\omega^2 = g/(l - h)$ and $r = l - h/2$, we have

$$k = 1 + \frac{g(l - h/2)}{(l - h)g} = 1 + \frac{l - h/2}{l - h} = 2.07.$$

73. A body slides down an inclined chute ending by a vertical loop of radius r = 40 cm (Fig. 76). What must be the height h of the chute for the body not to fall at the uppermost point of the loop? Friction should be neglected.

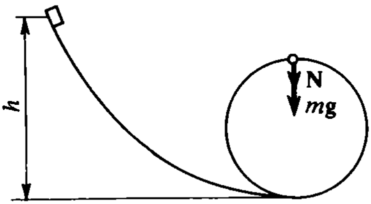


Fig. 76

Solution. At the uppermost point of the loop, the body is acted upon by the force of gravity mg and the normal reaction **N** of the loop. The resultant of these forces, $\mathbf{R} = \mathbf{N} + m\mathbf{g}$, imparts to the body a centripetal acceleration $a_c = v^2/r$, where v is the velocity of the body. By Newton's second law, $N + mg = mv^2/r$. The body is not detached from the loop if $N \geqslant 0$. In the limiting case, $N = 0$, i.e. $mg = mv^2/r$, or $v = \sqrt{gr}$.

Since only different kinds of mechanical energy are converted into each other in the motion under consideration, using the law of energy conservation, we can write $mg(h - 2r) = mv^2/2$. Substituting $v = \sqrt{gr}$ into this formula, we obtain

$$mg(h - 2r) = mgr/2, \quad \text{whence} \quad h = r/2 + 2r = 2.5r = 1 \text{ m}.$$

74. Solve the preceding problem under the assumption that the body is a cylindrical roller consisting of a massive rim and thin spokes whose mass can be neglected.

Solution. It was shown in solution to Problem 63 that the kinetic energy of such a roller is $W_k = mv^2$. Then $mg(h - 2r) = mgr$, which gives $h = 3r$ = 1.20 m.

75. Find the height of the chute mentioned in Problem 73 if the body is known to separate from the loop at an angle $\alpha = 60°$ (Fig. 77).

Solution. At point A (see Fig. 77), the body experiences the action of the force of gravity mg and the reaction **N** of the loop. The centripetal acceleration is imparted to the body by the resultant of **N** and the radial component of the force of gravity $m\mathbf{g}$. Therefore, by Newton's second law, $mg \cos\alpha + N = mv^2/r$ (m is the mass of the body). The body is separated from the loop when $N = 0$, i.e. when $mv^2/r = mg \cos\alpha$, whence $v^2 = gr \cos\alpha$. It follows from the law of energy con-

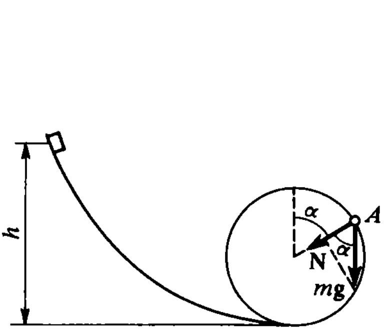


Fig. 77

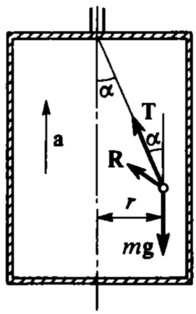


Fig. 78

servation that

$$mg[h - r(1 + \cos\alpha)] = mv^2/2 = m(gr\cos\alpha)/2,$$

whence $h = r + 3(r\cos\alpha)/2 = 1.75r = 70$ cm.

76. A vertical axis is fixed at the ceiling of a lift. A ball of mass $m = 800$ g, attached to a thread of length $l = 40$ cm, is suspended from this axis which is rotated at a frequency of 90 rpm. Find the tension **T** of the thread and the angle α formed by the thread with the vertical for the upward motion of the lift with an acceleration $a = 3.0$ m/s^2. The mass of the thread should be neglected.

Solution. The ball experiences the action of the force of gravity $m\mathbf{g}$ and the tension **T** of the thread (Fig. 78). The resultant **R** of these forces, equal to their vector sum, imparts to the ball both a centripetal (horizontal) acceleration $a_c = \omega^2 r$ and a vertical acceleration a. Projecting the force of gravity $m\mathbf{g}$ and the tension **T** onto the vertical and horizontal axes, we can write the equations of motion for the ball:

$$T\cos\alpha - mg = ma, \quad T\sin\alpha = m\omega^2 r.$$

The first of these equations gives the sought tension of the thread: $T = m(g + a)/\cos\alpha$. Substituting this quantity into the second equation, we obtain

$$[m(g + a)/\cos\alpha]\sin\alpha = m\omega^2 r, \quad \text{or} \quad (g + a)\tan\alpha = \omega^2 r.$$

But $r = l\sin\alpha$ and $\omega = 2\pi n$, where $n = 1.5$ rps. Hence $(g + a)\tan\alpha = 4\pi^2 n^2 l\sin\alpha$. Thus,

$$\cos\alpha = (g + a)/4\pi^2 n^2 l \simeq 0.36, \quad \text{i.e.} \quad \alpha = \arccos 0.36, \quad T \simeq 28.4 \text{ N}.$$

77. Two balls with masses $m_1 = 500$ g and $m_2 = 300$ g are connected by a nonstretchable thread and pinned on a horizontal rod so that they can move along it (Fig. 79). The rod is rotated about a vertical axis passing through its middle. At what distance from the rotational axis should the centre of the larger ball be to keep the balls at constant distances from the axis if the centres of the ball are separated by a distance $l = 20$ cm? Is the uniform rotation of the balls connected

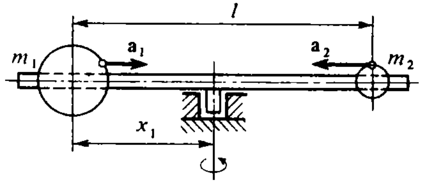


Fig. 79

by the thread stable? The masses of the thread and the rod, as well as the friction between the balls and the rod, should be neglected.

Solution. During the rotation of the balls connected by the thread, the latter acts on them with the forces $F_1 = m_1\omega^2 x_1$ and $F_2 = m_2\omega^2(l - x_1)$. These forces impart centripetal accelerations $\mathbf{a}_1$ and $\mathbf{a}_2$ to the balls (see Fig. 79). By Newton's third law, the balls act on the thread with the forces $F_1 = m_1\omega^2 x_1$ and $F_2 = m_2\omega^2(l - x_1)$, which are directed opposite to the forces acting on the balls. The thread (with the balls) is at rest when $F_1 = F_2$, i.e. when $m_1\omega^2 x_1 = m_2\omega^2(l - x_1)$, where ω is the angular velocity of rotation of the rod with the balls. This gives

$$m_1 x_1 = m_2 l - m_2 x_1, \quad (m_1 + m_2)x_1 = m_2 l,$$

whence $x_1 = [m_2/(m_1 + m_2)]l$. Since the expression in the brackets is dimensionless, we can substitute into it the quantities expressed in arbitrary but identical units. Thus $x_1 = 7.5$ cm.

If for some reason or other the thread with the balls is shifted, say, in the direction of the larger ball, the force $F_1 = m_1\omega^2 x_1$ acting on the thread to the left will increase (since x_1 has increased), while the force $F_2 = m_2\omega^2(l - x_1)$ acting on the thread to the right will accordingly decrease. Consequently, the thread with the balls will go on moving in the same direction. The same would be observed if the thread with the balls accidentally shifted towards the smaller ball. This means that the uniform rotation of the balls connected by the thread is unstable.

78*. Solve the preceding problem under the assumption that the smaller ball is connected with the rotational axis by a stretched spring (Fig. 80) whose length in

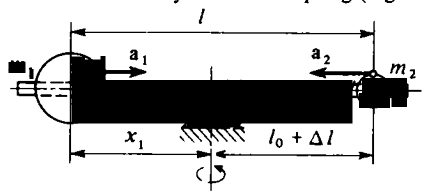


Fig. 80

the free state is $l_0 = 10$ cm and whose rigidity $k = 5$ N/m, and the rod with the balls rotates at a frequency $n = 45$ rpm. Find also the spring elongation.

Solution. Unlike the preceding problem, the centripetal acceleration here is imparted to the smaller ball by two forces, viz. the tension of the thread and the tensile force of the spring, which is equal to $k\,\Delta l$. Consequently, the tension of the thread is $T = m_2a_2 - k\,\Delta l$, where a_2 is the centripetal acceleration of the smaller ball. By Newton's thrid law, the smaller ball acts on the thread with the force $F_2 = m_2a_2 - k\,\Delta l$. (The force $\mathbf{F}_2$ is applied to the spring attached to the rotational axis.)

To attain an equilibrium motion of the thread with the ball, the following condition must be satisfied: $F_1 = F_2$, i.e.

$$m_1a_1 = m_2a_2 - k\,\Delta l, \quad \text{or} \quad m_1\omega^2x_1 = m_2\omega^2(l - x_1) - k\,\Delta l,$$

where $\Delta l = l - x_1 - l_0$ since $l_0 + \Delta l = l - x_1$, and the angular velocity of the rod is $\omega = 2\pi n/60 = 1.5\pi$ rad/s. Then

$$m_1\omega^2x_1 = m_2\omega^2l - m_2\omega^2x_1 - kl + kx_1 + kl_0,$$

whence

$$x_1 = \frac{m_2\omega^2l - k(l - l_0)}{(m_1 + m_2)\omega^2 - k} = 6.5\text{ cm}.$$

The spring elongation for the uniform rotation is $\Delta l = l - x_1 - l_0 = 3.5$ cm.

If the thread with the balls is accidentally shifted by a small distance Δx towards the larger ball, the force $\mathbf{F}_1$ acting on the thread to the left will increase by $m_1\omega^2\Delta x$, while the force $\mathbf{F}_2$ acting on the thread to the right will increase by $k\,\Delta x - m_2\omega^2\Delta x$. For a stable uniform rotation, the following condition should be satisfied:

$$m_1\omega^2\Delta x < k\,\Delta x - m_2\omega^2\Delta x, \quad \text{whence} \quad (m_1 + m_2)\omega^2 < k.$$

Under the conditions of this problem, $(0.5\text{ kg} + 0.3\text{ kg})\omega^2 < 5$ N/m, whence $\omega < 2.5$ rad/s. Since $\omega = 2\pi n/60$, we have $n = 30\omega/\pi$, and for stable equilibrium the inequality $n < 24$ rpm should be satisfied. In the given problem, $n = 45$ rpm > 24 rpm, which means that the motion becomes unstable at any however small displacement towards the larger ball.

If the thread with the balls is shifted by Δx towards the smaller ball, then after the perturbation stops, the thread with the balls returns to the previous position due to the additional action of the stretched spring. However, it will continue to move towards the larger ball by inertia.

79. The balls of a centrifugal ball governor are connected by a horizontal spring with a ring at the middle. The governor axis passes through this ring without touching it (Fig. 81). The mass of each ball is $m = 5$ kg, the length of the rods to which the balls are fixed is $l = 60$ cm, the length of the spring in the free state is $l_1 = 40$ cm, and its rigidity $k = 200$ N/m. What is the frequency n of governor rotation if the angle of its deviation from the vertical is $\alpha = 30°$? The mass of the rods should be neglected.

Solution. A ball (the right ball in Fig. 81) is acted upon by the force of gravity $m\mathbf{g}$, the rod reaction $\mathbf{N}$, and the tension $\mathbf{T}$ of the spring whose magnitude is $T = k(l_2 - l_1) = k(2l\sin\alpha - l_1)$ (l_2 is the length of the stretched spring). The resultant

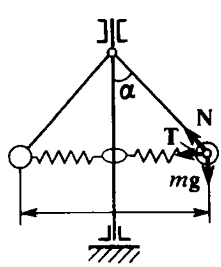


Fig. 81

of these forces imparts a centripetal acceleration to the ball. Consequently, the sum of the projections of all the forces (acting on the ball) onto the vertical axis should be zero, while the sum of the projections onto the horizontal axis is equal to the force required for creating the centripetal acceleration:

$$N\cos\alpha - mg = 0, \quad T + N\sin\alpha = m\omega^2 l\sin\alpha.$$

Substituting into this expression the value $T = k(2l\sin\alpha - l_1)$, we get

$$N\cos\alpha - mg = 0, \quad k(2l\sin\alpha - l_1) + N\sin\alpha - m\omega^2 l\sin\alpha = 0.$$

The first of these equations gives $N = mg/\cos\alpha$. Substituting this value into the second equation, we obtain

$$2kl\sin\alpha - kl_1 + mg\tan\alpha - m\omega^2 l\sin\alpha = 0,$$

whence

$$\omega^2 = \frac{2kl\sin\alpha - kl_1 + mg\tan\alpha}{ml\sin\alpha},$$

$$\omega = \sqrt{\frac{2k}{m} - \frac{kl_1}{ml\sin\alpha} + \frac{g}{l\cos\alpha}} = 6.8 \text{ rad/s}.$$

Thus, $n = \omega/2\pi = 1.08$ rps.

80. Find the velocity v of an artificial satellite moving in a circular orbit at a height $h = 1600$ km above the Earth's surface. The radius of the Earth $R_{\text{Earth}} = 6400$ km and the free-fall acceleration at the Earth's surface is $g = 9.8$ m/s^2.

Solution. At a height H above the surface of the Earth, the satellite experiences the action of the gravitational force

$$F_{\text{gr}} = GMm/(R_{\text{Earth}} + H)^2,$$

where M is the Earth's mass and m is the mass of the satellite. Since the satellite moves in a circular orbit of radius $R_{\text{Earth}} + H$, the force $\mathbf{F}$ imparts a centripetal acceleration $a_c = v^2/(R_{\text{Earth}} + H)$ to the satellite. According to Newton's second law,

$$GMm/(R_{\text{Earth}} + H)^2 = mv^2/(R_{\text{Earth}} + H), \quad \text{whence} \quad v = \sqrt{GM/(R_{\text{Earth}} + H)}.$$

At the surface of the Earth, the gravitational force $F_{gr} = GMm/R^2_{\text{Earth}}$ imparts the acceleration g to a body of mass m. Therefore, $GMm/R^2_{\text{Earth}} = mg$, i.e. $GM = gR^2_{\text{Earth}}$. Taking this into account, we obtain

$$v = \sqrt{gR^2_{\text{Earth}}/(R_{\text{Earth}} + H)} = R_{\text{Earth}}\sqrt{g/(R_{\text{Earth}} + H)} \simeq 7.1 \text{ km/s}.$$

Exercises

61. (a) A shaft whose diameter is 500 mm is machined on a lathe with a constant turning speed of 90.0 m/min. Find the angular velocity of the shaft, the period and frequency of rotation, and the acceleration of a point on the shaft surface.
Answer. 6.0 rad/s, $(\pi/3)$ s, $180/\pi$ rpm, 9.0 m/s^2.

(b) Find the linear velocity and the acceleration of the Earth relative to the Sun, assuming that the Earth is a material point, the radius of the Earth's orbit is 1.5×10^8 km, and the period of rotation is 365 days.
Answer. ~ 30 km/s, 0.006 m/s^2.

(c) Find the linear velocity and the centripetal acceleration of point on the surface of the globe at a latitude of 60°.
Answer. 230 m/s, 0.017 m/s^2.

(d) A flywheel whose mass is 0.8 kg and diameter is 180 cm rotates at a frequency of 300 rpm. Find its kinetic energy and express it in terms of angular velocity, assuming that the mass is concentrated on the flywheel rim.
Answer. 320 J, $W_k = mR^2\omega^2/2$.

(e) A 180-cm rod rotates about a vertical axis passing through its middle. Four loads are fixed on the rod such that two loads on the rod ends have a mass of 2.0 kg each, and two other 4.5-kg loads are at the mid-distances from the ends and the axis. Find the kinetic energy of the system if the frequency of rotation of the rod is 240 rpm, expressing it in terms of angular velocity. The mass of the rod should be neglected. At what distance from the rod axis should the middle loads be fixed for their kinetic energy to be equal to that of the loads at the ends?
Answer. $(m_1R_1^2 + m_2R_2^2)\omega^2 \simeq$ 1600 J, 60 cm.

62. A hoop rolls over a horizontal plane without slipping. The velocity of point A of the hoop (Fig. 82) is v_A. Find the velocity of the centre of the hoop.
Answer. $v_A/2\cos\varphi$.

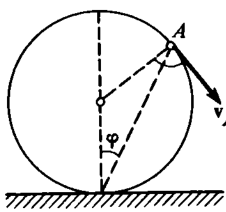


Fig. 82

63*. A hollow cylinder whose diameter is 1.0 m and mass is 100 kg rolls over the rails so that its rotational frequency is 120 rpm. Find its kinetic energy, assuming that the thickness of its walls is negligibly small as compared to its radius.
Answer. ~ 3940 J.
64. (a) A stone fixed to a 1.0 m-long rope is rotated in a vertical plane at a frequency of 180 rpm. At the moment the stone velocity forms an angle of 30° with the horizontal, the rope breaks. To what height relative to the point of breaking does the stone rise? The air resistance should be neglected.
Answer. 13.5 m.

(b*) A rod suspended from a horizontal axis with a 2-kg load attached to its lower end is deflected by an angle of 60° (from the vertical) and then released. Find the force acting on the rod at the moment when it is in the vertical position and when it is deflected by an angle of 30° from the vertical.
Answer. 39.6 N, 31.4 N.

(c*) A 2-kg load is suspended from a 20-cm spring attached to the upper point A of a rigid ring of radius 20 cm and at the initial instant is fixed to the ring so that the spring is in the undeformed state (Fig. 83). After the load has been released, it

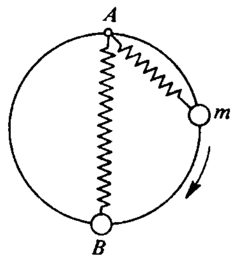


Fig. 83

falls sliding over the ring. What is its velocity at the lower point B of the ring if the rigidity of the spring is 50 N/m? What is the force of pressure of the load on the ring at this point? The mass of the spring and the friction between the load and the ring should be neglected.
Answer. 2.2 m/s, 68 N.
65. What are the ratios of the linear velocities and centripetal accelerations for the points of the minute and hour hands of a clock if the minute hand is 1.5 times longer than the hour hand?
Answer. 18, 216.
66. The resultant of the forces acting on a stone moving uniformly in a horizontal plane in a circle of diameter 1.50 m is eight times stronger than the resultant of the forces acting on a stone which moves in a circle whose diameter is 15 cm. Find the ratio of the linear velocities of the first and second stones if the mass of the first stone is five times larger than the mass of the second stone.
Answer. 4.
67. A rod with two equal loads pinned on it rotates in a horizontal plane about an

axis passing through one its end. One load is fixed at the free end of the rod and the other at 1/3 of its length from the axis of rotation. The horizontal force acting on the first load is stronger than the force acting on the second load by 100 N. Find the total tensile force acting on the rod. The mass of the rod should be neglected.
Answer. 200 N.
68. (a) A motorcar moves along a curve of radius 50 m with a velocity of 60 km/h. What must be the coefficient of friction between the wheels of the car and the road, for which the car does not slip?
Answer. $\geqslant$ 0.57.

(b) A load lies on a horizontal rotating platform at a distance of 115 cm from the vertical rotational axis. The coefficient of friction between the platform and the load is 0.20. At what frequency of rotation does the load start to slip?
Answer. 12 rpm.
69. (a) What is the maximum velocity of a cyclist on a road rounding of radius 25.0 m if the coefficient of friction between the wheels of the bicycle and the road is 0.15? By what angle to the vertical should the cyclist incline in this case?
Answer. 6.0 m/s, arctan 0.15.

(b) By what angle α are the loads of a centrifugal ball governor deflected (Fig. 84) if the length of the rods to which they are fixed is 20 cm, and the governor rotates at a frequency of 90 rpm? What tensile force acts on the governor rods if the mass of each load is 3 kg?

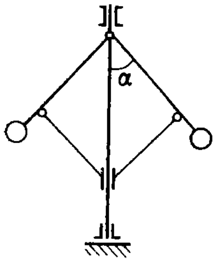


Fig. 84

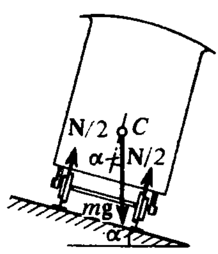


Fig. 85

Answer. ~57°, ~49.N.

(c*) A load is suspended from a 10-cm rope at the edge of a round table rotating at a frequency of 60 rpm. During the rotation, the rope with the load is deflected by an angle of 30° from the vertical. Find the diameter of the table.
Answer. 18 cm.

(d*) To what height should the outer rail be lifted relative to the inner rail (Fig. 85) on a curved part of the railroad of radius 400 m in order that the forces of pressure exerted by a train moving with a velocity of 54 km/h on the two rails be the same, and the rails do not experience a shear across railroad? The track width is 1524 mm.
Answer. 87 mm.

(e*) A ball is placed on a chute inclined at an angle of 30° to the horizontal and rotating at a frequency of 30 rpm about a vertical axis passing through its lower end. At what distance from this end of the chute is the ball kept? Friction should be neglected.

Answer. 0.66 m.

(f*) A vessel in the form of an inverted truncated cone with a diameter of the bottom of 40 cm and a slope of the walls of 60° rotates about its vertical axis. A chute is mounted in the vessel along its generatrix so that a small ball can move in it. The ball is kept at a height of 30 cm during the rotation. What is the rotational frequency?

Answer. 64.5 rpm.

70*. A motorcyclist moves over a cylindrical wall whose diameter is 12 m. At what coefficient of friction between the wall and the motorcycle wheels is the motion with a velocity of 54 km/h possible?

Answer. $\geqslant 0.262$.

71. (a) A 5-t motorcar moves with a velocity of 72 km/h over a road with a profile in the form of a concave arc of a circle of radius 40 m. At what point of the road is the force of pressure of the car maximum? Find this force. The position of the point on the road should be determined by the angle between the radius of the circle drawn to a given point and the vertical.

Answer. 0°, 99 kN.

(b) A motorcar moves over a convex bridge whose radius of curvature is 60 m. What is the minimum velocity of the car with which it does not exert a pressure on the bridge at its upper point?

Answer. 87 km/h.

72. (a) An aeroplane describes a wingover of radius 500 m. (A wingover is a circle in the vertical plane.) Find the minimum velocity with which the pilot is not separated from his seat at the upper point of the trajectory. Which overload does the pilot experience at the lower point of the loop with this velocity? (Overload is the ratio of the force pressing the pilot against the seat to the force of gravity acting on the pilot.)

Answer. 252 km/h, 2.

(b) An aeroplane comes out of a dive with a velocity of 300 km/h along a circular path of radius 100 m. Which overload does the pilot experience in this case?

Answer. 8.1.

(c*) What is a decrease in the weight of a body on the equator relative to its weight at a pole due to daily rotation of the Earth? Consider the radius of curvature of the Earth's surface to be the same at the pole and on the equator.

Answer. ~ 0.3%.

(d*) A 4-kg body is rotated in a vertical plane with the help of a rubber cord at a frequency of 120 rpm. Find the elongation and tension of the cord at the upper and lower points of the trajectory if its length in the unstretched state is 30 cm and the rigidity is 1.0 kN/m. The mass of the cord should be neglected.

Answer. 41 cm, 62 cm, 490 kN, 350 kN.

73 and 74*. A body slides down along an inclined chute which ends with a vertical circular loop of radius 40 cm. What should be the minimum height of the chute so

that the body does not fall at the upper point of the loop if the losses of mechanical energy due to friction amount to 20% of the difference in the potential energies of the body at the top of the chute and at the upper point of the loop? Solve the problem under the assumption that the body has the shape of a hollow cylinder, and losses amount to 5%.
Answer. 1.05 m, 1.22 m.
75*. A body slides down from the upper point on the outer surface of the hoop of radius r. At what height from the lower point of the hoop does the body separate from it? Find the force of pressure exerted by the body on the hoop at a point at which the radius of the hoop forms an angle of 30° with the vertical if the force of gravity acting on the body is mg. Friction should be neglected.
Answer. $5r/3$, $0.6mg$.
76*. A vertical axis is fixed to the ceiling of a lift moving downwards with an acceleration of 3.0 m/s^2. A 1.2-kg ball is attached to a 0.50-m long thread fixed to the axis which rotates at a frequency of 120 rpm. Find the tension of the thread and the angle it forms with the vertical.
Answer. 95 N, arccos 0.086.
77. Two balls connected by a 25-cm thread are pinned on a horizontal rod so that they can slide along it. When the distance between the 0.6-kg ball and the rotational axis becomes 10 cm, the thread with the balls does not change its position relative to the rod. Find the mass of the second ball. The masses of the rod and the thread should be neglected. Is this rotation stable?
Answer. 0.4 kg, no, it is not.
78*. A 2-kg load attached to a rubber cord is rotated in a horizontal plane at a frequency of 120 rpm. The length of the cord in the unstretched state is 60 cm. Find the kinetic energy of the load and the potential energy of deformation of the cord if its rigidity is 1.0 kN/m.
Answer. 124 J, ~40 J.
79*. The rods of a centrifugal ball governor to which 5-kg balls are fixed have a length of 60 cm and are connected at the middle by a horizontal spring with a ring at its centre, through which the governor axis passes without touching it (Fig. 86).

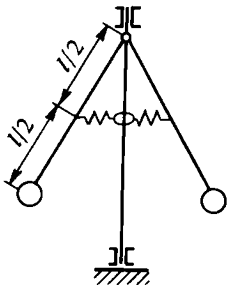


Fig. 86

The length of the spring in the unstretched state is 20 cm and its rigidity is 200 N/m. What is the rotational frequency of the governor if the angle formed by

the rods with the vertical is 30°? The masses of the rods should be neglected.
Answer. ~ 50 rpm.

80. Find the orbital velocity of an artificial satellite. What work should be done to launch a 500-kg satellite to a circular orbit? Assume that the Earth's radius is 6400 km and the free-fall acceleration at the ground is 9.8 m/s^2. The air resistance should be neglected.
Answer. 7.9 km/s, 47 GJ.

D. STATICS

1.28. Equilibrium of a Nonrotating Body. Equilibrium Conditions for a Body on an Inclined Plane

A body is **in equilibrium** *if it preserves the state of its motion with time.* Since any body can simultaneously be in translatory motion and rotate about a certain axis, its equilibrium takes place if the velocity of translatory motion and the angular velocity of rotation remain unchanged.

Let us first consider the case when a body cannot rotate for some reason.[13] The necessary condition for equilibrium here is that the *resultant* **R** *of all the forces acting on the body must be equal to zero.* If the resultant is equal to zero, its projection onto any axis also has zero value, i.e. the sums of the projections of all the forces onto any axis are equal to zero.

This condition gives in the general case three independent equilibrium equations for the body for three mutually perpendicular coordinate axes:

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum F_z = 0, \tag{1.28.1}$$

where F_x, F_y, and F_z are the projections of the force **F** onto the coordinate axes Ox, Oy, and Oz. If all the forces acting on a body lie in one plane, say, xOy, we can write two independent equilibrium equations:

$$\sum F_x = 0, \quad \sum F_y = 0. \tag{1.28.2}$$

[13] For example, the motion of a plane body along an inclined plane.

If $R = 0$ and the body is in equilibrium, this does not mean that it is at rest. The body can be either at rest or in a uniform motion in a straight line, but in both cases acceleration is absent.

While analyzing the equilibrium of a body on an inclined plane, it is expedient in most problems to direct one of the coordinate axes along the slope and the other axis, normal to the plane (Fig. 87).[14]

When a body **is at rest** on an inclined plane (Fig. 87*a*), it is acted upon by the force of gravity $m\mathbf{g}$, the normal reaction $\mathbf{N}$ of the plane, and the friction $\mathbf{F}_{fr}$. Assuming that the dimensions of the body are so small that it can be treated as a material point, we conditionally apply all the forces to its centre of mass.

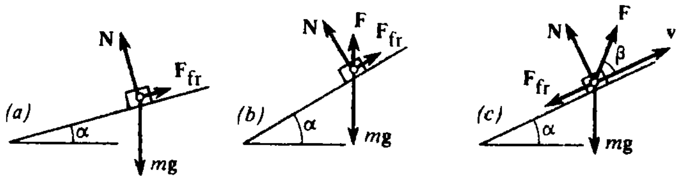


Fig. 87

Projecting all the forces onto the coordinate axes, we obtain the following system of equilibrium equations (1.28.2):

$$F_{fr} - mg\sin\alpha = 0, \quad N - mg\cos\alpha = 0.$$

As the angle α increases, the equilibrium conditions are satisfied as long as $F_{fr} \leqslant fN$, since the force of (normal) pressure on the plane is equal in magnitude to the normal reaction of the plane. From the second equation, we obtain $N = mg\cos\alpha$, which gives $F_{fr} \leqslant fmg\cos\alpha$. Using the first equation, we get

$$fmg\cos\alpha - mg\sin\alpha \geqslant 0, \quad \text{or} \quad \tan\alpha \leqslant f.$$

In the limiting case, $\tan\alpha = f$. With a further increase in the slope of the inclined plane, the body starts to slide down it, and in order to keep it at rest, a certain force $\mathbf{F}$ should be applied (Fig. 87*b*). In this case, the equilibrium equations (1.28.2) become

[14] There are some exceptions (see solution to Problem 91).

$$\max F_{fr} - mg\sin\alpha + F\cos\beta = 0, \quad N - mg\cos\alpha + F\sin\beta = 0.$$

The second of these equations gives

$$N = mg\cos\alpha - F\sin\beta, \quad \text{i.e.} \quad \max F_{fr} = f(mg\cos\alpha - F\sin\beta).$$

Then the first equation assumes the form

$$f(mg\cos\alpha - F\sin\beta) - mg\sin\alpha + F\cos\beta = 0.$$

Finally, if the body is uniformly lifted along the plane by a force $\mathbf{F}$ (with a velocity v = const), the friction $\mathbf{F}_{fr}$ is directed oppositely (Fig. 87*c*) and appears in the first equation with the minus sign.

Problems with Solutions

81. A 12-kg bar is pressed between two blocks (Fig. 88). The coefficient of friction between the blocks and the bar is $f = 0.3$. At what minimum forces F pressing the blocks against the bar would it not slide down? What additional vertical force F_{ad} is required to keep the block at rest if each of the pressing forces $F = 50$ N? What force F_{tr} should be applied to pull the block uniformly upwards?

Solution. In fact, we have three problems here.

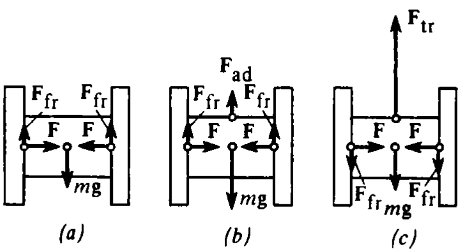


Fig. 88

The blocks act on the bar with equal frictions $F_{fr} \le fF$ (Fig. 88*a*). Since the bar is in equilibrium, the sum of the forces acting on it is equal to zero. This means that $2F_{fr} = mg$. But $F_{fr} \leqslant fF$. Consequently, $2fF \geqslant mg$, whence $F \geqslant mg/2f$ = 200 N.

Here, we must take into account the fact that in the absence of friction the bar would move downwards. This means that frictions are directed upwards (Fig. 88*b*). They help keep the load at rest. We can write the equilibrium condition for the bar: $F_{ad} + 2F_{fr} = mg$, whence $F_{ad} = mg - 2F_{fr} = mg - 2fF = 90$ N.

It follows from Fig. 88*c* that $F_{tr} = mg + 2F_{fr} = mg + 2fF = 150$ N.

82. An aeroplane tows two gliders with a constant velocity. The plane and the gliders fly in a horizontal plane so that the angles between the plane trajectory and the towing ropes are the same and equal to $\alpha = 30°$. The tension of each rope is $T_1 = T_2 = T = 500$ N. The air resistance to the motion of the plane with a given velocity is $F' = 400$ N. Find the tractive force of the plane engine.

Solution. For a uniform motion, the tractive force is equal in magnitude to the resultant of resistance forces and has the opposite direction. The forces $\mathbf{T}_1$ and $\mathbf{T}_2$ are added by the parallelogram rule (Fig. 89). Their resultant $\mathbf{R}$ is directed

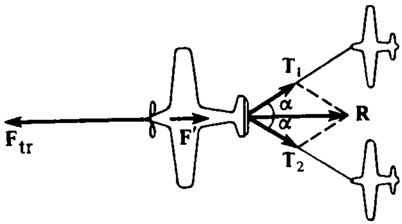


Fig. 89

along the flight axis. Then we add the forces $\mathbf{R}$ and $\mathbf{F}'$ directed along the same straight line and find the total resistance force:

$$F_{res} = R + F' = 2T\cos\alpha + F' \simeq 1266 \text{ N}.$$

The tractive force is equal in magnitude to this force: $F_{tr} = 1266$ N.

83. Three radial forces are applied to a disc so that the angles α between them (Fig. 90) are equal to 30°. The middle force $F_2 = 100$ N, while $F_1 = F_3 = 50$ N. Find the magnitude, direction, and the point of application of the balancing force.

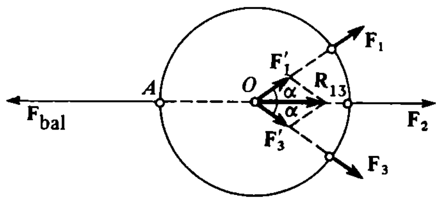


Fig. 90

Solution. We first determine the resultant $\mathbf{R}_{13}$ of the forces $\mathbf{F}_1$ and $\mathbf{F}_3$. To do this, we assume that the disc is a perfectly rigid body and transfer the forces $\mathbf{F}_1$ and $\mathbf{F}_3$ along the lines of their action to the point of intersection of these lines at the centre O of the disc. Then we add these forces by the parallelogram rule. Their resultant is the diagonal of the rhombus, which is directed along the line of action of the force $\mathbf{F}_2$: $R_{13} = 2F\cos\alpha \simeq 87$ N. Then we find the sum of the forces $\mathbf{R}_{13}$ and $\mathbf{F}_2$. The resultant of the three given forces is directed along the same straight line as $\mathbf{R}_{13}$ and $\mathbf{F}_2$ and is equal in magnitude to $R = R_{13} + F_2 = 187$ N. The balancing force acts along the same straight line but in the opposite direction, e.g. it is applied at point A of the disc and is equal in magnitude to the resultant force: $F_{bal} = 187$ N.

84*. Five forces whose magnitudes and directions are given by two sides and three diagonals of a regular hexagon with side a are applied at a point A of a vertical disc, lying at a certain distance from the centre of the disc along the horizontal (Fig. 91). The force of gravity acting on the disc is $10.4a$. Find the magnitude, the point of application, and the direction of the balancing force.

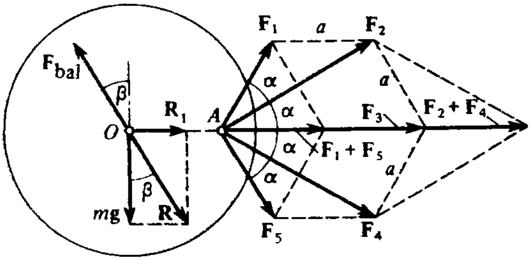


Fig. 91

Solution. Let us first find the forces F_1-F_5, considering that the angles between the directions of these forces are $\alpha = 30°$:

$$F_1 = F_5 = a, \quad F_2 = F_4 = \sqrt{3}a, \quad F_3 = 2a.$$

Let us first add symmetric forces pairwise. For the horizontal projections, we have

$$F_{1\,\mathrm{hor}} + F_{5\,\mathrm{hor}} = a, \quad F_{2\,\mathrm{hor}} + F_{4\,\mathrm{hor}} = 3a,$$

while the vertical projections are mutually compensated.

The resultant of all the forces is directed to the right along the horizontal diagonal of the hexagon. Its magnitude is $R_1 = a + 3a + 2a = 6a$. We transfer this force to the centre of the disc (point O) and add it to the force of gravity acting on the disc: $R = \sqrt{(6a)^2 + (10.4a)^2} = 12a$. The angle between this resultant and the vertical is $\beta = \arcsin(6a/12a) = \arcsin(1/2) = 30°$. The balancing force $F_{bal} = 12a$ is applied at the centre of the disc and is directed upwards and to the left at an angle of $\beta = 30°$ to the vertical.

85. A load whose force of gravity $mg = 100$ N is suspended from an arm BAC (Fig. 92). Find the forces acting on the rods AB and BC and the tension of the suspender if the angle $\alpha = 150°$.

Solution. We transfer the force of gravity mg along the line of its action to point A and consider the equilibrium of this point. The forces applied to this point are the tension $-\mathbf{T} = m\mathbf{g}$ of the suspender and the reactions $\mathbf{F}_1$ and $\mathbf{F}_2$ of the rods. Since point A is in equilibrium, the sum of the forces applied to it must be equal to zero. This means that the resultant of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ exerted by the rods must be equal in magnitude to the tension: $T = mg$. Thus,

$$F_1 = T/\cos(\alpha - 90°) = 200 \text{ N},$$

$$F_2 = T\tan(\alpha - 90°) = 170 \text{ N}.$$

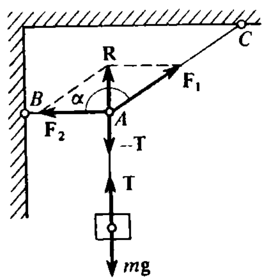


Fig. 92

86. A load of $m = 5$ kg is suspended from an arm formed by two rods whose lengths are $AC = 0.5$ m and $BC = 0.4$ m (Fig. 93). The distance $AB = 0.2$ m. Find the forces acting on the rods of the arm.

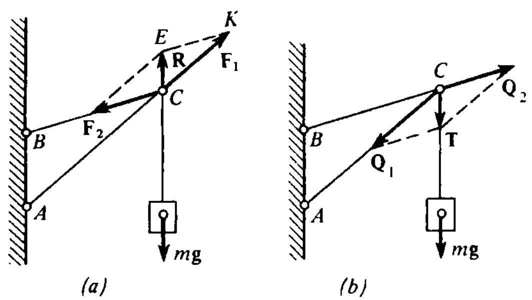


Fig. 93

Solution (see solution to Problem 85). Since the rope with the load is in equilibrium, the resultant $\mathbf{R}$ of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ exerted by the rods on the rope at point C should be such that $\mathbf{R} + m\mathbf{g} = 0$ (Fig. 93*a*), i.e. $\mathbf{R} = -m\mathbf{g}$ and $R = mg = 49$ N. It follows from the similarity of the triangles ABC and CEK that

$$CK/AC = EK/BC = EC/AB, \quad \text{or} \quad F_1/AC = F_2/BC = R/AB,$$

whence

$$F_1 = R \cdot AC/AB = 122.5\ \text{N}, \quad F_2 = R \cdot BC/AB = 98\ \text{N}.$$

The rod AC is compressed, while the rod BC is stretched.

This problem can be solved in a different way (Fig. 93*b*), i.e. by decomposing the tension $\mathbf{T} = m\mathbf{g}$ of the rope, acting on the rods at point C, into two components $\mathbf{Q}_1$ and $\mathbf{Q}_2$ parallel to the rods. The components $\mathbf{Q}_1$ and $\mathbf{Q}_2$ of the force $\mathbf{T}$ are just the forces acting on the corresponding rods at point C (see Fig. 93*b*).

87. A cylinder of radius $r = 1$ m, whose mass $m = 5$ t is uniformly distributed over volume, is at rest on the ledges of a masonry (Fig. 94). The distance a from one wall to the vertical passing through the centre of the cross section of the cylinder is 0.50 m, while the corresponding distance from the other wall to the vertical is $b = 0.87$ m. Find the forces of pressure exerted by the cylinder on the ledges of the masonry.

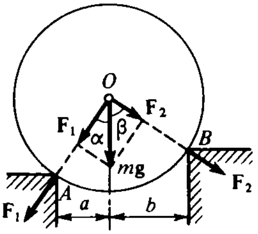


Fig. 94

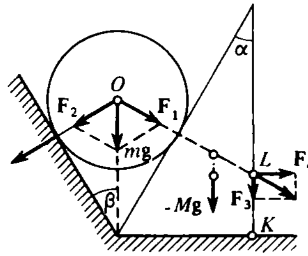


Fig. 95

Solution. We decompose the force of gravity $mg = 49$ kN along the directions of radii OA and OB. We note that $\sin\alpha = 1/2$ and hence $\alpha = 30°$, while $\sin\beta = 0.87 = \sqrt{3}/2$, i.e. $\beta = 60°$. Consequently, $\alpha + \beta = 90°$, and the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are at right angles. Hence

$$F_1 = mg\cos\alpha = 42.6 \text{ kN}, \quad F_2 = mg\sin\alpha = 24.5 \text{ kN}.$$

88. A prism of mass M, whose cross section is a right triangle with an angle $\alpha = 30°$, may slide over a horizontal plane with its smaller face as the base (Fig. 95). A ball of mass $m = 100$ kg is put between the inclined face of the prism and a wall forming an angle $\beta = 30°$ with the vertical. What horizontal force should be applied to the vertical face of the prism in order to prevent the prism from sliding over the horizontal plane? Friction should be neglected. What will change if the ball of the same mass has a smaller diameter? How will the solution of the problem change if the mass of the prism cannot be neglected?

Solution. We decompose the force of gravity mg acting on the ball along the two directions normal to the wall and to the inclined face of the prism: $F_1 = F_2 = 2mg\sin 30° = mg$. We translate the force $\mathbf{F}_1$ along the line of its action to point L on the vertical face of the prism and decompose it into the vertical and horizontal components. The vertical component $\mathbf{F}_3$ and the force of gravity Mg acting on the prism are balanced by the reaction of the floor. The magnitude of the horizontal component $\mathbf{F}_4$ of the force $\mathbf{F}_1$ is given by

$$F_4 = F_1\cos 30° = mg\cos 30° = 868 \text{ N}.$$

The required balancing force is equal in magnitude to F_4, applied at point L, and is opposite to the force $\mathbf{F}_4$.

As we decrease the diameter of the ball at the same mass, the balancing horizontal force does not change but the point of its application moves downwards to the base of the prism (point K). A further decrease in the ball diameter does not change this force or the point of its application. Since we neglected friction, an increase in the mass of the prism does not alter the result.

89. A 100-kg load is uniformly moved over a horizontal plane by a force **F** applied at an angle $\alpha = 30°$ to the horizontal. Find this force if the coefficient of friction between the load and the plane is $f = 0.3$.

Solution. The load experiences the action of the force of gravity $m\mathbf{g}$, the force **F** applied to it at an angle α to the horizontal, the normal reaction **N** of the plane, and the friction $\mathbf{F}_{fr}$ along the plane. Since the load moves uniformly, i.e. is in equilibrium, the sum of these forces (their resultant) should be zero. The sums of

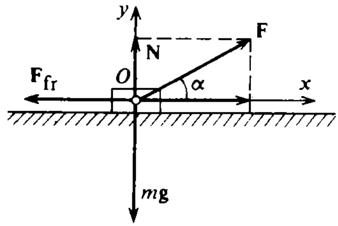


Fig. 96

the projections of all the forces onto any axes, in particular, the coordinate axes Ox and Oy, should also be zero (Fig. 96): $F\cos\alpha - F_{fr} = 0$, or

$$F\cos\alpha - fN = 0, \quad N + F\sin\alpha - mg = 0.$$

The second of these equations gives $N = mg - F\sin\alpha$. Substituting the value of N into the first equation, we get

$$F\cos\alpha - f(mg - F\sin\alpha) = 0,$$

whence $F = fmg/(\cos\alpha + f\sin\alpha) \approx 289$ N.

90. What should be the force F keeping the bar of mass m on a smooth inclined plane if the slope of the plane is α and the force acts along the inclined plane? The friction between the bar and the plane should be neglected. Find the normal reaction N of the plane.

Solution. Since the bar is in equilibrium, the sums of the x- and y-projections of all the forces acting on the bar must be zero (Fig. 97):

$$F - mg\sin\alpha = 0, \quad N - mg\cos\alpha = 0,$$

whence

$$F = mg\sin\alpha, \quad N = mg\cos\alpha.$$

91. Solve the preceding problem for the case when the force **F** keeping the bar on the plane is horizontal (Fig. 98).

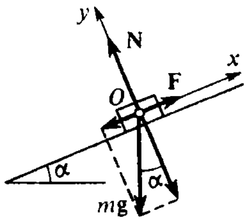


Fig. 97

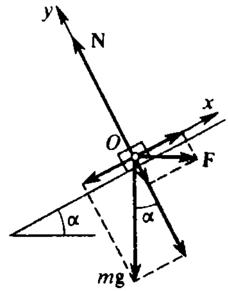


Fig. 98

Solution. Since the bar is in equilibrium, we have

$$\sum F_x = 0: \quad F\cos\alpha - mg\sin\alpha = 0,$$

$$\sum F_y = 0: \quad N - mg\cos\alpha - F\sin\alpha = 0.$$

Solving this system, we obtain

$$F = mg\tan\alpha, \quad N = mg(\cos\alpha + \sin\alpha\tan\alpha) = mg/\cos\alpha.$$

Remark. In this problem, the introduction of a coordinate system formed by the horizontal and vertical coordinate axes Ox' and Oy' would not complicate the solution. In this system, the equilibrium equations would have the form

$$F - N\sin\alpha = 0, \quad N\cos\alpha - mg = 0.$$

92. A body of mass $m = 20$ kg is kept on an inclined plane by a rope attached to a spring balance fixed above the plane. The spring balance indicates 113 N. The plane is inclined at an angle $\alpha = 30°$ to the horizontal. Find the angle β between the rope and the inclined plane, and the normal reaction N of the plane. Friction should be neglected.

Solution. Since the body is in equilibrium, the sums of the x- and y-projections of the forces $m\mathbf{g}$, $\mathbf{F}$, and $\mathbf{N}$ (Fig. 99) must be equal to zero:

$$-mg\sin\alpha + F\cos\beta = 0, \quad -mg\cos\alpha + N + F\sin\beta = 0.$$

The first equation gives

$$\cos\beta = (mg\sin\alpha)/F \simeq 0.86, \quad \text{i.e.} \quad \beta \simeq 30°.$$

From the second equation, we find

$$N = mg\cos\alpha - F\sin\beta \simeq 113\ \text{N}.$$

93. Solve Problem 90 under the condition that the coefficient of friction between the bar and the plane is f. Find also the tractive force that should be applied to the bar to lift it uniformly.

Solution. The friction is directed upwards along the inclined plane against the

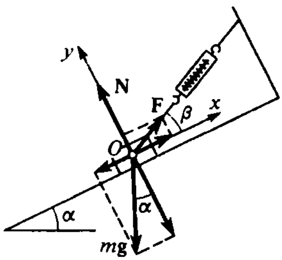


Fig. 99

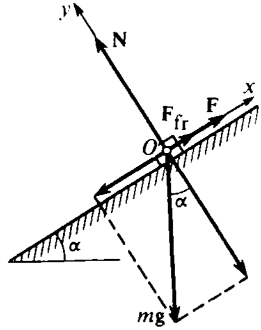


Fig. 100

direction of motion of the body in the absence of friction (Fig. 100). Since the bar is in equilibrium, the sums of the x- and y-projections of the forces acting on it are equal to zero:

$$F + F_{fr} - mg \sin \alpha = 0, \quad N - mg \cos \alpha = 0,$$

whence

$$N = mg \cos \alpha, \quad F = mg \sin \alpha - F_{fr}.$$

Substituting $F_{fr} = fN = fmg \cos \alpha$ into the last equation, we get $F = mg(\sin \alpha - f \cos \alpha)$.

If the body is pulled upwards along the inclined plane with a constant velocity, the friction is directed downwards along this plane (Fig. 101). In equilibrium, we have

$$F - F_{fr} - mg \sin \alpha = 0, \quad N - mg \cos \alpha = 0.$$

Considering that $F_{fr} = fN$, we obtain $F = mg(\sin \alpha + f \cos \alpha)$.

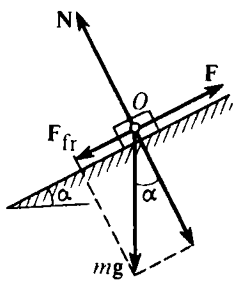


Fig. 101

94. A body of mass m is kept in equilibrium on an inclined plane by a horizontal force $\mathbf{F}$. The slope of the plane is α and the coefficient of friction is f. Find the force F keeping the body on the plane and the normal pressure N.

Solution. In comparison with Problem 91, we have here an additional force $F_{fr} = fN$ directed upwards along the plane (Fig. 102). Only the first of the equilibrium equations will change as a result:

$$F\cos\alpha - mg\sin\alpha + fN = 0, \quad N - mg\cos\alpha - F\sin\alpha = 0.$$

Solving this system, we obtain

$$F = mg\frac{\sin\alpha - f\cos\alpha}{\cos\alpha + f\sin\alpha}, \quad N = \frac{mg}{\cos\alpha + f\sin\alpha}.$$

95. A body of mass m is uniformly lifted along an inclined plane with a slope α by a force $\mathbf{F}$ directed at an angle $\beta > \alpha$ to the plane. The coefficient of friction is f. Find the tractive force F and the normal pressure N.

Solution. The forces acting on the body and their projections onto the coordinate axes are shown in Fig. 103. The equilibrium conditions have the form

$$F\cos(\beta - \alpha) - mg\sin\alpha - fN = 0, \quad N - mg\cos\alpha + F\sin(\beta - \alpha) = 0.$$

Solving this system of equations, we obtain

$$F = mg\frac{\sin\alpha + f\cos\alpha}{\cos\beta + f\sin\beta}, \quad N = mg\frac{\cos(\alpha + \beta)}{\cos\beta + f\sin\beta}.$$

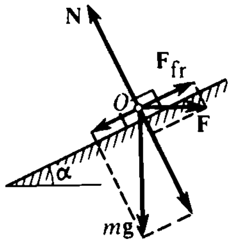


Fig. 102

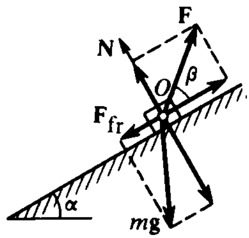


Fig. 103

Exercises

81. (a) A bar with flat faces, whose mass is 5 kg, is pressed against a vertical wall by a force of 100 N. What must be the coefficient of friction for lifting the bar uniformly by a vertical force of 80 N? What vertical force is required to keep the bar at rest?

Answer. 0.3, 20 N.

(b) A 7-kg bar is pressed with a certain force between two vertical walls. Find

the magnitude of the force if the coefficient of friction is 0.2 and it is known that the bar can be kept at rest by a vertical force of 30 N.
Answer. 100 N.

(c) A 6-kg bar is pressed between two vertical walls by horizontal forces of 200 N each. What vertical force is required to pull it down if the coefficient of friction is 0.25? Solve the problem for the case when the bar is pulled up.
Answer. $\geqslant 40$ N, 160 N.

82. (a) A boat is kept at the middle of a river by two ropes each 30-m long and attached to poles dug into the river banks (Fig. 104) so that the distance between them is 30 m. What is the force pulling the boat downstream (the direction of the flow is indicated by the arrows) if the tension of each rope is 50 N?
Answer. ~ 87 N.

(b) A 10-kg load is suspended at the middle of a rope whose length is 80 cm and the two ends are fixed at the ceiling. The point at which the body is suspended sags by 20 cm. What additional load can be suspended at the middle of the rope if the maximum admissible tension of the rope is 160 N? The mass of the rope should be neglected.
Answer. 6.2 kg.

83. (a) Three braces (rods) are attached to the rod AB of a truss with the help of a "gusset" *abcde* so that their longitudinal axes meet at point O and the angles between them are 60° (Fig. 105). The forces $F_1 = F_2 = 1$ kN acting on the rods are

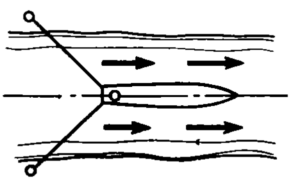


Fig. 104

Fig. 105

tensile, while the force $F_3 = 1.2$ kN is compressive. Find the magnitude and direction of the force exerted on the rod AB by the braces.
Answer. The upward force of 500 N.

(b) A homogeneous rod is suspended from the ceiling with the help of two wires forming angles of 30° with the vertical and experiencing tensile forces of 50 N. A load of mass $m = 6$ kg is suspended at the middle of the rod (Fig. 106). Find the mass of the rod. The masses of the wires should be neglected.
Answer. ~ 2.7 kg.

84*. Seven forces of equal magnitude, $F_1 = F_2 = \ldots = F_7 = 50$ N, are applied to the rim of a vertical disc with a uniformly distributed mass of 18.7 kg along the

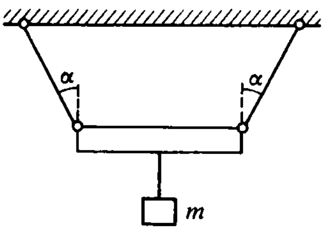


Fig. 106

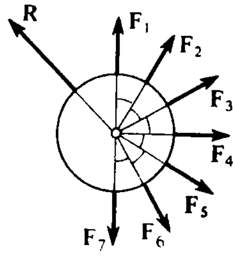


Fig. 107

disc radii (Fig. 107). A force balancing these forces is applied to the rim of the disc above its horizontal diameter. Find the magnitude, direction, and the point of application of the balancing force.

Answer. The upward force of 264 N is applied at the middle of the left upper quarter of the rim circle at an angle of 45° to the horizontal.

85. A 100-kg load is suspended from a bracket A_1AA_2 (Fig. 108). Find the forces acting on the rods AA_1 and AA_2 of the bracket and the tension of the suspender if the angle between the rods is 60°. The masses of the bracket and the suspender should be neglected.

Answer. $2\sqrt{3}/3$ kN, $\sqrt{3}$ kN, 1 kN.

86*. (a) A 100-kg load is suspended from the bracket fixed to a vertical wall. The upper and lower rods of the bracket are inclined to the vertical at 30° and 60°

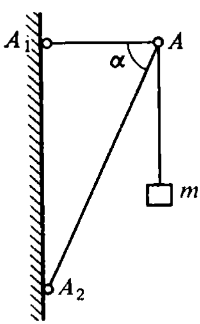


Fig. 108

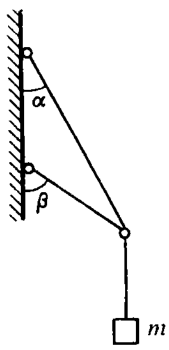


Fig. 109

respectively (Fig. 109). Find the forces acting on the rods. The masses of the bracket and the suspender should be neglected.

Answer. 980 N, 1.7 kN.

(b) A 45-kg load is suspended from the bracket fixed to a vertical wall at points A_1 and A_2 and consisting of two rods of lengths AA_1 = 50 cm and AA_2 =

70 cm (Fig. 110). The distance A_1A_2 = 30 cm. Find the forces acting on the rods. The mass of the bracket should be neglected.
Answer. 735 N, 1029 N.
87*. (a) A homogeneous 10-kg ball rests on two mutually perpendicular inclined planes one of which has a slope of 30° (Fig. 111). Find the forces of pressure of the ball on each plane. Friction should be neglected.
Answer. 49 N, 85 N.

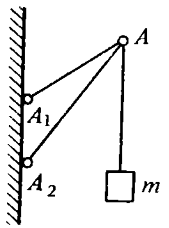


Fig. 110

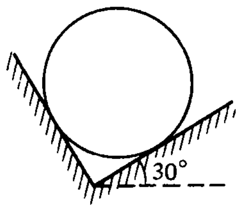


Fig. 111

Hint. In the absence of friction between two bodies in contact, the forces exerted on the body by the other body are directed along the normal to the plane of contact by the contact point.

(b) The piston of a steam engine experiences the action of the force of pressure of 500 N. The length of the connecting rod is AB = 1 m and the length of the crank is BO = 20 cm (Fig. 112). Find the force of pressure exerted by the crosshead on

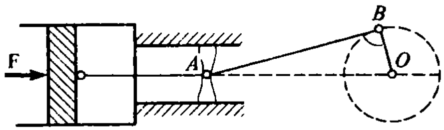


Fig. 112

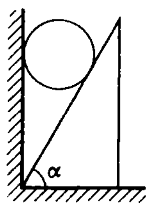


Fig. 113

the directing plane and the force transmitted from the connecting rod AB to the crank OB at the moment when $\angle ABO = 90°$.
Answer. 100 N, 510 N.
88*. A prism whose cross section is a right triangle with an angle of 60° at the base can slide over a horizontal plane. A 100-kg ball is placed between a vertical wall and the inclined plane of the prism (Fig. 113). What horizontal force should be applied to the vertical face of the prism in order to prevent it from sliding over the horizontal plane? The mass of the prism and friction should be neglected. How

will the problem change if the diameter of the ball is decreased without changing its mass? What will be if the mass of the prism is increased?
Answer. 173 N.
89. (a) A 100-kg load is uniformly moved over a horizontal plane by a force of 300 N applied at an angle of 60° to the horizontal. Find the coefficient of friction between the load and the plane.
Answer. 0.21.
(b) A load is uniformly moved over a horizontal plane by a force of 600 N applied at an angle of 30° to the horizontal. The coefficient of friction between the load and the plane is 0.3. Find the mass of the load.
Answer. 207 kg.
90. (a) A 50-kg load is moved up with a constant velocity along an inclined plane by a force of 250 N directed along the plane. Find the force of normal pressure and the slope of the plane. Friction should be neglected.
Answer. 43 N, 30°.
(b) A 60-kg body kept in equilibrium on an inclined plane presses on this plane with a force of 300 N. What is the slope of the plane? Find the force which acts parallel to the plane and keeps the body at rest. Friction should be neglected.
Answer. ~60°, 510 N.
(c) A load is kept in equilibrium on an inclined plane with a slope of 30° by a force of 200 N which is parallel to the plane. Find the mass of the load and the force of pressure exerted by it on the plane. Friction should be neglected.
Answer. 40 kg, 346 N.
91. (a) A 50-kg load is kept in equilibrium on an inclined plane by a horizontal force of 500 N. What is the slope of the plane? Find the force of normal pressure. Friction should be neglected.
Answer. ~45°, 700 N.
(b) A load is kept in equilibrium on an inclined plane with a slope of 60° by a horizontal force of 1 kN. Find the mass of the load and the force of normal pressure. Friction should be neglected.
Answer. 58 kg, 1.16 kN.
92*. (a) A 100-kg load is kept in equilibrium on an inclined plane with a slope of 30° by a rope forming an angle of 60° with the horizontal. Find the force of normal pressure and the tension of the rope. Friction should be neglected.
Answer. 562 N, 565 N.
(b) A 100-kg load is kept in equilibrium on an inclined plane with a slope of 30° by a rope stretched at 30° to the horizontal. Find the tension of the rope and the force of normal pressure. Friction should be neglected.
Answer. 980 N, 1700 N.
93. (a) A 100-kg load is kept on an inclined plane by a force of 320 N acting along the plane. Find the coefficient of friction if the slope of the plane is 30°. What force is required to pull the load uniformly to the top of the plane?
Answer. 0.19, 660 N.
(b) A 20-kg load is pulled uniformly up an inclined plane by a force of 200 N.

Find the slope of the plane if the coefficient of friction is 0.3. What force is required to keep the load in equilibrium?
Answer. 60°, 140 N.
94*. (a) A 20-kg load is kept in equilibrium on a plane with a slope of 30° by a horizontal force of 50 N. Find the coefficient of friction.
Answer. 0.28.
(b) A load is kept in equilibrium on a plane with a slope of 60° by a horizontal force of 200 N. What is the mass of the load if the coefficient of friction is 0.3?
Answer. 21 kg.
95*. (a) A 100-kg load is uniformly raised along an inclined plane by a force directed at an angle of 75° to the horizontal. The slope of the plane is 30° and the coefficient of friction is 0.3. Find the tractive force.
Answer. 52.3 N.
(b) A body is kept in equilibrium on a plane with a slope α by a force acting at the angle $-\alpha$ to the horizontal. Find the relation between the force of gravity mg acting on the body, the force F keeping the body in equilibrium, and the coefficient of friction f.
Answer. $F/mg = (\sin\alpha - f\cos\alpha)/(f\sin 2\alpha - \cos 2\alpha)$.

1.29. Moment of Force

Suppose that a force **F** is applied at a point A (Fig. 114). We choose arbitrarily a point O and drop the perpendicular from this point to the line of action of the force. The length l of this perpendicular is called the **arm of force F** about point O.

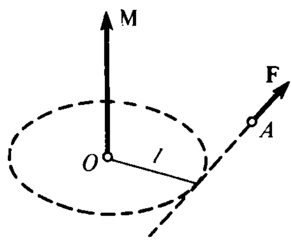


Fig. 114

The **moment of force F (torque)** about point O is a vector quantity **M** whose magnitude is equal to the product of the magnitude of the force and the arm:

$$M = Fl.$$

The moment of force **M** is directed along the straight line normal to the plane containing the line of action of the force and point O so that the direction of rotation that would be caused by the force **F** and the direction of the moment of force **M** form a right-handed system (rotating the head of a right screw which faces us clockwise, we cause the motion of the screw towards us).

The moment of force characterizes the ability of the force to rotate a body about a point relative to which it is taken. If the body can rotate about point O arbitrarily, the force will rotate the body about an axis coinciding with the direction of the moment of force **M**.

We shall assume that a moment of force is positive if it causes a counterclockwise rotation and negative if it causes a clockwise rotation. After this, we can add the moments of force algebraically.

1.30. Addition of Parallel Forces. A Couple

The resultant of two *parallel forces* applied to a body is parallel to these forces, is equal to their sum, and is applied at a point which divides the straight line connecting the points of application of the

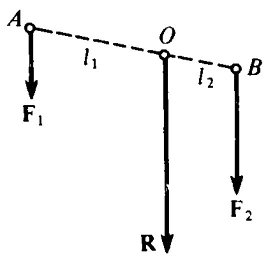


Fig. 115

component forces in the ratio inverse to the ratio of their magnitudes (Fig. 115):

$$\mathbf{F}_1 \| \mathbf{R} \| \mathbf{F}_2, \tag{1.30.1}$$

$$R = F_1 + F_2, \tag{1.30.2}$$

$$l_1/l_2 = F_2/F_1, \quad \text{or} \quad F_1 l_1 = F_2 l_2. \tag{1.30.3}$$

In view of Eq. (1.30.2), the acceleration imparted to the body by the force **R** is equal to the sum of the accelerations imparted by the forces $\mathbf{F}_1$ and $\mathbf{F}_2$, i.e. $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$.

The rotational action of the force **R** is equivalent to the rotational actions of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$. In particular, the torque of the force **R** about its point of application is equal to zero (since the arm is zero). On the other hand, the torques of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ about this point are equal in magnitude (in view of Eq. (1.30.3)) and rotate the body in opposite directions, thus their actions are mutually compensated.

The resultant of two *antiparallel forces* is parallel to these forces, is directed towards the larger force, and is equal in magnitude to their difference. This resultant is applied at a point lying on the extension of the straight line connecting the points of application of the component forces at the distances whose ratio is inverse to the ratio of their magnitudes (Fig. 116):

$$\mathbf{F}_1 \| \mathbf{R} \| \mathbf{F}_2,$$
$$R = F_2 - F_1,$$
$$l/l_2 = AC/BC = F_2/F_1.$$

If two antiparallel forces are equal in magnitude, their resultant is equal to zero. Two equal antiparallel forces are called a

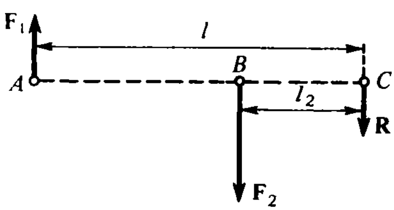


Fig. 116

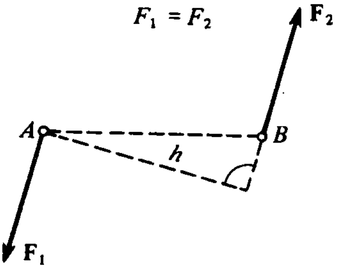


Fig. 117

couple (Fig. 117). The action of a couple on a rigid body is reduced only to a rotation of the body about a point lying in the same plane with this couple. The torque of a couple is equal to the product of the magnitude of one of the components and the distance separating the lines of action of the forces making a couple, which is called the **arm of couple**.

The sum of several parallel forces applied to a body at arbitrary points is determined consecutively by taking two forces at a time.

Several parallel forces *lying in the same plane* can be added on the basis of the following considerations. Since the resultant is equivalent to the action of its component forces, the moment of the resultant about any axis (or point) of rotation of the body is equal to the algebraic sum of the moments of the component forces about the same axis (or point) (see the end of Sec. 1.29):

$$M_R = \sum M_F, \quad \text{i.e.} \quad Rx = F_1 l_1 + F_2 l_2 + \dots + F_5 l_5$$

(Fig. 118). Hence we can easily find the arm of the resultant force, which determines its point of application:

$$x = (F_1 l_1 + F_2 l_2 + \dots + F_5 l_5)/R.$$

(In the general case, the moments of forces are taken with their signs (see Sec. 1.29).)

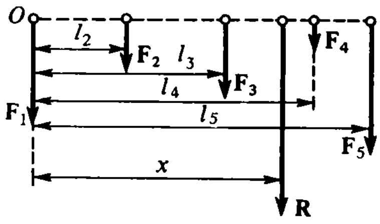


Fig. 118

A force can be decomposed into two parallel forces according to the same rules as we used for addition. The problem has a unique solution if, besides a given force, the points of application of the components, or one of the components and the point of its application, are specified.

1.31. Equilibrium of a Body with a Fixed Rotational Axis (Law of Torques)

If a body has a fixed rotational axis, it will be in equilibrium if the line of action of the resultant **R** of all the forces applied to the body passes through this axis. In this case, the moment of force **R** about the rotational axis is equal to zero since the arm of this force is zero.

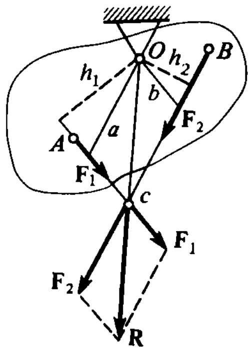


Fig. 119

Figure 119 represents a body whose rotational axis passes through point O (along the normal to the plane of the figure). Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are applied to the body so that their moments about this axis are equal and opposite:

$$F_1 h_1 = F_2 h_2. \qquad (1.31.1)$$

Let us prove that the resultant of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ should pass through point O. For this we draw from point O two straight lines Oa and Ob parallel to vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ till they intersect the lines of action of these forces. In the parallelogram thus obtained, the areas of the triangles Oac and Obc are equal:

$$ac \cdot h_1 = bc \cdot h_2. \qquad (1.31.2)$$

A comparison of Eqs. (1.31.1) and (1.31.2) shows that the segments ac and bc are proportional to the magnitudes of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$. We translate vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ to the point c of intersection of the lines of their action and plot at this point a parallelogram of forces on the same scale. It can be seen from the figure that the sides of this parallelogram are equal and parallel to the sides of the parallelogram *Oacb*. Consequently, vector $\mathbf{R}$ is a continuation of the diagonal *Oc*, i.e. it passes through the rotational axis (the body is in equilibrium).

On the basis of what has been proved above, we can formulate the **law of torques**: *a body having a fixed rotational axis is in equilibrium if the algebraic sum of the moments of all the forces acting on the body about the rotational axis is zero.*

If the algebraic sum of the torques acting on a body with a fixed rotational axis is equal to zero, this does not mean that the body is necessarily at rest. In can be either at rest or can rotate uniformly about this axis.

Let us consider a particular case of the equilibrium of one-dimensional bodies, viz. the rods called levers. Figure 120 shows

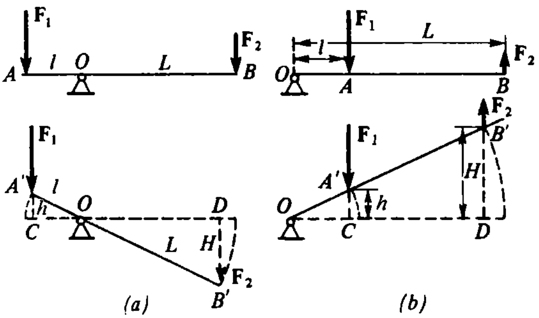


Fig. 120

two types of levers. On the left, there is a lever with the rotational axis (fulcrum) between the points of application of parallel forces $\mathbf{F}_1$ and $\mathbf{F}_2$ (such a lever is usually called the first-kind lever), while on the right, there is a lever with the rotational axis at its end and with antiparallel forces $\mathbf{F}_1$ and $\mathbf{F}_2$ (such a lever is called the second-kind lever).

In the absence of friction, the lever with a load is in equilibrium if the torque rotating it clockwise (in the case under consideration, the moment of force $\mathbf{F}_2$) about its rotational axis is equal to the torque $F_1 l$ of the force $\mathbf{F}_1$ about the same axis, which rotates the lever in the opposite direction: $F_2 L = F_1 l$, whence

$$F_2/F_1 = l/L. \qquad (1.31.3)$$

Consequently, the necessary condition for the lever equilibrium in the absence of friction is that the magnitudes of the forces applied to the lever ends be inversely proportional to the distances from the points of application of the forces to the rotational axis. In other words, it is necessary that the moments of the two forces about the rotational axis of the lever be equal and opposite. It follows from the similarity of the triangles COA' and DOB' that $l/L = h/H$, where h and H are the vertical displacements of points A and B respectively. Taking into account Eq. (1.31.3), we obtain $F_2/F_1 = h/H$. Thus, in the absence of friction, $F_2 = F_1 h/H$.

1.32. Equilibrium of a Rigid Body in the General Case

If the lines of action of all the forces applied to a body meet at one point, the necessary and sufficient condition for equilibrium of such a body is that the vector sum of all these forces be equal to zero. Indeed, the point of intersection of the straight lines along which the forces applied to the body act is in this case the point of application of the resultant of all these forces, which is equal to zero and does not impart any acceleration to the body.

If the lines of action of all the forces applied to a body do not intersect at a single point, the above condition is necessary but insufficient, since the action of the applied forces may turn out to be equivalent to the action of a certain couple. The torque of this couple causes rotation of the body in which its different parts have different centripetal accelerations, although the geometrical sum of a couple is zero. Hence we arrive at the following more general **condition for equilibrium** of a rigid body: *the necessary and sufficient condition for equilibrium of a body acted upon by forces lying in the same plane but such that their lines of action do* *not intersect at a single point is that the vector sum of all the forces be equal to zero and the algebraic sum of the moments of all the forces about an axis passing through any point of the plane containing the forces perpendicularly to this plane be also equal to zero.* Thus, the equilibrium conditions for a body can be written as follows:

$$\mathbf{R} = 0 \quad \text{and} \quad \sum M = 0.$$

If all the forces lie in one plane (say, xOy), the condition for equilibrium can be written in the form of three scalar equations:

$$\sum F_x = 0, \quad \sum F_y = 0, \quad \sum M = 0,$$

where F_x and F_y are the x- and y-projections of forces.

Problems with Solutions

96. Find the magnitude, direction, and the point of application of the resultant of parallel and antiparallel forces $\mathbf{F}_1, \ldots, \mathbf{F}_6$ acting on a beam AB if $F_1 = 30$ kN, $F_2 = 20$ kN, $F_3 = 60$ kN, $F_4 = 40$ kN, $F_5 = 60$ kN, and $F_6 = 50$ kN (Fig. 121).

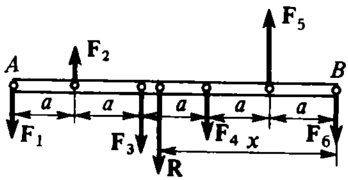


Fig. 121

Solution. Choosing the upward direction as positive, we find the resultant of these forces:

$$R = -F_1 + F_2 - F_3 - F_4 + F_5 - F_6 = -100 \text{ kN},$$

i.e. it is equal to 100 kN and is directed downwards.

In order to find the point of application of the resultant $\mathbf{R}$, we equate the sum of the moments of all the forces $\mathbf{F}_1$-$\mathbf{F}_6$ about point B to the moment of their resultant $\mathbf{R}$ about the same point:

$$F_1 5a - F_2 4a + F_3 3a + F_4 2a - F_5 a = Rx.$$

Hence the arm of force R is $x = 2.7a$.

97. A homogeneous beam whose mass $m = 50$ kg rests on two supports (Fig. 122). A load of mass $M = 100$ kg is placed at 1/4 of the beam length from the left support. Find the forces of pressure exerted by the beam on the supports.

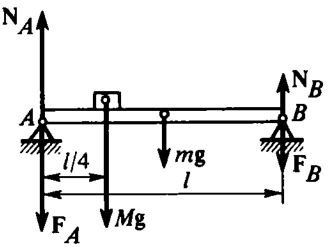


Fig. 122

Solution. We denote the length of the beam by l. The force of pressure exerted by the load, equal to $Mg = 980$ N, acts on the beam at a distance of $l/4$ from a support A. The force of gravity is applied to the beam at the middle of the span and is equal to $mg = 490$ N.

The problem can be solved by two methods.

Method I. Besides the forces indicated above, the beam experiences the action of the normal reactions $\mathbf{N}_A$ and $\mathbf{N}_B$ of the supports. Since the beam is in equilibrium, the sum of the forces acting on it is equal to zero: $\mathbf{N}_A + \mathbf{N}_B + M\mathbf{g} + m\mathbf{g} = 0$. For projections onto the vertical axis, we have

$$N_A + N_B - Mg - mg = 0. \quad (1)$$

Besides, the sum of the moments of these forces about any axis, for example, the horizontal axis passing perpendicularly to the plane of the figure through point A, is also equal to zero:

$$-Mgl/4 - mgl/2 + N_B l = 0, \quad (2)$$

whence

$$N_B = mg/2 + Mg/4 = 490 \text{ N}.$$

We can now find N_A from Eq. (1):

$$N_A = Mg + mg - N_B = 980 \text{ N}.$$

According to Newton's third law, the beam exerts the forces of pressure $\mathbf{F}_A = -\mathbf{N}_A$ and $\mathbf{F}_B = -\mathbf{N}_B$ on the supports. The magnitudes of these forces are $F_A = 980$ N and $F_B = 490$ N.

Method II. We decompose the force of gravity $m\mathbf{g}$ and the force of pressure $M\mathbf{g}$ of the load, which act on the beam and the load, into two components applied at points A and B. The components of the force of gravity mg, which is applied at the middle of the beam span, are $mg/2$ each. The components of the force of pressure Mg of the load are inversely proportional to the distance from the load to the supports, so the left support experiences the action of the force $0.75Mg$ and the right support, $0.25Mg$. Consequently, the required forces of pressure acting on the left and right supports are

$$F_A = 0.5mg/2 + 0.75Mg = 980\text{ N},\quad F_B = 0.5mg/2 + 0.25Mg = 490\text{ N}.$$

98. One end of a beam is embedded in a wall (Fig. 123). The mass M of the protruding part of the beam is 100 kg and its length $L = 1.5$ m. A load of mass $m =$

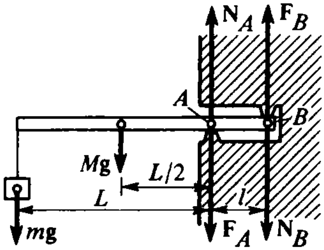


Fig. 123

50 kg is suspended from the end of the beam. The force F_A of pressure of the beam on the wall at point A must not exceed 8 kN. Find the minimum admissible distance l between the supports and the corresponding force of pressure at point B. The mass of the embedded part of the beam should be neglected.

Solution. The beam experiences the action of the force of gravity $M\mathbf{g}$, the force of pressure $m\mathbf{g}$ of the load, and the normal reactions $\mathbf{N}_A$ and $\mathbf{N}_B$ of the supports (see Fig. 123). According to Newton's third law, the beam acts on the supports with the forces $\mathbf{F}_A = -\mathbf{N}_A$ and $\mathbf{F}_B = -\mathbf{N}_B$. Since the beam is in equilibrium, the sum of the moments of these forces about any point is zero. In particular, the sum of the moments of forces acting on the beam about point A is equal to zero:

$$-N_B l + MgL/2 + mgL = 0. \tag{1}$$

Besides, the resultant of the forces $M\mathbf{g}$, $m\mathbf{g}$, $\mathbf{N}_A$, and $\mathbf{N}_B$ is also equal to zero. For the projections onto the vertical axis, we have

$$mg + Mg - N_A + N_B = 0, \tag{2}$$

whence

$$N_B = N_A - mg - Mg = 6530\text{ N}.$$

Consequently, $F_B = 6.53$ kN. We can now find l from Eq. (1):

$$l = L\,\frac{(Mg/2) + mg}{N_B} \simeq 0.225\text{ m}.$$

99. The length l of the 0.4-t car of the body is 3 m and the distance between the axes of the wheels is $a = 1.8$ m. Find the vertical force F_1 that should be applied to the right end of the car to lift it. What downward vertical force F_2 should be applied to the right end of the car to lift its left end?

Solution. In order to lift the right end, the force $\mathbf{F}_1$ should be directed upwards (Fig. 124*a*). We write the equation for the moments of forces about the axis of the left wheel:

$$-mga/2 + F_1(l + a)/2 = 0,$$

which gives $F_1 = mga/(l + a) = 1440$ N.

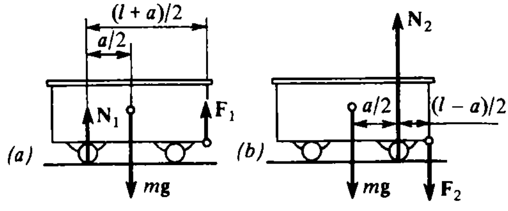


Fig. 124

In order to lift the left end, the force $\mathbf{F}_2$ should be directed downwards (Fig. 124*b*). We now write the condition for equilibrium for the moment of forces about the axis of the right wheel:

$$-F_2(l - a)/2 + mga/2 = 0,$$

whence $F_2 = mga/(l - a) = 5880$ N.

Remark. While solving this problem, we wrote the condition for equilibrium (equation for the moments of all the forces) so that they did not contain the unknown normal reactions N_1 and N_2 of the rails. This allowed us to do without another condition for equilibrium, viz. the equation for all the forces acting on the car.

100. The end A of a homogeneous rod AB of mass $m_1 = 40$ kg, which forms an angle $\alpha = 30°$ with the horizontal, is supported by a bearing having the horizontal rotational axis, while the other end B is attached to a rope fixed at point C lying on the same vertical with point A so that the triangle ABC is equilateral (Fig. 125). A

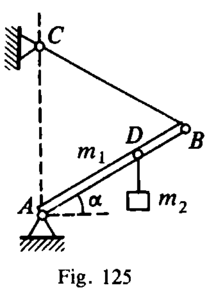


Fig. 125

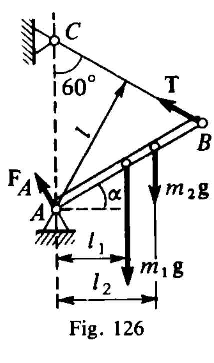


Fig. 126

load of mass $m_2 = 90$ kg is suspended from the rod AB at point D (at a distance equal to one third of its length from point B). Find the tension T of the thread. The friction in the bearing should be neglected.

Solution. The rod AB is a second-kind lever with the rotational axis at point A. The rod experiences the action of the forces of gravity $m_1\mathbf{g}$, $m_2\mathbf{g}$, the tension $\mathbf{T}$ of the rope, and the reaction $\mathbf{F}_A$ of the hinge (Fig. 126). Since the rod is in equilibrium, the sum of the moments of forces about point A must be zero:

$$-m_1gl_1 - m_2gl_2 + Tl = 0.$$

Introducing the notation $AB = BC = a$, $l = a\sin 60°$, $l_1 = (a/2)\cos 30°$, $l_2 = (2a/3)\cos 30°$, we obtain

$$m_1g(a/2)\cos 30° + m_2g(2a/3)\cos 30° - Ta\sin 60° = 0,$$

whence $T = m_1g/2 + 2m_2g/3 = 784$ N.

101. A horizontal homogeneous steel beam having a mass $m = 5$ t and a length l is hinged at one end to a massive wall. Its other end is supported by an iron rod attached to the same wall at an angle $\alpha = 30°$ to the vertical (Fig. 127). What must

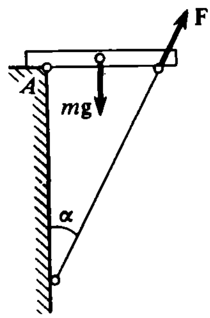


Fig. 127

be the cross-sectional area of the rod for its relative compression to be below $\varepsilon = 0.001$ if the modulus of elasticity of the rod material is $E = 200$ GPa and the elastic limit is $\sigma_{el} = 300$ MPa?

Solution. The compressive force F_{com} emerging in the rod is equal in magnitude to the force $\mathbf{F}$ with which the rod acts on the beam (in accordance with Newton's third law). In equilibrium, the sum of the moments of the force $\mathbf{F}$ about the hinge A and of the force of gravity $m\mathbf{g}$ acting on the beam about the same hinge is equal to zero. Since the beam is homogeneous, the force of gravity $m\mathbf{g}$ is applied at its middle. Hence $Fh - mgl/2 = 0$, or

$$Fl\cos\alpha = mgl/2, \quad \text{whence} \quad F_{com} = F = mg/(2\cos\alpha).$$

Let us find the stress in the rod:

$$\sigma = F_{\text{com}}/S, \quad \text{whence} \quad S = F_{\text{com}}/\sigma = mg/(2\sigma \cos\alpha).$$

Since $\sigma = E\varepsilon$, we have $S = mg/(2E\varepsilon \cos\alpha) = 1.5\ \text{cm}^2$. We must now verify the applicability of the formulas. Let us calculate the stress:

$$\sigma = E\varepsilon = 200\ \text{MPa} < 300\ \text{MPa}.$$

Therefore, σ is less than the elastic limit σ_{el}, and hence the formulas used by us for calculations are applicable.

102. A load of mass $m = 120$ kg is kept in equilibrium with the help of a system of rods $ABCD$ by a vertical rope DE fixed to the floor (Fig. 128*a*). Find the tension of the rope if the lengths of the rods $AB = CD = 3$ m, $AK = 0.6$ m, and $CL = 0.75$ m. The friction in the bearings should be neglected.

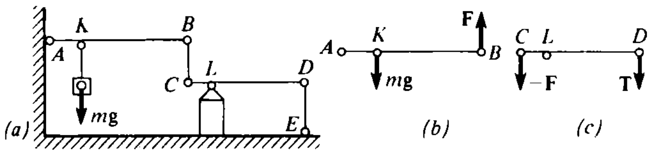


Fig. 128

Solution. The system of rods can be treated as two levers AB and CD connected by the rod BC (AB is the second-kind lever with the rotational axis passing through point A). To keep the load in equilibrium, we must apply at point B of the rod AB a force **F** whose magnitude can be found from the condition that the sum of the moments of all the forces about point A be equal to zero (Fig. 128*b*):

$$F \cdot AB - mg \cdot AK = 0, \quad \text{whence} \quad F = mg \cdot AK/AB = 235.2\ \text{N}.$$

The lever AB acts on the rod BC with the force equal and opposite to F. We translate this force along the line of its action to point C and consider the equilibrium of the first-kind lever CD (Fig. 128*c*). We write the equation for the moments about point L:

$$F \cdot CL - T \cdot DL = 0, \quad \text{whence} \quad T = F \cdot CL/DL = 78.4\ \text{N}.$$

Exercises

96*. Find the magnitude, direction, and the point of application of the resultant of six forces acting on a beam AB (Fig. 129), whose magnitudes are 30, 50, 20, 40, 70, and 20 kN respectively.

Answer. The force of 50 kN is directed downwards and is applied at a distance a to the left of point A.

97. (a) A 50-kg homogeneous beam rests on two supports. A 120-kg load is placed at 1/3 of the length of the beam from the left support, and an 80-kg load is placed at 0.6 of the length from it. Find the forces of pressure of the beam on the supports.

Answer. 1108 N and 1343 N.

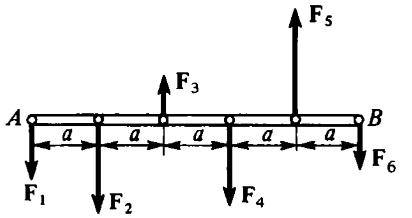


Fig. 129

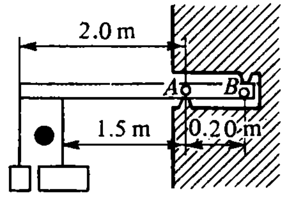


Fig. 130

(b) A homogeneous beam whose mass is 100 kg and length is $6a$ rests on two supports which are at a distance a from the ends of the beam. An 80-kg load lies on the left end of the beam, a 120-kg load lies on the right end, and a 220-kg load is placed at a distance a from the left support. An upward force of 600 N acts on the beam span at a distance a from the right support. Find the forces of pressure exerted by the beam on the supports.
Answer. 2900 N and 610 N.
98*. One end of a beam is embedded in the wall (Fig. 130). The mass of the protruding part of the beam is 200 kg and its length is 2.0 m. A 20-kg load is suspended from the beam end and an 80-kg load, at a distance of 1.5 m from the wall. The distance between the supports at points A and B is 0.20 m. Find the forces acting at these points on the supports. The mass of the embedded part of the beam should be neglected.
Answer. 20.6 kN and 17.6 kN.
99. (a) A horizontal rod whose length is 2 m and mass is 32 kg is supported by a vertical rod at a distance of 40 cm from its end. What force should be applied to the other end of the horizontal rod to keep it in the horizontal position? What force must be applied for this to the opposite end of the horizontal rod?
Answer. The upward force of 118 N, the downward force of 470 N.
(b) The length of the handle of a hand pump (Fig. 131) is $OA = 0.70$ m and

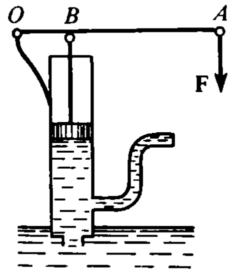


Fig. 131

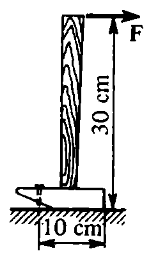


Fig. 132

$OB = 10$ cm. A force of 120 N is applied to the end A of the handle. What force is transmitted to the piston?
Answer. 720 N.

100*. (a) A 60-kg homogeneous rod is hinged to a vertical wall at its lower end and is kept in equilibrium by a horizontal rope attached to its upper end. Find the tension of the rope if the rod is at an angle of 30° to the wall.
Answer. 170 N.

(b) A nail is pulled out with a hammer as shown in Fig. 132. The dimensions of the hammer are given in the figure. Find the force of resistance offered by the nail if the force applied to the handle of the hammer is normal to it and equal to 120 N.
Answer. 360 N.

101*. A homogeneous bar having a mass of 16 t and a length of 2 m is hinged to a vertical wall at its upper end and is kept in equilibrium with the help of a 0.8-m long flexible rod forming an angle of 60° with the wall and hinged to the bar at 0.4 m from its lower end (Fig. 133). Find the force compressing the flexible rod. What must be the cross-sectional area of the rod in order that its relative compression should not exceed 0.001 and the stress be within the elastic limit? The modulus of elasticity of the rod material is 200 GPa and the elastic limit is 150 MPa. Assume that the rod does not experience a lateral bend.
Answer. 4.9 kN, 32.7 mm^2, in this case ε = 0.00075.

102*. A 150-kg load is uniformly lifted with the help of a system of levers by a force of 10 N (Fig. 134). Find the length of the upper lever AB. The friction in the bearings and the masses of the levers should be neglected.
Answer. $5b$.

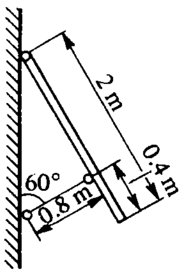


Fig. 133

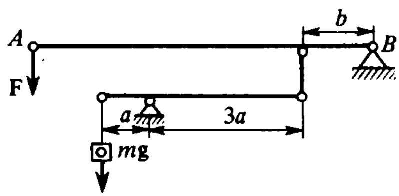


Fig. 134

1.33°. Types of Equilibrium

Three types of equilibrium are distinguished.

1. **Stable equilibrium** takes place when a body, after having been displaced slightly from the equilibrium position and left to itself, returns to this position.

2. **Unstable equilibrium** takes place when a body, after having been displaced slightly from the equilibrium position and left to itself, deviates from this position still further.

3. **Neutral equilibrium** occurs when a body, after having been displaced from the equilibrium position and left to itself, remains in the new position in equilibrium.

A body suspended at a point is in equilibrium if the point of application of the resultant of the forces of gravity, viz. the centre of mass C of a body (see Sec. 1.34), lies on the same vertical as the point of suspension O (Fig. 135), since in this case the line of action of the reaction of the suspender coincides with the line of action of the force of gravity.

The equilibrium of a body is (a) *stable* if the centre of mass in the equilibrium position is at the lowest level in comparison with all possible neighbouring positions (Figs. 136*a* and 137*a*); (b) *unstable* if its centre of mass is at the highest of all possible neighbouring positions (Figs. 136*b* and 137*b*), and (c) *neutral* if the centre of mass of the body is on the same horizontal level in all possible neighbouring positions of this body (Figs. 136*c* and 137*c*).

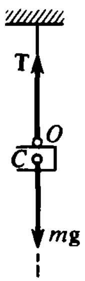


Fig. 135

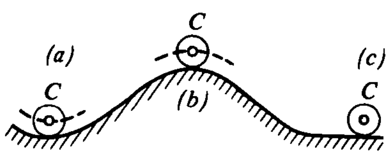


Fig. 136

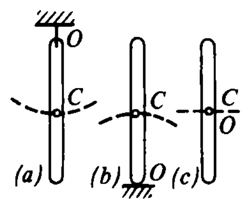


Fig. 137

The higher the centre of mass of a body, the higher its potential energy. Therefore, it can be said that the equilibrium of a body is (a) stable if the potential energy of the body in the equilibrium position has the *lowest* of all its possible values in the neighbouring positions of the body; (b) unstable if the potential energy of the body in the equilibrium position has the *maximum* value, and (c) neutral if the potential energy of the body *does not change* when the body is displaced from the equilibrium position to a neighbouring position.

The above statements can be easily explained by the fact that if the potential energy of a body in a given position is higher than in a neighbouring position, a certain work should be done to return the body from the neighbouring position to the initial position. On the other hand, if the potential energy of a body in a given position is lower than in a neighbouring position, after having been displaced from the equilibrium position, the body returns to it and does a certain work.

A body resting on a horizontal plane is in equilibrium if the vertical drawn through its centre of mass is *inside* (or on the boundary of) the bearing surface (Fig. 138). In this case, the force

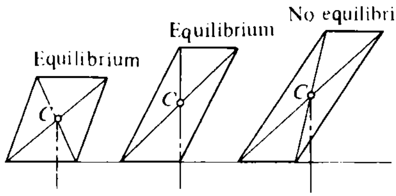


Fig. 138

of gravity is balanced by the reaction of the support. Otherwise, the force of gravity creates a torque about the boundary of the bearing surface and tips the body.

1.34°. Centre of Mass of a Body

The **centre of mass** of a body is the point of intersection of the straight lines along which the forces causing only translational motion of the body should be directed.

Experiments show that such a point exists in each body and any force which does not pass through this point causes a rotation of the body. In Fig. 139, the centre of mass is denoted by C. The forces $\mathbf{F}_1$-$\mathbf{F}_4$ cause only a translational motion of the body, while the force $\mathbf{F}_5$ causes, in addition to translation, a rotation of the body due to its torque about the centre of mass, $M_C = F_5h$.

It follows from the definition of translational motion (see Sec. 1.1) that in translational motion the accelerations of all parts of the body are equal in magnitude and have the same direction. In particular, the force of gravity causes a translational motion of

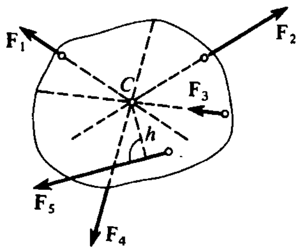


Fig. 139

a body since this force imparts the same acceleration to all parts of the body (if the size of the body is not too large). Consequently, in any position the resultant of the forces of gravity applied to all parts of a body passes through its centre of mass.

1.35°. Determination of the Centre of Mass for Bodies of Various Shapes

The position of the centre of mass of a body can be determined by dividing the body into parts of a simpler shape and then finding the points of application of the resultant of the forces of gravity acting on these parts.

Let us find the positions of the centres of mass of several simple figures.

1. *A thin homogeneous rectangular plate.* It follows from symmetry considerations that the centre of mass of such a plate coincides with its geometrical centre.

2. *A straight homogeneous rod* (Fig. 140). It is clear from symmetry considerations that the centre of mass of a rod is at its geometrical centre.

3. *A triangle* (Fig. 141*a*). Let us divide the area of the triangle into thin strips parallel to one of its sides. Since the centre of mass of each strip lies at its middle, the centre of mass of the system of all the strips, i.e. the centre of mass of the triangle, is on the median to the side of the triangle, which is parallel to the strips. Let us now divide the triangle into strips parallel to its other side. By arguing in a similar way, we find that the centre of mass of the triangle also lies on the median to this side of the triangle. Conse-

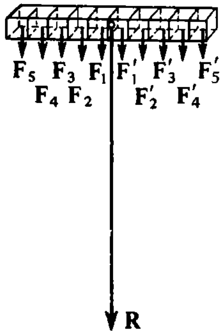


Fig. 140

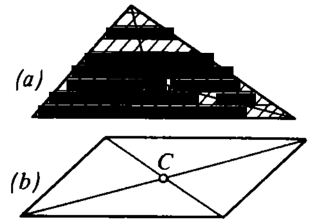


Fig. 141

quently, the centre of mass of a triangle lies at the point of intersection of its medians.

4. *A parallelogram* (Fig. 141*b*). We divide the parallelogram into two triangles by its diagonal. The centres of mass of each triangle lie on the other diagonal which is the median for both these triangles. Consequently, the resultant of the two forces of gravity acting on these triangles is on the same diagonal. We can now divide the parallelogram into two triangles by its other diagonal and show in a similar way that its centre of mass lies on the first diagonal. Consequently, the centre of mass of a parallelogram lies on the intersection of its diagonals.

5. *A homogeneous ring.* It is clear from symmetry considerations that the centre of mass of a homogeneous ring is at its geometrical centre.

6. *A homogeneous disc.* It follows from symmetry considerations that the centre of mass of a homogeneous disc is at its geometrical centre.

7. *A plate of an arbitrary shape.* We divide the plate into parts of simple shapes (triangles, rectangles or parallelograms) for which the positions of the centres of mass can be easily determined. Then we determine the centre of the parallel forces of gravity applied at the centres of mass of these simple figures, which are proportional to the areas of the corresponding figures, since the mass of a homogeneous plate is proportional to its area. In other words, we determine the point of application of the resul-

tant of the forces of gravity acting on individual parts of the plate. It was shown in Sec. 1.30 that the x-coordinate of this point, viz. the centre of mass of the plate, is

$$x_{c.m} = \frac{m_1 g x_1 + \dots + m_n g x_n}{m_1 g + \dots + m_n g} = \frac{m_1 x_1 + \dots + m_n x_n}{m_1 + \dots + m_n},$$

or

$$x_{c.m} = (m_1 x_1 + \dots + m_n x_n)/m, \tag{1.35.1}$$

where $m = m_1 + \dots + m_n$ is the mass of the entire plate. The y-coordinate of the centre of mass is determined similarly:

$$y_{c.m} = (m_1 y_1 + \dots + m_n y_n)/m. \tag{1.35.2}$$

As has been mentioned above, we can substitute the areas for the masses of the parts into formulas (1.35.1) and (1.35.2):

$$x_{c.m} = (s_1 x_1 + \dots + s_n x_n)/S, \quad y_{c.m} = (s_1 y_1 + \dots + s_n y_n)/S.$$

8. *Several simple homogeneous bodies.* It follows from symmetry considerations that the centres of mass of a parallelepiped (in particular, a cube), a ball, and a cylinder coincide with their geometrical centres.

These examples visually illustrate the following *general statements*: (a) if a body has a centre of symmetry, the centre of mass of the body coincides with its centre of symmetry; (b) if a body has a symmetry axis, its centre of mass lies on its axis; (c) if a body has a symmetry plane, the centre of mass of such a body lies in this plane, and (d) if a body has several axes or planes of symmetry, its centre of mass lies on their intersection. This is valid only for homogeneous bodies.

Problems with Solutions

103. A cylindrical rod whose length $L = 1.2$ m is made of three materials: the first part of $l_1 = 0.5$ m is made of iron, the second part of $l_2 = 0.3$ m is of copper, and the last part of $l_3 = 0.4$ m is of aluminium. Find the centre of mass of the rod. The densities of iron, copper, and aluminium are $\rho_1 = 7.8 \times 10^3$ kg/m^3, $\rho_2 = 8.9 \times 10^3$ kg/m^3, and $\rho_3 = 2.7 \times 10^3$ kg/m^3 respectively.

