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Selected Problems on Physics

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ПОСОБИЕ ПО ФИЗИКЕ

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Selected Problems on Physics

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Chapter 2. **Molecular Physics and Thermodynamics**

Chapter 3. **Electricity**

PREFACE TO THE FIFTH RUSSIAN EDITION

The main purpose of the book is to help those preparing for entrance examinations to engineering colleges in revising the high-school physics course and in further studies at the college.

The fourth edition of the book came out in 1981. Amendments to the physics curriculum at the high-school and polytechnic level have been incorporated as well as extra material on other branches of the physics course. The fifth edition was prepared by taking into account the modified style of problems set at the entrance examinations.

Each section begins with a brief description of the basic theory, physical laws, and formulas. This is followed by worked problems and a few descriptive problems. Exercises and questions for revision are given at the end of each section. The problems are solved according to the unified and optimal approach described in the introduction. By solving the problems, students will acquire a firm theoretical background and knowledge which will help them in their work in whichever sector of the economy they will be empoyed. The appendices contain tables required for solving problems, SI units of physical quantities, and the rules for approximate calculations.

In addition to the problems composed by the authors, this book also includes a selection of problems set for the aptitude tests and entrance examinations in physics at the N.E. Bauman Higher Technical School and other technical institutions in Moscow.

Intended for students of preparatory courses at engineering colleges, this book can also be used by high-school students, students of intermediate colleges, and those interested in self-education.

The author is indebted to Prof. A.N. Remizov and Asst. Prof. N.V. Tygliyan for their enormous help in preparing the manuscript for publication.

T.N. Osanova

INTRODUCTION

1. The Role of Physics in Comprehending Material World

Physics is one of the fundamental natural sciences. It studies the regularities of the most general forms of motion of matter.

According to Lenin, matter is a philosophical category to denote the objective reality, which is given to man in his sensations and which is reflected, photographed, and depicted by our perception, existing independent of it.

Matter exists only in motion. The physical forms of motion of matter include mechanical, molecular, electromagnetic, and nuclear motion. Moving matter exists in space and time. Hence space and time are forms of existence of matter.

The world of moving matter can be perceived. Physics plays a major role in the process of investigation of moving matter. The study of physics forms the basis of dialectic and materialistic views and helps develop the productive ability of society.

Physics is an experimental science. All studies of physical laws start with an experiment and the laws are confirmed (or disproved) by experiments. The results of new experiments refine the physical laws or define the limits of their applicability.

Physics is an exact science employing mathematical apparatus.

2. Mathematical Apparatus of Physics

ELEMENTS OF VECTOR ANALYSIS

Physical quantities can be either scalars or vectors. **Scalar quantities (scalars)** are characterized only by their numerical values. Examples of scalar quantities are time

t, mass m, temperature T, electric charge q, and potential φ. Scalars can be positive or negative and are added algebraically.

Example 1. Determine the total charge of a system consisting of $q_1 = 2$ nC, $q_2 = -7$ nC, and $q_3 = 3$ nC.

The total charge is $q = q_1 + q_2 + q_3 = (+2 \times 10^{-9} - 7 \times 10^{-9} + 3 \times 10^{-9})$ C $= -2$ nC.

Vector quantities (vectors) are characterized both by a numerical value and a direction. Examples of vectors are velocity **v**, acceleration **a**, force **F**, momentum m**v**, electric field strength **E**, magnetic induction **B**, and magnetic field strength **H**. Vector quantities are added geometrically.

Example 2. Compose two forces $F_1 = 3$ N and $F_2 = 4$ N if vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ form angles of 10° and 40° respectively with the X-axis (Fig. 1).

The composition of vectors is symbolically written as

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2.$$

The result of composition of these forces is a force **F** called the **resultant**. The vector **F** is directed along the diagonal of the parallelogram formed by vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ as its sides, its magnitude being equal to the length of this diagonal (see Fig. 1). The magnitude of vector **F** can be determined from the cosine law:

$$F = \sqrt{F_1^2 + F_2^2 + 2F_1F_2 \cos(\alpha_2 - \alpha_1)}$$
$$= \sqrt{3^2 + 4^2 + 2 \times 3 \times 4 \cos(40° - 10°)} \text{ N}$$
$$\simeq 6.8 \text{ N}.$$

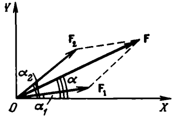


Fig. 1

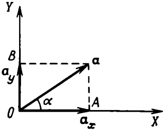


Fig. 2

The angle formed by vector **F** and the X-axis can be found from the formula

$$\alpha = \arctan \frac{F_1 \sin \alpha_1 + F_2 \sin \alpha_2}{F_1 \cos \alpha_1 + F_2 \cos \alpha_2},$$

$$\alpha = \arctan \frac{3 \times 0.17 + 4 \times 0.64}{3 \times 0.98 + 4 \times 0.77} = \arctan 0.51 \simeq 0.47 \text{ rad.}$$

The projections of vector **a** on the X- and Y-axes of a rectangular coordinate system are $a_x = a \cos \alpha$ and $a_y = a \sin \alpha$, where α is the angle formed by vector **a** and the X-axis. Knowing the projections of a vector, we can determine its magnitude and the angle with the X-axis (Fig. 2):

$$a = \sqrt{a_x^2 + a_y^2}, \quad \alpha = \arctan (a_y/a_x).$$

Multiplying a vector **A** by a positive scalar k, we obtain a new vector k**A** whose direction coincides with that of vector **A** and whose numerical value differs from that of **A** by a factor of k.

Example 3. Determine the momentum of a body of mass 2 kg, moving at a velocity of 5 m/s.

The momentum of the body $mv = 2 \times 5$ kg·m/s = 10 kg·m/s is directed along the velocity vector **v** (Fig. 3).

Multiplying a vector **A** by a negative scalar k, we obtain a new vector k**A** whose direction is opposite to that of vector **A** and whose numerical value differs from that of **A** by a factor of k.

Example 4. A charge $q = -7.5$ nC is in an electric field of strength $E = 400$ V/m. Determine the magnitude and direction of the force acting on the charge.

By definition, the force $\mathbf{F} = q\mathbf{E}$. Since the charge is negative, the force vector is directed against the vector **E** (Fig. 4). The magnitude of the force $F = |q| E = 7.5 \times 10^{-9}$ C $\times$ 400 V/m $= 3 \times 10^{-6}$ N $= 3$ μN.

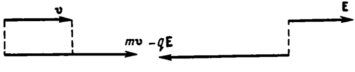


Fig. 3 Fig. 4

The **scalar product of vectors A and B** is a scalar equal to the product of the numerical values of vectors **A** and **B**, multiplied by the cosine of the angle between them:

$$C = AB\cos\alpha.$$

In symbolic form, a scalar product is written as

$$C = \mathbf{A}\cdot\mathbf{B}.$$

Example 5. Determine the work done by a constant force $F = 20$ N if the displacement of the body is $s = 7.5$ m and the angle α between the force and displacement is 120°.

By definition, the work done by a force is equal to the scalar product of the force and displacement: $A = \mathbf{Fs} = Fs\cos\alpha = 20 \times 7.5\cos 120°\ \mathrm{J} = -150 \times (1/2)\ \mathrm{J} = -75$ J.

The **vector product of vectors A and B** is defined as a vector **C** whose magnitude is equal to the product of the numerical values of **A** and **B**, multiplied by the sine of the angle between them:

$$C = AB\sin\alpha.$$

The vector **C** is normal to the plane containing the vectors **A** and **B**, its direction being connected with the directions of vectors **A** and **B** through the right-hand screw rule (Fig. 5).

The symbolic form of a vector product is

$$\mathbf{C} = \mathbf{A}\times\mathbf{B}.$$

ELEMENTS OF DIFFERENTIAL AND INTEGRAL CALCULUS

Let a function $f(x)$ exist in a certain domain x (Fig. 6). We shall use the notation $\Delta x = x_1 - x$, $\Delta f(x) = f(x_1) - f(x)$. The expression

$$\lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = f'(x) = \frac{df(x)}{dx}$$

is called the **first derivative** of the function $f(x)$ with respect to x.

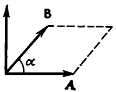


Fig. 5

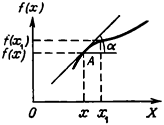


Fig. 6

Graphically, $df(x)/dx = \tan\alpha$, where α is the angle of inclination of the tangent to the curve at point A (see Fig. 6).

The derivative of the first derivative of a function is known as the **second derivative** and is denoted by $d^2f(x)/dx^2$.

Properties of Derivatives

1. If $f(x) = C\varphi(x)$, where $C = \text{const}$, then

$$\frac{df(x)}{dx} = C\frac{d\varphi(x)}{dx}.$$

2. If $f(x) = \varphi_1(x) + \varphi_2(x)$, then

$$\frac{df(x)}{dx} = \frac{d\varphi_1(x)}{dx} + \frac{d\varphi_2(x)}{dx}.$$

3. If $f(x) = \varphi_1(x)\varphi_2(x)$, then

$$\frac{df(x)}{dx} = \frac{d\varphi_1(x)}{dx}\varphi_2(x) + \frac{d\varphi_2(x)}{dx}\varphi_1(x).$$

4. If $f(x) = \frac{\varphi_1(x)}{\varphi_2(x)}$, then

$$\frac{df(x)}{dx} = \frac{\frac{d\varphi_1(x)}{dx}\varphi_2(x) - \frac{d\varphi_2(x)}{dx}\varphi_1(x)}{\varphi_2^2(x)}.$$

5. If $f(x) = f(\varphi(x))$, then

$$\frac{df(\varphi)}{dx} = \frac{df(\varphi)}{d\varphi}\,\frac{d\varphi(x)}{dx}.$$

The values of derivatives of some elementary functions are given in Table 1 of Appendices.

If $\frac{df(x)}{dx} > 0$, the function $f(x)$ increases with x.

If $\frac{df(x)}{dx} < 0$, the function $f(x)$ decreases with increasing x.

If $\frac{df(x)}{dx} = 0$, the function $f(x)$ has an extreme (minimum or maximum) value.

Let a function $f(x)$ exist in a certain range of x (Fig. 7). We divide the range ab of variation of x into

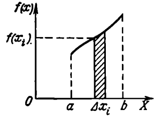


Fig. 7

infinitesimal segments $\Delta x_1, \Delta x_2, \ldots, \Delta x_i, \ldots, \Delta x_n$ and compose the sum $\sum_{i=1}^{n} f(x_i)\,\Delta x_i$. The expression

$$\lim_{\substack{\Delta x_i \to 0 \\ n \to \infty}} \sum_{i=1}^{n} f(x_i)\,\Delta x_i = \int_a^b f(x)\,dx$$

is called a **definite integral**.

An expression of the type $\int f(x)\,dx$ is called an **indefinite integral**.

Properties of Integrals

1. If $f(x) = \sum_{i=1}^{n} f_i(x)\,dx$, then

$$\int \sum_{i=1}^{n} f_i(x)\,dx = \sum_{i=1}^{n} \int f_i(x)\,dx.$$

2. If $f(x) = \alpha\varphi(x)$, then

$$\int \alpha\varphi(x)\,dx = \alpha \int \varphi(x)\,dx.$$

The values of integrals of some elementary functions are given in Table 2 of Appendices.

Procedure for Solving Problems

1. Read the formulation of a problem. Indicate the physical phenomena and processes involved.
2. Recollect the definitions of physical quantities characterizing these phenomena as well as the properties of bodies involved in them.
3. Recall the physical laws describing the phenomena encountered in the problem.
4. Clarify the physical meaning of the quantities characterizing the phenomena and processes mentioned in the problem.
5. Write down the given quantities (in SI units) and required quantities on the left-hand side.
6. Draw a diagram (plot, figure) of the problem according to adopted rules, taking into account the conditions of the problem.
7. Write the required physical laws and definitions of the physical quantities in analytic form, taking into account the conditions of the problem.
8. Write analytically the relations expressing the physical meaning of additional conditions specifying the phenomena mentioned in the problem.
9. Solve the obtained system of equations in general form to determine the required quantities.
10. Check the dimensions of the quantities in the obtained formula.
11. Calculate the values of the required quantities, using the rules for approximate calculations (see Appendices).

Chapter 1
MECHANICS

1.1. Kinematics

Kinematics studies various forms of mechanical motion of bodies without taking into account their causes.

UNIFORM RECTILINEAR MOTION

Uniform rectilinear motion is a motion in which a point mass (body) covers equal distances in equal intervals of time.

The **displacement** of a point mass (body) is a vector connecting its initial and final positions.

Any motion obeys a law defining the position of a body in space at a given instant. In order to describe a motion, we introduce a Cartesian system of coordinates XOY. Then the position of the body will be determined by its x- and y-coordinates. Consequently, the law of motion must specify the dependence of x- and y-coordinates on time t. It can be determined by calculating the projections of the displacement vector on the coordinate axes.

For a uniform motion along the X-axis, the projection of the displacement of the body on the X-axis is

$$s_x = x - x_0 = v_x t,$$

and the equation of motion has the form

$$x = x_0 + v_x t, \tag{1}$$

where x_0 is the coordinate of the body at the initial instant, v_x the projection of the velocity vector on the X-axis, and t the time of motion of the body.

The signs of the terms on the right-hand side of Eq. (1) are determined by the choice of the direction of the X-axis.

Figure 8 shows the displacement graph. It can be seen that the velocity

$$v = s_1/t_1 = \tan \alpha.$$

The velocity graph is shown in Fig. 9. It follows from the graph that the displacement $s_1 = v_1 t_1$ is numerically equal to the area of the rectangle $OABC$.

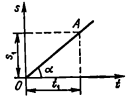


Fig. 8

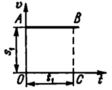


Fig. 9

* * *

1. Two bodies start to move in the same direction simultaneously from two points A and B separated by 90 m. The body starting from point A has a velocity of 5 m/s, while the body starting from point B has a velocity of 2 m/s. How long will it take the first body to catch up with the second? What will be the displacement of each body? Solve the problem analytically and graphically.

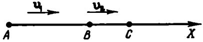


Fig. 10

Given: $x_{02} = 90$ m, $v_1 = 5$ m/s, $v_2 = 2$ m/s.

s_1 — ? s_2 — ? t_1 — ?

Solution. 1. We choose as the origin a point A of the X-axis and direct the axis along the motion of the body (Fig. 10). Then the equations of motion for the bodies will be written as follows:

$$x_1 = v_1 t, \quad x_{01} = 0, \quad x_2 = x_{02} + v_2 t, \tag{1}$$

where x_1 and x_2 are the coordinates of the first and second bodies. At point C where the first body catches up with the second, $x_1 = x_2$ and $t = t_1$. Using Eqs. (1), we can write

$$v_1 t_1 = x_{02} + v_2 t_1, \tag{2}$$

where t_1 is the time of motion of the bodies to point C. From this equation, we determine the time of motion of the bodies:

$$t_1 = \frac{x_{02}}{v_1 - v_2},$$

$$t_1 = \frac{90}{5-2} \text{ s} = 30 \text{ s}.$$

Then the displacements of the bodies are given by

$$s_1 = x_1 - x_{01} = v_1 t_1,$$
$$s_2 = x_2 - x_{02} = v_2 t_1,$$
$$s_1 = 5 \times 30 \text{ m} = 150 \text{ m},$$
$$s_2 = 2 \times 30 \text{ m} = 60 \text{ m}.$$

2. We plot the time t of motion on the abscissa axis and the values of the x-coordinate on the ordinate axis to a certain scale. Let us write the equations of motion for the bodies:

$$x_1 = v_1 t, \quad x_2 = x_{02} + v_2 t.$$

Then the time dependence of the coordinates can be represented by straight lines *1* and *2* (Fig. 11). Let us determine the coordinates of the point C of their intersection:

$t = 30$ s and $x_1 = x_2 = 150$ m. Consequently, it will take the first body 30 s to catch up with the second. The displacements of the bodies are $s_{1x} = x_1 = 150$ m and $s_{2x} = x_2 - x_{02} = 60$ m.

2. A high-speed lift in a skyscraper moves upwards uniformly at a velocity of 3 m/s. Plot the displacement graph and use it to determine the time in which the lift reaches a height of 90 m (26th floor).

Given: $v = 3$ m/s, $y = 90$ m.
t—?

Solution. Let the Y-axis coincide with the direction of motion of the lift, and the origin with the position of the

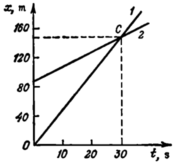


Fig. 11

lift at the initial instant. Then $y_0 = 0$ and $y = vt$. Consequently, $s = y - y_0 = vt$. In order to plot the displacement graph, we plot the time t on the abscissa axis and the displacement s on the ordinate axis (Fig. 12). The time dependence of the displacement will be presented by the straight line OA whose slope is numerically equal to the velocity v. Using the graph, we find that it will take the lift 30 s to reach the height of 90 m.

3. Using the displacement graph (Fig. 13a), plot the velocity graph and determine the nature of motion of the body relative to the X-axis (Fig. 13b).

Answer. Figure 13a shows that the body moves uniformly in the negative direction of the X-axis since the

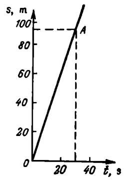


Fig. 12

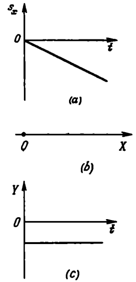


Fig. 13

x-projection s_x of the displacement vector is negative and increases in magnitude in direct proportion to time (Fig. 13c).

EXERCISES

4. A motor car moving uniformly at a velocity of 12 m/s covers the same distance in 10 s as another motor car does in 15 s. What is the velocity of the second motor car?

5. A hiker started from a point lying 2 km to the east and 1 km to the north of a crossing, took an hour to walk 5 km eastwards at an angle of 135°. Determine the final position of the hiker.

6. Using the rectangular system of coordinates, plot the displacement vector directed at 45° north-eastwards from a point lying 1 km to the east and 2 km to the north of a fork in the road. Determine the coordinates of the tip of the displacement vector whose magnitude is 25 km.

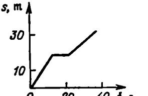


Fig. 14

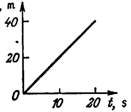


Fig. 15

7. A body has moved from a point with coordinates $x_0 = 1$ m and $y_0 = 4$ m to a point with coordinates $x_1 = 5$ m and $y_1 = 1$ m. Determine the magnitude of the displacement vector of the body and its projections on the coordinate axes.

8. Using the graph in Fig. 14, determine the nature of motion of a body.

9. Using the displacement graph in Fig. 15, plot the velocity graph.

UNIFORMLY VARYING RECTILINEAR MOTION

Uniformly varying rectilinear motion of a point mass (body) is a motion in which its velocity changes by the same value during equal time intervals. This motion can be either uniformly accelerated or uniformly decelerated.

In order to describe a uniformly varying motion, we direct the X- (or Y-)axis along the trajectory of the body. Then the equation of motion and the formula for the velocity can be written as follows:

$$x = +x_0 + v_0 t + at^2/2,$$
$$v = +v_0 + at,$$

where v_0 is the velocity of the body at the initial instant and a the acceleration. The plus or minus sign depends on the choice of the direction of the X-axis and of vectors $\mathbf{v}_0$ and $\mathbf{a}$. For freely falling bodies, the acceleration must be taken as $g = 9.8$ m/s^2.

The velocity vector of a nonuniform (in particular, of a uniformly varying) motion can also be defined as the first derivative of the displacement vector with respect to time:

$$\mathbf{v} = \frac{d\mathbf{s}}{dt}.$$

Then the x-projection of the velocity vector is $v_x = dx/dt$. The acceleration vector of a varying motion can be defined as the first derivative of the velocity vector with respect to time:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}.$$

Knowing the projections $v_x = v_x(t)$ and $a_x = a_x(t)$ of velocity and acceleration, we can determine the x-coordinate and the x-projection of the velocity vector as follows:

$$x = \int_{t_1}^{t_2} v_x \, dt, \qquad v_x = \int_{t_1}^{t_2} a_x \, dt.$$

In a varying motion, the concept of average velocity is used.

The **average velocity** of a varying motion of a body is the ratio of its displacement vector $\mathbf{s}$ to the time t during which this displacement occurs:

$$\langle \mathbf{v} \rangle = \mathbf{s}/t.$$

For a uniformly varying motion, the average velocity is defined as the arithmetic mean of the initial and final

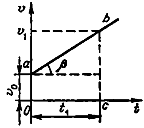


Fig. 16

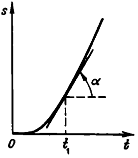


Fig. 17

velocities:

$$\langle v \rangle = (v_0 + v)/2.$$

The velocity graph for such a motion is shown in Fig. 16. It can be seen that

$$a = \frac{v_1 - v_0}{t_1} = \tan\beta,$$

and the displacement s is numerically equal to the area $Oabc$. The displacement graph for $v_0 = 0$ has the form shown in Fig. 17. The velocity of the body at a given instant t_1 is equal to the slope of the tangent to the graph at a given point:

$$v = \tan\alpha.$$

In all problems except 84, 101, and 146, the air resistance should be neglected.

