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COMBINATORIAL MATHEMATICS FOR RECREATION

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Specialists in a broad range of fields have to deal with problems that involve combinations made up of letters, numbers, or any other objects. The department head in a factory has to allocate production assignments to machine-tool operators, the agronomist has to decide on what crops to grow on a selected group of fields, the school principal draws up schedules of lessons, the investigating chemist analyzes relations involving atoms and molecules, the linguist examines the meanings of combinations of letters in an unknown language, and so forth. The field of mathematics that studies problems of how many different combinations (subject to certain restrictions) can be built out of a specific number of objects is called *combinatorial mathematics* (*combinatorics*).

This branch of mathematics has its origin in the 16th century, in the gambling games that played such a large part in high society in those times. Whole fortunes, from gold and precious stones to pedigreed horses, castles and estates, were won or lost in a game of cards or dice. All manner of lotteries were in vogue. It is quite natural that the first combinatorial problems had to do mainly with gambling, such as in how many ways can a certain sum in throws of two or three dice be scored, or in how many ways is it possible to get two kings in a card game. These and other problems in games of chance gave the initial impetus to develop combinatorial mathematics and the burgeoning theory of probability.

One of the first to enumerate the various combinations achieved in games of dice was the Italian mathematician Tartaglia. He drew up a table illustrating the number of ways r dice can fall. It was not taken into account, however, that the same sum can be obtained in different ways (say, $1 + 3 + 4 = 4 + 2 + 2$).

In the 17th century, the French scholars Pascal and Fermat made a theoretical investigation into the problems of combinatorics. Again, the starting point was in the form of problems of games of chance, particularly the so-called problem of points in determining the division of the stakes of an interrupted game of chance. This problem was posed to Pascal by his friend the Chevalier de Méré, an ardent gambler. Roughly, the problem was this: a match of coin tossing to six winning games is interrupted when one player has won five tosses and the other four tosses. How are the stakes to be divided? It was clear that a division of 5 to 4 would not be fair. Applying methods of combinatorics, Pascal solved the problem for the general case when one player has r games left to win and the other one has s games. An alternative solution was given by Fermat.

Further advances in the theory of combinations were connected with the names of Jakob Bernoulli, Leibniz and Euler. However, in these studies the main role was played by applications to various games (lotto, solitaire, etc.). During recent years, combinatorial mathematics has seen extensive developments associated with greater interest in problems of discrete mathematics. Combinatorial methods are employed in solving transport problems, in particular, scheduling; the scheduling of production facilities and of the sale of goods. Links have been established between combinatorics and problems of linear programming, statistics, etc. Combinatorial methods are used in coding and decoding and in the solution of other problems of information theory.

The combinatorial approach also plays a significant role in purely mathematical problems such as the theory of groups and their representations, in the study of the foundations of geometry, nonassociative algebras, and elsewhere.

In the present book, the aim has been to set forth a variety of combinatorial problems in popular form and understandable language. At the same time, an attempt is made to present some rather involved combinatorial problems and to give the reader an idea of the methods of recurrence relations and generating functions.

The first chapter is devoted to the general rules of combinatorics, the rules of sum and product. In the second chapter we investigate permutations and combinations. This traditionally grade-school material is accompanied by an analysis of some amusing examples. In the third chapter, a study is made of combinatorial problems in which certain restrictions are imposed on the combinations. Chapter IV considers problems involving partitions of numbers into integers and contains a description of certain geometrical methods in combinatorics. Chapter V is devoted to random-walk problems and to a variety of modifications of the arithmetic triangle. Chapter VI takes up recurrence relations, and Chapter VII discusses generating functions and, in particular, the binomial formula.

The last section of the book is devoted to combinatorial problems of which there are over 400. This material has been taken from a variety of sources, including Whitworth's *Choice and Chance* (London, 1901), John Riordan's *An Introduction to Combinatorial Analysis* (New York, 1958), an interesting book by A. M. Yaglom and I. M. Yaglom entitled *Nonelementary Problems in an Elementary Exposition* (Moscow, 1954), and various collections of problems given at mathematical olympiads in the USSR.

SUPERSTITIOUS CYCLISTS

"Another eight!" exclaimed the chairman of the cyclists' club with exasperation as he examined the twisted wheel of his bicycle. "And why? All because I was given this blasted No. 008 ticket when I became a member. Now hardly a month goes by without one of the wheels smashing into a figure eight. What I need is a ticket with a different number. To avoid accusations of superstition I'll reregister all the members, leaving out the digit eight altogether."

The next day he changed all the tickets. Now *how many members were there in the club if it is known that all three-digit numbers were used that did not contain 8*? (Say, 000 was used but not 836.)

To start with, let us determine the number of one-digit numbers without eight. Clearly, there are nine such numbers: 0, 1, 2, 3, 4, 5, 6, 7, 9. Now indicate all the two-digit numbers lacking the numeral 8. They can be found by taking all the one-digit numbers that we found and writing any one of the acceptable nine digits after it. This yields nine two-digit numbers for each one-digit number, and since there are 9 one-digit numbers, we get $9 \times 9 = 81$ two-digit numbers without any 8's:

00,	01,	02,	03,	04,	05,	06,	07,	09
10,	11,	12,	13,	14,	15,	16,	17,	19
20,	21,	22,	23,	24,	25,	26,	27,	29
30,	31,	32,	33,	34,	35,	36,	37,	39
40,	41,	42,	43,	44,	45,	46,	47,	49
50,	51,	52,	53,	54,	55,	56,	57,	59
60,	61,	62,	63,	64,	65,	66,	67,	69
70,	71,	72,	73,	74,	75,	76,	77,	79
90,	91,	92,	93,	94,	95,	96,	97,	99

Thus, there are $9^2 = 81$ two-digit numbers without 8. Each one of these numbers can be used to adjoin any one of the nine acceptable digits to. This yields $9^2 \times 9 = 9^3 = 729$ three-digit numbers. Therefore, the club had 729 registered cyclists. If we were to take four-digit

numbers, there would be $9^4 = 6{,}561$ numbers without any 8's.

In another club, the members were still more superstitious and they even threw out the number 0 as being too much like a wheel. They made do with eight digits: 1, 2, 3, 4, 5, 6, 7, 9. *How many members were there in the club if all ticket numbers were three-digit*?

The problem is similar to the first one, the sole difference being that in place of 9 digits we have 8 altogether. So the club had $8^3 = 512$ members.

PERMUTATIONS WITH REPETITIONS

The problem of the cyclists belongs in the following class of problems. We have objects referring to n different varieties. They are used to make up arrangements of k objects in each or, as we shall say in the future, *k-arrangements*. The arrangements can contain objects of the same type, and two arrangements will be considered distinct if they differ either as to objects or as to the order of the objects. The problem is to *find the total number of such arrangements.*

Arrangements of this type are called *k-permutations of n distinct things with repetitions.* The number of all such arrangements is denoted by $\overline{A}_n^k$.* In the first problem of the cyclists, the number of things (types of elements) was 9 (we took all digits except 8), and each permutation (each number) contained three elements. In this case, the number of permutations came to $\overline{A}_9^3 = 9^3$. It is natural to assume that if the number of things is n, and each permutation contains k elements, then the number of permutations with repetitions is n^k.

We wish to prove that the number of k-permutations of n distinct objects with repetitions is equal to

$$\overline{A}_n^k = n^k \qquad (1)$$

We carry out the proof by means of mathematical induction with respect to k, which is the number of elements in a permutation with a fixed value of n. For $k = 1$ the answer is obvious—each permutation (with repetitions) consists of one element only, and the different permutations are obtained by taking distinct objects. But since the number of types is n, the number of permutations is n. Thus, $\overline{A}_n^1 = n$, in accord with formula (1).

Now assume that we have the proof for $\overline{A}_n^{k-1} = n^{k-1}$ and we consider k-permutations with repetitions. We obtain all such permutations as follows. Let us take any $(k-1)$-permutation (with repetitions) $(a_1, \ldots, a_{k-1})$ and adjoin to it the element a_k of one of the available n objects. We get some k-permutation $(a_1, \ldots, a_{k-1}, a_k)$. It is now clear that out of each $(k-1)$-permutation we get as many k-permutations as there are distinct objects, that is, n permutations. It is obvious that in this manner we will not miss a single k-permutation and will not take any one twice [if $(a_1, \ldots, a_{k-1}) \neq (b_1, \ldots, b_{k-1})$ or if $a_k \neq b_k$, then $(a_1, \ldots, a_{k-1}, a_k) \neq (b_1, \ldots, b_{k-1}, b_k)$]. Therefore the number of k-permutations of n distinct objects with repetitions is n times greater than the number of $(k-1)$-permutations of the same objects with repetitions. Consequently, $\overline{A}_n^k = n\overline{A}_n^{k-1}$. But we already have the proof of $\overline{A}_n^{k-1} = n^{k-1}$. Therefore,

$$\overline{A}_n^k = n \cdot n^{k-1} = n^k$$

Thus, equation (1) has been proved for all values of k.

Formula (1) occurs in a wide range of problems, a few of which will occupy us in the forthcoming sections.

NUMBER SYSTEMS

Besides the decimal number system which we commonly use, there are other systems of numeration such as the binary, ternary, octal, etc. number systems. In the base-n number system we use n digits. Let us calculate how many natural numbers can be written in a base-n system of numeration by using exactly k digits (for the sake of convenience, we will regard 0 as a natural number). If we allow for numbers beginning with zero, then every k-digit number in a base-n number system may be regarded as a permutation, with repetitions, consisting of k digits of n distinct types. Using formula (1) we find that the number of numbers thus represented is n^k.

However, the natural numbers do not have representations beginning with zero. We will thus have to subtract from the value of n^k the number of numbers whose base-n representation begins with zero. Discarding the first digit (zero) in these numbers, we get a $(k-1)$-digit number (which possibly also begins with zero). By formula (1), there will be n^{k-1} such numbers. Hence, the total number of k-digit numbers in

* A permutation of n things taken r at a time can also be symbolized as ${}_nP_r$ or $P(n, r)$ [*Translator*].

a base-n system of numeration is equal to

$$n^k - n^{k-1} = n^{k-1}(n-1)$$

For example, in the decimal system there are $10^3 \times 9 = 9{,}000$ four-digit numbers: out of 10,000 numbers between 0 and 9,999, we discard one thousand, namely, those from 0 to 999.

There is another way to derive our formula. In a k-digit number represented in a base-n number system, the first digit can be any one of the digits $1, 2, \ldots, n-1$; the second digit and succeeding digits, any one of the digits $0, 1, 2, \ldots, n-1$. We thus have $n-1$ candidates for position one, and n candidates for each of the remaining $k-1$ positions. From this it is easy to see that the desired total number is $(n-1)\, n^{k-1}$.

SECRET LOCK

Safes and storage lockers make use of secret locks, called combination locks, that open up only when a specific combination of numbers or letters is dialled. *Suppose a dial has 12 letters and the combination has 5 letters. How many failing attempts can be made by a person who does not know the combination?*

Using formula (1), we find the total number of combinations to be

$$12^5 = 248{,}832$$

which means that 248,831 attempts can be failures. Incidentally, safes are frequently made to give a warning signal at the first failure to dial the right combination.

MORSE CODE

The Morse code is used in telegraph communications. In this code, the letters, numbers and punctuation marks are represented by dots and dashes. Some characters require only a single sign, like E (•) whereas others use all five signs, like zero, 0 (— — — — —).

Why the number 5? Couldn't we make do with a smaller number of dots and dashes, say four, to transmit all our communications? The answer is no, and the reason becomes clear if we apply the formula for the number of permutations with repetitions. From formula (1) it follows that $\bar{A}_2^1 = 2$. In other words, using one character, it is possible to transmit only two letters (E• and T—). Using two characters, it is possible to transmit $2^2 = 4$ letters, three yield $2^3 = 8$ letters and four, $2^4 = 16$. Thus, the total number of letters that can be transmitted by means of four characters is

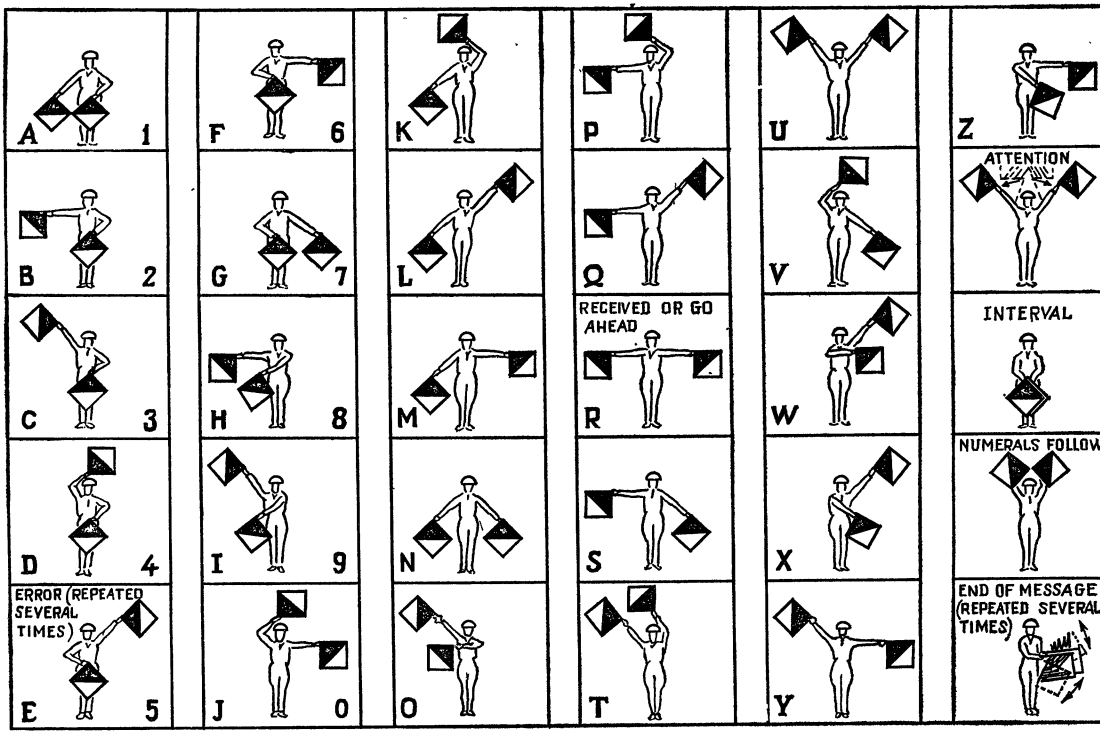
$$2+4+8+16=30$$

Taking an alphabet of 26 letters, and also numerals and marks of punctuation we see that symbols made up of four characters do not suffice. But if we take combinations of 5, then we get 32 symbols in addition to the 30 obtained from four. These 62 symbols are quite sufficient for telegraph communications.

There is also a five-digit code for telegraph communications that makes use of five symbols for every letter. Instead of the dash-and-dot system, use is made of alternating current flow or of sending current signals and noncurrent signals. This code yields exactly $2^5 = 32$ combinations, which is enough for all the letters. Numerals and punctuation utilize the same combinations, and so a telegraph apparatus operating on the five-digit code has a special device for shifting from letters to numerals and back again.

WIGWAG CODE

The navy has a visual signalling system by flags, called a semaphore. Each letter is represented by two flags in a specific arrangement.



For the most part, an individual letter is displayed by a pattern of one flag on each side of the signal-man. But some of the letters (say H, O, X) require both flags on one side. The reason is obvious if we examine the formula for permutations with repetitions. The point is that each flag has a total of five positions: straight down, inclined down, horizontal, inclined up and straight up. Since we have two flags, the total number of combinations of the basic positions comes to $\overline{A}_5^2 = 5^2 = 25$. One position has to be omitted (both flags down) since it serves to separate words. This yields 24 combinations, which is not sufficient to transmit all letters of the alphabet. That is why some letters have patterns with both flags on one side.

ELECTRONIC DIGITAL COMPUTER

Electronic computers are capable of handling an enormous range of problems. A single machine can decipher inscriptions in an unknown language, perform the design computations for a dam, and process the trajectory data of a space vehicle. How does one account for such versatility? Mainly the reason is that all these problems reduce to computations involving numbers. How does such a machine handle so many problems involving so many numerical data? How many combinations of numbers is a computer capable of processing?

Let us illustrate this case with a relatively small computer called "Strela" (Arrow). This machine has an immediate-access memory of 2,048 storage cells, each one of which can accommodate 43 binary digits. Each digit is either a 0 or a 1. Altogether we have $43 \times 2{,}048 > > 87{,}000$ distinct positions, and two ways (0 or 1) the cells can be filled. Using formula (1), we find that the "Strela" can be in any one of more than $2^{87{,}000}$ distinct states. This enormous number is far beyond the limits of our imagination. Suffice it to say that the number of neutrons packed side by side in a sphere the radius of which is equal to the distance to the most distant stellar systems does not exceed 2^{500}.

If we used just one memory cell, it would take nine years for a typist pool of 100,000 workers to type out all the numbers that this single cell can accommodate (we assume they work a seven-hour day and do one 43-digit number in 10 seconds).

GENETIC CODE

Breaking the genetic code has been one of the most remarkable achievements of twentieth century biology. Biologists now know how genetic (hereditary) information is passed on to the next generation. This information is recorded in the form of gigantic molecules of desoxyribonucleic acid (DNA). The various molecules of DNA differ in the order in which four nitrogenous bases are arranged: adenine, thymine, guanine, and cytosine. These bases determine the order in which the proteins of the organism are built up out of roughly twenty amino acids, each amino acid being in the form of a code made up of three nitrogenous bases.

It is easy to see where the number 3 came from. Using combinations of two bases, we could code only $4^2 = 16$ amino acids, which is not enough. Using three bases, we get $4^3 = 64$ combinations, which is far and away sufficient to encode some twenty amino acids. The intriguing question is how nature takes advantage of so much redundant information—the number of combinations is 64 while the number of amino acids is only one third that.

A single chromosome contains several tens of millions of nitrogenous bases. The number of distinct combinations which they can form is simply horrendous—it is equal to 4^N, where N is the number of bases in the chromosome; go back to formula (1). A minute portion of these combinations has been sufficient to ensure the extraordinary diversity of all living nature over the entire span of life here on the planet Earth. Note of course that only a very small fraction of the theoretically possible combinations lead to viable organisms.

GENERAL RULES OF COMBINATORICS

As we shall soon see, combinatorial problems offer a multiplicity of types. But most of them can be solved with the aid of two basic rules: the rule of sum and the rule of product.

It is often possible to partition the combinations under study into several classes, one combination appearing in one and only one class. It is then clear that *the total number of combinations is equal to the sum of the numbers of combinations in all classes*. This assertion goes by the name of *the rule of sum*. It is sometimes formulated thus:

If a certain object A can be chosen in m ways and another object B can be chosen in n ways, then the choice of "either A or B" can be accomplished in $m + n$ ways.

When employing the rule of sum in this latter formulation, take care to see that no choice of A should coincide with any choice of B (or, as we put it at the beginning, that no combination should appear in two classes at once). If there are such coincidences, then the rule of sum breaks down, and all we have is $m + n - k$ choices, where k is the number of coincidences.

The second rule, called *the rule of product*, is somewhat more complicated. When making up combinations of two objects, it is often known in how many ways the first object can be chosen and in how many ways the second one, the number of ways of choosing the second object being independent of how the first object was chosen. Suppose the first object can be chosen in m ways, the second in n ways. Then the pair can be chosen in mn ways. To put it differently:

If an object A can be chosen in m ways and if, after every such choice, an object B can be chosen in n ways, then the choice of the pair (A, B) in that order can be accomplished in mn ways.

To prove the rule of product, note that each one of the m ways of choosing A can be combined with n ways of choosing B, which brings us to mn ways of choosing the pair (A, B).

The rule of product can be pictorialized as follows:

Table 1

$$\begin{array}{ccc}
(A_1, B_{11}), & \ldots, & (A_1, B_{1n}) \\
(A_2, B_{21}), & \ldots, & (A_2, B_{2n}) \\
\ldots & \ldots & \ldots \\
(A_i, B_{i1}), & \ldots, & (A_i, B_{in}) \\
\ldots & \ldots & \ldots \\
(A_m, B_{m1}), & \ldots, & (A_m, B_{mn})
\end{array}$$

$A_1, \ldots, A_m$ indicate m ways of choosing A and $B_{i1}, \ldots, B_{in}$ denote n ways of choosing B, provided A has been chosen the ith way. It is clear that this table contains all ways of choosing the pair (A, B) and consists of mn elements.

If the choices of B are independent of how object A is chosen, then in place of Table 1 we get the following simpler table.

Table 2

$$\begin{array}{cccc}
(A_1, B_1), & (A_1, B_2), & \ldots, & (A_1, B_n) \\
(A_2, B_1), & (A_2, B_2), & \ldots, & (A_2, B_n) \\
\ldots & \ldots & \ldots & \ldots \\
(A_m, B_1), & (A_m, B_2), & \ldots, & (A_m, B_n)
\end{array}$$

It may, of course, happen that we need combinations of more than two elements (objects). Then we arrive at the following problem.

How many k-arrangements can be made if the first element can be one of n_1 distinct objects, the second, one of n_2 distinct objects, and the kth, one of n_k distinct objects. Here, two arrangements are considered distinct if at least one position is occupied by different elements.

This problem is solved in the same way as the problem of the cyclists. The first element may be chosen in n_1 ways. Each of the chosen elements can be combined with any one of the n_2 types of second elements, yielding $n_1 n_2$ pairs. Each pair can be combined with any one of the n_3 types of third elements, thus yielding $n_1 n_2 n_3$ triples. Continuing in this fashion, we finally get the k-tuple $n_1 n_2 \ldots n_k$ of arrangements of the desired type.

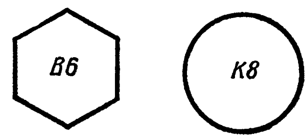


Fig. 1.

In the problem of the cyclists, we had to choose three elements (the hundreds digit, the tens digit and the units digit). At each stage we could choose one of nine permissible digits, and so we got $9 \times 9 \times 9 = 729$ numbers. Here is a harder problem.

Signs are being made which consist of a geometrical figure (a circle, square, triangle or hexagon), a letter and a number. How many signs can be made?

As a first step we can choose a geometric figure. This can be done in four ways (we have a total of four figures). Then we can choose one of 32 letters (Russian alphabet), and finally, one of 10 numerals. This brings the total to $4 \times 32 \times \times 10 = 1{,}280$ combinations.

DOMINO PROBLEM

A more complicated combinatorial problem is that in which the number of choices after each stage depends on the elements chosen in the preceding stages, an instance of which is the following.

In how many ways can two dominoes be chosen out of 28 pieces so that they can be put together (that is, so that some number occurs on both pieces)?

First choose one piece. This can be done in 28 ways. In 7 cases, the chosen piece will be a "double", i.e. one of the type 00, 11, 22, 33, 44, 55, 66, and in 21 cases the piece will have different numbers (say, 05, 13, etc.). In the first instance, the second piece may be chosen in 6 ways (for example, if at the first stage the piece 11 was chosen, then in the second stage, we can take one of the pieces with 01, 12, 13, 14, 15, 16). Now in the second instance, the second piece may be chosen in 12 ways (for piece 35 we can take pieces 03, 13, 23, 33, 34, 36, 05, 15, 25, 45, 55, 56). By the rule of product, we get $7 \times 6 = 42$ choices in the first case, and $21 \times 12 = 252$ choices in the second case. Hence, by the rule of sum, we get $42 + 252 = 294$ ways of choosing a pair.

In the above discussion we took into account the order in which the pieces were chosen, so each pair of pieces appeared twice (for instance, 01 and 16 the first time and 16 and 01 the second time). If the order of choice of the pieces is disregarded, then the number of choices is cut by one half: we have 147 choices.

THE CREW OF A SPACESHIP

For cases when at each stage the number of possible choices depends on which elements were chosen earlier, it is convenient to depict the process of building combinations in the form of a tree. Starting from one point, draw line segments illustrating the various choices that can be made in the first stage (here, each segment corresponds to a single element). The choices in the second stage are made from the endpoints of each of the line segments, if the given element was chosen in the first stage, etc.

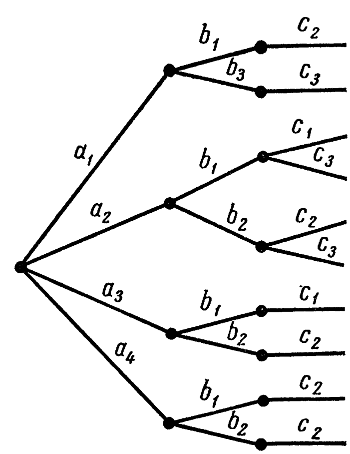


Fig. 2.

This construction yields a tree, an examination of which immediately gives the number of solutions to our problem.

By way of illustration let us take the problem of staffing a multi-seated spaceship where the problem comes up of psychological compatibility of the crew in an extended space mission. People that are suitable in every respect when taken separately may not fit into the pattern of a crew for prolonged space exploration. Suppose our problem is to make up a crew of three: commander, engineer, and physician. We have four candidates, a_1, a_2, a_3 and a_4 for the commander, three candidates for the engineer (b_1, b_2 and b_3) and three for the doctor (c_1, c_2 and c_3). A preliminary checkup has disclosed that commander a_1 is psychologically compatible with engineers b_1 and b_3 and with doctors c_2 and c_3, commander a_2 with engineers b_1 and b_2 and all doctors, commander a_3 with engineers b_1 and b_2 and doctors c_1, c_3, commander a_4 with all engineers and with doctor c_2. Also, engineer b_1 does not get along with doctor c_3, engineer b_2 with doctor c_1 and engineer b_3 with doctor c_2. *Given these conditions, in how many ways can a crew be made up?*

The conditions are illustrated in the tree in Fig. 2. We see that there are only 10 permissible combinations (if there were no compatibility restrictions, then the number of combinations would, by the rule of product, be $36 = 4 \times 3 \times 3$).

CHECKERBOARD PROBLEMS

Solve the following problem.

In how many ways can two checkers (or draughts) (white and black) be placed on a checkerboard so that the white can take the black?

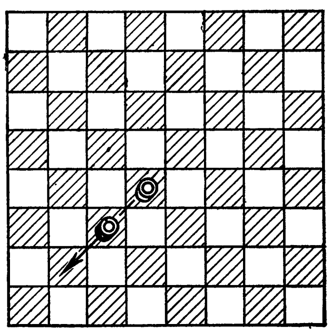
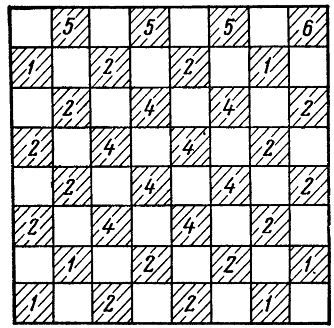


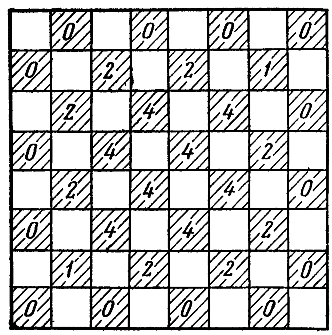
Fig. 3.

By the rules of the game, the pieces are placed on black squares and one checker takes another by jumping over it and occupying the next square (Fig. 3). If the piece reaches the last row, it becomes a king and can take all men on the same diagonal except those on the end squares.

This problem is complicated by the fact that for different positions of white there are different numbers of positions of black in which it can be taken. For instance, if white is on square $a1$, then there is only one position where black is under attack. Now if white is on square $c3$, then the number of desired positions for black



(a)

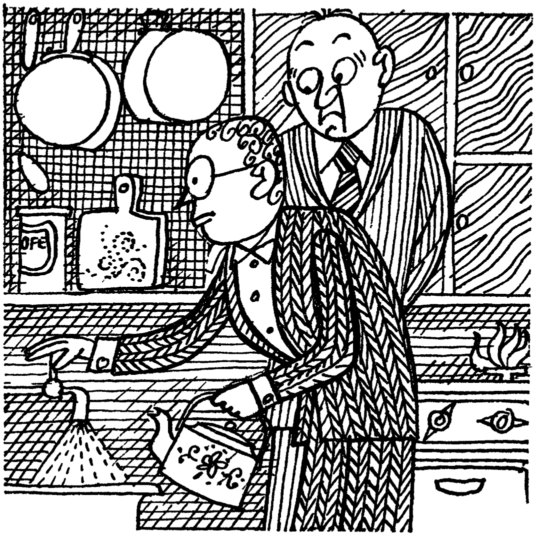


(b)

Fig. 4.

is 4. Finally, if white has become a king on square $h8$, then there are 6 positions in which black is under attack by the king.

It is therefore simpler to indicate for each position of white the number of possible positions of black and add the results obtained. Fig. 4*a* shows a checkerboard with the appropriate figures



indicated. Combining them, we get 87. Thus, the sought-for arrangement is possible in 87 ways.

It is clear that there are exactly the same number of positions in which black can take white, but there are fewer positions in which both men can take one another. For instance, if white stands on the edge of the board, black cannot take it no matter where black is positioned. Therefore, assign 0 to all squares along the edges of the board. In the same manner, we find numbers that correspond to the other black squares. They are shown in Fig. 4*b*. Adding up these numbers, we find that the desired arrangement is possible in 50 different ways.

Finally, let us find the number of positions of white and black in which neither can take the other. The problem can be solved in the same way by placing white on each of the black squares and counting the possible ways of positioning black so that not one of the pieces could take the other. However, it is simpler to make use of the "teakettle principle" * and reduce the problem to one that has already been solved. To do this, let us find the total number of positions for placing white and black on the board. White can be placed on any one of 32 black squares. That leaves 31 squares for black. Thus, by virtue of the rule of product, the arrangement is possible in $32 \times 31 = 992$ ways. But these include 87 in which white can take black, and 87 in which black can take white. We have to reject $2 \times 87 = 174$ ways. However, we must take into consideration that in doing so we have rejected some of the ways twice: because white can take black and because black can take white. We have seen that there are 50 positions in which both men can take each other. Consequently, the number of positions in which neither man can take the other is

$$92 - 174 + 50 = 868$$

HOW MANY PEOPLE DON'T KNOW FOREIGN LANGUAGES?

The method applied in solving the foregoing problem can often be used in dealing with combinatorial problems. Here is one.

A research institution has a staff of 67: 47 know Spanish, 35 German and 23 both languages. How many employees at this institution know neither Spanish nor German?

To solve this problem, we partition the entire staff into parts having no elements in common. The first part consists of those who know Spanish only, the second, those who know German only, the third, those who know both languages, and the fourth part, those who know neither language

* The story goes that a mathematician once asked a physicist: "You have an empty teakettle and an unlit gas range. How do you go about boiling water?" "Very simply," replied the physicist. "Fill the teakettle with water, light the gas and put the water on to boil."

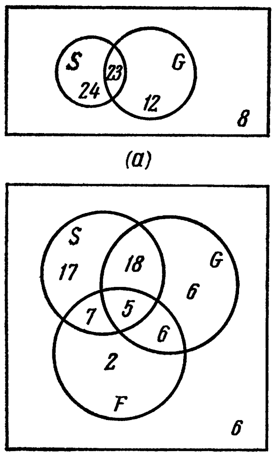
"Right," said the mathematician. "Now solve this problem: the gas is burning and the teakettle has water in it. How do you boil the water?"

"That's no problem at all," replied the physicist. "Just put the teakettle on the range."

"No," said the mathematician firmly. "You turn off the gas, pour out the water and we arrive at our first problem, which we know how to solve."

Now, when a new problem is reduced to an already solved one, we speak of applying the "teakettle principle".

(Fig. 5). What we know is that Part Three consists of 23 persons, but since 47 know Spanish, there are $47 - 23 = 24$ persons who know only Spanish. In the same way, we find that $35 - 23 = 12$ scientists know only German. From this



(a)

(b)

Fig. 5.

we conclude that the total number of persons knowing one of these two languages is equal to $23 + 24 + 12 = 59$. And since there are 67 in all, only $67 - 59 = 8$ workers remain who know neither language. In this institution, 8 know neither Spanish nor German.

We can write the answer in the form

$$8 = 67 - (23 + 24 + 12)$$

But we get 24 by subtracting 23 from 47, and 12 by subtracting 23 from 35. Therefore,

$$8 = 67 - 23 - (47 - 23) - (35 - 23) = 67 - 47 - 35 + 23$$

Here is the regularity: we subtract from the total number of scientific workers those who know Spanish and those who know German. In this way, some find themselves in both lists and are "subtracted" twice; these are precisely the polyglots with a knowledge of both languages. Adding on these, we obtain the number of persons who never studied either language.

Let us complicate the problem by adding another language—French. Suppose there are 20 scientific workers with French as their foreign language, then 12 with Spanish and French, 11 with German and French, and 5 polyglots that handle all three languages. It is then clear that those who know Spanish and French (no German) number $12 - 5 = 7$, whereas $11 - 5 = 6$ know only German and French. This leaves 2 who read French $(20 - 7 - 6 - 5 = 2)$. These workers are among the 8 persons who have no command of Spanish or German. Hence, the number of researchers here who cannot work in any one of the three languages is $8 - 2 = 6$.

We can write down the answer as follows:

$$6 = 8 - 2 = 67 - 47 - 35 + 23 - (20 - 7 - 6 - 5) = 67 - 47 - 35 + 23 - 20 + (12 - 5) + (11 - 5) + 5 = 67 - 47 - 35 - 20 + 23 + 12 + 11 - 5$$

Now the law is perfectly clear. From the total number of workers we first subtract those who have a knowledge of one foreign language (and, possibly, other languages). In this process, some of them are "subtracted" twice since they know two languages. Therefore, we add the numbers 23, 12, and 11 which indicate how many employees read two languages (and, possibly, a third language). But those with three languages are rejected three times and then added three times. And since they have to be subtracted, we reject 5.

THE PRINCIPLE OF INCLUSION AND EXCLUSION

The foregoing examples enable us to state a general law. Let there be N objects some of which have properties $\alpha_1, \alpha_2, \ldots, \alpha_n$. Each object can either have none of these properties, or one or several. Let us denote by $N(\alpha_i, \alpha_j, \ldots, \alpha_k)$ the number of objects having properties $\alpha_i, \alpha_j, \ldots, \alpha_k$ (and, possibly, certain other properties). If we wish to emphasize that only objects devoid of a certain property are taken, this property will be indicated by a dash. Say, $N(\alpha_1\alpha_2\alpha_4')$ indicates the number of objects having properties α_1 and α_2, but devoid of property α_4 (the question of the other properties remains open).

The number of objects not possessing a single one of the indicated properties is denoted, by this rule, by $N(\alpha_1'\alpha_2' \ldots \alpha_n')$. The general law consists in the fact that

$$N(\alpha_1'\alpha_2' \ldots \alpha_n') = N - N(\alpha_1) - N(\alpha_2) - \ldots - N(\alpha_n) + N(\alpha_1\alpha_1) + N(\alpha_1\alpha_3) + \ldots + N(\alpha_1\alpha_n) + \ldots + N(\alpha_{n-1}\alpha_n) - N(\alpha_1\alpha_2\alpha_3) - \ldots - N(\alpha_{n-2}\alpha_{n-1}\alpha_n) + \ldots + (-1)^n N(\alpha_1\alpha_2 \ldots \alpha_n) \qquad (2)$$

Here, the algebraic sum is extended to all combinations of properties $\alpha_1, \alpha_2, \ldots, \alpha_n$ (without regard for their order), the + sign being used if the number of properties accounted for is even and the — sign if that number is odd. For instance, $N(\alpha_1\alpha_3\alpha_6\alpha_8)$ enters with a + sign while $N(\alpha_3\alpha_4\alpha_{10})$ enters with a — sign. Formula (2) defines what is known as *the principle of inclusion and exclusion* (*the inclusion and exclusion formula*): first, all objects with at least one of the properties $\alpha_1, \alpha_2, \ldots, \alpha_n$ are excluded, then all objects having at least two of these properties are included, then those having at least three are excluded, etc.

Let us prove formula (2). The proof is by induction on the number of properties. For one property the formula is obvious. Each object either has this property or does not. Therefore,

$$N(\alpha') = N - N(\alpha)$$

Now suppose that formula (2) has been proved for the case when the number of properties is equal to $n-1$:

$$N(\alpha_1'\alpha_2' \ldots \alpha_{n-1}') = N - N(\alpha_1) - \ldots - N(\alpha_{n-1}) + N(\alpha_1\alpha_2) + \ldots + N(\alpha_{n-2}\alpha_{n-1}) - N(\alpha_1\alpha_2\alpha_3) - \ldots - N(\alpha_{n-3}\alpha_{n-2}\alpha_{n-1}) + \ldots + (-1)^{n-1} N(\alpha_1\alpha_2 \ldots \alpha_{n-1}) \qquad (3)$$

By hypothesis, this formula holds true for any collection. In particular, it is true for the collection $N(\alpha_n)$ of elements having the property α_n. For this collection, formula (3) takes the form

$$N(\alpha_1'\alpha_2' \ldots \alpha_{n-1}'\alpha_n) = N(\alpha_n) - N(\alpha_1\alpha_n) - \ldots - N(\alpha_{n-1}\alpha_n) + N(\alpha_1\alpha_2\alpha_n) + \ldots + N(\alpha_{n-2}\alpha_{n-1}\alpha_n) - N(\alpha_1\alpha_2\alpha_3\alpha_n) - \ldots + (-1)^{n-1} N(\alpha_1\alpha_2 \ldots \alpha_{n-1}\alpha_n) \qquad (4)$$

(the restriction is indicated that in each case we take only those objects having the property α_n).

Let us subtract (4) from (3). On the right we get what we desire: the right-hand side of formula (2). The left-hand member is the difference

$$N(\alpha_1'\alpha_2' \ldots \alpha_{n-1}') - N(\alpha_1'\alpha_2' \ldots \alpha_{n-1}'\alpha_n) \qquad (5)$$

But $N(\alpha_1'\alpha_2' \ldots \alpha_{n-1}')$ is the number of objects not possessing the properties $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ and, possibly, possessing the property α_n. Now, $N(\alpha_1'\alpha_2' \ldots \alpha_{n-1}'\alpha_n)$ denotes the number of objects that do not possess the properties $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ but definitely possess the property α_n. This means that the difference (5) is just equal to the number of objects that do not possess a single one of the properties ($\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, \alpha_n$). In other words,

$$N(\alpha_1'\alpha_2' \ldots \alpha_{n-1}') - N(\alpha_1'\alpha_2' \ldots \alpha_{n-1}'\alpha_n) = N(\alpha_1'\alpha_2' \ldots \alpha_{n-1}'\alpha_n')$$

Thus, after subtraction, the left-hand member of formula (2) also appears on the left. The formula is thus proved for the case when the number of properties is equal to n.

Thus, the relation (2) holds true for n properties if it holds for $n-1$ properties (for $n = 1$

it has already been proved). It follows that the relation holds for any set of properties.

Formula (2) may be represented symbolically as follows:

$$N(\alpha'\beta'\ldots\omega')=N(1-\alpha)(1-\beta)\ldots(1-\omega) \qquad (6)$$

After removing the brackets, the product $N\alpha\beta\ldots\lambda$ has to be written as $N(\alpha\beta\ldots\lambda)$. For example, in place of $N\,\alpha\beta\delta\omega$ we write $N(\alpha\beta\delta\omega)$.

WHERE'S THE MISTAKE?

The class president reports as follows: "Our class has 45 students, of which 25 are boys. Thirty students get marks of "good" and "excellent" (of this number, 16 are boys). A total of 28 students go in for athletics, of which number 18 are boys and 17 are students with marks of "good" and "excellent". Fifteen boys get good and excellent marks and also go in for athletics."

A couple of days later, the home-room teacher, who, ironically, was the mathematics instructor, summoned the class president and stated that his report contained an error. Let us try to find out what the mistake was. To do this, we compute the number of girls who do not engage in athletics and get passing marks (3's) or even failing marks (2's). Denote membership in the set of boys by α_1, students getting good marks (4' s and 5's) by α_2 and those who go in for athletics by α_3. We find $N(\alpha_1'\alpha_2'\alpha_3')$. By hypothesis, we have

$$N(\alpha_1)=25,\ N(\alpha_2)=30,\ N(\alpha_3)=28,$$
$$N(\alpha_1\alpha_2)=16,$$
$$N(\alpha_1\alpha_3)=18,\ N(\alpha_2\alpha_3)=17,\ N(\alpha_1\alpha_2\alpha_3)=15$$

By the principle of inclusion and exclusion, we find that

$$N(\alpha_1'\alpha_2'\alpha_3')=45-25-30-28+16+18+17-15=-2$$

But the answer can only be positive! So the report definitely contains an inconsistency and is incorrect.

THE SIEVE OF ERATOSTHENES

One of the greatest mysteries of mathematics is the distribution of the prime numbers among all the natural numbers. There are cases where two primes occur every other number (say, 17 and 19, 29 and 31), and then there are cases where a million composite numbers occur in an unbroken sequence. Today, mathematicians know a good deal about how many primes occur among the first N natural numbers. In these computations, there is a very useful method that goes back to the ancient Greek scholar Eratosthenes (third century B.C.) who lived in Alexandria.

Eratosthenes was a versatile man engaged in a wide range of problems. He excelled in mathematics, astronomy and many other fields. True, such breadth led to a certain superficiality. Contemporaries referred somewhat ironically to Eratosthenes as "Beta"—always second best (the second mathematician after Euclid, the second astronomer after Hipparchus, etc.).

In mathematics, Eratosthenes wanted to know how to find all the prime numbers from among the natural numbers up to N (Eratosthenes considered 1 to be prime; today, mathematicians consider 1 to be a special type not belonging either to the primes or to composite numbers). Here is how he went about it. First of all, he crossed out all numbers divisible by 2 (excluding the number 2 itself). Then he took the first of the remaining numbers (namely, 3). It is clear that this number is prime. Then he crossed out all numbers divisible by 3 that follow it. The first remaining number is 5. Then he deleted all successive numbers divisible by 5, and so on. The numbers that remain after all deletions are primes. Since writing was done on wax tablets in those days and the crossing out was done by punching the figures, a tablet took on the appearance of a sieve. Whence the method of Eratosthenes for finding prime numbers came to be known as the "sieve of Eratosthenes".

Let us compute the number of numbers that remain in the first hundred after deleting by the Eratosthenes method, dividing by 2, 3, and 5. In other words: how many numbers are there among the first hundred that are not divisible by 2, 3 or 5? This problem can be solved by the principle of inclusion and exclusion.

Denote by α_1 the property of a number to be divisible by 2, by α_2 the property of divisibility by 3, and by α_3 the property of divisibility by 5. Then $\alpha_1\alpha_2$ signifies that a number is divisible by 6, $\alpha_1\alpha_3$ means that it is divisible by 10, and $\alpha_2\alpha_3$, that it is divisible by 15. Finally, $\alpha_1\alpha_2\alpha_3$ signifies that a number is divisible by 30. Our task is to find the numbers from 1 to 100 that are not divisible either by 2, 3 or 5, that is, such as do not possess a single one of the properties α_1, α_2, α_3. By formula (2) we have

$$N(\alpha_1'\alpha_2'\alpha_3') = 100 - N(\alpha_1) - N(\alpha_2) - N(\alpha_3) + $$
$$+ N(\alpha_1\alpha_2) + N(\alpha_1\alpha_3) + N(\alpha_2\alpha_3) - N(\alpha_1\alpha_2\alpha_3)$$

However, in order to find the numbers up to N that are divisible by n, we have to divide N by n and take the integral part of the resulting quotient. Therefore,

$$N(\alpha_1) = 50,\ N(\alpha_2) = 33,\ N(\alpha_3) = 20,$$
$$N(\alpha_1\alpha_2) = 16,$$
$$N(\alpha_1\alpha_3) = 10,\ N(\alpha_2\alpha_3) = 6,\ N(\alpha_1\alpha_2\alpha_3) = 3$$

and, hence,

$$N(\alpha_1'\alpha_2'\alpha_3') = 26$$

Thus, 26 numbers between 1 and 100 are not divisible by 2, 3, or 5. These numbers remain after the first three stages in the Eratosthenes process. In addition, we will also have the numbers 2, 3, and 5 themselves. There will thus be 32 numbers left altogether.

After three stages in the elimination process of the sieve of Eratosthenes there will remain 269 numbers out of the first thousand. This follows from the fact that in this case

$$N(\alpha_1) = 500,\ N(\alpha_2) = 333,\ N(\alpha_3) = 200,$$
$$N(\alpha_1\alpha_2) = 166,\ N(\alpha_1\alpha_3) = 100,\ N(\alpha_2\alpha_3) = 66,$$
$$N = (\alpha_1\alpha_2\alpha_3) = 33$$

PERMUTATIONS AND COMBINATIONS

We have considered some general rules for solving combinatorial problems of a broad variety of types. However, as in geometry, it is not always convenient to reduce the solution of problems to axioms; it is preferable to take advantage of theorems. It is often better to use ready-made formulas instead of solving by general rules. The point is that certain types of problems occur much more often than others. The combination types that occur in such problems are called permutations and combinations.

Special formulas have been derived for such combinatorial problems. One such formula is already familiar: at the beginning of Chapter I it was demonstrated that the number of k-permutations of n distinct objects with repetitions is equal to n^k. Let us now find out how many such permutations can be formed if we do not allow for repetitions, that is to say, if all the elements in the permutations are distinct. First let us examine the following problem.

FOOTBALL CHAMPIONSHIP

In the USSR football championship, Group One of Class A consists of 17 teams that aspire to gold, silver and bronze medals. In how many ways can the medals be distributed?

This problem is solved by means of the rule of product. Any one of the 17 teams can get the gold medal. We thus have 17 possibilities. But if some one team gets the gold medal, then there remain 16 aspirants for the silver medal, since the same team cannot get both medals.

After first place with the gold medal has been settled, there remain 16 possibilities for the silver medal. In the same way, the bronze medal can go to any one of the remaining 15 teams after the gold and silver medals have been awarded. By the rule of product we find that the medals can be distributed in 4,080 ways ($17 \times 16 \times 15 = 4{,}080$).

PERMUTATIONS WITHOUT REPETITIONS

The above problem belongs to the class of combinatorial problems concerning permutations without repetitions. The general statement of the problem is this:

There are n distinct objects. How many k-arrangements can be made out of them? Two arrangements are considered distinct if they differ even by a single element or have a different order.

Such arrangements are termed *permutations without repetitions* and are symbolized by A_n^k. In building k-permutations out of n distinct things without repetitions, we have to make k choices. To start, we can choose any one of the n objects available. Once this choice has been made, the second step is to choose from the $n-1$ remaining objects, since the choice already made cannot be repeated (recall that in contrast to the case of permutations with repetitions there is only one thing of each type here). In the same way, for the third step we have a choice of $n-2$ free objects, at the fourth stage we have $n-3$ objects, ..., at the kth stage we have $n-k+1$ objects. Therefore, by the rule of product we find that the number of k-arrangements of n objects without repetitions is expressed as follows:

$$A_n^k = n(n-1)\dots(n-k+1) \tag{1}$$

A SCIENCE CLUB

Let us apply this newly derived formula to the following problem. *A science club consists of 25 members. The task at hand is to elect a president, a vice president, scientific secretary and treasurer. In how many ways can this choice be made if each member of the club can occupy only one post at a time*?

Here we have to find the number of permutations (without repetitions) of 25 elements taken four at a time. The point is that there is a difference in who heads the club and what persons occupy the subsidiary posts (say, a choice of Ivanov as president, Tatarinov as vice president, Timoshenko as scientific secretary and Alekseyev as treasurer differs from Timoshenko as president, Ivanov as vice president, Tatarinov as scientific secretary and Alekseyev as treasurer). The answer is therefore contained in the formula

$$A_{25}^4 = 25 \times 24 \times 23 \times 22 = 303{,}600$$

PERMUTATIONS OF N ELEMENTS

When constructing permutations of n elements without repetitions taking k at a time, we obtained arrangements that differed both as to elements and as to their order. But if we take arrangements involving all n elements, then they can differ solely in the order of the elements. Such arrangements are called *permutations of n elements*, or, briefly, *n-permutations.*

In other words, n-permutations are permutations of n elements without repetitions in which all elements participate. We can also say that permutations of n elements are all possible n-arrangements, each one of which contains all the elements once and all of which differ solely in the order of the elements. The number of n-permutations is denoted by P_n. The formula for P_n is readily obtained from the formula for the number of permutations without repetitions. Namely,

$$P_n = A_n^n = n(n-1)\ldots 2 \times 1 \qquad (2)$$

Thus, to find out how many permutations there are of n elements, multiply together all the natural numbers from 1 to n. This product is denoted by $n!$ (read: n-factorial). We thus have

$$P_n = n! = 1 \times 2 \times \ldots \times n$$

We agree that $1! = 1$.

In the future we will encounter 0!. Though it might seem that 0! should be equal to zero, we agree to consider 0! equal to unity ($0! = 1$).

The point is that factorials have the obvious property

$$n! = n(n-1)!$$

This equality holds true for $n > 1$. It is natural to define 0! so that it should remain true for the case $n = 1$ as well, that is so that $1! = 1 \times 0!$. But then we have to set $0! = 1$.

It will also be noted that formula (1), the number of permutations without repetitions, can be written as

$$A_n^k = \frac{n!}{(n-k)!} \qquad (3)$$

Indeed, in (3) all the factors $(1, 2, 3, \ldots, n-k)$ enter both the numerator and the denominator. Cancelling, we find that $A_n^k = n(n-1, \ldots, \ldots (n-k+1)$, in accordance with formula (1).

THE PROBLEM OF THE ROOKS

How many ways are there of placing 8 rooks on a chessboard so that they do not take each other?

It is clear that only one rook can occupy each rank (row) and file (column). Let us take one such position and use a_1 to denote the number of the occupied square on the first rank, a_2 to denote the position on the second rank, ..., a_8 the position on the eighth rank. Then $(a_1, a_2, \ldots, a_8)$ will be a certain permutation of the numbers 1, 2, ..., 8 (it is clear that among the numbers $a_1, a_2, \ldots, a_8$ there is no pair alike, for then two rooks would be occupying the same file). Conversely, if $a_1, a_2, \ldots, a_8$ is a certain permutation of the numbers 1, 2, ..., 8, then to it corresponds a certain nontaking arrangement of the rooks. Fig. 6 shows one arrangement of the rooks corresponding to the permutation 7 5 4 6 1 3 2 8. Thus, the number of desired arrangements of rooks is equal to the number of permutations of the numbers 1, 2, ..., 8, which is P_8. But

$$P_8 = 8! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 = 40{,}320$$

These rooks can therefore be positioned as required in 40,320 ways.

The same proof is used to demonstrate that *there are $n!$ ways of positioning n nontaking rooks on an n by n chessboard.*

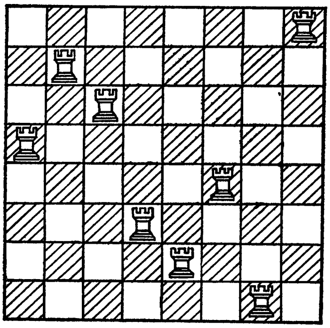


Fig. 6.

The result would be quite different if the rooks differed in some way, say, by colour or by some other label. Suppose the rooks are numbered. Then from each arrangement of non-numbered rooks we get $n!$ arrangements of numbered rooks. They are obtained by permuting the n rooks in all possible ways for the same occupied squares. There would then be $(n!)^2$ ways of positioning nontaking rooks.

We also arrive at the same conclusion by applying the rule of product directly. The first rook can be placed on any one of n^2 squares. Crossing out the rank and file of this rook, we have an $n-1$ by $n-1$ board left ($n-1$ ranks and $n-1$ files) with a total of $(n-1)^2$ squares. This means that the second rook can be placed in $(n-1)^2$ ways. In the same manner, the third rook can be positioned in $(n-2)^2$ ways, etc. Altogether we have

$$n^2 (n-1)^2 \dots 1^2 = (n!)^2$$

ways of placing the rooks.

LINGUISTIC PROBLEMS

One of the problems of linguistics is deciphering inscriptions in unknown languages. Suppose we have a text written with the aid of 26 unknown signs, each one depicting one of 26 sounds. *In how many ways can the sounds be correlated with the written symbols?*

Let us arrange the written symbols in a certain order. Then each correlation will yield a certain permutation of sounds. Using 26 sounds, we can form $P_{26} = 26!$ permutations. This number is roughly equal to 4×10^{26}. It is clearly impossible to verify all these possibilities. It is even beyond the capabilities of an electronic computer. Attempts are therefore made to reduce the number of possibilities. It is frequently possible to separate the vowels from the consonants (one more often encounters vowel-consonant combinations than vowel-vowel or consonant-consonant combinations; observing combinations of symbols that occur more often than others, it is possible to separate the vowels from the consonants). Suppose we have been able to find 7 symbols for vowels and 19 symbols for consonants. Let us now find out *how many times the number of possibilities has been reduced*. The 7 vowels can be permuted in $7!$ ways and the 19 consonants in $19!$ ways. The total number of combinations is equal to $7! \times 19!$. Which means the work has been reduced $26!/7! \times 19! \approx 650{,}000$ times. This is of course a gain, but $7! \times 19!$ is still an enormous number.

The next step is to compute the frequency of occurrence of the individual symbols. Comparing this frequency with that of letters in related languages, one can conjecture as to the significance of certain symbols. Other symbols are found by comparing the given text with the same text in another language (the kings of old delighted in describing their heroic deeds in several languages).

Suppose that in this way 4 vowels and 13 consonants have been identified. What are the remaining possibilities? Obviously, $3! \cdot 6! = 4{,}320$. Now this number of combinations can be handled by electronic computers.

Cryptologists meet similar difficulties in decoding operations.

ROUND DANCE

Seven girls form for a round dance. In how many different ways can they stand in the circle?

If they stood in one place, the result would be $7! = 5{,}040$ permutations. But since the dancing girls circle all the time, their position relative to surrounding objects is not essential, the only important thing being their mutual arrangement. For this reason, we consider the permutations appearing as the dancers move round the circle to be the same. However, from each permutation we can obtain another six by means of rotation, so the number 5,040 has to be divided by 7. This yields $5{,}040 : 7 = 720$ distinct permutations of girls in a round dance.

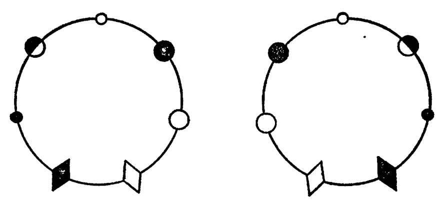


Fig. 7.

Generally, if we consider permutations of n objects in a circular arrangement and regard as identical those positions that go into one another in a rotation, then the number of distinct permutations is equal to $(n - 1)!$.

Now let's see how many necklaces we can make out of 7 distinct beads. By analogy with the problem that was just solved, it might be imagined that the number of distinct necklaces is just 720. But a necklace can also be turned upside down in addition to being moved round the circle (Fig. 7). So the answer is $720 : 2 = 360$.

PERMUTATIONS WITH REPETITIONS

Up to now we have been permuting objects that are pairwise distinct. But if some of the objects being permuted are identical, then we obtain fewer permutations, some of them coinciding with one another.

Let us illustrate with the word "part"; permuting, we obtain 24 different permutations:

part	rapt	ptra	rtpa
patr	ptar	ratp	rtap
prat	rpat	rpta	prta
trap	tpar	aptr	artp
tarp	atrp	atpr	tapr
aprt	arpt	trpa	tpra

If we take the word "papa" instead, then in all the permutations written out above we will have to replace "r" by "p" and "t" by "a". Some of our 24 permutations will be the same. For example, in the first row of permutations, we get one word, "papa". All four permutations in the second row give "paap". Generally, all 24 permutations split up into foursomes, in which the replacements ("r" by "p", "t" by "a") all yield the same result. In the table, these permutations stand in one row. Therefore, the number of distinct permutations that can be produced using the word "papa" is $24 : 4 = 6$. They are

papa, paap, ppaa, apap, aapp, appa

Stated generally, we have the following problem:

There are k elements of different types. How many permutations can be made out of n_1 elements of the first type, n_2 elements of the second type, ..., n_k elements of the kth type?

The number of elements in each permutation is equal to $n = n_1 + n_2 + \ldots + n_k$. Thus, if all the elements were distinct, then the number of permutations would be $n!$. But since some of them are the same, there is a smaller number of permutations. Indeed, take, for instance, the permutation

$$\underbrace{aa\ldots a}_{n_1}\quad\underbrace{bb\ldots b}_{n_2}\quad\ldots\underbrace{xx\ldots x}_{n_k}\qquad(4)$$

where we have all elements of the first type, then all elements of the second type, . . ., finally, all elements of the kth type. The elements of type one can be permuted in $n_1!$ ways. But since all these elements are the same, such permutations are useless. In the same way, it is pointless to generate the $n_2!$ permutations of the elements of type two . . ., $n_k!$ permutations of the kth type. For example, in the permutation "ppaa" we change nothing by interchanging the first and second elements or the third and fourth elements.

The permutations of the elements of type one, type two, etc., can be effected independently. Therefore (by the rule of product) the elements of permutation (4) can be permuted in $n_1!n_2! \ldots n_k!$ ways so that it remains unchanged. The same holds true for any other arrangement of elements. Thus, the set of all $n!$ permutations splits up into parts consisting of $n_1!n_2! \ldots n_k!$ identical permutations each. Hence, the number of distinct permutations with repetitions that can be generated from given elements is

$$P(n_1, n_2, \ldots, n_k) = \frac{n!}{n_1!n_2!\ldots n_k!}\qquad(5)$$

where, if you recall, $n = n_1 + n_2 + \ldots + n_k$.

Using formula (5), it is easy to state *the number of permutations that can be obtained from the word "Mississippi"*. A total of 11 letters: one m, four i's, four s's, and two p's. By formula (5) we have the number of permutations:

$$P(4,\ 4,\ 2,\ 1) = \frac{11!}{4!\times 4!\times 2!\times 1!} = 34{,}650$$

ANAGRAMS

Prior to the 17th century there were hardly any scholarly journals. Scholars learned about the work of their colleagues from books and from private communications (letters). This created great difficulties in the publication of new results because book printing often took many years, while to write about a discovery in a private letter might be quite a risk: go and prove that the idea was yours and not that of a colleague who had spoken out on the basis of your letter to him. Or it might easily happen that the addressee had been working on the same problem, had found a solution and your letter had added nothing to it, in fact he himself had been on the verge of writing to you a similar communication.

Priority quarrels were a matter of constant concern to scholars in those days, as witness the prolonged wrangling at the end of the 17th century over the question of priority regarding the discovery of differential and integral calculus by Newton and Leibniz, or over who first stated the law of universal gravitation between Newton and Hooke, and so on.

In ancient Greece, Archimedes even had to resort to guile. When some scholars of Alexandria took the credit for his results (which they learned from his letters), he wrote a final letter in which he gave the remarkable formulas for the areas and volumes of certain figures. The Alexandrians again said that these formulas were familiar and that Archimedes had not added anything new. This time, however, Archimedes caught them unawares, for the formulas given in his letter were erroneous.

So as to ensure priority and not allow for premature announcement of new results, scholars were used to stating them in the form of brief phrases with the letters transposed. In that form they reached other scholars. Such texts with transposed letters are called *anagrams*. For example, 'rebate' is an anagram of 'beater'. Finally, when a book came out with a detailed description of the new result, the anagram was decoded. Anagrams were also used in political quarrels. For instance, after the murder of the French king Henry III, an anagram was made up out of the name of the murderer frère Jacques Clément: C'est l'enfer qui m'a créé (I was created by hell). The king's opponents retorted with the name of Henri de Valois made into the anagram Vilain Herodés (Herode the villain). When Christian Huygens (1629-1695) discovered the ring of Saturn, he composed the anagram

aaaaaaa, ccccc, d, eeeee, g, h, iiiiiii, lll, mm, nnnnnnnnn, oooo, pp, q, rr, s, ttttt, uuuuu

When the letters are rearranged in the proper order, we get the following text:

"Annulo cingitur tenui, plano, nusquam cohaerente, ad eclipticam inclinato"

("Surrounded by a thin flat ring not suspended anywhere, inclined to the ecliptic)

However, anagrams were not always enough to keep the secret. When Huygens discovered the first satellite of Saturn (Titan) and found its period of revolution about the planet to be 15 days, he composed an anagram and sent it to his colleagues. Now one of them, Wallis, a great hand at decoding cryptic messages, figured it out and composed a counter-anagram which he sent to Huygens. When the two scholars exchanged codes of their anagrams, it appeared as if Wallis had made the same discovery prior to Huygens. Only later did Wallis admit that it was all a joke just to demonstrate the uselessness of anagrams in cryptography. Huygens did not see the point and was embittered....

Let us compute the number of permutations needed to reach the true meaning of the first anagram of Huygens. It contains 7 letters a, 5 c's, 1 d, 5 e's, 1 g, 1 h, 7 i's, 3 l's, 2 m's, 9 n's, 4 o's, 2 p's, 1 q, 2 r's, 1 s, 5 t's, and 5 u's, making a total of 61 letters. By formula (5) we thus get

$$\frac{61!}{7!\,5!\,1!\,5!\,1!\,1!\,7!\,3!\,2!\,9!\,4!\,2!\,1!\,2!\,1!\,5!\,5!}$$

permutations. This enormous number is about equal to 10^{60}.

To run through all these permutations, an electronic computer doing one million operations per second would be hard at work during the entire lifetime of the solar system and more.

In a certain sense, it is easier for a human being to solve this problem than for a machine. The point is that the human investigator does not take all permutations, but only those which yield meaningful words, he will also take account of grammatical rules, etc., all of which drastically reduce the number of attempts. Most important of all, he usually has a rough idea of the problems on which his correspondent is engaged. Still and all, it is a tremendous undertaking.

COMBINATIONS

We are not always interested in the order in which the elements (objects) are arranged. For example, in the semifinals of a chess tournament in the USSR with 20 participants, where only three enter the finals, the order in which these three come is not essential—the main thing is to be in the finals. There have been cases when the champion of the Soviet Union has come up from a lower position in the semifinals.

The same thing happens in football. In the championship of the USSR, four teams at the end of the group of 17 that make up the top league move into the lower league. There is little consolation in the fact that a team occupies 14th place instead of 17th place, since they all leave the top league.

When we are not interested in the order of the elements of a set but only in their composition, we speak of combinations. Thus, *k-combinations of n elements* are all possible k-arrangements made up of these elements such as differ in composition but not in the order of the elements. The number of k-combinations that can be generated out of n elements is denoted by C_n^k.

The formula for the number of combinations is readily obtained from the earlier derived formula for the number of permutations. Indeed, let us first consider all k-combinations of n elements, and then let us permute the elements of each combination in all possible ways. We get all the k-permutations of n elements taken one at a time. But it is possible to produce $k!$ permutations out of each k-combination; the number of these combinations is equal to C_n^k. The following formula is thus true:

$$k!C_n^k = A_n^k$$

From this formula we find that

$$C_n^k = \frac{A_n^k}{k!} = \frac{n!}{(n-k)!\,k!} \tag{6}$$

It is a remarkable fact that the equation we derived coincides with the formula for the number of permutations of k elements of one type and $n-k$ elements of a second type:

$$P(k,\ n-k) = \frac{n!}{k!\,(n-k)!}$$

In other words,

$$C_n^k = P(k,\ n-k) \tag{7}$$

This equality can be proved directly, without resorting to the formula $k!\ C_n^k = A_n^k$. To do this, arrange in order all n elements that make up the combinations, and label each combination by an n-arrangement of zeros and ones. Namely, if some element appears in a combination, then we put a 1 in its place, if it does not appear, we write a 0. For example, if we are making up combinations of the letters a, b, c, d, e, f, g, h, i, j, k, l, then the combination {a, c, g, i, j} will be associated with the arrangement 1 0 1 0 0 0 1 0 1 1 0 0 and to the arrangement 0 1 1 1 0 0 1 0 0 1 0 0 there will correspond the combination {b, c, d, g, j}. Clearly, to each k-combination there corresponds an arrangement of k ones and $n-k$ zeros, and to each arrangement of k ones and $n-k$ zeros there corresponds some k-combination, distinct k-combinations corresponding to distinct arrangements. Hence it follows that the number of k-combinations of n elements coincides with the number of permutations of k elements of one type (ones) and $n-k$ elements of another type (zeros).

Using formula (6) it is easy to solve the problems given at the beginning of this section. The number of distinct outcomes in the semifinals of the chess championship is given by the formula

$$C_{20}^3 = \frac{20!}{3!\ 17!} = 1{,}140$$

The number of distinct "sad" outcomes of the football championship is

$$C_{17}^4 = \frac{17!}{4!\ 13!} = 2{,}380$$

Here is another problem involving combinations:

In how many ways is it possible to place 8 rooks on a chessboard? Unlike the problem examined on page 23, here the restriction that the rooks are nontaking is lifted. All we have to do is to choose any 8 squares of the 64 squares of the chessboard. This can be done in

$$C_{64}^{8}=\frac{64!}{8!\,56!}=4{,}328{,}284{,}968$$

ways.

In exactly the same way, we can prove that an m by n chessboard (m ranks, or rows, n files, or columns) can accommodate k rooks in

$$C_{mn}^{k}=\frac{(mn)!}{k!\,(mn-k)!}$$

ways.

Now if instead of k identical rooks we take k distinct chessmen, then it all depends on which piece is placed where. And so instead of combinations, we obtain permutations, and the answer is given by the formula

$$A_{mn}^{k}=\frac{(mn)!}{(mn-k)!}$$

GENOESE LOTTERY

A few centuries ago, the so-called *Genoese* (or *number*) *lottery* was all the rage (it is even played in some countries today). In a nutshell it is this. Players buy tickets numbered from 1 to 90. Tickets with two, three, four and five numbers are also sold. On the day of drawings, five counters are drawn from a bag containing counters numbered 1 to 90. Winners are those whose ticket numbers coincide with the drawn numbers. For example, if a ticket has the numbers 8, 21, 49 and the numbers drawn are 3, 8, 21, 37, 49, this is a win, but if the drawn numbers are 3, 7, 21, 49, 63, the ticket is a loser because the number 8 is missing.

In the old days, if a player in the lottery bought a ticket with one number (simplum), he got 15 times the cost of the ticket, if it had two numbers (ambo), he got 270 times more, if three numbers (terno), it was 5,500 times greater, if four numbers (quaterno) he won 75,000 times more, and if there were five digits on the winning ticket (quinto), then the winner was paid 1,000,000 times the cost of the ticket.

Many tried to get rich fast by playing terno or ambo, but almost nobody won—the lottery was calculated to leave the winnings with the organizers.

To find out why this is so, let us try to compute the ratio of "lucky" outcomes to the total number of outcomes for different ways of playing the game. The total number of outcomes of the lottery is given immediately by formula (6). Five counters are extracted from a bag holding 90 counters, the order playing no role whatsoever. What we have is combinations of 90 elements taken 5 at a time, the number of which is

$$C_{90}^{5}=\frac{90!}{5!\,85!}=\frac{90\times 89\times 88\times 87\times 86}{1\times 2\times 3\times 4\times 5}$$

Suppose a player buys a ticket with one number. In how many cases does he win? To win, one of the numbers drawn from the bag must coincide with the number of the ticket. The other four numbers are irrelevant. But these four numbers are drawn from the remaining 89 numbers. Therefore, the number of favourable combinations is given by

$$C_{89}^{4}=\frac{89\times 88\times 87\times 86}{1\times 2\times 3\times 4}$$

From this we see that the ratio of the number of favourable combinations to the total number of combinations is

$$\frac{C_{89}^{4}}{C_{90}^{5}}=\frac{5}{90}=\frac{1}{18}$$

This means, speaking roughly, that a player will win once out of 18 times. In other words, he pays for 18 tickets and wins 15 times the cost of one ticket, and the price of three tickets will be pocketed by the owners of the lottery.

This does not mean, of course, that a player wins exactly once out of every 18 tries. It may happen that 20 or 30 games will be played between two winnings; on the other hand, it also may happen that he wins twice in a row. What we are talking about is the average number of winnings over a large interval of time or given a large number of players. If this were not so, we might make the mistake of the doctor who told his patient that one out of ten survives the particular illness from which his patient was suffering, and that since the last 9 whom the doctor had treated had died, he would definitely live.

Now let us compute the chances when playing ambo. Here we need two numbers to coincide with those drawn from the bag, the remaining three can be any numbers. Since they can be chosen from the remaining 88 numbers, the number of "lucky" outcomes when playing ambo is given by the formula

$$C_{88}^{3}=\frac{88\times 87\times 86}{1\times 2\times 3}$$

And the ratio of the number of "lucky" outcomes to the total number is

$$\frac{C_{88}^{3}}{C_{90}^{5}}=\frac{4\times 5}{90\times 89}=\frac{2}{801}$$

This time, only two out of 801 outcomes win. But since the winnings are only 270 times the price of the ticket, it follows that out of every 801 ambo tickets, the price of 261 tickets is pocketed by the owners of the lottery. It is clear that playing ambo offers still less of a chance to the player than playing simplum.

The situation is much worse when we come to terno, quaterno and quinto. In terno, the ratio of the number of favourable outcomes to the total number of outcomes is

$$\frac{C_{87}^{2}}{C_{90}^{5}}=\frac{3\times 4\times 5}{90\times 89\times 88}=\frac{1}{11{,}748}$$

In quaterno, it is

$$\frac{C_{86}^{1}}{C_{90}^{5}}=\frac{2\times 3\times 4\times 5}{90\times 89\times 88\times 87}=\frac{1}{511{,}038}$$

In quinto, it is

$$\frac{1}{C_{90}^{5}}=\frac{1\times 2\times 3\times 4\times 5}{90\times 89\times 88\times 87\times 86}=\frac{1}{43{,}949{,}268}$$

The winnings are, respectively, only 5,500, 75,000, and 1,000,000 times greater. The reader can easily calculate how much players lose under these conditions.

BUYING CAKES

A confectionary sells four types of cakes: napoleon, éclair, shortbread and cream puffs. In how many ways is it possible to buy 7 cakes?

This is a somewhat different problem from those we have already solved. It is not a problem involving permutations with repetitions because the order in which the cakes are arranged in the box is immaterial. It is therefore closer to the combination type. But here too there is a difference. The combinations may involve repeated elements (say, we could buy 7 éclairs). Such problems have to do with *combination with repetitions.*

To solve our problem, let us label each purchase by means of zeros and units. First, write the number of units indicating the number of napoleons bought. Then, to separate napoleons from éclairs, write a zero, then units to indicate the number of éclairs, then again a zero (if no éclairs are bought, there will be two zeros in succession). This is followed by units to indicate the number of shortbreads, again a zero, and, finally, units to indicate the purchase of cream puffs. For instance, if we buy 3 napoleons, 1 éclair, 2 shortbreads and 1 cream puff, then we have the following notation: 1110101101. But if we buy 2 napoleons and 5 shortbreads, then we write 1100111110. Different purchases clearly are associated with different combinations of 7 units and 3 zeros. Conversely, to each combination of 7 units and 3 zeros there corresponds a definite purchase. For example, a purchase of 3 éclairs and 4 shortbreads corresponds to the combination 0111011110.

Thus, the number of distinct purchases is equal to the number of permutations with repetitions that can be made up out of 7 units and 3 zeros. As we pointed out on page 26, this number is

$$P(7, 3)=\frac{10!}{7!\,3!}=\frac{10\times 9\times 8}{1\times 2\times 3}=120$$

We could arrive at the same result by a different approach. Arrange the cakes in each purchase in the following order: napoleons, éclairs, shortbreads and cream puffs, and then number them. But add 1 to the number labels of the éclairs, 2 to the number labels of the shortbreads and 3 to the number labels of the cream puffs (do not add anything to the number labels of the napoleons). For example, suppose we buy 2 napoleons, 3 éclairs, 1 shortbread and 1 cream puff. These cakes will then have the designations, or number labels 1, 2, 4, 5, 6, 8, 10. It is clear that the biggest number is 10 (the last cream puff gets the number label $7+3=10$), and the smallest number label is 1 (which is the designation of the first napoleon). Not a single number label is repeated. Conversely, to every increasing sequence of 7 numbers from 1 to 10 there corresponds a purchase. For instance, the sequence 2, 3, 4, 5, 7, 8, 9 indicates a purchase of 4 éclairs and 3 shortbreads. To see this, subtract the numbers 1, 2, 3, 4, 5, 6, 7 from the given number labels. We thus obtain the numbers 1, 1, 1, 1, 2, 2, 2, which is 4 units and 3 twos. But we added 1 to the number labels of the éclairs and 2 to the labels of the shortbreads, which means that we have 4 éclairs and 3 shortbreads.

We only get increasing sequences of numbers in this case, and, consequently, each sequence is completely determined by its composition. Therefore, the number of such 7-sequences is equal to the number of 7-combinations of 10 numbers (from 1 to 10). And the number of these combinations is given by

$$C_{10}^{7}=\frac{10!}{7!\,3!}=120$$

which is the same result as that obtained earlier.

COMBINATIONS WITH REPETITIONS

Above we mentioned that this was a type of problem that involves combinations with repetitions. Such problems are stated generally as follow. There are n distinct types. How many k-combinations of them can be made up if the order of the elements in the combinations is disregarded (in other words, distinct combinations must differ in at least one object)?

In general form, this problem can be solved in exactly the same way as the cake problem. Namely, label each combination by means of zeros and ones: for each type write as many units as there are objects of the given type in the combination, and separate the distinct types by means of zeros (if objects of a specific type are absent in the combination, then there will be two or more zeros in succession). We then obtain as many ones as there are objects in the combination, that is k, while the number of zeros will be 1 less than the number of types of objects, that is to say, $n-1$. We thus get permutations, with repetitions, made up of k ones and $n-1$ zeros. Here, to different combinations correspond distinct permutations with repetitions, and each permutation with repetitions has its specific combination. Thus, the number $\overline{C}_n^k$ of k-combinations (with repetitions) of elements of n types is equal to the number of $P(k, n-1)$ permutations of $n-1$ zeros and k ones (with repetitions). But

$$P(k, n-1)=\frac{(k+n-1)!}{k!\,(n-1)!}$$

And so

$$\overline{C}_n^k=\frac{(k+n-1)!}{k!\,(n-1)!}=C_{n+k-1}^k$$

This same formula may be proved in a different manner. In each combination, arrange the elements according to type (first, all elements of the first type, then all elements of the second type, and so on). Then number all elements in the combination, but add 1 to Type Two elements, 2 to Type Three elements, etc. Then from each combination with repetitions we get a combination, without repetitions, consisting of the numbers $1, 2, \ldots, n+k-1$, each combination involving k elements. Whence it again follows that

$$\overline{C}_n^k=C_{n+k-1}^k=\frac{(n+k-1)!}{k!\,(n-1)!} \qquad (8)$$

In some problems, we have combinations with repetitions with a supplementary restriction: they must include elements of r fixed types, where $r \leqslant n$. These problems can easily be reduced to the problem already solved. To ensure the presence of elements of the given r types, let us from the very beginning take one element of each such type. In this way there will be r positions occupied in a k-combination. The remaining $k-r$ positions can be filled by any elements, which, under the conditions of the problem, belong to n types. Therefore, there are just as many combinations of the desired type as there are combinations (with repetitions) of elements of n types containing $k-r$ elements each, that is,

$$\overline{C}_n^{k-r}=C_{n+k-r-1}^{k-r}$$

In particular, if $n \leqslant k$ and it is required that at least one element of each of the n types enter into k-combinations with repetitions, then there will be C_{k-1}^{k-n} combinations.

THE FOOTBALL CHAMPIONSHIP AGAIN

We have analyzed problems involving permutations and combinations. In many cases we have to deal with combinations of a variety of types. Consider the following problem.

Let us call two outcomes of the Soviet football championship *coincident in the main* if the holders of gold, silver and bronze medals coincide and also if the four teams leaving the top league coincide. *Find the number of noncoincident-in-the-main outcomes of a championship* (with 17 participating teams, as usual).

As we have already seen, the medals can be distributed in $A_{17}^3=17\times 16\times 15$ ways (see page 22). This leaves 14 teams, of which 4 leave the top league. Since the order of the leaving teams is immaterial, this operation can take place in $C_{14}^4=\frac{14!}{4!\,10!}$ ways. By the rule of product we find that the number of noncoincident-in-the-main outcomes of the championship is equal to

$$A_{17}^3\times C_{14}^4=17\times 16\times 15\times\frac{14!}{4!\,10!}=\frac{17!}{4!\,10!}=4{,}084{,}080$$

We can obtain the same result by a different approach. The total number of distinct outcomes of the championship (rejecting cases when two teams occupy the same position) is equal to $P_{17} = 17!$ But permutations of teams occupying places 4 to 13, and also permutations of teams occupying places 14 to 17 result in a coincident-in-the-main outcome. The number of such permutations is $10! \times 4!$. Hence, the number of distinct outcomes is given by the formula $\frac{17!}{10!\,4!}$

Suppose we wish to telegraph the results of the championship in a telegram consisting of k dots and dashes. *What is the least number of symbols needed*? We already know that it is possible to construct 2^k distinct combinations out of k dots and dashes. Therefore, the minimal number of elements (characters) needed to transmit the required information must be such as to satisfy the following inequality:

$$2^k \geqslant 4{,}084{,}080$$

Solving it, we find that $k \geqslant 22$. And so to transmit the results of the championship by means of dots and dashes we need at least 22 characters.

Calculations of this kind are of course not needed in such mundane matters as the outcome of a championship, but it is easy to imagine a case where the transmission of information involves considerable engineering difficulties (say, in transmitting a photograph from a space vehicle) and every element is practically worth its weight in gold. Then we have to consider various possibilities of such a transmission and choose the optimal ones. Problems of this nature are studied in a special division of mathematics called *information theory*.

PROPERTIES OF COMBINATIONS

(The end of this chapter may be skipped in a first reading, but the equations that are proved here, $C_n^k = C_n^{n-k}$ and $C_n^k = C_{n-1}^{k-1} + C_{n-1}^k$, will be used frequently in the sequel.)

The numbers C_n^k possess a wide range of remarkable properties, which may be demonstrated in a variety of ways. In some cases it is more convenient to make direct use of the formula

$$C_n^k = \frac{n!}{k!\,(n-k)!} \tag{9}$$

However, it is often possible to obtain a proof using combinatorial arguments: compute the number of combinations of a given type and partition these combinations into disjoint classes (that is, classes without common elements). We then find out how many combinations there are in each class. Combining the numbers obtained, we again get the number of all combinations of a given type. This yields the desired relation.

Let us begin with the simplest relation:

$$C_n^k = C_n^{n-k} \tag{10}$$

This follows directly from formula (9). The point is that if we replace k by $n - k$, then $n - k$ will be replaced by $n - (n - k) = k$ and as a result the factors in the denominator will be interchanged. However, (10) can easily be proved without resorting to any explicit type of number of combinations. If we choose some k-combination from n distinct elements, then there remains the complementary combination of $n - k$ elements, the combination complementary to the $(n - k)$-combination being the original k-combination. Thus, k-combinations and $(n - k)$-combinations constitute mutually complementary pairs, and so the number of these combinations is the same. Hence, $C_n^k = C_n^{n-k}$

The relation

$$C_n^k = C_{n-1}^{k-1} + C_{n-1}^k \tag{11}$$

can be proved in just as simple a fashion. To do this, form k-combinations out of n elements $a_1, \ldots, a_{n-1}, a_n$ and partition them into two classes. The first will contain combinations involving the element a_n, while the second will contain combinations which do not involve this element. If we drop element a_n from any combination of the first class, there will remain a $(k - 1)$-combination made up of the elements $a_1, \ldots, a_{n-1}$. There are C_{n-1}^{k-1} such combinations.

Therefore the first class includes C_{n-1}^{k-1} combinations. The combinations of the second class are k-combinations made up of $(n-1)$ elements $a_1, \ldots, a_{n-1}$. And so there are C_{n-1}^{k} of them. Since any k-combination of the elements $a_1, \ldots, a_n$ belongs to one and only one of these classes, and the total number of such combinations is C_n^k, we arrive at equation (11).

A similar proof is used with respect to the relation

$$C_n^0+C_n^1+C_n^2+\ldots+C_n^n=2^n \tag{12}$$

Recall that 2^n is the number of n-permutations of elements of two types with repetitions. Split these permutations into classes, referring to the kth class those involving k elements of the first type and $n-k$ elements of the second type. The permutations of the kth class are all possible permutations of k elements of the first type and $n-k$ elements of the second type. We know that the number of such permutations is $P(k, n-k)$ and $P(k, n-k) = C_n^k$ (see pages 26 and 28). Which means that the total number of permutations of all classes is equal to $C_n^0 + C_n^1 + \ldots + C_n^n$. On the other hand, this same number is equal to 2^n, which completes the proof of (12).

In exactly the same way we can prove that

$$\sum_{n_1+n_2+n_3=n} P(n_1, n_2, n_3)=3^n \tag{13}$$

where the sum is extended over all partitions of the number n into three integers [the order of the integers being taken into account; for example, both $P(n_1, n_2, n_3)$ and $P(n_2, n_3, n_1)$ are counted]. To prove this, we have to consider all n-permutations of elements of three types and split them into classes of the same composition (that is to say, we take permutations with one and the same number of elements of the first type, of the second type and of the third type).

Generally, we have the equality

$$\sum_{n_1+\ldots+n_k=n} P(n_1, \ldots, n_k)=k^n \tag{14}$$

where the sum is extended over all partitions of the numbers n into k integers (with regard for the order of the integers).