Solution. The problem can be solved by two methods.

Method I. The centre of mass of the rod lies on its axis since it is a symmetry axis. If the centre of mass is at point O (Fig. 142*a*), the rod is in equilibrium if we apply at this point a force **F** equal in magnitude to the sum of the forces of gravity

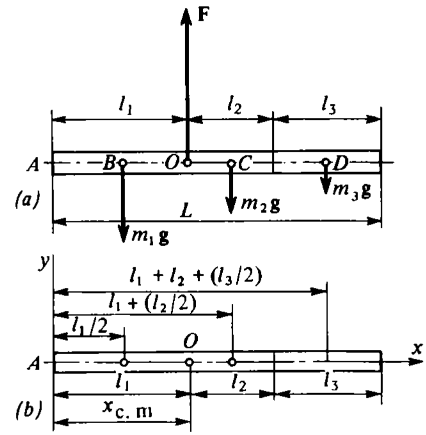


Fig. 142

$m_1\mathbf{g}$, $m_2\mathbf{g}$, and $m_3\mathbf{g}$, acting on its three parts, and having the opposite direction: $F = m_1 g + m_2 g + m_3 g$. Besides, the necessary condition for equilibrium is that the sum of the moments of all these forces about any point, say, point D, must be equal to zero:

$$m_1 g \cdot BD + m_2 g \cdot CD - F \cdot OD = 0.$$

This gives

$$OD = \frac{m_1 g \cdot BD + m_2 g \cdot CD}{F} = \frac{m_1 \cdot BD + m_2 \cdot CD}{m_1 + m_2 + m_3}.$$

The mass of the iron part of the rod is $m_1 = \rho_1 S l_1 = 0.390S$, the mass of the copper part is $m_2 = \rho_2 S l_2 = 0.267S$, and the mass of the aluminium part is $m_3 = \rho_3 S l_3 = 0.108S$. Consequently, $OD = 0.50$ m.

Method II. We choose the coordinate system as shown in Fig. 142*b* with the origin at point A. The abscissa of the centre of mass can be found from formula (1.35.1):

$$x_{c.m} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$= \frac{S l_1 \rho_1 l_1/2 + S l_2 \rho_2 (l_1 + l_2/2) + S l_3 \rho_3 (l_1 + l_2 + l_3/2)}{S l_1 \rho_1 + S l_2 \rho_2 + S l_3 \rho_3}$$

$$= \frac{l_1 \rho_1 l_1/2 + l_2 \rho_2 (l_1 + l_2/2) + l_3 \rho_3 (l_1 + l_2 + l_3/2)}{l_1 \rho_1 + l_2 \rho_2 + l_3 \rho_3} = 0.5 \text{ m}.$$

104. Find the centre of mass of a homogeneous square plate with a cut, shown in Fig. 143*a*.

Solution. Since the axis OO is an axis of symmetry of the plate, its centre of mass (point C) lies on this axis. Let us find the distance CO_1 from the centre of mass to the geometrical centre O_1 of the plate without cut. We make another cut in the plate, whose area is equal to that of the first cut and which is arranged symmetrically to the first cut about the vertical axis (Fig. 143*b*). We denote the mass

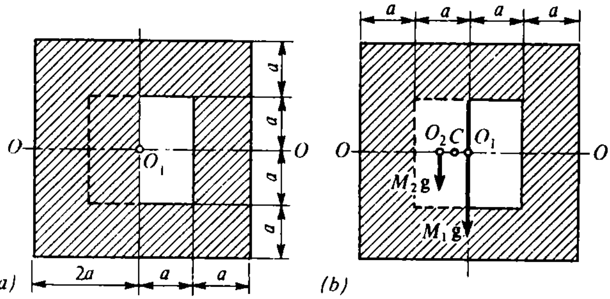


Fig. 143

of the remaining part of the plate by M_1 and the mass of the cut-out part of the plate by M_2. Then we can write

$$M_2 g \cdot O_2C = M_1 g \cdot O_1C, \quad \text{or} \quad M_2 \cdot O_2C = M_1 \cdot O_1C.$$

Since the masses of the parts of the plate are proportional to the areas of these parts, instead of masses we can take the areas $S_1 = (4a)^2 - (2a)^2 = 12a^2$ and $S_2 = a \cdot 2a = 2a^2$. Hence

$$\frac{O_1C}{O_2C} = \frac{S_2}{S_1}, \quad \text{or} \quad \frac{O_1C}{a/2 - O_1C} = \frac{2a^2}{12a^2} = \frac{1}{6}, \quad \text{whence} \quad O_1C = \frac{a}{14}.$$

105. A solid homogeneous cylinder whose mass $m = 50$ kg, height $h = 20$ cm, and the diameter of the base $d = 10$ cm is placed at the middle of a board (Fig. 144). The board has a rotational axis at one end, while the other end is lifted by a vertical force. Will the cylinder tip over or slide down the board if the coefficient of friction between the cylinder and the board is $f = 0.3$?

Solution. We denote by α the angle between the board and the horizontal at which the cylinder starts to slide down and by α_1 the angle at which it is tipped over. Let us find these angles. As the left end of the board is lifted and the angle α increases, the tangential force $F = mg \sin\alpha$ also increases, while the force $N = mg\cos\alpha$ of the normal pressure decreases. At the same time, the maximum possible friction fN also decreases:

$$F_{fr} \leq fN, \quad \text{or} \quad F_{fr} \leqslant fmg\cos\alpha.$$

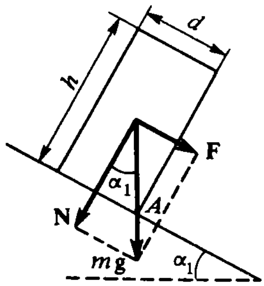


Fig. 144

The cylinder starts to slide down when the component of the force of gravity which is parallel to the board slightly exceeds the maximum friction:

$$F \geqslant F_{fr}, \quad \text{i.e.} \quad mg \sin \alpha \geqslant fmg \cos \alpha, \quad \text{whence} \quad \tan \alpha \geqslant f = 0.3.$$

The cylinder starts to tip over when the line of action of the force of gravity passes through the boundary point A of the bearing surface of the cylinder. It can be seen from Fig. 144 that $\tan \alpha_1 = 0.5d/0.5h = 0.5$. Consequently, $\alpha < \alpha_1$, and the cylinder slides down earlier.

Exercises

103. (a) A homogeneous stepped cylindrical rod of 100-cm length has a diameter of 10 cm over 30 cm of length, 15 cm over 40 cm of length, while the remaining part has a diameter of 20 cm. Determine the centre of mass of the rod if its longitudinal axis is straight.

Answer. The centre of mass lies on the cylinder axis at 65.3 cm from the end of the part with a smaller diameter.

(b) A 100-cm rod having a constant cross-sectional area is made of four materials: the first 20 cm are made of iron, the next 30 cm of lead, the following 20 cm of aluminium, and the remaining part is made of copper. Find the centre of mass of the rod. The densities of iron, lead, aluminium, and copper are 7.9×10^3 kg/m^3, 11.4×10^3 kg/m^3, 2.7×10^3 kg/m^3, and 8.9×10^3 kg/m^3 respectively.

Answer. At 44 cm from the end of the iron part.

104*. (a) Find the centre of mass of a rectangular plate with a cut, shown in Fig. 145.

Answer. At a distance of $0.125a$ to the left of point O.

(b) Find the centre of mass of a circular plate of radius r (Fig. 146) with a circular cut of radius $r/2$ and a cover plate of the same thickness as that of the plate, having the radius $r/2$.

Answer. At a distance of $r/4$ to the left of the centre of the plate.

105. (a) A cylinder having a radius of 10 cm and a height of 40 cm rests on an inclined board whose length is 30 cm. By what height should the end of the board be

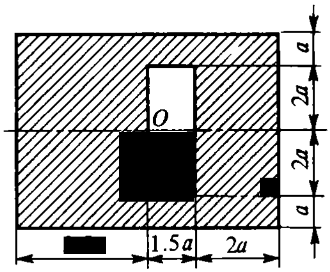


Fig. 145

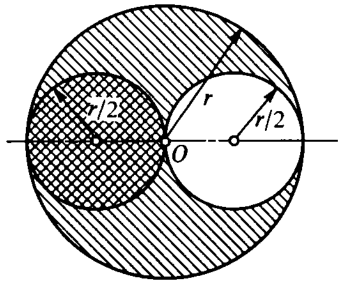


Fig. 146

lifted to make the cylinder tip over? It is assumed that the cylinder is held from sliding down.
Answer. ~ 13.4 cm.

(b) Find the work that must be done to tip a column having a square cross section of $50 \times 50\ \text{cm}^2$ and a height of 2.4 m about the edge of its base if the density of the column material is $3 \times 10^3\ \text{kg/m}^3$.
Answer. ~ 460 J.

2. FLUIDS

2.1. Pressure

If a force **F** acts on a certain surface element, the component of the force normal to this surface element is called the **force of normal pressure**, or simply the **force of pressure**:

$$F_{pr} = F \cos \alpha,$$

where α is the angle between vector **F** and the normal to the surface element (Fig. 147).

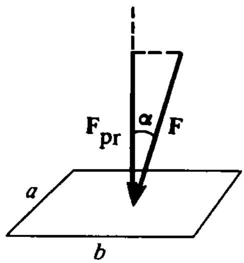


Fig. 147

Pressure is a scalar quantity equal to the ratio of the magnitude of the force of pressure to the area of the surface on which this force acts:

$$p = F_{pr}/S = (F \cos \alpha)/S.$$

The force of pressure of a *fluid* on a surface is always normal to this surface. Therefore, the pressure of a fluid is defined as

$$p = F/S.$$

The SI unit of pressure is a **pascal** (Pa): 1 Pa = 1 N/m^2.

Sometimes, the following out-of-system units are employed in physics and engineering: (a) a physical atmosphere (atm), 1 atm = 1.013×10^5 Pa (this is the so-called normal atmospheric pressure equal to the air pressure at a temperature of 15 °C or the pressure of a water column whose height is 10.33 m); (b) millimetre of mercury column (mmHg) or torr (after Torricelli): 1 atm = 760 mmHg, and (c) bar: 1 bar = 10^5 Pa, 1 atm = 1.013 bar.

2.2. Pascal's Law

Pascal's law is formulated as follows: *the pressure applied at any point of a fluid at rest is transmitted without loss to all other parts of the fluid.*

Figure 148 shows a vessel with a liquid. A piston of mass M is on the surface of the liquid. The pressure exerted on a unit area of

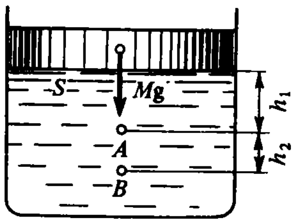


Fig. 148

this surface due to the weight of the piston is

$$p = Mg/S,$$

where S is the area of the surface of the liquid below the piston. The pressure at point A lying at a depth h_1 is equal to the sum of the pressure on the surface and the pressure exerted by the above-lying layer of liquid. If the force of gravity acting on this layer of liquid is $m_1 g$, we have

$$p_A = p + m_1 g/S = p + \rho g h_1.$$

Thus, the pressure on any surface element placed at point A, regardless of whether this element is vertical, horizontal or at a certain angle to the horizontal, is given by the above expression.

The pressure at point B is equal to the sum of the pressure at point A and the pressure of the column of liquid between points A and B, and so on.

2.3. Hydraulic Press

Hydraulic presses are used for obtaining strong compressive forces at small displacements, for example, for pressing various materials, for punching holes in metallic sheets, or for a strength test of materials.

A schematic diagram of a hydraulic press is shown in Fig. 149.

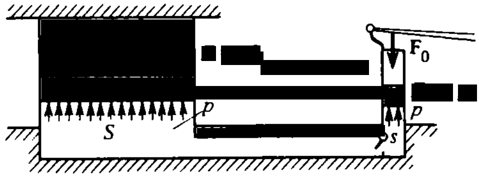


Fig. 149

It consists of a larger-diameter cylinder and a pump with a smaller-diameter cylinder. A body that must be compressed is placed above the larger-diameter piston, where it is pressed between the upper immobile plane, which is usually fixed to the support of the press, and the piston. The body is compressed with the help of the pump which delivers oil to the cylinder through a pipe of small diameter. After the end of compression, the liquid flows from the cylinder of the press back to the cylinder of the pump (this stage is not shown in the figure).

A hydraulic press operates on the basis of Pascal's law. A force F_0 acts on the piston of the pump. If the surface area of the piston is s, the pressure below the pump piston is

$$p = F_0/s.$$

The pressure on any area element in the larger cylinder of the press will be the same. If the cross-sectional area of this cylinder is S, the total force acting on the larger piston is

$$F = pS.$$

The gain in force, i.e. the ratio of the force compressing a body to the force acting on the piston of the pump is

$$F/F_0 = pS/ps = S/s.$$

The gain in force is equal to the ratio of areas S and s. The displacement of the larger piston is less than the displacement of the smaller piston by the same factor.

2.4. Pressure of a Fluid on the Bottom and Walls of a Vessel. Law of Communicating Vessels

The pressure of a fluid on a small area element lying at a depth h does not depend on its orientation and is equal in magnitude to the force of gravity acting on a vertical column of fluid with a unit area of the base and a height equal to the distance from the middle of the area element to the level of fluid in the vessel:

$$p = \rho g h,$$

where ρ is the density of the fluid. The pressure of a liquid on the walls A and B of a vessel (Fig. 150) and on any area element C lying at the same depth h is the same and equal to $\rho g h$. This explains

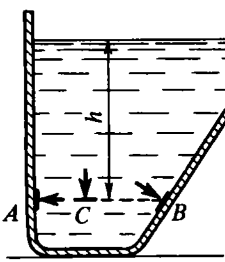


Fig. 150

the so-called "hydrostatic paradox": the pressure on the bottom of a vessel does not depend on the shape of the vessel and is determined by the level of liquid in the vessel.

The columns of liquids at rest in communicating vessels, balancing each other, have such heights that they exert the same pressure on their bases:

$$p_1 = p_2 = p.$$

Let us prove this. If we place a mobile plug into the pipe connecting two vessels, it will not move, since the liquid is in equilibrium. This is possible only if the pressure on both sides of the plug is the same, i.e. $p_1 = p_2$. Since $p_1 = \rho_1 g h_1$ and $p_2 = \rho_2 g h_2$, we have

$$\rho_1 h_1 = \rho_2 h_2, \quad \text{whence} \quad h_1/h_2 = \rho_2/\rho_1.$$

In a particular case of a homogeneous liquid, we have

$$\rho_1 = \rho_2 = \rho, \quad h_1 = h_2 = h.$$

These equalities express the **law of communicating vessels**: (a) *a homogeneous liquid in communicating vessels has the same level* (Fig. 151*a*) and (b) *the heights of mutually balanced columns of different liquids in communicating vessels are inversely proportional to the densities of the liquids* (Fig. 151*b*).

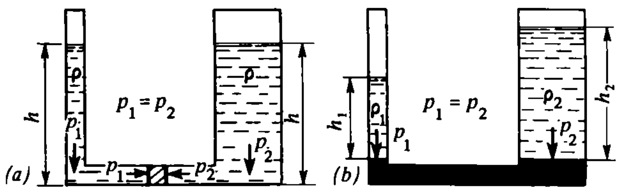


Fig. 151

2.5. Atmospheric Pressure. Barometers

Atmospheric air exerts on all bodies a pressure equal to the force of gravity acting on the air column with a unit area of the base. This pressure, as the pressure in a liquid, is also transmitted without loss in all directions (according to Pascal's law).

Torricelli's experiment demonstrated the presence of atmospheric pressure and made it possible to measure it for the first time. This experiment consists in the following procedure. A glass tube about one metre long, sealed at one end, is filled with mercury. The open end of the tube is chucked, after which the tube is turned upside down, is immersed by its chucked end in a mercury-containing vessel, and left to itself. The level of mercury in the tube becomes lower (a part of mercury flows into the vessel) and stays so that the mercury column in the tube is higher than the level of mercury in the vessel (Fig. 152). This level does not depend on whether the tube is strictly vertical or inclined by a certain angle. Hence it follows that the pressure in a fluid depends only on the level of fluid and does not depend on the shape of the vessel.

The atmospheric pressure is measured by instruments called **barometers**. A *mercury barometer* is a Torricellian tube with a scale attached to it to measure the height of the mercury column that balances the atmospheric pressure. In a *siphon barometer* (Fig. 153), the mercury column balancing the atmospheric

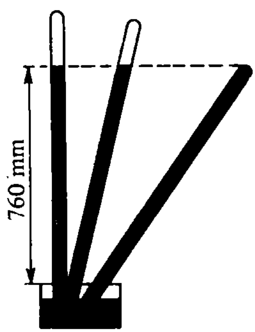


Fig. 152

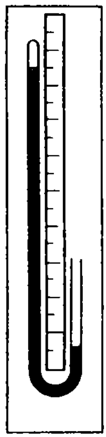


Fig. 153

pressure is determined by the difference in the levels of mercury in closed and open arms of a U-shaped tube. A *metallic barometer*, or *aneroid* (Fig. 154), consists of an evacuated flat cylindrical closed box with an elastic corrugated lid (intended for increasing its mobility). The lid is connected through a system of levers with a pointer that indicates the atmospheric pressure on a scale.

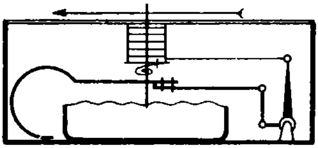


Fig. 154

2.6. Archimedean Principle

Archimedean principle consists in the following: *a body immersed in a fluid wholly or partially experiences an upward force equal to the force of gravity acting on the fluid which would fill the space occupied by the immersed part of the body.* This force is called the **buoyancy**:

$$F_b = \rho_{fl} g V_{d.fl}.$$

Here F_b is the buoyancy directed upwards, ρ_{fl} is the density of the fluid, and $V_{d.fl}$ is the volume of displaced fluid. The buoyancy, or Archimedean force, is applied at the centre of mass of the volume of the fluid displaced by the body.

A body *floats* in a fluid if the buoyancy is *equal* in magnitude to the force of gravity mg acting on the body (here m is the mass of the body):

$$F_b = mg = \rho g V$$

(here ρ is the average density of the body and V is its volume). Substituting the expression for F_b, we obtain

$$\rho_{fl} V_{d.fl} = \rho V, \quad \text{or} \quad \rho/\rho_{fl} = V_{d.fl}/V.$$

Since $V_{d.fl} \leqslant V$, the condition of floatation for a given body and fluid can be written in the form

$$\rho \leqslant \rho_{fl}, \quad \text{or} \quad \rho/\rho_{fl} \leqslant 1.$$

If a body floats in a fluid, the ratio of the part of the volume of the body submerged in the fluid to the entire volume of the body is equal to the ratio of the average density of the body to the density of the fluid.

This principle is used in the construction of *areometer* (or

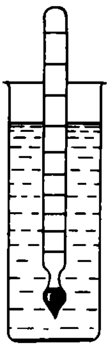


Fig. 155

hydrometer), viz. an instrument for measuring the density of a fluid. An areometer (Fig. 155) is a sealed glass tube floating in the vertical position due to a load (mercury or lead pellets) placed in its lower part. The depth of submergence of the instrument is the larger, the smaller the density of the fluid in which it is immersed. A graduated scale fixed to the areometer makes it possible to measure the density of the fluid in which it is immersed.

Problems with Solutions

106. Find the pressure in a lake having a depth $h = 4.5$ m and express it in the out-of-system units.

Solution. The pressure at the depth h is $p_0 + p_h$, where p_0 is the atmospheric pressure (the pressure of air at the water surface) and p_h is the pressure exerted by the water column of height h. The atmospheric pressure $p_0 = 1$ atm $= 1.013 \times 10^5$ Pa. The pressure p_h of the water column of height h is equal to $\rho g h$ (ρ is the density of water). In SI, $p_h = 4.41 \times 10^4$ Pa. Consequently,

$$p = p_0 + p_h = 1.454 \times 10^5 \text{ Pa} = 1.43 \text{ atm} = 1.454 \text{ bar}.$$

In millimetres of mercury column, we have $p = 760 \times 1.43 = 1090$ mmHg.

107. The height of water in a vessel is $h = 5$ m. The vessel wall of width $b = 1.5$ m is at an angle $\alpha = 60°$ to the vertical. Find the force of pressure of water on the wall.

Solution. The air pressure at the level of water in the vessel is $p_0 = 1$ atm $= 1.013 \times 10^5$ Pa. The pressure at the bottom of the vessel is $p = p_0 + \rho g h$. Since pressure varies with height linearly (Fig. 156), the average pressure is

$$p_{av} = (p_0 + p)/2 = p_0 + \rho g h/2.$$

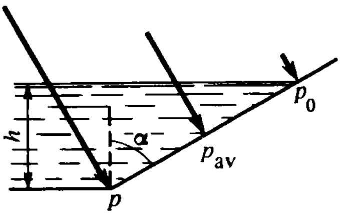


Fig. 156

The area of the wall is $S = bh/\cos\alpha$. According to Pascal's law, the force of pressure is

$$F = p_{av}S = \left(p_0 + \frac{\rho g h}{2}\right)\frac{bh}{\cos\alpha} = 1887\ \text{kN}.$$

108. A load of mass $m = 80$ t must be compressed during 0.5 min with the help of a hydraulic press with a piston area ratio $s/S = 1/50$ and an efficiency $\eta = 75\%$ so that the maximum force of pressure exerted by the load on the upper plate must attain the value $F = 1$ MN. The load is then compressed by $\Delta h = 30$ cm. Find the work done by the engine, assuming that the vertical strain of the load is proportional to the compressive force. Find the average and maximum powers of the engine and the number of strokes of the smaller piston if the smaller piston is lowered by $H = 10$ cm per stroke.

Solution. Since the vertical strain of the load linearly depends on the force acting on it, the work of the force of pressure exerted by the larger piston is $A = F_{av}\Delta h$, where $F_{av} = (F_{max} + F_{min})/2$ is the average force of pressure exerted by the larger piston. The minimum force of pressure exerted by the larger piston is $F_{min} = mg$, while the maximum force $F_{max} = F + mg$. Therefore, $F_{av} = mg + F/2$. The work done by the engine is

$$W = \frac{A}{\eta} = F_{av}\frac{\Delta h}{\eta} = \left(mg + \frac{F}{2}\right)\frac{\Delta h}{\eta} = 510\ \text{kJ}.$$

The average power developed by the engine is $N_{av} = W/t = 17$ kW. Since the engine power is proportional to the force acting on the load, the maximum power can be found from the following relation: $N_{max}/N_{av} = F_{max}/F_{av}$:

$$N_{max} = \frac{N_{av}F_{max}}{F_{av}} = \frac{N_{av}(F + mg)}{mg + F/2} = 23.6\ \text{kW}.$$

The height by which the larger piston is raised during a stroke of the smaller piston is $h_1 = Hs/S$. Therefore, the required number of strokes of the smaller piston is

$$n = \Delta h/h_1 = \Delta hS/Hs = 150.$$

109. Two communicating vessels of the same cross section are connected by a pipe whose cross-sectional area s is equal to 1/10 of the cross-sectional area S of the

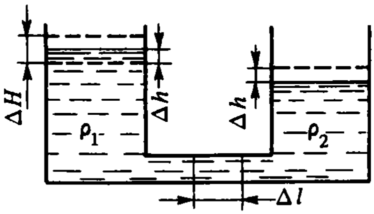


Fig. 157

vessels (Fig. 157). One of the vessels is filled with water and the other, with oil. The densities of water and oil are $\rho_1 = 1 \times 10^3$ kg/m^3 and $\rho_2 = 0.85 \times 10^3$ kg/m^3. By what distance Δl is the interface between the liquids in the connecting pipe shifted if the layer of the same oil having a thickness $\Delta H = 2.0$ cm is poured over the surface of water?

Solution. After the layer of oil has been poured over the surface of water, this surface lowers by Δh. Since the cross-sectional area of the communicating vessels is the same, the surface of oil in the other vessel rises by the same height. The displacement Δl can be determined from the condition that an increase in pressure on both sides should be the same: $\Delta p_1 = \Delta p_2$. Here $\Delta p_1 = \Delta H \rho_2 g - \Delta h \rho_1 g$, $\Delta p_2 = \Delta h \rho_2 g$. Hence $\Delta H \rho_2 - \Delta h \rho_1 = \Delta h \rho_2$, whence $\Delta h = \Delta H \rho_2/(\rho_1 + \rho_2)$. Since $\Delta l/\Delta h = S/s$, we have

$$\Delta l = \Delta h \frac{S}{s} = \Delta H \frac{\rho_2}{\rho_1 + \rho_2} \frac{S}{s} = 9.2 \text{ cm}.$$

110. A brass body of mass $m = 1$ kg is suspended from a spring balance and immersed in water. The spring balance indicates $F = 8.5$ N. Find the mass of copper contained in the brass if the densities of copper and zinc are $\rho_c = 8.8 \times 10^3$ kg/m^3 and $\rho_z = 7.2 \times 10^3$ kg/m^3.

Solution. A body immersed in water is acted upon by the buoyancy $\mathbf{F}_b$ and the force of gravity $m\mathbf{g}$. Since the reading of the spring balance is F, we have $F_b = mg - F$. On the other hand, $F_b = \rho_0 g(V_c + V_z)$, where V_c and V_z are the volumes of copper and zinc in the brass body. We equate the right-hand sides: $mg - F = \rho_0 g(V_c + V_z)$. Consequently, the volume of the body is $V_c + V_z = (mg - F)/\rho_0 g$, where ρ_0 is the density of water. The subsequent solution will be easier if we calculate the volume of the body at this stage: $V_c + V_z = 133$ cm^3. Further, $m_c/\rho_c + m_z/\rho_z = V_c + V_z$. Since $m_z = m - m_c$, we have $m_c/\rho_c + (m - m_c)/\rho_z = V_c + V_z$. Consequently,

$$m_c/\rho_c - m_c/\rho_z = (V_c + V_z) - m/\rho_z, \quad \text{whence} \quad m_c \approx 240 \text{ g}.$$

111. A piece of metal of mass $m = 0.8$ kg, suspended from a spring balance, is immersed in petrol. The spring balance indicates $F_1 = 7.2$ N. When some other solution is taken, the spring balance shows $F_2 = 7.5$ N. Find the density ρ_2 of the other solution and the density ρ of the metal if the density of petrol is $\rho_1 = 0.7 \times$

10^3 kg/m^3 and the density of metal is known to be higher than the densities of petrol and the other solution.

Solution. Since the metal is completely submerged in liquid in both cases, the volumes of displaced liquids are equal, the ratio of the buoyancies being equal to the ratio of their densities: $F_{b\,2}/F_{b\,1} = \rho_2/\rho_1$. On the other hand, $F_{b\,2} = mg - F_2$ and $F_{b\,1} = mg - F_1$. Consequently, $(mg - F_2)/(mg - F_1) = \rho_2/\rho_1$. Hence the density of the other solution is

$$\rho_2 = \rho_1(mg - F_2)/(mg - F_1) \approx 0.37 \times 10^3 \text{ kg/m}^3.$$

The density of the metal can be found from the fact that $mg = F_1 + F_{b\,1}$, $V_p = V_m$, where V_p is the volume of displaced petrol and V_m is the volume of metal. Thus we have $mg - F_1 = F_{b\,1}$, where $F_{b\,1} = \rho_1 g V_m = \rho_1 gmg/g\rho = mg\rho_1/\rho$. This gives $mg - F_1 = mg\rho_1/\rho$, whence the density of the metal is

$$\rho = \rho_1 mg/(mg - F_1) \approx 8.6 \times 10^3 \text{ kg/m}^3.$$

112. A piece of cork floats in a tank with kerosene. What part of its volume is submerged in kerosene? The densities ρ_c and ρ_k of the cork and kerosene are 0.2×10^3 kg/m^3 and 0.8×10^3 kg/m^3.

Solution. Since the cork floats (is in equilibrium), the buoyancy acting on the cork is equal to the force of gravity: $\rho_k g V_0 = \rho_c g V$, where V is the total volume of the cork and V_0 is the volume of the submerged part. Hence $V_0/V = \rho_c/\rho_k = 0.25$.

113. A hollow cylinder of cross-sectional area $S = 2.5$ m^2 floats in kerosene. To make this cylinder float in sea water with the same draught (the depth of submergence), a load of mass $m = 100$ kg must be placed into it. Find the mass M of the cylinder and the depth h to which it is submerged. The density of kerosene is $\rho_k = 0.8 \times 10^3$ kg/m^3.

Solution. The buoyancy acting on the cylinder in kerosene is equal in magnitude to the force of gravity acting on it: $\rho_k ghS = Mg$, while the buoyancy acting on the cylinder in water is equal to the force of gravity acting on the cylinder and the load: $\rho_0 ghS = (M + m)g$, where ρ_0 is the density of water. Dividing the second equality by the first, we obtain

$$\rho_0/\rho_k = (M + m)/M, \quad \text{whence} \quad M = m/(\rho_0/\rho_k - 1) = 400 \text{ kg}.$$

The draught of the cylinder is $h = (M + m)/\rho_0 S = 0.2$ m.

114. A cork lifebelt has a mass $m = 3.2$ kg. Find the lifting force F_{lif} of the lifebelt in sea water. The densities ρ_c and ρ_0 of the cork and sea water are 0.2×10^3 kg/m^3 and 1.03×10^3 kg/m^3.

Solution. The buoyancy must be equal in magnitude to the sum of the lifting force and the force of gravity acting on the lifebelt: $F_b = F_{lif} + mg$. Here $F_b = V_c\rho_0 g$, where the volume of the cork lifebelt is $V_c = m/\rho_c$. Consequently, $F_b = mg\rho_0/\rho_c$. Substituting this expression for the buoyancy into the first equation, we get

$$mg\rho_0/\rho_c = F_{lif} + mg, \quad \text{whence} \quad F_{lif} = mg(\rho_0/\rho_c - 1) = 130 \text{ N}.$$

115. The part of an iceberg above water has a volume $V_a = 500$ m^3. Find the total volume V of the iceberg if the densities ρ_i and ρ_0 of ice and sea water are 0.92×10^3 kg/m^3 and 1.03×10^3 kg/m^3.

Solution. The buoyancy acting on the iceberg is equal in magnitude to its force of gravity: $F_b = mg$. On the other hand, $F_b = (V - V_a)\rho_0 g$, while $mg = V\rho_i g$. Consequently,

$$(V - V_a)\rho_0 = V\rho_i, \quad \text{whence} \quad V = V_a\rho_0/(\rho_0 - \rho_i) = 4680 \text{ m}^3.$$

116. What is an increase in the draught of a motor ship at a sea wharf as a result of loading if the mass of the load is $\Delta m = 2.5$ t and the cross-sectional area of the motor ship at the waterline is $S = 4000$ m^2? The density of sea water is $\rho_0 = 1.03 \times 10^3$ kg/m^3.

Solution. The volume V_d of water displaced by the motor ship as a result of loading is $\Delta m/\rho_0$. On the other hand, $V_d = S\,\Delta h$, where Δh is an increase in the motor ship draught. Consequently, $S\,\Delta h = \Delta m/\rho_0$, whence

$$\Delta h = \Delta m/\rho_0 S = 0.62 \text{ m}.$$

117. A balloon of volume $V = 4000$ m^3 is filled with helium. The total mass M of the balloon, equipment, and crew is 3 t. At an altitude where the density of air is $\rho_a = 1.2$ kg/m^3, helium completely fills the balloon at a density $\rho_h = 0.18$ kg/m^3. Find the maximum mass of the load that can be lifted by the balloon.

Solution. The buoyancy is equal in magnitude to the sum of the forces of gravity acting on the balloon (with the crew and equipment), helium, and load: $F_b = Mg + m_h g + mg$. Here $F_b = V\rho_a g$ and $m_h = V\rho_h$. Consequently, $V\rho_a g = Mg + V\rho_h g + mg$, whence

$$m = [V(\rho_a - \rho_h) - M] = 1.058 \times 10^3 \text{ kg}.$$

118. A balloon volume $V = 20$ m^3 filled with helium rises to a height $h = 180$ m during a time $t = 0.5$ min. The mass M of the balloon with the equipment and cage is 12 kg. Find the mass of the load lifted by the balloon, assuming that the densities ρ_a and ρ_h of air and helium are constant up to the height $h = 180$ m and equal to 1.29 kg/m^3 and 0.18 kg/m^3.

Solution. Under the action of the resultant of all the vertical forces acting on the balloon, it moves upwards with an acceleration. Since the density of air is assumed to be constant over the entire distance, the buoyancy F_b is also constant. This means that the resultant R of all the vertical forces is also constant, and hence the motion of the balloon is uniformly accelerated. Its acceleration can be found from the formula $h = at^2/2$: $a = 2h/t^2 = 0.40$ m/s^2.

The resultant of all the vertical forces is directed vertically upwards:

$$R = F_b - Mg - m_h g - mg,$$

where the buoyancy $F_b = V\rho_a g$ and the mass of helium is $m_h = V\rho_h$. In accordance with Newton's second law,

$$a = \frac{R}{M + m + m_h} = \frac{F_b - (M + m_h + m)g}{M + m + m_h} = \frac{V\rho_a - M - V\rho_h - m}{M + V\rho_h + m}\, g,$$

whence

$$m = V\rho_a \frac{g}{a+g} - V\rho_h - M = 9.5 \text{ kg}.$$

119. A thick layer of liquid having a density $\rho_2 = 0.7 \times 10^3$ kg/m^3 is poured over the surface of a liquid having a density $\rho_1 = 1.0 \times 10^3$ kg/m^3. The liquids do not mix. Find the part of volume V of a body having a density $\rho_b = 0.9 \times 10^3$ kg/m^3 that will be submerged in the denser liquid (Fig. 158).

Solution. We write the condition of equilibrium for a floating body: $F_b = mg$, where $m = V\rho_b$. The buoyancy $F_b = V'\rho_1 g + (V - V')\rho_2 g$, where V' is the volume of the part of the body submerged in the denser liquid. Hence

$$V\rho_b = V'\rho_1 + V\rho_2 - V'\rho_2, \quad \text{whence} \quad V'/V = (\rho_b - \rho_2)/(\rho_1 - \rho_2) = 2/3.$$

120. A homogeneous rod of length $l = 1.2$ m, made of a material with a density $\rho = 0.8 \times 10^3$ kg/m^3, is hinged at one end at a height $h = 40$ cm above the water level, its other end being submerged in water (Fig. 159). At what depth h_0 is the

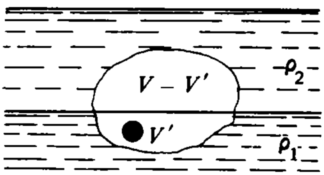


Fig. 158

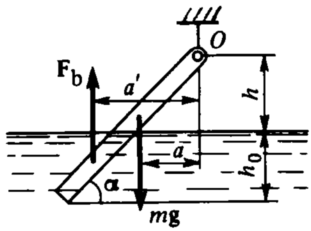


Fig. 159

lower end of the rod in equilibrium? What is the angle α formed by the rod with the horizontal?

Solution. In equilibrium, the moment of the buoyancy about the hinge O, which rotates the rod clockwise (see Fig. 159), must be equal to the moment of the force of gravity of the rod, which rotates the rod counterclockwise:

$$F_b a' = mga, \tag{1}$$

where the arm of the force of gravity of the rod is $a = [(h + h_0)/2] \cot\alpha$, while the arm of the buoyancy is $a' = (h + h_0/2)\cot\alpha$. The buoyancy $F_b = \rho_0 g S h_0/\sin\alpha$, where S is the cross-sectional area of the rod and $\rho_0 = 1 \times 10^3$ kg/m^3 is the density of water. The force of gravity $mg = \rho g S(h + h_0)/\sin\alpha$. Substituting these quantities into Eq. (1), we obtain

$$\rho_0 g S \frac{h_0}{\sin\alpha}\left(h + \frac{h_0}{2}\right)\cot\alpha = \rho g S \frac{h + h_0}{\sin\alpha}\frac{h + h_0}{2}\cot\alpha,$$

$$\rho_0 h_0(h + h_0/2) = \rho(h + h_0)^2/2.$$

After transformations, we obtain a quadratic equation

$$h_0^2 + 2hh_0 - \rho h^2/(\rho_0 - \rho) = 0.$$

Solving this equation, we get $h_0 = -h \pm h\sqrt{1 + \rho/(\rho_0 - \rho)}$. The value of h_0 cannot be negative. Consequently, $h_0 = h[-1 + \sqrt{\rho_0/(\rho_0 - \rho)}] \approx 0.50$ m; $\sin\alpha = (h + h_0)/l = 0.75$, and $\alpha = \arcsin 0.75$.

121. A thin aluminium rod of length $l = 50$ cm falls from the deck of a stationary sea motor ship which is at $h = 1.5$ m above the water level (Fig. 160). With what

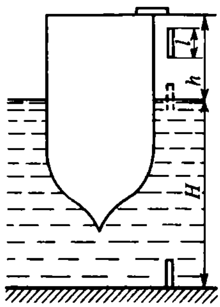


Fig. 160

velocity v does the rod reach the bottom if the depth $H = 3.0$ m? The densities ρ_0 and ρ of sea water and aluminium are 1.03×10^3 kg/m^3 and 2.65×10^3 kg/m^3. The resistance offered by air and water to the motion of the rod should be neglected. The rod remains in the vertical position during the fall.

Solution. The change in the kinetic energy of the rod is equal to the work of the resultant of all the forces acting on the rod. Before the rod enters water, it experiences the action of only the force of gravity equal to ρglS, where S is the cross-sectional area of the rod. The work done by this force over the first segment of the path (to the level of water) is

$$A_1 = \rho glhS.$$

When the rod enters water, it is acted upon by two forces, viz. the force of gravity and the buoyancy. Since they have opposite directions, their resultant $R = \rho glS - F_b$.

As the rod submerges in water, the buoyancy varies linearly from zero to $\rho_0 glS$. Therefore, the resultant R also varies linearly over this segment of the path from ρglS to $\rho glS - \rho_0 glS$. The work done by the force R over this segment can be calculated as the work done by the average force $R_{av} = (\rho glS + \rho glS - \rho_0 glS)/2 = (\rho - \rho_0/2)glS$. This work is

$$A_2 = R_{av}l = (\rho - \rho_0/2)gl^2S.$$

As the rod sinks further, the force R remains unchanged: $R = \rho glS -$

$\rho_0 glS = (\rho - \rho_0)glS$. The work done by this force is

$$A_3 = R(H - l) = (\rho - \rho_0)gl(H - l)S.$$

We equate the total work done by the forces acting on the rod to the change in its kinetic energy:

$$A_1 + A_2 + A_3 = mv^2/2,$$

$$\rho glhS + (\rho - \rho_0/2)gl^2S + (\rho - \rho_0)gl(H - l)S = \rho lSv^2/2,$$

whence $v = \sqrt{2g[h + H - \rho_0(H - 0.5l)/\rho]} = 8.4$ m/s.

122. A body of mass m, whose density is less than the density of water, is immersed in a cylindrical vessel containing water. What is the change in the level of water in the vessel if the area of its bottom is S and water does not overflow from the vessel? The density of water is ρ_0.

Solution. The force of pressure acting on the bottom of the vessel is $F = Mg$, where M is the mass of water in the vessel. When the body is immersed in it, the force of pressure changes by $\Delta F = mg$. Since the density of the body is less than the density of water, the body floats and does not sink to the bottom. This means that the change in the pressure on the bottom is due to the change in the level of water in the vessel, i.e. $\Delta p = \rho_0 g\, \Delta h$. On the other hand, $\Delta p = \Delta F/S = mg/S$. Consequently, $\rho_0 g\, \Delta h = mg/S$, whence $\Delta h = m/\rho_0 S$.

Exercises

106. (a) Find the pressure at a sea depth of 30 m and express it in out-of-system units. The density of sea water is 1.03×10^3 kg/m^3.

Answer. 3.03×10^5 Pa = 3.0 atm = 3.03 bar = 2280 mmHg.

(b) The pressure of water on the bottom of a lake is 1.27×10^5 Pa. Find the depth of the lake at this place.

Answer. 13.0 m.

107. (a) A hole of 10-cm^2 area has been formed in the underwater part of a river ship at a depth of 5.0 m. What is the force of pressure exerted by water on a patch covering the hole from inside?

Answer. 49 N.

(b) Find the force of pressure on the front wall of a dam, inclined at 30° to the vertical, which is transmitted from the upper reach if the depth of water is 20 m and the width of the wall is 50 m.

Answer. 230 MN.

(c*). A pipe whose diameter is 20 cm is inserted into the tightly fitting cover of a barrel. The barrel height is 2.0 m. Water is poured into the barrel with the pipe up to 0.5 m. Find the pressure of water on the barrel wall at the bottom. What is the pressure if 50 l of water are poured additionally into the pipe?

Answer. 24.5 kPa, 28.4 kPa.

108. (a) A bar having a thickness of 20 cm and a mass of 20 kg breaks under a pressure of 10^5 Pa. The crushing occurs in a hydraulic press in which the diameter of the larger piston is five times larger than the diameter of the smaller piston and the ratio of the lever arms of the smaller piston is 1:4. Find the force that must be

applied to the lever for crushing the bar if the efficiency of the press is 80%. The mass of the pistons should be neglected.
Answer. 52.5 N.

(b) A load is compressed in a mechanically driven hydraulic press having a ratio of piston diameters of 1:5 and an efficiency of 80%. The smaller piston makes 50 strokes 10 cm each. Determine the mass of the load if the force of pressure on the upper plate of the press is known to reach 100 kN, while the force of pressure on the smaller piston in this case is 7 kN. Find the deformation of the load.
Answer. 4 t, 20 cm.

109. (a) Water and some other liquid are poured into the arms of a U-shaped tube partially filled with mercury. Find the density of the second liquid if the level of mercury remains unchanged in the two arms when the heights of the liquid and water columns are 18 cm and 20 cm respectively.
Answer. 0.9×10^3 kg/m^3.

(b) The diameter of one of the two communicating vessels containing mercury is four times larger than the diameter of the other vessel. Water is poured into the smaller vessel so that the height of the water column is 70 cm. Find the change in the levels of mercury in the vessels.
Answer. 0.3 cm, 4.8 cm.

(c) Mercury is poured into a U-shaped tube of a constant cross-sectional area. When some amount of water is poured into one arm, the level of mercury in the other arm rises by 2 cm. Find the height of the water column. What must be the height of the column of alcohol poured into the second arm to make the level of mercury in it 2 cm higher than in the first arm? The densities of mercury and alcohol are 13.6×10^3 kg/m^3 and 0.8×10^3 kg/m^3 respectively.
Answer. 54.4 cm, 34.0 cm.

(d*) Two vessels of the same cross-sectional area are connected by a pipe whose cross-sectional area amounts to 1/10 of that of the vessels. One of the communicating vessels contains water, while the other contains oil whose density is 0.9×10^3 kg/m^3. When a layer of oil is poured above water in the water-containing vessel, the interface between the liquids in the pipe shifts by 18 cm. Find the thickness of the additional layer of oil.
Answer. 38 mm.

110. (a) A gold nugget enclosed in quartz stretches a spring balance with a force of 1.32 N. When it is immersed in water, the buoyancy acting on it is 0.20 N. Find the mass of the gold nugget if the densities of gold and quartz are 19.3×10^3 kg/m^3 and 2.6×10^3 kg/m^3 respectively.
Answer. 88 g.

(b) A hollow iron sphere stretches a spring balance with a force of 6 N. When immersed in water, the sphere stretches the spring balance with a force of 5 N. Find the inner volume of the sphere if the density of iron is 7.8×10^3 kg/m^3.
Answer. 24 cm^3.

(c) A hollow metallic sphere stretches a spring balance with a force of 4.0 N. Being immersed in water, the sphere stretches the spring balance with a force of 2.5 N. Find the volume of the cavity if the density of the material of the sphere is 8.0×10^3 kg/m^3.
Answer. ~ 102 cm^3.

111. (a) A piece of metal suspended from a spring balance is first immersed in water and then in kerosene. The readings of the spring balance are 2.0 kN and 2.5 kN respectively. Find the density of metal if the density of kerosene is 0.8×10^3 kg/m^3.
Answer. 1.8×10^3 kg/m^3.

(b) A body suspended from a spring balance is immersed in petrol whose density is 0.7×10^3 kg/m^3. The force indicated by the spring balance in petrol is equal to 1/8 of the value indicated in air. Find the density of the body.
Answer. 0.8×10^3 kg/m^3.

112. (a) A piece of alloy floats in mercury so that 75% of its volume are submerged in the liquid. Find the density of the alloy if the density of mercury is 13.6×10^3 kg/m^3.
Answer. 10.2×10^3 kg/m^3.

(b) A right cone whose height is 1 m floats in water so that its axis is vertical and its apex is turned up. What is the height of the underwater part of the cone if the density of the cone material is 0.5×10^3 kg/m^3?
Answer. 0.30 m.

113. (a) A cylinder whose height is 40 cm floats in a vessel with water so that its axis is vertical. A 20-cm layer of liquid having a density of 0.7×10^3 kg/m^3 and immiscible with water is poured into the vessel. What part of the cylinder is in water in the first and second cases? The density of the cylinder material is 0.8×10^3 kg/m^3.
Answer. 32 cm, 18 cm.

(b*) A vessel with vertical walls and horizontal bottom whose area is 0.4 m^2 floats in a reservoir. The vessel contains a layer of water whose thickness is 25 cm; its draught is 50 cm. Find the height of water in the vessel and its draught after a 6-kg block whose density is 0.75×10^3 kg/m^3 is immersed in it. The thickness of the vessel walls should be neglected.
Answer. 26.7 cm, 51.5 cm.

114. (a) Find the lifting force of a 4-kg cork lifebelt in sea water if the densities of cork and sea water are 0.2×10^3 kg/m^3 and 1.03×10^3 kg/m^3 respectively.
Answer. 163 N.

(b) A cork lifebelt keeps a man in sea water so that his head and shoulders (1/8 of his volume) are above the water surface. Find the volume of the lifebelt if the mass of the man is 70 kg and his volume is 65.6 dm^3 respectively.
Answer. 15.75 dm^3.

(c) Find the minimum area of a flat ice floe having a thickness of 0.35 m, which can keep a person whose mass is 70 kg. The density of ice is 0.9×10^3 kg/m^3.
Answer. 2.0 m^2.

115. (a) The part of an iceberg above water has a volume of 1000 m^3. At what density of ice is the volume of the underwater part of the iceberg 8000 m^3?
Answer. 0.89×10^3 kg/m^3.

(b) A block of ice floats in a fresh-water lake so that its 0.5-m part is above the water surface. Find the total height of the block of ice if the density of ice is 0.9×10^3 kg/m^3.
Answer. 4.5 m.

116. (a) A motor ship sails from the sea to a river. To keep the same draught, a 90-t load is removed from the ship. Find the mass of the loaded ship before it has been unloaded. The density of sea water is 1.03×10^3 kg/m^3.
Answer. 3030 t.

(b) The cross-sectional area of a motor ship at the waterline is 4000 m^2. After its loading at a wharf, the draught has increased by 1.5 m. Find the mass of the load taken on board the ship.
Answer. 6180 t.

117. A 4000-m^3 balloon is filled with helium. The mass of the structure, equipment, and crew is 3 t. Helium completely fills the balloon at a height where the density of air is 1.2×10^3 kg/m^3. Find the maximum mass that can be lifted by the balloon. The density of helium is 0.18×10^3 kg/m^3.
Answer. 10 kg.

118. A balloon filled with hydrogen rises from the ground with an acceleration of 1.0 m/s^2. The mass of the balloon with the crew, equipment, and load is 0.7 t. Find its volume if the densities of air and hydrogen are 1.29×10^3 kg/m^3 and 0.09×10^3 kg/m^3 respectively.
Answer. 656 m^3.

119. (a) A cube with an edge of 10 cm is immersed in a vessel with water. A layer of liquid immiscible with water and having a density of 0.8×10^3 kg/m^3 is poured above water. The interface between the liquids is at the middle of the cube height. Find the mass of the cube.
Answer. 0.90 kg.

(b) A cube made of a material having a density of 0.9×10^3 kg/m^3 floats between water and a liquid of density 0.7×10^3 kg/m^3, which is immiscible with water. What part of the cube is immersed in water?
Answer. 2/3.

(c) A metallic bar floats in a vessel containing mercury and water poured above it. The bar is submerged by 1/4 of its height in mercury and by 1/2 in water. Find the density of metal if the density of mercury is 13.6×10^3 kg/m^3.
Answer. 3.9×10^3 kg/m^3.

120. (a) A thin homogeneous rod can freely rotate about an axis passing through its upper end. The lower end of the rod is immersed in water so that equilibrium is attained when half of the rod is in water. Find the density of the material of the rod.
Answer. 0.75×10^3 kg/m^3.

(b*) A vessel contains water above which a 20-cm layer of a liquid immiscible with water and having a density of 0.7×10^3 kg/m^3 is poured. A 1.2-m homogeneous rod is hinged at its upper end to the vertical wall of the vessel. The other end of the rod is submerged in the vessel. The rod is in equilibrium when it is submerged by half of its length and is at an angle of 60° to the wall. Find the density of the rod material.
Answer. 0.62×10^3 kg/m^3.

121. (a) A ball whose density is 0.4×10^3 kg/m^3 falls into water from a height of 9 cm. To what depth does the ball sink?
Answer. 6 cm.

(b) A pile whose height is 10 m and diameter is 50 cm is mounted vertically in a swimming pool with water so that half of it is above the water surface. The diameter of the pool is 5.0 m. The pile is extracted from water by a crane which spends 370 kJ of energy for that. Find the density of the material of which the pile is made.
Answer. $0.7 \times 10^3\ \mathrm{kg/m^3}$.
122. A body is immersed in a vessel with vertical walls, containing kerosene. It is known that the density of the body is $\rho < \rho_k$, where ρ_k is the density of kerosene. The level of kerosene rises by Δh but kerosene does not overflow from the vessel. Find the mass m of the body if the cross-sectional area of the vessel is S.
Answer. $m = \Delta h \rho_k S$.

3. MOLECULAR PHYSICS. THERMAL PHENOMENA

A. MOLECULAR PHYSICS

3.1. Basic Concepts of Molecular-Kinetic Theory

According to the molecular-kinetic theory, all substances are composed of tiniest particles, viz. molecules, moving continuously and interacting with one another. A **molecule** is the smallest particle of a substance that possesses its chemical properties. Molecules consist of smaller particles, viz. **atoms** of chemical elements. Molecules of different substances have different atomic compositions.

Molecules possess kinetic energy as well as potential energy of interaction. The ratio between the average values of the kinetic energy W_k and the potential energy W_p of the interaction of molecules of a substance determines its state of aggregation. In the gaseous state, the kinetic energy of particles is high as compared with the energy of their interaction ($W_k \gg W_p$). In the condensed (solid or liquid) state, the kinetic energy of particles is comparable with the energy of their interaction ($W_k \sim W_p$). (The difference between the solid and liquid states lies in the nature of thermal motion of molecules.)

The molecular-kinetic theory is based on three fundamental statements.

1. All substances consist of the tiny particles—molecules, i.e. have a *discrete* (discontinuous) *structure*.

2. Molecules are in continuous *random* (chaotic) *motion*.

3. There exist *forces of interaction* between the molecules of a body.

The molecular-kinetic theory of matter is confirmed by many experiments and observations. Among them, there are the experiments on mixing fluids, on dissolving solids in liquids, the observations of the compressibilities of various substances, Brownian movement, diffusion, observations of gases, for example, the dependence of the gas pressure of the vessel walls on its density and temperature, and so on.

Let us consider some experiments and observations.

Proofs of the existence of molecules.

The existence of molecules is confirmed by the **law of multiple proportions**, which states that *when various compounds are formed from two elements, the ratio of the masses of one of the elements in different compounds is equal to the ratio of integers, i.e. they are in multiple proportions.* For example, nitrogen and oxygen form five compounds: N_2O, N_2O_2, N_2O_3, N_2O_4, and N_2O_5. In this compounds, oxygen appears in combination with the same amount of nitrogen in the amounts which are in multiple proportions: 1:2:3:4:5.

The law of multiple proportions can be easily explained by the fact that the same number of atoms of one element (nitrogen in the above example) is combined in the molecules of different substances (compounds) with a different number of atoms of the other element (oxygen). Any substance consists of identical molecules of a certain atomic composition. Since all the molecules of a given substance are identical, the ratio of the quantities of simple elements comprising the given body (in terms of mass) is the same as in a separate molecule. Consequently, it is a multiple of atomic masses, which is confirmed by experiments.

Another proof is **Dalton's law** which is valid for rarefied gases. This law states that *the total pressure of a gas mixture is equal to the sum of the partial pressures of the constituent gases.* In other words, the pressure of each gas in the space occupied by a gas mixture, i.e. the so-called partial pressure, is the same as if this gas or vapour occupied the entire space alone. This is possible only if each gas behaves as if it occupied the entire given volume. The molecules of other gases of the mixture occupy such a small volume that they practically do not hinder the motion of molecules of the given gas.

Some additional proofs of random (chaotic) motion of molecules are:

(1) the tendency of a gas to occupy the entire volume it is

brought in, for example, the spreading of a strong-smelling gas over the entire room;

(2) the existence of a gas pressure on the walls of a gas-containing vessel or on the surface of a body introduced into a gas; an increase in pressure with the gas density or temperature;

(3) Brownian movement (see Sec. 3.2);

(4) diffusion and osmosis (see Sec. 3.3).

The proofs of intermolecular interaction are:

(1) the deformation of bodies under the action of the forces applied to them;

(2) the fact that solid bodies retain their shape;

(3) the surface tension of liquids, and wetting and capillary phenomena as consequences of surface tension.

Modern experimental methods of physics, such as X-ray diffraction analysis and electron microscopy, made it possible to directly observe the microscopic structure and enriched our knowledge of the structure of matter.

3.2. Brownian Movement. Gas Pressure

Brownian movement is a random motion of the smallest visible particles of a substance suspended in a liquid and insoluble in it. This motion occurs under the action of random impacts of molecules of the liquid, which are in constant chaotic motion (Fig. 161).

At each instant of time, this motion is determined by the predominant direction of impacts of liquid molecules.

The intensity of the Brownian movement increases with decreasing size of particles suspended in the liquid. This is due to the fact that the larger the particles, the higher the probability that the impacts of liquid molecules on the entire particle turn out to be mutually balanced. It is more difficult to change the motion of coarse particles (larger particles have larger masses). The intensity of the Brownian movement also increases with the temperature of the liquid, since in this case the kinetic energy of liquid molecules, which determines the strength of impacts, becomes higher.

Experiments involving the Brownian movement have played a significant role in the evolution of the molecular-kinetic theory.

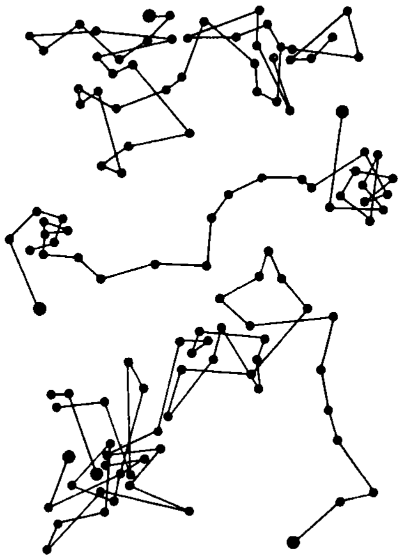


Fig. 161

The pressure of a gas on the walls of a vessel containing it or on the surface of a body introduced into the gas is explained by incessant impacts of moving molecules of the gas. An increase in pressure with gas density is due to a larger number of molecules in a denser gas. An increase in pressure with temperature can be explained by an increasing velocity of moving molecules. In both cases, the number of molecular impacts against the vessel walls or on the surface of the body increases.

3.3. Diffusion in Gases, Liquids, and Solids

Diffusion is the mutual penetration of molecules of substances in contact. In diffusion, the molecules of a substance, being in permanent motion, penetrate into the spaces between the molecules of another substance which is in contact with the first substance and are distributed among them. As a result of molecular motion, the concentration of a substance in an inhomogeneous material levels out, and it becomes homogeneous.

Diffusion is observed in all the states of aggregation (in gaseous, liquid, and solid bodies) but to different extents.

Diffusion in gases can be observed if, for example, a vessel containing an odorous gas is opened in a room. In a certain time, the gas will spread over the entire room.

Diffusion in liquids can also be easily observed, although it occurs at a much slower rate than in gases. If, for example, we first pour some blue vitriol into a glass and then add very carefully a layer of water and leave the glass in a vibration-free room at a constant temperature, the sharp boundary between the liquids will soon disappear, and in several days the liquids will be mixed in spite of the fact that the density of vitriol is higher than the density of water. Water and alcohol or some other liquids diffuse in the same way.

Diffusion in solids occurs at a still slower rate than in liquids. It can be observed only between well-polished bodies in contact, when the distances between the surfaces of the bodies are close to intermolecular distances ($\sim 10^{-8}$ cm). The rate of diffusion increases with temperature.

A modification of diffusion is **osmosis**, viz. the penetration of liquids and solutions through a porous membrane. Some membranes which are permeable to one type of liquids and solutions are partially or completely impermeable to other liquids and solutions. Such membranes are called semipermeable.

Diffusion and osmosis play an important role in nature and in engineering. Nutrition of plants, for example, is possible due to diffusion. Living organisms receive nutrients by absorbing them through the alimentary tract. In engineering, the surface layers of metallic articles are saturated by diffusion with carbon (cementation), and so on.

3.4. Motion of Molecules in Gases, Liquids, and Solids

The nature of molecular motion in gases, liquids, and solids exhibits some common features as well as considerable differences.

The common features of molecular motion are that (a) the average velocity of a molecule is the higher, the higher the temperature of the substance and (b) the velocities of individual molecules are spread over a wide range. Most of molecules, however, have the velocities close to the so-called most probable velocity which slightly differs from the average velocity.

The considerable difference in the nature of molecular motion in gases, liquids, and solids is due to different types of interactions among their molecules, which is explained by the difference in the average intermolecular distances.

The intermolecular distances in **gases** exceed many times the dimensions of the molecules. For this reason, the forces of intermolecular interaction in gases are weak, and the molecules can move over the entire vessel containing the gas almost independently, changing the direction and magnitude of their velocities upon collisions with other molecules and the vessel walls. The trajectory of a gas molecule is a broken line resembling the trajectory of a Brownian particle.

The mean free path of a gas molecule, i.e. the average distance between two successive collisions, depends on the pressure and temperature of a gas. At room temperature and normal atmospheric pressure, the mean free path amounts to 10^{-5} cm. Gas molecules collide with one another or with the vessel walls about 10^{10} times per second, each time changing the direction of their motion. For this reason, the diffusion rate in gases is small in comparison with the velocity of translatory motion of gas molecules, which under normal conditions is 1.5 times higher than the velocity of sound in a given gas.

The distances between **liquid** molecules are considerably smaller than the intermolecular distances in gases. The forces of interaction of each molecule with neighbouring ones are sufficiently strong, which makes the molecules vibrate about their equilibrium positions. At the same time, having acquired by chance an excessive kinetic energy, the molecules overcome the attraction of neighbouring particles and alter their equilibrium positions, since their average kinetic energy is comparable with the interaction energy. Vibrating particles of a liquid move jumpwise in space in very short time intervals (about 10^{-8} s), which is manifested as the *fluidity* of liquids.

Thus, a liquid consists of a very large number of microscopic regions within which there is a certain ordering in the arrangement of neighbouring particles. This ordering varies in space and time, i.e. it is not repeated in the entire volume of the liquid. Such a structure is said to have the *short-range order*.

The forces of interaction of neighbouring molecules in **solids** are so strong that a molecule can perform only small vibrations about a certain *fixed* equilibrium position, viz. a lattice site. In a crystalline body, a certain regular arrangement of lattice sites can be distinguished, which is called its *crystal lattice*. Such a structure has the *long-range order*. The form of a crystal lattice, i.e. the symmetry type of a solid, is determined by the nature of intermolecular interactions in a given substance.

All what has been said above applies to an ideal crystalline solid. In real crystals, the long range is violated in different ways in the process of crystallization.

Besides crystals, there exist amorphous bodies in nature, in which, like in liquids, the atoms vibrate about randomly arranged sites. The particles of an amorphous body move from one equilibrium position to another for such long time intervals, however, that practically amorphous bodies are solids rather than liquids.

3.5. Intermolecular Interaction

The dependence of the force F of intermolecular interaction on the intermolecular distance r is depicted in Fig. 162. Strictly

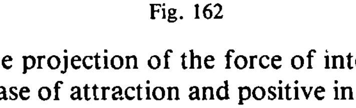


Fig. 162

speaking, F is the projection of the force of interaction, which is negative in the case of attraction and positive in the case of repulsion of molecules. When two molecules approach each other, the force of interaction first increases in magnitude and then

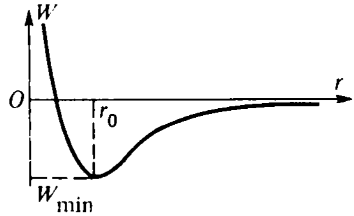


Fig. 163

decreases (the attraction region). As we go over through the point $r = r_0$, the force of interaction changes its sign and grows very rapidly with a further decrease in the intermolecular distance (the repulsion region).

The distance r_0 corresponds to a stable equilibrium of a system consisting of two molecules. If the intermolecular distance deviates from r_0 in any direction, a force appears that returns the molecules to the equilibrium state.

Figure 163 shows the graph of the potential energy of interaction of molecules as a function of the distance between them. The distance r_0 corresponds to the minimum of the potential energy, and hence to the state of stable equilibrium.

B. THERMAL PHENOMENA

3.6. Internal Energy of a Body

The **internal energy** of a body is the sum of the kinetic energies of molecules constituting the body, the potential energy of their interaction, and the intramolecular energy (i.e. the energy of motion and interaction of atoms, nuclei, ions, etc.). The internal energy of a body depends neither on its motion as a whole nor on its potential energy in an external force field.

In this chapter, we shall consider physical phenomena and processes which do not involve a change in the intramolecular energy. Hence, for the sake of convenience and simplicity, we shall treat the *internal energy* of a body as the sum of the kinetic energies of molecules constituting a substance and the potential energy of their interaction.

The internal energy of a body can be changed as a result of two kinds of effects on the body:

(1) *when a work is done on the body* (as a result of compression, extension, and so on),

(2) *when a heat is supplied to the body* (heating a gas in a closed vessel, heating a liquid, etc.).

The transport of internal energy from one body to another without a work being done by the bodies is called **heat transfer.** The amount of energy transported from body to body by heat transfer is called the **amount of heat**.

The unit of the amount of heat is a **joule** (J).

There are three types of heat transfer, viz. convection, conduction, and radiation.

1. **Convection** (in fluids) is a heat transfer by moving or mixing hot and cold layers of a fluid.

Consequently, in convection heat is transferred with a flow of heated fluid. The examples of convection are the circulation of air in a heated room, the heating of a liquid from the bottom of the vessel (if the vessel containing a liquid is heated at the top, no convection occurs), the draught in chimneys, central water heating, winds, and ocean currents.

2. **Conduction** (in gases, liquids, and solids) is the process of heat transfer from a hot part of a body to a cold part without a visible motion of any part of the body.

Different bodies have different thermal conductivities. Some solids (metals) are good conductors of heat, while others (wood, glass or leather) are poor conductors. Porous materials (wool, cork, paper, wood, etc.) have the lowest thermal conductivities. Most liquids (except mercury) are poor conductors of heat. Thermal conductivity of gases is still lower. This is the reason behind a low thermal conductivity of porous solids, since their pores are filled with air.

3. **Thermal radiation** is the process of heat transfer from an isolated body with the help of electromagnetic waves. For example, the entire energy received by the Earth from the Sun is transferred by radiation.

3.7. Law of Conservation and Transformation of Energy. First Law of Thermodynamics

The **law of conservation and transformation of energy** is a fundamental law of nature. It can be formulated as follows: *in all processes occurring in nature, energy is not created or destroyed. It is transferred from one body to another or converted from one kind to another in equivalent amounts.*

We shall consider physical phenomena associated with the mutual conversion of mechanical and internal energies and with the transfer of internal energy from one body to another. The branch of physics dealing with this problems is called **thermodynamics**. The law of energy conservation and conversion is expressed by the first law of thermodynamics, which is a fundamental law in this field.

An aggregate of bodies singled out for analysis form a thermodynamic system. In particular, a system may contain only one body (for example, a certain amount of gas or liquid or a solid body).

In thermodynamics, we usually consider the amount of heat Q received by a system from without, and the work A done by the system on external bodies.

The **first law of thermodynamics** is formulated as follows: *the heat supplied to a system is spent to increase its internal energy and to do work on external bodies*:

$$Q = \Delta U + A, \quad \textit{where} \quad \Delta U = U_2 - U_1. \tag{3.7.1}$$

According to Newton's third law, the forces of interaction of bodies in a system and external bodies differ only in sign. Hence the work A done by the system on external bodies in the course of some process also differs from the work A' done by external bodies on the system during the same process only in sign: $A' = -A$. For example, if a gas which expands and pushes a piston does a work $A = 2$ J on it, the piston does a work $A' = -2$ J on the gas.

Substituting $-A'$ for A in Eq. (3.7.1), we arrive at the following relation:

$$\Delta U = Q + A', \tag{3.7.2}$$

which can also be considered as a formulation of the first law of thermodynamics. It follows from (3.7.2) that the increase in the internal energy of the system is equal to the sum of the amount of heat received by the system and the work done on it. In a particular case, one of the quantities Q or A' may be equal to zero. It should be also borne in mind that the values of ΔU, Q, and A' are not necessarily positive. If, for example, a system gives an amount of heat $Q' = 2$ J to external bodies (i.e. receives the heat $Q = -2$ J) and does a work $A = 3$ J on the external bodies (i.e. the work $A' = -3$ J is done on the system), the increase in its internal energy is $\Delta U = -5$ J. This means that the final value of the internal energy is lower than the initial value by 5 J. These five joules were spent by the system to transfer two joules of heat to the external bodies and to do a work of three joules on them.

3.8. Temperature Gradient. Thermodynamic Temperature Scale. Absolute Zero

If there is no heat transfer between two bodies in contact, the bodies are said to be in thermal equilibrium. In other words, they are at the same temperature. The internal energy is transferred by heat conduction from a body at a higher temperature to a body at a lower temperature. **Temperature gradient** is a measure of deviation of bodies from the state of their mutual thermal equilibrium.

In everyday life and in engineering, the Celsius (centigrade) temperature scale is used, where temperature is measured in degrees centigrade (°C). This scale is based on the assumption that under normal atmospheric pressure, the melting point of ice is 0 °C, and the boiling point of water is 100 °C. This temperature interval was divided into 100 equal parts of one degree. On the Celsius scale, the temperature is denoted by t or θ.

The SI unit of temperature is a **kelvin** (K), which is equal to 1/273.16 of the thermodynamic temperature corresponding to the triple point of water.

The temperature scale adopted in this system is called the **thermodynamic temperature scale.** On this scale, the zero temperature is equal to −273.15 °C, and one kelvin is equal to one degree on the Celsius scale.

The temperature of 0 K is called the **absolute zero temperature.** Thermodynamic temperature is denoted by T. The relation between the thermodynamic temperature and the temperature on the Celsius scale is as follows: $T = t + 273.15$ K, $t = T - 273.15$ °C. The absolute zero cannot be attained in principle. The lowest temperature reached by now is 10^{-6} K.

3.9. Heat Capacity

The **heat capacity** C of a body is the amount of heat that should be supplied to it to raise its temperature by a kelvin.

In SI units, heat capacity has the dimensions of a **joule per kelvin** (J/K).

The **specific heat** c is the heat capacity of a body of unit mass.

The SI unit of specific heat is a **joule per kilogram-kelvin** (J/(kg·K)).

There exists the following obvious relation between C and c:

$$C = mc,$$

where m is the mass of a body.

Heat capacity depends on the conditions of heating or cooling a body. For gases, the processes of heating (cooling) at constant volume and at constant pressure are of special interest. In the former case, we speak of the specific heat at constant volume (c_V), when the gas does not perform work, and the entire amount of heat supplied to it is spent to raise its internal energy: $Q_V = \Delta U$. In the latter case, the specific heat at constant pressure is meant (c_p). The gas heated at constant pressure expands, and a part of heat supplied to the gas is spent to do the work of expansion: $Q_p = \Delta U + A$. Hence we conclude that $Q_p > Q_V$, and the heat capacity at constant pressure is higher than the heat capacity at constant volume: $C_p > C_V$. For specific heats, we have $c_p > c_V$.

3.10. Experimental Determination of Specific Heat of a Substance

The heat capacity of a body is determined with the help of a calorimeter and a thermometer. A simple calorimeter (Fig. 164) consists of a polished metallic cylinder placed into another

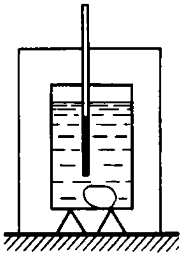


Fig. 164

metallic cylinder on corks (for thermal insulation). The inner cylinder is filled with water or some other liquid with the known specific heat. A body heated to a certain temperature t is immersed in the calorimeter in which the temperature is measured. Suppose that the temperature of liquid in the calorimeter is t' before the body is immersed in it, and when the temperatures of the liquid and the body level out, the temperature becomes θ.

It follows from the law of energy conservation that the amount of heat Q given away by the heated body is equal to the sum of the amounts of heat Q_1 and Q_2 received by water and the calorimeter:

$$Q = Q_1 + Q_2, \quad \text{or} \quad cm(t - \theta) = c_1 m_1(\theta - t') + c_2 m_2(\theta - t').$$

Here c_1 and m_1 are the specific heat and the mass of water in the calorimeter, and c_2 and m_2 are the specific heat and the mass of the calorimeter. This equation is called the **heat balance equation**. The required specific heat of the body can be found from this equation:

$$c = \frac{Q_1 + Q_2}{m(t - \theta)} = \frac{c_1 m_1(\theta - t') + c_2 m_2(\theta - t')}{m(t - \theta)}$$
$$= \frac{(c_1 m_1 + c_2 m_2)(\theta - t')}{m(t - \theta)}.$$

3.11. Heat of Combustion of a Fuel

The **heat of combustion** (heat value) of a fuel is the amount of heat liberated upon a complete burning a unit mass of a fuel.

In terms of SI units, the heat of combustion is expressed in **joules per kilogram** (J/kg).

Heat of Combustion of Some Fuels (in MJ/kg)	
Hard coal	21-34
Dry wood	12
Petroleum, kerosene, petrol	44-48
Fuel gas	34-38

In engineering, calculations are often made for the so-called theoretical standard fuel whose heat of combustion q = 29.4 MJ/kg.

3.12. Efficiency of a Heat Engine

A **heat engine** is a cyclic engine that performs work at the expense of heat received from without. The same working cycle is periodically repeated in such engines.

In such a cyclic operation, the amount of heat Q_1 received by the engine per cycle from a heat source (say, steam boiler) cannot be completely converted into work A. Some amount of heat Q_2 should be transferred to a sink (in the simplest case, the role of a sink is played by the ambient). It follows from the law of energy conservation that the work done by a heat engine per cycle $A = Q_1 - Q_2$.

The **efficiency** η of a heat engine is the ratio of the work A done per cycle to the amount of heat Q_1 received by the engine from the heat source in a cycle:

$$\eta = A/Q_1 = (Q_1 - Q_2)/Q_1.$$

According to the laws of thermodynamics, the efficiency of a heat engine cannot exceed the limiting ("ideal") value given by

$$\eta_{id} = (T_1 - T_2)/T_1,$$

where T_1 and T_2 are the thermodynamic temperatures of the heat source and the sink.

We take for T_1 the temperature of steam entering an engine and equal to 373 K (100 °C) and put T_2 equal to 273 K (0 °C). Then the maximum possible efficiency is

$$\eta_{id} = (373 - 273)/373 = 0.27, \quad \text{or} \quad 27\%.$$

In high-pressure boilers, the steam temperature reaches several hundred degrees centigrade, which makes it possible to elevate the efficiency of steam engines.

The real efficiency of heat engines is considerably lower than the ideal efficiency of engines. The efficiency of modern steam power plants with a steam engine attains 10-15% and those with a steam turbine, 20-30%. The efficiency of internal combustion engines varies between 25 and 35%.

3.13. Phase of a Substance. Fusion. Latent Heat of Fusion

A **phase** is a physically homogeneous part of a substance separated from other parts of the system by an interface (e.g. ice, water, steam). Fusion of a solid, crystallization of a liquid, evaporation and condensation are the examples of **phase transitions.** A transition from one phase to another at a given pressure occurs at a strictly constant temperature.

Fusion (or **melting**) is a translation from the solid phase to the liquid phase. For this transition, a certain amount of heat should be supplied to a body. On the contrary, a crystallizing body gives away some heat. The **fusion point**, or the **melting temperature**, is the temperature at which a crystalline body melts (or crystallizes) at constant pressure.

Sometimes, if a process is carried out very carefully, a body can be "supercooled". Supercooling is the cooling of a liquid to a temperature somewhat lower than the fusion point without its crystallization. In this case, a slightest shock leads to a partial crystallization with the temperature increasing to the melting point.

Amorphous bodies (glass, wax, paraffin or pitch) have no definite melting point. They gradually become malleable. Figure 165 shows the temperature variation due to a heat supply to crystalline and amorphous bodies.

Almost all materials increase in volume during fusion and decrease in volume (contract) during crystallization. As the pressure on a substance increases, its melting temperature becomes higher. (Pressure prevents from increasing the volume of

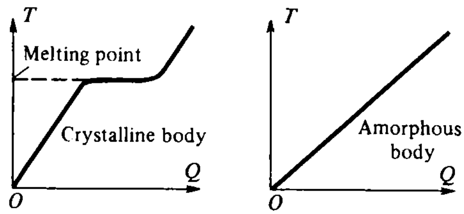


Fig. 165

a body.) The exceptions are water (ice), cast iron, and some other materials which increase in volume during crystallization and decrease in volume during fusion. Their melting temperature decreases with increasing pressure.

The **latent heat of fusion** λ is the amount of heat required to melt a unit mass of a solid crystalline substance at the fusion point at constant pressure.

The latent heat of fusion is measured in SI in **joules per kilogram** (J/kg).

Crystallization of a liquid is accompanied by the liberation of the heat of fusion.

3.14. Evaporation. Condensation. Vaporization and Boiling. Latent Heat of Vaporization

Vaporization proceeds in two ways: by evaporation and boiling.

Evaporation from the surface of a liquid occurs at any temperature. The intensity of evaporation increases with the free surface of the liquid (since the number of molecules leaving the liquid per unit time increases in this case), with the temperature of the liquid (since the velocity of molecular motion and hence the kinetic energy of molecules increase with temperature; the number of molecules that can overcome the molecular attraction of the liquid increases), and with decreasing external pressure on the free surface of the liquid. The evaporation rate also increases when the vapour formed above the surface of the liquid is removed. The evaporation intensity is higher for liquids in which the cohesive forces between molecules are weaker (volatile liquids).

The mechanism of boiling consists in the following. When a liquid is heated, the bubbles of air dissolved in it and containing the vapour of this liquid grow. As the temperature rises, the pressure of vapour in the bubbles increases. Under the action of buoyancy, the bubbles rise to the surface and partially condense as long as the upper layers of liquid are colder than the lower layers. When the whole liquid is sufficiently heated, steam bubbles reach the surface, the pressure in them attains the atmospheric pressure, and steam bursts out of the bubbles. Vaporization occurring simultaneously in the bulk of a liquid and on its surface is called **boiling.** At a given pressure, each substance boils at a quite definite temperature which remains constant during the process of boiling.

Boiling point is the temperature at which a given liquid boils at constant pressure. As the external pressure increases, the boiling temperature rises, and vice versa. The presence of a dissolved substance in a liquid alters its boiling temperature.

In evaporation and boiling, the vaporization is characterized by the latent heat of vaporization which has a definite value for each substance under given external conditions, viz. temperature and pressure.

The **latent heat of vaporization** r is the amount of heat required for converting a unit mass of liquid into vapour at the boiling temperature.

The latent heat of vaporization is expressed in SI in **joules per kilogram** (J/kg).

The amount of heat required for the vaporization of a certain mass of a liquid preliminarily heated to the boiling temperature is

$$Q = rm.$$

As the temperature of an evaporating liquid increases, the latent heat of vaporization decreases. In particular, if the boiling temperature is raised (for example, by increasing pressure), the latent heat of vaporization during boiling decreases. Water, for instance, has the following values of the latent heat of vaporization at different temperatures:

t, °C	0	100	200
r, MJ/kg	2.50	2.20	1.94

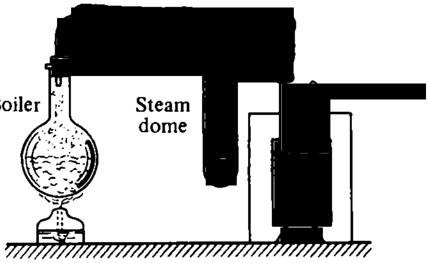


Fig. 166

The latent heat of vaporization is determined experimentally, with the help of a calorimeter with a steam dome (Fig. 166). The vapour of a liquid flows from a boiler through the steam dome to the calorimeter containing the same liquid. In the calorimeter, it is condensed, heating the liquid thereby. The steam dome is intended for separating liquid drops from vapour. It consists of a test tube with two pipes. The longer pipe connects the steam dome with the boiler, while the shorter pipe connects it with the calorimeter. Liquid drops settle on the bottom of the test tube, and dry vapour enters the calorimeter. Having measured the temperature of the liquid in the calorimeter, we can write the heat balance equation and find the latent heat of vaporization of the liquid.

During the **condensation** of a certain mass of vapour, the amount of heat liberated is exactly the same as is required for the vaporization of the same mass of the liquid.

Problems with Solutions

123. A body of mass $m = 100$ kg slides down an inclined plane with a slope $\alpha = 30°$. What is the change in the internal energies of the body and the plane upon the displacement of the body by a distance $h = 3.0$ m along the vertical? The coefficient of sliding friction is $f = 0.2$.

Solution. When the body slides down the inclined plane (Fig. 167), the body and the plane are heated. Hence their internal energies increase. This change in the internal energies is equal to the work of friction: $\Delta W = A_{fr}$. The friction is $F_{fr} = fN = fmg \cos\alpha$. The distance covered by the body is $l = h/\sin\alpha$. Consequently, the work of friction $A_{fr} = F_{fr}l = fmgh \cot\alpha$. The increase in the internal energies is $\Delta W = A_{fr} = 1020$ J.

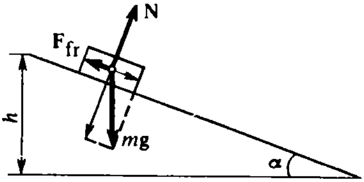


Fig. 167

124. A hole having a diameter $d = 4$ cm is tapped in a steel plate of diameter $D =$ 10 cm. The torque applied to the tap wrench is $M = 40$ N·m. The pitch of the thread is $h = 0.5$ mm. Find the increase in the temperature Δt of the plate if 60% of the spent energy are converted into the internal energy of the plate. The specific heat of steel is $c = 0.46$ kJ/(kg·K) and its density $\rho = 7.8 \times 10^3$ kg/m^3.

Solution. In order to tap a hole, H/h turns are made, where H is the thickness of the plate. The work done in one turn is $A_1 = 2\pi M$. The total work $A = 2\pi MH/h$. The increase in the internal energy of the plate during tapping is

$$W = 0.60A = 0.60 \times 2\pi MH/h. \tag{1}$$

On the other hand, the increase in the internal energy is given by

$$W = cm\,\Delta t = c\pi(D^2 - d^2)\rho H\,\Delta t/4. \tag{2}$$

Equating the right-hand sides of Eqs. (1) and (2), we obtain

$$\Delta t = 4.8M/hc(D^2 - d^2)\rho = 12.7\text{ K}.$$

125. A steel article is hardened by heating to a temperature $t_{st} = 800$ °C and by subsequent immersion in oil having a mass $m_{oil} = 2$ kg and a temperature $t_{oil} =$ 10 °C. As a result, oil is heated to $\theta = 40$ °C. Find the mass m_{st} of the steel article if it is cooled in oil by $\Delta t = 20$ °C. The specific heats of steel and oil are $c_{st} =$ 0.63 kJ/(kg·K) and $c_{oil} = 1.9$ kJ/(kg·K).

Solution. The amount of heat transferred to the oil by the heated article is $Q_{st} = c_{st}m_{st}(t_{st} - \Delta t - \theta)$, while the amount of heat received by the oil is $Q_{oil} =$ $c_{oil}m_{oil}(\theta - t_{oil})$. We can write the heat balance equation

$$c_{st}m_{st}(t_{st} - \Delta t - \theta) = c_{oil}m_{oil}(\theta - t_{oil}),$$

which gives

$$m_{st} = \frac{c_{oil}}{c_{st}}\,\frac{\theta - t_{oil}}{t_{st} - \Delta t - \theta}\,m_{oil} = 243\text{ g}.$$

126. A "Moskvich" motorcar consumes $m = 5.67$ kg of petrol over a distance $s = 50$ km. Find the power N developed by the car engine if the velocity of motion is $v = 72$ km/h and the engine efficiency $\eta = 22\%$. The heat of combustion of petrol $q = 45$ MJ/kg.

Solution. The amount of heat liberated during the combustion of petrol is $Q = mq$. The useful work constitutes $\eta = 0.22$ of this amount of heat: $A = 0.22Q$. The engine power $N = A/t = 0.22mq/t$, where $t = s/v$ is the time during which the mass m of fuel is consumed. Therefore, $N = 0.22mqv/s = 22.5$ kW.

127. An engine of power $N = 1$ kW consumes $m_f = 250$ g/h of fuel oil per unit time. Find the efficiency of the engine if the heat of combustion of the fuel is $q = 46.2$ MJ/kg.

Solution. The amount of heat released per unit time upon the combustion of the fuel is $Q_f = m_f q$. The efficiency $\eta = N/Q_f = N/m_f q = 0.31$, i.e. $\eta = 31\%$.

128. A piece of ice having a mass $m_i = 50$ g and a temperature $t_i = -10$ °C is placed into a brass calorimeter having a mass $m_b = 200$ g and containing $m_w = 0.5$ kg of water at $t_w = 20$ °C. Find the temperature θ of water in the calorimeter after the ice has melted. The specific heats of water, ice, and brass are $c_w = 4.19$ kJ/(kg·K), $c_i = 2.1$ kJ/(kg·K), and $c_b = 0.38$ kJ/(kg·K). The latent heat of fusion of ice is $\lambda = 0.33$ MJ/kg.

Solution. The amount of heat given away by water is $Q_1 = m_w c_w (t_w - \theta)$. The amount of heat released by the calorimeter is $Q_2 = m_b c_b (t_w - \theta)$. The amount of heat received by ice during its heating to 0 °C is $Q_3 = m_i c_i (0\ °\mathrm{C} - t_i)$. The amount of heat absorbed during melting ice at 0 °C is $Q_4 = \lambda m_i$. The amount of heat received by the water produced by melting ice during its heating from 0 °C to θ is $Q_5 = m_i c_w \theta$ (since the mass of water is equal to the mass of ice). Hence we can write the heat balance equation:

$$Q_1 + Q_2 = Q_3 + Q_4 + Q_5,$$

$$m_w c_w (t_w - \theta) + m_b c_b (t_w - \theta) = m_i c_i (0\ °\mathrm{C} - t_i) + \lambda m_i + m_i c_w \theta,$$

$$-m_w c_w \theta - m_b c_b \theta - m_i c_w \theta = m_i c_i |t_i| + \lambda m_i - m_w c_w t_w - m_b c_b t_w.$$

This gives

$$\theta = \frac{m_i c_i |t_i| + \lambda m_i - m_w c_w t_w - m_b c_b t_w}{-m_w c_w - m_b c_b - m_i c_w} = 10.9\ °\mathrm{C}.$$

129. Steam having the normal pressure and a temperature $t_s = 150$ °C is injected into a vessel containing $m_w = 2$ kg of water and $m_i = 0.5$ kg of ice at 0 °C. After the ice has melted, a temperature $\theta = 30$ °C sets in the vessel. What is the mass of steam if the heat capacity of the vessel is $C_{ves} = 630$ J/K? The specific heats of water and steam are $c_w = 4.19$ kJ/(kg·K) and $c_s = 1.97$ kJ/(kg·K). The latent heat of fusion for ice is $\lambda = 0.33$ MJ/kg, the latent heat of vaporization for water is $r = 2.26$ MJ/kg.

Solution. The amount of heat released by steam during its cooling to the condensation (vaporization) temperature is $Q_1 = m_s c_s (t_s - 100\ °\mathrm{C})$. The amount of heat liberated by steam during its condensation is $Q_2 = m_s r$. The amount of heat given away by water formed as a result of condensation during its cooling from the boiling point to θ is $Q_3 = m_s c_w (100\ °\mathrm{C} - \theta)$. The amount of heat received by ice during its melting is $Q_4 = m_i \lambda$. The amount of heat received by water contained in the vessel and that formed from melting ice during heating from 0 °C to θ is $Q_5 = (m_w + m_i) c_w \theta$. The amount of heat received by the vessel during heating to the temperature θ is $Q_6 = m_{ves}c_{ves}\theta = C_{ves}\theta$. We write the heat balance equation:

$$Q_1 + Q_2 + Q_3 = Q_4 + Q_5 + Q_6,$$

$$m_s c_s(t_s - 100\ ^\circ\mathrm{C}) + m_s r + m_s c_w(100\ ^\circ\mathrm{C} - \theta) = m_i\lambda + (m_w + m_i)c_w\theta + C_{ves}\theta.$$

Hence we can find

$$m_s = \frac{m_i\lambda + (m_w + m_i)c_w\theta + C_{ves}\theta}{c_s(t_s - 100\ ^\circ\mathrm{C}) + r + c_w(100\ ^\circ\mathrm{C} - \theta)} = 190\ \mathrm{g}.$$

130. A lead pellet flying with a velocity $v_1 = 100$ m/s pierces a board and after that its velocity becomes $v_2 = 60$ m/s. Find the increase in temperature of the pellet, assuming that the fraction $\alpha = 0.4$ of its lost kinetic energy is spent for increasing the internal energy. The specific heat of lead is $c = 125$ J/(kg·K).

Solution. The decrease in the kinetic energy of the pellet piercing the board is $\Delta W_k = m(v_2^2 - v_1^2)/2$. The amount of heat spent for heating the pellet is $Q = cm\,\Delta t$. Since by the condition of the problem $cm\,\Delta t = \alpha\,\Delta W_k$, we obtain

$$\Delta t = \alpha\,\Delta W_k/cm = \alpha(v_2^2 - v_1^2)/2c = 10.2\ ^\circ\mathrm{C}.$$

131. Water having a volume $V = 0.8$ l and a temperature $t = 15$ °C is heated to boiling on an electric hot plate of power $N = 500$ W and efficiency $\eta = 40\%$, and 10% of water vaporizes. What time does the process take?

Solution. The amount of heat required for heating water from 15 °C to the boiling point is $Q_1 = c_w m(100\ ^\circ\mathrm{C} - t)$. To vaporize 10% of water, the amount of heat $Q_2 = 0.10rm$ should be spent. We equate the total amount of heat to the useful work of the electric hot plate:

$$\eta N\tau = Q_1 + Q_2 = c_w m(100\ ^\circ\mathrm{C} - t) + 0.10rm,$$

where τ is the required time of heating and vaporizing water. Hence

$$\tau = [c_w m(100\ ^\circ\mathrm{C} - t) + 0.10r]/\eta N,$$

where $r = 2.26$ MJ/kg, $c_w = 4.19$ kJ/(kg·K), and the mass of water is $m = 0.8$ kg. Substituting these values into the last equation, we obtain $\tau = 2.34$ kJ/W $= 2340$ s $\simeq 39$ min.

132. Air is rapidly pumped out of a vessel containing $m = 100$ g of water at $t = 0$ °C. As a result of intense evaporation, a part of water that has no time to evaporate is frozen. Find the mass of ice thus formed if the latent heat of vaporization at 0 °C is $r = 2.49$ MJ/kg, and the latent heat of fusion for ice is $\lambda = 0.336$ MJ/kg. Assume that the vessel is thermally insulated.

Solution. During the freezing of m_1 of water, the amount of heat $Q_1 = \lambda m_1$ is liberated. During the evaporation of $m_2 = (m - m_1)$ of water, the amount of heat $Q_2 = rm_2$ is absorbed. Since $Q_1 = Q_2$, we have $\lambda m_1 = rm - rm_1$, whence $m_1 = rm/(\lambda + r) = 88$ g.

Exercises

123. (a) A sledge whose mass is 50 kg slides down a hill with a slope of 30°. At

what distance from the starting point is its velocity 10 m/s if 3 kJ of heat are liberated as a result of friction between the runners and snow?
Answer. 22.4 m.

(b) A 10-kg body is lifted along a plane with a slope of 60° with a constant velocity to a height of 1 m. The applied force of 102 N acts along the plane. Find the amount of heat liberated and the coefficient of friction between the body and the plane.
Answer. 20 J, 0.35.

124. Find the pitch of thread of a drill if a copper cylinder whose diameter is 15 cm is heated by 5 °C when a hole of 50-mm diameter is drilled in it. The torque applied to the tap wrench is 24.2 N · m, and 70% of the spent energy are converted into heat. The density of copper is 8.9×10^3 kg/m^3 and its specific heat is 0.38 kJ/(kg · K).
Answer. 0.4 mm.

125. (a) A mass m_1 of water at a temperature t_1 is mixed with a mass m_2 at a temperature t_2. Find the temperature of the mixture.
Answer. $(m_1t_1 + m_2t_2)/(m_1 + m_2)$.

(b) Find the average temperature of mixture of a mass m_1 of a liquid having a specific heat c_1 and a temperature t_1 and a liquid with parameters m_2, c_2, and t_2.
Answer. $(m_1c_1t_1 + m_2c_2t_2)/(m_1c_1 + m_2c_2)$.

(c) An iron article having a mass of 600 g and a temperature of 100 °C is carried into water whose temperature is 10 °C. After some time, a temperature of 15.3 °C is settled. Find the mass of water if the heat lost on the way to water amounts to 20%. The specific heats of water and iron are 4.19 kJ/(kg · K) and 0.46 kJ/(kg · K).
Answer. 840 g.

126. The power developed by an aeroplane engine with a velocity of 900 km/h is 30 kW. Over a distance of 10 km, 8 kg of petrol are consumed. The heat of combustion of petrol is 46 MJ/kg. Find the efficiency of the engine.
Answer. 32.5%.

127. (a) Find the consumption of fuel oil whose heat of combustion is 44 MJ/kg for one-hour operation of an engine having a power of 1 kW if its efficiency is 30%.
Answer. 274 g.

(b) A thermoelectric plant consumes 400 g of ideal fuel for generating 1 kWh of electric energy. Find the efficiency of the power plant if the heat of combustion of ideal fuel is 29.4 MJ/kg.
Answer. 30.7%.

(c) A spirit lamp having an efficiency of 40% consumes 3 g/min of alcohol. What time is required to heat 1.5 l of water from 10 °C to the boiling point (100 °C)? The heat of combustion of alcohol is 29.4 MJ/kg.
Answer. 16 min.

128. A 6-kg piece of ice at a temperature of -20 °C is immersed in water having a mass of 10 kg and a temperature of 60 °C. What is the temperature of water after the whole ice has melted? The effect of cooling the vessel walls should be

neglected. The specific heat of ice is 2.1 kJ/(kg·K), its latent heat of fusion is 0.33 MJ/kg.
Answer. 4.2 °C.

(b) A piece of ice having a temperature of −8 °C is immersed in a brass calorimeter having a mass of 300 g and containing 1 kg of water at 18 °C. After the ice has melted, the temperature of water becomes 10 °C. Find the mass of ice if the specific heats of water, ice, and brass are 4.19 kJ/(kg·K), 2.1 kJ/(kg·K), and 0.38 kJ/(kg·K), respectively, and the latent heat of fusion for ice is 0.33 MJ/kg.
Answer. 89 g.

129. (a) Steam having normal pressure, a mass of 0.5 kg, and a temperature of 150 °C is injected into a vessel containing 3 l of water and 1 kg of ice at 0 °C. What temperature is settled in the vessel after the whole ice has melted? The specific heats of steam and water are 1.96 kJ/(kg·K) and 4.19 kJ/(kg·K), the heat capacity of the vessel is 0.82 kJ/(kg·K), the latent heat of vaporization for water is 2.26 MJ/kg, and the latent heat of fusion for ice is 0.33 MJ/kg.
Answer. 67.3 °C.

(b) 30 g of steam at 100 °C are injected into a brass vessel having a mass of 400 g and containing water at 15 °C. As a result, the temperature of water becomes 25 °C. If 60 g of steam are injected into the same vessel, the settled temperature will be 34.7 °C. Find the latent heat of vaporization for water from this experiment if the specific heats of water, steam, and brass are 4.18 kJ/(kg·K), 1.96 kJ/(kg·K), and 0.38 kJ/(kg·K) respectively.
Answer. 2.22 MJ/kg.

(c) Molten lead at the fusion temperature (327 °C) is poured to a mixture containing 20 l of water and 1 kg of ice at 0 °C. The temperature of the mixture becomes 100 °C, and besides 200 g of water are vaporized at this temperature. Find the mass of lead. The specific heat of lead is 0.125 kJ/(kg·K), its latent heat of fusion is 21 kJ/kg. The remaining data should be taken from the previous problems.
Answer. 33.6 kg.

130. (a) A lead bullet having a temperature of 27 °C melts as a result of impact. Find its velocity assuming that the entire kinetic energy of the bullet is converted into the internal energies of the bullet and the obstacle, 80% being converted into the internal energy of the bullet. The temperature of fusion for lead is 327 °C, the latent heat of fusion is 21 kJ/kg, and the specific heat of lead is 0.125 kJ/(kg·K).
Answer. 300 m/s.

(b) A lead ball falls from a height of 51 m. What is the increase in its temperature as a result of impact if 50% of heat liberated during the impact are spent for heating the ball?
Answer. 2 °C.

(c) Find the increase in the temperature of water falling from a height of 15 m, assuming that 30% of the kinetic energy of falling water are converted into its internal energy.
Answer. 0.01 °C.

131. 0.8 kg of ice taken at a temperature of −20 °C is melted on a 1-kW electric

hot plate. The obtained water is heated to boiling, and 25% of water are vaporized. Find the efficiency of the hot plate if the entire process takes 40 min.
Answer. 47%.

132. Air is rapidly pumped out of a thermally insulated vessel containing water at 0 °C; 11.7% of water are vaporized, and the remaining water freezes due to the intense evaporation. Find the latent heat of vaporization in this experiment. The latent heat of fusion for ice is 0.33 MJ/kg.
Answer. 2.5 MJ/kg.

3.15. Temperature Coefficients of Linear and Cubic Expansion

Experiments show that most of bodies increase their volume upon heating.[1] The extent of expansion of various bodies is characterized by the **temperature coefficient of expansion**, or simply the **coefficient of expansion**. While considering solids which retain their shape during temperature variations, the distinction is made between (a) a change in linear dimensions (viz. the dimensions in a certain direction), i.e. linear expansion, and (b) a change in the volume of a body, i.e. cubic expansion.

The **coefficient of linear expansion** is the quantity α equal to the fraction of the initial length by which a body taken at 0 °C has elongated as a result of heating it by 1 °C (or by 1 K):

$$\alpha = (l_t - l_0)/l_0 t,$$

where l_0 is the initial length at 0 °C and l_t is the length at a temperature t. From this expression, we can find

$$l_t = l_0(1 + \alpha t).$$

The dimensions of α are K^{-1} (or $°C^{-1}$).

The **coefficient of cubic expansion** is the quantity β equal to the fraction of the initial volume by which the volume of a body taken at 0 °C has increased upon heating it by 1 °C (or by 1 K):

$$\beta = (V_t - V_0)/V_0 t,$$

where V_0 is the volume of the body at 0 °C and V_t is its volume at

[1] The exceptions are water in the temperature interval from 0 °C to 4 °C and some other materials.

a temperature t. From this equation, we obtain

$$V_t = V_0(1 + \beta t).$$

The quantity β has also the dimensions of K^{-1} (or $°C^{-1}$).