* * *

10. A motor car starts to move at zero velocity and passes the first kilometre with an acceleration a_1 and the second with an acceleration a_2. Its velocity increases by 10 m/s over the first kilometre and by 5 m/s over the second kilometre. Which acceleration is higher: a_1 or a_2? Determine these accelerations.

Given: $s_1 = s_2 = 10^3$ m, $v_0 = 0$, $v_1 = 10$ m/s, $v_2 = 15$ m/s.

$a_2 \gtrless a_1$?

Solution. Let the X-axis coincide with the direction of motion of the motor car. We choose the origin at the point from which the car starts. Let us write the equation of motion and the formula for the velocity of the car:

$$x = x_0 + v_0 t + at^2/2, \quad v = v_0 + at. \tag{1}$$

For the final point of the first segment, Eqs. (1) have the form

$$s_1 = x_1 - x_{01} = a_1 t_1^2/2, \quad v_1 = a_1 t_1.$$

Solving these equations together, we find that $v_1^2 = 2a_1 s_1$, whence

$$a_1 = \frac{v_1^2}{2s_1},$$

$$a_1 = \frac{10^2}{2 \times 10^3} \frac{\text{m}}{\text{s}^2} = 5 \times 10^{-2} \text{ m/s}^2.$$

For the final point of the second segment, Eqs. (1) assume the form

$$s_2 = x_2 - x_{02} = v_1 t_2 + a_2 t_2^2/2 \text{ (since } v_{02} = v_1),$$
$$v_2 = v_1 + a_2 t_2.$$

Solving these equations together, we find that $v_2^2 - v_1^2 = 2a_2 s_2$, whence

$$a_2 = \frac{v_2^2 - v_1^2}{2s_2},$$
$$a_2 = \frac{15^2 - 10^2}{2 \times 10^3} \frac{\text{m}}{\text{s}^2} = 6.25 \times 10^{-2} \text{ m/s}^2.$$

Consequently, $a_2 > a_1$.

11. Two cyclists move towards each other. One of them, whose velocity is 18 km/h, decelerates uniformly at a rate of 20 cm/s², while the other, whose velocity is

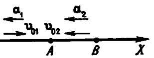


Fig. 18

5.4 km/h, accelerates uniformly at a rate of 0.2 m/s². How long will it take them to meet and what will be the displacements of the two cyclists when they meet if the initial separation between them is 130 m?

Given: $v_{01} = 18$ km/h $= 5$ m/s, $a_1 = 20$ cm/s² $=$ 0.2 m/s²,
$v_{02} = 5.4$ km/h $= 1.5$ m/s, $a_2 = 0.2$ m/s²,
$x_{02} = 130$ m.

s_1 — ? s_2 — ? τ — ?

Solution. We direct the X-axis along the motion of the first cyclist and choose the point O at which it is at the moment $t = 0$ as the origin (Fig. 18). Then the equations of motion for the cyclists will be

$$x_1 = v_{01}t - a_1t^2/2 \quad (x_{01} = 0), \quad x_2 = x_{02} - v_{02}t - a_2t^2/2. \tag{1}$$

At the moment of meeting ($t = \tau$) at point A,

$$x_1 = x_2. \tag{2}$$

Substituting expressions (1) into (2), we obtain $v_{01}\tau - a_1\tau^2/2 = x_{02} - v_{02}\tau - a_2\tau^2/2$, whence

$$\tau = \frac{x_{02}}{v_{01}+v_{02}},$$

$$\tau = \frac{130}{5+1.5}\ \mathrm{s} = 20\ \mathrm{s},$$

$$s_1 = x_1 - x_{01} = v_{01}\tau - \frac{a_1\tau^2}{2},$$

$$s_1 = 5\times 20 - \frac{0.2\times 20^2}{2}\ \mathrm{m} = 60\ \mathrm{m},$$

$$s_2 = |x_2 - x_{02}| = v_{02}\tau + \frac{a_2\tau^2}{2},$$

$$s_2 = 1.5\times 20 + \frac{0.2\times 20^2}{2}\ \mathrm{m} = 70\ \mathrm{m}.$$

12. A body moving with a uniform acceleration from the state of rest covers 90 cm during the fifth second following the onset of motion. Determine the displacement of the body during the seventh second.

Given: $s_5 = 90\ \mathrm{cm} = 0.9\ \mathrm{m}$.

$s_7 - ?$

Solution. We direct the X-axis along the trajectory of the body and choose the origin O at the point from which

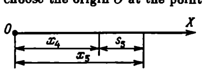


Fig. 19

the body starts to move (Fig. 19). According to the equations of motion, we then have $x_4 = at_4^2/2$ and $x_5 = at_5^2/2$, where $t_4 = 4$ s and $t_5 = 5$ s. Consequently, the displacement of the body during the fifth second is $s_5 = x_5 - x_4 = a\,(t_5^2 - t_4^2)/2$, whence

$$a = 2s_5/(t_5^2 - t_4^2). \tag{1}$$

Similarly, $s_7 = x_7 - x_6 = a(t_7^2 - t_6^2)/2$, where $t_7 = 7$ s and $t_6 = 6$ s. Taking Eq. (1) into account, we obtain

$$s_7 = \frac{s_5(t_7^2 - t_6^2)}{t_5^2 - t_4^2} = \frac{s_5(t_7 + t_6)(t_7 - t_6)}{(t_5 + t_4)(t_5 - t_4)},$$

$$s_7 = \frac{0.9\,(7+6)\,(7-6)}{(5+4)\,(5-4)}\ \mathrm{m} = 1.3\ \mathrm{m}.$$

13°. The equation of motion for a body has the form $x = 15t + 0.4t^2$. Determine the initial velocity and acceleration of the body, and also its coordinate and velocity in 5 s.

Given: $t = 5$ s.

a — ? v_0 — ? x — ? v — ?

Solution. Method 1. Let us compare the given equation of motion for the body with the equation of motion in general form:

$$x = x_0 + v_0 t + at^2/2, \quad x = 15t + 0.4t^2. \qquad (1)$$

Obviously, $x_0 = 0$, and the coefficients of t and t^2 are $v_0 = 15$ m/s and $a/2 = 0.4$ m/s^2, whence $a = 0.8$ m/s^2. The coordinate of the body in 5 s can be determined from Eq. (1):

$$x = (15 \times 5 + 0.4 \times 5^2)\ \mathrm{m} = 85\ \mathrm{m}.$$

The velocity in 5 s can be determined from the formula

$$v = v_0 + at,$$

$$v = (15 + 0.8 \times 5)\ \mathrm{m/s} = 19\ \mathrm{m/s}.$$

Method 2. The coordinate x at $t = 5$ s can be found from Eq. (1). By definition, the velocity is

$$v = \frac{dx}{dt} = \frac{d}{dt}(15t + 0.4t^2) = 15 + 0.8t,$$

$$v = (15 + 0.8 \times 5)\ \mathrm{m/s} = 19\ \mathrm{m/s},$$

and the acceleration is

$$a = \frac{dv}{dt} = \frac{d}{dt}(15 + 0.8t) = 0.8\ \mathrm{m/s^2}.$$

14. A load is dropped from a helicopter flying at an altitude of 300 m. How long will it take the load to reach the ground if the helicopter (1) is stationary, (2) descends at a velocity of 5 m/s, (3) ascends at a velocity of 5 m/s?

Given: $y_0 = 300$ m, $v_0 = 5$ m/s.

t—?

Solution. We direct the Y-axis vertically downwards and choose the origin O at an altitude y_0 above the ground (Fig. 20).

1. If the helicopter is stationary, the equation of motion for the load has the form

$$y = gt^2/2. \quad (1)$$

When the load touches the ground ($t = t_1$, $y = y_0$), Eq. (1) assumes the form $y_0 = gt_1^2/2$, whence the time of fall of the load to the ground is

$$t_1 = \sqrt{\frac{2y_0}{g}},$$

$$t_1 = \sqrt{\frac{2\times 300}{9.8}}\ \text{s} = 7.8\ \text{s}.$$

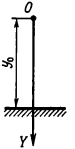


Fig. 20

2. Since the load descended with the helicopter at a velocity v_0 before being dropped, the equation of motion for the load has the form

$$y = v_0 t + gt^2/2. \quad (2)$$

When the load touches the ground ($t = t_2$, $y = y_0$), Eq. (2) assumes the form $y_0 = v_0 t_2 + gt_2^2/2$, whence

$$t_2^2 + 2v_0 t_2/g - 2y_0/g = 0.$$

Solving this equation for t_2, we obtain

$$t_2 = \frac{-v_0 \pm \sqrt{v_0^2 + 2gy_0}}{g},$$

$$t_2 = \frac{-5 \pm \sqrt{5^2 + 2\times 9.8\times 300}}{9.8}\ \text{s} \simeq (-0.5 \pm 7.8)\ \text{s}.$$

Consequently, $t_2 \simeq 7.3$ s (the negative root is rejected since $t > 0$).

3. Since the load ascended with the helicopter at a velocity v_0 before being dropped, the equation of motion

for the load has the form

$$y = -v_0 t + gt^2/2. \quad (3)$$

When the load touches the ground ($t = t_3$, $y = y_0$), Eq. (3) assumes the form $y_0 = -v_0 t_3 + gt_3^2/2$, whence

$$t_3^2 - \frac{2v_0 t_3}{g} - \frac{2y_0}{g} = 0.$$

Solving this equation for t_3, we obtain

$$t_3 = \frac{v_0 \pm \sqrt{v_0^2 + 2gy_0}}{g},$$

$$t_3 = \frac{5 \pm \sqrt{5^2 + 2 \times 9.8 \times 300}}{9.8} \text{ s} \simeq (0.5 \pm 7.8) \text{ s}.$$

Rejecting the negative root, we obtain $t_3 \simeq 8.3$ s.

15. A load is thrown vertically upwards from a balloon at a velocity of 18 m/s relative to the ground, descending at a constant velocity of 2 m/s. Determine the distance between the balloon and the load at the moment when the latter reaches the highest point of its ascent. How long will it take the falling load to pass by the balloon?

Given: $v_{01} = 2$ m/s, $v_{02} = 18$ m/s.

s — ? τ — ?

Solution. We direct the Y-axis vertically upwards and choose the origin O at the point where the balloon was at the moment of separation of the load from it (Fig. 21). Then the equations of motion for the balloon and the load will be

$$y_1 = -v_{01} t, \quad y_2 = v_{02} t - gt^2/2. \quad (1)$$

The velocity of the load varies according to the law $v = v_{02} - gt$. At the highest point A of its ascent, the velocity of the load is zero: $0 = v_{02} - gt_{max}$, and hence the time of ascent of the load is $t_{max} = v_{02}/g$. The coordinate of the load at point A is

$$y_{2A} = v_{02} t_{max} - \frac{gt_{max}^2}{2} = v_{02}\frac{v_{02}}{g} - \frac{gv_{02}^2}{2g^2} = \frac{v_{02}^2}{2g}.$$

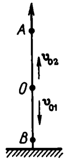


Fig. 21

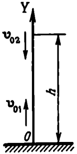


Fig. 22

During the same time t_{max}, the balloon descends to point B with the coordinate $y_{1B} = -v_{01}t_{max} = -v_{01}v_{02}/g$. The separation between points A and B is

$$s = |AB| = |AO| + |OB| = |y_{2A}| + |y_{1B}|$$

$$= \frac{v_{02}^2}{2g} + \frac{v_{01}v_{02}}{g} = \frac{v_{02}}{2g}(v_{02} + 2v_{01}),$$

$$s = \frac{18}{2 \times 9.8} \times (18 + 2 \times 2) \text{ m} \simeq 20 \text{ m}.$$

At the moment τ when the load passes by the balloon, the coordinates of the bodies will be equal: $y_1 = y_2$. Taking Eqs. (1) into account, we obtain $-v_{01}\tau = v_{02}\tau - g\tau^2/2$, whence

$$\tau = \frac{2(v_{01} + v_{02})}{g},$$

$$\tau = \frac{2(2+18)}{9.8} \text{ s} \simeq 4 \text{ s}.$$

16. A body is thrown vertically upwards at an initial velocity v_0, and another body falls freely from a height h. Determine the time dependence of the separation between the bodies if the bodies are known to start simultaneously.

Given: v_0, $y_0 = h$.

$\Delta y = f(t) - ?$

Solution. We direct the Y-axis vertically upwards and choose the origin O at the ground (Fig. 22). Since the bodies start to move simultaneously, the time of their motion is the same, and the equations of motion have the form $y_1 = v_{01}t - gt^2/2$ and $y_2 = y_0 - gt^2/2$, where y_0 is the initial coordinate of the second body.

Before one of the bodies falls, the separation between the bodies at any instant is expressed by $\Delta y = y_2 - y_1 = y_0 - gt^2/2 - v_{01}t + gt^2/2 = y_0 - v_0t$, or

$$\Delta y = h - v_{01}t.$$

17. Two bodies are thrown vertically upwards at the same initial velocity v_0 with a time interval τ. Determine the velocity of the second body relative to the first. What law describes their separation?

Given: v_0, τ.

$v - ?$ $\Delta y - ?$

Solution. We direct the Y-axis vertically upwards and choose the origin at the ground at the point from which the bodies are thrown. For the starting time we take the moment when the first body is thrown. The equations of motion for the bodies have the form $y_1 = v_0t - gt^2/2$ and $y_2 = v_0(t - \tau) - g(t - \tau)^2/2$. The separation between the bodies varies according to the law

$$\Delta y = y_1 - y_2 = v_0t - gt^2/2 - v_0(t - \tau) + g(t - \tau)^2/2$$
$$= v_0\tau + g\tau^2/2 - g\tau t.$$

In the chosen coordinate system, the velocities of the bodies vary according to the laws $v_1 = v_0 - gt$ and $v_2 = v_0 - g(t - \tau)$. Then the velocity of the second body relative to the first is

$$v = v_2 - v_1 = v_0 - g(t - \tau) - v_0 + gt = g\tau = \text{const}.$$

Therefore, the relative motion of the bodies is uniform.

18. A ball is thrown vertically upwards. It was observed at a height h twice with a time interval Δt. Determine the initial velocity of the ball.

Given: h, Δt.

$v_0 - ?$

Solution. We direct the Y-axis vertically upwards and choose the origin O at the ground (Fig. 23). The equations of motion for the ball for instants t and $t + \Delta t$ are

$$h = v_0 t - gt^2/2, \quad h = v_0 (t + \Delta t) - g (t + \Delta t)^2/2. \quad (1)$$

Solving them together, we obtain $v_0 t - gt^2/2 = v_0 t + v_0 \Delta t - gt^2/2 - gt \Delta t - g \Delta t^2/2$, whence

$$t = (2v_0 - g \Delta t)/(2g). \quad (2)$$

Substituting expression (2) into (1), we get

$$h = v_0 \left(\frac{2v_0 - g \Delta t}{2g} \right) - \frac{g}{2} \left(\frac{2v_0 - g \Delta t}{2g} \right)^2 = \frac{4v_0^2 - g^2 \Delta t^2}{8g},$$

whence

$$v_0 = \sqrt{8gh + g^2 \Delta t^2}/2.$$

19. A body falls freely from a height of 490 m. Determine the displacement of the body during the last second of its descent.

Given: $h = 490$ m, $\Delta t = 1$ s.
Δy — ?

Solution. We direct the Y-axis vertically downwards and choose the origin O at a height h above the ground (Fig. 24). Then the equation of motion for the body has

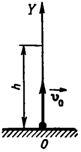


Fig. 23

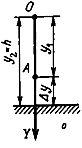


Fig. 24

Fig. 25

the form

$$y = gt^2/2. \tag{1}$$

When the body touches the ground ($t = t_2$, $y = y_2 = h$), Eq. (1) becomes $h = gt_2^2/2$, whence $t_2 = \sqrt{2h/g}$. At the moment $(t_2 - \Delta t)$, the body is at point A with the coordinate $y_1 = g\,(t_2 - \Delta t)^2/2$. Therefore,

$$\Delta y = y_2 - y_1 = \frac{g}{2}\,[t_2^2 - (t_2 - \Delta t)^2] = \frac{g\,\Delta t}{2}\,(2t_2 - \Delta t)$$

$$= \frac{g\,\Delta t}{2}\left(2\sqrt{\frac{2h}{g}} - \Delta t\right),$$

$$\Delta y = \frac{9.8\times 1}{2}\left(2\sqrt{\frac{2\times 490}{9.8}} - 1\right)\ \mathrm{m} = 93\ \mathrm{m}.$$

20°. A ball thrown vertically upwards falls to the ground in 3 s. Determine the initial velocity of the ball and the maximum height of its ascent.

Given: $t_1 = 3$ s.

v_0 — ? y_{max} — ?

Solution. We direct the Y-axis vertically upwards and choose the origin O at the ground (Fig. 25). Then the equation of motion for the ball and the formula for its velocity will be

$$y = v_0 t - gt^2/2. \tag{1}$$

$$v = v_0 - gt. \tag{2}$$

At the ground, $t = t_1$, $y = 0$, and Eq. (1) becomes $0 = v_0 t_1 - gt_1^2/2$, whence

$$v_0 = \frac{gt_1}{2},$$

$$v_0 = \frac{9.8\times 3}{2}\ \frac{\mathrm{m}}{\mathrm{s}} = 14.7\ \mathrm{m/s}.$$

At the highest point A of the ball trajectory, $t = t_{max}$, $y = y_{max}$, and $v = 0$. Let us write Eq. (2) for point A: $0 = v_0 - gt_{max}$, whence

$$t_{max} = v_0/g. \tag{3}$$

The time of ascent can be found in a different way. Since the y-coordinate at point A attains its maximum value, $dy_{max}/dt = 0$.

Using Eq. (1), we can write

$$\frac{dy_{max}}{dt}=\frac{d}{dt}\left(v_0t_{max}-\frac{gt_{max}^2}{2}\right)=v_0-gt_{max}=0,$$

whence

$$t_{max}=v_0/g.$$

Substituting Eq. (3) into (1), we obtain

$$y_{max}=v_0t_{max}-\frac{gt_{max}}{2}=v_0\frac{v_0}{g}-\frac{gv_0^2}{2g^2}=\frac{v_0^2}{2g},$$

$$y_{max}=\frac{14.7^2}{2\times 9.8}\text{ m}=11\text{ m}.$$

21. A body is thrown vertically upwards at a velocity of 4.9 m/s. Another body is thrown vertically downwards at the same initial velocity simultaneously from the maximum height that can be attained by the first body. Determine the time in which the bodies meet.

Given: $v_0=4.9$ m/s.

τ — ?

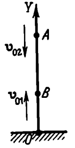


Fig. 26

Solution. We direct the Y-axis vertically upwards and choose the origin O at the ground (Fig. 26). Then the equations of motion for the first and second bodies can be written as

$$y_1=v_{01}t-gt^2/2,\quad y_2=y_0-v_{02}t-gt^2/2,$$

where $v_{01}=v_{02}=v_0$. At point B where they meet ($t=\tau$, $y_1=y_2$), we have

$$v_0\tau-g\tau^2/2=y_0-v_0\tau-g\tau^2/2,$$

whence $\tau=y_0/(2v_0)$, where y_0 is the maximum height of ascent of the first body, $y_0=y_{max}=v_0^2/(2g)$ (see Problem 20). Substituting this expression into the formula for τ, we obtain

$$\tau=\frac{v_0^2}{2g\cdot 2v_0}=\frac{v_0}{4g},$$

$$\tau=\frac{4.9}{4\times 9.8}\text{ s}\simeq 0.13\text{ s}.$$

22. A motor car moves the first half of its path at a velocity of 80 km/h and the second half at a velocity of 40 km/h. Determine the average velocity of the car.

Given: $v_1 = 80$ km/h $\simeq 22$ m/s, $v_2 = 40$ km/h $\simeq 11$ m/s.