Now consider m-combinations (with repetitions) consisting of elements of $n+1$ types, say $n+1$ letters $a, b, c, \ldots, x$. There are $\bar{C}_{n+1}^m = C_{n+m}^m$ such combinations. Split all these combinations into classes, placing in the kth class the combinations in which the letter a occurs k times; the remaining $m-k$ places may be occupied by the remaining letters $b, c, \ldots, x$, the number of which is n. Therefore, the kth class includes as many combinations as there are $(m-k)$-combinations (with repetitions) which can be formed from elements of n types, that is, $C_{n+m-k-1}^{m-k}$. Thus, the total number of such combinations is equal to

$$C_{n+m-1}^m+C_{n+m-2}^{m-1}+\ldots+C_n^1+C_{n-1}^0$$

On the other hand, we saw that this number is equal to C_{n+m}^m. We have thus proved the equation

$$C_{n-1}^0+C_n^1+C_{n+1}^2+\ldots+C_{n+m-1}^m=C_{n+m}^m \tag{15}$$

Replacing n by $n+1$ and m by $m-1$ and using (10), we find that

$$C_n^n+C_{n+1}^n+C_{n+2}^n+\ldots+C_{n+m-1}^n=C_{n+m}^{n+1} \tag{16}$$

Particular cases of formula (16), for $n=1, 2, 3$, are

$$1+2+\ldots+m=\frac{m(m+1)}{2} \tag{17}$$

$$1\times 2+2\times 3+\ldots+m(m+1)=\frac{m(m+1)(m+2)}{3} \tag{18}$$

$$1\times 2\times 3+2\times 3\times 4+\ldots+m(m+1)(m+2)=\frac{m(m+1)(m+2)(m+3)}{4} \tag{19}$$

Using formulas (17) to (19), it is easy to find the sum of the squares and the sum of the cubes of the natural numbers from 1 to m. Formula (18) may be rewritten as

$$1^2+2^2+\ldots+m^2+1+2+\ldots+m=$$
$$=\frac{m(m+1)(m+2)}{3}$$

But from formula (17)

$$1+2+\ldots+m=\frac{m(m+1)}{2}$$

and so

$$1^2+2^2+\ldots+m^2=\frac{m(m+1)(m+2)}{3}-$$
$$-\frac{m(m+1)}{2}=\frac{m(m+1)(2m+1)}{6} \qquad (20)$$

In exactly the same way, from (9) we conclude that

$$1^3+2^3+\ldots+m^3=\frac{m^2(m+1)^2}{4} \qquad (21)$$

It is left to the reader to obtain in this fashion formulas for the sums of higher powers of the natural numbers.

It is possible to classify m-combinations (with repetitions) of elements of n types by taking as the basis the number of distinct elements which enter into a given combination. In other words, the first class will involve combinations consisting of the same elements, the second class will be made up of elements of two types, . . ., the nth class will consist of elements of all n types (naturally, if $m<n$, there will only be m classes).

Let us compute the number of combinations in each class. We can take two stages to choose the combinations that belong to the kth class. Let us first choose precisely the k types that enter into a combination. Since the total number of types is equal to n, this choice can be made in C_n^k ways. After the types have been chosen, we have to form those m-combinations (with repetitions) of elements of these k types in which all the k types are represented. But we have proved (see page 32) that the number of such combinations with repetitions is equal to $C_{m-1}^{m-k}=C_{m-1}^{k-1}$.

From this it follows, by the rule of product, that $C_n^kC_{m-1}^{k-1}$ combinations are included in the kth class. Adding up the number of combinations of each class, we find the total number of m-combinations (with repetitions) of elements of n types, or C_{m+n-1}^m. Such is the proof of the equality

$$C_n^1C_{m-1}^0+C_n^2C_{m-1}^1+\ldots+C_n^nC_{m-1}^{n-1}=C_{m+n-1}^m \qquad (22)$$

If $m<n$, then $C_n^mC_{m-1}^{m-1}$ will be the last term in the sum. It is more convenient to write this equality with C_n^k replaced by C_n^{n-k} in each summand. We then get

$$C_n^{n-1}C_{m-1}^0+C_n^{n-2}C_{m-1}^1+\ldots+C_n^0C_{m-1}^{n-1}=$$
$$=C_{m+n-1}^{n-1} \qquad (23)$$

Herre, in each term on the left the sum of the superscripts is equal to $n-1$, the sum of the subscripts is $n+m-1$. The subscripts are constant and the superscripts vary. Another way of writing this equation is

$$C_p^0C_{n-p}^m+C_p^1C_{n-p}^{m-1}+\ldots+C_p^mC_{n-p}^0=C_n^m \qquad (23')$$

We will now derive a similar formula in which the subscripts in the summation vary too. To do this, take p distinct vowels and $n-p$ distinct consonants and form all possible m-combinations of them with repetitions. Split these combinations into classes, referring to the kth class such combinations as contain k vowels and $m-k$ consonants. Compute the number of combinations in the kth class. Each combination of this class is partitioned into a k-combination (with repetitions) made up of p vowels, and an $(m-k)$-combination (with repetitions) made up of $n-p$ consonants. Thus, the kth class includes $C_{k+p-1}^kC_{m+n-p-k-1}^{m-k}$ combinations. Consequently, the total number of combinations under consideration is equal to

$$C_{p-1}^0C_{m+n-p-1}^m+C_p^1C_{m+n-p-2}^{m-1}+\ldots+$$
$$+C_{m+p-1}^mC_{n-p-1}^0$$

On the other hand, these combinations yield all possible m-combinations (with repetitions) of elements of n distinct types and therefore their number is equal to C_{m+n-1}^{m}. We arrive at the identity

$$C_{p-1}^{0}C_{m+n-p-1}^{m}+C_{p}^{1}C_{m+n-p-2}^{m-1}+\dots+ +C_{m+p-1}^{m}C_{n-p-1}^{0}=C_{m+n-1}^{m} \qquad (24)$$

Let us rewrite this identity so that, in summing, only the subscripts vary. This requires applying to all terms the identity $C_r^q = C_r^{r-q}$. We get

$$C_{p-1}^{p-1}C_{m+n-p-1}^{n-p-1}+C_{p}^{p-1}C_{m+n-p-2}^{n-p-1}+ +\dots+C_{m+p-1}^{p-1}C_{n-p-1}^{n-p-1}=C_{m+n-1}^{n-1}$$

This formula may be rewritten as follows (we replace p by $p+1$, n by $n+2$, and m by $m-n$):

$$C_{p}^{p}C_{m-p}^{n-p}+C_{p+1}^{p}C_{m-p-1}^{n-p}+\dots+C_{m-n+p}^{p}C_{n-p}^{n-p}= =C_{m+1}^{n+1} \qquad (24')$$

We see that when summing, the superscripts remain fixed and the subscripts vary, the sum of the superscripts being equal to n and that of the subscripts, m.

Note a particular case of the earlier obtained formula (23′). If we put $n-p=m$ in it, we get

$$C_{p}^{0}C_{m}^{0}+C_{p}^{1}C_{m}^{1}+\dots+C_{p}^{m}C_{m}^{m}=C_{p+m}^{m} \qquad (25)$$

In particular, for $p=m$ we have

$$(C_{p}^{0})^2+(C_{p}^{1})^2+\dots+(C_{p}^{p})^2=C_{2p}^{p} \qquad (26)$$

The identities obtained above may be generalized. To do this let us consider a set consisting of elements of q types: n_1 elements of the first type, n_2 elements of the second type, . . ., n_k elements of the kth type; elements of one type are distinct (for instance, the type is determined by the colour of an object while objects of the same colour have different shapes).

Let us form all possible m-combinations of the elements of this set and let us classify them as to composition, that is, as to the number of elements of the first, second, . . ., qth type. Thus, each class is characterized by natural numbers $(m_1, m_2, \dots, m_q)$ satisfying the inequalities $0 \leqslant m_i \leqslant n_i$. It consists of m_1 elements of the first type, m_2 elements of the second type, ..., m_q elements of the qth type, and $m_1+m_2+\dots+m_q=m$. We shall denote such a class by $A(m_1, \dots, m_q)$.

From the rule of product it follows that class $A(m_1, \dots, m_q)$ includes $C_{n_1}^{m_1}C_{n_2}^{m_2}\dots C_{n_q}^{m_q}$ combinations. Summing the number of combinations over all classes, we obtain the identity

$$\sum C_{n_1}^{m_1}C_{n_2}^{m_2}\dots C_{n_q}^{m_q}=C_{n}^{m} \qquad (27)$$

where $n=n_1+n_2+\dots+n_q$ and the summation is extended over all possible combinations of the natural numbers $(m_1, m_2, \dots, m_q)$, where $m_1+m_2+\dots+m_q=m$.

If we take combinations with repetitions, we get a similar identity,

$$\sum C_{n_1+m_1-1}^{m_1}-C_{n_2+m_2-1}^{m_2}\dots \dots C_{n_q+m_q-1}^{m_q}=C_{n+m-1}^{m} \qquad (28)$$

where, again, $n=n_1+n_2+\dots+n_q$ and the summation is over the same combinations of numbers $(m_1, m_2, \dots, m_q)$.

Another property of combinations is established thus. We have the identity

$$C_{n}^{k}C_{n-k}^{m-k}=C_{m}^{k}C_{n}^{m} \qquad (29)$$

This identity can readily be verified combinatorially. To do this, take n distinct elements, select k elements from among them, and from the remaining $n-k$ elements choose another set of $m-k$. We thus get an m-combination of n elements. For k fixed, this process can be carried out in $C_n^k C_{n-k}^{m-k}$ ways. It is easy to verify then that each of the C_n^m combinations is obtained in C_m^k ways. Whence follows the equality (29).

Write (29) for $k = 0, \ldots, m$ and combine the resulting equalities. Since, by (12),

$$C_m^0 + C_m^1 + \ldots + C_m^m = 2^m$$

we get

$$C_n^0 C_n^m + C_n^1 C_{n-1}^{m-1} + \ldots + C_n^m C_{n-m}^0 = 2^m C_n^m$$

or

$$C_n^0 C_n^{n-m} + C_n^1 C_{n-1}^{n-m} + \ldots$$
$$\ldots + C_n^m C_{n-m}^{n-m} = 2^m C_n^m \qquad (30)$$

A PARTICULAR CASE OF THE PRINCIPLE OF INCLUSION AND EXCLUSION

Many properties of combinations are derived on the basis of the principle of inclusion and exclusion (see page 18). We will need a special case of this formula (the inclusion and exclusion formula). Let the number $N(\alpha_1 \ldots \alpha_k)$ of elements possessing the properties $\alpha_1, \ldots, \alpha_k$ be dependent not on the properties themselves but only on their number, that is, let

$$N(\alpha_1) = \ldots = N(\alpha_n),$$
$$N(\alpha_1\alpha_2) = N(\alpha_1\alpha_3) = \ldots = N(\alpha_{n-1}\alpha_n),$$
$$N(\alpha_1\alpha_2\alpha_3) = N(\alpha_1\alpha_2\alpha_4) = \ldots = N(\alpha_{n-2}\alpha_{n-1}\alpha_n)$$

and so on. Then all the terms in the sum $N(\alpha_1) + \ldots + N(\alpha_n)$ are equal to one and the same number, which we denote by $N^{(1)}$. Since there are n terms in this sum, it is equal to $nN^{(1)} = C_n^1 N^{(1)}$. In exactly the same way we prove that

$$N(\alpha_1\alpha_2) + N(\alpha_1\alpha_3) + \ldots + N(\alpha_{n-1}\alpha_n) = C_n^2 N^{(2)}$$

where $N^{(2)} = N(\alpha_1\alpha_2)$ and, generally,

$$N(\alpha_1\alpha_2 \ldots \alpha_k) + \ldots$$
$$\ldots + N(\alpha_{n-k+1} \ldots \alpha_n) = C_n^k N^{(k)} \qquad (31)$$

[it is clear that the sum (31) is extended naturally over all possible combinations of k properties out of n].

And so in this case the principle of inclusion and exclusion yields the following formula:

$$N^{(0)} = N - C_n^1 N^{(1)} + C_n^2 N^{(2)} - \ldots$$
$$\ldots + (-1)^n C_n^n N^{(n)} \qquad (32)$$

ALTERNATING SUMS OF COMBINATIONS

Now let us derive some further properties of combinations. These properties are much like the ones proved earlier, but differ from them in that the signs of the terms vary—they alternate from plus to minus to plus, etc.

The simplest of these formulas is

$$C_n^0 - C_n^1 + C_n^2 - \ldots + (-1)^n C_n^n = 0 \qquad (33)$$

This identity follows from equality (11). In order to prove this, note that $C_n^0 = C_{n-1}^0 = 1$. Replace the first term by C_{n-1}^0 and note that, by (11), $C_{n-1}^0 - C_n^1 = -C_{n-1}^1$. Furthermore, we have $-C_{n-1}^1 + C_n^2 = C_{n-1}^2$ and so on. All terms finally cancel out.

This formula may be proved combinatorially. Write out all combinations of the n elements $a_1, \ldots, a_n$ and make the following transformation: to a combination not containing the letter a_1 adjoin this letter, and delete it from combinations in which it appears. It is easy to check that again we thus obtain all the combinations, and, in this instance, one at a time. However, in the given transformation, all combinations having an even number of elements turn into combinations with an odd number of elements, and conversely. Hence, there are just as many combinations with an even number of elements as there are with an odd number (we include the empty combination which has no elements). That is what formula (33) expresses.

Now let us prove a more complicated formula:

$$C_n^0 C_n^m - C_n^1 C_{n-1}^{m-1} + C_n^2 C_{n-2}^{m-2} - \ldots$$
$$\ldots + (-1)^m C_n^m C_{n-m}^0 = 0 \qquad (34)$$

Consider m-combinations of the n elements $a_1, \ldots, a_n$. Denote by $(a_1, \ldots, a_k)$ the property of a combination to definitely include the elements $a_1, \ldots, a_k$. The number $N(a_1, \ldots, a_k)$ of such combinations is equal to C_{n-k}^{m-k} (here, k positions are occupied by the elements $a_1, \ldots, a_k$, and the remaining $m-k$ positions have $n-k$ aspirants). The total number of combinations is C_n^m, and there are no combinations devoid of any one of the properties $(a_1), \ldots, (a_n)$—every m-combination has some quantity of elements. Therefore, in our case, $N = C_n^m$, $N^{(0)} = 0$, $N^{(k)} = C_{n-k}^{m-k}$. Substituting these values into (32), we arrive at the identity (34).

The following relation is proved in exactly the same way:

$$C_n^0 C_{n+m-1}^m - C_n^1 C_{n+m-2}^{m-1} + C_n^2 C_{n+m-3}^{m-2} - \ldots \\ \ldots + (-1)^n C_n^n C_{m-1}^{m-n} = 0$$

if $m \geqslant n$,

$$C_n^0 C_{n+m-1}^m - C_n^1 C_{n+m-2}^{m-1} + \ldots \\ \ldots + (-1)^m C_n^m C_{n-1}^0 = 0 \qquad (35)$$

if $m < n$

Namely, we consider m-combinations (with repetitions) of n kinds of elements $a_1, a_2, \ldots, a_n$ and denote by (a_k), $1 \leqslant k \leqslant n$, the property of a combination to include elements of the kind a_k (and, possibly, elements of other kinds). Then $N(a_1, \ldots, a_k)$ is the number of combinations that definitely include elements of the types $a_1, \ldots, a_k$. It is possible to eliminate from each such combination one element each of the types $a_1, \ldots, a_k$. As a result, we get some $(m-k)$-combination (with repetitions) of n types of elements $a_1, \ldots, a_n$. Now, conversely, by adding to the $(m-k)$-combination (with repetitions) of elements $a_1, \ldots, a_n$ one element each of the types $a_1, \ldots, a_k$, we get an m-combination which definitely has the types $a_1, \ldots, a_k$. From this it follows that the number $N(a_1, \ldots, a_k)$ is equal to the number of $(m-k)$-combinations (with repetitions) of elements of n types, that is, $N(a_1, \ldots, a_k) = C_{n+m-k-1}^{m-k}$. Furthermore, the total number of m-combinations with repetitions is equal to C_{n+m-1}^m and there are no combinations not possessing a single property (a_k), $1 \leqslant k \leqslant n$. Substituting the values $N^{(0)} = 0$, $N = C_{n+m-1}^m$, $N^{(k)} = C_{n+m-k-1}^{m-k}$ thus found into the formula we arrive at the identity (35).

Finally, let us prove the identity

$$n^m - C_n^1 (n-1)^m + C_n^2 (n-2)^m - \ldots \\ \ldots + (-1)^{n-1} C_n^{n-1} \cdot 1^m = 0 \qquad (36)$$

which holds for $m < n$.

To do this, consider the m-permutations (with repetitions) of n distinct elements and denote by (a_k) the property of a permutation not to include elements of type a_k. Then $N(a_1, \ldots, a_k)$ is the number of m-permutations (with repetitions) that do not contain elements of the types $a_1, \ldots, a_k$, that is, the number consisting of $n-k$ types of elements $a_{k+1}, \ldots, a_n$. Now the number of such permutations is equal to $(n-k)^m$. Thus, $N^{(k)} = N(a_1, \ldots, a_k) = (n-k)^m$. The total number of permutations (32), is n^m.

Finally, there are no permutations not possessing a single one of the properties $(a_1), \ldots, (a_n)$. Indeed, if a permutation does not possess a single one of the properties (a_k), then it contains elements of all n types. But this is impossible because the number m of elements in a permutation is less than n. And so $N^{(0)} = 0$, and we arrive at the identity (36).

We have proved a number of relations for the numbers C_n^k. They can be proved in other ways. In Chapter V we will examine geometric proofs of these relations, and in Chapter VII we will give the most powerful proof, the method of generating functions. With this method we can prove not only all the relations of the present chapter, but also a wide range of other interesting relations.

Up to now we have considered problems in which the order of the elements in combinations was not restricted by any additional conditions. Either (as in permutations) any order was permissible, or (as in combinations) the order was disregarded completely. We shall now examine some problems in which the order of the elements is restricted.

LIONS AND TIGERS

A trainer of wild animals has to take 5 lions and 4 tigers out into the circus ring so that no tiger follows another one. In how many ways can he handle the job?

First line up the lions so that there are gaps between each pair. This can be done in $5! = 120$ ways. There are four gaps. Adjoining the two positions in front and in back, we get a total of 6 positions for placing the tigers, no two following one another. Since the order of the tigers is essential, the number of ways of placing them is equal to the number of 4-permutations of 6, or $A_6^4 = 360$. Combining each arrangement of the lions with one arrangement of the tigers, we get $120 \times 360 = 43{,}200$ ways of leading the animals into the ring.

If the animal trainer had n lions and k tigers, he could solve the problem in $P_n A_{n+1}^k = \frac{n!\,(n+1)!}{(n-k+1)!}$ ways. This is only possible provided $k \leqslant n + 1$, otherwise two tigers will come together.

BUILDING A STAIRWAY

The job is *to build a stairway from point A to point B* (Fig. 8). *The distance from A to C is 4.5 metres, from C to B, 1.5 metres. Each step is 30 cm high and the width is an integral multiple of 50 cm. In how many ways can the stairway be built?*

From the conditions of the problem it is evident that the stairway must have 5 steps, and, since $4.5 : 0.5 = 9$, there are 10 places where the step can be made. We thus have to pick 5 places out of 10. This can be done in

$$C_{10}^5 = \frac{10!}{5!\,5!} = 252$$

ways.

Generally, if there are to be k steps, and the length AC accommodates n steps, then the stairway can be built in C_{n+1}^k ways.

This problem is much like the animal-trainer problem in that the trainer did not want two

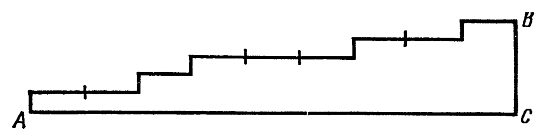


Fig. 8.

tigers together, and the builder cannot allow for steps of double height. But there is an essential difference between the problems. The trainer had to bear in mind the order in which the tigers marched: it is one thing to have tiger Shah come out first, and quite a different matter if tiger Akbar does. It makes no difference to the carpenter, since the rises are all the same. Besides, the animal trainer had to take into account the order of the lions as well, while all positions permitting a rise are the same for the carpenter. The stairway builder thus has fewer choices than the animal trainer. If the stairway were 1.2 metres in height and 2.5 metres in length, then there would be 4 steps and 6 sites for placing them, and the answer would be $C_6^4 = 15$, whereas the animal trainer would, in precisely the same situation, have 43,200 possibilities. This is because he could permute 5 lions in $5! = 120$ ways and 4 tigers in $4! = 24$ ways, making a total of $120 \times 24 = 2{,}880$ ways and $15 \times 2{,}880 = 43{,}200$.

The stairway problem can be stated thus:

In how many ways can n zeros and k ones be placed so that no two ones come together?

Each stairway can be labelled with a sequence of zeros and ones: a zero denoting the place where the polygonal line goes to the right and a one where it goes up. To illustrate, take the stairway in Fig. 8 and label it. We get the sequence 100101001010010. Here, no two ones are adjacent because there are no double-height steps in the stairway. And so the number of sequences of n zeros and k ones in which no two ones occur in succession is equal to the number of stairways, that is C_{n+1}^{k}.

A BOOKSHELF PROBLEM

A bookshelf has 12 books. In how many ways can 5 books be selected so that no two of them stand side by side?

This problem reduces to the one just solved. Label each choice of books by a sequence of zeros and units, namely, mark each remaining book with a 0 and each removed book with a 1. This yields a sequence of 5 units and 7 zeros. And since we are not allowed to take two adjacent books, the sequence will not have two units in succession. But the number of sequences made up of 5 units and 7 zeros, in which no two units appear in succession, is equal to $C_8^5 = 56$.

Generally, if a bookshelf accommodates n books and k books are removed so that no two are adjacent on the shelf, then this can be done in C_{n-k+1}^{k} ways. From this we see that the problem is solvable only when $2k - 1 \leqslant n$.

KING ARTHUR'S ROUND TABLE

Twelve knights are seated at the Round Table. Each one is hostile to his neighbours. Five knights are to be selected to release the enchanted princess.

In how many ways can this be done without including enemy knights?

This problem resembles the bookshelf problem, but differs in that the knights are in a circular arrangement, not a linear arrangement. True, it is easy to reduce the case to one in which the knights occur linearly. To do this, take a knight, say Lancelot. All chosen combinations of knights fall into two classes, one set of combinations including Lancelot, the other excluding him. Let us compute the number of combinations in each of the classes.

If Lancelot sets out to liberate the princess, then neither the man on his left nor the man on his right can participate. There remain 9 knights from which 4 are to accompany Lancelot. Since the knights next to Lancelot do not participate, our sole concern is to see that there are no enemies among the 4 knights, that is, that no two sit together. But by excluding Lancelot and the two knights on each side of him, we break the circle of knights and we can assume what we have to be a linear arrangement instead of a circular one. Now it is possible to choose 4 knights out of 9 in the desired fashion in $C_6^4 = 15$ ways. Which means there are 15 combinations in the first class.

Now we consider the number of combinations that enter into the second class. Since Lancelot does not participate, he is excluded. This breaks the circle of knights and their interrelationships, leaving 11 knights in a line. We have to choose 5 men for the mission, none of whom sit side by side. This can be done in $C_7^5 = 21$ ways. Thus, the total number of ways is equal to $15 + 21 = 36$.

Generally, *if there are n knights at the Round Table and we have to choose k knights so that no two of them sit next to one another, then the problem can be solved in* $C_{n-k-1}^{k-1} + C_{n-k}^{k}$ *ways.*

This assertion is proved in exactly the same way. All combinations of knights are partitioned into two classes depending on whether Lancelot does or does not participate. There will be C_{n-k-1}^{k-1} combinations in which he does participate, and C_{n-k}^{k} combinations in which he does not. It is easy to verify that

$$C_{n-k-1}^{k-1} + C_{n-k}^{k} = \frac{n}{n-k} C_{n-k}^{k}$$

For example, when $n=12$, $k=5$, we get

$$\frac{12}{7} \times C_7^5 = \frac{12}{7} \times 21 = 36$$

SHE'S GOT A DATE

That's the title of a film in which two pleasure seekers arrive at a resort town without any identification papers. The papers are then sent by mail, but the girl at the post office has a date and by mistake switches the two envelopes. It's lucky there aren't five envelopes involved or there would be five grumpy fellows spending the night on park benches.

True, that is not exactly so because she might, just accidentally, put some of the papers in the right envelopes. Let us now compute *in how many*

cases she could produce complete confusion (*not a single person gets his own papers*).

This problem can be stated as follows. *We take all permutations of 5 numbers 1, 2, 3, 4, 5. How many cases do we have in which not a single number occupies its original position*? The solution is carried out on the basis of the principle of inclusion and exclusion (see page 18). Denote by (α) the property of a permutation such that the number α is in its original position, and by N_α denote the number of permutations with this property. In the same way, denote by $N_{\alpha\beta}$ the number of permutations having properties (α) and (β) at the same time, that is such that both α and β occupy their original positions. The notations $N_{\alpha\beta\gamma}$, etc. have the same meaning. Finally, denote by $N^{(0)}$ the number of permutations that do not have a single one of the properties (1), (2,) (3), (4), (5), that is to say, permutations in which not a single number occupies its original position. By the principle of inclusion and exclusion, we have

$$N^{(0)} = N - N_1 - N_2 - N_3 - N_4 - N_5 + N_{12} + \dots$$
$$+ N_{45} - N_{123} - \dots - N_{345} + N_{1,234} + \dots$$
$$\dots + N_{2,345} - N_{12,345} \qquad (1)$$

where $N = P_5$ is the total number of permutations of 5 elements (see page 19).

This problem is simplified by the fact that the properties (1), (2), (3), (4), (5) are all of equal status. It is therefore clear that $N_1 = N_2 = \ldots = N_5$. In the same way, we have $N_{12} = N_{23} = \ldots = N_{45}$ because it is all the same whether 1 and 2 or 3 and 4 remain fixed. But the number of pairs that can be chosen from the numbers 1, 2, 3, 4, 5 is equal to C_5^2 [properties (1, 2) and (2, 1) coincide, and so we are not interested in the order of the numbers chosen in a pair].

Also, we have C_5^3 triples, C_5^4 quadruples and C_5^5 quintuples. And so we can rewrite formula (1) as

$$N^{(0)} = N - C_5^1 N^{(1)} + C_5^2 N^{(2)} - C_5^3 N^{(3)} + C_5^4 N^{(4)} - C_5^5 N^{(5)} \quad (2)$$

For brevity, we use $N^{(k)}$ to denote the number of permutations in which the given k numbers remain fixed. To complete the solution, we have yet to find the values of $N^{(k)}$, $k = 1, 2, 3, 4, 5$.

Use $N^{(1)}$ to denote the number of permutations in which a given number, say 1, remains fixed. But if the number 1 remains fixed, the other numbers can be permuted in $P_4 = 24$ ways. Hence, $N^{(1)} = P_4$. Also, if numbers 1 and 2 remain fixed, then the other three numbers can be permuted in $P_3 = 6$ ways. Thus, $N^{(2)} = P_3 = 6$. Similarly, we find that

$$N^{(3)} = P_2 = 2, \; N^{(4)} = P_1 = 1 \text{ and } N^{(5)} = P_0 = 1$$

Substituting the values of $N^{(1)}$, $N^{(2)}$, $N^{(3)}$, $N^{(4)}$, $N^{(5)}$ thus obtained into formula (2), we get

$$N^{(0)} = P_5 - C_5^1 P_4 + C_5^2 P_3 - C_5^3 P_2 + C_5^4 P_1 - C_5^5 P_0$$
$$= 120 - 5 \times 24 + 10 \times 6 - 10 \times 2 + 5 \times 1 - 1 \times 1 = 44$$

Consequently, in 44 cases out of 120 not a single addressee would have received his identification papers.

Using the same approach, we can find out in how many cases exactly one addressee would get his papers. If this lucky fellow were the first one, then all the other 4 would have the wrong papers. This could occur in

$$P_4 - C_4^1 P_3 + C_4^2 P_2 - C_4^3 P_1 + C_4^4 P_0 = 9$$

ways. But since any one of the addressees could be the lucky one, the total number of ways in which precisely one person gets the letter addressed to him is equal to $5 \times 9 = 45$.

Check for yourself that precisely two persons get their papers in 20 cases, three in 10 cases, four in 0 cases and five in 1 case. The result for four is due to the fact that if four receive their letters, then the fifth letter is correctly addressed too.

To summarize, 120 distinct permutations of 5 elements break up into 44 permutations in which not a single element remains fixed, 45 permutations in which exactly one element remains fixed, 20 permutations in which two elements remain fixed, 10 permutations in which three elements remain fixed, and 1 permutation in which all elements occupy their original positions.

A SESSION IN TELEPATHY

Some believe that it is possible to read a person's thoughts at a distance. To verify this we conduct experiments. In some room a person picks up so-called Zener figures (Fig. 9) in a certain sequence. The telepathist has to guess the order in which these figures are picked up.

We assume that the figures were picked up without repetition. Then the total number of possible permutations of these figures is $5! = 120$. In one session, only one of these permutations is utilized. The telepathist names a different permutation of the figures, and his success is the greater, the larger the number of figures he guesses. From the computations carried out on pages 41 to 42, it follows that random guessing would produce approximately the following results: not a single figure would be guessed in 44 cases out of 120, one figure in 45 cases, two in 20 cases, three in 10 cases and all five figures in one case.

On the average, random guessing turns up the following number of correctly chosen figures:

$$\frac{45+20\times 2+10\times 3+5}{120}=1$$

which is to say, one figure is guessed out of five. For n distinct figures, one out of n will, on the

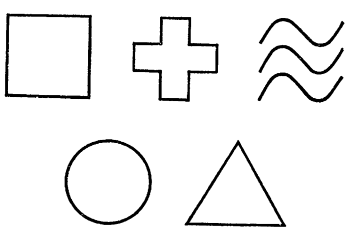


Fig. 9.

average, be guessed correctly. If a trend is revealed in which a larger number of figures is guessed, then a detailed investigation is called for to find out whether the person is cheating (as is often the case) or whether he actually does possess certain capabilities.

Let us see whether there is any change in average correct guessing if repetitions are allowed for. In this case we have permutations with repetitions. But the number of such permutations of n elements in which not a single element is in its "proper" position, is equal to $(n-1)^n$. Indeed, any element (except the first) can occupy the first position; the second position can be occupied by any element except the second one, etc. In other words, there are $n-1$ candidates for every position. By the rule of product, we conclude that the total number of possible combinations is $(n-1)^n$.

Let us find out in how many cases exactly one element occupies its proper position. If it does (say the first), then there are another $n-1$ positions to be occupied, with $n-1$ candidates aspiring to each position (all elements with the exception of the rightful owner, so to say). Hence, the number of permutations in which the first and only the first element is in its proper position is $(n-1)^{n-1}$. But since any one of the n elements can occupy its proper position, the number of permutations where precisely one element has not been displaced is equal to $n(n-1)^{n-1}$. In exactly the same way, we prove that the number of permutations in which exactly k elements have not been displaced is $C_n^k (n-1)^{n-k}$.

For example, in the case of five distinct elements we get the following result: the number of permutations with repetitions in which all elements have been displaced (deranged) is equal to $4^5 = 1{,}024$; there are $5 \times 4^4 = 1{,}280$ permutations in which exactly one element remains fixed, $10 \times 4^3 = 640$ permutations in which exactly two elements remain fixed, $10 \times 4^2 = 160$ in which three remain fixed, $5 \times 4 = 20$ in which four remain fixed, and $1 \times 4^0 = 1$ in which five remain fixed. This yields

$$1{,}024+1{,}280+640+160+20+1=3{,}125$$

permutations, which is in accord with the formula

$$\overline{A}_5^5=5^5=3{,}125$$

Guessing at random, we get a correct answer (on the average) of

$$\frac{1{,}280+640\times 2+160\times 3+20\times 4+1\times 5}{3{,}125}=1$$

The answer is the same: in random guessing we pick one correct figure out of five, irrespective of whether repetitions are allowed for or not. However, the distribution of the number of correctly guessed figures will be different, as we can see from the table below.

Number of correctly guessed figures	Without repetitions	With repetitions
0	0.366	0.328
1	0.375	0.410
2	0.167	0.205
3	0.083	0.051
4	0	0.006
5	0.009	0.000

GENERAL PROBLEM OF DERANGEMENTS
(may be skipped in a first reading)

We solve the general problem of derangements in the same way as the foregoing problems: *find the number D_n of derangements of n elements in which not a single element remains in its original position.* The answer is given by the formula

$$D_n=P_n-C_n^1P_{n-1}+C_n^2P_{n-2}-\dots+(-1)^n C_n^n$$
$$=n!\left[1-\frac{1}{1!}+\frac{1}{2!}-\dots+\frac{(-1)^n}{n!}\right] \qquad (3)$$

The reader familiar with the theory of series will recognize, in the brackets, the partial sum of the expansion of e^{-1}.

Generalizing formula (3) to the case of $n=0$, we find that it is natural to assume $D_0=1$

The number of permutations in which exactly r elements remain fixed and the remaining $n-r$ change their positions, is given by the formula

$$D_{n,\,r}=C_n^rD_{n-r} \qquad (4)$$

Indeed, it is first necessary to choose which r elements remain fixed. This can be done in C_n^r ways. The remaining $n-r$ elements can then be deranged in any way so long as none occupies its original position. This can be done in D_{n-r} ways. By the rule of product, we see that the total number of desired permutations is equal to $C_n^rD_{n-r}$.

Let us partition all permutations into classes depending on the number of elements that remain fixed under a given permutation. Since the total number of permutations is $n!$, we get the identity

$$n!=\sum_{r=0}^{n} D_{n,\,r}=\sum_{r=0}^{n} C_n^rD_{n-r} \qquad (5)$$

A different identity relating $n!$ and the numbers $D_{n,r}$ is obtained in the following manner. Take all $n!$ permutations of the elements $a_1, \dots, a_n$ and compute the number of numbers that remained fixed in all these permutations. This computation can be done in two ways. Firstly, note that if, say, element a_1 is in its position, the remaining elements can be permuted in $P_{n-1}=(n-1)!$ ways. Therefore, element a_1 resides in first position in $(n-1)!$ permutations. So also, element a_2 is in second position in $(n-1)!$ permutations, etc. In all, we obtain $n(n-1)!=n!$ elements holding their positions. But the number of these elements can be computed differently. The number of permutations of class r, that is such that r elements are in their proper places, is equal to $D_{n,r}$. Each such permutation yields r fixed elements. Therefore, the total number of fixed elements in the permutations of class r is equal to $rD_{n,r}$ and in all we get $\sum_{r=0}^{n} rD_{n,\,r}$ fixed elements.

This proves the identity

$$n! = \sum_{r=0}^{n} rD_{n,\,r} = \sum_{r=0}^{n} rC_n^r D_{n-r} \qquad (5')$$

We can solve the following problem by applying the principle of inclusion and exclusion: *find the number of permutations of n elements in which r given elements are deranged* (the remaining elements may either be deranged or fixed in their natural positions). The answer is given by the formula

$$n! - C_r^1 (n-1)! + C_r^2 (n-2)! - \dots \\ \dots + (-1)^r (n-r)! \qquad (6)$$

SUBFACTORIALS
(may be skipped in a first reading)

Some authors use the term *subfactorials* for the numbers D_n. These numbers have much in common with ordinary factorials. For instance, the following equality holds for factorials:

$$n! = (n-1)\,[(n-1)! + (n-2)!] \qquad (7)$$

Indeed,

$$(n-1)\,[(n-1)! + (n-2)!] = (n-1)\,(n-2)!\,n = n!$$

Now we will show that this equation holds true for the subfactorials D_n as well; that is, that

$$D_n = (n-1)\,[D_{n-1} + D_{n-2}] \qquad (8)$$

Replace D_{n-1} and D_{n-2} by their expansions according to (3). Separating the last summand in the expression of D_{n-1}, we find that

$$(n-1)\,[D_{n-1} + D_{n-2}] = (n-1)\,[(n-1)! + (n-2)!] \\ \times \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots \right. \\ \left. \dots + \frac{(-1)^{n-2}}{(n-2)!}\right] + (-1)^{n-1}\,(n-1)$$

But, by (7)

$$(n-1)\,[(n-1)! + (n-2)!] = n!$$

Besides,

$$(-1)^{n-1}\,(n-1) = n!\left[\frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}\right]$$

And so

$$(n-1)\,[D_{n-1} + D_{n-2}] \\ = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} - \dots \right. \\ \left. \dots + \frac{(-1)^{n-2}}{(n-2)!} + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}\right] = D_n$$

Relation (8) that we have just proved can, if we follow Euler, be derived by means of purely combinatorial arguments. Consider all permutations in which all elements have been deranged (derangements). Any element, except the first, can occupy the first position in such derangements. Since the number of remaining elements is $n - 1$, it follows that D_n derangements can be split up into $n - 1$ groups according to the element that occupies the first position. It is clear that all groups will have the same number of elements.

Let us compute the number of elements in one of these groups, say in the group where the first position accommodates the second element. This group is then split into two parts: those with first element in the second position, and all the others. If the first element takes the second position (and the second, as we recall, the first position), then the remaining $n - 2$ elements may be permuted in any fashion so long as not one of them occupies its natural position. This can be accomplished in D_{n-2} ways, which means the first part consists of D_{n-2} derangements.

Let us show that the second part consists of D_{n-1} derangements. Indeed, the second part will include all derangements in which the first element does not stand in the second position, and the remaining elements are deranged. If we temporarily consider the second position "proper" for the first element, then it turns out that the first, third, fourth, . . ., nth elements do not reside in their proper positions. Since there are

$n-1$ such elements, it follows that there are D_{n-1} derangements in the second part. But then the entire group consists of $D_{n-2}+D_{n-1}$ derangements. Since the whole set of derangements consists of the $(n-1)$ st group, it includes $(n-1)\,[D_{n-2}+D_{n-1}]$ derangements. This proves equation (8).

From formula (8) it follows that

$$D_n - nD_{n-1} = -[D_{n-1}-(n-1)\,D_{n-2}]$$

And so as n varies, the expression $D_n - nD_{n-1}$ only changes sign. Applying this relation several times, we find that

$$D_n - nD_{n-1} = (-1)^{n-2}\,[D_2 - 2D_1]$$

But $D_2 = 1$ and $D_1 = 0$, and so

$$D_n = nD_{n-1} + (-1)^n \tag{9}$$

This formula resembles the relation $n! = n\,(n-1)!$ for factorials.

Let us write down the values of the subfactorials for the first 12 natural numbers.

n	D_n	n	D_n	n	D_n	n	D_n
1	0	4	9	7	1,854	10	1,334,961
2	1	5	44	8	14,833	11	14,684,570
3	2	6	265	9	133,496	12	176,214,841

CARAVAN IN THE DESERT

A caravan consists of 9 camels. The journey has been in progress for many days and, finally, everyone is tired of seeing the same camel in front of him. In how many ways can we permute the camels so that each one has a different camel in front of him?

Such permutations surely exist. Say, we could reverse the order of the camels so that the last is first, etc. As the Arab proverb goes, "when the caravan turns around, the lame camel is the leader".

First, label the camels in the original order from the end to the beginning with the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9. The last camel gets No. 1, the second to the last, No. 2, and so on. We have to find all the permutations of the numbers 1 through 9 in which there are no pairs like (1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 9). Now we again apply the principle of inclusion and exclusion.

First compute the number of permutations including the pair (1, 2). In such permutations, we can take the pair to be one element, and so the total number of elements will be 8 instead of 9 and the number of permutations containing (1, 2) is P_8. We get the same result for all 8 pairs.

Now let us consider the permutations containing two given pairs. In this case we combine the elements that enter into each of these pairs. And if both pairs contain one and the same element [say, the pairs (1, 2) and (2, 3)], then we combine all three elements, otherwise [say for (1, 2) and (5, 6)] we combine two elements at a time. In both cases, there will be 7 new elements after the combining process (part of them represent a pair or triple of the original elements) which can be permuted in P_7 ways. Now, two pairs can be chosen from 8 pairs in C^2 ways.

In exactly the same way we prove that the number of permutations containing k given pairs is P_{9-k}. Here, k pairs may be chosen in C_8^k ways. Applying the principle of inclusion and exclusion, we find that the number of permutations not containing a single one of the given pairs is

$$P_9 - C_8^1P_8 + C_8^2P_7 - C_8^3P_6 + C_8^4P_5 - C_8^5P_4 + C_8^6P_3 - C_8^7P_2 + C_8^8P_1 = 8!\left[9 - \frac{8}{1!} + \frac{7}{2!} - \frac{6}{3!} + \frac{5}{4!} - \frac{4}{5!} + \frac{3}{6!} - \frac{2}{7!} + \frac{1}{8!}\right] = 148{,}329$$

In exactly the same way we can prove that the number of permutations of n numbers 1, 2, 3, ..., n that do not contain a single one of the pairs (1, 2), (2, 3), . . ., $(n-1, n)$ is given by the formula

$$E_n = P_n - C_{n-1}^1 P_{n-1} + C_{n-1}^2 P_{n-2} - C_{n-1}^3 P_{n-3} + \dots + (-1)^{n-1} C_{n-1}^{n-1} P_1$$

$$= (n-1)!\left[n - \frac{n-1}{1!} + \frac{n-2}{2!} - \frac{n-3}{3!} + \dots + \frac{(-1)^{n-1}}{(n-1)!}\right] \qquad (10)$$

Let us express the result in terms of subfactorials. To do this, split each summand in the right-hand member into two:

$$\frac{(-1)^k (n-k)}{k!} = \frac{(-1)^k n}{k!} + \frac{(-1)^{k-1}}{(k-1)!}$$

We get

$$E_n = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \dots + \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}\right] + (n-1)!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \dots + \frac{(-1)^{n-2}}{(n-2)!} + \frac{(-1)^{n-1}}{(n-1)!}\right]$$

[we added one term—the last one—in each of the brackets; it is obvious that these terms cancel out since they turn into $(-1)^n$ and $(-1)^{n-1}$ respectively when the brackets are removed]. But the first term is precisely D_n, while the second one is just D_{n-1}. Therefore

$$E_n = D_n + D_{n-1} \qquad (11)$$

Thus, the number of permutations of 1, 2, 3, ... that do not contain a single one of the pairs (1, 2), (2, 3), . . ., $(n-1, n)$ is equal to $D_n + D_{n-1}$.

In exactly the same way we can prove that the number of permutations of n elements that exclude given $r \leqslant n-1$ pairs, is

$$P_n - C_r^1 P_{n-1} + C_r^2 P_{n-2} - \dots + (-1)^r C_r^r P_{n-r} \qquad (12)$$

The answer is different if the number of forbidden pairs is greater than $n-1$. Suppose, for instance, that in addition to the pairs (1, 2), (2, 3), . . ., $(n-1, n)$, the permutations reject the pair $(n, 1)$ as well. Arguing in the same fashion, we find that the answer is given by the formula

$$F_n = P_n - C_n^1 P_{n-1} + C_n^2 P_{n-2} - \dots + (-1)^k C_n^k P_{n-k} + \dots + (-1)^{n-1} C_n^{n-1} P_1 =$$

$$= n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \dots + \frac{(-1)^{n-1}}{(n-1)!}\right] = nD_{n-1} \qquad (13)$$

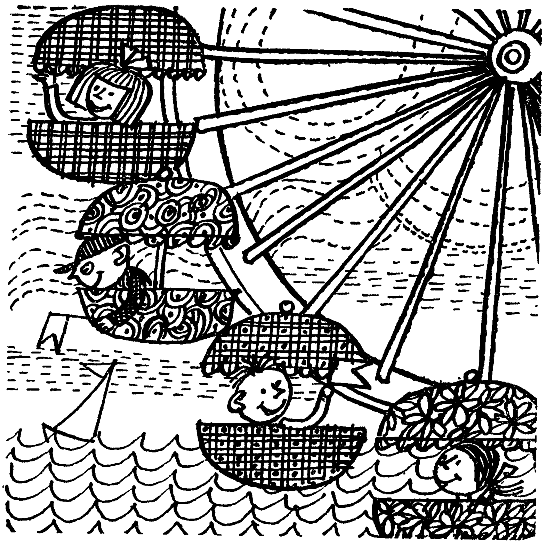
Indeed, in this case the number of forbidden pairs is n, while there is no permutation including all n pairs, because if, say, a permutation includes the pairs (1, 2), (2, 3), . . ., $(n-1, n)$, then the first element is 1 and the last is n, and so the pair $(n, 1)$ does not occur in the permutation. That is why the last term in (13) is $(-1)^{n-1} C_n^{n-1} P_1$ and not $(-1)^n C_n^n P_0 = (-1)^n$.

It would be interesting to substantiate the answer obtained, $F_n = nD_{n-1}$, with a purely combinatorial argument.

MERRY-GO-ROUND

There are n children on a merry-go-round. They decide to change places so that somebody else is in front of each one. In how many ways can they achieve this?

The problem is like the caravan problem we just solved. This time, however, the number of forbidden pairs is equal to n; the following pairs do not occur: (1, 2,) (2, 3), . . ., $(n-1, n)$ and $(n, 1)$. Besides, permutations obtained one from the other by reseating the children circularly will be dropped since they make no difference when the merry-go-round is in motion. Therefore, out of k elements we can generate only $P_{k-1} = (k-1)!$ essentially distinct permutations. Finally, the new problem provides for permutations in which all n pairs are included. Such, for example, is the original permutation. Taking all these circumstances into account, we see,



by applying the principle of inclusion and exclusion, that the number of desired permutations is

$$Q_n = P_{n-1} - C_n^1 P_{n-2} + C_n^2 P_{n-3} - \dots$$
$$\dots + (-1)^{n-1} C_n^{n-1} P_0 + (-1)^n C_n^n \qquad (14)$$

It is easy to verify that this expression may be written as

$$Q_n = D_{n-1} - D_{n-2} + D_{n-3} - \dots + (-1)^{n-3} D_2 \quad (15)$$

True enough, from (14), by virtue of the equality $C_n^k - C_{n-1}^{k-1} = C_{n-1}^k$ it follows that for $n \geqslant 1$

$$Q_n + Q_{n-1} = P_{n-1} - C_{n-1}^1 P_{n-2} + C_{n-1}^2 P_{n-3} - \dots$$
$$\dots + (-1)^n$$

Now this expression is equal to D_{n-1} (see page 45). Thus, $Q_n + Q_{n-1} = D_{n-1}$. Besides, from (14) it follows that $Q_2 = 0$. We thus have

$$\begin{aligned} Q_n + Q_{n-1} &= D_{n-1}, \\ -Q_{n-1} - Q_{n-2} &= -D_{n-2}, \\ Q_{n-2} + Q_{n-3} &= D_{n-3}, \\ &\dots\dots\dots\dots \\ (-1)^{n-3} Q_3 &= (-1)^{n-3} D_2 \end{aligned}$$

Adding these equalities, we arrive at the relation (15).

STANDING IN LINE AT A TICKET OFFICE

There is a line (queue) of $m + k$ persons at the ticket office of a cinema. m have roubles, k have fifty-copeck pieces. A ticket costs 50 copecks and the cashier has no change to begin with. In how many ways can the people line up with roubles and 50-copeck pieces so that the line keeps moving and no one has to wait for change?

For example, if $m = k = 2$, then there will only be two favourable cases: frfr and ffrr, where f stands for 50-copeck coins and r for roubles. In four cases—rrff, rfrf, rffr and frrf—there is a hold-up; in the first three cases, the first person in line does not get his change, and in the last instance, the third in line has change trouble.

When the values of m and k are small, the problem can be solved just by running through the cases. But if m and k are relatively large, this will not do, because, as you will recall, the number of distinct permutations of m roubles and k 50-copeck coins is equal to

$$P(m, k) = \frac{(m+k)!}{m!\,k!}$$

If, say, $m = k = 20$, then

$$P(20, 20) = \frac{40!}{20!\,20!}$$

which is a number exceeding 100,000 million.

Let us derive a formula that expresses the number of desired combinations in terms of m and k. What we need is to find the number of permutations of m r's and k f's having the following property: for any d, $1 \leqslant d \leqslant m + k$, the number of f's in the first d terms of the permutation is not less than the number of r's (the f's must not be fewer than the r's, for otherwise the line would come to a halt).

It is clear that for the problem to be solvable, it is necessary that the condition $m \leqslant k$ be met, otherwise there will be a stop due to a lack of 50-copeck pieces for change for those who have only roubles. We therefore assume that $0 \leqslant m \leqslant k$.

As in certain other combinatorial problems, it is better, here, to seek the number of "unfavourable" cases, that is, cases where there is an interruption in the movement of the line due to a lack of change. If we find that number, then, subtracting it from the number $P(m, k) = C_{m+k}^{m}$ of all permutations of m r's and k f's, we will have the answer to our problem.

Let us first prove the following assertion: the number of unfavourable cases for permutations of m r's and k f's is equal to $P(m-1, k+1) = C_{m+k}^{m-1}$, that is, to the number of all permutations of $m-1$ r's and $k+1$ f's. The proof is this. Take any unfavourable permutation of m r's and k f's. Let the line stop moving at some point. Then up to this point there will be an identical number of f's and r's (all 50-copeck coins will have gone to rouble owners), and we have the letter r standing here (otherwise the line would continue to move).

The position at which the hold-up occurred is of the form $2s+1$; in front of this position there are s f's and s r's. Now place the letter f in front of our permutation (if there is agitation in the line, we say that this is being done to get change). We have a permutation of m r's and $k+1$ f's, the first letter of the permutation being f and the same number of r's and f's among the first $2s+2$ letters (there were s f's and $s+1$ r's, we added one f and evened the score).

We now undertake an operation that will displease the rouble owners and please the 50-copeck piece owners: in the first $2s+2$ positions, we will change the r's to f's and the f's to r's. To illustrate, suppose the line was of this form,

frfrfrrfrrffrffr

It will stop at the bold-face **r**. After an f is placed in front, and the above-described operation is completed, the line looks like this:

rrrfrfrffrffffrffr.

Since in the first $2s+2$ positions there were identical numbers of roubles and 50-copeck pieces, there will be no change in the total number of coins of each kind, and we get a permutation of m r's and $k+1$ f's. This time the first letter is r. Thus, we associated with each "unfavourable" sequence of m r's and k f's a sequence of m r's and $k+1$ f's beginning with the letter r.

We will now demonstrate that in this way it is possible to obtain any sequence of m r's and $k+1$ f's beginning with the letter r. Indeed, take such a sequence. Since we assume that $m \leqslant k$, there will be a position at which the letters f and r even out. If, up to this position, we replace all f's by r's and all r's by f's and drop the first letter f, then we get an unfavourable arrangement of roubles and 50-copeck pieces in the line. The line will stop moving at precisely that position where, in the given sequence, the number of letters f and r is the same for the first time.

We have thus established that *the number of unfavourable distributions of roubles and 50-copeck* *pieces in a line is exactly equal to the number of all permutations of* m r's *and* $k+1$ f's, *the permutations beginning with the letter* r. If we discard the first letter, we get all possible permutations of $m-1$ r's and $k+1$ f's. Now the number of such permutations is

$$P(m-1,\ k+1)=C_{m+k}^{m-1}$$

Thus, the number of unfavourable permutations is equal to C_{m+k}^{m-1}. Since the number of all permutations of m r's and k f's is C_{m+k}^{m}, the number of favourable permutations is given by the formula

$$C_{m+k}^{m}-C_{m+k}^{m-1}=\frac{k-m+1}{k+1}\,C_{m+k}^{m} \tag{16}$$

In particular, if $k=m$, i.e. if there is the same number of roubles and 50-copeck pieces in the line, then it will move on in $\frac{1}{k+1}\,C_{2k}^{k}$ cases and will come to a halt in $\frac{k}{k+1}\,C_{2k}^{k}$ cases. Thus, the greater k is, the smaller the percentage of favourable cases.

That completes the solution of our problem. Let us now consider a related problem. Namely, *assume that the cashier had foreseen trouble and has put* q *50-copeck pieces in the register at the start. In how many cases will a line move without interruption if it consists of* m *rouble-owners and* k *50-copeck piece owners*?

It is clear that if $m \leqslant q$, then the line will most likely go smoothly since the reserve of 50-copeck pieces in the register at the start will suffice to satisfy all change-giving operations. But if $m > k+q$, then the line is sure to come to a halt: there will not be enough 50-copeck coins in the cash register and in the line to handle the situation and give change to all those with roubles. We can therefore confine ourselves to a consideration of the case when

$$q < m \leqslant k+q$$

We further assume that q 50-copeck pieces appeared in the cash register because q new persons with 50-copeck pieces were placed at the head of the line. The problem can therefore be restated as follows:

There are $k+q$ *persons in line with 50-copeck pieces and* m *persons with roubles, the first* q *positions being occupied by those with 50 copecks. In how many cases will no one have to wait for change*?

This problem can be solved in the same way as the above particular case for $q=0$. We shall seek the number of unfavourable cases. In each such case, the interruption will occur at the person holding a rouble, in front of whom are the same number of s roubles and 50-copeck pieces. Now put one person with 50 copecks at the head of the line and give roubles for 50-copeck pieces to the first $2s+2$ persons in line (also replace roubles with 50-copeck pieces). What we get is a permutation of m roubles and $k+q+1$ 50-copeck pieces, the first $q+1$ positions being roubles. Here, any such permutation can be uniquely obtained from the unfavourable arrangement of roubles and 50-copeck pieces. But the first $q+1$ roubles can be dropped and then we get all possible permutations of $m-q-1$ roubles and $k+q+1$ 50-copeck pieces. Now the number of such permutations is $P(m-q-1,\ k+q+1)=C_{m+k}^{m-q-1}$. We have proved that in the problem at hand there are C_{m+k}^{m-q-1} unfavourable permutations. And since the total number of permutations is C_{m+k}^{k}, the number of favourable permutations is given by

$$C_{m+k}^{m}-C_{m+k}^{m-q-1} \tag{17}$$

The foregoing approach enables one to solve many other problems. For example, using it, we readily obtain the following results.

If $m<k$, *then the number of permutations of* m r's *and* k f's *such that in front of each letter (except the first) there are more* f's *than* r's, ***is equal to***

$$C_{m+k-1}^{m}-C_{m+k-1}^{m-1}=\frac{k-m}{k}\,C_{m+k-1}^{m} \tag{18}$$

The reasoning is the same as before, only we do not need to add an f at the beginning.

This formula holds true for $m < k$. But if $m = k$, then the number of permutations with the indicated property is equal to $\frac{1}{k} C_{2k-2}^{k-1}$. This is easy to see. Each such permutation must begin with the letter f and terminate with r. If we drop these letters, we get a permutation of $k-1$ r's and $k-1$ f's. The line will clearly move along without a hitch for that permutation. Conversely, a permutation with the property we need is obtained from each permutation of $k-1$ f's and $k-1$ r's, for which the line moves without stopping, by adding the letter f at the beginning and the letter r at the end. But the number of permutations of $k-1$ f's and $k-1$ r's for which the line moves without stopping is precisely $\frac{1}{k} C_{2k-2}^{k-1}$.

THE PROBLEM OF THE TWO RANKS

It often happens in combinatorial mathematics that two disparate, at first glance, problems reduce to one another. Consider the following problem.

In how many ways is it possible to arrange $2n$ persons of different height in two ranks of n persons each so that in each rank they stand according to height and, besides, so that each man in the first rank is taller than the man behind him in the second rank?

We will demonstrate that the solution of this problem reduces to the problem already solved of the ticket line. Put all persons in two ranks as required and give a 50-copeck piece to each one in the first rank and a rouble to each one in the second rank and then arrange them by height in a single file. We get a line of n holders of 50-copeck pieces and n holders of roubles. From the statement of the problem, it follows that the line moves without interruption. Indeed, suppose someone takes up the kth position in the second rank. Then there will only be $k-1$ taller persons among the rouble-holders. Among the holders of 50-copeck pieces there will be at least k persons taller (the one in front and all persons to the right). So when he comes up to the cashier, there will be at least one 50-copeck piece available, and he will get his change.

Conversely, suppose we have an arrangement of n persons with 50-copeck pieces and n persons with roubles in which the line proceeds without stopping. Without any loss of generality, we can take it that all $2n$ persons are standing according to height. Now select all holders of 50-copeck pieces and put them according to height in the first rank, and rouble-holders in the second rank. We leave it to the reader to verify that the resulting arrangement satisfies the hypothesis. From this it follows that there are as many possible arrangements as there are favourable permutations of n f's and n r's, that is, $\frac{1}{n+1} C_{2n}^{n}$.

NEW PROPERTIES OF COMBINATIONS (may be skipped in a first reading)

The formulas developed in the preceding sections enable us to establish some more properties of the number of combinations C_m^k (see page 33). To do this split into classes all the "unfavourable" permutations of m r's and k f's. We have seen that for such permutations the line comes to a halt at the position with the number $2s+1$; note that in front of it are s r's and s f's, the position itself being occupied by the letter r and the line moving without interruption up to that position. Put in the sth class all unfavourable permutations for which there is a halt at the position $2s+1$. Clearly, s can assume the values $0, 1, 2, \ldots, m-1$.

Find the total number of permutations which enter into the sth class. In the first $2s$ positions there can be any favourable permutations of s r's and s f's; this is because the line does not stop until the position $2s+1$ is reached. We have seen that the number of such permutations is $\frac{1}{s+1}C_{2s}^{s}$. At the position $2s+1$ we have r, after which comes any permutation of the remaining $m-s-1$ r's and $k-s$ f's. The number of these permutations is $P(m-s-1,\ k-s)=$ $=C_{m+k-2s-1}^{m-s-1}$. Thus, by virtue of the rule of product the number of unfavourable permutations of the sth class is

$$\frac{1}{s+1}C_{2s}^{s}C_{m+k-2s-1}^{m-s-1}$$

Since the total number of unfavourable permutations is C_{m+k}^{m-1} and the number of classes is equal to $m-1$, we get, for $m\leqslant k$, the relation

$$C_0^0C_{m+k-1}^{m-1}+\frac{1}{2}C_2^1C_{m+k-3}^{m-2}+\frac{1}{3}C_4^2C_{m+k-5}^{m-3}+\dots$$
$$+\frac{1}{m}C_{2m-2}^{m-1}C_{k-m+1}^{0}=C_{m+k}^{m-1}\qquad(19)$$

This relation is a particular case of the formula

$$\sum_{s=p}^{m-1}[C_{2s-p}^{s}-C_{2s-p}^{s-p-1}]\,C_{m+k+p-2s-1}^{m-s-1}=C_{m+k}^{m-p-1}\qquad(20)$$

where $p<m\leqslant p+k$ (in the first summand, C_s^{-1} is taken to be zero). Formula (20) is proved in the same way as (19), by partitioning into classes the unfavourable permutations of m r's and $k+p$ f's, in which there are p r's at the beginning (see page 50).

Let us now consider the relations obtained by splitting into classes the *favourable* permutations made up of k r's. There are $\frac{1}{k+1}C_{2k}^{k}$ such permutations. After the whole line has passed through, there will again be no 50-copeck pieces in the cash register, all having been used up for change. However, in certain favourable permutations there occur situations in which the cashier has no change; it so happens that the next in line saves the situation by handing in a 50-copeck piece. Let us split up all favourable permutations into classes, putting in the sth class all permutations in which the cashier does not have a single 50-copeck piece for the first time at the position $2s$, $s=1,2,\dots,k$.

Let us find the number of permutations in the sth class. Each such permutation splits up into two parts. The first $2s$ letters form a permutation of s f's and s r's such that each letter is preceded by more f's than r's (otherwise the equalization would have occurred before the position $2s$). We have seen that there are $\frac{1}{s}C_{2s-2}^{s-1}$ such permutations (see page 51). After the sale of the first $2s$ tickets the cashier has no change. Thus, if the line is to move without interruptions, the last $k-s$ r's and the $k-s$ f's must form a favourable permutation. But there are $\frac{1}{k-s+1}C_{2k-2s}^{k-s}$ such permutations (see page 50). By the rule of product, we see that there are

$$\frac{1}{s(k-s+1)}C_{2s-2}^{s-1}C_{2k-2s}^{k-s}$$

permutations in the class. And since the total number of favourable permutations is $\frac{1}{k+1}C_{2k}^{k}$, we obtain the identity

$$\sum_{s=1}^{k}\frac{k+1}{s(k+s-1)}C_{2s-2}^{s-1}C_{2k-2s}^{k-s}=C_{2k}^{k}\qquad(21)$$

If we introduce the notation

$$\frac{1}{s+1}C_{2s}^{s}=T_s$$

formula (21) becomes

$$T_0T_{k-1}+T_1T_{k-2}+\dots+T_{k-1}T_0=T_k\qquad(22)$$

Another relation between the numbers C_n^m is obtained as follows. Specify the number l, $1\leqslant l\leqslant m$, and partition the set of all favourable permutations into classes such that the sth class has all permutations containing exactly s r's among the first l elements. Then the number of f's among the first l elements is equal to $l-s$. Since there must be at least as many f's as r's, it follows that s satisfies the inequalities $0 \leqslant 2s \leqslant l$.

Let us find the number of permutations in the sth class. Each such permutation splits up into two parts: one contains the first l letters, the other, the last $k+m-l$ letters. The first part includes $l-s$ f's and s r's. Here, since the entire permutation is favourable, so also is its part consisting of the first l letters. But using $l-s$ f's and s r's, we can form $\frac{l-2s+1}{l-s+1} C_l^s$ such permutations.

After the first part of the permutation has passed, the cashier will have $l-2s$ 50-copeck pieces. The second part of the permutation consists of $k-l+s$ letters f and $m-s$ letters r. The number of permutations under which this part of the line moves without stopping is computed from formula (17) (see page 50), in which it is necessary to replace q by $l-2s$, m by $m-s$ and k by $k-l+s$. From this formula it follows that the second part of the permutation can be chosen in $C_{m+k-l}^{m-s} - C_{m+k-l}^{m+s-l-1}$ ways. By the rule of product, we find that the number of permutations in the sth class is

$$\frac{l-2s+1}{l-s+1} C_l^s \left[C_{m+k-1}^{m-s} - C_{m+k-l}^{m+s-l-1}\right]$$

Since the total number of favourable permutations of k f's and m r's is $\frac{k-m+1}{k+1} C_{m+k}^m$, we obtain the identity*

$$\sum_{s=0}^{E\left(\frac{l}{2}\right)} \frac{l-2s+1}{l-s+1} C_l^s \left[C_{m-k-l}^{m-s} - C_{m+k-l}^{m+s-l-1}\right] = \frac{k-m+1}{k+1} C_{m+k}^m \qquad (23)$$

(Here, C_r^p is zero for $p<0$.) The reader will find it easy to derive similar relations establishing various methods of partitioning permutations into classes.

* $E\left(\frac{l}{2}\right)$ denotes the greatest integer in $\frac{l}{2}$.

In the preceding problems involving permutations and combinations of given elements we constructed various combinations and computed the number of such combinations under certain restrictions. We were hardly ever interested in the fate of the elements that remained after choosing the combinations. The problems we are about to investigate are of a different nature. Here the elements are separated into two or more groups and we have to find all possible ways of such partitioning.

A variety of cases arise. At times, the order of the elements in the groups plays an essential role: say, when signal flags are hung on masts, the order in which the flags are arranged is essential—and not only what flag is on which mast. In other cases, the order of the elements in the groups is completely irrelevant. When a domino player takes pieces from a pile, the order in which he draws them is immaterial; the important thing is the final result.

Problems also vary as to the role played by the order of the groups themselves. In dominoes, the players sit in a specific order; both how the pieces are distributed and who gets which pieces are of importance. If I am putting photographs into identical envelopes to send to a friend, the distribution of photos among the envelopes is important but not the order of the envelopes themselves because they will be mixed up at the post-office in any case.

Another factor of importance is whether we distinguish the elements among themselves or not, and whether we distinguish between the groups into which the elements fall. Finally, in some problems there will be empty groups (not containing a single element), in others such groups will be inadmissible. All this gives rise to a variety of *combinatorial problems involving partitions.*

DOMINOES

In a game of dominoes, 4 players split 28 pieces equally. In how many ways can this be done?

The pieces (men) can be divided as follows. First put all 28 pieces in a row. The first player then takes the first 7, the second the next 7, the third the following 7, and the fourth player takes the remaining 7. This is clearly a way to obtain all possible partitions of the pieces.

Since the number of permutations of 28 elements is 28!, it might appear that the total number of all ways of splitting up the dominoes is equal to 28!. This is not so because for the first player it is immaterial whether he picks up a double 6 first or a 3 : 4 domino; he is only interested in the final result. For this reason, any permutation of the first 7 pieces is of no consequence. The same goes for any permutation of the next batch of 7 pieces, the third batch and the last batch as well. By the rule of product, we have $(7!)^4$ permutations of the pieces that do not alter the result of the partition.

Thus, 28! permutations of the dominoes are divided into groups of $(7!)^4$ permutations in each group; permutations of any one group lead to the same distribution of the pieces. Hence, the number of ways of dividing the pieces is $\frac{28!}{(7!)^4}$. This number is approximately equal to 4.7×10^{15}.

The same result may be obtained by a different approach. The first player selects 7 pieces out of 28. Since the order of the pieces is immaterial, he has C_{28}^{7} choices. Then the second player chooses 7 pieces out of the remaining 21 pieces. He can do that in C_{21}^{7} ways. The third player chooses from 14 dominoes and so has C_{14}^{7} opportunities of choice. Finally, the fourth player takes C_{7}^{7}, which is the only choice left.

By the rule of product, we find that the total number of possibilities is

$$C_{28}^{7}C_{21}^{7}C_{14}^{7}C_{7}^{7}=\frac{28!}{21!\,7!}\times\frac{21!}{14!\,7!}\times\frac{14!}{7!\,7!}=\frac{28!}{(7!)^4}$$

The proof is exactly the same for the case when in the card game of preference, 32 cards are dealt to three players, each getting 10, and two cards for the widow. The number of distinct deals is

$$\frac{32!}{10!\ 10!\ 10!\ 2!} = 2{,}753{,}294{,}408{,}504{,}640$$

The reader may ask whether it is worth wasting time on the study of card games. If so, he will do well to recall that the study of games of chance served as the impetus to the initial development of combinatorial mathematics and the theory of probability. Such outstanding mathematicians as Pascal, Bernoulli, Euler and Chebyshev sharpened the ideas and methods of combinatorics and probability theory on problems involving coin tossing, dice and cards. Many of the ideas of the theory of games (a division of mathematics which finds wide application in economics and military affairs) originated in studies of the most elementary models of card games.

PLACING OBJECTS INTO CELLS

Problems involving dominoes and preference are in the class of combinatorial problems that deal with placing objects into cells. The general statement of such problems is:

Given n distinct objects and k cells, place into the first cell n_1 objects, into the second n_2 objects,. . . . . ., into the kth cell, n_k objects, where $n_1 + n_2 + \dots + n_k = n$. In how many ways can such a distribution be accomplished?

In the domino problem, the players were cells and the pieces were the objects. Reasoning in the same fashion, we get the answer in the general form: the number of distinct ways of placing objects into cells is

$$P(n_1, n_2, \dots, n_k) = \frac{n!}{n_1!\ n_2! \dots n_k!} \qquad (1)$$

This formula cropped up earlier when we were solving what appeared to be quite a different problem:

There are k distinct types of objects. How many distinct permutations can be made up out of n_1 objects of the first type, n_2 objects of the second type, . . ., n_k objects of the kth type?

Here too, the answer was given by the formula

$$P(n_1, n_2, \dots, n_k) = \frac{n!}{n_1!\ n_2! \dots n_k!}$$

where $n = n_1 + n_2 + \dots + n_k$ (see page 26). To see how these problems are related, number all n positions that our objects can occupy. Each permutation is associated with a distribution of the number labels of the places into k classes. The first class contains the labels of those positions occupied by objects of type one, the second class, the labels of positions of objects of type two, etc. We have thus established a correspondence between permutations with repetitions and the placing of number labels of positions into "cells". Naturally, the formulas for solving both problems are the same.

A BOUQUET OF FLOWERS

In the problem of placing objects into cells we considered as known the quantity of objects going into each cell (for instance, the number of pieces that each player takes). In most problems involving the division of objects, these numbers are not indicated.

Two girls have picked 10 bluebells, 15 bachelor's buttons and 14 forget-me-nots. In how many ways can they divide the flowers?

It is clear that the bluebells can be divided in 11 ways: the first girl can either take no bluebells at all, or 1 or two etc. up to 10 bluebells. In the same way, the bachelor's buttons can be divided in 16 ways, the forget-me-nots in 15 ways. Since the flowers of each kind can be divided independently of the flowers of another kind, we get, by the rule of product, $11 \times 16 \times 15 = 2{,}640$ ways of dividing up the flowers.

True, there will be some very unfair divisions, like when one of the girls gets no flowers at all. Let us therefore introduce a restriction, say that each of the girls is to receive at least 3 flowers of each kind. Then the bluebells can only be divided in five ways, the first girl taking 3, 4, 5, 6 or 7 flowers; the bachelor's buttons in 10 ways, the forget-me-nots in 9 ways. In this case, the flowers can be divided in $5 \times 10 \times 9 = 450$ different ways.

Generally, if we have n_1 objects of one kind, n_2 objects of another kind, . . ., n_k objects of a kth kind, then they can be divided between two people in

$$(n_1+1)(n_2+1)\ldots(n_k+1) \tag{2}$$

ways. In particular, if all the objects are distinct and their number is k, then $n_1 = n_2 = \ldots = n_k = 1$ and so there are 2^k modes of distribution.

But if a restriction is imposed that each of the participants in the distribution is to get at least s_1 objects of the first kind, s_2 of the second kind, . . ., s_k of the kth kind, then the total number of ways of dividing the objects is given by the formula

$$(n_1-2s_1+1)(n_2-2s_2+1)\ldots(n_k-2s_k+1) \tag{3}$$

We leave it to the reader to prove these assertions.

THE NUMBER-OF-DIVISORS PROBLEM

Formula (2) derived above enables us to solve the following problem which appears in the theory of numbers.

Find the number of divisors of a natural number N. To do this, factor N into prime factors: $N = p_1^{n_1}p_2^{n_2}\ldots p_k^{n_k}$ where $p_1, \ldots, p_k$ are distinct prime numbers. For instance, $360 = 2^3 \times 3^2 \times 5$. When N is factored into two factors, $N = N_1N_2$, the prime factors will be distributed between N_1 and N_2. If factor p_j appears in N_1 a total of m_j times, $j = 1, \ldots, k$, then the factorization is of the form

$$N=(p_1^{m_1}\ldots p_k^{m_k})(p_1^{n_1-m_1}\ldots p_k^{n_k-m_k})$$

Thus, the factorization of N into two factors reduces to dividing n_1 elements of one kind, n_2 elements of a second kind, . . ., and n_k elements of a kth kind into two parts. Now, formula (2) shows that this can be done in $(n_1+1)\ldots(n_k+1)$ ways. Hence, the number of divisors of the natural number $N = p_1^{n_1}\ldots p_k^{n_k}$ is equal to $(n_1+1)\ldots(n_k+1)$. We denote this number by $\tau(N)$.

PICKING APPLES

Three boys picked 40 apples. In how many ways can they divide them if all the apples are considered to be the same (that is, we are only interested in the number that each boy gets and not in the quality of the apples, etc.)?

We solve this problem by adding two identical pears and then permuting the 40 apples and 2 pears in all possible ways. Using the formula for permutations with repetitions, we get

$$P(40, 2) = C_{42}^2 = \frac{42!}{40!\,2!} = 861$$

permutations. But each permutation is associated with its mode of dividing the apples. The first boy gets all the apples from the first to the first pear, the second boy gets the apples lying between the first and second pear, and the third boy takes the apples that follow the second pear. The various permutations correspond to different ways of dividing the apples, of which there are thus 861 ways. True, it may turn out that one (or even two) of the boys will get no apples at all. For instance, this will happen if one of the pears comes at the beginning of a permutation, then the first boy loses out; if this occurs at the end, the third boy doesn't get any apples. And if the two pears come together, the second boy is the complete loser. Now figure out what happens if both pears come at the beginning or at the end.

In exactly the same way we prove that n identical objects can be partitioned in

$$P(n, k-1) = C_{n+k-1}^{n} = C_{n+k-1}^{k-1} \tag{4}$$

ways among k persons.

Now suppose, for the sake of equity, it is agreed that each participant gets at least r objects. Here, we begin by giving r objects to each person. There will then remain $n - kr$ objects which can now be distributed in arbitrary fashion, which, as we have alr ady seen, can be done in $C_{n-kr+k-1}^{k-1} = C_{n-k(r-1)-1}^{k-1}$ ways.

In particular, if each of the k participants is to receive at least one object, then the problem is solved in C_{n-1}^{k-1} *ways.*

The latter result may be derived differently. Let us arrange the given n objects in a row. There will then be $n - 1$ gaps. If arbitrary $k - 1$ of these gaps are filled by separating barriers, then all objects will be divided into k nonempty parts. Then the first part is given to the first person, the second to the second person, etc. Since it is possible to place $k - 1$ barriers in $n - 1$ gaps in C_{n-1}^{k-1} ways, the number of modes of division is C_{n-1}^{k-1}.

HUNTING MUSHROOMS

If objects of different kinds are being divided, then we have to find the number of modes of division for each kind, and then multiply the numbers. Solve the following problem.

In how many ways is it possible to divide 10 meadow mushrooms, 15 fairy-rings and 8 chanterelles among 4 children?

Using the results of the preceding section, we see that the answer looks like this

$C_{13}^{3}C_{18}^{3}C_{11}^{3} = 41{,}771{,}040$

Now if each one of the kids gets at least one mushroom of each kind, then we have

$C_{9}^{3}C_{14}^{3}C_{7}^{3} = 1{,}070{,}160$

When n distinct objects are divided among k persons without restrictions, then each object may be delivered in k ways (by giving it to one of the participants). The number of solutions is therefore k^n.

By way of illustration, 8 different cakes can be divided among 5 persons in $5^8 = 390{,}625$ ways.

MAILING PHOTOGRAPHS

I want to send a friend of mine 8 different photographs. In how many ways can I do it if I use 5 different envelopes?

This problem is similar to the one discussed at the end of the preceding section. The answer would seem to be $5^8 = 390{,}625$. But we will not accept empty envelopes, and so a fresh restriction is imposed—there must be no empty envelopes. To take into account this restriction, let us apply the principle of inclusion and exclusion (the answer C_{n-1}^{k-1} is wrong because the photographs are different).

Let us first find out in how many cases a given set of r envelopes are empty (the others may be empty or may contain photographs). In this case, the photos are placed without restriction into $5 - r$ envelopes and, by what has been proved above, the number of such distributions is $(5 - r)^8$.

But we can choose r envelopes out of five in C_5^r ways. Whence, using the principle of inclusion and exclusion, we conclude that the number of distributions (no envelopes empty) is

$$5^8 - C_5^1 \times 4^8 + C_5^2 \times 3^8 - C_5^3 \times 2^8 + C_5^4 \times 1^8 = 126{,}020$$

In the very same way we can prove that if we are sending n different photographs in k distinct envelopes, not a single envelope being empty, then the number of modes of distribution is expressed by the formula

$$k^n - C_k^1 (k-1)^n + C_k^2 (k-2)^n - \dots + (-1)^{k-1} \times \times C_k^{k-1} \times 1^n \qquad (5)$$

The reader should be able to handle the following problem.

There are n_1 objects of one kind, n_2 objects of a second kind, . . ., n_s objects of an sth kind. In how many ways can they be distributed to k persons so that each person gets at least one object?

The answer is

$$C_{n_1+k-1}^{k-1} C_{n_2+k-1}^{k-1} \dots C_{n_s+k-1}^{k-1} - C_k^1 C_{n_1+k-2}^{k-2} \times \times C_{n_2+k-2}^{k-2} \dots C_{n_s+k-2}^{k-2} + C_k^2 C_{n_1+k-3}^{k-3} \times \times C_{n_2+k-3}^{k-3} \dots C_{n_s+k-3}^{k-3} - \dots + (-1)^{k-1} C_k^{k-1} \qquad (6)$$

For example, if we divide 8 apples, 10 pears and 7 oranges among 4 children, each receiving at least one item, then the division is possible in

$$C_{11}^3 C_{13}^3 C_{10}^3 - C_4^1 C_{10}^2 C_{12}^2 C_9^2 + C_4^2 C_9^1 C_{11}^1 C_8^1 - C_4^3 = 5{,}464{,}800$$

ways.

FLAGS ON MASTS

Up to now we have not considered the order in which the elements of each part are arranged. Some problems require that the order of the elements be taken into account.

There are n distinct signal flags and k masts to attach them to. The meaning of a signal depends on the order in which the flags are arranged. In how many ways can they be arranged if all flags are used but some of the masts may be empty?

Each mode of hanging the flags can be accomplished in two stages. In the first stage, we permute the given n flags in all possible ways. This can be done in $n!$ ways. Then we take one of the modes of distribution of n identical flags among k masts (recall that the number of such modes is C_{n+k-1}^{k-1}). Suppose this mode consists in arranging n_1 flags on the first mast, n_2 flags on the second, . . ., n_k flags on the kth, where $n_1 + n_2 + \dots + n_k = n$. Then we take the first n_1 flags of the given permutation and hang them on the first mast in the order obtained; then the next n_2 flags on the second mast, etc. Clearly, if we use all permutations of n flags and all modes of distributing n identical flags among k masts, we will get all ways of solving this problem. By the rule of product, we find the number of ways of arranging the flags to be

$$n!\, C_{n+k-1}^{k-1} = \frac{(n+k-1)!}{(k-1)!} = A_{n+k-1}^n \qquad (7)$$

Generally, *if we have n distinct things, then the number of ways of distributing them into k distinct boxes, with regard for order of arrangement in the boxes, is A_{n+k-1}^n.*

The same result can be attained in a different manner. Add to the n things being distributed $k-1$ identical balls and consider all possible permutations of the resulting $n+k-1$ objects. Each permutation determines one of the distribution modes. Namely, the first box receives all objects in a permutation up to the first added ball (if the first object in the permutation is one of the added balls, then the first box remains empty). Then put in the second box all objects between the first and second ball, . . ., in the kth box, all objects following the $(k-1)$th ball. We then obviously have all distributions of the objects having the indicated properties. However, the number of permutations of n distinct objects and $k-1$ identical balls is

$$P(\underbrace{1, 1, \dots, 1}_{n \text{ times}}, k-1) = \frac{(n+k-1)!}{1! \dots 1!\,(k-1)!} = A_{n-k-1}^n$$

The solution is similar in the problem where every mast must have at least one flag (or, what is the same thing, each box must contain at least one object). Using the formula derived on page 57, we find that in this case we have $n!\,C_{n-1}^{k-1}$ modes of distribution. This same result may also be obtained by choosing division points among $n-1$ gaps.

TOTAL NUMBER OF SIGNALS

Up to now we assumed that all flags have to be used for the transmission of a signal. But some signals may require only a portion of the flags and some of the masts may be empty. *Let us find the total number of signals that can be transmitted by means of n signal flags hung on k masts.*

Partition the signals into classes according to the number of flags participating in a signal.

By formula (7), using s given flags we can transmit A_{s+k-1}^{s} signals (the number of masts is k). But there are C_n^s ways of choosing s flags out of n. Therefore the number of signals in the sth class is $C_n^s A_{s+k-1}^s$. Hence the total number of signals is given by the formula

$$C_n^0 A_{k-1}^0 + C_n^1 A_k^1 + C_n^2 A_{k+1}^2 + \ldots + C_n^n A_{n+k-1}^k \qquad (8)$$

For example, using 6 distinct flags on 3 masts, we can transmit

$$1 + C_6^1 A_3^1 + C_6^2 A_4^2 + C_6^3 A_5^3 + C_6^4 A_6^4 + C_6^5 A_7^5 + C_6^6 A_8^6 = \\ = 42{,}079$$

signals.

If no masts are allowed to be empty, then in place of (8) we have

$$C_n^k C_{k-1}^{k-1} k! + C_n^{k+1} C_k^{k-1}\,(k+1)! + C_n^{k+2} C_{k+1}^{k-1} \times \\ \times (k+2)! + \ldots + C_n^n C_{n-1}^{k-1} n! \qquad (9)$$

ways.

PARTICLE STATISTICS

Problems of placing objects into cells are of extreme importance in statistical physics, which deals with the distribution of physical particles according to properties: for instance, what part of the molecules of a given gas have such and such a velocity for a given temperature? Here, the set of all possible states is distributed over a large number k of tiny cells (phase states) so that each of the n particles lands in a cell.

The question of what particles obey what statistics depends on the particle type. In the classical statistical physics developed by Maxwell and Boltzmann, the particles are considered to be distinguishable. Molecules obey this statistics. We know that n distinct particles can be distributed into k cells in k^n ways. If for a given energy, all these k^n ways are of equal probability, then we speak of the *Maxwell-Boltzmann statistics.*

It turns out that not all physical entities obey this statistics. Photons, atomic nuclei and atoms containing an even number of elementary particles obey a different statistics developed by Einstein and Bose (of India). *In the Bose-Einstein statistics,* the particles are indistinguishable. Therefore, the only important thing is how many particles reside in a cell and not the type of particles there. This problem is similar to the apple-dividing problem (see page 56). We already know that in such an approach, we have $C_{n+k-1}^{k-1} \equiv C_{n+k-1}^{n}$ distinct modes of partitioning the elements. In the Bose-Einstein statistics, all these modes are considered to be of equal probability.

However, many particles, such as, for example, electrons, protons and neutrons, do not obey the Bose-Einstein statistics. Here, a single cell accommodates at most one particle and distinct distributions satisfying this condition have the same probability. In this case there can be C_k^n distinct distributions. This statistics is called the *Fermi-Dirac statistics.*

PARTITIONS OF INTEGERS

In most of the problems considered above, the objects to be divided were distinct. We shall now examine problems in which all the objects undergoing division are the same. We will now speak not of dividing objects but of partitioning positive integers into parts (summands which, of course, are also positive integers—the natural numbers).

Here we have a great variety of problems. In some we have regard for the order of the integers, in others, we disregard the order. We may consider partitions into an even number of parts or only into an odd number of them, into distinct summands, or into arbitrary summands, etc. The basic method for solving partition problems is to reduce them to problems of partitioning smaller integers or of partitioning into a smaller number of summands.

MAILING PACKAGES

We have to pay 18 copecks to post a package. In how many ways can we pay using postage stamps of 4, 6 and 10 copecks if two ways that differ as to the order of the stamps are considered distinct? (There is an unlimited supply of stamps.)

Denote by $f(N)$ the number of ways 4-, 6- and 10-copeck stamps can be used so that the total price of the postage stamps is N. Then for $f(N)$ we have the relation

$$f(N) = f(N-4) + f(N-6) + f(N-10) \qquad (10)$$

Indeed, suppose we have a method for pasting on stamps totalling N copecks, and let the last one be a 4-copeck stamp. Then all the remaining stamps cost $N-4$ copecks. Conversely, adjoining one 4-copeck stamp to any combination of stamps costing $N-4$ copecks, we obtain a combination of stamps totalling N copecks. Then from different combinations costing $N-4$ copecks we obtain different combinations costing N copecks. Thus, the number of desired combinations where the last stamp is a 4-copeck stamp is equal to $f(N-4)$.