The coefficient of cubic expansion is about three times larger than the coefficient of linear expansion:

$$\beta \simeq 3\alpha.$$

Proof. If we cut a cube of edge l from a solid material, its volume at 0 °C is $V_0 = l_0^3$. At a temperature t, the volume becomes

$$V_t = [l_0(1 + \alpha t)]^3 = l_0^3(1 + \alpha t)^3 = V_0(1 + \alpha t)^3.$$

But $V_t = V_0(1 + \beta t)$. Consequently, $1 + \beta t = (1 + \alpha t)^3$. Having opened the parentheses on the right-hand side of this expression, we obtain

$$1 + \beta t = 1 + 3\alpha t + 3(\alpha t)^2 + (\alpha t)^3,$$

or

$$\beta t = 3\alpha t + 3(\alpha t)^2 + (\alpha t)^3.$$

Since α is a very small quantity (ranging from 10^{-5} to 10^{-6}), αt is also small. Consequently, its square $(\alpha t)^2$, and the more so its cube $(\alpha t)^3$, are so small in comparison with αt that they can be neglected. Then the last equality becomes

$$\beta t \simeq 3\alpha t, \quad \text{or} \quad \beta \simeq 3\alpha.$$

The coefficients β of cubic expansion for liquids are somewhat higher than for solid bodies, ranging between 10^{-3} and 10^{-4} K^{-1}.

Water obeys the general laws of thermal expansion only at a temperature above 4 °C. From 0 °C to 4 °C, water contracts rather than expands. At 4 °C, water occupies the smallest volume, i.e. it has the highest density. At the bottom of deep lakes, there is denser water in winter, which retains the temperature of 4 °C even after the upper layer has been frozen.

Problems with Solutions

133. The lengths $l_{1\,i} = 100$ m of iron wire and $l_{1\,c} = 100$ m of copper wire are marked off at $t_1 = 20$ °C. What is the difference in the lengths of the wires at $t_2 =$

60 °C? The coefficients of linear expansion for iron and copper are $\alpha_i = 1.2 \times 10^{-5}\ K^{-1}$ and $\alpha_c = 1.7 \times 10^{-5}\ K^{-1}$.

Solution. $l_{1i} = l_{0i}(1 + \alpha_i t_1)$ and $l_{2i} = l_{0i}(1 + \alpha_i t_2)$. The elongation of the iron wire is $l_{2i} - l_{1i} = l_{0i}\alpha_i(t_2 - t_1)$. Substituting $l_{0i} = l_{1i}/(1 + f_i t_1)$, we find the elongations of the iron and copper wires:

$$l_{2i} - l_{1i} = l_{1i}\alpha_i(t_2 - t_1)/(1 + \alpha_i t_1), \quad (1)$$

$$l_{2c} - l_{1c} = l_{1c}\alpha_c(t_2 - t_1)/(1 + \alpha_c t_1), \quad (2)$$

Subtracting Eq. (1) from Eq. (2) and considering that $l_{1i} = l_{1c} = l_1$, we obtain

$$l_{2c} - l_{2i} = l_1 \frac{(\alpha_c - \alpha_i)(t_2 - t_1)}{(1 + \alpha_c t_1)(1 - \alpha_i t_1)} \simeq 19.9 \text{ mm}.$$

For low values of temperature t, when $\alpha t \ll 1$, it is not necessary to reduce l_1 and l_2 to l_{01} and l_{02} at $t = 0$ °C. To a sufficiently high degree of accuracy, we can assume that $\Delta l = l\alpha\,\Delta t$. Under this assumption, the problem can be solved in a simpler way:

$$\Delta l_i = l_{1i}\alpha_i(t_2 - t_i), \quad \Delta l_c = l_{1c}\alpha_c(t_2 - t_1).$$

Consequently, since $l_{1i} = l_{1c} = l_1$, we have

$$\Delta l = \Delta l_c - \Delta l_i = l_1(t_2 - t_1)(\alpha_c - \alpha_i) = 20 \text{ mm}.$$

It can be seen that the deviation from a more exact value of 19.9 mm amounts to 0.1 mm, i.e. the relative error $\Delta = 0.1/19.9 = 0.5\%$.

134. Find the force F with which a steel bar having a cross-sectional area $S = 20\ \text{cm}^2$ acts on the massive walls between which it is fixed as a result of an increase in its temperature by $\Delta t = 30$ °C. The elastic modulus for steel is $E = 2 \times 10^{11}$ Pa and the coefficient of linear expansion is $\alpha = 1.2 \times 10^{-5}\ K^{-1}$.

Solution. If the bar were free, its elongation as a result of the same increase in temperature would be $\Delta l = l\alpha\,\Delta t$, and the relative elongation $\varepsilon = \Delta l/l = \alpha\,\Delta t$. In order to retain the length of the bar, the longitudinal compressive forces should be applied to its ends, which satisfy the condition $\sigma = E\varepsilon$. Since $\sigma = F/S$, we have $F = \sigma S$, and then $F = E\alpha\,\Delta t\,S = 144$ kN. In accordance with Newton's third law, the bar acts on the walls with the same force.

135. The density of water is $\rho = 0.99 \times 10^3\ \text{kg/m}^3$. What is its temperature? The coefficient of cubic expansion for water is $\beta = 1.8 \times 10^{-4}\ K^{-1}$.

Solution. The density of water at 4 °C is $\rho_0 = 1.03 \times 10^3\ \text{kg/m}^3$. At a given temperature,

$$\rho = \frac{m}{V} = \frac{m}{V_{4°}[1 + \beta(t - 4\ °C)]} = \frac{\rho_0}{1 + \beta(t - 4\ °C)},$$

whence $\beta(t - 4\ °C) = \rho_0/\rho - 1$ and $t = (\rho_0 - \rho)/\rho\beta + 4\ °C = 54$ °C.

136. Petroleum is stored in a cylindrical tank of height $h = 10$ m. At a temperature $t_0 = -10$ °C, the separation between the level of petroleum and the top of the tank is $\Delta h = 50$ cm. At what temperature t does petroleum overflow from the tank? The coefficient of cubic expansion for petroleum is $\beta = 1.0 \times 10^{-4}\ K^{-1}$. The expansion of the tank walls due to heating should be neglected.

Solution. Since the cross-sectional area of the tank is the same over its height, the volume of petroleum in the tank is proportional to the height of petroleum above the bottom of the tank. Therefore, in the formulas for cubic expansion we shall use the height of petroleum in the tank instead of its volume. The height h_0 of the level of petroleum in the tank at 0 °C can be found from the equation $h - \Delta h = h_0(1 + \beta t_0)$:

$$h_0 = (h - \Delta h)/(1 + \beta t_0) = 9.595\ \text{m}.$$

(Although it is an intermediate quantity, its value should be calculated at this stage in order to simplify subsequent formulas.) As the temperature increases further, $h = h_0(1 + \beta t)$, whence $\beta t = h/h_0 - 1$ and $t = (h - h_0)/\beta h_0 = 42.2$ °C. It is assumed in the given solution that

$$h = (h - \Delta h)(1 + \beta t)/(1 + \beta t_0).$$

For comparison, we give the solution, based on an approximate formula $h = (h - \Delta h)\,[1 + \beta(t - t_0)]$. We obtain

$$\beta(t - t_0) = \Delta h/(h - \Delta h), \quad \text{or} \quad t = \Delta h/[\beta(h - \Delta h)] + t_0 = 42.6\ ^\circ\text{C}.$$

The absolute error $\delta = 42.6 - 42.2 = 0.4$ °C. The relative error $\Delta = 0.94\%$.

137. Solve Problem 136, taking into account the expansion of the steel tank whose diameter $d = 6$ m. The coefficient of linear expansion for steel is $\alpha_{st} = 1.1 \times 10^{-5}\ K^{-1}$.

Solution. Since hollow bodies expand upon heating in the same way as solid bodies, the volume of the steel tank at a temperature t is

$$V_1 = (\pi d^2/4)h\,[1 + \beta_{st}(t - t_0)], \quad \text{where} \quad \beta_{st} = 3\alpha_{st}.$$

The volume of petroleum at the same temperature is

$$V_2 = (\pi d^2/4)\{(h - \Delta h)[1 + \beta(t - t_0)]\}.$$

We equate the right-hand sides of these expressions:

$$h\beta_{st}(t - t_0) = (h - \Delta h)\beta(t - t_0) - \Delta h,$$

or

$$t - t_0 = \Delta h/[\beta(h - \Delta h) - \beta_{st} h],$$

whence $t_1 = \Delta t/[\beta(h - \Delta h) - 3\alpha_{st} h] + t = 44.1$ °C.

138. An iron sheet whose size at 0 °C is $2.0 \times 3.0\ m^2$ is heated to $t = 200$ °C. Find the increase in the area of the sheet as a result of heating. The coefficient of linear expansion for iron is $\alpha = 1.2 \times 10^{-5}\ K^{-1}$.

Solution. To a sufficiently high degree of accuracy, the coefficient of superficial expansion can be assumed to be equal to the doubled coefficient of linear expansion. For the sake of simplicity, we shall prove this for a square sheet with side l. At 0 °C the area $S_0 = l_0^2$. At $t = 200$ °C, the area is

$$S_t = l_t^2 = l_0^2(1 + \alpha t)^2 = l_0^2(1 + 2\alpha t + \alpha^2 t^2) \approx S_0(1 + 2\alpha t)$$

(since the coefficient α of linear expansion is small, its square can be neglected). Thus, $\Delta S = S - S_0 = 2S_0\alpha t = 288\ \text{cm}^2$.

139. Vessels connected by a thin pipe with a sliding plug contain kerosene at temperatures $t_1 = 20\ °\text{C}$ and $t_2 = 80\ °\text{C}$. What is the ratio of heights of kerosene columns in the vessels if the coefficient of cubic expansion for kerosene is $\beta = 1.0 \times 10^{-3}\ \text{K}^{-1}$?

Solution. The heights of liquid columns are inversely proportional to the densities of liquid in the vessels (see Sec. 2.4): $h_2/h_1 = \rho_1/\rho_2$. And since $\rho_1 = m/V_1 = m/V_0(1 + \beta t_1) = \rho_0/(1 + \beta t_1)$ and $\rho_2 = \rho_0/(1 + \beta t_2)$, we obtain

$$h_2/h_1 = (1 + \beta t_2)/(1 + \beta t_1) = 1.06.$$

140. A copper and a tungsten plates having a thickness $\delta = 2$ mm each are riveted together so that at 0 °C they form a flat bimetallic plate. Find the average radius of curvature of this plate at $t = 200\ °\text{C}$. The coefficients of linear expansion for copper and tungsten are $\alpha_c = 1.7 \times 10^{-5}\ \text{K}^{-1}$ and $\alpha_t = 0.4 \times 10^{-5}\ \text{K}^{-1}$.

Solution. The average length of the copper plate at a temperature $t = 200\ °\text{C}$ is $l_c = l_0(1 + \alpha_c t)$, where l_0 is the length of the copper plate at 0 °C. The length of the tungsten plate is $l_t = l_0(1 + \alpha_t t)$. We shall assume that the edges of the plates are not displaced during deformation and that an increase in the plate thickness

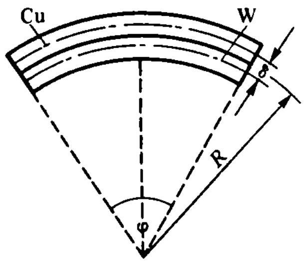


Fig. 168

due to heating can be neglected. From Fig. 168, we find

$$l_c = \varphi(R + \delta/2), \quad l_t = \varphi(R - \delta/2).$$

Consequently,

$$\varphi(R + \delta/2) = l_0(1 + \alpha_c t),$$
$$\varphi(R - \delta/2) = l_0(1 + \alpha_t t).$$

To eliminate the unknown quantities φ and l_0, we divide the first equation by the second termwise:

$$(R + \delta/2)/(R - \delta/2) = (1 + \alpha_c t)/(1 + \alpha_t t).$$

This gives

$$R = \delta[2 + (\alpha_c + \alpha_t)t]/[2(\alpha_c - \alpha_t)t].$$

The quantity $(\alpha_c + \alpha_t)t$ in the numerator can be neglected in comparison with the first term. Hence $R = \delta/(\alpha_c - \alpha_t)t = 0.769$ m.

141. A solid body floats in a liquid at a temperature $t = 50$ ° being completely submerged in it. What fraction δ of the volume of the body is submerged in the liquid after its cooling to $t_0 = 0$ °C if the coefficient of cubic expansion for the solid is $\beta_s = 0.3 \times 10^{-5}$ K^{-1} and of the liquid, $\beta_l = 8.0 \times 10^{-5}$ K^{-1}?

Solution. At $t_0 = 0$ °C, the buoyancy is

$$F_b = \delta V_0 \rho_0 g, \tag{1}$$

where V_0 is the volume of the body and ρ_0 is the density of the liquid at $t_0 = 0$ °C. At $t = 50$ °C, the volume of the body becomes $V = V_0(1 + \beta_s t)$ and the density of the liquid is $\rho_l = \rho_0/(1 + \beta_l t)$. The buoyancy in this case is

$$F_b = V_0 \rho_0 g(1 + \beta_s t)/(1 + \beta_l t). \tag{2}$$

Equating the right-hand sides of Eqs. (1) and (2), we obtain

$$\delta = (1 + \beta_s t)/(1 + \beta_l t) = 96\%.$$

Exercises

133. (a) An iron and a zinc rods have lengths of 604 mm and 600 mm at 0 °C. At what temperature do their lengths become equal? The coefficients of linear expansion for iron and zinc are 1.2×10^{-5} K^{-1} and 2.9×10^{-5} K^{-1}.

Answer. 390 °C.

(b) What must be the lengths of a steel and a copper rods at 0 °C for their difference be 10 cm at any temperature? The coefficients of linear expansion for steel and copper are 1.2×10^{-5} K^{-1} and 1.8×10^{-5} K^{-1}.

Answer. 30 cm for steel rod and 20 cm for copper rod.

(c) Two rods having lengths l_1 and l_2 and made of materials with the linear expansion coefficients α_1 and α_2 were soldered together. Find the coefficients of linear expansion for the obtained rod.

Answer. $\alpha = (l_1\alpha_1 + l_2\alpha_2)/(l_1 + l_2)$.

(d) A brass rod having a mass of 4.25 kg and a cross-sectional area of 5 cm^2 increases its length by 0.3 mm upon heating from 0 °C. What amount of heat is spent for heating the rod? The coefficient of linear expansion for brass is 2.0×10^{-5} K^{-1}, its specific heat is 0.39 kJ/(kg·K), and the density of brass is 8.5×10^3 kg/m^3.

Answer. 24 kJ.

134. (a) A steel rod having a cross-sectional area of 10 cm^2 pushes against two rigidly fixed massive slabs by its ends. Find the force exerted by the rod on the slabs when its temperature is increased by 15 °C. The elastic modulus for steel is 2×10^{11} Pa and the coefficient of linear expansion is 1.2×10^{-5} K^{-1}.

Answer. 37.8 kN.

(b) Find the longitudinal force that should be applied to a 10-m long steel rail having a cross-sectional area of 50 cm^2 to cause the same elongation of the rail as occurs upon increasing its temperature from 0 °C to 40 °C. The elastic modulus

and the coefficient of linear expansion for steel are 2×10^{11} Pa and 1.2×10^{-5} K^{-1}.
Answer. 515 kN.

135. (a) Find the density of water at a temperature of 20 °C if the coefficient of cubic expansion for water is 1.8×10^{-4} K^{-1}.
Answer. 0.997×10^3 kg/m^3.

(b) A mixture is prepared from three masses m_1, m_2, and m_3 of water taken at temperatures t_1, t_2, and t_3 respectively. Find the density and temperature of the mixture.
Answer. $\rho = \rho_0/[1 + \beta(t - 4)]$, where $\rho_0 = 1 \times 10^3$ kg/m^3 is the density of water at 4 °C and $t = (m_1t_1 + m_2t_2 + m_3t_3)/(m_1 + m_2 + m_3)$.

136. The level of kerosene contained in a cylindrical iron tank whose height is 6 m and diameter of the cross section is 5 m is 20 cm lower than the top of the tank at 0 °C. Find the temperature at which kerosene starts to overflow from the tank. The coefficient of cubic expansion for kerosene is 1×10^{-3} K^{-1}. The expansion of the tank should be neglected.
Answer. 34.5 °C.

137. (a) Solve the preceding problem, taking into account the expansion of the tank. The coefficient of linear expansion for the tank material is 1.2×10^{-5} K^{-1}.
Answer. 34.8 °C.

(b) A glass flask contains 330 g of mercury at 0 °C and 325 g at 100 °C. Find the coefficient of linear expansion for glass if the coefficient of cubic expansion for mercury is 1.8×10^{-4} K^{-1}.
Answer. 0.9×10^{-5} K^{-1}.

138. The area of a brass plate is 120 cm^2 at 10 °C and 120.5 cm^2 at 110 °C. Find the coefficient of linear expansion for brass.
Answer. 2.0×10^{-5} K^{-1}.

139. (a) Two vessels connected by a pipe with a sliding plug contain mercury. In one vessel, the height of mercury column is 39.3 cm and its temperature is 0 °C, while in the other, the height of mercury column is 40 cm and its temperature is 100 °C. Find the coefficient of cubic expansion for mercury. The volume of the connecting pipe should be neglected.
Answer. 1.8×10^{-4} K^{-1}.

(b) One of two vessels connected by a pipe with a sliding plug and filled with kerosene contains a 30-cm column at 10 °C. The temperature of kerosene in the other vessel is 100 °C. Find the difference in the levels of liquid in the communicating vessels. The volume of the connecting pipe and the thermal expansion of the vessels should be neglected. The coefficient of cubic expansion for kerosene is 1×10^{-3} K^{-1}.
Answer. 2.7 cm.

140. A copper and an iron plates having the same thickness of 2 mm have the same length at 0 °C. They are arranged in parallel and their ends are soldered so that there is a 1-mm gap between the plates. Assuming that the plates bend to form an arc upon heating, find the mean radius of the outer plate after the system has been heated to 200 °C. The coefficients of linear expansion for iron and copper are 1.2×10^{-5} K^{-1} and 1.7×10^{-5} K^{-1}.
Answer. 252 cm.

141. (a) A copper ingot is submerged in water. Find the change in the buoyancy due to an increase in the temperatures of water and copper by 50 °C. The coefficient of linear expansion for copper is $1.7 \times 10^{-5}\ K^{-1}$, and the coefficient of cubic expansion for water is $1.8 \times 10^{-4}\ K^{-1}$.
Answer. The buoyancy decreases by 0.6%.

(b) A solid body floats in a liquid at a temperature of 0 °C so that 98% of its volume are submerged in water. When the liquid is heated to 25 °C, the body is submerged in it completely. Find the coefficient of cubic expansion for the liquid if the coefficient of cubic expansion for the solid is $2.6 \times 10^{-6}\ K^{-1}$.
Answer. $8.2 \times 10^{-4}\ K^{-1}$.

C. GAS LAWS

The state of a certain mass of a gas is uniquely determined by its volume V, pressure p, and temperature T. The quantities V, p, and T are called the **parameters of state**. They are not independent and are related to one another in a certain way. The mathematical expression of this relationship is called an **equation of state.** The processes occurring in a gas so that one of the state parameters remains unchanged are called **isoprocesses.**[2]

3.16. Isobaric Process. Charles' Law

Like other bodies, gases expand upon heating.

A process of expansion or compression of a gas at constant pressure (p = const) is called an **isobaric process**. Such processes obey **Charles' law**: *the volume of a given mass of any gas increases for each degree rise in temperature by* 1/273 (to be more exact, by 1/273.15) *of the volume at* 0 °C, *the pressure being constant throughout*:

$$V = V_0(1 + \alpha t).$$

Here V_0 is the volume of the gas at 0 °C *and α is the temperature coefficient of cubic expansion of the gas*, which is proved experimentally to have the same value for all ideal gases (see Sec. 3.22):

$$\alpha \simeq 1/(273\ K) \simeq 3.66 \times 10^{-3}\ K^{-1}.$$

[2] From the Greek word *isos* meaning equal.

Substituting into the previous equation t °C = T K − 273.15, we obtain another expression for Charles' law:

$$V = V_0 \alpha T.$$

The volume of a certain mass of an ideal gas is proportional to its thermodynamic temperature at constant pressure.

Let us suppose that the temperature of a gas changes from T_1 to T_2. The volume of the gas at T_1 is $V_1 = V_0 \alpha T_1$, while the volume of the gas at T_2 is $V_2 = V_0 \alpha T_2$. Dividing the latter expression by the former, we obtain

$$V_2/V_1 = T_2/T_1, \quad \text{or} \quad V_2 = V_1 T_2/T_1.$$

Hence

$$V_2/T_2 = V_1/T_1 = \text{const} \; (= V_0/T_0).$$

In an isobaric process, the ratio of the volume of a gas to its thermodynamic temperature is constant.

Figures 169*a* and *b* show the graphs of an isobaric process in the p vs V and V vs T coordinates.

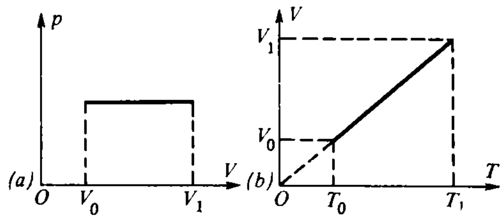


Fig. 169

An isobaric process can be carried out, for example, by heating a gas in a cylinder with easily movable piston to which a constant force is applied.

3.17. Isothermal Process. Boyle's Law. Dalton's Law

Boyle's law establishes the relationship between a change in the pressure of a gas and a change in its volume at constant temperature (T = const), i.e. in an **isothermal process** of gas expansion or compression: *the pressure of a given mass of a gas at constant temperature varies in inverse proportion to its volume*:

$$p_1/p_2 = V_2/V_1, \quad \text{or} \quad p_1V_1 = p_2V_2.$$

This equality shows that the product of the volume of a given mass of a gas and its pressure is constant in an isothermal process:

$$pV = \text{const.}$$

The constants of processes occurring at different temperatures are different. For any gas, an isothermal process is depicted by a family of curves (Fig. 170) called **isotherms**. These curves

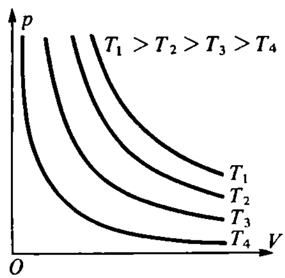


Fig. 170

describe the dependence between the volume and pressure of this gas, each curve corresponding to a certain temperature. For an ideal gas, these curves are hyperbolas.

The molecular-kinetic theory of matter gives the following explanation to Boyle's law. In random motion, the molecules of a gas collide with the vessel walls. A very large number of consecutive collisions is perceived as a permanent pressure of the gas on the vessel walls. If the volume is reduced several times, the number of molecules in a unit volume of the gas increases accordingly, and hence the number of molecular collisions with the vessel walls per unit time increases in the same proportion, since the average velocity of molecules remains unchanged if the temperature does not change. Consequently, the gas pressure caused by quite frequent molecular collisions increases the same number of times.

If a vessel contains a mixture of several gases, the pressure in the vessel is equal to the sum of the partial pressures, viz. the pressures existing in the vessel containing only one of the gases. This is the essence of **Dalton's law** for a gas mixture.

3.18. Isochoric Process. Gay-Lussac's Law

Gay-Lussac's law governs an **isochoric process**, i.e. the process occurring at constant volume (V = const): *the pressure of a given mass of a gas at constant volume increases for each degree rise in temperature by* 1/273 *of its pressure at* 0 °C.

The coefficient

$$\gamma = (p - p_0)/p_0 t = (1/273)\ \mathrm{K}^{-1}$$

characterizing the increase in pressure is called the *temperature coefficient of pressure of a gas*. In an isochoric process, the pressure of a gas varies in accordance with the formula

$$p = p_0(1 + \gamma t).$$

Since t °C = T K − 273.15, we have $p = p_0\gamma T$. Hence it follows that if a gas has a pressure p_1 at a temperature T_1 and a pressure p_2 at a temperature T_2, then

$$p_2 = p_1 T_2/T_1, \quad \text{and hence} \quad p_2/T_2 = p_1/T_1 = \text{const}\ (= p_0/T_0).$$

In an isochoric process, the ratio of the gas pressure to its thermodynamic temperature is constant.

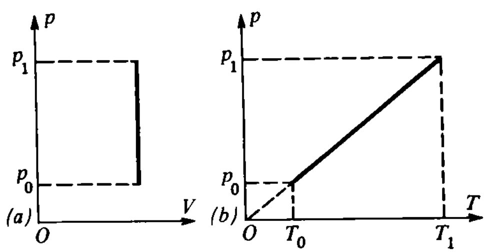


Fig. 171

The graphs of an isochoric process in the p vs V and p vs T coordinates are shown in Figs. 171*a* and *b*.

Gay-Lussac's law can be easily explained by the molecular-kinetic theory. The velocities of gas molecules increase with temperature. A molecule covers the distance between the vessel walls at a much shorter time at an elevated temperature. For this reason, it collides with the vessel walls more frequently. The pressure of gas molecules on the vessel walls increases accordingly.

3.19. Adiabatic Process

A process of compression or expansion of a gas *without heat exchange with the ambient* is called an **adiabatic process**. The curve depicting an adiabatic process is called an **adiabat**. To analyze this curve in the p vs V coordinates, let us consider the first law of thermodynamics (see Sec. 3.7): $\Delta U = A' + Q$, where A' is the work done by *external forces*.

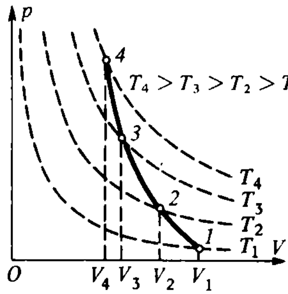


Fig. 172

Let us first consider an isothermal process. Figure 172 shows four isotherms for temperatures T_1, T_2, T_3, and T_4. Since according to Charles' law an increase in temperature of a given mass of a gas at constant pressure leads to an increase in its volume, $T_4 > T_3 > T_2 > T_1$ in Fig. 172. Isothermal processes occur at constant internal energy ($\Delta U = 0$), i.e. $A' + Q = 0$. For an isothermal compression, $A' > 0$, and hence $Q < 0$. This means that the process of isothermal compression is accompanied by a heat transfer from the gas to the ambient.

In an adiabatic compression of a gas, $Q = 0$, i.e. $\Delta U = A' >$

0. The work done by external forces is positive, the internal energy of the gas increases, and hence the gas temperature rises. This means that as the volume of the gas decreases, the adiabat intersects the isotherms corresponding to higher temperatures. Hence it follows that an adiabat is steeper than an isotherm.

In the theory of ideal gas, the *adiabatic equation* is derived:

$$p_1/p_2 = (V_2/V_1)^\gamma, \quad \text{or} \quad pV^\gamma = \text{const},$$

where $\gamma > 1$. The coefficient $\gamma = C_p/C_V$, where C_p and C_V are the heat capacities of a gas at constant pressure and at constant volume. The value $\gamma = 5/3$ corresponds to monatomic gases, $\gamma = 7/5$ for diatomic gases, and $\gamma = 4/3$ for polyatomic gases.

An adiabatic process can be realized with a perfect thermal insulation of a gas or for such a rapid process that the heat exchange with the ambient during a very short time interval taken by the process can be neglected.

Adiabatic processes play a significant role in nature and in engineering. Such processes take place, for example, in internal combustion engines where compression occurs during fractions of a second. Another example of an adiabatic process is the gas compression or expansion accompanying the propagation of an acoustic wave.

3.20. The Boyle-Charles Generalized Law. Equation of State for an Ideal Gas

The general relation connecting all the three state parameters of an ideal gas is expressed by the **Boyle-Charles generalized law** of gas state. Mathematically, this law is written as follows:

$$pV/T = \text{const},$$

i.e. *for a given mass of an ideal gas, the product of pressure and volume divided by the thermodynamic temperature is constant.*

The value of the constant can be calculated if we take a given mass of a gas under normal conditions ($T = T_0 = 273.15$ K and $p = p_0 = 1.013 \times 10^5$ Pa):

$$\text{const} = p_0V_0/T_0 = mp_0/\rho_0T_0.$$

Here ρ_0 is the density of the given gas under normal conditions.

Let us derive the equation of the generalized law of state for an ideal gas. Suppose that the same gas is characterized in two arbitrary states by the following state parameters: V_1, p_1, T_1 and V_2, p_2, T_2. Since Boyle's and Charles' laws are fulfilled independently of the sequence of different processes, we can imagine that the transition from the initial state to the final state occurs in two stages. Suppose, for example, that pressure p_2 is first created as a result of an isothermal process (T = const) and then temperature T_2 is attained as a result of an isobaric process (p = const). In accordance with Boyle's law, the volume of the gas would become

$$V_1' = p_1 V_1 / p_2 \tag{3.20.1}$$

as a result of the isothermal process. Further, on the basis of Charles' law, we have

$$V_1'/T_1 = V_2/T_2. \tag{3.20.2}$$

Substituting expression (3.20.1) for V_1' into this equation, we obtain the *equation of state for an ideal gas*:

$$p_1 V_1 / T_1 = p_2 V_2 / T_2. \tag{3.20.3}$$

This means that pV/T = const for a given mass of a gas.

Remark. The equation of state for an ideal gas can be derived in a different way, going over from the initial state to the final state first via an isobaric process and then via an isothermal process. Alternately, we could use an isochoric process (obeying Gay-Lussac's law) and then an isobaric process, and so on.

3.21. The Clapeyron-Mendeleev Equation. Avogadro's Law

The equation expressing the generalized law of state for an ideal gas can be written for a mole of the gas instead of an arbitrary mass.

A **mole** is the amount of substance containing the same number of structural units as the number of atoms in 0.012 kg of carbon ^{12}C. Thus, the number of structural units (molecules or atoms) in a mole of any substance is the same and equal to N_A =

$6.02 \times 10^{23}\ \mathrm{mol}^{-1}$. This quantity is called the **Avogadro number** (or **constant**).

According to **Avogadro's law** (or **hypothesis**), *a mole of any gas under the normal atmospheric pressure* ($p_0 = 1.013 \times 10^5$ Pa) *and a temperature of* 0 °C ($T_0 = 273.15$ K) *occupies the same volume* $V_\mu = 22.4 \times 10^{-3}\ \mathrm{m^3/mol}$, *which is called the molar volume*. Consequently, the gas constant in the equation of state for an ideal gas is the same for a mole of any gas. It is called the **gas constant**, is denoted by R, and can be calculated from the normal conditions.

In SI, we have

$$R = p_0 V_\mu / T_0 = 8.31\ \mathrm{J/(mol \cdot K)}. \qquad (3.21.1)$$

Consequently, the equation of state for a mole of any gas is the same and is written as follows:

$$pV_\mu / T = R, \quad \text{or} \quad pV_\mu = RT. \qquad (3.21.2)$$

This equation can be easily generalized for an arbitrary mass of a gas if we express the molar volume V_μ in terms of an arbitrary volume V as follows:

$$V_\mu = \mu/\rho = \mu V/m, \qquad (3.21.3)$$

where μ is the molar mass (the mass of a mole) of an ideal gas. Substituting this value of V_μ into Eq. (3.21.2), we obtain the equation of state for an ideal gas for an arbitrary mass, which is called the **Clapeyron-Mendeleev equation**:

$$pV = (m/\mu)RT.$$

3.22. Ideal Gas. Physical Meaning of Thermodynamic Temperature

An **ideal gas** is an imaginary gas that exactly obeys gas laws. It could be a gas (a) that is in "thermal equilibrium", i.e. has the same temperature and pressure in all parts of the vessel containing it, and (b) whose molecules do not interact.

"Thermal equilibrium" is established in a gas quite rapidly. As regards the second of the above properties, only at sufficiently low pressures and at moderate temperatures, the distances between molecules are so large in real gases that the forces of mutual attraction are vanishingly small. Under these conditions, real gases are close in their properties to an ideal gas.

At high pressures, the intermolecular distances in gases become smaller, the forces of mutual attraction increase, the volume of molecules becomes comparable with the total volume of a gas, and the laws described in the previous sections become insufficiently correct. For high pressures, corrections are introduced into the formulas expressing gas laws for ideal gases.

While monatomic molecules can only be in translational motion in gases, polyatomic molecules can be in addition in rotational and vibrational motion. It follows from the molecular-kinetic theory that the average value of the energy of translational motion of a gas molecule is equal to $(3/2)kT$, where the proportionality factor $k = 1.38 \times 10^{-23}$ J/K is called the **Boltzmann constant** and T is the thermodynamic temperature. Thus, thermodynamic temperature has a simple physical meaning. This is the quantity proportional to the average energy of translational motion of molecules.

3.23. Work Done by a Gas During Expansion

The work done by a gas during expansion can be easily calculated for an isobaric process, for example, for an isobaric expansion of a gas in a cylinder. If the area of the piston is S, the gas pressure is p, and the distance by which the piston is moved during expansion is l, the force acting on the piston during expansion of the gas is $F = pS$, and the work done by this force is

$$A = Fl = pSl.$$

Since the product Sl is equal to the increment ΔV of the gas volume, we have

$$A = p\Delta V.$$

Obviously, the work A is numerically equal to the area below the curve describing the pressure as a function of the volume.

In order to calculate the work done during an isothermal expansion of a gas, we divide the p-V diagram (Fig. 173) into ver-

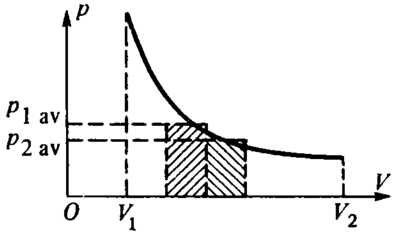


Fig. 173

tical strips, and within each strip we replace the isothermal process by an isobaric process, corresponding to the average pressure for a given strip: $p_{1\,\mathrm{av}}$, $p_{2\,\mathrm{av}}$, etc. Within each strip, the work of expansion of the gas is equal to its area. Thus, the entire isotherm is replaced by a step-like curve which is the closer to the isotherm, the narrower the strips. In the limit, the total area of all the rectangles is equal to the area bounded by the isotherm and the vertical straight lines V_1 and V_2 corresponding to the initial and final states.

For any other process, the work done by a gas is also numerically equal to the area below the p-V diagram in the corresponding scale.

Problems with Solutions

142. A cylinder having a volume $V = 200$ l contains oxygen at a temperature $t = 20$ °C and a pressure $p = 10^7$ Pa. Find the volume which this amount of oxygen would occupy under normal conditions.[3]

Solution. We use the equation of state in the form $pV/T = p_0V_0/T_0$, whence $V_0 = pVT_0/p_0T$. Since T K $= t$ °C + 273.15 K = 293.15, $V_0 = 18.6$ m^3.

143. Find the number of air molecules contained in a room having dimensions 8 × 4 × 3 m^3 at a temperature $t = 18$ °C and a pressure $p = 0.97 \times 10^5$ Pa.

Solution. The volume which would be occupied by air under normal conditions is $V_0 = pVT_0/p_0T = 88$ m^3. Since one mole of air occupies the volume $V_\mu = 22.4 \times 10^{-3}$ m^3/mol and contains $N_A = 6.02 \times 10^{23}$ mol^{-1} molecules, the number of molecules in the room is

$$n = V_0N_A/V_\mu = 2.35 \times 10^{27}.$$

[3] Under normal conditions, we assume the temperature $t_0 = 0$ °C ($T_0 = 273.15$ K) and the pressure $p_0 = 1.013 \times 10^5$ Pa.

144. A closed cylinder whose volume $V_1 = 2.0$ l contains air at a pressure $p_1 = 0.53 \times 10^5$ Pa at room temperature. Then the cylinder is submerged in water having the same temperature and at a depth $h = 1.2$ m is opened. What volume V_w of water enters the cylinder if the atmospheric pressure at this moment is $p = 0.99 \times 10^5$ Pa?

Solution. Here we have an isothermal process of air compression in the cylinder. In accordance with Boyle's law, $p_1V_1 = p_2V_2$, where p_2 is the pressure in water at the depth $h = 1.2$ m, $p_2 = 1.10 \times 10^5$ Pa. Therefore,

$$V_w = V_1 - V_2 = V_1 - p_1V_1/p_2 = V_1(1 - p_1/p_2) = 1.045 \text{ dm}^3.$$

145. An air column is closed in a tube sealed at one end by a mercury column having a height $h = 19$ cm. When the tube is placed with its open end downwards, the height of the air column is $l_1 = 10$ cm. If the tube is turned so that its open end is at the top, the height of the air column is $l_2 = 6$ cm. Find the atmospheric pressure p.

Solution. In the former case, the air pressure inside the tube (in mmHg) is $p_1 = p - h$ (Fig. 174). In the latter case, $p_2 = p + h$. According to Boyle's law,

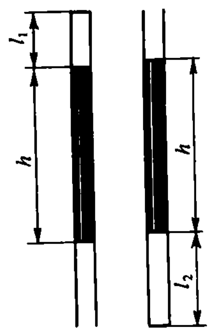


Fig. 174

we have $p_1V_1 = p_2V_2$ for the air column. Since the tube cross section is constant, $p_1l_1 = p_2l_2$, i.e. $(p - h)l_1 = (p + h)l_2$. Hence $p = h(l_1 + l_2)/(l_1 - l_2) = 760$ mmHg $= 1.013 \times 10^5$ Pa.

146. At a temperature $t_1 = 20$ °C, the air pressure in a car tyre is $p_1 = 6 \times 10^5$ Pa. Find the pressure in the tyre during the motion of the car if the temperature of air in it becomes $t_2 = 40$ °C. The change in the tyre volume should be neglected.

Solution. Since $V =$ const, the process is governed by Gay-Lussac's law: $p_1/T_1 = p_2/T_2$, where $T_1 = 293$ K and $T_2 = 313$ K. Hence $p_2 = p_1T_2/T_1 = 6.41 \times 10^5$ Pa.

147. A diving bell having a height $h = 3.0$ m and a constant cross section is submerged in sea at a depth $H = 80$ m. To what height h_0 does water rise in the bell at this depth if the temperature of water is $t_1 = 20$ °C at the surface and $t_2 =$ 7 °C at the depth H? What must be the pressure p_0 of air pumped into the bell to remove water from the bell completely? The density of sea water is $\rho = 1.03 \times 10^3$ kg/m^3.

Solution. Since the mass of air in the bell remains unchanged during submerging, we can apply the equation of state to this process in the form

$$p_1V_1/T_1 = p_2V_2/T_2. \quad (1)$$

Since the cross-sectional area of the bell is constant over its height, we can use the heights of the air column $h_1 = h$ and $h_2 = h - h_0$ instead of the volumes in Eq. (1):

$$p_1h/T_1 = p_2(h - h_0)/T_2. \quad (2)$$

Further, $p_1 = p$ and $p_2 = p + \rho gH$, where p is the atmospheric pressure. Then Eq. (2) becomes

$$ph/T_1 = (p + \rho gH)(h - h_0)/T_2.$$

Hence, assuming that $p = 1.013 \times 10^5$ Pa, $T_1 = 293$ K, and $T_2 = 280$ K, we obtain

$$h_0 = h\left[1 - \frac{pT_2}{(p + \rho gH)T_1}\right] \simeq 2.68 \text{ m}.$$

The air pressure required for the complete displacement of water from the bell can be found from Boyle's law, assuming that the air temperature in the bell remains unchanged in the process of displacement since the submergence depth is constant:

$$(p + \rho gH)(h - h_0) = p_0h, \quad \text{whence} \quad p_0 = (p + \rho gH)(h - h_0)/h.$$

Putting $h_0 = 0$, we obtain $p_0 = 9.1 \times 10^5$ Pa.

148. Two spherical flasks having a volume $V_0 = 1.0$ l each and containing air are connected by a tube of diameter $d = 6$ mm and length $l = 1.0$ m. A droplet of mercury contained in the tube is at its middle at 0 °C (Fig. 175). By what distance

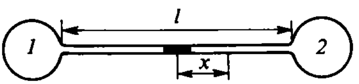


Fig. 175

does the mercury droplet move if flask *1* is heated by 2 °C while flask *2* is cooled by 1 °C? The expansion of the flask walls should be neglected.

Solution. When the air temperature in the flasks changes, the mercury droplet moves by such a distance that the air pressures on both its sides equalize. Since the masses of air in the process under consideration remain unchanged, we can apply the equation of state for a gas. For the left and right flasks we have

$$p_1V_1/T_1 = C_1, \quad p_2V_2/T_2 = C_2.$$

Since the pressures and masses of air in the two flasks are the same, $C_1 = C_2$ and $p_1 = p_2$. Consequently,

$$V_1/T_1 = V_2/T_2. \tag{1}$$

The total volume of the two flasks and the tube is

$$V_1 + V_2 = 2V_0 + \pi d^2 l/4. \tag{2}$$

Solving Eqs. (1) and (2) together, we obtain

$$V_1 = \frac{2V_0 + (\pi d^2/4)l}{T_1 + T_2} T_1 = 1.020 \text{ dm}^3,$$

$$V_2 = \frac{2V_0 + (\pi d^2/4)l}{T_1 + T_2} T_2 = 1.008 \text{ dm}^3.$$

The change in volume of each flask is $\Delta V = 6$ cm^3. Taking into account the fact that the cross-sectional area of the tube is 28 mm^2, we hence obtain $x = 21$ cm. By this distance the mercury droplet will be shifted towards the cooled flask.

149. A barometer gives faulty readings due to the pressure of air above the mercury column. At the atmospheric pressure $p_1 = 755$ mmHg, the barometer indicates 748 mmHg, while at $p_2 = 740$ mmHg, the barometer reading is 736 mmHg. What pressure does the barometer indicate at $p_3 = 760$ mmHg? The air temperature should be assumed constant. The change in the mercury level in the cup should be neglected.

Solution. The pressures p_1, p_2, and p_3 are true, and h_1, h_2, and h_3 are the corresponding readings of the barometer in millimetres. We denote by H the height of the barometer tube above the level of mercury in the cup, by S the cross-sectional area of the tube, by V_{a1}, V_{a2}, and V_{a3} the volumes of air in the tube, and by p_{a1}, p_{a2}, and p_{a3} the pressures of air in the tube. Here we have an isothermal variation of the volume of air in the tube. In accordance with Boyle's law, we have

$$p_{a1}V_{a1} = p_{a2}V_{a2} = p_{a3}V_{a3}, \tag{1}$$

where (in mmHg)

$$p_{a1} = p_1 - h_1, \quad p_{a2} = p_2 - h_2, \quad p_{a3} = p_3 - h_3,$$
$$V_{a1} = (H - h_1)S, \quad V_{a2} = (H - h_2)S, \quad V_{a3} = (H - h_3)S.$$

Substituting these quantities into expression (1) and dividing the result by S, we get

$$(p_1 - h_1)(H - h_1) = (p_2 - h_2)(H - h_2) = (p_3 - h_3)(H - h_3), \tag{2}$$

whence the height of the barometer tube above the mercury level is

$$H = \frac{h_1(p_1 - h_1) - h_2(p_2 - h_2)}{(p_1 - h_1) - (p_2 - h_2)} = 764 \text{ mm}.$$

We now equate the right and, say, the middle parts of expression (2):

$$(p_2 - h_2)(H - h_2) = (p_3 - h_3)(H - h_3),$$

whence

$$h_3^2 - h_3(H + p_3) + [Hp_3 - (p_2 - h_2)(H - h_2)] = 0.$$

It is easier to solve this equation with numerical coefficients:

$$h_3^2 - 1524h_3 + (764 \times 760 - 4 \times 28) = 0, \quad \text{i.e.} \quad h_3 = 751.4 \text{ mm}.$$

150. A cylinder having a height H and a cross-sectional area S is connected to the atmospheric air with the help of a tap (Fig. 176). A piston sliding without friction

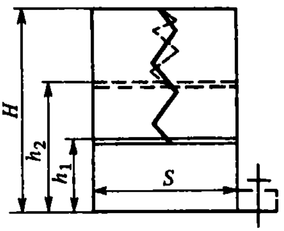


Fig. 176

is suspended from the top of the cylinder by a spring. When the tap is open, the piston is at a height h_1 from the bottom of the cylinder. The tap is closed at a temperature t_1 and an air pressure p_1. Then the air at the lower part of the cylinder is heated to t_2. As a result, the piston rises to a height h_2 from the bottom of the cylinder. Find the rigidity k of the spring, assuming that the spring deformation is proportional to the applied force. The piston should be assumed to be weightless and thermally insulated.

Solution. For the air contained in the lower part of the cylinder, we can write

$$p_1V_1/T_1 = p_2V_2/T_2, \quad \text{or} \quad p_1h_1/T_1 = p_2h_2/T_2. \tag{1}$$

For the air in the upper part, the following equation is valid:

$$p_1(H - h_1) = p_{a\,2}(H - h_2) \tag{2}$$

(isothermal process), where $p_{a\,2}$ is the pressure of compressed air in the upper part of the cylinder. Here $p_2 = p_{a\,2} + F/S$, where $F = k(h_2 - h_1)$ is the force with which the spring acts on the piston. Taking this into account, we obtain from Eq. (2)

$$p_1(H - h_1) = \left[p_2 - \frac{k}{S}(h_2 - h_1)\right](H - h_2),$$

whence

$$\frac{k}{S}(h_2 - h_1) = p_2 - p_1\frac{H - h_1}{H - h_2}. \tag{3}$$

From Eq. (1), we obtain $p_2 = h_1T_2p_1/h_2T_1$. Then Eq. (3) can be written as follows:

$$\frac{k}{S}(h_2 - h_1) = p_1\left(\frac{h_1T_2}{h_2T_1} - \frac{H - h_1}{H - h_2}\right).$$

Hence the rigidity of the spring is

$$k = \frac{Sp_1}{h_2 - h_1}\left(\frac{h_1T_2}{h_2T_1} - \frac{H - h_1}{H - h_2}\right).$$

151. Determine the number n of the strokes of a pump which delivers a volume $v = 40\ \text{cm}^3$ of air in each act, required to inflate a bicycle tyre having a volume $V = 2000\ \text{cm}^3$ so that its area S of contact with the road is $60\ \text{cm}^2$ when the force of pressure of the wheel on the road is $F = 350$ N. Initially, the tyre is filled to 0.75 of its volume by air at atmospheric pressure $p_0 = 1.013 \times 10^5$ Pa.

Solution. The inflation of the tyre is an isothermal process governed by Boyle's law:

$$p_0vn = (p_0 + p)(V - V'), \tag{1}$$

where $p = F/S$ is the pressure exerted by the wheel on the road. The volume V' of air at pressure $p_0 + p$ is related to the volume V_0 at atmospheric pressure ($V_0 = 0.75V$ according to the conditions of the problem) through the equation $p_0V_0 = (p_0 + p)V'$, which gives

$$V' = p_0V_0/(p_0 + p) = 0.75p_0V/(p_0 + p). \tag{2}$$

Substituting this value into Eq. (1), we obtain

$$p_0vn = (p_0 + p)\left(V - 0.75\frac{p_0}{p_0 + p}V\right),$$

whence the number of strokes is

$$n = \frac{V}{v}\left(0.25 + \frac{p}{p_0}\right), \quad \text{or} \quad n = \frac{V}{v}\left(0.25 + \frac{F}{Sp_0}\right) \simeq 41.$$

152. Find the number n of the strokes of a piston pump sucking in a volume $v = 400\ \text{cm}^3$ of air during a stroke, required to pump air out of a glass flask having a volume $V = 1$ l to decrease its pressure to $p_n = 10^2$ Pa if the initial pressure in the flask is $p_0 = 10^5$ Pa. The air temperature should be considered unchanged.

Solution. A process occurring at constant temperature obeys Boyle's law. Since the air initially occupies the volume V and after the first stroke its volume becomes $V + v$, we can write the following equation for the air after the first stroke of the piston

$$p_0V = p_1(V + v), \quad \text{or} \quad p_1/p_0 = V/(V + v). \tag{1}$$

After the second stroke of the piston, we have

$$p_1V = p_2(V + v), \quad \text{or} \quad p_2/p_1 = V/(V + v). \tag{2}$$

Multiplying Eqs. (1) and (2), we obtain

$$p_2/p_0 = [V/(V + v)]^2.$$

Consequently, after n strokes of the piston, we have

$$p_n/p_0 = [V/(V + v)]^n,$$

whence

$$n \log \frac{V}{V + v} = \log \frac{p_n}{p_0}, \quad \text{i.e.} \quad n = \log \frac{p_n}{p_0}\left(\log \frac{V}{V + v}\right)^{-1} = 20.$$

153. A vessel is divided by partitions into three parts having volumes V_1, V_2, and V_3. These parts contain gases under pressures p_1, p_2, and p_3 respectively. Find the pressure established in the vessel after the removal of the partitions if the temperature remains unchanged.

Solution. After the removal of the partitions, each gas occupies the entire volume. In accordance with Boyle's law, for the first, second, and third gases we can write

$$p_1'(V_1 + V_2 + V_3) = p_1V_1, \quad p_1' = p_1V_1/(V_1 + V_2 + V_3);$$
$$p_2'(V_1 + V_2 + V_3) = p_2V_2, \quad p_2' = p_2V_2/(V_1 + V_2 + V_3);$$
$$p_3'(V_1 + V_2 + V_3) = p_3V_3, \quad p_3' = p_3V_3/(V_1 + V_2 + V_3).$$

By Dalton's law, the total pressure is equal to the sum of the partial pressures of all the gases:

$$p = p_1' + p_2' + p_3' = (p_1V_1 + p_2V_2 + p_3V_3)/(V_1 + V_2 + V_3).$$

Exercises

142. A cylinder having a volume of 20 dm^3 contains a gas at 16 °C and a pressure of 10^7 Pa. Find the volume occupied by the gas under normal conditions.
Answer. 1.89 m^3.

143. (a) Find the number of air molecules in a room having a volume $8 \times 5 \times 4\ m^3$ at a temperature of 10 °C and a pressure of 1.031×10^5 Pa.
Answer. 42.2×10^{26}.

(b) Under the conditions of the previous problem, find the mass of air in a room. The density of air under normal conditions is 1.29 kg/m^3.
Answer. 203 kg.

144. (a) A 1-m long cylindrical tube closed at one end has its open end immersed in a bath with mercury to a depth of 0.5 m at the atmospheric pressure of 1.013×10^5 Pa. Find the height of the column of mercury entering the tube.
Answer. 249 mm.

(b) A 500-cm^3 flask containing air is heated to 227 °C. After that, it is turned upside down and its neck is immersed in water. Find the mass of water entering the flask by the moment when its temperature has decreased to 27 °C. The change in the flask volume should be neglected.
Answer. 0.2 kg.

145. (a) An air column is closed in a tube open at one end by a mercury column whose length is 216 mm. When the tube is in the horizontal position, the length of the air column is 307 mm. Find the length of the air column if the tube is kept with its open end upwards (downwards). The atmospheric pressure is 0.97×10^5 Pa.
Answer. 238 mm, 432 mm.

(b) Under the conditions of the previous problem, find the length of the air column in the tube placed at 30° to the horizontal with its open end downwards.
Answer. 359 mm.

146. The air pressure in a car tyre is 5×10^5 Pa at 15 °C. During the motion of the car, the pressure in the tyre has increased to 5.7×10^5 Pa due to an increase in temperature. Find the temperature of air in the tyre during its motion. The change in the tyre volume should be neglected.
Answer. 53 °C.

147. A rubber ball contains 2 dm^3 of air at 20 °C and the atmospheric pressure of 1.03×10^5 Pa. What is the volume of air in the ball after submerging it in water having a temperature of 4 °C at a depth of 10 m? To what level should the pressure of air in the ball be increased to keep its volume unchanged?
Answer. 0.97 dm^3, 2.11×10^5 Pa.

148. (a) Two identical balls are connected by a thin tube containing a droplet of mercury that separates the balls. At 0 °C, the droplet is at the middle of the tube (see Fig. 175). The volume of air contained in each ball and in the part of tube to the mercury droplet is 0.20 dm^3. The cross-sectional area of the tube is 20 mm^2. By what distance is the droplet shifted if one ball is heated by 2 °C and the other is cooled by the same temperature? The expansion of the walls should be neglected.
Answer. 74 mm.

(b) Two identical balls are connected by a thin tube having a cross-sectional area of 20 mm^2 and containing a mercury droplet that separates the balls. At 0 °C, the droplet is at the middle of the tube (see Fig. 175). The volume of air contained in each ball and in the part of the tube to the mercury droplet is 0.5 dm^3. When one ball is heated and the other cooled by the same number of degrees centigrade, the droplet is displaced by 100 mm. Find the increase and decrease in the temperatures of the balls. The expansion of the walls should be neglected.
Answer. 2.7 °C.

149. A barometer gives faulty readings due to the presence of air above the mercury column. At a pressure of 750 mmHg and a temperature of 15 °C, the barometer indicates 350 mmHg, while at 39 °C it shows 330 mmHg. Find the length of the barometer tube and the barometer reading at 15 °C and 700 mmHg and also at -33 °C and a pressure of 750 mmHg. The changes in the level of mercury in the cup and in the volume of the tube should be neglected.
Answer. 950 mm, 319 mmHg, 392 mmHg.

150. A cylinder with the area of the bottom of 100 cm^2 contains air. A piston is located at a height of 60 cm from the cylinder bottom. The atmospheric pressure is 1.013×10^5 Pa, the air temperature is 12 °C. Find the distance by which the piston is lowered after a 100-kg load is placed on it, while the air in the cylinder is heated to 15 °C. Find the distance by which the piston is lowered when the load is suspended from a spring whose rigidity is 980 N/m if the air temperature remains

unchanged. The friction of the piston against the cylinder walls and the mass of the piston should be neglected.
Answer. 29 cm, 22.7 cm.
151. The pressure of air in a car tyre is raised in 60 strokes of a piston pump from 10^5 Pa to a pressure at which the area of contact between the tyre and the road is 100 cm^2, and the force of pressure of the wheel on the road is 1.4 kN. Find the volume of the piston pump if the volume of the tyre is 155 dm^3.
Answer. 1 dm^3.
152. Air is pumped out of the tyre balloon having a volume of 16 dm^3 so that the pressure in the balloon is increased from 1.013×10^5 Pa to 50.0 Pa in 22 strokes of the piston. Find the volume of the balloon.
Answer. 40 dm^3.
153. A vessel of volume V is divided by a partition into two parts containing gases at pressures p_1 and p_2. After the partition has been removed, the pressure in the vessel becomes p. Find the volumes of the parts of the vessel.
Answer. $V_1 = V(p - p_2)/(p_1 - p_2)$, $V_2 = V(p - p_1)/(p_2 - p_1)$.

3.24. Saturated and Unsaturated Vapours. Temperature Dependence of Saturation Vapour Pressure

When a liquid evaporates, two processes occur: the transition of liquid molecules into the space above the surface of the liquid and the reverse transition. Some molecules in the surface layer of the liquid, which have higher velocities, overcome the forces of molecular attraction and go over to the vapour of this liquid above its surface. Here each molecule, having experienced a number of collisions with other molecules of the vapour, may approach the surface of the liquid, get into the range of molecular forces, and return to the liquid.

At the beginning of evaporation, the number of molecules leaving a liquid is larger than the number of returning molecules. As the number of molecules above the liquid increases, the number of returning molecules becomes larger and larger. Finally, at a sufficiently high density of molecules above the surface of the liquid, the numbers of escaping and returning molecules become equal. In this case, **saturation** sets in, viz. the concentration of vapour molecules above the surface of the liquid attains the maximum possible value at a given temperature. Any further increase in the vapour concentration at this temperature is impossible. It should be emphasized, however, that the equilibrium between the saturated vapour and its liquid is dynamic rather than static, since in the state of saturation the evaporation does not cease but is continued, being accompanied by the reverse process of condensation. Here the number of molecules returning to the liquid, i.e. condensing molecules of vapour, is exactly equal to the number of escaping molecules, i.e. evaporating molecules of the liquid. As a result, the vapour concentration in the space above the liquid remains unchanged.

A vapour which is in dynamic equilibrium with its liquid is called a **saturated vapour**. A vapour having a lower concentration is **unsaturated**.

The properties of saturated vapour.

1. The pressure of a saturated vapour of a given liquid at constant temperature is a constant quantity which does not depend on the volume of the space above the evaporating liquid. Consequently, the density of the saturated vapour is also constant at a given temperature.

2. The pressure and density of saturated vapours of different liquids are different at the same temperature. Volatile liquids have higher pressures and densities.

3. The pressure of a saturated vapour is the maximum possible pressure of the vapour of a given liquid at a given temperature.

4. As the temperature increases, the pressure of a saturated vapour rises, since the number of evaporating molecules and their kinetic energy increase.

5. The pressure of a saturated vapour is maximum at the boiling point.

6. The presence of other gases above an evaporating liquid does not affect the pressure (and density) of the saturated vapour of the given liquid and only slows down the evaporation to saturation.

7. The relation between the mass, volume, pressure, and temperature for a saturated vapour differs from that for gases, since if the vapour remains saturated in a certain process, its mass changes. For example, as a result of isothermal compression, a saturated vapour partially condenses so that its pressure remains unchanged. During an isochoric decrease in temperature, a saturated vapour also partially condenses, and its pressure

decreases and becomes equal to the saturation vapour pressure at a new (lowered) temperature.

An unsaturated vapour which is far from saturation obeys the laws of state for ideal gases the more precisely, the farther the vapour from saturation.

3.25. Absolute Humidity. Relative Humidity

The **absolute humidity** of air is the mass of water vapour contained in a unit volume of air, i.e. the density of water vapour contained in air:

$$f = \rho = m/V.$$

The **relative humidity** of air is the ratio of the density of water vapour, $\rho = f$, contained in the air to density of the saturated vapour, $\rho_{sat} = F$, at the same temperature in per cent:

$$B = (\rho/\rho_{sat})100\%.$$

The relative humidity of air increases when its temperature decreases. If the content of vapour in air is low and the vapour is unsaturated, the same vapour may become saturated as the temperature is lowered.

The temperature at which the vapour contained in air becomes saturated is called the **dew point** for air having a given humidity. A decrease in temperature below the dew point leads to the condensation of vapour. For temperatures above 30 °C, the saturation vapour density should be calculated with the help of the Clapeyron-Mendeleev equation:

$$pV = (m/\mu)RT.$$

Since $\rho = m/V$, it follows from this equation that

$$\rho = p\mu/RT.$$

At the normal atmospheric pressure, we have

$$\rho = p_0\mu/RT.$$

Let us calculate, for example, the density of water vapour at 100 °C, i.e. at 373 K:

$$\rho = \frac{1.013 \times 10^5 \text{ Pa} \times 0.018 \text{ kg/mol}}{8.31 \text{ J/(mol}\cdot\text{K)} \times 373 \text{ K}} = 0.598 \text{ kg/m}^3.$$

3.26. Instruments for Determining Humidity

We shall describe two types of instruments used for determining humidity, viz. hygrometers and psychrometers.

A simple **hygrometer** is an instrument for *determining the dew point* (Fig. 177). An ordinary glass contains ether with a thermometer immersed in it. If we blow air through a pipe immersed

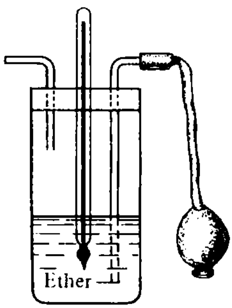


Fig. 177

in ether, the latter starts to evaporate, absorbing the heat of vaporization from the vessel and surrounding air. As a result, the temperature in the glass is lowered. As the temperature drops to the dew point, i.e. to the temperature at which the vapour contained in the air becomes saturated, the droplets of moisture (dew) appear on the walls of the glass. The temperature corresponding to the moment when dew precipitates is just the *dew point*.

The absolute humidity of air is equal to the humidity of a saturated vapour at the dew point. Therefore, having determined the dew point with the help of a hygrometer, we can find from the table the humidity of the saturated vapour corresponding to this temperature.

The operation principle of another simple hygrometer is based on the property of a degreased hair to elongate when humidity increases and to contract when it decreases. In a *hair hygrometer*

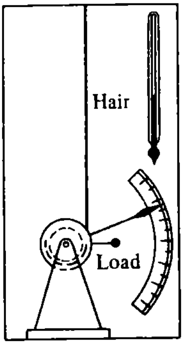


Fig. 178

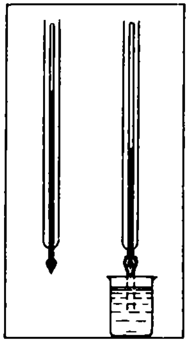


Fig. 179

(Fig. 178), a degreased hair is fixed at its upper end to a frame, and at its lower end, to a pulley that can rotate about its axis O. A load that stretches the hair is attached to the pulley by a lever. A pointer is also connected to the pulley. The pulley and the pointer are turned upon change in length of the hair as a result of a change in humidity. The scale of the instrument is graduated in percent of relative humidity.

Psychrometers are intended for a rapid and more exact determination of air humidity and are used, for instance, at meteorological stations. **Psychrometers** (Fig. 179) indicate the *relative humidity of air.* A psychrometer consists of two thermometers, one of which is wrapped by a piece of cloth. An end of this cloth is immersed in a vessel with water, which makes the piece of cloth wet. As a result of evaporation of water from the vessel, its temperature drops the stronger, the more intense the evaporation, i.e. the lower the relative humidity of air. The readings of the two thermometers, viz. the dry-bulb thermometer indicating the temperature of air and the wet-bulb thermometer showing the temperature of the wet cloth, differ. This so-called psychrometric difference is the larger, the more intense the evaporation of water, i.e. the lower the relative humidity of air.

The dependence of the readings of the two thermometers on the relative humidity is given in psychrometric tables with which the psychrometer is supplied. From this tables, the value of the relative humidity corresponding to the readings of the thermometers is determined.

Problems with Solutions

154. A closed vessel of volume $V = 200\ \text{dm}^3$ contains air at a temperature $t_1 = 25\ °\text{C}$ and a relative humidity $B_1 = 60\%$. Find the mass of water vapour that condenses as a result of cooling the vessel to $t_2 = 10\ °\text{C}$. The saturation vapour pressures are $p_1 = 3.13 \times 10^3$ Pa at 25 °C and $p_2 = 1.21 \times 10^3$ Pa at 10 °C.

Solution. The pressure of water vapour in the vessel is $p_{ves} = B_1 p_1 = 1.78 \times 10^3$ Pa. Consequently, the dew point corresponds to the temperature t_2 10 °C. It follows from the Clapeyron-Mendeleev equation that $\rho_1 = p_1\mu/RT_1$ and $m_1 = Vp_1\mu/RT_1$. Similarly, $m_2 = Vp_2\mu/RT_2$. The mass of water obtained as a result of condensation is

$$m = m_1 - m_2 = \frac{V\mu}{R}\left(\frac{p_1}{T_1} - \frac{p_2}{T_2}\right) = 1\ \text{g},$$

where the gas constant $R = 8.31$ J/(mol·K), and the molar mass of water is $\mu = 0.018$ kg/mol.

155. A closed vessel of volume $V = 200\ \text{dm}^3$ contains air at $t_1 = 100\ °\text{C}$ and a relative humidity $B_1 = 60\%$. Find the mass of water condensed upon cooling the vessel to $t_2 = 10\ °\text{C}$. The saturation vapour pressure $p_2 = 1.21 \times 10^3$ Pa at 10 °C.

Solution. The saturation vapour density is calculated from the Clapeyron-Mendeleev equation. For $t_1 = 100\ °\text{C}$, it is $\rho_1 = 0.598\ \text{kg/m}^3$. The density of water vapour in the vessel is $\rho_{ves} = B_1\rho_1 = 0.359\ \text{kg/m}^3$. The saturation vapour density $\rho_2 = 0.009\ \text{kg/m}^3$ at 10 °C. The mass of condensed water is

$$m = (\rho_{ves} - \rho_2)\,V = 69.9\ \text{g}.$$

156. The relative humidity of air in a closed vessel is $B_1 = 80\%$ at $t_1 = 10\ °\text{C}$. What is the relative humidity when the temperature is increased to $t_2 = 25\ °\text{C}$? The saturation vapour pressures are $p_1 = 1.21 \times 10^3$ Pa at 10 °C and $p_2 = 3.13 \times 10^3$ Pa at 25 °C.

Solution. The pressure of water vapour in the vessel is $p_{ves} = B_1 p_1$. Hence the relative humidity is

$$B_2 = (p_{ves}/p_2) \times 100\% = (B_1 p_1/p_2) \times 100\% = 31\%.$$

157. The air pressure in an open vessel at $t_1 = 17\ °\text{C}$ and a relative humidity $B_1 = 80\%$ is $p_{ves\,1} = 1.01 \times 10^5$ Pa. The vessel is closed and cooled to $t_2 = 7\ °\text{C}$. Find the relative humidity and pressure in the vessel. The saturation vapour pressures are $p_1 = 1.9 \times 10^3$ Pa at 17 °C and $p_2 = 0.99 \times 10^3$ Pa at 7 °C.

Solution. The pressure of water vapour is $p_{v\,1} = 1.52 \times 10^3$ Pa at 17 °C. Since according to Dalton's law, the total pressure in the vessel is the sum of the pressures of dry air and vapour, the pressure of dry air is $p_{d.a\,1} = 0.99 \times 10^5$ Pa at 17 °C. At a temperature of 7 °C, a part of water vapour condenses since the dew point corresponding to $p_{v\,1} = 1.52 \times 10^3$ Pa is above 7 °C. Consequently, water vapour in the vessel is saturated at 7 °C and its pressure is $p_{v\,2} = 0.99 \times 10^3$ Pa. The pressure of dry air can be found from the equation describing Gay-Lussac's law:

$$p_{d.a\,2}/T_2 = p_{d.a\,1}/T_1, \quad \text{whence} \quad p_{d.a\,2} = p_{d.a\,1}T_2/T_1 = 0.96 \times 10^5 \text{ Pa}.$$

The pressure in the vessel is $p_{ves\,2} = 0.97 \times 10^5$ Pa.

158. A vessel whose volume $V = 200$ dm^3 contains air at a temperature $t = 25$ °C and a relative humidity $B_1 = 40\%$. What is the relative humidity B_2 if a mass $\Delta m = 3$ g of water is poured into the vessel? The saturation vapour pressure is $p_{sat} = 3.13 \times 10^3$ Pa at 25 °C. The vapour should be treated as an ideal gas.

Solution. The mass of water vapour contained in the vessel before the additional mass of water has been poured can be found from the equation $m/V = B_1\rho_{sat}/100\%$. The saturation vapour density is found from the Clapeyron-Mendeleev equation: $\rho_{sat} = B_1\mu p_{sat}/RT$. If the entire mass of the additionally poured water evaporates, the absolute humidity becomes

$$f = \rho = \frac{m + \Delta m}{V} = B_1\frac{\mu p_{sat}}{RT} + \frac{\Delta m}{V} = 24 \text{ g/m}^3.$$

Since $\rho > \rho_{sat}$, water evaporates only partially to complete saturation of the vapour, $B_2 = 100\%$.

159. A mass $m = 3$ g of water is poured into a vessel of volume $V = 10$ dm^3 filled with dry air at a pressure of 10^5 Pa. After that the vessel is closed and heated to $t = 100$ °C. Find the pressure and the relative humidity of air in the vessel after heating. The vapour should be treated as an ideal gas.

Solution. We apply to water vapour the Clapeyron-Mendeleev equation:

$$pV = mRT/\mu, \quad \text{whence} \quad p = mRT/\mu V = 0.51 \times 10^5 \text{ Pa},$$

where the gas constant $R = 8.31$ J/(mol·K), the molar mass of water is $\mu = 0.018$ kg/mol, and $T = 373.15$ K. Consequently, the entire mass of introduced water evaporates. Since the saturation vapour pressure $p_{sat} = 10^5$ Pa at 100 °C, we have $B = (p/p_{sat}) \times 100\% = 51\%$. In accordance with Dalton's law, at 100 °C the pressure in the vessel is equal to the sum of the partial pressures of dry air and vapour. The pressure of dry air can be found from Gay-Lussac's law:

$$p_2/T_2 = p_1/T_1, \quad \text{whence} \quad p_2 = p_1T_2/T_1 = 1.37 \times 10^5 \text{ Pa}.$$

The total pressure in the vessel is $p_{ves} = p_1 + p_2 = 1.88 \times 10^5$ Pa at 100 °C.

Exercises

154. A closed vessel whose volume is 500 dm^3 contains air at a temperature of 20 °C and a relative humidity of 80%. Find the relative humidity of air after the vessel has been cooled to 18 °C. What mass of water vapour will condense upon further cooling to 15 °C? The saturation vapour pressures are 2.3×10^3 Pa at 20 °C, 2.0×10^3 Pa at 18 °C, and 1.7×10^3 Pa at 15 °C.
Answer. 90%, 0.6 g.

155. A closed vessel whose volume is 500 dm^3 contains air at 100 °C and a relative humidity of 40%. Find the mass of water vapour condensed upon cooling the vessel to 25 °C. The saturation vapour pressure is 3.1×10^3 Pa at 25 °C.
Answer. 108 g.
156. The relative humidity of air contained in a closed vessel is 80% at 20 °C. Find the relative humidity of air after the temperature in the vessel has been raised to 100 °C. The saturation vapour pressure is 2.3×10^3 Pa at 20 °C.
Answer. 2.3%.
157. The pressure of air in a closed vessel is 1.02×10^5 Pa at 25 °C and a relative humidity of 80%. Find the pressure and the relative humidity of air in the vessel after the temperature in it has decreased to 2 °C. The saturation vapour pressures are 3.1×10^3 Pa at 25 °C and 0.5×10^3 Pa at 2 °C.
Answer. 0.93×10^5 Pa, 100%.
158. A vessel whose volume is 500 dm^3 contains air at 20 °C and a relative humidity of 30%. Find the relative humidity of air after the addition of 5 g of water into the vessel. Solve the problem under the condition that, before adding water, the relative humidity has been 60%. The saturation vapour pressure is 2.3×10^3 Pa at 20 °C.
Answer. 87%, 100%.
159. Five grams of water are poured into a vessel having a volume of 20 dm^3 and filled with dry air at a pressure of 10^5 Pa. After that, the vessel is closed and heated to 100 °C. Find the relative humidity and pressure of air in the vessel after heating. Solve the problem for the case when 15 g of water are poured. The vapour should be treated as an ideal gas.
Answer. 43%, 1.8×10^5 Pa, 100%, 2.37×10^5 Pa.

4. FUNDAMENTALS OF ELECTRODYNAMICS

A. ELECTROSTATICS

4.1. Law of Electric Charge Conservation. Electric Field. Coulomb's Law. Effect of Medium on the Force of Interaction of Charges

There exist two types of electric charges in nature, conditionally called positive charges and negative charges. The carriers of electric charges are elementary particles, in particular, those constituting atoms. These are the electron (negative charge) and the proton (positive charge). Electrons and protons have the smallest indivisible charge known as an elementary charge (e). An electrically charged body has unequal numbers of negative and positive charges, its charge being measured by an integral number of elementary charges. Electrically neutral bodies contain equal numbers of unlike elementary charges.

The law of electric charge conservation is formulated as follows: *the algebraic sum of electric charges remains unchanged in an isolated system.* Charges can be transferred from one body of a given system to another or move within the same body. This means that the total charge of an electrically isolated system can be changed only by introducing charges from outside or by transporting them beyond the limits of the system.

Electric charges interact so that like charges repel, while unlike charges attract one another.

The interaction of electric charges is realized through electric field. **The electric field** is a form of matter existing in the space surrounding an electric charge and manifested in forces acting on other charges in this space. The force of interaction of two point charges at rest is determined by **Coulomb's law** that was established experimentally. A point charge is a charged body whose dimensions can be neglected in comparison with the distance from this body to other bodies carrying electric charges. **Coulomb's law** is formulated as follows: *the force of interaction of two electrostatic point charges in vacuum is directly proportional to the product of these charges and inversely proportional to the square of their separation,* i.e.

$$F = k_{q_1 q_2}/r^2. \qquad (4.1.1)$$

This force is directed along the straight line connecting the points of location of the charges.

The proportionality factor k in formula (4.1.1) has the following form in SI: $k = 1/4\pi\varepsilon_0$, where ε_0 is the **electric constant.** Consequently, Coulomb's law in this system is written as follows:

$$F = (1/4\pi\varepsilon_0)(q_1 q_2/r^2). \qquad (4.1.2)$$

The value of ε_0 is determined experimentally and has the dimensions of **farad per metre** (F/m) (see Sec. 4.7):

$$\varepsilon_0 = 1/(4\pi \times 9 \times 10^9)\ \mathrm{F/m}\ [\mathrm{C^2/(N \cdot m^2)}].$$

Substituting the value of ε_0 into (4.1.2), we arrive at the following formula convenient for calculations

$$F = (9 \times 10^9)(q_1 q_2/r^2)[\mathrm{N \cdot m^2/C^2}]. \qquad (4.1.3)$$

Here F is expressed in newtons, q_1 and q_2 in coulombs, and r in metres.

As regards their electric properties, all substances can be divided into dielectrics (insulators), conductors, and semiconductors. Experiments show that the force of interaction of electric charges in a dielectric medium is weaker than in vacuum. The quantity equal to the ratio of the force of interaction of electric charges in a given medium to the force of their interaction in vacuum is called the **relative permittivity** of the medium and is denoted by ε_r. The value of ε_r is equal to unity (to be more precise, 1.0006) for gases and air, $\varepsilon_r = 2$ for kerosene, $\varepsilon_r = 7$ for glass, and $\varepsilon_r = 81$ for water.

Coulomb's law for charges located in a liquid or gaseous dielectric with a relative permittivity ε_r has the form

$$F = (1/4\pi\varepsilon_0)(q_1 q_2/\varepsilon_r r^2). \qquad (4.1.4)$$

The SI unit of electric charge is a **coulomb** (C). It is a derived unit. The basic unit in this system is the unit of current, viz. an **ampere** (A). (This unit was established on the basis of the law of interaction of current-carrying conductors (see Sec. 4.26) rather than on the basis of the law of interaction of charges.)

4.2. Charge Equilibrium in Metals. Electrostatic Induction

In addition to electrons bound to nuclei, metals contain a large number of highly mobile free electrons. On the other hand, positively charged nuclei are "fixed" at the sites of the crystal lattice and perform only small vibrations. Free electrons in a metal form a "jelly" containing positive ions, viz. nuclei with electrons bound to them. A conductor is neutral as a whole, and electric field is absent in it.[1] There is no electric field inside a hollow metallic body such as a metallic sphere. The methods of protection (screening) of electric wires and systems from external electric fields (when an external electric field creates harmful disturbances) are based on this property. Electric wires and systems are enclosed in metallic networks.

If an uncharged (neutral) piece of metal is introduced into an electric field, the field causes a displacement of free electrons in this conductor. If the external field is created by a positive charge, electrons of the conductor are attracted to this charge and move to the end of the conductor which is closer to the charge creating the field. This end turns out to be charged negatively. The deficiency of electrons at the far end of the conductor creates a positive charge of the same magnitude. If the charge creating the external field is negative, the electrons of an insulated conductor introduced into this field are accumulated at the far end of the conductor, which thus acquires a negative charge. The end of the conductor which is closer to the charge creating the field turns out to be

[1] To be more precise, there is no field "on the average" if we consider the volume of a conductor which is much larger than the volume of an atom and determine the average field within this volume.

charged positively. The displacement of charges ceases when the field created by them inside the conductor exactly compensates the external electric field so that the field inside the conductor is as before equal to zero. When the insulated conductor is taken out of the field, electrons in it will be distributed so that they restore the initial neutral state.

The emergence of unlike charges at the ends of an insulated conductor introduced into an electric field (for example, by bringing a charged body to it) is called the **electrostatic induction.**

If a conductor is charged by transferring additional charges to it, the excess electric charges will move apart under the action of Coulomb forces to the largest possible distances. A stable equilibrium of electric charges is attained when the excess charges are accumulated on the outer surface of the conductor. This state corresponds to the mininum of the potential energy of charges.

Experiments show that in equilibrium the **surface density** σ **of electric charges,** i.e. the number of charges per unit surface area of a conductor, is higher in the regions with larger curvature of the surface and lower in the regions where the curvature is smaller. In conductors with the same curvature at all points of the surface (like on the surface of the sphere), the surface charge density is the same at all points.

4.3. Electroscope

An **electroscope** is an instrument intended for detecting an electric charge on a body and for determining the sign of this charge.

Let us consider the construction of a simple electroscope (Fig. 180). A metallic rod with a ball at the top has at its lower end two very thin metallic leaves (made, for instance, of aluminium). The rod is fixed to a glass flask with a neck (or to a metallic box with glass windows) with the help of an insulating cork. When a charged body is brought in contact with the ball, the leaves of the electroscope move apart since they acquire like electric charge. The angle by which they deviate is the greater, the larger the charge imparted to the electroscope.

The sign of the electroscope charge is determined by bringing in contact with it a charged body whose sign is known

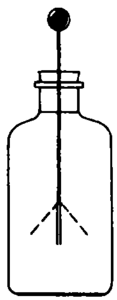


Fig. 180

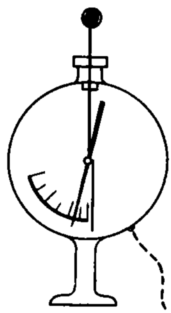


Fig. 181

beforehand. If the deflection of the electroscope leaves increases, the charge of the electroscope has the same sign as that of the body. A decrease in deflection of the leaves indicates that the electroscope charge and the charge of the body are unlike.

Figure 181 shows the schematic diagram of an electroscope with a scale. A metallic rod is placed inside a metallic shell with glass windows. A light metallic pointer can turn about an axis passing through the bend of the rod. When the electroscope is charged, the pointer is "repelled" from the rod and is deflected through an angle that can be measured on the scale. Such an electroscope is intended for measuring potentials and is called an **electrometer**.

4.4. Electric Field Strength. Electric Field Lines

The **electric field strength E** at a given point is a physical quantity measured by the force with which the electric field acts on a unit positive point charge placed at this point. If a force **F** acts on a charge q' at a certain point of the field, the electric field strength at this point is

$$\mathbf{E} = \mathbf{F}/q'.$$

Hence the magnitude of the electric field strength is given by

$$E = F/q'.$$

According to Coulomb's law, two point electric charges q and q separated by a distance r interact with a force

$$F = (1/4\pi\varepsilon_0)(q'q/\varepsilon_r r^2).$$

Hence the strength of the electric field created by the charge q at the distance r from it is

$$E = F/q' = (1/4\pi\varepsilon_0)(q/\varepsilon_r r^2). \qquad (4.4.1)$$

If the charge q is positive, the electric field strength **E** is directed away from it. If the charge q is negative, the field is directed towards it. Electric field can be represented graphically with the help of the **electric field lines** (which are also called the **lines of vector E**). These lines are drawn so that the tangent to a field line at any point coincides in direction with vector **E** at this point, and the density of lines is proportional to the magnitude of vector **E** at this point.

The pattern of electric field lines can be observed, for example, in the following experiment. Wheat groats are poured in a vessel containing a liquid dielectric (castor oil or turpentine). If an electric field is created in the dielectric, wheat groats get electrolized and move and turn until they form chains coinciding with the electric field lines.

Electric field lines should not be identified with the trajectories of motion of very light charged bodies in an electrostatic field (the so-called point charges). The tangent to such a trajectory at a certain point coincides in direction with the velocity of the body.

On the other hand, the tangent to a field line at any point coincides in direction with the force $q\mathbf{E}$, i.e. with the acceleration **a**. Vectors **a** and **v** are directed along the same straight line only in rectilinear motion. In the general case, their directions do not coincide.

An electric field whose strength has the same magnitude and direction at all points is called a **uniform field.** The direction and density of field lines of a uniform field are the same at all points (Fig. 182).

Figure 183 depicts the field lines of two insulated point charges, while Fig. 184 shows the field lines of two interacting point charges of equal magnitude. These fields are nonuniform.

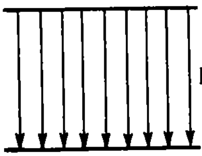


Fig. 182

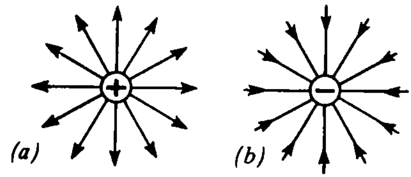


Fig. 183

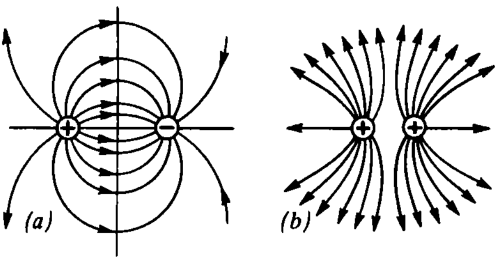


Fig. 184

The strength of electric field of a point charge varies in inverse proportion to the squared distance from the charge (since the area of the sphere pierced by the field lines increases in proportion to the square of its radius).

The number N of the field lines emerging from a positive point charge (or terminating on a negative charge) can be found from the principle underlying the plotting of the lines. Their density should be proportional to the magnitude of **E** at a given point of the field. According to (4.4.1), at all points of the sphere of radius r, we have

$$E = (1/4\pi\varepsilon_0)(q/\varepsilon_r r^2).$$

Multiplying this expression by the area of the surface of the sphere (equal to $4\pi r^2$), we obtain the number of lines of the field created by the point charge:

$$N = (1/4\pi\varepsilon_0)(q/\varepsilon_r r^2)4\pi r^2 = q/\varepsilon_0\varepsilon_r$$

(for a negative sign, we must take $|q|$ instead of q in this formula).

At the interface between two media with different permittivities, the field strength, and hence the density of the field lines, change abruptly. Since there is no electric field inside a conductor, the lines of an external field terminate and start on the conductor surface. There are no field lines inside a conductor.

Actually, the field lines do not exist. This is just a convenient method of graphical representation of a field.

4.5. Work Done on a Charge by the Forces of Electrostatic Field. Potential

When a charge moves in an electric field, the forces of the field do a work on it. If the angle α between the velocity of the charge and an electric force acting on it is acute ($\alpha < \pi/2$), a positive work is done by the field on this charge. As a result, the kinetic energy of the charge increases. If the angle is obtuse ($\alpha > \pi/2$), the work done by the forces of the field is negative, and the kinetic energy of the charge decreases during its motion.

It can be shown that the *work* done by the forces of an electrostatic field on a charge *does not depend on the path* and is determined only by the initial and final positions of the charge. This means that the forces of an electrostatic field are conservative. Consequently, a charge in an electrostatic field possesses a potential energy W_p. In this respect, electrostatic field is similar to gravitational field.

If the velocity of a charge remains unchanged, the work A' of external forces relative to the force field[2] is spent to increase the potential energy of the charge: $A' = W_{p2} - W_{p1} = \Delta W_p$. In accordance with Newton's third law, the forces of the field do in this case a work $A = -A' = W_{p1} - W_{p2} = -\Delta W_p$. Thus, the work A done by the forces of the electrostatic field on the charge q is equal to the difference between the initial and final values of the potential energy of the charge. To make it short, the

[2] For example, the force with which we act on a charged body by taking it with an insulated pincers and carrying it to another point of the field is external relative to the field.

work is equal to the decrease in the potential energy of the charge:

$$A = W_{p1} - W_{p2} = -\Delta W_p. \qquad (4.5.1)$$

The ratio of the potential energy W_p to a charge q is called the **potential** φ of the electrostatic field at the point of location of the charge:

$$\varphi = W_p/q.$$

Like potential energy, potential is defined to within an additive constant. In practice, it is counted off from an arbitrarily chosen initial level. In the theory of electromagnetism, the zero potential level is taken equal to the potential at an infinitely distant point. In engineering, the potential of the surface of the Earth is taken for the zero potential level.

Substituting the product $q\varphi$ for W_p into formula (4.5.1), we obtain the following expression for the work done by the forces of an electrostatic field on a charge:

$$A = q(\varphi_1 - \varphi_2) = -q\Delta\varphi$$

(it should be recalled that $\Delta\varphi = \varphi_2 - \varphi_1$ by definition). Thus, *the work done on a charge by the forces of an electrostatic field is equal to the product of the charge and the potential drop* (i.e. the difference in the potentials at the initial and final points of the path along which the charge moves).

Assuming that the potential at infinity is zero, we obtain the following expression for the work done by the forces of the field to carry a charge q from the point corresponding to the potential φ to infinity:

$$A = q(\varphi - 0) = q\varphi$$

(in this case, $\varphi_1 = \varphi$ and $\varphi_2 = 0$). If $q = 1$, the potential φ is numerically equal to A. Hence it can be said that the potential at a certain point of the field coincides with the work done by the forces of the field on a unit positive charge to remove it from the given point to infinity.

The following definition is also correct: the potential at a given point of the field is equal to the work A' that should be done by external forces on a unit positive charge (in other words, the work done against the field) to bring a unit positive charge from infinity to the given point of the field. It is assumed that the kinetic energy of the charge does not change during its transportation.

It should be noted that an electrostatic field has a potential irrespective of whether or not it contains a charge. Potential is a scalar characteristic of field, while the field strength **E** is its vector characteristic. A surface whose potential has the same value at all its points is called an equipotential surface. When a charge moves over an equipotential surface, the field forces do not perform any work.

Positive charges move in conductors from points at a higher potential to points at a lower potential, while negative charges move in the opposite direction. Charges move until the potentials of the field level out over the entire conductor, after which the transport of charges ceases. Consequently, if electric charges are in equilibrium in a conductor, its surface is always one of the equipotential surfaces of the field.

The SI unit of potential is a **volt** (V). The field potential at a certain point is equal to one volt if the potential energy of a one-coulomb charge at this point is one joule:

$$1\ \mathrm{V} = 1\ \mathrm{J}/1\ \mathrm{C}.$$

Calculations show that the potential of the field of a point charge q at a distance r from it is defined by the formula

$$\varphi = (1/4\pi\varepsilon_0)(q/\varepsilon_r r),$$

(it is assumed that the potential is zero when $r = \infty$).

4.6. Relation Between Potential and Field Strength for a Uniform Electric Field

The work done in the motion of a charge over an equipotential surface is zero. This means that the field vector **E** is directed along the normal at each point of this surface. Using this circumstance, we can plot equipotential surfaces from the pattern of field lines or, vice versa, knowing the form of equipotential surfaces, plot the electric field lines. It is expedient to plot equipotential surfaces in such a way that the potential difference between any pairs of neighbouring surfaces be the same. Then the density of the surfaces shows the rate of variation of potential during the motion along the field lines.

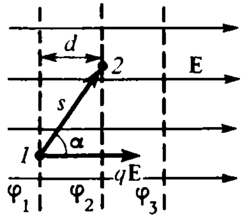


Fig. 185

Figure 185 depicts the lines of field **E** (solid lines) and equipotential surfaces (dashed lines) for a uniform field. Suppose that a charge q has shifted from point *1* to point *2* along a straight line segment of length s. The force $q\mathbf{E}$ does thereby a work $A = qEs \cos \alpha = qEd$, where d is the distance between the equipotential surfaces containing points *1* and *2*. On the other hand, $A = q(\varphi_1 - \varphi_2)$. Equating the two expressions for work and cancelling out q, we obtain the relation connecting the magnitude of the electric field strength and the potential:

$$E = (\varphi_1 - \varphi_2)/d. \tag{4.6.1}$$

The right-hand side of this formula gives the rate of variation of potential during the motion along the field lines (the change of φ per unit length of a field line).

The potential difference is measured in SI in volts and the distance, in metres. Accordingly, the SI unit of the field strength is a **volt per metre** (V/m). At the end of Sec. 4.5, it was established that 1 V = 1 J/1 C. Therefore,

$$1 \text{ V/m} = 1 \text{ J/(1 C}\cdot 1 \text{ m)} = 1 \text{ N/1 C}.$$

Thus, a volt per metre is the strength of the field in which the force of one newton acts on a charge of one coulomb.

4.7. Capacitance

The **capacitance** (or **capacity**) of an isolated conductor is a physical quantity measured by the ratio of the change in the charge of the conductor to the change in its potential:

$$C = \Delta q/\Delta\varphi.$$

If the charge and potential of a conductor are initially equal to zero, the capacitance is the ratio of the charge supplied to the conductor to the potential appearing as a result.

If a conductor under consideration is surrounded by other conductors, the electric fields of their own charges and the charges induced by the given conductor change the potential difference, and hence the capacitance of the conductor. The presence of a dielectric also affects the capacitance of a conducting body. Thus, the capacitance of a conductor depends on the presence of various bodies around it. If the surrounding conductors and dielectrics are sufficiently far from the conductor under consideration, their effect on the capacitance of the given conductor is weak, and the conductor can be treated as *isolated.*

The capacitance of an isolated conductor is determined by its shape and size, as well as by the dielectric medium surrounding it. The capacitance of a conductor placed in a medium with the relative permittivity ε_r is ε_r times higher than in a vacuum. This is due to the fact that the potential at any point of the electric field created by a charge q in a medium having the relative permittivity ε_r is ε_r times smaller than in a vacuum.

The SI unit of capacitance is a **farad** (F). This is the capacitance of an isolated conductor whose potential increases by one volt when a charge of one coulomb is imparted to it:

$$1\ \mathrm{F} = 1\ \mathrm{C}/1\ \mathrm{V}.$$

This is a very high capacitance. In practice, smaller units are being used, such as one mullionth of a farad-microfarad (μF) and one millionth of a microfarad. viz. a picofarad (pF).

4.8. Capacitors. Energy of a Charged Capacitor

A **capacitor** is a system of two closely arranged conductors in a vacuum or in a dielectric medium. These conductors are called the capacitor plates. A capacitor is usually charged by imparting equal unlike charges $+q$ and $-q$ to its plates. An electric field appears in the space surrounding the plates, and a voltage U emerges between the plates (this is the name given to the potential difference $\varphi_1 - \varphi_2$). The **capacitance** C of a capacitor is a quantity directly proportional to the charge q supplied to the capacitor and inversely proportional to the potential difference $\varphi_1 - \varphi_2$ that appears between its plates as a result:

$$C = q/(\varphi_1 - \varphi_2) = q/U \tag{4.8.1}$$

(here q is the magnitude of the charge on one of the capacitor plates).

If the plates are made in the form of two identical parallel plates, we have a parallel-plate capacitor. It will be proved below that the field between the plates of a parallel-plate capacitor is uniform, while outside the plates it is practically equal to zero. The following formula will be obtained for the capacitance of a parallel-plate capacitor:

$$C = \varepsilon_0 \varepsilon_r S/d, \tag{4.8.2}$$

where S is the area of a plate, d is the gap between the plates, and ε_r is the relative permittivity of the medium filling the gap (in a particular case, there can be a vacuum between the plates; then $\varepsilon_r = 1$).

To analyze the field between the plates of a parallel-plate capacitor, we first consider the field of an infinitely long plane plate uniformly charged with a density $+\sigma$ (Fig. 186; σ is the

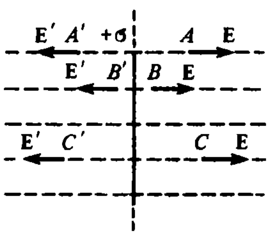


Fig. 186

charge per unit area on both sides of the plate). Since the plate extends infinitely in all directions relative to any point — A, or B, or

C, i.e. it is symmetric about any point, the field strength **E** must be directed along the normal to the plate at any point. Hence it follows that the lines of field **E** are straight lines normal to the plate. At points A and A' symmetric about the plate, vectors **E** and **E**$'$ differ only in sign.

It follows from the above arguments that the field of the plate is depicted by a system of equidistant (σ is the same everywhere) straight lines normal to the plate (the dashed lines in Fig. 186). These lines emerge from the plate and go to infinity.

It was shown in Sec. 4.4 that the field lines emerge from a charge q if it is positive and terminate on it if it is negative, their number being $N = |q|/\varepsilon_0\varepsilon_r$, where ε_r is the relative permittivity of the medium surrounding the charge. Consequently, $\sigma/\varepsilon_0\varepsilon_r$ lines emerge from a unit area of the plate (bearing the charge σ), half of which is directed to the right and the other half, to the left. The number of field lines on both sides of the plate is $\sigma/2\varepsilon_0\varepsilon_r$. By hypothesis, the number of lines is proportional to the magnitude of the field strength. Thus, the field of the plate is uniform, and its magnitude is

$$E = \sigma/2\varepsilon_0\varepsilon_r. \tag{4.8.3}$$

If a plate has a negative charge, the only difference will be that the field lines come from infinity and terminate on the plate.

Let us now arrange two identical infinite plates in parallel. The plates are uniformly charged with a density σ which has the same magnitude and opposite signs (Fig. 187; to make the diagram

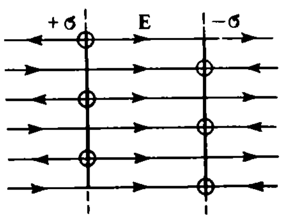


Fig. 187

more understandable, the field lines of the plates are displaced relative to one another, the crossed dots indicating the starting or terminal points of the lines). It can be seen that outside the plates, the fields neutralize each other, while in the gap between the plates they are added. As a result, the field turns out to be nonzero only in the gap between the plates. This field is uniform, and its magnitude is

$$E = \sigma/\varepsilon_0\varepsilon_r. \tag{4.8.4}$$

An actual parallel-plate capacitor is formed by the plates having finite dimensions. If, however, the linear dimensions of the plates are much larger than the gap between them, the result obtained above is valid for the most part of the capacitor volume. The field will gradually attenuate only at the edge of the plates, and the field lines are bent there.

Suppose that charges $+q$ and $-q$ supplied to the plates of a parallel-plate capacitor create a voltage U between the plates. According to formula (4.6.1), this voltage is $Ed = \sigma d/\varepsilon_0\varepsilon_r$, where d is the width of the gap between the plates:

$$U = \sigma d/\varepsilon_0\varepsilon_r.$$

Substituting $\sigma = q/S$ (S is the area of a plate) into this equation, we obtain the following relation:

$$U = qd/\varepsilon_0\varepsilon_r S,$$

from which we get the capacitance C of a parallel-plate capacitor:

$$C = q/U = \varepsilon_0\varepsilon_r S/d.$$

In order to increase capacitance, capacitors are connected in batteries. The plates of capacitors in batteries are connected *in parallel* (Fig. 188), i.e. the positively charged plates are combined

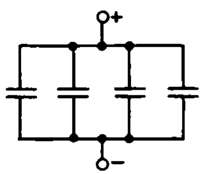


Fig. 188

in one group and the negatively charged plates, in the other. With such a connection all capacitors have the same potential difference between the plates (voltage), but their charges and capacitances can be different. The total charge of the capacitors is

$$q = q_1 + q_2 + \dots + q_n = (C_1 + C_2 + \dots + C_n)U$$

(here U is the voltage across the capacitor plates). Hence we get

$$C = q/U = C_1 + C_2 + \dots + C_n.$$

The capacitance of a battery of capacitors connected in parallel is equal to the sum of the capacitances of all the capacitors. The capacitance of a battery formed by n identical capacitors connected in parallel is n times larger than the capacitance of a single capacitor, since the distance between the plates of the capacitors remains unchanged, while the surface area of the plates is n times larger than the surface area of one capacitor.

Fig. 189

In the *series* connection of capacitors (Fig. 189), the plates with unlike charges are connected. In this case, the charges on the plates are equal in magnitude, and the total charge of connected plates is zero. (This follows from the charge conservation law.) The voltage across each capacitor is determined by its capacitance:

$$U_1 = q/C_1,\ U_2 = q/C_2,\ \dots,\ U_n = q/C_n.$$

The total potential difference between the outer plates of the entire system of capacitors is

$$U = U_1 + U_2 + \dots + U_n = q\left(\frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_n}\right).$$

Consequently, the capacitance of series-connected capacitors is

$$C = \frac{q}{U} = \left(\frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_n}\right)^{-1},$$

whence

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_n}.$$

In the series connection of capacitors, the reciprocals of their capacitances are added.

We can conclude on the basis of the formula for the capacitance of a parallel-plate capacitor that the capacitance of a system of series-connected capacitors having identical plates is equal to the capacitance of a single capacitor having identical plates and such that the thickness of a dielectric in it is equal to the sum of the thicknesses of all the dielectric layers of the system. Consequently, the capacitance of a battery formed by n identical series-connected capacitors is n times smaller than the capacitance of one capacitor of the system.

A typical capacitor (to be more precise, a battery of capacitors) used in radio engineering (Fig. 190) consists of tin foils

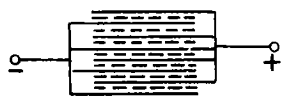


Fig. 190

(plates) separated by paraffined paper (dielectric). Odd tin foils are connected to form one plate and even tin foils form the other plate. Lead-tin or aluminium foils are often used for capacitor plates and mica sheets, for dielectric.

Ceramic capacitors are also widely used in radio engineering. They consist of two silver layers (plates) applied to the surface of special ceramics (dielectric) made in the form of a disc, cylinder, etc. (Fig. 191) and coated by a lacquer layer for protecting from damage. A telephone capacitor (Fig. 192) is made of two tin foils and two sheets of paraffined paper coiled together. There are many other types of capacitors. In physical experiments, the Leyden jar is used (Fig. 193). It is a glass cylindrical vessel with

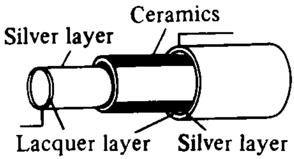


Fig. 191

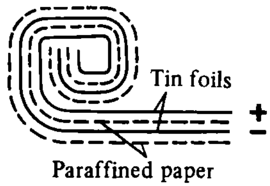


Fig. 192

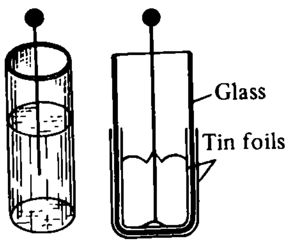


Fig. 193

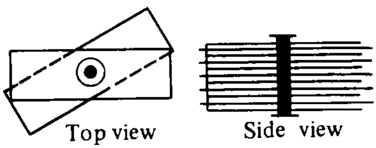


Fig. 194

two plates of tin foil. The outer plate is earthed and the inner plate is connected to a metallic ball.