$\langle v \rangle - ?$

Solution. Since for the uniform rectilinear motion the path length is equal to the magnitude of the displacement vector, we have

$$\langle v \rangle = s/t. \quad (1)$$

Here s is the path length of the car, $t = t_1 + t_2 = s_1/v_1 + s_2/v_2$ is the total time of motion, where s_1 and s_2 are the distances covered by the car moving during a time t_1 at a velocity v_1, and during a time t_2 at a velocity v_2 respectively. By hypothesis, $s_1 = s_2 = s/2$, and hence

$$t = s/(2v_1) + s/(2v_2). \quad (2)$$

Substituting expression (2) into (1), we obtain

$$\langle v \rangle = \frac{s}{s/2 \cdot v_1 + s/2 \cdot v_2} = \frac{2v_1v_2}{v_1 + v_2},$$

$$\langle v \rangle = \frac{2 \times 22 \times 11}{22 + 11} \frac{\text{m}}{\text{s}} \simeq 14.7 \text{ m/s}.$$

23. The cabin of a lift ascends with a constant acceleration for the first 4 s, attaining a speed of 4 m/s. The cabin moves at this velocity for 8 s, after which it moves with a constant deceleration for 3 s. Determine the displacement of the lift cabin and plot the velocity, displacement, and acceleration graphs.

Given: $\Delta t_1 = 4$ s, $\Delta t_2 = 8$ s, $\Delta t_3 = 3$ s, $v_2 = 4$ m/s.

$h - ?$

Solution. We direct the Y-axis vertically upwards and choose the origin O at the initial position of the cabin. Let us consider the motion of the cabin on three segments. The displacement corresponding to the first segment is given by

$$h_1 = \langle v_1 \rangle \Delta t_1, \quad (1)$$

where $\langle v_1 \rangle$ is the average velocity corresponding to this segment. Since the motion is uniformly accelerated, we have

$$\langle v_1 \rangle = (v_{1\text{in}} + v_{f1\text{n}})/2 = (0 + v_2)/2 = v_2/2.$$

Using this expression, we can write Eq. (1) in the form

$$h_1 = v_2 \, \Delta t_1/2.$$

The displacements corresponding to the second and third segments are given by

$$h_2 = v_2 \, \Delta t_2, \quad h_3 = \langle v_3 \rangle \, \Delta t_3.$$

Since $\langle v_3 \rangle = v_2/2$, $h_3 = v_2 \, \Delta t_3/2$. Consequently, the total displacement of the cabin is

$$h = h_1 + h_2 + h_3,$$

or

$$h = \frac{v_2 \Delta t_1}{2} + v_2 \Delta t_2 + \frac{v_2 \Delta t_3}{2} = \frac{v_2}{2} (\Delta t_1 + 2\Delta t_2 + \Delta t_3),$$

$$h = \frac{4}{2} (4 + 2 \times 8 + 3) \text{ m} = 46 \text{ m}.$$

We shall plot the velocity, displacement, and acceleration graphs for each segment separately. The complete velocity graph (Fig. 27*a*) is the broken line $OBCD$. The displacement graph (Fig. 27*b*) consists of three regions: OB' is a segment of parabola with the vertex at point O, $B'C'$ a line segment, and $C'D'$ a segment of parabola with the vertex at point D'. The acceleration graph (Fig. 27*c*) is the broken line $ABB'CC'D$.

24. A train gains speed from 36 to 54 km/h during 10 s. During the next 0.3 min, it moves at a uniform velocity. Determine the displacement and the average velocity of the train, and plot the velocity and displacement graphs.

Given: $v_0 = 36$ km/h $= 10$ m/s, $v = 54$ km/h $= 15$ m/s, $\Delta t_1 = 10$ s, $\Delta t_2 = 0.3$ min $= 18$ s.

$\langle v \rangle$ — ? s — ?

Solution. We direct the X-axis along the trajectory of motion of the train and choose the point O at which the velocity of the train is v_0 as the origin. We shall consider the motion of the train on two segments. The displacement corresponding to the first segment is given by

$$s_1 = \langle v_1 \rangle \, \Delta t_1.$$

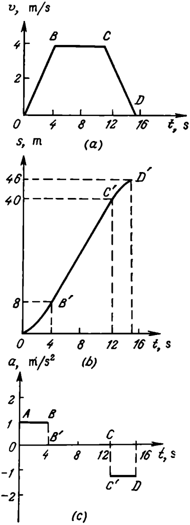

v, m/s
4
2
B
C
D
0
4
8
12
16
t, s
(a)
s, m
46
40
8
D′
C′
B′
0
4
8
12
16
t, s
(b)
a, m/s²
2
1
0
-1
-2
A
B
B′
C
C′
D
4
8
12
16
t, s
(c)

Fig. 27

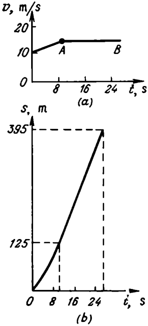

v, m/s
20
10
0
A
B
8
16
24
t, s
(a)
s, m
395
125
0
8
16
24
t, s
(b)

Fig. 28

Since the train moves on this segment with a uniform acceleration, its average velocity is $\langle v_1 \rangle = (v_0 + v)/2$. Therefore,

$$s_1 = \frac{v_0 + v}{2} \Delta t_1. \quad (1)$$

Since the motion on the second segment is uniform, we can write

$$s_2 = v \, \Delta t_2. \quad (2)$$

Consequently, the total displacement of the train is $s = s_1 + s_2$, or (see Eqs. (1) and (2))

$$s = \frac{v_0 + v}{2} \Delta t_1 + v \, \Delta t_2,$$

$$s = \left(\frac{10 + 15}{2} 10 + 15 \times 18 \right) \text{ m} = 395 \text{ m}.$$

The average velocity of the train is

$$\langle v \rangle = s/t,$$

where $t = \Delta t_1 + \Delta t_2$. Therefore,

$$\langle v \rangle = \frac{s}{\Delta t_1 + \Delta t_2},$$

$$\langle v \rangle = \frac{395 \text{ m}}{10 + 18 \text{ s}} \simeq 14.1 \text{ m/s}.$$

We construct the velocity (Fig. 28*a*) and displacement (Fig. 28*b*) graphs, considering each segment separately.

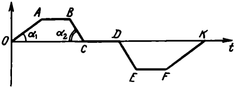


Fig. 29

25. Characterize the motion of a motorcyclist whose velocity graph is shown in Fig. 29.

Answer. The velocity graph shows that the motorcyclist starts to move from the state of rest (point O). Its motion is uniformly accelerated on segment OA, is uniform on segment AB, and is uniformly decelerated on segment BC, the magnitude of acceleration on this segment being larger than that on segment OA ($\tan \alpha_1 < \tan \alpha_2$). Segment CD corresponds to a stop. Segments DE, EF, and FK of the graph correspond to the motion of the motorcyclist in the opposite direction: uniformly accelerated (segment DE), uniform (segment EF), and uniformly decelerated (segment FK).

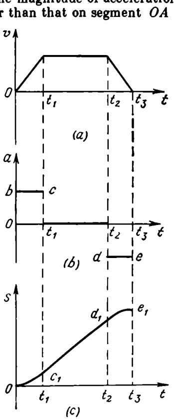


Fig. 30

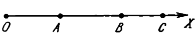


Fig. 31

26. Using the velocity graph shown in Fig. 30a, plot the acceleration and displacement graphs.

Answer. On segment Ot_1, the motion is uniformly accelerated in the positive direction of the X-axis (Fig. 31, segment OA of the trajectory). Therefore, the acceleration graph has the form of a straight line parallel to the t-axis (Fig. 30b, segment bc), and the displacement graph is a segment of parabola Oc_1 with the vertex at point O (Fig. 30c). On segment t_1t_2, the motion is uniform (Fig. 31, segment AB of the trajectory). Therefore, the acceleration graph is a line segment coinciding with the t-axis (Fig. 30b, segment t_1t_2), and the displacement graph is a straight line c_1d_1 touching the parabola at point c_1 (Fig. 30c). On segment t_2t_3, the motion is uniformly decelerated in the same direction (Fig. 31, segment BC of the trajectory). Therefore, the acceleration graph has the form of a straight line parallel to the t-axis (Fig. 30b, segment de), and the displacement graph is a segment of parabola with the vertex at point e_1, touching the straight line c_1d_1 at point d_1 (Fig. 30c).

27. Using the displacement graph consisting of two segments of parabolas (Fig. 32a), construct the velocity and acceleration graphs.

Answer. On segment Ot_1 (Fig. 32a), the motion is uniformly decelerated and directed against the X-axis (Fig. 33, segment OA of the trajectory). Therefore, the velocity is negative and decreases in magnitude (Fig. 32b, segment bc), while the acceleration is positive (Fig. 32c, segment b_1c_1). On segment t_1t_2, the motion is uniformly accelerated in the direction of the X-axis (Fig. 33, segment AO of the trajectory). Hence the velocity is positive and increases in magnitude (Fig. 32b, segment cd), while the acceleration remains the same in magnitude and direction (Fig. 32c, segment c_1d_1'). On segment t_2t_3, the

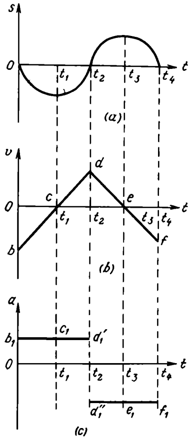


Fig. 32

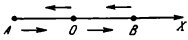


Fig. 33

motion is uniformly decelerated in the same direction (Fig. 33, segment OB of the trajectory). Therefore, the velocity is positive and decreases in magnitude (Fig. 32b, segment de), while the acceleration is negative (Fig. 32c, segment d_1e_1). On segment t_3t_4, the motion is uniformly accelerated and directed against the X-axis (Fig. 33, segment BO of the trajectory). Therefore, the velocity is negative and increases in magnitude (Fig. 32b, segment ef), while the acceleration remains the same (Fig. 32c, segment e_1f_1).

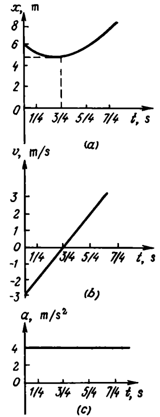


Fig 34

28°. A body moves along the X-axis according to the law $x = 6 - 3t + 2t^2$. Determine the average velocity of the body and acceleration during the time interval from 1 to 4 s. Plot the displacement, velocity, and acceleration graphs.

Given: $t_1 = 1$ s, $t_2 = 4$ s.

$\langle v \rangle$ — ? a — ?

Solution. By definition,

$$v = \frac{dx}{dt} = \frac{d}{dt}(6 - 3t + 2t^2) = -3 + 4t, \qquad (1)$$

$$\langle v \rangle = \frac{v_1 + v_2}{2}.$$

Substituting the values of t_1 and t_2 into Eq. (1), we obtain $v_1 = (-3 + 4 \times 1)$ m/s $= 1$ m/s and $v_2 = (-3 + 4 \times 4)$ m/s $= 13$ m/s. Then

$$\langle v \rangle = \frac{1+13}{2}\ \frac{\mathrm{m}}{\mathrm{s}} = 7\ \mathrm{m/s},$$

$$a = \frac{dv}{dt} = \frac{d}{dt}(-3 + 4t) = 4\ \mathrm{m/s^2} = \mathrm{const}.$$

The displacement graph is a parabola (Fig. 34a), where $x_0 = 6$ m is the initial coordinate of the body. The velocity graph is a straight line (Fig. 34b), where $v_0 = -3$ m/s is the initial velocity of the body. The acceleration graph is a straight line parallel to the abscissa axis (Fig. 34c).

29°. The velocity of a body is expressed by the formula $v = 2.5 + 0.2t$. Determine the displacement of the body 20 s after the beginning of motion.

Given: $t_1 = 0$, $t_2 = 20$ s.
$s - ?$

Solution. By definition, $s = \int_{t_1}^{t_2} v\, dt$. Then

$$s = \int_0^{20} (2.5 + 0.2t)\, dt = \left(2.5t + \frac{1}{2} \times 0.2t^2\right)\Big|_0^{20}$$

$$= 2.5 \times 20 \text{ m} + \frac{1}{2} \times 0.2\,(20)^2 \text{ m} = 90 \text{ m}.$$

EXERCISES

30. A train moving over a horizontal segment at a velocity of 36 km/h acquires a uniform acceleration over a distance of 600 m so that its velocity becomes 45 km/h at the end of the segment. Determine the acceleration and the time of the accelerated motion.

31. Two bodies start to move at the same time from the same point and in the same direction. The first body moves uniformly at a velocity of 98 m/s and the second body moves with a uniform acceleration of 980 cm/s^2, its initial velocity being zero. How long will it take the second body to catch up with the first?

32. A motor car covers 1.2 m during the second second after the beginning of motion. What is its acceleration? Determine the displacement of the car during the tenth second after the beginning of motion.

33. The equation of motion for a body is $x = 5t + 0.8t^2$. Determine the acceleration and the initial velocity of the body.

34. A ball falls from a height of 20 m on a plane surface and bounces to a height of 5 m. What is the velocity of the ball at the moment it strikes the plane surface? What is the time taken by the ball to reach the highest point from the beginning of the fall? What is the velocity of the ball at the moment when it leaves the plane surface?

35. A body falls from a height of 2000 m. How long will it take the body to traverse the last 100 m?

36. A train starts from a station with an acceleration of 20 cm/s^2. Having attained a velocity of 37 km/h, it moves uniformly for 2 min and then brakes over a distance of 100 m. Determine the average velocity of the train and plot the velocity graph.

37. A motor car moves at a velocity of 80 km/h for the first half of the duration of its motion and at a velocity of 40 km/h for the second half. Determine the average velocity of the car.

38. A point mass moves in a straight line. At a distance of 1 km from its initial position, it stops and then moves in the opposite direction over 1.2 km before it stops. What are the displacement and the total distance traversed by the point?

39. A motor car moving with a uniform acceleration covers two 100-m long adjacent segments of its path in 5 and 3.5 s respectively. Determine the acceleration and the average velocity of the car on each segment and on both segments taken together.

40. A body A is thrown vertically upwards at a velocity of 20 m/s. At what height will it collide with a body B, which was thrown horizontally at the same time at a velocity of 4 m/s? The horizontal distance between the initial

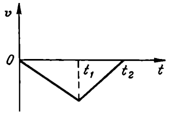


Fig. 35

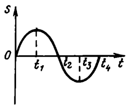


Fig. 36

positions of the bodies is 4 m. Find the time of motion of the bodies before the collision and the velocity of each body at the moment of collision.

41. Using the velocity graph shown in Fig. 35, plot the displacement and acceleration graphs and describe the type of motion of the body on various segments.

42. Using the displacement graph shown in Fig. 36, plot the velocity and acceleration graphs and describe the type of motion on each segment.

43. A box with a metal ball at the centre, which does not touch the walls of the box, falls from a certain height. Describe the motion of the ball relative to a wall of the box during falling, neglecting air resistance.

TYPES OF COMPOSITE MOTION

Uniform rectilinear motion. Any uniform motion at a constant velocity v along an arbitrary straight line AB (Fig. 37) can be decomposed into two independent uniform rectilinear motions along the X- and Y-axes at velocities v_x and v_y.

In order to describe this motion, we choose a Cartesian system of coordinates XOY. Then the equations of motion for the body can be written in the form

$$x = x_0 + v_x t,$$

$$y = y_0 + v_y t,$$

where $v_x = v \cos \alpha$ and $v_y = v \sin \alpha$.

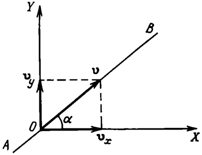


Fig. 37

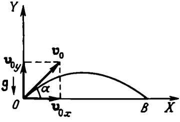


Fig. 38

The velocity of the body at any point of the trajectory is given by

$$v = \sqrt{v_x^2 + v_y^2}$$

and is directed along the trajectory.

Motion of a body thrown horizontally from a certain height. This motion can be decomposed into two independent motions: a uniform rectilinear motion in the horizontal direction at a velocity v_x equal to the initial velocity v_0 ($v_x = v_0$) and a free fall from the initial height of the body characterized by acceleration g.

In order to describe this motion, we choose a Cartesian system of coordinates XOY and direct the X-axis along the horizontal and the Y-axis vertically upwards. Then the equations of motion for the body along the X- and Y-axes will be

$$x = x_0 + v_0 t, \quad y = y_0 - gt^2/2.$$

The velocity of the body at any point of the trajectory is given by

$$v = \sqrt{v_x^2 + v_y^2}$$

and is directed along the tangent to the trajectory at the given point.

Motion of a body thrown at an angle to the horizontal. This motion can be decomposed into two independent motions: a uniform rectilinear motion in the horizontal direction at the initial velocity $v_{0x} = v_0 \cos \alpha$ and a free fall at the initial velocity $v_{0y} = v_0 \sin \alpha$, where α is the angle between the directions of the velocity vector $\mathbf{v}_0$ and the X-axis (Fig. 38).

In order to describe this motion, we choose a Cartesian system of coordinates XOY and direct the X-axis along the horizontal and the Y-axis vertically upwards. Then the equations of motion will be

$$x = x_0 + v_{0x} t, \quad y = y_0 + v_{0y} t - gt^2/2.$$

The velocity of the body at each point of the trajectory is

$$v = \sqrt{v_x^2 + v_y^2},$$

where $v_x = v_{0x}$, $v_y = v_{0y} - gt$, and is directed along the tangent to the trajectory at the given point.

The direction of the velocity vector at an arbitrary point of the trajectory is determined by the angle φ formed by the velocity vector $\mathbf{v}$ with the X-axis:

$$\tan \varphi = v_y/v_x.$$

The values of x_0 and y_0 and the signs of the terms on the right-hand sides of the above equations depend in all three cases on the choice of the origin and the directions of the X- and Y-axes.

* * *

44. A passenger on a train moving at a velocity of 40 km/h sees a 75-m long train passing from the opposite direction in 3 s. What is the velocity of the second train?

Given: $v_1 = 40$ km/h $\simeq 11$ m/s, $t = 3$ s, $l = 75$ m.

$v_2 - ?$

Solution. Let us consider the motion of the second train relative to the reference frame connected to the first train. Then the velocity of the second train relative to the chosen reference frame is $\mathbf{v} = \mathbf{v}_1' + \mathbf{v}_2$, where $\mathbf{v}_1' = -\mathbf{v}_1$ is the velocity of the ground relative to the first train and $\mathbf{v}_2$ the velocity of the second train relative to the ground. In scalar form, $v = v_1' + v_2 = v_1 + v_2$. Considering that $v = l/t$, we obtain $l/t = v_1 + v_2$, whence

$$v_2 = l/t - v_1,$$

$$v_2 = (75/3 - 11)\ \text{m/s} = 14\ \text{m/s}.$$

45. A boat floats across a river at right angles to the banks at a velocity of 2 m/s. Determine the angle to the chosen direction of the Y-axis and the velocity of the boat relative to water if the velocity of the flow is 5 km/h (see Fig. 38).

Given: $v = 2$ m/s, $v_1 = 5$ km/h $\simeq 1.4$ m/s.

$\alpha - ?$ $v_2 - ?$

Solution. Let us consider the motion of the boat relative to a bank. Then the velocity of the boat is $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1$ is the translational velocity due to the river flow and $\mathbf{v}_2$ the velocity of the boat relative to water.

Since, by hypothesis, $\mathbf{v} \perp \mathbf{v}_1$ and $\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1$ (Fig. 39), we have

$$v_2 = \sqrt{v_1^2 + v^2} = \sqrt{1.4^2 + 2^2}\ \text{m/s} \simeq 2.4\ \text{m/s}.$$

We can write $\tan \alpha = v_1/v$, whence

$$\alpha = \arctan\ (v_1/v),$$
$$\alpha = \arctan\ (1.4/2) \simeq 0.84\ \text{rad}.$$

46. What must be the velocity and direction of motion of an aeroplane to traverse 300 km northwards in 2 h if a northwesterly wind blows during the flight at an angle of 30° to the meridian at a velocity of 27 km/h?

Given: $t = 2\ \text{h} = 7.2 \times 10^3\ \text{s}$, $l = 300\ \text{km} = 3 \times 10^5\ \text{m}$, $\alpha = 30° \simeq 0.52\ \text{rad}$, $v_1 = 27\ \text{km/h} = 7.5\ \text{m/s}$.

v_2—? φ—?