In the same way we prove that the number of combinations ending in a six-copeck stamp is $f(N-6)$; and there are $f(N-10)$ combinations ending in a 10-copeck stamp. Since any combination terminates in one of the stamps indicated above, we get relation (10) by the rule of sum.

Relation (10) permits reducing the problem of pasting on postage stamps totalling N copecks to problems of pasting on stamps of smaller sums. For small values of N the problem may be solved directly. A simple computation shows that

$$f(0)=1,\ f(1)=f(2)=f(3)=0,\ f(4)=1,\ f(5)=0,$$
$$f(6)=1,\ f(7)=0,\ f(8)=1,\ f(9)=0$$

The equality $f(0) = 1$ means that the sum of 0 copecks may be paid in only one way: by not putting on any stamps at all. Sums of 1, 2, 3, 5, 7, and 9 copecks cannot be obtained in any way via stamps costing 4, 6, and 10 copecks apiece. Using the values $f(N)$ for $N = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$, we easily find $f(10)$:

$$f(10) = f(6) + f(4) + f(0) = 3$$

and then we find

$$f(11) = f(7) + f(5) + f(1) = 0,$$
$$f(12) = f(8) + f(6) + f(2) = 2$$

and so on. Finally, we get $f(18) = 8$, which means that the stamps can be put on in eight ways. These ways are:

10,4,4; 4,10,4; 4,4,10; 6,4,4,4; 4,6,4,4; 4,4,6,4; 4,4,4,6; 6,6,6.

It may be noted that the values of $f(N)$ for $N = 1, 2, 3, 4, 5, 6, 7, 8, 9$ are obtainable without direct verification. The point is that for $N < 0$ we have $f(N) = 0$, since a negative sum cannot be paid using a nonnegative quantity of stamps. Yet, as we have seen, $f(0) = 1$. And so

$$f(1) = f(-3) + f(-5) + f(-9) = 0$$

In exactly the same way we obtain $f(2) = 0$, $f(3) = 0$. Now for $N = 4$, we have

$$f(4) = f(0) + f(-2) + f(-6) = 1$$

GENERAL PROBLEM OF POSTAGE STAMPS

The above problem is a particular case of the following general problem:

We have postage stamps of $n_1, n_2, \ldots, n_k$ copecks each (all the numbers $n_1, \ldots, n_k$ are distinct and the supply of stamps is unlimited). In how many ways can they be used to pay a sum of N copecks if two modes of payment differing as to order are taken to be distinct?

In this case the number $f(N)$ of ways satisfies the relation

$$f(N)=f(N-n_1)+f(N-n_2)+\ldots+f(N-n_k) \tag{11}$$

Here, $f(N) = 0$ if $N < 0$ and $f(0) = 1$. Using (11), we can find $f(N)$ for any N by computing successively $f(1), f(2), \ldots, f(N-1)$.

Consider a particular case of this problem when $n_1 = 1, n_2 = 2, \ldots, n_k = k$. We get all possible partitions of N into the integers $1, 2, \ldots, k$, different orders of the integers being considered as distinct partitions. Denote the number of these partitions by $\varphi(k; N)$.* From (11) it follows that

$$\varphi(k;\ N)=\varphi(k;\ N-1)+\varphi(k;\ N-2)+\ldots+\varphi(k;\ N-k) \tag{12}$$

Here,

$$\varphi(k;\ 0)=1 \text{ and } \varphi(k;\ N)=0 \text{ if } N<0$$

The computation of $\varphi(N; k)$ may be simplified if we note that

$$\varphi(k;\ N-1)=\varphi(k;\ N-2)+\ldots+\varphi(k;\ N-k)+\varphi(k;\ N-k-1)$$

and therefore

$$\varphi(k;\ N)=2\varphi(k;\ N-1)-\varphi(k;\ N-k-1) \tag{13}$$

The integers cannot, clearly, exceed N. Therefore $\varphi(N, N)$ is equal to the number of all partitions of N into positive integers (including the "partition" $N = N$). If the number of integers is equal to s, then we get C_{N-1}^{s-1} partitions (see page 57). Therefore

$$\varphi(N,\ N)=C_{N-1}^{0}+C_{N-1}^{1}+\ldots+C_{N-1}^{N-1}=2^{N-1}$$

We have thus proven that the natural number N can be partitioned in 2^{N-1} ways. Note that the order of the terms is taken into account.

To illustrate, the integer 5 can be partitioned in $2^{5-1} = 16$ ways:

$5 = 5$	$5 = 3+1+1$	$5 = 1+2+2$
$5 = 4+1$	$5 = 1+3+1$	$5 = 2+1+1+1$
$5 = 1+4$	$5 = 1+1+3$	$5 = 1+2+1+1$
$5 = 2+3$	$5 = 2+2+1$	$5 = 1+1+2+1$
$5 = 3+2$	$5 = 2+1+2$	$5 = 1+1+1+2$
		$5 = 1+1+1+1+1$

COMBINATORIAL PROBLEMS OF INFORMATION THEORY

The theory of information offers a problem much like the one we have just solved. Suppose a message is being transmitted by means of signals of several types. The transmission time of the first type of signal is t_1, of the second type, $t_2, \ldots,$ of the kth type, t_k units of time. *How many different messages can be transmitted with the aid of these signals in T units of time?* Here we are only dealing with maximum messages; these are messages in which not a single signal can be added without going beyond the restrictions of our transmission-time limit.

Denote the number of messages that can be transmitted in time T by $f(T)$. Arguing as in the stamp problem, we find that $f(T)$ satisfies the relation

$$f(T)=f(T-t_1)+\ldots+f(T-t_k) \tag{14}$$

Here, again, $f(T)=0$ if $T<0$ and $f(0)=1$.

ENTRANCE-EXAMS PROBLEM

Entering a higher educational institution requires taking 4 examinations. Our student-to-be thinks he can make it by collecting a total mark of 17

* Here and henceforward we agree to indicate the number of summands (integers) first, the number being partitioned, second, and the restrictions on the number of summands, last.

(*marking is done on a scale of 5 in which 3 is passing and 5 is the highest mark*). *In how many ways can he pass the exams and enter*?

This is a lot like the stamp problem, but differs in that we indicate the number of "stamps" needed to "pay the sum of 17 points". Passing marks are 3, 4, and 5. Denote by $F(k;\ N)$ the number of ways of collecting N points after k exams. The following relation holds:

$$F(k;\ N)=F(k-1;\ N-3)+F(k-1;\ N-4)+ \\ +F(k-1;\ N-5)$$

the derivation is exactly analogous to that of (11) on page 61.

From this we get

$$F(4;\ 17)=F(3;\ 14)+F(3;\ 13)+f(3;\ 12)=$$
$$=F(2;\ 11)+2F(2;\ 10)+3F(2;\ 9)+2F(2;\ 8)+$$
$$+F(2;\ 7)=2+3F(2;\ 9)+2F(2;\ 8)+F(2;\ 7)$$

since 2 exams cannot yield 11 points and the only way to reach 10 points in two exams is to get the top mark of 5 twice.

Continuing the computation, we get

$$F(4;\ 17)=2+3F(1;\ 6)+5F(1;\ 5)+6F(1;\ 4)+ \\ +3F(1;\ 3)+F(1;\ 2)$$

But $F(1;\ 6)=F(1;\ 2)=0$ (there is no mark of 6, and 2 is failing), and $F(1;\ 5)=F(1;\ 4)=$ $=F(1;\ 3)=1$. Therefore $F(4;\ 17)=16$. In exactly the same way we conclude that

$$F(4;\ 18)=10,\ F(4;\ 19)=4 \text{ and } F(4;\ 20)=1$$

which gives us $16+10+4+1=31$ ways of passing the examinations.

There is another way of obtaining the same result. It is easy to check that 17 points may be obtained in two essentially different ways: either by getting 2 fives, 1 four, and 1 three, or by getting 1 five and 3 fours. These marks may be distributed in arbitrary fashion among the four subjects. Since

$$P(2,\ 1,\ 1)+P(1;\ 3)=\frac{4!}{2!\ 1!\ 1!}+\frac{4!}{3!\ 1!}=16$$

we get 17 points in 16 ways. The approach is the same for finding the number of ways of obtaining 18, 19, and 20 points.

Generally, let $F(m;\ N)$ be the number of ways of partitioning N into m parts, each of which is equal to one of the numbers $n_1, n_2, \ldots, n_k$. Then for $F(m;\ N)$ the following relation holds:

$$F(m;\ N)=F(m-1;\ N-n_1)+\ldots \\ \ldots+F(m-1;\ N-n_k) \qquad (15)$$

which is derived just like (11). The reader may like to derive it himself.

In particular if $n_1=1$, $n_2=2, \ldots, n_k=k$, then we get partitions of N into m summands, each of which is equal to one of the numbers $1, 2, \ldots, k$. Denote the number of these partitions by $F(m;\ N;\ k)$. Then for $F(m;\ N;\ k)$ we have the relation

$$F(m;\ N;\ k)=F(m-1;\ N-1;\ k)+ \\ +F(m-1;\ N-2;\ k)+\ldots+F(m-1;\ N-k;\ k) \qquad (16)$$

As on page 61, it follows from this relation that

$$F(m;\ N;\ k)=F(m,\ N-1;\ k)+ \\ +F(m-1;\ N-1;\ k)-F(m-1;\ N-k-1;\ k) \qquad (17)$$

Now let us investigate partitions in which those differing solely in the order of the integers are considered identical.

PAYING MONEY

In your purse you have the following coins (one of each): 1, 2, 3, 5, 10, 15, 20, and 50 copecks. In how many ways, using these coins, can you pay for a purchase of 73 copecks?

Here, the order of the coins is irrelevant, the important thing is the type of coin. Let us use the notation

$$F(n_1,\ n_2,\ \ldots,\ n_m;\ N)$$

to denote the number of ways of paying N copecks by means of coins of value $n_1, n_2, \ldots, n_m$ copecks each, using not more than one coin of each value. Split up all the modes of paying into two classes according as a coin of value n_m copecks is used or not. If it is, then we have $N-n_m$ copecks to pay by means of coins of value $n_1, n_2, \ldots, n_{m-1}$ copecks. Now this can be done in $F(n_1, n_2, \ldots, n_{m-1}; N-n_m)$ ways. If the n_m-copeck piece is not used, then the whole sum of N copecks has to be paid with the aid of the coins $n_1, n_2, \ldots, n_{m-1}$ copecks. This can be done in $F(n_1, n_2, \ldots, n_{m-1}; N)$ ways.

We thus have the relation

$$F(n_1, n_2, \ldots, n_m; N)= \\ =F(n_1, n_2, \ldots, n_{m-1}; N-n_m)+ \\ +F(n_1, n_2, \ldots, n_{m-1}; N) \quad (18)$$

This relation enables us to reduce the problem of choosing from among m coins to that of choosing from among $m-1$ coins. Repeating the argument, we reduce the problem to one of choosing from $m-2$ coins, and so on, until we arrive at the problem of paying a zero sum or of choosing from a total of one coin. Both problems have unique solutions. In the course of the computations, many terms are dropped. For instance in $n_1+n_2+\ldots+n_m<N$, then $F(n_1, n_2, \ldots, n_m; N)=0$ since there are not enough coins to pay. Besides, if $n_m>N$, then (18) is replaced by

$$F(n_1, n_2, \ldots, n_m; N)=F(n_1, n_2, \ldots, n_{m-1}; N)$$

since the coin n_m cannot participate.

Let us apply this method to solving our problem. By (18), we conclude that

$$F(1, 2, 3, 5, 10, 15, 20, 50; 73)=F(1, 2, 3, 5, 10, 15, 20; 23)+F(1, 2, 3, 5, 10, 15, 20; 73)= \\ =F(1, 2, 3, 5, 10, 15, 20; 23)$$

since $1+2+3+5+10+15+20<73$ and so $F(1, 2, 3, 5, 10, 15, 20; 73)=0$. Then we get

$$F(1, 2, 3, 5, 10, 15, 20; 23)=F(1, 2, 3, 5, 10, 15; 3)+F(1, 2, 3, 5, 10, 15; 23)$$

But

$$F(1, 2, 3, 5, 10, 15; 3)=F(1, 2, 3; 3) \\ =F(1, 2; 0)+F(1, 2; 3)=1+F(1; 3)+ \\ +F(1; 1)=2$$

Compute the second term

$$F(1, 2, 3, 5, 10, 15; 23)=F(1, 2, 3, 5, 10; 8)+ \\ +F(1, 2, 3, 5, 10, 23)=F(1, 2, 3, 5, 10; 8)$$

since $1+2+3+5+10<23$. But $F(1, 2, 3, 5; 8) =F(1, 2, 3; 3)=2$.

We finally get

$$F(1, 2, 3, 5, 10, 15, 20, 50; 73)=4$$

Thus the payment can be made in four ways, namely: 50, 20 and 3; 50, 20, 2 and 1; 50, 15, 5 and 3, and, finally, 50, 15, 5, 2 and 1.

BUYING CANDY

A shop sells different varieties of sweets: 3 kinds at 2 copecks apiece, 2 kinds at 3 copecks apiece. In how many ways can one buy 8 copecks worth of candy if he takes at most one item of each kind?

The solution is obtained from the following relations:

$$F(2, 2, 2, 3, 3; 8)=F(2, 2, 2, 3; 5)+F(2, 2, 2, 3; 8) \\ =F(2, 2, 2; 2)+2F(2, 2, 2; 5)+F(2, 2, 2; 8) \\ =F(2, 2, 2; 2)=F(2, 2; 0)+F(2, 2; 2) \\ =1+F(2; 0)+F(2; 2)=3$$

The purchase may be made in three ways: buy one of both kinds at 3 copecks apiece and add either one of the 2-copeck candies.

The following problem would appear to have the same number of solutions: *We have three 2-copeck coins and two 3-copeck coins in a purse. In how many ways can they be used to pay out a sum of 8 copecks?*

This depends on the kind of coins. If the 2-copeck and the 3-copeck pieces are considered distinguishable, then the problem coincides with the foregoing one and the payment is made in three ways. But if the 2-copeck coins are indistinguishable, then the only mode of payment is two coins of 3 copecks each and one of 2 copecks.

Thus the problem of payment differs depending on the distinguishability or otherwise of coins of the same value. The above-analyzed method of solution is good only if all the coins are considered distinguishable irrespective of whether they are of the same or different value. Now let us see how to solve the problem for the case when coins of one value are considered indistinguishable.

We have ten 2-copeck coins and five 3-copeck coins in a purse. In how many ways can they be used to pay out 22 copecks if coins of the same value are indistinguishable?

Denote the number of solutions of the problem by Φ $(10 \times 2,\ 5 \times 3;\ 22)$ (10×2 means that we have ten 2-copeck coins and 5×3 means that there are five 3-copeck coins). Partition all modes of solution into classes depending on how many three-copeck coins are used. If, say, two are used, then there remains 16 copecks to be paid out using 2-copeck coins, and if all 5 are used, then we have 7 copecks left to pay. Now if 3-copeck coins were not used at all, then the whole sum of 22 copecks will have to be paid with 2-copeck coins. We thus have the equation

$$\begin{aligned}\Phi(10\times 2,\ 5\times 3;\ 22) &= \Phi(10\times 2;\ 22)+\\ &+\Phi(10\times 2;\ 19)+\Phi(10\times 2;\ 16)+\\ &+\Phi(10\times 2;\ 13)+\Phi(10\times 2;\ 10)+\\ &+\Phi(10\times 2;\ 7) \qquad (19)\end{aligned}$$

We do not need to continue because we only have five 3-copeck coins. It is clear that ten 2-copeck coins are not enough to pay 22 copecks. Therefore Φ $(10 \times 2;\ 22) = 0$. It is furthermore obvious that an odd sum cannot be paid with 2-copeck coins, and an even sum can be paid in unique fashion. It therefore follows from (19) that

$$\Phi(10\times 2;\ 5\times 3;\ 22)=2$$

There are only 2 ways of paying:

$$22=8\times 2+2\times 3=5\times 2+4\times 3$$

GETTING CHANGE

What with slot machines at every turn today, people always need change. This raises the following question.

In how many ways can we get change for a 10-copeck piece (the equivalent of a U.S. dime) in the form of 1-, 2-, 3-, and 5-copeck coins?

This problem is similar to the one solved at the end of the preceding section. The only difference is that there are no restrictions on the number of coins of any value. And so we denote the number of solutions by Φ (1, 2, 3, 5; 10). Arguing in the same way as in the preceding section, we get the relation

$$\Phi(1,\ 2,\ 3,\ 5;\ 10)=\Phi(1,\ 2,\ 3;\ 10)+ +\Phi(1,\ 2,\ 3;\ 5)+\Phi(1,\ 2,\ 3;\ 0) \qquad (20)$$

(all modes of getting change are split into classes according to the number of 5-copeck pieces—or nickels if you like—in each class). Clearly, Φ $(1,\ 2,\ 3;\ 0) = 1$: 0 copecks can be paid in only one way.

In order to compute Φ (1, 2, 3; 5), let us split up all modes of changing 5 copecks into 1-, 2-, and 3-copeck pieces into classes depending on how many 3-copeck pieces are accepted. We get

$$\Phi(1,\ 2,\ 3;\ 5)=\Phi(1,\ 2;\ 5)+\Phi(1,\ 2;\ 2)$$

(the first term corresponds to the case of none, the second to that of only one 3-copeck coin).

Continuing the computation, we obtain

$$\Phi(1,\ 2,\ 3;\ 5)=\Phi(1;\ 5)+\Phi(1;\ 3)+\Phi(1;\ 1)+ +\Phi(1;\ 2)+\Phi(1;\ 0)$$

All these summands are equal to 1, since any sum is paid in only one way when using 1-copeck pieces. And so Φ $(1,\ 2,\ 3;\ 5) = 5$. In the same way we find that Φ $(1,\ 2,\ 3;\ 10) = 14$. Altogether, we get $14 + 5 + 1 = 20$ ways of obtaining change.

In place of relation (20), we could have started with the relation

$$\Phi(1,\ 2,\ 3,\ 5;\ 10)=\Phi(1,\ 2,\ 3;\ 10)+ +\Phi(1,\ 2,\ 3,\ 5;\ 5)$$

It shows that the modes of making change split up into those without 5-copeck pieces and those that make use of at least one such coin.

To put the matter generally, if one needs to pay out N copecks with coins of value $n_1, \ldots, n_k$ copecks, then we have the relation

$$\Phi(n_1, \ldots, n_{k-1}, n_k; N) = \Phi(n_1, \ldots, n_{k-1}; N) + \Phi(n_1, \ldots, n_{k-1}, n_k; N-n_k) \quad (21)$$

It shows that either we do not use a single n_k coin and then the whole sum of N has to be paid with the remaining $n_1, \ldots, n_{k-1}$-copeck coins, or at least one n_k-copeck coin is used and then we have to pay the remaining sum of $N-n_k$ copecks using coins of $n_1, \ldots, n_{k-1}, n_k$ copecks. However, if, as was the case on page 63, the coins must not repeat, then relation (21) is replaced by the earlier relation:

$$F(n_1, \ldots, n_{k-1}, n_k; N) = F(n_1, \ldots, n_{k-1}; N) + F(n_1, \ldots, n_{k-1}; N-n_k) \quad (22)$$

PARTITIONING INTEGERS

Let us consider a special case of making change when any coins from 1 to n copecks are allowed. In other words, solve the following problem.

In how many ways is it possible to partition a number N into parts (summands) each of which is equal to one of the numbers $1, 2, \ldots, n$ (disregard the order of the summands)?

Let us denote the number of such modes of partitioning by Π_N^n (we assume the value of Π_0^n to be 1). We then get the relation

$$\Pi_N^n = \Pi_N^{n-1} + \Pi_{N-n}^n \quad (23)$$

Indeed, if the number n is not used as a summand, then N is partitioned into the summands $1, 2, \ldots, n-1$, and this is possible in Π_N^{n-1} ways. But if n is used as one of the summands, then the number $N-n$ is partitioned into $1, 2, \ldots, n$ summands, which may be done in Π_{N-n}^n ways.

Now let us impose the restriction that *all summands must be distinguishable*. Then the number of solutions is denoted by Φ_N^n (here, $\Phi_0^n = 1$). We leave it to the reader to demonstrate that for Φ_N^n we have the relation

$$\Phi_N^n = \Phi_N^{n-1} + \Phi_{N-n}^{n-1} \quad (24)$$

(the number n cannot be used again as a summand).

It is readily seen that $\Phi_1^1 = 1$ and $\Phi_N^1 = 0$ for $N > 1$, and it is possible, with the aid of (24), to compute Φ_N^n successively for all n and N. For Π_N^n it is more convenient, in place of (23), to use the relation

$$\Pi_N^n = \Pi_N^{n-1} + \Pi_{N-n}^{n-1} + \Pi_{N-2n}^{n-1} + \cdots \quad (25)$$

which is obtained by successive application of (23). It is then sufficient to notice that $\Pi_N^1 = 1$ (any natural number can be decomposed uniquely into summands equal to 1). Using relation (25), we consecutively compute Π_N^2 for all N, then Π_N^3 and so on.

Note that the number of all possible ways of decomposing N into summands is Π_N^N —no summands greater than N will appear in the partition. In exactly the same way, the number of ways of partitioning N into distinct parts is equal to Φ_N^N.

ARRAYS OF DOTS

The original methods of proving theorems involving partitioning of integers were exceedingly complicated. As in many other problems of mathematics, by invoking geometrical reasoning we greatly simplify and pictorialize the proofs.

Each partition of a number N into parts may be depicted as an array of dots. Each row of the array consists of a number of dots equal to the units that make up the appropriate summand. For example, the array in Fig. 10 corresponds to the partition $7 = 1 + 1 + 2 + 3$.

Since the order of the summands in the partition is irrelevant, the rows may be arranged so that they do not decrease in length when moving downwards. Also, the first points of each row will be depicted in the same column. Such arrays will be termed *normal.*

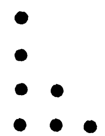


Fig. 10.

Diagrammatic arrays of dots enable us to prove a variety of properties of partitions with relative ease. Let us prove, for instance, that *the number of modes of partitioning N into at most m parts is the same as that of the ways of partitioning $N+m$ into m parts.* Indeed, the array depicting the partition of N into at most m parts consists of N points arranged in no more than m rows. Adjoin to each such array a column consisting of m points (see Fig. 11 where we have this transformation for $N=5$, $m=4$). We get an array made up of $N+m$ dots arranged in m rows.

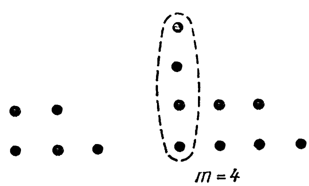


Fig. 11.

Conversely, removing the first column from each array consisting of $N+m$ dots arranged in m rows, we get an array of N dots, the number of rows not exceeding m.

We have thus established a one-to-one correspondence between the two kinds of arrays, which implies that the number of these arrays is the same, and this proves the assertion.

The proof of the following theorem (Euler's theorem) is a little more complicated.

The number of ways of partitioning N into at most m parts is equal to the number of partitions of $N+\frac{m(m+1)}{2}$ into m unequal parts.

Each partition of N into at most m parts is depicted as an array of N dots containing no more

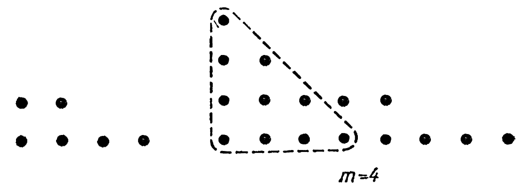


Fig. 12.

than m rows. To each array let us adjoin an isosceles right triangle of m rows, and then reduce the array to its normal form (Fig. 12), where we have the transformation for $N=6$, $m=4$. Since the number of dots in the triangle is $\frac{m(m+1)}{2}$, we obtain an array of $N+\frac{m(m+1)}{2}$ dots involving m rows, all the rows being of unequal length. Indeed, the lengths of the rows of the original array do not decrease, while the lengths of the rows of the triangle constantly increase, and so after the triangle is adjoined, we get an array whose rows increase in size all the time. Consequently, there will be no equal-length rows.

Conversely, from each array illustrating the partition of $N+\frac{m(m+1)}{2}$ into m unequal parts, we can remove an isosceles right triangle containing m rows and thus obtain an array for partitioning N into at most m parts. This correspondence between the two types of arrays indicates that their number is the same, which proves our assertion.

DUAL ARRAYS

We can transform our arrays so as to interchange rows and columns. To do this, rotate an array through 90° and reduce it to normal form (as illustrated in Fig. 13).

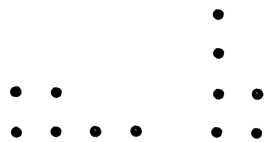


Fig. 13.

It is easy to see that if we repeat the process, we return to the original array. Hence, all arrays may be divided into dual pairs (note, too, that some are self-dual, as witness Fig. 14).

Utilizing the dual nature of such arrays of dots, we can compare partitions subject to certain restrictions regarding the size of the summands with other partitions restricted relative to the number of the summands. For instance, we have the assertion:

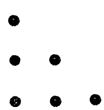


Fig. 14.

The number of partitions of N into summands not exceeding n is equal to the number of partitions of N into at most n summands.

This is true since the arrays for partitions of N into summands (parts) not exceeding n consist of N dots, with no more than n dots per row. Hence, such an array has at most n columns. But then the dual array has at most n rows, that is to say it corresponds to the partition of N into at most n summands.

In exactly the same way it may be proved that the number of partitions of N into n summands is equal to the number of partitions into summands not exceeding n, of which at least one is equal to n.

Now let us consider the partition of the number N into even parts. These partitions are depicted by arrays whose rows contain an even number of dots. But then the dual array will have an even number of parts of each kind (Fig. 15). We draw the following conclusion.

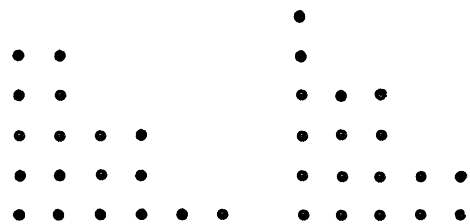


Fig. 15.

The number of partitions of N into even parts is equal to the number of partitions into which each of the numbers enters an even number of times (some summands may naturally not enter at all since zero is an even number).

The proof is the same for the following.

The number of partitions of N into odd summands is equal to the number of partitions into which each of the summands (except the largest) enters an even number of times and the largest summand enters an odd number of times.

EULER'S FORMULA
(may be skipped in a first reading)

In connection with certain problems of partitions, Euler made a study of the infinite product

$$A=(1-x)(1-x^2)(1-x^3)\dots(1-x^n)\dots \qquad (26)$$

Removing the first 22 brackets in this product, we get the expression

$$A=[1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+\ldots]\times \times(1-x^{23})(1-x^{24})\ldots(1-x^n)\ldots$$

where the dots stand for terms containing higher powers of x than 22. We did not write out these terms since they change when the square bracket is multiplied by $1-x^{23}$, $1-x^{24}$... and so on, while the terms that are written out will not change. And so if we remove all the brackets, we get an infinite series, the first terms of which are of the form

$$1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+\ldots \quad (27)$$

We see that two negative terms are followed by two positive terms, which in turn are followed by two negative terms, etc. But it is more difficult to detect the law governing the exponents of these terms. After a good deal of experimenting Euler established the following rule.

If the infinite product

$$(1-x)(1-x^2)(1-x^3)\ldots(1-x^n)\ldots$$

is converted into a series, then only terms like $(-1)^k x^{\frac{3k^2\pm k}{2}}$, *where* k *is a natural number, will be different from zero.*

Euler's theorem is of great importance not only in the theory of partitions, but also in the theory of elliptical functions and in other areas of mathematical analysis. However, most of the proofs of this theorem are rather involved. We give here only an extremely simplified geometrical proof of Euler's theorem, but first we will have to formulate the theorem in the language of partition theory.

When removing brackets in expression (26), the terms $\pm x^N$ appear as many times as there are ways of partitioning N into distinct summands. Also, x^N appears if the number of summands is even, and $-x^N$ if this number is odd. Say, to the partition $12=5+4+2+1$ corresponds the term $(-x^5)(-x^4)(-x^2)(-x)=x^{12}$ and to the partition $12=5+4+3$, the term $(-x^5)(-x^4)(-x^3)=-x^{12}$

Thus, in the expansion (27), the coefficient of x^N is equal to the difference between the number of partitions into an even number of distinct

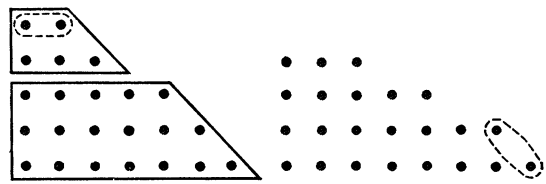


Fig. 16.

summands and the number of partitions into an odd number of distinct summands. The Euler theorem reads:

If a number N *can be represented in the form* $N=\frac{3k^2\pm k}{2}$, *then it has the same number of partitions into an even and into an odd number of distinct summands. For numbers of the type* $N=\frac{3k^2\pm k}{2}$, *the difference between these quantities* is $(-1)^k$ (that is to say, if k is even, then one more partition into an even number of summands, and if k is odd, then one more partition into an odd number of summands).

In order to prove Euler's theorem, let us illustrate one transformation of an array with an even number of rows into an array with the same number of dots having an odd number of rows, and conversely. Since we are considering only partitions into distinct parts, the arrays of such partitions consist of several trapezoids on top of each other. Denote the number of dots in the upper row of the array by m, the number of rows of the lower trapezoid by n. Fig. 16 portrays an array for which $m=2$ and $n=3$.

Let us suppose that an array has at least two trapezoids, $m \leqslant n$. In this case we discard the first row and extend the last m rows of the lower trapezoid by one dot, which does not alter the total number of points; all rows are then of different length, but the parity in the number of rows will change. The same kind of transformation may be effected if the array consists of a single

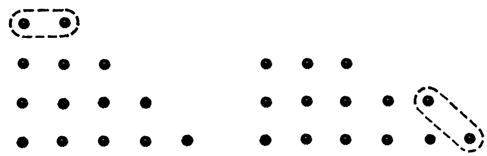


Fig. 17 a.

trapezoid and $m \leqslant n - 1$. Fig. 17a is an illustration of the result of such a transformation.

Now let the array contain at least two trapezoids, $m > n$. Then take one dot from each row of the latter trapezoid and use these dots to make the first row of a new array. This can be done because $m > n$ and therefore the generated row is shorter than the first row of the original array. Besides, since we took all the rows of the lower

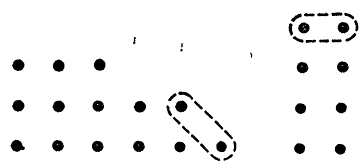


Fig. 17 b.

trapezoid, all the rows in the newly obtained array will have unequal length. Finally, the new array contains as many dots as the original one, but the parity of the number of rows has changed: the new array contains one more row. The same operation can be performed on arrays consisting of one trapezoid if $n \geqslant m - 2$. Fig. 17b is an illustration of the result of just such a transformation. A comparison of Figs. 17a and 17b convinces us that *the above-described transformations are inverses: performing one and then the other, we get the original diagram.*

Thus, *arrays of partitions of N which allow for one of these transformations split up into the same number of arrays with an even and an odd number of rows.* It now remains to find out which arrays do not allow for such a transformation. Clearly,

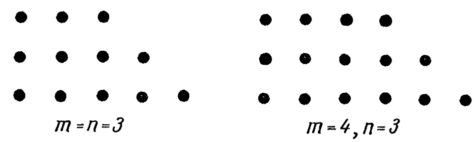


Fig. 18.

they consist of one trapezoid, and for them we either have $m = n$ or $m = n + 1$. In the former instance, the array contains $\frac{3n^2 - n}{2}$ dots, in the latter, $\frac{3n^2 + n}{2}$ dots (Fig. 18).

The foregoing reasoning shows that if N is not a number of the form $\frac{3n^2 \pm n}{2}$, then it has an equal number of partitions into an even and into an odd number of distinct summands. If $N = \frac{3n^2 \pm n}{2}$ and n is even, then there remains one array that does not allow for the transformation and has an even number of rows. And so there will be one more partition into an even number of summands than into an odd number. But if $N = \frac{3n^2 \pm n}{2}$ and n is odd, then there will be one more partition into an odd number of summands. The proof of the theorem is complete.

WANDERING ABOUT TOWN

Fig. 19 is a portion of the city plan of Canberra, Australia. Here, there are $n \times k$ rectangular city blocks separated by $n - 1$ "horizontal" and $k - 1$ "vertical" streets. A person is moving from A to B along the shortest possible route, that is, by going from "left to right" and "upwards". How many routes can he take?

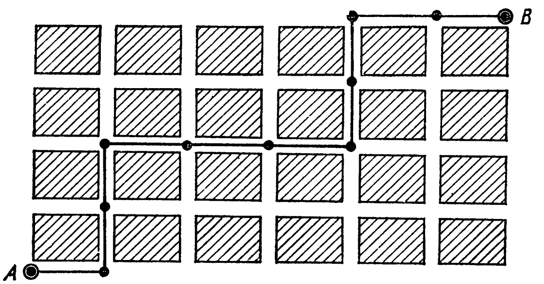
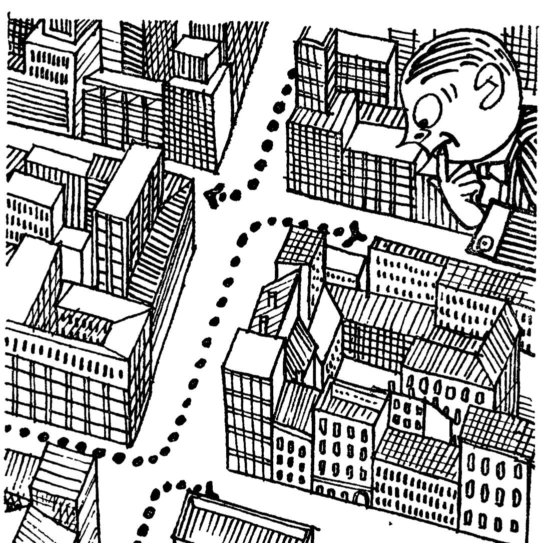


Fig. 19.



No matter what route he takes, it is obvious that he will pass through $k + n$ intersections (counting A but not B).

At each intersection he can go either to the right or upwards. All intersections accordingly divide into two classes. Label with 0 those intersections where he goes rightwards and with 1 those where he goes upwards. Since the number of intersections of the first class must be k, and of the second class, n (otherwise he will not reach B), we obtain a permutation of k zeros and n ones. Conversely, each such permutation represents a route. In Fig. 19 we have a route associated with the permutation 0110001100.

But the number of permutations of k zeros and n ones is

$$P(k, n) = C_{n+k}^{n} = \frac{(n+k)!}{n!\,k!} \tag{1}$$

which, also, is the number of shortest routes between A and B.

THE ARITHMETIC SQUARE

The meanderings of a person wandering about town resemble the movements of a chess rook. Take an infinite-sized chessboard bounded on two sides by perpendicular rays and place a rook in the corner. We assume the rook moves either downwards or from left to right. Combining such movements, we get a variety of pathways leading from the corner square to a given square of the chessboard. In each square we write the number of these routes. It is clear that this number depends on the coordinates of the square, that is, on the vertical (file) and horizontal (rank) intersection.

It will be convenient to label the verticals and horizontals with the numbers 0, 1, 2, ..., n, . . . In this notation, the corner square has the coordinates (0, 0). Using the result obtained in solving the above problem, we assure ourselves that at the intersection of the kth vertical line and nth horizontal line we have the number C_{n+k}^{k} (to reach this square we have to make k moves to the right and n moves downwards). In place of C_{n+k}^{k} substitute their numerical values. What we get is Table 3, which is called an *arithmetic square.*

Table 3

1	1	1	1	1	1	. . .
1	2	3	4	5	6	. . .
1	3	6	10	15	21	. . .
1	4	10	20	35	56	. . .
1	5	15	35	70	126	. . .
1	6	21	56	126	252	. . .
. . .	. . .	. . .	. . .	. . .	. . .	. . .

Let us investigate some of its properties. First of all, a study of the numerals in the squares shows that they are obtained by the following law: *each number is equal to the sum of the number written above it and the number to the left of it.* Say, $10 = 4 + 6$, because 4 stands above it and 6 is to the left of 10.

This rule follows readily from the equation that was proved earlier on page 33: $C_n^k = C_{n-1}^k + C_{n-1}^{k-1}$. It can be proved directly, however. A rook can reach square (k, n) either from square $(k-1, n)$ or $(k, n-1)$. And so, by virtue of the rule of sum, the number of ways of reaching square (k, n) is equal to the sum of the number of ways of reaching square $(k-1, n)$ and the number of ways of reaching square $(k, n-1)$ which is just our assertion.

From the relation $C_{n+k}^{k} = C_{n+k}^{n}$ it follows that the arithmetic square is symmetric about the diagonal going through the corner (we will call it the *principal diagonal*). Incidentally, this property can just as easily be proved geometrically: we have the same number of ways of getting to the intersection of the nth vertical and kth horizontal as to the intersection of the kth vertical and nth horizontal.

FIGURATE NUMBERS

When we calculated the elements of Table 3, we made use of the elements of the preceding row and the preceding column. However, it would have sufficed to use the elements of the preceding row. Indeed, on page 34 we proved formula (15):

$$C_{n+k}^{k} = C_{n+k-1}^{k} + C_{n+k-2}^{k-1} + \ldots + C_{n-1}^{0}$$

This formula shows that each element of our table is equal to the sum of the elements of the preceding row, beginning with the first and terminating with the element occurring directly above

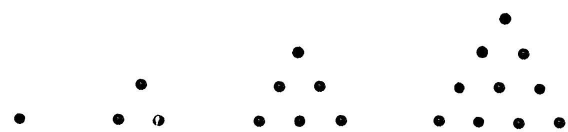


Fig. 20.

the one being computed. Thus, by adding in succession the elements of the $(n-1)$st row, we compute the succession of elements of the nth row.

This method of computing Table 3 goes back to the ancient Greek mathematicians Pythagoras and Nicomedes and their figurate numbers. The numbers 1, 2, 3, . . . may be depicted as rows of one, two, three, etc., dots, the rows being combined to form triangles (Fig. 20). Then the number of dots in every triangle will be equal to the corresponding number in the second row of the table*.

Whence the name *triangular numbers* for 1, 3, 6, 10, 15, 21, etc., the kth triangular number being

$$C_{k+1}^{2} = \frac{(k+1)k}{2}$$

Going another step, we can combine the triangles depicted in Fig. 20 into pyramids. The num-

* The rows are labelled 0, 1, 2, . . ., and so the top row is zero, the next, 1, then 2, etc.

ber of dots in each pyramid is equal to the corresponding number in the third row of our table. And so the numbers 1, 4, 10, 20, 35, etc. are termed *pyramidal numbers.* Their general form is

$$C_{k+2}^{3}=\frac{(k+2)(k+1)k}{1\times 2\times 3}$$

To continue similar interpretations of the numbers of succeeding rows, we would have to pass to pyramids in spaces of higher dimensions.

The theory of figurate numbers has enticed mathematicians for many centuries and at one time was an important division of the theory of numbers.

THE ARITHMETIC TRIANGLE

Now take a board bounded on one side only and put a checker on square A of the zeroth horizontal row (Fig. 21). Moving in accord with the

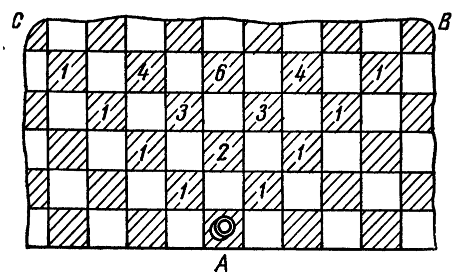


Fig. 21.

rules of checkers (draughts), this piece can reach any square in the region bounded by the straight lines AB and AC. Again write down in each square the number of ways of reaching it. We see that these numbers actually coincide with the numbers of the arithmetic square, only they are arranged differently, and no wonder: all we have to do is turn the board through 45° and our piece will move along horizontal and vertical lines, and the problem becomes that of the movements of a rook. The numbers in Fig. 21 are ordinarily depicted as a triangle (Table 4).

Table 4

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Here, each number is equal to the sum of the two numbers of the preceding row between which it lies. This triangle is often called Pascal's triangle after Pascal (1623-1662). But the Italian mathematician Tartaglia* (1500-1557) was familiar with it. But long before Tartaglia's time, this triangle was used in the works of the Arabian mathematicians al-Kashi and Omar Khayyam. We will therefore call it simply the arithmetic triangle.

The arithmetic triangle can also be written in tabular form:

Table 5

1	0	0	0	0	0	. . .
1	1	0	0	0	0	. . .
1	2	1	0	0	0	. . .
1	3	3	1	0	0	. . .
1	4	6	4	1	0	. . .
1	5	10	10	5	1	. . .
. . .	. . .	. . .	. . .	. . .	. . .	. . .

Here, the intersection of the kth vertical and the nth horizontal yields the number C_n^k (the extreme

* Tartaglia was a remarkable mathematician. Besides the arithmetic triangle, he discovered a formula for solving cubic equations. He mentioned this formula to another Italian mathematician G. Cardano, who promised not to divulge the secret. But Cardano soon published the solution in his algebra, and so the formula for solving cubic equations is unfairly called the "Cardano formula".

lines, it will be recalled, have zero numbers). Each number of the triangle is equal to the sum of the number above it and the number of the preceding row upward left. To illustrate, take 4: in the fourth row above it is 1, upward to the left, 3, and so $4 = 1 + 3$.

Other peculiarities of the arithmetic triangle are: *all elements above the principal diagonal are equal to zero, and the zeroth column consists of units.* The numbers in the nth row of the arithmetic triangle, that is the numbers C_n^k for fixed n, are coefficients in the binomial expansion of $(1 + x)^n$ in powers of x. They are therefore also called *binomial coefficients*, which will be discussed in more detail in Chapter VII.

THE EXTENDED ARITHMETIC TRIANGLE

The arithmetic triangle only occupies a portion of the plane. Let us extend it to the entire plane, while retaining the rule formulated above that *each element is equal to the sum of the element above it and the element of the preceding row upward left.* Here, since the zeroth column of the arithmetic triangle consists of units, we fill this column with units in the extended triangle as well.

Applying this rule to the elements of the zeroth column, we see that there should be a column in front filled with zeros. But then all columns to the left will consist of zeros as well. And so we have to find out what goes above the zeroth row of the triangle. The first element of the zeroth row is zero and at an angle from it upwards we find 1, so we have to write $-1(1 + (-1) = 0)$ above it. But then if we must obtain zero in the second position of the zeroth row, we have to place the number 1 above it. Continuing, we see that a new row appears over the zeroth row and it consists of an alternation of the numbers 1 and -1. The other rows, moving upwards, are filled in the same fashion.

This yields a table, a portion of which is shown below:

Table 6

. . . 0	1	—5	15	—35	70	—126	. . .
. . . 0	1	—4	10	—20	35	—56	. . .
. . . 0	1	—3	6	—10	15	—21	. . .
. . . 0	1	—2	3	—4	5	—6	. . .
. . . 0	1	—1	1	—1	1	—1	. . .
. . . 0	1	0	0	0	0	0	. . .
. . . 0	1	1	0	0	0	0	. . .
. . . 0	1	2	1	0	0	0	. . .
. . . 0	1	3	3	1	0	0	. . .
. . . 0	1	4	6	4	1	0	. . .
. . . 0	1	5	10	10	5	1	. . .
. . . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .

A glance at the portion above the zeroth row convinces us that it differs from the arithmetic square on page 71 solely in the signs of the terms. Namely, we have $(-1)^{k-1}C_{n+k-1}^k$ at the intersection of the $(-n)$th horizontal and the kth vertical. Quite naturally, a simple inspection of part of a table cannot serve as proof that this assertion holds for all rows and all columns. To see that this assertion holds true, notice that

$$(-1)^{k-1}\, C_{n+k-1}^k + (-1)^{k-2}\, C_{n+k-2}^{k-1}$$
$$= (-1)^{k-1}\,[C_{n+k-1}^k - C_{n+k-2}^{k-1}]$$
$$= (-1)^{k-1}\, C_{n+k-2}^k$$

[see formula (11) on page 33]. This equality shows that in a table made up of the numbers $(-1)^{k-1}$ C_{n+k-1}^k, the kth element of the row $n + 1$ is equal to the sum of the elements of the $(-n)$ th row with labels k and $k - 1$. In other words, the rule for filling up a table of the numbers $(-1)^{k-1}C_{n+k-1}^k$ coincides with the rule for filling in a table of the extended arithmetic triangle. Besides, since these tables have the same rows labelled -1 and a zeroth column, it follows that all their elements coincide.

In the original arithmetic triangle, we have the number C_n^k at the intersection of the nth horizontal and the kth vertical. In the extended triangle, we have the number $(-1)^{k-1}C_{n+k-1}^k$ at the intersection of the $(-n)$th horizontal and the kth vertical. We can therefore generalize the symbol C_n^k to negative values of n, putting

$$C_{-n}^k=(-1)^{k-1}\,C_{n+k-1}^k \tag{2}$$

As Table 6 shows us, the generalization of the symbol C_n^k to negative values of k is trivial: for $k<0$ we have $C_n^k=0$ (also see page 114). Also, $C_n^k=0$, if $0\leqslant n<k$.

THE CHESS KING

An arithmetic triangle can be generated in the following manner. Put a "one-sided chess king" (this is a piece that can only move one square forward and one square upward right) in the upper left corner of the table. Write in each square the number of ways it can be reached by the king. This yields the arithmetic triangle.

Now replace the "one-sided king" by an ordinary chess king, and restrict its movements in only one way: the king must always move forward to the next rank (horizontal). To enable the king to utilize its new opportunities, we have to extend the board and take a chessboard bounded only on one side by a straight line. Fig. 22 depicts such a chessboard, each square of which indicates the number of ways the king can reach it from square A.

Let us see how the new table is constructed. Suppose that we have already found the number of ways our wandering king can reach each square of horizontal line $n-1$. Let us find the ways of reaching squares on the nth horizontal. The king can reach every one of them from adjacent squares of horizontal $n-1$ (see Fig. 22: immediately below, right upward and left upward). By the rule of sum, we get the following result.

The number of ways in which the chess king can reach some square of the nth horizontal line is equal to the sum of the numbers of ways the three adjacent squares of horizontal $n-1$ can be reached.

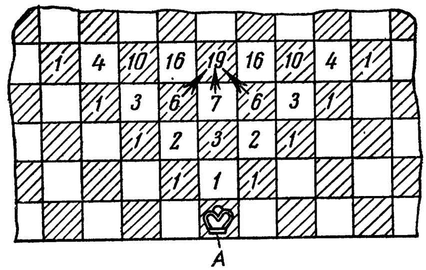


Fig. 22.

It is agreed that the square occupied by the king is reached in one way only—by remaining where it is; such ways don't exist at all for the other squares of the zeroth horizontal.

THE GENERALIZED ARITHMETIC TRIANGLE

The triangle in Fig. 22 may be depicted in another way by shifting all numbers to the right so that the table fits into the portion of the chessboard bounded by two perpendicular rays. Then the rule for obtaining any number of the array reads as follows.

Each number is equal to the sum of three numbers of the preceding row: one straight up and two adjacent ones to the left. The corner is occupied by the number 1, and all other elements of the zeroth row are zero.

For example, the number 16 in the fourth row is the sum of the numbers 3, 6 and 7 of the third row.

Further generalization of the arithmetic triangle is clear. Take some natural number m and fill in the array using the rule: put the number 1 in the upper left corner and adjoin zeros for all the remaining cells of the zeroth row. Then write, in each cell of the first row, the sum of the m elements of the zeroth row: the one directly above the desired element and the $m - 1$ elements to the left of it. Then, quite naturally, the first m elements of the first row will be equal to unity, and the remaining elements, zero (if certain summands are lacking when constructing a sum, the missing terms are considered equal to zero; in other words, the table is completed leftwards by an array of zeros (see Table 7).

Table 7

0	0	1	0	0	0	0	0	0	0	0
0	0	1	1	1	0	0	0	0	0	0
0	0	1	2	3	2	1	0	0	0	0
0	0	1	3	6	7	6	3	1	0	0
0	0	1	4	10	16	19	16	10	4	1
. . .		. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .

The remaining rows are filled in exactly the same way: each element of the table is equal to the sum of the m elements of the preceding row: the element directly above the desired element and the $m - 1$ elements to the left. In particular, the arithmetic triangle is obtained when $m = 2$, and the triangle in Table 7 when $m = 3$.

In order to differentiate between arithmetic triangles with distinct values of m, we will term them *m-arithmetic triangles*. The element of an m-arithmetic triangle lying at the intersection of the nth horizontal and kth vertical is denoted by $C_m(k, n)$. From the definition of an m-arithmetic triangle it follows that the numbers $C_m(k, n)$ satisfy the relation

$$C_m(k, n) = C_m(k, n-1) + C_m(k-1, n-1) + \ldots \\ \ldots + C_m(k-m+1, n-1) \qquad (3)$$

The boundary conditions are

$$C_m(k, 1) = \begin{cases} 1 & \text{if } 0 \leqslant k \leqslant m-1; \\ 0 & \text{if } k \geqslant m \end{cases}$$

GENERALIZED ARITHMETIC TRIANGLES AND A BASE-M NUMBER SYSTEM

The numbers $C_m(k, n)$ are related to the base-m number system. Namely, $C_m(k, n)$ *is equal to the number of n-digit numbers in a base-m system of numeration in which the sum of the digits is k.* The term "n-digit" will be taken to include numbers beginning with one or several zeros. To illustrate, 001, 215 will be regarded as a six-digit number the sum of whose digits is equal to 9.

In order to prove the stated assertion, denote by $B_m(k, n)$ the set of n-digit numbers in the base-m system of numeration, the sum of the digits of which is k. We will demonstrate that the numbers $B_m(k, n)$ satisfy the same relation (3) as $C_m(k, n)$ does. Indeed, the last digit of a number in the base-m system of numeration can assume one of the values $0, 1, \ldots, m - 1$. Accordingly, the sum of the digits of an $(n - 1)$-digit number obtained from an n-digit number by dropping the last digit can assume one of the values $k, k - 1, \ldots, k - m + 1$. Then, from the rule of sum, we obtain

$$B_m(k, n) = B_m(k, n-1) + \ldots \\ \ldots + B_m(k-m+1, n-1) \qquad (4)$$

Besides it is clear that $B_m(k, 1)$ is equal to 1 if $0 \leqslant k \leqslant m - 1$, and 0 otherwise (in a base-m system of numeration there is only one single-digit number with the sum of the digits equal to k, if $0 \leqslant k \leqslant m - 1$, and there is no such number if $k \geqslant m$). Thus, the first row of the table of numbers $B_m(k, n)$ coincides with the first row of the table of numbers $C_m(k, n)$. Since the rules (3) and (4) for constructing these tables also coincide, we have $B_m(k, n) = C_m(k, n)$ for arbitrary k and n.

SOME PROPERTIES OF THE NUMBERS $C_m(k, n)$

The numbers $C_m(k, n)$ have a range of properties resembling those of the numbers C_k^n. This should be no surprise since by virtue of the construction of the arithmetic triangle we have $C_2(k, n) = C_k^n$. Note first of all that $C_m(k, n)$ is *nonzero only when* $0 \leqslant k \leqslant n(m-1)$. This follows immediately from the fact that each succeeding row of the m-arithmetic triangle is longer than the preceding one by $m-1$.

We now show that the numbers $C_m(k, n)$ possess the following symmetry property:

$$C_m(k, n) = C_m(n(m-1)-k, n) \tag{5}$$

To do this, we associate with each n-digit number in the base-m system of numeration a "complementary number" obtained by replacing each digit by its complement with respect to $m-1$. For example, in the base-7 number system, the complement of 3, 140, 216 is the number 3, 526, 450. Clearly, if the sum of the digits of the given number is k, then the sum of the digits of the complementary number is equal to $n(m-1)-k$. For this reason, there are just as many n-digit numbers with the sum of digits k as there are with the digit sum $n(m-1)-k$. But that is what (5) expresses.

Since the total number of n-digit numbers in the base-m system of numeration is equal to m^n (see page 11), the following relation holds:

$$C_m(0, n) + C_m(1, n) + \ldots + C_m(n(m-1), n) = m^n \tag{6}$$

Let us now prove the relation

$$C_m(0, l)\,C_m(k, n-l) + C_m(1, l)\,C_m(k-1, n-l) + \ldots + C_m(k, l)\,C_m(0, n-l) = C_m(k, n) \tag{7}$$

where $0 \leqslant l \leqslant n$. To do this, partition into classes all n-digit numbers with digit sum equal to k. Put in the sth class those numbers whose sum of the first l digits is equal to s. Then the sum of the last $n-l$ digits will be equal to $k-s$. By the rule of product we find that the sth class includes $C_m(s, l)\,C_m(k-s, n-l)$ numbers. Since the total number of n-digit numbers with digit sum k is equal to $C_m(k, n)$, relation (7) follows by the rule of sum.

In particular, for $l = 1$, relation (7) leads to (3) (since $C_m(k, 1) = 1$ for $0 \leqslant k \leqslant m-1$ and $C_m(k, 1) = 0$ when $k \geqslant m$).

Finally, let us show that the following equation holds:

$$C_n^0 C_{m-1}(k-n, n) + C_n^1 C_{m-1}(k-n+1, n-1) + \ldots + C_n^s C_{m-1}(k-n+s, n-s) + \ldots + C_n^n C_{m-1}(k, 0) = C_m(k, n) \tag{8}$$

To do this, split into classes all n-digit numbers in the base-m system of numeration whose digit sum is k. Put in the sth class, $0 \leqslant s \leqslant n$, all numbers whose base-m notation exhibits exactly s zeros.

Let us see how many numbers enter into the sth class. Each number of the sth class can be chosen in two stages. First choose the positions of the zeros. Since n-digit numbers are being considered and the number of zeros is s, this can be done in C_n^s ways. Then cross out all zeros and reduce each remaining digit by unity. We get an $(n-s)$-digit number written with the digits $0, 1, \ldots, m-2$ [which is a number in the base-$(m-1)$ system of numeration], the sum of the digits of which is equal to $k-(n-s) = k-n+s$. There are $C_{m-1}(k-n+s, n-s)$ such numbers. From the foregoing reasoning, it is evident that the sth class includes $C_n^s C_{m-1}(k-n+s, n-s)$ numbers. Since the total number of n-digit numbers having digit sum k is equal to $C_m(k, n)$, relation (8) follows by the rule of sum.

Since $C_2(k, n) = C_k^n$, relation (8) implies

$$C_3(k, n) = C_n^0 C_{k-n}^n + C_n^1 C_{k-n+1}^{n-1} + \ldots + C_n^n C_k^0$$

Repeated application of formula (8) yields the expression of $C_m(k, n)$ in terms of binomial coefficients.

A CHECKER IN THE CORNER

Again take an infinite chessboard bounded by two perpendicular rays and put a checker in the corner as shown in Fig. 23 (the figure includes an extra column that will be needed later on). In each square of the chessboard write the number of ways the piece can reach it. The result

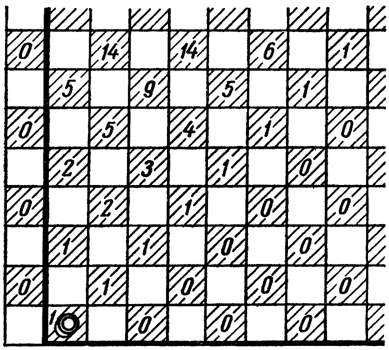


Fig. 23.

will differ from that obtained earlier when the chessboard was only bounded by a single straight line (see page 72), since now the piece cannot cross the vertical boundary. The opportunities for reaching some square are therefore fewer now: the checker piece must not go too far to the left. For example, the piece can reach squares along the boundary only from one square and not from two, as was the case on page 72, where it was stated that the number written in each black square is equal to the sum of the two numbers written on the neighbouring black squares of the preceding horizontal line. For this law to hold now too, it is necessary to draw another vertical line to the left of the boundary and write zero in each black square (it is impossible to reach these squares).

Let us compute the number of ways of reaching a certain square, bearing in mind the restriction. Each route can be denoted by a succession of zeros and ones, a zero defining a move leftwards, and a one, a move to the right. The number of zeros and ones is then determined solely by the square to be reached by the piece. For instance, any route made up of 4 zeros and 6 ones leads to a square at the intersection of the second vertical line and the tenth horizontal line (as before, the extreme lines are labelled zero).

However, not every succession of zeros and ones is permissible. It is forbidden, say, to begin with zero since it will take the piece off the board at once. Admissible sequences have the following characteristic features: there are at least as many ones as there are zeros in front of each position in the sequence at any instant, the number of moves rightwards must not be less than the number of moves leftwards, otherwise the piece will go beyond the limits of the board.

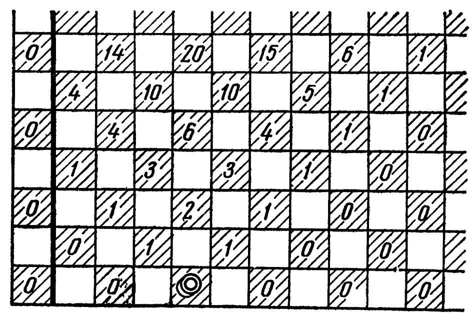


Fig. 24.

To summarize, then, *we have to find the number of sequences of k zeros and m ones possessing the following property*: preceding each position in the sequence there are at least as many ones as there are zeros. But we solved this problem on page 50 (there we were dealing with the letters r and f instead of zeros and ones). It was shown there that the number of such sequences is equal to $\frac{m-k+1}{m+1}C_{m+k}^{k}$. That is the number to be written at the intersection of the $(m+k)$th horizontal line and the $(m-k)$th vertical line.

Now put piece on the qth square of the zeroth horizontal line (contrary to checker rules, this square may be white too). Now our piece has q reserve routes to the left. This case corresponds to the problem examined on page 50, where the cashier put in a supply of q 50-copeck pieces. Utilizing the answer we obtained there, we conclude that if a checker reaches a square in k moves leftwards and m moves rightwards, $0 \leqslant k \leqslant m + q$, then the number of distinct ways of reaching this square is $C_{m+k}^{k} - C_{m+k}^{k-q+1}$. In Fig. 24 we have a table generated for $q = 3$.

THE ARITHMETIC PENTAGON

Rotate the chessboard through 45°. Then the checker will move along vertical and horizontal straight lines and the boundary will be inclined to them at a 45° angle. The problem of the checker in the corner then takes the following form.

Starting from the corner, in how many ways can a rook reach the square (m, k) *moving along the shortest route and not crossing the diagonal of the chessboard* (the rook can take up a position on the diagonal, however)?

From what was proved above it follows that for $k \leqslant m$, the number of ways is equal to $\frac{m-k+1}{m+1} C_{m+k}^{k}$ and for $k > m$, it is zero. However, if we translate the diagonal a total of q squares rightwards, then the answer will be as follows: for $0 \leqslant k \leqslant m + q$, the number of ways is equal to $C_{m+k}^{k} - C_{m+k}^{k-q-1}$, and for $k > m + q$, it is zero.

If the chessboard is finite, then the nonzero numbers of this array fill a pentagon (Fig. 25). It is called an *arithmetic pentagon*. This same term is used for the array obtained on an infinite board bounded by two perpendicular rays.

The basic property of the arithmetic pentagon coincides with that of the arithmetic square: each number of the arithmetic pentagon is equal to the sum of two numbers: the one directly above it and the one to the left of it. It differs from the arithmetic square, however, in that the diagonal of the pentagon positioned q lines above the principal diagonal consists of zeros (in this the pentagon resembles the arithmetic triangle considered on page 72).

Let us now take a chessboard bounded by two perpendicular rays and draw two lines (instead

1	1	1	1	0	0
1	2	3	4	4	0
1	3	6	10	14	14
1	4	10	20	34	48

Fig. 25.

of one) parallel to the principal diagonal: q lines above it and s lines below it. We will consider both lines as forbidden for the rook; in each square write the number of ways the rook can reach

1	1	1	1	1	0	0
1	2	3	4	5	5	0
1	3	6	10	15	20	20
1	4	10	20	35	55	75
0	4	14	34	69	124	199
0	0	14	48	117	241	340

Fig. 26.

it. The resulting array is an *arithmetic hexagon* (see Fig. 26 depicting such an array for $q = 4$, $s = 3$).

We can interpret the arithmetic hexagon as follows. Take a chessboard bounded by a segment of length $s + q$ squares and two rays perpendicular to it; put a checker on a square distant s squares from one corner and q squares from the other. Write in each square the number of ways our piece can reach it. Turning the array through 45°, we obtain an arithmetic hexagon.

GEOMETRIC PROOF OF PROPERTIES OF COMBINATIONS

In Chapter II we proved some of the properties of combinations. Here we shall pictorialize these properties with more vivid geometrical arguments.

Let us first demonstrate how the relation

$$C_n^0+C_n^1+C_n^2+\ldots+C_n^n=2^n \qquad (9)$$

can be derived. To do this we consider all routes leading from point A (0, 0) to a point of the type B_k $(k,\ n-k)$, $0\leqslant k\leqslant n$ (Fig. 27).

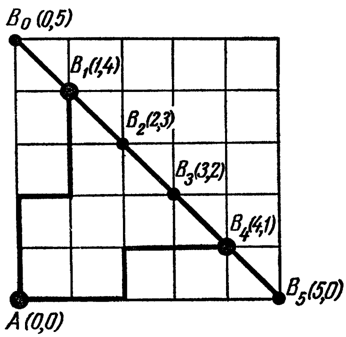


Fig. 27.

These routes split up into classes according to the point B_k, $0\leqslant k\leqslant n$, at which they terminate. There are $P\ (k,\ n-k)=C_n^k$ routes leading to B_k. Our job is to compute the total number of pathways under consideration. Each such route has length n. It can be labelled by an n-sequence of zeros and ones, associating zeros with horizontal segments of the route and ones with vertical segments. Now the number of all n-sequences of zeros and units is 2^n. This is the proof of relation (9).

On page 71 we gave geometric proof of the relations

$$C_n^k=C_{n-1}^k+C_{n-1}^{k-1} \text{ and } C_n^k=C_n^{n-k}$$

This method can be applied to solve more complicated equations as well. In Fig. 28 we draw

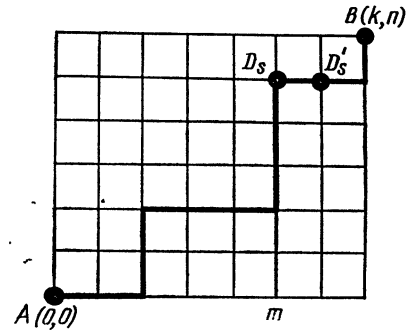


Fig. 28.

a vertical straight line with abscissa m, $0\leqslant m\leqslant k$. Every route leading from A (0, 0) to B $(k,\ n)$ crosses this line, some, in part, passing along it. Partition the set of all routes from A to B into classes, the sth class including routes for which the *last* common point with the straight line $x=m$ is $D_s\ (m,\ s)$.

Now let us compute the number of routes connecting A and B that belong to the sth class. Every such route consists of the section leading from A to D_s, the section from $D_s\ (m,\ s)$ to $D'_s\ (m+1,\ s)$ (D_s, you remember, is the last point of the straight line $x=m$ on this route!) and the section from $D'_s\ (m+1,\ s)$ to the point $B\ (k,\ n)$. By formula (1), there are $P\ (m,\ s)$ routes between A (0, 0) and $D_s\ (m,\ s)$, whereas there are $P\ (k-m-1,\ n-s)$ routes from $D'_s\ (m+1,\ s)$ to $B\ (k,\ n)$ (in order to go from D'_s to B we have to traverse $k-m-1$ unit segments to the right and $n-s$ unit segments upwards).

By the rule of product, the total number of routes in the sth class is

$$P(m, s)\,P(k-m-1, n-s)$$

Now the number of such routes from A to B is $P(k, n)$ and so, by the rule of sum, we find that

$$P(k, n) = P(m, 0)\,P(k-m-1, n) + \\ + P(m, 1)\,P(k-m-1, n-1) + \ldots + \\ + P(m, n)\,P(k-m-1, 0)$$

This equality can be written as follows:

$$C_{n+k}^{k} = C_{m}^{m} C_{n+k-m-1}^{k-m-1} + C_{m+1}^{m} C_{n+k-m-2}^{k-m-1} + \ldots \\ \ldots + C_{m+n}^{m} C_{k-m-1}^{k-m-1} \qquad (10)$$

[cf. (24) on page 36].

In particular, for $m = k-1$, we get

$$C_{n+k}^{k} = C_{k-1}^{k-1} + C_{k}^{k-1} + \ldots + C_{k+s-1}^{k-1} + \ldots \\ \ldots + C_{k+n-1}^{k-1} \qquad (11)$$

Note that the relations (10) and (11) can be derived by repeated use of the relation $C_n^k = C_{n-1}^{k} + C_{n-1}^{k-1}$.

The reader can prove by himself (using geometrical reasoning) formula (23) on page 35:

$$C_{n+k}^{n} = C_{n+k-s}^{n} C_s^0 + C_{n+k-s}^{n-1} C_s^1 + \ldots \\ + C_{n+k-s}^{n-m} C_s^m + \ldots + C_{n+k-s}^{n-s} C_s^s \qquad (12)$$

where $0 \leqslant s \leqslant k$, $0 \leqslant s \leqslant n$.

To do this, draw a straight line through the points $D\,(k-s, n)$ and $E\,(k, n-s)$ and split the set of all routes from $A\,(0, 0)$ to $B\,(k, n)$ into classes according to the point of this line that they pass through. Formula (12) differs from (23) on page 35 in notation alone.

This same geometrical approach can be used to prove a whole range of relations for the numbers C_{n+k}^{k}. This is done by partitioning the routes leading from $A\,(0, 0)$ to $B\,(k, n)$ into classes in various ways.

In order to prove in similar fashion the relations between the numbers $P\,(n_1, \ldots, n_k)$ [see formulas (27), (28), on page 36], we would have to appeal to multidimensional geometry. We shall not go into this question further here.

It may be noted that the relations between the numbers C_n^k that were derived on pages 51-53 also admit of a geometrical interpretation. Here, it is necessary to take a board with a line drawn parallel to the principal diagonal and consider only those routes that do not cross this line (they may, however, have common points with it). Partitioning the set of these routes into classes in a variety of ways, we arrive at the formulas derived in Chapter III.

The problem of the line (queue) at the ticket office admits of a very simple geometric solution. The movement of the line can be depicted graphically by associating every 50-copeck coin with a horizontal segment and every rouble with a vertical segment. From the statement of the problem it is clear that this graph must not cross the principal diagonal. The manipulations we carried out in the solution (adding one person with a 50-copeck coin and then replacing such coins with roubles and roubles with the 50-copeck pieces) take on a very simple geometrical meaning: they reduce to reflecting the graph of the movement of the line into a straight line parallel to the principal diagonal and one unit of length distant from it. We leave it to the reader to translate into geometrical language the reasoning used in solving this problem.

RANDOM WALKS

The above-considered problems on the movements of chess men are closely related to the problem of random walks which is so important in physics. Consider the following problem which was offered at the Eighth Moscow Mathematics Olympiad in 1945.

We have a grid of roads (Fig. 29). 2^N people start out from point A. Half of them proceed in the direction l, the other half in the direction m.

At the first intersection, half of each group proceed in the direction l, the other half in the direction m. Such splitting occurs at each intersection. Where will the people be after traversing N sections and how many will there be at each intersection?

Since the total number of sections traversed by each person is N, it is obvious that they will

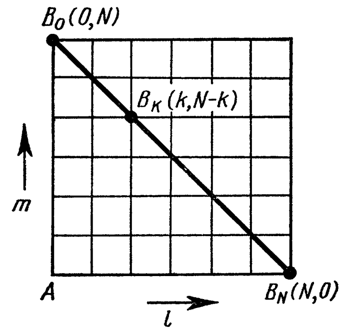


Fig. 29.

all reach B_k with coordinates of the form $(k, N-k)$, where k assumes the values $0, 1, \ldots, N$. All these points are located on the straight line passing through the points $B_0(0, N)$ and $B_N(N, 0)$ (see Fig. 29).

Our job now is to find out how many persons arrive at point $B_k(k, N-k)$. Label all routes leading from A (0, 0) to the points $B_k(k, N-k)$, $k = 0, 1, \ldots, N$, by means of zeros and ones. We thus obtain all possible N-sequences of zeros and ones. There are, as we know, 2^N such sequences, that is to say, just as many as there are people that left A. This implies that each route will be traversed by exactly one person. Therefore, point $B_k(k, N-k)$ will be reached by exactly as many people as there are shortest routes leading to it from A. But we have already computed the number of such minimal-distance routes. It is

$$P(k, N-k) = C_N^k = \frac{N!}{k!\,(N-k)!}$$

Hence, a total of $\frac{N!}{k!\,(N-k!)}$ persons will reach point $B_k(k, N-k)$. This number is equal to the kth number of the Nth row of the arithmetic triangle.

BROWNIAN MOTION

We can cast the foregoing problem in what is essentially an equivalent form.

A total of 2^N persons start out from point O on the straight line Ox. Half turn left, the other half turn right. In one hour, each group subdivides once again, the halves going right and left. These subdivisions occur at hourly intervals. How many persons will arrive at each point after a lapse of N hours?

We assume that in one hour they cover one half of a unit of the route. Arguing in a manner similar to the reasoning of the above problem, we get the following result: after N hours, the hikers will have reached the points $B_k\left(k - \frac{N}{2}\right)$, $k = 0, 1, \ldots, N$ (the starting point is O). A total of $C_N^k = \frac{N!}{k!(N-k)!}$ persons will have arrived at the point B_k.

It is highly improbable that people actually walk as described (true, in the original version, we are told, there was a decent bar at point O). But in certain problems of physics such wanderings occur quite naturally. Random walks are just such an elementary model of the Brownian motion executed by particles under the impacts of molecules.

Let us consider particles that can only move in a straight line. Since the molecular impacts are of a random nature, we can take it, as a first approximation, that in unit time half the particles will have moved $^1/_2$ unit length to the right and the other half, $^1/_2$ unit length to the left (actually, of course, the process is far more intricate and movements over a great variety of distances are possible). Therefore, if we take 2^N particles originally at point O, then they will move, approximately, as described in our problem. In physics, this is called *diffusion*. The problem we solved involving the random walks of a group of people enables us to find out how diffusing particles move during a certain time after the start of diffusion. Namely, in N units of time the particles are distributed according to the following law: there will be $C_N^k = \frac{N!}{k!(N-k)!}$ particles at point $B_k\left(k-\frac{N}{2}\right)$.

As we have already noted, the numbers C_N^k are elements of the Nth row of the arithmetic triangle. Diffusion of a different nature is described by the numbers of the Nth row of an m-arithmetic triangle. Namely, let there be m^N particles at the start at point O. They are divided into m equal parts and placed at m points on the straight line Ox, the distance between adjacent points being unity and the points themselves being symmetric about the point O. After that, each part splits up in exactly the same way (naturally, if the subdividing portion resides at B, then the particles are placed at m points symmetric about point B). After the lapse of N stages, the particles will be located at points B_k with coordinates $k-\frac{m-1}{2}N$, where $k=0$, $1, \ldots, (m-1)N$, and there will be $C_m(N, k)$ particles at the point B_k.

When the values of N are large, computing the number of particles at each point becomes too complicated. But, as often happens in mathematics, as the law of distribution becomes ever more complex, it begins to approach a simple limiting regularity, and this regularity describes the particle distribution the more precisely, the greater the number of particles (the more complex the exact regularity).

In the theory of probability it is proved that for large values of N on a line-segment $\left[x-\frac{a}{2},\ x+\frac{a}{2}\right]$, where a is small compared to N, there are roughly

$$\frac{12am^N}{\sqrt{2\pi}\,N(m^2-1)}\exp\left[-\frac{72x^2}{N^2(m^2-1)^2}\right]$$

particles (here, $\exp x$ is used to denote e^x). This assertion can be interpreted as follows. Construct a step-like line whose height at the point B_k $\left(k-\frac{m-1}{2}N\right)$ is $C_m(N, k)$. Reduce all the abscissas of the resulting line by a factor of $\frac{N(m^2-1)}{12}$ and all ordinates by a factor of $\frac{12am^N}{N(m^2-1)}$. Then, if N is great, we get a step curve that differs but slightly from the graph of the function

$$y=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$$

This is the famous *Gaussian function* introduced into probability theory by the great German mathematician Karl Gauss. It plays an important role not only in problems of diffusion of gases, but in the theory of heat conduction, the theory of errors, and elsewhere.

THE QUEEN'S REALM

Let us again return to the random walk along the x-axis. Now, to the left of O is a realm such that he who enters never returns (like the Queen of Shamakha who played such a fateful role in the life of Czar Dadon and his sons). We too assume that those who move off to the left of the axis remain there. The problem is to determine how many persons remain in the queen's realm and where the remainder of the group are after N hours following their emergence from point O.

It turns out that this problem can be reduced to the earlier one about the line of people at a ticket office. Consider the movements of some person starting out from O. These movements can be specified by a sequence of 1's and -1's: a movement to the right represents 1, to the left, -1. If this sequence involves k ones, then the person has moved k times to the right and $N-k$ times to the left. This should have gotten him to the point $B_k\left(k-\frac{N}{2}\right)$ (it will be recalled that each step is equal to $1/2$ unit of length). But this only happens if our walker steers clear of the queen's realm. If at some point of time the number of movements to the left is greater than to the right, he is lost.

If instead of movements leftwards and rightwards, we consider the line at the ticket office with roubles and 50-copeck pieces, we can say that landing in the queen's realm is equivalent to the ticket line coming to a halt. Hence, the number of persons arriving at point $B_k\left(k-\frac{N}{2}\right)$ is equal to the number of cases when the line in which there are k people with 50-copeck pieces and $N-k$ holders of roubles moves along without any interruptions. Now we know that this number is different from zero only if $k \geqslant N-k$. In that case (see page 50) it is

$$A(N-k,\ k)=C_N^{N-k}-C_N^{N-k-1}=$$

$$=\frac{N!\,(2k-N+1)}{(N-k)!\,(k+1)!}$$

Thus, 2^N persons start out from point O and after N hours of walking, a total of $C_N^{N-k}-C_N^{N-k-1}$ persons reach point $B_k\left(k-\frac{N}{2}\right)$, where $2k \geqslant N$. It is now easy to compute the number who get lost in the queen's realm. To do this, first add the numbers $C_N^{N-k}-C_N^{N-k-1}$ from $k=E\left(\frac{N}{2}\right)+$ $+1$ $\left[\text{where } E\left(\frac{N}{2}\right) \text{ is the greatest integer of } \frac{N}{2}\right]$ to N. We find that $C_N^{N-E\left(\frac{N}{2}\right)-1}$ persons did not get into the queen's realm, and since there were 2^N persons at the starting point O, there must be $2^N-C_N^{N-E\left(\frac{N}{2}\right)-1}$ persons lost in the queen's realm.

Now if the realm of the queen started to the left of point O_1 $\left(\text{with abscissa } -\frac{q}{2}\right)$ instead of O, then the result would be different. Namely, there would be $C_N^{N-k}-C_N^{N-k-q-1}$ persons at the points $B_k\left(k-\frac{N}{2}\right)$, $k \geqslant \frac{N-q}{2}$. The others would be lost in the queen's realm. This follows immediately from the results of the problem on page 50.

ABSORBING BARRIERS

We have already pointed out that random-walk problems are very important in physics, representing the most elementary models of particle diffusion. The problem in the preceding section can also be interpreted physically in a very simple way: there is a wall to the left of point O that absorbs all particles. If the wall adjoins the point O, then we have the case examined at the beginning. If it is $q/2$ units of length away from point O, then we get the problem analyzed at the end of last section.

In the days when combinatorial mathematics and probability theory found most of their practical applications in the theory of games of chance, the problem of random walks with absorbing barriers was formulated differently. There, it was the ruin problem. Picture a coin tossing contest between two players. After each trial, the loser pays the winner one rouble. The player who loses all his money is ruined. That's the end. The aim was to determine the probability of various outcomes if one player started out with p roubles and the other with q roubles. The connection between this problem and the problem of particle diffusion in a domain bounded on two sides by absorbing barriers is obvious.

RANDOM WALKS ON AN INFINITE PLANE

Up to now we have considered either the wanderings of a rook that can move only up or to the right, or, what is essentially the same thing, random walks along an infinite straight line. Now let us investigate the case when the rook moves in any direction over an infinite chessboard. To put it differently, we solve the following problem:

A rook starts out on square O (0, 0) of an infinite chessboard. In how many ways can it reach square A (p, q) in N moves (we agree that one move advances the rook to an adjacent square)?

For reasons of symmetry, it suffices to consider the case when $p \geqslant 0$, $q \geqslant 0$. If the rook were to take the shortest route, it would reach the square A (p, q) in $p+q$ moves. So the inequality $N \geqslant p+q$ must hold.

The difference between an N-move route and the shortest route lies in the fact that the rook makes several self-cancelling moves. It is obvious that the number of such moves has to be even. We call this $2k$. Let us denote by X the set of moves the rook makes to the left and down, and by Y the set of moves to the right and down. Knowing the sets X and Y, we have the entire route of the rook: the moves common to both sets are downward moves, those belonging to X but not to Y are moves to the left, and the moves belonging to Y but not to X are moves to the right. Finally, the moves that do not belong to either X or Y are upward moves.

Thus, counting the number of opportunities of reaching point A (p, q) reduces to counting the number of ways of choosing the sets X and Y. But any move down or to the left refers to some pair of self-cancelling moves, and so there are k elements in the set X. Every move to the left corresponds to a move to the right that cancels it. Besides, there are another p moves to the right that are not cancelled by moves to the left. And so the set Y contains $p+k$ elements. Hence, the set X may be chosen in C_N^k ways, and the set Y in C_N^{p+k} ways, which, by the rule of product, yields $C_N^k C_N^{p+k}$ ways of choosing the sets X and Y. Consequently, the total number T of ways of getting from point O (0, 0), to point A (p, q) is equal to $C_N^k C_N^{p+k}$,

$$T = C_N^k C_N^{p+k}$$

THE GENERAL PROBLEM OF THE ROOKS

We now take up a new cycle of combinatorial problems on the chessboard. These problems involve counting the number of ways of placing two chesspieces (kings, queens, and so forth) so that they can take one another. This clearly also indicates the number of ways of placing these pieces so that they are in nontaking positions: the total number of arrangements of two pieces is computed at once by the formula for permutations.

A few problems of this kind have already been solved, on page 23 we analyzed the problem of 8 rooks on an ordinary chessboard. Let us generalize this problem to an m by n chessboard (that is, one with m ranks, or horizontal lines, or rows, and n files, or vertical lines, or columns). We want to find out in how many ways we can place k nontaking rooks on an m by n chessboard.

To make this problem solvable it is clearly necessary that the conditions $k \leqslant m$ and $k \leqslant n$ hold, otherwise there will be two rooks on one rank or on one file. Suppose these conditions are met. Then we can position the rooks in two stages. First choose the ranks on which the rooks will stand. Since the total number of ranks is m, and we have to choose k ranks, the choice can be made in C_m^k ways. The same for the files, which will be occupied by the rooks; this can be done in

C_n^k ways. Since the choice of files does not depend on the choice of ranks, we get, by the rule of product, $C_m^k C_n^k$ ways of choosing the lines on which the rooks stand.

This however is not the end. The point is that k ranks and k files intersect in k^2 squares. By shifting these squares (if necessary) we obtain a new chessboard made up of k ranks and k files. But we already know that on such a chessboard we can place k rooks in $k!$ ways (for nontaking rooks). Therefore the total number of required positions of the rooks is

$$C_m^k C_n^k k! = \frac{n!\ m!}{k!\,(n-k)!\,(m-k)!} \tag{13}$$

For example, we can place 3 rooks on an ordinary chessboard in

$$\frac{8!\ 8!}{3!\ 5!\ 5!} = 17{,}696$$

ways.

For $k = m = n$, formula (13) yields $n!$ in accord with what was said on page 24.

If we removed the restriction that the rooks are nontaking, then the answer would be different. Namely, we would have had to choose any k squares out of $m \times n$ squares, and this can be done in

$$C_{mn}^k = \frac{(mn)!}{k!\,(mn-k)!}$$

ways. And if the k rooks were distinct, then we would have to multiply the answers by $k!$

SYMMETRIC ARRANGEMENTS

Now let us complicate the problem of the rooks and require that they be nontaking and, what is more, stand symmetrically on the chessboard. Here a host of problems arise depending on the type of symmetry restriction.

The simplest is the case when the rooks are symmetric about the centre of the chessboard. Denote by G_n the number of solutions for the case when n rooks occupy an n by n chessboard. We will now show that

$$G_{2n} = 2nG_{2n-2} \tag{14}$$

Let the chessboard consist of $2n$ rows and $2n$ columns. The rook in the first column can occupy any one of the $2n$ squares of that column. By hypothesis, this determines the position of the rook in the last column, it must be symmetric with the first rook about the centre of the board. Cross out the first and last columns and rows occupied by rooks (since the number of rows is even, the discarded rooks cannot occupy one and the same row). We get a board consisting of $2n—2$ columns and $2n—2$ rows. It is clear that each symmetric arrangement of the rooks on the new board is associated with a symmetric arrangement of the rooks on the original board. Whence it follows that $G_{2n} = 2nG_{2n-2}$ (recall again that the first rook could occupy any one of the $2n$ squares of the first column).

Using formula (14), we get $G_{2n} = 2^n n!$.