Variable capacitors required for radio engineering (Fig. 194) consist of a system of fixed plates and a system of plates that can be rotated or drawn out, which are separated by air gaps (dielectric).

To charge a capacitor, some work should be done. A discharging capacitor also performs a work. Consequently, a charged capacitor has an energy which, as will be shown later, is localized (concentrated) in the electric field of the capacitor.

Let us calculate the energy of a parallel-plate capacitor whose plates are in a vacuum. For this purpose, we determine the force with which the plates of a charged capacitor attract each other.

According to formula (4.8.3), the left plate (Fig. 195) creates a field of strength $E_+ = \sigma/2\varepsilon_0$, where $\sigma = q/S$ (S is the area of a plate). This field acts on the right plate with the force

$$F_- = E_+ q = (\sigma/2\varepsilon_0)\sigma S = \sigma^2 S/2\varepsilon_0.$$

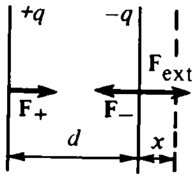


Fig. 195

Similarly, the right plate acts on the left plate with the force F_+ of the same magnitude.

Fixing the left plate and acting on the right plate with an external force $\mathbf{F}_{ext}$ which slightly exceeds in magnitude the force $\mathbf{F}_-$, we displace this plate to the right by a distance x. The work of the external force in this case is

$$A = F_{ext}x = F_-x = (\sigma^2 S/2\varepsilon_0)x = ((2E_+)^2\varepsilon_0/2)Sx$$

(we have used the fact that $E_+ = \sigma/2\varepsilon_0$).

The doubled strength of the field created by one plate is equal to the strength E of the field created in the gap by the two plates. The product Sx is equal to ΔV, viz. the increment of the capacitor volume. Thus,

$$A = \varepsilon_0 E^2 \Delta V/2.$$

According to the law of energy conservation, this work must be equal to the increment of the capacitor energy. The only result of the work done is the increase ΔV in the volume containing the field of strength $\mathbf{E}$ (it should be noted that the field strength in the gap has not changed upon moving the plates apart). Hence it follows that the field possesses an energy which, being recalculated per unit volume (energy density) is given by

$$w = \varepsilon_0 E^2/2.$$

It should be emphasized that the above calculations are valid only when the capacitor plates are in a vacuum. It would seem that if the gap between the plates is filled with a dielectric, the force of attraction between the plates decreases by a fraction of ε_r and becomes $\sigma^2 S/2\varepsilon_0\varepsilon_r$. However, this is not so. The calculation of the energy of a capacitor containing a dielectric between its plates is beyond the scope of this book because of its complexity.

It can be shown that in a dielectric having a relative permittivity ε_r, the energy density of the electric field is defined by the following formula:

$$w = \varepsilon_0\varepsilon_r E^2/2.$$

Multiplying the energy density w by the capacitor volume, which is equal to Sd, we obtain the energy of the charged capacitor (with a dielectric between its plates):

$$W_c = wSd = \varepsilon_0\varepsilon_r E^2 Sd/2.$$

This energy can be expressed in terms of the charge q on a plate and the capacitance C of the capacitor. For this we consider that the field strength in the gap between the plates is $E = \sigma/\varepsilon_0\varepsilon_r = q/\varepsilon_0\varepsilon_r S$. Substituting this value of E into the formula for W_c, we obtain

$$W_c = q^2 d/2\varepsilon_0\varepsilon_r S = q^2/2C.$$

By definition, $C = q/U$, where U is the voltage across the capacitor. Using this relation, we can write the formula for energy as follows:

$$W_c = q^2/2C = qU/2 = CU^2/2.$$

If q is measured in coulombs, U in volts, and C in farads, W_c is expressed in joules.

Problems with Solutions

160. A ball of mass $m = 0.5$ kg is suspended by a thread and a charge $q = 0.1\ \mu$C is supplied to it. When a ball having a diameter $d = 5$ cm and the like charge of the same magnitude is brought to the first ball from below, the tension of the thread decreases to 1/3 of its initial value. Find the distance between the centres of the balls and the surface density of electric charge on the balls.[3]

Solution. The force F with which the balls repel each other is equal in magnitude to 2/3 of the tension of the thread, i.e. to 2/3 of the force of gravity

[3] Here and below, unless other conditions are specially stipulated, assume that interaction occurs in air, i.e. the relative permittivity $\varepsilon_r = 1$ and does not appear in the formulas.

acting on the ball:

$$\frac{1}{4\pi\varepsilon_0}\frac{q^2}{\varepsilon_r r^2} = \frac{2}{3}mg, \text{ or } r = q\sqrt{\frac{1}{4\pi\varepsilon_0}\frac{3}{2\varepsilon_r mg}} = 0.52 \text{ m}.$$

The surface density of charges on the balls (if they are distributed uniformly over the surface) is $\sigma = q/\pi d^2 = 1.3 \text{ mC/m}^2$.

161. Two identical balls of mass $m_1 = m_2 = m = 600$ g and radius $R_1 = R_2 = R = 2$ cm bear identical negative charges. Find the surface charge density if the Coulomb force of their repulsion is known to be balanced by the force of universal gravitation. The distance between the balls is large in comparison with their radii.

Solution. Let us write the equation for the forces of universal gravitation and the Coulomb repulsion, assuming that the charges and masses are point-like:

$$\gamma\frac{m^2}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{r^2}, \text{ whence } q = m\sqrt{4\pi\varepsilon_0\gamma} = 0.052 \text{ nC}.$$

The required surface charge density $\sigma = q/4\pi R^2 \simeq 10 \text{ nC/m}^2$.

162. Two identical small balls having charges $q_1 = +1$ mC and $q_2 = -0.33$ mC are brought in contact and then moved apart to a distance $r = 20$ cm. Find the force of their interaction.

Solition. Since the capacitances of the balls are equal, after having been brought in contact, the balls acquire equal charges. The total charge of the balls remains unchanged (according to the charge conservation law). Hence after their contact, the charge of each ball becomes $q_3 = (q_1 + q_2)/2 = 0.33$ mC. The force of interaction of the balls is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_3^2}{r^2} = 25 \text{ kN}.$$

163. Two positive point charges are separated by a distance $l = 50$ cm. One charge is twice larger than the other. A charged ball is in equilibrium on the straight line connecting the charges. Find the distance from this ball to the larger charge. Is this equilibrium stable?

Solution. If the ball has a positive charge (Fig. 196*a*), the Coulomb repulsive

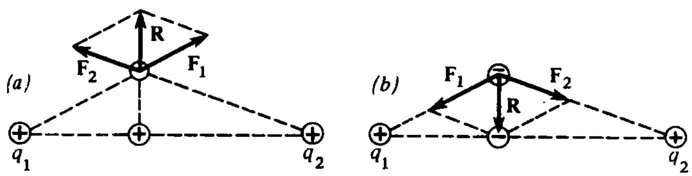


Fig. 196

forces $\mathbf{F}_1$ and $\mathbf{F}_2$ exerted by the two charges must be equal and opposite. This means that the ball is closer to the smaller charge. The forces of repulsion from the smaller and larger charges ($q_2 = 2q_1$) are

$$F_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q}{(l - l_1)^2} \quad \text{and} \quad F_2 = \frac{1}{4\pi\varepsilon_0}\frac{2q_1 q}{l_1^2}.$$

Equating F_1 and F_2, we obtain

$$1/(l - l_1)^2 = 2/l_1^2, \text{ whence } l_1^2 = 2(l - l_1)^2,$$
$$l_1 = l(2 \pm \sqrt{2}).$$

The solution with the plus sign of the radical does not fit. Hence $l_1 = l(2 - \sqrt{2}) = 0.586l = 29.3$ cm.

If the ball is displaced along the straight line connecting the charges, the repulsive force exerted by the charge to which the ball is displaced increases, while the force exerted by the other charge decreases. In other words, the resultant of the two forces returns the ball to the equilibrium position. Consequently, the equilibrium of the ball is stable over the straight line connecting the charges. If the ball is displaced along the normal to the straight line connecting the charges, the resultant R of the two repulsive forces moves the ball away from this line. Hence the equilibrium of the ball is unstable in this direction.

If the ball is charged negatively (Fig. 196*b*), the Coulomb forces $\mathbf{F}_1$ and $\mathbf{F}_2$ reverse their direction. In this case, however, the equilibrium along the straight line connecting the charges is unstable, since when the ball is displaced towards one of the charges, the attraction by this charge increases, while the attraction by the other charge decreases, i.e. the resultant force moves the ball away from the equilibrium position. On the other hand, the equilibrium is stable on the normal to the straight line connecting the charges, since the resultant emerging upon a displacement of the ball in this direction returns it to the equilibrium position.

164. Two identical small balls with a mass $m = 100$ g are suspended in air by two thin silk threads of the same length $l = 2$ m. A charge $q = -1.0\ \mu$C is imparted to the balls. Find the distance r to which the balls move apart.

Solution. Since the capacitances of the balls are equal, the charge imparted to the balls in contact is distributed equally between them:

$$q' = q/2 = -0.5\ \mu\text{C}.$$

After the balls have moved apart, each ball experiences the action of three forces (Fig. 197): the force of gravity $m\mathbf{g}$, the tension $\mathbf{T}$ of the thread, and the repulsive force $\mathbf{F}$ exerted by the other charge. Since the balls are in equilibrium, the vector sum of all the forces acting on them is zero. This means that $F = mg \tan\alpha$. If the angle α is small, $\tan\alpha \approx \sin\alpha = r/2l$. Then $F = mgr/2l$. In accordance with Coulomb's law, $F = (1/4\pi\varepsilon_0)(q'^2/r^2)$, i.e.

$$\frac{mgr}{2l} = \frac{1}{4\pi\varepsilon_0}\frac{q'^2}{r^2}, \quad \text{whence} \quad r = \sqrt[3]{\frac{1}{4\pi\varepsilon_0}\frac{2lq'^2}{mg}} = 0.21 \text{ m}.$$

165. Two likely charged small balls suspended by thin threads of the same length are in kerosene. What must be the density ρ_{ball} of the balls for the angle between

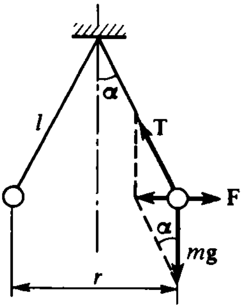


Fig. 197

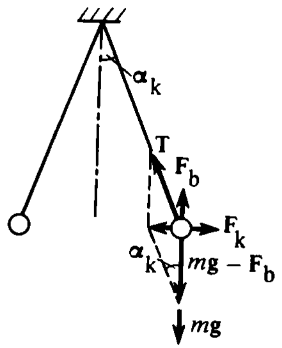


Fig. 198

the threads to be the same in air and in kerosene? The density of kerosene is $\rho_k = 0.8 \times 10^3$ kg/m^3 and the relative permittivity of kerosene is $\varepsilon_r = 2$.

Solution. It was shown in Problem 164 that in air $\tan\alpha = F/mg$. In kerosene (Fig. 198), $\tan\alpha_k = F_k/(mg - F_b)$, where buoyancy $F_b = V_{ball}\rho_k g = m\rho_k g/\rho_{ball}$ (V_{ball} is the volume of a ball). Consequently, $\tan\alpha_k = F_k/[mg(1 - \rho_k/\rho_{ball})]$. Considering that $\tan\alpha = \tan\alpha_k$, we obtain

$$\frac{F}{mg} = \frac{F_k}{mg(1 - \rho_k/\rho_{ball})}, \text{ whence } 1 - \frac{\rho_k}{\rho_{ball}} = \frac{F_k}{F}.$$

Since $F_k/F = 1/\varepsilon_r$, we have $1 - \rho_k/\rho_{ball} = 1/\varepsilon_r$, i.e.

$$\rho_{ball} = \frac{\rho_k}{1 - 1/\varepsilon_r} = 1.6 \times 10^3 \text{ kg/m}^3.$$

166. Two identical positive charges $q_1 = q_2 = q = 1$ μC are located at the vertices of an equilateral triangle with side $a = 0.5$ m. Find the potential and strength of the electric field at the third vertex of the triangle and at the midpoint between the charges.

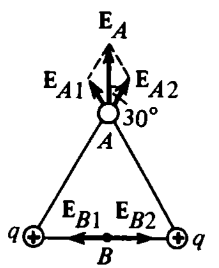


Fig. 199

Solution. The directions of the fields created by the charges at the third vertex A of the triangle are shown in Fig. 199. The magnitude of the field created by each charge at point A is

$$E_{A1} = E_{A2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{a^2} = 36\ \text{kV/m}.$$

Adding vectors $\mathbf{E}_{A1}$ and $\mathbf{E}_{A2}$ by the parallelogram rule, we obtain the strength of the resultant field at the third vertex: $E_A = 2E_{A1}\cos 30° = 36\sqrt{3}$ kV/m.

The potential of the field created by each charge at point A has the form

$$\varphi_{A1} = \varphi_{A2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{a} = 18\ \text{kV}.$$

The potential of the resultant field at the third vertex of the triangle is $\varphi_A = 2\varphi_{A1} = 36$ kV.

The fields created by each charge at the midpoint B are equal and opposite. Hence the strength of the resultant field at the midpoint between the charges is zero: $E_B = 0$. The potential of the field of each charge at point B is

$$\varphi_{B1} = \varphi_{B2} = \frac{1}{4\pi\varepsilon_0}\frac{q}{a/2} = 36\ \text{kV}.$$

The potential of the resultant field at the midpoint between the charges is $\varphi_B = 2\varphi_{B1} = 72$ kV.

167. Point charges $q_1 = q_2 = q_3 = q_4 = q = 1\ \mu$C are arranged at the vertices of a square with side $a = 0.5$ m. Find the potential energy of the system of charges.

Solution. The potential created at the first vertex by the charge located at the second vertex is $\varphi_{12} = q_2/4\pi\varepsilon_0 a$. Consequently, the potential energies of interaction of charges q_1 and q_2, as well as q_1 and q_3, q_1 and q_4, q_2 and q_3, q_2 and q_4, and q_3 and q_4 are given by

$$W_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{a} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a}, \qquad W_{13} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_3}{a\sqrt{2}} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a\sqrt{2}},$$

$$W_{14} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_4}{a} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a}, \qquad W_{23} = \frac{1}{4\pi\varepsilon_0}\frac{q_2q_3}{a} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a},$$

$$W_{24} = \frac{1}{4\pi\varepsilon_0}\frac{q_2q_4}{a\sqrt{2}} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a\sqrt{2}}, \qquad W_{34} = \frac{1}{4\pi\varepsilon_0}\frac{q_3q_4}{a} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a}.$$

The total energy of the system is equal to the sum of the energies obtained above:

$$W = \frac{1}{4\pi\varepsilon_0}\frac{4q^2}{a} + \frac{1}{4\pi\varepsilon_0}\frac{2q^2}{a\sqrt{2}} = \frac{4+\sqrt{2}}{4\pi\varepsilon_0}\frac{q^2}{a} = 0.10\ \text{J}.$$

168. Plot the potential and the electric field projection onto the straight line connecting two positively charged spheres as functions of the x-coordinate if the radius of the spheres is $r = 10$ cm, the surface charge density $\sigma = 80\ \mu C/m^2$, and the distance between the centres of the spheres is $L = 1$ m. The straight line connecting the centres coincides with the x-axis. Mark on the graphs the potential and the projections of the field at the points of intersection of the x-axis with the surfaces of the spheres, at the midpoint between them, and at a distance $l = 0.25$ m from the centre of one sphere, as well as at the centres of the spheres. Plot the graphs only for the points of the axis which lie between the spheres.

Solution. The charge of each sphere is $q = 4\pi r^2\sigma = 0.1\ \mu C$. It follows from symmetry considerations that inside the sphere the field strength created by the charge of the sphere is equal to zero, while outside the sphere, the field strength is the same as the one created by a point charge having the same magnitude and located at the centre of the sphere. If we direct the coordinate axis from the left sphere to the right sphere, the projection of the field created by the left sphere onto this axis is positive, while the projection of the field created by the right sphere is negative. The projection of the strength of the resultant field is equal to the algebraic sum of the two projections. Consequently, at the point of intersection of the coordinate axis with the surface of the left sphere, we have

$$E_{x\,1} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} - \frac{1}{4\pi\varepsilon_0}\frac{q}{(L-r)^2} = \frac{1}{4\pi\varepsilon_0}q\left[\frac{1}{r^2} - \frac{1}{(L-r)^2}\right] = 8.89\ \text{MV/m}.$$

At a distance $l = 0.25$ m from the centre of the sphere, we have

$$E_{x\,2} = \frac{1}{4\pi\varepsilon_0}q\left[\frac{1}{l^2} - \frac{1}{(L-l)^2}\right] = 1.28\ \text{MV/m}.$$

At the midpoint between the charges, the field strength is zero since the projections of the fields created by the two charges are equal and opposite: $E_{x\,\mathrm{m}} = 0$. At the points symmetric about the midpoint of the straight line connecting the centres of the spheres, the projections of the electric fields are equal in magnitude and have opposite signs.

The potentials at different points of the electric field are equal to the sum of the potentials created at these points by the two charged spheres:

$$\varphi_1 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} + \frac{1}{4\pi\varepsilon_0}\frac{q}{L-r} = \frac{1}{4\pi\varepsilon_0}q\left(\frac{1}{r} + \frac{1}{L-r}\right) = 1.0\ \text{MV},$$

$$\varphi_2 = \frac{1}{4\pi\varepsilon_0}q\left(\frac{1}{l} + \frac{1}{L-l}\right) = 0.48\ \text{MV}.$$

At the midpoint between the charges, the potential is equai to the doubled potential due to one charged sphere:

$$\varphi_{\mathrm{m}} = \frac{1}{4\pi\varepsilon_0}\frac{2q}{L/2} = 0.36\ \text{MV}.$$

Inside a sphere, the potential of the electric field created by the charge of this sphere is the same as at its surface. The potential of the electric field created by the charge of the other sphere is found by the same formula. Consequently,

$$\varphi_0 = \frac{1}{4\pi\varepsilon_0} q \left(\frac{1}{r} + \frac{1}{L}\right) = 0.99 \text{ MV}.$$

The plots of the potential and the projection of the field strength onto the straight line connecting the centres of the spheres are shown in Fig. 200. The curves are symmetric about the midpoint of the straight line connecting the centres.

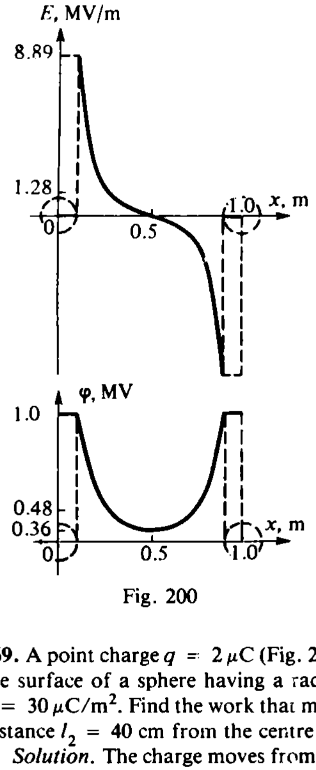


Fig. 200

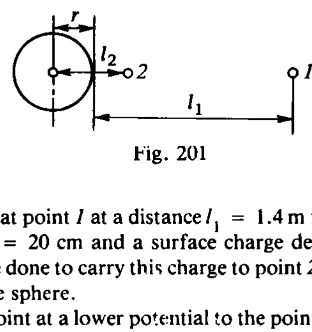


Fig. 201

169. A point charge $q = 2\,\mu$C (Fig. 201) is at point *1* at a distance $l_1 = 1.4$ m from the surface of a sphere having a radius $r = 20$ cm and a surface charge density $\sigma = 30\,\mu\text{C/m}^2$. Find the work that must be done to carry this charge to point *2* at a distance $l_2 = 40$ cm from the centre of the sphere.

Solution. The charge moves from the point at a lower potential to the point at a higher potential. For such a displacement, a work $A = q(\varphi_2 - \varphi_1)$ must be done against the forces of the field. The potentials at points *1* and *2* are

$$\varphi_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_{\text{sph}}}{l_1 + r} \text{ and } \varphi_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_{\text{sph}}}{l_2},$$

where $q_{sph} = \sigma \cdot 4\pi r^2$. Consequently,

$$A = q(\varphi_2 - \varphi_1) = \frac{1}{4\pi\varepsilon_0} q q_{sph} \left(\frac{1}{l_2} - \frac{1}{l_1 + r}\right) = \frac{q\sigma r^2}{\varepsilon_0} \left(\frac{1}{l_2} - \frac{1}{l_1 + r}\right) = 0.51 \text{ J}.$$

170. A thousand of similar electrified raindrops merge into one so that their total charge remains unchanged. Find the change in the total electric energy of the drops, assuming that the drops are spherical and that small drops are at large distances from one another.

Solution. We denote by r, C_1, w, and q the radius, capacitance, energy, and charge of a single drop (before merging) and by R, C, W, and Q the corresponding quantities for the large drop (obtained as a result of merging of small drops). We equate the volume of the drop after merging to the total volume of the drops before that: $4\pi R^3/3 = n \cdot 4\pi r^3/3$, whence $R/r = \sqrt[3]{n} = C/C_1$. The electric energy of a single drop before merging is $w = q\varphi/2 = q^2/2C_1$. The energy of n drops before merging is n times higher: $nw = nq^2/2C_1$. The energy of the large drop formed from merging n small drops is $W = Q^2/2C$. Thus,

$$\frac{W}{nw} = \frac{Q^2}{2C} \frac{2C_1}{nq^2} = \frac{1}{n} \left(\frac{Q}{q}\right)^2 \frac{C_1}{C} = 100,$$

i.e. the energy has increased 100 times.

171. Two spheres of diameters $d_1 = 0.20$ m and $d_2 = 0.80$ m are in a vessel with kerosene at a distance $l = 160$ cm (Fig. 202). The first sphere is charged to a

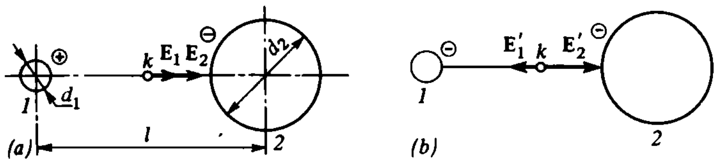


Fig. 202

potential $\varphi_1 = 250$ V and the second, to $\varphi_2 = -100$ V. The spheres are connected by a conductor which is then removed. Find the strength and potential of the electric field at the midpoint between the centres of the spheres. The relative permittivity of kerosene is $\varepsilon_r = 2$.

Solution. The capacitances of the spheres are

$$C_1 = d_1/2k = 11.1 \text{ pF and } C_2 = d_2/2k = 44.4 \text{ pF},$$

where $k = 1/4\pi\varepsilon_0 = 9 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$ is the value of the constant in Coulomb's law. Before the spheres are connected (Fig. 202*a*), their charges have been

$$q_1 = \varphi_1 C_1 = 2.75 \text{ nC and } q_2 = \varphi_2 C_2 = -4.40 \text{ nC}.$$

The magnitudes of the fields created by the charges of spheres *1* and *2* at point K

are

$$E_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{\varepsilon_r(l/2)^2} = \frac{q_1}{\pi\varepsilon_0\varepsilon_r l^2} \text{ and } E_2 = \frac{1}{4\pi\varepsilon_0}\frac{|q_2|}{\varepsilon_r(l/2)^2} = \frac{|q_2|}{\pi\varepsilon_0\varepsilon_r l^2}.$$

These fields are directed to the right. Consequently, the magnitude of the total field at point K is

$$E = E_1 + E_2 = (q_1 + |q_2|)/\pi\varepsilon_0\varepsilon_r l^2 = 100 \text{ V/m}.$$

This field is also directed to the right.

The potentials of the fields created by the charges of spheres *1* and *2* at point K are

$$\varphi_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{\varepsilon_r l/2} = \frac{q_1}{2\pi\varepsilon_0\varepsilon_r l} \text{ and } \varphi_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_2}{\varepsilon_r l/2} = \frac{q_2}{2\pi\varepsilon_0\varepsilon_r l}.$$

The potential of the resultant field is

$$\varphi = \varphi_1 + \varphi_2 = (q_1 + q_2)/2\pi\varepsilon_0\varepsilon_r l = -18.8 \text{ V}.$$

After the spheres have been connected through a conductor (Fig. 202*b*), their potential difference becomes zero, while the charges of the spheres are distributed in proportion to their capacitances. The total charge

$$q = q_1 + q_2 = -1.65 \text{ nC}.$$

After connection, the charges of the spheres become

$$q_1' = \frac{q}{C_1 + C_2}C_1 = -0.33 \text{ nC and } q_2' = \frac{q}{C_1 + C_2}C_2 = -1.32 \text{ nC}$$

(since l is large, we neglect the effect of one sphere on the charge distribution over the other sphere).

The magnitudes of the fields created by the charges of spheres *1* and *2* after connection at point K are

$$E_1' = \frac{1}{4\pi\varepsilon_0}\frac{4|q_1'|}{\varepsilon_r l^2} = \frac{|q_1'|}{\pi\varepsilon_0\varepsilon_r l^2} \text{ and } E_2' = \frac{1}{4\pi\varepsilon_0}\frac{4|q_2'|}{\varepsilon_r l^2} = \frac{|q_2'|}{\pi\varepsilon_0\varepsilon_r l^2}.$$

Vector $\mathbf{E}_1'$ is directed to the left, while $\mathbf{E}_2'$ is directed to the right. Since $E_2' > E_1'$, vector $\mathbf{E}'$ of the resultant field at point K is directed to the right. Its magnitude is given by

$$E' = (|q_2'| - |q_1'|)/\pi\varepsilon_0\varepsilon_r l^2 = 46 \text{ V/m}.$$

The potentials of the fields of the two charges at point K are summed up:

$$\varphi' = \frac{1}{4\pi\varepsilon_0}\frac{q_1' + q_2'}{\varepsilon_r l/2} = \frac{q_1' + q_2'}{2\pi\varepsilon_0\varepsilon_r l} = -18.8 \text{ V}.$$

172. Find the potential φ of a sphere if the potentials of its field are $\varphi_1 = 400$ V at a distance $l_1 = 50$ cm from its centre and $\varphi_2 = 800$ V at a distance $l_2 = 20$ cm from its surface.

Solution. It follows from symmetry considerations that the electric field of a uniformly charged sphere is equivalent to the field of a point charge located at the centre of the sphere and having the same magnitude. If the charge of the sphere is q, the potentials of its field at distances l_1 and l_2 are

$$\varphi_1 = \frac{1}{4\pi\varepsilon_0}\frac{q}{l_1} \text{ and } \varphi_2 = \frac{1}{4\pi\varepsilon_0}\frac{q}{l_2 + r},$$

where r is the radius of the sphere. Hence $\varphi_1/\varphi_2 = (l_2 + r)/l_1$, and the radius of the sphere $r = l_1\varphi_1/\varphi_2 - l_2 = 5$ cm. Consequently, the capacitance of the sphere is $C = r/k = 5.5$ pF. From the expression $\varphi_1 = kq/l_1$, we obtain $q = \varphi_1 l_1/k$, and hence the required potential of the sphere is $\varphi = q/C = \varphi_1 l_1/kC = 4$ kV.

173. Two metallic concentric spheres in air have the radii $r_1 = 20$ cm and $r_2 = 40$ cm. A charge $q_1 = -30$ nC is on the inner sphere, while the outer sphere is charged to the potential $\varphi_2 = 600$ V. Find the field strengths and potentials at

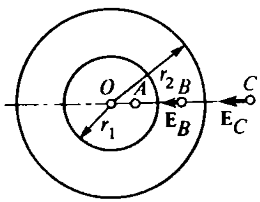


Fig. 203

points A, B, and C (Fig. 203), located on the same straight line at distances $l_A = 10$ cm, $l_B = 25$ cm, and $l_C = 50$ cm from the centre of the spheres.

Solution. The charge uniformly distributed over the surface of a sphere creates outside it the same field as a point charge having the same magnitude and located at the centre of the sphere. The field inside the sphere is zero, while the potential is equal to the potential on the surface of the sphere. The charge of the outer sphere is $q_2 = \varphi_2 C = \varphi_2 r_2/k = 26.6$ nC.

Let us calculate the projections of the electric field strength onto a horizontal axis passing through the centre of the spheres from left to right:

$$E_C = \frac{q_1}{l_C^2} + \frac{q_2}{l_C^2} = -120 \text{ V/m}, \quad \varphi_C = \frac{q_1}{l_C} + \frac{q_2}{l_C} = -60 \text{ V}$$

outside the spheres,

$$E_B = 0 + \frac{q_1}{l_B^2} = \frac{q_1}{l_B^2} = -4320 \text{ V/m}, \quad \varphi_B = \frac{q_2}{r_2} + \frac{q_1}{l_B} = -480 \text{ V}$$

between the spheres, and

$$E_A = 0, \quad \varphi_A = \frac{q_1}{r_1} + \frac{q_2}{r_2} = -750 \text{ V}$$

inside the smaller sphere. The minus sign of E_B and E_C indicates that the electric field strengths are directed to the centre of the spheres.

174. A charge dust particle whose mass $m = 10^{-8}$ g is in a uniform electrostatic field between two horizontal plates. The lower plate is charged to a potential $\varphi_1 = 3$ kV, while the upper plate, to a potential $\varphi_2 = -3$ kV. The separation of the plates is $d = 5$ cm. Being initially at a distance $d_0 = 1$ cm from the lower plate, the particle reaches the upper plate in $t = 0.1$ s. Find the charge of the particle. What must be the charge for the particle to remain in equilibrium?

Solution. The dust particle moves in the direction of the electric field (from the higher potential to the lower potential). This means that it has a positive charge. The particle experiences the action of two forces: the upward electric force $\mathbf{F} = q\mathbf{E}$ and the downward force of gravity $m\mathbf{g}$. The electric field strength between the plates is $E = (\varphi_1 - \varphi_2)/d$. Consequently, $F = q(\varphi_1 - \varphi_2)/d$. The resultant of the forces $\mathbf{F}$ and $m\mathbf{g}$ is directed upwards. Its magnitude is $R = q(\varphi_1 - \varphi_2)/d - mg$. This force imparts to the particle an acceleration

$$a = \frac{R}{m} = \frac{q}{m}\frac{\varphi_1 - \varphi_2}{d} - g. \tag{1}$$

On the other hand, since $a = \text{const}$, $d - d_0 = at^2/2$, and

$$a = \frac{2(d - d_0)}{t^2}. \tag{2}$$

Equating the right-hand sides of Eqs. (1) and (2), we obtain

$$\frac{q}{m}\frac{\varphi_1 - \varphi_2}{d} - g = \frac{2(d - d_0)}{t^2},$$

whence the charge of the particle is

$$q = \frac{md}{\varphi_1 - \varphi_2}\left[\frac{2(d - d_0)}{t^2} + g\right] = 1.5 \times 10^{-15} \text{ C}.$$

The particle is in equilibrium when $qE = mg$:

$$q(\varphi_1 - \varphi_2)/d = mg,$$

or

$$q = mgd/(\varphi_1 - \varphi_2) = 0.81 \times 10^{-15} \text{ C}.$$

175. An electron having a kinetic energy $W_k = 10^{-16}$ J flies into the space between two horizontal square metallic plates (Fig. 204) (at the midpoint between the plates) parallel to one side of the plates. The separation of the plates is $d = 0.4$ m

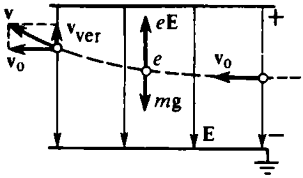


Fig. 204

and the side of a plate is $b = 0.5$ m. The lower plate is earthed, while the upper plate has a charge $q = 2$ nC. Find the velocity of the electron when it leaves the limits of the plates. At what distance from the lower plate is it at this moment? The electron charge $e = 1.6 \times 10^{-19}$ C and its mass $m = 9.1 \times 10^{-31}$ kg.

Solution. The capacitance of the capacitor formed by the plates and the potential of the upper plate have the form

$$C = \varepsilon_0\varepsilon_r S/d, \quad \varphi = q/C = qd/\varepsilon_0\varepsilon_r S.$$

The strength of the uniform electric field of the capacitor is

$$E = \varphi/d = q/\varepsilon_0\varepsilon_r b^2 = 905 \text{ V/m}.$$

Since $W_k = mv_0^2/2$, the magnitude of the velocity of the electron entering the space between the plates is

$$v_0 = \sqrt{2W_k/m} = 14.8 \times 10^6 \text{ m/s}.$$

Vector $\mathbf{v}_0$ has the horizontal direction parallel to a side of the capacitor plate. The time of the electron flight above the plate is $t = b/v_0$.

The electron moving between the plates is acted upon by the downward force of gravity $m\mathbf{g}$ and the upward force $e\mathbf{E}$ of electrostatic interaction. The resultant of these two forces is directed upwards: $F = eE - mg$. During the time t of the flight, the electron acquires a vertical component of velocity:

$$v_{ver} = Ft/m = (eE - mg)b/mv_0 = 5.4 \times 10^6 \text{ m/s}.$$

Therefore, the magnitude of the velocity of escaping electron is given by

$$v = \sqrt{v_0^2 + v_{ver}^2} = 15.8 \times 10^6 \text{ m/s}.$$

Vector $\mathbf{v}$ forms with the horizontal an angle α such that $\sin\alpha = v_{ver}/v = 0.340$, i.e. $\alpha \simeq 19.5°$. Since the electron moves with a constant acceleration, we have $\Delta h = v_{ver.av}t = v_{ver}b/2v_0 \simeq 9.0$ cm, and the height of the electron above the lower plate is $h = h_0 + \Delta h = 29.0$ cm.

176. A battery of capacitors is made of five mica sheets of thickness $d = 0.1$ mm and surface area $S = 100\ \text{cm}^2$ each and of tin foils. How many tin foils are required for the parallel connection of the capacitors in the battery? Draw a diagram of the battery. Find the capacitance of the battery and the energy stored in it if it is connected to a d.c. circuit at a voltage $U = 220$ V. The relative permittivity of mica is $\varepsilon_r = 7$.

Solution. In parallel connection, all positively charged mica sheets and negatively charged tin foils are connected. Each tin foil may serve as a plate for two neighbouring capacitors (Fig. 205). Consequently, the number $n_{t.f}$ of tin foils is 6. The capacitance of the battery is $C_b = nC$, where the capacitance of a capacitor is $C = \varepsilon_0\varepsilon_r S/d$. Thus, the capacitance of the battery is $C_b = n\varepsilon_0\varepsilon_r S/d = 0.031\ \mu F$. The energy of the battery of capacitors is $W_b = CU^2/2 = 0.75$ mJ.

177. A battery of five capacitors is connected as shown in Fig. 206*a*. The capacitors have the same capacitance $C = 3$ nF. Find the capacitance of the battery.

Solution. Figure 206*b* shows the signs of the charges on the capacitor plates. Judging from the charges, the capacitors C_1, C_2, and C_3 are connected in series, as

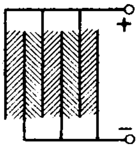


Fig. 205

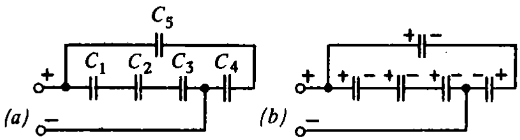


Fig. 206

well as the capacitors C_4 and C_5, the two groups being connected in parallel. Thus, if we denote by C the capacitance of a capacitor, then $C_{1\text{-}3} = C/3$ and $C_{4\text{-}5} = C/2$. The capacitance of the battery is

$$C_b = C/3 + C/2 = 5C/6 = 2.5 \text{ nF}.$$

178. Find the capacitance of the battery of capacitors connected as shown in Fig. 207. The space between the square capacitor plates with side l is filled with dielectric layers of the same thickness with relative permittivities $\varepsilon_{r\,1}$ and $\varepsilon_{r\,2}$. The distance between the plates is d.

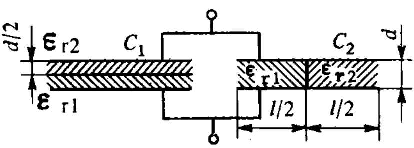


Fig. 207

Solution. The capacitors C_1 and C_2 are connected in parallel. Hence the total capacitance of the battery is $C_b = C_1 + C_2$. The capacitor C_1 can be regarded as two capacitors connected in series through the contact surface between the dielectrics, since the introduction of a thin conducting plate between the dielectrics ($\varepsilon_{r\,1}$ and $\varepsilon_{r\,2}$) does not change the capacitance of the capacitor. Consequently,

$$\frac{1}{C_1} = \frac{1}{\varepsilon_0\varepsilon_{r\,1}l^2/(d/2)} + \frac{1}{\varepsilon_0\varepsilon_{r\,2}l^2/(d/2)} = \frac{d}{2\varepsilon_0 l^2}\left(\frac{1}{\varepsilon_{r\,1}} + \frac{1}{\varepsilon_{r\,2}}\right),$$

whence

$$C_1 = \frac{2\varepsilon_0 l^2}{d} \frac{\varepsilon_{r1}\varepsilon_{r2}}{\varepsilon_{r1} + \varepsilon_{r2}}.$$

The capacitor C_2 can be treated as two capacitors connected in parallel. Consequently,

$$C_2 = \frac{\varepsilon_0 \varepsilon_{r1} l(l/2)}{d} + \frac{\varepsilon_0 \varepsilon_{r2} l(l/2)}{d} = \frac{\varepsilon_0 l^2}{2d}(\varepsilon_{r1} + \varepsilon_{r2}).$$

Thus, the capacitance of the battery of capacitors is

$$C_b = \frac{2\varepsilon_0 l^2}{d} \frac{\varepsilon_{r1}\varepsilon_{r2}}{\varepsilon_{r1} + \varepsilon_{r2}} + \frac{\varepsilon_0 l^2}{2d}(\varepsilon_{r1} + \varepsilon_{r2}) = \frac{\varepsilon_0 l^2}{2d} \frac{\varepsilon_{r1}^2 + \varepsilon_{r2}^2 + 6\varepsilon_{r1}\varepsilon_{r2}}{\varepsilon_{r1} + \varepsilon_{r2}}.$$

179. A battery of capacitors connected as shown in Fig. 208 is connected to a battery of cells with an e.m.f. $\mathcal{E} = 30$ V. The capacitances of the capacitors are $C_1 = C_2 = C_3 = C = 0.11$ nF, $C_4 = 0.05$ nF, and $C_5 = 0.11$ nF. Find the charge of each capacitor.

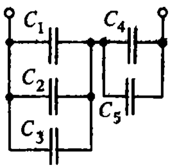


Fig. 208

Solution. The capacitance of the parallel-connected capacitors C_1, C_2, and C_3 is $C_{1\text{-}3} = 3C$, while the capacitance of the parallel-connected capacitors C_4 and C_5 is $C_{4\text{-}5} = C_4 + C_5$. The capacitance of the capacitor battery can be found from the expression

$$\frac{1}{C_b} = \frac{1}{C_{1\text{-}3}} + \frac{1}{C_{4\text{-}5}} = \frac{1}{3C} + \frac{1}{C_4 + C_5} = 10^9\ \mathrm{F}^{-1}, \text{ i.e. } C_b = 1\ \mathrm{nF}.$$

There are no charges on the inner plates. The charge on the outer plates of the left and right groups of capacitors are equal in magnitude and opposite in sign: $|q_{1\text{-}3}| = |q_{4\text{-}5}| = q = \mathcal{E} C_b = 30$ nC. In the left group of capacitors having equal capacitance, the charges are distributed uniformly: $q_1 = q_2 = q_3 = q/3 = 10$ nC. In the right group, the charges are distributed in proportion to the capacitances since the potential difference of likely charged capacitor plates is zero: $\varphi_4 - \varphi_5 = 0$, i.e. $q_4/C_4 - q_5/C_5 = 0$, whence

$$q_4/q_5 = C_4/C_5 = 1/2, \quad q_5 = 2q_4, \quad q_4 = 10\ \mathrm{nC}, \quad q_5 = 20\ \mathrm{nC}.$$

Exercises

160. A charged sphere whose mass is 300 g is suspended by a thread. When a likely charged sphere of radius 2 cm is brought to a distance of 0.40 m to the first sphere from below, the tension of the thread decreases to 1/4 of its initial value. Find the surface density of electric charge on the second sphere.
Answer. 125 mC/m^2.

161. Two charged spheres of the same radius and of equal mass of 0.3 kg are located at such a distance that the force of interaction between their charges is balanced by the force of universal gravitation. Find the radii of the spheres if their surface charge density is 1.25 nC/m^2.
Answer. 4 cm.

162. (a) Two identical balls having like charge and placed at 10 cm repel each other with a force of 0.12 mN. They are brought in contact and then moved apart to the initial positions. The balls repel each other with a force of 0.16 mN. Find the charges of the balls before they have been brought in contact.
Answer. 20 nC, 6.6 nC.

(b) Two identical balls having like charge and placed at a certain distance repel each other with a certain force. They are brought in contact and then moved apart to a distance equal to half their initial separation. The force of repulsion between them increases 4.5 times in comparison with the initial value. Find the ratio of the initial charges of the balls.
Answer. 2.

163. Two point charges, viz. a positive charge $(+q)$ and a negative charge $(-2q)$, are fixed at a distance of 50 cm from each other. A small positively charged ball is in equilibrium on the straight line connecting these charges. Determine the equilibrium position of the ball and find out whether or not this equilibrium is stable. Solve the same problem for a negatively charged ball in equilibrium.
Answer. At 120.7 cm behind the positive charge. The equilibrium of the positive charge is unstable and of the negative charge, stable, both in longitudinal and transverse directions.

164. (a) Two identical balls are suspended in air by thin threads 20 cm long and nailed at the same point so that the balls are in contact. When a charge of 0.4 μC is imparted to the balls, they move apart so that the threads form an angle of 60°. Find the masses of the balls, neglecting the mass of the threads.
Answer. 6.24 g.

(b) Two small 50-mg balls are suspended by 20-cm thin threads fixed at the same level at 10 cm from each other. When unlike charges of equal magnitude are imparted to the balls, they approach each other to a distance of 2 cm. Find the charges of the balls, neglecting the mass of the threads.
Answer. 2.1 nC.

(c*) Two small charged balls are suspended by long threads fixed at the same point. The charges and masses of the balls are such that the balls are in equilibrium at a separation of 10 cm. At what distance from each other are the balls in

equilibrium if their charges are increased four-fold? The mass of the threads should be neglected.
Answer. 25.2 cm.
165*. Two identical charged balls suspended by thin threads of equal length are in kerosene. The angle between the threads does not exceed $15°$. What must be the density of the balls for the angle between the threads in air to be 1.5 times as large as in kerosene? 1.5 times as small? What is the ratio of the angles in air and in kerosene if the density of the balls is 8×10^3 kg/m^3?. The density of kerosene is 0.8×10^3 kg/m^3 and its relative permittivity is 2. For angles smaller than $30°$, assume that $\tan \alpha \approx \alpha$. The mass of the threads should be neglected.
Answer. 3.2×10^3 kg/m^3, 1.2×10^3 kg/m^3, 1.8.
166. Three identical positive point charges of 5 nC each are arranged at the vertices of a square with a side of 40 cm. Find the electric field strength and potential at the fourth vertex. Solve the problem for the case when the charge located on the diagonal with the fourth vertex is negative and equal in magnitude to the positive charges.
Answer. $E_1 = 535$ V/m, vector $\mathbf{E}_1$ is directed along the extension of the diagonal, $\varphi_1 \approx 305$ V, $E_2 = 254$ V/m, vector $\mathbf{E}_2$ has the same direction as $\mathbf{E}_1$, $\varphi_2 \approx 145$ V.
167. Find the potential energy of a system of three identical point charges of 3.3 μC each, located at the vertices of an equilateral triangle with a side of 1 m.
Answer. 0.3 J.
168. Plot the potential and the projections of the field strength onto the straight line connecting two spheres having a radius of 10 cm and bearing unlike charges of the same surface density $\pm 16\ \mu$C/m^2 as functions of the coordinate if the distance between their centres is 1 m. Mark the required quantities at the points of intersection of the surfaces of the spheres with the straight line connecting their centres, at the midpoint of this straight line, and at a distance of 25 cm from the centre of one sphere, as well as at the centres of the spheres. Draw the curves only within the distance between the centres of the spheres.
Answer. Fig. IV, 1.82 MV/m, 0.32 MV/m, 0.02 MV/m, 0.14 MV/m, 1.60 MV, 0.48 MV, 0.
169. (a) A positive point charge of 7 μC is at a distance of 0.9 m from the surface of a sphere of radius 10 cm, bearing a positive charge with a surface density of 30 μC/m^2. What work must be done to carry the charge to a point at 50 cm from the centre of the sphere? The surrounding medium is kerosene whose relative permittivity is 2.
Answer. 0.24 J.

(b) Find the velocity acquired by an electron that has flown through an electric field between two points at the potentials of 100 V and 300 V if its initial velocity is 5×10^6 m/s. The electron charge is 1.6×10^{-19} C and its mass is 9.1×10^{31} kg.
Answer. 9.7×10^6 m/s.

(c) The work of electric field done during the displacement of a negatively charged particle towards a fixed positively charged particle is 9 J. As a result, the distance between the charges has been decreased by half. What work is done by the electric field over the first half of this distance?
Answer. 3 J.

170. (a) Eight mercury droplets having a radius of 1 mm and a charge of 0.066 pC each merge into one droplet. Find its potential.
Answer. 2.4 V.

(b) A spherical droplet having a potential of 2.5 V is obtained as a result of merging of 125 identical droplets. Find the potential of a constituent droplet.
Answer. 0.1 V.

171. Two spheres having radii of 5 cm and 10 cm bear identical charge of 6.6 nC. Find the potentials and charges of the spheres after they have been connected by a conductor. Assume that the spheres are at a large distance from each other.
Answer. 8 V, 4.4 nC, 8.8 nC.

172. At distances of 5 cm and 10 cm from the surface of a sphere, the potentials are 600 V and 420 V. Find the potential of its surface.
Answer. 1050 V.

173. (a) Two charged concentric spheres are in air. The potentials of the electric field at distances of 5 cm, 40 cm, and 60 cm from the centre of the spheres are 2100 V, -150 V, and -250 V. Find the charges of the two spheres and the radius of the larger sphere if the radius of the smaller sphere is 10 cm, while the radius of the larger sphere is greater than 40 cm and smaller than 60 cm.
Answer. 33 nC, 50 nC, 50 cm.

(b) Two charged concentric spheres in air have radii of 20 cm and 60 cm. At a distance of 80 cm from the centre of the spheres, the electric field strength is 230 V/m and is directed away from the centre. At a distance of 40 cm from the centre, the field strength is 940 V/m and is directed to the centre. Find the charges of the spheres and the field potentials at distances of 80 cm, 40 cm, and 10 cm from the centre.
Answer. -16 nC, 33 nC, 187.5 V, 125 V, -250 V.

174. A dust particle whose mass is 10^{-8} g is a uniform electrostatic field between two charged horizontal plates separated by 5 cm. The lower plate is charged to a potential of 8000 V, while the upper plate, to a potential of 2000 V. What is the charge of the particle if it is in equilibrium? What must be the potential of the lower plate (at the same value of the potential of the upper plate) for the particle, after having lost a charge equal to 1000 electron charges, to remain in equilibrium? The electron charge is 1.6×10^{-19} C.
Answer. 8.2×10^{-16} C, 9500 V.

(b*) A charged dust particle whose mass is 6×10^{-9} g is in a uniform electrostatic field between two charged horizontal plates separated by 2 cm. The charge of the particle is 4.8×10^{-16} C. The lower plate is charged to 900 V and the upper plate, to 300 V. Find the time during which the particle reaches the upper plate if initially it has been at the midpoint between the plates.
Answer. 0.13 s.

175. (a) An electron flies into the space between two parallel charged plates separated by 16 mm. The electron velocity of 2×10^6 m/s is parallel to the plates. The potential difference across the plates is 4.8 V. Find the deviation of the electron from the initial path over a distance of 3 cm. The effect of the force of gravity should be neglected. The electron charge is 1.6×10^{-19} C and its mass is 9.1×10^{-31} kg.
Answer. 6 mm.

(b) An electron flies with a velocity v_0 into the space between two parallel charged plates of an air capacitor, parallel to the plates. Over a distance l, the electron velocity deviates through an angle α from its initial direction. Find the field strength of the capacitor.

Answer. $2mv_0^2 \tan \alpha / el$.

176. (a) A battery of capacitors is formed by seven mica sheets, whose thickness is 0.2 mm and whose area is 200 cm^2, and eight tin foils. Draw a schematic diagram of the parallel connection. Find the capacitance of the battery and the voltage of a d.c. circuit if the battery connected to this circuit stores an energy of 1.05 mJ. The relative permittivity of mica is 7.

Answer. Fig.V, 43 nF, 220 V.

(b) A battery of capacitors is made of the same elements as in Problem 176 (a). Draw a diagram of the series connection. Find the capacitance of the battery and the energy stored in it after it has been connected to a d.c. circuit at a voltage of 220 V.

Answer. Fig. VI, 885 pF, 21.3 μJ.

177. (a) A battery of capacitors having the same capacitance C is connected as shown in Fig. 209*a*. Find the capacitance of the battery.

Answer. 1.36C.

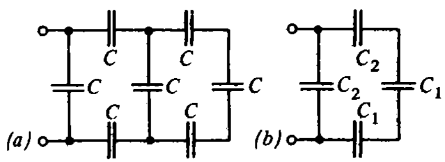


Fig. 209

(b) Capacitors are connected in a battery as shown in Fig. 209*b*. The capacitances of the capacitors are $C_1 = 2\,\mu$F and $C_2 = 1\,\mu$F. Find the capacitance of the battery.

Answer. 1.5 μF.

178. Find the capacitance of a battery of two capacitors in which the spaces between the plates are partially filled with a dielectric having a relative permittivity ε_r

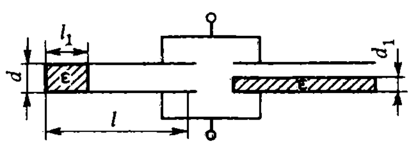


Fig. 210

as shown in Fig. 210. The area of a plate is S and their separation is d.

Answer.

$$\varepsilon_0 S \left[\frac{l + l_1(\varepsilon_r - 1)}{ld} + \frac{\varepsilon_r}{d_1 + \varepsilon_r(d - d_1)} \right].$$

179. Find the charges of the capacitors depicted in Fig. 211 if $C = 1\ \mu F$ and the voltage across the capacitor plates is $\mathscr{E} = 2$ V.
Answer. $2\ \mu C$, $1\ \mu C$, $1\ \mu C$.

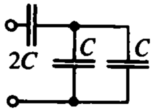


Fig. 211

B. DIRECT CURRENT

4.9. Electric Current. Current Intensity. Electromotive Force

Electric current is a directed motion of electric charges. In metals, electric current is due to the motion of negatively charged particles (electrons) in a direction opposite to that of the field strength **E** In other words, electrons move from points having a lower potential to points having a higher potential. In physics and engineering, however, the direction of current is conventionally determined by the direction of motion of *positive* charges.

According to the electron theory, when atoms combine to form a crystal lattice of a metal, the weakly bound (valence) electrons are detached from atoms and begin to move freely over the whole metallic body. These electrons are called free, or conduction, electrons. They are in a random motion similar to the motion of gas molecules. For this reason, the aggregate of free electrons in metals is called an electron gas.

If an external electric field is applied to a conductor, the random thermal motion of free electrons is replaced by an ordered motion under the action of electric field forces. This is the so-called electron drift that causes an electric current. Since electrons are the carriers of electric current in metals, the conduction of metallic conductors is called the **electron conduction.**

The idea behind numerous experiments carried out to prove the above statement is that since an electron has a mass, it should possess an inertia of motion. Consequently, if a metallic conductor is set in motion and then abruptly stopped, its electrons will continue their drift motion by inertia. The electric current pulse created as a result can be registered and measured by highly sensitive instruments.

The **current intensity** (or simply **current**) I flowing through a certain surface is a physical quantity equal to the charge carried through the surface per unit time. If a charge Δq is transported over a very short time interval Δt, then the current

$$I = \Delta q/\Delta t.$$

To be more precise,

$$I = \lim_{\Delta t \to 0} \Delta q/\Delta t.$$

When the current does not change with time, it is called **direct current**. In this case,

$$I = q/t,$$

where t is the (arbitrarily long) time over which the charge q is transferred. If the current varies with time, it is called **alternating current**. In a particular case of an alternating current its intensity varies according to the sine law.

When considering current in wires, the wire cross section is assumed to be the surface through which a charge is transported.

The SI unit of current, an **ampere** (A), is one of the fundamental units. It will be defined later (see Sec. 4.25). Fractional units of current, viz. mulliampere (mA) and microampere (μA), are also used:

$$1\ \text{mA} = 10^{-3}\ \text{A},\ 1\ \mu\text{A} = 10^{-6}\ \text{A}.$$

Figure 212 shows a segment of a current-carrying conductor. If the velocity of ordered motion (drift velocity) of charge carriers (free electrons) is $\mathbf{v}$, all the carriers at a distance smaller than $v\Delta t$ from the surface S will pass through it over the time Δt. The number of these carriers is $nSv\ \Delta t$, where n is the number of free electrons per unit volume of the conductor. Multiplying the number of passing charge carriers by the electron charge e, we find the charge Δq transported through the cross section S during

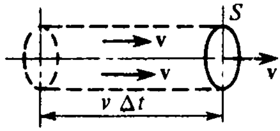


Fig. 212

the time Δt. Consequently, we obtain the following equation for the current:

$$I = nevS.$$

The current per unit cross-sectional area of a conductor is called the **current density** j. It follows from the above expression for I that

$$j = I/S = nev.$$

Current density is measured in **amperes per square metre** (A/m^2).

The necessary condition for a current to appear in a closed conducting circuit is that extraneous forces, i.e. the forces of nonelectrostatic origin, must act in the entire circuit or in its subcircuits. Subcircuits in which extraneous forces act are called nonuniform, while those in which extraneous forces do not act are called uniform. In uniform subcircuits, charge carriers move under the action of the electric field generated by nonuniform subcircuits.

Extraneous forces emerge, for example, at the boundary between an electrode and an electrolyte of a galvanic cell (or accumulator), in the rotor winding of a generator, and so on.

The action of extraneous forces is characterized by a physical quantity called **electromotive force** $\mathcal{E}$. This quantity is defined as the work done by extraneous forces in moving a unit positive charge over a closed circuit or over some subcircuit. In the former case, we speak of an electromotive force (e.m.f.) acting in the closed circuit, and in the latter case, an e.m.f. acting over the given subcircuit.

If A is the work done by extraneous forces in moving a charge q, the e.m.f. is

$$\mathcal{E} = A/q.$$

Like voltage, e.m.f. is measured in volts.

4.10. Ohm's Law for a Subcircuit. Resistance of Conductors

If an electric field is created in a uniform subcircuit (i.e. the subcircuit containing no extraneous forces), a current will appear in the subcircuit. The electric field created in it can be characterized by the voltage U applied to the subcircuit or, which is the same, by the potential difference $\varphi_1 - \varphi_2$ across its ends ($U = \varphi_1 - \varphi_2$).

In 1826, **Ohm's law** was experimentally established, according to which the *current I in a uniform subcircuit is directly proportional to the voltage U applied to the subcircuit and inversely proportional to the characteristic of the subcircuit called its electric resistance R*:

$$I = U/R.$$

Figure 213 shows the conventional notation for resistance on diagrams.

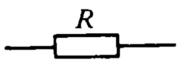


Fig. 213

According to the electron theory, the resistance offered by metallic conductors to electric current is due to the fact that charge carriers, viz. conduction electrons, collide in their motion with ions in the crystal lattice. During these collisions, moving electrons transfer to the ions a part of their energy acquired during their free path in the electric field. The energy transferred to the ions is converted into the energy of random vibration of lattice ions (the amplitude of vibrations increases), i.e. into internal energy. The difference in the resistance of different metals is due to different mean free paths of electrons and the different number of free electrons per unit volume of metal.

The SI unit of resistance is an **ohm** (Ω), equal to the resistance of a conductor in which the current of 1 A flows at a voltage of 1 V. The reciprocal of an ohm is a **siemens** (S).

The resistance of a homogeneous cylindrical conductor is

$$R = \rho l/S,$$

where l is the length of the conductor, S is its cross-sectional area, and ρ is the **resistivity** of the material of which the conductor is made. Numerically, ρ is equal to the resistance of a 1-m long conductor whose cross-sectional area is 1 m^2.

The SI unit of ρ is an **ohm-metre** ($\Omega \cdot m$).

The quantity reciprocal to ρ is called the **electrical conductivity** of the material:

$$\sigma = 1/\rho.$$

Conductivity σ is measured in **siemens per metre** (S/m).

The presence of impurities in a metallic conductor increases its resistivity. For example, impurities in copper may increase the resistivity of a copper wire several times. When a wire of low resistance is required, chemically pure copper is used. Metal alloys usually have a considerably higher resistivity than pure metals constituting them. Alloys are used whenever high resistances are required.

Let us consider a segment of a conductor (Fig. 214). Suppose that there is a potential difference $U = \varphi_1 - \varphi_2$ between its ends.

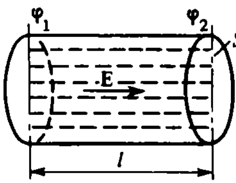


Fig. 214

Then a current $I = jS$ flows in the conductor, where j is the surface density of current. According to Ohm's law, $I = U/R$, i.e. $jS = U/R$.

The field in the conductor is uniform (the dashed lines in the figure). Hence $U = \varphi_1 - \varphi_2$ can be represented as the product El (see formula (4.6.1)). The resistance of the segment is $R = \rho l/S$. Substituting the values of U and R into the expression for Ohm's law, we obtain

$$jS = El/(\rho l/S), \text{ whence } j = E/\rho = \sigma E.$$

This relation is valid at each point of a conductor having an arbitrary shape and for any field configuration. It expresses Ohm's law in the differential form.

4.11. Temperature Dependence of Resistance. Semiconductors

The resistance of conductors varies with temperature. As the temperature increases, the resistance of metallic conductors grows. On the other hand, the resistance of coal, solutions and melts of salts and acids drops with increasing temperature.

The resistance of most metals varies in proportion to the absolute temperature except at very low temperatures:

$$\rho = bT.$$

We represent the proportionality factor in the form $b = \rho_0/T_0$, where ρ_0 is the resistivity at $T_0 = 273$ K (i.e. at 0°C). Then

$$\rho = \rho_0 T/T_0.$$

Let us go over from the absolute temperature to the Celsius scale by replacing T by $T_0 + t$:

$$\rho = \rho_0(T_0 + t)/T_0 = \rho_0(1 + \alpha t),$$

where $\alpha = 1/T_0$ is called the **temperature coefficient of resistance.** It can be easily shown that α can be represented in the form

$$\alpha = (\rho_t - \rho_0)/\rho_0 t, \tag{4.11.1}$$

where ρ_t is the value of ρ at the temperature t. The value $\alpha = 1/T_0$ is approximate. In actual practice, the coefficient α is determined by formula (4.11.1) from the experimental values obtained for ρ_t and ρ_0.

Some metals and alloys abruptly lose their resistance when cooled to 1-10 K. This phenomenon is known as **superconductivity.**

Semiconductors have a different temperature dependence of resistance. The resistance of semiconductors sharply drops upon heating (from 3% to 5% for a temperature increase of 1 K), while the resistance of metals increases by about 0.3% for the corresponding increase in temperature.

Semiconductors include silicon, germanium, selenium, tellurium, boron, arsenic, phosphorus, and some other elements and compounds.

Semiconductor instruments whose operation is based on a strong temperature dependence of resistance are called **thermistors** (temperature-sensitive resistors), or **thermometer resistors.** The instruments which employ the dependence of semiconductor resistance on illumination intensity are called **photoresistors.**

4.12. Series Connection of Conductors

In **series connection**, the end of one conductor is connected to another conductor (Fig. 215).

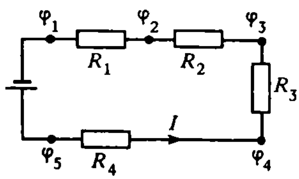


Fig. 215

When n conductors are connected in series, the same current I (I = const) flows through all the conductors in the circuit, while the total voltage is the sum of the voltages across individual segments:

$$U = U_1 + U_2 + U_3 + \dots + U_n.$$

The total resistance of the circuit is

$$R = \frac{U}{I} = \frac{U_1 + U_2 + \dots + U_n}{I}$$
$$= \frac{U_1}{I} + \frac{U_2}{I} + \frac{U_3}{I} \dots + \frac{U_n}{I}$$
$$= R_1 + R_2 + R_3 + \dots + R_n.$$

In series connection, the resistance of the circuit is equal to the sum of the resistances of individual conductors (subcircuits) connected in it. In this case, the potential drop across the conductors is proportional to their resistance:

$$U_1 : U_2 : U_3 : \dots : U_n = R_1 : R_2 : R_3 : \dots : R_n.$$

4.13. Parallel Connection of Conductors

In **parallel connection**, all the conductors are connected as shown in Fig. 216.

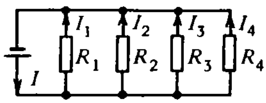


Fig. 216

In parallel connection, the current is divided among the subcircuits. The sum of the currents in all n conductors connected in parallel is equal to the current before and after division:

$$I = I_1 + I_2 + I_3 + \dots + I_n.$$

On the other hand, the voltage across all the conductors is the same and equal to the potential drop across the junction. The total conductance of the circuit is

$$\frac{1}{R} = \frac{1}{U} = \frac{I_1 + I_2 + I_3 + \dots + I_n}{U}$$

$$= \frac{I_1}{U} + \frac{I_2}{U} + \frac{I_3}{U} + \dots + \frac{I_n}{U}$$

$$= \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \dots + \frac{1}{R_n}.$$

The conductance of all parallel-connected conductors is equal to the sum of the conductances of individual conductors.

Consequently, the resistance of a branched circuit is less than the resistance of each of the conductors connected in parallel. In particular, if n identical conductors (having the same resistance) are connected in parallel, the total resistance R is smaller than the resistance R_1 of each conductor by a factor of n:

$$R = R_1/n.$$

In parallel connection of n conductors, the currents in individual parallel-connected conductors are inversely proportional to their resistances:

$$I_1 : I_2 : I_3 : \ldots : I_n = \frac{1}{R_1} : \frac{1}{R_2} : \frac{1}{R_3} : \ldots : \frac{1}{R_n},$$

while the products of the currents in individual parallel-connected conductors and their resistances are the same and equal to the potential drop (potential difference) between the junctions of the conductors:

$$U = I_1R_1 = I_2R_2 = I_3R_3 = \ldots = I_nR_n.$$

4.14. Rheostats

Rheostats are intended for varying the resistance in a circuit. A change in the resistance at constant voltage leads to a corresponding change in the circuit current.

Depending on their duty, rheostats are divided into *rheostatic controllers* and *starting rheostats*. Starting rheostats are intended for a smooth variation of the current while connecting or disconnecting a circuit, in particular, while starting or switching off electric motors. Rheostatic controllers serve to change current in an operating circuit.

Rheostat constructions are of different kinds.

1. A *sliding-contact rheostat* (Fig. 217) is a high-resistivity wire wound on an insulator to form a coil. A metallic rod is fixed

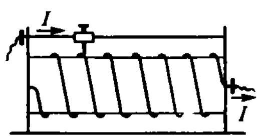


Fig. 217

above the coil as a guide for a slider with a contact. The highest resistance, i.e. the smallest current on the diagram, corresponds to the extreme left position of the slider. To connect the circuit, the slider is moved from left to right, while the slider motion in the opposite direction disconnects the circuit. This type of rheostat is used in laboratories, medical equipment, and so on.

2. A *rotaty-switch rheostat* (Fig. 218) is used for starting or switching off electric motors. The switch handle (see the figure) is turned from left to right (clockwise) during starting and counterclockwise during switching off.

3. A *lamp resistor* (Fig. 219) consists of a certain number of parallel-connected lamps. It is used in laboratories. To connect

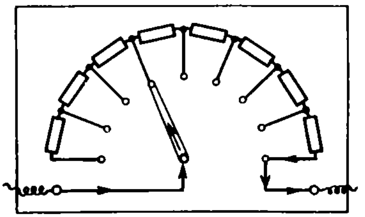


Fig. 218

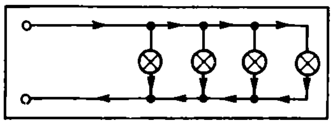


Fig. 219

the circuit, it is sufficient to switch on a lamp. As other lamps are consecutively switched on, the resistance of the circuit gradually decreases, and the current increases accordingly.

4. A *liquid rheostat* (Fig. 220) is a cast-iron vessel filled with a soda solution (or acidified water). A handle with an iron sheet is hinged to it. As the handle is turned down, the depth of submergence of the sheet increases and the resistance of the circuit decreases since the resistivity of metal is lower than the resistivity of electrolyte.

5. A *plus rheostat,* or *resistance box* (Fig. 221), is used in laboratories and consists of a wooden box containing a set of standard coils of different resistances. Copper plates on the cover of the box are separated by gaps into which the plugs connecting the corresponding coils are inserted. In Fig. 221, the resistors R_2 and R_5 are disconnected, while the coils R_1, R_3, and R_4 are connected in series so that the total resistance is $R_1 + R_3 + R_4$.

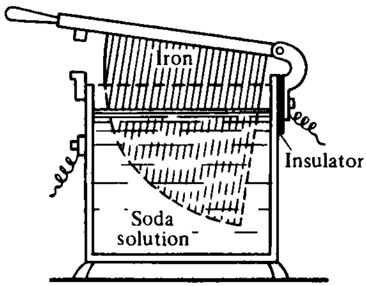


Fig. 220

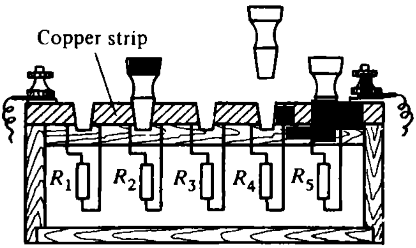


Fig. 221

4.15. Current Sources. Ohm's Law for a Closed Circuit

Direct current can flow only in a closed circuit. An actual electric circuit contains current sources (in which extraneous forces act) and uniform subcircuits formed by one or several resistors. Figure 222 shows by way of an example a circuit consisting of a source of e.m.f. $\mathcal{E}$ and three resistors R_1, R_2, and R_3, which form the *external resistance* of the circuit. Any current source has a resistance r which is called the *internal resistance.*

The current in a closed circuit is determined by the formula

$$I = \mathcal{E}/(R + r), \tag{4.15.1}$$

where $\mathcal{E}$ is the e.m.f. acting in the circuit, R is the total external resistance of the circuit, and r is the internal resistance of the cur-

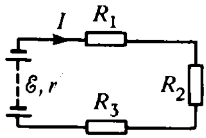


Fig. 222

rent source. The sum $R + r$ forms the **total resistance of the circuit.**

Formula (4.15.1) expresses **Ohm's law for a closed circuit**: *the current in a closed circuit is equal to the e.m.f. divided by the total resistance of the circuit.*

The product of the current and the resistance of a subcircuit is equal to the potential drop across this subcircuit. Consequently, IR is equal to the potential drop U across the external resistance of the circuit, while Ir is the potential drop u in the current source. According to formula (4.15.1), we have

$$\mathscr{E} = IR + Ir, \text{ i.e. } \mathscr{E} = U + u. \qquad (4.15.2)$$

If the external resistance R is very high, the current I is extremely small. In this case, the term u in formula (4.15.2) can be neglected in comparison with U, and we arrive at the relation

$$\mathscr{E} \approx U,$$

which is the more accurate, the higher R. In the limit as $R \to \infty$, the approximate equality becomes exact: $\mathscr{E} = U$. An infinite resistance corresponds to a broken circuit. Therefore it can be said that the e.m.f. is equal to the voltage across the terminals of the current source for a broken circuit.

4.16. Parallel and Series Connection of Current Sources

Figure 223 shows the notation for different current sources in circuit diagrams. Let us consider the parallel and series connection of current sources.

In the parallel connection of cells (Fig. 224*a*), the positive terminals of the cells form one junction and the negative terminals

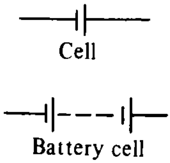


Fig. 223

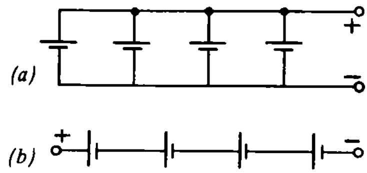


Fig. 224

form the other junction. If the cells forming a battery have the same e.m.f., the total e.m.f. of the battery is equal to the e.m.f. of a single cell. The total internal conductance is equal to the sum of the conductances of individual cells. If n individual cells have the same conductance, the total internal conductance is n times higher. Consequently, the internal resistance of the battery of n cells is lower than the internal resistance of each cell by a factor of n:

$$R_{\text{int}} = r/n,$$

where r is the internal resistance of a cell. Then the current is

$$I_{\text{par}} = \mathscr{E}/(R + r/n),$$

where R is the external resistance of the circuit.

In the series connection of cells (Fig. 224b), the positive terminal of each cell is connected to the negative terminal of the next cell. The total e.m.f. of the battery in this case is equal to the sum of the e.m.f.s of individual cells. The total internal resistance is equal to the sum of the internal resistances of individual cells:

$$R_{\text{int}} = nr.$$

The current is then

$$I_{\text{ser}} = n\mathscr{E}/(R + nr).$$

In the mixed (series-parallel) connection of k parallel groups consisting of m series-connected elements each (Fig. 225), the total e.m.f. of the battery is equal to the e.m.f. of a parallel group, i.e. $m\mathscr{E}$, where $\mathscr{E}$ is the e.m.f. of a cell. The internal resistance of each parallel group is mr, while the total internal resistance of k such groups is mr/k. The current is given by

$$I_{\text{mix}} = m\mathscr{E}/(R + mr/k).$$

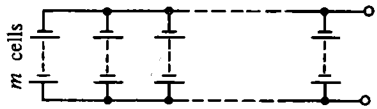


Fig. 225

4.17. Direct Current Power. Joule's Law

When a charge q is transferred along a conductor with the potential difference $U = \varphi_1 - \varphi_2$ across the ends, the electric field force does a work $A = Uq$ on the charge. If the current is I, the charge transferred during time t from one end of the conductor to the other is $q = It$. Consequently, the forces of the field during time t do the work

$$A = Uq = UIt.$$

According to Ohm's law, $I = U/R$, or $U = IR$. Hence the expression for the work can also be written in the form

$$A = I^2Rt, \text{ or } A = (U^2/R)t. \qquad (4.17.1)$$

Dividing the work A by the time t during which it was done, we obtain the power N developed by the current over this subcircuit:

$$N = UI = I^2R = U^2/R.$$

In these formulas, R is the external resistance, and hence N is the useful power.

The total (supplied) power is given by

$$N_{sup} = I^2(R + r).$$

Therefore, the efficiency of an electric generator whose circuit contains only resistors is

$$\eta = N/N_{sup} = I^2R/I^2(R + r) = R/(R + r).$$

The unit of work is

$$1\text{ J} = 1\text{ V} \times 1\text{ A} \times 1\text{ s} = 1\text{ V} \times 1\text{ C} = 1\text{ W} \times 1\text{ s}.$$

In engineering, out-of-system units are often employed:

$$1\ \text{Wh} = 3600\ \text{J},\ 1\ \text{kWh} = 3600\ \text{kJ}.$$

The units of power are

$$1\ \text{W} = 1\ \text{V} \times 1\ \text{A} = 1\ \text{J/s},\ 1\ \text{kW} = 1\ \text{kJ/s}.$$

The power developed by a current can be supplied to rotate electric motors (in this case electric energy is converted into mechanical energy), to maintain chemical reactions, and finally to heat wires in special heating appliances. In lighting equipment, the current either heats the elements to a very high temperature, as a result of which they become sources of light waves (incandescent lamps), or excites gas atoms, which emit light while returning to the ground state (glow-discharge lamps).

In subcircuits having no moving parts and in which no chemical reaction occurs, the entire power developed by the current is spent for heating the conductors of the subcircuit. In other words, it is liberated in the form of heat. In this case, the amount of liberated heat is equal to the work done on charge carriers by the electric field forces: $Q = A$. Substituting expression (4.17.1) for A, we obtain the following expression for the amount of heat liberated in the subcircuit over time t:

$$Q = I^2Rt$$

(I is the current in the circuit and R is the resistance of the subcircuit). This formula expresses **Joule's law** which was established experimentally at the end of the last century: *the heat produced by an electric current flowing through a conductor is proportional to the square of the current, the conductor resistance, and the time during which the current flows.*

The amount of heat is given by

$$Q = I^2Rt = U^2t/R = UIt.$$

Problems with Solutions

180. Find the average velocity v of the directed motion (drift) of electrons in a metallic conductor at a current $I = 12$ A if a unit volume of the conductor contains $n = 5 \times 10^{21}\ \text{cm}^{-3}$ free electrons. The cross-sectional area of the conductor is $S = 0.5\ \text{cm}^2$. The electron charge $e = 1.6 \times 10^{-19}$ C.

Solution. The current $I = neSv$, whence $v = I/neS = 0.03$ A·cm/C. Since 1 C = 1 A·s, we have $v = 0.03$ cm/s.

181. The resistance of a copper wire is twice that of an aluminium wire, while its mass is one fourth of the mass of the aluminium wire. Find the ratio of their lengths. The densities of copper and aluminium are $\gamma_c = 8.9 \times 10^3$ kg/m^3 and $\gamma_{al} = 2.7 \times 10^3$ kg/m^3, the resistivities of these materials being $\rho_c = 0.01$ $\mu\Omega\cdot$m and $\rho_{al} = 0.028$ $\mu\Omega\cdot$m.

Solution. We have two equations: $R_c = 2R_{al}$ and $m_c = m_{al}/4$, whence

$$\rho_c l_c/S_c = 2\rho_{al}l_{al}/S_{al}, \; \gamma_c l_c S_c = \gamma_{al}l_{al}S_{al}/4.$$

Multiplying the right- and left-hand sides of these equations, we obtain

$$\rho_c l_c^2 \gamma_c = \rho_{al} l_{al}^2 \gamma_{al}/2, \text{ or } l_{al}/l_c = \sqrt{2\rho_c\gamma_c/\rho_{al}\gamma_{al}} = 2,$$

i.e. the length of the aluminium wire is twice the length of the copper wire.

182. An arc lamp is rated for a voltage $U_l = 50$ V at a current $I_l = 3.5$ A. It must be connected to a circuit under a voltage $U = 120$ V with the help of a series resistor made of nickeline wire having a cross-sectional area $S = 0.1$ mm^2. Find the length of the wire if the resistivity of nickeline is $\rho = 0.40$ $\mu\Omega\cdot$m.

Solution. The current in the subcircuit *ab* (Fig. 226) is the same and equal to I_l. The potential drop across *ac* is $U - U_l$. The resistance R of the nickeline wire must be equal to $(U - U_l)/I_l$. In the case under consideration, it is convenient to calculate the numerical value of the resistance: $R = 20\,\Omega$. The formula $R = \rho l/S$ gives $l = RS/\rho = 5.0$ m.

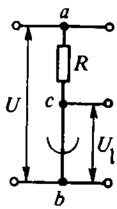


Fig. 226

183. A user is at $l = 20$ km from a d.c. source. They are connected by a two-conductor transmission line having a resistance $R = 400\,\Omega$. The line is short-circuited and the voltmeter at the source indicates a voltage $U' = 10$ V, while the milliammeter indicates $I' = 40$ mA. At what distance from the current source does the short circuit occur?

Solution. The external resistance of the closed subcircuit after the short circuit is $R' = U'/I' = 250\,\Omega$. Since $l'/l = R'/R$, we have $l' = lR'/R$, whence $l = 12.5$ km.

184. The tungsten filament of an incandescent lamp has a length $l = 20$ cm and a resistance $R_t = 200\,\Omega$ at a temperature $t = 2500$ °C. Find the diameter of the filament if the resistivity of tungsten $\rho_0 = 0.056$ $\mu\Omega\cdot$m and the temperature resistance coefficient $\alpha = 4.2 \times 10^{-3}$ K^{-1}.

Solution. Using the formula $R_t = R_0(1 + \alpha t)$, we find the resistance of the filament at 0 °C: $R_0 = R_t/(1 + \alpha t) = 17.4\ \Omega$. Since

$$R_0 = \rho_0 l/S = \rho_0 \cdot 4l/\pi d^2, \text{ we get } d = \sqrt{4l\rho_0/\pi R_0} = 28.6\ \mu\text{m}.$$

185. Find the resistance of the circuit shown in Fig. 227, if $r_1 = 2\ \Omega$, $r_2 = 4\ \Omega$, $r_3 = 1.5\ \Omega$, $r_4 = 2\ \Omega$, $r_5 = 3\ \Omega$, $r_6 = 4\ \Omega$, $r_7 = 1\ \Omega$, $r_8 = r_9 = r_{10} = 7.5\ \Omega$, and $r_{11} = 1.5\ \Omega$.

Solution. We replace the resistances r_1 and r_2 by the equivalent resistance R_1:

$$\frac{1}{R_1} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{r_2 + r_1}{r_1 r_2}, \quad R_1 = 1.3\ \Omega.$$

The resistances from r_4 to r_7 are replaced by the equivalent resistance R_2:

$$\frac{1}{R_2} = \frac{1}{r_4} + \frac{1}{r_5} + \frac{1}{r_6} + \frac{1}{r_7}, \quad R_2 = 0.48\ \Omega.$$

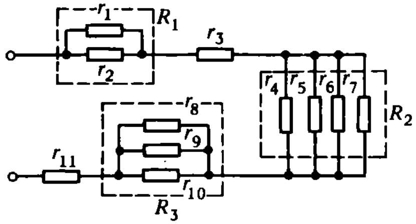


Fig. 227

The resistances from r_8 to r_{10} are replaced by the equivalent resistance R_3. Since $r_8 = r_9 = r_{10}$, we have $R_3 = r_8/3 = 2.5\ \Omega$. The resistances R_1, r_3, R_2, R_3, and r_{11} are connected in series. The total resistance is

$$R = R_1 + r_3 + R_2 + R_3 + r_{11} = 7.31\ \Omega.$$

186. Find the resistance of the circuit shown in Fig. 228 if $r_1 = r_2 = 1.8\ \Omega$, $r_3 = 1.0\ \Omega$, $r_4 = 3\ \Omega$, $r_5 = 1.0\ \Omega$, $r_6 = 1.2\ \Omega$, $r_7 = 0.8\ \Omega$, $r_8 = 1.2\ \Omega$, $r_9 = r_{10} = 3\ \Omega$, $r_{11} = r_{12} = 0.3\ \Omega$, and $r_{13} = 0.7\ \Omega$.

Solution. It is more convenient to solve this problem numerically since the general formula would be cumbersome. We replace the resistances r_9 and r_{10} by the equivalent resistance $r_{9,\ 10} = r_9/2 = 1.5\ \Omega$. Then we note that the series-connected resistances $r_{9,\ 10}$, r_8, and r_{11} and the series-connected resistances r_6 and r_7 form two parallel-connected groups of resistances. They can be replaced by the equivalent resistance R_1 connected between points c and d of the circuit:

$$\frac{1}{R_1} = \frac{1}{r_{9,\,10} + r_8 + r_{11}} + \frac{1}{r_6 + r_7} = \frac{5}{6}, \quad R_1 = 1.2\ \Omega.$$

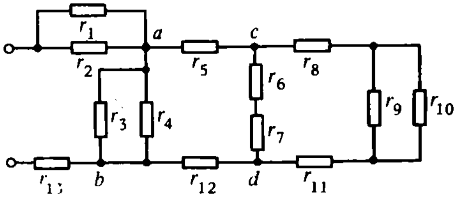


Fig. 228

Further, we replace the series-connected resistances R_1, r_5, and r_{12} by the equivalent resistance R_2:

$$R_2 = R_1 + r_5 + r_{12} = 2.5\,\Omega.$$

The parallel-connected resistances r_3 and r_4 are replaced by the equivalent resistance R_3:

$$\frac{1}{R_3} = \frac{1}{r_3} + \frac{1}{r_4} = \frac{4}{3}, \quad R_3 = 0.75\,\Omega.$$

The parallel-connected resistances R_2 and R_3 can be replaced by the equivalent resistance R_4 connected between points a and b:

$$\frac{1}{R_4} = \frac{1}{R_2} + \frac{1}{R_3} = \frac{26}{15}, \quad R_4 = 0.58\,\Omega.$$

Finally, the total resistance of the circuit is

$$R = R_4 + r_{13} + r_1/2 = 2.18\,\Omega.$$

187. A ring is made of a wire having a resistance $R_0 = 10\,\Omega$. Find the points at which current-carrying conductors should be connected so that the resistance R of the subcircuit between these points is equal to $1\,\Omega$.

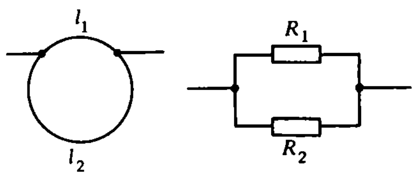


Fig. 229

Solution. After the wires have been connected (Fig. 229), we have two parallel-connected resistances whose sum is equal to the resistance of the ring. Thus we have two equations:

$$R_1 + R_2 = R_0,$$
$$1/R_1 + 1/R_2 = 1/R.$$

Solving them together, we find $R_1 = 5 - \sqrt{15}\ \Omega$ and $R_2 = 5 + \sqrt{15}\ \Omega$. Since the resistance of a conductor is proportional to its length, $l_1/(l_1 + l_2) = R_1/R_0 = 0.113$ and $l_1 = 0.113(l_1 + l_2)$.

188. What is the number of equal parts into which a conductor having a resistance $R_0 = 100\ \Omega$ should be cut to obtain the resistance $R = 1\ \Omega$ if the parts are connected in parallel?

Solution. If the conductor is cut into n equal parts, the resistance r of each part is R_0/n. The total resistance of n parallel-connected conductors having a resistance r each is $R = r/n = R_0/n^2$, whence $n = \sqrt{R_0/R} = 10$. Thus, the conductor should be cut into 10 parts.

189. A voltmeter connected in series with a resistance R_1 to a circuit indicates a voltage $U_1 = 198$ V. When a series resistor $R_2 = 2R_1$ is used, the voltmeter indicates a voltage $U_2 = 180$ V. Find the voltage in the circuit and the series resistance R_1 if the resistance of the voltmeter is $R_v = 900\ \Omega$.

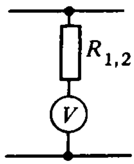


Fig. 230

Solution. Assuming that the potential drop in the subcircuit is proportional to its resistance, we write the following equations for the two cases (Fig. 230):

$$U_1/U = R_v/(R_v + R_1),\ U_2/U = R_v/(R_v + R_2).$$

Dividing the first equation of this system by the second equation, we get

$$U_1/U_2 = (R_v + 2R_1)/(R_v + R_1),$$
$$R_1 = R_v(U_1 - U_2)/(2U_2 - U_1) = 100\ \Omega.$$

Using this result, we obtain from the first equation of the system

$$U = U_1(R_v + R_1)/R_v = U_1(1 + R_1/R_v) = 220\text{ V}.$$

190. Find the resistance of the circuit depicted in Fig. 231*a*.

Solution. The figure shows that the potential difference between points *a, b,* and *c* is zero since the potential drops across subcircuits *Aea, Afc, Aeb,* and *Afb* are equal. Consequently, there is no current along *abc* through resistances $3r$ and $4r$. These resistances can be excluded, and the total resistance can be calculated by using either of the circuits shown in Figs. 231*b* and *c*. In the circuit shown in Fig. 231*b*, points *a, b,* and *c* are disconnected and the resistances between them are excluded. In Fig. 231*c*, points *a, b,* and *c* are connected into a junction.

In circuit (*b*), the resistance between points *e* and *k* and between *f* and *h* is $R_{ef} = (r + 2r)/2 = 1.5r$. The total resistance of the circuit is $R = (r + 1.5r + 2r)/2 = 2.25r$.

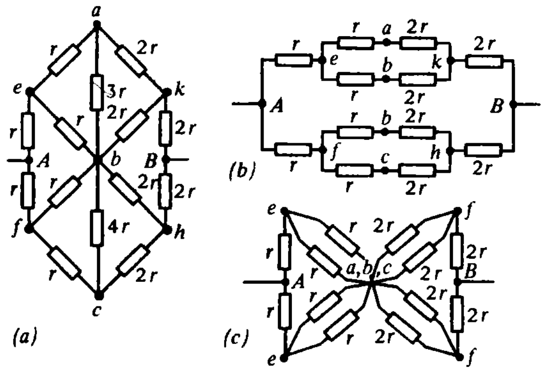


Fig. 231

In circuit (c), the resistance between point e and junction abc is $R_1 = r/2$, while the resistance between junction abc and point f is $R_2 = 2r/2 = r$.

The resistance between point A and junction abc is $R_3 = 0.5(r + r/2) = 0.75r$, while the resistance between junction abc and point B is $R_4 = 0.5(r + 2r) = 1.5r$.

The total resistance of the circuit is

$$R = R_3 + R_4 = 0.75r + 1.5r = 2.25r.$$

191. Find the resistance of a cubic frame formed by pieces of wire having the same resistance.

Solution. Current I branches at points A and B into three parts (Fig. 232a).

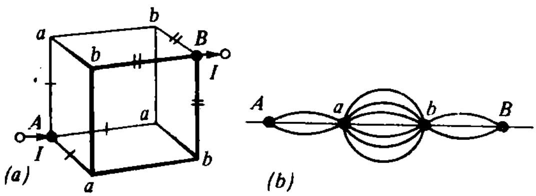


Fig. 232

Since the conditions for each branch Aa and Bb are the same, the current in them is $I/3$, and the voltage drop across them is the same: $V = IR/3$. Consequently, all points a and all points b have equal potentials and can be connected to form two junctions a and b, as is shown in Fig. 232b. It turns out that six identical conductors ab are connected between these junctions. The total resistance of the circuit is

$$R_{tot} = R/3 + R/6 + R/3 = 5R/6.$$

192. Find the resistance of the circuit shown in Fig. 233a.

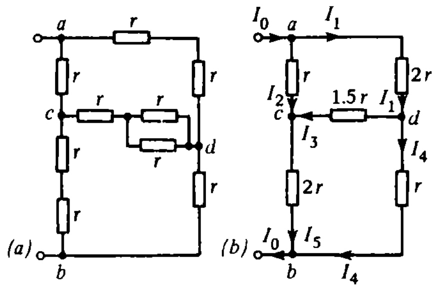


Fig. 233

Solution. The circuit diagram can be simplified by adding the resistances between points a and d, c and b, and c and d. The simplified diagram is shown in Fig. 233*b*. It contains neither series nor parallel connections in pure form. It does not contain points having equal potentials either, since it has no symmetry. For this reason, a more general method of calculation is applicable here.

We assume that current I_0 comes to point a where it branches into currents I_1 and I_2. Consequently,

$$I_0 = I_1 + I_2.$$

Similarly, at points d, b and c we have

$$I_1 = I_3 + I_4, \; I_0 = I_4 + I_5,$$
$$I_5 = I_2 + I_3.$$

Thus, we have four equations in six unknowns.

The second group of equations reflects the fact that the work done by electric field forces does not depend on the path along which a charge is displaced. We denote by R the total resistance of the circuit. Then we obtain the following equations for subcircuits *acb*, *adb*, and *adcb*:

$$I_0R = I_2 \cdot r + I_5 \cdot 2r,$$
$$I_0R = I_1 \cdot 2r + I_4 \cdot r, \; I_0R = I_1 \cdot 2r + I_3 \cdot 1.5r + I_5 \cdot 2r.$$

Thus, we now have seven equations in seven unknowns. For comparison, one more equation can be written for subcircuit *acdb*:

$$I_0R = I_2 \cdot r - I_3 \cdot 1.5r + I_4 \cdot r.$$

Solving this system of equations, we obtain $R = 1.4r$.

193. Find the internal resistance and voltage across the terminals of a cell having an e.m.f. $\mathscr{E} = 2.1$ V and located at a distance $l = 20$ cm from the load if the resistance of the load is $R = 2\ \Omega$ and the current in the circuit is $I = 0.7$ A. Copper connecting wires have a diameter $d = 1.2$ mm, the resistivity of copper is $\rho = 0.017\ \mu\Omega \cdot \text{m}$.

Solution. The resistance of connecting wires of length $2l$ is

$$R_0 = \rho \frac{2l}{\pi d^2/4} = \rho \frac{8l}{\pi d^2} = 0.6\ \Omega.$$

The current in the circuit is $I = \mathscr{E}/(R + R_0 + r)$, whence we obtain the internal resistance of the cell: $r = \mathscr{E}/I - (R + R_0) = 0.4\ \Omega$. The voltage across its terminals is $U = (R + R_0)I = 1.82$ V.

194. The current in a circuit with an external resistance R_1 is I_1. When the external resistance is R_2, the current is I_2. Find the e.m.f. and the internal resistance of the current source.

Solution. We write the system of two equations:

$$I_1 = \mathscr{E}/(R_1 + r),\ I_2 = \mathscr{E}/(R_2 + r).$$

Dividing the first equation by the second equation, we obtain

$$I_1/I_2 = (R_2 + r)/(R_1 + r).$$

Hence we can find the internal resistance of the power source:

$$r = (I_2R_2 - I_1R_1)/(I_1 - I_2).$$

Substituting this value of r, say, into the first equation, we obtain the e.m.f. of the power source:

$$\mathscr{E} = I_1(R_1 + r) = I_1I_2(R_2 - R_1)/(I_1 - I_2).$$

195. A battery consists of $k = 3$ parallel groups each of which contains $m = 20$ series-connected cells. The e.m.f. of a cell is $\mathscr{E}' = 1.5$ V and the internal resistance $r' = 0.3\ \Omega$. The battery is connected through an external resistance $R = 2.5\ \Omega$. Find the current in the external circuit and in each cell, as well as the voltage across the battery terminals. Prove that with such an arrangement, the current in the external circuit is maximum. What must be the arrangement of the cells in the battery to obtain the maximum current if the external resistance is doubled?

Solution. To obtain the maximum current in the external circuit, the cells in the battery must be grouped in such a way that the internal resistance of the battery be equal to the resistance of the external circuit. Let us prove this. The e.m.f. of the battery is $\mathscr{E} = m\mathscr{E}'$. The total internal resistance is $r = mr'/k$. The current in the external circuit is

$$I = km\mathscr{E}'/(kR + mr'). \qquad (1)$$

It is well known from algebra that the sum of non-negative addends whose product is constant is minimum when the addends are equal. The product $kmr'R$ of the two addends in the denominator of expression (1) is constant. Consequently, the denominator $kR + mr'$ has the minimum value when $kR = mr'$, or $R = mr'/k$, i.e. when the external resistance of the circuit is equal to the total internal resistance of the battery. In this case the maximum current is:

$$I_{max} = m\mathscr{E}'/2R.$$

If we could group the cells in such a way as to make their internal resistance exactly equal to the external resistance of the circuit, the current in the external circuit would be $I = 6$ A. In actual practice, it is not always possible to obtain such a grouping. In the case under consideration, the internal resistance of the battery is $r = mr'/k = 2\ \Omega$ and the current $I \simeq 6.7$ A. Since $r < R$, we must check the current for an arrangement corresponding to a somewhat higher internal resistance. The nearest possible arrangement is $k = 2$, $m = 30$. The internal resistance of such a battery is $r = mr'/k = 4.5\ \Omega > R$. For such an arrangement, the current is

$$I = m\mathscr{E}'/(R + r) \simeq 6.4 \text{ A} < I.$$

Consequently, the current in the external circuit has the maximum value for the arrangement specified in the conditions of the problem: $I_{max} \simeq 6.7$ A. This current branches into three equal parts flowing through parallel-connected groups of resistances: $I_{1,2,3} = I/3 \simeq 2.22$ A. The voltage across the battery terminals is $U = IR \simeq 16.7$ V.

When $R = 5\ \Omega$, the most advantageous arrangement is the one involving two parallel-connected groups of 30 cells since in this case $r = 4.5\ \Omega \simeq R$. Consequently, $m = 30$, $k = 2$.

196. An accumulator is connected for charging a battery whose e.m.f. $\mathscr{E}_b = 12.5$ V (Fig. 234). The internal resistance of the accumulator is $r_a = 1\ \Omega$. The charging current $I = 0.5$ A. Find the e.m.f. of the accumulator.

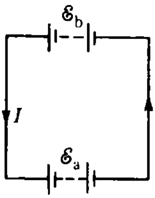


Fig. 234

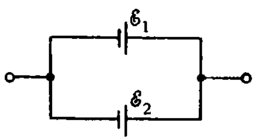


Fig. 235

Solution. The current in the circuit is determined by the joint action of e.m.f.s $\mathscr{E}_b$ and $\mathscr{E}_a$ of the battery and accumulator. The direction of the current is opposite to the action of $\mathscr{E}_a$. Ohm's law for this circuit is $\mathscr{E}_b - \mathscr{E}_a = I(r_b + r_a)$. Since $\mathscr{E}_b - Ir_b = U_b$, we have $U_b - \mathscr{E}_a = Ir_a$, whence $\mathscr{E}_a = U_b - Ir_a = 12$ V.

197. Two cells are connected in parallel (Fig. 235). The e.m.f.s and internal resistances of the cells are $\mathscr{E}_1 = 2.0$ V, $\mathscr{E}_2 = 1.5$ V and $r_1 = 0.3\ \Omega$, $r_2 = 0.2\ \Omega$. Find the voltage across the battery terminals.

Solution. The direction of the resultant current is determined by the direction of the higher e.m.f. ($\mathscr{E}_1$), and the magnitude of the current, by the difference $\mathscr{E}_1 - \mathscr{E}_2$ and the total resistance $r_1 + r_2$ of the closed circuit. We write Ohm's law for the closed circuit: $I = (\mathscr{E}_1 - \mathscr{E}_2)/(r_1 + r_2)$. The voltage across the terminals of the cells is given by

$$U = \mathscr{E}_1 - Ir_1 = (\mathscr{E}_1 r_2 + \mathscr{E}_2 r_1)/(r_1 + r_2) = 1.7 \text{ V}.$$

198. Two cells having e.m.f.s $\mathscr{E}_1 = 2$ V and $\mathscr{E}_2 = 1.5$ V and internal resistances $r_1 = 0.3\ \Omega$ and $r_2 = 0.2\ \Omega$ are connected in parallel to a circuit with an external resistance $R = 2\ \Omega$. Find the current in the external circuit and in each cell, as well as the voltage across the battery terminals.

Solution. The directions of current are shown in Fig. 236 by arrows. It can be seen that $I_1 = I + I_2$. The voltage across the battery terminals is $U = IR$, or $U = \mathscr{E}_1 - I_1 r_1$. Further, considering the branch containing the cell $\mathscr{E}_2$ as a load for the current source with voltage U across the terminals, we obtain

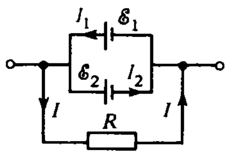


Fig. 236

$U - \mathscr{E}_2 = I_2 r_2$. As a result, we obtain a system of four equations in four unknowns: I, I_1, I_2, and U. Solving this system, we get

$$I = \frac{\mathscr{E}_1 r_2 + \mathscr{E}_2 r_1}{R(r_1 + r_2) + r_1 r_2} = 0.80\ \text{A},$$

$$I_1 = \frac{\mathscr{E}_1 - IR}{r_1} = 1.33\ \text{A},$$

$$I_2 = I_1 - I = 0.53\ \text{A},\ U = IR = 1.6\ \text{V}.$$

199. In the circuit shown in Fig. 237, each cell of a battery has an e.m.f. $\mathscr{E} = 10$ V.

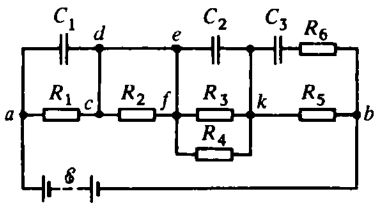


Fig. 237

All external resistances are identical: $R = 5\ \Omega$. The capacitances of the capacitors are also equal: $C = 900$ cm each. Find the charges on the plates of the capacitors if the current through the battery increases $n = 9$ times during a short circuit.