Solution. Let us consider the motion of the plane in the reference frame fixed to the Earth. We direct the X-axis eastwards and the Y-axis northwards (Fig. 40). Then the velocity of the plane in the chosen reference frame is

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2,$$

or (see Fig. 40)

$$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1. \quad (1)$$

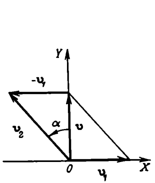


Fig. 39

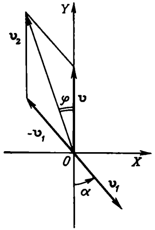


Fig. 40

From the cosine law, we calculate the velocity:

$$v_2 = \sqrt{v_1^2 + v^2 + 2v_1 v \cos\alpha},$$

where $v = l/t$. Therefore,

$$v_2 = \sqrt{v_1^2 + l^2/t^2 + 2v_1 l \cos\alpha/t},$$

$$v_2 = \sqrt{7.5^2 + \left(\frac{3\times10^5}{7.2\times10^3}\right)^2 + \frac{2\times7.5\times0.86\times3\times10^5}{7.2\times10^3}} \text{ m/s}$$

$$\simeq 48.3 \text{ m/s}.$$

Since the projection of the sum is equal to the sum of projections, we determine the projections of Eq. (1) on the X- and Y-axes:

$$v_2 \sin\varphi = v_1 \sin\alpha, \quad v_2 \cos\varphi = v_1 \cos\alpha + v, \qquad (2)$$

whence $\tan\varphi = \frac{v_1 \sin\alpha}{v_1 \cos\alpha + l/t}$, or

$$\varphi = \arctan\frac{v_1 \sin\alpha}{v_1 \cos\alpha + l/t},$$

$$\varphi = \arctan\frac{7.5\times0.5}{7.5\times0.86 + 3\times10^5/(7.2\times10^3)}$$

$$\simeq \arctan 0.078 \simeq 0.078 \text{ rad}.$$

47. An aeroplane flies along the horizontal at a velocity of 360 km/h at an altitude of 490 m. When it flies

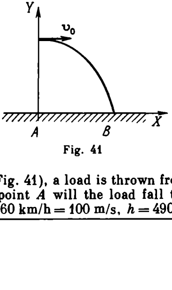


Fig. 41

over point A (Fig. 41), a load is thrown from it. At what distance from point A will the load fall to the ground?

Given: $v_0 = 360$ km/h $= 100$ m/s, $h = 490$ m.

s—?

Solution. We direct the X-axis along the horizontal and the Y-axis upwards along the vertical and choose point A as the origin (see Fig. 41). Let us write the equations of motion for the load along the X- and Y-axes:

$$x = v_0 t, \quad y = y_0 - gt^2/2, \qquad (1)$$

where $y_0 = h$. For point B where the load falls ($t = t_1$, $x = x_B$, and $y = y_B = 0$), Eqs. (1) have the form

$$x_B = v_0 t_1, \qquad (2)$$

$$0 = h - gt_1^2/2. \qquad (3)$$

The duration of motion of the load to point B can be determined from Eq. (3)

$$t_1 = \sqrt{2h/g}. \qquad (4)$$

The required distance $s = x_B$ can be calculated from Eq. (2) taking Eq. (4) into account:

$$s = v_0 t_1 = v_0 \sqrt{\frac{2h}{g}},$$

$$s = 100 \sqrt{\frac{2 \times 490}{9.8}} \text{ m} = 10^3 \text{ m}.$$

48. A jet of water is ejected from a hydraulic giant at a velocity of 50 m/s at an angle of 35° to the horizontal. Determine the horizontal range of the water jet and the maximum height the jet can reach.

Given: $v_0 = 50$ m/s, $\alpha = 35° \simeq 0.61$ rad.

h—? s—?

Solution. We choose the coordinate system with the origin at point O from which water is ejected (Fig. 42) and write the equations of motion for the water jet:

$$x = v_{0x} t, \qquad (1)$$

$$y = v_{0y} t - gt^2/2. \qquad (2)$$

The velocity of the jet varies along the Y-axis according to the law

$$v_y = v_{0y} - gt. \qquad (3)$$

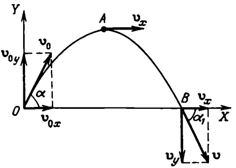


Fig. 42

For point A (the vertex of the parabola), $t = t_{max}$, $y = h$, and $v_y = 0$. Then Eq. (3) assumes the form $0 = v_{0y} - gt_{max}$, whence

$$t_{max} = v_{0y}/g. \qquad (4)$$

Using Eq. (2) for point A, we obtain $h = v_{0y}t_{max} - gt^2_{max}/2$, or taking Eq. (4) into account,

$$h = v_{0y}\frac{v_{0y}}{g} - \frac{gv^2_{0y}}{2g^2} = \frac{v^2_{0y}}{2g}.$$

Considering that $v_{0y} = v_0 \sin \alpha$, we obtain

$$h = \frac{v_0^2 \sin^2 \alpha}{2g},$$

$$h = \frac{50^2 \times 0.57^2}{2 \times 9.8} \text{ m} \simeq 41.3 \text{ m}.$$

Let us write Eq. (2) for point B at which the jet falls to the ground ($t = t_B$, $y = 0$, and $x = s$):

$$0 = v_{0y}t_B - gt^2_B/2.$$

Then the duration of motion of the jet to point B is given by

$$t_B = 2v_{0y}/g = (2v_0/g) \sin \alpha. \qquad (5)$$

We write Eq. (1) for point B:

$$s = v_{0x}t_B. \qquad (6)$$

Substituting Eq. (5) into (6) and considering that $v_{0x} = v_0 \cos\alpha$, we obtain

$$s = v_0 \cos\alpha \frac{2v_0 \sin\alpha}{g} = \frac{v_0^2 \sin 2\alpha}{g},$$

$$s = \frac{50^2 \times 0.94}{9.8} \text{ m} \simeq 240 \text{ m}.$$

49. At what angle to the horizontal must a body be thrown for its maximum height of ascent to be equal to its horizontal range?

Given: h, s.

α — ?

Solution. The height of ascent of the body (see Problem 48) is

$$h = v_0^2 \sin^2\alpha/(2g),$$

and the horizontal range is

$$s = (2v_0^2/g) \sin\alpha \cos\alpha.$$

By hypothesis, $h = s$, and hence $v_0^2 \sin^2\alpha/(2g) = 2v_0^2 \sin\alpha \cos\alpha/g$, which gives

$$\tan\alpha = 4,$$

$$\alpha = \arctan 4 \simeq 1.3 \text{ rad}.$$

50. A body is thrown at an initial velocity v_0 at an angle α to the horizontal. Determine the velocity of the body at the point of maximum ascent and at the point of its fall to the horizontal plane.

Given: v_0, α.

v_A — ? v_B — ?

Solution. Let us construct the trajectory of the body in the chosen coordinate system (see Fig. 42). At any point of the trajectory, the total velocity of the body can be determined from the formula $v = \sqrt{v_x^2 + v_y^2}$, where $v_x = v_{0x} = v_0 \cos\alpha$ and $v_y = v_{0y} - gt$ are the horizontal and vertical velocity components at the given point. At point A of maximum ascent, we have

$$v_x = v_{0x} = v_0 \cos\alpha, \quad v_y = 0.$$

Consequently, the total velocity of the body at point A is

$$v_A = \sqrt{v_x^2 + v_y^2} = \sqrt{v_x^2} = v_x = v_0 \cos\alpha.$$

The velocity vector at point A is obviously directed along the horizontal. Similarly, for point B at which the body falls to the ground, we have

$$v_x = v_{0x} = v_0 \cos\alpha, \quad v_y = v_{0y} - gt_B, \qquad (1)$$

where the duration of motion to point B can be calculated by the formula (see Problem 48)

$$t_B = (2v_0/g) \sin\alpha. \qquad (2)$$

Substituting the second formula of (1) into (2) and considering that $v_{0y} = v_0 \sin\alpha$, we obtain $v_y = v_0 \sin\alpha - g(2v_0/g) \sin\alpha = -v_0 \sin\alpha$. Consequently, the total velocity at point B is

$$v_B = \sqrt{v_x + v_y^2} = \sqrt{v_0^2 \cos^2\alpha + v_0^2 \sin^2\alpha} = v_0.$$

In order to determine the direction of the velocity vector at point B, we shall use Fig. 42 which shows that

$$\tan\alpha_1 = v_y/v_x = -v_0 \sin\alpha/(v_0 \cos\alpha) = -\tan\alpha,$$

i.e. the velocity vector at point B at which the body falls forms with the horizontal an angle α_1 equal to the angle α at which the body was thrown.

51. A body falls freely from a height of 4 m. At a height of 2 m, it strikes elastically against a small surface element fixed at an angle of 30° to the horizontal. Determine the total time of motion of the body and its horizontal range.

Given: $H = 4$ m, $h = 2$ m, $\alpha = 30° \simeq 0.52$ rad.

t — ? s — ?

Solution. We choose the coordinate system with the origin at point O (Fig. 43) and write the equation of motion for the body on the first segment AB of the trajectory:

$$y = y_0 - gt^2/2, \qquad (1)$$

where $y_0 = H$. The velocity of the body on this segment is

$$v = -gt. \qquad (2)$$

For point B ($t = t_1$, $y = h$, and $v = v_B$), Eqs. (1) and (2) become $h = H - gt_1^2/2$ and $|v_B| = gt_1$, whence

$$t_1 = \sqrt{2(H-h)/g}, \tag{3}$$

$$|v_B| = g\sqrt{2(H-h)/g} = \sqrt{2g(H-h)}. \tag{4}$$

On the second segment BD of the trajectory, the body moves along a parabola, and the equations of motion have the form

$$x = v_{B_x} t, \tag{5}$$

$$y = h + v_{B_y} t - gt^2/2, \tag{6}$$

where $v_{B_x} = v_B \cos\gamma$ and $v_{B_y} = v_B \sin\gamma$. Considering that the impact is elastic and that $\alpha = 30°$, we have

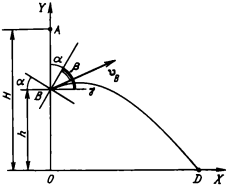


Fig. 43

$\alpha = \beta = \gamma$. Then $v_{B_x} = v_B \cos\alpha$ and $v_{B_y} = v_B \sin\alpha$, or, taking Eq. (4) into account, we obtain

$$v_{B_x} = \sqrt{2g(H-h)}\cos\alpha, \quad v_{B_y} = \sqrt{2g(H-h)}\sin\alpha. \tag{7}$$

At point D where the body falls ($t = t_2$, $y = 0$, and $x = s$), Eq. (6) becomes $0 = h + v_{B_y} t_2 - gt_2^2/2$, whence

$t_2 = \frac{v_{B_y} \pm \sqrt{v_{B_y}^2 + 2gh}}{g}$, or, taking Eq. (7) into account,

$$t_2 = \frac{\sqrt{2g(H-h)} \sin\alpha + \sqrt{2gh + 2g(H-h)\sin^2\alpha}}{g} \quad (8)$$

(the negative root is rejected). Then the total duration of motion of the body is

$$t = t_1 + t_2 = \sqrt{\frac{2(H-h)}{g}} + \frac{\sqrt{2g(H-h)}\sin\alpha + \sqrt{2gh + 2g(H-h)\sin^2\alpha}}{g},$$

$$t = \sqrt{\frac{2(4-2)}{9.8}} + \frac{\sqrt{2\times 9.8(4-2)}\times 0.5 + \sqrt{2\times 9.8\times 2 + 2\times 9.8(4-2)\times 0.25}}{9.8}\ \text{s}$$

$$\simeq 10.1\ \text{s}.$$

The horizontal range can be determined from Eq. (5), taking Eq. (8) into account:

$$s = \sqrt{2g(H-h)}\cos\alpha \times \frac{\sqrt{2g(H-h)}\sin\alpha + \sqrt{2gh + 2g(H-h)\sin^2\alpha}}{g},$$

$$s = \sqrt{2\times 9.8(4-2)} \times 0.87 \times \frac{\sqrt{2\times 9.8(4-2)}\times 0.5 + \sqrt{2\times 9.8\times 2 + 2\times 9.8(4-2)\times 0.25}}{g}\ \text{m}$$

$$\simeq 5.6\ \text{m}.$$

52. What will be the change in the time and horizontal range of a body thrown horizontally from a certain height if its initial velocity is doubled?

Answer. The horizontal range is $s = v_x t$, where the time t of motion is determined only by the height of the body above the ground. Therefore, if the height remains unchanged and the initial velocity is doubled, the horizontal range must increase by a factor of two.

53. A water jet escapes from a hose at an angle to the horizontal. Why is the ascending branch of the jet con-

tinuous, while the descending splits into fractions?

Answer. Let us consider two adjacent regions of the ascending branch. The velocity of water on any segment is lower than that on the preceding segment, which leads to a compaction of the jet. On the other hand, in the descending branch the difference in velocity on two neighbouring segments leads to a splitting of the jet.

EXERCISES

54. An aeroplane flies at a velocity of 800 km/h relative to air. A westerly wind is blowing at a velocity of 15 m/s. At what velocity will the plane fly relative to the Earth in the southward direction and at what angle to the meridian should it be routed?

55. The traces of raindrops falling vertically on the window of a motor car moving at a velocity of 45 km/h form an angle of 30° with the vertical. Determine their velocity.

56. The range of a body thrown along the horizontal at a velocity of 10 m/s is equal to the height from which the body is thrown. Determine the height.

57. What will be the change in the time of flight and the range of a body thrown along the horizontal if the height from which it is thrown increases fourfold? The initial velocity remains unchanged.

58. A shell is fired from a gun at an initial velocity of 1000 m/s at an angle of 30° to the horizontal. Determine the horizontal range and the flight time of the shell if the gun and the point where the shell falls to the ground are on the same horizontal line.

59. A body is thrown at an angle α to the horizontal at a velocity v_0 from a height h. What will be the velocity at which the body falls to the ground?

60. A stone is thrown at an angle of 30° to the horizontal at a velocity $v_0 = 5$ m/s from the top of an inclined plane which forms an angle of 36° with the horizontal. At what distance from the initial point will the stone fall?

61. At what angle to the horizontal should a body be thrown so that the maximum height of its ascent is equal to the horizontal range? Assume that a tail wind imparts a horizontal acceleration a to the body.

62. A ball is thrown at an angle of 30° to the horizontal at an initial velocity of 14 m/s. At a distance of 11 m from the initial point, the ball elastically bounces off a vertical wall. At what distance from the wall will the ball fall to the ground?

63. A carriage moves at a velocity of 72 km/h. Raindrops leave traces on the carriage window at an angle of 60° to the vertical in the absence of wind. What is the velocity of the raindrops?

64. A motor boat passes the same distance downstream and upstream in 4 and 6 h respectively. What time would it take the boat to traverse this distance in still water?

MOTION OF A POINT MASS IN A CIRCLE. ROTATIONAL MOTION OF A RIGID BODY

When a point mass moves in a circle, the direction of its linear acceleration vector does not coincide with the direction of its linear velocity vector. At any point of

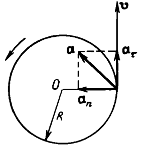


Fig. 44

the trajectory, the linear velocity **v** of the point mass is directed along the tangent to the circle, while the acceleration vector **a** can be decomposed into two: $\mathbf{a}_\tau$ and $\mathbf{a}_n$ (Fig. 44), where $\mathbf{a}_\tau$ is the component of the vector **a** along the tangent to the trajectory at this point, known as the **tangential acceleration**, and $\mathbf{a}_n$ is the component of the vector **a** along the normal (radius) to the trajectory at this point, known as the **normal (centripetal) acceleration**,

$a_n = v^2/R$. It follows from the figure that

$$a = \sqrt{a_\tau^2 + a_n^2}.$$

For a uniform motion in a circle ($a_\tau = 0$), $a = a_n$ and is directed along the radius to the centre of the circle. In this case,

$$v = l/t = 2\pi R/T, \quad a_n = 4\pi^2 R/T^2,$$

where l is the length of the circumference, T the period of revolution (the time of one revolution), and R the radius of the circle.

Rotational motion of a rigid body is characterized by the following quantities: φ is the angular displacement (the angle of rotation of an arbitrary radius from the initial position), ω the angular velocity, ν the frequency of rotation, and T the period of revolution. The quantities ω, ν, and T are connected through the relation

$$\omega = 2\pi/T = 2\pi\nu.$$

For a uniform rotation of a body, the equation of motion is

$$\varphi = \omega t.$$

For a uniformly varying rotation (uniformly accelerated or uniformly decelerated) of a body, the equation of motion and the dependence of the angular velocity on time have the form

$$\varphi = \omega_0 t \pm \varepsilon t^2/2, \quad \omega = \omega_0 \pm \varepsilon t,$$

where ω_0 is the initial angular velocity and ε the angular acceleration.

In these formulas, the plus and minus signs are taken for a uniformly accelerated and uniformly decelerated rotation respectively. For a nonuniform (in particular, uniformly varying) rotation, the angular velocity can be expressed as the first derivative of the rotational angle with respect to time:

$$\omega = \frac{d\varphi}{dt}.$$

The angular acceleration for a uniformly varying rotation can be defined as the first derivative of the angular velocity with respect to time:

$$\varepsilon = \frac{d\omega}{dt}.$$

Knowing $\omega = \omega(t)$ and $\varepsilon = \varepsilon(t)$, we can determine φ and ω:

$$\varphi = \int_{t_1}^{t_2} \omega\, dt, \quad \omega = \int_{t_1}^{t_2} \varepsilon\, dt.$$

The angular quantities φ, ω, and ε are connected with the corresponding linear quantities l, v, and a through the following relations:

$$l = \varphi R, \quad v = \omega R, \quad a_\tau = \varepsilon R, \quad a_n = \omega^2 R.$$

* * *

65. Determine the centripetal acceleration of the points on the equator, at a latitude of 45°, and at a pole due to diurnal rotation of the Earth.

Given: $T = 24\ \text{h} = 8.64 \times 10^4$ s, $\varphi_1 = 0$ rad, $\varphi_2 = 45° \simeq 0.79$ rad, $\varphi_3 \simeq 1.57$ rad.

a_c — ?

Solution. All points on the Earth's surface participate in the diurnal rotation of the Earth at an angular velocity

$$\omega = 2\pi/T. \tag{1}$$

Consequently, their centripetal acceleration is

$$a_c = \omega^2 r, \tag{2}$$

where $r = R \cos\varphi$ is the radius of the circle in which a point moves, φ the latitude of the point, and R the Earth's radius (Fig. 45).

From Eqs. (1) and (2), we obtain $a_c = (4\pi^2/T^2) R \cos\varphi$. Hence the acceleration of points on the equator, at a latitude of 45°, and at a pole are respectively given by

$$a_{c1}=\frac{4\pi^2}{T^2}R,$$

$$a_{c1}=\frac{4\times 3.14^2\times 6.37\times 10^6}{(8.64\times 10^4)^2}\simeq 3.4\times 10^{-2}\ \text{m/s}^2$$

$(\cos\varphi_1=1)$,

$$a_{c2}=a_{c1}\cos\varphi_2,$$

$$a_{c2}\simeq 3.4\times 10^{-2}\times 0.7\ \text{m/s}^2\simeq 2.4\times 10^{-2}\ \text{m/s}^2,$$

$$a_{c3}=0\ \text{ since }\ \cos\varphi_3=\cos\pi/2=0.$$

66. A pulley 20 cm in diameter completes 300 revolutions in 3 min. Determine the period of revolution, the

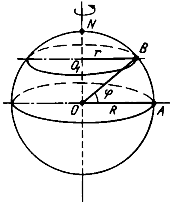


Fig. 45

angular and linear velocities on the rim of the pulley.

Given: $D=20\ \text{cm}=0.2\ \text{m},\ N=300,\ t=3\ \text{min}=180\ \text{s}.$

T—? ω—? v—?

Solution. The uniformly rotating pulley completes N revolutions in t seconds. Therefore, the time of one complete revolution (period) is

$$T=t/N,$$

$$T=(180/300)=0.6\ \text{s}.$$

The angular velocity is

$$\omega = 2\pi/T,$$

$$\omega = (2 \times 3.14/0.6)\ \text{rad/s} = 10.5\ \text{rad/s}.$$

The linear velocity is given by

$$v = \omega R = \omega D/2,$$

$$v = (10.5 \times 0.2/2)\ \text{m/s} = 1.05\ \text{m/s}.$$

67. Determine the radius of a rotating wheel if the linear velocity of a point on the rim is known to be 2.5 times the linear velocity of a point lying 5 cm closer to the wheel axle.

Given: $v_1 = 2.5 v_2$, $\Delta R = 5\ \text{cm} = 5 \times 10^{-2}\ \text{m}$.

R — ?

Solution. Since the angular velocity of all points of a rigid body is the same, the linear velocities of the two points are given by

$$v_1 = \omega R, \quad v_2 = \omega (R - \Delta R). \tag{1}$$

Substituting expressions (1) into the condition $v_1 = 2.5 v_2$, we obtain $\omega R = 2.5\omega (R - \Delta R)$, whence

$$R = \frac{2.5 \times \Delta R}{1.5},$$

$$R = \frac{2.5 \times 5 \times 10^{-2}}{1.5}\ \text{m} \simeq 8.3 \times 10^{-2}\ \text{m}.$$

68. A shaft starts to rotate, making 50 revolutions during the first 10 s. Assuming that the rotation of the shaft is uniformly accelerated, determine the angular acceleration and the final angular velocity.