Now let us consider a chessboard of $2n + 1$ columns and $2n + 1$ rows. Here there is a square without any symmetric ones—this is the centre square of the board. It must have a rook. Deleting the central column and the central row, we get a symmetric arrangement for $2n$ rooks on a $2n \times 2n$ board. We get the equality

$$G_{2n+1} = G_{2n} = 2^n n! \tag{15}$$

Let us now examine a somewhat more involved problem concerning arrangements that remain invariant under a rotation of the board through 90° (Fig. 30 depicts one such arrangement on an 8 by 8 board). Suppose a board has $4n$ columns and $4n$ rows and the number of rooks is also $4n$. In this case the rook in the first column can occupy any one of the squares except the corner ones, that is, any one of $4n —2$ squares (one cannot put a rook on a corner square because then after a rotation through 90° we would have two rooks capturing each other). To this rook correspond another three rooks standing in the last row, the last column and the first row (they are obtained from the chosen one by rotations through 90°, 180° and 270°). Deleting the rows and columns occupied by these rooks, we get on a $(4n-4)$ by $(4n-4)$ board a rook placement with the same symmetry. And so we have the equality

$$R_{4n}=(4n-2)\,R_{4n-4}$$

where R_n is the number of solutions of the problem for n by n boards. It is now clear that

$$R_{4n}=2^n\,(2n-1)\,(2n-3)\,\ldots\,1 \tag{16}$$

The number of solutions of the problem for a $(4n+1)$ by $(4n+1)$ board is the same as for

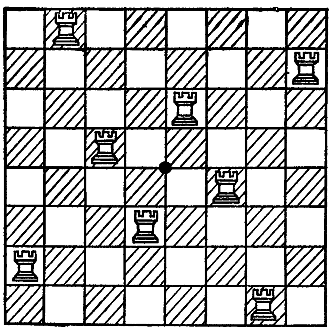


Fig. 30.

a $4n$ by $4n$ board: this is so because on a $(4n+1)$ by $(4n+1)$ board one rook must be in the centre and the central row and column can be crossed out. Therefore

$$R_{4n+1}=R_{4n} \tag{17}$$

Now for $(4n+2)$ by $(4n+2)$ and $(4n+3)$ by $(4\,n+3)$ boards, the number of solutions is zero. Indeed, there are two cases for each rook: either the centre of the board or not the centre. In the latter case, the rook is one of a foursome of rooks that pass into one another in rotations of the board through 90°. For this reason, the total number of rooks must be either of the form $4n$ (when the board has no central square) or $4n+1$. We have thus proved that $R_{4n+2}=R_{4n+3}=0$.

Finally, let us find the number of arrangements of n rooks symmetric about the diagonal passing through the lower left corner square. Denote the number of solutions by Q_n for an n by n board. Then we have the relation

$$Q_n=Q_{n-1}+(n-1)\,Q_{n-2} \tag{18}$$

Indeed, the rook in the first row is either in the lower left-hand corner, or it is not. In the former case, cross out the first column and the first row and obtain a symmetric placement of $n-1$ rooks on an $(n-1)$ by $(n-1)$ board. There are Q_{n-1} such arrangements. In the latter case, the given rook will be found to have another rook symmetric to it relative to the chosen diagonal. Delete the columns and rows in which these rooks stand. This yields a symmetric placement of $n-2$ rooks on the $(n-2)$ by $(n-2)$ board. Since there are Q_{n-2} such placements, and we can put a rook in the $n-1$ square of the first column, we get $(n-1)\,Q_{n-2}$ ways. Hence follows the relation (18).

We have the equality

$$Q_n=1+C_n^2+\frac{1}{1\times 2}\,C_n^2C_{n-2}^2+$$
$$+\frac{1}{1\times 2\times 3}\,C_n^2C_{n-2}^2C_{n-4}^2+\ldots \tag{19}$$

It is derived by partitioning all placements of the rooks into classes, the sth class including positionings in which s pairs of rooks do not occupy the diagonal.

In the same way it is demonstrated that the number B_n of ways of putting n rooks on an n by n board such that the rooks are nontaking and are symmetric about both diagonals satisfies the relations

$$B_{2n}=2B_{2n-2}+(2n-2)\,B_{2n-4},\quad B_{2n+1}=B_{2n}$$

TWO KNIGHTS

In how many ways can we put a white and a black knight on an m by n chessboard so that they do not capture one another?

The solution of this problem is complicated by the fact that on different squares of the board the knight has different numbers of moves: if $m \geqslant 5$ and $n \geqslant 5$, there are only two moves in the corner of the board, three moves on some extreme squares, and four moves on others, while there are 8 moves in the centre. This is due to the fact that the knight has 8 kinds of moves in all. They can be specified by indicating how many squares it moves horizontally and how many vertically. Thus, these moves are: (2, 1), (1, 2), (−1, 2), (−2, 1), (−2, −1), (−1, −2), (1, −2), (2, −1).

To master this situation, let us agree that the knight is a combination of 8 chessboard pieces, each one of which has moves of one type only. Let us see in how many ways we can put a (2, 1)-knight on the board so that it keeps a certain square of the board under attack. Clearly, it can be in any column except the last two, and in any row, except the very last one. This means that we can choose a column in $n-2$ ways, and a row in $m-1$ ways, making a total of $(m-1)(n-2)$ ways to put a white (2, 1)-knight on the chessboard. By virtue of symmetry, it is clear that there are just as many ways to put any one of the white $(\pm 2, \pm 1)$-knights so as to capture the black knight. For the white $(\pm 1, \pm 2)$-knights, there are $(m-2)(n-1)$ ways. From this we get the total number of ways of putting two knights on the board so that they capture one another. It is given by the formula

$$4[(m-1)(n-2)+(m-2)(n-1)]=$$
$$=2[(2m-3)(2n-3)-1]$$

If we were to place knights of the same colour on the board so that they could defend each other, we would get half the number of ways (due to the possibility of interchanging the knights). Now the number of ways of placing two knights of different colours so that they do not capture each other is

$$m^2n^2-9mn+12m+12n-16$$

(We can put two knights on an m by n board in $mn(mn-1)$ ways.)

Composers of chess problems sometimes introduce "new" chessmen that move in unusual ways. Let us introduce a new piece and call it the (p, q)-*knight*, $p \geqslant 0$, $q \geqslant 0$. One move of this piece consists in traversing p squares horizontally and q squares vertically. For instance, the ordinary knight is a combination of the (1, 2)- and (2, 1)-knights. Reasoning as before, we conclude that if $0 < p \leqslant n$, $0 < q \leqslant m$, then we can put two (p, q)-knights of different colours on an m by n board in $4(n-p)(m-q)$ ways so that they are nontaking. But if p or q is zero, then there are one half as many ways. The number of ways is also cut by one half if the knights are of one colour.

Any chessboard piece can be regarded as a combination of several (p, q)-knights with a variety of values for p and q. For instance, the king is a combination of (0, 1)-, (1, 0)- and (1,1)-knights. And so two different-colour kings can be placed on an m by n board in

$$2[n(m-1)+(n-1)m+2(n-1)(m-1)]=$$
$$=8mn-6m-6n+4$$

taking ways. Consequently, there are $m^2n^2-9mn+6m+6n-4$ nontaking ways of placing them.

The bishop is a combination of (1,1)-, (2,2)-, . . . , (p, p)-knights, where p is the smallest of the numbers $m-1$, $n-1$. Suppose for the sake of definiteness, that $m \leqslant n$. Then $p=m-1$, and two bishops (black and white) may be placed in

$$4[(n-1)(m-1)+(n-2)(m-2)+\dots$$
$$\dots+(n-m+1)\times 1]$$

taking ways. Removing brackets and using the formula for the sum of positive integers from 1 to $m-1$ and the sum of the squares of these numbers, we find that the number of ways may be written thus: $\frac{2m(m-1)(3n-m-1)}{3}$. For $m \geqslant n$, interchange m and n. In particular, if $m = n$, we have $\frac{2m(m-1)(2m-1)}{3}$ ways.

For rooks it is easier to take a different approach in enumerating the ways of placing rooks. The white rook can be placed on any one of mn squares. After that, it holds $m + n - 2$ squares under attack, on any one of which we can put the black rook. We thus obtain $mn\ (m + n - 2)$ ways of placing taking rooks.

Since the queen may be regarded as a combination of a rook and a bishop, an m by n board (for $m \leqslant n$) can accommodate two taking queens in

$$\frac{2}{3}\, m\,(m-1)\,(3n-m-1) + mn\,(m+n-2)$$

ways. When $m = n$, this expression takes on the form $\frac{2}{3}\, m\,(m-1)\,(5m-1)$. We leave it to the reader to enumerate the ways to put these chessboard pieces in nontaking positions.

In solving a number of combinatorial problems we have already made use of the method of reducing the given problem to one involving a smaller number of objects. An instance is the formula for the number of permutations with repetitions which we derived on page 10. This same method was used to solve almost all the partition problems in Chapter IV. The method of reduction to a similar problem for a smaller number of objects is termed the *method of recurrence relations* (from the Latin *recurrere*, to run back). Using a recurrence relation, we can reduce a problem involving n objects to one involving $n-1$ objects, then to one dealing with $n-2$ objects, etc. By consecutively reducing the number of objects, we arrive at a problem we are able to solve. In many cases it is possible, from the recurrence relation, to obtain an explicit formula for solving the combinatorial problem.

For instance, in Chapter II (see page 23), we derived the formula $P_n = n!$ for the number of permutations of n elements using the formula for the number of permutations without repetitions. But the same formula may be derived in a different way, by first finding the recurrence relation which P_n satisfies.

Suppose we have n objects $a_1, \ldots, a_{n-1}, a_n$. Any permutation can be obtained as follows: take a permutation of the elements $a_1, \ldots, a_{n-1}$ and adjoin element a_n. Element a_n can clearly occupy distinct positions. It can be placed at the very beginning, between the first and second elements of a permutation, between the second and third, and also at the very end. The number of distinct positions that a_n can occupy is equal to n, and so each permutation of the elements $a_1, \ldots, a_{n-1}$ yields n permutations of the elements $a_1, \ldots, a_{n-1}, a_n$. Which means that there are n times more permutations of n elements than there are permutations of $n-1$ elements. We thus have the recurrence relation

$$P_n = nP_{n-1}$$

Using this relation, we find, successively, that

$$P_n = nP_{n-1} = n(n-1)P_{n-2} = n(n-1)\ldots 2P_1$$

But $P_1 = 1$ since one element can be permuted in one way only. Therefore

$$P_n = n(n-1)\ldots 2\times 1 = n!$$

And we again have the formula $P_n = n!$.

We encountered numerous recurrence relations when solving problems involving partitions, chessmen on a chessboard, etc. We will now examine some more problems of this type and at the end of the chapter we will investigate the general theory of recurrence relations.

FIBONACCI NUMBERS

In his book "Liber abaci", which appeared in 1202, the Italian mathematician Fibonacci gives this problem (the Rabbit Problem):

How many pairs of rabbits can be produced from a single pair in a year if every month each pair begets a new pair which from the second month on becomes productive?

It follows that in one month's time there will be two pairs of rabbits. In two months only the first pair will produce, and we have 3 pairs. Within another month, the original pair will produce and so will the pair of rabbits that appeared two months before. There will then be 5 pairs of rabbits.

Let us denote by $F(n)$ the number of pairs after n months starting at the beginning of a year. We see that in $n+1$ months there will be $F(n)$ pairs and as many more newly born pairs as there were at the end of the month $n-1$, which is to say, $F(n-1)$ pairs of rabbits. In other words, we have the recurrence relation

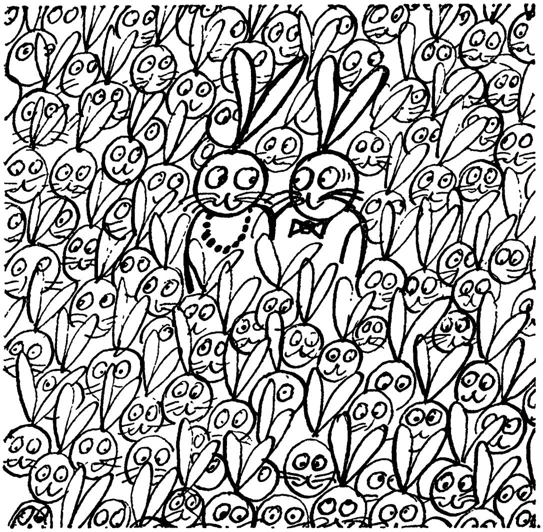
$$F(n+1) = F(n) + F(n-1) \tag{1}$$

Since, by hypothesis, $F(0) = 1$, and $F(1) = 2$, we find, in succession,

$$F(2) = 3,\ F(3) = 5,\ F(4) = 8, \text{ etc.}$$

In particular, $F(12) = 377$.

The numbers $F(n)$ are called *Fibonacci numbers*. They possess a variety of remarkable properties.



We will now derive an expression for these numbers in terms of C_m^k. To do this we establish a link between the Fibonacci numbers and the following combinatorial problem.

Find the number of n-sequences (consisting of zeros and ones) in which no two ones are consecutive.

To establish this connection, let us take any such sequence and associate it with a pair of rabbits according to the rule: ones correspond to the months when one pair is born of the "ancestors" of the given pair (including the original pair), and zeros represent all the other months. For example, the sequence 010010100010 establishes the following genealogy: the pair itself appeared at the end of the 11th month, its parents, at the end of the 7th month, "grandfather" at the end of the 5th month, and "great grandpa" at the end of the second month. The original pair of rabbits is labelled by the sequence 000000000000.

It is clear that there will not be a single sequence with two ones in succession, since a newly born pair cannot produce in one month. What is more, the rule states that to different sequences correspond distinct pairs of rabbits, and, conversely, two distinct pairs of rabbits will always have different genealogies, since, by hypothesis, a she-rabbit gives birth to only one pair of rabbits.

The relationship thus established shows that the number of n-sequences with this property is equal to $F(n)$.

Now let us demonstrate that

$$F(n)=C_{n+1}^0+C_n^1+C_{n-1}^2+\dots+C_{n-p+1}^p \qquad (2)$$

where $p=\frac{n+1}{2}$ if n is odd and $p=\frac{n}{2}$ if n is even. In other words, p is the largest integer in $\frac{n+1}{2}$ [from now on we will denote the largest integer of α by $E(\alpha)$; thus, $p=E\left(\frac{n+1}{2}\right)$].

Indeed, $F(n)$ is the number of n-sequences of 0 and 1, in which no two ones come together. Now the number of such sequences in which there are exactly k ones and $n-k$ zeros is equal to C_{n-k+1}^k (see page 40). Since the inequality $k \leqslant n-k+1$ must hold here, it follows that k varies from 0 to $E\left(\frac{n+1}{2}\right)$. Using the rule of sum, we get relation (2).

Equation (2) may be proved differently. Set

$$G(n)=C_{n+1}^0+C_n^1+C_{n-1}^2+\dots+C_{n-p+1}^p$$

where $p=E\left(\frac{n+1}{2}\right)$. From $C_n^k=C_{n-1}^k+C_{n-1}^{k-1}$ it follows immediately that

$$G(n)=G(n-1)+G(n-2) \qquad (3)$$

Besides, it is clear that $G(1)=2=F(1)$ and $G(2)=3=F(2)$. Since both sequences $F(n)$ and $G(n)$ satisfy the recurrence relation $X(n)= =X(n-1)+X(n-2)$, we have

$$G(3)=G(2)+G(1)=F(2)+F(1)=F(3)$$

and, generally, $G(n)=F(n)$.

AN ALTERNATIVE PROOF

In the preceding section we established a direct link between the Fibonacci problem and a combinatorial problem. This relationship could have been established in another way, by proving directly that the number $T(n)$ of solutions of the combinatorial problem satisfies the same recurrence relation

$$T(n+1)=T(n)+T(n-1) \tag{4}$$

as the Fibonacci numbers do.

This becomes evident if we take any $(n+1)$-sequence of zeros and ones satisfying the condition that no two ones come together. It can terminate either in 0 or 1. If it terminates in 0, then, dropping the zero, we get an n-sequence satisfying our condition. Conversely, if we take any n-sequence of zeros and ones in which no two ones come together and if we adjoin a zero, then we have an $(n+1)$-sequence with the same property. We have proved that the number of "good" sequences ending in zero is equal to $T(n)$.

Now suppose the sequence terminates in a 1. Since two ones cannot be in succession, this 1 will be preceded by a zero. In other words, the sequence terminates in 01. Now the $(n-1)$-sequence left after dropping the 0 and 1 can be any so long as no two ones come in succession. And so there are $T(n-1)$ "good" sequences ending in one. But every sequence terminates either in 0 or in 1. By virtue of the rule of sum, we have that $T(n+1) = T(n) + T(n-1)$.

This is the same recurrence relation. This does not yet imply however that the numbers $T(n)$ and $F(n)$ coincide. We recall, for example, that for factorials and subfactorials (see page 45) we had the same recurrence relation

$$X(n+1)=n[X(n)+X(n-1)] \tag{5}$$

But for factorials the first terms of the sequence are $0! = 1$, $1! = 1$, while for subfactorials they are $D(0) = 1$, $D(1) = 0$. And so the third, fourth, etc. terms of the sequence differed.

To prove the coincidence of $T(n)$ and $F(n)$, we still have to demonstrate that $T(1) = F(1)$ and $T(2) = F(2)$; then by the recurrence relation we have, also, $T(3) = F(3)$, $T(4) = F(4)$, etc. There exist two 1-sequences satisfying this condition: 0 and 1, and three 2-sequences: 00, 01 and 10. Therefore, $T(1) = 2 = F(1)$ and $T(2) = 3 = F(2)$. This proves the assertion.

THE PROCESS OF SUCCESSIVE PARTITIONS

Combinatorial problems are frequently solved by the method used in the above section. A recurrence relation is set up for the given problem and then it is demonstrated that it coincides with the recurrence relation of another problem whose solution is known. If, now, a sufficient number of the initial terms of the sequences coincide (later on we will go into detail about how many terms must coincide), then both problems have the same solutions.

Let us apply this approach to the following problem. Suppose we have a certain set of n objects arranged in a definite order. We partition the set into two nonempty parts so that one of them lies to the left of the other [say, one part consists of elements from 1 to m, the other, of elements from $(m+1)$ to n]. Then take these two subsets and split them in the same fashion into two nonempty parts (if one of the parts now consists of a single object, it is not further partitioned). This process is continued until we have parts consisting of one object each. *How many partition processes are there of this kind* (two processes are considered distinct if at least one step produces different results)?

Denote the number of ways of partitioning a set of $n+1$ objects by B_n. In the first stage, this set can be partitioned in n ways (the first part, or subset, can contain one object, two objects, . . ., n objects). Accordingly, the set of all partition processes breaks up into n

classes, the sth class including processes in which the first part consists of s objects.

Let us compute the number of processes in the sth class. The first part consists of s elements. It can therefore be further partitioned by B_{s-1} distinct processes. The second part contains $n-s+1$ elements and it can be further partitioned by B_{n-s} processes. By the rule of product, we find that the sth class consists of $B_{s-1}B_{n-s}$ distinct processes. By the rule of sum, it then follows that

$$B_n = B_0 B_{n-1} + B_1 B_{n-2} + \ldots + B_{n-1} B_0 \tag{6}$$

We have obtained a recurrence relation for B_n, which occurred (see page 52) when solving the problem of the line at the ticket office. There, it was shown that this relation is satisfied by the numbers

$$T_n = \frac{1}{n+1} C_{2n}^{n}$$

To prove

$$B_n = T_n = \frac{1}{n+1} C_{2n}^{n} \tag{7}$$

we have to show that the initial terms T_0 and B_0 of the sequences $T_0, T_1, \ldots, T_n, \ldots$ and $B_0, B_1, \ldots, B_n, \ldots$ coincide.

We have $T_0 = C_0^0 = 1$. On the other hand, $B_0 = 1$, since the set consisting of a single element is partitioned in unique fashion. Thus, $B_0 = T_0$. But by the recurrence formula, we have $B_1 = B_0^2 = 1$. Since T_0 satisfies the same recurrence formula, it follows that $T_1 = T_0^2 = 1$. We then establish that

$$B_2 = B_0 B_1 + B_1 B_0 = 2 \text{ and } T_2 = T_0 T_1 + T_1 T_0 = 2$$

and so on. Thus, all the terms of both sequences coincide. This, then, proves the following result.

The number of processes of successive division of a set of $n+1$ elements arranged in a specific order is equal to

$$T_n = \frac{1}{n+1} C_{2n}^{n}$$

MULTIPLYING AND DIVIDING NUMBERS

We have n numbers $a_1, \ldots, a_n$ in a given order. By virtue of the associative law for multiplication, the product of these numbers may be computed in different ways (preserving the order of the factors). For example, three numbers can be multiplied together in two ways $(ab)\,c = a\,(bc)$, four numbers, in five ways, etc. It is required to *find the number of the ways of multiplying together n numbers arranged in a given order.*

It is clear that each mode of multiplication reduces to the process of partitioning the given n numbers into parts of one element each. For example, the multiplication of four numbers by the formula $(ab)(cd)$ reduces to the partition process $a \mid b \mid c \mid d$, and the multiplication of these numbers by the formula $((ab)\,c)\,d$, to the partition process $a \mid b \mid c \mid d$. Therefore the number of distinct modes of multiplication is equal to the number of distinct modes of partitioning a set of n elements, or $T_{n-1} = \frac{1}{n} C_{2n-2}^{n-1}$.

However, besides the associative property, the commutative property also holds for multiplication. If this is taken into account, then the number of processes of multiplication increases by a factor of $n!$ because n numbers can be permuted in $n!$ ways; then we have to subject the permuted numbers to certain partitions. Whence it follows that the total number of modes of multiplying any n numbers is equal to $(n-1)!\,C_{2n-2}^{n-1}$.

This same result can be reached directly, without resorting to the formula for the number of partition processes. This conclusion yields a new method for obtaining a formula for partition processes and thus also for the problem of the ticket-office line (provided the number of roubles in the line is equal to the number of 50-copeck pieces).

The immediate conclusion is this. Suppose we have found the number $\Phi(n)$ of ways of multiplying n numbers. We adjoin one more factor, a_{n+1}. Let us find out in how many ways we can adjoin this factor to one of the products of the numbers $a_1, \ldots, a_n$.

We can multiply the entire product by this number a_{n+1} taking it either as the multiplicand or as the multiplier. This gives us two modes of adjunction. But a_{n+1} can also be adjoined at one of the intermediate stages. The multiplication of n numbers reduces to $n-1$ successive multiplications in each of which two numbers are multiplied together. In each case of multiplication, we can adjoin the number a_{n+1} in 4 ways: by multiplying it into the first factor either as multiplicand or multiplier and also into the second factor either as multiplicand or multiplier. But since there are $n-1$ multiplications to which we can adjoin a_{n+1}, we obtain a total of $4n-4$ modes. Adding in the two ways mentioned earlier, we get $4n-2$ ways of adjoining a_{n+1} to each of the $\Phi(n)$ ways of multiplying together the numbers $a_1, \ldots, a_n$. This implies that

$$\Phi(n+1) = (4n-2)\,\Phi(n)$$

But $\Phi(1) = 1$. And so

$$\Phi(n) = 2 \times 6 \ldots (4n-6) = 2^{n-1} \times 1 \times 3 \ldots (2n-3)$$

This coincides with the earlier obtained answer since

$$\Phi(n) = 2^{n-1} \times 1 \times 3 \ldots (2n-3) = \frac{(2n-2)!}{(n-1)!} = (n-1)!\, C_{2n-2}^{n-1}$$

Let us now consider the operation of division. We write the expression

$$\cfrac{a_1}{\cfrac{a_2}{\cfrac{a_3}{\cfrac{\vdots}{a_n}}}} \tag{8}$$

This notation is meaningless unless we indicate the order in which the divisions are to be performed. Let us enumerate the ways in which this expression becomes meaningful. To do so, note that each way of indicating the order of division can also be regarded as a process of splitting n elements into parts containing one element each, as described above. We have seen that the number of processes is $\frac{1}{n} C_{2n-2}^{n-1}$.

This means that expression (8) is meaningful in $\frac{1}{n} C_{2n-2}^{n-1}$ ways.

PROBLEMS INVOLVING POLYGONS

In certain areas of quantum chemistry, problems like the following crop up.

A regular $2n$-gon is inscribed in a circle. In how many ways can the vertices be joined in pairs so that the resulting line segments do not intersect?

For $n = 1$, there is only one mode of connection (here, the diameter is taken to be a "regular 2-gon"). For $n = 2$ we get two modes; they are shown in Fig. 31. In order to find the number of ways $F(n)$ for arbitrary n, let us derive the recurrence relation for $F(n)$. Take one of the vertices A of the polygon. It can be joined to any of the vertices B such that there are an even number of vertices between A and B (Fig. 32). Accordingly, all methods of joining vertices fall into classes

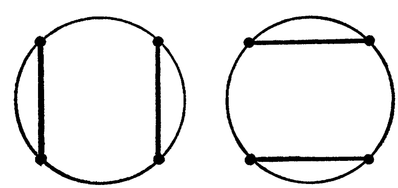


Fig. 31.

depending on how many vertices remain to the left of the segment drawn from point A.

If there are $2s$ vertices here, then there will be $2(n-s-1)$ vertices on the other side. Thus, a $2n$-gon splits up into a $2s$-gon and a $2(n-s-1)$-gon. But it is possible, in a $2s$-gon, to draw the line segments in $F(s)$ ways so that they do not intersect. Now in a $2(n-s-1)$-gon, this can be done in $F(n-s-1)$ ways. By the rule of product, we find that the sth class includes $F(s)F(n-s-1)$ ways of drawing line-segments.

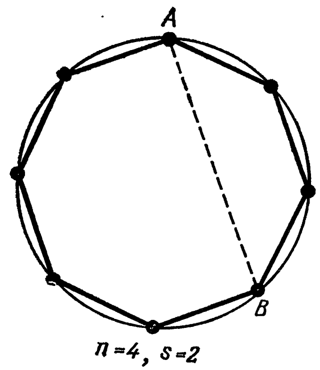


Fig. 32.

Hence, the total number of ways is $F(0)F(n-1)+F(1)F(n-2)+\ldots+F(n-1)F(0)$. We have derived the recurrence relation

$$F(n)=F(0)F(n-1)+F(1)F(n-2)+\ldots+F(n-1)F(0)$$

This is the very same relation which the numbers $T_n=\frac{1}{n+1}C_{2n}^{n}$ satisfy. Since $F_0=T_0=1$, we have $F(n)=T_n$ for all n. Thus, in a $2n$-gon there are $T_n=\frac{1}{n+1}C_{2n}^{n}$ ways of drawing diagonals so that they do not intersect in pairs.

The answer is the same for the following problem.

In how many ways can a convex $(n+2)$-gon be partitioned into triangles by means of diagonals that do not intersect within the polygon?

Denote the number of ways by $\Phi(n)$. Take one of the sides of the polygon and classify all partitions depending on which vertex of the polygon coincides with the vertex of the triangle whose base is the chosen side (Fig. 33). If this triangle is removed, the polygon breaks up into an $(s+2)$-gon and an $(n-s+1)$-gon. Splitting these polygons into triangles and combining the partitions, we get all the partitions of the original polygon, which partitions include the removed triangle. Then, applying the rule of product and the rule of sum, we get the recurrence relation

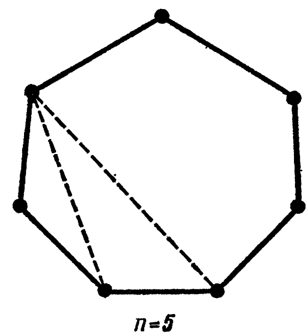


Fig. 33.

$$\Phi(n)=\Phi(0)\Phi(n-1)+\Phi(1)\Phi(n-2)+\ldots+\Phi(n-1)\Phi(0)$$

where we set $\Phi(0)=1$. We leave it to the reader to convince himself, using this relation, that $\Phi(n)=T_n=\frac{1}{n+1}C_{2n}^{n}$

DIFFICULTIES OF A MAJORDOMO

Combinatorial problems crop up in which one has to set up a whole system of recurrence relations that link several sequences. These relations express the $(n+1)$th terms of the sequences via the preceding terms not only of the given sequence, but of the other sequences as well.

It so happened that the majordomo of King Arthur noticed that six pairs of hostile knights had been invited to dine at the Round Table. In how many ways can they be seated so that no two adversaries sit side by side?

If we find some seating arrangement for the knights, then by circulating them round the table we get another 11 seatings. For the time being we will not consider as distinct those modes obtained by such circular rearrangements.

Let us introduce the following notations. The number of knights is $2n$. A_n will denote the number of ways of seating them so that no two opponents sit side by side, B_n will denote the number of ways of seating exactly one pair of enemies, and C_n, the number of ways of seating exactly two pairs of warring neighbours.

First, let us derive a formula expressing A_{n+1} in terms of A_n, B_n and C_n. Let $n+1$ pairs of knights be seated so that no two enemies sit next to each other. We assume that all the hostile pairs of knights are labelled. We ask the pair of knights with the number $n+1$ to rise. There are then three possibilities: there is not a single pair of hostile neighbours seated at the table, there is one such pair, and, there are two such pairs (the knights who got up could have been separating these pairs).*

Now let us find out in how many ways we can again seat the knights who left the table so that there is no pair of warring knights.

Seating them is elementary if there are two pairs of enemies at the table. Then one of the newcomers takes a seat between the knights of the first pair, the other between those of the second pair. This can be done in two ways. But since the number of ways of seating $2n$ knights in which two pairs of neighbours are enemies is C_n, there are $2C_n$ ways in all.

Now suppose there is only one pair of enemies sitting side by side. One of the returning knights has to sit down between them. Then there will be $2n+1$ knights between which there are $2n+1$ places. Of these, there are two (next to the guest who just took his seat) forbidden to the second knight, and so he has $2n-1$ places left. Since either of the two knights who left can come in first, there are $2(2n-1)$ seating arrangements. But there are B_n cases when $2n$ knights took seats so that exactly two enemies were side by side. And so we get $2(2n-1)B_n$ seating arrangements for the guests in the manner required.

Finally, suppose that no two enemies were seated side by side. Then the first knight takes a seat between any two guests, which he can do in $2n$ ways. His enemy is then left with a choice of $2n-1$ seats: he can take any place, with the exception of the two places adjacent to the knight who just sat down. Thus, if $2n$ knights are already seated as required, then the returning guests can be seated in $2n(2n-1)$ ways. This makes a total of $2n(2n-1)A_n$ ways.

As we have already pointed out, these cases exhaust all the possibilities, and so we have the recurrence relation

$$A_{n+1} = 2n(2n-1)A_n + 2(2n-1)B_n + 2C_n \qquad (9)$$

This relation is still not enough to be able to find A_n for all values of n. We have yet to indicate how we are to express B_{n+1} and C_{n+1} in terms of A_n, B_n, C_n.

Suppose that among the $2n+2$, $n>1$, knights there proved to be exactly one pair of enemy neighbours. We know that this can occur in B_{n+1} cases. To avoid unpleasantness, we ask them to get up and leave the table. This leaves $2n$ knights, and one of two things is possible: either there are no enemy neighbours among them, or there is exactly one pair of such enemies—they were on either side of the two who left

* Here and henceforth we assume that $n>1$ as well. When $n=1$, the arguments that follow become meaningless.

and now find themselves next to one another. In the latter case, those who left can only be seated in their old places, otherwise another hostile pair will appear. But since $2n$ knights can be seated in B_n ways so that there is only one hostile pair together, we obtain $2B_n$ variants (the returning knights can change places). But in the first case, we can seat those who left between any two knights, which is to say in $2n$ ways; and since they can be interchanged, we have $4n$ ways. Combining them with all the other seating arrangements of n pairs of knights, with no adjacent enemies, we get $4nA_n$ ways. Finally, the label of the pair that left and returned could be any one from 1 to $n + 1$, which implies that the recurrence relation for B_{n+1} is of the form

$$B_{n+1} = 4n(n+1)A_n + 2(n+1)B_n \qquad (10)$$

Finally, let us take up the case when there are two hostile adjacent pairs among $2n + 2$ knights. The labels of these pairs can be chosen in $C_{n+1}^2 = \frac{n(n+1)}{2}$ ways. Let us replace each pair by a new knight and we consider the two new knights as enemies. Then there will be $2n$ knights seated at the table, and among them there will be either no pair of enemy neighbours (if the new knights are not sitting side by side), or there will be only one such pair.

The first version occurs in A_n cases. We can revert to the original company in 4 ways because of the possibility of changing the order of the knights in each pair. Therefore, the first version leads to $4C_{n+1}^2 A_n = 2n(n+1)A_n$ seating arrangements.

The second version, however, can occur in $\frac{1}{n}B_n$ cases*. Here too we can revert to the original company in 4 ways, and we get a total of $2(n+1)B_n$ ways, which implies that, for $n \geqslant 1$,

$$C_{n+1} = 2n(n+1)A_n + 2(n+1)B_n \qquad (11)$$

We now have a system of recurrence relations

$$A_{n+1} = 2(2n-1)(nA_n + B_n) + 2C_n, \qquad (9)$$

$$B_{n+1} = 2(n+1)(2nA_n + B_n), \qquad (10)$$

$$C_{n+1} = 2(n+1)(nA_n + B_n) \qquad (11)$$

which hold true for $n \geqslant 2$. However, a simple computation shows that $A_2 = 2$, $B_2 = 0$, $C_2 = 4$. It therefore follows from the relations (9)-(11) that $A_3 = 32$, $B_3 = 48$, $C_3 = 24$. Continuing, we find that the guests may be seated in the required fashion in a total of $A_6 = 12{,}771{,}840$ ways.

The foregoing problem is much like the following one, which is often referred to as the "problèm des ménages".

In how many ways is it possible to seat n married couples at a circular table so that men and women are in alternate position and no wife sits next to her husband?

This problem is solved in roughly the same way as the majordomo problem. First seat the women. If the seats are labelled, then either all the women are on even seats, or on odd seats. But the number of even places is n, and the women can thus be seated in $n!$ ways. The number of ways is the same for the odd seats. Thus, the women can be seated in $2 \times (n!)$ ways. Then consider the cases when no husband can sit next to his wife, when a couple is seated together, and, finally, when two couples come together. We leave it to the reader to set up the appropriate system of recurrence relations.

LUCKY TROLLEYBUS TICKETS

There are people who believe that there are lucky trolleybus tickets, such as when the sum of the digits occupying the even positions is equal to the sum of the digits in odd sites. Say, the ticket 631 752 is "lucky" since $6 + 1 + 5 =$

* There are B_n cases when two hostile knights are side by side. If we indicate the precise two that have to sit together, we get n times fewer cases.

$= 3 + 7 + 2 = 12$. Our task now is to find the "lucky" numbers between 000000 and 999999.

To do this, first find out how many three-digit numbers have the given sum N of digits (here we consider numbers of the form 075 and even 000 as three-digit numbers). This problem resembles the one solved on page 62: the number of summands is 3, the sum is N, and the summands are from 0 to 9. Let us denote the number of its solutions by $F(3, 9; N)$. Then we have the recurrence relation

$$F(3, 9; N) = F(2, 9; N) + F(2, 9; N-1) + F(2, 9; N-2) + F(2, 9; N-3) + F(2, 9; N-4) + F(2, 9; N-5) + F(2, 9; N-6) + F(2, 9; N-7) + F(2, 9; N-8) + F(2, 9; N-9)$$

In exactly the same way, we have

$$F(2, 9; N) = F(1, 9; N) + F(1, 9; N-1) + \ldots + F(1, 9; N-9)$$

It is clear that $F(1, 9; N) = 1$ if $0 \leqslant N \leqslant 9$, and $F(1, 9; N) = 0$ otherwise. Using these relations, we can easily fill in the following table.

Table 8

k \ N	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0
2	1	2	3	4	5	6	7	8	9	10	9	8	7	6	5
3	1	3	6	10	15	21	28	36	45	55	63	69	73	75	75

k \ N	15	16	17	18	19	20	21	22	23	24	25	26	27
1	0	0	0	0	0	0	0	0	0	0	0	0	0
2	4	3	2	1	0	0	0	0	0	0	0	0	0
3	73	69	63	55	45	36	28	21	15	10	6	3	1

Now we can find the lucky tickets by squaring the numbers of the third row and adding the results. True enough, every lucky ticket has the same sum of digits in even and odd sites. Suppose this sum is N. The number in the Nth site of the third row of our table indicates how many three-digit numbers have the sum N of digits. In other words, this number indicates in how many ways it is possible to choose the digits occupying even sites (that is, second, fourth and sixth). In the same number of ways we can choose the digits in the odd sites (first, third, and fifth). Since these choices are independent of one another, it follows, by the rule of product, that there are $[F(N)]^2$ lucky tickets with the sum N of digits in even sites. But then, by the rule of sum, the total number of lucky numbers is

$$2[1^2 + 3^2 + 6^2 + 10^2 + 15^2 + 21^2 + 28^2 + 36^2 + 45^2 + 55^2 + 63^2 + 69^2 + 73^2 + 75^2]$$

This yields the answer: 55,252.

RECURRENCE TABLES

In combinatorics, one often deals with quantities that depend on several numbers, not one. For example, the number C_n^k is a function both of n and of k. If a quantity at hand, $F(n, k)$, say, depends on two natural numbers n and k, then its values may be arranged in the form of a table, positioning $F(n, k)$ at intersections of the nth row and kth column. We have already encountered such quantities a number of times in Chapter V: the arithmetic square, arithmetic triangles and generalized arithmetic triangles were just such tables.

In all the examples studied in Chapter V, there existed certain relationships between the elements of the table. These relationships enabled us to compute the elements of the nth row of the table on the basis of elements of the preceding row and, possibly, on the basis of the first several elements of the nth row. For this reason, if the first row of the table was specified and also the first elements of the other rows, the remaining rows could be computed one after the other. Such tables are reminiscent of recurrence relations and we shall call them *recurrence tables* from now on.

For the arithmetic square, the recurrence relation is of the form

$$F(n, k)=F(n-1, k)+F(n, k-1) \qquad (12)$$

and the boundary conditions were specified as $F(n, 0) = 1$, $F(0, k) = 0$ for $k > 0$ (it will be recalled that relative to the arithmetic square we speak of the zeroth column or row, not the first).

The arithmetic pentagon and hexagon have recurrence relations that are of the same form as (12). The point is that these figures appeared when we counted the number of ways a rook can get to a certain square by moving over a board bounded by two perpendicular rays and one or two lines parallel to the principal diagonal. But a rook can reach square (n, k) either from $(n-1, k)$ or from $(k-1, n)$. And so no matter what the restrictions imposed on its motions, relation (12) will always hold. Now restrictions result in certain of the elements of the table being zero for sure. Such elements in the arithmetic pentagon are those lying above some straight line parallel to the principal diagonal; for the arithmetic hexagon, these are the elements outside a domain cut off by two straight lines parallel to the principal diagonal.

Quite different is the recurrence relation for the arithmetic triangle and the m-arithmetic triangle. Namely, for the m-arithmetic triangle,

$$F(n, k)=F(n-1, k-m+1)+$$
$$+F(n-1, k-m+2)+\ldots+F(n-1, k) \qquad (13)$$

Here, $F(0, 0)=1$ and $F(0, k)=0$ if $k>0$.

ALTERNATIVE SOLUTION OF THE MAJORDOMO PROBLEM

As yet another instance of the use of recurrence tables, we offer an alternative solution to the problem of the majordomo (see page 95). As the reader recalls, we were seeking the number of seating arrangements for $2n$ knights about the Round Table so that no two enemy knights sat together (there were n pairs of enemies among the $2n$ knights).

We denote by $F(m, n)$ the number of ways of seating the knights so that exactly m pairs of enemies come together. We will now derive a recurrence formula expressing $F(m, n+1)$ in terms of $F(k, n)$, $k = m-1$, m, $m+1$, $m+2$.

We assume that at first there were n pairs of knights at the table, then pair $n+1$ came and sat down. Now let us compute the number of cases of m pairs of adjacent enemies seated at the table. This can occur in the following manner:

(a) There were $m-1$ pairs of enemies seated together at the table. This could occur in $F(m-1, n)$ ways. In order to have m pairs of hostile neighbours, the new pair would have to sit down together without breaking up any one of the already seated enemy pairs. But there are $2n$ gaps between $2n$ knights and $m-1$ gaps are forbidden. This leaves $2n-m+1$ gaps where the newly arrived knights can take seats. Since each gap can be filled in two ways (the newcomers can change places), we get a total of

$$2(2n-m+1)F(m-1, n) \qquad (14)$$

ways.

(b) There were m pairs of enemies seated together. In this case, the newcomers could choose one of two things: either sit separately, without separating any pair of neighbouring enemies, or sit down together between two hostile neighbours. It is easy to compute that the first solution can be accomplished in $(2n - m)(2n - m - 1)$ ways, while the second can be done in $2m$ ways, making a total of $(2n - m)^2 - 2n + 3m$ ways. Since n pairs of knights can sit in $F(m, n)$ ways where there are m pairs of enemies, we get a total of

$$[(2n-m)^2-2n+3m]\,F(m, n) \qquad (15)$$

ways.

(c) We further consider the case when there were $m + 1$ pairs of enemies together among $2n$ knights [this occurs in $F(m + 1, n)$ ways]. In this case, one of the newcomers has to sit between one of the enemy pairs, and the other has to sit so that he does not upset any such pair. The former can be done in $m + 1$ ways, and the latter in $2n - m - 1$ ways. We get a total of $2(m + 1)(2n - m - 1)$ possibilities (the factor 2 appears because either of the newcomers can sit down between enemies). This case therefore offers a a total of

$$2(m+1)(2n-m-1)\,F(m+1, n) \qquad (16)$$

opportunities.

(d) Finally, suppose that there were $m + 2$ pairs of enemy neighbours. This could occur in $F(m + 2, n)$ ways. In order to obtain only m pairs of warring neighbours, each of the newcomers has to break up one such pair. The first knight can sit down in $m + 2$ ways, and the second one has only $m + 1$ seats to pick from. In all, there are

$$(m+1)(m+2)\,F(m+2, n) \qquad (17)$$

possibilities.

It is easy to see that we have exhausted all possibilities of seating at the Round Table m pairs of hostile neighbours from among $2n + 2$ knights. Therefore, $F(m, n)$ satisfies the following recurrence relation:

$$\begin{aligned} F(m, n+1) &= 2(2n-m+1)\,F(m-1, n) \\ &+ [(2n-m)^2-2n+3m]\,F(m, n) \\ &+ 2(m+1)(2n-m-1)\,F(m+1, n) \\ &+ (m+1)(m+2)\,F(m+2, n) \end{aligned} \qquad (18)$$

A direct computation shows that

$$F(0, 2)=2, \quad F(1, 2)=0, \quad F(2, 2)=4$$

(we do not consider as distinct the seating arrangements obtained by circular permutations).

Applying formula (18), we find that $F(0, 12) = 12{,}771{,}840$.

SOLUTION OF RECURRENCE RELATIONS

We shall say that a recurrence relation has order k if it permits expressing $f(n + k)$ in terms of $f(n)$, $f(n + 1)$, ..., $f(n + k - 1)$. Say,

$$f(n+2)=f(n)\,f(n+1)-3f^2(n+1)+1$$

is a recurrence relation of the second order, and

$$f(n+3)=6f(n)\,f(n+2)+f(n+1)$$

is a third-order recurrence relation.

If a recurrence relation of order k is given, then it is satisfied by an infinite number of sequences. The point is that the first k elements of the sequence can be specified quite arbitrarily, there being no relations between them. But if the first k elements are specified, then all the remaining elements are determined in unique fashion: element $f(k + 1)$ is expressed by virtue of the recurrence relation in term of $f(1), \ldots, f(k)$, element $f(k + 2)$ in terms of $f(2), \ldots, f(k + 1)$, etc.

Using recurrence relations and the initial terms, it is possible to write down the terms of a sequence one after the other, and sooner or later we can obtain any term. However, in the process we will have to write out all the preceding terms since without them we cannot write down the subsequent terms. Now in many cases we are only interested in one definite term of a sequence, the other terms being of no interest at all. In these cases, it is more convenient to have an explicit formula for the nth term of the sequence. We shall say that a certain sequence is a *solution* of a given recurrence relation if upon substitution of the sequence the relation is identically satisfied. To illustrate, say, the sequence

$$2,\ 4,\ 8,\ \ldots,\ 2^n,\ \ldots$$

is one of the solutions of the recurrence relation

$$f(n+2)=3f(n+1)-2f(n)$$

Indeed, the general term of this sequence is of the form $f(n) = 2^n$. Hence, $f(n+2) = 2^{n+2}$, $f(n+1) = 2^{n+1}$. But for any n, we have the identity $2^{n+2} = 3 \times 2^{n+1} - 2 \times 2^n$. And so 2^n is a solution of the indicated relation.

A solution of a kth order recurrence relation is termed general if it depends on k arbitrary constants $C_1, \ldots, C_k$, and a selection of these constants yields any solution of the given relation. For instance, the general solution of the relation

$$f(n+2)=5f(n+1)-6f(n) \tag{19}$$

is

$$f(n)=C_1 2^n+C_2 3^n \tag{20}$$

It is easy to verify that in fact the sequence (20) converts relation (19) into an identity. And so all we have to do is demonstrate that any solution of our relation can be represented as (20). But any solution of (19) is uniquely determined by the values of $f(1)$ and $f(2)$. We therefore have to prove that for arbitrary numbers a and b there exist values C_1 and C_2 such that

$$2C_1+3C_2=a$$

and

$$2^2C_1+3^2C_2=b$$

Now it is easy to see that for any values of a and b, the system of equations

$$\begin{cases} 2C_1+3C_2=a, \\ 4C_1+9C_2=b \end{cases} \tag{21}$$

has a solution. Therefore, (20) is indeed the general solution of the relation (19).

LINEAR RECURRENCE RELATIONS WITH CONSTANT COEFFICIENTS

Generally speaking, there are no general rules for solving recurrence relations. However, there is a class of relations very often encountered that is solvable by a uniform method. These are recurrence relations of the form

$$f(n+k)=a_1 f(n+k-1)+a_2 f(n+k-2)+\ldots+ a_k f(n) \tag{22}$$

where $a_1, a_2, \ldots, a_k$ are certain numbers. These relations are called *linear recurrence relations with constant coefficients.*

Let us first see how they are solved for $k = 2$, that is let us make a study of relations of the form

$$f(n+2)=a_1 f(n+1)\ a_2 f(n) \tag{23}$$

The solution of these relations is based on the following two assertions:

(1) If $f_1(n)$ and $f_2(n)$ are solutions of the recurrence relation (23), then for arbitrary numbers A and B the sequence $f(n) = Af_1(n) + Bf_2(n)$ is also a solution of this relation.

Indeed, by hypothesis, we have

$$f_1(n+2)=a_1 f_1(n+1)+a_2 f_1(n)$$

and

$$f_2(n+2)=a_1 f_2(n+1)+a_2 f_2(n)$$

Multiplying these equalities by A and B, respectively, and adding the resulting identities, we get

$$Af_1(n+2)+Bf_2(n+2)=a_1[Af_1(n+1)+ Bf_2(n+1)] + a_2[Af_1(n)+Bf_2(n)]$$

But this means that $Af_1(n)+Bf_2(n)$ is a solution of (23).

(2) If the number r_1 is a root of the quadratic equation

$$r^2=a_1r+a_2$$

then the sequence

$$1,\ r_1,\ r_1^2,\ \ldots,\ r_1^{n-1},\ \ldots$$

is a solution of the recurrence relation

$$f(n+2)=a_1f(n+1)+a_2f(n)$$

True enough, for if $f(n)=r_1^{n-1}$, then $f(n+1)=r_1^n$ and $f(n+2)=r_1^{n+1}$. Substituting these values into (23), we get

$$r_1^{n+1}=a_1r_1^n+a_2r_1^{n-1}$$

It holds true since, by hypothesis, we have $r_1^2=a_1r+a_2$.

Note that in addition to the sequence $\{r_1^{n-1}\}$ any sequence of the form

$$f(n)=r_1^{n+m},\quad n=1,\ 2,\ \ldots$$

is also a solution of (23). To prove this, it suffices to use assertion (23), setting $A=r_1^{m+1}$, $B=0$.

From the assertions (1) and (2) there follows a rule for the solution of linear recurrence relations of the second order with constant coefficients:

Given a recurrence relation

$$f(n+2)=a_1f(n+1)+a_2f(n) \tag{23}$$

Form the quadratic equation

$$r^2=a_1r+a_2 \tag{24}$$

which is called the characteristic equation of the given relation. If this equation has two distinct roots r_1 and r_2, then the general solution of relation (23) *is of the form*

$$f(n)=C_1r_1^{n-1}+C_2r_2^{n-1}$$

To prove this rule, note first that by assertion (2), $f_1(n)=r_1^{n-1}$ and $f_2(n)=r_2^{n-1}$ are solutions of our relation. But then, by assertion (1), $C_1r_1^n+$ $+\ C_2r_2^n$ too is a solution. It need only be demonstrated that any solution of (23) can be written in this form. But any solution of a second-order relation is determined by the values $f(1)$ and $f(2)$. It therefore suffices to show that the system of equations

$$\begin{cases} C_1+C_2=a, \\ C_1r_1+C_2r_2=b \end{cases}$$

has a solution for arbitrary a and b. Verify that these solutions are

$$C_1=\frac{b-ar_2}{r_1-r_2},\qquad C_2=\frac{ar_1-b}{r_1-r_2}$$

Later we shall take up the case when both roots of equation (24) coincide. Meanwhile we give an example of this rule.

When we studied the Fibonacci numbers, we arrived at the recurrence relation

$$f(n)=f(n-1)+f(n-2) \tag{25}$$

Its characteristic equation is of the form

$$r^2=r+1$$

The roots of this quadratic equation are the numbers

$$r_1=\frac{1+\sqrt{5}}{2},\qquad r_2=\frac{1-\sqrt{5}}{2}$$

And so the general solution of the Fibonacci relation is of the form

$$f(n)=C_1\left(\frac{1+\sqrt{5}}{2}\right)^n+C_2\left(\frac{1-\sqrt{5}}{2}\right)^n \tag{26}$$

(we took advantage of the remark made above and took n instead of $n-1$ as the exponents).

We used the term Fibonacci numbers for the solution to the relation (25) that satisfies the initial conditions $f(0)=1$ and $f(1)=2$, that is for the sequence 1, 2, 3, 5, 8, 13, ... It is often more convenient to adjoin the numbers 0 and 1 at the beginning, that is to consider the sequence 0, 1, 1, 2, 3, 5, 8, 13, ... It is clear that this sequence satisfies the same recurrence relation (25) and the initial conditions $f(0)=0$,

$f(1)=1$. Putting $n=0$ and $n=1$ (26), we get a system of equations for C_1 and C_2:

$$\begin{cases} C_1+C_2=0, \\ \dfrac{\sqrt{5}}{2}(C_1-C_2)=1 \end{cases}$$

From this we find that $C_1=-C_2=\dfrac{1}{\sqrt{5}}$ and therefore

$$f(n)=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right] \qquad (27)$$

At first glance it seems remarkable that this expression assumes integral values for all the natural values of n.

THE CASE OF EQUAL ROOTS OF A CHARACTERISTIC EQUATION

Let us now examine the case when both roots of a characteristic equation coincide: $r_1=r_2$. In this case, the expression $C_1r_1^{n-1}+C_2r_2^{n-1}$ will no longer be the general solution, the point being that since $r_1=r_2$, this solution can be written as

$$f(n)=(C_1+C_2)r_1^{n-1}=Cr_1^{n-1}$$

We only have one arbitrary constant C and it is impossible, speaking generally, to choose it so as to satisfy the two initial conditions $f(1)=a$ and $f(2)=b$.

And so we have to find some second solution distinct from $f_1(n)=r_1^{n-1}$. It turns out that $f_2(n)=nr_1^{n-1}$ is such a solution. Indeed, if the quadratic equation $r^2=a_1r+a_2$ has two coincident roots $r_1=r_2$, then by Vieta's theorem, $a_1=2r_1$, $a_2=-r_1^2$. And so our equation is written thus:

$$r^2=2r_1r-r_1^2$$

But then the recurrence relation is of the form

$$f(n+2)=2r_1f(n+1)-r_1^2f(n) \qquad (28)$$

Let us verify that $f_2(n)=nr_1^{n-1}$ is indeed its solution. We have

$$f_2(n+2)=(n+2)r_1^{n+1} \text{ but } f_2(n+1)=(n+1)r_1^n.$$

Substituting these values into (28), we get the obvious identity

$$(n+2)r_1^{n+1}=2(n+1)r_1^{n+1}-nr_1^{n+1}$$

Hence nr_1^{n-1} is a solution of our relation.

Now we already know two solutions $f_1(n)=r_1^{n-1}$ and $f_2(n)=nr_1^{n-1}$ of the given relation. Its general solution is written thus:

$$f(n)=C_1r_1^{n-1}+C_2nr_1^{n-1}=r_1^{n-1}(C_1+C_2n)$$

Now, by choosing C_1 and C_2 we can satisfy any initial conditions.

Linear recurrence relations with constant coefficients, the order of the relations exceeding two, are solved in the same manner. Let the relation have the form

$$f(n+k)=a_1f(n+k-1)+\ldots+a_kf(n) \qquad (29)$$

Set up the characteristic equation

$$r^k=a_1r^{k-1}+\ldots+a_k$$

If all the roots $r_1, \ldots, r_k$ of this kth degree algebraic equation are distinct, then the general solution of (29) is of the form

$$f(n)=C_1r_1^{n-1}+C_2r_2^{n-1}+\ldots+C_kr_k^{n-1}$$

But if, say, $r_1=r_2=\ldots=r_s$, then to this root there correspond the solutions

$$f_1(n)=r_1^{n-1},\ f_2(n)=nr_1^{n-1},\ f_3(n)=n^2r_1^{n-1},\ \ldots,\ f_s(n)=n^{s-1}r_1^{n-1}$$

of the recurrence relation (29). In the general solution, to this root corresponds the part

$$r_1^{n-1}[C_1+C_2n+C_3n^2+\ldots+C_sn^{s-1}]$$

Constructing such expressions for all the roots and combining them, we get the general solution to relation (29).

By way of illustration, let us solve the recurrence relation

$$f(n+4)=5f(n+3)-6f(n+2)-4f(n+1)+8f(n)$$

The characteristic equation here is of the form

$$r^4-5r^3+6r^2+4r-8=0$$

Solving it, we get the roots

$$r_1=2,\ r_2=2,\ r_3=2,\ r_4=-1$$

Hence, the general solution of our relation is of the form

$$f(n)=2^{n-1}[C_1+C_2n+C_3n^2]+C_4(-1)^{n-1}$$

APPLICATION OF THE THEORY OF RECURRENCE RELATIONS TO PROBLEMS OF TRANSMITTING INFORMATION

We have already considered the problem of the number of distinct communications that can be transmitted in time T if the transmission time of the separate signals is known (see page 61). In doing so, we arrived at the recurrence relation

$$f(T)=f(T-t_1)+f(T-t_2)+\ldots+f(T-t_n) \qquad (30)$$

here, $f(0)=1$ and $f(T)=0$ if $T<0$.

We consider the numbers $T, t_1, \ldots, t_n$ as integral, and we denote by $\lambda_1, \ldots, \lambda_k$ the roots of the characteristic equation (30). Then the general solution to the equation is of the form

$$f(T)=C_1\lambda_1^T+\ldots+C_k\lambda_k^T$$

Let λ_1 be the largest, in absolute value, root of the characteristic equation. Then for large values of T, all the terms will be negligibly small compared with the first one, and we get

$$f(T)\backsim C_1\lambda_1^T$$

This equality enables us to give an approximate estimate of the number of communications that can be transmitted in time T by means of the given system of signals.

A THIRD SOLUTION TO THE MAJORDOMO PROBLEM

Both of the foregoing solutions of the majordomo problem lead to recurrence relations. We will now derive a formula for the solution of these relations such that permits us to compute at once the number of seating arrangements of hostile knights at the Round Table. We take advantage of the principle of inclusion and exclusion. Denote by α_k an event such that the kth pair of hostile knights are together. Compute $N(\alpha_1, \ldots \alpha_k)$, which is the number of cases when k pairs of enemies are seated together. The first pair may be seated in $4n$ ways (choose a place for one in $2n$ ways, seat the second one adjacently clockwise and note that the knights can change places). For the rest of the knights there remain $2n-2$ seats, and they must be occupied so that the second, third,. . ., kth pairs of enemies are side by side. Combine these pairs of knights into a single "entity". These $k-1$ pairs of knights and the $2n-2k$ remaining knights may be permuted in $(2n-k-1)!$ ways. If we take one of these permutations and seat the knights in order in the vacant places, then the $k-1$ pairs of enemies that we chose will be together. This condition is not violated even if we have some of the enemies that are sitting together change places. Since such reseatings can be done in 2^{k-1} ways, we get a total of $4n2^{k-1}(2n-k-1)!$ seating arrangements. Thus

$$N(\alpha_1 \ldots \alpha_k)=2^{k+1}n(2n-k-1)!$$

We want to find out in how many cases not a single pair of enemies are neighbours at the Round Table, that is, we wish to compute $N(\alpha_1', \ldots \alpha_n')$. Taking into account that k pairs may be chosen in C_n^k ways, we find, by the inclusion and exclusion formula, that

$$A_n=N(\alpha_1' \ldots \alpha_n')$$
$$=(2n)!-C_n^1 2^2 n(2n-2)!+C_n^2 2^3 n(2n-3)!-\ldots$$
$$+(-1)^k C_n^k 2^{k+1} n(2n-k-1)!+\ldots$$
$$\ldots+(-1)^n 2^{n+1} n!$$

CHAPTER VII

COMBINATORICS AND SERIES

The method of recurrence relations permits solving many combinatorial problems. But in a large number of cases the recurrence relations are hard to set up and still more difficult to solve. At times these difficulties are overcome through the use of generating functions. Since the concept of a generating function is connected with infinite power series, we will have to investigate these series first.

DIVIDING POLYNOMIALS

The reader of course knows how to divide polynomials. If we have two polynomials $f(x)$ and $\varphi(x)$, then there always exist the polynomials $q(x)$ (*quotient*) and $r(x)$ (*remainder*) such that $f(x) = \varphi(x)\, q(x) + r(x)$, the degree of $r(x)$ being less than the degree of $\varphi(x)$ or $r(x) = 0$. Here, $f(x)$ is called the *dividend*, and $\varphi(x)$ the *divisor*. But if we want the division to be exact, then we have to admit, for the quotient, not only polynomials but also infinite power series. To obtain the quotient, we have to arrange the polynomials in increasing powers of x and divide beginning with the lower terms. Consider, by way of illustration, the division of 1 by $1-x$. We have

$$
\begin{array}{r|l}
 & 1+x+x^2+\dots \\
\hline
1-x & 1 \\
 & \underline{\mp 1 \pm x} \\
 & \quad x \\
 & \quad \underline{\mp x \pm x^2} \\
 & \qquad x^2 \\
 & \qquad \underline{\mp x^2 \pm x^3} \\
 & \qquad\quad x^3 \dots
\end{array}
$$

The division process will clearly never end (as for instance when we convert $\frac{1}{3}$ into a nonterminating decimal). Arguing by induction, it is easy to see that all coefficients of the quotient are equal to unity. And so for the quotient we get an infinite series:

$$1+x+x^2+\dots+x^n+\dots$$

Generally, if $f(x)$ and $\varphi(x)$ are two polynomials:

$$f(x)=a_0+\dots+a_nx^n, \quad \varphi(x)=b_0+\dots+b_mx^m$$

and the constant term b_0 of $\varphi(x)$ is different from zero, $b_0 \neq 0$, then division of $f(x)$ by $\varphi(x)$ yields the infinite series

$$c_0+c_1x+\dots+c_kx^k+\dots \tag{1}$$

For example, if we take polynomials $f(x) = 6x^3-2x^2+x+3$ and $\varphi(x)=x^2-x+1$, the new mode of division yields

$$
\begin{array}{r|l}
 & 3+4x-x^2+x^3+2x^4+\dots \\
\hline
1-x+x^2 & 3+x-2x^2+6x^3 \\
 & \underline{\mp 3 \pm 3x \mp 3x^2} \\
 & \quad 4x-5x^2+6x^3 \\
 & \quad \underline{\pm 4x \pm 4x^2 \mp 4x^3} \\
 & \qquad -x^2+2x^3 \\
 & \qquad \underline{\pm x^2 \mp x^3 \pm x^4} \\
 & \qquad\quad x^3+x^4 \\
 & \qquad\quad \underline{\mp x^3 \pm x^4 \mp x^5} \\
 & \qquad\qquad 2x^4-x^5 \\
 & \qquad\qquad \dots\dots
\end{array}
$$

The situation will be the same in all cases when $b_0 \neq 0$ and $r(x) \neq 0$. Only when $f(x)$ is exactly divisible by $\varphi(x)$ does the series (1) terminate to yield a polynomial.

ALGEBRAIC FRACTIONS AND POWER SERIES

When we divided the polynomial $f(x)$ by the polynomial $\varphi(x)$, we got an infinite power series. The question arises as to how this series is related to the algebraic fraction $\frac{f(x)}{\varphi(x)}$, that is, what meaning can be attached to the notation

$$\frac{f(x)}{\varphi(x)} = c_0+c_1x+\dots+c_nx^n+\dots \tag{2}$$

Let us consider, for instance, the expansion

$$\frac{1}{1-x} \simeq 1+x+x^2+\ldots+x^n+\ldots \qquad (3)$$

We do not use the equal sign because we do not know what meaning the sum of an infinite number of terms on the right has. To find out, let us try to substitute various values of x into both members of relation (3). To begin with, set $x = \frac{1}{10}$. Then the left member becomes $\frac{10}{9}$, the right turns into an infinite number series:

$$1+0.1+0.01+\ldots+0.000\ldots01+\ldots$$

Since we are not able to add an infinite number of terms, let us try first one, then two, then three, etc. summands. This operation yields the sums: $1, 1.1, 1.11, \ldots, \underbrace{1.111\ldots1}_{n \text{ units}}, \ldots$. It is clear that as n increases, these sums approach the value $\frac{10}{9} = 1.11, \ldots$, which the left member assumes in (3) for $x = \frac{1}{10}$.

The picture is the same if in place of x we substitute the number $\frac{1}{2}$ into both sides of (3). The left-hand member becomes 2, the right-hand member turns into an infinite number series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} + \ldots$. Taking, in succession, one, two, three, four, ... terms, we get the numbers $1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, \ldots, 2-\frac{1}{2^n}$. Clearly, as n increases, these numbers tend to the number 2.

However, if we take $x = 4$, then the left-hand member of (3) becomes $-\frac{1}{3}$, and the right-hand member yields the series $1 + 4 + 4^2 + \ldots + 4^n + \ldots$. If we add the terms of this series in succession, we get the sums 1, 5, 21, 85, ... These sums increase without bound and do not approach the number $-\frac{1}{3}$.

We have thus encountered two cases. To distinguish them, we introduce the concepts of convergence and divergence of a number series. Suppose we have an infinite number series

$$a_1+a_2+\ldots+a_n+\ldots \qquad (4)$$

We say that it *converges to the number* b if the difference $b-(a_1+a_2+\ldots+a_n)$ tends to zero as n increases without bound. In other words, no matter what number $\varepsilon > 0$ we take, the deviation of the sum $a_1+\ldots+a_n$ from b, from some number N onwards, is less than ε:

$$|b-(a_1+\ldots+a_n)| < \varepsilon \quad \text{if} \quad n \geqslant N$$

In this case, the number b is termed the *sum of the infinite series* $a_1+\ldots+a_n+\ldots$ and we write

$$b = a_1+\ldots+a_n+\ldots$$

If no number b exists to which the given series (4) approaches, then the series is termed *divergent*.

The investigation carried out above shows that

$$\frac{10}{9} = 1+0.1+0.01+\ldots+0.00\ldots01+\ldots,$$

$$2 = 1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}+\ldots$$

whereas the series $1 + 4 + 16 + \ldots + 4^n + \ldots$ diverges.

A more detailed investigation shows that if $|x| < 1$, then the series $1 + x + \ldots + x^n + \ldots$ converges to $\frac{1}{1-x}$, and if $|x| \geqslant 1$, then it diverges.

To prove this assertion, it suffices to note that

$$1+x+\ldots+x^n = \frac{1-x^{n+1}}{1-x}$$

and that as $n \to \infty$ the expression x^{n+1} tends to zero if $|x| < 1$ and to infinity if $|x| \geqslant 1$. For $x = \pm 1$ we get the divergent number series $1 + 1 + \ldots + 1 + \ldots$ and $1 - 1 + \ldots + 1 - 1 + \ldots$.

Thus, if $|x|<1$, then

$$\frac{1}{1-x}=1+x+\ldots+x^n+\ldots \tag{5}$$

It will be noticed that (5) is the familiar formula for the sum of an infinitely decreasing geometric progression.

We have thus figured out the meaning of the notation

$$\frac{1}{1-x}=1+x+\ldots+x^n+\ldots$$

It shows that for values of x lying in some interval, namely for $|x|<1$, the series on the right converges to $\frac{1}{1-x}$. We say that for $|x|<1$ the function $\frac{1}{1-x}$ expands into the power series $1+x+\ldots+x^n+\ldots$.

We can now elucidate a more general question. Suppose in the division of a polynomial $f(x)$ by a polynomial $\varphi(x)$, we get the power series

$$c_0+c_1x+\ldots+c_nx^n+\ldots \tag{6}$$

It then turns out that for sufficiently small values of x, (6) converges to $f(x)/\varphi(x)$.

The size of the interval of convergence depends on the roots of the denominator, that is, on numbers for which the denominator vanishes. Namely, if these numbers are $x_1, \ldots, x_k$ and r is the least of the numbers $|x_1|, \ldots, |x_k|$, then the series converges in the interval $|x|<r$. For instance, the function $1-x$ vanishes for $x=1$, and therefore the expansion of $\frac{1}{1-x}$ holds true only when $|x|<1$. Now the function $x^2-7x+10$ vanishes for $x_1=2$, $x_2=5$, and therefore the expansion of $\frac{x-1}{x^2-7x+10}$ converges for $|x|<2$.

Note that not one of the roots of the denominator is zero, since we assumed that the constant term of the denominator is nonzero, and therefore $\varphi(0)=b_0\neq 0$.

In other words, there is always an interval $|x|<r$ in which the following equality holds true:

$$\frac{f(x)}{\varphi(x)}=c_0+c_1x+\ldots+c_nx^n+\ldots \tag{7}$$

Not only algebraic fractions, but also many other functions can be expanded in power series. In mathematical analysis, for example, proof is given that

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots \tag{8}$$

and

$$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots \tag{9}$$

We will be interested in the expansion

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots \tag{10}$$

From (10), we see that

$$e=1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots \tag{11}$$

Taking a sufficient number of terms in the series (11), we get the value of e to any desired degree of accuracy. To 15 decimal places, e is of the form

2.718281828459045...

Note that the series (8), (9), and (10) converge for all values of x.

Note likewise the following important assertion.

A function $f(x)$ cannot have two distinct power-series expansions.

In other words, if

$$f(x)=a_0+a_1x+\ldots+a_nx^n+\ldots$$

and

$$f(x)=b_0+b_1x+\ldots+b_nx^n+\ldots$$

then

$$a_0=b_0,\quad a_1=b_1,\quad \ldots,\quad a_n=b_n,\ldots$$

OPERATIONS ON POWER SERIES

Now let us take a look at operations involving power series. Let the functions $f(x)$ and $\varphi(x)$ be expanded in the power series

$$f(x)=a_0+a_1x+\ldots+a_nx^n+\ldots \qquad (12)$$

and

$$\varphi(x)=b_0+b_1x+\ldots+b_nx^n+\ldots \qquad (13)$$

Then

$$f(x)+\varphi(x)=(a_0+a_1x+\ldots+a_nx^n+\ldots)+$$
$$+(b_0+b_1x+\ldots+b_nx^n+\ldots)$$

It turns out that the terms on the right can be interchanged and grouped together with terms of equal powers of x (this assertion is not so obvious as it might seem at first glance; the point is that in the right-hand member we have infinite sums, and it is by far not always possible to rearrange terms in infinite sums). After this rearrangement, we get

$$f(x)+\varphi(x)=(a_0+b_0)+(a_1+b_1)x+\ldots+$$
$$+(a_n+b_n)x^n+\ldots \qquad (14)$$

the series on the right of (14) is called the sum of the power series (12) and (13).

Now let us see what the power-series expansion is of the product of the functions $f(x)$ and $\varphi(x)$. We have

$$f(x)\varphi(x)=(a_0+a_1x+\ldots+a_nx^n+\ldots)\times$$
$$\times(b_0+b_1x+\ldots+b_nx^n+\ldots) \qquad (15)$$

As in the case of polynomials, the series on the right of (15) can be multiplied termwise (we omit the proof of this assertion). Let us find the series resulting from such a term-by-term multiplication. The constant term of the series is a_0b_0. Terms in x appear twice: when multiplying a_0 by b_1x and when multiplying a_1x by b_0. They yield

$$a_0b_1x+a_1b_0x=(a_0b_1+a_1b_0)x$$

In exactly the same way, we compute the terms containing x^2:

$$a_0b_2x^2+a_1b_1x^2+a_2b_0x^2=(a_0b_2+a_1b_1+a_2b_0)x^2$$

Generally, the coefficient of x^n has the form

$$a_0b_n+a_1b_{n-1}+\ldots+a_kb_{n-k}+\ldots+a_nb_0$$

Thus

$$f(x)\varphi(x)=a_0b_0+(a_0b_1+a_1b_0)x+\ldots+(a_0b_n+$$
$$+\ldots+a_nb_0)x^n+\ldots \qquad (16)$$

The series on the right of (16) is called the *product of the series* (12) and (13).

In particular, squaring (12), we get

$$f^2(x)=a_0^2+2a_0a_1x+(a_1^2+2a_0a_2)x^2+2(a_0a_3+$$
$$+a_1a_2)x^3+\ldots \qquad (17)$$

Now let us see how power series can be divided. Let the constant term of the series (13) be nonzero. We then show that there is a power series

$$c_0+c_1x+\ldots+c_nx^n+\ldots \qquad (18)$$

such that

$$(b_0+b_1x+\ldots+b_nx^n+\ldots)\times(c_0+c_1x+\ldots+$$
$$+c_nx^n+\ldots)=a_0+a_1x+\ldots+a_nx^n+\ldots \qquad (19)$$

To prove this, multiply together the series in the left member of this equation. We get the series

$$b_0c_0+(b_0c_1+b_1c_0)x+\ldots+(b_0c_n+\ldots$$
$$\ldots+b_nc_0)x^n+\ldots$$

For this series to coincide with the series (12), it is necessary and sufficient that the following equations hold:

$$b_0c_0=a_0,$$
$$b_0c_1+b_1c_0=a_1,$$
$$\cdots\cdots$$
$$b_0c_n+\ldots+b_nc_0=a_n,$$
$$\cdots\cdots$$

These equations yield an infinite system of equations for finding the coefficients $c_0, c_1, \ldots, c_n, \ldots$. From the first equation of the system we get $c_0=\frac{a_0}{b_0}$. Substituting this value into the second equation, we get

$$b_0c_1=a_1-\frac{b_1a_0}{b_0}$$

from which we find that $c_1 = \frac{a_1 b_0 - b_1 a_0}{b_0^2}$. Generally, if we have already found the coefficients $c_0, \ldots, c_{n-1}$, then for c_n we have

$$b_0 c_n = a_n - b_1 c_{n-1} - \ldots - b_n c_0$$

This equation is solvable since $b_0 \neq 0$.

We have thus demonstrated the existence of the series (18) which satisfies relation (19). The series (18) is called the quotient obtained from the division of the series (12) and (13). It can be proved that it is obtained in the expansion of the function $f(x)/\varphi(x)$. To summarize, power series can be added, multiplied and divided (division is permissible provided the constant term of the divisor is nonzero). These operations correspond to operations on expansible functions.

Note that now we can give a different interpretation to the expansion

$$\frac{a_0 + \ldots + a_n x^n}{b_0 + \ldots + b_m x^m} = c_0 + c_1 + \ldots + c_k x^k + \ldots \qquad (20)$$

The expansion signifies that the series $c_0 + c_1 x + \ldots + c_k x^k + \ldots$ is obtained upon the division of the finite power series $a_0 + \ldots + a_n x^n$ by the finite power series $b_0 + \ldots + b_m x^m$. In other words, this equation means that

$$(b_0 + \ldots + b_m x^m)(c_0 + c_1 x + \ldots + c_k x^k + \ldots) = a_0 + \ldots + a_n x^n \qquad (21)$$

where the product in the left-hand member is given by a formula of the type (16).

USING POWER SERIES TO PROVE IDENTITIES

Power series can be used to prove a great many identities. This is done by taking a function and expanding it in a power series in two ways. Since a function is uniquely representable as a power series, the coefficients of like powers of x in both series must coincide. This then leads to the identity being proved.

To illustrate, let us consider the familiar expansion

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots$$

Squaring both members of this expansion, we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots + (n+1)x^n + \ldots \qquad (22)$$

If we now replace x by $-x$, we get

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - \ldots + (-1)^n (n+1) x^n + \ldots \qquad (22')$$

Multiplying the expansions (22) and (22′), we obtain

$$\frac{1}{(1-x)^2}\frac{1}{(1+x)^2} = 1 + [1(-2) + 2 \times 1]x + [1 \times 3 + 2(-2) + 3 \times 1]x^2 + \ldots$$
$$\ldots + [1(-1)^n (n+1) + 2(-1)^{n-2} n + \ldots + (-1)^n (n+1) \times 1] x^n + \ldots \qquad (23)$$

It is obvious that the coefficients of odd powers of x vanish (each term appears twice in these coefficients with opposite sign). Now the coefficient of x^{2n} is equal to

$$1(2n+1) - 2 \times 2n + 3(2n-1) - \ldots + (2n+1)$$

But the function $\frac{1}{(1-x)^2(1+x)^2}$ may be expanded in a power series differently. We have

$$\frac{1}{(1-x)^2(1+x)^2} = \frac{1}{(1-x^2)^2}$$

Now the expansion for $\frac{1}{(1-x^2)^2}$ is obtained from the expansion (22) by replacing x by x^2:

$$\frac{1}{(1-x^2)^2} = 1 + 2x^2 + 3x^4 + \ldots + (n+1)x^{2n} + \ldots \qquad (24)$$

We know that no function can have two distinct power-series expansions. Therefore the coefficient of x^{2n} in the expansion (23) must be equal to the coefficient of x^{2n} in the expansion (24). This yields the following identity:

$$1(2n+1)-2\cdot 2n+3(2n-1)-\ldots$$
$$\ldots+(2n+1)\times 1=n+1$$

GENERATING FUNCTIONS

Now we are in a position to take up the basic topic of this chapter, the concept of a generating function. Suppose we have a sequence of numbers $a_0, a_1, \ldots, a_n, \ldots$. Form the power series

$$a_0+a_1x+\ldots+a_nx^n+\ldots$$

If this series converges in some interval to the function $f(x)$, then this function is called the *generating function* of the sequence of numbers $a_0, a_1, \ldots, a_n, \ldots$. For instance, from the formula

$$\frac{1}{1-x}=1+x+\ldots+x^n+\ldots$$

it follows that the function $\frac{1}{1-x}$ is the generating function of the sequence of numbers $1, 1, 1, \ldots, 1, \ldots$. And formula (22) shows that the function $\frac{1}{(1-x)^2}$ is the generating function of the sequence of numbers $1, 2, 3, 4, \ldots, n, \ldots$.

We will be interested in the generating functions of sequences $a_0, a_1, \ldots, a_n, \ldots$ which are connected in one way or another with combinatorial problems. Using these functions we are able to obtain a great variety of properties of the sequences. Also, we will consider the connection between generating functions and the solution of recurrence relations.

NEWTON'S BINOMIAL THEOREM

We will now derive the generating function of a finite sequence of numbers.

From school algebra we know that

$$(a+x)^2=a^2+2ax+x^2$$

and

$$(a+x)^3=a^3+3a^2x+3ax^2+x^3$$

These equations are special cases of the more general formula which yields the expansion of $(a+x)^n$. Let us write $(a+x)^n$ as

$$(a+x)^n=\underbrace{(a+x)(a+x)\ldots(a+x)}_{n \text{ times}} \tag{25}$$

Removing the brackets in the right-hand member, we will write all the factors in the order in which they come. For instance, $(a+x)^2$ will be written as

$$(a+x)^2=(a+x)(a+x)=aa+ax+xa+xx \tag{26}$$

and $(a+x)^3$ as

$$(a+x)^3=(a+x)(a+x)(a+x)$$
$$=aaa+aax+axa+axx+xaa+xax+xxa+xxx \tag{27}$$

It will be seen that formula (26) includes all permutations, with repetitions, made up of the letters x and a, two letters each time, and formula (27) includes the permutations, with repetitions, of the same letters taken three at a time. The same idea holds in the general case: *after removing the brackets in* (25) *we get all possible n-element permutations, with repetitions, of the letters x and a.*

Now collect like terms. Like terms are those containing the same number of letters x (then there will be the same number of a's too). Let us find the number of terms in which there are k x's and, hence, $n-k$ a's. These terms are permutations (with repetitions) made up of k x's and $n-k$ a's. Therefore, by formula (5) of Chapter II, their number will be equal to

$$P(k, n-k)=C_n^k=\frac{n!}{k!\,(n-k)!}$$

This implies that after collecting like terms, the expression x^ka^{n-k} will enter with the coefficient $C_n^k=\frac{n!}{k!(n-k)!}$. We have thus demonstrated that

$$(a+x)^n = C_n^0 a^n + C_n^1 a^{n-1}x + \ldots + C_n^k a^{n-k} x^k + \ldots + C_n^n x^n \quad (28)$$

Equation (28) is generally referred to as the binomial formula. If we put $a=1$, then we get

$$(1+x)^n = C_n^0 + C_n^1 x + \ldots + C_n^k x^k + \ldots + C_n^n x^n \quad (29)$$

We see that $(1+x)^n$ is the generating function of the numbers C_n^k, $k = 0, 1, \ldots, n$.

Using this generating function, it is comparatively easy to prove a large range of properties of the numbers C_n^k which were obtained earlier by means of rather ingenious reasoning.

Let us first prove that

$$C_{n+1}^k = C_n^k + C_n^{k-1} \quad (30)$$

For this, all we need to do is multiply both members of (29) by $1+x$ to get

$$(1+x)^{n+1} = (C_n^0 + C_n^1 x + \ldots + C_n^k x^k + \ldots + C_n^n x^n)(1+x)$$

Again expand the left member by the binomial theorem. Here, n is replaced by $n+1$ in the binomial formula. And so the coefficient of x^k will be C_{n+1}^k. In the right member, the term containing x^k appears twice when we remove the brackets: when multiplying $C_n^k x^k$ by 1 and $C_n^{k-1}x^{k-1}$ by x. Therefore, the coefficient of x^k in the right member is of the form $C_n^k + C_n^{k-1}$. But the polynomial should be the same on the left and on the right, and so the coefficients of x^k on the left and right must be the same. This proves that $C_{n+1}^k = C_n^k + C_n^{k-1}$.

On page 33 we proved this equality with the aid of combinatorial arguments. On that same page we had to use rather complicated reasoning to prove that

$$2^n = C_n^0 + C_n^1 + \ldots + C_n^k + \ldots + C_n^n \quad (31)$$

Using formula (29) the proof is practically instantaneous: all we have to do is put $x=1$. And if we put $x=-1$, we get

$$0 = C_n^0 - C_n^1 + C_n^2 - C_n^3 + \ldots + (-1)^k C_n^k + \ldots + (-1)^n C_n^n$$

In other words, the sum of the values C_n^k with even k is equal to the sum of the values C_n^k with odd k:

$$C_n^0 + C_n^2 + C_n^4 + \ldots + C_n^{2m} + \ldots = C_n^1 + C_n^3 + \ldots + C_n^{2m+1} + \ldots \quad (32)$$

Both sums are finite and terminate when $2m$ (or $2m+1$) exceeds n.

A curious result is obtained if we put $x = i$, $n = 4m$ in (29). A simple computation shows that $(1+i)^4 = -4$. And so $(1+i)^{4m} = (-4)^m$. We thus obtain the equation

$$(-4)^m = C_{4m}^0 + C_{4m}^1 i + C_{4m}^2 i^2 + C_{4m}^3 i^3 + C_{4m}^4 i^4 + \ldots + C_{4m}^{4m} i^{4m}$$
$$= C_{4m}^0 + C_{4m}^1 i - C_{4m}^2 - C_{4m}^3 i + C_{4m}^4 + \ldots + C_{4m}^{4m}$$

Separating the real part and the imaginary parts, we arrive at the identities

$$C_{4m}^1 - C_{4m}^3 + C_{4m}^5 - \ldots - C_{4m}^{4m-1} = 0 \quad (33)$$

$$C_{4m}^0 - C_{4m}^2 + C_{4m}^4 + \ldots + C_{4m}^{4m} = (-4)^m \quad (34)$$

Find out for yourself what identities result if we put $n = 4m+1$, $4m+2$, $4m+3$.

Another proof made simple by the generating function is that of

$$C_{n+m}^s = C_n^0 C_m^s + C_n^1 C_m^s + \ldots + C_n^k C_m^{s-k} + \ldots + C_n^n C_m^{s-n} \quad (35)$$

(here, we put $C_m^{s-k} = 0$ for $s-k<0$; for this reason, k actually varies from 0 to the smallest of the numbers m, n). To prove this, we have to take the expansions

$$(1+x)^n = C_n^0 + C_n^1 x + \ldots + C_n^k x^k + \ldots + C_n^n x^n$$

and

$$(1+x)^m = C_m^0 + C_m^1 x + \ldots + C_m^s x^s + \ldots + C_m^m x^m$$

and multiply the left and right members of these equations. We then find that

$$(1+x)^{n+m} = [C_n^0 + C_n^1 x + \ldots + C_n^k x^k + \ldots + C_n^n x^n] \times [C_m^0 + C_m^1 x + \ldots + C_m^s x^s + \ldots + C_m^m x^m]$$

Now apply the binomial formula to the left member (for the exponent $n + m$), and remove brackets in the right member. Comparing coefficients of x^s on the left and on the right, we obtain (35). A particular case of this equality is

$$C_{2n}^n = (C_n^0)^2 + (C_n^1)^2 + \ldots + (C_n^n)^2 \qquad (35')$$

(recall that $C_n^k = C_n^{n-k}$).

THE MULTINOMIAL THEOREM

Using the binomial formula, we can expand more involved expressions such as, for example, $(x + y + z)^4$. Namely,

$$(x+y+z)^4 = [(x+y)+z]^4$$
$$= (x+y)^4 + C_4^1 (x+y)^3 z + C_4^2 (x+y)^2 z^2 + $$
$$+ C_4^3 (x+y) z^3 + C_4^4 z^4$$

Now expand $(x + y)^4$, $(x + y)^3$, $(x + y)^2$, again using the binomial formula. We get

$$(x+y+z)^4 = x^4 + y^4 + z^4 + 4x^3y + 4x^3y + 4xy^3$$
$$+ 4y^3z + 4xz^3 + 4yz^3 + 6x^2y^2 + 6x^2z^2$$
$$+ 6y^2z^2 + 12x^2yz + 12xy^2z + 12xyz^2 \qquad (36)$$

But this method is too complicated. What is more, it is hard to say what coefficient the term $x^2y^4z^3$ has in the expansion of $(x + y + z)^9$. It is therefore desirable to derive a formula that directly gives the expansion of the expression

$$(x_1 + x_2 + \ldots + x_m)^n \qquad (37)$$

It is not so difficult to guess what this formula is. In proving the binomial theorem, we saw that in the expansion of $(a + x)^n$ the term $x^k a^{n-k}$ has the coefficient $P(k, n - k)$. We can conjecture that in the expansion of $(x_1 + x_2 + \ldots + x_m)^n$ the coefficient of $x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$ will be $P(k_1, k_2, \ldots, k_m)$. We will now prove that such is the case.