Solution. It can be seen from the diagram that there is no current through the resistance R_2 since it is shunted by a very small resistance of wire *de*. There is no current through the resistance R_6 either since direct current cannot pass through a capacitor. In subcircuit *cf*, a current flows through *cdef* whose resistance is negligibly small. Therefore, the resistances R_2 and R_6 can be disregarded. Thus, the current flows through *acdefkb*. The equivalent external resistance is

$$R_{ext} = R_1 + R_3/2 + R_5 = 2.5\,R.$$

The e.m.f. of the battery is

$$\mathscr{E} = I(r + 2.5R). \tag{1}$$

During the short circuit,

$$\mathscr{E} = I_0 r, \tag{2}$$

where r is the internal resistance of the battery. We equate the right-hand sides of Eqs. (1) and (2) to obtain

$$I(r + 2.5R) = I_0 r, \; r = 2.5R/(n - 1).$$

The current in the circuit is given by

$$I = \mathscr{E}/(r + 2.5R) = \mathscr{E}(n - 1)/2.5Rn.$$

The potential drops across *ac*, *kb*, and *fk* are

$$U_1 = U_3 = U_{1,3} = IR = \mathscr{E}(n - 1)/2.5n, \; U_2 = IR/2 = \mathscr{E}(n - 1)/5n.$$

The charges on the plates of capacitors C_1, C_2, and C_3 are

$$q_1 = q_3 = U_{1,3}C = \mathscr{E}C(n - 1)/2.5n = 3.56\text{ nC},$$
$$q_2 = U_2C = \mathscr{E}C(n - 1)/5n = 1.78\text{ nC}.$$

200. A lift of mass $M = 0.8$ t is raised to a height $h = 40$ m during a time $t = 0.5$ min. The voltage across the motor terminals is $U = 120$ V and the motor efficiency $\eta = 90\%$. Find the power N consumed by the motor, the current I in it, and the price of one hoisting if 1 gW·h costs 1.2 kopecks.

Solution. When the lift is raised to a height h, its potential energy increases by $W_p = mgh$. The energy consumed by the motor during one hoisting is $W_1 = W_p/\eta = mgh/\eta$. The power consumption of the motor is $N = W_1/t = mgh/\eta t = 11.6$ kW. The current is given by the formula $I = N/U = 90.7$ A. The energy consumed for one hoisting is $W_1 = Nt = 0.97$ gW·h. The hoisting costs $k = 1.2 \times 0.97 = 1.16$ kop.

201. Find the e.m.f. and the internal resistance of an accumulator if its output power is $N_1 = 9.5$ W for the current $I_1 = 5$ A and $N_2 = 12.6$ W for the current $I_2 = 7$ A.

Solution. The voltage across the accumulator terminals is $U = \mathscr{E} - Ir$. On the other hand, $U = N/I$. Equating the left-hand sides of these expressions, we get

$$\mathscr{E} - Ir = N/I.$$

According to the conditions of the problem, we can write two equations:

$$\mathscr{E} - I_1 r = N_1/I_1, \; \mathscr{E} - I_2 r = N_2/I_2.$$

Solving them together, we obtain

$$r = \frac{N_1/I_1 - N_2/I_2}{I_2 - I_1} = 0.05\ \Omega,$$

$$\mathcal{E} = I_1 r + N_1/I_1 = 2.15\ \text{V}.$$

202. An electric iron rated for the voltage $U_0 = 120$ V has a power $N = 400$ W. When it is switched on, the voltage across the socket drops from $U_1 = 127$ V to $U_2 = 115$ V. Find the resistance of the leads.

Solution. The (rated) resistance of the iron is $R = U_0^2/N$. For the leads, we can write $U_1 - U_2 = IR_1$. The current in the circuit is $I = U_1/(R + R_1)$. Consequently, $U_1 - U_2 = U_1R_1/(R + R_1)$. Hence we can find the resistance R_1 of the leads:

$$R_1 = \frac{U_1 - U_2}{U_2} R = \frac{U_1 - U_2}{U_2} \frac{U_0^2}{U} = 3.76\ \Omega.$$

203. Electric energy has to be supplied over a distance $l = 10$ km with the help of copper wires so that the energy loss must be less than 10%. The power of the electric power plant is $N = 100$ kW and the voltage is $U_1 = 220$ V. Find the mass of copper wires. What amount of copper is required at the voltage $U_2 = 440$ V? The resistivity of copper is $\rho = 0.017\ \mu\Omega\cdot\text{m}$ and its density $\gamma = 8.9 \times 10^3\ \text{kg/m}^3$.

Solution. Since the current in the wires is the same over their length, we can write $\Delta W/W = \Delta U \cdot I/UI = \Delta U/U$, whence we can find the voltage drop in the wires: $\Delta U = \Delta W/WU = 0.10U$. Since $\Delta U = IR$, while $I = N/U$, we obtain $0.10U = NR/U$. Hence we can find the resistance of the wires:

$$R = 0.10\,U^2/N = 0.05\ \Omega. \tag{1}$$

On the other hand, the resistance of the wires is $R = \rho \cdot 2l/S$, whence the cross-sectional area $S = 2l\rho/R$, and the volume of copper is $V = 2ls = 4l^2\rho/R$. The mass of copper is

$$m_1 = V\gamma = 4l^2\rho\gamma/R = 1250\,\text{t}. \tag{2}$$

It follows from this expression that the mass of copper is inversely proportional to the wire resistance, while expression (1) shows that the resistance of the wires is proportional to the squared voltage of the power plant. Consequently, the mass of copper is inversely proportional to the squared voltage: $m_2/m_1 = (U_1/U_2)^2 = 1/4$, whence $m_2 = m_1/4 = 312.5$ t.

204. What must be the length l of a Ni-Cr conductor having a cross-sectional area $S = 0.1\ \text{mm}^2$ and used as a heater which can heat the volume $V = 1.5$ l of water from a temperature $t_1 = 20$ °C to the boiling point during $\tau = 5$ min? The mains voltage $U = 220$ V, the efficiency of the heater is $\eta = 90\%$, and the resistivity of Ni-Cr alloy is $\rho = 1.10\ \mu\Omega\cdot\text{m}$.

Solution. The mass of water is $m = 1.5$ kg. The amount of heat that must be supplied for heating water is $Q = cm(t_2 - t_1)$, where c is the specific heat for water. The energy consumed for heating motor is $W = Q/\eta$. The power of the heater is

$$N = W/\tau = Q/\eta\tau. \tag{1}$$

On the other hand, the power developed by the heater at a given mains voltage is

$$N = U^2/R. \tag{2}$$

Equating the right-hand sides of Eqs. (1) and (2), we obtain $Q/\eta\tau = U^2/R$. Considering that $Q = cm(t_2 - t_1)$, we get

$$R = U^2\tau\eta/Q = U^2\tau\eta/cm(t_2 - t_1) = 26\,\Omega.$$

Since $R = \rho l/S$, the required length of the Ni-Cr conductor is

$$l = RS/\rho = 2.4\text{ m}.$$

Remark. Actually, l must be somewhat smaller since at the temperature of heating, the resistivity of Ni-Cr is greater than ρ_0.

205. A lead fuse has to be connected to the main circuit formed by copper wires whose cross-sectional area $S_1 = 5\text{ mm}^2$ at a temperature $t = 25\ ^\circ\text{C}$. The fuse must melt when the temperature of the wires increases by $\Delta t = 20\ ^\circ\text{C}$. Find the cross-sectional area of the fuse if the specific heats of copper and lead are $c_1 = 0.36\text{ kJ/(kg}\cdot\text{K)}$ and $c_2 = 0.12\text{ kJ/(kg}\cdot\text{K)}$, the melting point of lead is $t_m = 327\ ^\circ\text{C}$, the latent heat of fusion for lead is $\lambda = 2.4 \times 10^4$ J/kg, the resistivities of copper and lead are $\rho_1 = 0.017\ \mu\Omega\cdot\text{m}$ and $\rho_2 = 0.21\ \mu\Omega\cdot\text{m}$, and the densities of copper and lead are $\gamma_1 = 8.9 \times 10^3\text{ kg/m}^3$ and $\gamma_2 = 11.4 \times 10^3\text{ kg/m}^3$ respectively.

Solution. In accordance with Joule's law, the amount of heat liberated in the copper wire due to the passage of current through it is

$$Q_1 = I^2R_1\tau = I^2\rho_1 l_1\tau/S_1, \tag{1}$$

where τ is the time during which the current flows in the conductor. The amount of heat required for heating the copper wire by Δt is

$$Q_1' = c_1 l_1 S_1 \gamma_1 \Delta t. \tag{2}$$

Equating the right-hand sides of expressions (1) and (2) we obtain

$$I^2\rho_1 l_1\tau/S_1 = c_1 l_1 S_1\gamma_1\Delta t, \text{ or } I^2\tau = c_1 S_1^2\gamma_1\Delta t/\rho_1. \tag{3}$$

The amount of heat required for heating the fuse to the melting point and for its further melting is given by

$$Q_2' = c_2 l_2 S_2\gamma_2(t_m - t) + \lambda l_2 S_2\gamma_2 = l_2 S_2\gamma_2[c_2(t_m - t) + \lambda]. \tag{4}$$

The amount of Joule heat liberated in the fuse is

$$Q_2 = I^2R_2\tau = I^2\rho_2 l_2\tau/S_2. \tag{5}$$

Equating the right-hand sides of expressions (4) and (5), we obtain

$$I^2\rho_2 l_2\tau/S_2 = l_2 S_2\gamma_2[c_2(t_m - t) + \lambda],$$

whence

$$I^2\tau = S_2^2\gamma_2[c_2(t_m - t) + \lambda]/\rho_2. \tag{6}$$

Since the product $I^2\tau$ is the same for the wire and the fuse, we can equate the right-hand sides of expressions (6) and (3):

$$S_2^2\gamma_2[c_2(t_m - t) + \lambda]/\rho_2 = c_1 S_1^2\gamma_1\Delta t/\rho_1.$$

Hence we can find the cross-sectional area of the fuse:

$$S_2 = S_1\sqrt{\frac{\rho_2}{\rho_1}\,\frac{\gamma_1}{\gamma_2}\,\frac{c_1\Delta t}{c_2(t_m - t) + \lambda}} = 5.4\ \text{mm}^2.$$

Exercises

180. A current of density 100 A/cm^2 flows through a conductor having a cross-sectional area of 4 mm^2. Find the number of electrons passing through the conductor cross section during 2 min and the number of free electrons per unit volume of the conductor if the drift velocity of electrons is 10^{-4} m/s and the electron charge is 1.6×10^{-19} C.
Answer. 3×10^{21}, 6.25×10^{22} cm^{-3}.
181. Steel and aluminium wires have equal resistance and mass. Which of the wires is longer? What is the ratio of their lengths? The densities of steel and aluminium are 7.8×10^3 kg/m^3 and 2.7×10^3 kg/m^3 and their resistivities are 0.15 $\mu\Omega\cdot$m and 0.028 $\mu\Omega\cdot$m respectively.
Answer. The aluminium wire is 3.9 times longer.
182. (a) Four arc lamps rated for a voltage of 40 V and a current of 15 A each are connected in series with one another and a resistor in a circuit at a voltage of 220 V. Find the resistance of the resistor.
Answer. 4 Ω.
(b) A series resistance of 8.5 Ω is required for connecting an arc lamp having the rated voltage and current of 42 V and 10 A respectively to a circuit. Find the voltage in the circuit.
Answer. 127 V.
183. A two-conductor transmission line having a resistance of 500 Ω connects a power source and a load. At a distance of 10 km from the source, a short circuit occurs, during which the voltmeter connected at the source indicates the voltage of 5 V, the current shown by an ammeter being 25 mA. Find the distance from the load to the source.
184. (a) A telegraph line is made of iron wires whose cross-sectional area is 10 mm^2. The length of the wire in winter is 100 km. Find the change in the wire resistance due to an increase in temperature from -30 °C in winter to $+30$ °C in summer if the change in the wire volume is insignificant. The resistivity of wire at 20 °C is 0.098 $\mu\Omega\cdot$m and the temperature coefficient of resistance is 6×10^{-3} K^{-1}. Solve the problem taking into account the change in the wire volume upon heating. The coefficient of thermal expansion of iron is 1.2×10^{-5} K^{-1}.

Answer. The resistance in summer is higher by 315 Ω, the correction is −0.6 Ω.

(b) The cap of a tungsten incandescent lamp has an indication : 220 V, 150 W. Find the resistance of the filament at a temperature of 20 °C if the temperature of the glowing filament is 2500 °C. The temperature coefficient of resistance for tungsten is $5.1 \times 10^{-3}\ K^{-1}$.

Answer. 25 Ω.

(c) Find the temperature of the filament of a tungsten incandescent lamp in operation if the switch-on current (at $t_1 = 20$ °C) is 12.5 times stronger than the operating current. The temperature coefficient of resistance for tungsten is $5.1 \times 10^{-3}\ K^{-1}$.

Answer. 2500 °C.

185. Find the resistance of the circuit shown in Fig. 238 if each resistance is 2 Ω.

Answer. 7.0 Ω.

186. (a) Find the resistance of the circuit depicted in Fig. 239 if each resistance is 2 Ω.

Answer. 4.85 Ω.

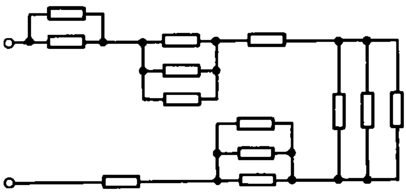


Fig. 238

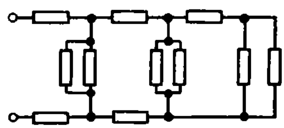


Fig. 239

(b) Find the resistance of the circuit shown in Fig. 240 if each resistance is 1 Ω.

Answer. 1.19 Ω.

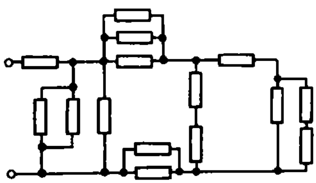


Fig. 240

187. Leads are connected to a wire ring. What is the ratio of the arcs of the circumference between the points of junction if the resistance of the obtained circuit is smaller than the resistance of the ring by a factor of 6.25?

Answer. 1:4.

188. (a) Find the number of parts into which a conductor having a resistance of 12.8 Ω should be cut so that the resistance of the circuit formed by parallel-connected parts be 0.2 Ω.

Answer. 8.

(b) The resistance of two wires connected in parallel is 3.43 Ω, while the resistance of the same wires connected in series is 14 Ω. Find the resistance of each wire.
Answer. 6 Ω, 8 Ω.

(c) How can the resistance of 15 Ω be obtained by using three conductors of 10 Ω each?
Answer. By connecting two conductors in parallel and the third one in series.

189. After the external resistance of a circuit has been increased five times, the voltage across the battery terminals increases from 10 V to 30 V. Find the e.m.f. of the battery.
Answer. 60 V.

190. Find the resistance of the circuit shown in Fig. 241.
Answer. $3r$.

191. Find the resistance of the hexagon circuit represented in Fig. 242.
Answer. $0.5r$.

192. Find the resistance of the circuit shown in Fig. 243.
Answer. $2.8r$.

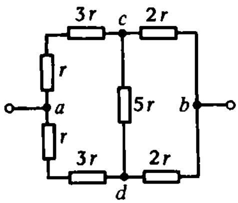


Fig. 241

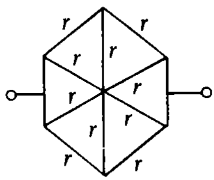


Fig. 242

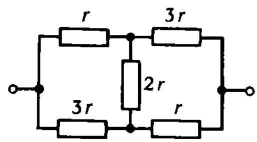


Fig. 243

193. A battery formed by cells having an e.m.f. of 1.22 V each is at 325 km from the load. The resistance per unit length of the wire is 6 Ω/km, load resistance is 900 Ω, and the internal resistance of each cell is 2.5 Ω. Find the number of cells in the battery if the current in the circuit is 0.008 A with series connection of the cells.
Answer. 32.

194*.(a) The current in a circuit having an external resistance of 3.75 Ω is 0.5 A. When a resistance of 1 Ω is introduced into the circuit, the current becomes 0.4 A. Find the e.m.f. and the internal resistance of the power source.
Answer. 2 V, 0.25 Ω.

(b) Upon a six-fold increase in the external resistance of a circuit, the voltage across the terminals of the battery has increased from 5 V to 10 V. Find the e.m.f. of the battery.
Answer. 12.5 V.

(c) When the external resistance of a circuit is 10 Ω, the current in it is 10 A. When the resistance becomes 20 Ω, the current drops to 8 A. Find the external resistance corresponding to a 9-A current.
Answer. 14.44 Ω.

(d) A circuit consists of five indentical cells and an external resistance of 1 Ω. When the cells are connected in series, the current in the circuit is 1.5 A, and when the cells are connected in parallel the current becomes 1.25 A. Find the e.m.f. and the internal resistance of a cell.
Answer. 1.44 V, 0.76 Ω.
195. There are 40 identical cells having an e.m.f. of 1.6 V and an internal resistance of 0.3 Ω each. In which way should these elements be grouped to obtain the maximum current if the external resistance is 3 Ω? Find this maximum current.
Answer. Two parallel groups of 20 series-connected cells, 5.3 A.
196. An accumulator battery having the output voltage of 12 V is connected for charging to a circuit at a voltage of 15 V. What series resistor should be connected to the circuit for the charging current be less than 1 A if the internal resistance of the battery is 2 Ω?
Answer. 1 Ω.
197. Two cells having the internal resistances of 0.2 Ω and 0.4 Ω are connected in parallel. The voltage across the battery terminals is 1.5 V. The e.m.f. of one element is 1.2 V. Find the e.m.f. of the second element.
Answer. 2.1 V.
198. Two cells having internal resistances of 0.4 Ω and 0.2 Ω are connected in parallel, and the circuit is closed through a resistance of 4 Ω. The voltage across the battery terminals is 1.36 V. The e.m.f. of one cell is 1.80 V. Find the e.m.f. of the other cell, the current in the external subcircuit and through each cell.
Answer. 1.21 V, 0.34 A, 1.10 A, 0.76 A.
199. Find the charge on the plates of a capacitor in the circuit shown in Fig. 244 (the internal resistance of the power source is $0.4r$).
Answer. 0.2 ʹC.

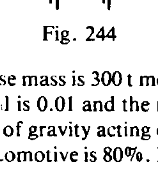


Fig. 244

200. An electric locomotive whose mass is 300 t moves down the hill with a velocity of 36 km/h. The slope of the hill is 0.01 and the resistance to the locomotive motion amounts to 3% of the force of gravity acting on it. The voltage in the circuit is 3 kV and the efficiency of the locomotive is 80%. Find the current flowing through its motor.
Answer. 245 A.
201. (a) Find the e.m.f. and the internal resistance of an accumulator if its output power is 9.5 W at a current of 5 A and 14.4 W at an external resistance of 0.25 Ω.
Answer. 2 V, 0.02 Ω.
(b) When a power source having an internal resistance of 2 Ω is connected to a

resistance of 4 Ω, the voltage across its terminals becomes 6 V. Find the total power of the source.
Answer. 13.5 W.
202. An electric instrument rated for a voltage of 210 V is connected to a circuit at a voltage of 220 V. As a result, the voltage of the socket drops to 200 V. Find the power consumed by the instrument if the resistance of leads is known to be 5 Ω.
Answer. 882 W.
203. (a) Find the maximum power that can be supplied to a user over a distance of 3 km through copper wires having a cross-sectional area of 18 mm^2 if the voltage at the electric power plant is 230 V. The admissible voltage loss in the transmission line should be equal to 10%. The resistivity of copper is 0.017 $\mu\Omega \cdot$m.
Answer. 1680 W.

(b) Electric energy should be transmitted over a distance of 10 km through copper wires so that energy losses in the wires be below 5%. The power of the electric plant is 12 MW. The energy is transmitted first at a voltage of 120 V and then at 12 kV. Find the diameter of the wires for both cases. The resistivity of copper is 0.017 $\mu\Omega \cdot$m.
Answer. 90 mm, 0.9 mm.

(c*) Electric energy has to be transmitted over a distance of 10 km through copper wires so that the energy loss be within 3%. The voltage at the electric power plant is 3 kV and the current power in the leads at the load must be 10 kW. Find the mass of copper required for the transmission line. The resistivity of copper is 0.017 $\mu\Omega \cdot$m and its density is 8.9×10^3 kg/m^3.
Answer. 5780 kg.
204. (a) 800 g of turpentine are heated in a calorimeter with the help of an electric heater made of a wire having a resistance of 30 Ω. A voltmeter connected to the ends of the wire indicates 10 V. After a 10-min passage of current, turpentine is heated by 1.4 °C. Find the specific heat of turpentine in this experiment.
Answer. 1.79 kJ/(kg·K).

(b) A current from a 50-V battery having an internal resistance of 4 Ω liberates 1.5 kJ/min of heat upon passing through a wire spiral. Find the resistance of the wire and the current through the battery.
Answer. 96 Ω, 0.5 A.

(c) An electric tea-kettle has two windings. When one of them is switched on, water boils in the kettle in 15 min, and with the other winding, in 30 min. Find the time required for boiling water if two windings are connected (a) in series; (b) in parallel.
Answer. (a) 45 min; (b) 10 min.
205*. A fuse made of a lead wire having a cross-sectional area of 1 mm^2 is connected to the main circuit of a copper wire whose cross-sectional area is 2 mm^2. When the current has reached 30 A and the temperature has become 20 °C, a short circuit occurs. Find the time interval between the beginning of the short circuit and the moment when the fuse starts to melt. What is the increase in the temperature of the copper wire during this time? The specific heats of copper and lead are 0.38 kJ/(kg·K) and 0.13 kJ/(kg·K), the resistivities of copper and lead are 0.017 $\mu\Omega \cdot$m and 0.21$\mu\Omega \cdot$m, the melting point of lead is 327 °C, and the densities of copper and lead are 8.9×10^3 kg/m^3 and 11.4×10^3 kg/m^3.
Answer. 2.4 ms, 2.7 °C.

4.18. Electrolysis

Electrolysis is the process of liberation of constituents of chemical compounds at the electrodes due to the passage of current through a solution or a melt of these compounds.

Solutions of some chemical compounds in water or other solvents, as well as melts conducting electric currents, are called **electrolytes.** Electrolytes include the solutions of many salts, acids, and alkalis, as well as the melts of salts and metal oxides.

The conduction observed in electrolyte solutions is called the **ionic conduction.** Its mechanism is explained as follows. The molecules of many substances (e.g. salts) are the combinations of positive and negative ions kept together as a single entity by the force of mutual attraction:

$$AgNO_3 = Ag^+ + NO_3^-,\ CuSO_4 = Cu^{++} + SO_4^{--},$$
$$NaCl = Na^+ + Cl^-.$$

In water, which has a very high permittivity ($\varepsilon = 81$), the force of mutual attraction between unlike ions decreases (by a factor of 81). With such a weakening of the ionic bond, the thermal motion of a molecule may cause its decomposition (dissociation) into positive and negative ions:

$$Ag_{NO_3} \rightarrow Ag^+ + NO_3^-,\ CuSO_4 \rightarrow Cu^{++} + SO_4^{--},$$
$$NaCl \rightarrow Na^+ + Cl^-.$$

In their random thermal motion, some ions "meet" one another to form molecules again (the process of association of ions into molecules is called recombination). With the passage of time, a situation arises when the rates of dissociation and recombination of ions become equalized, and dynamic equilibrium sets in. Although the decomposition and association processes do not cease, the ratio of dissociated and nondissociated molecules of the substance in the solution remains unchanged:

$$AgNO_3 \rightleftarrows Ag^+ + NO_3^-,\ CuSO_4 \rightleftarrows Cu^{++} + SO_4^{--},$$
$$NaCl \rightleftarrows Na^+ + Cl^-.$$

Like molecules, ions move at random in a solution. If a current source is connected to the electrodes immersed in such a solution, a directed motion of charged ions is observed (Fig. 245).

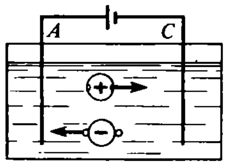


Fig. 245

Positive ions move towards the negative electrode (**cathode**), and are therefore called **cations**, while negative ions move towards the positive electrode (**anode**) and are therefore called **anions.** Having reached the surface of the cathode, a cation takes from it the electrons required for neutralization and becomes a molecule of a given substance. At the same time, an anion that has reached the surface of the anode gives to it its excess electrons and also becomes a neutral molecule. Vanishing ions in electrolyte are replaced by newly dissociated ions. It appears that the process occurs in such a way as if electrons move in the source-electrolyte-source closed circuit.

Consequently, unlike electric current, which is generated in metallic conductors by electrons moving in the same direction, electric current in electrolytes is a directed motion of ions in a solution in both directions under the effect of the electric field: positive ions move to the cathode and negative ions, to the anode. Positive ions are the ions of metals and hydrogen, while negative ions are the radicals of acids (SO_4^{--}, NO_3^-, Cl^-) or the hydroxyl ion OH^-.

4.19. Faraday's Laws of Electrolysis

Faraday's laws of electrolysis determine the mass of a substance liberated on the electrodes during electrolysis.

Faraday's first law: *the mass of a substance liberated at an electrode during electrolysis is proportional to the amount of electricity (charge) that has passed through the electrolyte*:

$$m = Kq, \qquad (4.19.1)$$

where m is the mass of the liberated substance, q is the charge that has passed through the electrolyte, and K is a proportionality factor called the electrochemical equivalent of a given substance.

The **electrochemical equivalent** of a substance is equal to the mass of this substance deposited from the electrolyte on an electrode upon the passage of a unit charge through the electrolyte. The electrochemical equivalent is expressed in kilograms per coulomb.

Faraday's second law: *the electrochemical equivalents of different substances are proportional to their chemical equivalents*:

$$K = CX,$$

where X is the chemical equivalent of a substance.

The **chemical equivalent** of a substance is equal to the molar mass divided by the valence, i.e. by the number equal to the number of hydrogen atoms that can be admitted or substituted by an atom of the given substance:

$$X = \mu/n, \tag{4.19.2}$$

where μ is the molar mass and n is the valence. Consequently, the electrochemical equivalent

$$K = C\mu/n.$$

The coefficient C in this expression is written in the form $C = 1/F$. The quantity F is called the **Faraday constant**. Then the electrochemical equivalent is given by

$$K = (1/F)(\mu/n).$$

Substituting this expression for K into formula (4.19.1), we can express the Faraday laws by a single formula

$$m = (\mu/n)(q/F), \text{ or } m = X(q/F). \tag{4.19.3}$$

The Faraday constant F is given by the following relation:

$$F = (X/m)q. \tag{4.19.4}$$

This formula shows that the Faraday constant is numerically equal to the charge that must be passed through any electrolytic solution for obtaining $1/n$ moles of the substance. Obviously, this quantity does not depend on the substance (either on its molar mass or on the valence). It was established experimentally that the Faraday constant is

$$F = 96.5 \times 10^3 \text{ C/mol}.$$

The application of electrolysis in technology is based on the fact that ions approaching electrodes are neutralized and either are deposited on the electrodes from the solution or take part in secondary reactions with the substance of electrodes or electrolyte.

The most important technical applications of electrolysis are the obtaining and refining of pure metals (especially copper) by the electrolysis of melts of their salts, electrodeposition of metals (electroplating), manufacturing duplicates from a matrix (galvanoplasty) and many others. The operating principle of chemical sources of electric energy is also based on the interaction between metals and electrolytes.

Problems with Solutions

206. Plates are silvered at a current density $j = 0.5$ A/dm^2. The mass of silver $m = 2$ kg is liberated during 5 h. Find the surface of the plate. The electrochemical equivalent of silver is $K = 1.118 \times 10^{-6}$ kg/C.

Solution. According to Faraday's first law of electrolysis, $m = KIt$. Since $I = jS$, we have $m = KjSt$, whence $S = m/Kjt = 2.0$ m^2.

207. Using the conditions of the previous problem, find the voltage across the terminals of circuit if the voltage across the terminals of each bath is $U_0 = 5$ V and silver plating is carried out in $n = 60$ baths connected in two parallel groups which contain 30 series-connected baths. Find the energy consumption (in hWh) during one hour of silvering. The efficiency of the set-up is $\eta = 90\%$.

Solution. The voltage at the terminals of the circuit is $U = nU_0/2 = 150$ V. The energy consumption is $W = 2IUt/\eta = 2jSUt/\eta = 120$ MJ $= 333$ hWh.

208. A plate is nickeled at a current density $j = 0.4$ A/dm^2. What is the rate of increasing the thickness of the nickel layer? The molar mass of nickel is $\mu = 58.71$ g/mole, its valence $n = 2$, and the density $\rho = 8.8 \times 10^3$ kg/m^3.

Solution. We denote the area of the plate by S and the thickness of coating by b. Then the rate of increase in the thickness of the layer is $v = b/t$. According to Faraday's first law, the mass of liberated nickel is $m = KIt = KjSt$. On the other hand, $m = Sb\rho$. We equate the right-hand sides of these expressions: $Sb\rho = KjSt$, whence

$$v = b/t = Kj/\rho. \tag{1}$$

The electrochemical equivalent of nickel is

$$K = (1/F)(\mu/n),$$

where the Faraday constant $F = 96.5 \times 10^3$ C/mole. Consequently, $K = 0.3 \times 10^{-6}$ kg/C. Substituting the results into expression (1), we obtain the rate of increase in the layer thickness: $v = 1.36$ nm/s.

209. Electrolysis of water is carried out at a current $I = 2.6$ A. As a result, the volume $V = 0.5$ l of oxygen at a pressure $p = 1.3 \times 10^5$ Pa is obtained during an hour. Find the temperature of oxygen. The gas constant $R = 8.31$ J/(mole·K).

Solution. Since the molar mass of oxygen is $\mu = 16$ g/mole and its valence $n = 2$, the electrochemical equivalent of oxygen is

$$K = (1/F)(\mu/n) = 8.3 \times 10^{-8}\ \text{kg/C}.$$

The mass of liberated oxygen is $m = KIt = 0.78$ g. Using the Clapeyron-Mendeleev equation $pV = (m/\mu)RT$ (where μ is the molar mass), we obtain the temperature:

$$T = (pV/R)(\mu/m) = 330\ \text{K}.$$

210. Electrolysis of water is carried out at the same current that is used to silver a plate of an area $S = 25\ \text{cm}^2$ at a rate $v = 0.05$ mm/s. Find the volume V of hydrogen liberated per hour if the process occurs under a pressure $p = 1.05 \times 10^5$ Pa at a temperature $t = 41$ °C. The electrochemical equivalents of silver and hydrogen are $K_s = 1.118 \times 10^{-6}$ kg/C and $K_h = 1.04 \times 10^{-8}$ kg/C, the densities of silver and hydrogen (under standard conditions) are $\rho_s = 10.5 \times 10^3\ \text{kg/m}^3$ and $\rho_{0h} = 0.09\ \text{kg/m}^3$.

Solution. Before solving this problem, we must find the density of hydrogen under the conditions of the problem. For this, we make use of the generalized gas law: $pV/T = p_0V_0/T_0$. Dividing both sides of this equation by the mass m_h of the liberated hydrogen and substituting $m_h/V_h = \rho_h$ and $m_h/V_0 = \rho_{0h}$, we obtain

$$p/T\rho_h = p_0/T_0\rho_{0h}, \text{ whence } \rho_h = (p/p_0)(T_0/T)\rho_{0h} = 0.0822\ \text{kg/m}^3.$$

We can now write the following equations for the two processes mentioned in the problem:

$$m_h = K_h I t_h, \tag{1}$$

$$m_s = K_s I t_s. \tag{2}$$

We divide expression (1) by (2):

$$m_h/m_s = K_h t_h/K_s t_s. \tag{3}$$

Let us find the mass of silver deposited per unit time: $m_s/t_s = Sv\rho$. Then expression (3) can be written as

$$m_h/Sv\rho_s = (K_h/K_s)t_h \text{ or } V_h\rho_h/Sv\rho_s = (K_h/K_s)t_h,$$

whence

$$V_h = (K_h/K_s)t_h(\rho_s/\rho_h)Sv = 0.55\ \text{m}^3.$$

Exercises

206. A plate is nickeled at a current density of 0.4 A/dm². What time is required for depositing a nickel layer of thickness 0.05 mm? The electrochemical equivalent of nickel is 0.3×10^{-6} kg/C and its density is $8.8 \times 10^3\ \text{kg/m}^3$.
Answer. 10 h 11 min.

207. Find the energy consumed for the production of 100 kg of refined copper if the electrolysis is carried out at a voltage of 8 V, and the efficiency of the set-up is 80%. The electrochemical equivalent of copper is 0.33×10^{-6} kg/C.
Answer. 3 GJ.
208. (a) Electrolysis of $FeCl_3$ is carried out for 2 h at a current of 10 A. Find the masses of iron and chlorine liberated as a result of the process. The molar masses of iron and chlorine are 55.85 g/mole and 35.46 g/mole. The valences of iron and chlorine are 3 and 1 respectively.
Answer. 14 g, 26 g.

(b) 430 g of nickel are liberated during the nickel-plating of 100 plates, having an area of 9.7 cm^2 each, in 100 parallel-connected baths having a resistance of 3 Ω each. The rate of increase in the nickel layer thickness is 0.007 mm/s. The electrolysis is carried out at a voltage of 6 V. Find the valence of nickel if its molar mass is 58.71 g/mole and its density is 8.8×10^3 kg/m^3.
Answer. 2.
209. Electrolysis of water is carried out at a temperature of 300 K and a pressure of 0.92×10^5 Pa. In this process, 2 l of oxygen are liberated during 15 min. Find the power of the current if the electrolysis takes place at a voltage of 12 V, and the efficiency of the set-up is 80%. The molar mass of oxygen is 16 g/mole, its valence is 2, and its density (under standard conditions) is 1.43 kg/m^3.
Answer. 7.5 W.
210. A plate having an area of 20 cm^2 is silvered by the same current that causes the liberation of 10 cm^3/s of hydrogen during electrolysis of water. The process occurs at a pressure of 0.95×10^5 Pa and a temperature of 27 °C. Find the time required to obtain a 5-mm thick silver layer. The electrochemical equivalents of silver and hydrogen are 1.118×10^{-6} kg/C and 1.04×10^{-8} kg/C. The densities of silver and hydrogen (under standard conditions) are 10.5×10^3 kg/m^3 and 0.09 kg/m^3.
Answer. 21 min.

4.20°. Electric Current in Gases

Electric current in gases is a directed motion of electric charges whose carriers are *free electrons* and *ions*. Under the normal pressure and not very high temperatures, the number of ions in gases is so insignificant that they practically do not conduct current. The number of free charge carriers is insufficient for creating current in such a gas. For example, the air gap in air capacitors "operates" as a dielectric.

A gas cannot conduct current unless it is ionized. **Ionization** consists in knocking out electrons from atoms. As a result, a formerly neutral molecule becomes a positive ion. Some electrons remain in the free state, while others combine with (adhere to) other molecules to form negative ions. Under the action of an electric field on the gas ionized in this way, two counterflows of charged particles are created, viz. the flow of electrons and negative ions and the flow of positive ions. Therefore, the conduction in gases is of the **ion-electron** type. In this respect, it differs from the electronic conduction of metals and the ionic conduction of electrolytes.

In order to detach an electron from a neutral molecule, a certain amount of energy, called the **ionization energy**, has to be spent. A gas can be ionized (a) by ultraviolet or X-rays, radioactive or cosmic radiation (it is the ionizing actions of cosmic rays that led to their discovery); (b) by heating the gas to high temperatures (thermal ionization), and (c) by producing a strong electric field in the gas.

Electric conduction in gases resulting from external effects is called the **induced conduction**, while electric conduction due to the field between the electrodes is called the **intrinsic conduction.** The passage of the current in the former case is called a **non-self-sustained discharge**, and in the latter case, a **self-sustained discharge.**

Induced conduction of gases can be observed in many experiments. Two of them are represented in Fig. 246. A circuit consists of a current source, a galvanometer, and a capacitor whose

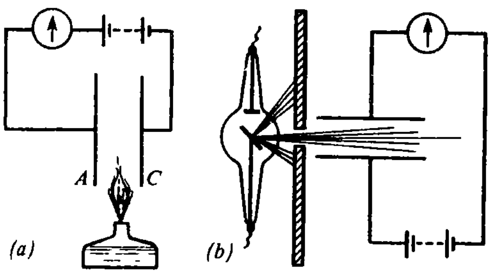


Fig. 246

plates are separated by an air gap. The flame of a burner is introduced into the air gap between the plates (Fig. 246*a*) or a beam of rays (X-rays, ultraviolet rays, etc.) is passed through a hole in a diaphragm (Fig. 246*b*). In both cases the galvanometer indicates a current in the circuit. If the flame or the source of ionizing rays is removed, the current ceases.

The basic characteristic of a gas discharge is the dependence of the current in the electrode gap on the voltage applied to the electrodes. The curve expressing the dependence of current I on voltage U is called the **voltage-current characteristic.** Figure 247 shows the voltage-current characteristic for a non-self-sustained discharge.

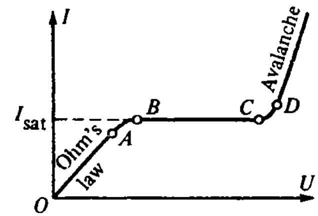


Fig. 247

Along with the ionization of molecules, the process of recombination occurs, which consists in combining ions into neutral molecules. If the voltage is not very high, only an insignificant fraction of formed ions reaches the electrodes and hence participates in the current. The remaining ions recombine before reaching the electrodes. The higher the voltage, the larger the number of ions that manage to reach the electrodes and the larger the current through the gas. In other words, the current increases with the voltage. At low voltages, the voltage-current characteristic is a straight line (OA in Fig. 247). This means that Ohm's law is satisfied in this region (the current is proportional to the voltage). As voltage U increases, the characteristic becomes more and more curved. Finally, at a certain voltage, all the ions formed under the action of the ionizer are able to reach the electrodes without recombining (point B in Fig. 247). A further increase in U cannot cause an increase in current I, and saturation sets in (the horizontal region BC in the figure). The corresponding value I_{sat} of the current is called the saturation current.

Then the voltage attains the value corresponding to point C on the voltage-current characteristic (see Fig. 247), the current starts to grow abruptly. This is due to the fact that the ions accelerated by such a strong field acquire the energy sufficient to ionize the molecules with which they collide. The number of ions increases in avalanche, and the electric breakdown occurs in the gas.

A breakdown is observed when the kinetic energy W_k acquired by an ion under the action of an electric field becomes equal to the ionization energy W_i, i.e. the energy sufficient to knock out the electron from a neutral molecule as a result of collision.

Let us suppose that the distance covered by an electron between two collisions is l (this distance is called the **mean free path**). The field forces do over this distance a work $A = eEl$ (E is the field strength). This work is equal to the increase in the kinetic energy of the electron, $W_k = eEl$ (we assume that the electron velocity at the beginning of a free path is zero). Hence we obtain the breakdown condition:

$$W_i = eEl.$$

The mean free path l of particles in a gas is inversely proportional to the gas pressure. Therefore,

$$W_i \propto E/p.$$

Self-sustained discharge may be of different types. Let us consider several types of self-sustained discharge: the glow discharge, corona discharge, spark discharge, and arc discharge.

1. The **glow discharge** occurs in rarefied gases. It has been mentioned above that the lower the gas pressure, the lower the voltage required to induce a self-sustained discharge. Thus, the glow discharge emerges in a tube in which a significant rarefaction is created with the help of a vacuum pump. This is the so-called gas-discharge tube (Fig. 248). It has two electrodes, viz. the anode

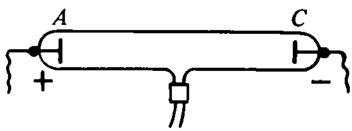


Fig. 248

A and the cathode C. The electrodes are connected to a source of electric current having a voltage of several hundred volts. Figure

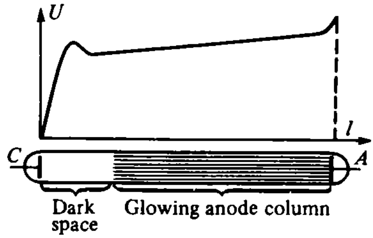


Fig. 249

249 shows the potential drop in the tube as a function of the interelectrode distance. Near the cathode, the voltage decreases abruptly (this is called the cathode potential drop). In this region, the electric field strength is very high while in the remaining part of the tube, the field strength is not very high.

In the immediate vicinity of the cathode, there is a nonglowing region called the dark cathode space. The remaining part of the tube is filled by a homogeneous glow called the positive (anode) column. In this region, collision ionization occurs. Electrons formed as a result of ionization move to the anode, while positive ions fly to the cathode. Passing through the region of the cathode potential drop, they acquire a high kinetic energy near the cathode. Hitting the cathode, they knock electrons out of it. The electrons emitted by the cathode move towards the anode. It follows from what has been said above that the glow discharge is caused by two types of ionization: the *collision ionization* and the knocking of electrons out of the cathode by positive ions (*secondary ionization*).

In this type of discharge, the glow is due to the fact that the energy liberated during recombination of gas molecules, which accompanies their ionization, is released in the form of radiation. This is the so-called *recombination glow*. Gas-discharge tubes glowing with different colours are used as decoration for advertisements. In nature, glow discharge of rarefied gases is observed in aurora borealis. This is the glow of the upper, highly rarefied layers of the atmosphere caused by the flow of charged particles from the Sun.

The concentrations of electrons and positive ions in the positive column of a glow discharge are virtually equal. Such a gas, neutral as a whole, in the state corresponding to a high degree of ionization is called the *electron-ion plasma.* If this gas contains neutral particles also, it is called *partially ionized plasma.* The positive column of a glow discharge is a partially ionized plasma.

In nature, a partially ionized plasma is encountered in the upper layers of the atmosphere called the *ionosphere.* A high-temperature fully ionized plasma is in the atmosphere of the Sun and hot stars.

2. The **corona discharge** is a discharge accompanied by a weak violet glow in the form of a crown enveloping an electrode. It emerges both at the normal and elevated pressure at sharp ends of electrodes. The charge of such an electrode produces a nonuniform electric field whose strength is very high at the tip and abruptly decreases with distance. The strong electric field in the region directly adjoining the electrode causes the collision ionization in this region (accompanied by a partial recombination) under the atmospheric pressure.

Corona discharges sometimes emerge in nature under the action of the atmospheric electric field on the branches of trees, tips of masts (so-called Saint Elmo's fire), etc.

A corona discharge may also emerge on thin wires at a high voltage, which leads to a leakage of electric energy. This circumstance has to be taken into account in engineering when designing the thickness of the wires for high-voltage transmission lines.

The operating principle of the *lightning rod* that protects buildings and transmission lines from lightning stroke is based on the appearance of a corona dischage on the tips of conductors. The lightning rod is a vertical conductor whose upper end is sharpened and arranged above the buildings nearby, while the lower end is earthed. During a thunderstorm, charges induced by charged clouds appear on the surface of the Earth, and an electric field is created. If a lightning rod is available, the field strength is maximum at the tip of the rod, as a result of which a corona discharge emerges, and induced charges leak from the surface of the Earth. Thus, induced charges cannot be accumulated on buildings and constructions, and this prevents the emergence of a lightning. In those rare cases when a lightning still appears, it strikes the lightning rod on whose tip the electric field strength is maximum, and the charges are conducted to the ground, causing no damage.

3. The **spark discharge** is a discontinuous self-maintained discharge occurring under the normal or elevated pressure of a gas in a high-strength electric field.

Spark discharges are ion or electron avalanches caused by collision ionization. In this case, ionized gas channels (*streamers*) appear in the gas, along which the spark discharge propagates. The gas in the streamers is heated to a high temperature, and the pressure in it rises. Tending to expand, this gas generates acoustic waves, giving rise to the well-known sonic effects accompanying a spark discharge (in particular, the thunder accompanying a lightning). A spark has a zigzag shape with branches due to inhomogeneities of the medium. Streamers are formed in the permanently varying direction of the easiest ionization. A short-term spark appears when the power of the source is insufficient to sustain a continuous discharge.

Spark discharges are used in engineering, for example, in magneto for "igniting" petrol internal combustion engines. The voltage created in magneto amounts to 12-15 kV. The spark gap of a spark plug is less than 1 mm.

Lightning is a kind of the spark discharge between two charged clouds or between a cloud and the Earth. Charge carriers in clouds are raindrops or snowflakes.

4. The **arc discharge** is the discharge between the electrodes heated to a high temperature under the atmospheric or elevated pressure. The electrodes at a potential difference of 30-50 V are brought in contact, and a large current flows in the closed circuit. Since the resistance at the contact is comparatively high, according to Joule's law the maximum amount of heat is liberated in this region, and the tips of the electrodes are red-heated. Then the electrodes are gradually moved apart. The current continues to flow since the entire space between the electrodes is filled with a high-temperature plasma which is a good conductor. The electrode tips heated to 3000-4000 °C start to evaporate. With a

passage of time, the cathode is sharpened, while the anode acquires a depression called the positive arc crater.

In the case of an alternating current, the two electrodes burn similarly.

The position of the electrodes is controlled manually or with the help of some mechanism to ensure the continuous burning of the arc. This inconvenience has been overcome in constructing an

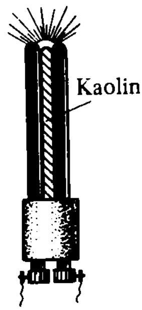


Fig. 250

electric lamp. In the Yablochkov candle (Fig. 250), carbon electrodes are arranged in parallel and separated by an insulating pad (e.g. a layer of kaolin) which evaporates as the electrodes burn out.

The high temperature of the cathode is maintained during the burning of an arc by the bombardment of the cathode with positive ions, which causes thermionic emission (see Sec. 4.22). The gas in the arc also becomes heated due to collisions between electrons and ions accelerated by the electric field. This leads to the thermal ionization of the gas (i.e. the ionization at an elevated temperature) and chemical processes accompanying it.

Let us consider some applications of the electric arc. It is used in lamps, melting furnaces (Fig. 251), in medical equipment (in the devices of the type of "artificial mountain Sun"), and for electric arc welding. The medical lamp is a quartz balloon with mercury electrodes (Fig. 252). To switch on this lamp, it is inclined to obtain a mercury jet between the electrodes. As soon as the lamp is returned to the initial vertical position, an arc is formed in place

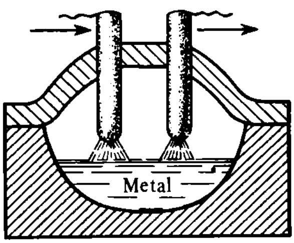


Fig. 251

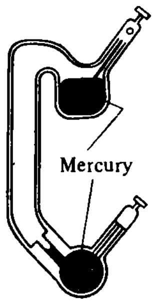


Fig. 252

of the mercury jet, which radiates high-intensity ultraviolet rays.

The electric arc is also used in chemical technology processes when high temperatures are required. For example, electric arcs are used to obtain from air nitric oxide NO with subsequent oxidizing it to NO_2 in the production of nitric acid, or to obtain calcium carbide CaC_2 from lime and coke.

Arc discharges can also be employed in high-power mercury-arc rectifiers. A mercury-arc rectifier unit is a gas-discharge tube in which the discharge occurs in mercury vapour between the surface of liquid mercury, which serves as a cathode, and an anode

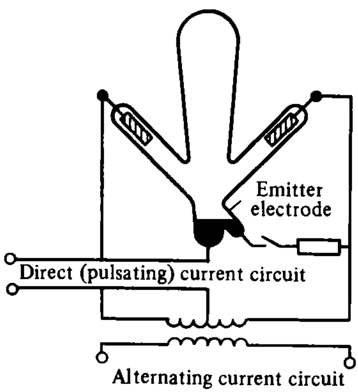


Fig. 253

made of graphite or iron (Fig. 253). The operation of such a rectifier is based on the property of mercury vapour (formed at a high temperature) to conduct electrons only in one direction, viz. from mercury to graphite. Each time when mercury acquires a potential of the cathode, a discharge emerges and it is discontinued when the current alters its direction.

4.21°. Electron and Ion Beams, Their Properties and Application

An **electron beam** is a flow of electrons accelerated by some method. Electron beams emerging in a glow discharge are called **cathode rays.**

The electron nature of cathode rays can be proved by various experiments including those with a (Crookes) gas-discharge tube, i.e. the tube containing electrodes and evacuated to a pressure of 0.1 Pa. Cathode rays are deflected by an electric field (Fig. 254). A positively charged plate introduced into a Crookes tube attracts cathode rays, while a negatively charged plate repels them. Cathode rays are also deflected by a magnetic field (Fig. 255). The

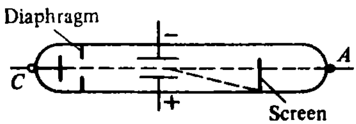


Fig. 254

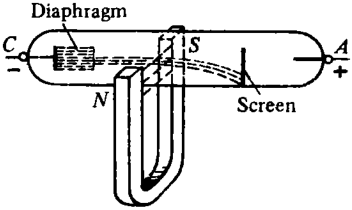


Fig. 255

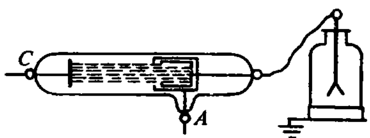


Fig. 256

direction of deflection indicates that cathode rays are the flows of negatively charged particles. The cathode ray flow produces, like an electric current, a magnetic field. In Perrin's experiments, the negative charge of cathode rays is observed directly (Fig. 256). A cylinder connected to an electroscope is introduced into a Crookes tube. Cathode rays directed into the cylinder supply a negative charge to the electroscope.

Properties of cathode rays

1. Cathode rays propagate in a straight line perpendicular to the surface of the cathode irrespective of the position of the anode (Fig. 257). The fact that cathode rays propagate in a straight line

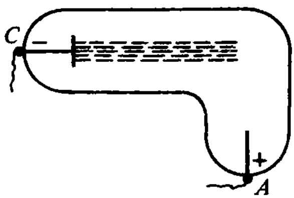


Fig. 257

is explained by the circumstance that the electrons emitted by the cathode acquire a high kinetic energy in the immediate vicinity of the cathode, in the region of the cathode potential drop, where they move along the normal to the cathode surface. The further motion of the electrons proceeds by inertia.

2. While passing through gases, cathode rays ionize them and make them glow. Cathode rays themselves are invisible.

3. Impinging at solid bodies, cathode rays cause their glow called *fluorescence.* Glass and diamond, for example, fluoresce in a bright-green light.

4. Like light beams, cathode rays blacken photographic plates.

5. Cathode rays have a kinetic energy. Getting to the surface of a metal, they heat it. Cathode rays rotate a light propeller installed on their way, and so on.

6. Cathode rays cause the X-ray radiation when they impinge at a plate made of a heavy metal (tungsten or platinum) placed on their way.

7. The velocity of electrons in cathode rays increases with the voltage applied to the gas-discharge tube and may attain the values close to the speed of light.

Electric beams are used in cathode-ray tubes (see Sec. 5.16), electron microscopes, etc. The electron beam in an electron microscope passes through a specially selected electric and magnetic fields deflecting electron beams in the same way as lenses in an ordinary microscope deflect light beams.

Ion beams include **anode rays,** i.e. the *flow of positively charged particles in the space behind the cathode,* on the side opposite to the anode.

Anode rays can be observed by placing a perforated cathode in a tube (Fig. 258). The flow of positive particles has such a direction as if it were emitted by the anode and passed through the perforation of the cathode. For this reason, anode rays are also called **canal rays**, or **positive rays.**

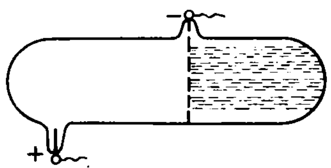


Fig. 258

The properties of anode rays are: (1) anode rays are deflected by an electric and a magnetic field; (2) they cause the glow of certain substances; (3) anode rays blacken a photographic plate; (4) they ionize gases, and (5) anode rays are intensely absorbed by gases and other bodies.

4.22. Thermionic Emission. Electron Work Function

Vacuum is such a rarefaction of a gas in a vessel in which the mean free path of gas particles exceeds the vessel size. In vacuum gas-discharge tubes, conduction is due to electrons emitted by a heated cathode.

Free electrons in metals, including those contained in the electrodes of a vacuum tube, move at random. In this motion, some electrons may acquire a kinetic energy sufficient to escape from the metal and leave it. A part of electrons escaping from the metal return to it again. Thus, a dynamic equilibrium is established, at which a thin layer of charged particles — *electron cloud* — is formed above the electrode surface. This layer prevents from removing the electrons leaving the electrode.

At normal temperatures, the electron concentration above the surface of a cathode is negligibly small. As the cathode is heated, the energy of free electrons in it increases, and their emission is intensified. The emission of electrons of a cathode caused by its heating is called the thermionic (or thermoelectron) emission.

If a vacuum tube is connected to an electric circuit (Fig. 259)

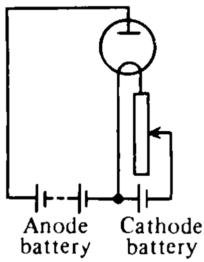


Fig. 259

so that the red-hot conductor is connected to the negative pole of the power source (battery in Fig. 259), i.e. surves as a cathode, a fraction of electrons escaping from it is attracted by the anode and forms a direct thermionic current, while a fraction of electrons having a lower kinetic energy is repelled by the electron cloud back to the cathode. As the voltage increases, an increasingly larger amount of electrons from the cloud are attracted by the anode and the current increases. With a further increase in the voltage between the vacuum tube electrodes, the situation arises when all the electrons leaving the cathode filament are carried by the field to the anode, the electron cloud disappears, and the current assumes the maximum possible value under the given conditions. If the voltage is increased still further, this does not lead to any increase in the current since the number of electrons participating in the electric current does not increase, i.e. the current remains constant. This is a **saturation current** (Fig. 260) I_{sat}, viz. the current corresponding to the voltage at which all the electrons emitted by the cathode at a given filament temperature are en-

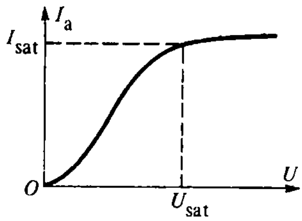


Fig. 260

trained by the electric field to the anode. The saturation current is the larger, the higher the cathode temperature.

The curve (Fig. 260) describing the dependence of the anode current I_a on the voltage U between the anode and the cathode of a tube is called the **anode characteristic** of the tube.

An electron escapes from the cathode when its kinetic energy is high enough to overcome the attraction by the atoms and the repulsion of the electron cloud. This energy is measured by the so-called electron work function. The **electron work function** is the work that has to be done to extract an electron from a metal:

$$A = e\varphi_c,$$

where e is the electron charge and φ_c is the potential difference between the metal and a vacuum, called the **contact potential** of a given metal.

Problems with Solutions

211. The saturation current in a non-self-sustained discharge is $I_{sat} = 4.8 \times 10^{-12}$ A. Find the number of ion pairs formed per unit time with the help of an external ionizer. The electron charge $e = 1.6 \times 10^{-19}$ C.

Solution. The saturation current $I_{sat} = en$, where n is the total number of ions created by the external ionizer per unit time. Hence $n = I_{sat}/e = 3 \times 10^7$. The number of ion pairs is $n/2 = 15 \times 10^6$.

212. The ionization energy for air molecules is $W_i = 15$ eV. Find the mean free path λ of an electron in air. The electron charge $e = 1.6 \times 10^{-19}$ C. Under the normal pressure, the spark discharge in air appears at the electric field strength $E = 3$ MV/m.

Solution. The work done by the electric field forces over the mean free path is $A = eE\lambda$. This work must be equal to or larger than the ionization energy:

$A \geqslant W_i$, i.e. $eE\lambda \geqslant W_i$, whence the mean free path $\lambda \geqslant W_i/eE$. Since W_i = 15 eV = 24×10^{-19} J, $\lambda \geqslant 5\ \mu$m.

213. A cathode beam comprising $n = 10^6$ electrons is emitted with a velocity $v_0 = 10^5$ km/s into the space between the plates of an air parallel-plate capacitor in parallel to them. The potential difference between the plates is $\varphi = 400$ V, their separation $d = 2$ cm, and the area of a plate is $l^2 = 10 \times 10\ \text{cm}^2$. Find the deflection Δy of the beam and the direction of its outlet velocity. The electron mass $m = 9.1 \times 10^{-31}$ kg and its charge $e = 1.6 \times 10^{-19}$ C. The relativistic effect should be disregarded.

Solution. Let us choose the coordinate system so that its origin coincides with point O at which the beam enters the air gap of the capacitor. We direct the x-axis along the velocity v_0 with which the beam enters the capacitor, and the y-axis, towards the positively charged plate (Fig. 261). As the beam passes through the

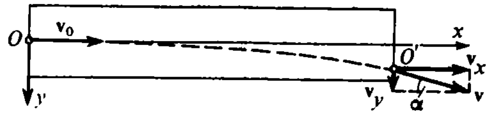


Fig. 261

capacitor, it experiences the action of the electric force directed normally to the positively charged plate: $F = Ene = \varphi ne/d$. This force imparts to the cathode beam an acceleration

$$a = F/nm = \varphi ne/dnm = \varphi e/dm,$$

where m is the electron mass. On account of this expression, the deflection $\Delta y = at^2/2$ of the beam is equal to $\varphi et^2/2dm$. The x-component of the outlet velocity of the beam is $v_x = v_0$ = const, since the electric field is directed along the normal to the x-axis. Therefore, the time during which the beam traverses the capacitor is $t = l/v_0$. This gives

$$\Delta y = \varphi el^2/2dmv_0^2 = 1.76\ \text{mm}.$$

Let us now calculate the v_y-component of the outlet velocity of the cathode beam:

$$v_y = 2v_{y\,\text{av}} = 2\Delta y/t = 2\Delta yv_0/l = 3.52 \times 10^6\ \text{m/s}.$$

The angle by which the beam velocity deviates from the horizontal at the capacitor outlet is given by

$$\tan\alpha = v_y/v_x = 3.52 \times 10^2,\ \text{i.e.}\ \alpha \simeq 2^\circ.$$

Remark. The v_y-component of the outlet velocity of the cathode beam can be found in a different way, viz. by equating the increment of the kinetic energy of the beam to the work done by the electric field over the displacement Δy of the beam in the direction of the field:

$$W - W_0 = \Delta\varphi ne,\ \text{or}\ (nm/2)(v_x^2 + v_y^2 - v_0^2) = \Delta\varphi ne.$$

Since $v_x = v_0$, we have $nmv_y^2/2 = \Delta\varphi ne$, whence $v_y = \sqrt{2\Delta\varphi e/m}$. The potential drop over the distance Δy is $\Delta\varphi = \varphi\Delta y/d$. This gives

$$v_y = \sqrt{2\varphi e\Delta y/md} = 3.52 \times 10^6 \text{ m/s}.$$

214. The electron work function $A = 3$ eV. With which velocity does an electron having a kinetic energy $W_k = 10^{-18}$ J leave a given metal? The electron mass $m = 9.1 \times 10^{-31}$ kg and the charge $e = 1.6 \times 10^{-19}$ C.

Solution. The energy of the escaping electron is $mv^2/2 = W_k - A$. Hence we can find the velocity of the escaping electron:

$$v = \sqrt{2(W_k - A)/m}.$$

The energy of 1 eV is equal to the product of the electron charge e and 1 V, i.e. 1 eV $= 1.6 \times 10^{-19}$ C·V. Consequently, $v = 1.07 \times 10^6$ m/s.

Exercises

211. Find the current in a non-self-sustained discharge induced by an ionizer which produces 2×10^6 ion pairs per second. The electron charge $e = 1.6 \times 10^{-19}$ C.
Answer. 6.4×10^{-13} A.
212. Find the ionization energy for air molecules (in joules and electron volts) if under normal conditions the spark discharge in air takes place at an electric field strength of 3 MV/m, and the mean free path of electrons in air is 1 μm.
Answer. 4.8×10^{-19} J, 3 eV.
213*. A cathode beam enters the space between the plates of an air parallel-plate capacitor in parallel to the plates. Having passed a distance of 5 cm during 5×10^{-9} s, the beam is deflected by 1 mm. The field strength between the capacitor plates is 150 V/cm. Find the change in the energy of an electron from the cathode beam per unit time and the angle of deviation of its velocity vector over the specified distance (the angle should be found by two methods). The electron mass and charge are 9.1×10^{-31} kg and 1.6×10^{-19} C.
Answer. 4.55×10^{-17} W, $\sim 8'$.
214. The electron work function is 5 eV. What must be the kinetic energy of an electron escaping from a metal with a velocity of 1×10^6 m/s? The electron mass and charge are 9.1×10^{-31} kg and 1.6×10^{-19} C.
Answer. 1.255×10^{-14} J.

C. MAGNETIC PHENOMENA

4.23. Interaction of Currents. Magnetic Field. Magnetic Induction. Magnetic Field Lines

Stationary electric charges interact through the electric field. This interaction also takes place for moving charges, but in this case the **magnetic interaction** is also observed. The magnetic interaction is realized through the magnetic field.

A magnetic field is generated by moving charges and currents. The emerging field acts on other moving charges and currents. Thus, the interaction of currents and moving charges takes place. The magnetic field at a given point in space is characterized by a vector quantity **B** called the **magnetic induction.** Experiments show that *unidirectional currents attract each other* (Fig. 262*a*), *while countercurrents repel each other* (Fig. 262*b*).

Figure 263 illustrates the experiment carried out by the Danish physicist Oersted who was the first to discover the action of an

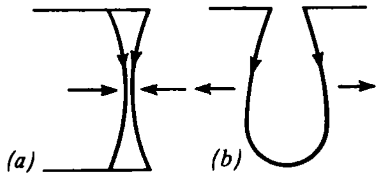


Fig. 262

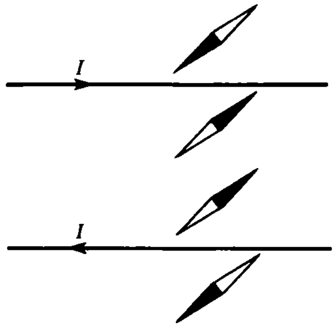


Fig. 263

electric current on a magnet. In this experiment, a magnetic needle turned when an electric current was passed through a rectilinear conductor in the vicinity of the needle.

Magnetic field can be visually represented by the **magnetic field lines.**[4] These lines are plotted in the same way as the electric field lines: the tangent to a field line coincides at each point with the direction of vector **B**, while the density of lines is proportional to the magnitude of vector **B** at a given point.

The magnetic field of a *straight current* (i.e. the current in a rectilinear conductor) is represented by concentric circles with centres on the axis of the conductor, lying in the plane normal to

[4] They are called the **lines of magnetic induction**, or the **lines of vector B**. (*Editor's note.*)

its axis (Fig. 264). The direction of the magnetic field of such a current can be determined by one of the following methods:

(1) the *right-hand rule* (Fig. 265); if the screw moves in the same direction as the current, the direction of rotation of its handle coincides with the direction of the field lines;

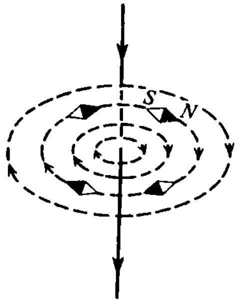


Fig. 264

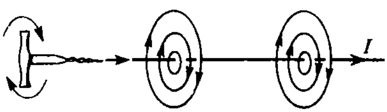


Fig. 265

(2) if we look in the direction of the current, the field lines are directed clockwise (Fig. 266*a*); if we look "against" the current, the field lines are directed counterclockwise (Fig. 266*b*).

These methods can also be used for determining the direction of the magnetic field of a *circular current* or of a *solenoid* (Fig.

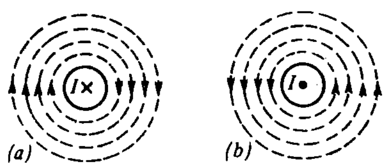


Fig. 266

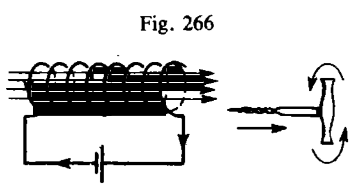


Fig. 267

267). The magnetic field of a *current-carrying coil* (*solenoid*) is similar to the field of a permanent magnet (Fig. 268).

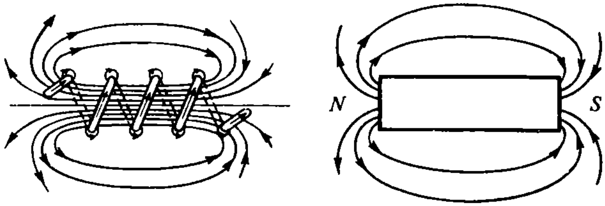


Fig. 268

Magnetic field lines are *always closed* (they can also start or terminate at infinity). In this respect, they differ from electric field lines which always start and terminate on electric charges or go off to infinity.

4.24. Force Acting on a Current-Carrying Conductor in a Magnetic Field. Magnetic Forces

A current-carrying conductor in a magnetic field experiences the action of a force that is determined only by the properties of the field in the region where the conductor is, and does not depend on the system of currents or permanent magnets which generates the field.

We shall consider a small element of a current-carrying conductor of length Δl as a vector $\Delta \mathbf{l}$ whose direction coincides with the direction of the current in the conductor. Experiments show that the magnetic force $\mathbf{F}$ acting on this element is normal to the plane containing vector $\Delta \mathbf{l}$ and the magnetic induction vector $\mathbf{B}$, the rotation from $\Delta \mathbf{l}$ to $\mathbf{B}$ over the shortest path being related to the direction of $\mathbf{F}$ through the right-hand rule (Fig. 269, vectors $\Delta \mathbf{l}$ and $\mathbf{B}$ lie in the plane of the figure and vector $\mathbf{F}$ is directed away from us along the normal to the plane of the figure).

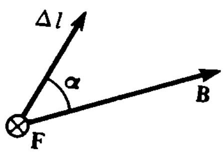


Fig. 269

According to **Ampère's law**, *the magnitude of the magnetic force* **F** *is defined by the formula*

$$F = I \Delta l B \sin \alpha, \qquad (4.24.1)$$

where I is the current in the conductor, the remaining quantities being the same as in Fig. 269. It follows from formula (4.24.1) that when a conductor is arranged along a field line ($\alpha = 0$), the magnetic force is zero.

In the simplest case when the current-carrying conductor and the magnetic field are mutually perpendicular ($\alpha = \pi/2$), the direction of the magnetic force can be determined by the *left-hand rule*. Hold the thumb, first finger, and middle finger of the left hand at right angles to one another. Point the first finger in the direction of the field, the second finger in the direction of the current; then the thumb points in the direction of motion (force) on the conductor.

The SI unit of magnetic induction is a **tesla** (T). In a uniform magnetic field having an induction of one tesla, a force of one newton acts on a metre of a straight conductor perpendicular to vector **B** and carrying a current of one ampere.

$$1\ \mathrm{N} = 1\ \mathrm{A} \times 1\ \mathrm{m} \times 1\ \mathrm{T}, \text{ whence } 1\ \mathrm{T} = 1\ \mathrm{N/(A \cdot m)}.$$

In Sec. 4.9, we obtained the following expression for the current in a metallic conductor: $I = nevS$, where n is the number of free electrons per unit volume, e is the electron charge, v is the electron velocity, and S is the cross-sectional area of the conductor. Substituting this expression into formula (4.24.1), we get

$$F = nevS\ \Delta l\ B \sin \alpha.$$

The product $nS\,\Delta l$ gives the number N of charges moving in the element Δl of the conductor. The magnetic force acts just on these charges and only afterwards is it transmitted to the crystal lattice of a substance of which the conductor is made. Therefore, dividing F by the number N of charges, we obtain the magnetic force acting on an individual charge e moving with a velocity v:

$$F_{\mathrm{m}} = evB \sin \alpha .$$

This formula is valid not only for electrons but also for any point charges q:

$$F_{\mathrm{m}} = qvB \sin \alpha . \tag{4.24.2}$$

The direction of this force is related to vectors **v** and **B** in the same way as the direction of the force **F** is related to vectors $\Delta\mathbf{l}$ and **B** in Fig. 269 (it should be borne in mind that for the direction of $\Delta\mathbf{l}$ in Fig. 269 we took the direction of the current, i.e. the direction of motion of positive charges; for negative charges, the direction of the force $\mathbf{F}_{\mathrm{m}}$ determined by formula (4.24.2) should be reversed).

The forces whose magnitudes are given by formulas (4.24.1) and (4.24.2) can be represented as vector products:

$$\mathbf{F} = I\Delta\mathbf{l} \times \mathbf{B}, \quad \mathbf{F}_{\mathrm{m}} = q\mathbf{v} \times \mathbf{B}. \tag{4.24.3}$$

If a point charge is in electric and magnetic fields simultaneously, the force acting on it is equal to the sum of the electric and magnetic forces:

$$\mathbf{F}_{\mathrm{L}} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}.$$

This total force is called the **Lorentz force** (sometimes the magnetic force is erroneously called the Lorentz force).

4.25. Permeability of a Medium. Magnetic Field Strength

All materials placed in a magnetic field are magnetized and generate their own magnetic field whose action is superimposed on the action of the external field. Experiments show that a homogeneous continuous medium can either enhance or weaken a magnetic field. The effect of a medium on a magnetic field is characterized by a quantity μ_r called the **relative permeability** of the medium. It is equal to the ratio of the magnetic induction in a given medium to the magnetic induction in a vacuum.

The substances that weaken an external field are called **diamagnetics.** The substances enhancing an external magnetic field are known as **paramagnetics.** The value of μ_r is greater than unity ($\mu_r > 1$) for paramagnetics and less than unity ($\mu_r < 1$) for diamagnetics. In both cases, the value of μ_r differs from unity only slightly, by the values of the order of 10^{-4}-10^{-5}.

Several materials (such as iron, nickel, cobalt, and some alloys) cause a considerable enhancement of the magnetic field. They are called **ferromagnetics** (from Latin *ferrum* meaning iron). The relative permeability of ferromagnetics is of the order of 10^3-10^5.

In addition to the *basic* (force) characteristic **B** of a magnetic field, an auxiliary quantity **H** is used for describing a magnetic field in a substance. This quantity is called the **magnetic field strength**[5] and is defined by the following relation:

$$\mathbf{H} = \mathbf{B}/\mu_0\mu_r, \tag{4.25.1}$$

where μ_r is the relative permeability of a substance and μ_0 is the **magnetic constant**.

The value of μ_0 determined experimentally is expressed in a **henry per metre** (H/m), see Sec. 4.32:

$$\mu_0 = 4\pi \times 10^{-7}\ \text{H/m} = 12.56 \times 10^{-7}\ \text{H/m}\ [\text{N/A}^2].$$

According to formula (4.25.1), we have

$$\mathbf{B} = \mu_0\mu_r\mathbf{H}, \text{ or } \mathbf{B} = \mu_0\mu_r H.$$

Substituting this expression for B into formula (4.24.1) we obtain the following expression for the force exerted by a magnetic field on an element of the current:

$$F = \mu_0\mu_r I\,\Delta l\,H \sin\alpha.$$

[5] Sometimes it is erroneously stated that vector **H** is a characteristic of a magnetic field, which does not depend on the properties of the medium in which the field is created.

4.26. Forces of Interaction Between Parallel Current-Carrying Conductors

Suppose that we have two parallel thin very long conductors with currents I_1 and I_2 respectively. In accordance with Ampère's law, the force of interaction between the conductors per unit length is given by

$$F_l = (\mu_0/4\pi)\,(2\mu_r I_1 I_2/r), \tag{4.26.1}$$

where r is the separation of the conductors and μ_r is the relative permeability of the medium containing the conductors. A conductor is assumed to be thin if its thickness is much less than the distance r between the wires. For finite-length conductors, formula (4.26.1) is valid for the regions that are separated from the conductor ends by a distance much longer than r.

Multiplying the force F_l by the length l of a segment of the conductor, we obtain the force F acting on this segment:

$$F = (\mu_0/4\pi)(2\mu_r I_1 I_2 l/r). \tag{4.26.2}$$

According to Ampère's law, the force in (4.26.2) can be represented either in the form $F = I_2 l B_1$, where B_1 is the magnetic induction of the field generated by current I_1 in the region where current I_2 flows, or in the form $F = I_1 l B_2$, where B_2 is the magnetic induction of the field generated by current I_2 in the region where current I_1 flows (in both cases, the directions of vector $\mathbf{B}$ and the current-carrying conductor are at right angles so that $\sin\alpha = 1$). Equating, say, the right-hand sides of the former formula and of (4.26.2), we get

$$\left(\frac{\mu_0}{4\pi}\frac{2\mu_r I_1}{r}\right) I_2 l = I_2 l B_1.$$

Cancelling out $I_2 l$ and omitting the subscript 1 on both sides of the expression, we obtain

$$B = \mu_0\mu_r(I/2\pi r). \tag{4.26.3}$$

This formula determines the magnetic induction of the field created by an infinitely long straight current-carrying conductor at a distance r from it (to make it short, the magnetic induction of a straight current). Formula (4.26.3) can be applied in practical cases to the points located near the middle of a conductor of a finite length l for which the distance r is much smaller than the conductor length ($r \ll l$).

In view of formula (4.25.1), we can divide relation (4.26.3) by $\mu_0\mu_r$ to obtain the following expression for the magnetic field strength of a straight current:

$$H = I/2\pi r. \tag{4.26.4}$$

It follows from this formula that the SI unit of magnetic field strength is an **ampere per metre** (A/m).

Formula (4.26.4) can be used for a definition of the unit of current, viz. an ampere, which is a basic SI unit.

Ampere is defined as a constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross section, and placed one metre apart in a vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per metre of length.

Since conductors of infinite length and of negligible circular cross section cannot be manufactured in practice under real conditions the standard of current is reproduced by measuring the force of interaction of current-carrying coils (using the so-called current balance), and then the theoretical corrections taking into account the size and shape of conductors are introduced.

The unit of electric charge is a coulomb (C) which is a derived unit. **Coulomb** is defined as an electric charge passing per second through the cross section of a conductor carrying a constant current of one ampere.

4.27. Magnetic Flux

Suppose that in a uniform magnetic field of induction **B** there is a plane surface S oriented so that its normal **n** forms an angle α with the direction of **B** (Fig. 270). The **flux of magnetic induction,** or simply the **magnetic flux,** through the surface S is the quantity

$$\Phi = BS\cos\alpha.$$

If we draw the magnetic field lines with a density numerically equal to B, it can be seen from Fig. 270 that the flux Φ is

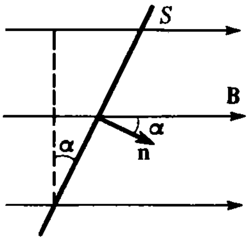


Fig. 270

numerically equal to the number of field lines piercing the surface S.

If the field is nonuniform and the surface is bent, the flux through the surface S is calculated by dividing the surface into such small elements ΔS that each of them can be treated as flat and the field in the vicinity of ΔS can be assumed to be uniform. Then the elementary flux through the surface element is $\Delta\Phi = B\Delta S \cos\alpha$, and the total flux is obtained by summing up elementary fluxes:

$$\Phi = \sum B \,\Delta S \cos\alpha.$$

Generally, the values of B and $\cos\alpha$ are different for different surface elements.

The SI unit of magnetic flux is a **weber** (W). The relation $\Phi = BS$ shows that

$$1 \text{ W} = 1 \text{ T} \times 1 \text{ m}^2.$$

4.28. Ammeter and Voltmeter

Ammeter is an instrument for measuring current, while **voltmeter** is an instrument for measuring voltage (potential difference). A high-sensitivity electrical measuring instrument intended for measuring small currents is called a **galvanometer.**

The most widely used are *moving-coil instruments.* Their operation principle is based on the action of the field of a permanent magnet on a loop with a current whose direction is indicated in Fig. 271. The loop is placed in a plane parallel to magnetic field

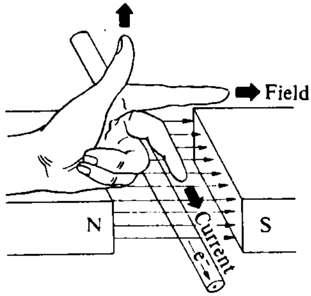


Fig. 271

lines. The field does not act on the sides of the loop, which are parallel to the field lines. The directions of forces $\mathbf{F}_1$ and $\mathbf{F}_2$ exerted by the field on the sides crossing the field is determined, for example, by the left-hand rule (see Sec. 4.24). The loop as a whole is acted upon by a couple $\mathbf{F}_1$, $\mathbf{F}_2$ that tends to rotate it.

In an *electrical measuring instrument* (Fig. 272), the loop is replaced by a coil of insulated wire, fixed to a rotating axis. The

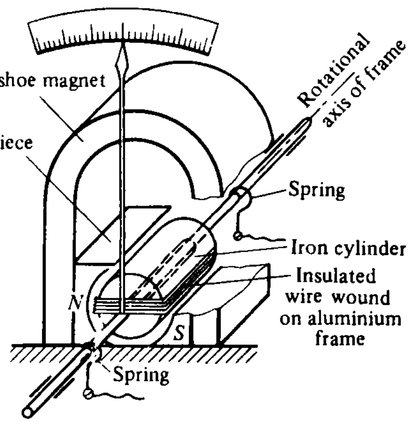


Fig. 272

current to be measured is fed to spiral springs that keep the coil in a certain position. The magnetic field is created by a horse-shoe magnet with two pole pieces. An iron cylinder fixed between them is intended to amplify the magnetic field. As a current is passed, the coil rotates together with the cylinder about its longitudinal axis until the torque of the couple $\mathbf{F}_1$, $\mathbf{F}_2$ is balanced by the counteracting torque of twisted springs. The stronger is the current, the larger the angle by which the coil turns. A pointer fixed to the coil axis indicates the current or voltage on the scale graduated accordingly. The moving-coil instruments provide a high accuracy but can be used only for direct current.

The operation of *moving-iron instruments* is based on the effect of pulling an iron core into a current-carrying coil. Such an instrument (Fig. 273) consists of a fixed coil and an iron plate that

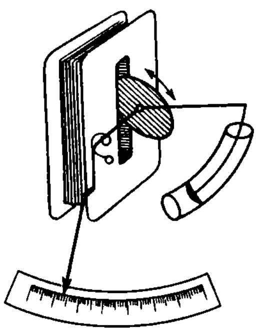


Fig. 273

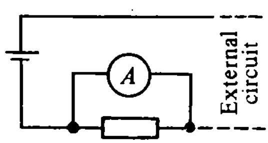


Fig. 274

can rotate about an axis to which a pointer and a spring balancing the plate are fixed. When an electric current of any direction is passed through the coil, the iron plate is pulled into it, turning about its axis and moving the pointer. To suppress the oscillations of the pointer, a damper is used. It consists of a cylinder with a piston which is connected to the iron plate. The moving-iron instruments are less accurate than the moving-coil instruments, but they have a simpler construction and can be used both for direct and alternating currents.

1. An **ammeter** is obtainted by connecting a *shunt* (resistor with a very low resistance) in parallel to the coil. The ammeter is connected in series with a circuit (Fig. 274).

If the resistance R_{sh} of the shunt is lower than the internal resistance R_c of the coil, say, by a factor of 99, i.e. $R_{sh} = R_c/99$, the total resistance of the ammeter can be found from the relation

$$\frac{1}{R_a} = \frac{1}{R_c} + \frac{1}{R_{sh}} = \frac{1}{R_c} + \frac{1}{R_c/99} = \frac{100}{R_c},$$

whence $R_a = R_c/100$. In other words, the total resistance of the ammeter amounts to 1/100 of the internal resistance of the coil, which itself is very low. Consequently, the ammeter connected to a circuit does not noticeably affect the current in it.

When the current splits at the points where the coil is connected, the currents in the parallel branches are inversely proportional to their resistances:

$$I_c/I_{sh} = R_{sh}R_c,$$

or (derived ratio)

$$I_c/(I_{sh} + I_c) = R_{sh}/(R_c + R_{sh}).$$

Since $I_{sh} + I_c = I$, $R_c + R_{sh} = R$, and $R_{sh} = R_c/99$ as before, we have

$$I_c/I = R_{sh}/(99R_{sh} + R_{sh}) = 1/100,$$

i.e. only 1/100 of the current passes through the instrument. If another shunt is connected, for example, such that $R_{sh} = R_c/999$, we obtain

$$I_c/I = R_{sh}/(999R_{sh} + R_{sh}) = 1/1000,$$

i.e. only 1/1000 of the current passes through the instrument.

In the second case, the scale factor is 10 times larger than in the first case. If the instrument is graduated for the first shunt, the reading of the galvanometer should be increased ten times when the second shunt is connected instead of the first shunt. Having a set of shunts with different resistances, we can considerably broaden the range of currents that can be measured with the help of a given instrument.

2. A **voltmeter** is obtained by connecting a very large *series resistance* to the coil. A voltmeter is connected in parallel to a circuit (Fig. 275). Since the voltmeter offers a high resistance to the

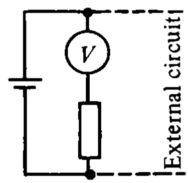


Fig. 275

current, the fraction of current splitting from the external circuit to the circuit of the instrument is very small. As a result, the potential difference of the external circuit is changed insignificantly by connecting the voltmeter.

As an ammeter, a voltmeter actually measures the current passing through it. But since the current is proportional to the voltage, $U = R_v I_v$, and the proportionality factor R_v is known (it is the resistance of the voltmeter), the voltmeter scale can be graduated in volts.

D. ELECTROMAGNETIC PHENOMENA

4.29. Electromagnetic Induction

The phenomenon of **electromagnetic induction** consists in that an e.m.f. is always induced in a closed loop when the magnetic flux piercing it changes. This induced e.m.f. causes the appearance of an **induced current.** An e.m.f. can also be induced in an unclosed conductor during its motion in a magnetic field so that the conductor crosses the magnetic field lines.

Electromagnetic induction was experimentally discovered by M. Faraday, in 1831. Faraday (as well as many other contemporary scientists) based his experiments on the assumption that since an electric current generates a magnetic field, the converse should also be true, i.e. a current can be created with the help of a magnetic field.

Faraday's experiments on electromagnetic induction can be divided into three groups.

1. A magnet is pulled into a coil connected to a galvanometer

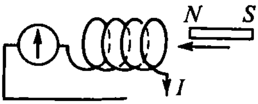


Fig. 276

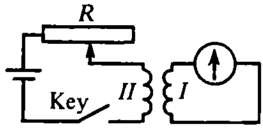


Fig. 277

(Fig. 276). As a result, the pointer of the galvanometer is deflected. As soon as the magnet is stopped, the pointer indicates zero. It is also possible to draw a coil onto a fixed magnet.

2. Coil *I* connected to a galvanometer is pulled on coil *II* connected to a d.c. circuit (Fig. 277). When the current in coil *II*

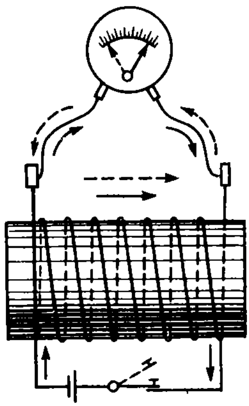


Fig. 278

changes, an induced current appears in coil *I*. The direction of this current depends on whether the current in coil *II* increases or decreases. The same is observed while connecting and disconnecting the circuit of coil *II* since connection and disconnection are particular cases of variation of the current.

3. The schematic diagram represented in Fig. 278 demonstrates self-induction (see Sec. 4.32). The current is passed through a coil connected to a d.c. source and a galvanometer, which is parallel to the source, in the direction shown by solid arrows. The pointer of the galvanometer is deflected to the right. When the circuit is disconnected, a strong short-time current of

the same direction as the current from the source appears in the coil. Since the subcircuit containing the source is disconnected, the current passes through the subcircuit containing the galvanometer in the opposite direction, as is shown by dashed arrows. This causes the deflection of the pointer to the left.

4.30. Induced Electromotive Force

It was established in Faraday's experiments that the **induced e.m.f.** is proportional to the rate of variation of the magnetic flux piercing the current loop:

$$\mathscr{E}_{\text{ind}} = -\Delta\Phi/\Delta t.$$

The minus sign in this formula indicates that the induced e.m.f. causes a current whose magnetic field counteracts the change in the magnetic flux. In other words, if $\Delta\Phi/\Delta t > 0$, the induced e.m.f. $\mathscr{E}_{\text{ind}} < 0$, and vice versa.

An e.m.f. is induced in a straight conductor of length l, moving at a constant velocity $\mathbf{v}$ perpendicular to the conductor in a

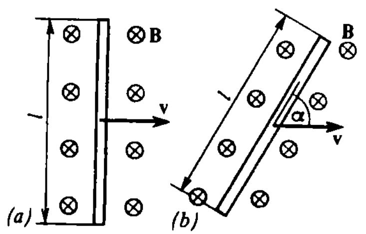


Fig. 279

uniform magnetic field of induction **B** (Fig. 279*a*; the magnetic field lines are directed normally to the plane of the figure away from us) is given by the formula

$$\mathscr{E}_{\text{ind}} = Blv.$$

If the axis of the conductor forms an angle $\alpha \neq 90°$ with the direction of the velocity (Fig. 279*b*), the number of field lines crossed by the conductor of length l is proportional to $l \sin \alpha$ and not to l. Then

$$\mathscr{E}_{\text{ind}} = Blv \sin \alpha.$$

Induced currents are also excited in solid massive conductors upon a variation of the magnetic field in them. This is observed, for example, in the cores of electromagnets. In this case, these currents are called the **Foucault** (or **eddy**) **currents.** In bodies having a high conductance, the Foucault currents may reach a large magnitude. As a result, a large amount of heat is liberated.

The Foucault currents are employed in metallurgy, for heating metallic parts in tubes to degas a metal, and so on. The Foucault currents are harmful in electromagnets and transformers since they cause energy expenditures for undesirable heating of cores. In order to reduce eddy currents, the cores are made of iron plates. This reduces the conductance of the core and, accordingly, the eddy currents in it.

4.31. Lenz's Law

The direction of induced current is determined by the law established by Lenz. **Lenz's law** states: *the induced current is always in such a direction as to oppose the change producing it.* For example, if a current is induced by a varying magnetic field, the direction of the current is such that its magnetic field *opposes* the change of the magnetic field inducing the current. The magnetic field of an induced current hampers the motion causing electromagnetic induction. When a magnet or a current-carrying conductor is displaced relative to a conducting loop, the current induced in the loop has such a direction that it opposes the motion due to which it is induced. This statement is illustrated by Fig. 280.

When a current-carrying solenoid approaches another solenoid connected to a galvanometer, the latter indicates a back current whose magnetic field opposes the approaching of the solenoids. When the current-carrying solenoid is removed from the solenoid connected to the galvanometer, the current of the same direction (cocurrent) appears, whose magnetic field opposes the removal of the first solenoid. When a magnet approaches a solenoid connected to a galvanometer, the current appearing in the solenoid has such a direction that the pole like that of the nearest end of the magnet is induced in the upper part of the

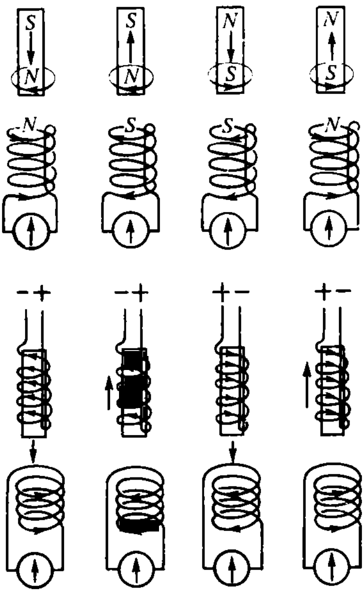


Fig. 280

solenoid. When the magnet is removed, the opposite pole is induced in this part. If we have two fixed conductors, one of which is connected to a circuit, the current of the opposite direction is induced in the second conductor when the circuit of the first conductor is closed or the current in it increases. When this circuit is disconnected or the current in it is reduced, the current of the same direction is induced in the second conductor.

The direction of a current induced in a moving conductor can be determined by the rule similar to that determining the direction of motion of a conductor in a magnetic field with the only difference that the right hand is used instead of the left hand. The *right-hand rule* states (Fig. 281): hold the thumb, first finger, and middle finger of the right hand at right angles to one another. Point the thumb in the direction of motion, the first finger along the lines of the field; then the second finger points in the direction of the induced current (i.e. the induced e.m.f. in the unclosed conductor).

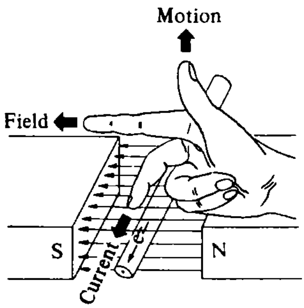


Fig. 281

4.32. Self-Induction. Inductance

When the magnetic field of a current in a conductor changes, an e.m.f. is induced not only in neighbouring conductors but also in the conductor itself since conductor is in the same varying magnetic field (see the diagram illustrating Faraday's third experiment in Sec. 4.29). The appearance of an e.m.f. in a conductor due to a change in the current flowing through it is called **self-induction.** The current induced in this conductor is called the **self-inductance current.** Due to self-induction, the current in a circuit attains its steady-state value not immediately after closing the circuit, but in a certain time. When the circuit is disconnected, the e.m.f. does not vanish immediately, and, as a result, a spark appears at the site of disconnection. If there is another closed circuit in the vicinity of the first circuit, an attenuating current continues to flow in it (Fig. 282).

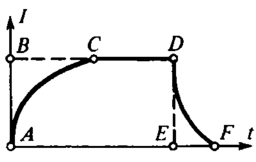


Fig. 282

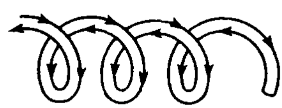


Fig. 283

Self-induction is quite undesirable in the instruments in which a rapid variation of current is required, since it hampers the change in current. A bifilar winding is used in the coils of such instruments (Fig. 283). The wire used in coils is folded before winding. Another method of preventing self-induction consists in winding half the wire in one direction and the other half in the opposite direction.

The e.m.f. of self-induction is determined by the rate of variation of the magnetic flux. Since the magnetic flux is proportional to the current, the e.m.f. of self-induction is defined by the following formula:

$$\mathcal{E} = -L(\Delta I/\Delta t). \tag{4.32.1}$$

The minus sign in this formula indicates that the e.m.f. has such a direction that it opposes the change in the current.

The proportionality factor L is called the **coefficient of self-inductance,** or simply the **inductance,** of the conductor. It is determined by the conductor geometry (its shape and size), as well as by the relative permeability of the medium surrounding the conductor.

The inductance of a straight conductor is not high. If the same conductor forms a loop, its inductance becomes much higher.

The inductance of a solenoid, viz. a very long coil with a one-layer winding (the expression "very long" means that the length l of the coil is much larger than its diameter d), is defined by the formula

$$L = \mu_0 \mu_r n^2 V,$$

where μ_r is the relative permeability of the solenoid core, n is the number of turns per unit length, and V is the solenoid volume equal to $\pi d^2 l/4$.

The SI unit of inductance is a **henry** (H). A conductor has the inductance of one henry if the change in the current by one ampere per second induces an e.m.f. of one volt in it. It follows from formula (4.32.1) that

$$1 \text{ H} = 1 \text{ V}\cdot\text{s/A}.$$

Problems with Solutions

215. A force $F = 5$ N acts on a straight conductor of length $l = 2$ m, carrying a current $I = 50$ A, and located in a uniform magnetic field at an angle $\alpha = 30°$ to the direction of the field lines. Find the magnetic induction and the strength of the magnetic field [6].

Solution. The current-carrying conductor in a magnetic field experiences the action of the force $F = BIl \sin\alpha$. Hence the magnetic induction $B = F/Il \sin\alpha = 0.1$ T. The magnetic field strength $H = B/\mu_0 \simeq 79.6$ kA/m.

216. A 40-cm conductor carrying a current $I = 2$ A moves in a uniform magnetic field of strength $H = 0.16 \times 10^7$ A/m over a distance $L = 50$ cm. The conductor is at an angle $\alpha = 60°$ to the direction of the field. The direction of motion is at right angles to the field and the current. Find the work of the current source.

Solution. The force acting on the conductor is at right angles to the directions of the field and the current, i.e. it acts in the direction of the motion:

$$F = BIl \cos(90° - \alpha) = \mu_0 HIl \sin\alpha.$$

The work $A = FL = \mu_0 HLIl \sin\alpha \simeq 0.35$ J.

217. Two parallel wires are suspended at a distance $l = 40$ cm from each other. A current $I = 200$ A flows through each wire in the same direction. Find the force of interaction of the wires between two neighbouring points of suspension separated by a distance $L = 100$ m.

Solution. Using Ampère's formula, we obtain

$$F = \mu_0 I_1 I_2 L/2\pi l = \mu_0 I^2 L/2\pi l = 2.0 \text{ N}.$$

218. A magnetic flux $\Phi = 1$ mWb pierces a strip having an area $S = 200$ cm^2 and placed in a vacuum at an angle $\alpha = 60°$ to the direction of a uniform magnetic field. Find the magnetic induction and the strength of the field.

Solution. The projection of the given surface area onto the plane normal to the direction of the field is $S_0 = S \sin\alpha$. The magnetic flux $\Phi = BS_0 = BS \sin\alpha$. Hence the magnetic induction $B = \Phi/S \sin\alpha = 0.058$ T. The magnetic field strength $H = B/\mu_0 = 4.6$ kA/m.

219. An electron flying in a vacuum into a uniform magnetic field of strength $H = 32$ kA/m at right angles to the direction of the magnetic field starts to move in a circle of radius $R = 2$ cm. Find the potential difference U which has accelerated the electron before it enters the magnetic field. The force of gravity acting on the electron should be neglected. The electron mass and charge are $m = 9.1 \times 10^{-31}$ kg and $e = 1.6 \times 10^{-19}$ C.

Solution. The electron in a magnetic field experiences the action of the Lorentz force

$$F_L = Bev \sin\alpha = \mu_0 Hev \sin\alpha,$$

which is directed at right angles to the velocity of the charged particle and the magnetic induction (Fig. 284). Consequently, $\sin\alpha = 1$ and $F_L = \mu_0 Hev$. This force imparts a centripetal acceleration $a = v^2/R$ to the electron. According to

[6] Here and below, unless stipulated especially, we will assume that interaction takes place in air, i.e. the relative permeability $\mu_r = 1$ and does not appear in formulas (*Editor's note*).

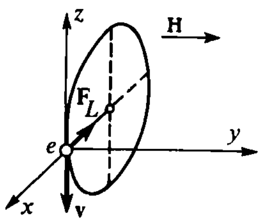


Fig. 284

Newton's second law, $F_L = ma$, i.e.

$$\mu_0 Hev = mv^2/R, \text{ whence } v = \mu_0 HeR/m. \tag{1}$$

On the other hand, the kinetic energy of the electron having passed the potential difference U is

$$mv^2/2 = Ue, \text{ whence } U = mv^2/2e. \tag{2}$$

Substituting formula (1) for the velocity v of the electron into this expression, we obtain $U = R^2\mu_0^2H^2e/2m = 14$ kV.

220. After having passed an accelerating potential difference $U = 10$ kV, an electron flies in a vacuum into a uniform magnetic field of strength $H = 79.6$ kA/m at an angle $\alpha = 53°$ to the field lines. Find the radius and the lead[7] of a helix in which the electron moves in the magnetic field. The force of gravity acting on the electron should be neglected. The electron mass and charge are $m = 9.1 \times 10^{-31}$ kg and $e = 1.6 \times 10^{-19}$ C.

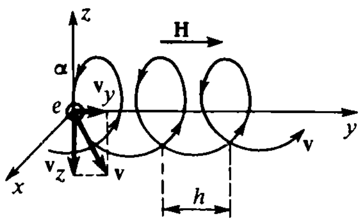


Fig. 285

Solution. We decompose the velocity of the electron entering the magnetic field in two directions: along the field lines ($\mathbf{v}_y$) and normally to them ($\mathbf{v}_z$) (Fig. 285). According to the principle of independent action of forces, the projection of the displacement onto the plane normal to the magnetic field lines is the same as the path of an electron flying normally to the field at the velocity $\mathbf{v}_z$, while along the

[7] The lead of a helix is the distance the electron covers along the axis during one turn.

field the electron moves with the velocity $\mathbf{v}_y$. For the first motion, we can write the following two equations:

(1) the condition that the Lorentz force creates a centripetal acceleration (see Eq. (1) in Problem 219):

$$\mu_0 Hev_z = mv_z^2/R, \text{ i.e. } \mu_0 He = mv_z/R; \tag{1}$$

(2) the condition that the kinetic energy of the electron is equal to the work done by the electric field forces before the electron enters the magnetic field (see Eq. (2) of Problem 219): $mv^2/2 = Ue$ or, since $v = v_z/\sin\alpha$,

$$Ue = mv_z^2/2\sin^2\alpha. \tag{2}$$

Dividing both sides of Eq. (2) by the squared sides of Eq. (1), we get

$$Ue/\mu_0^2 H^2 e^2 = mv_z^2 R^2/2m^2 v_z^2 \sin^2\alpha,$$

i.e.

$$U/\mu_0^2 H^2 e = R^2/2m \sin^2\alpha, \text{ whence } R(\sin\alpha/\mu_0 H)\sqrt{2mU/e} = 2.7 \text{ mm}.$$

Here $\sin\alpha = 4/5 = 0.8$ since $53°$ is an angle of the so-called Egyptian triangle, viz. a right triangle with the legs multiple of 3 and 4 and the hypotenuse multiple of 5.