Given: $t = 10$ s, $N = 50$.

ω — ? ε — ?

Solution. Since the initial angular velocity is zero, the equation of motion and the formula for angular velocity will be

$$\varphi = \varepsilon t^2/2, \quad \omega = \varepsilon t.$$

Since the angular displacement during one revolution is 2π, the total angular displacement of the shaft corresponding to N revolutions is $\varphi = 2\pi N$. Substituting this expression into the equation of motion, we obtain $2\pi N = \varepsilon t^2/2$, whence

$$\varepsilon = \frac{4\pi N}{t^2}, \quad \varepsilon = \frac{4 \times 3.14 \times 50}{10^2} \frac{\text{rad}}{\text{s}^2} = 6.28 \ \text{rad/s}^2.$$

Knowing ε, we can calculate the final angular velocity:

$$\omega = \varepsilon t, \quad \omega = 6.28 \times 10 \ \text{rad/s} = 62.8 \ \text{rad/s}.$$

69. The initial rotational frequency of a wheel is 5 s^{-1}. After the brakes have been applied, its frequency decreases to 3 s^{-1}. Determine the angular acceleration of the wheel and the number of turns made by it during this time.

Given: $\nu_0 = 5 \ \text{s}^{-1}$, $\nu = 3 \ \text{s}^{-1}$.

ε — ? N — ?

Solution. The equations of motion for the wheel have the form

$$\varphi = \omega_0 t - \varepsilon t^2/2, \quad (1)$$

$$\omega = \omega_0 - \varepsilon t. \quad (2)$$

Since $\varphi = 2\pi N$ and $\omega = 2\pi\nu$, we transform Eq. (2) as follows: $2\pi\nu = 2\pi\nu_0 - \varepsilon t$, whence

$$\varepsilon = \frac{2\pi(\nu_0 - \nu)}{t},$$

$$\varepsilon = \frac{2 \times 3.14 \times (5-3)}{60} \frac{\text{rad}}{\text{s}^2} = 0.21 \ \text{rad/s}^2.$$

Taking Eqs. (1) and (2) into account, we determine the total number of turns:

$$N = \nu_0 t - \frac{\varepsilon t^2}{4\pi},$$

$$N = 5 \times 60 - \frac{0.21 \times 60^2}{4 \times 3.14} = 240.$$

70°. A wheel rotates according to the law $\varphi = 4 + 5t - t^3$. Determine the angular velocity of the wheel at the end of the first second of rotation, as well as the linear velocity and the total acceleration of the points on the rim. The radius of the wheel is 2 cm.

Given: $R = 2$ cm $= 0.2$ m, $t = 1$ s.
$\omega - ?\ v - ?\ a - ?$

Solution. By definition, the angular velocity is

$$\omega = \frac{d\varphi}{dt} = \frac{d}{dt}(4 + 5t - t^3) = 5 - 3t^3,$$
$$\omega = (5 - 3 \times 1)\ \text{rad/s} = 2\ \text{rad/s}. \qquad (1)$$

The linear velocity is

$$v = \omega R,$$
$$v = 2 \times 0.2\ \text{m/s} = 0.4\ \text{m/s}.$$

By definition, the angular acceleration is $\varepsilon = \frac{d\omega}{dt}$. Taking Eq. (1) into account, we obtain

$$\varepsilon = \frac{d}{dt}(5 - 3t^2) = -6t,$$
$$\varepsilon = -6\ \text{rad/s}^2.$$

The total linear acceleration is

$$a = \sqrt{a_\tau^2 + a_n^2},$$

where $a_\tau = \varepsilon R$ and $a_n = \omega^2 R$. Then

$$a = R\sqrt{\varepsilon^2 + \omega^4},$$
$$a = 0.2\sqrt{(-6)^2 + (2)^4}\ \text{m/s}^2 \simeq 1.44\ \text{m/s}^2.$$

71°. A body rotates so that the time dependence of the angular velocity is given by the equation $\omega = 2 + 0.5t$. Determine the total number of turns completed by the body during the first 20 s.

Given: $t_1 = 0$, $t_2 = 20$ s.
$N - ?$

Solution. By definition, $\varphi = \int_{t_1}^{t_2} \omega\, dt$. Then

$$\varphi = \int_{t_1}^{t_2} (2 + 0.5t)\, dt = \left(2t + \frac{0.5t^2}{2}\right)\Big|_0^{20}$$
$$= 2 \times 20 + \frac{0.5 \times (20)^2}{2} = 140\ \text{rad}.$$

The total number of turns made by the body is

$$N=\frac{\varphi}{2\pi},$$

$$N=\frac{140}{2\times3.14}\simeq 22.$$

72. A point moves in a circle of radius 20 cm with a constant tangential acceleration of 5 cm/s². How much will it take the normal acceleration of the point to be twice as large as its tangential acceleration?

Given: $R = 20$ cm $= 0.2$ m, $a_\tau = 5$ cm/s² $= 5\times10^{-2}$ m/s², $a_n = 2a_\tau$.

t — ?

Solution. By definition, $v = a_\tau t$. Substituting this expression into the formula for the normal acceleration, we obtain $a_n = v^2/R = (a_\tau t)^2/R = a_\tau^2 t^2/R$. By hypothesis, $a_n = 2a_\tau$, or $a_\tau^2 t^2/R = 2a_\tau$, whence

$$t=\sqrt{\frac{2R}{a_\tau}},$$

$$t=\sqrt{\frac{2\times0.2}{5\times10^{-2}}}\ \text{s}=2.86\ \text{s}.$$

73. Are the distances traversed by the right and left wheels of a motor car during a turn equal?

Answer. The right and left wheels cover different distances since their linear velocities are different: the farther a wheel from the centre of curvature of the road, the higher its linear velocity.

74. Why did the carrier rocket of the first artificial satellite of the Earth get ahead of the satellite after its separation?

Answer. After the separation, the satellite acquires a velocity higher than that of its carrier rocket (see Problem 150). For this reason, the semi-axis of the elliptical orbit of the satellite is larger than that of the carrier rocket, and they move in different orbits. According to Kepler's third law, the period of revolution of the carrier rocket is smaller than that of the satellite, and the carrier rocket gets ahead of the satellite.

EXERCISES

75. The minute hand of the clock on the Spasskaya Tower of the Moscow Kremlin is 3.5 m long. What is the displacement of its tip during 1 min?

76. Determine the average orbital velocity of a satellite if the mean radius of its orbit from the Earth's surface is 1200 km and the period of revolution is 105 min.

77. What are the angular and linear velocities of the points on the Earth's surface at a latitude of 45°?

78. The turbine of a hydroelectric power plant has a working wheel diameter of 9 m and completes 68.2 turns during 1 min. Determine the velocity of the tips of the turbine blades.

79. A flywheel rotating at a frequency of 2 s^{-1} is stopped during 1.5 min. Assuming that its motion is uniformly decelerated, determine the number of turns the flywheel makes before stopping and its angular acceleration.

80. A wheel rotating with a uniform acceleration attains an angular velocity of 20 rad/s having made 10 turns after the beginning of motion. Determine the angular acceleration of the wheel.

81. A fan rotates at a frequency of 15 s^{-1}. After switching off, the fan rotates with a uniform deceleration and completes 75 turns before it stops. How long will it take the fan to stop after switching?

82. A wheel rotates with a constant angular acceleration of 2 rad/s^2. The total acceleration of the wheel becomes 13.6 m/s^2 0.5 s after the beginning of motion. Determine the radius of the wheel.

QUESTIONS FOR REVISION

1. List the main branches of mechanics and characterize each of them briefly. 2. What motion is uniform? uniformly varying? 3. Name the types of uniformly varying motion. 4. Write the equation of a uniform motion and plot the graph of the motion. 5. Define the average velocity, instantaneous velocity, and acceleration of a varying motion. 6. Write the equation of motion and the formula for velocity for a uniformly accelerated and a uniformly decelerated motion (with an initial velocity and without it). 7. Plot the velocity graphs for a uniform, uniformly varying, uniformly accelerated, and uniformly decelerated motion. 8. What is

the form of the acceleration graph for a uniformly varying motion? 9. Characterize the motion of a body thrown vertically upwards. 10. What is the formula for the acceleration of a body thrown vertically upwards? What is the relation between the times of ascent and descent of the body? 11. Formulate the principle of independence of motions. 12. Plot the trajectories of motion of a body thrown along the horizontal and at an angle to the horizontal. In what simpler motions can each of these motions be decomposed? 13. What is the horizontal range? What does it depend on? 14. Define a ballistic curve. 15. At what point of the trajectory does a body thrown at an angle to the horizontal have the minimum velocity? 16. What are the directions of linear velocity and linear acceleration vectors in a curvilinear motion? 17. What is the relation between the linear and the angular velocity of a point mass moving in a circle? 18. Define the centripetal acceleration.

1.2. Dynamics

Dynamics deals with the factors causing a change in the state of motion of bodies.

APPLICATION OF THE LAWS OF DYNAMICS TO RECTILINEAR MOTION OF A BODY (POINT MASS)

In mechanics, we deal with three types of force:

(1) the **elastic** force emerging upon deformation of a support (**N**) or a string (**T**);

(2) the **force of gravity** $\mathbf{P} = m\mathbf{g}$;

(3) the **frictional force** $\mathbf{F}_{fr} = \mu \mathbf{N}$, where μ is the coefficient of friction and **N** the normal reaction of the support.

While solving problems in dynamics, special attention should be paid to correct application of Newton's second law.

The solution of a problem must begin with a drawing in which a reference frame must be indicated, all the forces acting on the given body must be depicted, and the directions of velocity and acceleration vectors pointed out if necessary. Then Newton's second law should be written in vector form:

$$\sum_{i=1}^{n} \mathbf{F}_i = m\mathbf{a}, \qquad (1)$$

where $\sum_{i=1}^{n} \mathbf{F}_i = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n$, m is the mass of the body, and **a** its acceleration.

Finally, the vector equation (1) should be projected onto the chosen directions of the X- and Y-axes:

$$\sum_{i=1}^{n} F_{ix} = ma_x, \quad \sum_{i=1}^{n} F_{iy} = ma_y \tag{2}$$

and the obtained system of equations (2) must be solved for unknown quantities.

If several bodies connected to one another participate in a motion, all the above procedures should be carried out for each body separately, and the obtained equations should be solved simultaneously.

* * *

83. A carriage whose mass is 20 t moves with a uniform deceleration of 0.3 m/s² at an initial velocity of 54 km/h.

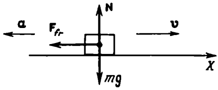


Fig. 46

Determine the braking force acting on the carriage, the time of motion before it stops, and the displacement.

Given: $m = 20\ \text{t} = 2 \times 10^4$ kg, $a = 0.3$ m/s²,
$v_0 = 54$ km/h $= 15$ m/s.

F_{fr} — ? t — ? s — ?

Solution. The carriage experiences the action of the force of gravity $m\mathbf{g}$, the friction $\mathbf{F}_{fr}$, and the normal reaction **N** (Fig. 46). Writing Newton's second law in vector form, we obtain

$$m\mathbf{g} + \mathbf{F}_{fr} + \mathbf{N} = m\mathbf{a}. \tag{1}$$

We choose the direction of motion of the carriage as the positive direction of the X-axis. Since the motion of the carriage is uniformly decelerated, the acceleration vector is opposite to the direction of motion. Projecting both sides of Eq. (1) onto the X-axis, we obtain $-F_{tr} = -ma$, or

$$F_{tr} = ma,$$
$$F_{tr} = 2 \times 10^4 \times 0.3 \text{ N} = 6 \text{ kN}.$$

The stopping time and the distance covered by the carriage can be determined from the kinematic equations: $0 = v_0 - at$ and $s = v_0^2/(2a)$, whence

$$t = v_0/a, \quad t = 15/0.3 \text{ s} = 50 \text{ s},$$
$$s = 15^2 \times (2 \times 0.3) \text{ m} = 375 \text{ m}.$$

84. A body of mass 3 kg falls in air with an acceleration of 8 m/s². Determine the air drag.

Given: $m = 3$ kg, $a = 8$ m/s².
F — ?

Solution. The body falling in air is under the action of the force of gravity mg and the air drag **F** (Fig. 47). Since the motion is uniformly accelerated, the acceleration vector is directed along the trajectory of motion. Writing Newton's second law for the body in vector form, we obtain

$$m\mathbf{g} + \mathbf{F} = m\mathbf{a}.$$

Projecting the forces and acceleration onto the Y-axis directed along the motion, we get

$$mg - F = ma,$$

whence

$$F = mg - ma = m(g - a),$$
$$F = 3 \times (9.8 - 8) \text{ N} = 5.4 \text{ N}.$$

85. A load of mass 50 kg hoisted by a rope in the vertical direction with a uniform acceleration reaches a height of 10 m in 2 s. Determine the tension of the rope.

Given: $m = 50$ kg, $t = 2$ s, $h = 10$ m.
T — ?

Solution. The load is acted upon by the force of gravity $m\mathbf{g}$ and the tension $\mathbf{T}$ of the rope (Fig. 48). Since the load moves upwards with a uniform acceleration, the vector $\mathbf{a}$ is directed vertically upwards. Writing Newton's second law for the load in vector form, we obtain

$$\mathbf{T} + m\mathbf{g} = m\mathbf{a}. \qquad (1)$$

Directing the Y-axis along the motion of the load and projecting the forces and accelerations onto this axis, we get $T - mg = ma$, whence

$$T = mg + ma = m(g + a).$$

Since the motion is uniformly accelerated without an initial velocity, $h = at^2/2$ and $a = 2h/t^2$. Then

$$T = m\left(g + \frac{2h}{t^2}\right),$$

$$T = 50 \times \left(9.8 + \frac{2 \times 10}{2^2}\right) \text{N} = 740 \text{ N}.$$

Consequently, for a uniformly accelerated upward motion of the load, the tension exceeds the force of gravity.

86. A man weighing 70 kg is in a lift moving vertically upwards with a uniform deceleration of 1 m/s^2. Determine the force of pressure exerted by the man on the cabin floor.

Given: $m = 70$ kg, $a = 1$ m/s^2.
F — ?

Solution. The man in the lift is under the action of the force of gravity $m\mathbf{g}$ and the reaction $\mathbf{N}$ of the cabin floor

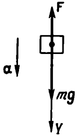


Fig. 47

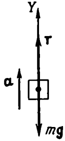


Fig. 48

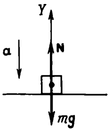


Fig. 49

(Fig. 49). The acceleration is directed vertically downwards. Writing Newton's second law for the man in vector form, we obtain

$$\mathbf{N} + m\mathbf{g} = m\mathbf{a}. \qquad (1)$$

Directing the Y-axis along the motion of the lift and projecting the forces and accelerations onto this axis, we get

$$N - mg = -ma, \qquad (2)$$

whence

$$N = mg - ma = m(g - a),$$
$$N = 70 \times (9.8 - 1)\ \mathrm{N} = 616\ \mathrm{N}.$$

According to Newton's third law, the force of pressure F exerted by the man on the cabin floor is equal in magnitude to the reaction N of the cabin floor: $F = N = 616$ N.

87. The cage of an elevator of mass 5000 kg operates in a 900-m pit. When the cage is at the bottom of the pit, a driving force of 60 kN starts to set on it in the vertical direction. At a distance of 150 m from the bottom, the driving force changes so that the motion of the cage becomes uniform over the next 600 m. Finally, the driving force changes once again so that the cage stops when it reaches the surface. Assuming that friction is 5 kN, consider the motion of the cage on these segments and determine the time of ascent.

Given: $m = 5 \times 10^3$ kg, $h = 9 \times 10^2$ m, $F_{fr} = 5$ kN $= 5 \times 10^2$ N, $F_d = 60$ kN $= 6 \times 10^4$ N, $h_1 = 1.5 \times 10^2$ m, $h_2 = 6 \times 10^2$ m.

t — ?

Solution. Let us consider the motion of the cage on each segment. On the first segment of length h_1, the cage moves with a uniform acceleration at zero initial velocity; on the second segment of length h_2, it moves uniformly, while on the third segment of length

$$h_3 = h - (h_1 + h_2), \qquad (1)$$

the cage moves with a uniform deceleration and stops at the end of the segment.

In order to describe the motion of the cage, we direct the Y-axis vertically upwards and choose the origin at the bottom of the pit. The equation of motion for the cage on the first segment is

$$y_1 = a_1 t^2/2. \qquad (2)$$

If $t = t_1$, then $y_1 = h_1$, and Eq. (1) assumes the form $h_1 = a_1 t_1^2/2$, whence

$$t_1 = \sqrt{2h_1/a_1}. \qquad (3)$$

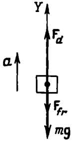


Fig. 50

Writing Newton's second law in projections on the Y-axis (Fig. 50) $F_d - mg - F_{fr} = ma_1$, we find that

$$a_1 = (F_d - mg - F_{fr})/m. \qquad (4)$$

Substituting Eq. (4) into (3), we obtain

$$t_1 = \sqrt{2h_1 m/(F_d - mg - F_{fr})}. \qquad (5)$$

The velocity of the cage at the end of the first segment is $v_1 = a_1 t_1$, or, taking Eqs. (4) and (5) into account,

$$v_1 = \frac{F_d - mg - F_{fr}}{m} \sqrt{\frac{2h_1 m}{F_d - mg - F_{fr}}} = \sqrt{\frac{2h_1 (F_d - mg - F_{fr})}{m}}.$$

On the second segment, the cage moves uniformly at the velocity acquired by the end of the first segment:

$$v_2 = v_1 = \sqrt{\frac{2h_1 (F_d - mg - F_{fr})}{m}}. \qquad (6)$$

Choosing the origin of the Y-axis for the second segment at the height h_1 from the bottom of the pit, we write the equation of motion for the cage:

$$y_2 = v_2 t. \qquad (7)$$

If $t = t_2$, then $y_2 = h_2$, and Eq. (7) becomes $h_2 = v_2 t_2$, whence $t_2 = h_2/v_2$, or, taking Eq. (6) into account,

$$t_2 = h_2 \sqrt{m/[2h_1 (F_d - mg - F_{fr})]}.$$

The time of motion on the third segment can be found by using the concept of average velocity: $\langle v_3 \rangle = (v_2 + 0)/2 =$

$v_2/2$, or, taking Eq. (6) into account,

$$\langle v_3\rangle = \sqrt{[h_1(F_d - mg - F_{fr})]/(2m)}.$$

Then the time of motion on the third segment is

$$t_3 = h_3/\langle v_3\rangle. \qquad (8)$$

Transforming Eq. (8) by applying Eqs. (1) and (7), we obtain

$$t_3 = [h-(h_1+h_2)]\sqrt{\frac{2m}{h_1(F_d-mg-F_{fr})}}.$$

The total time of the ascent of the cage is

$$t = t_1+t_2+t_3 = \sqrt{\frac{2h_1m}{F_d-mg-F_{fr}}}$$

$$+h_2\sqrt{\frac{m}{2h_1(F_d-mg-F_{fr})}}$$

$$+[h-(h_1+h_2)]\sqrt{\frac{2m}{h_1(F_d-mg-F_{fr})}}$$

$$=(2h-h_2)\sqrt{\frac{m}{2h_1(F_d-mg-F_{fr})}},$$

$$t = (2\times 9\times 10^2 - 6\times 10^2)$$

$$\times\sqrt{\frac{5\times 10^3}{2\times 1.5\times 10^2\times(60-50-5)\times 10^3}}\ \text{s} \simeq 69\ \text{s}.$$

88. A load of mass 45 kg is moved over a horizontal plane by a force of 294 N directed at an angle of 30° to the horizontal. The coefficient of friction between the load and the plane is 0.1. Determine the acceleration of the load.

Given: $F = 294$ N, $m = 45$ kg, $\mu = 0.1$, $\alpha = 30° \simeq 0.52$ rad.

a — ?