Write $(x_1 + x_2 + \ldots + x_m)^n$ as a product of n factors and remove the brackets, writing out all factors in the order of their appearance. What we get, clearly, is all possible permutations (with repetitions) made up of the letters $x_1, x_2, \ldots, x_m$ such that in each permutation there are n letters. But some of these permutations will yield like terms. This will occur if each letter appears in the first permutation as many times as it does in the second. Therefore, in order to find the coefficient of $x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$, we have to count the number of permutations, with repetitions, that contain the letter x_1 k_1 times, the letter x_2 k_2 times $\ldots$, the letter x_m k_m times. It is clear that every such permutation is a permutation, with repetitions, made up of k_1 letters x_1, of k_2 letters $x_2, \ldots$, and k_m letters x_m. We denote the number of such permutations by $P(k_1, k_2, \ldots, k_m)$. Thus, indeed, the coefficient of $x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}$ in the expansion of expression (37) is $P(k_1, k_2, \ldots, k_m)$ (where naturally, $k_1 + k_2 + \ldots + k_m = n$, this is because every term of the expansion includes one element from every bracket, and the total number of brackets being multiplied is n).

The formula that we have just proved can be written as

$$(x_1 + x_2 + \ldots + x_m)^n = \sum P(k_1, k_2, \ldots, k_m)\, x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m} \qquad (38)$$

where the sum is extended over all possible partitions $k_1 + k_2 + \ldots + k_m$ of the number n into m nonnegative parts. Recall that

$$P(k_1, k_2, \ldots, k_m) = \frac{(k_1 + k_2 + \ldots + k_m)!}{k_1! k_2! \ldots k_m!} \qquad (39)$$

Clearly, if the numbers $s_1, \ldots, s_m$ are obtained from the numbers $k_1, \ldots, k_m$ by permutation, then $P(s_1, \ldots, s_m) = P(k_1, \ldots, k_m)$. Therefore, to illustrate, the coefficients of x^2yz and xyz^2 in the expansion (36) are the same. This remark simplies writing out the terms of the expansion (37). It is sufficient to find the coefficients of partitions $n = k_1 + k_2 + \ldots + k_m$ such that $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_m$, and then permute the exponents in all possible ways.

For example, compute $(x + y + z)^5$. Disregarding the order of the summands, the number

5 may be partitioned into three parts in five ways:

$5=5+0+0$, $5=4+1+0$, $5=3+2+0$, $5=3+1+1$, $5=2+2+1$

But $P(5, 0, 0)=1$, $P(4, 1, 0)=5$, $P(3, 2, 0)=10$, $P(3, 1, 1)=20$, $P(2, 2, 1)=30$. Therefore,

$$(x+y+z)^5=x^5+y^5+z^5+5x^4y+5xy^4+5x^4z+ \\ +5xz^4+5y^4z+5yz^4+10x^3y^2+10x^2y^3+10x^3z^2 \\ +10x^2z^3+10y^3z^2+10y^2z^3+20x^3yz+20xy^3z \\ +20xyz^3+30x^2y^2z+30x^2yz^2+30xy^2z^2$$

Formula (38) enables us, with ease, to prove certain properties of the numbers $P(k_1, k_2, \ldots, k_m)$. If we put $x_1 = x_2 = \ldots = x_m = 1$ in this formula, we have

$$m^n=\sum P(k_1, \ldots, k_m) \qquad (40)$$

Here, the sum is extended over all partitions of the number n into m nonnegative parts: $n = k_1 + k_2 + \ldots + k_m$ with regard for the order of the summands.

To continue, if we multiply both members of (38) by $x_1 + x_2 + \ldots + x_m$ and apply a similar expansion to the left member, and then remove brackets in the right-hand member, we get the following recurrence relation for $P(k_1, \ldots, k_m)$:

$$P(k_1, k_2, \ldots, k_m)=P(k_1-1, k_2, \ldots, k_m) \\ +P(k_1, k_2-1, \ldots, k_m)+\ldots \\ \ldots+P(k_1, k_2, \ldots, k_m-1) \qquad (41)$$

However, if we multiply both sides of the expansions

$$(x_1+x_2+\ldots+x_m)^n=\sum P(k_1, k_2, \ldots, k_m)\, x_1^{k_1}x_2^{k_2}\ldots x_m^{k_m}$$

and

$$(x_1+x_2+\ldots+x_m)^s=\sum P(l_1, l_2, \ldots, l_m)\, x_1^{l_1}x_2^{l_2}\ldots x_m^{l_m}$$

and compare the coefficients of $x_1^{r_1}x_2^{r_2}\ldots x_m^{r_m}$ in both members, we get the identity

$$P(r_1, r_2, \ldots, r_m)=\sum_{k_p+l_p=r_p} P(k_1, k_2, \ldots \\ \ldots k_m)\, P(l_1, l_2, \ldots, l_m) \qquad (42)$$

Here, the summation in the right member is extended over all nonnegative integers $k_1, k_2, \ldots, k_m$; $l_1, l_2, \ldots, l_m$ such that $k_1+k_2+\ldots+k_m=n$, $l_1+l_2+\ldots+l_m=s$ and $k_1+l_1=r_1$, $k_2+l_2=r_2, \ldots, k_m+l_m=r_m$

We leave it to the reader to carry out the computations in detail.

The formulas (40) to (42) could quite naturally be obtained without resorting to the generating function (38). But then we would have had to invoke geometrical arguments like those on page 79, and not in the plane but in n-dimensional space. The generating function allows us to obtain these identities in automatic fashion via simple algebraic manipulations.

NEWTON'S SERIES

The formula for $(a + x)^n$ is often connected with Newton's name, but historically speaking this is not true. The mathematicians of Central Asia, Omar Khayyam, al-Kashi and others knew the formula for $(a + x)^n$ very well. In Western Europe, Blaise Pascal used it long before Newton. Newton's service here is of a different nature: he generalized the formula for $(x + a)^n$ to the case of nonintegral exponents. Precisely, what he did was to prove that *if a is a positive number and $|x| < a$, then for any real value of α we have the equation*

$$(x+a)^\alpha=a^\alpha+\alpha a^{\alpha-1}x+\frac{\alpha(\alpha-1)}{1\times 2}a^{\alpha-2}x^2 \\ +\ldots+\frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{1\times 2\ldots k}a^{\alpha-k}x^k \qquad (43)$$

Only this time we get an infinite series and not a finite number of summands. When n is a positive integer, the bracket $(n - n)$ vanishes. But this bracket enters into the coefficients of all terms beginning with the $(n+2)$th term, and so all these terms of the expansion are zero. That is why for n a positive integer, the series (43) becomes a finite sum.

We will not prove the formula (43) for all values α and will consider only the case when α is a negative integer, $\alpha = -n$. Then the formula to be proved takes the form

$$(x+a)^{-n} = a^{-n} - na^{-n-1}x + \frac{n(n+1)}{1\times 2}a^{-n-2}x^2 - \frac{n(n+1)(n+2)}{1\times 2\times 3}a^{-n-3}x^3 + \dots + (-1)^k\frac{n(n+1)\dots(n+k-1)}{1\times 2\dots k}a^{-n-k}x^k + \dots \quad (44)$$

This equation can also be written as

$$\left(1+\frac{x}{a}\right)^{-n} = 1 - C_n^1\left(\frac{x}{a}\right) + C_{n+1}^2\left(\frac{x}{a}\right)^2 - C_{n+2}^3\left(\frac{x}{a}\right)^3 + \dots + (-1)^k C_{n+k-1}^k\left(\frac{x}{a}\right)^k + \dots \quad (44')$$

$\left(\text{for it is true that } C_{n+k-1}^k = \frac{n(n+1)\dots(n+k-1)}{1\times 2\dots k}\right)$.

It will be more convenient for us to replace $\frac{x}{a}$ by $-t$ and prove the following equation instead of (44′):

$$(1-t)^{-n} = 1 + C_n^1 t + C_{n+1}^2 t^2 + \dots + C_{n+k-1}^k t^k + \dots \quad (45)$$

We carry out the proof by means of induction with respect to n. For $n=1$ we have $C_{n+k-1}^k = C_k^k = 1$ and so the relation becomes

$$\frac{1}{1-t} = 1 + t + t^2 + \dots + t^k + \dots \quad (46)$$

But this is the familiar formula for the sum of an infinite decreasing geometric progression (recall that here $|t| = \left|\frac{x}{a}\right| < 1$).

Now suppose that (45) has already been proved, we will show that from it follows the equation

$$(1-t)^{-n-1} = 1 + C_{n+1}^1 t + C_{n+2}^2 t^2 + \dots + C_{n+k}^k t^k + \dots \quad (47)$$

To do this, multiply both sides of (47) by $1-t$. If we obtain a true equality, then (47) holds true as well. But multiplication by $1-t$ yields

$$(1-t)^{-n} = [1 + C_{n+1}^1 t + C_{n+2}^2 t^2 + \dots + C_{n+k-1}^{k-1} t^{k-1} + C_{n+k}^k t^k + \dots](1-t)$$

Let us remove the brackets in the right member and collect terms. Terms in t^k appear twice: when $C_{n+k}^k t^k$ is multiplied by 1 and when $C_{n+k-1}^{k-1} t^{k-1}$ is multiplied by $-t$. Therefore, the coefficient of t^k on the right is

$$C_{n+k}^k - C_{n+k-1}^{k-1} = C_{n+k-1}^k$$

[see formula (11) on page 33].

But by the induction hypothesis the coefficient of t^k in the expansion of $(1-t)^{-n}$ is precisely equal to C_{n+k-1}^k. Since after multiplication by $1-t$ we obtain a true equality, it follows that the equality being proved, (45), is true as well.

If the reader does not want to proceed from the equality being proved to the already familiar relation and prefers the reverse approach, he must multiply both sides of (45) by the corresponding terms of relation (46). This yields

$$(1-t)^{-n-1} = (1 + C_n^1 t + C_{n+1}^2 t^2 + \dots + C_{n+k-1}^k t^k + \dots) \times (1 + t + t^2 + \dots + t^k + \dots)$$

He must now remove the brackets and take advantage of the identity

$$C_{n-1}^0 + C_n^1 + C_{n+1}^2 + \dots + C_{n+k-1}^k = C_{n+k}^k$$

(see page 34). We then arrive at the relation (47), which is what we are proving.

Thus, (45) is proved. Note once again that it holds true only for $|t| < 1$. If the incautious reader attempts to put $t = -1$ in both members of the equality and, on this basis, derives the "remarkable" formula

$$\frac{1}{2^n} = 1 - C_n^1 + C_{n+1}^2 - C_{n+2}^3 + \ldots$$

$$\ldots + (-1)^k C_{n+k-1}^k + \ldots \quad (48)$$

he will be seriously in error, for the right member is the sum of integers and such a sum cannot be equal to the fraction $1/2^n$.

In the 18th century, when the theory of infinite series had not yet been thoroughly investigated, eminent mathematicians were capable of making such mistakes. Decades of intensive study were needed to grasp the meaning of the sum of an infinite series, to realize when it exists and when it does not. Incidentally, it must be added that at the end of the 19th century the concept of the sum of an infinite series was appreciably generalized and there exist definitions for which formula (48) holds true. But such problems take us far beyond the scope of this book.

Let us compare the expansion we have proved,

$$(1+t)^{-n} = 1 - C_n^1 t + C_{n+1}^2 t^2 - \ldots$$

$$\ldots + (-1)^k C_{n+k-1}^k t^k + \ldots \quad (49)$$

with the formula

$$(1+t)^n = 1 + C_n^1 t + C_n^2 t^2 + \ldots$$

$$\ldots + C_n^k t^k + \ldots + t^n \ldots \quad (50)$$

We again conclude that when generalizing the symbol C_n^k to negative values of n, we have to put

$$C_{-n}^k = (-1)^k C_{n+k-1}^k$$

(see page 74). Now, for negative values of k, $C_n^k = 0$, since terms with negative powers of t do not enter into the expansions (49) and (50). For the same reasons, $C_n^k = 0$ when $0 \leqslant n < k$.

EXTRACTING SQUARE ROOTS

We proved the binomial theorem for integral values of the exponent. But, as has already been mentioned, the formula holds true not only for integral but for fractional (even for irrational) values of the exponent. We will not prove it for such values, and will only write down the expansions for $n = \frac{1}{2}$ and $n = -\frac{1}{2}$.

For $n = \frac{1}{2}$ the binomial formula becomes

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\times 2}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1\times 2\times 3}x^3 + \ldots$$

$$\ldots + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\ldots\left(\frac{1}{2}-k+1\right)}{1\times 2\ldots k}x^k + \ldots \quad (51)$$

Simplifying we get

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{2\times 4}x^2 + \frac{1\times 3}{2\times 4\times 6}x^3 - \ldots$$

$$\ldots + (-1)^{k-1}\frac{1\times 3\ldots(2k-3)}{2\times 4\ldots 2k}x^k + \ldots$$

In the same way, when $n = -\frac{1}{2}$ we find

$$(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{1\times 3}{2\times 4}x^2 - \ldots$$

$$\ldots + (-1)^k \frac{1\times 3\ldots(2k-1)}{2\times 4\ldots 2k}x^k + \ldots \quad (52)$$

These formulas may be written in a different way. Note that

$$\frac{1\times 3\ldots(2k-1)}{2\times 4\ldots 2k} = \frac{(2k)!}{2^{2k}(k!)^2} = \frac{1}{2^{2k}}C_{2k}^k$$

And so

$$(1+x)^{-\frac{1}{2}}=1-\frac{1}{2^2}C_2^1x+\frac{1}{2^4}C_4^2x^2-\dots$$
$$\dots+\frac{(-1)^k}{2^{2k}}C_{2k}^k x^k+\dots \qquad (53)$$

In the same way we get

$$(1+x)^{\frac{1}{2}}=1+\frac{1}{2}x-\frac{1}{2\times 2^3}C_2^1x^2+\frac{1}{3\times 2^5}C_4^2x^3-\dots$$
$$\dots+\frac{(-1)^{k-1}}{k\times 2^{2k-1}}C_{2k-2}^{k-1}x^k+\dots \qquad (54)$$

These expressions converge in the interval $|x|<$ <1. They can be used to extract square roots to any desired degree of accuracy. For example,

$$\sqrt{30}=\sqrt{25+5}=5\sqrt{1+0.2}=5(1+0.2)^{\frac{1}{2}}$$
$$=5\left[1+\frac{1}{2}\times 0.2-\frac{1}{2\times 4}\times 0.2^2+\right.$$
$$\left.+\frac{1\times 3}{2\times 4\times 6}\times 0.2^3-\dots\right]=5.4775\dots$$

However, we will not be interested in using these formulas in the extraction of roots; we will examine the relationships among the binomial coefficients that follow from the expansions obtained. To obtain these relations, square both sides of (53). By the rule for multiplying power series we see that the coefficient of x^k on the right is of the form

$$\frac{(-1)^k}{2^{2k}}\left[C_{2k}^k+C_2^1C_{2k-2}^{k-1}+C_4^2C_{2k-4}^{k-2}+\dots+C_{2k}^k\right]$$

On the left, we get

$$[(1+x)^{-\frac{1}{2}}]^2=\frac{1}{1+x}$$

But we know that

$$\frac{1}{1+x}=1-x+x^2-\dots+(-1)^k x^k+\dots$$

Comparing the coefficients of the powers x^k in both members of the equation, we arrive at the identity

$$C_{2k}^k+C_2^1C_{2k-2}^{k-1}+C_4^2C_{2k-4}^{k-2}+\dots+C_{2k}^k=2^{2k} \qquad (55)$$

Similar reasoning applied to (54) yields the identity

$$\frac{C_{2k-4}^{k-2}}{1\times(k-1)}+\frac{C_2^1C_{2k-6}^{k-3}}{2(k-2)}+\frac{C_4^2C_{2k-8}^{k-4}}{3(k-3)}+\dots$$
$$\dots+\frac{C_{2k-4}^{k-2}}{(k-1)\times 1}=\frac{C_{2k-2}^{k-1}}{k} \qquad (56)$$

which holds true for $k\geqslant 2$.

Now, multiplying the expansions (53) and (54) termwise, we get

$$1=\left[1+\frac{1}{2}x-\frac{1}{2\times 2^3}C_2^1x^2+\frac{1}{3\times 2^5}C_4^2x^3-\dots\right.$$
$$\left.\dots+\frac{(-1)^{k-1}}{k\times 2^{2k-1}}C_{2k-2}^{k-1}x^k+\dots\right]\left[1-\frac{1}{2^2}C_2^1x+\right.$$
$$\left.+\frac{1}{2^4}C_4^2x^2+\dots+\frac{(-1)^k}{2^{2k}}C_{2k}^kx^k+\dots\right] \qquad (57)$$

Remove the brackets in the right-hand member. This yields a power series, and from (57) it follows that all coefficients of this series (except the constant term) are zero. From this we have the identity

$$C_{2k-2}^{k-1}+\frac{1}{2}C_2^1C_{2k-4}^{k-2}+\frac{1}{3}C_4^2C_{2k-6}^{k-3}+\dots$$
$$\dots+\frac{1}{k}C_{2k-2}^{k-1}=\frac{1}{2}C_{2k}^k \qquad (58)$$

which holds for $k\geqslant 1$.

Finally, note that

$$(1+x)^{\frac{1}{2}}(1+x)^{-1}=(1+x)^{-\frac{1}{2}}$$

This implies that

$$\left(1+\frac{1}{2}x-\frac{1}{2\times 2^3}C_2^1x^2+\dots\right.$$
$$\left.\dots+\frac{(-1)^{k-1}}{k\times 2^{2k-1}}C_{2k-2}^{k-1}x^k+\dots\right)\times$$
$$\times(1-x+x^2-x^3+\dots+(-1)^kx^k+\dots)$$
$$=1-\frac{1}{2^2}C_2^1x+\frac{1}{2^4}C_4^2x^2-\dots$$
$$\dots+\frac{(-1)^k}{2^{2k}}C_{2k}^kx^k+\dots$$

Removing brackets in both members and comparing the coefficients of x^k in both members, we arrive at the identity

$$1-\frac{1}{2\times 2^2}C_2^1-\frac{1}{3\times 2^4}C_4^2-\dots$$
$$\dots-\frac{1}{k\times 2^{2k-2}}C_{2k-2}^{k-1}=\frac{1}{2^{2k-1}}C_{2k}^{k} \quad (59)$$

GENERATING FUNCTIONS AND RECURRENCE RELATIONS

We have already mentioned the fact that the theory of generating functions is closely bound up with recurrence relations. Let us go back once again to the division of polynomials. Let

$$f(x)=a_0+a_1x+\dots+a_nx^n$$

and

$$\varphi(x)=b_0+b_1x+\dots+b_mx^m$$

be two polynomials, $b_0\neq 0$. Besides, we assume that $n<m$, that is that the algebraic fraction $\frac{f(x)}{\varphi(x)}$ is proper (otherwise we can always extract the integral part).

We know that if

$$\frac{f(x)}{\varphi(x)}=c_0+c_1x+\dots+c_kx^k+\dots \quad (60)$$

then

$$a_0+a_1x+\dots+a_nx^n=(b_0+b_1x+\dots$$
$$\dots+b_mx^m)(c_0+c_1x+\dots+c_kx^k+\dots)$$

Remove the brackets on the right and compare the coefficients of like powers of x on the left and on the right.

First we get m relations of the type

$$\begin{aligned}&b_0c_0=a_0,\\&b_0c_1+b_1c_0=a_1,\\&b_0c_2+b_1c_1+b_2c_0=a_2,\qquad (61)\\&\dots\dots\dots\dots\\&b_0c_{m-1}+b_1c_{m-2}+\dots+b_{m-1}c_0=a_{m-1}\end{aligned}$$

(if $n<m-1$, then we assume that $a_{n+1}=\dots$ $\dots=a_{m-1}=0$). From then on, all the relations are of the same type:

$$b_0c_{m+k}+b_1c_{m+k-1}+\dots+b_mc_k=0,\ k=0,1,\dots \quad (62)$$

The point is that there are no terms in $f(x)$ containing x^m, x^{m+1} and so on. Thus, the coefficients $c_0, c_1, \dots, c_k, \dots$ of the series (60) satisfy the recurrence relation (62). The coefficients of this relation depend solely on the denominator of the fraction, the numerator being necessary to find the first terms $c_0, c_1, \dots, c_{m-1}$ of the recurrence sequence.

Conversely, if we have the recurrence relation (62) and the terms $c_0, c_1, \dots, c_{m-1}$, then the first thing to do is to use the formulas (61) to compute the values of $a_0, \dots, a_{m-1}$. Then the generating function of the sequence of numbers $c_0, c_1, \dots, c_k, \dots$ is the algebraic fraction

$$\frac{f(x)}{\varphi(x)}=\frac{a_0+a_1x+\dots+a_{m-1}x^{m-1}}{b_0+b_1x+\dots+b_mx^m} \quad (63)$$

It would appear, at first glance, that we gained but little in replacing the recurrence relation by the generating function, since we still have to divide the numerator by the denominator, which leads to the very same recurrence relation (62). The point, however, is that the fraction (63) admits certain algebraic manipulation, and this will facilitate our finding the numbers c_k.

DECOMPOSITION INTO PARTIAL FRACTIONS

We will now show how it is possible, using algebraic manipulations of the generating function, to solve recurrence relations. Suppose the denominator of the fraction (63) has been decomposed into linear factors:

$$\varphi(x)=b_m(x-\alpha_1)^r\dots(x-\alpha_k)^s$$

Note that this requires first solving the equation $b_0 + \ldots + b_m x^m = 0$, that is, the characteristic equation of the relation (62).

It is then clear that the fraction (63) was obtained via reduction to a common denominator of the following partial fractions:

$$\frac{A_{11}}{(x-\alpha_1)^r}, \quad \frac{A_{12}}{(x-\alpha_1)^{r-1}}, \quad \ldots, \quad \frac{A_{1,\, r-1}}{(x-\alpha_1)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{A_{k1}}{(x-\alpha_k)^s}, \quad \frac{A_{k2}}{(x-\alpha_k)^{s-1}}, \quad \ldots, \quad \frac{A_{k,\, s-1}}{x-\alpha_k}$$

In other words,

$$\frac{a_0 + \ldots + a_{m-1}x^{m-1}}{b_m (x-\alpha_1)^r \ldots (x-\alpha_k)^s} = \frac{A_{11}}{(x-\alpha_1)^r} + \ldots$$

$$\ldots + \frac{A_{1,\, r-1}}{x-\alpha_1} + \ldots + \frac{A_{k1}}{(x-\alpha_k)^s} + \ldots$$

$$\ldots + \frac{A_{k,\, s-1}}{x-\alpha_k} \tag{64}$$

The only unknowns here are the coefficients $A_{11}, \ldots, A_{k,\, s-1}$. To find these coefficients we have to multiply both sides of (64) by the denominator $(x-\alpha_1)^r \ldots (x-\alpha_k)^s$, remove the brackets and compare the coefficients of like powers of x. We find the desired coefficients from the resulting system of equations.

It is sometimes possible to get around solving the system of equations. Say, let it be required to decompose the fraction

$$\frac{x^3 - 2x^2 + 6x + 1}{x^4 - 5x^2 + 4}$$

Since

$$x^4 - 5x^2 + 4 = (x^2-1)(x^2-4)$$
$$= (x-1)(x+1)(x-2)(x+2)$$

it follows that this decomposition must be of the form

$$\frac{x^3 - 2x^2 + 6x + 1}{(x-1)(x+1)(x-2)(x+2)} = \frac{A}{x-1} +$$
$$+ \frac{B}{x+1} + \frac{C}{x-2} + \frac{D}{x+2}$$

Reducing to a common denominator, we find that

$$x^3 - 2x^2 + 6x + 1 = A(x+1)(x-2)(x+2)$$
$$+ B(x-1)(x-2)(x+2)$$
$$+ C(x-1)(x+1)(x+2) + D(x-1)(x+1)(x-2)$$

This equality should hold for all values of x. But for $x = 1$ all the terms in the right-hand member, except the first, vanish, and we get $-6A = 6$. Therefore $A = -1$. In the same way, setting $x = -1$, $x = 2$, $x = -2$, we find $B = -\frac{4}{3}$, $C = \frac{13}{12}$, $D = \frac{9}{4}$.

Thus

$$\frac{x^3 - 2x^2 + 6x + 1}{x^4 - 5x^2 + 4} = -\frac{1}{x-1} -$$
$$-\frac{4}{3(x+1)} + \frac{13}{12(x-2)} + \frac{9}{4(x+2)} \tag{65}$$

For fractions of the type $\frac{A}{(x-\alpha)^r}$, the series expansion is obtained by the binomial formula. For instance,

$$\frac{13}{12(x-2)} = -\frac{13}{24}\left(1 - \frac{x}{2}\right)^{-1}$$
$$= -\frac{13}{24}\left[1 + \frac{x}{2} + \frac{x^2}{2^2} + \ldots + \frac{x^n}{2^n} + \ldots\right]$$

Applying such an expansion to all fractions in (65), we get

$$\frac{x^3 - 2x^2 + 6x + 1}{x^4 - 5x^2 + 4} = (1 + x + x^2 + \ldots + x^n + \ldots)$$
$$-\frac{4}{3}(1 - x + x^2 - \ldots + (-1)^n x^n + \ldots)$$
$$-\frac{13}{24}\left(1 + \frac{x}{2} + \frac{x^2}{2^2} + \ldots + \frac{x^n}{2^n} + \ldots\right)$$
$$+\frac{9}{8}\left(1 - \frac{x}{2} + \frac{x^2}{2^2} - \ldots + \frac{(-1)^n x^n}{2^n} + \ldots\right)$$

Grouping terms with like powers of x, we find that the coefficient of x^n is expressed by

$$c_n = 1 - \frac{4}{3}(-1)^n - \frac{13}{24 \times 2^n} + \frac{9(-1)^n}{8 \times 2^n}$$

We already know that the problem of the expansion of an algebraic fraction in a power series is equivalent to the problem of solving some recurrence relation for given initial conditions. It is thus possible, by means of a partial-fraction decomposition and a subsequent power-series expansion of the resulting partial fractions, to solve linear recurrence relations with constant coefficients.

To summarize, then, if we have a recurrence relation (62) and the values $c_0, \ldots, c_{m-1}$, it is first necessary, using (61), to find the values of $a_0, \ldots, a_{m-1}$. They yield the coefficients of the polynomial in the numerator of the fraction:

$$\frac{f(x)}{\varphi(x)} = c_0 + c_1x + \ldots + c_kx^k + \ldots$$

The denominator of the same fraction is equal to $b_0 + \ldots + b_mx^m$.

The fraction $\frac{f(x)}{\varphi(x)}$ has then to be decomposed into partial fractions, each one of which is expanded in a power series by the binomial theorem. The coefficient of x^k in the resulting series yields the value of c_k.

By way of illustration, let us solve the recurrence relation

$$c_{k+2} - 5c_{k+1} + 6c_k = 0 \tag{66}$$

given the initial conditions $c_0 = 1$, $c_1 = -2$. Here, $b_0 = 1$, $b_1 = -5$, $b_2 = 6$. From (61) we get

$$a_0 = b_0c_0 = 1, \quad a_1 = b_0c_1 + b_1c_0 = -7$$

For this reason, the numerator of the fraction

$$\frac{f(x)}{\varphi(x)} = c_0 + c_1x + \ldots + c_kx^k$$

is equal to $1 - 7x$. The denominator of this fraction is obtained directly from (60). It is of the form $x^2 - 5x + 6$. Hence, to find the solution we have to expand the fraction

$$\frac{1-7x}{x^2-5x+6}$$

in a power series. But $x^2 - 5x + 6 = (x-2)(x-3)$ and so

$$\frac{1-7x}{x^2-5x+6} = \frac{1-7x}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$$

Clearing fractions, we get

$$1 - 7x = A(x-3) + B(x-2)$$

Putting $x = 3$, we get $B = -20$, and putting $x = 2$, we get $A = 13$. Hence,

$$\frac{1-7x}{x^2-5x+6} = \frac{13}{x-2} - \frac{20}{x-3}$$

$$= -\frac{13}{2}\left(1 - \frac{x}{2}\right)^{-1} + \frac{20}{3}\left(1 - \frac{x}{3}\right)^{-1}$$

$$= -\frac{13}{2}\left(1 + \frac{x}{2} + \ldots + \frac{x^n}{2^n} + \ldots\right) + \frac{20}{3}\left(1 + \frac{x}{3} + \ldots + \frac{x^n}{3^n} + \ldots\right)$$

And therefore

$$c_n = -\frac{13}{2} \times \frac{1}{2^n} + \frac{20}{3} \times \frac{1}{3^n} = -\frac{13}{2^{n+1}} + \frac{20}{3^{n+1}}$$

ON A SINGLE NONLINEAR RECURRENCE RELATION

In the solution of the problem of the partition of a sequence we arrived at the recurrence relation

$$T_n = T_0T_{n-1} + T_1T_{n-2} + \ldots + T_{n-1}T_0 \tag{67}$$

where $T_0 = 1$ (see page 92). This equation was solved in a very artificial way: we reduced the problem to that of the ticket-office line (see page 52) which we knew how to solve. But the line (queue) problem was itself rather awkward.

We shall now demonstrate how to solve (67) directly. First form the generating function

$$f(x) = T_0 + T_1x + T_2x^2 + \ldots + T_nx^n + \ldots \tag{68}$$

Put

$$F(x) \equiv xf(x) = T_0x + T_1x^2 + \dots$$
$$\dots + T_nx^{n+1} + \dots \qquad (69)$$

and square $F(x)$. This yields

$$F^2(x) = T_0^2x^2 + (T_0T_1 + T_1T_0)\,x^3 + \dots$$
$$\dots + (T_0T_{n-1} + \dots + T_{n-1}T_0)\,x^{n+1} + \dots$$

But by the recurrence relation (67),

$$T_0T_{n-1} + \dots + T_{n-1}T_0 = T_n$$

Hence,

$$F^2(x) = T_1x^2 + T_2x^3 + \dots + T_nx^{n+1} + \dots$$

The series thus obtained is nothing but $F(x) - T_0x$; since $T_0 = 1$, it is equal to $F(x) - x$. Thus,

$$F^2(x) = F(x) - x \qquad (70)$$

For the function $F(x)$ we get the quadratic equation (70), which yields

$$F(x) = \frac{1 - \sqrt{1-4x}}{2}$$

We choose the minus sign in front of the radical since otherwise, for $x = 0$, we would have $F(0) = 2$, and from the expansion (69) we see that $F(0) = 0$.

By (54) we get

$$\sqrt{1-4x} = (1-4x)^{\frac{1}{2}} = 1 - 2x - \frac{2}{2}C_2^1x^2$$
$$-\frac{2}{3}C_4^2x^3 - \dots - \frac{2}{n+1}C_{2n}^nx^{n+1} - \dots$$

Thus,

$$F(x) = \frac{1}{2}\Big[1 - \Big(1 - 2x - \dots$$
$$\dots - \frac{2}{n+1}C_{2n}^nx^{n+1} - \dots\Big)\Big]$$
$$= x + C_2^1x^2 + \dots + \frac{1}{n+1}C_{2n}^nx^{n+1} + \dots \qquad (71)$$

Comparing formulas (69) and (71), we get $T_n = \frac{1}{n+1}C_{2n}^n$. This is in total agreement with the solution obtained earlier by a combinatorial method (see page 92).

GENERATING FUNCTIONS AND PARTITIONS OF INTEGERS

In Chapter IV we solved a variety of combinatorial problems involving partitions of integers. These problems have very simple solutions when generating functions are employed. Denote by a_n the number of ways of partitioning for n and form the series

$$a_0 + a_1x + \dots + a_nx^n + \dots$$

In many cases it is possible to form an algebraic expression $f(x)$ such that after removing brackets the term x^n is repeated exactly a_n times. Then

$$f(x) = a_0 + a_1x + \dots + a_nx^n + \dots$$

and, hence, $f(x)$ is the generating function of the sequence $a_0, a_1, \dots, a_n, \dots$.

To illustrate, suppose we are considering the *partitions of N into parts, each of which is equal to one of the numbers $n_1, \dots, n_k$. The terms must not be repeated in the sum and their order is of no importance.*

We first form the expression

$$(1 + x^{n_1})(1 + x^{n_2}) \dots (1 + x^{n_k}) \qquad (72)$$

Removing brackets, we get summands of the form $x^{m_1}, \dots, x^{m_s}$, where $m_1, \dots, m_s$ are some of the numbers $n_1, \dots, n_k$. Therefore, x^N occurs in the sum as many times as there are ways of partitioning N into parts in the required fashion.

For example, if it is required to find the number of ways of paying 78 copecks by using at most one coin of each denomination, then we have to form the expression

$$(1+x)(1+x^2)(1+x^3)(1+x^5)\times$$
$$\times(1+x^{10})(1+x^{15})(1+x^{20})(1+x^{50}) \qquad (73)$$

Remove the brackets and find the coefficient of x^{78}.

Now let us use generating functions to solve the following problem.

In how many ways can we pay 29 copecks using 3- and 5-copeck pieces?

The task is to find the number of ways of partitioning 29 into the summands 3 and 5, the order of the integers being irrelevant. In other words, we have to find the number of nonnegative solutions of the equation $3m + 5n = 29$.

To do this, we form the expression

$$f(x) = (1 + x^3 + x^6 + \ldots + x^{3m} + \ldots) \times \\ \times (1 + x^5 + x^{10} + \ldots + x^{5n} + \ldots) \qquad (74)$$

The exponents on x in the first bracket run through all nonnegative multiples of 3, in the second bracket, through all nonnegative multiples of 5. It is evident that after the brackets are removed, x^N will occur with a coefficient equal to the number of solutions of the equation $3m + 5n = N$. In particular, the coefficient of x^{29} supplies the answer.

Instead of removing the brackets, you can take advantage of the formula for an infinite geometric progression. Then (74) appears as

$$f(x) = \frac{1}{1 - x^3}\,\frac{1}{1 - x^5} = \frac{1}{1 - x^3 - x^5 + x^8}$$

Now divide the numerator by the denominator by the rule for division of polynomials (only this time arrange the polynomials in ascending powers of x instead of descending powers). The beginning of the division process looks like this:

$$\begin{array}{l|l}
1 - x^3 - x^5 + x^8 & 1 + x^3 + x^5 + x^6 + x^8 + \ldots \\
\hline
 & 1 \\
 & x^3 + x^5 - x^8 \\
 & \quad x^5 + x^6 - x^{11} \\
 & \qquad x^6 + x^8 + x^{10} - x^{11} - x^{13} \\
 & \qquad\quad x^8 + x^9 + x^{10} - x^{13} - x^{14}
\end{array}$$

Continuing the division, we get the desired coefficient of x^{29}.

The general problem is:

To find the number of ways to partition a number N into parts equal, respectively, to $a, b, \ldots, m$, the order of the summands being disregarded.

In this case, the generating function looks like

$$f(x) = (1 + x^a + x^{2a} + \ldots + x^{ta} + \ldots) \times \\ \times (1 + x^b + x^{2b} + \ldots + x^{sb} + \ldots) \times \\ \times (1 + x^m + x^{2m} + \ldots + x^{qm} + \ldots) \\ = \frac{1}{(1 - x^a)(1 - x^b)\ \ldots\ (1 - x^m)} \qquad (75)$$

For instance, in the problem of getting change for a ten-copeck piece (see page 64), we have to form the generating function

$$f(x) = \frac{1}{(1 - x)(1 - x^2)(1 - x^3)(1 - x^5)}$$

Multiplying together the expressions in the denominator of the fraction, we get

$$f(x) = \frac{1}{1 - x - x^2 + x^7 - x^9 - x^{10} + x^{11}}$$

Performing the division, we get

$$\begin{array}{l|l}
1 - x - x^2 + x^4 + x^7 - x^9 - x^{10} + x^{11} & 1 + x + 2x^2 + 3x^3 + 6x^4 + \ldots \\
\hline
 & 1 \\
 & x + x^2 - x^4 - x^7 + x^9 + x^{10} - x^{11} \\
 & \quad 2x^2 + x^3 - x^4 - x^5 - x^7 - x^8 + x^9 + 2x^{10} - x^{12} \\
 & \qquad 3x^3 + 3x^4 - x^5 - 2x^6 - x^7 - x^8 - x^9 + 2x^{10} + 2x^{11} + x^{12} - 2x^{13} \\
 & \qquad\quad 6x^4 + 2x^5
\end{array}$$

The coefficient of x^{10} contains the answer to our question.

It is of course rather complicated to carry out the division in the ordinary way. There is a different approach. Write the result of the division in the form of an infinite series with undetermined coefficients:

$$\frac{1}{1 - x - x^2 + x^7 - x^9 - x^{10} + x^{11}} = A_0 + A_1 x + \\ + A_2 x^2 + \ldots + A_n x^n + \ldots$$

Multiply both sides by the denominator. Then the coefficient of x^n on the right will turn out to be

$$A_n - A_{n-1} - A_{n-2} + A_{n-4} + A_{n-7} - A_{n-9} - A_{n-10} + A_{n-11}$$

But on the left the coefficient of x^n, $n \geqslant 1$, is zero. Thus for $n \geqslant 1$ the coefficients A_n must satisfy the recurrence relation

$$A_n = A_{n-1} + A_{n-2} - A_{n-4} - A_{n-7} + A_{n-9} + A_{n-10} - A_{n-11}$$

The initial conditions are $A_n = 0$ for $n < 0$ and $A_0 = 1$. Using these conditions, it is easy to find all the coefficients A_n in succession.

By way of illustration, let us consider the entrance-exams problem on page 61. There the task was to find the number of ways of representing the number 17 as a sum of 4 integers assuming the values 3, 4, and 5, the order of the terms playing a role. Here, for the generating function we have to take $(x^3 + x^4 + x^5)^4$. This is because when the brackets are removed in the expression $f(x) = (x^3 + x^4 + x^5)^4$, each term x^N will occur as many times as there are ways of partitioning N into a sum of 4 integers that assume the values 3, 4, 5. There will also be terms that are obtained from each other via a permutation of the integers in the exponent (say, $x^3x^4x^5x^3$ and $x^4x^3x^3x^5$).

In the expression $(x^3 + x^4 + x^5)^4 = x^{12}(1 + x + x^2)^4$ the brackets may be removed by using, say, the multinomial theorem. But there is an easier way. We note that $1 + x + x^2 = \frac{1-x^3}{1-x}$. Therefore, $f(x)$ can be written as

$$f(x) = \frac{x^{12}(1-x^3)^4}{(1-x)^4} = x^{12}(1-x^3)^4(1-x)^{-4}$$

But by the binomial theorem we have

$$(1-x^3)^4 = 1 - 4x^3 + 6x^6 - 4x^9 + x^{12}$$

and by the formula of the Newton series (see page 112),

$$(1-x)^{-4} = 1 + 4x + 10x^2 + 20x^3 + \ldots + \frac{4\times 5 \ldots (n+3)}{1\times 2 \ldots n} x^n + \ldots$$

Therefore

$$f(x) = x^{12}(1 - 4x^3 + 6x^6 - 4x^9 + x^{12}) \times (1 + 4x + 10x^2 + 20x^3 + 35x^4 + 56x^5 + \ldots)$$

Multiplying together these expansions term-by-term, we find that the coefficient of x^{17} in the expansion is equal to 16, which means the partitions can be done in 16 ways.

Generally, if we have to find the number of ways of partitioning a number N into k summands which assume the values $n_1, \ldots, n_s$, with regard for the order of the terms, then the generating function is of the form

$$f(x) = (x^{n_1} + x^{n_2} + \ldots + x^{n_s})^k \tag{76}$$

The problem is simplified if the numbers $n_1, \ldots, n_s$ form an arithmetic progression: in this case $x^{n_1}, \ldots, x^{n_s}$ form a geometric progression, and this enables us to simplify the expression for $f(x)$.

For example, let us find the number of ways of obtaining 25 points in throwing 7 dice. Here we have to form the generating function

$$f(x) = (x + x^2 + \ldots + x^6)^7 \tag{77}$$

By the formula for the sum of a geometric progression, this function may be written as

$$f(x) = \frac{x^7(1-x^6)^7}{(1-x)^7} = x^7(1-x^6)^7(1-x)^{-7}$$

Now expand $(1-x^6)^7$ by the binomial theorem, and $(1-x)^{-7}$ by the formula for the Newton series. We get

$$f(x) = x^7(1 - 7x^6 + 21x^{12} - 35x^{18} + 35x^{24} - 21x^{30} + 7x^{36} - x^{42})(1 + 7x + 28x^2 + 84x^3 + 210x^4 + 462x^5 + \ldots)$$

Multiplying these expansions, we can easily compute the coefficient of x^{25}, which is the answer to our problem.

The other problems that we discussed in Chapter IV can also be solved in similar fashion with the aid of generating functions.

SUMMARY OF THE COMBINATORICS OF PARTITIONS

1. The number of ways of partitioning n distinct things into r distinguishable groups, empty groups allowed, is r^n.

2. The number of ways of partitioning n distinct things into r distinguishable groups, all groups nonempty, is equal to the coefficient, multiplied by $n!$, of x^n in the power-series expansion of the function $(e^x - 1)^r$. This number can be written in the form

$$r^n - \frac{r}{1}(r-1)^n + \frac{r(r-1)}{1\times 2}(r-2)^n - \dots$$

3. If in the same problem the groups are indistinguishable, then there will be $r!$ fewer ways.

4. The number of ways of partitioning n indistinguishable things into r distinguishable groups, all groups nonempty, is C_{n-1}^{r-1}.

5. The number of ways of partitioning n indistinguishable things into r distinguishable groups, empty cells admissible, is C_{n+r-1}^{r-1}.

6. The number of ways of partitioning n indistinguishable things into r distinguishable groups, each group containing at least q things, is $C_{n-1-r(q-1)}^{r-1}$.

7. The number of ways of partitioning n indistinguishable things into r distinguishable groups, the number of elements in each group lying between q and $q+s-1$, $q \leqslant x \leqslant q+s-1$, is equal to the coefficient of x^{n-rq} in the power-series expansion of the function $\left(\frac{1-x^s}{1-x}\right)^r$.

8. Denote the number of ways of partitioning n indistinguishable things into r indistinguishable groups, no group empty, by Π_n^r. Then we have the recurrence formula

$$\Pi_n^r = \Pi_{n-1}^{r-1} + \Pi_{n-r-1}^{r-1} + \Pi_{n-2r-1}^{r-1} + \dots$$

The equation

$$\Pi_n^r = \Pi_{n-1}^{r-1} + \Pi_{n-r}^r$$

holds. For $n - r < r$ we have $\Pi_n^r = \Pi_{n-1}^{r-1}$.

Along with partitions, we consider orderings of elements in which the order of the groups is significant, and also the order of the elements in the groups. For ordering, the following assertions hold true:

9. If n distinct things are ordered into r distinguishable groups, empty groups allowed, then the number of orderings is

$$r(r+1)\dots(r+n-1)$$

10. If n distinct things are ordered into r distinguishable groups, all groups nonempty, the number of orderings is

$$n!\,C_{n-1}^{r-1} = \frac{n!\,(n-1)!}{(n-r)!\,(r-1)!}$$

If the groups are indistinguishable, however, and, hence, the order plays no role, then the number of orderings is $\frac{n!}{r!}C_{n-1}^{r-1}$.

11. Suppose n distinct elements are used to form, in all possible ways, r ordered groups (not all elements need be taken, empty groups are allowed, the order of the groups is taken into account). The number of such groups is

$$n!\left[\frac{1}{n!} + \frac{r}{1!\,(n-1)!} + \frac{r(r+1)}{2!\,(n-2)!} + \dots\right]$$

This expression is the coefficient, multiplied by $n!$, of x^n in the power-series expansion of the function $e^x(1-x)^{-r}$.

12. If, in the same problem, empty groups are forbidden, then the answer is equal to the coefficient, multiplied by $n!$, of x^{n-r} in the expansion of the function $e^x(1-x)^{-r}$ in powers of x.

1.

There are five roads leading from city A to city B, and three from B to C. How many routes passing through B lead from A to C?

2.

Two sport clubs with 100 fencers each have to send one fencer each to a competition. In how many ways can this choice be made?

3.

There are five types of envelopes without postage stamps and four types of postage stamps of the same value. In how many ways can we choose an envelope with a postage stamp?

4.

In how many ways can a vowel and a consonant be chosen from the word "almost"?

5.

The same for the word "orange".

6.

A six-faced die is thrown and a teetotum having eight faces is spun. In how many different ways can they fall?

7.

Five roads lead to the top of a mountain. In how many ways can a climber ascend and descend? The same, provided that the ascent and descent are accomplished by different routes.

8.

A farmer has 20 sheep and 24 pigs. In how many ways can he choose one sheep and one pig? If a choice has been made, in how many ways can he choose the next time?

9.

In how many ways can two squares (white and black) be selected on a chessboard? The same, if there are no restrictions as to thier colour.

10.

In how many ways is it possible to choose a white square and a black square on a chessboard if the squares must not lie in the same row or column?

11.

Choose one word of each gender out of 12 masculine words, 9 feminine and 10 neuter. In how many ways can this choice be made?

12.

There are 6 pairs of gloves of different sizes. In how many ways can you choose one glove for the left hand and one for the right without taking a pair?

13.

Choose one textbook each out of 3 algebras, 7 geometries and 7 trigonometry books. In how many ways can this be done?

14.

A bookshop has 6 copies of Tolstoy's *War and Peace*, 3 copies of Dostoyevsky's *Crime and Punishment* and 4 copies of Turgenev's *Fathers and Sons*. Besides that there are 5 separate sets containing the first two books and 7 sets containing the second and third books. In how many ways can a purchase be made so that one copy of each novel is obtained?

15.

The same problem, only there are three more sets including *War and Peace* and *Fathers and Sons*.

16.

A basket contains 12 apples and 10 oranges. John takes an apple or an orange, then Tom takes an apple and an orange. In which case does Tom have more choice: when John takes an apple or when he takes an orange?

17.

There are three teetotums having 6, 8 and 10 sides respectively. In how many ways can they fall? The same, if we know that at least two of the teetotums fell showing the number 1.

18.

In how many ways is it possible to choose three different colours out of five?

19.

In how many ways is it possible to make a tricolour flag if there is bunting of 5 different colours? The same, only one of the strips has to be red.

20.
How many dictionaries are needed to translate directly from one of five languages: Russian, English, French, German, and Italian into any one of the remaining four languages?

21.
How many more dictionaries will be needed if the number of distinct languages is 10?

22.
In how many ways can one card of each suit be selected from a full pack of cards? The same provided that no two cards drawn form a pair (say, two kings, two tens, etc.).

23.
In how many ways can we choose one card of each suit from a full pack of 52 cards if the selected cards make a red pair and a black pair (say, a nine of spades and clubs and a jack of diamonds and hearts)?

24.
A child is given at most 3 names. In how many ways can this be done if the total number of names to pick from is 300?

25.
Several persons take seats at a round table. We consider that two seating arrangements coincide if each person has the same neighbours in each case. In how many ways can we seat four persons? Seven persons? In how many cases are two assigned persons neighbours out of seven? In how many cases will a given person (out of seven) have two given neighbours?

26.
Five girls and three boys are going to play croquet. In how many ways can they form two sides of 4 each if each side is to have at least one boy?

27.
Six urgent letters are to be delivered. In how many ways can this be done if there are three messengers and each letter can be given to any one of them?

28.
One person has 7 mathematics books, another has 9 books. In how many ways can they exchange their books, one for one?

29.
The same problem, only two books are exchanged for two.

30.
Five persons, A, B, C, D, E, are to speak at a meeting. In how many ways can they take their turns without B speaking before A?

31.
The same, only A must speak immediately before B.

32.
In how many ways can we seat 5 men and 5 women at a circular table so that no two men or no two women come together?

33.
The same problem, only they are seated at a merry-go-round and the seating arrangements that pass into one another as they turn are considered coincident.

34.
Ten cards are drawn from a pack of 52. In how many cases is there at least one ace? In how many cases is there exactly one ace? In how many cases are there at least two aces? Exactly two aces?

35.
There are m light signals at a railway station. How many distinct messages can be conveyed if each light signal has three distinct states: red, yellow and green?

36.
In a small country there were no two persons with the same set of teeth. What is the largest population the country can have (32 teeth forming a maximum set)?

37.
In a railway car compartment there are two rows of facing seats, five in each. Out of 10 passengers, four wish to sit looking forward and three looking towards the rear of the train. The other three are indifferent. In how many ways can the passengers take seats?

38.
A committee of 9 is elected. They elect a chairman, vice-chairman, secretary and treasurer. In how many ways can this be done?

39.
A delegation of 5 members is to be elected by a conference of 52 persons. In how many ways can this be done?

40.
Automobile licence plates consist of one, two, or three letters and four digits. How many number-letter combinations can be formed using 32 letters of the Russian alphabet?

41.
Mother has 2 apples and 3 pears. Every day, for five days running, she gives me one piece of fruit. In how many ways can this be done?

42.
The same, for m apples and n pears.

43.
The same for 2 apples, 3 pears and 4 oranges.

44.
Father has 5 pairwise distinct oranges which he gives his eight sons so that each receives either one orange or none. In how many ways can this be done?

45.
The same, only the number of oranges each son gets is unlimited.

46.
How many distinct words can be generated by permuting the letters of the words "mathematics" (regard 'th' as bound—one letter), "parabola", and "ingredient"?

47.
A club of 30 members makes up a team of 4 for the 1,000 metres race. In how many ways can this be done? In how many ways can we form a team of 4 for a relay race of $100 + 200 + 400 + + 800$ metres?

48.
In how many ways can we place white pieces (2 knights, 2 bishops, 2 rooks, a queen and a king) in the first row of a chessboard?

49.
There are n telephone subscribers. In how many ways is it possible to connect three pairs simultaneously?

50.
Ten kinds of picture postcards are on sale. In how many ways can I buy 12? 8? In how many ways can I buy 8 distinct postcards?

51.
Out of a group of 7 men and 4 women we have to choose 6 persons so that there are at least 2 women. In how many ways can this be done?

52.
How many distinct four-digit numbers divisible by 4 can be generated out of the digits 1, 2, 3, 4, 5 if each digit can occur several times in the representation?

53.
A train carrying n passengers is to make m stops. In how many ways can the passengers be distributed among the stops? The same, provided we only count the passengers that get out at a definite stop.

54.
How many permutations can be generated out of n elements in which two given elements, a and b, do not come together? Three given elements, a, b, c, do not come together (in any order)? No two elements out of a, b, c come together?

55.
Ten persons compete in gynmastics. Three referees number them independently in accordance with their performance. The winner is the one named first by at least two referees. In how many cases (%) will the winner be named?

56.
Four students are taking exams. In how many ways can the marks be given if it is known that all received passing marks (that is, 3, 4, or 5)?

57.

How many necklaces can be made out of seven beads of different sizes (all seven have to be utilized)?

58.

How many necklaces can be made out of five identical beads and two large-size beads?

59.

A village has a population of 2,000. Prove that at least two of them have the same initials (if 29 letters are available for initials).

60.

A group of seven boys and ten girls are at a dance If in some dance all the boys participate, how many ways are there for the girls to take part? How many versions are there if we take into account only those girls that were not invited? The same, if with respect to two girls we can definitely say that they will be invited.

61.

A company of soldiers consists of 3 officers, 6 sergeants and 60 privates. In how many ways can a detachment be made consisting of one officer, two sergeants and 20 privates? The same, provided the detachment must contain the captain and the senior sergeant.

62.

There are 12 girls and 15 boys at a school ball. In how many ways can we form 4 pairs in a dance?

63.

How many combinations can be made up of 3 hens, 4 ducks and 2 geese so that each combination has hens, ducks and geese?

64.

In how many ways can we split up $m+n+p$ objects into three groups so that there are m objects in one, n in another and p in the third?

65.

There are $m+n$ different books on a bookshelf, of which m are in black bindings and n in red. How many permutations are there of these books in which the black books occupy the first m places? How many positions are there in which all books in black bindings are together?

66.

In how many ways can a group of men be selected for a job? The group may consist of 1, 2, 3, ..., 15 men. The same when the group is chosen from n men?

67.

Let $p_1, \ldots, p_n$ be distinct prime numbers. How many divisors has the number

$$q=p_1^{\alpha_1} \ldots p_n^{\alpha_n}$$

where $\alpha_1, \ldots, \alpha_n$ are certain natural numbers (including the divisors 1 and q). What is their sum?

68.

In how many ways can 12 identical coins be put into five different envelopes if no empty envelopes are allowed?

69.

In how many ways can 20 books be arranged in a bookcase with five shelves if each shelf holds 20 books?

70.

In how many ways can 5 different rings be put on four fingers of one hand?

71.

30 persons vote on 5 proposals. In how many ways can the votes be distributed if each person votes for one proposal and only votes cast for each proposal are counted?

72.

A bookbinder has to bind 12 distinct books using the colours red, green and brown. In how many ways can this be done if each colour is used for at least one book?

73.

In how many ways is it possible to form 6 words out of 32 letters if in the 6 words taken together each letter is used once and only once?

74.

In how many ways can we choose 12 persons out of 17 if two given persons of these 17 cannot be chosen together?

75.
How many different bracelets can be made using 5 identical emeralds, six identical rubies and seven identical sapphires (a bracelet consists of all 18 stones)?

76.
In how many ways is it possible to choose three stones for a ring from the same number of precious stones?

77.
Three students share a room. They have 4 cups, 5 saucers and 6 spoons (all distinct). In how many ways can they set the table for tea, each receiving one cup, one saucer and one spoon?

78.
The husband has 12 acquaintances: 5 women and 7 men, and the wife has 12: 7 women and 5 men. In how many ways can they get together in a company of 6 men and 6 women so that the husband invites 6 persons and the wife 6?

79.
A boat accommodates four persons on each side. In how many ways can a crew be made up if there are 31 candidates, 10 of whom want to be on the port side, 12 on the starboard side, and 9 are indifferent?

80.
An urn contains counters with the numbers 1, 2, 3, . . ., 10. Three counters are drawn. In how many cases will the sum they form be 9? At least 9?

81.
In how many ways can you choose six cards out of a full pack of 52 and have all four suits present?

82.
A chorus group consists of 10 persons. In how many ways can you choose 6 participants each time during three days so that the groups differ?

83.
A person has 6 friends and invites 3 in the course of 20 days (the company is never the same). In how many ways can he do this?

84.
Three boys and two girls choose jobs. The town has three factories requiring workers in foundry shops (only men required), two weaving factories (only women), and two factories employing both men and women. In how many ways can they take jobs at these factories?

85.
How many 5-letter words can be made out of 33 letters if repetitions are permitted, but no two adjacent letters can be the same (for example, "press" would be excluded)?

86.
At a mathematics contest, the prizes are three copies of one book, 2 copies of another and 1 copy of a third. In how many ways can they be awarded if there are 20 participants and none receives two books? The same, but none receives two copies of the same book (he may however get two or three different books).

87.
Dominoes from (0, 0) to (n, n) are taken. Show that the number of dominoes with the sum $n-r$ is equal to the number of dominoes with the sum $n+r$ and this number is equal to $\frac{1}{4}(2n-2r+3)$. Find the total number of all the dominoes.

88.
In how many ways can 7 men and 7 women be seated at a circular table so that no two women come together?

89.
In how many ways can six horses be chosen out of 16 for a team so that there are 3 horses out of the sextuplet ABCA′B′C′, but not a single one of the pairs AA′, BB′, CC′?

90.
In how many ways, using 9 consonants and 7 vowels, is it possible to make words with 4 distinct consonants and 3 distinct vowels? How many of these words do not have 2 adjacent consonants?

91.

In a department of a research institution, each of the employees knows at least one foreign language. Six know Spanish, six German, and seven French. Four know Spanish and German, three know German and French, and two know French and Spanish. One person knows all three languages. How many are there employed in the department? How many know only Spanish? How many only French?

92.

A hike was organized with 92 persons participating. Sandwiches were taken for lunch: 47 had sausage, 38 had cheese, 42 ham, 28 cheese and sausage, 31 a combination of sausage and ham, and 26 persons had a combination of cheese and ham. A total of 25 persons took all three types; then there were a few who took meat patties instead of sandwiches. How many took patties?

93.

There are 10 couples on a boating trip divided into 5 groups, 4 in a group. In how many ways can they split up so that there are two men and two women in each boat?

94.

In how many cases will a given man be in the same boat together with his wife?

95.

In how many cases will two given men be with their wives?

96.

How many distinct four-digit numbers can be made out of the *digits* 0, 1, 2, 3, 4, 5, 6, unlimited repetitions allowed?

97.

Find the number of six-digit numbers such that the sum of a three-digit number formed out of the first three digits and a three-digit number constructed out of the last three digits is less than 1,000.

98.

In how many ways can you place 12 white and 12 black draughtmen on the black squares of a draughtboard?

99.

In how many ways can the letters of the Russian word "Юпитер" be permuted so that the vowels are in alphabetic order (namely, 'е', 'и', 'ю')?

100.

In now many ways can the letters of the Russian word "перешеек" be permuted so that four e's do not come together?

101.

In how many ways can the letters of "opossum" be permuted so that the letter "p" comes immediately after "o"?

102.

In how many ways can the letters of the Russian word "обороноспособность" be permuted so that no two o's come together?

103.

In how many ways can the letters of the Russian word "каракули" be permuted so that no two vowels (а, у, or и) are in succession?

104.

In how many ways can the letters of the Russian word "фацетия" be permuted so that the order of the vowels (а, е, и, я) is preserved?

105.

In how many ways can we permute the letters in the word "parallelism" so as to preserve the order of the vowels?

106.

In how many ways can the letters of the phrase "sell it" be permuted so that two vowels come between two consonants?

107.

In how many ways can you permute the letters of the Russian word "логарифм" (=logarithm) so that the second, fourth and sixth places are taken by consonants (л, г, р, ф, м are consonants)?

108.

In how many ways can you choose two consonants and one vowel from the Russian word "логарифм"? The same problem, if the chosen letters include the letter "ф".

109.

In how many ways can the letters of the Russian word "огород" be permuted so that three o's do not come together?

110.

The same as in 109, only no two o's are allowed to come together.

111.

In how many ways can several letters be chosen from the Russian phrase "око за око, зуб за зуб" without regard for the order of the letters?

112.

In how many ways can three letters be chosen from the phrase of Problem 111.

113.

In how many ways can three letters be chosen from the phrase of Problem 111 if the order of the chosen letters is taken into consideration?

114.

In how many ways can the letters of the Russian word "пастухи" be permuted so that vowels and consonants are in alphabetic order (vowels а, и, у and consonants п, с, т, х)?

115.

In how many ways can the letters of the Russian word "кофеварка" be permuted so that vowels (о, е, а) and consonants (к, ф, в, р, к) alternate? The same for "samovar".

116.

In how many ways can the letters of "Abakan" be permuted so that the consonants are in alphabetic order? The same, with the added restriction that no two a's come together.

117.

In how many ways can the letters of "fulfil" be permuted so that no two identical letters come together? The same for the word "murmur".

118.

In how many ways can 4 letters be selected from the word "tartar" if the order of the chosen letters is disregarded? How many four-digit numbers can be made out of the digits of the number 132,132?

119.

How many nonnegative integers less than a million contain all the digits 1, 2, 3, 4? How many numbers consist of these four digits alone?

120.

Find the sum of the four-digit numbers obtained in all possible permutations of the digits 1, 2, 3, 4.

121.

The same for 1, 2, 2, 5.

122.

The same for 1, 3, 3, 3.

123.

The same for 1, 1, 4, 4.

124.

The same for all five-digit numbers which can be obtained by permuting the digits 0, 1, 2, 3, 4. Zero must not come first.

125.

How many numbers less than a million can be written with the aid of the digits 8 and 9?

126.

The same with the aid of 9, 8, 7.

127.

The same, using 9, 8, 0 (numbers beginning with 0 are forbidden).

128.

Find the sum of all three-digit numbers that can be written using the digits 1, 2, 3, 4.

129.

Find the sum of all possible five-digit numbers that can be written with the digits 1, 2, 3, 4, 5 and which contain each digit once and only once. The same for five-digit numbers that can be written using the digits 1, 2, 3, 4, 5, 6, 7, 8, 9.

130.

How many odd numbers can be formed out of the digits of the number 3,694 (each digit being used at most once)? How many even numbers?

131.

How many six-digit numbers are there in which three digits are even and three are odd?

132.

The same, provided "six-digit" numbers that begin with zero are also allowed.

133.

How many six-digit numbers are there in which the sum of the digits is even (the first digit being nonzero)? The same if we take all numbers from 1 to 999,999?

134.

How many ten-digit numbers are there with sum of digits equal to three (nonzero first digit)? The same, but take all numbers from 1 to 9,999,999,999.

135.

How many nine-digit numbers are there in which all digits are distinct?

136.

How many integers are there between 0 and 999 that are not divisible either by 5 or 7?

137.

How many integers are there between 0 and 999 which are not divisible by 2, 3, 5, or 7?

138.

How many numbers from 0 to 999 have the digit 9? How many have it twice? How many numbers have 0? How many have it twice? How many numbers have 0 and 9? 8 and 9? How many numbers are there between 0 and 999,999 with no two identical digits coming together?

139.

How many four-digit numbers can be formed from the digits of the number 123,153?

140.

How many five-digit numbers can be formed from the digits of the number 12,335,233?

141.

How many six-digit numbers can be formed from the digits of the number 1,233,145,254 so that no two identical digits come together?

142.

How many five-digit numbers can be generated by using the digits of the number 12,312,343 so that no three 3's come together?

143.

In how many ways can we permute the digits of the number 12,341,234 so that no two identical digits come together?

144.

The same for the number 12,345,254.

145.

In how many ways can the digits of the number 1,234,114,546 be permuted so that no three identical digits come in succession?

146.

In how many ways can this be done so that no two identical digits follow one another in succession?

147.

In how many ways can you pick two numbers out of the natural numbers from 1 to 20 so that their sum is odd?

148.

In how many ways can three numbers be chosen from the natural numbers 1 to 30 so that their sum is even?

149.

There are two high roads from London to Brighton and ten cross roads connecting the two high roads (see Fig. 34). In how many different ways can the journey be made without traversing the same ground twice in the same journey?

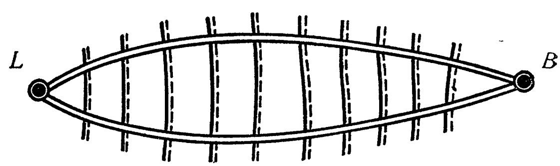


Fig. 34.

150.
If two travellers start from London one on each highway, in how many ways can they finish the journey without both of them traversing in the same direction any part of the road?

151.
There are three high roads from London to Cambridge and four cross roads connecting all the high roads (see Fig. 35). By how many routes can the journey be made without traversing any part of a high road in the direction towards London or any part twice?

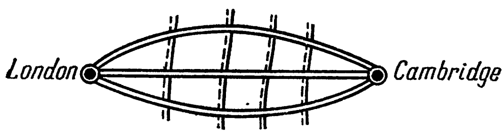


Fig. 35.

152.
Out of an unlimited number of coins, 10, 15 and 20 copecks each, in how many ways can 20 coins be selected?

153.
A person holds five coins and asks you to guess what they are. Possible coins are 1, 2, 3, 5, 10, 15, 20, 50 copecks each and 1 rouble. How many wrong answers is it possible to give?

154.
How many numbers (in decimal notation) are there consisting of five digits? In how many of these is every digit an even number? In how many, an odd number? In how many is there no digit lower than 6? How many have digits that do not exceed 3? How many of them contain all the digits 1, 2, 3, 4, 5? How many contain all the digits 0, 2, 4, 6, 8?

155.
The faces of each of two dice are numbered 0, 1, 3, 7, 15, 31. How many distinct sums can be thrown with these dice?

156.
The faces of three dice are numbered 1, 4, 13, 40, 121, 364. How many different sums can be thrown with them?

157.
Six dice are thrown with the numbers 1, 2, 3, 4, 5, 6. In how many cases will they show one type of sum? Two types? Three types? Four types? Five types? Six types? (All dice are distinguishable.)

158.
n dice are thrown. How many different outcomes are possible (outcomes differing in order of points alone are considered the same; each die is numbered 1, 2, 3, 4, 5, 6)?

159.
In how many distinct ways is it possible to represent the number 1,000,000 in the form of a product of three factors? Representations differing as to order of the factors are considered distinct.

160.
The same, provided that the order of the factors is disregarded.

161.
In how many ways can you put 9 coins of different value into two pockets?

162.
In how many ways can $3n$ distinct objects be distributed among three people so that each gets n objects?

163.
Given $2n$ elements. We consider all possible partitions of them into pairs; all partitions differing as to the order of the elements within the pairs and the order of the pairs are considered coincident. How many distinct partitions are there?

164.
The same problem, only there are nk elements being partitioned into n groups of k elements each.

165.
In how many ways is it possible to divide 30 workers into 3 teams of 10 each? Into 10 groups of 3 in each?

166.
In how many ways can a pack of 36 cards be split in half so that each half contains two aces?

167.
In how many ways can 10 books be wrapped up in 5 packages of two each (the order of the packages is irrelevant)?

168.
In how many ways can 9 books be wrapped up in 4 packages, 2 books in each and one package containing one book?

169.
The same problem, three packages with three books in each.

170.
In how many ways can three persons divide among themselves 6 identical apples, 1 orange, 1 plum, 1 lemon, 1 pear, 1 quince and 1 date?

171.
In Problem 170, make the division so that each gets 4 items. In how many ways can this be done?

172.
A, B and C have three apples each; besides, A has 1 pear, 1 plum and 1 quince, B has 1 orange, 1 lemon and 1 date, while C has 1 tangerine, 1 peach and 1 apricot. In how many ways can they distribute the fruit among themselves so that each gets 6 pieces of fruit?

173.
In how many ways can I deal a pack of 52 cards to 13 players, 4 cards to each player? The same, provided however that each gets one card of each suit. The same, provided that one has cards of all four suits, while the others have cards of one and the same suit.

174.
In how many ways is it possible to draw 4 cards from a full pack so that three suits are in evidence? So that there are two suits?

175.
In how many ways can 52 cards be dealt to four players so that each gets three cards of three suits and four cards of the fourth suit?

176.
In how many ways can 18 distinct objects be distributed among 5 participants so that four get 4 objects each and the fifth gets 2 objects? The same, only three get 4 objects each while two get 3 objects each.

177.
There are 14 pairs of different objects. Find the total number of samples of these objects. (Two samples differ in composition but not in the order of the objects.)

178.
In how many ways can 4 black balls, 4 white balls, and 4 blue balls be placed into 6 different packages (some of the packages may remain empty)?

179.
In how many ways can 3 rouble-bills and 10 50-copeck pieces be put into 4 different envelopes?

180.
Prove that the number of partitions of the number n into parts (summands) is equal to the number of partitions of the number $2n$ into n parts (without regard for order of the parts).

181.
There are n objects in a row. In how many ways can three objects be selected so that no two adjacent ones are taken?

182.
A child places black and white chessmen on the first two lines of a chessboard (two knights, two bishops, two rooks, a queen and a king, both white and black). In how many ways can he do this?

183.
In how many ways can this be done on the whole chessboard?

184.

Solve the same problem with the pawns included (8 of each colour).

185.

In how many ways can you put 15 white and 15 black draughtmen on 24 squares so that each square is occupied only by white or only by black? (This is the arrangement of draughtmen in the game of "nardy" played in the East.)

186.

In how many ways can you place 20 white draughtmen on a chessboard so that rotations of the board through 90° bring the pieces into the same position?

187.

In how many ways is it possible to place 20 white draughtmen on a chessboard so that the arrangement is symmetric about the line dividing the board in half?

188.

The same, provided the draughtmen are placed on black squares.

189.

In how many ways can you put 12 black and 12 white draughtmen on the black squares of the board so that the arrangement is symmetric about the centre of the board?

190.

The same, only symmetry is retained by changing the colour of the draughtmen.

191.

In how many ways can 20 white draughtmen be placed on the extreme lines of a chessboard so that the arrangement remains unchanged when the board is turned through 90°?

192.

In how many ways can 20 white draughtmen be placed on the extreme lines of a chessboard so that the pieces on opposite sides of the board are symmetric about the lines dividing the board in half?

193.

In how many ways can 7 white balls and 2 black balls be placed in 9 pockets? Some of the pockets may be empty; all pockets are considered distinct.

194.

In how many ways can 7 white balls, 1 black ball and 1 red ball be put into 9 pockets?

195.

In how many ways can 27 books be given to A, B, and C so that A and B together get twice as many books as C?

196.

Eight persons get into a lift. In how many ways can they leave on four floors so that at least one person gets out on each floor?

197.

In how many ways can three numbers be chosen from the numbers 1 to 100 so that their sum is divisible by 3?

198.

In how many ways can three numbers be chosen from $3n$ consecutive integers so that their sum is divisible by 3?

199.

We have one black ball and n white balls. In how many ways can some of these balls be put into $n+1$ pockets if each pocket holds at most one ball?

200.

In how many ways can n black and m white balls be arranged so that there are $2r-1$ contacts between black and white ($2r$ contacts)?

201.

In how many ways can 8 marks of 3, 4, or 5 be obtained in a series of subjects so that their sum is equal to 30?

202.

Prove that $m+n$ objects can be permuted in $\frac{(m+n)!\,D_m}{m!\,n!}$ ways so that exactly n remain fixed (see page 44).

203.

Prove that r distinct things can be distributed in

$$S_r = (n+p)^r - n(n+p-1)^r + C_n^2(n+p-2) - \ldots + (-1)^n p^r$$

ways among $n+p$ persons so that a given n receive at least one object each.

204.

Prove that the number of partitions of the number $2r+x$ into $r+x$ nonzero parts is the same as that of the number of partitions of r into nonnegative summands.

205.

A society of n members elects one member as a representative of the society. In how many ways can the voting take place if each votes for one person (including, possibly, for himself)? The same, provided that only the number of votes obtained by each candidate are taken into account, and not the names of his supporters.

206.

Prove that the number of ways of splitting up $2n$ indistinguishable objects into three indistinguishable parts so that the sum of any two is greater than the third is equal to the number of ways of partitioning $2n-3$ objects in that fashion.

207.

Prove that an odd number of objects can be chosen from n objects in 2^{n-1} ways.

208.

Prove that the number of ways in which two persons can divide $2n$ objects of one kind, $2n$ objects of another kind, and $2n$ objects of a third kind so that each receives $3n$ objects is equal to $3n^2+3n+1$.

209.

If we adjoin $2n$ objects of a fourth kind, the number of ways of dividing them so that each gets $4n$ objects is

$$\frac{1}{3}(2n+1)(8n^2+8n+3)$$

210.

If the objects are divisible into indistinguishable parts, the answers will be

$$\frac{1}{2}(3n^2+3n+2) \text{ and } \frac{1}{3}(n+1)(8n^2+4n+3)$$

211.

Prove that if there are m kinds of objects, $2n$ of each kind, then the number of ways of dividing them into two equal parts is given by the formula

$$C_{mn+m-1}^{m-1} - C_m^1 C_{mn+m-2n-2}^{m-1} + C_m^2 C_{mn+m-4n-3}^{m-1} - \ldots \ldots \pm C_m^x C_{mn+m-1-x(2n+1)}^{m-1} \mp \ldots$$

212.

In how many ways can five white balls, five black balls and five red balls be placed into three different boxes, five balls in each box?

213.

If there are three kinds of things, n of each kind, then they can be distributed among three persons A, B, C in

$$C_{n+2}^2 C_{n+2}^2 - 3C_{n+3}^4 = \frac{1}{8}(n+1)(n+2)(n^2+3n+4)$$

ways so that each gets n objects.

214.

In how many ways can 3 Englishmen, 3 Frenchmen and 3 Turks be seated in a row so that no three compatriots sit together?

215.

The same, provided that no two compatriots sit together.

216.

In how many ways can 3 Englishmen, 3 Frenchmen and 3 Turks be seated at a circular table so that no two compatriots sit together?

217.

In how many ways can postage stamps totalling 40 copecks be put on a package by using 5-, 10-, 15- and 20-copeck stamps arranged in a row? (Arrangements with different orders of stamps are considered distinct; the supply of stamps is unlimited.)

218.

In how many ways can you change a rouble (= 100 copecks) using 10-, 15-, 20-, and 50-copeck pieces?

219.

In how many ways can 78 grams be attained using eight weights of 1, 1, 2, 5, 10, 10, 20 and 50 grams each? The use of two different weights (even though weighing the same) forms a new combination.

220.

There are six balls: 3 black, 1 red, 1 white and 1 blue. In how many ways can they form a row of 4 balls?

221.

In how many ways can a natural number n be represented in the form of a sum of three positive integers (representations involving different orders of the integers are considered distinct)?

222.

How many digits (and what kind) are needed to write all the numbers from 1 to 999,999 inclusive? From 1 to 10^n-1 inclusive?

223.

How many different ten-digit numbers can be written using three digits 1, 2, 3 with the provision that the digit 3 is used exactly twice in each number? How many of the numbers thus written are divisible by 9?

224.

We shall say that two numbers in a permutation form an inversion if the larger one comes before the smaller one. How many inversions are there in all permutations of the numbers $1, 2, \ldots, n$?

225.

Prove that the number of partitions of n into 3 parts such that no two parts are equal is

$$E\left[\frac{1}{12}(n^2-6n+12)\right]$$

226.

Prove that the number of partitions of $12n+5$ into 4 parts such that no part exceeds $6n+2$ is

$$\frac{1}{2}(n+1)(12n^2+9n+2)$$

227.

Prove that the number of partitions of $12n+5$ into 4 parts such that none exceeds $6n+2$ and no two are equal is

$$\frac{n}{2}(12n^2+3n-1)$$

228.

Find the number of triples of positive integers forming a geometric progression and not exceeding 100.

229.

In how many ways can 6 Englishmen, 7 Frenchmen and 10 Turks be arranged in a row so that each Englishman is between a Frenchman and a Turk, but no Frenchman and Turk stand together?

230.

The same for 5 Englishmen, 7 Frenchmen and 10 Turks.

231.

How many solutions has the following problem: Find two numbers such that their greatest common divisor is equal to G, the least common multiple is $M=Ga^{\alpha}b^{\beta}c^{\gamma}d^{\delta}$ (a, b, c, d prime numbers).

232.

Solve the same problem dropping the words "greatest" and "least".

233.

How many combinations can be formed out of 20 letters taking 6 at a time so that no letter appears more than twice in each combination?

234.

There are $p+q+r$ letters: p α's, q β's and r γ's. They are permuted in all possible ways so that the α's appear before the β's and the β's before the γ's. How many permutations are possible?

235.

A pole 30 cm long is to be painted in bands in the following order: red, white, blue, red, white, blue, etc. Red at the bottom, blue at the top. Each colour occupies 10 cm, the bands occupy an integral number of centimetres not less than two. How many ways of painting them do we have? The same, if we remove the restriction that blue is the last colour? Show that if no band must be less than 3 cm, then there will be 153 arrangements ending with blue, 71 with white and 81 with red.

236.

I have 6 friends with each of whom I have met at dinner 8 times. I have met every two of them 5 times, every three of them 4 times, every four of them 3 times, every five, twice, all six once, and I have dined out 8 times without meeting any of them. How many times have I dined out alone?

237.

Two examiners working together examine a class of 12 in two subjects. Each is examined for 5 minutes in each subject. In how many ways can a suitable arrangement be made so that no boy may be wanted by both examiners at once?

238.

Out of six pairs of gloves, in how many ways can six persons take each a right-handed and a left-handed glove without any person taking a pair? The same for 9 pairs and 6 persons.

239.

The letters in the expression $\alpha^2\beta^2\gamma^2$ are permuted in all ways so that an α must have another α next to it (the same for the other letters). Prove that the number of such permutations is 6. For $\alpha^3\beta^3\gamma^3$ also 6. For $\alpha^4\beta^4\gamma^4$, 90, and for $\alpha^5\beta^5\gamma^5$, 426.

240.

In a chess tournament there are 4 representatives from each of n countries. In how many ways can they stand in a row so that every man has a compatriot next to him?

241.

The squares of a chessboard are painted with 8 colours so that each horizontal line contains all 8 colours and each vertical line is so arranged that no two adjacent squares have the same colour. In how many ways can this colour pattern be arranged?

242.

There are n things alike and n others all different. In how many ways can n things be chosen from them? In how many ways can all $2n$ things be ordered?

243.

There are m Frenchmen and n Englishmen in a row so that at least one compatriot stands next to another. Show that the number of possible orders is

$$m!n!\,[1+(C^0_{m-2}+C^1_{m-3})(C^0_{n-2}+C^1_{n-3})$$
$$+(C^1_{m-3}+C^2_{m-4})(C^1_{n-3}+C^2_{n-4})$$
$$+(C^2_{m-4}+C^3_{m-5})(C^2_{n-4}+C^3_{n-5})+\ldots]$$

244.

How many six-digit numbers contain exactly three distinct digits?

245.

How many m-digit numbers contain exactly k distinct digits?

246.

Consider all k-permutations of the numbers 1, 2, . . ., n under which even numbers occupy even positions and odd numbers, odd positions. How many permutations of this kind are there in the order of increasing numbers (say, of the form 3,678)?

247.

Given $2n$ elements $a_1, a_1, a_2, a_2, \ldots, a_n, a_n$, and $a_i \neq a_j$, if $i \neq j$. In how many permutations of these $2n$ elements do we find that no two identical elements come together?

248.

Given n sets, each of which includes q identical elements, the elements of distinct sets being distinct. In how many permutations of these nq elements are there no two identical elements together?

249.

Solve Problem 248 when the elements are arranged in a circle.

250.
A bookshelf has n books. In how many ways can p books be chosen from them so that between any two selected books (and also after the pth selected book) there are at least s books?

251.
The numbers expressing the number of contestants at a mathematics olympiad of pupils of the 5th, 6th, 7th, 8th, 9th and 10th classes are in arithmetic progression. The number of prizes for each class is equal to the common difference of the progression. Prove that the number of ways of awarding the prizes (all different) remains unchanged if all prizes go to the 10th class.

252.
A square grid $ABCD$ is built, with 4 cells on a side, and all the shortest routes from vertex A to vertex C are drawn along the sides of the cells. Show that there are 70 routes, 35 routes going along 4 segments, 20 routes along 8 segments, 18 routes along 4, 15 routes along 4, 12 routes along 4, 10 routes along 4, 5 routes along 4, 4 rontes along 4, and 1 route along 4 segments. Iuvestigate the intersections in similar fashion: 1 is crossed 36 times, 4, 35 times, 4, 30 times, 4, 15 times, 4, 5 times, 4, 40 times, and 2 once (the endpoints are excluded).

253.
How many triangles are there whose vertices are the vertices of a given convex hexagon?

254.
How many triangles are there with the lengths of the sides assuming one of the values 4, 5, 6, 7?

255.
How many different rectangular parallelepipeds can be constructed, the length of each edge of which is an integer from 1 to 10?

256.
Draw 4 straight lines in a plane, no two lines being parallel and no three passing through one point. How many triangles are there?

257.
Given, in the plane, n points of which p lie on one straight line; aside from these, no three points lie on one straight line. How many triangles are there with these points as vertices?

258.
On a straight line take p points, on a parallel line take q points. How many triangles can be made using these points as vertices?

259.
The same conditions, but one more parallel line is added with r points, no three points lying on one straight line which intersects all three parallels. How many more triangles appear?

260.
Each side of a square is divided into n parts. How many triangles can be built whose vertices are the points of division?

261.
n straight lines are drawn in a plane, no two lines being parallel and no three intersecting in one point. How many points of intersection do these lines have?

262.
In a plane are n straight lines, of which p pass through point A and q pass through point B; besides, no three lines pass through one point, no line passes through both points A and B, and no two are parallel. How many intersection points do the lines have?

263.
Into how many parts is a plane divided by n straight lines of which no two are parallel and no three concurrent?

264.
Into how many parts do n planes divide a space, no 4 planes passing through one and the same point, no 3 passing through one and the same straight line, and no 2 being parallel?

265.
There are five points in a plane. Among the straight lines connecting these five points there are no parallel lines, perpendicular lines or coincident lines. Draw through each point perpendiculars to all the straight lines that can be built, joining the remaining four points in pairs. What is the largest number of intersection points of these perpendiculars (among themselves) if we discount the given five points?

266.

In how many ways can we construct triangles, whose sides are integers greater than n and not exceeding $2n$? How many isosceles triangles and equilateral triangles will there be?

267.

Prove that the number of triangles with integral sides, the length of the sides not exceeding $2n$, is $\frac{1}{6} n(n+1)(4n+5)$. If we exclude isosceles triangles, this number is equal to $\frac{1}{6} n(n-1) \times \times (4n-5)$.

268.

Prove that the number of triangles the length of the sides of which does not exceed $2n-1$ is $\frac{1}{6} n(n+1)(4n-1)$ and that, after excluding isosceles triangles, there remain $\frac{1}{6}(n-1) \times \times (n-2)(4n-3)$ triangles.

269.

n straight lines are drawn in a plane, no three of them concurrent. Prove that the number of unordered groups of n intersection points, of which no three lie on one straight line, is equal to $\frac{1}{2}(n-1)!$

270.

There are n points in a plane, no three of which lie on a single straight line. How many r-segment closed polygonal lines are there with vertices at these points?

271.

n points are taken on a straight line and m points on a parallel line. These points are joined by straight lines. Prove that the number of intersection points of the lines is $\frac{mn(m-1)(n-1)}{2}$. (We consider that no three of the drawn lines intersect in a single point; the given $m+n$ points are not counted.)

272.

There are n points in a plane no three of which are collinear and no 4 are concyclic. A straight line is drawn through every two of these points, and a circle through every three. Find the largest number of intersection points of all drawn lines with all circles.

273.