The lead of the helix can be found from the following two equations:

$$2\pi R = v_z t, \; h = v_y t, \text{ whence } h = 2\pi R v_y/v_z.$$

But $v_y/v_z = \cot\alpha = \cot 53° = 3/4$. Consequently, $h = 1.5\pi R = 12.7$ mm.

221. A resistance $\Delta R = 1980\ \Omega$ is connected in series with a voltmeter, after which the scale division becomes $n = 100$ times larger. Find the resistance R_v of the voltmeter.

Solution. If the voltage in the circuit is U, the current through the voltmeter is $I_1 = U/R_v$. After the connection of the series resistance, the current through the voltmeter becomes $I_2 = U/(R_v + \Delta R)$. The deviation of the pointer in the second case must be n times less, i.e. n times smaller current passes through the voltmeter: $I_2/I_1 = 1/n$, or $R_v/(R_v + \Delta R) = 1/n$. We can write the proportion by addition and subtraction:

$$R_v(R_v + \Delta R - R_v) = 1/(n - 1), \text{ or } R_v/\Delta R = 1/(n - t),$$

whence $R_v = \Delta R/(n - 1) = 20\ \Omega$.

222. The resistance of an ammeter is $R = 13\ \Omega$ and its scale is graduated for a current $I = 100$ A. After an additional shunt has been connected, it becomes possible to measure the current $I' = 750$ A by this instrument. Find the resistance R_{sh} of the shunt.

Solution. After the additional shunt has been connected, the current $I = 100$ A passes through the ammeter (with the previous shunt) at the current $I' = 750$ A in the circuit, the current $I' - I = 650$ A passing through the additional shunt. These currents are inversely proportional to the resistances connected in parallel:

$$I/(I' - I) = R_{sh}/R, \text{ whence } R_{sh} = I/(I' - I)R = 2\ \Omega.$$

223. Find the potential difference appearing at the edges of an aeroplane wing during a horizontal flight with a velocity $v = 1200$ km/h if the wing span $l = 40.0$ m and the vertical component of the magnetic field of the Earth $H = 40$ A/m.

Solution. The potential difference at the wing edges is equal to the induced e.m.f.: $\varphi_2 - \varphi_1 = \mathcal{E}_{ind}$. We can assume to a sufficiently high degree of accuracy, that the e.m.f. induced in air is the same as the e.m.f. induced in a vacuum:

$$\varphi_2 - \varphi_1 = \mathcal{E}_{ind} = \mu_0 Hlv \sin \alpha = \mu_0 Hlv = 0.67 \text{ V}.$$

224. A straight conductor of length $l = 8$ m moves in a vertical magnetic field at an angle $\alpha = 30°$ to the horizontal with a velocity $v = 200$ m/s (Fig. 286). The

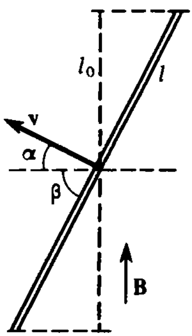


Fig. 286

angle between the longitudinal axis of the conductor and the horizontal component of its velocity is $\beta = 60°$. An e.m.f. $\mathcal{E}_{ind} = 2$ kV is induced in the conductor. Find the magnetic induction of the field and the work done by the magnetic field forces per minute if the resistance of the conductor is $R = 10\ \Omega$.

Solution. The induced e.m.f. $\mathcal{E}_{ind} = Bl_0 v \sin \varphi$, where $l_0 = l \sin \beta$ is the projection of the conductor length onto the direction of the magnetic field and $\varphi = \pi/2 - \alpha$ is the angle between the velocity and this direction. Since $\sin \varphi = \cos \alpha$, we have $\mathcal{E}_{ind} = Blv \sin \beta \cos \alpha$. Hence the magnetic induction $B = 1.7$ T. The work done by the magnetic field forces is equal to the energy lost in the conductor:

$$A = \mathcal{E}^2_{ind} t/R = 24 \text{ MJ}.$$

225. A coil of diameter $d = 20$ cm, containing $n = 50$ turns, is in a variable magnetic field. Find the rate of variation of the magnetic induction at the moment when the e.m.f. induced in the winding is $\mathcal{E}_{ind} = 100$ V.

Solution. Since the magnetic flux $\Phi = BS$, where $S =$ const is the area of a turn, $\Delta B/\Delta t = \Delta\Phi/\Delta t S = \mathcal{E}_{1\,ind}/S$, where $\mathcal{E}_{1\,ind}$ is the e.m.f. induced in one turn: $\mathcal{E}_{1\,ind} = \mathcal{E}_{ind}/n$. Consequently,

$$\Delta B/\Delta t = \mathcal{E}_{ind}/nS = 4\mathcal{E}_{ind}/\pi n d^2 = 63.7 \text{ T/s}.$$

226. An e.m.f. $\mathscr{E} = 20$ V is induced in a coil having an inductance $L = 0.4$ H. Find the average rate of variation of current in the coil.

Solution. The e.m.f. of self-induction is $\mathscr{E} = L\,\Delta I/\Delta t$, whence

$$\Delta I/\Delta t = \mathscr{E}/L, \quad \Delta I/\Delta t = 20\ \text{V}/0.4\ \text{H} = 50\ \text{A/s}.$$

227. In an a.c. circuit (Fig. 287), the external resistance $R = 100\ \Omega$ and the capacitance $C = 20\ \mu$F. At the moment when the e.m.f. of the current source is

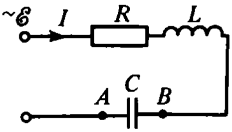


Fig. 287

$\mathscr{E} = 100$ V, the current $I = 0.5$ A, whose direction is shown in Fig. 287, increases at a rate $\Delta I/\Delta t = 200$ A/s, and the charge on the capacitor is $q = 0.6$ mC. Find the iductance L of the coil. The internal resistance of the power source and the resistance of the coil should be neglected. Assume that Ohm's law is valid for a slowly varying alternating current.

Solution. We equate the difference between the e.m.f. of the power source and the e.m.f. of self-induction to the potential drop in the circuit (the induced e.m.f. is directed opposite to the current):

$$\mathscr{E} - L\Delta I/\Delta t = IR + (\varphi_B - \varphi_A).$$

The potential drop across the capacitor is $\varphi_B - \varphi_A = q/C$. Consequently,

$$\mathscr{E} - L\Delta I/\Delta t = IR + q/C, \text{ or } L = (\mathscr{E} - IR - q/C)/\Delta I/\Delta t = 0.1\ \text{H}.$$

Exercises

215. A straight 20-cm long conductor carrying a current of 10 A is placed into a magnetic field at an angle of 30° to the direction of the field. The magnetic induction of the field is 5 T. Find the magnetic field strength and the force acting on the conductor.
Answer. 4 MA/m, 5 N.

216. When a conductor carrying a current of 10 A has been displaced in a magnetic field having an induction of 1.5 T over a distance of 0.25 m along the normal to the directions of the field and current, the work done by Ampère's forces is 0.38 J. The conductor is at 30° to the direction of the magnetic field. Find the length of the conductor.
Answer. 20 cm.

217. The force of interaction between two parallel conductors each 50-m long and separated by 20 cm is 1 N. Find the currents in the conductors if the current in one conductor is known to be twice stronger than the current in the other conductor.
Answer. 100 A, 200 A.

218. The magnetic induction of a uniform magnetic field is 0.5 T. Find the magnetic flux through an area element of 25 cm^2, located at 30° to the magnetic lines. Find the magnetic field strength.
Answer. 0.625 mWb, 400 kA/m.
219*. After having passed through a certain potential difference, an electron flies in a vacuum into a uniform magnetic field of strength 12.5 kA/m at right angles to the magnetic field lines. In the magnetic field, the electron moves in a circle of radius 114 mm. Find the potential difference that has accelerated the electron before it enters the magnetic field. The electron mass and charge are 9.1×10^{-31} kg and 1.6×10^{-19} C.
Answer. 28.4 kV.
220*. After having passed the potential difference of 71 kV, an electron flies in a vacuum into a magnetic field at 37° to the magnetic field lines and describes there a helix of radius 54 mm. Find the magnetic field strength and the lead of the helix in which the electron moves in the magnetic field. The electron mass and charge are 9.1×10^{-31} kg and 1.6×10^{-19} C.
Answer. 8 kA/m, 452 mm.
221. (a) What series resistance should be connected to a voltmeter having a resistance of 100 Ω to use it with a multiplying factor of 10?
Answer. 900 Ω.
(b) A voltmeter having a resistance of 1.1 kΩ and connected to a circuit indicates 220 V. After a certain resistance has been connected in series with the instrument, its reading remains unchanged. Find this resistance.
Answer. 110 Ω.
222. (a) The resistance of an ammeter whose scale is rated for 12 A is 0.2 Ω. Find the resistance of an additional shunt that should be used to measure a current up to 60 A.
Answer. 0.05 Ω.
(b) A copper conductor of length 10 cm and diameter 1.5 mm is connected in parallel to an ammeter having a resistance of 0.03 Ω. Find the current in the circuit if the ammeter indicates 0.4 A. The resistivity of copper is 0.017 $\mu\Omega \cdot m$.
Answer. 50.4 A.
223. A plane in a horizontal flight has a velocity of 900 km/h. The potential difference of 0.6 V appears at the edges of its wing. The vertical component of the magnetic field strength of the Earth is 80 A/m. What is the wing span?
Answer. 48 m.
224. A straight copper rod whose length is 5 m and the cross-sectional area is 0.85 cm^2 moves in a magnetic field at 37° to its lines with a velocity of 100 m/s. The angle between the longitudinal axis of the conductor and the horizontal component of its velocity is 53° (Fig. 288). The magnetic field power spent for this motion is 2300 kW. Find the magnetic induction of the field. The resistivity of copper is 0.017 $\mu\Omega \cdot m$.
Answer. 0.15 T.
225. A coil whose diameter is 0.40 m is in a variable magnetic field. As the magnetic induction of the field changes by 127.4 T during 2 s, an e.m.f. of 200 V is induced in the coil. What is the number of turns in the coil?
Answer. 25.

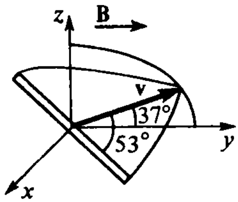


Fig. 288

Fig. 289

226. An e.m.f. of 25 V is induced in a coil when the current in it varies at a rate of 100 A/s. Find the inductance of the coil.
Answer. 0.25 H.
227. (a) In an a.c. subcircuit shown in Fig. 289, the resistance $R = 0.2\,\Omega$. At a certain instant, $\varphi_A - \varphi_B = 0.5$ V, $I = 0.5$ A, and $\Delta I/\Delta t = 8$ A/s. Find the inductance of the coil.
Answer. 0.05 H.
(b*) In a circuit shown in Fig. 290, the external resistance $R = 400\,\Omega$, the inductance of the coil is $L = 0.1$ H, and the capacitance of the capacitor is $C = 1\,\mu$F. At the moment when the e.m.f. of the source is 200 V, the current is 0.2 A and its variation rate is 500 A/s. The current is in the direction of the e.m.f. of the source. Find the charge of the capacitor at this moment.
Answer. 70 μC.

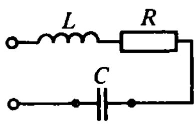


Fig. 290

5. OSCILLATIONS AND WAVES

5.1. Oscillatory Motion. Amplitude, Period, and Frequency of Oscillations

Oscillations are motions or processes which are periodic in time to a certain extent.

Examples of mechanical oscillations (*vibrations*) are: (1) the motion of a pendulum; (2) the motion of a spring fixed at one end, which is stretched or compressed and then released; (3) the motion of the leaf spring of a car caused by jolts due to roughness of a road; (4) vibrations of a factory chimney; (5) vibrations of sounding bodies (like tuning fork or string); (6) the vibration of the air columns in organ tubes and in wind instruments, and (7) the motion of a piston in an operating piston engine.

Examples of nonmechanical oscillations (vibrations) are: (1) alternating current in an electric circuit; (2) oscillations of current in an oscillatory circuit containing a capacitor and an induction coil, and (3) the variation of vectors **E** and **B** of electric and magnetic fields in an electromagnetic wave.

We shall consider only *mechanical vibrations* of a material point about a position of stable equilibrium, i.e. such a motion of the material point in which its trajectory, as well as the velocity and acceleration at any point, are periodically repeated.

The basic quantities characterizing a periodic motion are the amplitude, period, and frequency of vibrations.

The **amplitude** A **of vibration** is the maximum deviation of a vibrating body from the position of equilibrium.

The **period** T **of vibration** is the time between two successive identical positions passed by the body in the same direction.

The **frequency** f (or ν) **of vibration** is the number of vibrations of a body per second. The frequency is related to the period of vibrations as follows:

$$fT = 1, \text{ whence } f = 1/T, \ T = 1/f.$$

The unit of frequency is a **hertz** (Hz) which is equal to one vibration per second:

$$1 \text{ Hz} = 1 \text{ s}^{-1}.$$

Besides, multiple units of frequency are also used, such as **kilohertz** (kHz) and **megahertz** (MHz):

$$1 \text{ kHz} = 1000 \text{ Hz} = 10^3 \text{ Hz}, \ 1 \text{ MHz} = 1000 \text{ kHz} = 10^6 \text{ Hz}.$$

5.2. Harmonic Oscillations. Phase of Oscillation

A **harmonic oscillation** is a vibrational motion in which the coordinate of a body varies with time according to the law

$$x = A \sin \varphi = A \sin (\omega t + \varphi_0)$$

or

$$x = A \cos \varphi = A \cos (\omega t + \varphi_0).$$

The quantity $\varphi = \omega t + \varphi_0$ expressed in radians is called the phase of oscillation, φ_0 is the initial phase, ω is the **cyclic** (or **circular**) **frequency**, which is equal to frequency ν multiplied by 2π:

$$\omega = 2\pi\nu = 2\pi/T.$$

Harmonic oscillations can be represented geometrically in the form of the motion of the projection of a point moving in a circle with the cyclic frequency ω onto certain axes. Let us suppose that a material point M moves uniformly in a circle counterclockwise from a position M_0 (Fig. 291). As the point moves from M_0 to M_1 and from M_1 to M_0, its projections onto the coordinate axes, viz. points K and N, will be in oscillatory motion. Point K moves along the horizontal diameter of the circle (along the x-axis), while point N moves along the vertical diameter (along the y-axis). Both points move with the same amplitude equal to the radius A of the circle.

The position of point M at an arbitrary moment of time is determined by the angle φ. The position of its projection K onto

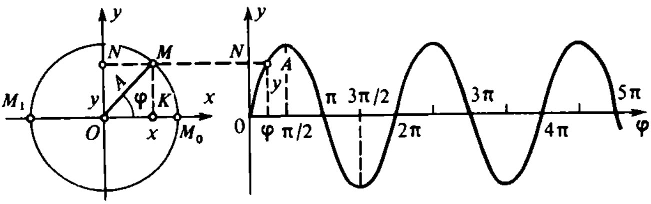


Fig. 291

the horizontal axis is determined by the abscissa $x = A \cos \varphi$, while the position of its projection N onto the vertical axis is determined by the ordinate $y = A \sin \varphi$. Consequently, points K and N are exactly in a harmonic motion.

Let us consider the motion of point N (Fig. 291, right). If we plot the angle φ on the abscissa axis and the quantity y on the ordinate axis, we obtain the dependence of y on φ. This curve is a sinusoid with the period 2π and amplitude A. The phase φ of oscillations determines the position of the oscillating point (displacement y) as well as the direction of its motion at a given instant of time. Similarly, we can plot the dependence of the angle φ on the x-coordinate (this is a cosinusoid).

If the motion of a material point starts from the position M_0, i.e. the initial phase $\varphi_0 = 0$, the angle $\varphi = \omega t$ and the coordinates of points K and N are connected through the following relations:

$$x = A \cos \omega t, \quad y = A \sin \omega t.$$

Since $\omega = 2\pi/T$, these equations of harmonic oscillations can be written in the form

$$x = A \cos (2\pi/T)t, \quad y = A \sin (2\pi/T)t.$$

It follows from the formula $\omega = 2\pi/T$ that the cyclic frequency is equal to the number of complete oscillations of a vibrating point performed during 2π seconds (while the ordinary frequency is equal to the number of complete oscillations per second).

If the motion of a material point in a circle starts not from the position M_0 but, say, from a position M (Fig. 292), the angle φ_0 which determines the position of the point and of its projections onto the coordinate axes differs from zero ($\varphi_0 \neq 0$). In this case,

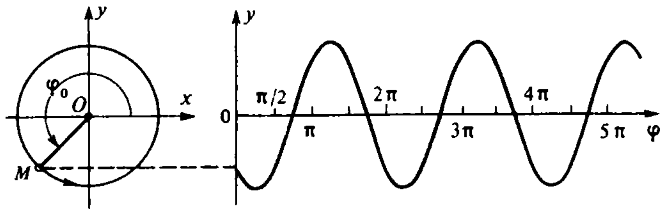


Fig. 292

the vibrational motion of points K and N is described by the formulas

$$x = A \cos(\omega t + \varphi_0),\ y = A \sin(\omega t + \varphi_0).$$

If follows from the equations of oscillatory motion that the *phase* can be defined as the argument of the function describing the oscillatory process.

5.3. Pendulum. Period of Oscillations of a Mathematical Pendulum

A **physical** (or **compound**) **pendulum** is a rigid body oscillating about a horizontal axis passing through the point of suspension located above its centre of mass.

A **mathematical** (or **simple**) **pendulum** is a material point attached to one end of a "nonstretchable" and "weightless" thread whose other end is fixed and swinging along an arc of the circle under the action of the force of gravity. A mathematical pendulum is an idealized mathematical system which is used to simplify the calculation of characteristics of a physical pendulum.

For any physical pendulum, we can choose a mathematical pendulum such that the frequency of its oscillations is equal to that for the physical pendulum. The length of such a mathematical pendulum is called the **reduced length** for the physical pendulum.

If the amplitude of a swing of a mathematical pendulum is small, we have a harmonic oscillation. The period and frequency of such oscillations of the mathematical pendulum are calculated as follows:

$$T = 2\pi \sqrt{l/g},\ f = (1/2\pi)\sqrt{g/l},$$

where l is the pendulum length and g is the acceleration of free fall. The period and frequency of harmonic oscillations of a mathematical pendulum do not depend on its mass.

The oscillatory motion of a pendulum occurs under the action of two forces: the force of gravity $m\mathbf{g}$ and the tension $\mathbf{T}$ of the thread. We decompose the force of gravity into two components, viz. the force $\mathbf{R}$ directed along the tangent to the path of the pendulum and the force $\mathbf{F}$ directed along the thread (Fig. 293). The

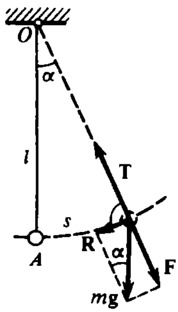


Fig. 293

resultant of forces $\mathbf{F}$ and $\mathbf{T}$ is directed to the centre of the circular path of the pendulum and imparts a centripetal acceleration to it. The magnitude of this force is

$$T - F = T - mg \cos \alpha.$$

The force $R = mg \sin \alpha$ imparts to the pendulum an acceleration tangential to the path and thus called the tangential acceleration. This force is directed to the position of equilibrium. For small angles of deviation of the pendulum, $\sin \alpha \approx \alpha$, and hence

$$R \approx mg\,\alpha = mgs/l.$$

The force imparting to a pendulum an acceleration in the direction to the equilibrium position in harmonic oscillation is proportional to the deviation of the pendulum from the equilibrium position.

5.4. Free and Forced Oscillations. Resonance

If an oscillatory system is brought out of equilibrium and left to itself, it oscillates at a frequency determined by the properties of the system.

Such oscillations of a system (body) are called **free oscillations.** When friction is present, free oscillations are damped. Their amplitude continuously decreases as a result of energy losses (Fig. 294).

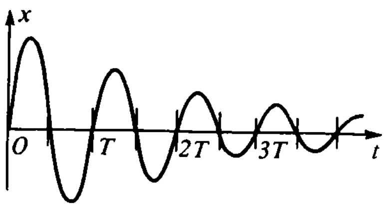


Fig. 294

If free mechanical oscillations occur without energy losses, they are called **natural oscillations,** and their frequency is termed **natural frequency.** In the presence of resistance, the period of oscillations increases, and if the resistance is sufficiently high (high viscosity of the medium), the motion becomes aperiodic. This effect is used for damping (suppressing) undesirable oscillations.

If a system oscillates under a periodically varying external effect, such oscillations are called **forced**. A periodic force causing mechanical oscillations is known as a **driving force.**

The frequency of forced mechanical oscillations (vibrations) coincides with the frequency of the driving force. For example, the foundation plate of a motor or a support to which the motor is fixed vibrates at a frequency determined by the rotational frequency of the motor, which does not depend on the size of the base or support. The membrane of a microphone performs forced vibrations whose frequency is determined by the frequency of the sound being transmitted, and so on.

Each oscillatory system has a frequency which is called a

resonance frequency. When the frequency of a driving force coincides with the resonance frequency, the amplitude of forced oscillations attains its maximum value. This phenomenon is known as **resonance**. The resonance frequency is lower than the natural frequency of an oscillatory system, the difference between the two frequencies being the larger, the stronger the resistance forces acting in the system. In the limiting case, when there is no resistance, the resonance frequency coincides with the natural frequency.

In engineering, resonance plays a positive role in some cases and a negative role in others.

The negative effect of resonance is manifested in the destruction of a vibrating body under the action of insignificant periodic forces. For example, the foundation of a motor can be destroyed if the motor operates at a frequency of rotation coinciding with the resonance frequency of the foundation. Cracks and destruction of some rods in the truss of an aeroplane located under its motor are observed when its piston engine operates at a certain rotational frequency. The small motor of a fan suspended from a ceiling may cause the crumbling of the ceiling plaster. A bridge can be destroyed by a military squadron keeping step while marching through it as a result of the coincidence of the resonance frequency of the bridge and the frequency of pace.

At the same time, resonance is used in engineering in the construction of **frequency meters**, viz. the instruments intended for measuring the frequency of vibrations. The "sensitive" part in such instruments is a resonator with a readily varying frequency. Resonance is widely used in acoustics, optics, and radio engineering.

In radio engineering, electric oscillations are amplified in the oscillatory circuit of a receiver by turning it to attain the resonance with electromagnetic waves sent by a transmitter (see Sec. 5.8).

5.5. Waves. Velocity and Wavelength

An oscillation caused at a certain point of a medium (air, water, spring, rope, etc.) propagates in the medium with a certain velocity determined by the properties of the medium. Mechanical oscillations propagate in elastic media, which is accompanied by the transmission of energy of a vibrating body from point to point in the elastic medium. Since oscillations propagate not instantaneously but with a finite velocity, different points in the medium have different phases of oscillation at the same instant.

The propagation of oscillations in space is called a **wave motion.** A **wave** is the process of propagation of oscillations in a medium, each particle of the medium oscillating about its equilibrium position.

Transverse waves are those in which the particles of a medium oscillate in planes *perpendicular* to the direction of propagation (for example, the waves on the surface of water and in a string).

Longitudinal waves are those in which the particles of a medium oscillate *along* the line of propagation of the wave (like waves in a compressed or stretched and then released steel spring, sonic (or acoustic) waves in air and in the tubes of wind instruments). The longitudinal oscillations propagate in an elastic medium, for instance, in air, in the form of consecutive condensations and rarefactions of the medium.

The **velocity v** of a wave is the velocity of propagation of a fixed value of the phase of oscillations (for example, the velocity of propagation of the regions of condensation and rarefaction in a sonic wave).

The **wavelength** λ is the distance between two nearest points in a wave in the same phase of oscillations (in particular, the distance between two neighbouring humps or sags in a transverse wave or between neighbouring condensations or rarefactions in a longitudinal wave).

During a period, an oscillatory process propagates over a distance equal to the wavelength. Therefore,

$$\lambda = vT$$

(T is the period of oscillation, i.e. the time required for one oscillation). Since $T = 1/f$, where f is the frequency of oscillations, we can also write

$$\lambda = v/f.$$

5.6. Sonic Waves

The sound we hear are the result of perceiving by our ear the oscillations of an elastic medium, in particular, air, at a frequency from 16 to 20 000 Hz.

Oscillations of an elastic medium at a frequency higher than the one perceived by ear are called **ultrasonic oscillations**, or **ultrasound**. Oscillations of an elastic medium at a frequency lower than the one perceived by ear are known as **infrasonic oscillations**, or **infrasound**.

The vibrations of a "sounding" body are transmitted to an elastic medium surrounding the body, in particular, to air, in the form of condensations and rarefactions of the elastic medium (air), which propagate in the form of waves in all directions. These are longitudinal sonic (acoustic) waves.

A sonic wave propagates in different media with different velocities. For example, $v_s = 340$ m/s in air at 15 °C, $v_s = 1450$ m/s in water, and it is still higher in metals, e.g. $v_s = 4900$ m/s in iron.

Acoustic waves (sound) are characterized by the sound intensity (loudness) and the pitch of sound.

The **sound intensity** is determined by the sound energy (energy of oscillations) passing per unit time through unit surface area of an elastic medium. The sound intensity is inversely proportional to the squared distance from the ear to the sound source, since the sound energy passing through unit surface area of a sphere surrounding the sound source varies in inverse proportion to the squared radius of this sphere.

The **pitch of sound** is determined by the frequency of sound vibrations. The higher the frequency, i.e. the shorter the sound wave, the higher the pitch.

The reflection of an elastic wave (and hence a sound wave as well) at the interface between two media is governed by the following law: *the angle of reflection is equal to the angle of incidence*. Reflected sound is weaker in comparison with the incident sound since along with reflection, a sound wave undergoes refraction, i.e. it goes into the second medium, carrying away a fraction of energy.

In a closed room, sound is multiply reflected from the walls, ceiling, and floor of the room, as well as from the objects contained in it. A human ear retains the perception of a sound during 0.1 s. If reflected sounds reach the ear with a smaller lag of time, they are not perceived as separate sounds but reinforce and prolong the main sound. If, however, a reflecting surface is at such a large distance from the point where the sound is received that the time interval between the moments of perception of the main and reflected sounds, as well as the intervals between the perception of consecutively reflected sounds, exceed 0.1 s, the reflected sounds are perceived separately as an echo.

The laws of mechanical resonance are applicable to sound vibrations which are a particular case of mechanical vibrations. The acoustic resonance is a special case of the mechanical resonance.

Problems with Solutions

228. A material point vibrates harmonically at a cyclic frequency $\omega = 0.5\ \text{s}^{-1}$. The amplitude of vibrations is $A = 80$ cm. What is the time dependence of displacement s of the material point if the vibrational motion starts from the equilibrium position?

Solution. Since the vibrations of the point are harmonic, we can write

$$s = A \sin(\omega t + \varphi_0) = A \sin((2\pi/T)t + \varphi_0),$$

where φ_0 is the initial phase of vibrations. When $t = 0$, the displacement $s = 0$. Consequently, $0 = A \sin(0 + \varphi_0)$, whence $\sin \varphi_0 = 0$ and $\varphi_0 = 0$ (or $2\pi n$, where n is an integer). Therefore, $s = A \sin \omega t = 0.80 \sin 0.5t$.

229. A material point is in harmonic vibrations with a period $T = 0.5$ s. The vibrational amplitude $A = 0.90$ m. The motion of the point starts from a position $x_0 = 30$ cm. Write the equation of motion for the point.

Solution. We write the equation of motion in the general form:

$$x = A \cos((2\pi/T)t + \varphi_0),$$

where φ_0 is the initial phase. When $t = 0$, the displacement $x_0 = A \cos \varphi_0$, whence $\cos \varphi_0 = 1/3$. The sought equation has the form

$$x = 0{,}90 \cos(4\pi t + \arccos(1/3)).$$

230. Two pendulums whose lengths differ by $l_1 - l_2 = 22$ cm oscillate at the same place so that one of them makes $N_1 = 30$ oscillations and the other $N_2 = 36$ oscillations during the same time. Find the lengths of the pendulums.

Solution. Obviously, the longer pendulum makes the smaller number of oscillations. The periods of oscillations of the pendulums are $T_1 = 2\pi\sqrt{l_1/g}$ and

$T_2 = 2\pi\sqrt{l_2/g}$. Their ratio is

$$T_1/T_2 = \sqrt{l_1/l_2}. \tag{1}$$

On the other hand, we have

$$T_1/T_2 = N_2/N_1. \tag{2}$$

Equating the right-hand sides of these expressions, we obtain

$$\sqrt{l_1/l_2} = N_2/N_1, \text{ or } l_1/l_2 = (N_2/N_1)^2 = 1.44,$$

i.e.

$$l_1 = 1.44 l_2. \tag{3}$$

Besides, from the condition of the problem we have

$$l_1 - l_2 = 22\,\text{cm}. \tag{4}$$

Solving Eqs. (3) and (4) together, we obtain $l_1 = 72$ cm and $l_2 = 50$ cm.

231. A clock with a simple pendulum is adjusted in Moscow. What will be the daily rate of this clock on the equator? The free-fall accelerations in Moscow and on the equator are $g_1 = 9.816\ \text{m/s}^2$ and $g_2 = 9.780\ \text{m/s}^2$.

Solution. The cyclic frequencies of oscillations of the pendulum in Moscow and on the equator are

$$\omega_1 = \sqrt{g_1/l} \text{ and } \omega_2 = \sqrt{g_2/l}.$$

The ratio of time indicated by the clock in a day in Moscow to the same time interval indicated on the equator is equal to the ratio of the frequencies of oscillations of the pendulum: $t_2/t_1 = \omega_2/\omega_1$. We write the proportion by addition and subtraction: $(t_2 - t_1)/t_1 = (\omega_2 - \omega_1)/\omega_1$, whence

$$t_2 - t_1 = (\omega_2/\omega_1 - 1)t_1 \text{ or } t_2 - t_1 = (\sqrt{g_2/g_1} - 1)t_1.$$

If $t_1 = 24 \times 3600$ s $= 86\,400$ s, then $t_2 - t_1 = -156$ s. The clock on the equator will lag behind by 2 min 36 s per day.

232. A seconds pendulum ($T_0 = 2$ s) is adjusted at a temperature $t_0 = 0$ °C. At what temperature t does the pendulum lag behind by $\Delta t = 0.5$ min per day if the temperature coefficient of linear expansion of the pendulum is $\alpha = 2 \times 10^{-5}$ K^{-1}?

Solution. The pendulum completes $n_0 = 43\,200$ oscillations per day. At the required temperature, the pendulum makes $\Delta n = 15$ oscillations less. The ratio of the periods of oscillations is inversely proportional to the number of oscillations made during the same time: $T/T_0 = 1.000\,34$. The period of oscillations of the pendulum varies in proportion to the square root of its length. Consequently, $l/l_0 = 1.000\,68$. But $l = l_0(1 + \alpha t)$, and hence

$$l/l_0 = 1 + \alpha t \text{ and } t = (l/l_0 - 1)/\alpha = 34\ °\text{C}.$$

233. A seconds pendulum ($T = 2$ s) consists of a metallic charged ball of mass $m = 16$ g, suspended by a thread whose mass can be neglected. When the pendulum is placed in the field of another ball bearing an unlike charge and located above the point of suspension at a large distance from the pendulum, the period of oscillations of the pendulum becomes $T' = 1.6$ s. Find the force of mutual attraction of the balls. The change in the separation of the balls and the change in the direction of the force of electric attraction should be neglected.

Solution. Since the electric charge is at a large distance from the pendulum, the force exerted on the charge of the pendulum by the other charge can be considered vertical and having a constant magnitude F. This force imparts an additional acceleration $a = F/m$ to the pendulum. Consequently, we must substitute $g' = g + a = g + F/m$ for g into the formula for the period of oscillations of the pendulum in the electric field. This gives two equations:

$$T = 2\pi\sqrt{l/g},\ T' = 2\pi\sqrt{l/g'} = 2\pi\sqrt{lm/(mg + F)}.$$

Dividing the first equation by the second, we obtain

$$T/T' = \sqrt{(mg + F)/mg},\ \text{or}\ F = mg[(T/T')^2 - 1] = 88\ \text{mN}.$$

234. A pendulum whose length $l = 1.2$ m is suspended from the ceiling of a carriage moving over a horizontal plane in a straight line with an acceleration $a = 2.2$ m/s^2. Determine the position of equilibrium and the period of oscillations of the pendulum.

Solution. Let us first consider the equilibrium position of the pendulum in the reference system fixed to the point of suspension. In this position, the resultant $\mathbf{R}$ of the force of gravity $m\mathbf{g}$ and the tension $\mathbf{T}$ of the thread must impart to the pendulum an acceleration equal to the acceleration $\mathbf{a}$ of the carriage. Therefore, $\mathbf{R} = m\mathbf{a}$, where m is the mass of the pendulum. Then the equilibrium angle α between the thread and the vertical is determined by the condition

$$\tan\alpha = R/mg = a/g,\ \tan\alpha = 0.2205.$$

For small angles, $\alpha \approx \tan\alpha = 0.2205$ rad $\simeq 13°$. The pendulum oscillates about the equilibrium position at an angle $\alpha = 13°$ to the vertical. The period of oscillations of the pendulum is the same as that for a pendulum of the same length, which oscillates under the action of the effective force

$$F = \sqrt{(mg)^2 + R^2}$$

with an acceleration

$$g' = F/m = \sqrt{(mg^2 + ma^2)/m} = \sqrt{g^2 + a^2}.$$

The period of oscillations of the pendulum is

$$T = 2\pi\sqrt{l/g'} = 2\pi\sqrt{l/\sqrt{g^2 + a^2}} = 0.69\pi\ \text{s}.$$

235. A 60-cm pendulum is fixed in an aeroplane taking off at an angle $\alpha = 30°$ to the horizontal with an acceleration $a = 4.0$ m/s^2. Determine the period of oscillations of the pendulum.

Solution. From the solutions of the two previous problems, we may conclude that if we call g' the reduced free-fall acceleration that should be substituted for g

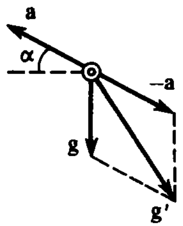


Fig. 295

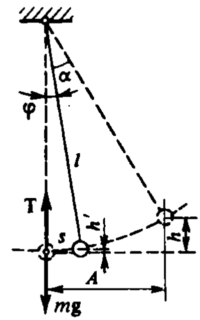


Fig. 296

into the formula of the period of oscillations of a simple pendulum, then $\mathbf{g}' = \mathbf{g} - \mathbf{a}$, i.e. vector $\mathbf{g}$ should be added to vector $-\mathbf{a}$. This gives (Fig. 295)

$$g'^2 = g^2 + a^2 - 2ag\cos(90^\circ + \alpha) = g^2 + a^2 + 2ag\sin\alpha.$$

The required period of oscillations is

$$T = 2\pi\sqrt{l/g'} = 1.4\ \text{s}.$$

236. A simple pendulum whose mass $m = 5$ kg, suspended by a thread of length $l = 0.80$ m, oscillates with an amplitude $A = 0.40$ m. Find the velocity v of the pendulum at the moment by which it has covered a distance $s = 10$ cm from the equilibrium position and the maximum tension T of its thread. The mass of the thread should be neglected.

Solution. The kinetic energy of the pendulum at an arbitrary position is equal to the difference in the potential energies of the pendulum in the extreme and in the given position: $mv^2/2 = mg(h - h')$, i.e.

$$v = \sqrt{2g(h - h')},$$

where $h = l - l\cos\alpha$, $h' = l - l\cos\varphi$ (Fig. 296). Hence

$$h - h' = l(\cos\varphi - \cos\alpha),$$

$$\sin\alpha = A/l = 0.40/0.80 = 0.5,\ \cos\alpha = 0.866,$$
$$\sin\varphi \approx \varphi = s/t = 0.10/0.80 = 0.12,\ \cos\varphi = 0.992.$$

Consequently, $v = 1.4$ m/s.

In the equilibrium position, two forces act on the pendulum, viz. the force of gravity $m\mathbf{g}$ and the tension $\mathbf{T}$ of the thread. The resultant of these forces imparts a centripetal acceleration $a = v^2/l$ to the pendulum. Therefore, by Newton's second law, we have $mv^2/l = T - mg$. Hence we can find the tension of the thread:

$$T = mg + mv^2/l = m(g + v^2/l).$$

The squared velocity of the pendulum can be found from the equality of the potential energy of the pendulum in the extreme position to its kinetic energy in the equilibrium position:

$$mv^2/2 = mgh,\ v^2 = 2gh = 2gl(1 - \cos\alpha).$$

Hence $T = mg[1 + 2(1 - \cos\alpha)] = 62.1$ N.

237. A wave whose frequency $f = 5$ Hz propagates in space with a velocity $v = 3$ m/s. Find the phase difference of the wave at two points separated in space by a distance $l = 20$ cm and located on the straight line coinciding with the direction of propagation of the wave.

Solution. The wavelength

$$\lambda = v/f = 0.60 \text{ m}.$$

Since the phase difference over the distance equal to the wavelength λ is 2π, the phase difference over the distance l is

$$\Delta\varphi = 2\pi l/\lambda = 2\pi/3.$$

238. The wheel of a siren has $k = 30$ holes and rotates at a frequency $n = 600$ rpm. Find the wavelength of the sound produced by the siren if the velocity of sound is $v = 340$ m/s.

Solution. The frequency of sound is equal to the number of coincidences of the holes in the inner and outer wheels of the siren per second: $f = kn$. The wavelength of sound is

$$\lambda = vT = v/f = v/kn = 1.13 \text{ m}.$$

Exercises

228. (a) A material point vibrates harmonically with a period of 0.8 s. The amplitude of vibrations is 1.5 m. Write the equation of motion for the point if it starts to move from the equilibrium position.

Answer. $x = 1.5 \sin 2.5\pi t$.

(b) A material point vibrates harmonically with an amplitude of 0.60 m, having started motion from the equilibrium position. Write the equation of motion for the point if $x = 0.30$ m after 1/3 of the period from the beginning of motion. What is the frequency of vibration of the point?

Answer. $x = 0.60 \sin 0.5\pi t$, $f = 0.25$ Hz.

229. (a) A material point vibrates harmonically at a frequency of 5 Hz. The amplitude of vibration is 50 cm. The motion starts from a position $x_0 = 30$ cm. Write the equation of motion for the point.

Answer. $x = 0.50 \sin (10\pi t + \arcsin 0.6)$.

(b*) A material point vibrates harmonically with a period of 2 s. The motion starts from a position $x_0 = 0.5A$, where A is the amplitude of vibration. After 1/2 of the period from the beginning of motion, the point is in a position $x = 0.50$ m. Write the equation of motion for the point.

Answer. $x = (\sqrt{3}/3) \sin (\pi t + \pi/6)$.

(c) A material point vibrates in accordance with the law $x =$

$4 \sin 2\pi(t + 0.25)$. Find the amplitude, period, frequency, cyclic frequency, and the initial phase of oscillations.
Answer. 4 cm, 1 s, 1 Hz, 2π s^{-1}, 0.5π.
230. One of two pendulums located at the same place completes during the same time 30 oscillations more than the other pendulum. The ratio of their lengths is 4/9. Find the number of oscillations made by each pendulum during this time.
Answer. 60, 90.
231. (a) A clock is adjusted at a latitude where the free-fall acceleration is 9.79 m/s^2. At what magnitude of the free-fall acceleration is the clock 1 min per day ahead?
Answer. 9.804 m/s^2.
(b) A pendulum consisting of a 106-cm wire and a metallic ball of 6 cm in diameter completes 100 oscillations during 3 min 29 s. Find the free-fall acceleration at the site where the experiment is carried out and the length of a seconds pendulum at this site.
Answer. 9.811 m/s^2, 99.8 cm.
232. (a) A clock with a simple pendulum is adjusted for the accurate rate. What is the daily rate of gaining if the length of the pendulum is reduced by 1%?
Answer. 7 min 12 s.
(b) A seconds pendulum is adjusted at a temperature of 20 °C. What is the change in the period of oscillations of the pendulum at 40 °C if the temperature coefficient of linear expansion of the pendulum is 2×10^{-5} K^{-1}?
Answer. The period increases by 1%.
233. (a) The ball of a simple pendulum of mass m is suspended by a weightless thread of length l. Find the period of oscillations of this pendulum placed in an electric field of strength **E** directed vertically upwards, after its ball has acquired a negative charge q.
Answer. $T = 2\pi \sqrt{lm/(gm + qE)}$.
(b) A 60-cm pendulum is suspended in a lift cabin descending with an acceleration of 2.45 m/s^2. Find the frequency of oscillations of the pendulum, assuming that the free-fall acceleration is 9.8 m/s^2.
Answer. 0.56 Hz.
(c) What is the acceleration of an ascending lift cabin if a 50-cm pendulum suspended in it completes 3 oscillations during 4 s?
Answer. 1.2 m/s^2.
(d) A clock with a simple pendulum is adjusted at the ground level. What is the daily rate of the clock at 25th floor? Assume that the average height of the storey is 3 m and the radius of the Earth is 6400 km.
Answer. The rate of losing is 1.01 s per day.
234*. (a) A 50-cm pendulum is suspended in the cabin of an aeroplane in a horizontal flight. Find the frequency of oscillations of the pendulum with 3.0-m/s^2 acceleration of the plane, assuming that the free-fall acceleration is 9.8 m/s^2.
Answer. 0.72 Hz.
(b) An 80-cm pendulum suspended in the cabin of an aeroplane which flies horizontally makes 4 oscillations in 7 s. What is the acceleration of the plane?
Answer. 3.2 m/s^2.

235*. (a) A pendulum of length l is mounted on a cart which moves uniformly up an inclined plane with a slope α. Find the period of oscillations of the pendulum.
Answer. $T = 2\pi\sqrt{l/(g\cos\alpha)}$.

(b) A pendulum of length l is mounted on a cart sliding down an inclined plane with a slope α. The coefficient of friction between the cart and the plane is $k < \tan\alpha$. Find the oscillation frequency of the pendulum.
Answer. $f = (1/2\pi)\sqrt{(g/l)(1 - k\sin 2\alpha + k^2\cos^2\alpha)}$.

236*. (a) A pendulum consisting of a 2-kg ball suspended by a weightless 1.0-m thread oscillates with an amplitude of 60 cm. Find the values of the kinetic energy of the pendulum when it passes through the equilibrium position and when it is displaced by 40 cm relative to it.
Answer. 3.92 J, 2.35 J.

(b) A pendulum consisting of a load suspended by a weightless 1.0-m thread oscillates with an amplitude of 50 cm. The maximum tension of the thread is 100 N. Find the mass of the load.
Answer. 8 kg.

(c) A pendulum consisting of a 5-kg load suspended by a weightless 1.0-m thread oscillates with an amplitude of 50 cm. Find the horizontal displacement of the pendulum from the equilibrium position at the moment when its velocity is 1.5 m/s.
Answer. 0.19 m.

237. (a) Find the velocity of propagation of sound in a material in which vibrations with a period of 0.01 s cause a sound whose wavelength is 10.0 m.
Answer. 1 km/s.

(b) A sound wave having a period of 0.01 s propagates in air. Determine the wavelength and the phase difference of two points separated by 1.70 m from each other and lying on the same straight line with the source of the wave. The velocity of sound in air should be taken as 340 m/s.
Answer. 3.40 m, π.

(c) A wave having a frequency of 10 Hz propagates in a certain medium so that the phase difference of two points separated by 100 cm from each other and lying on the same straight line with the source of vibrations is $\pi/4$. Find the velocity of propagation of the wave in this medium.
Answer. 80 m/s.

238*. (a) A siren wheel rotating at a frequency of 510 rpm generates a sound at a wavelength of 2 m. What is the number of holes in the siren wheel?
Answer. 20.

(b) What is the frequency of rotation of a siren wheel having 40 holes and generating a sound at a wavelength of 0.5 m?
Answer. 1020 rpm.

5.7. Electromagnetic Oscillations and Waves

Variable electric and magnetic fields cannot exist separately from each other since an electric field is excited in a space where there exists a variable magnetic field, and vice versa. Variable electric and magnetic fields produce each other and form a unique **electromagnetic field.**

A variable electric field which is a constituent of electromagnetic field differs from an electrostatic field in that its lines are closed like magnetic field lines. The fields whose lines are closed are called **vortex fields.**

A simultaneous periodic variation of coupled electric and magnetic fields is called **electromagnetic oscillations.**

The property of variable electric and magnetic fields to excite each other leads to the propagation of electromagnetic field in space. An alternating current I creates around a conductor a variable magnetic field with induction **B**, which in turn causes a variable electric field of strength **E**. The latter again creates a variable magnetic field, and so on (Fig. 297). As a result, the electromagnetic field propagates in space.

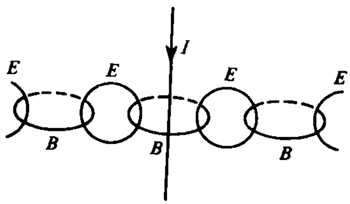


Fig. 297

An electromagnetic field propagating in space forms an electromagnetic wave. Vectors **E** and **B** of electric and magnetic fields are at right angles to each other and to the direction of propagation of the wave. Consequently, electromagnetic waves are transverse. A harmonic electromagnetic wave is graphically represented by two sinusoids lying in mutually perpendicular planes. Figure 298 shows that the electric field strength and magnetic induction coincide in phase.

J. Maxwell was the first to theoretically investigate and predict the properties of electromagnetic waves, which were then experimentally investigated by H. Hertz.

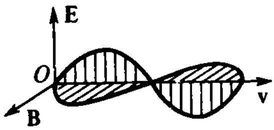


Fig. 298

5.8. Oscillatory Circuit

An **oscillatory circuit** is a closed circuit having a capacitance C and an inductance L. Figure 299 shows a simple closed oscillatory circuit consisting of a capacitor and an inductance coil. If we charge the capacitor and then connect it to the inductance coil, a rapidly varying current appears in the circuit, whose amplitude decreases with time as a result of energy losses associated with heating the wires of the circuit (Fig. 300). These are free damped oscillations.

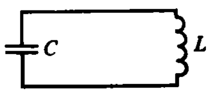


Fig. 299

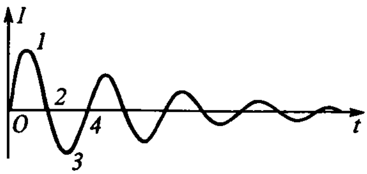


Fig. 300

The origin of these oscillations can be explained as follows (Fig. 301). In position *0*, the capacitor is uncharged. In position *1*, a charge is supplied to the capacitor, which creates an electric field in the space between the capacitor plates. Since the plates are connected by a conductor, the capacitor immediately starts to discharge through the coil. The motion of charges forms an electric current which is accompanied by a magnetic field induced in the coil and opposing the increase in the current. As a result, at the moment the capacitor is discharged, the current has not yet attained its maximum value (position *2*). At the moment the capacitor is discharged, the current would have to cease.

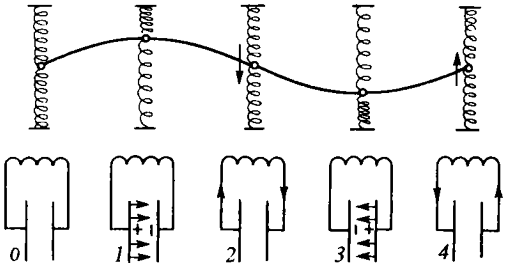


Fig. 301

However, the decreasing magnetic field of the coil opposes the ceasing of the current. As a result, a damped current continues to flow in the circuit, and the capacitor plates acquire a charge opposite in sign to the initial charge (position *3*). At the next moment, the capacitor starts to discharge in the reverse direction until the capacitor charge vanishes (position *4*).

In Fig. 300, the numbers from *1* to *4* mark different moments corresponding to the first oscillation in Fig. 301. Position *4* is the initial position (*0*) for the next oscillation, and so on.

In an oscillatory circuit, the mutual conversion of the energies of electric and magnetic fields is observed. As the capacitor discharges (interval *1-2*), the energy of the electric field of the capacitor is transformed into the energy of the magnetic field of the coil. The subsequent recharging of the capacitor (interval *2-3*) is due to the transformation of the energy of the magnetic field of the coil into the energy of the electric field of the capacitor, and so on.

The top of Fig. 301 represents a mechanical analog for the processes occurring in an oscillatory circuit. Here the electric energy of the capacitor corresponds to the potential energy of the deformed spring, while the magnetic energy of the coil corresponds to the kinetic energy of the load.

The mutual conversion of the energies of electric and magnetic fields is accompanied by energy losses for heating the conductors (in the mechanical model, this corresponds to energy losses due to friction). As a result of energy losses, the oscillations of an alternating current in a circuit with a singly charged capacitor damp.

In other words, the amplitude of each oscillation of alternating current is somewhat smaller than the amplitude of the preceding oscillation. The die-away time is the shorter, the higher the (ohmic) resistance of the circuit since in this case the energy losses due to heating are higher.

The period of natural oscillations of an oscillatory circuit depends on the inductance and capacitance of the circuit and is defined by the Thomson formula

$$T = 2\pi \sqrt{LC}.$$

If an a.c. generator is connected to an oscillatory circuit, its e.m.f. causes forced oscillations of the alternating current at a frequency equal to the frequency of the generator. In the forced oscillations, energy is continuously supplied to the circuit, due to which oscillations in it are undamped. The amplitude of forced oscillations is determined by the ratio of the frequencies of the generator and the oscillatory circuit. As the frequency of the external e.m.f. (generator) approaches the frequency of natural oscillations of the circuit, the amplitude sharply increases. This phenomenon is known as resonance.

Problems with Solutions

239. In an oscillatory circuit, two capacitors are connected in parallel. One of them has a capacitance $C_1 = 10^3$ pF, while the capacitance C_2 of the other (trimmer) capacitor varies between 100 pF and 1000 pF. The inductance of the circuit is $L = 1$ mH. Find the range of natural frequencies of the oscillatory circuit.

Solution. The total capacitance of the capacitors varies in the interval 1.1 nF $\leqslant C \leqslant$ 2nF. The natural frequency of the oscillatory circuit is $f = 1/(2\pi\sqrt{LC})$. Substituting the values of capacitances, we obtain the frequency range for the oscillatory circuit: 0.11 MHz $\leqslant f \leqslant$ 0.15 MHz.

240. The frequency of an oscillatory circuit varies in the interval 200 Hz $\leqslant f \leqslant$ 400 Hz. The capacitance of the capacitor is $C = 5\ \mu$F. Find the inductance of the coil in the circuit.

Solution. From the expression $f = 1/(2\pi\sqrt{LC})$, we obtain $L = 1/(4\pi^2 C f^2)$. Substituting the numerical values, we obtain

$$L = 0.128 \text{ H } (f = 200 \text{ Hz}),\ L = 0.032\ H\ (f = 400 \text{ Hz}).$$

Consequently, the inductance of the coil varies in the limits 32 mH $\leqslant L \leqslant$ 128 mH.

Exercises

239. (a) Find the frequency of oscillations in a circuit whose capacitance is 0.55 nF and inductance is 1 mH.
Answer. 0.21 MHz.

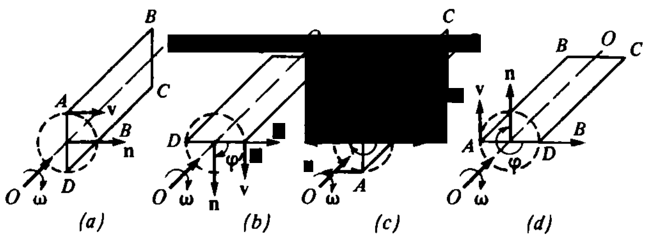
(b*) When the capacitor of an oscillatory circuit is connected in parallel to a capacitor having twice as large capacitance, the oscillation frequency in the circuit decreases by 300 Hz. Find the initial oscillation frequency of the circuit.
Answer. 710 Hz.

240. The inductance of the coil in an oscillatory circuit varies between 50 mH and 100 mH. The capacitance of the circuit is 20 μF. Find the frequency range of the circuit.
Answer. From 113 Hz to 159 Hz.

5.9. Alternating Current. A.C. Generator

An **alternating current** is a current whose magnitude and direction vary periodically.

The principle of obtaining an alternating current is as follows (Fig. 302). A current loop $ABCD$ rotates in a magnetic field with a



constant angular velocity $\omega = 2\pi/T$, where T is the time of one complete turn of the loop. The magnetic flux through the loop varies periodically. The rate of variation of the magnetic flux is proportional to the angular velocity of rotation of the loop and to the sine of the angle φ between a magnetic field line and the normal to the plane of the loop. An e.m.f. and current induced in the loop vary in proportion to sin φ.

The angle $\varphi = \omega t$. We shall measure time t from the moment when the loop is in the vertical position (Fig. 302a and c), i.e.

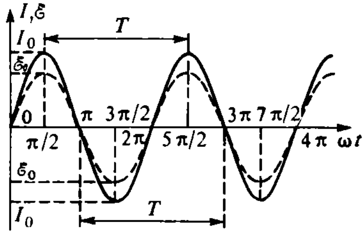


Fig. 303

$\varphi = 0$. Then

$$\mathscr{E} \propto \sin \omega t, \ I \propto \sin \omega t.$$

The e.m.f. and current induced in the loop attain their maximum values when the loop is in the horizontal position (Fig. 302*b* and *d*), i.e. when $\varphi = \omega t = (2k + 1)\pi/2$ and $\sin \varphi = \sin \omega t = \pm 1$. Consequently, when a loop is rotated uniformly in a uniform magnetic field, alternating e.m.f. and current are induced in it, which vary according to the sine law (Fig. 303):

$$\mathscr{E} = \mathscr{E}_0 \sin \omega t, \ I = I_0 \sin \omega t,$$

where $\mathscr{E}_0 = \mathscr{E}_{max}$ and $I_0 = I_{max}$ are the e.m.f. and current induced at $\varphi = \omega t = \pi/2$, i.e. when $\sin \omega t = 1$. The values $\mathscr{E}_0$ and I_0 of the e.m.f. and current are called the amplitude values. An alternating current varying in accordance with the sine law is called a

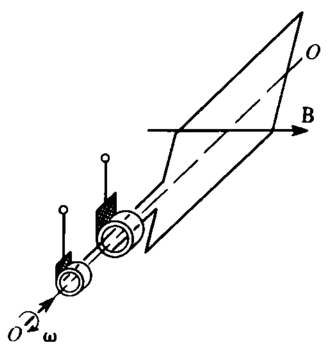


Fig. 304

sinusoidal current. The oscillations of the induced e.m.f. and alternating current are harmonic oscillations.

The device for obtaining an alternating current is called an **a.c. generator.** In an electric generator, the mechanical energy is transformed into the electric energy. A simple a.c. generator can be obtained by cutting the current loop shown in Fig. 302 and connecting its ends to the terminals of an external circuit with the help of sliding contacts made, for example, in the form of two separate rings brought in contact with two brushes connected to the terminals of the external circuit (Fig. 304). In actual practice, a very large numbers of series-connected loops are rotated in a generator in order to increase the induced e.m.f.

5.10. Period and Frequency of Alternating Current. Effective Current and Voltage

The **period** T of an laternating current is the time interval during which the current and voltage complete one oscillation:

$$T = 2\pi/\omega.$$

The **frequency** ν of an alternating current is the number of a.c. periods per unit time. The frequency is the reciprocal to the period:

$$\nu = 1/\mathrm{T}.$$

In the Soviet Union, the current having a frequency of 50 periods per second (i.e. 50 oscillations per second: $\nu = 50$ Hz) is normally used in electrical engineering. The quantity $\omega = 2\pi/T = 2\pi\nu$ is called the **cyclic frequency** of an alternating current.

Along with a simple sinusoidal current characterized by a single e.m.f. and called a **single-phase current,** the three-phase current is widely used in engineering. It was obtained for the first time in 1890.

A **three-phase current** is a system of three single-phase currents having the same amplitude and frequency and shifted in phase by 1/3 of the period (Fig. 305). The phase shift of the three currents is obtained due to three loops (windings) with the planes at 120° relative to one another, which are used instead of one rotating loop in a generator.

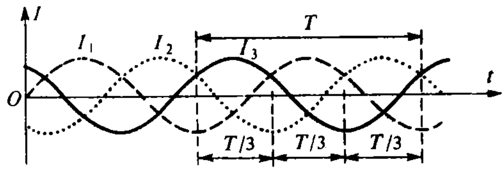


Fig. 305

The intensity of an alternating current can be judged from its thermal effect since the latter does not depend on the direction of current. The thermal effect of an alternating current is used for determining the effective current.

The **effective current** and **voltage** of an alternating sinusoidal current are the current and voltage of a direct current that has the same thermal effect as the given alternating current. The effective current and voltage of an alternating current amount to $1/\sqrt{2}$ of their maximum values:

$$I_{\text{eff}} = I_0/\sqrt{2}, \quad \mathscr{E}_{\text{eff}} = \mathscr{E}_0/\sqrt{2}.$$

5.11. Transmission and Distribution of Electric Energy

A transmission of electric energy is accompanied by energy losses in conductors mainly due to heating the wires (Joule heat):

$$W_{\text{loss}} = kI^2Rt,$$

where R is the resistance of the wires. This formula points to two possible ways of reducing heat losses in conductors: (a) to reduce the resistance of the wires and (b) to use as weak currents as possible.

The first way is not advantageous since it involves an increase in the cross-sectional area of the wire, i.e. an increase in its weight. In actual practice, heat losses are effectively reduced by decreasing current since heat losses are proportional to the squared current. In order to preserve the power $N = UI$ being transmitted, the voltage in the transmission line should be accordingly increased. The voltage supplied to the consumer should be reduced to a value which is admissible from the point of view of safety requirements for domestic appliances.

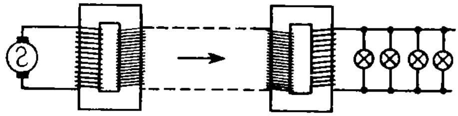


Fig. 306

Voltage can be increased or reduced with the help of transformers (Fig. 306). The electric energy is supplied by a generator to a line at a voltage up to 1 kV. Electric energy from low-power stations (up to 1 MW) can be transmitted without considerable losses only over short distances at such a voltage. Energy transmission over long distances is carried out at a voltage from 110 kV to 220 kV, which is ensured by step-up transformer substations. For example, the voltage of the transmission lines from the Kuibyshev and Volgograd state power plants to Moscow is 400 kV. At present, a unique system of high-voltage transmission lines is being developed in the USSR, which will combine all large-scale power plants and make it possible to introduce a centralized automatic control of production and distribution of electric energy and its transmission over very long distances (up to several thousand kilometres) at a voltage up to 1.5 MV.

5.12. Transformer

A **transformer** is an instrument intended for transforming the a.c. voltage. A step-up transformer transforms a current so that its voltage increases (with decreasing current), while a step-down transformer reduces voltage.

A transformer consists of two coils of an insulated wire with a common core made of separate soft-iron plates (Fig. 307). An alternating current being transformed is passed through the first winding (primary). This current creates a variable magnetic flux in the core, which pierces the two windings. As a result, an e.m.f. of self-induction appears in each turn of the primary, which is equal to $-\Delta\Phi/\Delta t$. In each turn of the secondary, the induced e.m.f. is also equal to $-\Delta\Phi/\Delta t$. Consequently, the ratio of the e.m.f.s induced in the windings is equal to the ratio of the numbers of turns in them:

$$\mathscr{E}_1/\mathscr{E}_2 = w_1/w_2 = K.$$

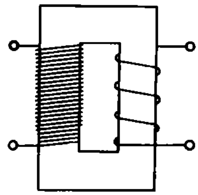


Fig. 307

The quantity K is called the **trasformation coefficient.** Its value is determined in *no-load operation* of a transformer, when its secondary circuit is disconnected. This can be explained by the following arguments.

During no-load operation, when the terminals of the secondary are not connected to a load, the so-called **no-load current** flows in the primary, whose intensity I is very low (about 5% of the rated current)[1]. As a result, the voltage drop in the primary is small, and the e.m.f. of self-induction in the primary is equal to the voltage across the terminals ($\mathscr{E}_1 = U_1$). The secondary circuit is disconnected. Hence there is no current in the circuit, and the voltage across its terminals is equal to the e.m.f. induced in it ($U_2 = \mathscr{E}_2$). Therefore,

$$K = U_1/U_2.$$

The *transformation coefficient of a transformer* is the ratio of the voltages across the terminals of the primary and secondary windings in no-load operation. In a step-up transformer $K < 1$ (accordingly, $w_2 > w_1$), while in a step-down transformer $K > 1$. The same device can be used as a step-up or a step-down transformer depending on the winding used as a primary.

During the operation of a transformer, energy is lost due to heating the transformer windings, dissipation of magnetic flux, eddy currents in the core, and magnetization reversal in it. The

[1] For an ideal transformer (with zero losses in the core), it follows from the energy conservation law that in no-load operation $UI = I^2R$, where UI is the power supplied to the primary and I^2R are the heat losses in this winding. Hence, it follows that $I = 0$ in no-load operation.

following measures are taken to reduce energy losses in transformers: (a) the lower-voltage winding is made of a thick wire since strong current flows through it; (b) the core is made closed since with such a shape the dissipation of the magnetic flux is minimum and almost the entire magnetic flux of the primary is concentrated in the core and pierces the secondary, and (c) the core is built up from laminations. The laminated structure of the core reduces eddy (Foucault) currents. Due to these measures, the efficiency of modern transformers reaches 95-99%.

In the on-load operation of a transformer, the energy is transferred from the primary circuit to the secondary. According to the law of conservation and transformation of energy, the power of the current in the secondary is equal to the power of the primary minus the power losses in the transformer: $N_2 = N_1 - \Delta N$. Since the transformer efficiency is close to unity, the power losses in transformers can be neglected in approximate calculations and it can be assumed that $N_2 \approx N_1$, or $\mathcal{E}_2 I_2 \approx \mathcal{E}_1 I_1$. Hence we have

$$I_2/I_1 \approx \mathcal{E}_1/\mathcal{E}_2 = w_1/w_2 = K.$$

The ratio of the currents in the secondary and primary is inversely proportional to the e.m.f.s in the windings and is equal to the transformation coefficient.

For many practical problems, the voltage across the terminals of windings of a transformer can be calculated to a sufficiently high degree of accuracy, by taking into account only the voltage drop due to the resistance of the windings. The voltage across the terminals of the secondary is

$$U_2 = \mathcal{E}_2 - I_2 R_2,$$

while the voltage supplied to the terminals of the primary is

$$U_1 \approx \mathcal{E}_1 + I_1 R_1.$$

Concluding the section, we can also mention autotransformers the armature winding, i.e. retaining the induced e.m.f. The winding of electromagnet poles is called the excitation winding. In a 308).

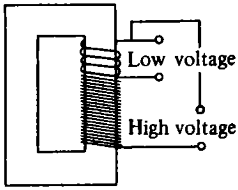


Fig. 308

5.13. D.C. Generator

D.C. generators with a magnetic field created by a permanent magnet are called **magnetos.** Generators of this type have a low power and are used for ignition in internal combustion engines, for ringing calls in intercommunication telephone lines, and generally everywhere when small currents are required.

D.c. generators with a magnetic field created by an electromagnets are sometimes called **dynamos.** A dynamo consists of a stationary inductance coil, rotating armature with a commutator, and a casing.

An **inductor** creating the magnetic field is a two-pole (Fig. 309*a*), four-pole (Fig. 309*b*) or multipole electromagnet.

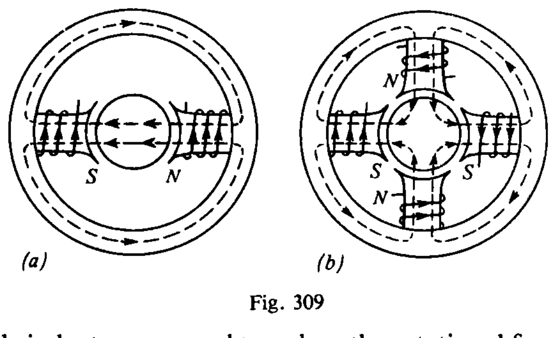


Fig. 309

Multipole inductors are used to reduce the rotational frequency of a machine, retaining a sufficiently high frequency of the current in the armature winding, i.e. retaining the induced e.m.f. The winding of electromagnet poles is called the excitation winding. In a multipole dynamo the poles of the electromagnet alternate.

Power is supplied to the electromagnet of a dynamo from the winding of its armature, i.e. by the current generated by the machine itself. At the initial moment of starting a d.c. generator, when there is no current in the armature winding, it is induced due to a residual magnetization of magnet poles. As soon as the armature is set in motion, its winding crosses a weak magnetic field caused by the residual magnetization, and a small e.m.f. is induced in it. This e.m.f. generates a small current in the windings of the armature and magnet poles, which somewhat increases the magnetic field of the poles and the magnetic flux. This, in turn, increases the induced e.m.f. and so on.

Depending on the way of connection of the excitation winding to the armature winding, three types of generators are distinguished (Fig. 310).

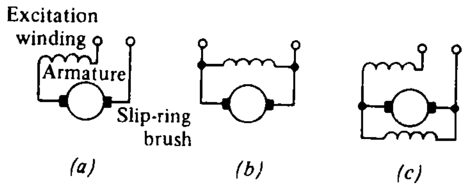


Fig. 310

1. The *series generator,* in which the excitation winding is connected in series with the armature winding (Fig. 310*a*). This type of connection is used rather seldom. A series generator cannot operate with a disconnected circuit.

2. The *shunt generator* with the parallel connection of windings (Fig. 310*b*). A field rheostat is connected in series with the armature winding to excite and control the current in the winding.

3. The *compound generator,* in which the excitation winding consists of two parts: one winding is connected in series with the winding of the armature, while the other is connected in parallel to it (Fig. 310*c*).

An **armature** is a rotating winding. In modern dynamos, drum armatures are used. A drum armature consists of an iron core made in the form of a cylinder with slots along generatrices, in which the sections of the winding are mounted. The armature core is made of stamped sheets of soft iron insulated by a varnish or paper separators in order to reduce eddy currents.

A **commutator** (Fig. 311) is intended for the rectification of current. It is mounted on the common shaft with the armature

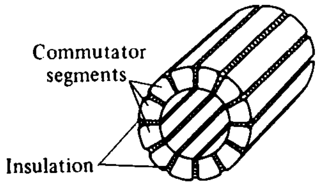


Fig. 311

and rotates with it. To explain the operational principle of the commutator of a dynamo, we consider again the diagram of generation of an alternating current in a conducting frame (see Sec. 5.9). If we make such a contact between the frame (shown in Fig. 302) and an external circuit that the ends of the frame are switched over when the direction of the e.m.f. is reversed, the current in the external circuit preserves its direction during the rotation of the frame, although its magnitude varies continuously (pulsating current). The switching over is attained, for example, by using isolated half-rings connected to the ends of the frame. The brushes connected to the terminals of the external circuit slide over the surface of these half-rings (Fig. 312), and a pulsating current (Fig. 313*a*) is generated in the circuit.

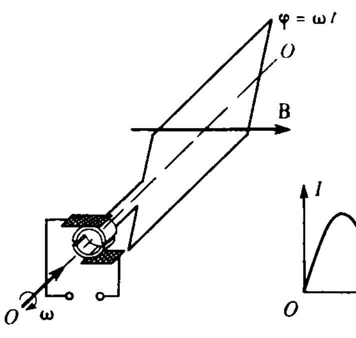


Fig. 312

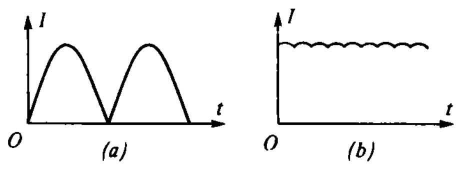


Fig. 313

Current pulses in a d.c. generator can be smoothed by grouping the winding into several sections arranged at an angle to one another. The phases of the current induced in different sections are different, and the pulses in the resultant current are smoothed the better, the larger the number of sections used. At a very large number of sections, the current is close to direct (Fig. 313*b*).

The current is removed from the commutator with the help of brushes made in the form of carbon plates with a small admixture of copper and mounted in the brush holders of the machine casing. The brushes are pressed against the commutator by springs fixed to the brush holders and connected to the generator clamps.

Problems with Solutions

241. The e.m.f. of an alternating current is given by the equation $\mathscr{E} = 100 \sin 20\pi t$. Find the maximum and effective values of the e.m.f. and its magnitude corresponding to the phase $\pi/6$. Determine the frequency and period of the current.

Solution. The maximum value of the e.m.f. (for $\sin 20\pi t = 1$) is $\mathscr{E}_0 = 100$ V. The effective value of the e.m.f. is $\mathscr{E}_{eff} = 100/\sqrt{2} = 100\sqrt{2}/2 = 70.7$ V. When $\varphi = \pi/6$, the e.m.f. $\mathscr{E}_\varphi = 100 \sin(\pi/6) = 50$ V. The period of the current can be determined from the condition $20\pi T = 2\pi$. This gives $T = 0.1$ s. The current frequency $\nu = 1/T = 10$ Hz.

242. A series generator with resistances $R_a = 0.8\ \Omega$ and $R_i = 3\ \Omega$ of the windings of the armature and of the inductor produces a current $I = 10$ A when the voltage across the brushes is $U_b = 120$ V (see Fig. 310*a*). Find the e.m.f. of the generator, the voltage across the terminals, the external resistance, and the generator efficiency.

Solution. The e.m.f. of the generator (see Fig. 310*a*) is $\mathscr{E} = U_b + IR_a = 128$ V. The voltage across the terminals is $U = U_b - IR_i = 90$ V. The external resistance $R = U/I = 9\ \Omega$. The generator efficiency $\eta = UI/\mathscr{E}I = U/\mathscr{E} = 70.3\%$.

243. A shunt generator has an e.m.f. $\mathscr{E} = 380$ V, the resistance of the armature $R_a = 0.5\ \Omega$, and the resistance of the inductor $R_i = 80\ \Omega$ (see Fig. 310*b*). Find the current in the external circuit, the voltage across the brushes, and the generator efficiency if the external resistance $R_1 = 20\ \Omega$. Find the generator efficiency for the external resistance $R_2 = 40\ \Omega$.

Solution. Since the external circuit and the inductor are connected in parallel (see Fig. 310*b*), their total resistance can be found from the condition $1/R_{01} = 1/R_i + 1/R_1 = 1/16$, which gives $R_{01} = 16\ \Omega$. When the resistance of the external circuit is equal to R_2, we have

$$1/R_{02} = 3/80, \text{ whence } R_{02} \simeq 26\ \Omega.$$

The current $I = \mathscr{E}/(R_{01} + R_a)$, and the voltage across the brushes is

$$U = \mathscr{E} - IR_a = \mathscr{E} - \mathscr{E}R_a/(R_{01} + R_a) = \mathscr{E}\,[1 - R_a/(R_{01} + R_a)].$$

The current in the external circuit is $I_R = U/R$. The generator efficiency is

$$\eta = \frac{U^2}{R_1} : \frac{\mathscr{E}^2}{R_{01} + R_a} = \left(\frac{U}{\mathscr{E}}\right)^2 \frac{R_{01} + R_a}{R_1} = \frac{R_{01}^2}{R_1(R_{01} + R_a)}.$$

For $R_1 = 20\,\Omega$, we have

$$I_{R_1} = U_1/R_1 = 18.4\ \text{A},\ U_1 = 368.5\ \text{V},\ \eta_1 = 77.6\%.$$

For $R_2 = 40\,\Omega$, we have

$$\eta_2 = \left(\frac{U_2}{\mathscr{E}}\right)^2 \frac{R_{02} + R_a}{R_2} = \frac{R_{02}^2}{R_2(R_{02} + R_a)} = 65.3\%.$$

244. A compound generator supplies to an external circuit a current $I = 60$ A at a voltage $U = 96$ V across its terminals. The e.m.f. of the generator $\mathscr{E} = 100$ V. The resistance of the thin inductor winding is $R_1 = 50\,\Omega$, while the resistance of the thick winding is $R_2 = 0.05\,\Omega$ (see Fig. 310*c*). Find the resistance R_a of the armature and the generator efficiency.

Solution. The thick winding of the inductor is obviously connected in series with the generator armature and with the external resistance, while the thin winding is connected in parallel to them. The resistance of the external circuit and of the thick winding is $R' = R + R_2$, where $R = U/I$ is the external resistance. This gives

$$R' = U/I + R_2. \tag{1}$$

The current in the armature can be found from the equation $I_a/I = (R' + R_1)/R_1 = (U/I + R_1 + R_2)/R_1$:

$$I_a = [U + I(R_1 + R_2)]/R_1. \tag{2}$$

The voltage drop in the armature is

$$U_a = \mathscr{E} - U - IR_2. \tag{3}$$

Dividing Eq. (3) by Eq. (2) termwise, we obtain the resistance of the armature:

$$R_a = \frac{U_a}{I_a} = \frac{(\mathscr{E} - U - IR_2)R_1}{U + I(R_1 + R_2)} = 0.016\,\Omega.$$

The useful power of the generator is $N = UI$. The power losses in the thick and thin windings of the generator inductor are given by

$$\Delta N_2 = I^2R_2,$$
$$\Delta N_1 = (\mathscr{E} - U_a)^2/R_1 = (U + R_2I)^2/R_1.$$

The total power loss is

$$\Delta N = I^2R_2 + (U + IR_2)^2/R_1.$$

The efficiency of the generator is calculated as follows:

$$\eta = \frac{N}{N + \Delta N} = \frac{UIR_1}{UIR_1 + I^2R_2R_1 + (U + IR_2)^2} = 93.8\%$$

Remark. This problem has a simpler numerical solution: $R' = 1.65\ \Omega$, $I_a = 62$ A. The voltage drop in the thick winding of the inductor is $U_2 = IR_2 = 3$ V. The voltage drop in the armature is $U_a = \mathscr{E} - U - U_2 = 1$ V. The resistance of the armature is $R_a = U_a/I_a = 0.016\ \Omega$. Then the useful power $N = UI = 5.76$ kW. The power losses in the thick and thin windings of the inductor are

$$\Delta N_2 = I^2R_2 = 180\ \text{W},\ \Delta N_1 = U_1^2/R_1 = (U + IR_2)^2/R_1 = 196\ \text{W}.$$

The generator efficiency is

$$\eta = \frac{N}{N + \Delta N_1 + \Delta N_2} = 93.8\%.$$

245. The voltage should be reduced from $U_1 = 6$ kV to $U_2 = 120$ V with the help of an autotransformer having $w_1 = 3000$ turns. The resistance of the secondary is $r_2 = 0.5\ \Omega$, the resistance of the external circuit (where the voltage is lower) is $R = 12\ \Omega$. What should be the number of turns w_2 connected to the low-voltage circuit? The resistance of the winding connected to the high-voltage circuit should be neglected.

Solution. The current in the secondary circuit is $I_2 = U_2/(r_2 + R)$. The voltage drop in the secondary is $\Delta\mathscr{E}_2 = I_2r_2 = U_2r_2/(r_2 + R)$. The e.m.f. induced in the secondary is

$$\mathscr{E}_2 = U_2 + \Delta\mathscr{E}_2 = U_2\left(1 + \frac{r_2}{r_2 + R}\right) = U_2\frac{R + 2r_2}{r_2 + R}.$$

The e.m.f. in the primary is equal to the voltage supplied to it since the losses in the primary are negligibly small: $\mathscr{E}_1 = U_1$. The transformation coefficient is

$$K = \frac{\mathscr{E}_1}{\mathscr{E}_2} = \frac{U_1}{U_2}\left(\frac{R + r_2}{R + 2r_2}\right) = \frac{w_1}{w_2}.$$

Hence the number of turns in the secondary is given by

$$w_2 = \frac{U_2}{U_1}\frac{R + 2r_2}{r_2 + R} = 62.$$

Exercises

241. The frequency of an alternating current is 50 Hz, the effective e.m.f. is 100 V,

and the initial phase is 60°. Write the equation for the time variation of the e.m.f. of the alternating current and find its period.
Answer. $\mathscr{E} = 141.1 \sin (100\pi t + 2\pi/3)$, 0.02 s.
242. (a) A series generator with the total resistance of the inductor and armature windings of 6 Ω generates a current of 10 A at a voltage across the terminals of 420 V. Find the e.m.f., the external resistance, and the efficiency of the generator.
Answer. 480 V, 42 Ω, 87.5%.

(b) A series generator generates a current of 6 A at a voltage across the brushes of 63 V. The resistance of the armature winding is 0.5 Ω and of the inductor winding, 1.5 Ω. Find the e.m.f. of the generator, the voltage across its terminals, the external resistance, and the generator efficiency.
Answer. 66 V, 54 V, 9 Ω, 81.8%.
243. (a) A shunt generator whose e.m.f. is 120 V supplies a current of 30 A to an external circuit. The current in the inductor winding is 1.5 A and the generator efficiency is 90%. Determine the voltage across the generator terminals and the resistance of the armature and inductor.
Answer. 113.7 V, 0.2 Ω, 75.8 Ω.

(b) A shunt generator has an e.m.f. of 100 V and the resistances of the armature and of the inductor of 0.5 Ω and 45 Ω respectively. The external circuit consists of three parallel-connected resistances: 10 Ω, 12 Ω, and 60 Ω. Find the voltage across the generator brushes, the power of the current in the external circuit and in the inductor, the current through each resistor in the external circuit, and the generator efficiency.
Answer. 90 V, 1620 W, 180 W, 9 A, 7.5 A, 1.5 A, 81%.
244*. A compound generator supplies to an external circuit a current of 55 A at a voltage of 80 V across its terminals. The resistances of the armature, thin parallel winding, and thick winding are 0.018 Ω, 40 Ω, and 0.04 Ω respectively. Find the e.m.f. and the efficiency of the generator.
Answer. 83.23 V, 92.8%.
245. (a) A transformer whose transformation coefficient is 10 reduces voltage from 10 kV to 800 V. The current in the secondary is 2 A. Determine the resistance of the secondary. Energy losses in the primary should be neglected.
Answer. 100 Ω.

(b) An autotransformer having 2000 turns reduces voltage from 5 kV to 220 V when 100 turns of the winding are connected to the low-voltage circuit. The resistance of the external circuit is 11 Ω. Find the transformation coefficient and the resistance of the secondary. The resistance of the primary should be neglected.
Answer. 20, 1.5 Ω.

5.14. Electron Tubes (Valves)

An **electron tube** (or **valve**) is a glass of metallic evacuated vessel with electrodes mounted in it. The nomenclature of tubes corresponds to the number of electrodes.

A **diode**, viz. a *two-electrode tube* (Fig. 314), has a cathode

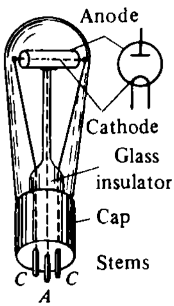


Fig. 314

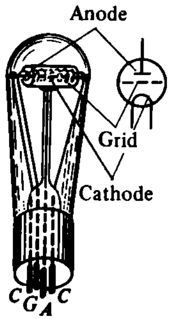


Fig. 315

and an anode. The cathode is made in the form of a filament manufactured from a refractory metal, say, tungsten. The two ends of the filament are brought through the cap of the tube in the form of two stems. The anode is made in the form of a tube of a circular, elliptic or rectangular cross section, which envelops the cathode. The anode terminal is also brought through the cap in the form of a stem.

The cathode filament is heated by the current from a filament battery or from a low-voltage transformer to initiate thermionic emission.

When the cathode and the anode are connected to a voltage supply, an electric field appears in the space between them. If the potential of the anode is higher than the potential of the cathode, the electrons emitted by the cathode start to move towards the anode, and a current appears in the tube and in the external circuit. If, however, the potential of the anode is lower than the potential of the cathode, there is no current through the tube since the electric field "repels" electrons from the anode.

A **triode** viz. a *three-electrode tube* (Fig. 315), can be obtained from a diode by inserting the third electrode—grid—between the cathode and the anode. The grid is made in the form of a spiral enveloping the cathode filament with a stem as the terminal. If the grid and the cathode are connected to a voltage source, an additional electric field appears between these electrodes, which is superimposed on the main field. Since the grid is considerably closer to the cathode than the anode, insignificant changes in the grid potential noticeably change the electric field strength in the tube. Thus, the anode current can be controlled by varying the grid potential.

Multielectrode tubes, viz. the *tubes with several grids,* are in principle modified triodes and operate on the same basis.

5.15. Diode as a Rectifier of Alternating Current

Figure 316 represents a rectifier circuit with a diode (R is the load resistance). An a.c. source is connected in the anode circuit of the diode either directly (Fig. 316*a*) or through a transformer (Fig. 316*b*). In the latter case, the cathode is heated from the current

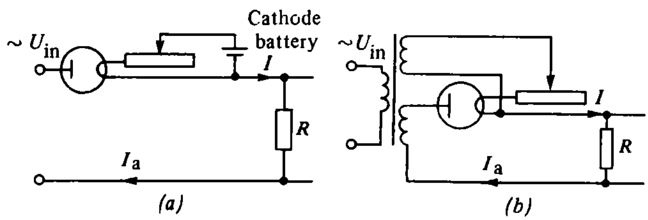


Fig. 316

source through a step-down transformer instead of the filament battery.

Figure 317*a* shows the input voltage U_{in} of the current source as a function of time. When the anode potential is positive (i.e. it is higher than the cathode potential), electrons emitted by the cathode are attracted by the anode, and a current flows through the tube as shown in Fig. 317*b*. When the anode potential is negative, there is no current in the circuit. Thus, the current flows in the external circuit only during a half-period of the alternating current. It is an intermittent, pulsating, and unidirectional current. The rectifier shown in Fig. 316 is known as a half-wave rectifier. .

To utilize both half-periods of an alternating current, a full-wave rectifier circuit is used (Fig. 318), which operates with two tubes. The two diodes function alternately, generating a unidirec-

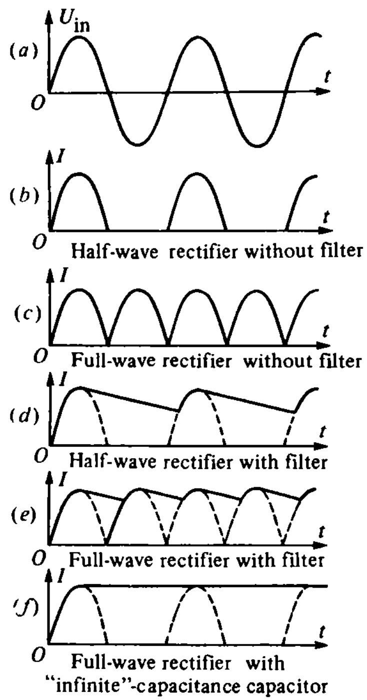


Fig. 317

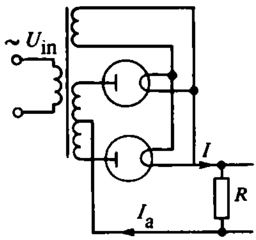


Fig. 318

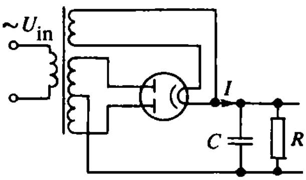


Fig. 319

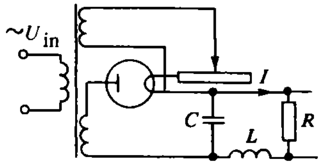


Fig. 320

tional current in the external circuit (one tube during the first half-period and the other, during the second half-period). Pulsating current flowing in the external circuit has the form shown in Fig. 317*c*.

Instead of two diodes, one tube with two anodes can be used for the full-wave rectification (Fig. 319).

Rectification of an alternating current can be improved by smoothing current pulses, for which capacitor filters are employed. The capacitors are connected in parallel to the load circuit. Chokes, high-inductance coils, connected in series with the load, also improve rectification (Fig. 320).

Each current pulse appearing in the load circuit charges the capacitor plates. When the pulse from the external current source damps and ultimately ceases, the capacitor discharges through the load resistor R, maintaining the current of nearly constant intensity (Fig. 317*d, e*) in it. If the capacitance of the capacitor were "infinitely high", the current through the load would be direct (Fig. 317*f*).

5.16. Cathode-Ray Tube

Cathode-ray tubes (Fig. 321) are used for obtaining a narrow controlled electron beam. Such a tube is a glass vessel evacuated to a

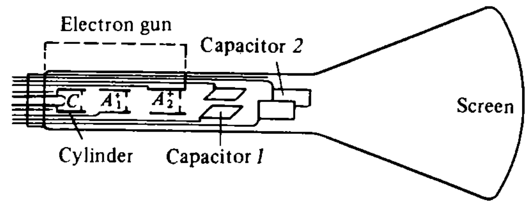


Fig. 321

high degree of rarefaction. The narrow part of the tube contains a hot cathode C which serves as an electron source. A cylinder enveloping the cathode lets the electron beam pass through the opening in its end face. Supplying a negative voltage of various magnitudes to this controlling cylinder, we can vary the velocity of electrons in the beam. Behind the controlling cylinder, two cylindrical anodes A_1 and A_2 are mounted, to which a positive voltage is supplied. This part of the cathode-ray tube is called the electron gun. The electron beam produced by the gun passes through two capacitors, *1* and *2*, with horizontal and vertical plates respectively and reaches the fluorescent screen. The voltage supplied to one of the capacitors, say, *1*, deflects the electron beam in the vertical direction. The electric field applied to the second capacitor is tripped in certain (very short) time intervals. As a result, we can judge about the time variation of the process from the horizontal deflections of the electron beam.

The low inertia of the electron beam makes it possible to use a cathode-ray tube for investigating rapidly varying processes.

5.17. Electron Tubes as Generators and Amplifiers

In radio engineering, three-electrode (and multielectrode) tubes fed by a d.c. source are used as high-frequency *generators* of undamped electric oscillations.

A simple circuit for generating undamped electric oscillations with the help of a triode is shown in Fig. 322. When the potential

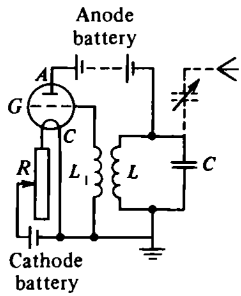


Fig. 322

of the triode grid is higher than the cathode potential, a d.c. source (anode battery) supplies a current through the tube, simultaneously charging capacitor C. When the grid potential is lower than the potential of the cathode, there is no current through the tube, and capacitor C is charged through the inductance coil L. Thus, oscillations of the grid potential cause oscillations in the LC-circuit. The oscillation appearing in the LC-circuit is transferred through coil L_1 to the grid, and the charge oscillation on the grid causes the oscillation of the anode current, i.e. amplifies the current oscillation in the LC-circuit (feedback of the grid in the LC-circuit), and so on. Thus, undamped oscillations are generated in the LC-circuit.

Coil L_1 matches the operation of the tube with the oscillatory process in the circuit and ensures the automatic energy supply at a frequency equal to the natural oscillation frequency in the circuit. Thus, *self-excited oscillatory system* is obtained.

A triode may serve as an *amplifier* of electromagnetic oscillations. Figure 323 shows a circuit where a triode amplifies elec-

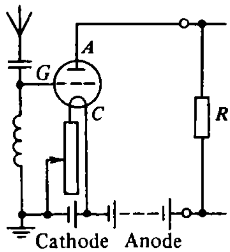


Fig. 323

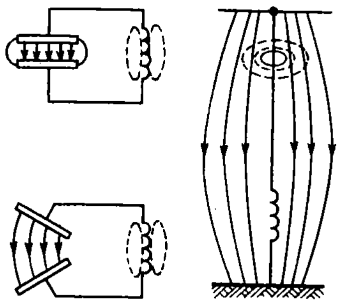
tromagnetic oscillations received by an aerial. Weak oscillations received by the aerial are amplified at the expense of the energy of the anode battery.

The current from the anode battery is amplified when the potential of the triode grid is higher than the cathode potential and is weakened or suppressed altogether when the grid potential is lower than the potential of the cathode. Since the grid is close to the cathode filament, insignificant changes in the potential of the grid as a result of the currrent oscillations in the aerial generate synchronous oscillations in the anode circuit. The power of these oscillations depends on the power of the d.c. source (anode battery).

5.18. Open Oscillatory Circuit. Emission and Reception of Electromagnetic Waves

The aim of the transformation of the electric energy of a high-frequency generator into the energy of electromagnetic oscillations in a circuit is the wireless energy transmission in space in the form of electromagnetic waves. However, this cannot be done with a closed oscillatory circuit since the electric field in it is concentrated in a narrow gap between the capacitor plates, and the radiation of energy into space is very weak.

To intensify energy emission, it is necessary to move apart the capacitor plates, i.e. to go over from a closed oscillatory circuit (Fig. 324*a*) to an open circuit of the type of an earthed aerial (Fig. 324*b*).



(a) (b)

Fig. 324

The energy produced continuously by a generator is transmitted in space in the form of undamped electromagnetic oscillations. For this purpose, the coil of the aerial circuit is inductively coupled with a high-frequency generator (Fig. 325*a*). The natural

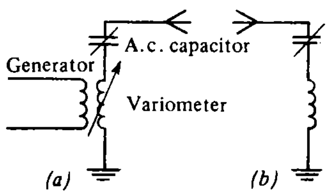


Fig. 325

frequency of oscillations of the transmitting aerial can be varied either by a variable capacitor or by a coil with variable inductance (variometer) connected to the aerial circuit. The reception of electromagnetic waves also requires an open oscillatory circuit, viz. aerial, similar to the radiating circuit (Fig. 325*b*).

For good reception it is necessary that a receiving oscillatory circuit be tuned in resonance with a transmitting circuit, i.e. that the natural frequency of oscillations of the receiving aerial circuit be close to the oscillation frequency of the transmitting aerial circuit. The tuning is realized by a variable capacitor or a variometer.

5.19. Scale of Electromagnetic Waves

The wavelength of electromagnetic waves (including radio, infrared, light and ultraviolet waves, X- and gamma-rays) varies over wide intervals. We shall indicate approximate limits for electromagnetic waves of different ranges:

radio waves	λ = 30 km to 1 mm
infrared waves	λ = 1 mm to 0.7 μm
light waves	λ = 0.7 μm to 0.4 μm
ultraviolet waves	λ = 0.4 μm to 5 nm
X-rays	λ = 5 nm to 4 pm
gamma-rays	λ = 4 pm to 0.1 pm

Radio waves have the following division depending on the wavelength: long waves with λ = 30 km to 3 km, medium waves with λ = 3000 m to 200 m, short waves with 200 m to 10 m, and ultrashort waves with λ = 10 m to 1 mm.

The velocity of propagation of electromagnetic waves (including radio waves) in a vacuum is $c \simeq$ 300 000 km/s = 3×10^8 m/s. The wavelength of an electromagnetic wave is equal to the product of the velocity of propagation of the wave and the duration of one period: $\lambda = cT$, whence the period of oscillation of the wave is $T = \lambda/c$, while the oscillation frequency $\nu = 1/T = c/\lambda$. This formula is used for calculating the oscillation frequency of radio waves in the range from 10 000 to 30 000 000 oscillations per second: long waves correspond to ν = 10 kHz to 100 kHz, medium waves to ν = 100 kHz to 1500 kHz, short waves to ν = 1.5 MHz to 30 MHz, and ultrashort waves to ν = 30 MHz to 300 GHz.

Thomson's formula indicates that in order to reduce the period of oscillations, i.e. to increase the oscillation frequency for generating high-frequency waves, it is necessary to reduce the capacitance C and the inductance L of an oscillatory circuit.

The capacitance can be reduced either by moving the plates apart or by decreasing their surface area. The inductance is reduced by decreasing the number of turns in the inductance coil. A further increase in the oscillation frequency is attained by using a straight conductor (Fig. 326), i.e. a simple vibrator of small capacitance and inductance.

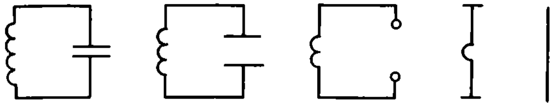


Fig. 326

The velocity of propagation of an electromagnetic wave in some medium is lower than the velocity in a vacuum by a factor of $\sqrt{\varepsilon_r \mu_r}$:

$$u = c/\sqrt{\varepsilon_r \mu_r}$$

where $c = 3 \times 10^8$ m/s is the velocity of propagation of the electromagnetic wave in a vacuum, ε_r is the relative permittivity of the medium, and μ_r is its relative permeability.

Problem with Solution

246. An oscillatory circuit consists of a capacitor with a capacitance $C = 500$ pF and a variometer whose inductance L varies in the limits between 0.5 mH and 1.5 mH. Find the range of wavelengths emitted by the circuit.

Solution. The wavelength of an electromagnetic wave is $\lambda = cT$, where $c = 3 \times 10^8$ m/s is the velocity of propagation of electromagnetic waves in a vacuum. Since

$$T = 2\pi \sqrt{LC},$$

we have

$$2\pi c \sqrt{L_1 C} \leqslant \lambda \leqslant 2\pi c \sqrt{L_2 C},$$

where $L_1 = 0.5$ mH and $L_2 = 1.5$ mH. Consequently,

$$942 \text{ m} \leqslant \lambda \leqslant 1632 \text{ m}.$$

Exercise

246. The wavelength λ of an oscillatory circuit varies from 1 km to 2 km. Find the range of variation of the inductance of a variometer if the capacitance of the circuit is 1000 pF.

Answer. 2.82 mH $\leqslant L \leqslant$ 11.2 mH.

6. OPTICS

6.1. Light Sources. Propagation of Light in a Straight Line

Light sources are bodies that emit light into the surrounding space. A **point source** is a source of light whose size is small in comparison with the distance from the point of observation.

It is known from experience that a human eye has a certain **resolving power**, and objects seen at an angle of vision of about one angular minute (and less) are perceived by the observer as points. Therefore, although a luminous point does not exist actually, the concept of a point source is introduced as a mathematical image that simplifies an investigation of phenomena and allows solving problems with an accuracy sufficient for practical purposes.

In a homogeneous medium, *the light from a point source propagates in a straight line*[1] *uniformly in all directions.* The direction of propagation of light is depicted by straight lines, viz. **light rays.** While solving problems, light rays represent actual light beams of small, although very finite, dimensions.

Thus, a *point source* and a *light ray* are mathematical concepts, i.e. *abstractions,* corresponding to actually existing light sources and light beams of finite, although very small, dimensions.

The fact that light propagates in a straight line is confirmed by the formation of umbras behind opaque bodies illuminated by a

[1] The law of rectilinear propagation of light is violated due to diffraction of light, which takes place when a light beam passes through small apertures or when it meets an obstacle whose size is commensurate with the wavelength of light. This phenomenon will not be considered in this book.

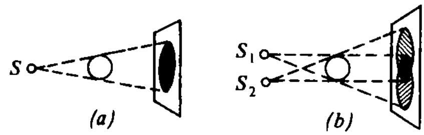


Fig. 327

point source. The size of an umbra depends on the mutual arrangement of the light source, body, and screen (Fig. 327*a*).

If a body is illuminated by several light sources (or by a light source having finite dimensions, which can be treated as a set of point sources), umbra is formed on a screen on the other side of the body (Fig. 327*b*) at the spots which are not illuminated by light from any source and penumbra at the spots illuminated only by some of the point sources or by a part of the finite-dimensional source.

6.2. Velocity of Light. Michelson's Experiment

The velocity of light in a vacuum is $c = 2.997\,924\,58 \times 10^8$ m/s (according to measurements made in 1977). In various transparent media, the velocity of light is lower. For example, it is equal to 225 000 km/s in water and to 200 000 km/s in glass. The velocity of light in air is 299 711 km/s, i.e. is slightly lower than that in a vacuum. In practical calculations, it can be assumed that the velocity of light in air and in a vacuum is $c \approx 300\,000$ km/s $= 3 \times 10^8$ m/s. When two media are compared with respect to the velocity of light in them, the medium in which the light velocity is lower is called the optically denser medium.

Let us consider **Michelson's experiment** which allowed one to measure the velocity of light with a high accuracy. The schematic diagram of the set-up is shown in Fig. 328. An octahedral mirror prism is rotated by a motor about its axis *O*. A light beam from source *S* is directed to a face of the mirror prism. A system of mirrors is used to make the beam reflected from the prism impinge on another face, after which it is reflected through a telescope to the eye of an observer. The observer sees light through the telescope only if the time required for light to pass from the source to the

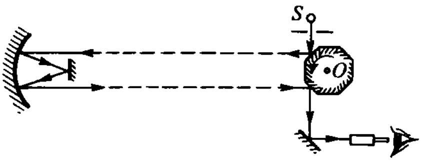


Fig. 328

mirrors and back is exactly equal to the time the prism is turned by 1/8 (2/8, 3/8, etc.) of its complete revolution.