Solution. The load experiences the action of the force of gravity mg, the normal reaction **N** of the plane, the driving force $\mathbf{F}_d$, and the friction $\mathbf{F}_{fr}$. The vector **a** is directed parallel to the plane to the right (Fig. 51). Writing Newton's second law for the load in vector form, we obtain

$$m\mathbf{g} + \mathbf{N} + \mathbf{F}_d + \mathbf{F}_{fr} = m\mathbf{a}. \qquad (1)$$

Choosing the directions of the X- and Y-axes and projecting the forces and accelerations onto the axes, we get

$$F_d \cos \alpha - F_{fr} = ma, \quad (2)$$

$$N + F_d \sin \alpha - mg = 0. \quad (3)$$

From Eq. (3), we find that $N = mg - F_d \sin \alpha$. Then $F_{fr} = \mu N = \mu (mg - F_d \sin \alpha)$. Substituting Eq. (3)

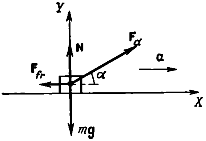


Fig. 51

into (2), we obtain $F_d \cos \alpha - \mu (mg - F_d \sin \alpha) = ma$, whence

$$a = \frac{F_d \cos \alpha - \mu (mg - F_d \sin \alpha)}{m},$$

$$a = \frac{294 \times 0.87 - 0.1 \times (45 \times 9.8 - 294 \times 0.5)}{45} \ \frac{\text{m}}{\text{s}^2} \simeq 5.9 \text{ m/s}^2.$$

89. A body slides uniformly down an inclined plane forming an angle of 40° with the horizontal. Determine the coefficient of friction of the body against the plane.

Given: $\alpha = 40° \simeq 0.7$ rad.

μ — ?

Solution. The body sliding down the plane is acted upon by the force of gravity $m\mathbf{g}$, the normal reaction $\mathbf{N}$ of the plane, and the friction $\mathbf{F}_{fr}$. In a uniform motion, acceleration is zero (Fig. 52). Writing Newton's second law for the body in vector form, we obtain

$$m\mathbf{g} + \mathbf{N} + \mathbf{F}_{fr} = 0.$$

Projecting the forces onto the X- and Y-axes, we get

$$mg \sin \alpha - F_{fr} = 0, \quad (1)$$

$$N - mg \cos \alpha = 0. \quad (2)$$

Equation (2) gives $N = mg \cos \alpha$. Using the expression for sliding friction $F_{fr} = \mu N = \mu mg \cos \alpha$, we write Eq. (1) in the form $mg \sin \alpha = \mu mg \cos \alpha = 0$, whence

$$\mu = \tan \alpha,$$

$$\mu = \tan 40° \simeq 0.84.$$

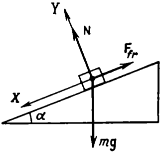


Fig. 52

90. A motor car of mass 1 t climbs a road with a slope of 30° under the action of a driving force of 7 kN. Determine the acceleration of the car, assuming that the force of resistance is independent of velocity and constitutes 0.1 of the normal reaction.

Given: $m = 1\ \text{t} = 10^3$ kg, $F = 7\ \text{kN} = 7 \times 10^3$ N, $\alpha = 30° \simeq 0.52$ rad, $\mu = 0.1$.

a — ?

Solution. The car experiences the action of the force of gravity mg, the normal reaction **N** of the road, the driving force $\mathbf{F}_d$, and the friction $\mathbf{F}_{fr}$. Choosing the direction of vector **a** up the inclined plane (Fig. 53) and writing Newton's second law for the car in vector form, we obtain

$$m\mathbf{g} + \mathbf{F}_d + \mathbf{N} + \mathbf{F}_{fr} = m\mathbf{a}.$$

Projecting the forces and accelerations onto the X- and Y-axes, we get

$$-mg \sin \alpha + F_d - F_{fr} = ma, \qquad (1)$$

$$-mg \cos \alpha + N = 0. \qquad (2)$$

Equation (2) gives $N = mg \cos \alpha$. Considering that $F_{fr} = \mu N = \mu mg \cos \alpha$, we write Eq. (1) in the form $-mg \sin \alpha + F_d - \mu mg \cos \alpha = ma$, whence

$$a = \frac{F_d - mg (\sin \alpha + \mu \cos \alpha)}{m},$$

$$a = \frac{7 \times 10^3 - 10^3 \times 9.8 \times (0.5 + 0.1 \times 0.87)}{10^3} \frac{m}{s^2} = 1.2 \text{ m/s}^2$$

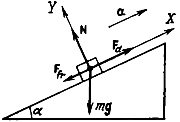


Fig. 53

91. A block of mass 2 kg slides over a horizontal plane under the action of a load of mass 0.5 kg fixed to the end of an inextensible string passing over a stationary pulley. The coefficient of friction between the block and the surface of the plane is 0.1. Determine the acceleration of the load and the tension of the string. The masses of the pulley and the string, as well as the friction in the pulley, should be neglected.

Given: $m_1 = 2$ kg, $m_2 = 0.5$ kg, $\mu = 0.1$.

a — ? T — ?

Solution. Let us consider the motion of each body separately. The block is under the action of the force of gravity $m_1\mathbf{g}$, the normal reaction $\mathbf{N}$ of the plane, the tension $\mathbf{T}_1$ of the string, and the friction $\mathbf{F}_{fr}$ (Fig. 54). Writing Newton's second law for the block in vector form, we obtain

$$m_1\mathbf{g} + \mathbf{N} + \mathbf{T}_1 + \mathbf{F}_{fr} = m_1\mathbf{a}_1. \quad (1)$$

Projecting Eq. (1) onto the chosen directions of the X- and Y-axes, we get

$$T_1 - F_{fr} = m_1 a_1, \quad (2)$$

$$m_1 g - N = 0. \quad (3)$$

Equation (3) gives $N = m_1 g$, and hence $F_{fr} = \mu N = \mu m_1 g$. Then (see Eq. (2)):

$$T_1 - \mu m_1 g = m_1 a_1. \quad (4)$$

The load is acted upon by the force of gravity $m_2\mathbf{g}$ and the

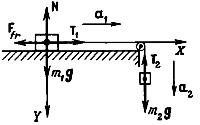


Fig. 54

tension $\mathbf{T}_2$ of the string. Writing Newton's second law for the load in vector form, we obtain

$$m_2\mathbf{g} + \mathbf{T}_2 = m_2\mathbf{a}_2. \quad (5)$$

Projecting Eq. (5) onto the Y-axis, we get

$$m_2 g - T_2 = m_2 a_2. \quad (6)$$

Summing Eqs. (4) and (6) and considering that $T_1 = T_2 = T$ and $a_1 = a_2 = a$, we obtain $m_2 g - T + T - \mu m_1 g = (m_2 + m_1)\, a$, whence

$$a = \frac{m_2 g - \mu m_1 g}{m_2 + m_1} = \frac{g\,(m_2 - \mu m_1)}{m_2 + m_1},$$

$$a = \frac{9.8 \times (0.5 - 0.1 \times 2)}{0.5 + 2}\ \frac{\text{m}}{\text{s}^2} \simeq 1.2\ \text{m/s}^2.$$

The tension of the string can be determined from Eq. (6):

$$T = m_2 g - m_2 a = m_2 (g - a),$$
$$T = 0.5 \times (9.8 - 1.2)\ \text{N} \simeq 4.3\ \text{N}.$$

92. A load of mass 5 kg connected to another load of mass 2 kg by an inextensible string passing over a stationary pulley attached to the top of an inclined plane slides down the plane forming an angle of 36° with the horizon-

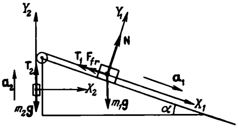


Fig. 55

tal. Determine the tension of the string and the acceleration of the loads if the coefficient of friction between the first load and the plane is 0.1. The masses of the string and the pulley, as well as the friction in the pulley, should be neglected.

Given: $m_1 = 5$ kg, $m_2 = 2$ kg, $\mu = 0.1$, $\alpha = 36° \simeq 0.63$ rad.
T—? a—?

Solution. We consider the motion of each load separately. The first load is acted upon by the force of gravity $m_1\mathbf{g}$, the normal reaction $\mathbf{N}$ of the inclined plane, the tension $\mathbf{T}_1$ of the string, and the friction $\mathbf{F}_{fr}$ (Fig. 55). By hypothesis, the acceleration vector $\mathbf{a}_1$ of the first load is directed down the inclined plane. Writing Newton's second law for the first load in vector form, we obtain

$$m_1\mathbf{g} + \mathbf{N} + \mathbf{T}_1 + \mathbf{F}_{fr} = m_1\mathbf{a}_1. \qquad (1)$$

Projecting Eq. (1) onto the X_1- and Y_1-axes, we get

$$m_1 g \sin\alpha - T_1 - F_{fr} = m_1 a_1, \qquad (2)$$
$$-m_1 g \cos\alpha + N = 0. \qquad (3)$$

Equation (3) gives $N = m_1 g \cos\alpha$, and hence

$$F_{fr} = \mu N = \mu m_1 g \cos\alpha. \qquad (4)$$

From Eqs. (2) and (4), we obtain

$$m_1 g \sin\alpha - T_1 - \mu m_1 g \cos\alpha = m_1 a_1. \qquad (5)$$

The second load experiences the action of the force of gravity $m_2\mathbf{g}$ and the tension $\mathbf{T}_2$ of the string. The acceleration vector $\mathbf{a}_2$ of the second load is directed vertically upwards (see Fig. 55). Writing Newton's second law for the second load in vector form, we obtain

$$m_2\mathbf{g} + \mathbf{T}_2 = m_2\mathbf{a}_2. \qquad (6)$$

Projecting Eq. (6) onto the Y_2-axis, we get

$$-m_2 g + T_2 = m_2 a_2. \qquad (7)$$

Summing Eqs. (5) and (7), we obtain $m_1 g \sin\alpha - T_1 - \mu m_1 g \cos\alpha + T_2 - m_2 g = m_1 a_1 + m_2 a_2$. Considering that $T_1 = T_2 = T$ and $a_1 = a_2 = a$, we obtain

$$a = \frac{m_1 g (\sin\alpha - \mu\cos\alpha) - m_2 g}{m_1 + m_2}$$

$$= \frac{m_1 (\sin\alpha - \mu\cos\alpha) - m_2}{m_1 + m_2} g,$$

$$a = \frac{5\times(0.59 - 0.1\times 0.81) - 2}{5+2} \times 9.8\,\frac{\text{m}}{\text{s}^2} \simeq 0.77\ \text{m/s}^2.$$

The tension of the string can be determined from Eq. (7):

$$T = m_2 g + m_2 a = m_2 (g + a),$$

$$T = 2\times(9.8 + 0.77)\ \text{N} \simeq 21\ \text{N}.$$

93. A massless pulley is fixed at the apex of two inclined planes forming angles of 30 and 45° with the horizontal. The loads A and B of mass 1 kg each are connected by a string passing over the pulley. Determine the acceleration of the loads and the tension of the string, assuming that the string is weightless and inextensible and neglecting friction.

Given: $\alpha = 30° \simeq 0.52$ rad, $\beta = 45° \simeq 0.79$ rad,

$m_1 = m_2 = 1$ kg.

T—? a—?

Solution. From the assumption that the masses of the string and the pulley are zero it follows that the tension of the string is the same in each cross section, while the condition of inextensibility of the string implies that the accelerations of the loads are equal in magnitude:

$$T_1 = T_2 = T, \quad a_1 = a_2 = a. \tag{1}$$

Let us consider the motion of each load separately. The load A is acted upon by the force of gravity $m_1\mathbf{g}$, the

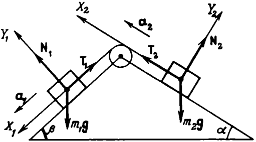


Fig. 56

normal reaction $\mathbf{N}_1$, and the tension $\mathbf{T}_1$ of the string (Fig. 56). Assuming that the load A slides down the inclined plane, we determine the direction of acceleration vector $\mathbf{a}_1$. Writing Newton's second law for the load A in vector form, we obtain

$$m_1\mathbf{g} + \mathbf{N}_1 + \mathbf{T}_1 = m_1\mathbf{a}_1. \tag{2}$$

Projecting Eq. (2) onto the X_1- and Y_1-axes, we get

$$m_1 g \sin\beta - T_1 = m_1 a_1, \tag{3}$$

$$-m_1 g \cos\beta + N_1 = 0. \tag{4}$$

The load B experiences the action of the force of gravity $m_2\mathbf{g}$, the normal reaction $\mathbf{N}_2$ of the inclined plane, and the tension $\mathbf{T}_2$ of the string. Writing Newton's second law for the load B in vector form, we obtain

$$m_2\mathbf{g} + \mathbf{N}_2 + \mathbf{T}_2 = m_2\mathbf{a}_2. \tag{5}$$

Projecting Eq. (5) onto the X_2- and Y_2-axes, we get

$$-m_2 g \sin\alpha + T_2 = m_2 a_2, \quad (6)$$

$$-m_2 g \cos\alpha + N_2 = 0. \quad (7)$$

Equations (4) and (7) are not used in this problem since we neglect friction. Solving the system of equations (3) and (6) by summing them, we obtain $m_1 g \sin\beta - T_1 - m_2 g \sin\alpha + T_2 = m_1 a_1 + m_2 a_2$. Taking Eqs. (1) into account, we find that

$$a = \frac{m_1 g \sin\beta - m_2 g \sin\alpha}{m_1 + m_2} = \frac{(m_1 \sin\beta - m_2 \sin\alpha)\, g}{m_1 + m_2},$$

$$a = \frac{(1 \times 0.7 - 1 \times 0.5) \times 9.8}{1+1}\ \frac{\text{m}}{\text{s}^2} = 0.98\ \text{m/s}^2.$$

The tension of the string can be determined from Eq. (3), taking Eqs. (1) into account

$$T = m_1 g \sin\beta - m_1 a = m_1 (g \sin\beta - a),$$

$$T = 1 \times (9.8 \times 0.7 - 0.98)\ \text{N} \simeq 5.9\ \text{N}.$$

94. Two loads of mass 100 g each are attached to the ends of a weightless inextensible string passing over a massless stationary pulley. Determine the force exerted by a small load of mass 10 g placed on one of the loads and the force of pressure on the axle of the pulley.

Given: $m_1 = m_2 = 100\ \text{g} = 0.1\ \text{kg}, \quad m = 10\ \text{g} = 0.01\ \text{kg}.$

F — ? F_{pr} — ?

Solution. The conditions of weightlessness and inextensibility of the string imply that the tension of the string is the same in all segments and that all the loads move with the same acceleration:

$$T_1 = T_2 = T, \quad a_1 = a_2 = a. \quad (1)$$

Let us consider the forces acting on each load.

The first load experiences the action of the force of gravity $m_1\mathbf{g}$, the tension $\mathbf{T}_1$ of the string, and the force of pressure $\mathbf{F}$ of the small load (Fig. 57). Writing Newton's second law for the first load in projections on the Y-axis, we obtain

$$m_1 g = F - T_1 = m_1 a_1. \quad (2)$$

The second load is acted upon by the force of gravity $m_2\mathbf{g}$ and the tension $\mathbf{T}_2$ of the string. Writing Newton's second law for the second load in projections on the Y-axis, we get

$$T_2 - m_2 g = m_2 a_2. \quad (3)$$

The small load is under the action of the force of gravity $m\mathbf{g}$ and the normal reaction N. Writing Newton's second law for this load in projections on the Y-axis, we obtain

$$mg - N = ma_1. \quad (4)$$

Summing Eqs. (2)-(4) and considering that $m_1 g = m_2 g = Mg$ and $F = N$, we get $Mg + N - T + T - Mg + mg - N = Ma + Ma + ma$, whence

$$a = mg/(2M + m). \quad (5)$$

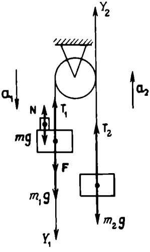


Fig. 57

Let us determine the force of pressure F exerted by the small load. According to Newton's third law, this force is numerically equal to the normal reaction N. Solving Eqs. (4), (5), and (1) together, we obtain

$$F = N = mg - ma = \frac{2mMg}{2M+m},$$

$$F = \frac{2\times 0.01\times 0.1\times 9.8}{2\times 0.1+0.01}\ \text{N} = 9.3\times 10^{-2}\ \text{N}.$$

The force of pressure on the axle of the pulley is

$$F_{\text{pr}} = 2T. \quad (6)$$

From Eq. (3), we determine the tension of the string: $T = M(a + g)$. Taking Eq. (5) into account, we obtain $T = 2Mg(m + M)/(2M + m)$. Then Eq. (6) become

$$F_{\text{pr}} = 4Mg\,\frac{m+M}{2M+m},$$

$$F_{\text{pr}} = 4\times 0.1\times 9.8\times \frac{0.01+0.1}{2\times 0.1+0.01}\ \text{N} \simeq 2.1\ \text{N}.$$

95. Three loads of mass 1 kg each are tied by strings and move over a horizontal plane under the action of a force of 10 N, directed at an angle of 30° to the horizontal. Determine the acceleration of the system and the tensions in the strings if the coefficient of friction is 0.1.

Given: $m_1 = m_2 = m_3 = m = 1$ kg, $F = 10$ N,
$\alpha = 30° \simeq 0.52$ rad, $\mu = 0.1$.

a — ? T_{12} — ? T_{23} — ?

Solution. Let us consider the forces acting on each load. The first load is acted upon by the force of gravity mg,

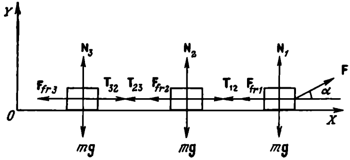


Fig. 58

the normal reaction $\mathbf{N}_1$, the applied force $\mathbf{F}$, and the friction $\mathbf{F}_{fr1}$ (Fig. 58). Writing Newton's second law for the first load in projections on the X- and Y-axes, we obtain

$$F \cos\alpha - T_{12} - F_{fr1} = ma, \quad N_1 + F \sin\alpha - mg = 0. \quad (1)$$

From the second equation in (1), we find that $N_1 = mg - F \sin \alpha$, and hence $F_{fr1} = \mu N_1 = \mu (mg - F \sin \alpha)$. Consequently, the first equation in (1) assumes the form

$$F \cos \alpha - T_{12} - \mu (mg - F \sin \alpha) = ma. \quad (2)$$

Similarly, for the second load, we obtain

$$T_{21} - F_{fr2} - T_{23} = ma, \quad N_2 - mg = 0. \quad (3)$$

After transforming these equations, we get

$$T_{21} - \mu mg - T_{23} = ma. \quad (4)$$

For the third load, we can write

$$T_{32} - F_{fr3} = ma, \quad N_3 - mg = 0, \tag{5}$$

whence

$$T_{32} - \mu mg = ma. \tag{6}$$

Summing Eqs. (2), (4), and (6) and considering that $T_{12} = T_{21}$ and $T_{23} = T_{32}$, we obtain $F \cos\alpha - \mu (mg - F \sin\alpha) - 2\mu mg = 3ma$, whence

$$a = \frac{F(\cos\alpha + \mu \sin\alpha) - 3\mu mg}{3m},$$

$$a = \frac{10 \times (0.87 + 0.1 \times 0.5) - 3 \times 0.1 \times 1 \times 9.8}{3 \times 1} \frac{\text{m}}{\text{s}^2} \simeq 2.1 \text{ m/s}^2.$$

From Eqs. (4) and (6), we find that

$$T_{21} = T_{12} = 2m(a + \mu g),$$

$$T_{12} = 2 \times 1 \times (2.1 + 0.1 \times 9.8) \text{ N} \simeq 6.2 \text{ N},$$

$$T_{32} = T_{23} = m(a + \mu g),$$

$$T_{23} = 1 \times (2.1 + 0.1 \times 9.8) \text{ N} \simeq 3.1 \text{ N}.$$

96. A body is weighed on a spring balance and on a beam balance at a pole and on the equator. What are the readings of the instruments?

Answer. The mass of the body determined with the help of the beam balance is compared with the mass of standard weights. Since the weight of the body and of the standard weights varies identically as we move from one point on the Earth's surface to another, the readings of the beam balance at the pole and on the equator will be the same. The weight of the body determined with the help of the spring balance is compared with the elastic force of the spring. The spring constant does not depend on the latitude, and hence the readings of the spring balance at the pole will be larger than on the equator. Indeed, using Newton's second law for the body suspended on the spring at the pole and on the equator, we obtain

$$mg - T_p = 0, \quad mg - T_{eq} = mv^2/R,$$

where T_p and T_{eq} are the tensions of the spring at the pole and on the equator respectively. Then $T_p = mg$, $T_{eq} = mg - mv^2/R$, i.e. $T_p > T_{eq}$.

97. A load is pulled along the surface of a table with the help of a string attached to a spring balance indicating 30 N. Then the same load is set in motion with the help of a string passing over a stationary pulley with a load of mass 3 kg at the other end. In which case does the load move faster?