Given n points in space, no four of which lie in one plane. A plane is drawn through every three points, no two planes being parallel. Find the number of straight lines resulting from the intersection of the planes, and also the number of straight lines not passing through a single given point.

274.

Out of n segments of length $1, 2, \ldots, n$ choose 4 so as to obtain a circumscribed quadrilateral. Prove that this can be done in $\frac{2n(n-2)(2n-5)-3+3(-1)^n}{48}$ ways. How many quadrilaterals result if sides of equal length can be taken?

275.

Given n points, no four of which lie on one circle. A circle is drawn through every three. What is the greatest number of intersection points of these circles?

276.

Prove that if n planes pass through the centre of a sphere, then in the general case they divide the sphere into at most n^2-n+2 parts.

277.

In how many distinct ways (geometrically) can the faces of a cube be painted with six different colours? Two modes of painting are considered geometrically coincident if one can be carried into the other by rigid motions of the cube.

278.

In how many geometrically different ways is it possible to paint the faces of a tetrahedron using four different colours?

279.

In how many geometrically different ways can the faces of an octahedron be painted with eight distinct colours?

280.

Solve similar problems for a regular dodecahedron and a regular icosahedron.

281.

In the preceding problems, consider cases when the number of colours is less than the number of faces (say, a cube is painted with two colours, three colours, four colours and five colours).

282.

How many triangles are there with integral sides and perimeter 40? With perimeter 43?

283.

Prove that the number of triangles with integral sides and with perimeter $4n + 3$ is $n + 1$ more than the number of triangles with integral sides and perimeter $4n$.

284.

Prove that the number of triangles with integral sides and perimeter N is given by the table

N	Number of triangles	N	Number of triangles
$12n$	$3n^2$	$12n+6$	$3n^2+3n+1$
$12n+1$	$n(3n+2)$	$12n+7$	$(n+1)(3n+2)$
$12n+2$	$n(3n+1)$	$12n+8$	$(n+1)(3n+1)$
$12n+3$	$3n^2+3n+1$	$12n+9$	$3n^2+6n+3$
$12n+4$	$n(3n+2)$	$12n+10$	$(n+1)(3n+2)$
$12n+5$	$(n+1)(3n+1)$	$12n+11$	$3n^2+7n+4$

285.

In a city the bus routes are arranged as follows:
(1) No changes are needed to get from any stop to any other stop;
(2) For any two routes, there is one and only one stop where the change can be made from one route to the other;
(3) Each route has exactly n stops.
How many bus routes are there in the city?

286.

A city has 57 bus routes, such that
(1) No changes are needed to get from one stop to any other stop;
(2) For any two routes, there is one and only one stop where the change can be made from one route to the other;
(3) Each route has at least three stops.
How many stops has each of the 57 routes?

287.

Is it possible to set up 10 autobus routes and arrange the stops so that no matter what 8 routes are taken, there is a stop not involving any one of them, and any 9 routes pass through all stops?

288.

What is the maximum number of distinct spheres that can be built in space so that they contact three given planes and a given sphere?

289.

Through each of three given points draw m straight lines so that no two are parallel and no three are concurrent. Find the number of intersection points of these lines.

290.

Given n points in space of which m lie in the plane P and the others are arranged so that no four lie in one plane. How many planes can be drawn so that each contains three given points?

291.

A plane contains three points A, B, C. Draw m straight lines through A, n through B and p through C. Among these lines, no three are concurrent and no two parallel. Find the number of triangles whose vertices are the intersection points of the lines and do not coincide with the given points A, B, C.

292.

How many triangles are there whose vertices are the vertices of a given convex n-gon, but whose sides do not coincide with the sides of the n-gon?

293.

n straight lines are drawn on a plane and p points are taken on each one so that no point is a point of intersection of the lines and no three points lie on one nongiven line. Find the number of triangles with vertices at these points.

294.
Prove that the number of points of intersection of the diagonals of a convex n-gon exterior to the n-gon is $\frac{1}{12} n(n-3)(n-4)(n-5)$ and the number interior to it is $\frac{1}{24} n(n-1)(n-2) \times \times (n-3)$ (it is assumed that no two diagonals are parallel and no three are concurrent).

295.
There are n points on a circle. How many different polygons (not necessarily convex) can be inscribed in the circle, the vertices of the polygons being the given points? How many convex polygons?

296.
There are m parallel straight lines drawn on a plane. Also, on the same plane are n lines not parallel among themselves or to the earlier drawn lines. No line passes through the point of intersection of two other lines. Into how many regions do the straight lines divide the plane?

297.
Given 11 points, of which 5 lie on one circle. Other than these 5, no 4 lie on one circle. How many circles can be drawn so that each contains at least 3 of the given points?

298.
Given, in a plane, 10 lines that intersect in pairs; no 3 lines pass through one point and no 4 are tangent to one and the same circle. How many circles can be built such that each one contacts 3 of the given 10 lines?

299.
Find the total number of convex k-gons whose vertices are k of the n vertices of a convex n-gon; two adjacent vertices of the k-gon must be separated by at least s vertices of the n-gon.

300.
A parallelogram is cut by two rows of straight lines parallel to its sides; each row consists of r lines. How many parallelograms are there in the resulting figure?

301.
Into how many regions is a convex n-gon split by its diagonals if no three diagonals intersect in a single point inside the n-gon?

302.
Suppose there is one card labelled 1, two cards, 2, three cards, 3, etc. Prove that the number of ways of drawing two cards so as to obtain a sum of n is $\frac{n}{12}(n^2-1)$ or $\frac{n}{12}(n^2-4)$ depending on whether n is odd or even.

303.
There are $3n+1$ objects of which n are identical and the remaining are distinct. Prove that n objects can be selected from them in 2^{2n} ways.

304.
Given a sequence of numbers $1, 2, 3, \ldots, 2n$. In how many ways can three numbers be extracted to form an arithmetic progression? The same for the sequence of numbers $1, 2, 3, \ldots, 2n+1$.

305.
A number of closed curves are drawn on a plane each of which intersects all the others in at least two points. Let n_r be the number of points at which r curves meet. Prove that the number of closed regions bounded by arcs of these curves and not containing within them such arcs is

$$1+n_2+2n_3+\ldots+rn_{r+1}+\ldots$$

306.
Two pencils of straight lines are drawn on a plane with centres at A and B; one contains m lines, the other, n lines. Suppose no two lines are parallel and no line passes through both points A and B. Into how many regions do the straight lines of these pencils divide the plane?

307.
Can each one of 77 telephones make connections with exactly 15 others?

308.
Find the sum of the coefficients of the polynomial obtained after removing the brackets in the expression

$$(7x^3-13y^2+5z^2)^{1964}(y^3-8y^2+6y+z)^2+ \\ +(2x^2+18y^3-21)^{1965}$$

309.

A box contains 100 balls of different colours: 28 red, 20 green, 12 yellow, 20 blue, 10 white and 10 black. What is the smallest number of balls that can be drawn so as to obtain 15 balls of one colour?

310.

The faces of a cube may be painted as follows: all white, all black, or part white and part black. How many distinct black and white patterns are there? (Two cubes are considered distinct if they are distinguishable no matter how they are turned.)

311.

Solve the same problem when black and white are used for the vertices and not the faces of the cube.

312.

Models of polyhedrons are made out of plane developments. In a development, faces adjoin along edges, and the model is built by bending the cardboard development along the edges. A regular tetrahedron has two such distinct developments. How many does a cube have?

313.

A regular dodecahedron can be painted in four colours so that any two adjoining faces have different colours. How many geometrically distinct ways of solving this problem are there?

314.

Out of six edges of a tetrahedron it is possible to choose four edges that form a closed space tetragon, which contains all the vertices of the tetrahedron. The same may be done with the cube (we obtain an octagon containing all the vertices of the cube). Can the same be achieved with an octahedron, a dodecahedron, an icosahedron? How many solutions will there be for each polyhedron?

315.

A particle is located at the origin of coordinates. In unit time, it decays into two particles, one of which moves one unit of length to the left, the other to the right. This process is repeated every unit of time, and two particles at any one point mutually annihilate (so that, for instance, in two units of time, there are two particles). How many particles will there be in 129 units of time? in n units of time?

316.

A certain alphabet consists of six letters which are coded for telegraph communications as follows:

·, —, ··, — —, · —, — ·

In the transmission of one word, no gaps (blanks) separating the letters were made and the result was a continuous chain of dots and dashes consisting of 12 characters. In how many ways can the word be read?

317.

Of the numbers from 1 to 10,000,000, which are more numerous, those that contain a unit or those that do not?

318.

Dots and dashes are used to construct all possible "words" of exactly 7 characters. What is the largest number of words that can be chosen from them so that any two selected words differ in at least three characters?

319.

In how many ways can a circle divided into p parts (p prime) be painted with n colours? Modes that coincide under a rotation of the circle about its centre are considered to be coincident.

320.

An n by n grid of cells is built and the numbers $1, 2, 3, \ldots, n^2$ are placed, one in each cell, so that the numbers on each vertical line and horizontal line form an arithmetic progression. Find the number of such arrangements.

321.

A man has no more than 300,000 hairs on his head. Prove that there are at least 10 persons in Moscow who have the same number of hairs (Moscow has a population of about 6 million).

322.

Given $2n + 1$ objects, prove that an odd number of objects can be chosen from them in as many ways as an even number of objects.

323.

Prove that 1 rouble can be changed in a greater number of ways using coins of 2 and 5 copecks than when using 3- and 5-copeck coins.

324.

In how many ways can a 20-copeck piece be changed using 1-, 2-, and 5-copeck pieces?

325.

Prove that with the aid of a standard set of weights: 1 mg, 2 mg, 2 mg, 5 mg, 10 mg, 20 mg, 20 mg, 50 mg, 100 mg, 200 mg, 200 mg, 500 mg, and 1 g, etc. it is possible to make up any weight expressed as an integral number of milligrams.

326.

Given six digits: 0, 1, 2, 3, 4, 5. Find the sum of all even four-digit numbers that can be written with these digits (repetitions are allowed).

327.

A pack of $2n$ cards is shuffled by the following process. Divide the pack into two equal parts. Push one half-pack into the other in such a way that the cards of the first half go singly into the interstices between the cards of the second half. Thus the $(n+1)$th card will become top; the 1st card, second; the $(n+2)$th, third; the 2nd, fourth; and so on. Prove that after shuffling r times, the card which was originally in the pth place will now be in the xth place, where x is the remainder when $p2^r$ is divided by $2n+1$.

328.

Prove that if, given the conditions of Problem 327, the pack has $6m+2$ cards, then the $(2m+1)$th and $(4m+2)$th cards will exchange places at every shuffle.

329.

If a pack of $14m+6$ cards is shuffled three times by the process described in Problem 327, the $(2m+1)$th, $2(2m+1)$th, $3(2m+1)$th, $4(2m+1)$th, $5(2m+1)$th and $6(2m+1)$th cards will regain their original positions.

330.

If 2^x-1 is divisible by $2n+1$, a pack of $2n$ cards will be restored to its original order after x shuffles by the process described in Problem 327.

331.

A pack of cards is shuffled as follows: take the first card, put the second on top of it, the third under it, etc. Prove that if the pack contains $6n-2$ cards, then the card $2n$ remains in its place.

332.

22 cards are shuffled as indicated in Problem 331. Prove that card 8 remains fixed, 5 and 14 change places, and 3, 13, 18 move circularly, one into another.

333.

Prove that, under the same conditions, a pack of 16 cards will regain its original order in 5 shuffles, a pack of 32 cards in 6 shuffles, 42 cards in 8 shuffles, 28 and 36 cards in 9, 12, 20, 46 cards in 10 shuffles, 22 and 52 cards in 12 shuffles, 14 cards in 14 shuffles, 18 cards in 18 shuffles, 26 cards in 26 shuffles, 30 cards in 30 shuffles, and 50 cards in 50 shuffles.

334.

A square is divided into 16 equal squares. In how many ways can it be painted using white, black, red and blue colours so that each horizontal row and each vertical column has four colours?

335.

15 children line up in 5 rows, 3 in a row. In how many ways can this be done without any 2 children coming together twice?

336.

Prove that if n is an integer, then $(n^2)!/(n!)^{n+1}$ is an integer too; if m and n are odd, then $(mn)!/(m!)^{\frac{n+1}{2}}(n!)^{\frac{m+1}{2}}$ is an integer.

337.

n objects are arranged in a circle. Prove that if f_n is the number of permutations of these objects under which no object follows the one it originally followed, then $f_n+f_{n+1}=D_n$ (see page 44).

338.

Find the number of integral solutions of the equation $x_1+x_2+\ldots+x_p=m$ if all the unknowns satisfy the inequality $0 \leqslant l \leqslant x_k \leqslant n$.

339.

There are 7 copies of one book, 8 of another book, and 9 of a third. In how many ways can they be distributed between two persons so that each gets 12 books?

340.

All n-combinations (with repetitions) made up of n letters are written down. Show that each letter will appear C_{2n-1}^{n} times.

341.

The distance between A and B is 999 km. Poles are put up along the road at 1-km intervals indicating the distances to A and to B (0,999), (1,998), . . ., (999,0). How many signs have only two distinct digits?

342.

All possible permutations, with repetitions, are formed out of m white balls and n black balls. Show that there are $P(m+1, n+1)-2$ permutations.

343.

We have all possible permutations of m white balls and n black balls (with repetitions). Show that the total number of white balls in all permutations is

$$1+\frac{mn+m-1}{n+2}P(m+1, n+1)$$

and the total number of black balls is

$$1+\frac{mn+n-1}{m+2}P(m+1, n+1)$$

Verify your answer using the Russian word "Гаага".

344.

Show that the number of permutations that can be generated from m white balls and n black balls and one red ball, 1, 2, . . ., $m+n+1$ at a time (in which the red ball is included) is

$$1+\frac{mn+m+n}{m+n+4}P(m+2, n+2)$$

345.

The total number of permutations that can be generated using m white balls, n black balls and one red ball, is

$$\frac{(m+1)(n+1)}{m+n+3}P(m+2, n+2)-1$$

Verify your answer using the Russian word "окорок".

346.

I have 7 friends. In how many ways can I invite them to dinner, three at a time, in the course of 7 days so that no 3 come twice?

347.

Prove that if I want to have 7 different gatherings of 3 persons each and nobody left uninvited, this can be done in

$$A_{35}^{7}-7A_{20}^{7}+21A_{10}^{7}$$

ways.

348.

Prove that if I want to have 7 different groups of 3 each and no friend comes every day, then this can be done in $A_{35}^{7}-7A_{15}^{7}$ ways.

349.

Show that the total number of permutations of $n \geqslant 2$ objects (1, 2, . . ., n at a time) is the closest integer to $en!-1$.

350.

Prove that if all permutations are written out, the number of times each object appears is the closest integer to $e(n-1)(n-1)!$.

351.

A coin is tossed $2n$ times. Prove that the number of times of heads never once falling more often than tails is

$$1+(C_{n}^{1})^{2}+\ldots+(C_{n}^{n})^{2}=C_{2n}^{n}$$

352.

In how many ways can $3n$ different books be distributed among three persons so that the numbers of books are in arithmetic progression?

353.

There are n pairs consisting of identical letters, different pairs consisting of distinct letters. These letters are ordered in all possible ways so that no two identical letters come in succession. Prove that the number of distinct orders is

$$\frac{1}{2^n}\left[(2n)!-\frac{n}{1}2(2n-1)!+\frac{n(n-1)}{1\times 2}2^2(2n-2)!-\ldots\right]$$

354.

There are r distinct things that are distributed among $n+p$ persons so that at least n of them receive at least one thing. Prove that the number of modes of dividing the things is

$$(n+p)^r-n(n+p-1)^r+\frac{n(n-1)}{1\times 2}(n+p-2)^r-\ldots$$

355.

Denote by Π_n^k the number of ways of dividing n distinct things into k groups. Prove that for $n>1$,

$$1-\Pi_n^2+2!\,\Pi_n^3-3!\,\Pi_n^4+\ldots=0$$

356.

There are m cells, in the first of which are n objects, in the second, $2n$ objects, . . ., in the mth, mn objects. In how many ways can n objects be selected from each cell?

357.

A basket contains $2n+r$ apples and $2n-r$ pears. Prove that for a given n, the number of choices of n apples and n pears will be greatest if $r=0$.

358.

1,000 points are the vertices of a convex 1,000-gon inside of which are another 500 points arranged so that no three of these 1,500 points lie on one straight line. The given 1,000-gon is divided into triangles so that all indicated 1,500 points are vertices of triangles, and the triangles do not have any other vertices. How many triangles will there be?

359.

Five persons play a number of games of dominoes (two against two), each player having each of the others as partner once and, as opponent, twice. Find the number of games played and all possible ways of arranging the players.

360.

From point O, on a plane, all closed polygonal lines of length $2n$ are drawn, the sides of which lie on the lines of squared paper, the side of a cell being 1. Find the number of these polygonal lines if one such line is allowed to traverse the same segment several times.

361.

On a piece of paper, a grid is constructed having n horizontal and n vertical lines. How many distinct $2n$-segment closed polygonal lines can be drawn along the lines of the grid so that each polygonal line has segments on all horizontal and all vertical lines?

362.

A factory manufactures rattles in the form of a ring with three red balls and 7 blue balls. How many different rattles can be made (two rattles are considered to be the same if one of them can be obtained from the other simply by shifting the balls around the ring and turning it over)?

363.

n persons gather together. Some of them are acquainted; every two persons who are unacquainted have exactly two acquaintances in common, and every two acquaintances have no acquaintances in common. Prove that every person present is acquainted with the same number of people.

364.

Several points are chosen on a circle; some are labelled A, others B. The arcs thus formed which divide the circle are labelled in the following manner: if both endpoints have the letters A, then the arc is labelled 2; if both endpoints have the letters B, we write $\frac{1}{2}$; if the endpoints of an arc are designated by different letters, then we write the number 1. Prove that the product of all indicated numbers is equal to 2^{a-b}, where a is the number of points denoted by A and b is the number of points denoted by B.

365.

The horizontal lines of a chessboard are designated by the digits 1 to 8, vertical lines, by the letters a to h. Now let a, b, c, d, e, f, g, h be arbitrary numbers. In each square of the board write the product of the numbers denoting the appropriate horizontal and vertical line, and place 8 nontaking rooks on the board. What is the product of the covered numbers?

366.

The organizing committee of a competition consists of 11 persons. The materials of the competition are kept in a safe. How many locks must the safe have and how many keys must each member of the committee have so that the safe may be opened when any six members of the committee are assembled but cannot be opened if there are less than six members present?

367.

A piece of chain has 60 links. Each link weighs 1 gram. What is the smallest number of links that must be opened so that, by using the split links and the resulting pieces, it is possible to obtain any integral weight from 1 to 60? Solve the same problem if a balance with two pans is available.

368.

How many pairs of integers x, y lie between 1 and 1,000 such that $x^2 + y^2$ is divisible by 49?

369.

How many two-digit numbers yield a perfect square? When added to a number with the same digits but in the reverse order?

370.

Find the sum of all four-digit numbers which are made up of these digits from 1 to 6 and are divisible by 3.

371.

Find the sum of all even four-digit numbers that can be generated from the digits 0 to 5.

372.

How many distinct integral solutions does the inequality $|x| + |y| \leqslant 1{,}000$ have?

373.

The points $A_1, A_2, \ldots, A_{16}$ are indicated on a circle. Construct all possible convex polygons whose vertices are among the points $A_1, A_2, \ldots, A_{16}$. Divide these polygons into two groups, the first of which includes all polygons one of the vertices of which is point A_1, the second, all the other polygons. Which group has more polygons?

374.

There is a knight on an infinite chessboard. Find the number of squares it can reach in $2n$ moves.

375.

There are 1,955 points. What is the largest number of triplets of points that can be chosen so that every two triplets have one point in common?

376.

The numbers from 1 to 100,000,000 are written down in succession so that we have the sequence of digits 123456 ... 100,000,000. Prove that the number of all the digits of this sequence is equal to the number of zeros in the sequence 1, 2, 3, ..., 10^9.

377.

How many four-digit numbers are there from 0000 to 9,999 such that the sum of the first two digits is equal to the sum of the last two digits?

378.

A total of $2n$ subjects are taught at school. All the students have marks of 4 and 5 (5 is the highest mark). No two students have the same marks, yet no two students are such that one studies better than the other. Prove that the number of students in the school does not exceed C_{2n}^{n} (we will assume that one student studies better than another if in all subjects his marks are not worse than another's, and in some subjects he has higher marks).

379.

Let M_r be the number of permutations (without repetitions) of m elements r at a time, and N_r the number of permutations (without repetitions) of n elements r at a time. Prove that the number of permutations of $m+n$ elements taken r at a time is given by the formula $(M+N)^r$, where, after raising to the power, we have to replace all exponents by indices.

380.

Find the coefficient of x^8 in the expansion of $(1+x^2-x^3)^9$

381.
Find the coefficient of x^m in the expansion of
$(1+x)^k+(1+x)^{k+1}+\ldots+(1+x)^n$
in powers of x. Investigate the cases $m<k$, $m \geqslant k$ separately.

382.
Find the coefficients of x^{17} and x^{18} after removing brackets and collecting terms in the expression $(1+x^5+x^7)^{20}$.

383.
After removing brackets and collecting terms, in which of the expressions $(1+x^2-x^3)^{1,000}$ or $(1-x^2+x^3)^{1,000}$ is the coefficient of x^{17} greater?

384.
Let $a_0, a_1, a_2, \ldots$ be the coefficients in the expansion of $(1+x+x^2)^n$ in ascending powers of x. Prove that

(a) $a_0a_1-a_1a_2+a_2a_3-\ldots-a_{2n-1}a_{2n}=0$,

(b) $a_0^2-a_1^2+a_2^2-\ldots+(-1)^{n-1}a_{n-1}^2=$
$$=\frac{1}{2}a_n+\frac{1}{2}(-1)^{n-1}a_n^2,$$

(c) $a_r-na_{r-1}+C_n^2a_{r-2}-\ldots+(1)^rC_n^ra_0=0$ if r is not a multiple of 3,

(d) $a_0+a_2+a_4+\ldots=\frac{1}{2}(3^n+1)$,

$a_1+a_3+a_5+\ldots=\frac{1}{2}(3^n-1)$

385.
Find the number of distinct terms in the expansion of
$(x_1+x_2+\ldots+x_n)^3$
which are obtained after raising to the power.

386.
Find the coefficient of x^k in the expansion of
$(1+x+x^2+\ldots+x^{n-1})^2$

387.
Prove that
$$\frac{[C_{n+1}^{r+1}-C_n^r]\,C_{n-1}^{r-1}}{(C_n^r)^2-C_{n+1}^{r+1}C_{n-1}^{r-1}}=r$$

388.
Prove that
$C_n^1+6C_n^2+6C_n^3=n^3$,
$1+7C_n^1+12C_n^2+6C_n^3=(n+1)^3$

389.
Prove that
$1+14C_n^1+36C_n^2+24C_n^3=(n+1)^4-n^4$
$C_n^1+14C_n^2+36C_n^3+24C_n^4=n^4$

390.
Prove that
$$1-3C_n^2+9C_n^4-27C_n^6+\ldots=(-1)^n2^n\cos\frac{2n\pi}{3}$$
$$C_n^1-3C_n^3+9C_n^5-\ldots=\frac{(-1)^{n+1}2^{n+1}}{\sqrt{3}}\sin\frac{2n\pi}{3}$$

391.
Prove that

(a) $C_n^0+C_n^3+C_n^6+\ldots=\frac{1}{3}\left(2^n+2\cos\frac{n\pi}{3}\right)$,

(b) $C_n^1+C_n^4+C_n^7+\ldots=$
$$=\frac{1}{3}\left(2^n+2\cos\frac{(n-2)\pi}{3}\right),$$

(c) $C_n^2+C_n^5+C_n^8+\ldots=$
$$=\frac{1}{3}\left(2^n+2\cos\frac{(n+2)\pi}{3}\right),$$

(d) $C_n^0+C_n^4+C_n^8+\ldots=$
$$=\frac{1}{2}\left(2^{n-1}+2^{\frac{n}{2}}\cos\frac{n\pi}{4}\right)$$

392.
Prove that for $n \geqslant 2$ and $|x|<1$ we have
$(1+x)^n+(1-x)^n \leqslant 2^n$

393.
Prove that for $m>n$,
$$\sum_{x=0}^{n}\frac{n(n-1)\ldots(n-x+1)}{m(m-1)\ldots(m-x+1)}=\frac{m+1}{m-n+1}$$

and

$$\sum_{x=0}^{n} \frac{C_n^x C_n^r}{C_{2n}^{x+r}} = \frac{2n+1}{n+1}$$

394.

Prove that

$$\frac{m}{1} + \frac{m(m+1)}{1\times 2} + \ldots + \frac{m(m+1)\ldots(m+n-1)}{1\times 2\ldots n} = \\ = \frac{n}{1} + \frac{n(n+1)}{1\times 2} + \ldots \\ \ldots + \frac{n(n+1)\ldots(n+m+1)}{1\times 2\ldots m}$$

395.

Prove that

$$\sum_{x=1}^{n} \frac{C_{n-1}^{x-1}}{C_{2n-1}^{x}} = \frac{2}{n+1}$$

396.

Prove that

$$\sum_{x=1}^{n} \frac{C_{n-1}^{x-1}}{C_{n+q}^{x}} = \frac{n+q+1}{(q+1)(q+2)}$$

397.

Prove that

$$\sum_{x=1}^{n} \frac{C_{n-2}^{x-2}}{C_{n+q}^{x}} = \frac{2(n+q+1)}{(q+1)(q+2)(q+3)}$$

398.

Prove that

$$(C_n^1)^2 + 2(C_n^2)^2 + 3(C_n^3)^2 + \ldots + n(C_n^n)^2 = \\ = \frac{(2n-1)!}{[(n-1)!]^2}$$

399.

Prove that

$$\frac{1}{[(n-1)!]^2} + \frac{1}{1!\,2!}\,\frac{1}{[(n-2)!]^2} + \\ + \frac{1}{2!\,3!}\,\frac{1}{[(n-3)!]^2} + \ldots = \frac{(2n-1)!}{[n!\,(n-1)!]^2}$$

400.

Prove that

$$\frac{(n+r-1)!}{r!} - \frac{n}{1}\,\frac{(n+r-3)!}{(r-2)!} + \\ + \frac{n(n-1)}{1\times 2}\,\frac{(n+r-5)!}{(r-4)!} - \ldots = \frac{n!\,(n-1)!}{r!\,(n-r)!}$$

401.

Compute the following sums:

(a) $C_n^1 + 2C_n^2 + 3C_n^3 + \ldots + nC_n^n$,

(b) $C_n^0 + 2C_n^1 + 3C_n^2 + \ldots + (n+1)\,C_n^n$,

(c) $C_n^2 + 2C_n^3 + 3C_n^4 + \ldots + (n-1)\,C_n^n$,

(d) $C_n^0 + 3C_n^1 + 5C_n^2 + \ldots + (2n-1)\,C_n^n$,

(e) $C_n^0 - 2C_n^1 + 3C_n^2 - \ldots + (-1)^n (n+1)\,C_n^n$,

(f) $3C_n^1 + 7C_n^2 + 11C_n^3 + \ldots + (4n-1)\,C_n^n$,

(g) $C_n^1 - 2C_n^2 + 3C_n^3 - \ldots + (-1)^{n-1}\,nC_n^n$,

(h) $\frac{C_n^0}{1} + \frac{C_n^1}{2} + \frac{C_n^2}{3} + \ldots + \frac{C_n^n}{n+1}$,

(i) $\frac{C_n^0}{2} + \frac{C_n^1}{3} + \frac{C_n^2}{4} + \ldots + \frac{C_n^n}{n+2}$,

(j) $\frac{C_n^0}{1} - \frac{C_n^1}{2} + \frac{C_n^2}{3} - \ldots + (-1)^n \frac{C_n^n}{n+1}$,

(k) $(C_n^0)^2 - (C_n^1)^2 + (C_n^2)^2 - \ldots + (-1)^n (C_n^n)^2$

402.

Find the largest coefficient in the expansions of $(a+b+c)^{10}$, $(a+b+c+d)^{14}$

403.

Denote by Y_n the coefficients of the expansion of the function $(1-4x)^{-\frac{1}{2}}$ in a power series:

$$(1-4x)^{-\frac{1}{2}} = 1 + Y_1 x + Y_2 x^2 + \ldots$$

Express Y_n in terms of the binomial coefficients. Find the expansion of $(1-4x)^{\frac{1}{2}}$.

404.
Prove that the numbers Y_n satisfy the relations:

(a) $Y_n+\frac{1}{2}Y_1Y_{n-1}+\frac{1}{3}Y_2Y_{n-2}+\dots$
$\dots+\frac{1}{n+1}Y_n=\frac{1}{2}Y_{n+1}$,

(b) $Y_0Y_n+Y_1Y_{n-1}+Y_2Y_{n-2}+\dots+Y_nY_0=4^n$,

(c) $\frac{Y_0Y_n}{1(n+1)}+\frac{Y_1Y_{n-1}}{2\times n}+\frac{Y_2Y_{n-2}}{3(n-1)}+\dots$
$\dots+\frac{Y_nY_0}{(n+1)\times 1}=\frac{Y_{n+1}}{n+2}$

405.
In the numerical triangle

```
      1
    1 1 1
  1 2 3 2 1
1 3 6 7 6 3 1
. . . . . . .
```

each number is equal to the sum of the numbers located in the preceding row above that number and above the adjacent numbers to the right and to the left (if some of these numbers are absent, they are taken to be zero). Prove that in every row, beginning with the third, there is an even number.

406.
The first row of the numerical triangle

```
0 1 2 3 . . . . . . . . . . . . . 1957 1958
  1 3 5 . . . . . . . . . . . . . . 3915
    . . . . . . . . . . . . . . . . . .
```

consists of the numbers 0, 1, . . ., 1958. The elements of each subsequent row consist of the sums of the elements of the preceding row on the left and on the right of the given number. Prove that the element of the last row of the triangle is divisible by 1958.

407.
Consider the Fibonacci sequence of numbers u_n: $u_0=0$, $u_1=1$, $u_2=1$, $u_3=2$, $u_4=3$, $u_5=5$, etc. (we began it with the terms 0 and 1, and not 1 and 2, as in Chapter VI). Prove that
(a) For any m and n we have

$$u_{n+m}=u_{n-1}u_m+u_nu_{m+1}$$

(b) For any m and $n=km$ the number u_n is divisible by u_m.
(c) Two successive terms of the Fibonacci sequence are relatively prime.

408.
Find the largest common divisor of the 1,000th and 770th terms of the Fibonacci sequence.

409.
Is there a number ending in four zeros among the first 100,000,001 terms of the Fibonacci sequence?

410.
Eight numbers are taken in succession from the Fibonacci sequence. Prove that their sum is not in the sequence.

411.
Prove that

(a) $u_2+u_4+\dots+u_{2n}=u_{2n+1}-1$,

(b) $u_1+u_3+\dots+u_{2n-1}=u_{2n}$,

(c) $u_1^2+u_2^2+\dots+u_n^2=u_nu_{n+1}$,

(d) $u_{n+1}^2=u_nu_{n+2}+(-1)^n$,

(e) $u_1u_2+u_2u_3+\dots+u_{2n-1}u_{2n}=u_{2n}^2$,

(f) $u_1u_2+u_2u_3+\dots+u_{2n}u_{2n+1}=u_{2n+1}^2-1$,

(g) $nu_1+(n-1)u_2+(n-2)u_3+\dots+2u_{n-1}+$
$+u_n=u_{n+4}-(n+3)$,

(h) $u_3+u_6+\dots+u_{3n}=\frac{u_{3n+2}-1}{2}$,

(i) $u_{3n}=u_{n+1}^3+u_n^3-u_{n-1}^3$.

412.
Prove that any natural number N can be represented as a sum of Fibonacci numbers, each number entering the sum at most once and no two successive numbers entering together.

413.
Let $p\geqslant q\geqslant r$ be integers such that $p<q+r$ and $p+q+r=2s$. There are p black, q white and r red balls. Show that the number of ways of dividing these balls between two persons so that each gets s balls is

$$s^2+s+1-\frac{1}{2}(p^2+q^2+r^2)$$

414.
If $q + r < p$, then the answer in the preceding problem is increased by $\frac{1}{2}(p-s)(p-s-1)$.

415.
There are $pq + r$ distinct objects, where $0 \leqslant r < p$. They are divided among p persons as equally as possible (all receive either q or $q+1$ objects). Show that the number of ways of making such a division is equal to

$$C_p^r \frac{(pq+r)!}{(q+1)^r (q!)^p}$$

416.
Compute the sum

$$\sum_{i_n=1}^{m} \sum_{i_{n-1}=1}^{i_n} \cdots \sum_{i_1=1}^{i_2} \sum_{i_0=1}^{i_1} 1$$

417.
Prove the identity

$$C_{n+m}^m = \sum P(k_1, \ldots, k_m, n-k_1-\ldots-k_m+1)$$

where the summation is extended over all nonnegative integral solutions of the equation $k_1 + 2k_2 + \ldots + mk_m = m$.

418.
Find the general solution of the recurrence relations:

(a) $a_{n+2} - 7a_{n+1} + 12a_n = 0$,

(b) $a_{n+2} + 3a_{n+1} - 10a_n = 0$,

(c) $a_{n+2} - 4a_{n+1} + 13a_n = 0$,

(d) $a_{n+2} + 9a_n = 0$,

(e) $a_{n+2} + 4a_{n+1} + 4a_n = 0$,

(f) $a_{n+3} - 9a_{n+2} + 26a_{n+1} - 24a_n = 0$,

(g) $a_{n+3} + 3a_{n+2} + 3a_{n+1} + a_n = 0$,

(h) $a_{n+4} + 4a_n = 0$

419.
Find a_n, knowing the recurrence relation and the initial terms:

(a) $a_{n+2} - 5a_{n+1} + 6a_n = 0$, $a_1 = 1$, $a_2 = -7$,

(b) $a_{n+2} - 4a_{n+1} + 4a_n = 0$, $a_1 = 2$, $a_2 = 4$,

(c) $a_{n+2} + a_{n+1} + a_n = 0$, $a_1 = -\frac{1}{4}$, $a_2 = -\frac{1}{2}$,

(d) $a_{n+3} - 9a_{n+2} + 26a_{n+1} - 24a_n = 0$, $a_1 = 1$, $a_2 = -3$, $a_3 = -29$

420.
Find a sequence such that $a_1 = \cos\alpha$, $a_2 = \cos 2\alpha$ and

$$a_{n+2} - 2\cos\alpha\, a_{n+1} + a_n = 0$$

421.
Prove that a sequence with the general term $a_n = n^k$ satisfies the relation

$$a_{n+k} - C_k^1 a_{n+k-1} + C_k^2 a_{n+k-2} - \ldots + (-1)^k C_k^k a_n = 0$$

422.
Find a sequence such that

$$a_{n+2} + 2a_{n+1} - 8a_n = 2^n$$

423.
From the identity $(1+x)^p (1+x)^{-k-1} = (1+x)^{p-k-1}$ infer that

$$\sum_{s=0} (-1)^s C_{k+s}^s C_p^{n-s} = C_{p-k-1}^n$$

(here and henceforth the sum is extended over nonnegative integral values of s for which the left member of the equality is defined).

424.
From the identity $(1-x)^{-m-1}(1-x)^{-q-1} = (1-x)^{-m-q-2}$ infer that

$$\sum_{s=0} C_{p-s}^m C_{q-s}^q = C_{p+q+1}^{p-m}$$

425.
From the identity $(1+x)^n = (1-x^2)^n (1-x)^{-n}$ infer that

$$\sum_{s=0} (-1)^s C_{n+k-2s}^n C_{n+1}^s = C_{n+1}^k$$

426.
From the identity $(1+x)^n (1-x^2)^{-n} = (1-x)^{-n}$ infer that

$$\sum_{s=0} C_n^{k-2s} C_{n+s-1}^s = C_{n+k-1}^k$$

427.

From the identity $(1-x^2)^{-p-1}=(1+x)^{-p-1}\times$ $\times(1-x)^{-p-1}$ infer that

$$\sum_{s=0} (-1)^s C^p_{p+2k-s} C^p_{p+s} = C^k_{p+k}$$

428.

From the identities

$$(1-x)^{-2p}\left[1-\left(\frac{x}{1-x}\right)^2\right]^{-p}=(1-2x)^{-p}$$

and

$$(1-x)^{2p}\left[1-\left(\frac{x}{1-x}\right)^2\right]^{p}=(1-2x)^{p}$$

infer that

$$\sum_{s=0} C^s_{p+s} C^{2p+2s+1}_{2p+m} = 2^{m-1} C^p_{m+p-1}$$

and

$$\sum_{s=0} (-1)^s C^s_p C^{m-2s}_{2p-2s} = 2^p C^m_p$$

429.

Prove that

$$\sum_{s=0} C^s_{p+s} C^{2p+2s}_{2p+m} = 2^{m-1}\,\frac{2p+m}{m}\,C^p_{m+p-1}$$

430.

From the identities

$$(1-x)^{\pm 2p}\left[1+\frac{2x}{(1-x)^2}\right]^{\pm p}=(1+x^2)^{\pm p}$$

derive the formulas

$$\sum_{s=0} (-1)^s C^s_{p+s-1} C^{2m+1-s}_{2m+2p+s} 2^s = 0,$$

$$\sum_{s=0} (-1)^s C^s_{p+s-1} C^{2m-s}_{2m+2p+s-1} 2^s = (-1)^m C^m_{p+m-1},$$

$$\sum_{s=0} (-1)^s C^s_p C^{2m+1-s}_{2p-2s} 2^s = 0,$$

$$\sum_{s=0} (-1)^s C^s_p C^{2m-s}_{2p-2s} 2^s = C^m_p$$

Using them, prove that

$$\sum_{s=0} C^{2s}_{2p+2m} C^p_{p+m-s} = 2^{2m}\,(p+m)\,\frac{(p+2m-1)!}{p!\,(2m)!}$$

$$\sum_{s=0} C^{2s+1}_{2p+2m+1} C^p_{p+m-s} = 2^{2m}\,(2p+2m+1)\,\frac{(p+2m)!}{p!\,(2m+1)!},$$

$$\sum_{s=1} C^{2s-1}_{2p+2m} C^p_{p+m-s} = 2^{2m-1} C^p_{p+2m-1},$$

$$\sum_{s=0} C^{2s}_{2p+2m+1} C^p_{p+m-s} = 2^{2m} C^p_{p+2m}$$

431.

Considering the formulas

$$[(1+x)^p \pm (1-x)^p]^2 =$$
$$=(1+x)^{2p}+(1-x)^{2p}\pm 2\,(1-x^2)^p,$$
$$[(1+x)^p+(1-x)^p]\,[(1+x)^p-(1-x)^p]=(1+x)^{2p}$$
$$-(1-x)^{2p}$$

for positive and negative values of p, prove that

$$2\sum_{s=0} C^{2s}_p C^{2m-2s}_p = C^{2m}_{2p} + (-1)^m C^m_p,$$

$$2\sum_{s=0} C^{2s+1}_p C^{2m-2s+1}_p = C^{2m+2}_{2p} + (-1)^m C^{m+1}_p,$$

$$2\sum_{s=0} C^{2s}_p C^{2m-2s+1}_p = C^{2m+1}_{2p},$$

$$2\sum_{s=0} C^p_{p+2s} C^p_{p+2m-2s} = C^{2p+1}_{2p+2m+1} + C^p_{p+m},$$

$$2\sum_{s=0} C^{p-1}_{p+2s} C^{p-1}_{p+2m-2s} = C^{2p-1}_{2p+2m+1} - C^p_{p+m},$$

$$2\sum_{s=0} C^p_{p+2s} C^p_{p+2m-2s+1} = C^{2p+1}_{2p+2m+2}$$

432.

Considering the expression

$$[(1+x)^{p+1} \pm (1-x)^{p+1}]\,[(1+x)^p \pm (1-x)^p]$$

for all possible combinations of signs, derive the formulas

$$2\sum_{s=0} C^{2s}_{p+1} C^{2m-2s}_p = C^{2m}_{2p+1} + (-1)^m C^m_p,$$

$$2\sum_{s=0} C^{2s}_{p+1} C^{2m-2s+1}_p = C^{2m+1}_{2p+1} - (-1)^m C^m_p,$$

$$2\sum_{s=0} C^{2s+1}_{p+1} C^{2m-2s}_p = C^{2m+1}_{2p+1} + (-1)^m C^m_p,$$

$$2\sum_{s=0} C^{2s+1}_{p+1} C^{2m-2s+1}_p = C^{2m+2}_{2p+1} + (-1)^m C^{m+1}_p,$$

$$2\sum_{s=0} C^{p-1}_{p+2s-1}C^{p}_{p+2m-2s}=C^{2p}_{2p+2m}+C^{p}_{p+m},$$

$$2\sum_{s=0} C^{p-1}_{p+2s-1}C^{p}_{p+2m-2s+1}=C^{2p}_{2p+2m+1}+C^{p}_{p+m},$$

$$2\sum_{s=0} C^{p-1}_{p+2s}C^{p}_{p+2m-2s}=C^{2p}_{2p+2m+1}-C^{p}_{p+m},$$

$$2\sum_{s=0} C^{p-1}_{p+2s}C^{p}_{p+2m-2s+1}=C^{2p}_{2p+2m+2}-C^{p}_{p+m+1}$$

433.
From the relation

$$\left(1-\frac{1}{x}\right)^{m}(1-x)^{-n-1}=\frac{(-1)^{m}}{x^{m}}(1-x)^{m-n-1}$$

infer that

$$\sum_{s=0}(-1)^{s}\,C_{m}^{m-k+s}C^{n}_{n+s}=C^{k}_{m-n-1}$$

434.
Prove that

$$\sum_{s=0}(-1)^{s}\,C^{s}_{m}C^{n}_{s}=\begin{cases}0 & \text{if } m\neq n,\\ (-1)^{n} & \text{if } m=n\end{cases}$$

435.
From the equation

$$(1-x)^{-n}(1-x^{h})^{n}=(1+x+\ldots+x^{h-1})^{n}$$

infer that

$$\sum_{s=0}(-1)^{s}\,C^{n-1}_{m-sh}C^{s}_{n}=\begin{cases}0 \text{ if } m>hn-1,\\ 1 \text{ if } m=hn-1\end{cases}$$

436.
From the identity

$$(1-x)^{-n-1}(1-x^{h})^{n}=\frac{(1+x+\ldots+x^{h-1})^{n}}{1-x}$$

infer that for $m\geqslant hn$

$$\sum_{s=0}(-1)^{s}\,C^{n}_{m-sh}C^{s}_{n}=h^{n}$$

437.
From the identity

$$(1+x)^{\pm p}(1-x)^{\pm p}=(1-x^{2})^{\pm p}$$

infer that

$$\sum_{s=0}(-1)^{s}\,C_{p}^{m-s}C_{p}^{s}=\begin{cases}(-1)^{\frac{m}{2}}\,C_{p}^{\frac{m}{2}} & \text{if } m \text{ is even},\\ 0 & \text{if } m \text{ is odd},\end{cases}$$

$$\sum_{s=0}(-1)^{s}\,C^{p}_{p+m-s}C^{p}_{p+s}=$$

$$=\begin{cases}(-1)^{\frac{m}{2}}\,C^{\frac{m}{2}}_{p+\frac{m}{2}} & \text{if } m \text{ is even},\\ 0 & \text{if } m \text{ is odd}.\end{cases}$$

438.
Prove that

$$\sum_{s=0}(-1)^{s}\,[C^{s}_{m}]^{2}=\begin{cases}(-1)^{\frac{m}{2}}\,C_{m}^{\frac{m}{2}} & \text{if } m \text{ is even},\\ 0 & \text{if } m \text{ is odd}.\end{cases}$$

439.
Denote the expression

$$a(a+1)(a+2)\ldots(a+n-1)$$

by $(a)_n$. Prove that

$$(a+b)_{n}=\sum_{m=0}^{n}C^{m}_{n}\,(a+m)_{n-m}\,(b-m+1)_{m}$$

1.
By the rule of product, we get $5 \times 3 = 15$ routes.

2.
By the same rule, we have $100^2 = 10{,}000$ choices.

3.
20.

4.
8.

5.
9.

6.
48.

7.
25; 20.

8.
480; 437.

9.
1,024; 4,032.

10.
We choose the white square in 32 ways and cross out the corresponding row and column. There remain 24 black squares on the board. In all there are $32 \times 24 = 768$ choices.

11.
By the rule of product, there are $12 \times 9 \times 10 = 1{,}080$ ways.

12.
In $6 \times 5 = 30$ ways.

13.
$3 \times 7 \times 7 = 147$.

14.
Either buy one copy of each novel or a set containing two novels and one copy of the third novel. By the rules of sum and product, we get $6 \times 3 \times 4 + 5 \times 4 + 7 \times 6 = 134$ ways.

15.
You can buy one more set containing the novels *War and Peace* and *Fathers and Sons* and one copy of *Crime and Punishment*. We thus add $3 \times 3 = 9$ ways, bringing the total to 143 ways.

16.
There is a greater number of choices when an apple is taken since $11 \times 10 > 12 \times 9$.

17.
$6 \times 8 \times 10 = 480$; if the first two teetotums fell on "1", then the third can fall in 10 ways; similarly for the cases when the other two teetotums fall on the same side; in all, we get $6 + 8 + 10$ ways, but one way (when all three teetotums fall on "1") is counted three times; and so there are 22 ways left.

18.
Since the order of the colours plays no role, we have $C_5^3 = 10$ ways.

19.
Here, the order of the colours is important; and so we have $A_5^3 = 60$ ways. If one of the bands is red, then we have $3 \times A_4^2 = 36$ ways.

20.
$A_5^2 = 20$ dictionaries.

21.
$A_{10}^2 - A_5^2 = 70$.

22.
We obtain permutations (with repetitions) of 13 cards four at a time. In all, $13^4 = 28{,}561$ ways. If no pairs are allowed, we have permutations without repetitions: in all, $A_{13}^4 = 17{,}160$.

23.
Since it suffices to draw one black and one white card, we have $13^2 = 169$ choices.

24.
A child receives either one, two or three names, all different. The total is $300 + 300 \times 299 + 300 \times 299 \times 298 = 26{,}820{,}600$ distinct names.

25.

Neighbours are preserved in cyclic permutations and in symmetric reflection. In the case of 4 persons, we have $2 \times 4 = 8$ transformations that preserve the relation of being neighbours. Since the total number of permutations of 4 persons is $4! = 24$, we have $24/8 = 3$ distinct seating arrangements. If there are 7 persons at the table, we have $7!/14 = 360$ ways; generally, in the case of n persons, $(n - 1)!/2$ ways. The number of ways in which 2 given persons are together is twice the number of seating arrangements of 6 persons (since the people can change places). Hence, it is $5! = 120$. In exactly the same manner, the number of ways for a given person to have two given neighbours is $4! = 24$.

26.

One boy plays for one team, two for the other. The boys can be made into teams in 3 ways. Then 3 girls out of 5 have to be chosen for the first team. This is done in $C_5^3 = 10$ ways. In all, by the rule of product, we have $3 \times 10 = 30$ ways of forming teams.

27.

The number of ways of partitioning n distinct objects into k groups is k^n. In our case we have $3^6 = 729$ ways.

28.

By the rule of product, $7 \times 9 = 63$ ways.

29.

The first can choose his books for exchanging in $C_7^2 = 21$ ways, the second in $C_9^2 = 36$ ways. In all, $21 \times 36 = 756$ ways for exchanging books.

30.

Divide all modes of ordering the speakers into pairs, the modes consisting of the ways in which A and B are obtained from one another by permutation. In each pair there is only one mode that satisfies the stipulated condition. We therefore have $5!/2 = 60$ ways.

31.

If A speaks just before B, we can count them as one speaker. Therefore we get $4! = 24$ ways.

32.

The choice of seats for the men and for the women can be made in two ways. Then the men can be seated in their chosen places in 5! ways. There are just as many ways for seating the women. We get a total of $2\,(5!)^2 = 28{,}800$ ways.

33.

We get 10 times fewer ways than in the preceding problem, or 2,880 ways.

34.

The total number of ways of drawing 10 cards is C_{52}^{10}. The number of ways in which not a single ace is drawn is C_{48}^{10}. Therefore, at least one ace will appear in $C_{52}^{10} - C_{48}^{10}$ cases. Exactly one ace in $C_4^1 C_{48}^9$ cases, at least two aces in $C_{52}^{10} - C_{48}^{10} - 4C_{49}^9$ cases, and exactly two aces in $C_4^2 C_{48}^8$ cases (we choose two aces in C_4^2 ways and another 8 cases out of 48 in C_{48}^8 ways).

35.

3^m signals (see Problem 27).

36.

Denote each set of teeth by a sequence of zeros and ones (a zero indicates no tooth at a given site, a one indicates the presence of a tooth). There are 2^{32} such sequences. Since each inhabitant has his own sequence, the number of inhabitants cannot exceed $2\ ^2$.

37.

First choose those passengers, of three, who are indifferent, and will be facing the front. This can be done in 3 ways. On each row of seats there are 5! ways of reseating the passengers. Hence, we get a total of $3\,(5!)^2 = 43{,}200$ ways.

38.

$A_9^4 = 3{,}024$.

39.

$C_{52}^5 = 2{,}598{,}960$.

40.

There are 32×10^4 combinations containing one letter, $32^2 \times 10^4$ containing two letters, and $32^3 \times 10^4$ three letters. In all, by the rule of sum, there are $33{,}820 \times 10^4$ licence plate numbers.

41.

Of the five days, we have to choose two on which apples are given. There are $C_5^2 = 10$ ways in all.

42.

C_{m+n}^{m}.

43.

$P(2, 3, 4) = 1{,}260$.

44.

Since the oranges are distinct, we have $A_8^5 =$ $= 6{,}720$ ways.

45.

Each orange goes to any one of the 8 sons. Therefore we get $8^5 = 32{,}768$ ways.

46.

$P(4, 3, 3, 2, 1, 1)$; $P(3, 1, 1, 1, 1, 1)$; $P(2, 2, 2, 1, 1, 1)$.

47.

$C_{30}^4 = 27{,}405$; $A_{30}^4 = 657{,}720$.

48.

$P(2, 2, 2, 1, 1) = 5{,}040$.

49.

First choose 6 subscribers in C_n^6 ways. Arrange these subscribers in any order and divide them up into pairs (first, second, then third, fourth and, finally, fifth and sixth). This can be done in 6! ways. Since the subscribers can be interchanged in each pair, and the order of the pairs is not important, the total number of ways must be divided by $2^3 \times 3! = 48$. We then get a total of $\dfrac{n!}{48(n-6)!}$ ways.

50.

$\overline{C}_{10}^{12} = C_{21}^{12}$; $\overline{C}_{10}^{8} = C_{17}^{8}$; C_{10}^{8}.

51.

We can choose two, three or four women. Two women may be chosen in C_4^2 ways. Then 4 men have to be chosen, which can be done in C_7^4 ways. By the rule of product, we get $C_4^2C_7^4$ ways. If three women are chosen, we get $C_4^3C_7^3$ ways, and if four women, then $C_4^4C_7^3$ ways. The total is

$$C_4^2C_7^4 + C_4^3C_7^3 + C_4^4C_7^3 = 371 \text{ ways}$$

52.

A number has to terminate in one out of 5 combinations: 12, 24, 32, 44, 52. The first two digits may be arbitrary. We get $5^2 \times 5 = 125$ numbers in all.

53.

Each one of n passengers can choose any one of m stops. We therefore have m^n modes of distribution. If we take into account only the number of passengers that get off at each of the stops, we have C_{m+n-1}^{m-1} ways.

54.

If a and b are adjacent, then we can combine them into a single symbol. Noting that a and b can be interchanged, we get $2(n-1)!$ permutations in which a and b appear together. Therefore, they fail to come together in $n! - 2(n-1)!$ permutations. Analogously, we find that a, b, and c do not come together in $n! - 6(n-2)!$ permutations. No two elements a, b, c, stand together in $n! - 6(n-1)! + 6(n-2)!$ permutations (by the inclusion and exclusion formula).

55.

Three referees can choose the winner in 10^3 ways. They name three distinct candidates in $A_{10}^3 =$ $= 720$ cases. For this reason, in 280 cases at least two referees will have coincident ratings. The portion of such cases is 0.28.

56.

Since each student can receive three kinds of marks, we get $3^4 = 81$ ways of passing the examinations.

57.

Since the necklaces remain unchanged in cyclic permutations of the beads and when turned over, we can make $\dfrac{7!}{14} = 360$ types of necklaces.

58.

The kinds of necklaces differ as to the number of small beads between the two large beads. So we get three types of necklace.

59.

The total number of different initials does not exceed $29^2 = 841$, which is less than 2,000.

60.

$A_{10}^7 = 604{,}800$, $C_{10}^3 = 120$. If two girls are definitely invited to a dance, then there are A_7^2 ways of choosing their partners; the remaining 5 boys choose a partner out of 8 girls, which is done in A_8^5 ways; the total is $A_7^2 A_8^5 = 282{,}240$ ways. Finally, if two given girls have been invited, then another five girls can be chosen in C_8^5 ways.

61.

The officer may be chosen in C_3^1 ways, the sergeants in C_6^2 ways and the privates in C_{60}^{20} ways. By the rule of product, we get a total of $C_3^1 C_6^2 C_{60}^{20}$ choices. If the captain is included in the detachment and the senior sergeant as well, we have $C_5^1 C_{60}^{20}$ choices.

62.

Four girls may be chosen in C_{12}^4 ways. We then choose boys in A_{15}^4 ways (here, the order is important). In all, there are $C_{12}^4 A_{15}^4 = 17{,}417{,}400$ choices.

63.

Each hen is either chosen or is not chosen. We thus get 2^3 choices of hens. Since at least one hen has to be chosen we get 7 choices of hens. Analogously, there are $2^4 - 1 = 15$ choices of ducks and $2^2 - 1 = 3$ choices of geese, making a total of $7 \times 15 \times 3 = 315$ choices.

64.

This number is equal to $P(m, n, p) = \dfrac{(m+n+p)!}{m!\,n!\,p!}$.

65.

The books in black bindings can be permuted in $m!$ ways, those in red, $n!$ ways. In all, by the rule of product, $m!\,n!$ ways. If the black books stand together, then we have also to choose for them a place between the books in red bindings. This can be done in $n+1$ ways. The total is

$m!\,n!\,(n+1) = m!\,(n+1)!$ ways

66.

Each one of the 15 men may either be included or not included. Since the group cannot be empty, we have $2^{15} - 1 = 32{,}767$ ways. For n men, there are $2^n - 1$ ways.

67.

The number p_k can enter into the given divisor α with exponents $0, 1, \ldots, \alpha_k$ in a total of $\alpha_k + 1$ ways. By the rule of product, the number of divisors is equal to $(\alpha_1 + 1) \ldots (\alpha_n + 1)$. To find the sum of the divisors, consider the expression

$$(1 + p_1 + \ldots + p_1^{\alpha_1}) \ldots (1 + p_n + \ldots + p_n^{\alpha_n})$$

Removing brackets, we get a sum which includes each divisor exactly once. By the formula for the sum of a geometric progression, we find that this sum is

$$\frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \cdots \frac{p_n^{\alpha_n+1} - 1}{p_n - 1}$$

68.

First put one coin in each envelope. Then we have to distribute 7 coins into 5 envelopes. This can be done in $C_{11}^4 = 330$ ways (see page 122).

69.

Add to the 20 books 4 identical dividing objects and consider all permutations of the obtained entities. There are $24!/4!$. To each permutation corresponds a definite arrangement of the books.

70.

As in the preceding problem, we find the total number of ways to be $8!/3! = 6{,}720$.

71.

Since only the number of votes cast for each proposal is taken into account, we have to distribute 30 identical "objects" into 5 "cells". To do this, append four identical dividing objects and take all permutations of the entities obtained. The total number is $P(30, 4) = 46{,}376$. With every permutation is associated a distinct distribution of votes.

72.

12 books can be bound in three colours in 3^{12} ways. Of this number, there will be 3×2^{12} cases when the books appear in at most two colours, and three cases in one colour. By the inclusion and exclusion formula, the books will appear in bindings of all three colours in $3^{12} - 3 \times 2^{12} + 3 = 519{,}156$ cases.

73.

Append to the 32 letters 5 identical "dividing elements" and consider all permutations of the resulting entities, in which not a single dividing element comes at the beginning or at the end and no two come together. The letters can be permuted in 32! ways, and there are 31 positions for the dividers, which can be placed in C_{31}^5 ways. Noting that the order of the words is of no importance, we get $32!\, C_{31}^5/6!$ ways of forming words.

74.

12 persons can be chosen in C_{17}^{12} ways. The two given persons are included in the chosen group in C_{15}^{10} cases. This leaves $C_{17}^{12} - C_{15}^{10}$ admissible choices.

75.

The precious stones can be permuted in $P(5, 6, 7)$ ways. The bracelet remains unchanged under cyclic permutations and symmetries. We obtain $P(5, 6, 7)/36 = \dfrac{18!}{36 \times 5!\,6!\,7!}$ ways.

76.

If all the chosen precious stones are of one kind, then in three ways, if two kinds are chosen, then in $2C_3^1 = 6$ ways, and if all three are distinct, then in 1 way: a total of 10 ways.

77.

The cups can be arranged in A_4^3 ways, the saucers in A_5^3 ways and the spoons in A_6^3 ways, which, by the rule of product, makes a total of $A_4^3 \times A_5^3 A_6^3 = 172{,}800$ ways.

78.

If the husband invites k women, then he invites $6 - k$ men. Then the wife invites $6 - k$ women and k men. By the rules of sum and product, this choice can be made in

$$\sum_{k=0}^{5} (C_5^k)^2 (C_7^{6-k})^2 = 267{,}148 \text{ ways.}$$

79.

The port side can accommodate 0, 1, 2, 3, or 4 persons from among those who are indifferent to the choice. But if k persons are chosen among them, then it is necessary to choose another $4 - k$ persons from the 10 that prefer the port side. Then there remain $12 + (9 - k)$ candidates, from among which we choose the 4 rowers for the starboard side. We have a total of $C_9^k C_{10}^{4-k} C_{21-k}^4$ choices. Summing with respect to k, we get the answer:

$$\sum_{k=0}^{4} C_9^k C_{10}^{4-k} C_{21-k}^4 =$$

$$= \frac{9!\,10!}{4!} \sum_{k=0}^{4} \frac{(21-k)!}{k!\,(9-k)!\,(4-k)!\,(6+k)!\,(17-k)!}$$

80.

The number 9 may be partitioned into three distinct parts in three ways: $9 = 1 + 2 + 6 = 1 + 3 + 5 = 2 + 3 + 4$. A sum less than 9 will occur in 4 cases: $1 + 2 + 3 = 6$, $1 + 2 + 4 = 7$, $1 + 2 + 5 = 1 + 3 + 4 = 8$. Since 3 counters may be drawn in C_{10}^3 ways, the sum is not less than 9 in $C_{10}^3 - 4 = 116$ cases.

81.

First choose one card of each suit. This can be done in 13^4 ways. Then choose another two cards. If they are of different suits, this can be done in $C_4^2 \times 12^2 = 864$ ways. Combining these choices with the distinct ways of choosing the first 4 cards and taking into account the permutations of the order of choice of two cards of each suit, we obtain 216×13^4 ways. If the two new cards are of the same suit, then we get $4 \times C_{12}^2 = 264$ choices. The same reasoning leads to 88×13^4 choices of all cards. This yields 304×13^4 ways in all.

82.

On the first day, the participants may be chosen in $C_{10}^6 = 210$ ways, on the second, in $C_{10}^6 - 1 = 209$ ways, on the third, in $C_{10}^6 - 2 = 208$ ways, making a total of $210 \times 209 \times 208 = 9{,}129{,}120$ ways.

83.

Since $C_6^3 = 20$, each choice of company will be utilized exactly once. The number of permutations of these choices is equal to 20!

84.

Each boy can choose from 5 jobs and each girl from 4 jobs. We get a total of $5^3 \times 4^2 = 2{,}000$ choices.

85.

Any one of the 33 letters can come first, and any one of the 32 letters (the preceding one being excluded) can come next. We get a total of $33 \times 32^4 = 34,503,008$ words.

86.

First choose the prize-winners and then distribute the books among them. By the rule of product, we get $C_{20}^{6} P\ (3, 2, 1)$ ways. In the second case, first choose the person who received the first book, then the one who received the second book, finally, the one who got the third book. The total number of modes of distribution is $C_{20}^{3} C_{20}^{2} C_{20}^{1}$.

87.

Match each domino (p, q) with a domino $(n - p, n - q)$. If $p + q = n - r$, then $(n - p) + (n - q) = n + r$. Thus, the number of pieces with sum $n - r$ is equal to the number of pieces with sum $n + r$. The total number of all pieces is C_{n+1}^{2}.

88.

It is given that the women's seats and the men's seats alternate. We therefore get $2\ (7!)^2$ seating arrangements.

89.

Choose one horse out of every pair AA', BB', CC' (8 choices), three horses out of the remaining 10 ($C_{10}^{3} = 120$ choices) and choose the order of harnessing them (6! ways). The total is $8 \times 6!\ C_{10}^{3} = 691,200$ ways.

90.

The consonants may be chosen in C_9^4 ways, the vowels in C_7^3 ways. The 7 selected letters can be permuted in 7! ways. The total is $C_9^4 C_7^3 \times 7!$ ways. If no two consonants are adjacent, the order of the letters is CVCVCVC. Here we have only $3!4!$ permutations and $C_9^4 C_7^3 3!4!$ words.

91.

By the inclusion and exclusion formula, the number of employees is $6 + 6 + 7 - 4 - 3 - 2 + 1 = 11$. There are $6 - 4 - 2 + 1 = 1$ who know only Spanish and $7 - 3 - 2 + 1 = 3$ who know only French.

92.

By the inclusion and exclusion formula, $92 - 47 - 38 - 42 + 28 + 31 + 26 - 25 = 25$ took patties.

93.

The men may be divided into pairs in $\frac{10!}{(2!)^5 5!}$ ways (taking into account the permutations within pairs and the permutations of the pairs themselves). The women may be divided in $10!/(2!)^5$ ways (here, the order of pairs is important). In all, $(10!)^2/2^{10}5!$ ways.

94.

First choose one man and one woman who appear together in the same boat as the earlier chosen pair (9^2 ways). Then split the remaining ones into 4 groups in $\frac{(8!)^2}{2^8 \times 4!}$ ways. The total is $\frac{(9!)^2}{2^8 \times 4!}$ ways.

95.

If the given two men are in the same group (and their wives are with them), then the others can be split into groups in $\frac{(8!)^2}{2^8 \times 4!}$ ways. But if they are in different groups, then these groups can be augmented in $(A_8^2)^2$ ways, after which the remaining people are divided into groups in $\frac{6!}{2^6 \times 3!}$ ways. In all, we have $17(8!)^2/2^8 \times 4!$ ways.

96.

Since the numbers cannot begin with zero, we have $7^4 - 7^3 = 2,058$ numbers.

97.

If the number represented by the first three digits is equal to x, then the number represented by the last three digits can have the values $0, 1, \ldots, 999 - x$, or a total of $1,000 - x$ values. Since x varies from 100 to 999, we have to find the sum of the natural numbers from 1 to 900. It is equal to 405,450.

98.

The white draughtmen can be placed in C_{32}^{12} ways. After choosing 12 squares for the white pieces, there remain 20 squares for black, which can be placed in C_{20}^{12} ways, making a total of $C_{32}^{12} C_{20}^{12}$ ways.

99.

Partition all permutations of the letters of "Юпитер" into classes so that permutations within a class differ solely in the order of the vowels. The number of classes is $P_6/P_3 = 120$. Only one permutation of each class satisfies the stipulated condition. Hence the answer is 120.

100.

Four "*e*'s" in succession in any permutation may be taken to be a single letter. There are thus 5! such permutations, which leaves $P(4, 1, 1, 1, 1) - 5! = 1560$ permutations.

101.

If "*p*" comes immediately after "*o*", these letters can be combined. And so the number of desired permutations is $P(2, 1, 1, 1, 1) = 360$.

102.

First permute all the "non-*o*" letters of the word in $P(3, 2, 2, 1, 1, 1, 1)$ ways. Then choose 7 out of 12 positions into which we can put the letter "*o*". This yields a total of $P(3, 2, 2, 1, 1, 1, 1) \times C_{12}^7$ ways.

103.

Both vowels and consonants can be permuted in $P(2, 1, 1) = 12$ ways. If the consonants are fixed, then five positions remain for the vowels. Hence, their positions can be chosen in $C_5^4 = 5$ ways. Altogether we have $5 \cdot 12^2 = 720$ ways.

104.

Write out the vowels in a given order. Then there are 5 positions for "ф". After "ф" is written down, there are 6 positions for "ц" and, finally, 7 positions for "т", making a total of $5 \cdot 6 \cdot 7 = 210$ ways.

105.

As in the preceding problem, we find the number of ways to be equal to $A_{11}^7/P_3 = 277,200$ (note that the letter "l" appears thrice in the word).

106.

First register the sequence of vowels (2 ways), then insert 2 consonants between the vowels ($A_4^2 = 12$ ways). The first of the remaining consonants may be placed either in front of or after both vowels (2 ways), and for the second one we have three positions. The total is $2 \times 12 \times 2 \times 3 = 144$ ways.

107.

Choose 3 letters out of the 5 consonants and place them in the indicated sites (A_5^3 ways). Then take the 5 remaining letters and place them in the remaining 5 sites in arbitrary fashion (5! ways). The total is $5!\,A_5^3 = 7{,}200$ ways.

108.

By the rule of product, $C_5^2 C_3^1 = 30$ ways; $C_4^1 C_3^1 = 12$ ways.

109.

$P(3, 1, 1, 1) - 4! = 96$ ways (see Problem 100).

110.

First arrange the consonants (3! ways). For the three "o's" we have 4 places left which may be arranged in 4 ways, making the total 24 ways.

111.

The letter "o" can appear among the chosen ones 0, 1, 2, 3 or 4 times (5 ways), the letter "k", in 3 ways, etc. We obtain a total of $5 \cdot 3 \cdot 5 \cdot 3 \cdot 3 \cdot 3 = 2{,}025$ combinations.

112.

There are $C_6^3 = 20$ combinations in which all three letters differ, $6 \cdot 5 = 30$ combinations with exactly 2 distinct letters, and 2 combinations containing only one letter. There are 52 choices altogether.

113.

If the order of the letters is also taken into account, we get $A_6^3 + 3A_6^2 + 2 = 212$ choices.

114.

Since the order of both vowels and consonants is fixed, all that remains is to choose 3 positions out of 7 for the vowels. This can be done in C_7^3 ways.

115.

For the word "кофеварка", the first and last letters must be consonants. The consonants can be permuted in $P(2, 1, 1, 1)$ ways, the vowels in $P(2, 1, 1)$ ways. In all we have $P(2, 1, 1, 1) \times P(2, 1, 1) = 720$ ways. For the word "samovar" we have $P_4 \cdot P(2, 1) = 72$ permutations.

116.

Out of the 6 positions we have to choose 3 for the letter "a". This can be done in $C_6^3 = 20$ ways. If there is the added restriction that no two a's can come together, then we have only four places for them, and we get $C_4^3 = 4$ ways.

117.

The letters of "fulfil" can be permuted in 180 ways. In these permutations, two f's come together in 60 cases, there are two l's in 60 cases, and both in 24 cases. By the inclusion and exclusion formula we get $180 - 60 - 60 + 24 = 84$ admissible permutations. For the word "murmur" we have $90 - 30 - 30 - 30 + 12 + 12 + 12 - 6 = 30$ admissible permutations.

118.

There are three combinations containing all three letters "t", "a", "r" and 3 combinations containing two distinct letters each. We have a total of 6 combinations. The number of distinct four-digit numbers that can be generated out of the digits of the number 123,123 is $3P(2, 1, 1) + 3P(2, 2) = 54$.

119.

By the inclusion and exclusion formula, we get that the digits 1, 2, 3, 4, contain $10^6 - 4 \times 9^6 + 6 \times 8^6 - 4 \times 7^6 + 6^6 = 23{,}160$ numbers. The digits 1, 2, 3, 4 alone generate $4 + 4^2 + 4^3 + 4^4 + 4^5 + 4^6 = \frac{4^7 - 4}{3} = 5{,}460$ numbers.

120.

Each digit appears in each order (digit place) 6 times ($P_4/4$). Therefore, combining the digits of the ones place, we obtain the sum $6(1 + 2 + 3 + 4) = 60$, of the tens place, 600, etc. In all, we have $60 + 600 + 6{,}000 + 60{,}000 = 66{,}660$.

121.

Here, the total number of permutations is 12, the digits 1 and 5 appearing in each place 3 times, the digit 2, 6 times each. Thus, the sum of the digits in the ones place is $3 \times 1 + 3 \times 5 + 6 \times 2 = 30$. The total sum is therefore $30 + 300 + 3{,}000 + 30{,}000 = 33{,}330$.

122.

Analogously, we find the sum to be 11,110.

123.

The total sum is 16,665.

124.

If we remove the restriction that 0 does not appear in the leading position, the sum is 2,666,640. The sum of the numbers with leading 0 is 66,660. Therefore the sum of five-digit numbers without leading 0 is 2,599,980.

125.

Since it is possible to write 2^k k-digit numbers using the digits 8 and 9, the total number of desired numbers is $\sum_{k=1}^{6} 2^k = 126$.

126.

Similarly, we obtain $\sum_{k=1}^{6} 3^k = 1{,}092$.

127.

Since 0 cannot be the leading digit, we get $2 \sum_{k=1}^{5} 3^k = 728$ numbers.

128.

Each digit is repeated $4^2 = 16$ times in each place. Therefore the sum of the digits in the ones place is $16(1 + 2 + 3 + 4) = 160$, in the tens place, 1,600, in the hundreds place, 16,000. The total sum is 17,760.

129.

In the first case, the sum is 3,999,960. In the second case, each digit is repeated A_8^4 times in each place, and we get the sum of the digits in the ones place: $A_8^4(1 + 2 + \ldots + 9) = 75{,}600$. The total sum is 839,991,600.

130.

Either 3 or 9 occupies the last place, and the remaining digits can be permuted in 3! ways. We get a total of 12 odd numbers. In exactly the same way, the number of even numbers is found to be 12.

131.

The positions for odd digits can be chosen in $C_6^3 = 20$ ways. Each position can be occupied by one of 5 digits (either even or odd). We get 20×5^6 numbers in all. But 10×5^5 of them have leading zero. That leaves $20 \times 5^6 - 10 \times 5^5 = 281{,}250$ numbers.

132.

$C_6^3 \times 5^6 = 312{,}500$ numbers.

133.

One out of 9 digits can occupy the leading position, any one of 10 digits can go into the 2nd, 3rd, 4th and 5th positions, and any one of 5 digits, into the last (it must be even). We get a total of $9 \times 10^4 \times 5 = 450{,}000$ numbers. If all numbers from 1 to 999,999 are taken, we get 499,999 numbers.

134.

Discarding zeros, we see that the remaining digits yield one of the following sequences: 3; 2, 1; 1, 2; 1, 1, 1. It remains to arrange the zeros so that the first digit is nonzero. This can be done in one way for 3, in 9 ways for 2, 1 and 1, 2 (according to the number of zeros between these digits), and in C_9^2 ways for 1, 1, 1, making a total of $1 + 9 + 45 = 55$ numbers. If all numbers from 1 to 9,999,999,999 are taken, then we have to choose places for digits different from zero. For 3 this can be done in C_{10}^1 ways, for 2, 1 and 1, 2, in C_{10}^2 ways, and for 1, 1, 1, in C_{10}^3 ways. The total is $C_{10}^1 + 2C_{10}^2 + C_{10}^3 = 340$ numbers.

135.

Any one of 9 digits 1, 2, . . ., 9 (zero cannot lead) can take first place, any one of the nine remaining digits can occupy the second place, any of 8 digits the third, etc. We get $9 \times 9!$ numbers in all.

136.

The number of numbers from 0 to 999 that are divisible by 5 is $E\left(\frac{1{,}000}{5}\right)$, where $E(x)$ is the largest integer in x. In the same way, there are $E\left(\frac{1{,}000}{7}\right)$ numbers divisible by 7, and $E\left(\frac{1{,}000}{35}\right)$ numbers divisible by 35. By the inclusion and exclusion formula we have

$$1{,}000 - E\left(\frac{1{,}000}{5}\right) - E\left(\frac{1{,}000}{7}\right) + E\left(\frac{1{,}000}{35}\right) = 686$$

numbers that are not divisible by 5 or by 7.

137.

Proceeding in the same way, we find the total number of desired numbers to be 228.

138.

The number of numbers without the digit 9 is 729. Therefore, the digit 9 appears in $1{,}000 - 729 = 271$ numbers. The digit 9 appears exactly twice in 27 numbers (099,909,990,199, etc.). Zero appears in one single-digit number, 9 two-digit numbers and 171 three-digit numbers, making a total of 181 numbers. Zero appears twice in 9 numbers. Both digits, 0 and 9, appear in 36 numbers (if the third digit is different from 0 and 9, then we have $2 \times 2 \times 8$ versions, and if it is equal to 0 or 9, we have another 4 versions). The digits 8 and 9 appear in 54 numbers. The number of n-digit numbers not containing two identical digits in succession is 9^n for $n > 1$ and 10 for $n = 1$. Hence, the number of such numbers from 0 to 999,999 is $10 + 9^2 + 9^3 + 9^4 + 9^5 + 9^6 = 597{,}871$.

139.

A four-digit number can consist either of four distinct digits (1, 2, 3, 5) or of two identical and two distinct digits (1, 1, 2, 3; 1, 1, 2, 5; 1, 1, 3, 5; 1, 2, 3, 3; 1, 3, 3, 5; 2, 3, 3, 5) or, finally, of two pairs of the same digits (1, 1, 3, 3). The total number of such numbers is thus

$$P_4 + 6P(2, 1, 1) + P(2, 2) = 24 + 6 \times 12 + 6 = 102$$

140.

As in the preceding problem, we get

$$2P(2, 1, 1, 1) + 3P(3, 1, 1) + 2P(2, 2, 1) + 3P(4, 1) = 255$$

141.

A six-digit number can have one, two or three pairs of the same digits. One pair can be chosen in C_5^1 ways. The number of permutations of 4 distinct and 2 identical digits is $P(2, 1, 1, 1, 1) = 6!/2! = 360$. Of them, there are two identical digits in succession in $5! = 120$ permutations. Hence, in this case we get $5(360 - 120) =$

$= 1,200$ six-digit numbers. Two pairs of the same digits can be chosen in $C_5^2 = 10$ ways, after which two more digits can be chosen in $C_3^2 = 3$ ways. The total number of permutations of these digits is equal to $P(2, 2, 1, 1) = 180$, and in $2 \times \frac{5!}{2!} =$ $= 120$ of them there is at least one pair of the same digits in succession, and in $4! = 24$ permutations there are two such pairs. By the inclusion and exclusion formula we find that this case yields the $10 \times 3(180 - 120 + 24) = 2,520$ numbers we need. In similar fashion, we find that three pairs of identical digits have the

$$C_5^3\left(\frac{6!}{(2!)^3} - 3\times\frac{5}{(2!)^2} + 3\times\frac{4!}{2!} - 3!\right) = 300$$

numbers we need. We get a total of 4,020 numbers.

142.

In all, there are

$$3\times\frac{5!}{2!} + C_3^2 C_2^1\frac{5!}{(2!)^2} + C_3^1\times\frac{5!}{3!} + C_2^1\times\frac{5!}{3!\,2!} = 440$$

five-digit numbers that can be generated from the given digits. Of that number, in $3P_3 + 2\frac{P_3}{2!} =$ $= 24$ cases the digit 3 occurs three times in succession. We get 416 desired numbers.

143.

The total number of permutations of the given digits is $P(2, 2, 2, 2)$. From among them, $P(2, 2, 2, 1)$ permutations have a given digit twice in succession, $P(2, 2, 1, 1)$ have 2 given digits in succession, $P(2, 1, 1, 1)$ have 3 given digits and $P(1, 1, 1, 1)$ have 4 given digits together. By the inclusion and exclusion formula, we find that no 2 digits are repeated in

$$P(2, 2, 2, 2) - 4P(2, 2, 2, 1) + 6P(2, 2, 1, 1)$$
$$- 4P(2, 1, 1, 1) + P(1, 1, 1, 1) = 864$$

permutations.

144.

Similarly we find that the number of permutations is

$$\frac{8!}{(2!)^3} - 3\times\frac{7!}{(2!)^2} + 3\times\frac{6!}{2!} - 5! = 2,220$$

145.

Again, in the same way, we get

$$\frac{10!}{(3!)^2} - 2\times\frac{8!}{3!} + 6! = 88,080$$

146.

Similarly, we get 20,040.

147.

If one number is chosen, then the second can be chosen in 10 ways (since the fact that it is even is already known). Taking into account the possibility of interchanging these two numbers, we get $\frac{20\times 10}{2} = 100$ choices.

148.

Either all three chosen numbers are even, or one is even and two are odd. We therefore get $C_{15}^3 +$ $+ C_{15}^1 C_{15}^2 = 2,030$ choices.

149.

At 11 points of the journey there is a choice between two alternatives. The number of ways the journey can be made is $2^{11} = 2,048$.

150.

Since the choice at the starting point has been made, there remain $2^{10} = 1,024$ possibilities.

151.

In the same manner, we find that the number of ways is equal to $3^5 = 243$.

152.

If p 10-copeck pieces have been chosen, then the 15-copeck pieces can be chosen in $0, 1, \ldots$ $\ldots, 20 - p$ ways (or $21 - p$ ways altogether). Since p varies from 0 to 20, we have $1 + 2 +$ $+ 3 + \ldots + 21 = 231$ choices in all.

153.

There are $C_{13}^5 = 1,287$ distinct combinations of the coins, and so there may be 1,286 wrong answers.

154.

There are 90,000 five-digit numbers, of which each digit is an even number in $4 \times 5^4 =$ $= 2,500$ cases, an odd number in $5^5 = 3,125$ cases. Digits less than 6 do not appear in $4^5 = 1{,}024$ cases, and greater than 3, in $3 \times 4^4 = 768$ cases. All the digits 1, 2, 3, 4, 5, are contained in $5! = 120$ numbers, and all the digits 0, 2, 4, 6, 8 are contained in $4 \times 4! = 96$ numbers.

155.

It is seen, from the statement of the problem, that distinct throws yield the same sum if they are obtained by permutations of the dice. The number of distinct sums is therefore $C_6^2 + 6 = 21$.

156.

We get $C_6^3 + 2C_6^2 + 6 = 56$ in exactly the same way.

157.

One sum will turn up in 6 cases. Two sums can appear in the following three ways: one die is of one type, and 5 dice of another type, or two dice are of one kind, and four dice of a second kind, or there are three dice of each kind. In the first instance, the type of dice can be chosen in A_6^2 ways, and any one of the 6 dice may be assigned a sum of the first kind. This yields $6A_6^2 = 180$ cases. In the same way, the version $2 + 4$ yields $A_6^2 P(2, 4) = 450$ cases, and the version $3 + 3$ yields $C_6^2 P(3, 3) = 300$ cases. Thus, two types of sum are obtained in $180 + 450 + 300 = 930$ cases. For three types of sum we first find all partitions of 6 into 3 summands: $6 = 1 + 1 + 4 = 1 + 2 + 3 = 2 + 2 + 2$. We accordingly get

$$\frac{1}{2!} A_6^3 P(1, 1, 4) = 1{,}800,$$

$$A_6^3 P(1, 2, 3) = 7{,}200,$$

$$\frac{1}{3!} A_6^3 P(2, 2, 2) = 1{,}800$$

or a total of 10,800 cases when exactly 3 types of sums occur.

The partitions of 6 into 4 summands are: $6 = 1 + 1 + 1 + 3 = 1 + 1 + 2 + 2$. These versions yield $\frac{1}{3!} A_6^4 P(1, 1, 1, 3) = 7{,}200$ and $\frac{1}{(2!)^2} A_6^4 P(1, 1, 2, 2) = 16{,}200$ or a total of 23,400 cases when four types of sum appear.

For 5 types of sum we have $\frac{1}{4!} A_6^5 P(1, 1, 1, 1, 2) = 10{,}800$ cases, and for 6 types, $6! = 720$ cases. Note that $6 + 930 + 10{,}800 + 23{,}400 + 10{,}800 + 720 = 6^6$.

158.

The dice are partitioned into groups according to the sum that appears in a throw. We must therefore find the number of ways of partitioning n dice into 6 groups. This number is C_{n+5}^5 (see page 122).

159.

Since $1{,}000{,}000 = 2^6 \times 5^6$, any factorization of one million into three factors is of the form

$$1{,}000{,}000 = (2^{\alpha_1} \times 5^{\beta_1})(2^{\alpha_2} \times 5^{\beta_2})(2^{\alpha_3} \times 5^{\beta_3})$$

where $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ are nonnegative integers such that $\alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 + \beta_3 = 6$. Since 6 can be partitioned into 3 nonnegative integers in $C_8^2 = 28$ ways, the number of factorizations (with regard for the order of the factors) is $28^2 = 784$.

160.

The factorizations obtained in Problem 159 fall into three classes: either all three factors coincide, or two coincide and the third is distinct, or all three are distinct. The first class consists of the single factorization $1{,}000{,}000 = 100 \times 100 \times 100$. Let us find the number of factorizations of the second class. If coincident factors are of the form $2^\alpha \cdot 5^\beta$, then we have $2\alpha + \alpha_3 = 2\beta + \beta_3 = 6$. But the equation $2x + y = 6$ has four solutions in nonnegative integers: $x = 0$, $y = 6$, $x = 1$, $y = 4$, $x = 2$, $y = 2$; $x = 3$, $y = 0$. Since any α may be combined with any β, we get 16 variants for $2^\alpha \times 5^\beta$. One of them, namely $2^2 \times 5^2$, must be discarded as resulting in a factorization of the first class. That leaves 15 variants. Each leads to three factorizations depending on the position of the third factor. Hence, the second class consists of 45 factorizations. Disregarding the order of the factors, we get 15 factorizations. Finally, the number of factorizations of the third class is equal to $784 - 1 - 45 = 738$. They fall into groups that differ solely in the order of the factors and that consist of 6 factorizations each. Therefore, disregarding the order of the factors, we have $1 + 15 + 123 = 139$ factorizations.

161.

Each coin goes into one of two pockets, and so we have 2^9 ways.