Gradually increasing the speed of rotation of a prism, Michelson obtained an illuminated field of view in the telescope. The velocity of light was determined as the length of the path from the light source to the eye of an observer divided by the time taken to turn the prism by 1/8 of a revolution. At present, there exist other, more accurate, methods of measurement of the velocity of light.

A. PHOTOMETRY

6.3. Luminous Flux. Luminous Intensity

Light is a type of electromagnetic waves with wavelengths ranging between 400 nm and 760 nm (1 nm = 10^{-9} m). Electromagnetic waves carry an energy.

The **luminous energy flux** N through a surface is the amount of energy passing through this surface per unit time.

The **luminous (light) flux** Φ is the flux of luminous energy estimated by the visual perception of a normal human eye. The effect of light on the eye is determined, in addition to the luminous energy flux, by the wavelength of light. Therefore, identical luminous energy fluxes differing in the wavelength correspond to different luminous fluxes. Electromagnetic waves with wavelengths shorter than 400 nm and longer than 760 nm do not produce any visual sensation, and the corresponding luminous fluxes are equal to zero. The strongest visual sensation corresponds to a green light with a wavelength equal to 550 nm.

The SI unit of luminous flux is a lumen (see below).

The ratio of the luminous energy flux N to the luminous flux Φ is called the **mechanical equivalent** A **of light** and is expressed in **watts per lumen** (W/lm).

The quantity reciprocal to it ($\eta = 1/A$) is known as the **luminous efficiency** of a light source and is expressed in **lumens per watt** (lm/W).

Before introducing the concept of luminous intensity, we shall give the definition of auxiliary SI units, viz. the unit of plane angle—radian and the unit of solid angle—steradian.

A **radian** (rad) is a central angle resting on the arc of a circle, whose length is equal to the radius of this circle. The angle of 2π radians corresponds to 360°, 1 rad being equal to about 57°.

A solid angle Ω is a part of space bounded by a conical surface (Fig. 329). A solid angle cuts from a sphere circumscribed from

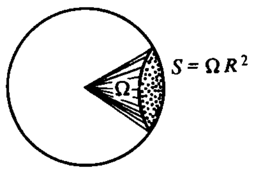


Fig. 329

the apex of the cone a surface S whose area is proportional to the squared radius R of the sphere. A solid angle is measured by the ratio of the area of the surface S to the squared radius of the sphere:

$$\Omega = S/R^2.$$

A **steradian** (sr) is an angle that cuts from a sphere a surface equal to squared radius of the sphere. Since the surface of a sphere is equal to $4\pi R^2$, the complete solid angle is equal to 4π sr (while the complete plane angle is equal to 2π rad).

A point source of light is characterized by the **luminous intensity** I, i.e. by the luminous flux emitted by the point source within a solid angle of one steradian. If a point source emits a luminous flux Φ uniformly distributed in all directions (such a source is called isotropic), the luminous intensity is connected with the luminous flux through the following relation:

$$I = \Phi/4\pi.$$

For an anisotropic (i.e. nonisotropic) source, the luminous intensity of light is different in different directions. In order to determine the luminous intensity in a given direction, we must take a small solid angle $\Delta\Omega$ such that this direction is within this angle. By measuring the luminous flux $\Delta\Phi$ emitted by the source and the solid angle $\Delta\Omega$, we obtain the luminous intensity:

$$I = \Delta\Phi/\Delta\Omega.$$

The SI unit of luminous intensity, viz. a **candela** (cd), is the luminous intensity of light emitted by a monochromatic source of radiation intensity of 1/683 W/sr in this direction at a frequency of 540×10^{12} Hz. Candela is a basic SI unit.

The SI unit of luminous flux, viz. a **lumen** (lm), is the luminous flux emitted by an isotropic source whose luminous intensity is 1 cd within a solid angle of 1 sr:

$$1 \text{ lm} = 1 \text{ cd} \cdot 1 \text{ sr}.$$

6.4. Illuminance (Illumination Intensity)

Illuminance E is a luminous flux per unit area of an illuminated surface. If a surface S is illuminated uniformly (i.e. with the same intensity at all points) by a luminous flux Φ, then

$$E = \Phi/S.$$

If the flux is distributed nonuniformly over a given surface, illuminance is different at different points. To determine the illuminance of a surface at a given point, we must isolate a very small area element ΔS in the vicinity of this point. The ratio of the luminous flux $\Delta\Phi$ incident on this element to its area ΔS gives the illuminance at the given point:

$$E = \Delta\Phi/\Delta S$$

(strictly speaking, we must take the limit of this ratio as ΔS tends to zero).

The SI unit of illuminance, viz. a **lux** (lx), is equal to the illuminance of a unit surface area by a uniformly distributed luminous flux of 1 lm:

$$1 \text{ lx} = 1 \text{ lm}/1 \text{ m}^2.$$

Let us place a point light source at the centre of a spherical surface of radius R (Fig. 330a). On this sphere, we isolate a surface

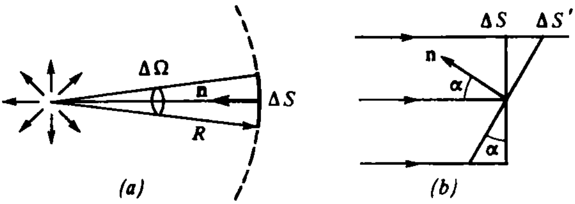


Fig. 330

element ΔS corresponding to the solid angle $\Delta\Omega$ connected with ΔS through the relation $\Delta S = R^2\Delta\Omega$ (if the dimensions of the surface element are small as compared to R, the solid angle can be treated as a plane angle and the rays incident on it as parallel). If the luminous intensity of the source in the direction of the surface element is I, the luminous flux impinging on ΔS is $\Delta\Phi = I\,\Delta\Omega$. Consequently, the illuminance of the surface element normal to the rays is

$$E_0 = \Delta\Phi/\Delta S = I\,\Delta\Omega/R^2\,\Delta\Omega = I/R^2.$$

Thus, the illuminance decreases with the increasing distance R from the source as $1/R^2$.

We considered the surface element ΔS normal to the incident rays and such that the normal $\mathbf{n}$ to it is directed towards the light source (Fig. 330a). Let us now take a surface element $\Delta S'$ illuminated by the rays at an angle α to the normal (Fig. 330b). We choose this surface element so that the same luminous flux $\Delta\Phi$ as that incident on ΔS illuminates it. It can be seen from Fig. 330b that $\Delta S' = \Delta S/\cos\alpha$. Hence the illuminance of such a surface element is smaller than the illuminance of a normal surface element by a factor of $1/\cos\alpha$:

$$E = E_0 \cos\alpha = (I\cos\alpha)/R^2. \qquad (6.4.1)$$

It should be recalled that I is the luminous intensity of the point light source in the direction to a given point of the surface, R is the distance from the source to the given point, and α is the angle at which the rays fall on the surface, i.e. the angle between the direction of the rays and the normal to the surface.

Illuminance is a basic quantity for calculations in lighting engineering, since for each room there exist certain norms of required illuminance depending on the duty, which are elaborated in practical life. For example, for reading, the illuminance of 50 lx is required while for high-precision operations (engraving or drawing), as well as in the workshops where small cutting tools are used, the illuminance of 100 lx is needed.

Knowing illuminance, we can easily determine the luminous intensity of light sources required for its creation:

$$I = ER^2/\cos\alpha.$$

6.5. Comparison of Luminous Intensity of Different Sources. Photometers

Photometers are the instruments for measuring the luminous intensity of light emitted by various sources.

The luminous intensities of two light sources can be compared by attaining the same illuminance from these sources of the same surface element or surface elements arranged at equal angles to the direction of light rays from the two sources. In this case,

$$I_1/I_2 = R_1^2/R_2^2,$$

i.e. *the luminous intensities of two sources producing the same illuminance of a surface are proportional to the squared distances from the sources to this surface.* If the luminous intensity I_0 of one of the sources is known, the intensity of the light source being compared to it is given by

$$I = I_0R^2/R_0^2.$$

This is the so-called *photometric formula.*

One of the simplest photometers, the *Joly photometer,* is shown schematically in Fig. 331*a*. The light sources under com-

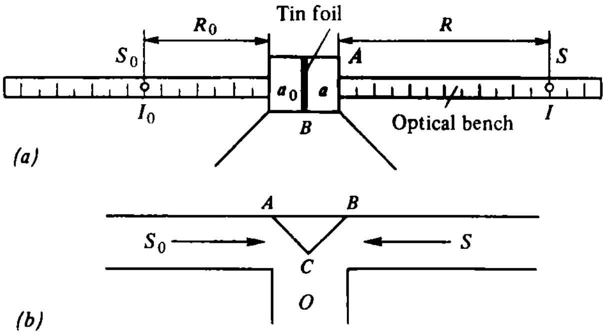


Fig. 331

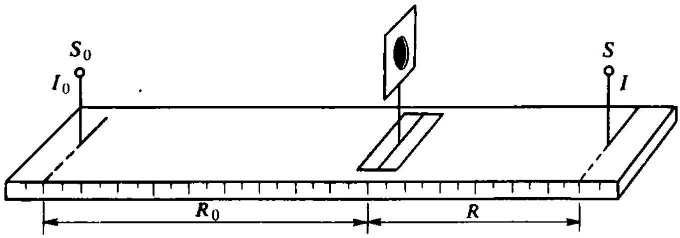


Fig. 332

parison (the one being investigated S and the standard S_0) are moved along a ruler with subdivisions until paraffin plates a_0 and a enclosed in an opaque box A with windows and separated by a thin tin foil are illuminated identically. The observations are carried out through the window B in the opaque box A that protects the eye of an observer from the direct influence of the light sources being compared.

A modification of this photometer is represented in Fig. 331*b*. The paraffin plates in this instrument are replaced by a trihedral prism ABC painted white and placed into a blackened tube. An observer looks at the prism through the tube O. If the faces AC and BC are illuminated identically, their common edge C disappears.

The construction of the *Bunsen photometer* is even simpler (Fig. 332). A paper screen with a grease spot is displaced on a ruler until the grease spot "vanishes". This occurs at the same illuminance on both sides of the screen.

Problems with Solutions

247. What time is required for a Sun beam to reach the Earth? The radius of the Earth's orbit is $R = 1.50 \times 10^{11}$ m.

Solution. The time $t = R/c = 8.3$ min, where the velocity of light $c = 3 \times 10^8$ m/s.

248. Determine the distance covered by a light signal during one year.

Solution. $S = ct \approx 9.5 \times 10^{15}$ m = 1 light year, where $t = 3600 \times 24 \times 365$ s.

249. An electric bulb is enclosed in an opaque glass ball of radius $R = 20$ cm and suspended at a height $h_1 = 5$ m above the floor. An opaque ball of radius $r = 10$ cm is suspended below the bulb at a height $h_2 = 1$ m from the floor. Find the dimensions of the umbra and penumbra on the floor.

Solution. The umbra is a circle of diameter D, while the penumbra is a ring with the inner diameter equal to the diameter of the umbra. Using the similarity of the figures illustrated in Fig. 333, we can write two proportions[2]:

$$(d/2 - D/2)/2R = h_2/(h_1 - h_2), \quad (d/2 + D/2)/2r = h_1/(h_1 - h_2),$$

from which we obtain

$$d - D = 4Rh_2/(h_1 - h_2) = 20\ \text{cm}, \quad d + D = 4rh_1/(h_1 - h_2) = 50\ \text{cm}.$$

Consequently, the diameter of the umbra is $D = 15$ cm and the outer diameter of the penumbra is $d = 35$ cm.

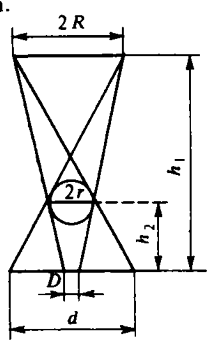


Fig. 333

[2] Since the dimensions of the balls are small in comparison with the heights of their points of suspension, we can assume with a sufficiently high degree of accuracy that the rays are tangent to the balls at the ends of their horizontal diameters.

250. An electric bulb enclosed in an opaque glass ball of diameter $D_1 = 50$ cm is suspended at a height $h_1 = 4.0$ m above the floor. At what height below the bulb should an opaque ball of diameter $D_2 = 25$ cm be suspended so that only a penumbra is formed on the floor? Determine the dimensions of the penumbra.

Solution. There is no umbra on the floor if the apex of the umbra cone is on the floor (Fig. 334). Therefore, we can write the following proportion: $D_1/h_1 = D_2/h_2$, whence $h_2 = h_1D_2/D_1 = 2.0$ m. From the similarity of triangles we have $d/2D_2 = h_1/(h_1 - h_2)$, i.e. the diameter of the penumbra is $d = 2D_2h_1/(h_1 - h_2) = 1.0$ m.

Remark. The same result can be obtained with the help of the formulas derived in the previous problem for a particular case $D = 0$.

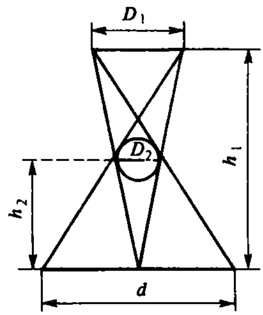


Fig. 334

251. The luminous efficiency[3] of a bulb is $\eta = 18.80$ lm/W. The bulb emits the flux $N = 12$ kJ/h of luminous energy into the surrounding space. Find the luminous intensity and the mechanical equivalent of light from the bulb if it consumes the power $N_0 = 100$ W from the circuit.

Solution. Since $\Phi = 4\pi I$ and $\eta = \Phi/N_0$, we have $\eta = 4\pi I/N_0$, whence the luminous intensity of the bulb is $I = \Phi/4\pi \simeq 150$ cd. The mechanical equivalent of light is $A = N/\Phi = 0.18$ mW/lm.

252. A bulb is at the apex of a cone whose solid angle $\Omega = 1.2$ sr. The luminous flux through the cone is $\Phi = 60$ lm. What is the luminous intensity of the bulb? Determine the total luminous flux emitted by the bulb.

Solution. Using the formula $\Phi = I\Omega$, we obtain the luminous intensity of the bulb: $I = \Phi/\Omega = 50$ cd. The total luminous flux of the bulb is $\Phi = 4\pi I = 628$ lm.

253. Three bulbs consuming a power $N_0 = 1$ kW each are suspended at a height $h = 4$ m above the ground on the posts arranged in a straight line at a distance $l = 20$ m from one another. Find the illuminance of the point at the base of the

[3] The luminous efficiency of a bulb is the ratio of the luminous flux emitted by it to the power consumed from the circuit. (*Editor's note.*)

first post if the luminous flux from each bulb is $\Phi = 15 \times 10^3$ lm. Find the fractions (in percent) of the illuminances from the bulbs on the second and third posts of the illuminance on the first post.

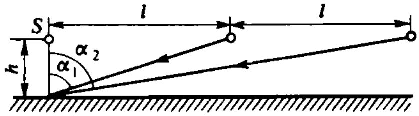


Fig. 335

Solution. The illuminance of the point located at the base of the first post from the bulb suspended from it is (Fig. 335)

$$E_1 = I/h^2.$$

The illuminances from the second and third bulbs are

$$E_2 = \frac{I \cos \alpha_1}{r_1^2} = \frac{Ih}{r_1^3} = \frac{Ih}{(l^2 + h^2)^{3/2}},$$

$$E_3 = \frac{I \cos \alpha_2}{r_2^2} = \frac{Ih}{r_2^3} = \frac{Ih}{(4l^2 + h^2)^{3/2}}.$$

The total illuminance at the base of the first post is

$$E = E_1 + E_2 + E_3 = Ih \left[\frac{1}{h^3} + \frac{1}{(l^2 + h^2)^{3/2}} + \frac{1}{(4l^2 + h^2)^{3/2}}\right].$$

Since $I = \Phi/4\pi$, we have

$$E = \frac{\Phi h}{4\pi} \left[\frac{1}{h^3} + \frac{1}{(l^2 + h^2)^{3/2}} + \frac{1}{(4l^2 + h^2)^{3/2}}\right] = 75 \text{ lx}.$$

The fractions of the illuminances of this point from the second and third bulbs are

$$\beta = [h^2/(l^2 + h^2)]^{3/2} \times 100\% = 0.75\%,$$

$$\beta \simeq [h^2/(4l^2 + h^2)]^{3/2} \times 100\% = 0.1\%.$$

254. Two bulbs having luminous intensities $I_1 = 15$ cd and $I_2 = 60$ cd are separated by $l = 180$ cm from each other. At what distance from the more intensive bulb should a paper screen with a grease spot be placed on the straight line connecting the bulbs to make the spot invisible?

Solution. The spot becomes invisible when the illuminances on both its sides are equal: $E_1 = E_2$, i.e. (Fig. 336) $I_1/l_0^2 = I_2/(l - l_0)^2$, or

$$(l - l_0)/l_0 = \pm \sqrt{I_2/I_1},$$

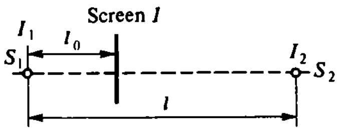


Fig. 336

whence

$$l_0 = \frac{l}{\pm\sqrt{I_2/I_1} + 1}.$$

The minus sign of the radical in the denominator does not fit since in this case $l_0 < 0$ as if the screen were illuminated by the two sources from one side. Consequently, $l_0 = 0.60$ m.

Exercises

247. (a) The radius of Venus' orbit is 1.08×10^{11} m. Find the time required to the light from Venus to reach the Earth when the Earth, Venus, and the Sun lie on the same straight line. The radius of the Earth's orbit is 1.50×10^{11} m.
Answer. 2 min 20 s.

(b) A radio signal sent to the Moon is reflected by it and received in 2.56 s. What is the radius of the Moon's orbit?
Answer. 3.84×10^8 m.

248. Sirius is separated from the Earth by about 8.4×10^{16} m. Express this distance in light years.
Answer. $\simeq 8.9$ light years.

249. (a) A telegraph post having a height of 4.0 m is illuminated by the Sun so that the length of its umbra is 3.0 m. Find the angle of incidence of the Sun rays.
Answer. arctan (3/4) = 37°.

(b) Two 1.20-m poles are arranged near a lighting post so that the distances between the bases of the poles and the post differ by 0.80 m. The umbras of the poles differ in length by 0.40 m. Find the height of the post.
Answer. 3.60 m.

(c) An opaque bulb having the shape of an 8-cm ball illuminates an opaque ball of diameter 40 cm suspended at the same height as the bulb so that their centre-to-centre distance is 1 m and the distance from the bulb to the screen is 3.5 m. Find the diameters of the umbra and penumbra on the screen.
Answer. 1.20 m, 1.60 m.

(d*) A linear horizontal lamp of 2.0-m length is fixed on a 5.0-m post and illuminates a vertical rectangular screen having a 3.0-m side and mounted on the ground at 4.0 m from the post. Find the dimensions of the umbra and penumbra if the lamp is parallel to the plane of the screen.
Answer. 6.0×4.5 m^2, 6.0×10.5 m^2.

250. Find the height of the cone of the Earth's umbra and the radius of its penum-

bra at the level of the cone apex of the umbra assuming that the radii of the Earth and of the Sun are 6.4×10^6 m and 0.7×10^9 m and the radius of the Earth's orbit is 1.5×10^{11} m.
Answer. 1.4×10^9 m, 1.3×10^7 m.
251. (a) The total luminous flux from a 100-W bulb is 1884 lm. Find the luminous intensity and the luminous efficiency of the bulb.
Answer. 150 cd, 18.8 lm/W.

(b) The mechanical equivalent of light from a bulb is $A = 0.011$ W/lm and its luminous intensity $I = 100$ cd. Find the luminous energy emitted by the bulb per unit time.
Answer. 830 J/min.
252. (a) A central solid angle cuts an area of 2250 cm^2 from a spherical surface of radius 1.5 m. What is the magnitude of this angle? What is the area of the surface cut by the same angle from a sphere whose radius is larger by 1.0 m?
Answer. 0.1 sr, 0.625 m^2.

(b) A bulb emitting the total luminous flux of 1256 lm is enclosed in a cone through which a luminous flux of 80 lm is emitted. Find the luminous intensity of the bulb and the solid angle of the cone.
Answer. 100 cd, 0.8 sr.

(c) The illuminance of a surface whose area is 100 cm^2 is 10^5 lx. What is the magnitude of the luminous flux incident on this surface?
Answer. 10^3 lm.
253. (a) A bulb emits light of luminous intensity of 200 cd, which illuminates the middle of a book lying on the table at an angle of 60° at the illuminance of 70 lx. What is the height of the bulb above the table and its distance from the book?
Answer. 0.6 m, 1.2 m.

(b) The illuminance of a surface due to a parallel beam of light incident on it at 30° is 50 lx. What is the illuminance if the angle of incidence becomes 60°?
Answer. 29 lx.

(c) The exposure time required for photographing an object illuminated by a 100-W bulb at a distance of 1 m is 8 s. What is the exposure time if two 100-W bulbs at distances of 3 m and 4 m from the object being illuminated are used if the total luminous energy flux on the photographic plate remains unchanged?
Answer. 46 s.

(d) A table is illuminated by two identical bulbs suspended at the same height, one of them being on the normal passing through the middle of the table and the other, on a straight line forming an angle of 60° with this normal. The second bulb has fused. By what distance should the first bulb be brought closer to the table to retain the same illuminance at the middle of the table?
Answer. By $0.057h$, where h is the height of the bulbs above the table.

(e) A point source of light creating a luminous flux of 628 lm is placed above a hemisphere of radius 1 m (Fig. 337) at a height equal to its diameter. Find the illuminance at a point of the hemisphere surface such that an angle between the direction of the beam and the normal to the tangent at this point is $\alpha = 37°$.
Answer. $E = \Phi/16\pi R^2 \cos\alpha = 15.6$ lx.

(f) A 100-W bulb is suspended at the upper point of a horizontal tube of 2-m

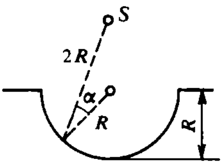


Fig. 337

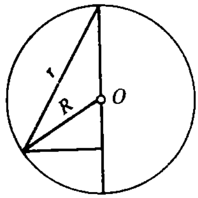


Fig. 338

diameter (Fig. 338). Find the illuminance of the surface of the tube at a height equal to 1/4 of its diameter if the luminous efficiency of the bulb is 18.8 lm/W.
Answer. 43.3 lx.

(g*) A bulb is fixed on a post at 4 m above the ground. Then it is lowered to 3 m above the ground. Find the distance from the base of the post to the point on the ground at which the illuminance has remained unchanged.
Answer. 2.05 m.

254. (a) A standard bulb having a luminous intensity of 20 cd and a tested bulb are placed on a photometer bench at a distance of 2 m from each other. A screen arranged between the bulbs is illuminated uniformly on both sides when the tested bulb is at 1.20 m from the screen. Find the luminous intensity of this bulb.
Answer. 45 cd.

(b) Two bulbs having luminous intensities of 25 cd and 100 cd are on a photometer bench at a distance of 3 m from each other. At what distance from a weaker bulb should a screen be placed to obtain identical illuminance from the two bulbs? Consider two cases.
Answer. 1.0 m, 3.0 m.

(c*) Light sources S_1 and S_2 are mounted on a horizontal bench (Fig. 339) at a distance of 1.0 m between them. The luminous intensity of source S_2 is twice

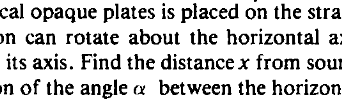


Fig. 339

higher than the luminous intensity of source S_1. The edge of a right dihedron formed by two identical opaque plates is placed on the straight line connecting the sources. The dihedron can rotate about the horizontal axis, and the horizontal bench can slide along its axis. Find the distance x from source S_2 to the edge of the dihedron as a function of the angle α between the horizontal and the plate facing source S_2, for which the edge is invisible.
Answer. $x = \sqrt{2} \tan \alpha/(1 + \sqrt{2} \tan \alpha)$.

(d*) The edge of a right dihedron formed by two identical opaque plates is on the straight line connecting two identical light sources S_1 and S_2 (Fig. 340). Find the ratio l_1/l_2 of the distances from the sources to the dihedron edge as a function

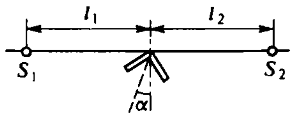


Fig. 340

of the angle formed by the bisector of the dihedron with the normal to the straight line S_1S_2, at which the edge of the dihedron is invisible.
Answer. $l_1/l_2 = \sqrt{(1 - \tan\alpha)/(1 + \tan\alpha)}$.

B. GEOMETRICAL OPTICS

To a first approximation, we can consider the propagation of light disregarding its wave nature and assuming that light propagates in straight lines called rays. This allows us to formulate the laws of optics in the language of geometry. Accordingly, the branch of optics where the wave nature of light is neglected is called **geometrical** (or **ray**) **optics.**

Geometrical optics is based on four laws: (a) the law of rectilinear propagation of light, (b) the law of independence of light rays, (c) the law of reflection, and (d) the law of refraction of light.

The **law of rectilinear propagation of light** states that *light propagates in straight lines in homogeneous media.*

The **law of independence of light rays** states that *rays do not perturb each other upon intersection.*

6.6. Law of Reflection of Light. Construction of Image Formed by a Plane Mirror

When light reaches the interface between two transparent media, a part of it passes to the second medium (is refracted), while the other is reflected to the first medium.

The **law of reflection of light** states that *the reflected ray lies in the same plane with the incident ray and with the normal to the reflecting surface at the point of incidence, the angle of reflection being equal to the angle of incidence* (Fig. 341).

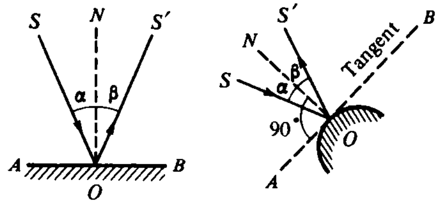


Fig. 341

The *angle of incidence* is the angle α between the normal and the incident ray, while the *angle of reflection* is the angle β between the normal and the reflected ray.

Light rays are reversible. This means that a ray passed against another ray that has covered the distance between points *1* and *2* propagates along the same path but in the opposite direction. In particular, if the ray propagating along SO is reflected in the direction OS', the ray incident along $S'O$ is reflected along OS.

The image of a point source formed by a mirror (or lens) is found at the intersection or continuation of light rays emerging from this point and reaching the eye of an observer.

If the rays emerging from a point source and getting to the eye do not intersect but their continuations do, the image is called *virtual*.

Figure 342 illustrates the construction of the image of a point source S formed by a *plane mirror AB*. For this purpose, it is sufficient to take (see Fig. 342) two rays SO_1 and SO_2 incident at points O_1 and O_2 of the mirror. The reflected rays do not intersect, but their continuations intersect at point S'. The triangles SO_1O_2 and $S'O_1O_2$ are equal since O_1O_2 is their common side, $\angle SO_1O_2 = (90° + \alpha_1) = \angle S'O_1O_2 = (90° + \beta_1)$ and $\angle SO_2O_1 = (90° - \alpha_2) = \angle S'O_2O_1 = (90° - \beta_2)$. Consequently, the heights of the triangles are also equal, and the image of a point source (object) formed by a plane mirror is virtual and is *symmetric* to the point source (object).

If a surface is not a mirror for given rays, diffuse reflection is observed since rays incident on such a surface are reflected in all directions from the nonuniformities (Fig. 343).

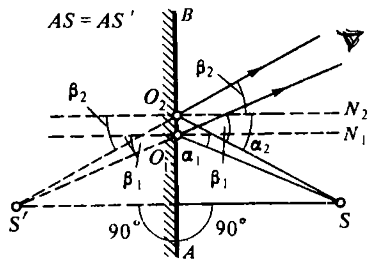


Fig. 342

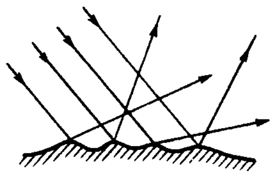


Fig. 343

6.7. Construction of Image Formed by a Spherical Mirror. Spherical Aberration

A *spherical mirror* is a polished surface of a spherical segment reflecting light rays. A concave mirror (Fig. 344) is a spherical segment with the inner mirror surface, while a convex mirror has the outer mirror surface. Let us define the main points, lines, and planes for a spherical mirror.

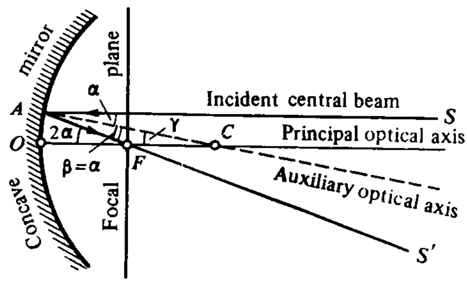


Fig. 344

The centre C of the spherical surface is the **optical centre** of the mirror. The point O at the middle of the mirror is called its **pole**. The normal to the mirror surface passing through the pole is called the **principal optical axis**. The normals drawn at other points of the mirror are called **auxiliary** (or **secondary**) **optical axes**. The rays passing in parallel to the principal optical axis near it are called **central rays**.

The point F at which ray SA parallel to the principal optical axis intersects it after having been reflected from the mirror is called the **focal point** (or **principal focus**) of the mirror. The distance OF from the focal point to the mirror pole is known as the **focal length** of the mirror and is denoted by F. The plane passing through the focal point normally to the principal optical axis is called the **focal plane**.

The focal point of a concave spherical mirror lies at the middle of the radius of the mirror, i.e. its focal length is

$$F = R/2.$$

The focal point of a convex mirror is virtual and lies on the principal optical axis behind the mirror at a distance $F = R/2$ from its pole.

The positions of a point source and of its image formed by a concave spherical mirror are connected through a relation known as the *formula for a concave spherical mirror*:

$$1/d + 1/f = 1/F. \tag{6.7.1}$$

Here (Fig. 345) d is the distance OS from the mirror pole to the

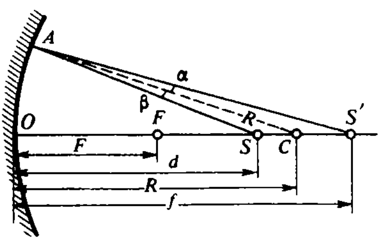


Fig. 345

point source along the principal optical axis of the mirror and f is the distance OS' from the mirror pole to the image of this point also along the principal optical axis.

The *formula for a convex spherical mirror* has the same form as (6.7.1), the only difference being that the distances from the mirror pole to points behind the mirror, i.e. the distances from the pole to the image, are assumed to be negative ($f < 0$). It follows from the formula for the spherical mirror that in this case $F < 0$ as well, i.e. the focal length of a convex mirror is negative.

Formula (6.7.1) shows that if a light source S is transferred to point S', its image coincides with point S.

The image of a point formed by a spherical mirror is obtained graphically as the point of intersection of any two reflected rays or their continuations. In the former case, the image is actual and in the latter, virtual. It is convenient to use any *two* rays shown in Fig. 346 for constructing the image.

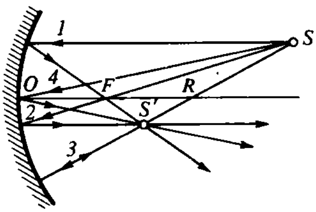


Fig. 346

In Fig. 346, ray *1* is an incident ray parallel to the principal optical axis. Having been reflected, it passes through the focal point (this is clear from formula (6.7.1), where $f = F$ when $d = \infty$).

Ray *2* is an incident ray passing through the focal point. Having been reflected, the ray passes in parallel to the principal optical axis.

Ray *3* is an incident ray passing through the centre of curvature of the mirror, i.e. along any auxiliary optical axis. Having been reflected, it returns along the same path since an optical axis is normal to the spherical surface of the mirror (hence it follows that the images of point sources lying on optical axes are also on the same axes).

Ray *4* is an incident ray passing through the mirror pole. Having been reflected, it passes symmetrically about the principal optical axis.

As has been mentioned above, it is sufficient to take any two of these rays to construct the image of a point formed by a spherical mirror.

The type and the position of the image of an object in a *concave spherical mirror* are mutually related (Fig. 347).

1. An object A is behind the centre of the mirror ($d > 2F$).

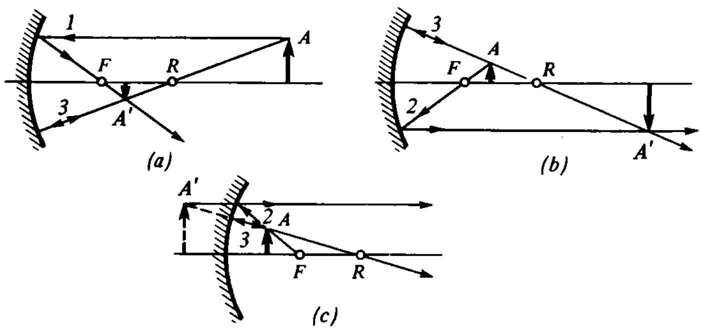


Fig. 347

The image A' is real, reversed, and diminished, and lies between the mirror centre and the focal point ($F < f < 2F$).

2. An object A is between the centre of the mirror and the focal point. The image A' is real, reversed, and magnified, and lies behind the centre of the mirror.

3. An object A is between the mirror and the focal point ($d < F$). The image A' is virtual, erect, and magnified, and lies behind the mirror ($f < 0$).

Some special cases can be easily substantiated by formula (6.7.1). For example, if an object is at a very large distance from the mirror ($d \to \infty$), the rays from the object are converged at the focal point ($f = F$). If an object is at the centre of the mirror ($d = 2F$), the image is real and is at the same distance from the mirror ($f = 2F$). If an object is at the focal point ($d = F$), there is no image since it goes to infinity ($f = \infty$).

The image of an object formed by a *convex spherical mirror* (Fig. 348) is virtual, erect, and diminished.

Remark. Strictly speaking, the above formulas and rules for constructing images can only be used for the rays incident near the pole of the mirror. However, the accuracy of constructing the intersections of such rays is low and inadequate for solving problems. Therefore, a formal method is used while solving problems. This method is based on the assumption that the reflection of light rays by a mirror takes place in a plane passing through the mirror pole and normal to the principal optical axis. Such a solution is the more exact, the smaller the mirror surface: $l \ll R$.

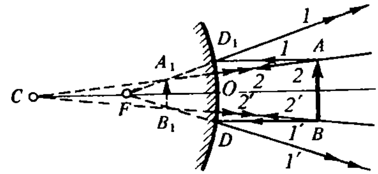


Fig. 348

The relation $F = R/2$ given above (where F is the focal length and R is the radius of a mirror) is not exact. It is the more exact, the closer the incident central ray to the principal optical axis of the mirror. Indeed, for a concave spherical mirror in Fig. 344, in $\triangle AFC$, the angles $\beta = \alpha$ and $\gamma = \alpha$, i.e. $\beta = \gamma$. Consequently, $\triangle AFC$ is isosceles and $AF = FC$.

The focal length $OF \approx AF \cos \angle AFO = FC \cos 2\alpha$, i.e. $F \approx (R - F) \cos 2\alpha$, whence

$$F = R \frac{\cos 2\alpha}{1 + \cos 2\alpha} = R \frac{\cos^2 \alpha - \sin^2 \alpha}{2 \cos^2 \alpha}, \text{ or } F = R \frac{1 - \tan^2 \alpha}{2}.$$

Thus, the spherical mirror does not exactly converge central rays at a single point (focal point). For a concave mirror, the further is a ray incident on a spherical mirror from the optical axis, the larger the displacement of the reflected ray from the source, for a convex mirror, the opposite is true.

The impossibility to converge the reflected rays at one point in this case is known as the **spherical aberration** of the mirror, which makes the image blurred. Spherical aberration can be reduced by using parabolic mirrors in which the radius of curvature increases for points of the surface at larger distances from the pole. But since the manufacture of parabolic mirrors involves considerable difficulties, spherical aberration is normally reduced by combining converging and diverging mirrors into a single optical system.

Problems with Solutions

255. An object is at a distance $d = 20$ cm from a plane mirror. Then it is displaced by $\Delta d_1 = 10$ cm from the mirror in the normal direction and by $\Delta d_2 = 50$ cm in the direction parallel to the mirror surface. What are the initial and final distances between the object and its image?

Solution. The distance from the image to the plane mirror is equal to the distance from the object to the mirror. Consequently (Fig. 349), $l = 2d = 40$ cm. A displacement of the object in the direction parallel to the mirror surface does not change the distance between the object and its image. Therefore, $l_1 = 2(d + \Delta d_1) = 60$ cm.

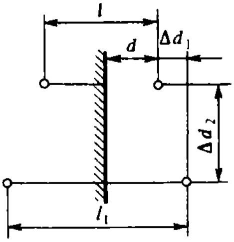


Fig. 349

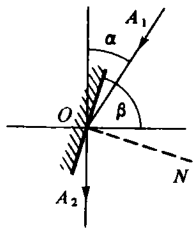


Fig. 350

256. It is necessary to illuminate the bottom of a well by reflected solar beam when the light is incident at an angle $\alpha = 40°$ to the vertical. At what angle β to the horizontal should a plane mirror be placed?

Solution. If A_1O is an incident ray (Fig. 350), while OA_2 is the reflected ray, $\angle A_1OA_2 = 180° - \alpha = 140°$. The normal ON to the mirror plane at the point of incidence of the solar ray forms an angle $\angle NOA_2 = \angle NOA_1 = 70°$ with the vertical. This angle is equal to the required angle β (as the angles with mutually perpendicular sides), i.e. $\beta = 70°$.

257. Two plane mirrors AB and CD are arranged in vertical positions at an angle of 60° to each other (Fig. 351). Construct all images of a point source S formed by these mirrors. What is the number of images? How many images can an observer located at point M see? Draw the borders of the area within which the observer can see all the images.

Solution. The first two images are S_1 formed by mirror AB and S_2 formed by mirror CD. The virtual point source S_1 has image S_3 formed by mirror CD, and point S_2 has image S_4 formed by mirror AB. The images of point S_3 formed by mirror AB and of point S_4 formed by mirror CD coincide at point S_5. Point S_5 is behind the two mirrors, and hence is not reflected by them and does not give any new images. Thus, we obtain five images in all.

Connecting (mentally) point M where the observer is located with images from S_1 to S_5, we see that only the straight line MS_5 passes outside mirror AB which produces image S_5. Hence at point M the observer see all the images but S_5.

To answer the last question, we proceed as follows. We connect points S_1 and S_4, which are the virtual images of the point source formed by mirror AB, with the edges of this mirror and continue the straight lines S_1A_1, S_1B_1 and S_4A_4, S_4B_4. The observer sees two points S_1 and S_4 if he is in the space bounded by the vertical planes containing the straight lines AA_1 and BB_4. To be able to see points S_2 and S_3 formed by mirror CD, the observer must be in the space bounded by planes CC_2

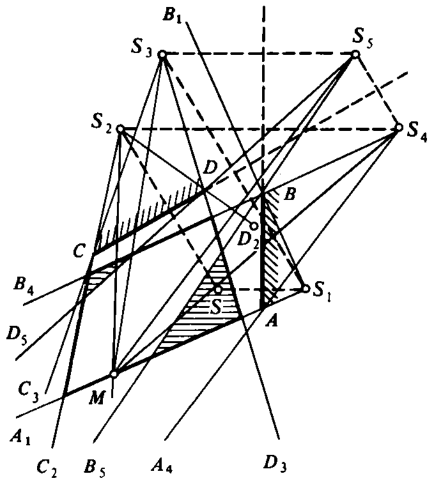


Fig. 351

and DD_3. Thus, we obtain a tetrahedron whose base is bounded by the planes shown in Fig. 351 by thickened lines. Finally, we can see point S_5 formed by mirror AB in front of the vertical plane whose base is the straight line BB_5 and formed by mirror CD behind the plane whose base is the straight line DD_5. Consequently, the observer can see all the five images when he is in the space bounded by two trihedrons whose bases are hatched in Fig. 351.

258. A vertical rod of length $l = 5$ cm is placed at $d = 30$ cm in front of a concave mirror whose radius of curvature $R = 40$ cm. Where is the image of the rod located? What is the type of the image? What is its height? Where should the rod be placed to obtain its virtual image of height $h_1 = 10$ cm? Construct the image.

Solution. The focal length of the mirror is $F = R/2 = 20$ cm. Substituting the values of d and F into the formula $1/d + 1/f = 1/F$ for a concave mirror, we obtain $f = Fd/(d - F) = 60$ cm. Since the rod is between the centre of the mirror and the focal point (Fig. 352*a*), its image is real ($f > 0$), reversed, and magnified. The magnification $k = f/d = 2$. Consequently, the height of the image is $h = kl = 10$ cm.

To obtain a virtual image, the rod should be placed between the mirror and the focal point (Fig. 352*b*). The magnification $k_1 = h_1/l = 2$. On the other hand, $k_1 = |f/d|$ (where $f < 0$). Hence $f = -k_1 d = -2d$. Substituting this value of f into the formula for the mirror, we obtain

$$1/F = 1/d - 1/2d = 1/2d, \text{ whence } d = F/2 = 10 \text{ cm}.$$

259. A vertical rod having a height $h = 5$ cm is at a distance $d = 60$ cm from the pole of a convex mirror whose radius of curvature $R = 40$ cm. Where is the image

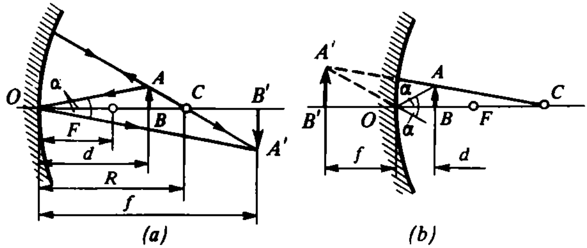


Fig. 352

of the rod located? What is the type of the image? What is its height? Construct the paths of the rays.

Solution. The image is virtual and erect (Fig. 353). The focal length $F = -R/2 = -20$ cm. Using the formula for a spherical mirror, we obtain

$$1/f = 1/F - 1/d = -1/15, \text{ whence } f = -15 \text{ cm}.$$

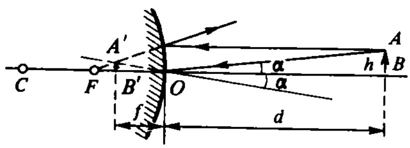


Fig. 353

The magnification $k = |f/d| = 1/4$. The image height $h' = kh = 1.25$ cm.

260. A convex mirror is placed in the path of a convergent light beam so that the point of convergence of the beam lies on the principal optical axis of the mirror at a distance $d = 30$ cm behind its pole. After the mirror has been mounted, the reflected light beam converges in front of the mirror at a distance $f = 50$ cm from the pole. Find the radius of curvature of the mirror. What must be the radius of curvature of the mirror for not the reflected rays themselves but their continuations to converge at the same distance behind the mirror?

Solution. The point of convergence of the light beam behind the mirror can be treated as a virtual light source. The distance from this point and the mirror is ascribed the minus sign. Then the formula for a convex mirror for convergent reflected rays can be written as follows:

$$1/F = 1/d + 1/f_1 = -1/75 \text{ cm}^{-1}, \text{ whence } F = -75 \text{ cm}.$$

The radius of curvature of the mirror is $R = 1.5$ m. In the second case, when the continuations of the reflected rays converge, the formula for a spherical mirror has the form

$$1/F = 1/d + 1/f_2 = -8/150 \text{ cm}^{-1}, \text{ whence } F = -18.75 \text{ cm}.$$

The radius of curvature of the mirror is $R = 0.38$ m.

261. A point light source is placed on the principal optical axis of a concave spherical mirror with the radius of curvature $R = 60$ cm at a distance $d = 40$ cm from its pole. At what distance from the concave mirror should a plane mirror be mounted for the light beam reflected from the latter to return to the point of location of the source?

Solution. The optical diagram in Fig. 354 indicates that point S_1 can be treated

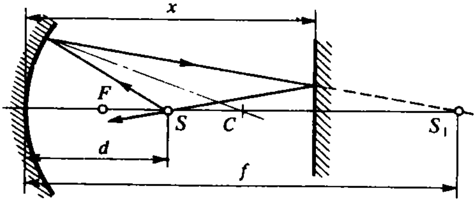


Fig. 354

as the image of source S formed by the plane mirror. At the same time, point S_1 is also the image of source S formed by the spherical mirror in the absence of the plane mirror. Therefore, we can write for the plane and spherical mirrors

$$x = (f + d)/2, \; 2/R = 1/d + 1/f.$$

Solving these equations together, we obtain $x = d^2/(2d - R) = 0.80$ m.

262. A point light source is placed at the middle between a screen and a plane mirror parallel to it. What is the ratio of illuminances at the point of the screen lying on the perpendicular drawn from the point source to the mirror in the presence and absence of the mirror? The losses of luminous energy in reflection from the mirror should be neglected.

Solution. In the absence of the mirror, the illuminance at the given point is $E_1 = I/r^2 = 4I/l^2$. When the mirror is mounted, the illuminance of the surface of the mirror is equal to the sum of illuminances E_1 and E_2, where E_1 is the illuminance due to the light from the source and E_2 is the illuminance created by the light reflected from the mirror. The distance from the virtual image of the source to the screen is $R = l + l/2 = 3l/2$. For an ideally reflecting mirror, the luminous flux from the light source is equal to the luminous flux from its virtual image: $\Phi_2 = \Phi_1$. Consequently, the luminous intensity $I_2 = I_1$ since $I = \Phi/4\pi$. The illuminance at the given point of the screen due to the reflected light is $E_2 = I/R^2 = 4I/9l^2$. The total illuminance at this point is $E = E_1 + E_2 = 40I/9l^2$. Therefore, $E/E_1 = 10/9$.

Exercises

255. An object in front of a plane mirror is displaced by 0.40 m along a straight line at an angle of 30° to the mirror plane. What is the change in the distance between the object and its image?

Answer. 0.40 m.

256. (a) At what angle to the horizontal surface of a table should a plane mirror be placed to obtain the image of an object lying on the table in a vertical plane?
Answer. 45°.

(b) At what angle to the surface of a desk should a plane mirror be arranged to obtain the image of an object lying on the desk in a vertical plane? The desk is inclined at an angle of 20° to the horizontal.
Answer. Two cases: 35° and 55°.

(c) Find the angle between the incident and reflected light rays if reflection occurs twice from two mirrors forming an acute angle φ.
Answer. 2φ.

(d) A plane mirror reflecting a beam of light is turned through an angle α about an axis lying in the plane of the mirror at right angles to the beam. By what angle is the reflected beam turned?
Answer. 2α.

(e*) A plane mirror rotating at an angular velocity of 3 rad/s reflects a light beam. Find the angular velocity of the reflected beam.
Answer. 6 rad/s.

257. (a) A point light source and its two primary images formed by two plane mirrors arranged at an angle to each other lie at the vertices of an isosceles triangle with the angles of 75° at the base. Find the angle between the mirrors and determine their positions relative to the source of light. Do the mirrors form some other images?
Answer. Two cases: 150° and 105°. The mirrors are arranged at the middles of the triangle sides at right angles to them. In the first case, there are no other images; in the second case, there is one more image, Fig. VII.

(b) Object S_1S_2 and mirror AB are arranged as shown in Fig. 355. Where should an observer be to be able to see the complete image of the object?

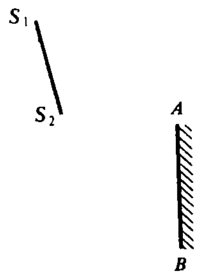


Fig. 355

Answer. Between lines AA and BB on the continuation of $S_1'A$ and $S_2'B$, where S_1' and S_2' are the images of points S_1 and S_2, Fig. VIII.

(c) A person should be able to see his full image in a plane vertical mirror without turning his head. What must be the minimum height of the mirror and the level of its upper edge? Plot an optical diagram.
Answer. The height of the mirror should be half that of the person, and its upper

edge must be at about the middle between the level of the eyes and the top of the head, Fig. IX.

(d) Determine the number of images of a point source located between two mirrors at right angles to each other and the positions of these images. Determine graphically the space where an observer should be to be able to see all the images.
Answer. Three images, Fig. X.

(e*) Two light sources S_1 and S_2 are at a distance of 105 cm from each other. Two plane mirrors, one at 60 cm from source S_1 and the other at 37.5 cm from source S_2, are arranged so that images S_1' and S_2' of the sources coincide. Find the angle between the mirrors. Construct an optical diagram.
Answer. 120°, Fig. XI.

258. (a) A concave spherical mirror forms a 40-cm high real image of an object whose height is 10 cm. The radius of the mirror is 60 cm. Find the distance from the object to its image. Determine the same distance for the case of virtual image. Plot an optical diagram.
Answer. 112.5 cm, 112.5 cm, Fig. XII.

(b) An object is at 80 cm from its image formed by a concave spherical mirror. The image is real and diminished to 1/3 of the object height. Find the radius of curvature of the mirror and the distance from the object to the mirror. Plot an optical diagram.
Answer. 0.60 m, 1.20 m, Fig. XIII.

(c) An object is at 80 cm from its virtual image formed by a concave spherical mirror, the image being magnified thrice. Find the radius of curvature of the mirror and the distance from the object to the mirror. Plot an optical diagram.
Answer. 0.60 m, 0.20 m, Fig. XIV.

(d*) The real image of an object formed by a concave spherical mirror is magnified three times in comparison with the object. If the mirror is displaced by 40 cm along its principal optical axis, the magnification turns out to be the same. Find the radius of curvature of the mirror.
Answer. 1.20 m.

259. (a) A convex spherical mirror forms a 10-cm image of an object whose height is 40 cm. The radius of curvature of the mirror is 60 cm. Find the distance from the image to the object. Plot an optical diagram.
Answer. 112.5 cm, Fig. XV.

(b) An object is at 80 cm from its image formed by a convex spherical mirror, the image being diminished three times in comparison with the object. Find the radius of curvature of the mirror and the distance from the object to the mirror. Plot an optical diagram.
Answer. 60 cm, 60 cm, Fig. XVI.

(c*) The image of an object formed by a convex spherical mirror is three times smaller than the object. If the mirror is displaced by 20 cm along its principal optical axis, the image becomes five times smaller than the object. Find the radius of curvature of the mirror.
Answer. 20 cm.

260*. (a) A convergent light beam is incident on a convex mirror whose radius of curvature is 1.20 m. The reflected rays are converged in front of the mirror on its

principal optical axis at a distance of 0.40 m from the mirror. At what distance from the mirror do the continuations of incident rays converge?
Answer. 0.24 m.

(b) Convergent rays are incident on a convex mirror with the focal length of 30 cm so that their continuations intersect behind the mirror on its axis at a distance of 15 cm from it. Find the distance from the mirror to the point of convergence of these rays after the reflection.
Answer. 30 cm.

(c) Convergent rays are incident on a concave mirror with a focal length of 30 cm so that their continuations intersect behind the mirror on its axis at a distance of 15 cm from it. Find the distance from the mirror to the point of convergence of these rays after the reflection.
Answer. 10 cm.

(d) A convergent light beam is incident on a convex mirror whose radius of curvature is 0.40 m. The reflected rays are converged on the principal optical axis of the mirror at 0.30 m from its pole. At what distance from the mirror pole do the continuations of the incident rays converge?
Answer. 0.12 m.

261*. (a) A point light source is placed between a plane and a concave spherical mirror on the principal optical axis of the latter so that, after having been reflected from the spherical mirror and then from the plane mirror, the rays from the source converge at the point where the source is located. The radius of curvature of the spherical mirror is 0.75 m and the distance between the mirrors is 1.00 m. Find the distance from the pole of the spherical mirror to the point source.
Answer. 0.50 m.

(b) Two identical concave mirrors face each other in such a way that their principal optical axes and focal points coincide. A point light source is on their principal optical axis at a distance d from one mirror. Find the image of the source after the reflection from the two mirrors.
Answer. The image coincides with the point source.

(c) A point source of light is located on the principal optical axis of a concave spherical mirror with the radius of curvature of 60 cm at a distance of 40 cm from its pole. At what distance from this mirror should a convex spherical mirror of the same radius be placed to return the light beam reflected from it to the point where the source is located? Plot an optical diagram.
Answer. 1.00 m. It is more convenient to solve the problem numerically.

(d) A point light source is placed on the principal optical axis of a concave spherical mirror with the radius of curvature of 60 cm at a distance of 40 cm from the mirror pole. When another concave spherical mirror whose principal optical axis coincides with that of the first mirror is arranged at 70 cm from the latter, the light beam reflected from it returns to the point where the source is located. Find the radius of curvature of the second mirror.
Answer. 0.50 m.

262. A light source emitting a luminous flux of 800 lm is placed between a plane mirror and a screen parallel to the mirror and arranged at 2.0 m from it. The light source is at 0.5 m from the mirror. Find the illuminance of the screen at the point

of its intersection with the normal dropped from the light source on the screen if the luminous energy losses in reflection amount to 25% of the total energy flux incident on the mirror. Plot an optical diagram.
Answer. 36 lx, Fig. XVII.

6.8. Laws of Refraction of Light. Refractive Index

Laws of refraction of light state that the *refracted ray lies in the same plane with the incident ray and the normal to the refracting surface, erected at the point of incidence, and the ratio of the sines of the angle of incidence α and of the angle of refraction i is a constant quantity for two given media.* The *angle of refraction i* is the angle between the normal and the refracted ray (Fig. 356).

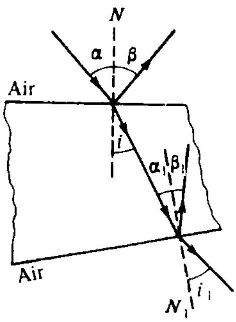


Fig. 356

The ratio of the sines of the angle of incidence and the angle of refraction is known as the **relative refractive index** n_{21} of the second medium with respect to the first medium:

$$n_{21} = \sin \alpha / \sin i.$$

The refractive index of a substance relative to a vacuum (a ray passes through the vacuum to the given medium) is called the **absolute refractive index** of the substance.

The relative refractive index for two media is equal to the ratio of their absolute refractive indices:

$$n_{21} = n_2 / n_1,$$

where n_2 and n_1 are the absolute refractive indices of the second and first media. Obviously, we can write

$$n_{21} = 1/n_{12}.$$

In refraction, as well as in reflection, light rays are reversible.

The absolute refractive index of air (under the atmospheric pressure) differs from unity very slightly (it is equal to 1.0003). For this reason, we can assume that the absolute refractive index of a substance is practically equal to its refractive index relative to air.

The larger the absolute refractive index, the higher the optical density of the medium.

Absolute Refractive Indices for Several Substances

Diamond		2.4
Water		1.33
Ice		1.31
Alcohol		1.36
Light crown glass, quartz, rock salt, and sugar		1.5
Dense flint glass[4]	up to	1.8

The absolute refractive index is equal to the ratio of the velocity of light c in a vacuum to the velocity of light u in a medium:

$$n = c/u.$$

Since light waves are of electromagnetic nature, the light velocity in a medium is expressed by the formula given in Sec. 5.19:

$$u = c/\sqrt{\varepsilon_r \mu_r},$$

where ε_r is the relative permittivity of the medium and μ_r is the relative permeability. Consequently,

$$n = c/u = \sqrt{\varepsilon_r \mu_r}.$$

For dielectrics (including glass, water, oil, and kerosene) $\mu_r = 1$. Consequently, $n = \sqrt{\varepsilon_r}$ for these materials.

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[4] The refractive index of glass is the higher, the larger the lead admixture in it.

The relative refractive index of medium *2* with respect to medium *1* is equal to the ratio of the velocities of light in the first and second media:

$$n_{21} = u_1/u_2.$$

6.9. Total Internal Reflection. Critical Angle

A light beam incident on the interface between two media having different optical densities splits into two beams, viz. the reflected and refracted beams.

The energy of a light beam is characterized by its **intensity** which is defined as the luminous energy transferred by the beam per unit time through a unit surface. If the angle of incidence $\alpha = 0°$, the intensity of the reflected beam is minimum and does not exceed 10% (this energy is the higher, the larger the ratio of refractive indices of media in contact). *As the angle of incidence increases, the intensity of the reflected beam increases and that of the refracted beam decreases.* Accordingly, the fraction of reflected luminous energy increases and the fraction of luminous energy passing through the interface decreases.

If light passes from a medium with a higher optical density to a less dense medium, e.g. from water to air (Fig. 357), the angle of

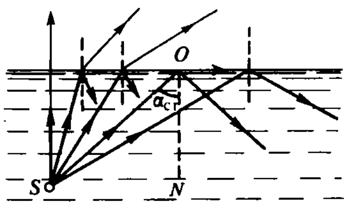

Fig. 357

refraction is larger than the angle of incidence ($i > \alpha$) and attains 90° when the angle of incidence $\alpha < 90°$. At such an angle of incidence, the refracted ray slides over the interface (ray *SO* in Fig. 357). As the angle of incidence α increases further, the light beam is not refracted but is reflected from the interface into the medium with the higher optical density.

The angle of incidence α_{cr} at which the angle of refraction $i = 90°$ and the entire luminous energy is reflected from the interface is known as the **critical angle of the total internal reflection.** The reflection of rays from the medium with the lower optical density at an angle $\alpha \geqslant \alpha_{cr}$ is called the **total internal reflection**.

The value of the angle α_{cr} can be easily calculated if the refractive index of a medium is known. For example, the critical angle for a transition from some medium to air is obtained from the ratio

$$n = \sin 90°/\sin \alpha_{cr}, \text{ whence } \sin \alpha_{cr} = 1/n,$$

where n is the refractive index of the given medium. Consequently, the sine of the critical angle is the reciprocal to the refractive index of the medium.

Critical Angles of Total Internal Reflection for Several Materials

Diamond	24°
Water	49°
Light crown glass, quartz, rock salt, and sugar	42°
Dense flint glass	35°

The total internal reflection can be observed when light propagates from an optically denser medium to a less dense medium. This phenomenon is used for manufacturing right prisms of total internal reflection.

6.10°. Ray Path in a Plane-Parallel Plate. Ray Path in a Prism

When a light ray is incident on the face of a plane-parallel plate, it is partially both reflected and refracted. The refracted ray leaves the plate with a certain displacement but in parallel to the incident ray (Fig. 358). An object observed through such a plate seems to

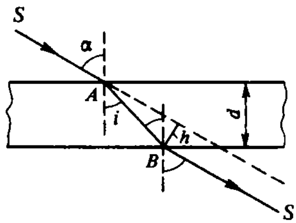

Fig. 358

be displaced relative to its actual position (since it is observed along the ray leaving the plate).

Using the diagram in Fig. 358, we obtain $h = AB \sin(\alpha - i)$, where $AB = d/\cos i$. Consequently, $h = d \sin(\alpha - i)/\cos i = d(\sin\alpha - \cos\alpha \tan i)$. This expression and Fig. 358 indicate that the displacement of the ray increases with the thickness d of the plate, the refractive index n, and the angle of incidence α.

Let us consider the ray paths in a prism. A ray of light is deflected to the base of a trihedral prism (Fig. 359*a*) made of an

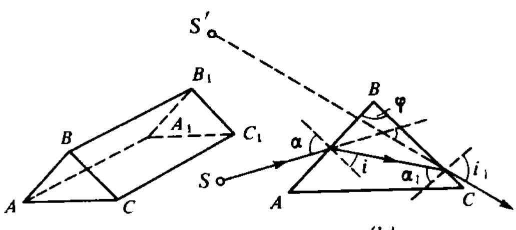

Fig. 359

optically denser substance than the surrounding medium (Fig. 359*b*). The image S' of object S seen through such a trihedral prism is virtual and deflected towards the vertex of the refracting angle.

Right totally reflecting prisms are of two types: (a) *deflecting prism* (Fig. 360*a*) intended for rotating a ray through 90° (and used, in particular, in prismatic periscopes) and (b) *inverting* (or *erecting*) *prism* (Fig. 360*b*) which turns the image by 180° (used, for example, in prismatic binoculars). Total internal reflection in these prisms is due to the fact that the angle of incidence

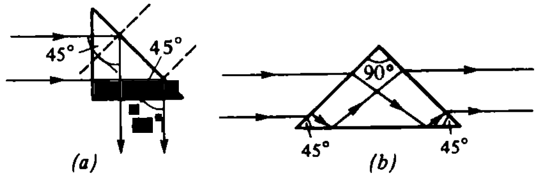

Fig. 360

$\alpha = 45° > \alpha_{cr} = 42°$ (for glass). Totally reflecting glass prisms have a simpler structure than mirrors and are more advantageous since they retain their reflecting properties for a longer time.

6.11. Converging and Diverging Lenses

A **lens** is a transparent polished body bounded on two sides by curved surfaces. In particular, one of the surfaces may be plane.

Spherical lenses have spherical surfaces as bounds.

Converging lenses convert a parallel beam of incident rays into a convergent beam. Converging lenses are convex, i.e. such that the thickness at the middle is larger than the thickness of edges. They include convexo-convex, plano-convex, and cancavo-convex lenses (Fig. 361).

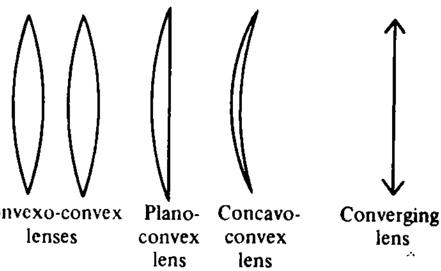

Fig. 361

Diverging lenses convert a parallel beam of rays into a divergent beam. Diverging lenses are concave, i.e. such that the thickness at their edges is larger than the thickness at the middle. They include concavo-concave, plano-concave, and convexo-concave lenses (Fig. 362).

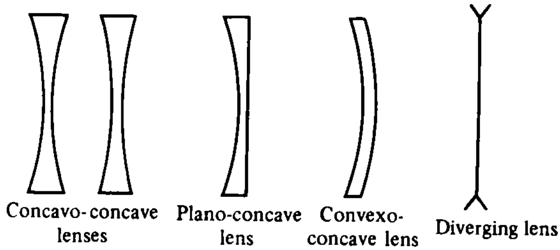

Fig. 362

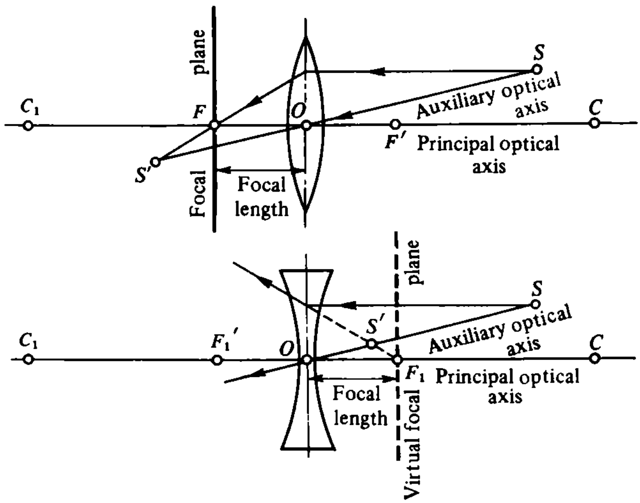

Fig. 363

The straight line passing through the centres C and C_1 of the spherical surfaces of a lens is called its **principal optical axis** (Fig. 363). Point O on the optical axis of a lens through which a ray passes without changing its direction is called the **optical centre** of the lens. Any straight line passing through the optical centre at an angle to the principal optical axis is called an **auxiliary optical axis**.

Point F on the principal optical axis of a converging lens, at which rays incident on the lens in parallel to its principal optical axis converge, is known as the **focal point** of the lens. Point F_1 on the principal optical axis of a diverging lens, at which the continuations of parallel rays incident on the lens and divergent after refraction intersect, is known as the **virtual focal point** of the lens. Each lens has two focal points on both sides of it.

The distance from the centre of a lens to the focal point is called the **focal length** of the lens. It is assumed that the focal length of a converging lens is positive and that of a diverging lens, negative. The planes passing through the focal points of a lens and perpendicular to the principal optical axis are the **focal planes** of the lens.

6.12. Lens Formula. Lens Power[5]

The lens formula has the form

$$1/d + 1/f = 1/F, \tag{6.12.1}$$

where d is the distance from the point source to the optical centre of the lens along the principal optical axis (in the same way as for a spherical mirror, d is the distance from the point source to the mirror pole), f is the distance from the image of the source to the optical centre of the lens (also along the principal optical axis), and F is the focal length of the lens, which is positive for converging lenses and negative for diverging lenses (Fig. 364).

Formula (6.12.1) indicates that a point source and its image are conjugate points. Formula (6.12.1) for a convex lens is similar to the formula for a concave mirror.

The reciprocal to the focal length of the lens in metres is known as the **lens power** and measured in diopters:

$$D = 1/F.$$

A **diopter** (D) is the power of a lens whose focal length F is equal

[5] In this and the following sections, formulas and arguments do not refer to the case when a convergent beam is incident on a lens, but can be applied to convergent beams also if a source is considered to be a virtual source located at the point of intersection of the rays of the convergent beam or of their continuations.

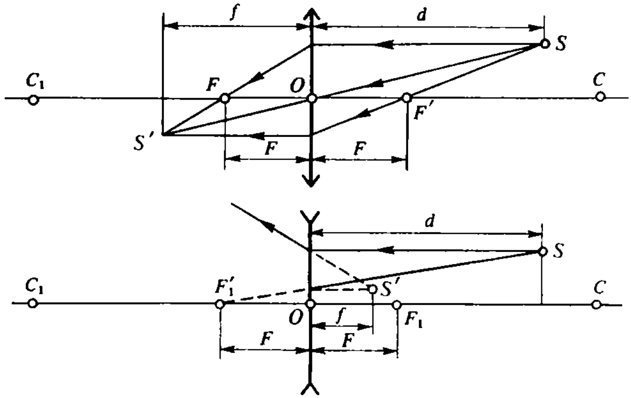

Fig. 364

to one metre. The lens power is calculated by the following formula:

$$1/F = (n - 1)(1/R_1 + 1/R_2),$$

where n is the refractive index of the material of the lens relative to the medium in which the lens is placed, and R_1 and R_2 are the radii of curvature of the lens surfaces, which are assumed to be positive for convex surfaces and negative for concave surfaces ($1/R = 0$ for a plane surface of a lens). This formula shows that the lens power is positive for a converging lens and negative for a diverging lens.

The power of a complex optical system consisting of several lenses in contact, which have the common principal optical axis, is equal to the sum of the powers of all the lenses in the system.

6.13. Image Formation by a Lens

The image of a point formed by a lens can be obtained graphically as the point of intersection of two refracted rays. It is convenient to use for the construction any two rays from those shown in Fig. 365: ray *1* is an incident ray parallel to the principal optical axis, which after having been refracted by the lens, passes through its focal point ($f = F$ for $d = \infty$); ray *2* is an incident ray passing

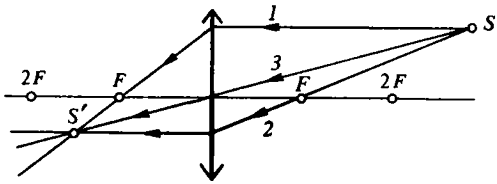

Fig. 365

through the focal point, which after having been refracted, propagates in parallel to the principal optical axis; ray *3* passes through the optical centre of the lens without changing its direction, since the surfaces of the lens are parallel at the optical centre. Like a ray passing through a plane-parallel plate, this ray, retaining its direction of propagation, is slightly displaced. For thin lenses, however, this displacement is so small that it can be disregarded in practice, and the images of point sources lying on optical axes of a lens can be assumed to be on the same axes.

While constructing an image formed by a lens, the latter is assumed to be thin. The ray path can be plotted in the plane passing through the optical centre of the lens at right angles to the principal optical axis. It is convenient to use the conventional representation of the lens (see Figs. 361 and 362).

The direction of the third ray shown in Fig. 365 coincides with an auxiliary optical axis. A point source lying on any optical axis has the image of the same axis.

The method of constructing images formed by lenses, shown in Fig. 366, is a special case of the general method of construction with the help of auxiliary optical axes, i.e. the axes of any direction, passing through the optical centre. We take an auxiliary optical axis of any direction and draw the ray from the point source parallel to this axis. The refracted ray passes though the point F' of intersection of the auxiliary optical axis with the focal plane. The second ray is drawn through the point F'' of intersection of the auxiliary optical axis with the second (image-side) focal plane. After having been refracted, it propagates in parallel to the given auxiliary axis. This general method allows us to construct the image of any point, including the one lying on the principal optical axis.

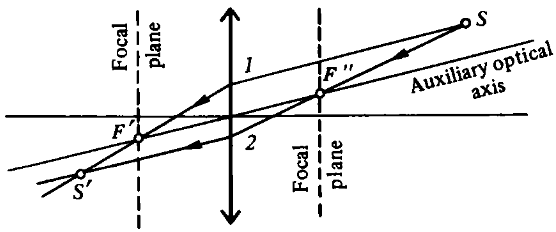

Fig. 366

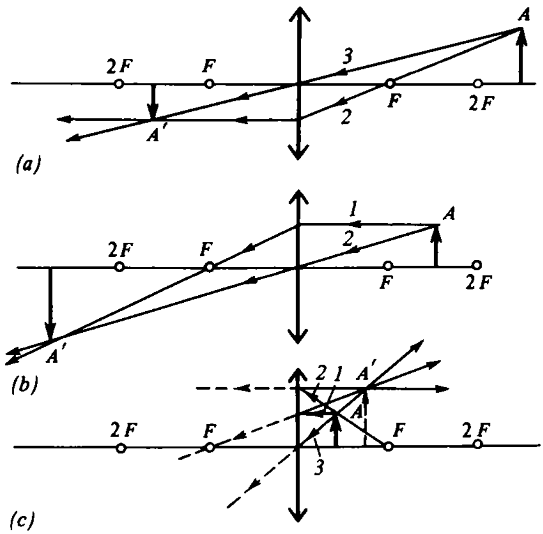

Fig. 367

The types and positions of the image of an object formed by a *converging lens* are shown in Fig. 367.

1. An object is behind the double focal length of a lens ($d > 2F$ as, for example, in photographing). The image is real, reversed, and diminished, and lies on the other side of the lens between the focal point and the point at the double focal length ($F < f < 2F$, Fig. 367*a*).

2. An object is between the focal point and the point at the

double focal length (as in a projection lantern). The image is real, reversed, and magnified, and lies on the other side of the lens behind the double focal length ($f > 2F$, Fig. 367*b*).

3. An object is between the focal point and the lens ($d < F$). The image is virtual, erect, and magnified, and lies on the same side of the lens as the object is ($f < F$, Fig. 367*c*). The virtual image is always behind the object, which can be easily shown by analyzing the converging lens formula:

$$1/d - 1/|f| = 1/F.$$

We get

$$1/|f| = 1/d - 1/F, \text{ i.e. } 1/|f| < 1/d, \text{ or } |f| > d.$$

When $d \to 0$, $f \to 0$ as well, since otherwise the difference between the infinitely large quantity $1/d$ and the finite quantity $1/|f|$ would be equal to the finite quantity $1/F$.

Several special cases are possible here [see formula (6.12.1)]: if an object is at a very large distance from a lens ($d \to \infty$), the rays from the object converge at the focal point of the lens ($f = F$); if an object is at the double focal length of a lens ($d = 2F$), the image is real, reversed, and nonmagnified, and lies on the other side of the lens at the same distance ($f = 2F$); if an object is at the focal point ($d = F$), the lens forms no image, it "goes to infinity" ($f \to \infty$).

The image formed by a *diverging lens* (Fig. 368) is virtual, erect, and diminished, and always lies on the same side of the lens

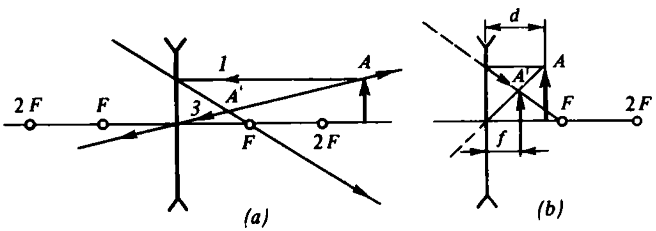

Fig. 368

as the object does. The virtual image of an object is always closer to the lens than the object itself. This can be proved by considering the formula for a diverging lens in the form

$$1/d - 1/|f| = -1/|F|.$$

We get

$$1/|f| = 1/d + 1/|F|, \text{ i.e. } 1/|f| > 1/d, \text{ or } |f| < d.$$

When $d \to 0$, $f \to 0$ for the reason which has been mentioned above for the virtual image formed by a converging lens.

The ratio k of the linear dimensions h of the image of an object to the linear dimensions H of the object itself is called the *magnification* of the lens:

$$k = h/H = |f/d|.$$

This formula can be easily derived from the similarity of triangles in any optical diagram in Figs. 367 and 368.

Problems with Solutions

263. Find the angle of deflection of a narrow light beam from its initial direction as it passes from glass to air if the angle of incidence is (a) $\alpha_1 = 30°$ and (b) $\alpha_2 = 45°$. The refractive index of glass is $n = 1.5$. Find the velocity of light in this glass.

Solution. As the light beam passes from glass to air, it is deflected from the normal to the interface so that the angle of refraction is larger than the angle of incidence (Fig. 369*a*). From the formula $n = \sin i/\sin \alpha$, we obtain $\sin i = n \sin \alpha$.

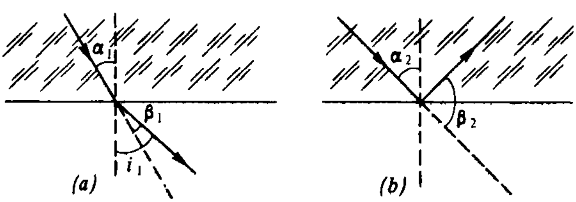

Fig. 369

For $\alpha_1 = 30°$ (Fig. 369*a*), $\sin i_1 = 1.5 \sin \alpha_1 = 0.75$. The angle $\beta_1 = i_1 - \alpha_1$, and hence

$$\sin \beta_1 = \sin i_1 \cos \alpha_1 - \cos i_1 \sin \alpha_1 = 0.32,$$

whence $\beta_1 = \arcsin 0.32$. For sufficiently small angles, we can assume that $\beta_1 \approx 0.32$ rad $\approx 18°20'$ (according to the table of sines, $\beta_1 = 18°40'$).

For $\alpha_2 = 45°$ (Fig. 369*b*), $\sin i_2 = 1.5 \sin \alpha_2 = 1.06$, which is impossible. Obviously, the given angle exceeds the critical angle of the total internal reflection.

Indeed, $\sin \alpha_{cr} = 1/n = 1/1.5 = 0.667$, whence $\alpha_{cr} = 42° < 45°$. Consequently, in this case the total internal reflection takes place, and $\beta_2 = 180° - 2\alpha_2 = 90°$.

Using the formula $n = c/u$, we obtain the velocity of light in the glass having the given refractive index: $u = c/n \approx 2 \times 10^8$ m/s.

264. A beam of parallel rays of width $b = 20$ cm propagates in glass at an angle $\varphi = 60°$ to its plane face. Find the beam width after it goes over to air through this face. The refractive index of glass is $n = 1.8$.

Solution. $b_1 = AB \cos i$ (Fig. 370), $AB = b/\cos \alpha = b/\sin \varphi$, where α is the angle of incidence. Consequently, $b_1 = b \cos i/\sin \varphi$. Here $\cos i = \sqrt{1 - \sin^2 i}$, while $\sin i/\sin \alpha = n$, i.e. $\sin i = n \sin \alpha = n \cos \varphi$, whence $\cos i = \sqrt{1 - n^2 \cos^2 \varphi}$. Therefore, $b_1 = b\sqrt{1 - n^2 \cos^2 \varphi}/\sin \varphi = 10.1$ cm.

265. A point source S placed in a medium with a refractive index n_1 is observed from a medium with a refractive index $n_2 > n_1$ at a small angle β to the normal to the interface between the media. What is the apparent distance from point S to the interface if the actual distance is h?

Solution. Figure 371 shows the ray path from the point source to the observer's eye. Since $n_2 > n_1$, the refracted ray is closer to the normal: $\sin \beta/\sin \alpha = n_1/n_2$.

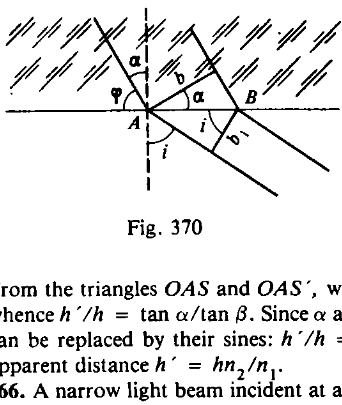

Fig. 370

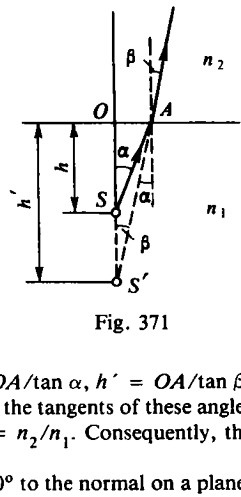

Fig. 371

From the triangles OAS and OAS', we obtain $h = OA/\tan \alpha$, $h' = OA/\tan \beta$, whence $h'/h = \tan \alpha/\tan \beta$. Since α and β are small, the tangents of these angles can be replaced by their sines: $h'/h = \sin \alpha/\sin \beta = n_2/n_1$. Consequently, the apparent distance $h' = hn_2/n_1$.

266. A narrow light beam incident at an angle $\alpha = 60°$ to the normal on a plane-parallel plate having a thickness $d = 20$ cm and a refractive index $n = 1.5$ is partially reflected from the front face and partially refracted. The refracted ray reaches the rear face, where it is again divided. One ray is refracted and leaves the plate, while the other is reflected and goes to the upper (front) face, where, after having been refracted, it emerges from the plate (Fig. 372). Find the displacement of the light beam emerging from the rear face of the plate and the distance BD between the rays reflected from the two faces.

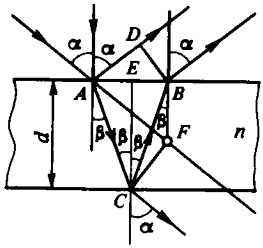

Fig. 372

Solution. Figure 372 shows that

$$CF = AC \sin(\alpha - \beta) = d \sin(\alpha - \beta)/\cos\beta$$
$$= d(\sin\alpha\cos\beta - \cos\alpha\sin\beta)/\cos\beta = d(\sin\alpha - \cos\alpha\tan\beta), \quad (1)$$
$$BD = AB\sin(90° - \alpha) = AB\cos\alpha = 2AE\cos\alpha = 2d\tan\beta\cos\alpha. \quad (2)$$

Since $\sin\alpha/\sin\beta = n$,

$$\sin\beta = (\sin\alpha)/n = \sqrt{3}/3;$$

we can easily find

$$\tan\beta = \sqrt{2}/2.$$

Substituting these values into Eqs. (1) and (2), we obtain $CF = 10.25$ cm and $BD = 14.1$ cm.

267. Two transparent plane-parallel plates are stacked face-to-face. The plate whose thickness $d_1 = 4$ cm has a refractive index $n_1 = 2.0$, while the other plate whose thickness $d_2 = 6$ cm has a refractive index $n_2 = 1.5$. A narrow light beam is incident on the first plate at an angle $\alpha = 37°$ to the normal. Find the lateral shift of the beam emerging from the second plate into air space.

Solution. The ray path through the plates is shown in Fig. 373. The required displacement is the sum of two displacements: *Bb* in the first plate and *Cc* in the

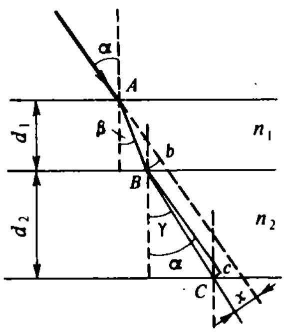

Fig. 373

second:

$$Bb = AB \sin(\alpha - \beta) = \frac{d_1}{\cos\beta}\sin(\alpha - \beta) = d_1(\sin\alpha - \cos\alpha\tan\beta),$$

$$Cc = BC \sin(\alpha - \gamma) = \frac{d_2}{\cos\gamma}\sin(\alpha - \gamma) = d_2(\sin\alpha - \cos\alpha\tan\gamma).$$

The total displacement is

$$x = d_1(\sin\alpha - \cos\alpha\tan\beta) + d_2(\sin\alpha - \cos\alpha\tan\gamma). \quad (1)$$

The second law of refraction yields

$$\frac{\sin\alpha}{\sin\beta} = n_1,\ \sin\beta = \frac{\sin\alpha}{n_1} = \frac{3}{10},\ \tan\beta = \frac{\sin\beta}{\sqrt{1 - \sin^2\beta}} = \frac{3}{\sqrt{91}},$$

$$\frac{\sin\beta}{\sin\gamma} = \frac{n_2}{n_1},\ \sin\gamma = \frac{n_1}{n_2}\sin\beta = \frac{2}{5},\ \tan\gamma = \frac{2}{\sqrt{21}}.$$

Substituting the obtained values into Eq. (1), we obtain $x = 2.9$ cm.

268. A monochromatic ray is incident on a vertical face of a transparent prism having a right triangle in its cross section. The refractive index of the prism material is $n = 1.6$. The angle of refraction of the prism is $\varphi = 30°$. Find the angle of deflection of the beam from the initial direction if it is incident on the face at right angles.

Solution. Figure 374 shows that $\beta = i - \alpha$. Using the equality $n = \sin i/\sin\alpha$, we can find the angle of refraction: $\sin i = n\sin\alpha$. Since $\alpha = \varphi$, $\sin i = n\sin\varphi = 0.8$, i.e. $i \simeq 53°$. Therefore, the angle of deflection is $\beta = 53° - 30° = 23°$.

269. Solve the previous problem for the case when the angle of incidence of the ray on a vertical face of the prism is $\alpha = 30°$, the refractive index of the prism is $n = 1.5$, and the ray is directed vertically (a) upwards and (b) downwards.

Solution. (a) The angle of refraction of a ray upon a transition from air to the prism on its vertical face can be found from the condition $\sin i = \sin\alpha/n = 1/3$ (Fig. 375*a*). For the angle of incidence on the inclined face of the prism

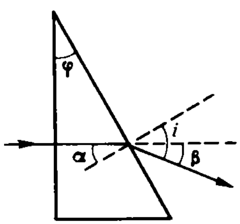

Fig. 374

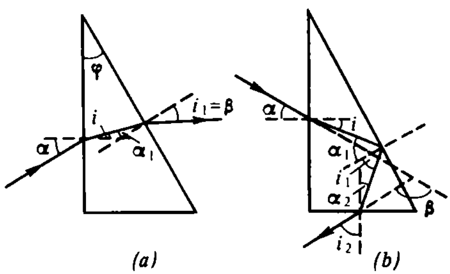

Fig. 375

($\alpha_1 = \varphi - i$), we have

$$\sin \alpha_1 = \sin (\varphi - i) = \sin \varphi \cos i - \cos \varphi \sin i = 0.183.$$

Using the condition $n = \sin i_1/\sin \alpha_1$, we obtain $\sin i_1 = n \sin \alpha_1 = 0.275$. Assuming arcsin $0.275 \approx 0.275$, we get $i_1 \approx 0.275 = 16^\circ$. Since the normal to the inclined face of the prism is parallel to the incident ray, the angle of deflection is $\beta = i_1 \approx 16^\circ$.

(b) As in case (a), $\sin i = 1/3$ (Fig. 375*b*). For the angle of incidence on the inclined face of the prism ($\alpha_1 = \varphi + i$), we have

$$\sin \alpha_1 = \sin \varphi \cos i + \cos \varphi \sin i = 0.758.$$

Like in case (a), $\sin i_1 = n \sin \alpha_1$. But in this case $\sin i_1 > 1$, which is impossible. Therefore, when a ray is incident on the inclined face, the total internal reflection takes place: $i_1 = \alpha_1$.

The angle of incidence of the reflected ray on the lower horizontal face is $\alpha_2 = 90^\circ - (2\alpha_1 - i)$, and since $\alpha_1 = \varphi + i$, we obtain $\alpha_2 = 90^\circ - (2\varphi + i) = 30^\circ - i$. For case (a), we find that $\sin (30^\circ - i) = 0.183$, hence $\sin \alpha_2 = 0.183$. Further, from the equation $n = \sin i_2/\sin \alpha_2$, we find $\sin i_2 = n \sin \alpha_2 = 0.275$, whence $i_2 = 16^\circ$. As can be seen from Fig. 375*b*, the angle of deflection is $\beta = 90^\circ - \alpha + i_2 = 76^\circ$.

270. A luminous object of height $h = 40$ cm is at a distance $d = 1$ m from a vertical diverging lens with a focal length $F = -25$ cm. Where is the image of the object? Find the height of the image and the lens power.

Solution. Using the lens formula $1/F = 1/d + 1/f$, we obtain the distance from the image to the lens: $f = Fd/(d - F) = -20$ cm. The image height $h' = hk = h\,|f/d| = 8$ cm. The lens power $D = 1/F = -4$ D.

271. A light source is at a distance $l = 420$ cm from a screen. Where should a converging lens be placed to obtain a 20-fold magnified image of the object? Find the lens power.

Solution. The position of the lens is determined by distance f to the screen. We write the lens formula $1/F = 1/d + 1/f$. Since $1/F = D$, we have

$$D = 1/d + 1/f. \tag{1}$$

From the condition of the problem, we have

$$l = d + f. \tag{2}$$

The magnification of the lens is

$$k = f/d. \tag{3}$$

Solving Eqs. (2) and (3) simultaneously, we obtain the distance between the lens and the screen: $f = lk/(k + 1) = 4$ m. The lens power can be calculated from Eq. (1):

$$D = k/f + 1/f = (k + 1)/f = 5.25 \text{ D}.$$

272. A converging lens forms a five-fold magnified image of an object. The screen is moved to the object by $\Delta d = 0.50$ m, and the lens is shifted so that the image of the object has the same size as the object. Find the lens power and the initial distance between the object and the screen.

Solution. Before the screen and the lens have been shifted, we have

$$D = 1/d_1 + 1/(l_1 - d_1), \tag{1}$$
$$k_1 = (l_1 - d_1)/d_1. \tag{2}$$

After their displacement, we get

$$D = 1/d_2 + 1/(l_1 - \Delta d - d_2), \tag{3}$$
$$k_2 = (l_1 - \Delta d - d_2)/d_2. \tag{4}$$

We have four equations in four unknowns D, d_1, d_2, and l_1. The quantities d_1 and d_2 should be eliminated. This can be done, for example, by using Eqs. (2) and (4):

$$d_1 = l_1/(k_1 + 1), \; d_2 = l_1 - \Delta d/(k_2 + 1).$$

Substituting these expressions for d_1 and d_2 into Eqs. (1) and (3), we obtain a system of two equations in two unknowns:

$$D = (k_1 + 1)^2/k_1 l_1, \tag{5}$$
$$D = (k_2 + 1)^2/(l_1 - \Delta d)k_2. \tag{6}$$

Equating their right-hand sides, we get

$$l_1 = \Delta d \frac{k_2(k_1 + 1)^2}{k_2(k_1 + 1)^2 - k_1(k_2 + 1)^2} = 1.125 \text{ m}.$$

Using Eq. (5), we find that the lens power $D = 6^2/5l_1 = 6.4$ D.

273. A converging lens is placed between a light source and a screen which are separated by $L = 120$ cm. When the lens is in some position, a clear magnified image of the object is formed on the screen. When the lens is shifted by $l = 90$ cm, a clear diminished image is formed on the screen. Find the focal length of the lens.

Solution. Method I. Since the magnification $k = f/d$, the former position of the lens is obviously closer to the object, while the latter position is closer to the screen. Since the distances from the lens to the object (d) and from the lens to the screen (f) are conjugate, the segment l is symmetric relative to the object and its image. If we denote by d and f the distances from the object and from the image to the lens in the former position, it can be seen from Fig. 376 that $L = d + f$ and

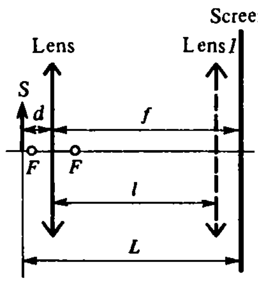

Fig. 376

$= f - d$. Adding and subtracting consecutively these equations, we get $+ l = 2f$ and $L - l = 2d$, whence $d = (L - l)/2$ and $f = (L + l)/2$. Con-equently,

$$/F = 2/(L - l) + 2/(L + l) = 4L/(L^2 - l^2), \text{ or } F = (L^2 - l^2)/4L = 0.131 \text{ m}.$$

Method II. Using the conditions of the problem, we write the following system of equations:

$$1/d + 1/f = 1/F, \quad 1/(d + l) + 1/(f - l) = 1/F, \quad d + f = 120 \text{ cm}.$$

Solving these equations simultaneously, we obtain

$$(d + f)/df = 1/F, \quad 4(d + f)/(d + l)(f - 3d) = 1/F.$$

Equating the denominators of the left-hand sides of equations, we get

$$df = (d + l) = (d + l)(f - 3d)/4.$$

Considering that $d + f = 120$ cm, we obtain $d = 15$ cm and $f = 105$ cm. This gives $F = 0.131$ m.

274. Find the smallest possible distance between a luminous object and its real image formed by a converging lens having a focal length F.

Solution. If the distance between the object and its image is l, the lens formula $1/d + 1/(l - d) = 1/F$ gives

$$(l - d + d)/d(l - d) = 1/F, \text{ i.e. } d^2 - ld + lF = 0,$$

whence $d = [l \pm \sqrt{l(l - 4F)}]/2$. The distance from the object to the lens is real ($d > 0$) when $l - 4F \geqslant 0$, i.e. $l \geqslant 4F$. The minimum distance $l_{min} = 4F$. For this value of l_{min}, the condition that the image is real is also satisfied: $d > F$ and $f > F$, i.e. $d + f > 2F$.

275. A convexo-concave diverging lens with the radii of curvature $R_1 = 80$ cm and $R_2 = 16$ cm is made of glass with a refractive index $n = 1.8$. Find the lens power.

Solution. Since the lens is diverging by the condition of the problem, its thickness increases from the centre to edges. Hence, the concave surface of the lens has the smaller radius of curvature than the convex surface. The minus sign should be ascribed to this radius, i.e. $R_2 = -0.16$ m. This gives

$$D = (n - 1)(1/R_1 + 1/R_2) = -4 \text{ D}.$$

276. An optical system consists of a converging and a diverging lens in contact, whose principal optical axes coincide. The focal lengths of these lenses are $F_1 = 50$ cm and $F_2 = -0.80$ m. Find the position of the image of a point source lying on the optical axis of the system at a distance $d = 80$ cm from its optical centre. What type of image is this?

Solution. The optical power of the system is

$$D = D_1 + D_2 = 1/F_1 + 1/F_2 = 0.75 \text{ D}.$$

Using the lens formula $1/d + 1/f = D$, we obtain $f = d/(dD - 1) = -2.0$ m. The image is virtual and lies on the optical axis of the system.

277. An optical system consists of two converging lenses whose principal optical axes coincide. The focal lengths of the lenses are $F_1 = 10$ cm and $F_2 = 8$ cm. The separation between the lenses is $l = 50$ cm. A luminous object is in front of the first lens at a distance $d_1 = 15$ cm. Determine the number of images, their positions, and types. Verify the solution of the problem by constructing an optical diagram.

Solution. For the image formed by the first lens, we can write $1/d_1 + 1/f_1 = 1/F_1$, whence $f_1 = F_1 d_1/(d_1 - F_1) = 30$ cm. The magnification $k_1 = f_1/d_1 = 2$. The first image is reversed and magnified. The distance from this image to the second lens is $d_2 = l - f_1$. Consequently, for the second lens we can write

$$1/(l - f_1) + 1/f_2 = 1/F_2, \text{ whence}$$
$$f_2 = (l - f_1)F_2/(l - f_1 - F_2) = 13.3 \text{ cm}.$$

The overall magnification of the object is

$$k = k_1 k_2 = f_1 f_2/d_1 d_2 = f_1 f_2/d_1(l - f_1) = 4/3.$$

The second image is erect and magnified. Figure 377 shows that two images are formed altogether.

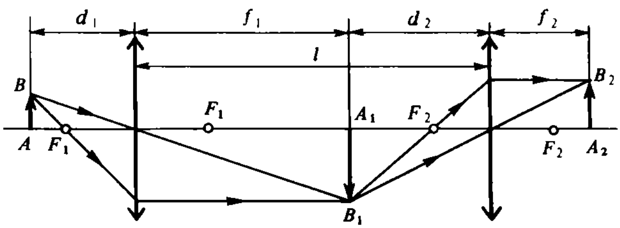

Fig. 377

278. An optical system consists of converging lens and a small concave mirror at a distance $l = 75$ cm from it. The principal optical axes of the lens and the mirror coincide. The focal length of the lens is $F_1 = 20$ cm and the radius of curvature of the mirror is $R_2 = 50$ cm. A luminous source is at a distance $d_1 = 30$ cm in front of the lens. Determine the number of images, their positions, and types. Verify the solution by constructing an optical diagram.

Solution. For the image formed by the lens, we can write

$$1/d_1 + 1/f_1 = 1/F_1, \text{ whence } f_1 = F_1 d_1/(d_1 - F_1) = 60 \text{ cm}.$$

The distance from this image to the optical centre of the mirror is $d_2 = l - f_1 = 15 \text{ cm} < R_2/2$. Since by the condition of the problem the mirror is small, we can write the following expression for the mirror image (see the remark made at the end of Sec. 6.7):

$$1/d_2 + 1/f_2 = 2/R_2, \text{ i.e. } f_2 = R_2 d_2/(2d_2 - R_2) = -37.5 \text{ cm}.$$

The luminous flux from the second image is again incident on the lens, which refracts the beam and forms the third image. For this image, we can write

$1/(|f_2| + l) + 1/f_3 = 1/F_1$, whence

$$f_3 = F_1(|f_2| + l)/(|f_2| + l - F_1) \simeq 24.3 \text{ cm}.$$

Thus, there are three images, the first being reverse relative to the object, the second erect relative to the first image, i.e. reversed relative to the object, and the third reversed relative to the second image, i.e. erect relative to the object. The latter (third) image is real. The overall magnification is

$$k = k_1k_2k_3 = \left|\frac{f_1}{d_1}\frac{f_2}{d_2}\frac{f_3}{d_3}\right| = \left|\frac{f_1}{d_1}\frac{f_2}{l - f_1}\frac{f_3}{(|f_2| + l)}\right| = 1.08.$$

The optical ray path is shown in Fig. 378.

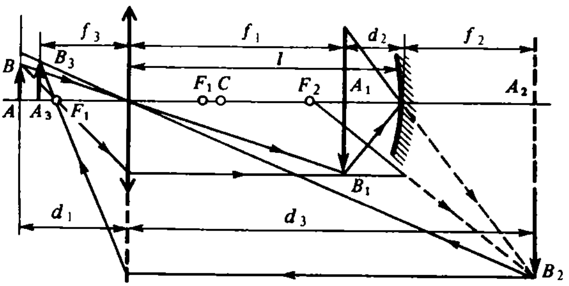

Fig. 378

279. A plane surface of a plano-convex lens having a focal length $F = 60$ cm is silver plated. A luminous object is at a distance $d_1 = 20$ cm from the lens in front of the convex surface. Determine the number of images, their position, and type. Verify the solution by constructing an optical diagram.

Solution. The lens can be treated as an optical system consisting of a plano-convex lens and a plane mirror in contact with it. For the lens, we can write

$$1/d_1 + 1/f_1 = 1/F, \text{ whence } f_1 = Fd_1(d_1 - F) = -30 \text{ cm}.$$

It can be seen that the image is virtual (and hence erect) and located at a distance of 30 cm in front of the lens. Light beams, after having been refracted by the lens, are reflected from the silvered surface as from a plane mirror (Fig. 379, rays B_1KB_0 and B_1OB') and form a virtual image at a distance $f_1 = 30$ cm from the silvered surface. The second image, virtual and erect, is formed behind the lens at a distance $f_2 = 30$ cm, which forms a new, third, image for which we can write $1/d_3 + 1/f_3 = 1/F$, where $d_3 = 30$ cm. Hence we obtain $f_3 = -60$ cm. The third image is also virtual and erect. The overall magnification is

$$k = \left| \frac{f_1}{d_1} \frac{f_2}{d_2} \frac{f_3}{d_3} \right| = 3.$$

The optical diagram of the system is shown in Fig. 379.

Remark. It should be noted that intermediate images cannot be actually observed.