Answer. In the second case, a force of 30 N acts on the system of bodies with a larger mass than in the first. Therefore, the load moves faster in the first case.

98. Why does the velocity of a train on a horizontal segment not increase indefinitely although the driving force of the engine acts constantly?

Answer. As the velocity of the body increases, the resistance forces become stronger and ultimately balance the driving force. Therefore, the motion of the train becomes uniform after a certain time.

99. Bodies fall due to the attraction of the Earth. In what respect is this statement inaccurate?

Answer. There are no bodies in nature which only exert an action or are only subjected to an action. On the contrary, bodies experience equal action and reaction (Newton's third law). Consequently, the correct way of putting this is that the body and the Earth attract each other, but since the mass of the Earth is huge in comparison with the mass of the body, we do not notice its motion towards the body.

100. Why does a running person fall in the direction of motion when he stumbles and in the opposite direction when he slips?

Answer. This phenomenon can be explained easily by using Newton's first law. In the former case, the feet of the person are slowed down, while the body retains its previous state of motion by inertia, and the person falls in the direction of motion. In the latter case, the body of the person retains its previous state of motion, while the feet slip in the forward direction at a higher velocity, and the person falls backwards.

EXERCISES

101. A body of mass 200 g falls vertically downwards with an acceleration of 920 cm/s^2. What is the air drag?

102. What is the force of pressure exerted by a load of mass 100 kg on the floor of the cage in a mine, ascending vertically with an acceleration of 24.5 cm/s^2?

103. At what distance from a crossroad must the driver of a motor car apply brakes at a red traffic light if the car

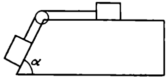


Fig. 59

moves uphill along a slope of 30° at a velocity of 60 km/h? The coefficient of friction between the tyres and the road is 0.1.

104. An inclined board forming an angle of 60° with the horizontal leans against a table (Fig. 59). Two equal loads of mass 1 kg each and connected through a light string passing over a massless stationary pulley can move over the board and the table respectively. Determine the tension of the string and the acceleration of the system if the coefficient of friction between the loads and the surfaces of the board and the table is the same and equal to 0.3.

105. A funicular railway forms an angle of 30° with the horizontal and has two cabins of mass 4600 kg each, connected by a steel rope passing over a pulley at the upper station. The descending cabin carries an additional load of mass 600 kg. Determine the acceleration of the system and the distance traversed by each cabin if the motion starts from the state of rest, and the maximum velocity attained by the cabins is 14.4 km/h. Determine the tension of the rope, neglecting friction and the masses of the rope and the pulley.

106. Two loads having a mass m each and connected by a string rest on a horizontal surface. Another string with the same load at its end is tied to the system of loads and passes over a stationary pulley. Determine the acceleration of the system of loads and the tension of the string connecting the loads on the surface, neglecting friction.

107. A load of mass 500 g is suspended on a string which passes over a pulley and has a load of mass 300 g at its other end. Determine the acceleration of the system, the displacement and the velocity of each load 1.2 s after the beginning of motion. Friction and the masses of the string and the pulley should be neglected.

108. Two bodies connected by a string move over a horizontal plane under the action of a horizontal force of 100 N. If the force is applied to the right body ($m_1 = 7$ kg), the tension of the string is 30 N. Determine the tension of the string if the force is applied to the left body ($m_2 =$ 3 kg), assuming that in both cases the bodies move in the direction of the applied force and neglecting friction.

109. A man pulls two sledges of mass 15 kg each and connected by a rope, applying a force of 120 N at an angle of 45° to the horizontal. Determine the acceleration of the sledges and the tension of the rope connecting the sledges if the coefficient of friction between the runners and snow is 0.02.

110. A wooden block is initially at rest on a table. Two strings pass over pulleys fixed at two ends of the table and are tied to the block. When loads of mass 0.85 and 0.2 kg are suspended on the free ends of the strings, the block starts to move and in 3 s covers a distance of 0.81 m. Determine the coefficient of friction and the tension of the strings if the mass of the block is 2 kg.

111. Two weights of mass 2 and 1 kg are connected by a string passing over a stationary pulley. Determine the acceleration of the weights, the tension of the string, and the force of pressure on the axle of the pulley.

APPLICATION OF THE LAWS OF DYNAMICS TO THE MOTION OF A BODY (POINT MASS) IN A CIRCLE

The solution of problems on dynamics of circular motion does not differ in principle from the solution of problems on dynamics of translational motion.

In the case under consideration, the projection of acceleration of a body on the Y-axis directed along the radius to the centre of the circle is known to be the centripetal acceleration. Therefore, according to Newton's second law, we have

$$mv^2/R = F, \quad \text{or} \quad m\omega^2 R = F,$$

where F is the sum of the Y-projections of the forces acting on the body.

* * *

112. A truck with a load of mass 5 t moves over a convex bridge at a velocity of 21.6 km/h. What is the force of pressure exerted by the truck at the midpoint of the bridge if the radius of curvature of the bridge is 50 m?

Given: $m = 5\ \text{t} = 5 \times 10^3$ kg, $v = 21.6$ km/h $= 6$ m/s, $R = 50$ m.

F—?

Solution. The truck is under the action of the force of gravity $m\mathbf{g}$ and the normal reaction $\mathbf{N}$ (Fig. 60). Directing the Y-axis vertically downwards along the bridge radius and writing Newton's second law for the truck in vector form, we obtain

$$m\mathbf{g} + \mathbf{N} = m\mathbf{a}.$$

Projecting this equation onto the Y-axis, we get

$$mg - N = ma_y,$$

where $a_y = a_c = v^2/R$. Then $mg - N = mv^2/R$, whence $N = mg - mv^2/R = m(g - v^2/R)$. According to Newton's third law, the same force is exerted by the truck on the bridge, i.e. $F = N$, or

$$F = m\left(g - \frac{v^2}{R}\right),$$

$$F = 5 \times 10^3 \times \left(9.8 - \frac{6^2}{50}\right) \text{N} \simeq 4.5 \times 10^4 \text{ N}.$$

113. A bucket with water is rotated in a vertical plane on a 0.5-m long rope. What is the minimum velocity of rotation at which water does not flow out of the bucket at the upper point of the trajectory?

Given: $l = 0.5$ m.

$v - ?$

Solution. At the upper point of the trajectory, the water in the bucket is acted upon by the force of gravity mg and the normal reaction N of the bottom (Fig. 61). Directing the Y-axis vertically downwards to the centre C of the circle and writing Newton's second law for the water in the bucket in projections on this axis, we obtain

$$mg + N = ma_y,$$

where $a_y = a_c = v^2/l$. This gives

$$mg + N = mv^2/l.$$

At the moment of separation of the water from the bottom of the bucket, $N = 0$, and hence $mg = mv^2/l$, which gives

$$v = \sqrt{gl},$$

$$v = \sqrt{9.8 \times 0.5} \text{ m/s} \simeq 2.2 \text{ m/s}.$$

114. At the lowest point of a wingover (Nesterov's loop), the pilot exerts a force of pressure of 7100 N on the seat of an aeroplane. The weight of the pilot is 80 kg and the radius of the loop is 250 m. Determine the velocity of the plane.

Given: $F = 7.1 \times 10^3$ N, $m = 80$ kg, $R = 2.5 \times 10^2$ m.

$v - ?$

Solution. The pilot experiences the action of the force of gravity mg and the normal reaction N of the seat

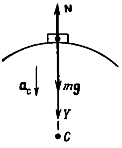


Fig. 60

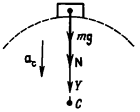


Fig. 61

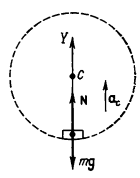


Fig. 62

(Fig. 62). Directing the Y-axis to the centre C of the circle and writing Newton's second law for the pilot in projections on this axis, we obtain

$$N - mg = ma_c,$$

or

$$N - mg = mv^2/R,$$

whence $v = \sqrt{(N - mg)R/m}$. According to Newton's third law, $N = F$, and hence

$$v = \sqrt{\frac{(F - mg)R}{m}},$$

$$v = \sqrt{\frac{(7.1 \times 10^3 - 80 \times 9.8) \times 2.5 \times 10^2}{80}} \ \frac{\text{m}}{\text{s}} \simeq 140 \text{ m/s}.$$

115. A ball of mass 200 g suspended on a string describes a circle in a horizontal plane at a constant velocity. Determine the velocity of the ball and the period of its rotation if the length of the string is 1 m and its angle with the vertical is 60°.

Given: $m = 200\,\text{g} = 0.2\,\text{kg}$, $l = 1\,\text{m}$, $\alpha = 60° \simeq 1.05$ rad.

v—? T—?

Solution. The ball experiences the action of the force of gravity mg and the tension **T** of the string (Fig. 63). Writing Newton's second law for the ball in vector form,

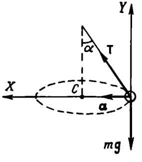


Fig. 63

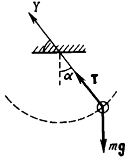


Fig. 64

we obtain

$$mg + \mathbf{T} = m\mathbf{a}. \quad (1)$$

Projecting Eq. (1) onto the chosen directions of the X- and Y-axes, we get

$$T \sin \alpha = ma_x, \quad -mg + T \cos \alpha = ma_y. \quad (2)$$

Assuming that $a_y = 0$ (the ball does not move in the vertical direction), $a_x = v^2/R$, $R = l \sin \alpha$, and substituting the expressions for a_x, a_y, and R into Eqs. (2), we obtain

$$T \sin \alpha = mv^2/(l \sin \alpha), \quad T \cos \alpha = mg. \quad (3)$$

Dividing the equations (3) termwise, we obtain

$$v = \sin \alpha \sqrt{\frac{gl}{\cos \alpha}},$$

$$v = 0.87 \sqrt{\frac{9.8 \times 1}{0.5}} \ \frac{\text{m}}{\text{s}} \simeq 3.8 \text{ m/s}.$$

The period of uniform motion of the ball in a circle is

$$T = 2\pi R/v = (2\pi l/v) \sin \alpha,$$

$$T = \frac{2 \times 3.14 \times 1 \times 0.87}{3.8} \text{ s} \simeq 1.4 \text{ s}.$$

116. A ball of mass 500 g suspended on a 1-m long inextensible string performs oscillations in a vertical plane. Determine the tension of the string at the instant when it forms an angle of 60° with the vertical if the velocity of the ball at this moment is 1.5 m/s.

Given: $m = 500 \text{ g} = 0.5 \text{ kg}$, $l = 1 \text{ m}$, $\alpha = 60° \simeq 1.05 \text{ rad}$, $v = 1.5 \text{ m/s}$.

T — ?

Solution. The ball is acted upon by the force of gravity $m\mathbf{g}$ and the tension $\mathbf{T}$ of the string (Fig. 64). Writing Newton's second law for the ball in vector form, we get

$$m\mathbf{g} + \mathbf{T} = m\mathbf{a}. \quad (1)$$

Directing the Y-axis along the radius and projecting Eq. (1) onto this axis, we obtain

$$T - mg \cos \alpha = ma_y. \quad (2)$$

Considering that $a_y = a_c = v^2/l$, we write Eq. (2) in the form $T - mg\cos\alpha = mv^2/l$, whence

$$T = m\left(\frac{v^2}{l} + g\cos\alpha\right),$$

$$T = 0.5 \times \left(\frac{1.5^2}{1} + 9.8 \times 0.5\right)\text{ N} \simeq 3.6 \text{ N}.$$

117. A hemispherical cup of radius 20 cm rotates about a vertical axis at a frequency of 2 s^{-1}. It contains a small ball rotating with it. Determine the angle between the radius drawn to the position of the ball and the vertical.

Given: $R = 20$ cm, $\nu = 2$ s^{-1}.

$\alpha - ?$

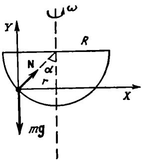


Fig. 65

Solution. The ball experiences the action of the force of gravity mg and the normal reaction **N** of the inner surface of the cup (Fig. 65). Writing Newton's second law for the ball in vector form, we get

$$m\mathbf{g} + \mathbf{N} = m\mathbf{a}. \quad (1)$$

Directing the X-axis towards the centre of the circle of radius r in which the ball moves and the Y-axis along the vertical and projecting Eq. (1) onto these axes, we obtain

$$N\sin\alpha = ma_x, \quad N\cos\alpha - mg = 0.$$

Since $a_x = a_c = v^2/r = \omega^2 r$, $\omega = 2\pi\nu$, and $r = R\sin\alpha$, we get

$$N\sin\alpha = 4\pi^2\nu^2 Rm\sin\alpha, \quad N\cos\alpha = mg. \quad (2)$$

Dividing Eqs. (2) termwise, we find that $\tan\alpha = (4\pi^2\nu^2 R/g)\sin\alpha$, whence $\cos\alpha = g/(4\pi^2\nu^2 R)$ and

$$\alpha = \arccos\frac{g}{4\pi^2\nu^2 R},$$

$$\alpha = \arccos\frac{9.8}{4 \times 3.14^2 \times 2^2 \times 0.2} \simeq 1.26 \text{ rad}.$$

118. What must be the height of the outer rail relative to the inner rail on a bend in a railway track of radius

300 m if the width of the track is 1524 mm? Assume that the normal velocity at which the force of pressure on the rails is perpendicular to them is 54 km/h.

Given: $R = 300$ m, $l = 1524$ mm $= 1.524$ m,
$v = 54$ km/h $= 15$ m/s.

h — ?

Solution. In order to impart a velocity v to the train moving in a circle of radius R', the forces acting on it must create a centripetal acceleration in the horizontal direc-

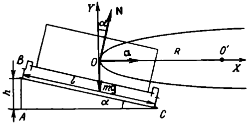


Fig. 66

tion (by hypothesis) (Fig. 66). The track is banked in order to improve the stability of the train and to reduce the wear of the rails. Then the resultant of the force of gravity mg and the normal reaction **N** of the rails imparts the required centripetal acceleration to the train. Writing Newton's second law for the train in projections on the X- and Y-axes (see Fig. 66), we obtain

$$N \sin\alpha = ma_x, \quad N\cos\alpha - mg = 0,$$

where $a_x = a_c = v^2/R$, whence $\tan\alpha = v^2/(Rg)$ and $\tan\alpha = 15^2/(9.8 \times 300) \simeq 7.7 \times 10^{-2}$. Since $\tan\alpha$ is a small quantity, $\sin\alpha \approx \tan\alpha$.

On the other hand, from $\triangle ABC$ we have $\sin\alpha = h/l$. Consequently, we obtain

$$h = \frac{lv^2}{gR},$$

$$h = 1.524 \times 7.7 \times 10^{-2} \text{ m} \simeq 0.12 \text{ m}.$$

119. Why is a passenger standing at a door to the right (relative to the direction of motion) of a moving car of a metro train pressed against the door as the train turns to the left round a bend?

Answer. As the train turns to the left round the bend, the person retains the previous motion by inertia and is pressed against the door.

120. Why do rivers of the northern hemisphere erode their right banks?

Answer. The Earth rotates from west to east. It is well known that the linear velocity of rotation of points on the Earth's surface decreases from the equator to the poles. Therefore, water in a river flowing to north will be deflected eastwards by inertia, retaining its previous velocity, and will erode the right bank.

EXERCISES

121. Determine the velocity of a lorry of mass 2 t moving over a concave bridge of radius 100 m if it exerts a force of pressure of 25 kN on the middle of the bridge.

122. A weight suspended on a 30-cm long string describes a circle of radius 15 cm in a horizontal plane. Determine the frequency of its rotation.

123. A ball is suspended on a string from the ceiling of a tram moving at a velocity of 9 km/h round a bend of radius 36.4 m. Through what angle will the string with the ball be deflected?

124. A boy is swinging on a giant stride at a frequency of 16 min^{-1}. The length of the ropes is 5 m. Determine the tension of the ropes if the weight of the boy is 45 kg.

125. An aeroplane flying at a velocity of 720 km/h makes a loop of radius 400 m in a vertical plane. Determine the force of pressure exerted by the fuel on the bottom of the tank of area 1 m^2, filled with fuel to 0.8 m, at the lowest point of the loop.

126. A skater runs at a velocity of 12 m/s in a circle of radius 50 m. At what angle to the horizontal must he incline his body to maintain the equilibrium?

127. A disc rotates in a horizontal plane at a frequency of 30 min^{-1}. What must be the coefficient of friction between the disc and a body lying on it for the body to remain on the disc? The distance from the axis of the disc to the body is 20 cm.

128. A ball suspended on a 50-cm long string rotates uniformly in a vertical plane. Determine the frequency at which the string breaks if the ultimate strength of the string is $9mg$, where m is the mass of the ball.

APPLICATION OF BASIC LAWS OF DYNAMICS TO SPACE FLIGHTS

Along with the laws of dynamics considered above, the **law of universal gravitation** will be used in this section:

$$F = Gm_1m_2/r^2,$$

where F is the force of interaction **(gravitational force)** between two point masses m_1 and m_2 separated by a distance r, and G the **gravitational constant**.

A gravitational force acts on a body in a gravitational field irrespective of whether or not there is a support. In a free fall, the body is not acted upon by the normal reaction, and the state of **weightlessness (zero gravity)** is observed. For example, zero gravity sets in when a satellite orbits round a planet.

The **orbital** (or **circular**) **velocity** is the minimum initial velocity at which a body starts to move parallel to the surface of a planet and becomes its artificial satellite. This velocity is different at different altitudes and for different celestial bodies. Near the Earth's surface, the orbital (circular) velocity $v_1 = 7.91$ km/s is given by

$$v_1 = \sqrt{g_0R},$$

where g_0 is the **free-fall acceleration** at the surface of the planet and R the radius of the planet.

The minimum initial velocity that must be imparted to a body starting to move near the surface of a planet to overcome its attraction is known as the **escape velocity:**

$$v_2 = \sqrt{2g_0R}.$$

This velocity varies with altitude and is different for different celestial bodies. For the Earth, $v_2 = 11.19$ km/s $\simeq$ 11.2×10^3 m/s.

If a body moves at the escape velocity at the initial instant and does not experience the action of any force other than the gravitational force, it will move relative to the celestial body in a parabola.

* * *

129. Calculate the free-fall acceleration for a body at a distance of 100 km above the Earth's surface.

Given: $h = 100\ \text{km} = 10^5\ \text{m}$.

g — ?

Solution. The body at a distance h above the Earth's surface is acted upon only by the gravitational force directed towards the centre of the Earth and equal to

$$F = GMm/r^2, \tag{1}$$

where M is the mass of the Earth, m the mass of the body, and r the distance from the centre of the Earth to the body.

Under the action of this force, the body will move vertically downwards with acceleration g. Writing Newton's second law for the body in projections on the Y-axis directed vertically downwards, we obtain

$$F = mg,$$

or, taking into account formula (1), we get

$$GMm/r^2 = mg,$$

whence

$$g = GM/r^2.$$

Here $r = R + h$, where R is the Earth's radius and h the altitude of the body. Then at the altitude h

$$g = G\frac{M}{(R+h)^2}. \tag{2}$$

At the Earth's surface $h \simeq 0$, and hence $r = R$ and $g = GM/R^2$. The value of this acceleration at the Earth's surface will be henceforth denoted by g_0:

$$\begin{aligned} g_0 &= \frac{GM}{R^2}, \\ g_0 &= \frac{6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{(6.37 \times 10^6)^2}\ \frac{\text{m}}{\text{s}^2} \simeq 9.8\ \text{m/s}^2. \end{aligned} \tag{3}$$

Dividing Eq. (2) by (3) termwise, we obtain $g/g_0 = R^2/(R+h)^2$, whence $g = g_0R^2/(R+h)^2$. Since $h/R \ll 1$, we can use the approximate relation $(1 + h/R)^{-2} \approx 1 - 2h/R$. This gives

$$g = g_0 \frac{R^2}{R^2(1+h/R)^2} \approx g_0\left(1-\frac{2h}{R}\right),$$

$$g = 9.8 \times \left(\frac{1-2\times10^5}{6.37\times10^6}\right)\frac{\text{m}}{\text{s}^2} \simeq 9.51\ \text{m/s}^2.$$

130. A spaceship of mass 10^6 kg is launched vertically upwards. The driving force of its engines is 2.94×10^7 N. Determine the acceleration of the spaceship and the weight of a body in it if the force of gravity acting on this body on the Earth's surface is 5.88×10^2 N.

Given: $m_1 = 10^6$ kg, $F_d = 2.94 \times 10^7$ N, $m_2g_0 = 5.88 \times 10^2$ N.

a—? P_2—?