162.

Arrange the objects in some order and give the first n objects to the first person, the second n objects to the second, and the remaining objects to the third. Since the order of the elements in a group is irrelevant, we get $C_{3n}^{n}C_{2n}^{n}=\frac{(3n)!}{(n!)^3}$ modes of distribution.

163.

As in the preceding problem, we find the number of partitions to be $\frac{(2n)!}{2^n n!}$.

164.

Similarly, we get $\frac{(nk)!}{(k!)^n n!}$.

165.

$\frac{30!}{(10!)^3 3!}$, $\frac{30!}{(3!)^{10} 10!}$

166.

Four aces may be split in half in $\frac{4!}{(2!)^3}=3$ ways, and the remaining 32 cards in $\frac{32!}{(16!)^2 2!}$ ways. Since these partitions may be combined in two ways one with the other, we get $\frac{3\times 32!}{(16!)^2}$ modes of division.

167.

There are $\frac{10!}{2^5\times 5!}=945$ ways.

168.

945.

169.

$9!/(3!)^4=280$.

170.

Three persons can split 6 apples in C_8^2 ways and each of the other fruits can go to any one of the three persons; they can be divided in 3^6 ways. We thus get a total of $3^6C_8^2=20{,}412$ ways of dividing the fruit.

171.

First distribute the apples. Since each gets at most 4 apples, this distribution (to within permutations) can be made in one of the following ways: $6=4+2+0=4+1+1=3+3+0=3+2+1=2+2+2$. If the apples are distributed in the pattern $4+2+0$, then we have to choose 2 pieces of fruit out of 6 for the second, and give the rest to the third. This can be done in C_6^2 ways. Taking into account the possibility of permutations of the people, we get $3!C_6^2$ modes of distribution. Using the $4+1+1$ pattern, we have to choose 3 pieces of fruit for the second out of 6 (C_6^3 ways). Since two persons have the same number of apples, the number of permutations of the people equals $P(2,1)=3$. By the $3+3+0$ pattern, we have to choose one piece of fruit out of 6 for the first person and one piece out of the remaining 5 for the second. Here too we have three permutations of the people. The other patterns are considered in the same way. We obtain a total of

$$6C_6^2+3C_6^3+3C_6^1C_5^1+6C_6^1C_5^2+C_6^2C_4^2=690$$

modes of distribution.

172.

Since $9=6+3+0=6+2+1=5+4+0=5+2+2=4+3+2=3+3+3$, it follows that, as in the preceding problem, we have

$$6\,[C_9^3+C_9^4+C_9^1C_8^2+C_9^1C_8^3+C_9^2C_7^3]+3\,(C_9^1C_8^4+C_9^2C_7^2)+C_9^3C_6^3=19{,}068$$

modes of distribution.

173.

A pack of cards can be dealt to 13 players in $\frac{52!}{(4!)^{13}}$ ways (see Problem 162). If each is to receive one card of every suit, then for each suit we get a permutation of 13 cards; since permutations of suits are independent, then by the rule of product we have $(13!)^4$ ways. In the third case, one player can choose one card of each suit in 13^4 ways. Then, the remaining 12 cards of each suit can be divided into 3 groups in $\frac{12!}{(4!)^3 3!}$ ways, and the remaining cards, in $\frac{(12!)^4}{(4!)^{12}(3!)^4}$ ways. These groups can be dealt to 12 players in 12! ways.

Taking into account that a player having all suits can be chosen in 13 ways, we get $\frac{(13!)^5}{(4!)^{12}(3!)^4}$ for the third case.

174.

Four cards may be drawn from a full pack in C_{52}^4 ways. There will be exactly 3 suits in $A_4^2(C_{13}^1)^2C_{13}^2 = 518{,}184$ cases: we choose the absent suit and the repeated suit in A_4^2 ways, and then choose two cards of the repeated suit in C_{13}^2 ways, and one card each of two more suits in $(C_{13}^1)^2$ ways. There will be exactly two suits in $C_4^2\,(C_{13}^2)^2 + A_4^2C_{13}^3C_{13}^1 = 81{,}120$ cases. Actually, this is possible either if we have two cards each of two suits or one card of one suit and three of another. In the first case we have to choose two suits and two cards each of these suits, and in the second, we choose the first and second suits (here the order of the suits is important), and then take three cards of the first suit and one card of the second suit.

175.

Split the 13 cards of each suit using the pattern $3+3+3+4$. This can be done in $\frac{13!}{4!\,(3!)^4}$ ways. Groups of 4 cards each can be dealt to the players in 4! ways, and groups of 3 cards of each suit, in 3! ways. Altogether, we get $(3!)^4 4!$ ways of distributing the groups. The cards can be dealt in

$$\left(\frac{13!}{4!\,(3!)^4}\right)^4 4!\,(3!)^4 = \frac{(13!)^4}{(4!)^3\,(3!)^{12}}$$

ways.

176.

Arrange the recipients in some order. Then arrange the 18 objects in order in all ways and divide them into 4 groups of 4 objects each and 1 group of 2 objects. Give the 2-object group to one of the 5 recipients, then give the other groups to the remaining participants (the first group to the first, the second to the second, etc.). Since the order of the elements in the groups is immaterial, we get $\frac{5\times 18!}{(4!)^4 2!}$ modes of division. In the same way, we get $\frac{18!\,C_5^2}{(4!)^3(3!)^2}$ modes in the second case.

177.

There are three possibilities for each pair of objects: a sample can contain two objects, one, or none of the pair. Therefore, the number of samples is $3^{14} = 4{,}782{,}969$.

178.

Four black balls can be put into 6 packages in C_9^5 ways. We have the same number of ways for white and blue balls. By the rule of product, we get $(C_9^5)^3 = 2{,}000{,}376$ ways.

179.

In the same manner we obtain $C_6^3C_{13}^3 = 5{,}720$.

180.

Depict each partition of the number n into parts in the form of an array of dots. Adjoining to each array one column of n dots, we get an array for the partition of the number $2n$ into n parts.

181.

Choose three arbitrary positive integers from 1 to $n-2$ and add 2 to the greater of them and 1 to the second in magnitude. We obtain three numbers, no two of which follow in succession. They thus yield the number labels of the chosen objects. Hence the choice can be made in C_{n-2}^3 ways.

182.

In $P\,(2, 2, 2, 2, 2, 2, 1, 1, 1, 1) = \frac{16!}{2^6}$ ways.

183.

We can occupy the empty squares with identical pieces and obtain a permutation of 48 pieces and of the pieces indicated in the problem. There are $P\,(48, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1) = \frac{64!}{2^6 48!}$ such permutations.

184.

Analogously, we get $P\,(32, 8, 8, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1)$.

185.

Let p squares be held by white draughtmen and q squares by black pieces. 15 white pieces can be placed on p squares (so that all squares are occupied) in C_{14}^{p-1} ways, and 15 black pieces can be placed on q squares in C_{14}^{q-1} ways. It is possible to choose p squares for white and q squares for black in $P(p, q, 24-p-q)$ ways. The total number of ways is therefore

$$\sum_{p,q} P(p, q, 24-p-q)\, C_{14}^{p-1} C_{14}^{q-1}$$

where the summation is extended over all p and q such that

$$1 \leqslant p \leqslant 15,\ 1 \leqslant q \leqslant 15,\ p+q \leqslant 24$$

186.

Combine into one group the cells that pass into one another in rotations of the board through 90°. It is given that 5 such groups are filled, and the total number of groups is 16. We therefore have $C_{16}^{5} = 4{,}368$ possible placements.

187.

Same type of solution as in Problem 186. We have C_{32}^{10} placements.

188.

The number of squares is reduced by half and we have C_{16}^{10} placements.

189.

Put 6 white and 6 black pieces on 16 black squares on one half of the board. This can be done in

$$P(6, 6, 4) = \frac{16!}{6!6!4!} \text{ ways.}$$

190.

On one half of the board choose 12 squares out of 16 and place any pieces on them, on the other half use symmetric squares and put on pieces of opposite colour. The squares can be chosen in C_{16}^{12} ways, and the colour of the pieces occupying these 12 squares can be chosen in 2^{12} ways. We obtain a total of $2^{12}C_{16}^{12} = 7{,}454{,}720$ ways.

191.

The position of the pieces is determined by which 5 squares out of 7 on the first horizontal line are occupied by white pieces. We therefore have $C_7^5 = 21$ ways.

192.

The positions are divided into two classes depending on whether a corner square is occupied or not. If the corner squares are taken, then there are another 8 pieces on the 12 noncorner squares on the first vertical line and the first horizontal line. These pieces can be placed in $C_{12}^{8} = 495$ ways. But if the corner squares are vacant, then the 12 noncorner squares of the first vertical line and first horizontal line accommodate 10 pieces, which can be placed in $C_{12}^{10} = 66$ ways. We thus have a total of 561 ways of placing the pieces.

193.

7 white balls can be placed in 9 pockets in C_{15}^{8} ways, and 2 black balls in C_{10}^{8} ways. This yields $C_{15}^{8} \times C_{10}^{8} = 289{,}575$ ways in all.

194.

Similarly, we have $C_{15}^{8}\,(C_9^8)^2 = 521{,}235$ ways.

195.

First choose 9 books for C. This can be done in C_{27}^{9} ways. The remaining 18 books can be divided between A and B in 2^{18} ways. This gives us a total of $2^{18}C_{27}^{9}$ ways of distributing the books.

196.

The 8 persons can be distributed among the floors in 4^8 ways. Of this number, no one gets out on a given floor in 3^8 cases, on any given two floors in 2^8 cases, and on any given three floors in one case. Using the inclusion and exclusion formula, we have $4^8 - 4 \times 3^8 + 6 \times 2^8 - 4 = 40{,}824$.

197.

The following cases are possible: all three summands are divisible by 3, one summand is, and none is. In the first case, the summands can be chosen in C_{33}^{3} ways. In the second case, one summand yields a remainder of 1, and the other, a remainder of 2. Since there are 34 numbers from 1 to 100 that yield a remainder of 1, and there are 33 numbers divisible by 3 and yielding 2 as a remainder, in the second case we have $C_{34}^{1}(C_{33}^{1})^2$ ways. If all three summands are not divisible by 3, then they have the remainders 1, 1 and 1, or the remainders 2, 2 and 2. We accordingly get C_{34}^{3} or C_{33}^{3} cases. In all, we have $2C_{33}^{3} + C_{34}^{3} + C_{34}^{1}(C_{33}^{1})^2 = 53{,}922$ choices.

198.

The solution is similar to that of Problem 197. The answer is

$$3C_n^3 + (C_n^1)^3 = \frac{n}{2}(3n^2 - 3n + 2)$$

199.

If p white balls are placed, then the occupied pockets can be chosen in C_{n+1}^{p} ways. There then remain $n - p + 1$ pockets for the black ball; also, it need not be placed at all. We get $n - p + 2$ possibilities. The answer is therefore of the form $\sum_{p=0}^{n} (n - p + 2) \times C_{n+1}^{p} = \sum_{s=1}^{q} sC_q^s + \sum_{p=0}^{q-1} C_q^p$. Since $\sum_{s=1}^{q} sC_q^s = q2^{q-1}$ and $\sum_{p=0}^{q-1} C_q^p = 2^q - 1$ (see Problem 401a), we get $(q + 2)\, 2^{q-1} - 1$.

200.

Let us denote a nonempty set of white balls by W, and a similar set of black balls by B. From the statement of the problem it follows that the balls are arranged in one of the following patterns: BWBW . . . BW or WBWB . . . WB, each pattern accommodating r pairs. But m white balls can be distributed among r nonempty sets in C_{m-1}^{r-1} ways. For the black balls we have C_{n-1}^{r-1} ways, making the total $2C_{m-1}^{r-1} C_{n-1}^{r-1}$ ways. In the same manner, we conclude that $2r$ contacts will occur in $C_{m-1}^{r} C_{n-1}^{r-1} + C_{m-1}^{r-1} C_{n-1}^{r}$ cases.

201.

Denote by $A\ (m, n)$ the number of ways of collecting a total of m marks in the course of n examinations (without getting a 2, which is a failing mark). It is then clear that $A\ (30,8) = A\ (25,7) + A\ (26,7) + A\ (27,7)$, etc. Continuing to reduce m, in a few steps we get the answer: 784.

202.

First choose n objects that remain fixed. This can be done in C_{m+n}^{n} ways. The remaining m objects are deranged so that none remains in its original position. This can be done in D_m ways (see page 44). Altogether we get $\frac{(m+n)!}{m!n!} D_m$ ways.

203.

r things can be distributed among $n + p$ persons in $(n + p)^r$ ways; in $(n + p - 1)^r$ cases a given person gets no object at all; in $(n + p - 2)^r$ cases two given persons get nothing, etc. Using the principle of inclusion and exclusion, we arrive at the desired result.

204.

The first column of the partition of $2r + x$ into $r + x$ nonzero summands contains $r + x$ elements. Discarding it, we obtain the array of the partition of r into nonnegative summands.

205.

Since each can vote for any one of n persons, we have n^n ways of voting. In the second case, it is necessary to divide n votes among n candidates, which can be done in C_{2n-1}^{n-1} ways.

206.

Let the number $2n$ be divided into three parts as required: $2n = a + b + c$, and $a \leqslant b \leqslant c$. Then $a \neq 1$, otherwise we would have $b + c = 2n - 1$, and for this reason $b < c$, which is impossible since $b + 1 > c$. Besides, $a + b > c$, and the numbers $a + b$ and c have the same parity. Hence, $a + b \geqslant c + 2$. But then the numbers $a - 1$, $b - 1$, $c - 1$ form a partition for $2n - 3$, and $(a - 1) + (b - 1) > c - 1$. In this way is established a one-to-one correspondence between the partitions of the numbers $2n$ and $2n - 3$.

207.

This follows from the equality

$$C_n^1 + C_n^3 + C_n^5 + \ldots = 2^{n-1}$$

208.

Let the first person receive x objects of the first kind, y objects of the second kind, and z objects of the third kind. Then $x + y + z = 3n$, and $0 \leqslant x, y, z \leqslant 2n$. It is thus required to find the number of solutions of the equation $x + y + z = 3n$ in nonnegative integers not exceeding $2n$. If we lift the restriction that $x \leqslant 2n$, $y \leqslant 2n$, $z \leqslant 2n$, then the number of solutions is equal to the number of ways of splitting $3n$ identical objects among three persons, or C_{3n+2}^{2}. Let us now find the number of solutions in which $x > 2n$. It is equal to the total number of solutions in nonnegative integers of the equations $y + z = k$, $0 \leqslant k < n$ or $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$. In the same number of solutions, $y > 2n$ and $z > 2n$. Discarding them, we get $3n^2 + 3n + 1$ solutions.

209.

The solution is like that of Problem 208. We get

$$C^3_{4n+3}-4\sum_{k=0}^{2n-1} C^2_{k+2}=C^3_{4n+3}-4C^3_{2n+2}=$$

$$=\frac{1}{3}(2n+1)(8n^2+8n+3)$$

210.

Since the parts are indistinguishable, the solutions x, y, z and $2n-x$, $2n-y$, $2n-z$ of the equation $x+y+z=3n$ can be identified. One solution, namely, $x=n$, $y=n$, $z=n$, is identified with itself, and the remaining, with solutions different from them. The answer is therefore of the form

$$\frac{3n^2+3n}{2}+1$$

Similarly for the case when we have things of 4 kinds.

211.

Here we have to find the number of integral solutions to the equation $x_1+x_2+\ldots+x_m=mn$ that satisfy the conditions $0\leqslant x_k\leqslant 2n$, $1\leqslant k\leqslant m$. If we lift the restrictions $0\leqslant x_k\leqslant 2n$, $1\leqslant k\leqslant n$, we get C^{m-1}_{mn+m-1} solutions. Let us find the number of solutions for which $x_1>2n$; it is equal to the total number of solutions of all equations

$$x_2+x_3+\ldots+x_m=k$$

where $0\leqslant k\leqslant mn-2n-1$, that is,

$$\sum_{k=0}^{mn-2n-1} C^{m-2}_{k+m-2}=C^{m-1}_{mn-2n+m-2}$$

There are just as many solutions for which $x_2>2n$, etc. Hence, we have to reject $C^1_m C^{m-1}_{mn+m-2n-2}$ solutions. In the process, some solutions (namely those for which, say, both $x_1>2n$ and $x_2>2n$) are discarded twice. Using the inclusion and exclusion formula, we get the desired result.

212.

In 231 ways. For the solution, see the next problem.

213.

Use x_1, x_2, x_3 to denote the quantity of things of the first kind, and y_1, y_2, y_3, those of the second kind received by A, B and C, respectively. We then have the equations $x_1+x_2+x_3=n$ and $y_1+y_2+y_3=n$ with the restrictions $x_k+y_k\leqslant n$, $1\leqslant k\leqslant m$. If these restrictions are lifted, then we get C^2_{n+2} solutions to the first equation and C^2_{n+2} solutions to the second, making a total of $(C^2_{n+2})^2$ solutions. Here, the number of solutions in which the restriction $x_1+y_1\leqslant n$ is violated is equal to the total number of nonnegative integral solutions of the systems of equations $x_2+x_3=r$, $y_2+y_3=s$, where $0\leqslant r<n$, $0\leqslant s<n$ and $r+s<n$. The system $x_2+x_3=r$, $y_2+y_3=s$ has $(r+1)(s+1)$ nonnegative integral solutions. Hence the total number of solutions of our systems is

$$\sum_{s=0}^{n-1}\sum_{r=0}^{n-s-1}(r+1)(s+1)=$$

$$=\frac{1}{2}\sum_{s=0}^{n-1}(s+1)(n-s)(n-s+1)=$$

$$=\sum_{s=0}^{n-1}C^1_{s+1}C^2_{n-s+1}=C^4_{n+3}$$

(see page 36). There are just as many solutions that do not satisfy the conditions $x_2+y_2\leqslant n$ and $x_3+y_3\leqslant n$. Discarding these solutions, we get

$$(C^2_{n+2})^2-3C^4_{n+3}$$

solutions. For $n=5$ there are 231 solutions.

214.

9 persons can be seated in 9! ways. Let us find the number of permutations in which 3 Englishmen are seated together. All such permutations are obtained from one by reseating the Englishmen among themselves (3! ways) and reseating the 6 Frenchmen and Turks and the company of 3 Englishmen (7! ways). We get a total of 3!7! permutations. In the same number of permutations we have 3 Frenchmen together and in the same number 3 Turks together. Furthermore, in $(3!)^2 5!$ permutations both Englishmen and Frenchmen sit together, and in $(3!)^4$ permuta-

tions the English, French and Turks are together. Using the inclusion and exclusion formula, we get

$$9!-3\times 3!\,7!+3\,(3!)^2 5!-(3!)^4=283{,}824$$

215.

The total number of permutations is 9! Find the number of permutations in which two given Englishmen are together. If we combine them, we get permutations of 8 entities. But, besides, we can rearrange them among themselves. And so we have a total of 2!8! permutations. Here, two given Englishmen can be chosen in C_3^2 ways, and we have three nationalities in all. Therefore, the appropriate term in the inclusion and exclusion formula is equal to $3C_3^2 2!8!$. Now let us find the number of permutations in which two given Englishmen sit together and the two given Frenchmen as well. Combining into a single pair the adjacent compatriots, we get 7 entities to be permuted. What is more, the compatriots sitting together can be permuted among themselves. The total is $(2!)^2 7!$ permutations. In addition, two pairs of compatriots can be chosen in $(C_3^2)^3$ ways. For this reason, the appropriate term in the inclusion and exclusion formula is $(C_3^2)^3(2!)^2 7!$. The following cases are then considered: sitting together are

(a) three compatriots,
(b) two of each nationality,
(c) three of one nationality and two of another,
(d) three of one nationality and three of another,
(e) three of one nationality, two of another and two of the third,
(f) three of one nationality, three of another and two of the third,
(g) three of each nationality.

Using the principle of inclusion and exclusion, we get

$$9!-9\times 2!\,8!+27\,(2!)^2\,7!+3\times 3!\,7!-(2!)^3\,6!-$$
$$-18\times 3!\,2!\,6!+3\,(3!)^2\,5!+27\times 3!\,(2!)^2\,5!-$$
$$-9\,(3!)^2\,2!\,4!+(3!)^4$$

216.

The solution is similar to that of Problem 215, but the number of permutations in which the given compatriots sit together is computed differently. Two Englishmen can be seated together in 2!9 ways, and then all the others can be reseated in 7! ways. If we take two Englishmen and two Frenchmen, then the number of seating arrangements in which these compatriots sit together is $(2!)^2 9\times 6!$. Namely, we can choose seats for the Englishmen in 9 ways, and then combine two Frenchmen and take all possible permutations of this pair and the remaining 5 persons. Taking into account the possibility of interchanging places (with respect to the Englishmen sitting together and the Frenchmen together), we get the indicated number of permutations. The remaining possibilities are considered in similar fashion. Altogether, we have

$$9!-9\times 2!\,9\times 7!+27\,(2!)^2\,9\times 6!+3\times 3!\,9\times 6!$$
$$-(2!)^3\,9\times 5!-18\times 3!\,2!\,9\times 5!+3\,(3!)^2\,9\times 4!$$
$$+27\times 3!\,(2!)^2\,9\times 4!-9\,(3!)^2\,2!\,9\times 3!+(3!)^3\,9\times 2!$$

ways.

217.

Use $F(N)$ to denote the number of ways of putting N copecks worth of postage stamps on a package. Divide these ways into classes in accordance with the value of the last stamp. We get the recurrence relation

$$F(N)=F(N-5)+F(N-10)+F(N-15)+$$
$$+F(N-20)$$

Using this relation and the equality $F(5)=1$, we obtain $F(40)=108$.

218.

Denote by $F(n_1, \ldots, n_m; N)$ the number of ways of paying out a sum of N copecks with the coins $n_1, \ldots, n_m$. Then the recurrence relation

$$F(n_1, \ldots, n_m; N)=F(n_1, \ldots, n_{m-1}; N)+$$
$$+F(n_1, \ldots, n_m; N-n_m)$$

holds (see page 63). Utilizing this relation and similar relations, we find that $F(10, 15, 20, 50; 100)=20$.

219.

Using a recurrence relation, we find that the problem has 4 solutions.

220.

The row can contain 3, 2, or 1 black ball. If it contains 3 black balls, then the fourth ball may be chosen in three ways; then permute the 3 black balls and 1 ball of a different colour in $P(3, 1)=4$ ways. There are twelve ways in all.

Similarly, if we take 2 black balls, we get $C_3^2P(2, 1, 1) = 36$ possibilities, and if we take 1, then 4! possibilities. It is possible to generate a total of $12 + 36 + 24 = 72$ rows.

221.

The number of such representations is equal to the number of partitions of n identical balls into 3 nonempty groups, that is C_{n-1}^2.

222.

Let us first find out how many zeros are needed to write down all the numbers from 1 to 999,999. The zero comes last in 99,999 numbers (10, 20,... 999,990), it comes second in 99,990 numbers, third in 99,900, etc. In all, we have $99{,}999 + 99{,}990 + 99{,}900 + 99{,}000 + 90{,}000 = 488{,}889$. The total number of digits is equal to $9 + 2 \times 90 + 3 \times 900 + 4 \times 9{,}000 + 5 \times 90{,}000 + 6 \times 900{,}000 = 5{,}888{,}889$. Since all the digits, except zero, enter the same number of times, each of them appears

$$\frac{5{,}888{,}889 - 4{,}888{,}889}{9} = 600{,}000 \text{ times}$$

223.

First choose the positions occupied by the digit 3 (the choices are C_{10}^2). Then place the digits 1 or 2 in the remaining 8 positions; this can be done in 2^8 ways. We get a total of $2^8C_{10}^2 = 11{,}520$ ways.

The sum of the digits of any one of the numbers lies between $8 \times 1 + 2 \times 3 = 14$ and $8 \times 2 + 2 \times 3 = 22$. Thus, if the number is divisible by 9, the sum of its digits is 18. Hence, 1's and 2's have a sum of 12. This sum is obtained if we take 4 1's and 4 2's. Thus, our number contains 4 1's, 4 2's and 2 3's. Out of these digits we can form

$$P(4, 4, 2) = \frac{10!}{4!\,4!\,2!} = 3{,}150$$

different numbers.

224.

Let the numbers a and b form an inversion in a given permutation. If they change places, we get a new permutation in which they no longer form an inversion. We have $n!$ permutations in each of which there are C_n^2 ways of choosing the numbers a and b. In half of the cases, these numbers form inversions. Hence, the number of inversions is $\frac{n!}{2} C_n^2$.

225.

The number n may be represented as the sum of three positive integers (representations differing as to the order of the integers are considered distinct) in $C_{n-1}^2 = \frac{n^2-3n+2}{2}$ ways. Of them, two integers are equal in $\frac{n-2}{2}$ representations for even n and in $\frac{n-1}{2}$ representations for odd n. Besides, if n is divisible by 3, we have a representation in which all three integers are equal. Applying the principle of inclusion and exclusion, we readily see that the number of representations with pairwise distinct summands is given by the following formulas:

$$\frac{n^2-3n+2}{2} - \frac{3}{2}(n-2) + 2 = \frac{n^2-6n+12}{2} \quad \text{if } n = 6k,$$

$$\frac{n^2-3n+2}{2} - \frac{3}{2}(n-1) = \frac{n^2-6n+5}{2} \quad \text{if } n = 6k+1,$$

$$\frac{n^2-3n+2}{2} - \frac{3}{2}(n-2) = \frac{n^2-6n+8}{2} \quad \text{if } n = 6k+2,$$

$$\frac{n^2-3n+2}{2} - \frac{3}{2}(n-1) + 2 = \frac{n^2-6n+9}{2} \quad \text{if } n = 6k+3,$$

$$\frac{n^2-3n+2}{2} - \frac{3}{2}(n-2) = \frac{n^2-6n+8}{2} \quad \text{if } n = 6k+4,$$

$$\frac{n^2-3n+2}{2} - \frac{3}{2}(n-1) = \frac{n^2-6n+5}{2} \quad \text{if } n = 6k+5$$

If the order of the summands is disregarded, we get 6 times fewer representations. It is easy to verify that the expressions thus obtained are simply the greatest integers in $\frac{n^2-6n+12}{12}$ for appropriate values of n.

226.

The number $12n+5$ can be represented as four summands in C^3_{12n+4} ways (considering as distinct the representations differing in the order of the summands). The number of representations in which $x=y$ is equal to the number of solutions of the equation $2x+z+t=12n+5$ in positive integers. Since the equation $z+t=12n-2k+5$ has $12n-2k+4$ solutions in positive integers, the total number of such solutions is

$$\sum_{k=1}^{6n+1}(12n-2k+4)=(6n+1)(6n+2)=2C^2_{6n+2}$$

The number of solutions in which $x=y=z$ is equal to the number of solutions of the equation $3x+t=12n+5$. that is, $4n+1$.

Let us find the number of solutions in which there are terms exceeding $6n+2$. Let $x=k\geqslant 6n+3$. Then $y+z+t=12n+5-k$. But the number $12n+5-k$ may be represented as a sum of three positive integers in $C^2_{12n+4-k}$ ways. Therefore, there are

$$\sum_{k=6n+3}^{12n+2} C^2_{12n+4-k}=C^3_{6n+2}$$

solutions for which $x\geqslant 6n+3$. Since we could take any other summand in place of x, we have $C^3_{12n+4}-4C^3_{6n+2}$ solutions in which the summands do not exceed $6n+2$.

Furthermore, the number of solutions of the equation $2x+z+t=12n+5$ in which $z\geqslant 6n+3$ is $3n(3n+1)\ 2C^2_{3n+1}$. Therefore, the number of solutions in which $x=y$ and all terms do not exceed $6n+2$ is equal to $2[C^2_{6n+2}-2C^2_{3n+1}]$. Since in place of x and y we can take any other pair of letters, the total number of solutions in which two terms are equal and all summands do not exceed $6n+2$ is

$$2C^2_4[C^2_{6n+2}-2C^2_{3n+1}]$$

Finally, the number of solutions of the equation $3x+t=12n+5$ for which $t\geqslant 6n+3$ is $2n$. And so the total number of solutions in which three terms are equal and all terms do not exceed $6n+2$ is $4(2n+1)$.

If, of all representations, we reject those for which two summands coincide, then the representations in which three summands coincide will have been rejected thrice. For this reason, their number in the inclusion and exclusion formula has to be multiplied by two. In all, we get

$$[C^3_{12n+4}-4C^3_{6n+2}]-2C^2_4[C^2_{6n+2}-2C^2_{3n+1}]+8(2n+1)=12n(12n^2+3n-1)$$

representations in which all summands are distinct,

$$2C^2_4[C^2_{6n+2}-2C^2_{3n+1}]-12(2n+1)=12n(9n+4)$$

representations containing exactly three distinct summands, and $4(2n+1)$ representations in which there are two distinct summands.

Partition all representations into classes such that two representations of one class differ in the order of the summands alone. Then the representations of the first type will fall into classes consisting of 24 elements, those of the second type, into classes consisting of 12 elements, those of the third type, into classes consisting of 4 elements. Therefore the number of partitions of the required type is

$$\frac{n}{2}(12n^2+3n-1)+n(9n+4)+2n+1=\frac{n+1}{2}(12n^2+9n+2)$$

227.

In the course of our solution of Problem 226 we found that the number of representations in which all summands are distinct is equal to $12n(12n^2+3n-1)$. Since now we do not have regard for the order of the terms, we get $\frac{n}{2}(12n^2+3n-1)$ partitions.

228.

A geometric progression is determined by the first term and the common ratio q. If the progression is increasing, then the inequality $aq^2\leqslant 100$ must hold, whence it follows that $a\leqslant\frac{100}{q^2}$. Hence, the number of increasing three-term progressions with ratio q is $E\left(\frac{100}{q^2}\right)$. The total number of progressions is

$$2\left[E\left(\frac{100}{4}\right)+E\left(\frac{100}{9}\right)+E\left(\frac{100}{16}\right)+\dots+E\left(\frac{100}{100}\right)\right]=102$$

(the factor 2 has to do with the fact that one and the same number triple may be regarded either as an increasing or a decreasing progression).

229.

Denote by F the set of several Frenchmen in succession, by T the set of several Turks in succession. Use E for Englishmen. From the statement of the problem it follows that one of the following patterns is possible: FETEFETEFETEF or TEFETEFETEFET. In the first type of pattern, we have to divide the 7 Frenchmen into 4 nonempty groups (this can be done in C_6^3 ways), the 10 Turks into 3 nonempty groups (C_9^2 ways), and then place these groups in order in their appropriate places and permute compatriots in all possible ways. We obtain $6!7!10!C_6^3C_9^2$ seating arrangements. In the same way, the second type of pattern yields $6!\ 7!\ 10!\ C_6^2C_9^3$ seating arrangements. In all we get

$$6!\,7!\,10!\,[C_6^3C_9^2+C_6^2C_9^3]=6!\,7!\,10!\,1980$$

solutions.

230.

As in the preceding problem, we get $5!7!10!\times$ $\times\ 1{,}080$ solutions.

231.

The desired two numbers differ in the factors a^α, b^β, c^γ, d^δ; each of these four factors appears in one of the numbers and does not appear in the other. Since 4 factors may be distributed between the two numbers in $2^4 = 16$ ways, the problem has 16 solutions. If the order of the numbers is disregarded, we have 8 solutions.

232.

The required numbers are of the form GA and GB, where A and B are divisors of the number $a^\alpha b^\beta c^\gamma d^\delta$. This number has $N=(\alpha+1)\ (\beta+1)\times$ $\times\ (\gamma+1)\ (\delta+1)$ divisors (see Problem 67). Therefore, A and B may be chosen in C_{N+1}^2 ways if we do not distinguish the pairs (GA, GB) and (GB, GA), and in N^2 ways if such pairs are distinguished.

233.

There are C_{20}^6 combinations in which all letters are distinct, $C_{20}^1C_{19}^4$ combinations in which two letters coincide, and so forth. We have a total of

$$C_{20}^6+C_{20}^1C_{19}^4+C_{20}^2C_{18}^4+C_{20}^3=146{,}400$$

combinations.

234.

The desired permutations begin with several letters α, which are then followed by the letter β, subsequent letters in any order. If there are k starting letters α and one letter β, then the remaining letters may be permuted in $P\ (p-k,$ $q-1,\ r)$ ways. Summing from $k=1$ to $k=p$, we find that the number of required permutations is

$$\sum_{k=1}^{p}\frac{(p+q+r-k-1)!}{(p-k)!\,(q-1)!\,r!}=C_{q+r-1}^{r}\sum_{k=1}^{p}C_{p-k+q+r-1}^{p-k}$$

But

$$\sum_{k=1}^{p}C_{p-k+q+r-1}^{p-k}=C_{p+q+r-1}^{p-1}$$

We therefore have $C_{q+r-1}^{r}C_{p+q+r-1}^{p-1}$ permutations.

235.

The numbers expressing the lengths of the bands of each colour form a representation of the number 10 in the form of a sum of integers assuming values from 2 to 10, the order of the integers being important. The number of such partitions into k summands is equal to the coefficient of x^{10} in the expansion of the expression

$$(x^2+x^3+\ldots+x^{10})^k=\left(\frac{x^2-x^{11}}{1-x}\right)^k=$$

$$=x^{2k}\,(1-x^9)^k\,(1-x)^{-k}$$

$$=x^{2k}\left(1-kx^9-\frac{k\,(k-1)}{1\times 2}\,x^{18}-\ \ldots\right)\times$$

$$\times\left(1+kx+\frac{k\,(k+1)}{1\times 2}\,x^2\right.$$

$$+\frac{k\,(k+1)\,(k+2)}{1\times 2\times 3}\,x^3+\ldots$$

$$\left.\ldots+\frac{k\,(k+1)\,\ldots\,(k+9)}{10!}\,x^{10}+\ldots\right)$$

From this we immediately find that for $k=1$ the desired coefficient is equal to 1, for $k=2$, it is 7, for $k=3$ it is 15, for $k=4$ it is 10, and for $k=5$, it is equal to 1. Since the number of bands of each colour for a given mode of painting is the same, and the lengths of the bands of different colours may be combined in arbitrary fashion, we get $1^3 + 7^3 + 15^3 + 10^3 + 1^3 = 4{,}720$ ways of painting.

We now lift the restriction that the last colour is blue. If the painting ends in red, then it occurs once more than blue and white. Then the number of ways is equal to $1 + 7 \times 1^2 + 15 \times 7^2 + 10 \times 15^2 + 1 \times 10^2 = 3{,}093$. Likewise, if the last colour is white, then the number of painting patterns is $1^2 + 7^2 \times 1 + 15^2 \times 7 + 10^2 \times 15 + 1^2 \times 10 = 3{,}135$, making a total of 10,948 ways.

If no single band is less than 3 cm, then the problem reduces to counting the number of representations of 10 as a sum of k positive integers assuming values from 3 to 10. For $k=1$ we have one representation, for $k=2$, five representations, for $k=3$, three representations. And so the painting ends in blue $1^3 + 5^3 + 3^3 = 153$ times, in red $1 + 5 \times 1^2 + 3 \times 5^2 = 81$ times, and in white, $1^2 + 5^2 \times 1 + 3^2 \times 5 = 71$ times.

236.

Since I dined with all six friends once and with every five, twice, it follows that I dined with every five friends once in the absence of the sixth. But then I dined three times with every four (two dinners in a group of five and one in a group of six), four times with every three and five times with every two. At all these dinners I met each one of my friends 7 times, which means that I dined out once together with each friend. Each friend was absent at 6 dinners (at 5 dinners as a couple and at one dinner as a group of five). And since I dined 8 times without one of my friends, I dined twice alone.

237.

The 12 students can line up to each examiner in 12! ways, to the two examiners in $(12!)^2$ ways. Then, in $C_{12}^1 \times 11!$ cases at least one student will be answering the questions of both examiners, in $C_{12}^2 \times 10!$ cases this will be the fate of two students, etc. Applying the inclusion and exclusion formula, we find that the number of reasonable ways of distribution is

$$(12!)^2 \left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + \frac{1}{12!}\right] = 12! \times 176{,}214{,}841$$

238.

Similarly, we get the answers

$$(6!)^2 \left[1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{1}{6!}\right] = 190{,}800$$

and

$$(A_9^6)^2 - C_6^1 A_9^1 (A_8^5)^2 + C_6^2 A_9^2 (A_7^4)^2 - C_6^3 A_9^3 (A_6^3)^2 + C_6^4 A_9^4 (A_5^2)^2 - C_6^5 A_9^5 (A_4^1)^2 + C_6^6 A_9^6$$

239.

From the statement of the problem it follows that the same letters of the expression $\alpha^2\beta^2\gamma^2$ enter pairwise into the permutations. We therefore get permutations of 3 elements $a = \alpha^2$, $b = \beta^2$, $c = \gamma^2$, which are 6 in number. The same occurs in the case of permutations of the letters of the expression $\alpha^3\beta^3\gamma^3$. Identical letters also occur in pairs in the permutations of the letters of the expression $\alpha^4\beta^4\gamma^4$. For the time being, put $\alpha^2 = a_1$, $\alpha^2 = a_2$; $\beta^2 = b_1$, $\beta^2 = b_2$, and $\gamma^2 = c_1$, $\gamma^2 = c_2$ (we temporarily disregard the fact that actually $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$). Then we have permutations of the 6 elements, a_1, a_2, b_1, b_2, c_1, c_2, the number of which is 720. But these permutations fall into groups of permutations which differ as to the permutations of the elements a_1 and a_2, b_1 and b_2, c_1 and c_2. Each group includes 8 permutations, and all these permutations correspond to one and the same permutation of the letters of the expression $\alpha^4\beta^4\gamma^4$. Thus, the number of permutations of the letters of this expression is $\frac{720}{8} = 90$.

Finally, consider the permutations of the letters of the expression $\alpha^5\beta^5\gamma^5$. If we temporarily put $\alpha^2 = a_1$, $\alpha^3 = a_2$, $\beta^2 = b_1$, $\beta^3 = b_2$, $\gamma^2 = c_1$, $\gamma^3 = c_2$ then each admissible permutation of the letters of the expression $\alpha^5\beta^5\gamma^5$ is a permutation of the letters a_1, a_2, b_1, b_2, c_1, c_2. However, some permutations of the letters a_1, a_2, b_1, b_2, c_1, c_2 yield one and the same permutation of the letters α, β, γ. For example, $a_1a_2b_1c_2b_2c_1$ and $a_2a_1b_1c_2b_2c_1$ give us $\alpha^5\beta^2\gamma^3\beta^3\gamma^2$. This occurs if some pair of letters (a_1, a_2), (b_1, b_2), or (c_1, c_2) come together. The letters a_1 and a_2 stand together in $2 \times 5!$ permutations, and also the letters (b_1, b_2) and (c_1, c_2). The letters of both pairs (a_1, a_2) and (b_1, b_2) stand together in $(2!)^2 4!$ permutations [just as the pairs (a_1, a_2), (c_1, c_2) or (b_1, b_2), (c_1, c_2)]. Finally, all three pairs stand in succession in $(2!)^3 3! = 48$ permutations. By the inclusion and exclusion formula, we find that in $6! - 6 \times 5! + 3\,(2!)^2 \times 4! - (2!)^3 3! = 240$ permutations not a single pair of letters stand together; in

$$3\,[2 \times 5! - 2\,(2!)^2\, 4! + (2!)^3\, 3!] = 288$$

permutations exactly one of the pairs of letters (a_1, a_2), (b_1, b_2), (c_1, c_2) stand together, and in

$$3\,[(2!)^2\, 4! - (2!)^3\, 3!] = 144$$

permutations exactly two such pairs of letters stand together. Whence it follows that the desired number of permutations of the letters α, β, γ is

$$240+\frac{288}{2!}+\frac{144}{(2!)^2}+\frac{48}{(2!)^3}=426$$

(if the letters a_1 and a_2 are together, then their permutation does not alter the order of the letters α, β, γ).

240.

First let us divide the players of each country into ordered pairs. For each country, this can be done in $\frac{4!}{2}=12$ ways (the order of the pairs themselves is inessential). We then have 12^n modes of partition in all. The pairs may be permuted in $(2n)!$ ways. And so the total number of admissible permutations is equal to $12^n\,(2n)!$.

241.

The first horizontal line may be painted in 8! ways. Each subsequent horizontal line must be painted so that the colour of each square differs from that of the one under it. This can be done in

$$8!\left[1-1+\frac{1}{2!}-\frac{1}{3!}+\cdots+\frac{1}{8!}\right]=14{,}833$$

ways. By the rule of product we find the total number of ways of painting to be $8!\,(14{,}833)^7$.

242.

A sample (including possibly 0, 1, . . ., n things) of n distinct things can be obtained in 2^n ways. After this sample has been taken, adjoin the missing things of the number n identical things. We thus get 2^n choices. The number of permutations of all $2n$ things is equal to $\frac{(2n)!}{n!}$.

243.

In every admissible permutation, both the Englishmen and the Frenchmen occur in groups consisting of at least two men. Here, the number of groups of Frenchmen differs from the number of groups of Englishmen by no more than 1. We compute the number of ways of partitioning n Englishmen into p ordered groups so that each group contains at least two men. To do this, arrange them in some order ($n!$ ways), and then take the second, the third, . . ., etc. up to the interval $n-2$ and place in them $p-1$ "dividers" so that no two dividers are adjacent. According to the results of page 57, this can be done in C_{n-p-1}^{p-1} ways. Altogether we have $n!C_{n-p-1}^{p-1}$ ways. In the same way, we can divide m Frenchmen into s groups of the indicated type in $m!C_{m-s-1}^{s-1}$ ways. These modes can be combined in the following manner:

(a) p groups of Englishmen and $p-1$ groups of Frenchmen,

(b) p groups of Englishmen and p groups of Frenchmen, the English leading,

(c) p groups of Englishmen and p groups of Frenchmen, the French leading,

(d) p groups of Englishmen and $p+1$ groups of Frenchmen. Hence, the total number of modes is given by the formula

$$m!\,n!\,[2\,(C_{m-2}^{0}C_{n-2}^{0}+C_{m-3}^{1}C_{n-3}^{1}+C_{m-4}^{2}C_{n-4}^{2}+\ldots)$$
$$+(C_{m-2}^{0}C_{n-3}^{1}+C_{m-3}^{1}C_{n-4}^{2}+\ldots)$$
$$+(C_{m-3}^{1}C_{n-2}^{0}+C_{m-4}^{2}C_{n-3}^{1}+\ldots)]$$

Removing the brackets in the formula on page 136, we get the same result.

244.

First find the number of numbers without the digit 0. The three digits in a number can be chosen in $C_9^3=84$ ways. Out of three digits we can generate 3^6 six-digit numbers, out of two, 2^6, and out of one, 1^6. By the inclusion and exclusion formula, there are

$$3^6-C_3^1 2^6+C_3^2 1^6=540$$

six-digit numbers which have all the three digits that we chose. Therefore the total number of six-digit numbers that have exactly three nonzero digits is equal to $84\times 540=45{,}360$.

If a zero appears in the number, we have to choose another two digits in it. This can be done in $C_9^2=36$ ways. Suppose the chosen digits are 0, 1 and 2. Then the first digit of the number must be either 1 or 2. If, say, the first is 1, then the other five digits can be any one of 0, 1 or 2, provided that 0 and 2 occur. By the inclusion and exclusion formula, we find that these five digits may be chosen in

$$3^5-C_2^1 2^5+1^6=180$$

ways. But then the total number of six-digit numbers made up of the digits 0, 1, 2 and containing all these digits is equal to $2\times 180=360$, and the total number of six-digit numbers made up of three digits, of which zero is one digit, is equal to $36 \times 360 = 12{,}960$. We get $45{,}360 + 12{,}960 = 58{,}320$ numbers in all.

245.

As in Problem 244, we find

$$C_9^k [k^m - C_k^1 (k-1)^m + C_k^2 (k-2)^m - \ldots + (-1)^{k-1} C_k^{k-1} 1^m] + (k-1) C_9^{k-1} [k^{m-1} - C_{k-1}^1 (k-1)^{m-1} + C_{k-1}^2 (k-2)^{m-1} - \ldots \ldots + (-1)^{k-2} C_{k-1}^{k-2} 1^{m-1}]$$

246.

We denote the number of permutations of this kind by $\Gamma_n^{(k)}$. These permutations fall into two classes. In the first class are those that begin with 1, in the second, all the rest. If the permutation begins with 1, then subtract 1 from all the numbers in it and discard the zero in the leading position (for example, 14,589 first goes into 03,478 and then into 3,478). This yields a $(k-1)$-permutation of the same type but consisting of the numbers $1, 2, \ldots, n-1$. Therefore the number of permutations of the first class is $\Gamma_{n-1}^{(k-1)}$. Every permutation of the second class begins with a number greater than 1. Subtract the number 2 from all numbers of such a permutation. We get a k-permutation of that same type which includes the numbers $1, 2, \ldots, (n-2)$. Therefore the number of permutations of the second class is $\Gamma_{n-2}^{(k)}$. Thus, the following recurrence relation holds true:

$$\Gamma_n^{(k)} = \Gamma_{n-1}^{(k-1)} + \Gamma_{n-2}^{(k)}$$

Set $F_n^{(k)} = C_N^k$, where $N = E\left(\frac{n+k}{2}\right)$. We have

$$F_{n-1}^{(k-1)} + F_{n-2}^{(k)} = C_{N-1}^{k-1} + C_{N-1}^{k} = C_N^k = F_n^{(k)}$$

Thus, the numbers $F_n^{(k)}$ satisfy the same recurrence relation as the number $\Gamma_n^{(k)}$.

We now demonstrate that $F_n^{(n)} = \Gamma_n^{(n)}$ and $F_{n+1}^{(n)} = \Gamma_{n+1}^{(n)}$. Note that the numbers $1, 2, \ldots, n$ can be uniquely arranged in increasing order, and therefore $\Gamma_n^{(n)} = 1 = C_n^n = F_n^{(n)}$. From the numbers $1, 2, \ldots, n+1$ it is also possible, in unique fashion, to choose n numbers in accord with the indicated conditions. And so, likewise, $\Gamma_{n+1}^{(n)} = 1 = C_n^n = F_{n+1}^{(n)}$. From what has been proved, it follows that for all values of n and k we have the equality

$$\Gamma_n^{(k)} = F_n^{(k)} = C_N^{(k)}$$

where, it will be recalled, $N = E\left(\frac{n+k}{2}\right)$

247.

The given elements may be permuted in

$$P(2, 2, \ldots, 2) = \frac{(2n)!}{2^n}$$

ways. Let us find the number of permutations in which the elements of the given k pairs occur together. In such permutations, we can combine the adjacent elements of one and the same pair. We then get a permutation of k distinct elements and of elements belonging to $n-k$ pairs. There are $\frac{(2n-k)!}{2^{n-k}}$ such permutations. Since k pairs may be chosen in C_n^k ways, it follows, by the inclusion and exclusion formula, that in

$$\frac{(2n)!}{2^n} - C_n^1 \frac{(2n-1)!}{2^{n-1}} + C_n^2 \frac{(2n-2)!}{2^{n-2}} - \ldots \ldots + (-1)^n C_n^n n!$$

permutations no two identical elements come together.

248.

Just as in the preceding problem, we get

$$\frac{(qn)!}{(q!)^n} - C_n^1 \frac{(qn-q+1)!}{(q!)^{n-1}} + C_n^2 \frac{(qn-2q+2)!}{(q!)^{n-2}} - \ldots$$

249.

The given elements may be permuted in $\frac{(qn)!}{(q!)^n}$ ways. Now compute the number of permutations in which the elements of the given k q-sets occur together. Choose one of these q-sets. Its elements may be arranged in succession round a circle in qn ways. After this has been done, combine the elements of each of the remaining $k-1$ sets and consider all possible permutations of the new $k-1$ elements thus obtained and the remaining $(n-k)q$ elements. Their number is $\frac{(qn-qk+k-1)!}{(q!)^{n-k}}$ and it is easy to see that with each such permutation is associated a specific arrangement of the elements round the circle. Therefore, the number of permutations in which the given k-sets occur in succession is $\frac{qn(qn-qk+k-1)!}{(q!)^{n-k}}$. Since the sets themselves can be chosen in C_n^k ways, it follows that by the inclusion and exclusion formula we obtain the number of desired permutations:

$$qn\left[\frac{(qn-1)!}{(q!)^n}-C_n^1\frac{(qn-q)!}{(q!)^{n-1}}+\right.$$
$$\left.+C_n^2\frac{(qn-2q+1)!}{(q!)^{n-2}}-\dots+(-1)^nC_n^n(n-1)!\right]$$

250.

Adjoin to each chosen book the s books following it. We must then choose p objects out of $n-ps$. This can be done in C_{n-ps}^p ways.

251.

If the common difference of the progression is d, and the number of contestants of the 5th class is a, then the prizes may be distributed in $A_a^dA_{a+d}^dA_{a+2d}^d\dots A_{a+5d}^d$ ways. However, if all prizes are given to students of the 10th class, then they may be distributed in A_{a+5d}^{6d} ways. The equality

$$A_a^dA_{a+d}^dA_{a+2d}^d\dots A_{a+5d}^d=A_{a+5d}^{6d}$$

follows from the obvious identity $A_n^mA_{n+k}^k = A_{n+k}^{m+k}$.

252.

The solution calls for considering the segments and intersections of various positions and for computing the number of routes passing through

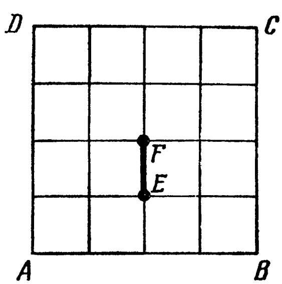


Fig. 36.

them. For instance, a total of 18 routes pass through EF (Fig. 36) ($C_3^1=3$ routes go from A to E and $C_4^2=6$ routes lead from F to C). There are 30 routes passing through point E: 3 routes from A to E and $C_5^2=10$ routes from E to C. The other segments and points are considered analogously.

253.

$C_6^3=20$.

254.

We are dealing with combinations with repetitions of elements of 4 kinds taken three at a time. Their number is

$\bar{C}_4^3=C_6^3=20$

255.

$\bar{C}_{10}^3=C_{12}^3=220$.

256.

Four triangles.

257.

If no three out of n points lay on one straight line, there would be C_n^3 triangles with vertices at these points. But p points lie on one line and so C_p^3 triangles have to be discarded. This leaves $C_n^3-C_p^3$ triangles.

258.

We can take two vertices on one straight line and a third on the other. We thus get $C_p^2C_q^1+C_p^1C_q^2=\frac{pq}{2}(p+q-2)$ triangles.

259.

Additional

$$C_r^2(C_p^1+C_q^1)+C_r^1(C_p^2+C_q^2)+C_r^1C_p^1C_q^1=$$
$$=\frac{r}{2}(p+q)(p+q+r-2)$$

triangles result.

260.

The triangles may be of two kinds; either all three vertices lie on different sides of the square or two vertices lie on one side of the square and the third on some other side. In the first case we have to choose three sides of the square out of four ($C_4^3=4$ choices), and then one point each out of $n-1$ on each of the three sides. In all we have $4\,(C_{n-1}^1)^3$ choices. In the second case, we have to choose the side with two vertices (4 choices) and two points out of $n-1$ (C_{n-1}^2 ways), and then choose one of the remaining three sides (three choices) and a point on it (C_{n-1}^1 choices). In the second case we get a total of $12C_{n-1}^1\ C_{n-1}^2$ choices. In all there are

$$4\,(C_{n-1}^1)^3+12C_{n-1}^1C_{n-1}^2=2\,(n-1)^2\,(5n-8)$$

choices.

261.

C_n^2 points of intersection.

262.

In the general position, n straight lines have C_n^2 points of intersection. But p straight lines passing through point A yield one intersection point instead of C_p^2 and q lines passing through point B yield one point instead of C_q^2. This leaves $C_n^2-C_p^2-C_q^2+2$ points of intersection.

263.

Let $k-1$ straight lines be drawn on a plane. Draw one more. It is divided, by the points of intersection with the earlier drawn lines, into k parts, each of which corresponds to one new piece of the plane. Therefore, n straight lines divide the plane into $1+1+2+\ldots+n=$
$=\frac{1}{2}(n^2+n+2)$ parts.

264.

Let $k-1$ planes be already drawn. Draw one more. This plane intersects the earlier drawn planes along $k-1$ straight lines, which divide it into $\frac{1}{2}(k^2-k+2)$ parts. Each of these parts corresponds to a new portion of space. And so n planes divide the space into

$$1+\frac{1}{2}\sum_{k=1}^{n}(k^2-k+2)=\frac{1}{6}(n+1)(n^2-n+6)$$

parts.

265.

A total of $C_5^2=10$ straight lines are drawn. 4 lines pass through each point (say point C). Hence, 6 perpendiculars emanate from this point. Consider any two points (say B and C). The perpendiculars dropped from B onto the straight lines passing through C intersect all the perpendiculars dropped from C. There are 3 lines emanating from C that do not pass through B. Hence, from B we can drop 3 perpendiculars onto them. These perpendiculars intersect with the perpendiculars dropped from C at $3\times 6=18$ points. Each of the perpendiculars dropped from B onto the other three lines passing through C intersects only 5 perpendiculars dropped from C, since it is parallel to one of these perpendiculars, for it is dropped onto the same line. We thus have another 15 points. Consequently, the perpendiculars dropped from two points intersect in $18+15=33$ points. But 10 pairs can be generated out of 5 points. This would yield $33\times$ $\times 10=330$ points of intersection, but some are coincident points. Namely, any 3 out of 5 given points form a triangle. The altitudes of this triangle (which are some of our perpendiculars) intersect in a single point, but we counted that point 3 times. Since there are $C_5^3=10$ such triangles, we have to reject 20 points, which leaves 310 possible points of intersection.

266.

Any three integers x, y, z that satisfy the inequalities $n+1\leqslant x, y, z\leqslant 2n$ can be the sides of a triangle. Therefore there are $\overline{C}_n^3=C_{n+2}^3$ triangles with such sides. To find the number of isosceles triangles, note that for a given base we have n isosceles triangles. Hence, the total number of them is n^2. The number of equilateral triangles is n.

267.

We have to find the number of triples of positive integers x, y, z such that $x\leqslant y\leqslant z\leqslant 2n$ and $x+y>z$. Let there be $x=p$. Then y assumes values from p to $2n$. When y runs through the values from p to $2n-p+1$, every value of y is associated with p values of z satisfying the inequalities $y\leqslant z<y+p$, $z\leqslant 2n$. But if y takes on the values from $2n-p+2$ to $2n$, then the number of corresponding values of z is $2n-y+1$. For $x=p$ we get a total of

$$2p\,(n-p+1)+\sum_{y=2n-p+2}^{2n}(2n-y+1)=2pn-\frac{3}{2}p^2+\frac{3}{2}p$$

pairs (y, z) such that x, y, z satisfy the indicated conditions. From this it follows that the total number of triangles for which $1\leqslant x\leqslant n$ and

$1 \leqslant y,\ \ z \leqslant 2n$ is

$$\sum_{p=1}^{n}\left(2pn-\frac{3}{2}p^2+\frac{3}{2}p\right)=\frac{n}{2}(n+1)^2$$

By virtue of Problem 266, there are C_{n+2}^3 triangles for which $x \geqslant n+1$. Therefore we have

$$\frac{n}{2}(n+1)^2+\frac{n(n+1)(n+2)}{6}=\frac{n(n+1)(4n+5)}{6}$$

triangles in all.

There are $2n-k$ isosceles triangles with base $x=2k$ and also $2n-k$ with base $2k+1$. Hence, the total number of isosceles triangles is

$$\sum_{k=1}^{n}(2n-k)+\sum_{k=0}^{n-1}(2n-k)=3n^2$$

Eliminating them, we get

$$\frac{n(n+1)(4n+5)}{6}-3n^2=\frac{n(n-1)(4n-5)}{6}$$

triangles.

268.

The solution is similar to that of Problem 267. The number of triangles with given value $x = p \leqslant n-1$ is $2np-\frac{3}{2}p^2+\frac{p}{2}$ and there are

$$\sum_{p=1}^{n-1}\left(2np-\frac{3}{2}p^2+\frac{p}{2}\right)=\frac{n(n+1)(n-1)}{2}$$

triangles for which $x \leqslant n-1$. Now the number of triangles for which $x \geqslant n$ is C_{n+2}^3 and so, in all, we have

$$\frac{n(n+1)(n-1)}{2}+\frac{n(n+1)(n+2)}{6}=\frac{n(n+1)(4n-1)}{6}$$

triangles. The number of isosceles triangles is

$$\sum_{k=1}^{n-1}(2n-k-1)+\sum_{k=0}^{n-1}(2n-k-1)=3n^2-3n+1$$

of scalene triangles,

$$\frac{n(n+1)(4n-1)}{6}-3n^2+3n-1=\frac{1}{6}(n-1)\times(n-2)(4n-3)$$

269.

Since we take n points of intersection and no three lie on a single straight line, it follows that there are two and only two points of the chosen group that lie on each line. And so, in order to specify such a group, we number the given straight lines and on the first choose the point of intersection with the second line, on the second, the point of intersection with the third, . . ., on the nth, the point of intersection with the first straight line. We obtain the desired group of points; all groups may be obtained as described above. Noting that a cyclic permutation of the points and any change in their order of traversal do not alter the group of points, we find that the number of groups is $\frac{P_n}{2n}=\frac{1}{2}(n-1)!$.

270.

We can choose r vertices having a given order in A_n^r ways. Since a cyclic permutation of the vertices and any change in the order of traversal do not alter a polygon, we get $\frac{1}{2r}A_n^r$ polygons.

271.

Choose two points on one line and two points on the other. To them correspond two points of intersection of the straight lines passing through these points (the intersection point of the diagonals of the trapezoid and the intersection point of the lateral sides). Since it is possible to choose C_n^2 pairs of points on the first straight line, and C_m^2 on the second, the number of intersection points is $2C_n^2C_m^2$.

272.

n points determine C_n^3 circles. Of them, C_{n-1}^2 pass through a given point and C_{n-2}^1 through two given points. And so the straight line passing through two given points has at most $2C_{n-2}^3 + (2C_{n-1}^2 - C_{n-2}^1)+2$ points of intersection with the circles. Since C_n^2 lines pass through n points, we have at most

$$C_n^2\,[2C_{n-2}^3+2C_{n-1}^2-C_{n-2}^1+2]$$

points of intersection.

273.

Each intersection line is determined by two planes, and each plane, by three given points. The straight lines fall into classes according to the number of points (of those specifying the first plane) that enter into the number of points specifying the second plane. All the points are distinct in $\frac{1}{2}C_n^3C_{n-3}^3$ cases (choose three points out of n and another three points out of the remaining $n-3$, the order of choice being irrelevant). If one point appears in both triples, then we get $\frac{3}{2}C_n^3C_{n-3}^2$ choices, and if two points occur in both triples, then $\frac{3}{2}C_n^3C_{n-3}^1$ choices. In all we have

$$\frac{1}{2}C_n^3(C_{n-3}^3+3C_{n-3}^2+3C_{n-3}^1)=$$
$$=\frac{n(n-1)(n-2)(n-3)(n^2+2)}{72}$$

straight lines. Of this number,

$$\frac{1}{2}C_n^3C_{n-3}^3=\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{72}$$

do not pass through a single given point.

274.

Denote the sides of a quadrilateral by a, b, c, d. Without loss of generality, we can assume that a is the smallest side, c the side opposite to it, and that $b<d$. Then $a<b<d$ and $a<c$. Besides, since the quadrilaterals are circumscribed about circles, it follows that $a+c=b+d$. Whence $a+c>2b$. Therefore, for given values of a and b, the length of c can assume values from $2b-a+1$ to n and the inequality $2b-a\leqslant n-1$ must hold.

We have thus proved that $b\leqslant\frac{a+n-1}{2}$ and that $2b-a+1\leqslant c\leqslant n$. Denote $E\left(\frac{a+n-1}{2}\right)$ by s. Then for a given value of a we have

$$\sum_{b=a+1}^{s}(n+a-2b)=(s-a)(n-s-1)$$

quadrilaterals.

Let n be an even number, $n=2m$. Then for odd values of a, $a=2k-1$, we have $s=E\left(\frac{n+a-1}{2}\right)=m+k-1$ and, consequently, $(m-k)^2$ quadrilaterals, and for even values of a, $a=2k$, we have $s=E\left(\frac{n+a-1}{2}\right)=m+k-1$ and, hence, $(m-k-1)(m-k)$ quadrilaterals. Summing with respect to a, we get the total number of quadrilaterals:

$$\sum_{k=1}^{m}(m-k)^2+\sum_{k=1}^{m}(m-k)(m-k+1)=$$
$$=\frac{m(m-1)(4m-5)}{6}=\frac{n(n-2)(2n-5)}{24}$$

The case when n is odd is considered in similar fashion.

If it is assumed that the quadrilaterals have identical sides, then a, b, c, d must satisfy the relations $a\leqslant b\leqslant d\leqslant n$, $a\leqslant c$ and $a+c=b+d$, whence it follows that $b\leqslant\frac{a+n}{2}$ and $2b-a\leqslant c\leqslant n$. If we put $E\left(\frac{a+n}{2}\right)=s$, then the number of quadrilaterals with a given value of a is $(n-s+1)(s-a+1)$.

For an even value of n, we get $\frac{n(n+2)(2n+5)}{24}$ quadrilaterals, and for odd n, $\frac{(n+1)(2n^2+7n+3)}{24}$.

275.

The number of circles drawn is C_n^3, with C_{n-1}^2 circles passing through a given point, and C_{n-2}^1 circles passing through two given points. Take one of these circles drawn through the points A, B, C. We have $C_n^3-3C_{n-1}^2+3C_{n-2}^1-1$ circles not passing through a single one of these points. The chosen circle intersects each of them in two points. Furthermore, we have $3(C_{n-1}^2-2C_{n-2}^1+1)$ circles passing through one of the points A, B, C and not passing through two of them. They yield one intersection point each that is different from A, B, C. The other circles intersect with the chosen one in two of the points A, B, C. Thus, the given circle yields

$$2(C_n^3-3C_{n-1}^2+3C_{n-2}^1-1)+3(C_{n-1}^2-2C_{n-2}^1+1)$$
$$=\frac{(n-3)(n-4)(2n-1)}{6}$$

intersection points different from A, B, C. Altogether we get

$$\frac{1}{2}C_n^3\,\frac{(n-3)(n-4)(2n-1)}{6}=\frac{5(2n-1)}{3}\,C_n^5$$

points of intersection that differ from the given points. Appending these n points, we find that the greatest number of intersection points is

$$\frac{5(2n-1)}{3}\,C_n^5+n$$

276.

Adding the plane $k+1$ to the earlier drawn k ($k=1, 2, \ldots$) planes, we get $2k$ new parts, and the total is $2+2+4+6+\ldots$ $\ldots+2(n-1)=n^2-n+2$.

277.

The total number of ways of painting 6 faces in 6 different colours is $6!=720$. Divide these modes into classes consisting of patterns that can be brought to coincidence via motions. The cube can be brought to coincidence with itself in 24 ways (a face into which a fixed face of the cube goes can be chosen in 6 ways; there then remain 4 rotations of the cube under which the given face goes into itself). For this reason, each class consists of 24 patterns, and the number of geometrically distinct ways of painting the cube is $720/24=30$.

278.

The solution is obtained along the same lines as that of Problem 277. The number of ways of painting the faces is $4!/12=2$.

279.

We have $8!/24=1{,}680$ ways of painting the faces.

280.

For a dodecahedron there are $12!/60$ patterns, and for an icosahedron, $20!/60$ colour patterns.

282.

We have to find the number of triplets of positive integers x, y, z such that $x\leqslant y\leqslant z$, $x+y+$ $+z=40$ and $x+y>z$. From these inequalities it follows that z can assume values which satisfy the inequalities $14\leqslant z\leqslant 19$. If $z=19$, then $x+y=21$, $x\leqslant y\leqslant 19$. Therefore $11\leqslant$ $\leqslant y\leqslant 19$, and we have 9 triangles with $z=19$. In exactly the same way, we find that the number of triangles for which $z=18, 17, 16, 15, 14$ is equal, respectively, to 8, 6, 5, 3, 2, making 33 triangles in all. In the same way we find the number of triangles with perimeter 43 to be 44.

283.

Take a triangle with perimeter $4n$. Let its sides be equal to x, y, z. Adding 1 to the lengths of these sides, we get the numbers $x+1$, $y+1$, $z+1$, which are the lengths of the sides of a triangle with perimeter 43. But besides we have triangles with sides $(1, 2n+1, 2n+1)$, $(2, 2n, 2n+1)$, $\ldots$, $(n+1, n+1, 2n+1)$, which cannot be obtained in the manner just described.

284.

Let $N=12n$. We have to find the number of triplets of the positive integers x, y, z such that $x\leqslant y\leqslant z$, $x+y+z=12n$ and $x+y>z$. From these inequalities it follows that $4n\leqslant$ $\leqslant z\leqslant 6n-1$. Here, if $z=2k$, then $x+y=$ $=12n-2k$ and the number of integral solutions of this equation such that $x\leqslant y\leqslant z=2k$ is $3k-6n+1$. But if $z=2k+1$, then we have $3k-6n+2$ solutions. The number of triangles is therefore

$$\sum_{k=2n}^{3n-1}(3k-6n+1)+\sum_{k=2n}^{3n-1}(3k-6n+2)=3n^2$$

The remaining cases are analyzed analogously. When passing from N to $N+3$, arguments similar to those used in Problem 282 may be applied.

285.

We will demonstrate that there are exactly n routes passing through each stop. Let l be one of the routes and let B be a stop located off the route (Fig. 37). By virtue of Condition 1, B can be reached by one of the routes to each of the n stops $A_1, \ldots, A_n$ of route l. Then, by virtue of Condition 2, each of the routes passing through B passes through some one of the stops $A_1, \ldots$ $\ldots, A_n$ (otherwise the change could not be made to route l) and only one (otherwise it would be possible to change to route l at two stops). Also, no two routes passing through B pass through the same stop of route l (otherwise it would be possible, from one of these routes, to change to another at two stops: at B and at the stop of route l

through which both pass). Whence it follows that there are just as many routes going through stop B as there are stops on route l, that is to say, exactly n routes.

It remains to prove that there are also exactly n routes that pass through each one of the stops

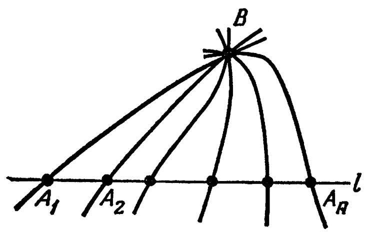


Fig. 37.

$A_1, \ldots, A_n$ situated on route l. It suffices to show that for any one of these stops there is a route, l', which does not pass through it (as stated, it has n stops and then, as we know, n routes pass through it). Since the total number of routes is at least two, it follows that aside from l there is at least one route l' (Fig. 38) that crosses route

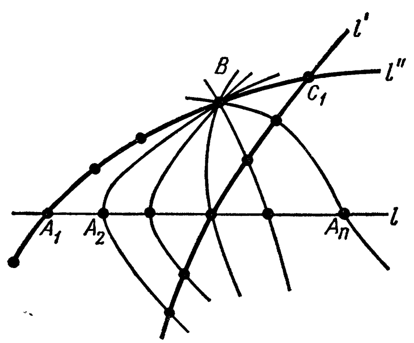


Fig. 38.

l at a unique point, say A_1. Then the stops $A_2, \ldots, A_n$ are located outside route l' and so n routes go through them. Let B be another stop on route l'. The route passing through B and A_2 does not pass through A_1 and therefore exactly n routes pass through A_1 as well. Thus, exactly n routes pass through any stop.

Since for each stop we can find a route not passing through it, and on each route there are n stops, it follows that there are n routes going through each stop. Take one of the routes, l. There are $n - 1$ routes different from l passing through each stop of this route; by virtue of Condition 2, no two of these routes coincide (otherwise they would have two stops in common), and any route appears among those obtained in this way. Thus, the number of routes different from l is equal to $n(n - 1)$; in all we have $n(n - 1) + 1$ routes.

286.

Suppose there are n stops on one of the routes, l. From the solution of Problem 285 it is evident that there are exactly n routes going through any stop, B, lying outside route l. We will show that there are exactly n stops on an arbitrary route l', which is different from l. By virtue of Condition 3, l' has at least three stops, and by Condition 2, one of these stops is at the same time a stop of route l. There are $n - 1$ stops of route l lying outside route l'. We will show that besides, there is at least one stop outside l' not lying on l. Indeed, let A_1 be one of the $n - 1$ stops of route l outside l', and let C_1 be one of the stops of route l' outside l (there are at least 2 such stops). By virtue of Condition 1, there is a route l'' passing through the stops A_1 and C_1, and by virtue of Condition 3, there is, on this route, aside from A_1 and C_1, at least one stop B_1 which will lie outside l' and l. As we know from the solution of Problem 285, n routes pass through stop B_1. Each one of these n routes crosses route l' at one single point. And through each stop of route l' there is at least one route connecting it with stop B. Therefore, the number of stops on route l' is equal to the number of routes passing through stop B, that is, it is equal to n.

As we saw in the solution of Problem 285, in this case the number of routes is given by the formula $n(n - 1) + 1$. Since, by hypothesis, this number is 57, we have to solve the equation $n^2 - n + 1 = 57$. Its solution yields $n = 8$.

287.

Yes. Let us consider, say, 10 straight lines on a plane such that no two lines are parallel and no three intersect in one point; we will assume that the straight lines are autobus routes and the points of intersection are stops. Here, it is possible to go from any stop to any other stop without changing if they lie on one line, and with one change if they lie on different lines. Even if we drop one of the straight lines, it will still be possible to reach any stop from each stop without making more than one change en route. But if we drop two lines, then one stop (the intersection point of the two lines) will not be serviced by the remaining routes, and it will be impossible to go from this stop to any other stop.

288.

A sphere may be in contact with any one of the planes on one of two sides and with the given sphere, either inside or outside. We can therefore construct 16 distinct spheres.

289.

Each of the m straight lines drawn through point A intersects with $2m$ lines. And so the lines passing through A yield $2m^2$ intersection points. The total number of points of intersection distinct from the given three is $3m^2$.

290.

Denote the points lying in one plane by $A_1, \ldots, A_m$ and the others by $B_1, \ldots, B_{n-m}$. Each plane is determined by a set of three points, which can include three, two, one or zero points out of the points $A_1, \ldots, A_m$. Accordingly, we find that the number of planes is

$$1+C_m^2C_{n-m}^1+C_m^1C_{n-m}^2+C_{n-m}^3$$

291.

There are $n+p$ points of intersection lying on each of the straight lines passing through A, $m+p$ passing through B, and $m+n$ through C. Since m lines pass through A, n lines through B, and p lines through C, the total number of intersection points is

$$\frac{1}{2}[m(n+p)+n(m+p)+p(m+n)]=mn+mp+np$$

There are $C_{mn+mp+np}^3$ ways of choosing from them a triple of points, but in $mC_{n+p}^3+nC_{m+p}^3+$ $+pC_{m+n}^3$ cases we get points on a single straight line. And so the number of triangles is

$$C_{mn+mp+np}^3-mC_{n+p}^3-nC_{m+p}^3-pC_{m+n}^3$$

292.

Arbitrarily choose the first vertex of a triangle. This is done in n ways. We then have to choose another two vertices among the $n-3$ points not adjacent to the given one so that they are not mutually adjacent either. This can be done in C_{n-4}^2 ways (see page 57). Since any one of the three vertices can be taken as the first, we have $\frac{n}{3}C_{n-4}^2=\frac{n(n-4)(n-5)}{6}$ choices.

293.

Divide all the triangles into two classes: those whose vertices all lie on distinct lines and those two vertices of which lie on one straight line. The number of triangles of the first class is $p^3C_n^3$ (we choose three lines on which the vertices lie—this is done in C_n^3 ways; then on each of the lines we choose one point out of p). The number of triangles of the second class is $\frac{1}{2}p^2(p-1)\times$ $\times n(n-1)$ (we choose a straight line on which there are two vertices and then two points on this line; we then take a straight line, which has one vertex, and a point on this line). In all, there are

$$p^3C_n^3+\frac{1}{2}p^2(p-1)n(n-1)=$$
$$=\frac{n(n-1)p^2(pn+p-3)}{6}$$

triangles.

294.

Each interior point of intersection of the diagonals is uniquely determined by 4 vertices of the n-gon—the endpoints of intersecting diagonals. There are C_n^4 such points. Now let us find the total number of points of intersection of the diagonals. $n-3$ diagonals emerge from each vertex of an n-gon; we have $\frac{n(n-3)}{2}$ diagonals altogether. Each diagonal AB intersects with all diagonals connecting the vertices distinct from A and B. And so we have

$$\frac{n(n-3)}{2}-2(n-3)+1=\frac{(n-3)(n-4)}{2}+1$$

points of intersection of the diagonal AB with all other diagonals. Since there are $\frac{n(n-3)}{2}$ diagonals in all, and each point of intersection is counted twice, we finally get

$$\frac{n(n-3)[(n-3)(n-4)+2]}{8}$$

points of intersection of the diagonals. Subtracting from this number the number of interior points of intersection, we find the number of exterior points of intersection to be

$$\frac{n(n-3)(n-4)(n-5)}{12}.$$

295.

Each r-gon is determined by choosing r points out of n taken in a specific order; a cyclic permutation of the points does not change the r-gon, nor does a change in orientation. Therefore, there are $\frac{1}{2r}A_n^r$ r-gons and the total number of polynomials generally is $\sum_{r=3}^{n}\frac{1}{2r}A_n^r$. The number of convex polygons is $\sum_{r=3}^{n}C_n^r$.

296.

m parallel straight lines divide the plane into $m+1$ strips. Each new line adds as many pieces as the number of parts it is divided into by the already drawn lines. Since we draw another n lines, we get

$$m+1+(m+2)+\ldots+(m+n)=\frac{n(2m+n+1)}{2}$$

parts.

297.

Divide the circles into classes according to the number of specified points lying on a given circle. One circle (namely the given one) contains all these points, $C_5^2C_6^1$ contain two points, $C_5^1C_6^2$ contain one point and C_6^3 has no point. Altogether we have $1+C_5^2C_6^1+C_5^1C_6^2+C_6^3=156$ circles.

298.

To every three straight lines there correspond 4 circles tangent to them. And so we have $4C_{10}^3=$ $=480$ circles.

299.

Choose s successive vertices of the n-gon, $A_1, \ldots, A_s$ and divide all the k-gons satisfying the stated condition into two classes. One class includes all the k-gons, one of the vertices of which coincides with one of the chosen vertices, the second class includes all the remaining k-gons. Then divide the k-gons of the first class into s subclasses according to which of the vertices A_m, $1\leqslant m\leqslant s$, belongs to the k-gon (it is obvious that these subclasses do not have any elements in common).

Let us find the number of k-gons for which A_m is one of the vertices. To do this, discard the vertex A_m and the successive s vertices going clockwise (not one of them is a vertex of a k-gon). Out of the remaining $n-s-1$ vertices, we have to choose $k-1$ vertices so that after each one we have at least s chosen vertices. This can be done in C_{n-ks-1}^{k-1} ways (see Problem 250). And so the number of k-gons with vertex A_m is C_{n-ks-1}^{k-1}, while the total number of k-gons of Class One is sC_{n-ks-1}^{k-1}.

Now let us find the number of k-gons in the second class. To do this, "cut" the circle between vertices A_s and A_{s+1}. We have to choose k vertices so that after each chosen vertex there come at least s remaining vertices (and not a single one of the vertices $A_1, \ldots, A_s$ is chosen). This can be done in C_{n-ks}^k ways. Thus, the total number of k-gons satisfying the stated condition is $sC_{n-ks-1}^{k-1}+$ $+C_{n-ks}^k$.

300.

Each parallelogram is determined by two pairs of parallel lines. Hence, there are $(C_{r+2}^2)^2$ parallelograms.

301.

Draw a succession of diagonals from the vertices $A_1, A_2, \ldots, A_n$. Each new diagonal yields as many new regions as there are pieces into which it is split by earlier drawn diagonals; that is to say, one part more than the number of its points of intersection with earlier drawn diagonals. Since, in the process, each intersection point is obtained once, the total number of new regions is equal to the sum of the number of intersection points and the number of diagonals. Since at the start we had one part, altogether we have

$$1+\frac{n(n-3)}{2}+\frac{n(n-1)(n-2)(n-3)}{24}=\frac{n(n-3)(n^2-3n+14)}{24}+1$$

parts (see Problem 294).

302.

Let n be even, $n = 2k$. Then we can represent n as the sum of two integers in the following ways:

$$n=1+(2k-1)=2+(2k-2)=\ldots=k+k$$

But the card labelled 1 can be drawn in only one way, the card labelled 2, in two ways, etc. And two cards with the number $k>1$ can be chosen in C_k^2 ways. Therefore, altogether we have

$$1(2k-1)+2(2k-2)+\ldots$$
$$\ldots+(k-1)(k+1)+\frac{k(k-1)}{2}$$
$$=\sum_{s=1}^{k-1} s(2k-s)+\frac{k(k-1)}{2}=\frac{2k(k^2-1)}{3}$$
$$=\frac{n(n^2-4)}{12}$$

ways of obtaining the sum $n=2k$. But if n is odd, $n=2k-1$, then

$$n=1+(2k-2)=2+(2k-3)=\ldots=(k-1)+k,$$

and the number of ways is

$$\sum_{s=1}^{k-1} s(2k-s-1)=\frac{k(k-1)(2k-1)}{3}=\frac{n}{12}(n^2-1)$$

303.

Split the samples into classes according to the number of identical objects in each. The number of samples containing k identical objects is C_{2n+1}^{n-k}. Therefore, the total number of samples is

$$C_{2n+1}^{n}+C_{2n+1}^{n-1}+\ldots+C_{2n+1}^{0}=$$
$$=\frac{1}{2}\sum_{k=0}^{2n+1} C_{2n+1}^{k}=2^{2n}$$

304.

If the first term of the progression is a and its common difference is d, then the third term is $a+2d$. By hypothesis, $a+2d\leqslant 2n$. This inequality has $2n-2d$ solutions for a given d. In all, we get

$$(2n-2)+(2n-4)+\ldots+2=n(n-1)$$

solutions. Since each of the progressions obtained may be regarded both as an increasing and a decreasing progression, we get $2n(n-1)$ progressions. We have $2n^2$ progressions for the sequence of numbers $1, 2, 3, \ldots, 2n+1$.

305.

We prove the assertion by means of induction with respect to the number s of curves. If $s=1$, then it is obvious since there are no points of intersection and the number of regions is 1. Let the assertion be already proved for s curves. Take a system of s curves having n_r intersection points of multiplicity r, $r=2, 3, \ldots$ (a point is of multiplicity r if r curves intersect in it). Then they bound $1+n_2+\ldots+rn_{r+1}+\ldots$ closed regions. Draw the curve $s+1$ and let it have k_r intersection points of multiplicity r with the earlier drawn curves, $r=2, 3, \ldots$, altogether a total of $k_2+k_3+\ldots+k_{r+1}+\ldots$ points of intersection. These points of intersection divide it into $k_2+k_3+\ldots+k_{r+1}+\ldots$ parts. Each part of the drawn curve corresponds to one new region, and so the number of regions is now

$$(1+n_2+\ldots+rn_{r+1}+\ldots)+$$
$$+(k_2+k_3+\ldots+k_{r+1}+\ldots) \qquad (*)$$

But if the new curve has passed through a point (multiplicity r) of intersection of the earlier drawn curves, this point now becomes an intersection point of multiplicity $r+1$.

Denote by n_r' the number of intersection points of multiplicity r in the new system of curves. It is clear that $n_{r+1}'=n_{r+1}-k_{r+1}+k_r$ (from the earlier points of intersection of multiplicity $r+1$ we must subtract k_{r+1} points which had multiplicity $r+1$ and increased it, and we must add k_r points that had multiplicity r and increased it). But then

$$1+n_2'+2n_3'+\ldots+rn_{r+1}'=1+(n_2-k_3+k_2)+$$
$$+2(n_3-k_4+k_3)+\ldots+r(n_{r+1}-k_{r+1}+k_r)+\ldots$$
$$=(1+n_2+2n_3+\ldots+rn_{r+1}+\ldots)+$$
$$+(k_2+k_3+\ldots+k_{r+1}+\ldots)$$

We have thus proved that $1+n_2'+2n_3'+\ldots+rn_{r+1}'+\ldots$ is equal to the number of regions for the new system of curves [see formula (*)]. By virtue of the principle of mathematical induction, the assertion holds true for any number of curves.

306.

The straight lines of the first pencil split the plane into $2m$ parts. The first line of the second pencil intersects all m lines of the first pencil and yields $m + 1$ new parts. All the other lines of the second pencil have $m + 1$ points of intersection with the earlier drawn lines. And so we have a total of

$$2m + m + 1 + (n - 1)(m + 2) = nm + 2n + 2m - 1$$

parts.

307.

No, for otherwise the number of connections would be equal to the fraction $\frac{77 \times 15}{2}$.

308.

The sum of the coefficients is equal to the value of the expression for $x = y = z = 1$. Substituting these values, we find the sum to be -1.

309.

The maximal number of balls among which there are no 15 alike is 74 (10 white balls, 10 black, 12 yellow, and 14 red, green and blue each). But if we take 75 balls, there will be 15 balls of one colour among them.

310.

Classify the colour patterns according to the number of white faces. There is a unique way of painting so that no face is white and one pattern that contains one white face. For the case of two white faces, there are two colour patterns: either the white faces have a common edge or are opposite. For three white faces, we again have two patterns: either there are two opposite white faces or all white faces adjoin one and the same corner. The cases of 4, 5 and 6 white faces reduce to earlier considered cases by interchanging the colours. We obtain a total of $1 + 1 + 2 + 2 + 2 + 1 + + 1 = 10$ ways of painting the object.

311.