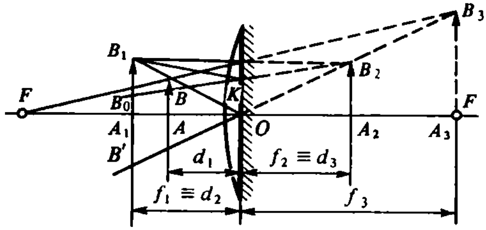

Fig. 379

280. A screen is illuminated by a light source placed at the focal point of a lens whose power $D = 0.5$ D. The lens is between the source and the screen (Fig. 380). Find the illuminance at the centre of the screen if the luminous intensity I of the source is 100 cd. The light absorption in the lens should be neglected.

Solution. The illuminance at point A of the screen is equal to the illuminance of the lens at point O. The focal length $F = 1/D = 2.0$ m. The illuminance at the centre of the screen is $E = E_0 = I/F^2 = 25$ lx.

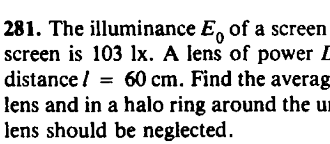

Fig. 380

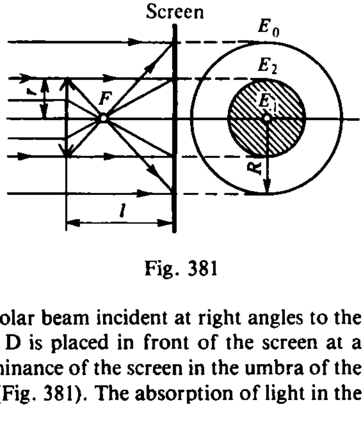

Fig. 381

281. The illuminance E_0 of a screen by a solar beam incident at right angles to the screen is 103 lx. A lens of power $D = 5$ D is placed in front of the screen at a distance $l = 60$ cm. Find the average illuminance of the screen in the umbra of the lens and in a halo ring around the umbra (Fig. 381). The absorption of light in the lens should be neglected.

Solution. The focal length of the lens is $F = 1/D = 20$ cm. Figure 381 shows that $E_1/E_0 = r^2/R^2 = F^2/(l - F)^2$, whence the illuminance of the screen in the umbra from the lens is $E_1 = E_0F^2/(l - F)^2 = 250$ lx. The illuminance in the halo ring is $E_2 = E_1 + E_0 = 1250$ lx.

282. A light source is at a distance $L = 1.5$ m from a screen. The luminous intensity I of the source is 90 cd. A converging lens having a focal length $F = 30$ cm is placed at the midpoint between the source and the screen (Fig. 382). Find the illuminance at the centre of the screen. The losses of luminous energy in the lens should be neglected.

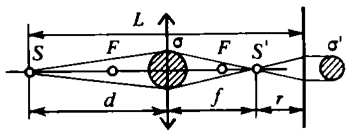

Fig. 382

Solution. The distance from the source to the lens is $d = L/2 = 75$ cm. Using the lens formula $1/F = 1/d + 1/f$, we obtain the distance from the image of the source to the lens: $f = 50$ cm. The distance from the image of the source to the screen is $r = 25$ cm. Figure 382 shows that the same luminous flux is incident on the area σ of the lens and the area σ' of the screen at its centre. Hence the ratio of the illuminances is $E'/E = \sigma/\sigma' = (f/r)^2$, whence $E' = E(f/r)^2$. The illuminance of the lens is $E = I/d^2$. Consequently, the illuminance at the centre of the screen is

$$E' = (I/d^2)(f^2/r^2) = 640 \text{ lx}.$$

Exercises

263. (a) A light ray passes from glass having a refractive index of 1.6 to air. Find the value of the angle of incidence for which the angle of refraction is twice larger than the angle of incidence.

Answer. arcsin (3/5) = 37°.

(b) The refractive index of glass relative to water is 1.16. Find the velocity of light in water if the absolute refractive index of glass is 1.54.

Answer. 2.26×10^8 m/s.

(c) A ray reflected from the surface of water forms an angle of 90° with the refracted ray. Find the angle of incidence and the angle of refraction if the refractive index of water is 1.33.

Answer. arctan (4/3) ≃ 53°, 37°.

(d) The angle of incidence of a light beam passing from glass having a refractive index of 1.67 to water with a refractive index of 1.33 is 60°. Find the angle of deflection of the beam at the interface. Solve the problem for an angle of incidence of 53°.

Answer. 60°, 74°.

264. (a) A parallel beam of light passes from air to water at an angle of incidence

of 30°. Find the width of the beam in water if its width in air is 10 cm. The refractive index of water is 1.33.
Answer. ~ 15 cm.

(b) A parallel beam of light whose width is 10 cm propagates from water whose refractive index is 1.33 to glass having a refractive index of 1.67. The beam width in glass is 20 cm. Find the angle of incidence on the interface between water and glass and the angle of refraction.
Answer. arcsin $(5/2\sqrt{7})$, arcsin $(2/\sqrt{7})$.

265*. (a) A diver in water sees the Sun at an angle of 60° to the horizontal. Find the actual angle formed by the light ray with the horizontal.
Answer. arccos (2/3).

(b) An observer is looking in a direction close to the vertical at a basin having a depth of 1.00 m. What is the apparent depth of the basin? The observer is assumed to have a good visual estimation.
Answer. 0.75 m.

266. (a) A parallel beam of light is incident on a plane-parallel plate of thickness 42 mm at an angle of 53° (sin 53° = 4/5). Find the displacement of the beam if the refractive index of the plate is 1.7.
Answer. 20 mm.

(b*) Find the refractive index of a transparent plate of thickness 4 cm if a parallel light beam incident on it at an angle of 60° emerges from the plate with a displacement of 2.4 cm.
Answer. 1.84.

(c*) A narrow light beam is incident on a plane-parallel plate having a refractive index of 1.6 at an angle of 30° to the normal. As a result it is partially reflected and refracted. The refracted beam is reflected by the rear surface of the plate and then undergoes refraction again, emerging from the plate with a displacement of 4.0 cm parallel to the primarily reflected beam. Determine the thickness of the plate.
Answer. 7.0 cm.

267*. Two plane-parallel plates having thicknesses of 3 cm and 5 cm and refractive indices of 1.5 and 1.8 respectively are piled together. A light beam is incident on the first plate at an angle of 60° to the normal, passes through the plates, and emerges at the same angle. Find the displacement of the beam.
Answer. 4.5 cm.

268. (a) A monochromatic ray is incident normal to a face of a prism having an equilateral triangle in its cross section. The refractive index of the material of the prism is 1.1. Find the angle of deflection of the ray emerging from the prism from its initial direction.
Answer. 60°.

(b) A monochromatic ray is incident normal to a face of a transparent trihedral prism with an angle of refraction of 53° (tan 53° = 4/3). The refractive index of the material of the prism is 1.25. What is the direction of the ray emerging from the prism?
Answer. The emerging ray slides along the opposite face of the prism.

269*. (a) A monochromatic ray is incident at an angle of 60° to a face of a

transparent trihedral prism having a refractive index of 1.75. Having undergone double refraction, the ray emerges in a direction normal to this face. Find the angle of refraction of the prism.
Answer. 60°.

(b) The cross section of a transparent prism having a refractive index of $\sqrt{2}$ is a right triangle with the angle of refraction of 90° and the acute angles of 30° and 60°. Find the angle of deflection of a ray incident normal to the steeper face of the prism.
Answer. 45°.

270. (a) A converging lens having a focal length of 40 cm forms a virtual image of a rod located on the principal optical axis at right angles to it. The image is at 10 cm from the lens, its length being 20 cm. Find the length of the rod, its separation from the lens, and the lens power.
Answer. 8 cm, 16 cm, 2.5 D.

(b) A diverging lens with a focal length of 50 cm forms a virtual image of a rod arranged normally to the optical axis. The image is at 10 cm from the lens, the image length being 20 cm. Find the length of the rod and its separation from the lens, as well as the lens power.
Answer. 12.5 cm, 25 cm, -2 D.

271. (a) A luminous object is at a distance of 3.0 m from a screen. At what distance from the screen should a converging lens having a power of 4 D be placed to obtain an image of the object on the screen? Find the magnification of the image.
Answer. 2.725 m, 9.9, 0.275 m, 0.1.

(b) A luminous object is at a distance of 1.0 m from its virtual image formed by a diverging lens. Determine the distance from the object to the lens and the lens power if the magnification is 0.6.
Answer. 2.5 m, 4/15 D.

272*. (a) A converging lens forms a doubly magnified image of an object. When the lens is shifted by 20 cm to the object, the magnification becomes equal to five. Find the lens power and the distances between the object and the lens in both cases if the images are real.
Answer. 1.5 D, 3.0 m, 4.8 m.

(b) A diverging lens forms a virtual image of an object with a magnification of 1/5. The lens is shifted so that it forms an image of half the size of the object, the distance between the object and the image being reduced by 54 cm. Find the lens power and the initial distance between the object and its virtual image.
Answer. -5 D, 0.64 m.

273*. A converging lens having a focal length of 20 cm is placed between a luminous object and a screen. At a certain position of the lens, a clear magnified image of the object is formed on the screen. When the lens is shifted from this position by 1 m, a clear diminished image is formed on the screen. Find the distance between the object and the screen.
Answer. 1.48 m.

274*. The distance between a luminous object and a screen is 2.0 m. Can an image of the object be obtained on the screen with the help of a converging lens having a

power of 1.25 D? If not, what lens power is required for obtaining an image?
Answer. No, it cannot, $\geqslant 2$ D.
275. A concavo-convex converging lens has the radii of curvature of its surfaces of 80 cm and 16 cm and a power of 4 D. Find the refractive index of the material of the lens.
Answer. 1.8.
276. An optical system consists of a converging lens with a focal length of 40 cm and a diverging lens with a focal length of 25 cm, which are brought in contact and have the common optical axis. At what distance from the optical centre of the system is a point light source located if its image is at 50 cm from this centre? What type of image is it? Can this system be used for obtaining an image at 80 cm from the optical centre?
Answer. 2.0 m, the image is virtual, no, it cannot.
277. An optical system consists of two converging lenses with focal lengths of 12 cm and 6 cm, whose principal optical axes coincide. The lens-to-lens distance is 40 cm. A luminous object is placed at 20 cm in front of the first lens. Determine the number of images, the distance from the optical centre of the second lens to the last image of the object, the type of this image, and the magnification. Verify the result by constructing an optical diagram.
Answer. Two images, 15 cm, the second image is real, erect, and magnified, the magnification is 2.25, Fig. XVIII.
278*. (a) A concave mirror having the radius of curvature of 20 cm is placed behind a convexo-convex lens at a distance of 40 cm. The focal length of the lens is 8 cm. The principal optical axes of the lens and of the mirror coincide. A 2-cm high luminous object is in front of the lens at 16 cm from its optical centre. Determine the number of images, the distance from the lens to the last image, the type and the height of this image. Verify the results by constructing an optical diagram.
Answer. Three images, 12.3 cm, the first two images are invisible, the third image is real, reversed, and diminished, its height is 7.7 mm, Fig. XIX.

(b) A plane mirror is arranged at right angles to the optical axis of a convexo-convex lens at 40 cm behind it. The focal length of the lens is 8 cm. A luminous object is in front of the lens at 16 cm from its optical centre. Determine the number of images, the distance from the lens to the last image, the type of this image, and the magnification. Verify the results by constructing an optical diagram.
Answer. Three images, 91/7 cm, the image is real, erect, and diminished, 1.7, Fig. XX.
279*. The plane surface of a plano-concave lens having a focal length of 40 cm is silver plated. A luminous object is placed at 60 cm from the lens on the side of the concave surface. Determine the number of images, the position and the type of the last image, and the overall magnification. Verify the results by constructing an optical diagram.
Answer. Three images, -15 cm, the image is virtual, erect, and diminished, 0.25, Fig. XXI.
280. A screen is illuminated by a light source arranged at the focal point of a lens of 2-D power. The lens is between the source and the screen. The illuminance at the

centre of the screen is 45 lx. Find the luminous intensity of the source, assuming that 10% of the luminous flux are absorbed by the lens.
Answer. 12.5 cd.
281. The illuminance of a screen due to solar rays incident normal to the screen is 800 lx. When a converging lens having a power of 4 D is placed in front of the screen, the average illuminance in the umbra on the screen is 200 lx. Determine the distance between the lens and the screen and the illuminance in the halo around the umbra (Fig. 383). The absorption of light in the lens should be neglected.
Answer. 0.75 m, 10^3 lx.

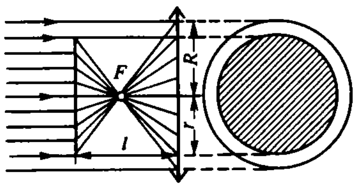

Fig. 383

282. A point light source is at 1-m distance from a screen. A converging lens having a focal length of 20 cm is placed between the source and the screen at 40 cm from the latter. The illuminance at the centre of the screen has become 200 lx. Determine the luminous intensity of the source. The absorption of light in the lens should be neglected.
Answer. 8 cd.

C. OPTICAL INSTRUMENTS

6.14. Searchlight. Projection Lantern

A **searchlight** (Fig. 384) consists of a parabolic concave mirror and a high-intensity light source arranged at the focal point of the

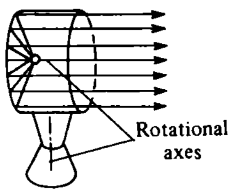

Fig. 384

mirror. The rays from the source are incident on the mirror and are reflected in the form of a central beam of parallel light rays. The searchlight has two rotational axes, vertical and horizontal, which makes it possible to send the light beam in any direction.

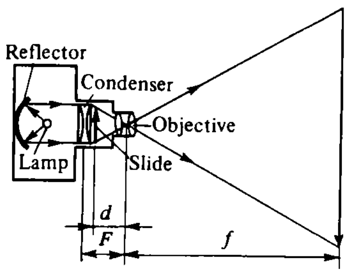

Fig. 385

A **projection lantern** (Fig. 385) is intended for obtaining on a remote screen a magnified image of a transparency placed between the focal point and a point at the double focal length. It consists of a box containing a powerful source of light (a special incandescent projector lamp having a power of 300-1000 W or an arc lamp) arranged at the focal point of a reflector, and a condenser, viz. a short-focus system of two lenses converging the light beam which illuminates a slide (or transparency) enclosed in a frame. An objective is arranged in the focal plane of the condenser so that light rays from the condenser passing through the slide converge in it. The position of the objective lens is controlled by a special mechanism.

In a motion-picture projector, a reel is moved with a speed of 24 frames per second instead of a slide. Since a human eye is able to preserve visual perception for about 0.1 s, the images of consecutive frames of moving objects form a single moving image.

The ray path in a projection lantern shows that the magnification is

$$\kappa = f/d \approx f/F_c$$

due to the fact that the slide is close to the condenser and $d \approx F_c$.

Consequently, in order to increase the image on the screen, it is necessary, on the one hand, to increase the distance f from the screen to the projection lantern and, on the other hand, to use a condenser with a shorter focal length.

An **epidiascope** (or **episcope**) is a modified projection lantern intended for projecting on a screen both transparent (diaprojection) and opaque (epiprojection) objects (Fig. 386). It has two objectives, a hinged blind, a reflector, and a table for opaque objects. When the blind is in the horizontal position shown in Fig. 386*a*, the epidiascope operates as a projection lantern.

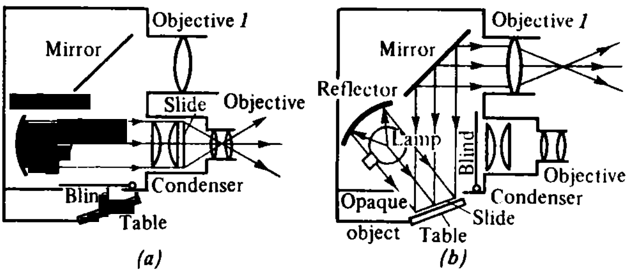

Fig. 386

If the blind is in the vertical position (Fig. 386*b*), the condenser and the objective are shielded, and the table on which an opaque object is placed is exposed to the light from the same light source supplied with a reflector, which is turned through a certain angle. Light reflected from the opaque object is incident on a mirror and reflected from it to the objective *1*.

6.15. Photographic Camera

A **photographic camera** (Fig. 387) is a box impervious to light with an objective in the front wall and a screen in the rear wall. In a simple camera, the screen is an opaque glass which is replaced by a light-sensitive film. A negative image is formed on the film in film cameras.

A photographic film (screen) in a camera is placed between the focal point and the point at the double focal length from the objective. When remote objects are photographed, the film is virtually in the focal plane of the objective lens. The position of the photographic film (plate) should be changed with the distance to the object. This is attained either by changing the distance between walls in a folding camera, made in the form of bellows (see Fig. 387), or by moving the objective in a box camera.

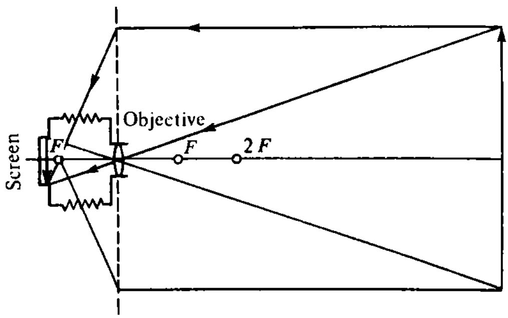

Fig. 387

Remark. While constructing an image in Fig. 387, besides the rays passing through the optical centre of the objective, the rays parallel to the principal optical axis are taken, which actually do not pass through the objective but propagate outside it. Such a formal approach is not incorrect since not only two rays used for the construction converge at a point on the screen, which is the image of a certain luminous point, but also an infinite number of other rays. If we replace a lens by a very large one so that its refractive properties remain unchanged, the point of convergence of all these rays on the screen remains the same. Therefore, while constructing an image, we can assume that the diameter of the lens is as large as desired and take any two rays that are conveniently constructed.

Besides the parts described above, modern photographic cameras have a number of other elements for improving performance: a diaphragm, a shutter, a viewfinder, a range finder, etc.

A movie camera differs in principle from a photographic camera only in that instead of a stationary photographic plate (film), a cine film is moved in it, exposed 24 times per second.

Modern projecting and photographic systems employ complex objectives consisting of several lenses. In such optical systems, drawbacks typical of a single lens are partially eliminated.

Main failures of a single lens are as follows.

1. *Spherical aberration* (see Sec. 6.7).

2. *Chromatic aberration,* which consists in that a white ray passing through a lens is decomposed into its components, viz. rays of different colours that are refracted through different angles and hence are focused at different distances from the objective (violet rays undergo the strongest refraction and are converged closer to the objective, while red rays are the least refracted ones). This leads to the appearance of colour fringes.

3. *Astigmatism,* which consists in that a lens focuses inclined beams at different distances.

4. *Curvature of field,* which consists in that a sharp image is formed on a sphere instead of on the surface, since the focal points for inclined rays are located in different focal planes.

5. *Distortion of straight lines* when a diaphragm is located on the side of a lens.

6.16. Magnifying Glass. Human Eye as an Optical Instrument

A **magnifying glass** is the simplest optical system consisting of a single or several lenses (which are used less frequently). An object being observed is placed between the lens and the focal point so that its image (virtual, erect, and magnified) is formed at a distance of best vision for a normal eye: $L = 25$ cm (Fig. 388).

It is most convenient to observe an object through a magnifying glass with an unstrained eye accommodated to infinity. For this purpose, the object under observation is placed near the focal plane of the lens.

The angle α' at which the image of the object is seen is determined from the condition

$$\tan \alpha' = h/F$$

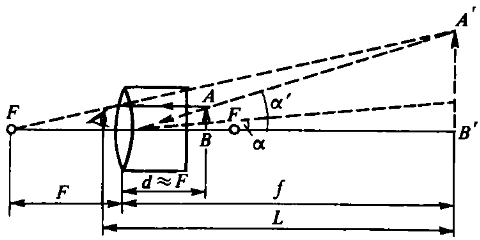

Fig. 388

(h is the height of the object). If the object were observed by a naked eye at the distance of best vision, it would be seen at an angle α such that

$$\tan \alpha = h/L.$$

Since the ratio of small angles can be replaced by the ratio of their tangents, the angular magnification for a given magnifying glass is determined as follows:

$$K = \alpha'/\alpha \approx \tan \alpha'/\tan \alpha = hL/Fh = L/F.$$

Figure 388 shows that the linear magnification has the same value:

$$k = A'B'/AB \approx L/F.$$

Magnifying glasses with focal lengths varying between 1 cm and 10 cm are used, their magnification ranging from 2.5 to 25 (for normal vision).

From the point of view of optics, a *human eye* (Fig. 389) is an optical system resembling a photographic camera.

A compound refractive system consists of a lens located behind a small chamber filled by an aqueous humour and covered by a transparent envelope, cornea, and a jelly-like substance called the vitreous body which is located behind the lens. The optical centre O of this system is inside the lens near its rear surface. The refractive index of the system is about 1.4.

The vitreous body is bounded by a light-sensitive surface of the fundus, called a retina, with its most sensitive part, the yellow spot, located against the pupil above the blind spot which covers the optic nerve and is insensitive to light. In front of the lens there

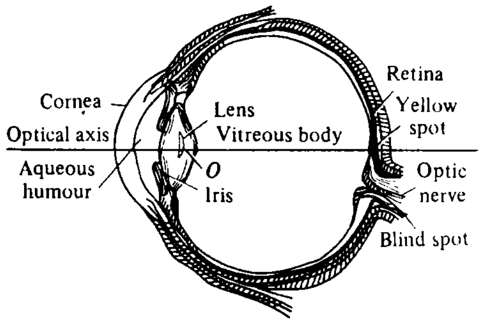

Fig. 389

is the iris with an aperture of varying size, the pupil, playing the role of a diaphragm. The optical axis of the system, called the visual axis of the eye, passes through the centre of the lens and the yellow spot. A real, reversed, and diminished image of an object is formed on the retina which is between the focal point of the system and a point at a double focal length, playing the role of a screen.

The angle formed by the rays emerging from the edges of an object and converging at the optical centre of the eye is called the *angle of vision*. It can be seen from Fig. 390 that the angle of vision 2α determines the size of the image formed on the retina. Consequently, the larger the angle of vision, the larger the number of details on the surface of the object that can be distinguished by the eye.

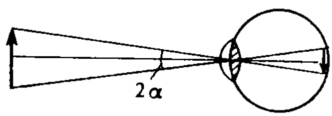

Fig. 390

Experience shows that the limiting angle of resolution for a normal eye at a good illumination is $1'$. The details of an object seen at a smaller angle cannot be resolved by the eye, and the object is perceived as a point. Optical instruments increasing the angle of vision (magnifying glass, microscope, telescope, binoculars, etc.) increase the so-called *resolving power* of the eye.

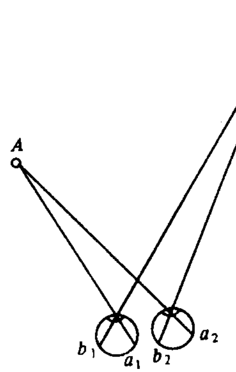

Fig. 391

In a binocular vision (Fig. 391), the images formed on the retinas of the right and left eyes are different. The difference in the images of two points is the larger, the further is one point from another.

6.17. Accommodation of Eye. Myopia and Hyperopia. Spectacles

In order that the image of an object be formed on the retina, it is necessary that the focal length of the eye vary depending on the distance to the objects. This is attained by changing the convexity of the eye lens with the help of the ciliary muscle.

The ability of an eye to change the curvature of the lens surfaces with a change in the distance to a viewed object is called **accommodation**.

Accommodation of an eye is limited. The limits of accommodation for a normal eye are from infinity to 10 cm. The distance of best vision for a normal eye is 25 cm.

A *short-sighted eye* has a strong optic power. It focuses rays in front of the retina due to a very strong refraction, and as a result the images of objects on the retina become blurred. The distance of best vision for a short-sighted eye is less than 25 cm.

A *long-sighted eye* has an insufficient optic power and focuses rays behind the retina, which also leads to unclear images. The distance of best vision for a long-sighted eye is more than 25 cm.

Myopia (or **short-sightedness**) and **hyperopia** (or **long-sightedness**) can be partially corrected by spectacles, which are simple lenses, diverging (concave) for short-sighted and converging (convex) for long-sighted eyes.

Problems with Solutions

283. A point light source having a luminous intensity $I = 100$ cd is placed at the focal point of a searchlight whose mirror has the radius of curvature $R = 2$ m (Fig. 392). A screen at a distance $l = 5$ m from the light source is perpendicular to

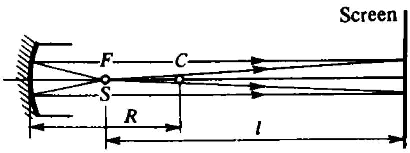

Fig. 392

the principal optical axis of the searchlight. Find the illuminance of the screen at a point on the principal optical axis of the searchlight if the loss of luminous energy upon the reflection from the mirror amounts to 25% of the entire energy incident on the mirror ($\alpha = 0.25$).

Solution. The illuminance at the given point is the sum of two illuminances: $E = E_1 + E_2$, where E_1 is the illuminance due to the light propagating directly from the source and E_2 is the illuminance due to the reflected light. It is known that $E_1 = I/l^2$ and $E_2 = \Phi'/\sigma = (1 - \alpha)\Phi/\sigma$, where Φ' is the luminous flux reflected from the mirror, Φ is the flux incident on the mirror from the source, and σ is the small surface area of the screen near the principal optical axis of the searchlight.

The luminous flux incident on the illuminated area of the mirror is $\Phi = \Omega I$, where Ω is the solid angle equal to the ratio of the area of the spherical segment having a radius equal to the focal length $R/2$ to the square of this distance. If the area element σ is small, the surface area of the spherical segment can be assumed to be equal to the area of this surface element. Then

$$\Omega = \sigma/(R/2)^2 = 4\sigma/R^2, \; E_2 = [4(1 - \alpha)I]/R^2.$$

The total illuminance $E = I[1/l^2 + 4(1 - \alpha)/R^2] = 79$ lx.

284. Solve the previous problem under the condition that the source of light is at a distance $d = 1.5$ m from the searchlight.

Solution. The illuminance due to the source of light proper is $E_1 = I/l^2 = 4.0$ lx. In order to determine the illuminance due to the reflected light, we must

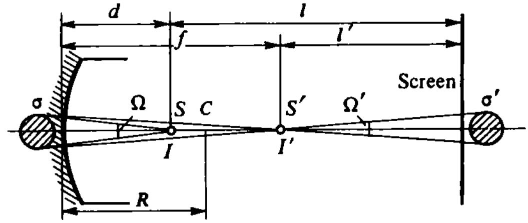

Fig. 393

find the position of the image of the source (Fig. 393). For this we shall use the formula for a concave mirror:

$$1/F = 1/d + 1/f, \text{ whence } 1/f = 1/F - 1/d = (1/3)\,\text{m}^{-1}.$$

The distance from the real image of the source of light to the screen is $l' = (l + d) - f = 3.5$ m. Taking into account the losses in the reflection, the luminous flux $\Phi' = (1 - \alpha)\Phi$.

We can now proceed in the following two ways.

Method I. The luminous flux Φ from the source S within a small solid angle Ω is reflected from the searchlight as the flux Φ' and propagates into a solid angle Ω'. Consequently, the ratio of the illuminance of the central surface element σ' of the screen by the reflected flux to the illuminance of the central surface element σ of the mirror by the light from the source has the form

$$E'/E = (\Phi'/\sigma')/(\Phi/\sigma) = (1 - \alpha)(\sigma/\sigma') = (1 - \alpha)(f/l')^2,$$

and since $E = I/d^2$, we get

$$E' = I(1 - \alpha)(f/l'd)^2 = 24.5 \text{ lx}.$$

Method II. The illuminance from the image S' of the source within a solid angle Ω' corresponds to the illuminance of a source having the luminous intensity $I' = \Phi'/\Omega'$, where $\Phi' = (1 - \alpha)\Phi$ and $\Phi = I\Omega$. Thus, $I' = I(1 - \alpha)\Omega/\Omega' = I(1 - \alpha)(f/d)^2$. The illuminance of the centre of the screen is

$$E' = I'/l'^2 = I(1 - \alpha)(f/l'd)^2,$$

i.e. it is the same as that obtained by method I. The total illuminance at the centre of the screen is $E = E_1 + E' = 28.5$ lx.

285. Solve the previous problem under the condition that the source of light is at a distance $d = 0.5$ m from the searchlight.

Solution. The illuminance created directly by the source is $E_1 = 4.0$ lx. The image of the source is virtual. Using the formula $1/F = 1/d + 1/f$ for a concave mirror, we obtain (Fig. 394)

$$1/f = 1/F - 1/d = -1, \text{ i.e. } f = -1 \text{ m}.$$

The illuminance from the virtual source within a solid angle Ω' is the same as the illuminance from a source having a luminous intensity $I' = \Phi'/\Omega'$, and $\Phi' = (1 - \alpha)\Phi$, where $\Phi = I\Omega$. Thus, $I' = I(1 - \alpha)\Omega/\Omega' = I(1 - \alpha)(f/d)^2$.

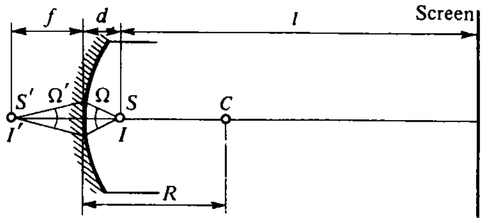

Fig. 394

The illuminance at the centre of the screen is

$$E' = \frac{I'}{(l + |f| + d)^2} = I(1 - \alpha)\left[\frac{f}{d(l + |f| + d)}\right]^2 = 7.1 \text{ lx}.$$

The total illuminance at the centre of the screen is $E = E_1 + E' = 11.1$ lx.

286. A photographic camera has an objective with a focal length $F = 5$ cm and uses a film with a frame size 6×8 cm^2. A drawing whose dimensions are 60×60 cm^2 should be photographed. At what distance from the drawing should the objective of the camera be placed to obtain the highest magnification? At what distance from the objective should the film be located?

Solution. Since the drawing has the form of a square, while the frame of the film is rectangular, it is impossible to fill the frame completely. The image of the 6×6 cm^2 size is formed on the film, while the fraction of the frame of the 2×6 cm^2 size is clean. The linear magnification of the drawing is $k = 1/10$. We write the lens formula: $1/F = 1/d + 1/f$ and consider that $f/d = k = 1/10$. Solving these equations simultaneously, we determine the distance from the objective to the drawing: $d = 55$ cm. The distance between the objective and the film is $f = 5.5$ cm.

287. The image formed by a convexo-convex lens is initially $k_1 = 4$ times larger than the object. Then the lens is displaced away from the object by $l = 0.4$ cm, after which the image becomes $k_2 = 5$ times larger than the object. Find the lens power if both images are virtual and are at the distance of best vision from the lens.

Solution. Since the lens is used as a magnifying glass, the object is placed between the lens and the focal point, and the image is virtual. This means that $f < 0$. We write the lens formula: $D = 1/d + 1/f$. In the former case, $k_1 = |f_1/d_1| = 4$, i.e. $f_1 = -k_1 d_1$, and

$$D = 1/d - 1/k_1 d = 3/4d_1. \tag{1}$$

In the second case, $k_2 = |f_2/d_2| = 5$, i.e. $f_2 = -k_2 d_2$, and

$$D = 1/d - 1/k_2 d = 4/5d_2. \tag{2}$$

Considering that $d_2 = d_1 + l$, where $l = 0.004$ m, and solving Eqs. (1) and (2) simultaneously, we obtain $D = 12.5$ D.

288. The distance of best vision for an eye is $L = 100$ cm. Find the optical power of the spectacles compensating the defect of vision for this eye.

Solution. The eye is long-sighted. The spectacles are converging lenses through which at a distance $L = 100$ cm the eye sees the virtual image of an object located at the distance of best vision for a normal eye (25 cm). Consequently, $d = 0.25$ m and $f = -1.00$ m. The lens formula yields

$$D = 1/d + 1/f = 3.0 \text{ D}.$$

The focal length of the lens is $F = 1/D \approx 33$ cm. Hence the observed object is between the lens and the focal point, i.e. the eye sees a virtual image through the spectacles.

Exercises

283. A point light source is at the focal point of a searchlight with a 4-m radius of curvature of the mirror. A screen is placed perpendicular to the principal optical axis of the searchlight at 10 m behind the source. The illuminance of the screen at the point lying on the optical axis of the searchlight is 105 lx. Find the luminous intensity of the source if the losses of luminous energy in reflection from the mirror amount to 20% of the entire luminous energy incident on the mirror.

Answer. 500 cd.

284. Solve Problem 283 under the condition that the source is at a distance of 3 m from the searchlight.

Answer. 1400 cd.

285. Solve Problem 283 under the condition that the source is at a distance of 1.0 m from the searchlight.

Answer. ~ 3600 cd.

286. A 40×60 cm^2 drawing is photographed on a film with the frame size of 6×8 cm^2. The shortest distance from the drawing to the objective lens for which the image of the entire drawing is obtained on the film is 90 cm. Determine the focal length of the objective of the camera and the distance from the objective to the film.

Answer. 12 cm, 10.6 cm.

287. An object is viewed through a magnifying glass. The most clear image is

formed when the object is at 12.5 cm from the lens. Determine the lens power and the magnification of the object.
Answer. 4 D, 2.
288. The distance of best vision for a short-sighted person is 15 cm. Find the lens power of spectacles compensating the defect of vision.
Answer. -2.67 D.

D. COMPOSITION OF LIGHT. INVISIBLE RAYS

6.18. Dispersion of Light. Spectrum. Spectroscope

Dispersion of light is the dependence of the refractive index of a material on the wavelength of light. Waves of different lengths propagate in a medium with different velocities and are refracted differently. Since light rays of different colours have different wavelengths, they are refracted through different angles by a glass prism placed on their way. As a result of dispersion, a beam of white light produces a rainbow strip on a screen behind a refracting prism (Fig. 395). Seven basic colours are conventionally

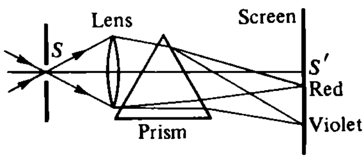

Fig. 395

distinguished in this strip. These colours are arranged in the following order: red (which corresponds to the longest wavelength $\lambda \simeq 700$ nm and the smallest refractive index), orange, yellow, green, blue, indigo, and violet (which has the shortest wavelength $\lambda \simeq 400$ nm and the largest refractive index).

The set of wavelengths constituting a given radiation is called its **spectrum**. A simple experiment on the synthesis of colours consists in a rapid rotation of a circle with sectors painted by different colours of a spectrum (Fig. 396). Since the persistence of vision is

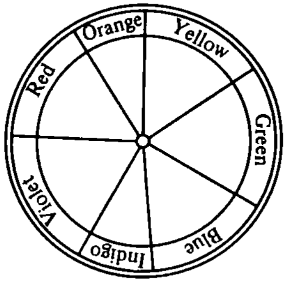

Fig. 396

about 0.1 s (i.e. the eye "memorizes" the colour of a sector for 0.1 s), the colours merge upon a rapid rotation of the disc, and it seems to be white.

A **spectroscope** (Fig. 397) is an instrument intended for analyzing spectra (for spectral analysis of substances).

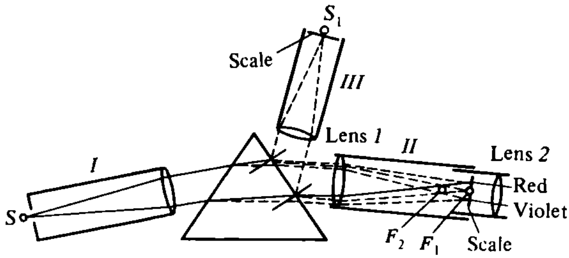

Fig. 397

A slit illuminated by a light source S is at the focal point of a lens in tube *I*. A divergent beam from the source thus becomes a parallel beam after passing through the lens. Being refracted by a prism, these parallel rays are decomposed into the rays of spectral colours which are converged by lens *1* of telescope *II* at its focal point F_1 in the form of spectrum. The spectrum is observed through lens *2* of the telescope. The position of lens *2* is controlled in such a way that the focal point F_1 of lens *1* is between lens *2* and its focal point F_2. An observer sees a virtual magnified image of the spectrum against the same image of a scale that is projected by tube *III*, illuminated by source S_1, onto the same focal plane F_2. If telescope *II* is replaced by a photographic camera having the focal point at the same point F_1, the spectrum can be recorded on a photographic film.

An instrument for obtaining photographs of a spectrum is called a **spectrograph**.

6.19. Infrared and Ultraviolet Radiation

Beyond the red boundary of visible spectrum, there is the region of **infrared radiation** ranging between 760 nm and 0.3 mm.

Infrared radiation is not perceived by the human eye but produces a heating effect. If, for example, the rays from a heated body are focused with the help of a concave mirror on a thermometer, the latter registers a noticeable increase in temperature (Fig. 398).

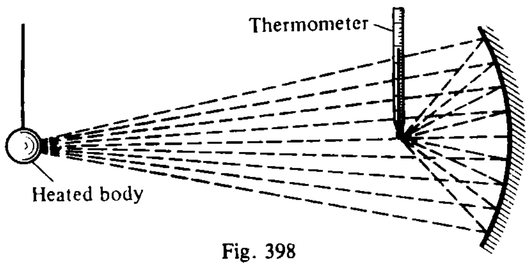

Fig. 398

Beyond the violet boundary of the optical spectrum, there is the region of **ultraviolet radiation** ranging between 400 nm and 10 nm. This radiation also does not produce a visual sensation, but has a strong chemical and biological effect. The tan acquired by the skin of a human body exposed to solar radiation is due to the ultraviolet part of the solar radiation spectrum. Ultraviolet rays may kill living microscopic organisms and cells and produce a harmful effect on the retina of the eye. Glass and water are virtually opaque for ultraviolet rays, while quartz absorbs these rays only slightly. For this reason, the bulbs of the lamps for obtaining ultraviolet radiation are made of quartz glass.

Beyond the region of ultraviolet radiation, there are the regions of X-rays and gamma-radiation.

6.20. Emission and Absorption Spectra. Fraunhofer Lines. Spectral Analysis

Spectra formed by the radiation of luminous bodies are called **emission spectra**.

Solids and liquids, when heated, give a continuous spectrum. Heated gases and vapours under a pressure slightly exceeding the normal pressure produce a line spectrum consisting of individual lines separated by dark spaces. Each chemical element is characterized by its own line spectrum.

The **absorption spectrum** of a substance is a radiation spectrum formed when a radiation having a continuous spectrum passes through a layer of this substance.

When white light passes through a vapour or a gas, a continuous spectrum containing dark lines is formed. These lines in the absorption spectrum are located at the same sites where the lines in the emission spectrum of the same vapour or gas are. The positions of dark lines in the absorption spectrum of every element are exactly the same as the positions of lines in the emission spectrum. This means that *each substance absorbs light rays of the same wavelengths which it emits under given conditions.*

The law of reversibility of spectral lines was discovered by Kirchhoff. The schematic diagram of a simple experiment illustrating this law is represented in Fig. 399. Figure 399*a* shows the formation of an emission spectrum of sodium vapour, while Fig. 399*b* gives an idea of obtaining its absorption spectrum. The light from a high-power source is passed through sodium vapour. The dark line in the absorption spectrum of sodium is formed at the same site of the screen where the yellow line in the emission spectrum is located. Thus, sodium absorbs the same yellow rays that it emits.

The solar spectrum is a continuous emission spectrum containing almost 20 000 dark absorption lines in the visible and invisible spectral regions. The source of the continuous spectrum is the surface of the Sun called the *photosphere.*

The absorption lines correspond to light rays absorbed by

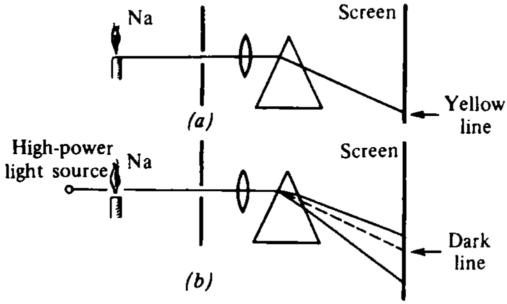

Fig. 399

gases and vapours of substances in the *chromosphere* (viz. the gaseous envelope of the Sun) and in the Earth's atmosphere. From these lines it was established that the chromosphere contains hydrogen, calcium, sodium, iron, and other elements encountered on the Earth. The absorption lines are called the **Fraunhofer lines** after the German scientist who was the first to describe these lines.

Spectral analysis is a method of determining the chemical composition of bodies from their emission or absorption spectra.

Spectral analysis is the finest tool of physico-chemical investigation, which makes it possible to detect insignificant amounts of impurities. This is its main advantage over ordinary chemical methods of analysis. Another advantage of spectral analysis is that it can be used for determining the composition of remote bodies. In particular, helium was discovered in the Sun by a spectroscopic method.

Quantitative spectral analysis has been developed in the last decades. It is based on the fact that the intensity of spectral lines depends on the concentration of an element in the substance under investigation. Comparing the intensities of the spectral lines of a standard sample with the spectral lines of a given substance, it is possible to determine the content of the given element in the substance.

6.21. On the Wave and Quantum Nature of Light

Two theories of light, viz. the Newton *corpuscular theory* and the Huygens *wave theory,* were put forward almost simultaneously.

According to the corpuscular theory formulated by Newton at the end of the 17th century, luminous bodies emit tiniest particles (corpuscles) propagating along straight lines in all directions and causing a visual perception in an eye.

According to the wave theory, a luminous body causes elastic oscillations in a special medium (ether) filling the entire space. These oscillations propagate in the ether like sonic waves in air.

At the time of Newton and Huygens, most scientists supported the Newton corpuscular theory which explained fairly well all optical phenomena known at that time. The reflection of light was explained in the same way as the reflection of elastic bodies impinging a plane. The refraction of light was explained by the action of stronger attraction exerted by a denser medium on the corpuscles. Under the action of these forces, light corpuscles acquire an acceleration in a direction normal to the interface between the media. As a result, the corpuscles change the direction of motion and at the same time increase their velocity. Other optical phenomena were explained similarly.

New observations made later could not be explained within the framework of this theory. In particular, the theory failed to explain the value of the velocity of light in water. The experimental value turned out to be smaller and not larger than the velocity of light in air.

At the beginning of the 19th century, the Huygens wave theory, which was refuted by contemporaries, was developed and perfected by Young and Fresnel and received a wide recognition.

In 1860's, after the formulation of the theory of electromagnetic field by Maxwell, it became clear that light is a kind of electromagnetic waves. Thus, the *mechanistic wave theory of light* was replaced by the *electromagnetic wave theory*. Light waves (visible spectrum) occupy the range from 0.4 μm to 0.7 μm on the scale of electromagnetic waves.

The Maxwell wave theory of light, interpreting radiation as a continuous process, also failed to explain some newly discovered optical phenomena. It was supplemeted by the *quantum theory of light,* according to which the energy of a light wave is emitted, propagates, and is absorbed not continuously but in certain portions, viz. *quanta,* or photons, which depend only on the length of the light wave. Thus, according to modern ideas, light has wave properties as well as corpuscular properties.

The energy of a light quantum depends on the frequency of oscillations of a light wave:

$$\varepsilon = h\nu, \text{ or } \varepsilon = hc/\lambda,$$

where ν is the oscillation frequency of the light wave, λ is its wavelength, c is the velocity of light in a vacuum, and h is the proportionality factor called **Planck's constant**. The value of this constant is $h \approx 6.6 \times 10^{-34}$ J·s.

6.22°. Interference of Light

Waves creating at each point of space oscillations with a phase difference invariable in time are called **coherent**. In this case, the phase difference has the same value, which is generally different for different points in space. Obviously, only the waves of the same frequency can be coherent.

When several coherent waves propagate in space, oscillations generated by these waves enhance one another at some points and suppress one another at other points. This phenomenon is known as the **interference** of waves. Interference is observed for waves of various types: sonic waves, electromagnetic waves, and so on. We shall consider the interference of light waves.

The sources of coherent waves are also called coherent. When a surface is illuminated by coherent sources of light, generally alternating dark and light fringes appear on it.

Two independent sources of light cannot be coherent. The waves emitted by such sources result from the addition of a large number of waves emitted by individual atoms. The emission of waves by atoms is a random process; therefore, there are no permanent relations between the phases of waves emitted by two sources.

When a surface is illuminated by incoherent sources, no interference pattern of alternation of light and dark fringes is observed. The illuminance at each point turns out to be the sum of the illuminances due to each individual source.

Coherent waves can be obtained by splitting a beam of light

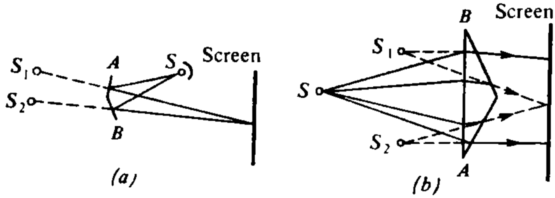

Fig. 400

from the same source into two or more separate beams. Figure 400 illustrates the two ways of obtaining coherent light beams.

Method I (Fig. 400*a*). The light from a point source S is incident on two plane mirrors A and B arranged at an angle slightly smaller than 180° and is reflected in the form of two coherent light beams propagating in different directions. The pattern formed on the screen will be the same as that obtained by illuminating the screen by two coherent light sources S_1 and S_2, whose position is determined by the equalities $SA = S_1A$ and $SB = S_2B$.

If the primary source S is sufficiently far from the screen, the light beams reflected from each mirror can be treated as practically parallel light beams. Such instruments are called the *Fresnel mirrors.*

Method II (Fig. 400*b*). The light from a point source S passes through a biprism and illuminates a screen in the same way as if we had two virtual coherent sources S_1 and S_2.

An interference pattern is observed by illuminating a transparent plate of varying thickness, e.g. an optical wedge (Fig. 401), by monochromatic rays. An observer's eye will

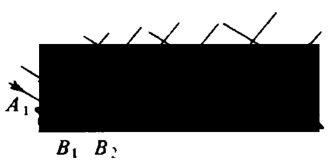

Fig. 401

perceive waves reflected both from the front and rear surfaces of the plate. An interference pattern is determined by the phase difference between these waves, which gradually changes with the thickness of the plate. The illuminance varies accordingly: if the path difference of interfering waves at a certain point is equal to an even number of half-waves, the observer sees a light fringe, and if the phase difference is equal to an odd number of half-waves, a dark fringe is observed.

If a parallel beam of light illuminates a plane-parallel plate (Fig. 402), the phase difference of light waves reflected from the front and rear surfaces is the same at all points, and the plate seems to be illuminated uniformly.

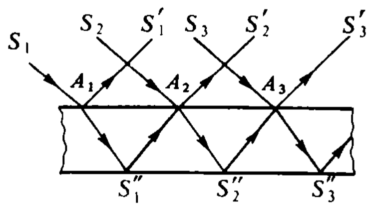

Fig. 402

An interference pattern (Fig. 403*b*) in the form of the so-called *Newton rings* is observed around the point of contact of a slightly convex glass and a plane plate (Fig. 403*a*) illuminated by a monochromatic light. Here, the thin air wedge between the glasses plays the role of a reflecting film having uniform concentric rings.

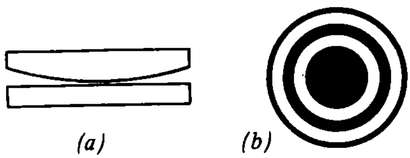

Fig. 403

6.23°. Diffraction of Light

When light propagates in a medium with clearly manifested inhomogeneities (e.g. at the edges of opaque bodies or through small holes), deviations from the laws of geometrical (ray) optics are observed. In particular, light wave may penetrate the zone of the umbra. These phenomena are called the **diffraction of light**. Figure 404 illustrates the propagation of a plane wave through an orifice whose diameter is smaller than the wavelength ($d < \lambda$).

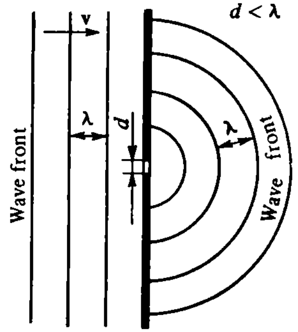

Fig. 404

In order to understand this phenomenon and construct the path of the wave, we shall make use of the **Huygens principle**. According to this principle, *each point where a light wave arrives becomes a source of a secondary wave.* Constructing the envelope of the secondary waves emitted by the sources at the points where the wave arrives at the same instant, we can determine the position of the wave front in a certain subsequent instant of time.

Each point of the orifice at which a wave has arrived becomes, in accordance with the Huygens principle, a source of secondary waves. Since the orifice is small in comparison with the wavelength, all these secondary sources can be considered point sources to a sufficiently high degree of accuracy. These sources emit spherical spreading waves filling the entire space behind the screen with the orifice.

When light passes an obstacle of a sufficiently large size (Fig. 405), the secondary sources located at the edge of the obstacle emit the secondary waves that penetrate the zone of the umbra of the obstacle. The interference of the secondary waves from these sources leads to a bending of the wave front towards the umbra. If an obstacle has dimensions commensurate with the wavelength ($d \sim \lambda$) (Fig. 406), the envelope wave completely fills the space behind the obstacle. If light passes through large holes,

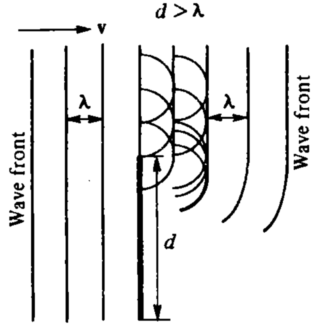

Fig. 405

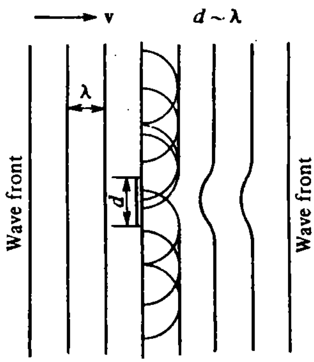

Fig. 406

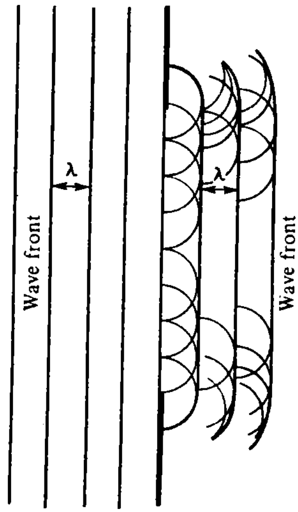

Fig. 407

only a slight bending around the brims of the hole is observed (Fig. 407).

In the following two experiments the diffraction of light is manifested.

1. When a monochromatic light is passed through a puncture in an opaque screen, a bright circle surrounded by alternating dark and bright rings is obtained on another screen behind the first screen.

2. The umbra of a thin wire on a screen has the form of a dark fringe with narrow light and dark fringes on both its sides (Fig. 408).

If similar experiments are carried out with a white light, the dispersion of light also takes place, and each fringe turns out to be coloured by spectral colours.

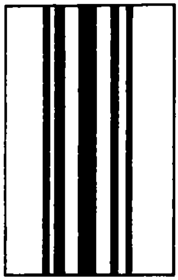

Fig. 408

Diffraction sets a limit on the resolving power of optical instruments, i.e. on the ability of an instrument to form sharp images of two closely spaced points.

6.24. Photoelectric Effect

The **photoelectric (photoemissive) effect** consists in the emission of electrons by a substance under the action of light. If was discovered in 1887 by the German physicist Hertz. The first fundamental investigation of the photoelectric effect was carried out in 1888 by the Russian physicist Stoletov.

Figure 409 represents a schematic diagram of the set-up used by Stoletov for studying the photoelectric effect. A solid plate C

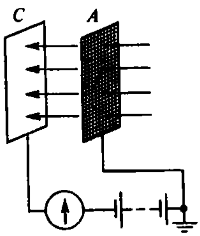

Fig. 409

(made of aluminium, copper, zinc, silver or nickel) and a thin grid A were arranged at a small distance from each other and connected to the terminals of a current source. The solid plate was illuminated by light passing through the grid. When the plate was connected to the negative pole, i.e. when it was a cathode, the ammeter indicated a current in the circuit. Naturally, it can be concluded that in this case the electric circuit was closed by negatively charged particles extracted from the illuminated plate. In 1899 Lenard determined that these particles are electrons.

The electron beam emitted by the cathode under the action of light is called the *photoelectric,* or *photoemissive, current.*

The further investigations of the photoelectric effect were carried out with the help of an instrument shown schematically in Fig. 410 (this instrument is called a *photoemissive cell,* or

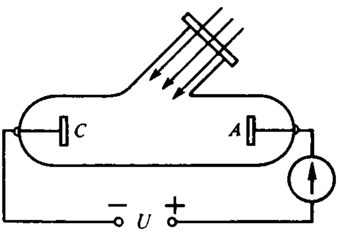

Fig. 410

phototube). An evacuated vessel contained two electrodes: cathode C and anode A. The cathode was illuminated through a quartz window by rays of different wavelengths, including ultraviolet rays. The voltage U was supplied to the clamps "−" and "+". The magnitude (and, if necessary, direction) of this voltage could be smoothly varied. The photoelectric current was measured by a galvanometer.

Figure 411*a* shows a voltage-current characteristic, i.e. the curve describing the photoelectric current as a function of the applied voltage U. Even at zero voltage, a certain number of electrons reaches the anode and forms a small current I_0. As the voltage increases, more and more electrons reach the anode, and the photoelectric current increases to a certain limiting value I_{sat}, after which any further increase in the voltage does not change the current. The limiting value of the photoelectric current is called the *saturation current.* At a low voltage, only a part of electrons escaping from the cathode flies to the anode, while the remaining

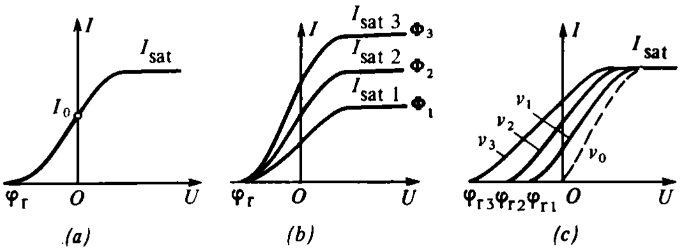

Fig. 411

part is returned to the cathode. As the voltage increases, the number of electrons reaching the anode increases, and the photoelectric current rises. At a sufficiently high voltage, all the electrons escaping from the cathode get to the anode, and the photoelectric current reaches the saturation.

If the illuminated plate is connected to the positive pole of the power source and the grid is connected to the negative pole, i.e. if the negative voltage is applied to the clamps, it can be seen from Fig. 411*a* that the photoelectric current decreases and becomes zero at a certain reverse voltage φ_r, which is called the *cut-off potential*.

Having measured the cut-off potential φ_r, we can easily determine the maximum velocity v_{max} of photoelectrons. For this we must equate the work done by the electric field forces on the electrons to the maximum kinetic energy of the electrons:

$$e\varphi_r = mv_{max}^2/2. \qquad (6.24.1)$$

Figure 411*b* shows the voltage-current characteristics for three values of the luminous flux ($\Phi_1 < \Phi_2 < \Phi_3$) incident on the cathode. In all the three cases, the radiation frequency is the same. Figure 411*c* gives the voltage-current characteristics for experiments where the frequency ν of light causing the photoelectric effect is varied.

The experiments described above led to the following conclusions.

1. The saturation photoelectric current is directly proportional to the luminous flux incident on the cathode:

$$I_{sat} = k\Phi,$$

where k is the proportionality factor which depends on the material of the cathode.

2. The maximum kinetic energy of photoelectrons depends only on the frequency of the incident radiation (increases with it).

In 1905, Einstein proved that the laws of photoelectric effect are easily explained by assuming that light is absorbed in quanta $h\nu$ (h is Planck's constant and ν is the frequency of light). Later these quanta were called photons. When a photon penetrates a substance, it is absorbed by an electron. A part of the energy absorbed by the electron is spent to reach the surface of the material, and the other part is required to escape from it. This part of the energy is called the *electron work function* A, while the remaining part of the energy is carried away by the electron in the form of kinetic energy. The electron located near the surface of the material escapes with a maximum velocity v_{max} since no energy is spent to reach the surface. In this case, the following relation holds:

$$h\nu = mv_{max}^2/2 + A, \tag{6.24.2}$$

which is called **Einstein's relation**. It follows from this formula that when $h\nu \leqslant A$, the electron cannot leave the material, and the photoelectric effect is not observed. The limiting frequency, $\nu_0 = A/h$ is known as the *photoelectric threshold*.

6.25. Photocells and Their Application

Figure 412 is a schematic diagram of a modern **photocell**. A light-sensitive layer (of sodium, potassium or caesium), which serves as a cathode, coats almost the entire inner surface of a glass evacuated bulb. Only a small window lets light rays in. An anode is made in the form of a wire ring fixed in the bulb. The photocell and a battery form an open electric energized circuit.

When the cathode is illuminated, a photoelectric current appears. To increase this current, the bulb is filled with an inert gas (neon or argon). In this case, the electrons moving with a high velocity to the anode ionize gas atoms, which generates the ion current which is 25-50 times stronger than the photoelectric cur-

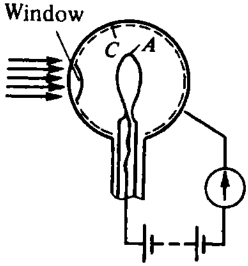

Fig. 412

rent. Besides, the photoelectric current can be enhanced by ordinary valve amplifiers.

In combination with electronic amplifiers, photocells are employed in automatic circuit breakers that are highly sensitive to the action of light and are called *photoelectric relay* (or *photorelay*). Depending on the type of connection to the circuit being controlled, a photorelay responds either when light is incident on the cell or when the illumination ceases. Photocells are also used in television and in sound film technique, as well as in teleautomation (telemetry), viz. the remote control of operation of machines.

The advantage of photocells over other automatic control systems is the *lack of inertia*. Photocells are used as the objectives for photometers whose operation is based on the linear dependence of the saturation photoelectric current on the luminous intensity.

6.26°. Effects of Light

Luminous radiation causes various effects in illuminated bodies.

1. *Thermal effect* is the heating of a body which always takes place to a certain extent when light is absorbed. It consists in the transformation of luminous energy into the internal energy of the body.

2. *Photoelectric effect* is the emission of electrons by a substance under the action of light.

3. *Chemical effect* involves chemical reactions as a result of

transformation of luminous energy into chemical energy (bleaching, decomposition of carbon dioxide by plants under the action of solar radiation, and so on).

4. *Mechanical effect* is manifested in *light pressure*. The fact that light must exert a pressure on the illuminated surface followed from the electromagnetic theory of light developed by Maxwell and was proved theoretically. Since the value of light pressure is insignificant, experimental verification and measurement of light pressure was fraught with considerable difficulties.

Experiments carried out by Lebedev showed that the force of light pressure is proportional to the energy of the incident beam and does not depend on frequency.

5. *Luminescence* is the conversion of one type of luminous energy into another, as well as the conversion of other kinds of energy into luminous energy.

Two types of luminescence under the action of light are distinguished: (a) the glow of bodies (called *fluorescence*) during illumination, which disappears when illumination ceases (e.g. if a bottle with kerosene which has a yellow colour is illuminated by white light, kerosene emits a bluish glow, while alcohol solution of chlorophyll emits light of red colour) and (b) the glow (called *phosphorescence*) which persists when the source of light is removed (e.g. the glow of calcium sulphite and sulphites of some other metals).

Light waves emitted by a luminescent substance mostly have longer wavelength than the absorbed waves causing luminescence. This property is used for detecting invisible ultraviolet rays from coloured luminescence caused by ultraviolet radiation. Luminescence is also employed in the so-called luminescence analysis aiming at detecting negligible impurities (10^{-10} g) of luminescent substances from a typical glow.

Phosphorescent materials are used for manufacturing glowing screens and objects like dials of instruments, watches, etc. Fluorescent materials are used for manufacturing glow-discharge luminescent lamps, viz. glass tubes filled with mercury or sodium vapour. The inner surface of a tube is coated by a layer of phosphor, i.e. a fluorescent substance emitting a bright glow under the action of ultraviolet radiation. The efficiency of gas-discharge lamps is much higher than the efficiency of incandescent lamps mainly due to the absence of heat losses.

Problem with Solution

289. The photoelectric threshold for caesium is $\lambda_0 = 653$ nm. Find the velocity of photoelectrons knocked out due to irradiation of caesium by violet light having a wavelength $\lambda = 400$ nm. The mass of the electron is $m = 9.1 \times 10^{-31}$ kg.

Solution. We write Einstein's equation

$$h\nu = A + mv_{max}^2/2.$$

Since the frequency of a light wave is $\nu = c/\lambda$, where c is the velocity of light and λ is the wavelength, this equation can be written as follows:

$$hc/\lambda = A + mv_{max}^2/2.$$

The electron work function can be expressed through the photoelectric threshold: $A = hc/\lambda_0$, where Planck's constant $h \simeq 6.6 \times 10^{-34}$ J·s. Then Einstein's equation has the form

$$hc/\lambda = hc/\lambda_0 + mv_{max}^2/2,$$

whence $v = \sqrt{2hc(\lambda_0 - \lambda)/m\lambda_0\lambda} = 6.5 \times 10^5$ m/s.

Exercise

289. Find the electron work function for a certain material if the velocity of electrons knocked out from the surface by a yellow light is 0.28×10^6 m/s. The wavelength of yellow light is 590 nm, the electron mass is 9.1×10^{-31} kg, and Planck's constant is 6.6×10^{-34} J·s.

Answer. 3.02×10^{-19} J = 1.89 eV.

7. STRUCTURE OF THE ATOM

7.1. Structure of the Atom and Its Energy

The **atom** consists of a heavy positively charged *nucleus* and negatively charged *electrons* moving around it. The electron is an elementary particle having a mass $m_e \simeq 9.1 \times 10^{-31}$ kg and a charge $-e$, e being an elementary charge approximately equal to 1.60×10^{-19} C.

The nuclear charge is equal to $+Ze$, where Z is the atomic number (the number of an element in the Mendeleev Periodic Table). The atom contains Z electrons, their total charge being $-Ze$. Consequently, the atom is an electrically neutral system. The size of the nucleus varies depending on Z from 10^{-13} cm to 10^{-12} cm. The size of the atom is a quantity of the order of 10^{-8} cm.

The energy of the atom is *quantized*. This means that it can assume only discrete (i.e. separated by finite gaps) values: E_1, E_2, E_3, ..., which are called the *energy levels* of the atom ($E_1 < E_2 < E_3 < \ldots$). Atoms with different Z's have different sets of energy levels.

In a normal (unexcited) state, the atom is on the lowest possible energy level. In such a state, the atom may stay for an infinitely long time. By imparting an energy to the atom, it is possible to transfer it to an excited state with an energy higher than the energy of the ground state. A transition of the atom to a higher energy level may occur as a result of absorption of a photon or as a result of a collision with another atom or a particle, say, an electron.

Excited states of the atom are unstable. The atom can stay in an excited state for about 10^{-8} s. After that the atom spontaneously (by its own) goes over to a lower energy level, emitting in this process a photon with an energy

$$h\nu_{ik} = E_i - E_k \quad (i > k),$$

where i is the number of the energy level in the initial state and k is the number of the level to which a spontaneous transition of the atom occurred. For example, an atom which is in an excited state with the energy E_3 can return to the ground state either directly, by emitting a photon of frequency $\nu_{31} = (E_3 - E_1)/h$, or through an intermediate state with the energy E_2, as a result of which two photons with frequencies $\nu_{32} = (E_3 - E_2)/h$ and $\nu_{21} = (E_2 - E_1)/h$ are emitted.

7.2. Atomic Nucleus

The **atomic nucleus** consists of two types of elementary particles, viz. *protons* and *neutrons*. These particles are called *nucleons*.

The **proton** (denoted by p) has a charge $+e$ and a mass $m_p \simeq 1.6726 \times 10^{-27}$ kg, which is approximately 1840 times larger than the electron mass. The proton is the nucleus of the simplest atom with $Z = 1$, viz. the hydrogen atom.

The **neutron** (denoted by n) is an electrically neutral particle (its charge is zero). The neutron mass $m_n \simeq 1.6749 \times 10^{-27}$ kg. The fact that the neutron mass exceeds the proton mass by about 2.5 electron masses is of essential importance. It follows from this that the neutron in free state (outside the nucleus) is unstable (radioactive). During the time equal on the average to 12 min, the neutron spontaneously transforms to the proton by emitting an electron (e^-) and a particle called the *antineutrino* ($\tilde{\nu}$). This process can be schematically written as follows:

$$n \rightarrow p + e^- + \tilde{\nu}.$$

The most important characteristics of the nucleus are the *charge number Z* (coinciding with the atomic number of the element) and the *mass number A*. The charge number Z is equal to the number of protons in the nucleus, and hence it determines the nuclear charge equal to $+Ze$. The mass number A is equal to the number of nucleons in the nucleus (i.e. to the total number of protons and neutrons). This number determines the mass of the nucleus in atomic mass units, which is approximately equal to A amu.[1] The approximate equality $m_p \approx m_n \simeq 1$ amu is valid with an accuracy of 1%. Obviously, the number of neutrons in the nucleus is $N = A - Z$.

Nuclei are symbolically designated as

$$^{A}_{Z}\mathrm{X},$$

where X stands for the symbol of a chemical element. For example, the nucleus of the hydrogen atom is symbolically written as $^{1}_{1}\mathrm{H}$, the nucleus of nitrogen atom, as $^{14}_{7}\mathrm{N}$, the nucleus of oxygen atom, as $^{16}_{8}\mathrm{O}$, etc.

Most of chemical elements have several types of atoms differing in the number of neutrons in their nuclei. These varieties are called *isotopes*. For example, oxygen has three stable isotopes: $^{16}_{8}\mathrm{O}$, $^{17}_{8}\mathrm{O}$, and $^{18}_{8}\mathrm{O}$, tin has ten stable isotopes, and so on. In addition to stable isotopes, there also exist unstable (radioactive) isotopes.

7.3. Radioactivity

Radioactivity is the ability of atomic nuclei of some elements to disintegrate spontaneously, being transformed to nuclei of another elements. There are two types of radioactivity: ***natural radioactivity*** observed in unstable elements in nature, and ***artificial radioactivity*** of artificially obtained isotopes. The rate of disintegration is different for different isotopes and is characterized by the half-life.

The **half-life** is the time interval during which a radioactive substance decays to half its original value. The half-lives of different elements vary from fractions of a second to billion years. The half-life is 1600 years for radium, about four days for radon, etc.

Natural radioactive processes are of two kinds:

(1) α-decay associated with the emission of α-particles, viz. nuclei $^{4}_{2}\mathrm{He}$ of helium. Alpha particles are heavy positively charged

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[1] The **atomic mass unit** (amu) is equal to 1/12 mass of the $^{12}\mathrm{C}$ atom: 1 amu $\simeq 1.6606 \times 10^{-27}$ kg.

particles having a mass $m_\alpha \approx 4$ amu and a charge $q_\alpha = +2e$. The velocity of α-particles is relatively low: $v_\alpha = (1/30 - 1/15)c$, where c is the velocity of light.

(2) β^--decay (beta-minus-decay) associated with the emission of electrons formed at the instant of decay.

Both processes are accompanied by γ-radiation, i.e. the flow of photons having a very small wavelength, and hence a very high energy. Like other electromagnetic waves, γ-rays propagate at a velocity of light. The penetrability of γ-rays is 10-100 times higher than the penetrability of β-rays and 1000-10 000 times higher than the penetrability of α-rays. It also exceeds the penetrability of X-rays.

In a magnetic field, a beam of α-, β-, and γ-rays splits into three beams (Fig. 413).

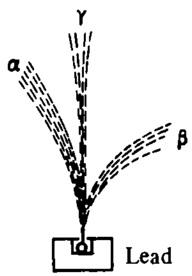

Fig. 413

Nuclei possessing the artificial radioactivity are obtained by bombarding stable nuclei of heavy elements by α-particles, neutrons, or (sometimes) protons and other particles. Nuclear transformations occur in two stages in this case. First a particle hits a target nucleus and causes its transformation into another, unstable (radioactive), nucleus. This newly formed nucleus spontaneously emits a particle and is transformed either into a stable nucleus or into a new radioactive nucleus. Artificial radioactivity obeys the same laws as natural radioactivity.

Radioactive processes occur in accordance with the laws of energy conservation, electric charge, and mass number (amount of nucleons).

In α-decay, the mass number of the nucleus decreases by four and the charge decreases by two units, as a result of which two electrons are removed from the atomic shell. The element transforms into another element with the atomic number which is two unis lower.

In β^--decay, a neutron in the nucleus transforms into a proton. Such a transformation of the neutral neutron into the positive proton is accompanied by the birth of an electron, i.e. by β-radiation. The mass number of the nucleus does not change in this process, while the charge increases by $+e$.

7.4. Uranium Nuclear Fission. Chain Reaction

The nucleus of the uranium isotope $^{235}_{92}U$ irradiated by slow (thermal) neutrons splits into nuclei of two different elements with approximately equal masses. A so-called *nuclear fission reaction* occurs. **Uranium fission** may result in different pairs of fragments and is accompanied by the emission of 2-3 neutrons. For example, when $^{235}_{92}U$ absorbs a neutron, it may split into xenon and strontium with the liberation of two neutrons:

$$^{235}_{92}U + ^{1}_{0}n \rightarrow ^{139}_{54}Xe + ^{95}_{38}Sr + 2^{1}_{0}n.$$

Both nuclei formed as a result, $^{139}_{54}Xe$ and $^{95}_{38}Sr$, have an enormous excess of neutrons, since the heaviest stable isotopes of xenon and strontium are $^{136}_{54}Xe$ and $^{90}_{38}Sr$. Therefore, these nuclei are unstable and undergo a number of consecutive transformations through β-decay, consisting in the transformation of excess neutrons into protons and the liberation of the corresponding number of electrons. The mass number of the element is conserved in this case.

The emission of several neutrons during the fission of the $^{235}_{92}U$ nucleus gives rise to a **chain reaction**. The essence of this reaction is that k neutrons emitted as a result of the disintegration of a nucleus may cause the fission of k nuclei, as a result of which k^2 new neutrons are emitted, which cause the fission of k^2 nuclei, and so on. Consequently, the number of neutrons born in each generation increases in the geometric progression. The process on the whole is of the avalanche nature, it occurs very rapidly and is accompanied (as it will be shown in the next section) by the liberation of an enormous amount of energy.

7.5. Binding Energy of Atomic Nucleus

According to the special theory of relativity developed by Einstein, the energy of a particle is defined by the following relation:

$$E = m_0c^2/\sqrt{1 - v^2/c^2},$$

where m_0 is the rest mass of the particle, c is the velocity of light in a vacuum, and v is the velocity of the particle. If a particle is at rest (i.e. $v = 0$), it has an energy

$$E_0 = m_0c^2, \tag{7.5.1}$$

which is called the *rest energy*.

In nuclear physics, energy is usually measured in electron volts (eV). An **electron volt** is the work done by electric field forces on a charge equal in magnitude to the electron charge (i.e. on the elementary charge e) falling freely through a potential difference of one volt:

$$1 \text{ eV} = 1.60 \times 10^{-19} \text{ C} \cdot 1 \text{ V} = 1.60 \times 10^{-19} \text{ J}.$$

The units multiple to an electron volt are also used:

$$\begin{aligned} 1 \text{ keV (kiloelectron volt)} &= 10^3 \text{ eV}, \\ 1 \text{ MeV (megaelectron volt)} &= 10^6 \text{ eV}, \\ 1 \text{ GeV (gigaelectron volt)} &= 10^9 \text{ eV}. \end{aligned}$$

According to formula (7.5.1), the rest energy of an electron is

$$\begin{aligned} E_{0\,e} &= (9.1095 \times 10^{-31}) \cdot (3 \times 10^8)^2 \approx 8.20 \times 10^{-14} \text{ J} \\ &\approx (8.20 \times 10^{-14}) : (1.60 \times 10^{-19}) \approx 0.51 \text{ MeV} \end{aligned}$$

(to be more precise, 0.511 MeV). Similar calculations for the proton and neutron give $E_{0\,p} = 938.3$ MeV and $E_{0\,n} = 939.6$ MeV.

The rest mass of the nucleus is smaller than the sum of the rest masses of nucleons constituting it. This is due to the fact that when nucleons combine to form a nucleus, the **binding energy** of nucleons is liberated. The binding energy is equal to the work that must be done to split the nucleus into particles constituting it.

The difference between the total mass of the nucleons and the mass of the nucleus is called the *mass defect* of the nucleus:

$$\Delta m = [Zm_p + (A - Z)m_n] - m_{nuc}.$$

Multiplying the mass defect by the square of the velocity of light, we can find the binding energy of the nucleus:

$$E_b = \Delta m \cdot c^2.$$

Dividing the binding energy by the number A of nucleons in the nucleus, we obtain the binding energy per nucleon. Figure 414 shows the dependence of the binding energy per nucleon E_b/A on the mass number A of the nucleus. Nucleons in nuclei with mass numbers from 50 to 60 have the highest binding energy. The binding energy per nucleon for these neclei amounts to 8.7 MeV/nucleon and gradually decreases with increasing A. For the heaviest natural element—uranium—it amounts to 7.5 MeV/nucleon. Figure 414 shows that when a heavy nucleus

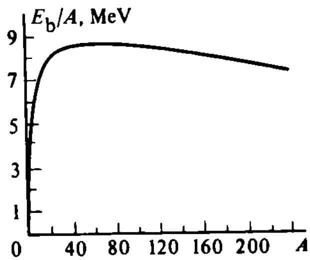

Fig. 414

(with $A \simeq 240$) splits into two nuclei with $A \simeq 120$, the released energy is of the order of 1 MeV per nucleon, i.e. 240 MeV per parent nucleus. It should be mentioned for comparison that when a carbon atom is oxidized (burnt) to CO_2, the energy of the order of 5 eV is liberated, which is smaller than the energy released in fission of a uranium nucleus by a factor of 50 millions.

It also follows from Fig. 414 that the fusion (synthesis) of light nuclei into one should be accompanied by the liberation of a huge energy. For example, the fusion of two nuclei of heavy hydrogen ^2_1H (this nucleus is called a deuteron) into a helium nucleus ^4_2He would yield an energy equal to 24 MeV.

The forces binding nucleons in a nucleus manifest themselves at distances below 10^{-13} cm. In order to bring together two positively charged deuterons to such a distance, their Coulomb repulsion should be overcome. For this, the deuterons must have a kinetic energy equivalent to their mean energy of thermal motion at a temperature of the order of 10^9 K. For this reason, the fusion reaction of nuclei is also called a **thermonuclear reaction**. Actually, some thermonuclear reactions may occur at a temperature of the order of 10^7 K. This is due to the fact that there is always a certain number of nuclei whose energy considerably exceeds the mean value.

Problem with Solution

290. Write the missing symbols in the following nuclear reactions:

$$^{19}_{9}F + ^{1}_{1}H \rightarrow ^{16}_{8}O + ?, \quad ^{55}_{25}Mn + ^{1}_{1}H \rightarrow ^{55}_{26}Fe + ?,$$
$$^{25}_{12}Mg + ? \rightarrow ^{22}_{11}Na + ^{4}_{2}He.$$

Solution. According to the law of electric charge conservation, the sum of the subscripts of the reaction product must be equal to the sum of the subscripts of the nuclei entering the reaction. The sum of the mass numbers, i.e. superscripts, must also be the same before the reaction and after it. Hence we can first write the missing indices:

$$^{19}_{9}F + ^{1}_{1}H \rightarrow ^{16}_{8}O + ^{4}_{2}?, \quad ^{55}_{25}Mn + ^{1}_{1}H \rightarrow ^{55}_{26}Fe + ^{1}_{0}?,$$
$$^{25}_{12}Mg + ^{1}_{1}? \rightarrow ^{22}_{11}Na + ^{4}_{2}He.$$

Then the symbols of elements can be written:

$$^{4}_{2}? \equiv ^{4}_{2}He, \quad ^{1}_{1}? \equiv ^{1}_{1}H.$$

Thus, we have the following reactions:

$$^{19}_{9}F + ^{1}_{1}H \rightarrow ^{16}_{8}O + ^{4}_{2}He, \quad ^{55}_{25}Mn + ^{1}_{1}H \rightarrow ^{55}_{26}Fe + ^{1}_{0}n,$$
$$^{25}_{12}Mg + ^{1}_{1}H \rightarrow ^{22}_{11}Na + ^{4}_{2}He.$$

Exercise

290. Write the missing symbols in the following nuclear reactions:

$$^{27}_{13}Al + ^{4}_{2}He \rightarrow ^{30}_{15}P + ?, \quad ^{198}_{80}Hg + ^{1}_{0}n \rightarrow ^{198}_{79}Au + ?,$$
$$^{27}_{13}Al + ? \rightarrow ^{30}_{14}Si + ^{1}_{1}H.$$

Answer. $^{1}_{0}n$, $^{1}_{1}H$, $^{4}_{2}He$.

GRAPHICAL SOLUTIONS TO EXERCISES

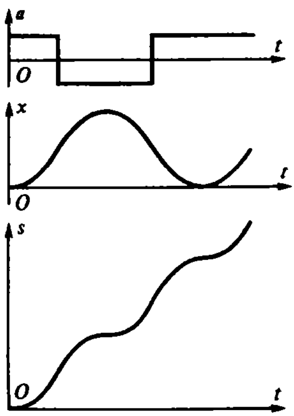

Fig. I

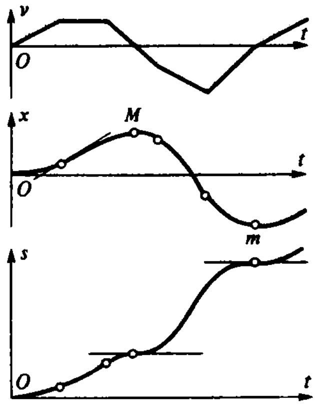

Fig. II

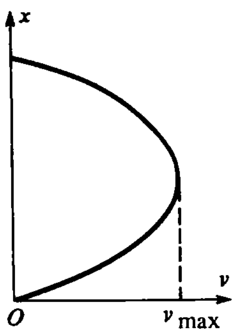

Fig. III

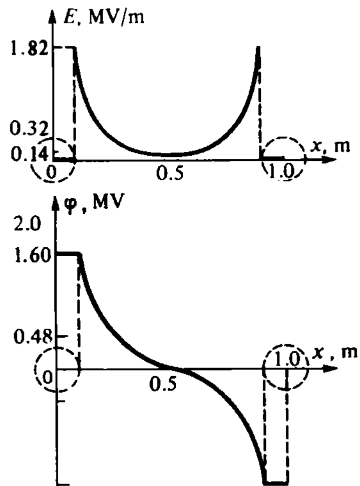

Fig. IV

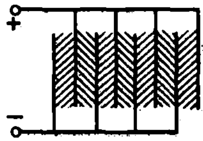

Fig. V

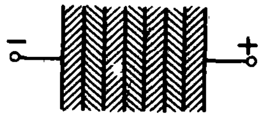

Fig. VI

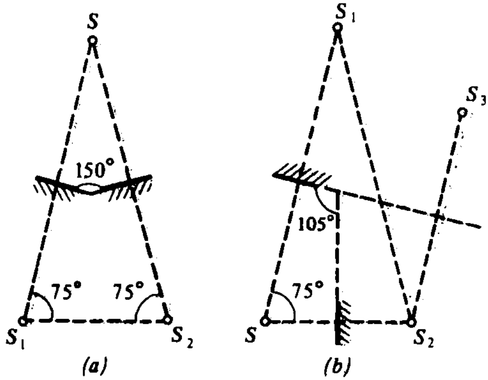

Fig. VII

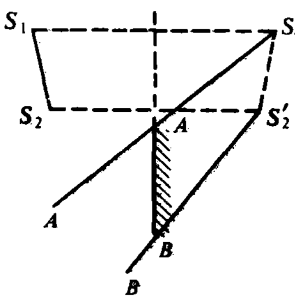

Fig. VIII

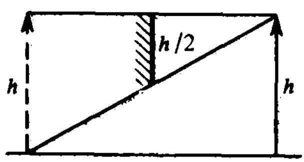

Fig. IX

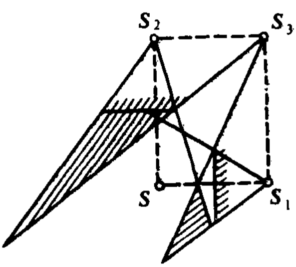

Fig. X

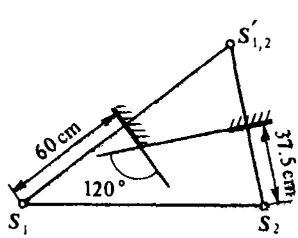

Fig. XI

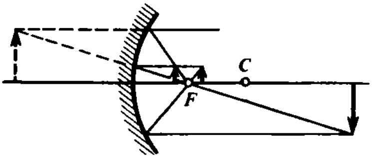

Fig. XII

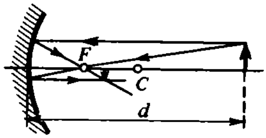

Fig. XIII

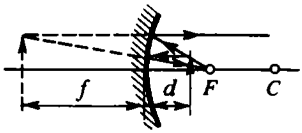

Fig. XIV

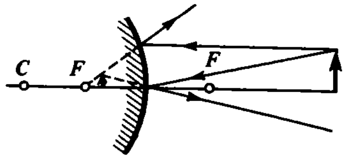

Fig. XV

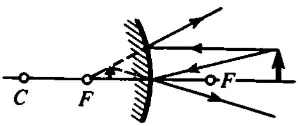

Fig. XVI

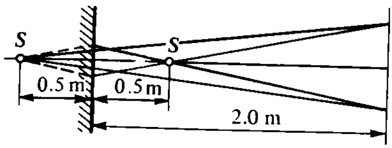

Fig. XVII

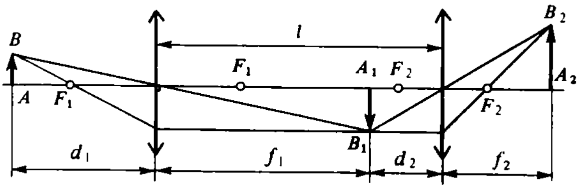

Fig. XVIII

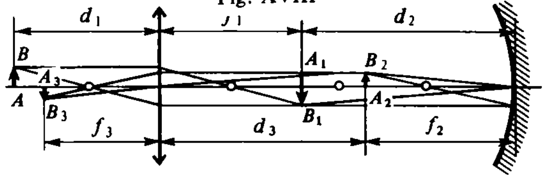

Fig. XIX

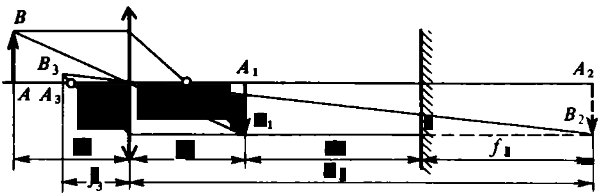

Fig. XX

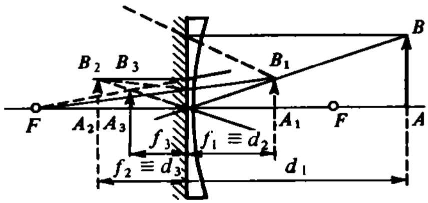

Fig. XXI

APPENDICES

I. Fundamental Physical Constants

Gravitational constant	$G = 6.6720 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$
Free-fall acceleration (normal)	$g = 9.806\ 65\ \mathrm{m/s^2}$
Velocity of light in a vacuum	$c = 2.997\ 924\ 58 \times 10^{8}\ \mathrm{m/s}$
Megnetic constant	$\mu_0 = 4\pi \cdot 10^{-7}\ \mathrm{H/m} = 12.566\ 371 \times 10^{-7}\ \mathrm{H/m}$
Electric constant	$\varepsilon_0 = 8.854\ 188 \times 10^{-12}\ \mathrm{F/m}$
Rest mass of electron	$m_e = 9.109\ 534 \times 10^{-31}\ \mathrm{kg}$
Rest mass of proton	$m_p = 1.672\ 648\ 5 \times 10^{-27}\ \mathrm{kg}$
Charge of electron	$e = 1.602\ 189\ 2 \times 10^{-19}\ \mathrm{C}$
Ratio of electron charge to electron mass	$e/m_e = 1.758\ 804\ 7 \times 10^{11}\ \mathrm{C/kg}$
Planck's constant	$h = 6.626\ 176 \times 10^{-34}\ \mathrm{J \cdot s}$
Avogadro constant	$N_A = 6.022\ 045 \times 10^{23}\ \mathrm{mol^{-1}}$
Boltzmann constant	$k = 1.380\ 662 \times 10^{-23}\ \mathrm{J/K}$
Molar gas constant	$R = 8.314\ 41\ \mathrm{J/(mol \cdot K)}$
Faraday's constant	$F = 96.484\ 56 \times 10^{3}\ \mathrm{C/mol}$
Volume of a mole of ideal gas under standard conditions ($p_0 = 101\ 325$ Pa, $T_0 = 273.15$ K)	$V_0 = 22.413\ 83 \times 10^{-3}\ \mathrm{m^3/mol}$

II. SI Units and Their Dimensions

Quantity		Unit		Relation to base units
Length	L	**metre**	m	**Base unit** (see definition on p. 16)
Area	L^2	square metre	m^2	
Volume	L^3	cubic metre	m^3	
Plane angle	—	radian	rad	*Supplementary unit*
Solid angle	—	steradian	sr	*Supplementary unit*
Time	T	**second**	s	**Base unit** (see definition on p. 17)
Velocity	LT^{-1}	metre per second	m/s	
Acceleration	LT^{-2}	metre per second per second	m/s^2	
Angular velocity	T^{-1}	radian per second	rad/s	$1\ rad/s = 1\ s^{-1}$
Angular acceleration	T^{-2}	radian per second per second	rad/s^2	$1\ rad/s^2 = 1\ s^{-2}$
Frequency of periodic process	T^{-1}	hertz	Hz	$1\ Hz = 1\ s^{-1}$
Rotational frequency	T^{-1}	inverse second	s^{-1}	
Mass	M	**kilogram**	kg	**Base unit** (see definition on p. 16)
Density	$L^{-3}M$	kilogram per cubic metre	kg/m^3	

Quantity		Unit		Relation to base units
Mass flow rate	MT^{-1}	kilogram per second	kg/s	
Volumetric flow rate	L^3T^{-1}	cubic metre per second	m^3/s	
Force	LMT^{-2}	newton	N	$1\ N = 1\ m \cdot kg \cdot s^{-2}$
Pressure	$L^{-1}MT^{-2}$	pascal	Pa	$1\ Pa = 1\ N/m^2 = 1\ m^{-1} \cdot kg \cdot s^{-2}$
Rigidity	MT^{-2}	newton per metre	N/m	$1\ N/m = 1\ kg \cdot s^{-2}$
Momentum	LMT^{-1}	kilogram-metre per second	kg·m/s	
Moment of inertia	L^2M	kilogram-square metre	$kg \cdot m^2$	
Angular momentum	L^2MT^{-1}	kilogram-square metre per second	$kg \cdot m^2/s$	
Impulse	LMT^{-1}	newton-second	N·s	$1\ N \cdot s = 1\ m \cdot kg \cdot s^{-1}$
Moment of force	L^2MT^{-2}	newton-metre	N·m	$1\ N \cdot m = 1\ m^2 \cdot kg \cdot s^{-2}$
Viscosity (dynamic)	$L^{-1}MT^{-1}$	pascal-second	Pa·s	$1\ Pa \cdot s = 1\ m^{-1} \cdot kg \cdot s^{-1}$
Work, energy, amount of heat (heat)	L^2MT^{-2}	joule	J	$1\ J = 1\ N \cdot m = 1\ m^2 \cdot kg \cdot s^{-2}$
Power, energy flux	L^2MT^{-3}	watt	W	$1\ W = 1\ J/s = 1\ m^2 \cdot kg \cdot s^{-3}$
Energy flux density	MT^{-3}	watt per square metre	W/m^2	$1\ W/m^2 = 1\ kg \cdot s^{-3}$
Temperature (thermodynamic)	Θ	**kelvin**	K	**Base unit** (see definition on p. 204) $t/°C = T/K - 273.15$

Amount of substance	N	**mole**	mol	**Base unit** (see definition on p. 17)
Molar mass	MN^{-1}	kilogram per mole	kg/mol	
Heat per unit mass	L^2T^{-2}	joule per kilogram	J/kg	$1\ \mathrm{J/kg} = 1\ \mathrm{m^2 \cdot s^{-2}}$
Molar heat	$L^2MT^{-2}N^{-1}$	joule per mole	J/mol	$1\ \mathrm{J/mol} = 1\ \mathrm{m^2 \cdot kg} \times \mathrm{s^{-2} \cdot mol^{-1}}$
Heat capacity	$L^2MT^{-2}\Theta^{-1}$	joule per kelvin	J/K	$1\ \mathrm{J/K} = 1\ \mathrm{m^2 \cdot kg \cdot s^{-2}} \times \mathrm{K^{-1}}$
Specific heat	$L^2T^{-2}\Theta^{-1}$	joule per kilo-gram-kelvin	J/(kg·K)	$1\ \mathrm{J/(kg \cdot K)} = 1\ \mathrm{m^2 \cdot s^{-2} \cdot K^{-1}}$
Molar heat capacity	$L^2MT^{-2}\Theta^{-1}N^{-1}$	joule per mole-kelvin	J/(mol·K)	$1\ \mathrm{J/(mol \cdot K)} = 1\ \mathrm{m^2 \cdot kg \cdot s^{-2} \cdot K^{-1}} \times \mathrm{mol^{-1}}$
Concentration (number density of particles)	L^{-3}	metre to minus third power	$\mathrm{m^{-3}}$	
Electric current	I	**ampere**	A	**Base unit** (see definition on p. 346)
Current density	$L^{-2}I$	ampere per square metre	$\mathrm{A/m^2}$	
Amount of electricity (electric charge)	TI	coulomb	C	$1\ \mathrm{C} = 1\ \mathrm{A \cdot s}$
Electric voltage, electric potential, potential difference, e.m.f.	$L^2MT^{-3}I^{-1}$	volt	V	$1\ \mathrm{V} = 1\ \mathrm{W/A} = 1\ \mathrm{m^2 \cdot kg \cdot s^{-3} \cdot A^{-1}}$
Electric field strength	$LMT^{-3}I^{-1}$	volt per metre	V/m	$1\ \mathrm{V/m} = 1\ \mathrm{W/(A \cdot m)} = 1\ \mathrm{m \cdot kg \cdot s^{-3} \cdot A^{-1}}$

Quantity		Unit		Relation to base units
Electric resistance	$L^2MT^{-3}I^{-2}$	ohm	Ω	$1\ \Omega = 1\ \mathrm{V/A} = 1\ \mathrm{m^2 \cdot kg \cdot s^{-3} \cdot A^{-2}}$
Resistivity	$L^3MT^{-3}I^{-2}$	ohm-metre	Ω·m	$1\ \Omega\cdot\mathrm{m} = 1\ \mathrm{m^3 \cdot kg \cdot s^{-3} \cdot A^{-2}}$
Electric conductance	$L^{-2}M^{-1}T^3I^2$	siemens	S	$1\ \mathrm{S} = 1\ \Omega^{-1} = 1\ \mathrm{m^{-2} \cdot kg^{-1} \cdot s^3 \cdot A^2}$
Electric conductivity	$L^{-3}M^{-1}T^3I^2$	siemens per metre	S/m	$1\ \mathrm{S/m} = 1\ \Omega^{-1}\cdot\mathrm{m^{-1}} = 1\ \mathrm{m^{-3} \cdot kg^{-1} \cdot s^3 \cdot A^2}$
Capacitance	$L^{-2}M^{-1}T^4I^2$	farad	F	$1\ \mathrm{F} = 1\ \mathrm{C/V} = 1\ \mathrm{m^{-2} \cdot kg^{-1} \cdot s^4 \cdot A^2}$
Absolute permittivity	$L^{-3}M^{-1}T^4I^2$	farad per metre	F/m	$1\ \mathrm{F/m} = 1\ \mathrm{m^{-3} \cdot kg^{-1} \cdot s^4 \cdot A^2}$
Magnetic flux	$L^2MT^{-2}I^{-1}$	weber	Wb	$1\ \mathrm{Wb} = 1\ \mathrm{V \cdot s} = 1\ \mathrm{T \cdot m^2} = 1\ \mathrm{m^2 \cdot kg \cdot s^{-2} \cdot A^{-1}}$
Magnetic induction (magnetic flux density)	$MT^{-2}I^{-1}$	tesla	T	$1\ \mathrm{T} = 1\ \mathrm{V \cdot s/m^2} = 1\ \mathrm{Wb/m^2} = 1\ \mathrm{kg \cdot s^{-2} \cdot A^{-1}}$
Inductance	$L^2MT^{-2}I^{-2}$	henry	H	$1\ \mathrm{H} = 1\ \mathrm{m^2 \cdot kg \cdot s^{-2} \cdot A^{-2}}$
Absolute permeability	$LMT^{-2}I^{-2}$	henry per metre	H/m	$1\ \mathrm{H/m} = 1\ \mathrm{m \cdot kg \cdot s^{-2} \cdot A^{-2}}$
Magnetic field strength	$L^{-1}I$	ampere per metre	A/m	
Radiant energy	L^2MT^{-2}	joule	J	$1\ \mathrm{J} = 1\ \mathrm{m^2 \cdot kg \cdot s^{-2}}$
Radiation power (radiation flux)	L^2MT^{-3}	watt	W	$1\ \mathrm{W} = 1\ \mathrm{J/s} = 1\ \mathrm{m^2 \cdot kg \cdot s^{-3}}$

Radiation intensity (radiation flux density)	MT^{-3}	watt per square metre	W/m^2	$1\ W/m^2 = 1\ kg \cdot s^{-3}$
Particle flux	T^{-1}	inverse second	s^{-1}	
Particle flux density	$L^{-2}T^{-1}$	inverse second-metre to minus second power	$s^{-1} \cdot m^{-2}$	
Luminous intensity	J	**candela**	cd	**Base unit** (see definition on p. 17)
Luminous flux	J	lumen	lm	$1\ lm = 1\ cd \cdot sr$
Luminous energy	TJ	lumen-second	$lm \cdot s$	$1\ lm \cdot s = 1\ s \cdot cd \cdot sr$
Luminous emittance	$L^{-2}J$	lumen per square metre	lm/m^2	$1\ lm/m^2 = 1\ m^{-2} \cdot cd \cdot sr$
Illuminance	$L^{-2}J$	lux	lx	$1\ lx = 1\ lm/m^2 = 1\ m^{-2} \cdot cd \cdot sr$
Luminance	$L^{-2}J$	candela per square metre	cd/m^2	
Radiant intensity	L^2MT^{-3}	watt per steradian	W/sr	$1\ W/sr = 1\ m^2 \cdot kg \cdot s^{-3} \cdot sr^{-1}$
Radiant emittance	MT^{-3}	watt per square metre	W/m^2	$1\ W/m^2 = 1\ kg \cdot s^{-3}$
Irradiance	MT^{-3}	watt per square metre	W/m^2	$1\ W/m^2 = 1\ kg \cdot s^{-3}$
Radiance	MT^{-3}	watt per steradian per square metre	$W/(sr \cdot m^2)$	$1\ W/(sr \cdot m^2) = 1\ kg \cdot s^{-3} \cdot sr^{-1}$

III. Multipliers and Prefixes for Decimal Multiple and Fractional Units

Multiplier	Prefix name	Prefix notation	Multiplier	Prefix name	Prefix notation
10^{18}	exa	E	10^{-1}	deci	d
10^{15}	peta	P	10^{-2}	centi	c
10^{12}	tera	T	10^{-3}	milli	m
10^{9}	giga	G	10^{-6}	micro	μ
10^{6}	mega	M	10^{-9}	nano	n
10^{3}	kilo	k	10^{-12}	pico	p
10^{2}	hecto	h	10^{-15}	femto	f
10^{1}	deca	da	10^{-18}	atto	a

IV. Units of Measurements Used Along with the SI Units

Quantity	Unit and its relation to SI units
Length	Astronomical unit of length: 1 au = $1.495\,978\,70 \times 10^{11}$ m
	Light year: 1 light year = $9.460\,530 \times 10^{15}$ m
	Parsec: 1 pc = $3.085\,678 \times 10^{16}$ m
Area	Hectare: 1 ha = 10^4 m^2
Volume	Litre: 1 l = 10^{-3} m^3
Plane angle	Degree: $1^\circ = (\pi/180)$ rad
	Minute: $1' = (\pi/10\,800)$ rad
	Second: $1'' = (\pi/648\,000)$ rad
Time	Minute: 1 min = 60 s
	Hour: 1 h = 3600 s
	Day: 1 day = 86 400 s
Mass	Tonne: 1 t = 10^3 kg
	Atomic mass unit: 1 amu = $1.660\,565\,5 \times 10^{-27}$ kg
Energy	Electron volt: 1 eV = $1.602\,189\,2 \times 10^{-19}$ J
Lens power	Diopter: 1 D = 1 m^{-1}

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