We now paint the vertices of a cube. There is one pattern without white vertices, one pattern with one white vertex, 3 with two white vertices (the white vertices lie on one edge, on one diagonal of the face or on one diagonal of the cube), 3 patterns with three white vertices (three vertices lie on one face or two lie on one edge, and the third one on one diagonal of a face with some one of these two vertices, or three vertices lie pairwise on one diagonal of a face, see Fig. 39). With four white vertices, there are 7 patterns (all four vertices lie on one face, or three vertices A, B, C lie on one face, and the fourth lies on one edge with vertex A, or with vertex B, or with

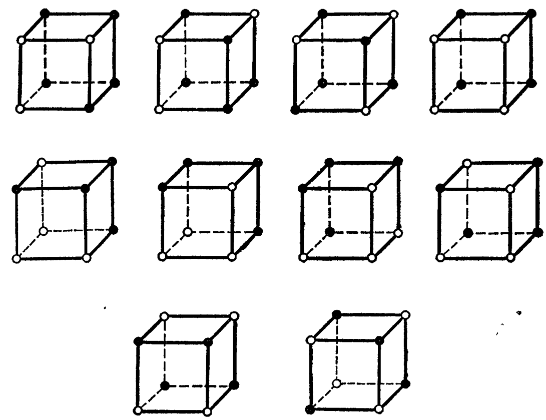


Fig. 39.

vertex C, or is diametrically opposite to vertex B; or two vertices lie on one edge and the other two lie on the edge diametrically opposite to it, or there is not a single edge with identically coloured ends). The cases of 5, 6, 7, and 8 white vertices reduce to the earlier analyzed cases by interchanging the colours. Altogether we get $1 + 1 + 3 + 3 + 7 + 3 + 3 + 1 + 1 = 23$ colour patterns.

312.

A cube has 11 developments (Fig. 40). The first six solutions yield those developments in which four faces of the cube are located in one strip of the development. The four succeeding developments are those which have three faces in one strip but no fourth faces. And, finally, in the last solution there are no three faces in any one strip.

313.

There are only 4 ways of painting it. For the proof, see Hugo Steinhaus's *Sto Zadach* (One Hundred Problems), Panstwowe Wydawnictwo Naukowe (Poland), Problem No. 40.

314.

See Steinhaus's book, Problem No. 44 (in Problem 313).

315.

We will first prove that after the lapse of 2^n units of time there are only two particles left, located respectively at points with coordinates 2^n and -2^n. When $n = 1$, this assertion is obvious. Assume that it has been proved for $n = k$. During the subsequent $2^k - 1$ steps these particles

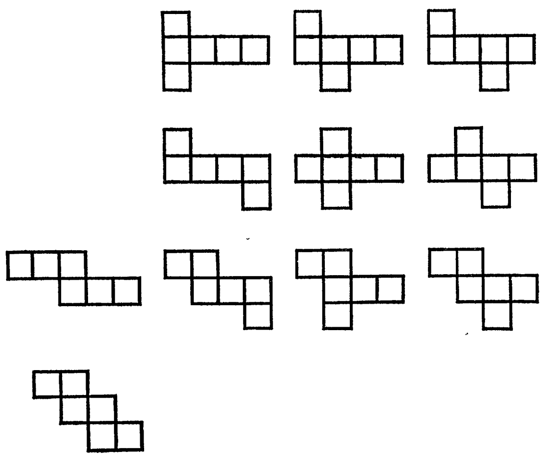


Fig. 40.

do not interact, and, by the induction hypothesis, each of them will yield (after 2^k steps) only two particles distant leftwards and rightwards 2^k. In other words, we get one particle at point 2^{k+1} and two at point 0, and one at point -2^{k+1}. The particles at point 0 annihilate each other, leaving two particles, which proves our assertion.

Thus, after 128 steps there remain two particles at points with coordinates 128 and -128. After 129 steps we get four particles at points 129, 127, -127, and -129.

If $n = 2^{k_1} + 2^{k_2} + \ldots + 2^{k_s}$, $k_1 > k_2 > \ldots > k_s$, then we get 2^s particles whose coordinates are of the form $\pm 2^{k_1} \pm 2^{k_2} \pm \ldots \pm 2^{k_s}$ (any combinations of signs are admissible). This assertion is easily proved by induction on s: for $s = 1$ it has already been demonstrated. Let it already be proved for $s < m$, and $n = 2^{k_1} + 2^{k_2} + \ldots + 2^{k_m}$. Then after the $(n - 2^{k_m})$th step we get 2^{m-1} particles located at points with coordinates $\pm 2^{k_1} \pm 2^{k_2} \pm \ldots \pm 2^{k_{m-1}}$. The distance between the nearest particles is not less than $2^{k_{m-1}+1}$. Therefore, in the course of $2^{k_{m-1}} - 1$ steps, particles generated by various "centres" do not interact, and after 2^{k_m} steps each centre yields two particles at a distance of $\pm 2^{k_m}$ from it. In other words, we obtain particles at points of the form $\pm 2^{k_1} \pm \ldots \pm 2^{k_m}$. The proof is complete.

316.

To decode the word, it suffices to indicate the gaps (blanks) between characters which are initial for letters coded with two characters. These blanks have to be chosen from among 11, no two being adjacent (we have a total of 13 blanks if we include the initial and terminal ones, but from the statement of the problem it is clear that we cannot take either the terminal blank or the one preceding it). If the word contains p "binary" letters, then we have to choose p blanks. This can be done in C^p_{12-p} ways. We therefore get

$$C^0_{12} + C^1_{11} + C^2_{10} + C^3_9 + C^4_8 + C^5_7 + C^6_6 = 233$$

ways of reading the given word.

317.

The number of p-digit numbers without any ones is $8 \times 9^{p-1}$. Hence between 1 and 10,000,000 there are

$$8\,(1 + 9 + 9^2 + 9^3 + 9^4 + 9^5 + 9^6) = 9^7 - 1 = 4{,}782{,}968$$

numbers whose representations do not contain 1. This is less than half of 10^7.

318.

Let us consider the first three characters of each word. These signs form at most $2^3 = 8$ combinations. We will show that each combination of these characters is associated with at most two words. In other words, we will show that if three words have the first three characters in common, then at least two of them have another two characters in common. Indeed, construct a table consisting of the last four characters of each of these three words. In each column there are at least two coincident characters. Since the number of pairs composed of three words is 3 and the number of columns is four, it follows that coincidence will occur in two columns for at least two words. But this means that the words have another two coincident characters, making a total of 5 coincident characters, which is a contradiction.

Thus, to every combination of the first three characters there correspond at most two words.

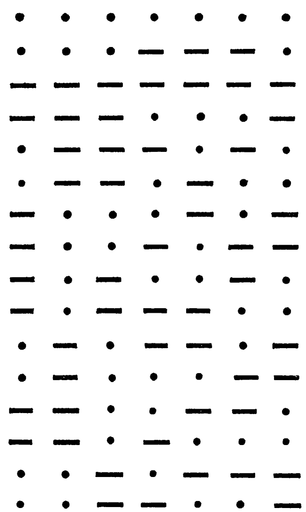


Fig. 41.

And so the total number of words does not exceed 16. Fig. 41 illustrates the 16 words that satisfy the indicated requirement.

319.

Since p is prime, only one-colour patterns go into themselves under rotations of the circle. There are n such patterns. The remaining patterns (of which there are $n^p - n$) fall into classes of p colour patterns in each, patterns of one class going into one another upon rotation of the circle. Therefore, they yield $\frac{n^p - n}{p}$ ways of painting the circle. In all, we have $\frac{n^p - n}{p} + n$ ways. In solving this problem, we proved the so-called lesser theorem of Fermat: if p is prime, then for any integer n, the number $n^p - n$ is divisible by p.

320.

Since 1 is the least of the given numbers, it must occupy the corner; 2 must stand next to it in the same vertical or horizontal line. Then the numbers $1, 2, \ldots, n$ will occur on a single vertical line or horizontal line. The smallest of the remaining numbers is $n + 1$. It must occupy a position next to 1. Continuing to reason in this fashion, we see that the numbers are arranged in a unique manner. But to begin with we can put 1 in any one of the corners of the board and we can choose either a horizontal line or a vertical line for the sequence $1, 2, \ldots, n$. We thus get 8 arrangements.

321.

Otherwise the population of Moscow would exceed 9,300,000.

322.

If we choose an odd number of objects, the number of objects left is even.

323.

The number of ways of changing 1 rouble using 2- and 5-copeck coins is equal to the number of nonnegative integral solutions of the equation $2x + 5y = 100$.

It is clear that y can assume any even value from 0 to 20. If we use 5- and 3-copeck pieces, we have to solve the equation $3x + 5y = 100$. Here, y can only assume the values 2, 5, 8, 11, 14, 17, and 20, the number of which is less than 21.

324.

It is necessary to find the number of nonnegative integral solutions to the equation $x + 2y + 5z = 20$ or, what is the same thing, to the inequality $2y + 5z \leqslant 20$. Clearly, z can assume only the values 0, 1, 2, 3, and 4, to which correspond 11, 8, 6, 3, and 21 possible values of y, making a total of 49 solutions.

325.

Since $3 = 2 + 1$, $4 = 2 + 2$, $6 = 5 + 1$, $7 = 5 + 2$, $8 = 5 + 2 + 1$, $9 = 5 + 2 + 2$, we can make up any integral weight from 1 to 9 mg using the indicated weights. In the same way we make up weights expressed in tens, hundreds, etc. of milligrams.

326.

The mean value of the last digit is 2, of the second and third, 2 and 5, and of the first, 3. The total number of numbers is $5 \times 6 \times 6 \times 3 = 540$. Therefore their sum is $540\ (3{,}000 + 250 + 25 + 2) = 1{,}769{,}580$.

327.

The assertion is obvious for $r = 1$: after the first step, the card in the pth position, $p \leqslant n$, goes to position $2p$, and for $p > n$, to position $2p - 2n - 1$. In both cases, the number label of the new position is the remainder left from dividing $2p$ by $2n + 1$. Let our assertion be proved for r, that is, let a card labelled p occupy position x in r steps, where $2^r p = k\,(2n + 1) + x$. At the next step it will take up position y, where $2x = l\,(2n + 1) + y$, $l = 0$ or 1. But then

$$2^{r+1}p = 2k\,(2n+1) + 2x = (2k + l)\,(2n+1) + y$$

where $y < 2^{r+1}p$. This means that y is the remainder after dividing $2^{r+1}p$ by $2n + 1$. Our assertion follows by the principle of mathematical induction.

328.

The answer follows directly from the result of Problem 327.

329.

The answer follows from Problem 327.

330.

Indeed, in this case the remainder left after dividing $2^x p$ by $2n + 1$ is p.

331.

In fact, after the card labelled $2n$ there come $2n - 1$ cards with even numbers; they will lie above the card $2n$.

332.

The assertion concerning card No. 8 follows from the result of Problem 331. The verification of the others is straight forward.

333.

Under the number of each card write the number it will have after the indicated shuffle:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 9 & 8 & 10 & 7 & 11 & 6 & 12 & 5 & 13 & 4 & 14 & 3 & 15 & 2 & 16 & 1 \end{matrix} \quad (*)$$

From this array it is evident that, for instance, in the first shuffle, 1 goes into 9, in the next one, 9 goes into 13, then 13 into 15, 15 into 16, and, finally, 16 goes into 1. This can be depicted as a cycle (1, 9, 13, 15, 16, 1). The whole permutation breaks up into such cycles. Besides the above-indicated cycle, we also have the cycles (2, 8, 5, 11, 14, 2), (3, 10, 4, 7, 12, 3) and a cycle consisting of the number 6 alone. Each cycle consists of one or 5 distinct numbers, and so all the cards will regain their original positions after 5 shuffles. The other cases are analyzed in similar fashion.

334.

In the first row we can arrange the colours in any order (24 ways); then in the first column we can arrange three colours, distinct from the colour of the corner square, in any manner (6 ways). Suppose the colours chosen are as shown in the accompanying table. Since all colours have to be represented in vertical and horizontal lines, the second row can have one of the following combinations: black, white, blue, red; black, red, blue, white; black, blue, white, red. In the first of these versions, the colour pattern of the cells of the second vertical column is determined in unique fashion, and there are two remaining possibilities for colouring the remaining 4 cells. Each of the remaining two versions leads only to one possible pattern. In all we get $4! \times 3! \times (2 + 1 + 1) = 576$ colour pattern possibilities.

W	B	R	Bl
B	W	Bl	R
R	Bl	W	B
Bl	R	B	W

335.

Divide the children into triples in some way. Three nonordered pairs can be chosen from each triple: (say, from the triple *abc* we can choose the pairs *ab*, *ac*, *bc*). This mode of partition embraces a total of 15 pairs, none of which can be found in any other mode of partition. But out of 15 children we can make $C_{15}^2 = 105$ pairs. Therefore the number of distinct modes of partition cannot exceed $105:15 = 7$. The following table shows that the value 7 is attained (this means that the children can be divided up into triples in the indicated manner in the course of 7 days):

klo	*ino*	*jmo*	*ilm*	*jln*	*ijk*	*kmn*
iab	*jac*	*lad*	*nae*	*kaf*	*mag*	*oah*
ncd	*mdb*	*kbc*	*ocg*	*mch*	*lce*	*icf*
mef	*keg*	*ieh*	*jfb*	*obe*	*ofd*	*jde*
jgh	*lhf*	*nfg*	*khd*	*idg*	*nhb*	*lbg*

336.

The number $\frac{(n^2)!}{(n!)^{n+1}}$ is equal to the number of ways of choosing from n^2 objects n unordered groups of n objects each and is therefore an integer. Another integer is the number $\frac{(mn)!}{(m!)^n n!}$, it is the number of ways of splitting mn objects into n unordered groups of m objects in each group. For the same reason, $\frac{(mn)!}{(n!)^m m!}$ is an integer. But then $\left[\frac{(mn)!}{(m!)^{\frac{n+1}{2}}(n!)^{\frac{m+1}{2}}}\right]^2$ is integral because it is the product of two integers. Since m and n are odd, it follows that $\frac{(mn)!}{(m!)^{\frac{n+1}{2}}(n!)^{\frac{m+1}{2}}}$ is a rational number the square of which is an integer. Hence, the number itself is also an integer.

337.

See page 48.

338.

This number is equal to the coefficient of x^m in the polynomial

$$(x^l+x^{l+1}+\ldots+x^n)^p = x^{lp}(1-x^{n-l+1})^p(1-x)^{-p}$$

Applying the binomial theorem, we find this coefficient to be

$$C_{m-(l-1)p-1}^{m} - C_p^1 C_{m-(l-1)(p-1)-n-1}^{m+l-n-1} + C_p^2 C_{m-(l-1)(p-2)-2n-1}^{m+2(l-n-1)} + \ldots$$

339.

Denote by x, y, z the number of books of the first, second and third type received by the first participant. By hypothesis, $x+y+z=12$, $0 \leqslant x \leqslant 7$, $0 \leqslant y \leqslant 8$, $0 \leqslant z \leqslant 9$. The number of integral solutions to the equation that satisfy the given inequalities is equal to the coefficient of t^{12} in the expansion of the product

$$(1+t+\ldots+t^7)(1+t+\ldots+t^8)(1+t+\ldots+t^9)$$

This product may be rewritten as

$$\frac{(1-t^8)(1-t^9)(1-t^{10})}{(1-t)^3} =$$
$$=(1-t^8-t^9-t^{10}+t^{17}+\ldots)\times$$
$$\times(1+3t+6t^2+10t^3+15t^4+\ldots+91t^{12}+\ldots)$$

Removing the brackets, we clearly find the coefficient of t^{12} to be 60. And so the distribution can be accomplished in 60 ways.

340.

The number of all n-combinations (with repetitions) of n letters is C_{2n-1}^n, hence, they include nC_{2n-1}^n letters. Since all letters occur the same number of times, each letter occurs C_{2n-1}^n times.

341.

The sum of the numbers written on a pole is 999. Therefore, if both numbers are three-digit numbers and one is of the form $\overline{abc}$, then the other has the form $\overline{9-a,\ 9-b,\ 9-c}$. But if one is single-digit or two-digit, then the other begins with the digit 9, and such poles have a, $\overline{99\,(9-a)}$ or $\overline{ab}$, $\overline{9\,(9-a)\,(9-b)}$. Since the poles can only have two distinct digits, it follows that either $a=b=c$, or two numbers out of a, b, c coincide, and the third one is the complement of these numbers in forming the number 9. The number of numbers of the first type is 10 (111, 222, ..., 999 and 0). Now each number of the second type is determined by choosing a pair of distinct digits of a three-digit number which includes both of these digits. The pair of distinct digits can be chosen in $C_{10}^2 = 45$ ways. To each pair there correspond 6 three-digit numbers (for instance, 221, 212, 122, 112, 121, 211). Therefore, the total number of numbers of the second kind is $6 \times 45 = 270$, and we have a total of 280 poles with numbers in which only two distinct digits appear.

342.

The required number is $\sum_{p=0}^{m} \sum_{q=0}^{n} P(p, q) - 1$. (We omit the empty permutation). Since $\sum_{q=0}^{n} P(p, q) = P(p+1, q)$, it follows that

$$\sum_{p=0}^{m} \sum_{q=0}^{n} P(p, q) - 1 = \sum_{p=0}^{m} P(p+1, n) - 1$$

$$= P(m+1, n+1) - 2$$

343.

By virtue of the preceding problem, the number of permutations containing exactly k white balls is $P(k+1, n+1) - P(k, n+1)$. Therefore, white balls occur $\sum_{k=1}^{m} k[P(k+1, n+1) - P(k, n+1)]$ times. This expression can be transformed as follows:

$$\sum_{k=1}^{m} kP(k+1, n+1) - \sum_{k=0}^{m-1} (k+1) P(k+1, n+1)$$

$$= mP(m+1, n+1) - \sum_{k=0}^{m-1} P(k+1, n+1)$$

$$= mP(m+1, n+1) - P(m, n+2) + 1$$

$$= 1 + \frac{mn+m-1}{n+2} P(m+1, n+1)$$

The proof of the assertion concerning black balls is similar.

344.

The desired number is equal to the sum

$$\sum_{p=0}^{m} \sum_{q=0}^{n} (p+q+1) P(p, q)$$

But

$$\sum_{q=0}^{n} (p+q+1) P(p, q) = (p+1) \sum_{q=0}^{n} P(p, q)$$

$$+ \sum_{q=1}^{n} qP(p, q)$$

$$= (p+1) [P(p+1, n) + \sum_{q=1}^{n} P(p+1, q-1)]$$

$$= (p+1) [P(p+1, n) + P(p+2, n-1)] =$$

$$= (p+1) P(p+2, n)$$

And so the sum is

$$\sum_{p=0}^{m} (p+1) P(p+2, n) = \sum_{p=0}^{m} (p+2) P(p+2, n) -$$

$$- \sum_{p=0}^{m} P(p+2, n)$$

$$= (n+1) P(m+1, n+2) - P(m+2, n+1) + 1$$

$$= 1 + \frac{mn+m+n}{m+n+4} P(m+2, n+2)$$

345.

Summing the results obtained in Problems 342 and 344, we arrive at the desired result.

346.

The total number of pairs that can be formed out of 7 persons is $C_7^2 = 21$. There are 3 pairs—(a, b), (a, c) and (b, c) — in each triple (a, b, c). Therefore, in the course of 7 days all the pairs will appear once each time. Since 21 persons will dine in the course of 7 days, each friend will come 3 times, and hence will appear in three triples.

First choose the triples that include the first friend. This can be done in $\frac{6!}{(2!)^3 3!}$ ways (the number of ways of splitting 6 into 3 pairs). When these triples have been chosen, there remain two possible choices of triples that include the second guest (say, if the first appears in the triples 1, 2, 3; 1, 4, 5; 1, 6, 7, then the second will appear either in the triples 2, 4, 6; 2, 5, 7 or in the triples 2, 4, 7; 2, 5, 6). After this, the distribution of the other guests is determined uniquely. Taking into account the permutations of triples of guests, we have

$$\frac{6!}{(2!)^3\,3!}\times 2\times 7!=151{,}200 \text{ ways.}$$

347.

$C_7^3=35$ triples can be made up out of 7 persons, $C_6^3=20$ out of 6, $C_5^3=10$ out of five and $C_4^3=4$ out of four. Hence, the total number of ways of inviting them is A_{35}^7. In $7A_{20}^7$ cases one friend is not invited and in $21A_{10}^7$ cases, two friends are left out. Applying the inclusion and exclusion formula, we get the desired result.

348.

If one of the friends comes every day, then the others can form $C_6^2=15$ pairs. And so the total number of get-togethers in which one and the same person participates is $7A_{15}^7$. There remain $A_{35}^7-7A_{15}^7$ modes of invitation.

349.

The permutations can consist of 1, 2, . . ., n objects. Thus, the total number of permutations is

$$A_n^n+A_n^{n-1}+\ldots+A_n=n!+\frac{n!}{1!}+$$
$$+\frac{n!}{2!}+\ldots+\frac{n!}{(n-1)!}$$
$$=n!\left[2+\frac{1}{2!}+\ldots+\frac{1}{(n-1)!}\right]$$

On the other hand,

$$en!-1=n!\left(2+\frac{1}{2!}+\ldots+\frac{1}{(n-1)!}\right)$$
$$+\left[\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\right.$$
$$\left.+\frac{1}{(n+1)(n+2)(n+3)}+\ldots\right] \qquad (*)$$

But for any positive integer $n\geqslant 2$,

$$\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+$$
$$+\frac{1}{(n+1)(n+2)(n+3)}+\ldots$$
$$<\frac{1}{n+1}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}+\ldots=\frac{1}{n}$$

Therefore the expression in square brackets in formula (*) is less than $\frac{1}{2}$. This completes the proof of our assertion.

350.

The total number of objects in all permutations is

$$nA_n^n+(n-1)A_n^{n-1}+\ldots+A_n^1=$$
$$=n!\left[n+\frac{n-1}{1!}+\ldots+\frac{1}{(n-1)!}\right]=$$
$$=(n-1)\,n!\left[\left(1+1+\frac{1}{2!}+\ldots+\frac{1}{(n-1)!}\right)\right.$$
$$\left.+\frac{1}{n-1}\left(1-\frac{1}{2!}-\frac{2}{3!}-\ldots-\frac{n-2}{(n-1)!}\right)\right]$$

It is easy to verify that

$$1-\frac{1}{2!}-\frac{2}{3!}-\frac{3}{4!}-\ldots-\frac{n-2}{(n-1)!}=\frac{1}{(n-1)!}$$

Since all objects occur the same number of times, each occurs

$$N=(n-1)(n-1)!\left[\left(1+1+\frac{1}{2!}+\ldots\right.\right.$$
$$\left.\left.\ldots+\frac{1}{(n-1)!}\right)\right]+1$$

times.

On the other hand,

$$(n-1)(n-1)!\,e=(n-1)(n-1)!\times$$
$$\times\left[1+1+\frac{1}{2!}+\ldots+\frac{1}{(n-1)!}+\frac{1}{n!}+\right.$$
$$\left.+\frac{1}{(n+1)!}+\ldots\right]$$
$$=(n-1)(n-1)!\left[1+1+\frac{1}{2!}+\ldots+\frac{1}{(n-1)!}\right]$$
$$+(n-1)\left[\frac{1}{n}+\frac{1}{n(n+1)}+\ldots\right]$$

and so

$$N-(n-1)(n-1)!\,e=$$
$$=1-(n-1)\left[\frac{1}{n}+\frac{1}{n(n+1)}+\ldots\right]$$
$$=\frac{1}{n}\left[1-\frac{1}{n+1}-\frac{1}{(n+1)(n+2)}-\ldots\right]<\frac{1}{2}$$

Hence, N is the closest integer to $(n-1)(n-1)!\,e$.

351.

See page 50.

352.

One of the three receives n books. These n books can be chosen in C_{3n}^{n} ways. The remaining $2n$ books are then distributed to the remaining two persons. Each of the books goes either to one or to the other, and so the number of distribution modes of these books is 2^{2n}. Since n books can be given to any one of three persons, we get $3 \times 2^{2n}C_{3n}^{n}$ ways of distributing them.

353.

The number of distinct orders in which k given pairs of letters are not upset is $2^k(2n-k)!$. These k pairs may be chosen in C_n^k ways. Applying the principle of inclusion and exclusion, we get the desired result.

354.

The number of ways of splitting up things so that the k given persons do not receive a single thing is $(n+p-k)^r$. Applying the principle of inclusion and exclusion yields the required result.

355.

$r!\,\Pi_n^r$ is equal to the number of ways of dividing n distinct objects into r cells. This number is equal to the coefficient of x^n in the expansion of $(e^x-1)^r$ multiplied by $n!$. Whence it follows that

$$n!\,[1-\Pi_n^2+2!\,\Pi_n^3-3!\,\Pi_n^4+\ldots]$$

is the coefficient of x^n in the expansion of the sum of the series

$$(e^x-1)-\frac{1}{2}(e^x-1)^2+\frac{1}{3}(e^x-1)^3-$$
$$-\frac{1}{4}(e^x-1)^4+\ldots$$

Since

$$x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\ldots=\ln(1+x) \qquad (*)$$

the sum of this series is equal to $\ln[1+(e^x-1)]=x$. Therefore, when $n>1$, expression (*) is zero.

356.

In one way from the first cell, in C_{2n}^{n} ways from the second, . . ., and in C_{kn}^{n} ways from the kth. This yields a total of

$$C_{2n}^{n}C_{3n}^{n}\ldots C_{mn}^{n}=\frac{(mn)!}{(n!)^m}$$

ways.

357.

We have to prove the inequality

$$C_{2n+r}^{n}C_{2n-r}^{n}\leqslant(C_{2n}^{n})^2$$

It can be rewritten as

$$\frac{(2n+r)(2n+r-1)\ldots(2n+1)}{(n+r)(n+r-1)\ldots(n+1)}\leqslant$$
$$\leqslant\frac{2n(2n-1)\ldots(2n-r+1)}{n(n-1)\ldots(n-r+1)}$$

This inequality follows from the fact that for $0\leqslant k<n$ we have $\dfrac{2n+k}{n+k}<\dfrac{2n-k}{n-k}$.

358.

Compute the sum of the angles of all resulting triangles. The sum of the angles having a vertex in one of the interior points is equal to 360°. Since there are 500 such points, they are associated with angles whose sum is $360°\times500$. Now consider the angles whose vertices coincide with the vertices of the 1,000-gon. Their sum is equal to the sum of the interior angles of the 1,000-gon, or $180°\times998$. We have $180°\times1{,}998$ in all. Since the sum of the angles of a triangle is 180°, we get, 1,998 triangles.

359.

Each of the players plays 4 games; 5 games will be played altogether. Suppose that in the first game the pair (a, c) played against the pair (b, d). Then in the next three games, a will have partners b, d, e, respectively, and will not participate in the fifth. Player e will take part in all games, except the first, and in the second and third he will be opposite a. In the second game, the vacant place may be filled either by player c or player d, and in the third game, by b or c. But if in the second game we choose d, then in the third we have to choose c (otherwise c will have passed up two games), and then in the fourth game d will be absent and b and c will be partners. But then in the fifth game, b and e will be partners, on the one hand, and c and d, on the other. But if in the second game we choose c, then we will have to take c in the third as well (otherwise e

and c will be partners twice); in the fourth, c and d, and in the fifth, b and c, will play against d and e. Thus, each choice for the players of the first game determines two possible divisions of the players in subsequent events. Since the order of the subsequent 4 games can be arranged in 24 ways, we get a total of 48 possibilities. For the first game we can choose the players in 15 ways (the number of ways of dividing 5 persons into 2 pairs and one reserve player). Each of these modes determines 48 possibilities for subsequent events, a total of 720 possibilities. If the order of the games is disregarded we have 6 possibilities left.

360.

The number of closed polygonal lines is $(C_{2n}^{n})^2$ (see page 84).

361.

Each polygonal line is specified by the coordinates of its vertices. These coordinates form a finite sequence of the form

$$(a_1, b_1), (a_1, b_2), (a_2, b_2), (a_2, b_3), \ldots, (a_n, b_1)$$

or

$$(a_1, b_1), (a_2, b_1), (a_2, b_2), (a_3, b_2), \ldots, (a_1, b_n)$$

These sequences are determined by specifying the permutations $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ and indicating to which of the two types the sequence belongs. Since a cyclic permutation of the coordinates does not alter the polygonal line, the number of such lines is $\frac{(n!)^2}{2n}$.

362.

Divide the rattles into classes, putting in the mth class the rattles for which the smallest number of blue balls between two red balls is m. For $m = 0$ we have 4 types of rattles (the third red ball is adjacent to two others or is separated from them by one, two, or three blue balls). When $m = 1$ we have two red balls separated by a blue ball. The third red ball can be separated from the nearest red one by one, two, or three blue balls. And so there are three kinds of rattles for $m = 1$. For $m = 2$ we have only one kind of rattle. The total is 8 types of rattles.

363.

Suppose that someone in the company, call him X, has m acquaintances $a_1, \ldots, a_m$. By hypothesis, no two persons from among $a_1, \ldots, a_m$ are acquainted (since they are acquaintances of X). And so for any two persons a_i, a_j there must be another common acquaintance other than X. This person cannot be acquainted with X, and different pairs are associated with different persons [if somebody were a common acquaintance of two distinct pairs (a_i, a_j) and (a_k, a_l), then he and X would have at least three common acquaintances]. Thus, the number of persons not acquainted with X is not less than the number of all couples from among $a_1, \ldots, a_m$, that is to say, not less than C_m^2.

On the other hand, every person not acquainted with X has exactly two acquaintances in common with him, from among $a_1, \ldots, a_m$, naturally. Here, different people are associated with different pairs [if one pair (a_i, a_j) corresponded to two different persons, then a_i and a_j would have more than two common acquaintances, since they are acquainted with X as well]. From this it follows that the number of people who do not know X does not exceed C_m^2 and so is equal to $C_m^2 = \frac{m(m-1)}{2}$. But then the total number n of people present is $1 + m + \frac{m(m-1)}{2}$.

Regarding the equation $n = 1 + m + \frac{m(m-1)}{2}$ as a quadratic equation in m, we see that it has only one positive root, and this means that the number m of acquaintances is the same for all the people.

364.

A verification shows that an interchange of two adjacent letters A and B does not alter the product (it suffices to consider the combinations $AABA$, $BABB$ and $AABB$). We can therefore assume that first come all the letters A and then all the B's. But then the assertion becomes obvious.

365.

There is one rook on each vertical line and on each horizontal line. Therefore, each of the numbers a, b, c, d, e, f, g, h and also each of the numbers 1, 2, 3, 4, 5, 6, 7, 8 will appear in the product exactly once. The product is therefore $8!\,abcdefgh$.

366.

Suppose 5 members of the organizing committee are assembled. By hypothesis, there must be at least one lock they cannot open, and each of the other 6 members have a key to this lock. Since this occurs in the case of any combination of 5 members, we see the total number of locks to be equal to $C_{11}^5 = 462$. Since there are six keys to each lock, the total number of keys is $462 \times \times 6 = 2{,}772$, and each member of the committee has $2{,}772:11 = 252$ keys.

If there were n members and the number of members necessary and sufficient to open the safe were m, then the number of locks would be C_n^{m-1}, while the number of keys in the hands of each member of the committee would be $\frac{n-m+1}{n} C_n^{m-1}$.

367.

Let us first find out what the largest length of the chain is such that after k links have been opened it is possible to obtain any weight from 1 to n. We consider the optimal arrangement of split links. Since the number of split links is k, we can, using them, obtain any weight from 1 to k. But we will not be able to obtain the weight $k+1$ if we lack one more portion. Clearly, the best way out is for this part to consist of $k+1$ links, then we can obtain any weight from 1 to $2k+1$. After that we will need parts weighing $2(k+1)$, $4(k+1)$, ..., $2^k(k+1)$. They can be used to yield any weight from 1 to

$$n = k + [(k+1) + 2(k+1) + \\ + 4(k+1) + \ldots + 2^k(k+1)] = k + \\ + (k+1)(2^{k+1} - 1) = 2^{k+1}(k+1) - 1$$

Thus, if $2^k k \leqslant n < 2^{k+1}(k+1)$, then we can get by with k opened links, but not with $k-1$ opened links. In particular, since $2^3 \times 3 \leqslant \leqslant 60 \leqslant 2^4 \times 4 - 1$, it will be required, in a chain of 60 links, to open 3, getting pieces weighing 4, 8, 16 and 29 grams.

Using a two-pan balance, it is necessary to adjoin to the k opened links a piece of weight $2k+1$ (putting it in one pan and the other links in the other, we can obtain any weight from $k+1$ to $2k$, and putting it together with the other links, we get any weight from $2k+1$ to $3k+1$). The subsequent pieces must have weight $3(2k+1)$, $9(2k+1)$, ..., $3^k(2k+1)$. Using them, it is possible to obtain any weight from 1 to

$$k + [(2k+1) + 3(2k+1) + \ldots + 3^k(2k+1)] \\ = \frac{1}{2}[(2k+1)\,3^{k+1} - 1]$$

In particular, for a chain weighing 60 grams, it is necessary to open two links and obtain pieces weighing 5, 15 and 38 grams.

368.

If when x is divided by 7 we get remainders 0, 1, 2, 3, 4, 5, 6, then x^2 yields, respectively, the remainders 0, 1, 4, 2, 2, 4, 1. Therefore, $x^2 + y^2$ is divisible by 7 (and all the more so by 49) only when x and y are divisible by 7. Therefore, the number of pairs (with regard for order) is $\left[E\left(\frac{1{,}000}{7}\right)\right]^2 = 142^2 = 20{,}164$. If we disregard order, we get $\overline{C}_{142}^2 = 10{,}153$ pairs.

369.

If the given number is $10a + b$, then, combining it with its reversal (digits in reverse order), we get $11(a+b)$. Since this is a perfect square, and $2 \leqslant a + b \leqslant 18$, it follows that $a + b = 11$. We get 8 possibilities: 29, 38, 47, 56, 65, 74, 83, 92.

370.

The first three digits of the number are arbitrary, and the last digit assumes one of two values (determined by the remainder obtained in dividing the sum of the first three digits by 3). Therefore, if at some position we specify the digit, then the remaining digits may be chosen in $6^2 \times 2 = = 72$ ways. Hence, the sum of the units digits is equal to $72(1+2+3+4+5+6) = = 1{,}512$, and the sum of all numbers is $1{,}512 + + 15{,}120 + 151{,}200 + 1{,}512{,}000 = 1{,}679{,}832$.

371.

The last position can be occupied by one of the digits 0, 2, 4. If one of these digits is specified, then the second and third places can be taken by any one of six digits, and the first position by any one of the five digits 1, 2, 3, 4, 5. In all we get $3 \times 5 \times 6 \times 6 = 180$ possibilities. Hence, the sum of the units digits is $(2+4) \cdot 180 = = 1{,}080$. In the same way we find the sum of the tens digits to be $(1+2+3+4+5)\,900 = = 13{,}500$, of the hundreds digits, 135,000, and of the thousands digits, 1,620,000. Altogether we obtain a sum of 1,769,580.

372.

The equation $x + y = k$ has $k - 1$ integral solutions satisfying the condition $1 \leqslant x$, $1 \leqslant y$.

Therefore the inequality $|x| + |y| \leqslant 1{,}000$ has

$$4 \sum_{k=2}^{1,000} (k-1) = 1{,}998{,}000$$

solutions for which $|x| \neq 0$ and $|y| \neq 0$. Besides, it has 3,996 solutions for which one of the unknowns is equal to zero, and one solution is of the type $x = 0$, $y = 0$. There are 2,001,997 solutions altogether.

373.

If this point is adjoined to the vertices of any polygon not containing the point A_1, we get a polygon containing A_1. This establishes a one-to-one correspondence between the set of all polygons not containing A_1 and a subset of the polygons containing A_1. Under this correspondence, there are no polygons corresponding to triangles, one of whose vertices is A_1. There are for this reason more polygons containing A_1.

374.

In an even number of moves, the knight can reach squares of the same colour it was on originally. We will find it more convenient to turn the board 45° and depict only the squares of that colour, replacing each square by its centre point. Then

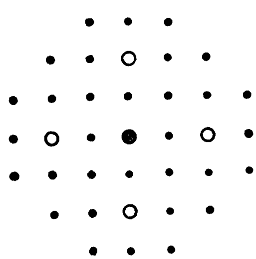


Fig. 42.

Fig. 42 gives us the squares the knight can reach in 2 moves. There are 33 such squares. Each square is the centre of the same kind of figure indicating where the knight will be in two more moves. Combining these figures, we get a display that is shown in Fig. 43. It breaks up into a square containing $9^2 = 81$ points and four trapezoids, each containing $7 + 5 = 12$ points. This yields a total of $81 + 4 \times 12 = 129$ points.

In $2n$ moves we get a display that breaks up into a square with side $4n$ which contains

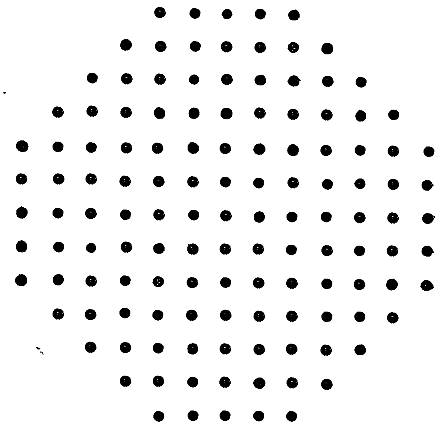


Fig. 43.

$(4n + 1)^2$ points and four trapezoids, each of which has

$$(4n-1)+(4n-3)+\ldots+(2n+1)=3n^2$$

points. We obtain a total of

$$12n^2+(4n+1)^2=28n^2+8n+1$$

points. Thus, in $2n$ moves ($n > 1$), the knight can reach any one of $28n^2 + 8n + 1$ squares.

375.

If we take triples containing one and the same element, say a, then they satisfy the required condition, and their number is $C_{1,954}^2 = 1{,}907{,}481$. We will show that it is impossible to choose a larger number of triples having, pairwise, one element in common. Suppose that we choose $N > C_{1,954}^2$ such triples and (a, b, c) is one of them. Since any one of the $N - 1$ triples left has at least one element in common with the chosen one, then for at least one of the chosen elements a, b, c (say a) there will be $\frac{N-1}{3}$ triples containing it, $\frac{N-1}{3} > 635{,}808$. There will be at most 3,906 triples containing one of the elements b or c besides a. We thus get a triple of he form

(a, d, e), where d and e are distinct from b and c. In similar fashion we find triples of the form (a, f, g) and (a, h, j), f and g being different from b, c, d, e, and h, j different from b, c, d, e, f, g.

Any one of the given N triples has at least one element in common with each of the four triples (a, b, c), (a, d, e), (a, f, g), (a, h, j). It is clear that the element a must be one of these common elements, for otherwise the triple would contain four distinct elements, which is impossible. Thus, all triples contain the element a, and for this reason their number does not exceed $C^2_{1,954}$, which contradicts the assumption.

376.

The given sequence contains $9 + 2 \times 90 + 3 \times \times 900 + \ldots + 8 \times 90{,}000{,}000 + 9$ digits. Compute the number of zeros in the sequence $1, 2, \ldots, 10^9$. Write down all the numbers from 1 to $10^9 - 1$ in the form of nine-digit numbers, appending in front the requisite number of zeros (say, 000,000, 003), and replace the number 10^9 by 000,000,000. As a result we get 9×10^9 digits, each digit appearing as many times as any other one. We thus have 9×10^8 zeros. But these zeros include the zeros we appended: 8×9 for one-digit numbers, 7×90 for two-digit numbers, etc. If they are discarded, we get $9 \times 10 - 8 \times \times 9 - 7 \times 90 - \ldots - 9 \times 10^7$ zeros left. It is easy to see that this sum is equal to $2 \times 9 + + 3 \times 90 + \ldots + 8 \times 9 \times 10^7$, which is the number of digits of the first sequence.

377.

If the sum of the first two digits is k, then for $k \leqslant 9$ we have $(k + 1)^2$ numbers with the indicated property, and for $k > 9$, $(19 - k)^2$ such numbers. Altogether, we obtain

$2(1^2 + \ldots + 9^2) + 10^2 = 670$ numbers.

378.

Denote by A_a the set of subjects in which student a gets marks of 5. All these sets contain at most $2n$ elements, and, by hypothesis, none is a part of another one. Partition these sets into classes, putting in the kth class the sets consisting of k elements. Let the smallest number of elements in the sets of our collection be r. We will show that if $r < n$, then the given collection of sets can be replaced by another one so that

(a) no set of the new collection is a subset of another set;

(b) the number of sets in the new collection is greater than in the original collection;

(c) the smallest number of elements in the sets of the new collection is $r + 1$.

To do this, take all the sets consisting of r elements and adjoin to each one of them, in all possible ways, one element not belonging to them. Leave the remaining sets of our collection unchanged. It is clear that after that operation we get a collection in which the smallest number of elements of the sets is equal to $r + 1$. And no set of the new collection is a part of another set, for if set B contained the new set A', then it would also contain the set A of the rth class from which A' was obtained by adjoining one element, but this contradicts the hypothesis. Also note that not a single one of the new sets coincides with the originally specified sets. For example, let a new set be obtained by adjoining to set A an element x. If it coincided with the originally given set B, then this would mean that B contains A, which runs counter to the assumption.

It remains to show that the number of new sets is greater than the number of original sets. Note, in this connection, that out of every set A of class r there are $2n - r$ elements not belonging to it, and therefore it gives rise to $2n - r$ new sets. But some of these sets coincide with one another [for instance, taking the sets (a, b) and (b, c), we can obtain one and the same set (a, b, c) by adjoining one element]. But the given set of $r + 1$ elements can be obtained from sets containing r elements in only $(r + 1)$ ways. For this reason, if the number of sets of the rth class were equal to m and if p distinct new sets were obtained from them, then $m(2n - r) \leqslant p(r + 1)$. Since for $r < n$ we have $2n - r > r + 1$, it follows that $m < p$, that is that the number of sets has increased.

Repeating the device described above, we can replace all sets containing less than n elements by sets composed of n elements if we retain the condition (a) and obtain a greater number of sets than originally. In the very same manner, we can replace all sets containing more than n elements (they are replaced successively by sets obtained by rejecting one element). We thus obtain a collection of sets consisting of n elements and containing more sets than the originally given collection. But using $2n$ elements it is possible to construct only C^n_{2n} sets of n elements each. Hence, the number of sets was not greater than C^n_{2n}, in other words, the school had at most C^n_{2n} students.

379.

We will call the first m elements first-grade elements, the second n elements second-grade elements. Split all permutations of $m+n$ elements, taken r at a time, into classes, putting in the kth class the permutations containing exactly k elements of the first grade. Then the kth class contains $C_r^k A_m^k A_n^{r-k}$ permutations. Indeed, we can choose the positions of the first-grade elements in C_r^k ways, and then in A_m^k ways we can fill these positions with elements of the first grade, and in A_n^{r-k} ways we can fill the remaining $r-k$ positions with elements of the second grade.

Thus, the number of permutations of $m+n$ elements taken r at a time is $\sum\limits_{k=0}^{r} C_r^k A_m^k A_n^{r-k}$ or, in accepted notation, $\sum\limits_{k=0}^{r} C_r^k M_k N_{r-k}$.

However this is nothing but the result of removing brackets in the expression $(M+N)^r$ and a subsequent replacement of exponents by indices.

Note that the number of permutations of the kth class may be counted in the following way as well: we choose k elements of the first grade, $r-k$ elements of the second grade, and permute these elements in all possible ways. This can be done in $P(k, r-k)\, A_m^k A_n^{r-k} = C_r^k A_m^k A_n^{r-k}$ ways.

380.

The exponent 8 can be built up out of the exponents 2 and 3 in the following ways: $8 = 2 + 2 + 2 + 2 = 2 + 3 + 3$. This means that if we denote x^2 in terms of y and x^3 in terms of z, then the desired coefficient is equal to the sum of the coefficients of y^4 and yz^2 in the expansion of $(1+y-z)^9$. By the formula for raising a polynomial to a power, this coefficient is equal to $P(2, 2, 2, 2, 1) + P(3, 3, 2, 1) = 378$.

381.

We have

$$(1+x)^k + \ldots + (1+x)^n = \frac{(1+x)^{n+1} - (1+x)^k}{x}$$

And so the coefficient of x^m is $C_{n+1}^{m+1} - C_k^{m+1}$ if $m < k$, and C_{n+1}^{m+1} if $m \geqslant k$.

382.

We have $17 = 7 + 5 + 5$, and 18 cannot be partitioned into a sum of positive multiples of 5 and 7. For this reason, x^{17} has the coefficient $C_{20}^1 C_{19}^2 = 3{,}420$ and x^{18} has the coefficient zero.

383.

We have $17 = 2 + 2 + 2 + 2 + 2 + 2 + 2 + 3 = 2 + 2 + 2 + 2 + 3 + 3 + 3 = 2 + 3 + 3 + 3 + 3 + 3$. And so in the expansion of $(1 + x^2 - x^3)^{1{,}000}$ the term in x^{17} has the coefficient

$$-C_{1{,}000}^7 C_{993}^1 - C_{1{,}000}^4 C_{996}^3 - C_{1{,}000}^1 C_{999}^5$$

and in the expansion of $(1 - x^2 + x^3)^{1{,}000}$ the coefficient

$$-C_{1{,}000}^7 C_{993}^1 + C_{1{,}000}^4 C_{996}^3 - C_{1{,}000}^1 C_{999}^5$$

It is clear that the second coefficient is greater.

384.

Given that

$$(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n} x^{2n} \qquad (*)$$

We will first show that $a_k = a_{2n-k}$. Put $x = \frac{1}{y}$ and multiply both sides of the equation by y^{2n}. This yields

$$(y^2 + y + 1)^n = a_0 y^{2n} + a_1 y^{2n-1} + \ldots + a_{2n} \qquad (**)$$

Comparing the expansions (*) and (**), we find that $a_k = a_{2n-k}$.

Now replace x by $-x$. This gives

$$(1 - x + x^2)^n = a_0 - a_1 x + a_2 x^2 - \ldots + a_{2n} x^{2n} \qquad (***)$$

Multiplying together the expansions (*) and (***), we get

$$(1 + x^2 + x^4)^n = \sum_{k=0}^{4n} (-1)^k (a_0 a_k - a_1 a_{k-1} + \ldots + a_k a_0)\, x^k \qquad (****)$$

Clearly, the expansion of the left member contains only terms with even powers of x and therefore the coefficient of x^{2n-1} is zero. But in the right-hand member the coefficient of x^{2n-1} is

$$-(a_0 a_{2n-1} - a_1 a_{2n-2} + a_2 a_{2n-3} - \ldots - a_{2n-1} a_0)$$
$$= -(a_0 a_1 - a_1 a_2 + a_2 a_3 - \ldots - a_{2n-1} a_{2n})$$

This completes the proof of equality (a).

Note now that the expansion (****) can be represented, by formula (*), as

$$(1+x^2+x^4)^n = a_0+a_1x^2+a_2x^4+\ldots+a_{2n}x^{4n}$$

It follows from this that the coefficient of x^{2n} in this expansion is equal to a_n. On the other hand, by formula (****), it is equal to

$$a_0a_{2n}-a_1a_{2n-1}+a_2a_{2n-2}-\ldots+a_{2n}a_0=$$

$$=2a_0^2-2a_1^2+2a_2^2-\ldots+(-1)^na_n^2$$

Whence immediately follows equality (b).

Rewrite equality (*) as

$$(1-x^3)^n=(1-x)^n(a_0+a_1x+a_2x^2+\ldots+a_{2n}x^{2n})$$

From this it follows that

$$1-C_n^1x^3+C_n^2x^6-\ldots+(-1)^nC_n^nx^{3n}$$

$$=(1-C_n^1x+C_n^2x^2-\ldots+(-1)^nC_n^nx^n)$$

$$\times(a_0+a_1x+a_2x^2+\ldots+a_{2n}x^{2n})$$

If r is not divisible by 3, then the coefficient of x^r in the left-hand member is zero. But in the right-hand member the coefficient of x^r is equal to

$$a_r-C_n^1a_{r-1}+C_n^2a_{r-2}-\ldots+(-1)^rC_n^ra_0$$

This means that this expression is zero if r is not divisible by 3, and is $(-1)^kC_n^k$ if $r=3k$. This proves relation (c).

Assuming $x=1$ in the expansion (*), we get

$$a_0+a_1+a_2+\ldots+a_{2n}=3^n$$

Putting $x=1$ in the expansion (**), we have

$$a_0-a_1+a_2-\ldots+a_{2n}=1$$

Adding and subtracting these equations, we arrive at relations (d).

385.

We have C_n^1 terms of the form x_k^3, $2C_n^2$ terms of the form $x_j^2x_k$, $j\neq k$, and C_n^3 of the form $x_ix_jx_k$, $i\neq j$, $i\neq k$, $j\neq k$, making a total of $C_n^1+2C_n^2+C_n^3$ terms.

386.

We have

$$(1+x+\ldots+x^{n-1})^2=\frac{(x^n-1)^2}{(x-1)^2}=(x^n-1)^2\times$$

$$\times(x-1)^{-2}=(x^{2n}-2x^n+1)(1+2x+3x^2+\ldots$$

$$\ldots+mx^{m-1}+\ldots)$$

And so the coefficient of x^k is equal to $k+1$ if $0\leqslant k\leqslant n-1$, and to $2n-k-1$ if $n\leqslant k\leqslant 2n-2$. The answer may be written as follows: $n-|n-k-1|$.

387.

Since $C_{n+1}^{r+1}=\frac{n+1}{r+1}C_n^r$, $C_n^r=\frac{n}{r}C_{n-1}^{r-1}$, the left-hand member of the equality may be written as

$$\frac{\frac{n}{r}\left(\frac{n+1}{r+1}-1\right)(C_{n-1}^{r-1})^2}{\left(\frac{n^2}{r^2}-\frac{(n+1)n}{(r+1)r}\right)(C_{n-1}^{r-1})^2}=\frac{\frac{n(n-r)}{r(r+1)}}{\frac{n(n-r)}{r^2(r+1)}}=r$$

388.

The number of permutations, with repetitions, of n elements taken 3 at a time is n^3. Divide these permutations into classes, putting in the kth class the permutations containing exactly k distinct types of elements. The number of permutations of the first class is equal to C_n^1, that of the second class is $6C_n^2$ (there are n choices of an element that appears twice in a permutation, $n-1$ choices of an element that appears once, and after that, three permutations of these elements), while the number of permutations of the third class is equal to $A_n^3=6C_n^3$. In all, we have $C_n^1+6C_n^2+6C_n^3$ permutations. Whence follows the first relation. To prove the second relation, divide into classes (in similar fashion) the permutations with repetitions containing at least one element of a fixed type. We see that

$$(n+1)^3-n^3=1+6C_n^1+6C_n^2$$

whence follows the relation we wish to prove.

389.

The proof is exactly the same as that of Problem 388, but here we consider permutations, with repetitions, of elements of n types taken 4 at a time.

390.

Consider the equality

$$\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^n=\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)^n=$$

$$=\cos\frac{2n\pi}{3}+i\sin\frac{2n\pi}{3}$$

By the binomial theorem we have

$$\frac{(-1)^n}{2^n}\{1+C_n^1(-i\sqrt{3})+C_n^2(-i\sqrt{3})^2+$$
$$+C_n^3(-i\sqrt{3})^3+\ldots\}=$$
$$=\frac{(-1)^n}{2^n}\{1-3C_n^2+9C_n^4-\ldots-i\sqrt{3}\times$$
$$\times[C_n^1-3C_n^3+\ldots]\}$$

Equating the real and imaginary parts of both members of the resulting equality, we obtain the required relations.

391.

Consider the identity

$$(1+x)^n=C_n^0+C_n^1x+C_n^2x^2+C_n^3x^3+\ldots+C_n^nx^n$$

and in it set successively $x=1$, ε, ε^2, where $\varepsilon=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$ and therefore $\varepsilon^2+\varepsilon+1=0$. This yields

$$2^n=C_n^0+C_n^1+C_n^2+\ldots+C_n^n,$$
$$(1+\varepsilon)^n=C_n^0+C_n^1\varepsilon+C_n^2\varepsilon^2+\ldots+C_n^n\varepsilon^n,$$
$$(1+\varepsilon^2)^n=C_n^0+C_n^1\varepsilon^2+C_n^2\varepsilon^4+\ldots+C_n^n\varepsilon^{2n}$$

But $1+\varepsilon^k+\varepsilon^{2k}=0$ if k is not divisible by 3, and $1+\varepsilon^k+\varepsilon^{2k}=3$ if k is divisible by 3. Hence

$$2^n+(1+\varepsilon)^n+(1+\varepsilon^2)^n=3(C_n^0+C_n^3+C_n^6+\ldots)$$

Since

$$1+\varepsilon=-\varepsilon^2=-\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)=$$
$$=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3},$$
$$1+\varepsilon^2=-\varepsilon=\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}$$

it follows that

$$2^n+(1+\varepsilon)^n+(1+\varepsilon^2)^n=2^n+2\cos\frac{n\pi}{3}$$

Whence it follows that

$$C_n^0+C_n^3+C_n^6+\ldots=\frac{1}{3}\left(2^n+2\cos\frac{n\pi}{3}\right)$$

The two other equalities are obtained similarly by considering the sums

$$2^n+\varepsilon(1+\varepsilon)^n+\varepsilon^2(1+\varepsilon^2)^n,\quad 2^n+\varepsilon^2(1+\varepsilon)^n+$$
$$+\varepsilon(1+\varepsilon^2)^n$$

Equality (d) is derived analogously from a consideration of the expression $(1+i)^n$.

392.

We have

$$(1+x)^n+(1-x)^n=2\sum_{k=0}^{E\left(\frac{n}{2}\right)}C_n^{2k}x^{2k}$$

The coefficients of this polynomial are positive. Therefore, the polynomial assumes its greatest value for $x=1$. This value is equal to 2^n.

393.

We have

$$\sum_{x=0}^{n}\frac{n!\,(m-x)!}{m!\,(n-x)!}=\frac{1}{C_m^n}\sum_{x=0}^{n}C_{m-x}^{m-n}=\frac{C_{m+1}^{m-n+1}}{C_m^n}=$$
$$=\frac{m+1}{m-n+1}$$

and

$$\sum_{x=0}^{n}\frac{C_n^xC_n^r}{C_{2n}^{x+r}}=\frac{n!\,C_n^r}{(2n)!}\sum_{x=0}^{n}\frac{(x+r)!\,(2n-x-r)!}{x!\,(n-x)!}$$
$$=\frac{(n!)^2}{(2n)!}\sum_{x=0}^{n}C_{x+r}^{r}C_{2n-x-r}^{n-r}=\frac{(n!)^2}{(2n)!}C_{2n+1}^{n+1}=\frac{2n+1}{n+1}$$

394.

The sum in the left-hand member reduces to

$$\sum_{k=1}^{n}C_{m+k-1}^{k}=C_{m+n}^{m}-1$$

The sum in the right member has the same value.

395.

We have

$$\sum_{x=1}^{n}\frac{C_{n-1}^{x-1}}{C_{2n-1}^{x}}=\frac{(n-1)!}{(2n-1)!}\sum_{x=1}^{n}\frac{x\,(2n-x-1)!}{(n-x)!}$$
$$=\frac{2n}{(2n-1)\,C_{2n-2}^{n-1}}\sum_{x=1}^{n}C_{2n-x-1}^{n-1}-\frac{1}{C_{2n-1}^{n-1}}\sum_{x=1}^{n}C_{2n-x}^{n}$$
$$=\frac{2nC_{2n-1}^{n-1}}{(2n-2)\,C_{2n-2}^{n-1}}-\frac{C_{2n}^{n-1}}{C_{2n-1}^{n-1}}=\frac{2}{n+1}$$

396.
We have

$$\sum_{x=1}^{n}\frac{C_{n-1}^{x-1}}{C_{n+q}^{x}}=\frac{(n-1)!}{(n+q)!}\sum_{x=1}^{n}\frac{x\,(n+q-x)!}{(n-x)!}$$

$$=\frac{(n+q+1)\,(n-1)!\,q!}{(n+q)!}\sum_{x=1}^{n}C_{n+q-x}^{n-x}-$$

$$-\frac{(n-1)!\,(q+1)!}{(n+q)!}\sum_{x=1}^{n}C_{n+q-x+1}^{n-x}$$

$$=\frac{(n+q+1)\,(n-1)!\,q!}{(n+q)!}C_{n+q}^{n-1}-\frac{(n-1)!\,(q+1)!}{(n+q)!}\times$$

$$\times C_{n+q+1}^{n-1}=\frac{n+q+1}{q+1}-\frac{n+q+1}{q+2}=$$

$$=\frac{n+q+1}{(q+1)(q+2)}$$

397.
We have

$$\sum_{x=1}^{n}\frac{C_{n-2}^{x-2}}{C_{n+q}^{x}}=\frac{(n-2)!}{(n+q)!}\sum_{x=1}^{n}\frac{x\,(x-1)\,(n+q-x)!}{(n-x)!}$$

Furthermore, using the identity

$$x\,(x-1)=(n+q-x+1)\,(n+q-x+2)$$
$$+(n+q+1)\,[n+q-2\,(n+q-x+1)]$$

we find that our sum is equal to

$$\frac{(n-2)!}{(n+q)!}\Big[(q+2)!\sum_{x=1}^{n}C_{n+q-x+2}^{n-x}-$$

$$-2\,(n+q+1)\,(q+1)!\sum_{x=1}^{n}C_{n+q-x+1}^{n-x}$$

$$+(n+q)\,(n+q+1)\,q!\sum_{x=1}^{n}C_{n+q-x}^{n-x}\Big]$$

$$=\frac{(n-2)!\,q!}{(n+q)!}\,[(q+1)\,(q+2)\,C_{n+q+2}^{n-1}$$

$$-2\,(n+q+1)\,(q+1)\,C_{n+q+1}^{n-1}+$$

$$+(n+q)\,(n+q+1)\,C_{n+q}^{n-1}]$$

Substituting the values C_{n+q+2}^{n-1}, C_{n+q+1}^{n-1}, C_{n+q}^{n-1} and performing the necessary manipulations, we get the required formula.

398.
We know that $C_{n-1}^{k-1}=\frac{k}{n}\,C_n^k$. Since

$$(1+x)^n=1+C_n^1x+\ldots+C_n^kx^k+\ldots+C_n^nx^n \qquad (*)$$

it follows that

$$n\,(1+x)^{n-1}=C_n^1+\ldots+kC_n^kx^{k-1}+\ldots+nC_n^nx^{n-1} \qquad (**)$$

[the reader who is acquainted with differential calculus can obtain this formula by termwise differentiation of both sides of equality (*)].

Multiplying the expansions (*) and (**), we get

$$n\,(1+x)^{2n-1}=(1+C_n^1x+\ldots+C_n^nx^n)\,(C_n^1+\ldots$$
$$\ldots+nC_n^nx^{n-1})$$

Comparison of coefficients of x^{n-1} in both members yields the desired relation.

399.
Consider all n combinations, with repetitions, of elements of n types. There are C_{2n-1}^{n} such elements. Divide all these combinations into classes, putting in the kth class those combinations which include elements of exactly k distinct types. The kth class has $C_n^k\,C_{n-1}^{n-k}$ combinations (in C_n^k ways we choose k types of elements appearing in the combinations of this class, and from the elements of the given k types we can form C_{n-1}^{n-k} n-combinations with repetitions that include the elements of all k types). Thus, we have

$$C_{2n-1}^{n}=\sum_{k=1}^{n}C_n^kC_{n-1}^{n-k}$$

Expressing the numbers C_{2n-1}^{n}, C_n^k, C_{n-1}^{n-k} in terms of factorials, we arrive at the desired relation.

400.
The equality to be proved may be written as follows:

$$C_{n+r-1}^{r}-C_n^1C_{n+r-3}^{r-2}+C_n^2C_{n+r-5}^{r-4}-\ldots=C_n^r$$

To prove it, take all r-combinations, with repetitions, of elements of n types and find, in two ways, the number of all such combinations consisting solely of the elements of distinct types. On the one hand, this number is C_n^r. On the other hand, the number of r-combinations, with repetitions, of elements of n kinds which at least twice include elements of the given k kinds is equal to $C_{n+r-2k-1}^{r-2k}$. Since these k kinds may be chosen in C_n^k ways, we get the desired relation by applying the principle of inclusion and exclusion.

401.

(a) Put $S_n = C_n^1 + 2C_n^2 + 3C_n^3 + \ldots + nC_n^n$. By virtue of the equality $C_n^k = C_n^{n-k}$ we have $S_n = nC_n^0 + (n-1)C_n^1 + \ldots + C_n^{n-1}$. Adding, we get

$$2S_n = n[C_n^0 + C_n^1 + \ldots + C_n^n] = 2^n n.$$

and so $S_n = 2^{n-1}n$. (b) In the same way we establish that $S_n = (n+1)2^{n-1}$. (c) $S_n = (n-2)2^{n-1} + 1$. (d) $S_n = (n+1)2^n$. (e) $S_n = 0$. (f) We have

$$S_n = 4(C_n^1 + 2C_n^2 + \ldots + nC_n^n) -$$
$$-(C_n^1 + C_n^2 + \ldots + C_n^n) = 2^{n-1}n - 2^n + 1$$

(g) We have $C_n^k = C_{n-1}^{k-1} + C_{n-1}^k$. Therefore

$$S_n = C_{n-1}^0 + C_{n-1}^1 - 2(C_{n-1}^1 + C_{n-1}^2)$$
$$+3(C_{n-1}^2 + C_{n-1}^3) - \ldots + (-1)^{n-1}nC_{n-1}^{n-1}$$
$$= C_{n-1}^0 - C_{n-1}^1 + C_{n-1}^2 - \ldots + (-1)^{n-1}C_{n-1}^{n-1}$$

This sum is equal to 1 for $n=1$ and to 0 for $n > 1$. (h) This sum is

$$S_n = \frac{1}{n+1}[C_{n+1}^1 + C_{n+1}^2 + \ldots + C_{n+1}^{n+1}] =$$
$$= \frac{2^{n+1}-1}{n+1}$$

(i) Since $C_n^k = \dfrac{(k+2)(k+1)}{(n+1)(n+2)} C_{n+2}^{k+2}$, it follows that the sum is

$$S_n = \frac{1}{(n+1)(n+2)}(C_{n+2}^2 + 2C_{n+2}^3 + \ldots$$
$$\ldots + (n+1)C_{n+2}^{n+2})$$
$$= \frac{1}{(n+1)(n+2)}[(C_{n+2}^1 + 2C_{n+2}^2 + \ldots$$
$$\ldots + (n+2)C_{n+2}^{n+2})$$
$$-(C_{n+2}^1 + \ldots + C_{n+2}^{n+2})]$$

Applying the results of Problems (a) and (b), we find that

$$S_n = \frac{1}{(n+1)(n+2)}[2^{n+1}(n+2) - 2^{n+2} + 1] =$$
$$= \frac{2^{n+1}n+1}{(n+1)(n+2)}$$

(j) Rewrite the sum as

$$S_n = \frac{1}{n+1}[C_{n+1}^1 - C_{n+1}^2 + \ldots + (-1)^n C_{n+1}^{n+1}] =$$
$$= \frac{1}{n+1}$$

since the expression in the square brackets is equal to 1. (k) If n is odd, then $S_n = 0$, and if $n = 2k$ is even, then $S_n = (-1)^k C_{2k}^k$. To prove this, multiply together the expansions $(1+x)^n$ and $(1-x)^n$, and then find the coefficient of x^n.

402.

The largest coefficient in the first expansion is the coefficient of $a^3b^3c^4$ (or $a^3b^4c^3$, $a^4b^3c^3$). It is equal to $P(3, 3, 4) = 4{,}200$. In the second expansion the largest coefficient is the coefficient $P(4, 4, 3, 3)$ of $a^3b^3c^4d^4$.

403.

The binomial formula yields

$$(1-4x)^{-\frac{1}{2}} = 1 +$$
$$+\sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(-\frac{1}{2}-n+1\right)}{n!}(-4x)^n \qquad (*)$$

And so the coefficient Y_n of x^n is equal to

$$Y_n = \frac{1 \times 3 \ldots (2n-1) \times 2^n}{n!} = \frac{(2n)!}{(n!)^2} = C_{2n}^n$$

For $(1-4x)^{\frac{1}{2}}$ we have, by the binomial theorem, the expansion

$$(1-4x)^{\frac{1}{2}}=1+$$
$$+\sum_{n=1}^{\infty}\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\dots\left(-\frac{1}{2}-n+1\right)}{n!}$$
$$\times(-4x)^n=1+\sum_{n=1}^{\infty}\frac{Y_n}{1-2n}x^n$$

But

$$\frac{Y_n}{1-2n}=-\frac{C_{2n}^{n}}{2n-1}=-\frac{(2n)!}{(n!)^2(2n-1)}$$
$$=-\frac{2}{n}\frac{(2n-2)!}{[(n-1)!]^2}=-\frac{2}{n}Y_{n-1}$$

Therefore

$$(1-4x)^{\frac{1}{2}}=1-2\sum_{n=1}^{\infty}\frac{Y_{n-1}}{n}x^n \qquad (**)$$

where we set $Y_0=1$.

404.

(a) Multiply together the expansions (*) and (**). This yields

$$1=\left(1+\sum_{n=1}^{\infty}Y_nx^n\right)\left(1-2\sum_{n=1}^{\infty}\frac{Y_{n-1}}{n}x^n\right)$$
$$=1+\sum_{n=1}^{\infty}\left[Y_n-2\left(Y_{n-1}+\frac{1}{2}Y_{n-2}Y_1+\dots\right.\right.$$
$$\left.\left.\dots+\frac{1}{n}Y_{n-1}\right)\right]x^n$$

Whence follows the desired equality.

(b) Square the expansion (*) to get

$$(1-4x)^{-1}=\left(1+\sum_{n=1}^{\infty}Y_nx^n\right)^2$$
$$=1+(Y_0Y_1+Y_1Y_0)x+$$
$$+(Y_0Y_2+Y_1Y_1+Y_2Y_0)x^2+\dots$$
$$+(Y_0Y_n+Y_1Y_{n-1}+\dots+Y_nY_0)x^n+\dots$$

Since

$$(1-4x)^{-1}=1+4x+4^2x^2+\dots+4^nx^n+\dots$$

the desired equality follows immediately.

(c) Square the expansion (**).

405.

Denote the even numbers by the letter E and the odd numbers by O. The first 4 elements of the third row have the notation OEOE, the fourth, OOEO, the fifth, OEEE, the sixth, OOOE, and the seventh, OEOE. The cycle then repeats (the first 4 elements of each row are determined by the first four elements of the preceding row). There will therefore be at least one even number in every row.

406.

We will show that each row of the triangle is an arithmetic progression, and the sum of elements equidistant from the ends is divisible by 1,958. We will argue by induction with respect to the number of the row. For the first row, the assertion is obvious. Let it be proved for the nth row. We take three adjacent elements a, $a+d$, $a+2d$ of the nth row. In the row $n+1$ they correspond to the elements $2a+d$, $2a+3d$, the difference of which is equal to $2d$. Hence, we have a progression in the $(n+1)$th row with common difference $2d$. To find the sum of elements of this row equidistant from the ends, it suffices to find the sum of the first and last elements. But if the first two elements of the nth row are equal to a and b, and its last two elements are equal to c and d, then the sum of the first element and last element of the $(n+1)$th row is equal to $(a+b)+$ $+(c+d)=2(a+d)$ and for this reason, by the induction hypothesis, is divisible by 1,958. Hence, for any row, the sum of the first and last elements is divisible by 1,958, but then this property is also possessed by the sum of the two elements of the penultimate row, that is the last element of the array.

407.

(a) We prove the equality by induction with respect to $n+m$. Let equality (a) be proved for all k and s such that $k+s<n+m$. We then have

$$u_{n+m}=u_{n+m-1}+u_{n+m-2}=u_{n-1}u_{m-1}+u_nu_m+$$
$$+u_{n-1}u_{m-2}+u_nu_{m-1}$$
$$=u_{n-1}(u_{m-1}+u_{m-2})+u_n(u_m+u_{m-1})=$$
$$=u_{n-1}u_m+u_nu_{m+1} \qquad (*)$$

Since for $m + n = 1$ the equality (*) is verified directly, it holds true for arbitrary m and n.
(b) We carry out the proof by means of induction on k. For $k = 1$ the assertion is trivial. Let it be proved that u_{km} is divisible by u_m. By (*) we have

$$u_{(k+1)m} = u_{km+m} = u_{km-1}u_m + u_{km}u_{m+1}$$

and therefore $u_{(k+1)m}$ is also divisible by u_m. By induction we infer that all u_{nm} are divisible by u_m.
(c) Let u_n and u_{n+1} be divisible by $k \neq 1$. Then also $u_{n-1} = u_{n+1} - u_n$ would be divisible by k. Continuing this reasoning, we would find that $u_1 = 1$ is divisible by k, but this is impossible.

408.

We will denote the largest common divisor of the numbers a and b by (a, b). From the equality $u_{m+n} = u_{n-1}u_m + u_n u_{m+1}$ it follows that (u_{m+n}, u_n) is a divisor of $u_{n-1}u_m$ and, since u_n and u_{n-1} are relatively prime, a divisor of u_m. Conversely, (u_m, u_n) is a divisor of u_{m+n}. Therefore, $(u_m, u_n) = (u_{m+n}, u_n)$. But then if $n = km + q$, it follows that $(u_m, u_n) = (u_m, u_q)$. Applying the Euclidean algorithm, we see that $(u_m, u_n) = u_{(m, n)}$. In particular, $(u_{1,000}, u_{770}) = u_{10} = 55$.

409.

Consider the sequence composed of the last four digits of the Fibonacci numbers. Since the number of four-digit numbers of the form 0000, 0001, . . ., 9999 is equal to 10^4, the number of pairs of such numbers is equal to 10^8. Hence, there will be two pairs (u_m, u_{m+1}) and (u_n, u_{n+1}), $n > m$, among the first 100,000,001 Fibonacci numbers such that u_m and u_n and also u_{m+1} and u_{n+1} have the same last four digits. But then the numbers $u_n - u_m$ and $u_{n+1} - u_{m+1}$ terminate in four zeros. Since

$$u_{n-1} - u_{m-1} = (u_{n+1} - u_{m+1}) - (u_n - u_m)$$

it follows that $u_{n-1} - u_{m-1}$ also terminates in four zeros. Continuing to reduce the index and taking into account that $u_0 = 0$, we find that the number u_{n-m} terminates in four zeros.

410.

Suppose the numbers $u_n, u_{n+1}, u_{n+2}, \ldots, u_{n+7}$ have been chosen; let us express them in terms of u_n and u_{n+1}:

$$u_{n+2} = u_n + u_{n+1}, \quad u_{n+3} = u_n + 2u_{n+1},$$
$$u_{n+4} = 2u_n + 3u_{n+1},$$
$$u_{n+5} = 3u_n + 5u_{n+1}, \quad u_{n+6} = 5u_n + 8u_{n+1},$$
$$u_{n+7} = 8u_n + 13u_{n+1}$$

Hence, the sum of these numbers is equal to $21u_n + 33u_{n+1}$. But $u_{n+8} = 13u_n + 21u_{n+1}$, $u_{n+9} = 21u_n + 34u_{n+1}$. From the inequality $u_{n+8} < 21u_n + 33u_{n+1} < u_{n+9}$ it is clear that $21u_n + 33u_{n+1}$ is not a Fibonacci number.

411.

Assertion (a) is proved by induction. For $n = 1$ it is obvious. Let it hold for n,

$$u_2 + u_4 + \ldots + u_{2n} = u_{2n+1} - 1$$

Add u_{2n+2} to both sides. Since $u_{2n+2} + u_{2n+1} = u_{2n+3}$, we get $u_2 + u_4 + \ldots + u_{2n+2} = u_{2n+3} - 1$, which proves our assertion. (b) is proved in exactly the same manner.

Assertion (c) is also proved by means of mathematical induction.

To prove assertion (d) note that

$$u_{n+1}^2 - u_n u_{n+2} = u_{n+1}^2 - u_n^2 - u_n u_{n+1}$$
$$= u_{n+1}(u_{n+1} - u_n) - u_n^2 = u_{n-1}u_{n+1} - u_n^2.$$

And so $u_{n+1}^2 - u_n u_{n+2} =$

$$= (-1)^n [u_1^2 - u_0 u_2] = (-1)^n.$$

Assertion (e) and (f) will be proved together. For $n = 1$ they are obvious. Let them be proved for $n = k$. By (d), we then have

$$u_1u_2 + u_2u_3 + \ldots + u_{2k}u_{2k+1} + u_{2k+1}u_{2k+2}$$
$$= u_{2k+1}^2 - 1 + u_{2k+1}u_{2k+2} = u_{2k+1}u_{2k+3} - 1 = u_{2k+2}^2$$

and

$$u_1u_2 + \ldots + u_{2k+1}u_{2k+2} + u_{2k+2}u_{2k+3}$$
$$= u_{2k+2}^2 + u_{2k+2}u_{2k+3} = u_{2k+2}u_{2k+4} = u_{2k+3}^2 - 1$$

Hence, these assertions are also true for $n = k + 1$, and thus for all n.

In proving (g), note that by virtue of (a) and (b), $u_1 + u_2 + \ldots + u_{n+1} = u_{n+3} - 1$. Therefore, if (g) holds, then

$$(n+1)u_1 + nu_2 + \ldots + 2u_n + u_{n+1}$$
$$= u_{n+4} - (n+3) + u_{n+3} - 1 = u_{n+5} - (n+4)$$

Since equation (g) is true for $n = 1$, it is also true for all n.

Relation (h) follows readily from the fact that

$$\frac{u_{3n+2} - 1}{2} + u_{3n+3} = \frac{u_{3n+5} - 1}{2}$$

To prove (i), put $m = n$ in the formula $u_{n+m} = u_{n-1}u_m + u_n u_{m+1}$. We find that $u_{2n} = u_{n-1}u_n + u_n u_{n+1} = u_{n+1}^2 - u_{n-1}^2$. The proof is the same for $u_{2n+1} = u_n^2 + u_{n+1}^2$. Putting $m = 2n$ here, we have

$$u_{3n} = u_{n-1}u_{2n} + u_n u_{2n+1}$$
$$= u_{n-1}(u_{n+1}^2 - u_{n-1}^2) + u_n(u_n^2 + u_{n+1}^2) =$$
$$= u_{n+1}^3 + u_n^3 - u_{n-1}^3$$

412.

Let $u_n \leqslant N < u_{n+1}$. Then $0 \leqslant N - u_n < u_{n-1}$ and so there is an $s < n - 1$ such that $u_s \leqslant N - u_n < u_{s+1}$. But then $0 \leqslant N - u_n - u_s < u_{s-1}$ and $s - 1 < n - 2$. In a few steps we find $N = u_n + u_s + u_p + \ldots + u_r$; the successive indices $n, s, p, \ldots, r$ differ from one another by at least 2.

413.

This number of ways is equal to the coefficient of x^s in the expansion of the expression

$$(1+x+\ldots+x^p)(1+x+\ldots+x^q)$$
$$(1+x+\ldots+x^r)$$
$$=(1-x^{p+1})(1-x^{q+1})(1-x^{r+1})(1-x)^{-3}$$
$$=(1-x^{p+1}-x^{q+1}-x^{r+1}-\ldots)(1+3x+6x^2+\ldots$$
$$\ldots+C_{n+2}^2x^n+\ldots)$$

Since $p < q+r$, it follows that $p < s$, $q < s$, $r < s$, and this coefficient is of the form

$$C_{s+2}^2 - C_{s-p+1}^2 - C_{s-q+1}^2 - C_{s-r+1}^2$$
$$= \frac{(s+2)(s+1)}{2} - \frac{(s-p+1)(s-p)}{2}$$
$$- \frac{(s-q+1)(s-q)}{2} - \frac{(s-r+1)(s-r)}{2}$$

Remove the brackets and take into account that $p + q + r = 2s$. After some manipulations we get $s^2 + s + 1 - \frac{1}{2}(p^2 + q^2 + r^2)$.

414.

If $q + r < p$, then $q < s$, $r < s$, but $p \geqslant s$ and therefore the coefficient is equal to $C_{s+r}^2 - C_{s-q+1}^2 - C_{s-r+1}^2$. Whence follows our assertion.

415.

All objects may be permuted in $(pq + r)!$ ways. Then we choose r persons out of p that get $q + 1$ objects (C_p^r ways) and distribute the objects to them in order, giving q or $q + 1$ objects, respectively. Since the result does not depend on the order of the elements in the groups, it follows that $C_p^r\,(pq + r)!$ must be divided by $(q!)^{p-r} \times [(q + 1)!]^r = (q!)^p\,(q + 1)^r$.

416.

Since $\sum_{i_0=1}^{i_1} 1 = i_1 = C_{i_1}^1$, then $\sum_{i_1=1}^{i_2}\sum_{i_0=1}^{i_1} 1 = \sum_{i_1=1}^{i_2} C_{i_1}^1 = C_{i_2+1}^2$. Furthermore, we have

$$\sum_{i_2=1}^{i_3}\sum_{i_1=1}^{i_2}\sum_{i_0=1}^{i_1} 1 = \sum_{i_2=1}^{i_3} C_{i_2+1}^2 = C_{i_3+2}^3.$$ Whence

it is clear that the sum being computed is equal to C_{n+m}^{n+1}.

417.

Divide all permutations of m white balls and n black balls into classes. Include in the class $(k_1, \ldots, k_m)$ all permutations in which k_1 is a separate white ball, k_2 pairs, k_3 triples, $\ldots$, k_m successive m white balls. Clearly $k_1 + 2k_2 + \ldots + mk_m = m$. We compute the number of permutations of the class $(k_1, \ldots, k_m)$. If n black balls are in order, then we have $n + 1$ positions for placing the white balls. Of these positions, k_1 will be occupied by one white ball, k_2 by two, $\ldots$, k_m positions by m white balls and $n - k_1 - \ldots - k_m + 1$ positions will remain vacant. Therefore the number of ways of distributing positions for the white balls, that is, the number of permutations of the class $(k_1, \ldots, k_m)$ is equal to $P(k_1, \ldots, k_m, n - k_1 - \ldots - k_m + 1)$. Since the total number of permutations of m white balls and n black balls is C_{n+m}^m, we get the desired relation.

418.

(a) Solving the characteristic equation $r^2 - 7r + 12 = 0$, we find the roots $r_1 = 3$, $r_2 = 4$. Therefore the general solution is of the form $a_n = C_1 3^n + C_2 4^n$. (b) In the same way we get $a_n = C_1 2^n + C_2(-5)^n$. (c) We have $a_n =$

$= C_1 (2+3i)^n + C_2 (2-3i)^n$, (d) $a_n = C_1 (3i)^n + C_2 (-3i)^n$. (e) $r_1 = r_2 = -2$. For this reason, $a_n = (-2)^n (C_1 + C_2 n)$. (f) The characteristic equation is $r^3 - 9r^2 + 26r - 24 = 0$. Its roots are $r_1 = 2$, $r_2 = 3$, $r_3 = 4$ and so $a_n = C_1 2^n + C_2 3^n + C_3 4^n$. (g) $r_1 = r_2 = r_3 = -1$. Therefore $a_n = (-1)^n (C_1 + C_2 n + C_3 n^2)$. (h) The characteristic equation has the form $r^4 + 4 = 0$. Its roots are $r_{1,2} = 1 \pm i$, $r_{3,4} = -1 \pm i$.

For this reason,

$$a_n = C_1 (1+i)^n + C_2 (1-i)^n + C_3 (-1+i)^n + C_4 (-1-i)^n$$

419.

(a) Solving the characteristic equation $r^2 - 5r + 6 = 0$, we get $r_1 = 2$, $r_2 = 3$ and therefore $a_n = C_1 2^n + C_2 3^n$. Putting $n = 1$ and $n = 2$, we get the system of equations

$$2C_1 + 3C_2 = 1, \quad 4C_1 + 9C_2 = -7$$

for finding C_1 and C_2. It yields $C_1 = 5$, $C_2 = -3$ and so $a_n = 5 \times 2^n - 3^{n+1}$.

(b) We have $a_n = 2^n (C_1 + C_2 n)$. Putting $n = 1$, $n = 2$, we get the system of equations $C_1 + C_2 = 1$, $C_1 + 2C_2 = 1$, from which we conclude that $C_1 = 1$, $C_2 = 0$ and so $a_n = 2^n$.

(c) $a_n = \dfrac{1}{2^{n+2}} [(-1 + i\sqrt{3})^n + (-1 - i\sqrt{3})^n]$.

(d) $a_n = 2^n + 3^n - 4^n$.

420.

The characteristic equation is of the form $r^2 - 2r \cos\alpha + 1 = 0$. Its roots are $r_{1,2} = \cos\alpha \pm i \sin\alpha$. Therefore $a_n = C_1 (\cos\alpha + i \sin\alpha)^n + C_2 (\cos\alpha - i \sin\alpha)^n$. Putting $n = 1, 2$, we get the system of equations

$$\begin{cases} (C_1 + C_2)\cos\alpha + (C_1 - C_2)\, i \sin\alpha = \cos\alpha, \\ (C_1 + C_2)\cos 2\alpha + (C_1 - C_2)\, i \sin 2\alpha = \cos 2\alpha \end{cases}$$

Whence $C_1 = C_2 = \dfrac{1}{2}$, $a_n = \dfrac{1}{2}[(\cos\alpha + i \sin\alpha)^n + (\cos\alpha - i \sin\alpha)^n]$. By De Moivre's formula, $a_n = \cos n\alpha$.

421.

This follows from the fact that the characteristic equation $r^k - C_k^1 r^{k-1} + C_k^2 r^{k-2} + \ldots + (-1)^k = 0$ may be written as $(r-1)^k = 0$. It has a root $r = 1$ of multiplicity k. Therefore, one of the solutions of the recurrence relation is $a_n = n^k$ (see page 103).

422.

$$a_n = \frac{n}{12} \times 2^n + C_1 (-4)^n + C_2 2^n.$$

423.

We have

$$(1+x)^p = 1 + C_p^1 x + C_p^2 x^2 + \ldots + C_p^m x^m + \ldots + C_p^p x^p, \qquad (*)$$

$$(1+x)^{-k-1} = 1 - C_{k+1}^1 x + C_{k+2}^2 x^2 - \ldots + (-1)^s C_{k+s}^s x^s + \ldots \qquad (**)$$

$$(1+x)^{p-k-1} = 1 - C_{p-k-1}^1 x + \ldots + (-1)^n C_{p-k-1}^n x^n + \ldots$$

Multiplying together the expansions (*) and (**), we find the coefficient of x^n. It is $\sum_s (-1)^{n-s} \times C_{k+n-s}^{n-s} C_p^s = \sum_s (-1)^s C_{k+s}^s C_p^{n-s}$. Whence immediately follows the required identity. The remaining identities up to Problem 438 inclusive are proved in the same way.

439.

The proof is by induction with respect to n.