Solution. The spaceship is acted upon by the driving force $\mathbf{F}_d$ of its engines and the gravitational force $\mathbf{F}$ (Fig. 67). According to the law of universal gravitation,

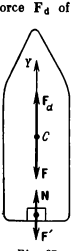


Fig. 67

$$F = GMm_1/r^2,$$

where $r = R + h$ (see Problem **129**). Since $h \ll R$ at the beginning of the flight, neglecting h in comparison with R, we can assume that $r \approx R$, and hence

$$F = GMm_1/R^2.$$

Writing Newton's second law for the spaceship in projections on the Y-axis, we obtain

$$F_d - F = m_1a, \quad \text{or}$$

$$F_d - GMm_1/R^2 = m_1a,$$

whence

$$a = \frac{F_d}{m_1} - \frac{GM}{R^2} = \frac{F_d}{m_1} - g_0$$

since $GM/R^2 = g_0$. This gives

$a = (2.94 \times 10^7 \times 10^{-6} - 9.8)\ \text{m/s}^2 = 19.6\ \text{m/s}^2 = 2g_0$.

Let us determine the weight of the body in the flying spaceship. It should be recalled that the weight of a body is the force exerted by it on a constraint (the force with which it stretches a spring or presses on a support). In the case under consideration, the weight of the body in the launched spaceship is the force of pressure F_{pr} exerted by the body on the ship. In order to determine this force, let us consider the forces acting on the body in the ship. There are two such forces, viz. the normal reaction N of the ship and the gravitational force $F' = GMm_2/r^2 \approx GMm_2/R^2$. Writing Newton's second law for the body in projections on the Y-axis, we obtain

$$N - F' = m_2 a,$$

whence

$$N = F' + m_2 a = GMm_2/R^2 + m_2 a = m_2 (GM/R^2 + a).$$

Considering that the ship and the body move upwards with acceleration $a = 2g_0$, we find that $N = m_2 (g_0 + 2g_0) = 3m_2 g_0$. According to Newton's third law,

$$F_{pr} = N = 3m_2 g_0,$$

$$F_{pr} = 3 \times 5.88 \times 10^2 \text{ N} \simeq 1.76 \text{ kN}.$$

131. Having approached an unknown planet, the astronauts impart a horizontal velocity of 11 km/s to their spacecraft. This velocity ensures the flight of the spacecraft in a circular orbit of radius 9100 km. What is the free-fall acceleration at the surface of the planet if its radius is 8900 km?

Given: $v = 11$ km/s $= 1.1 \times 10^4$ m/s, $r = 9100$ km $= 9.1 \times 10^6$ m, $R = 8900$ km $= 8.9 \times 10^6$ m.

g_0 — ?

Solution. The spacecraft is under the action of only the gravitational force exerted by the planet and directed to its centre. According to the law of universal gravitation, this force is

$$F = GMm/r^2,$$

where M is the mass of the planet, m the mass of the spacecraft, and r the distance from the centre of the planet to the spacecraft (the radius of the orbit). Multiplying the numerator and the denominator by R^2, we obtain

$$F = G\frac{Mm}{r^2} = G\frac{M}{R^2}m\frac{R^2}{r^2} = g_0 m\frac{R^2}{r^2}. \quad (1)$$

As for the Earth, here $g_0 = GM/R^2$ is the free-fall acceleration at the surface of the planet. Writing Newton's second law for the spacecraft in projections on the Y-axis directed to the centre of the planet, we obtain

$$F = ma_y, \quad (2)$$

where

$$a_y = a_c = v^2/R. \quad (3)$$

Substituting Eqs. (1) and (3) into (2), we obtain $g_0 mR^2/r^2 = mv^2/r$, whence

$$g_0 = \frac{v^2 r}{R^2},$$

$$g_0 = \frac{(1.1\times 10^4)^2\times 9.1\times 10^6}{(8.9\times 10^6)^2}\ \frac{\text{m}}{\text{s}^2} \simeq 14\ \text{m/s}^2.$$

132. A body on the equator of a planet weighs half its weight at the pole. The density of matter of the planet is

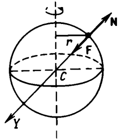


Fig. 68

3 g/cm³. Determine the period of rotation of the planet about its axis.

Given: $P_{eq} = P_p/2$, $\rho = 3\ \text{g/cm}^3 = 3\times 10^3\ \text{kg/m}^3$.

T — ?

Solution. The body on the surface of the planet is acted upon by the gravitational force **F** exerted by the planet and the normal reaction **N** of the surface of the planet (Fig. 68). At the pole, the body is at rest relative to an inertial reference frame. In this case,

$$\mathbf{F} + \mathbf{N}_p = 0 \quad \text{or} \quad F = N_p.$$

According to the law of universal gravitation,

$$F = GMm/R^2,$$

where m is the mass of the body, M the mass of the planet, and R the radius of the planet. By the definition of density, $M = \rho V$, where V is the volume of the planet. Assuming that the planet is spherical, we can write $V = (4/3)\pi R^3$ and $M = (4/3)\pi R^3\rho$. Then the gravitational force is

$$F = G\frac{4\pi R^3\rho m}{3R^2} = (4/3)\pi G\rho m R, \tag{1}$$

whence $N_p = (4/3)\pi G\rho m R$.

On the equator, the body rotates together with the planet, i.e. has an acceleration $a_c = \omega^2 R = 4\pi^2 R/T^2$, where T is the period of rotation of the planet. According to Newton's second law, $F - N_{eq} = ma_c$, or

$$(4/3)\pi G\rho m R - N_{eq} = 4\pi^2 Rm/T^2. \tag{2}$$

Since according to Newton's third law the normal reaction is equal to the weight of the body and, by hypothesis, $P_{eq} = P_p/2$, $N_{eq} = N_p/2 = (2/3)\pi G\rho m R$, we obtain

$$(4/3)\pi G\rho m R - (2/3)\pi G\rho m R = 4\pi^2 Rm/T^2. \tag{3}$$

Finally,

$$T = \sqrt{\frac{6\pi}{G\rho}},$$

$$T = \sqrt{\frac{6 \times 3.14}{6.67 \times 10^{-11} \times 3 \times 10^3}}\ \text{s} \simeq 9.7 \times 10^3\ \text{s}.$$

133. A satellite moves in a circular orbit in the equatorial plane at an altitude equal to the Earth's radius. At what velocity must a terrestrial observer move for the satellite to appear above him every 5 s? Consider the cases when the directions of motion of the satellite and the rotation of the Earth coincide and are opposite.

Given: $h = R$, $T = 24\ \text{h} = 8.64 \times 10^4\ \text{s}$, $t = 5\ \text{h} = 1.8 \times 10^4\ \text{s}$.

v — ?

Solution. The satellite moving in a circular orbit is acted upon by the gravitational force $F = GMm/r^2$ exerted by the Earth. Considering that $r = 2R$ by hypothesis, we transform the expression for the force:

$$F = GMm/(4R^2) = (GM/R^2)(m/4) = g_0 m/4. \quad (1)$$

Here $g_0 = GM/R^2$ is the free-fall acceleration at the Earth's surface. Writing Newton's second law for the

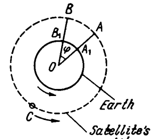


Fig. 69

satellite in projections on the Y-axis directed to the centre of the Earth, we obtain

$$F = ma_y, \quad (2)$$

where

$$a_y = a_c = v^2/r = v^2/(2R). \quad (3)$$

Substituting Eqs. (1) and (3) into (2), we obtain

$$g_0 m/4 = mv^2/(2R),$$

whence $v = \sqrt{g_0 R/2}$. The angular velocity of the satellite is $\omega = v/r = v/(2R) = \sqrt{g_0/(8R)}$

We shall consider two cases.

1. The satellite moves in the direction of rotation of the Earth (Fig. 69). During a time t, the satellite traverses a distance $ABCAB = s = vt$ along its orbit and reaches point B separated from point A by the distance

$$|AB| = vt - 2\pi r = \omega r t - 2\pi r = r(\omega t - 2\pi)$$
$$= 2R(\omega t - 2\pi). \quad (4)$$

For the satellite to be above the observer, the latter must reach point B in the same time, i.e. must cover the distance $|A_1B_1|$ at a velocity $v_1 + v_2$, where v_1 is the velocity of the observer relative to the Earth and v_2 the linear velocity of the Earth. Consequently,

$$|A_1B_1| = (v_1 + v_2)\,t. \qquad (5)$$

The figure shows that the angular displacements of the satellite and the observer during the time t are the same, i.e. $\varphi_1 = \varphi_2$. Since $\varphi_1 = |A_1B_1|/R$ and $\varphi_2 = |AB|/(2R)$, taking into account formulas (4) and (5), we obtain

$$\frac{(v_1+v_2)\,t}{R} = \frac{2R\,(\omega t - 2\pi)}{2R}.$$

Considering that $v_2 = 2\pi R/T$ and $\omega = \sqrt{g_0/(8R)}$, we find that

$$\frac{(v_1+2\pi R/T)\,t}{R} = \sqrt{\frac{g_0}{8R}}\,t - 2\pi,$$

whence

$$v_1 = \left(\sqrt{\frac{g_0}{8R}} - \frac{2\pi}{t} - \frac{2\pi}{T}\right)R,$$

$$v_1 = \left(\sqrt{\frac{9.8}{8\times 6.37\times 10^6}} - \frac{2\times 3.14}{1.8\times 10^4} - \frac{2\times 3.14}{8.64\times 10^4}\right) \times 6.37\times 10^6 \text{ m/s} \simeq 1.04\times 10^2 \text{ m/s}.$$

2. The satellite moves against the direction of rotation of the Earth. We leave it to the reader to verify that in this case

$$v_1 = \left(\sqrt{\frac{g_0}{8R}} - \frac{2\pi}{t} + \frac{2\pi}{T}\right)R \simeq 1.1\times 10^3 \text{ m/s}.$$

134. The average altitude of a satellite above the Earth's surface is 1700 km. Determine its orbital velocity and the period of revolution.

Given: $h = 1700$ km $= 1.7\times 10^6$ m.

v — ? T — ?

Solution. The motion in a circular orbit occurs only under the action of the gravitational force exerted by the Earth:

$$F = GmM/(R + h)^2, \tag{1}$$

where R is the Earth's radius. Writing Newton's second law for the satellite in projections on the Y-axis directed to the centre of the Earth, we obtain

$$F = ma_y, \tag{2}$$

where

$$a_y = a_c = v^2/(R + h). \tag{3}$$

Taking into account Eqs. (1) and (3), we transform Eq. (2): $GmM/(R + h)^2 = mv^2/(R + h)$, whence

$$v^2 = GM/(R + h). \tag{4}$$

Multiplying the numerator and the denominator of the right-hand side of Eq. (4) by R^2, we obtain

$$v^2 = (GM/R^2)\,(R^2/(R + h)),$$

where $GM/R^2 = g_0$ is the free-fall acceleration at the Earth's surface. Consequently,

$$v^2 = g_0R^2/(R + h), \tag{5}$$

whence

$$v = R\sqrt{\frac{g_0}{R+h}},$$

$$v = 6.37\times10^6\sqrt{\frac{9.8}{6.37\times10^6+1.7\times10^6}}\ \frac{\text{m}}{\text{s}} \simeq 7.01\times10^3\ \text{m/s}.$$

The period of revolution of the satellite in the circular orbit of radius $R + h$ is

$$T = \frac{2\pi\,(R+h)}{v},$$

$$T = \frac{2\times3.14\times(6.37\times10^6+1.7\times10^6)}{7.01\times10^3}\ \text{s} \simeq 7.24\times10^3\ \text{s}.$$

135. Can a satellite be launched so that it remains all the time above the same location on the Earth's surface?

Answer. This is possible if the period of revolution of the satellite is equal to the period of rotation of the Earth about its axis, and the plane of the orbit coincides with the equatorial plane of the Earth.

136. Can a match struck in a spacecraft orbiting the Earth burn?

Answer. All the objects on board the spacecraft are in the zero-gravity state (as long as its engines are switched off). For this reason, there is no convection in the spacecraft, and the combustion products will accumulate near the flame and extinguish it.

EXERCISES

137. Determine the gravitational force exerted by the Earth on a body of mass 1 kg on the Moon's surface. The distance between the centres of the Earth and the Moon should be taken as 384 000 km.

138. Determine the density of a planet whose day lasts for T hours if the bodies on the equator of the planet are known to be weightless.

139. A satellite completes 16 revolutions during one turn of the Earth. Determine the period, the altitude, and the velocity of the satellite, assuming that its orbit is circular.

140. The velocity of a satellite decreases with altitude from 7.79 to 7.36 km/s. Determine the change in the period of its revolution and in the separation from the Earth's surface.

141. An artificial satellite orbits a planet A with a period of revolution T_1. What will be the change in the period of revolution of the satellite if it orbits a planet B having the same density as that of the planet A but a twice as large radius? The satellite moves in both cases in circular orbits close to the surfaces of the planets.

142. Determine the g-factor for an astronaut rotating in a horizontal plane in a centrifuge of diameter 12 m at an angular velocity of 4.04 rad/s.

143. The period of revolution of a satellite moving in a circular orbit around the Earth is 240 min. Determine the separation of the orbit from the Earth's surface.

144. Determine the period of revolution and the orbital velocity of an artificial satellite moving around the

Moon at a distance of 200 km from its surface if the Moon's mass is 7.3×10^{22} kg and its radius is 1.7×10^{6} m.

145. Determine the period of revolution of an artificial satellite of a planet which can be regarded as a homogeneous sphere of density ρ.

MOMENTUM OF A BODY. MOMENTUM CONSERVATION LAW

Newton's second law can be written in the form

$$\mathbf{F}\,\Delta t = \Delta\,(m\mathbf{v}),$$

where $\mathbf{F}\,\Delta t$ is the **impulse of force** and $m\mathbf{v} = \mathbf{p}$ is the **momentum** of a body.

Let us write Newton's second law in differential form:

$$\mathbf{F}\,\mathrm{d}t = \mathrm{d}\,(m\mathbf{v}), \quad \text{or} \quad \mathbf{F} = \frac{\mathrm{d}\,(m\mathbf{v})}{\mathrm{d}t}.$$

If a system of interacting bodies is under the action of external forces, the following relation holds:

$$\sum_{i=1}^{n} \Delta\,(m_i\mathbf{v}_i) = \sum_{i=1}^{n} \mathbf{F}_i\,\Delta t,$$

where $\Delta m_i\mathbf{v}_i$ is the change in the momentum of a body of mass m_i of the system acted upon by an external force $\mathbf{F}_i$, and $\mathbf{F}_i\,\Delta t$ the impulse of this force.

In the absence of external forces (if the system is closed), the **momentum conservation law** is valid:

$$\sum_{i=1}^{n} m_i\mathbf{v}_i = \text{const},$$

where m_i is the mass of an individual body in the system of interacting bodies, $\mathbf{v}_i$ the velocity of this body, and n the number of bodies interacting in the system.

For a system consisting of two interacting bodies, the momentum conservation law has the form

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = m_1\mathbf{u}_1 + m_2\mathbf{u}_2$$

for elastic interaction and

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\,\mathbf{u}$$

for inelastic interaction. Here $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocities of the bodies before their interaction, and $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}$ the velocities of the bodies after their interaction.

While solving problems, we shall write the momentum conservation law in vector form, choose the directions of coordinate axes, and project both parts of the vector equation onto them.

* * *

146. A body of mass 0.2 kg falls from a height of 1 m with an acceleration of 8 m/s². Determine the change in the momentum of the body.

Given: $m = 0.2$ kg, $h = 1$ m, $a = 8$ m/s².

$\Delta(mv) - ?$

Solution. The body falls with an acceleration a, and its velocity changes. The change in momentum is

$$\Delta(m\mathbf{v}) = m\mathbf{v} - m\mathbf{v}_0 = m(\mathbf{v} - \mathbf{v}_0). \qquad (1)$$

Projecting both sides of Eq. (1) onto the Y-axis directed vertically downwards, we obtain

$$\Delta(mv) = m(v - v_0). \qquad (2)$$

Since the body starts to fall from the state of rest, its initial velocity $v_0 = 0$. The final velocity v can be determined from the equation $v = \sqrt{2ah}$. Using this expression in Eq. (2), we obtain

$$\Delta(mv) = m\sqrt{2ah},$$

$$\Delta(mv) = 0.2\sqrt{2 \times 8 \times 1}\ \text{kg}\cdot\text{m/s} = 0.8\ \text{kg}\cdot\text{m/s}.$$

147. A molecule of mass 5×10^{-26} kg flying at a velocity of 500 m/s elastically impinges on a wall at an angle of 30° to the normal. Determine the impulse of the force transferred to the wall during the impact.

Given: $m = 5 \times 10^{-26}$ kg, $v = 500$ m/s, $\alpha = 30° \simeq 0.52$ rad.

$F\Delta t - ?$

Solution. We direct the X-axis along the normal to the wall and the Y-axis vertically upwards (Fig. 70). Newton's third law implies that the impulse of the force acting on the wall is numerically equal to the impulse of

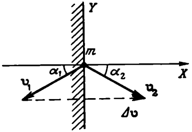


Fig. 70

the force acting on the molecule. According to Newton's second law, the impulse of the force acting on the molecule is

$$F\,\Delta t = m\,(\mathbf{v}_2 - \mathbf{v}_1),$$

or, in projections on the X- and Y-axes,

$$(F\,\Delta t)_x = m\,(v_{2x} - v_{1x}),$$
$$(F\,\Delta t)_y = m\,(v_{2y} - v_{1y}).$$

Since the impact is elastic, $v_1 = v_2 = v$ and $\alpha_1 = \alpha_2 = \alpha$. Therefore, $v_{1x} = -v\cos\alpha$, $v_{2x} = v\cos\alpha$ and $v_{1y} = -v\sin\alpha$, $v_{2y} = -v\sin\alpha$, whence

$$(F\,\Delta t)_x = 2mv\cos\alpha, \quad (F\,\Delta t)_y = 0.$$

By definition, the impulse of a force is

$$F\,\Delta t = \sqrt{(F\,\Delta t)_x^2 + (F\,\Delta t)_y^2}.$$

Taking into account the expressions for $(F\,\Delta t)_x$ and $(F\,\Delta t)_y$, we obtain

$$F\,\Delta t = \sqrt{(2mv\cos\alpha)^2} = 2mv\cos\alpha,$$

$$F\,\Delta t = 2\times 5\times 10^{-26}\times 500\times 0.87\ \text{N}\cdot\text{s} = 4.35\times 10^{-23}\ \text{N}\cdot\text{s}.$$

148. A shell of mass 100 kg flying along a railroad horizontally at a velocity of 500 m/s hits a carriage with sand of mass 10 t and gets stuck in it. Determine the velocity of the carriage after the impact if it moved towards the shell at a velocity of 36 km/h.

Given: $m_1 = 100$ kg, $v_1 = 500$ m/s, $m_2 = 10$ t $= 10^4$ kg, $v_2 = 36$ km/h $= 10$ m/s.

u — ?

Solution. Writing the momentum conservation law for the inelastic collision between the shell and the carriage with sand, we obtain

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\,\mathbf{u}. \qquad (1)$$

Choosing the direction of the X-axis along the motion of the shell and projecting both sides of Eq. (1) onto it, we get $m_1v_1 - m_2v_2 = (m_1 + m_2)\,u$, whence

$$u = \frac{m_1v_1 - m_2v_2}{m_1 + m_2},$$

$$u = \frac{100 \times 500 - 10^4 \times 10}{100 + 10^4}\ \frac{\text{m}}{\text{s}} = -5\ \text{m/s}.$$

Consequently, the direction of motion of the carriage has not changed.

149. A grenade flying at a velocity of 15 m/s explodes into two fragments of mass 6 and 14 kg. The velocity of the larger fragment has increased to 24 m/s in the direction of motion. Determine the velocity and the direction of the smaller fragment.

Given: $v = 15$ m/s, $u_2 = 24$ m/s, $m_1 = 6$ kg, $m_2 = 14$ kg.

u_1 — ?

Solution. Directing the X-axis along the motion of the grenade and writing the momentum conservation law in projections on this axis, we obtain

$$(m_1 + m_2)\,v = m_1u_1 + m_2u_2, \qquad (1)$$

where $m_1 + m_2$ is the mass of the grenade. From this relation, we get

$$u_1 = \frac{(m_1 + m_2)\,v - m_2u_2}{m_1},$$

$$u_1 = \frac{(6 + 14) \times 15 - 14 \times 24}{6}\ \frac{\text{m}}{\text{s}} = -6\ \text{m/s}.$$

The minus sign indicates that the smaller fragment will fly in the direction opposite to the direction of motion of the grenade.

150. The third stage of a rocket consists of a carrier rocket of mass 500 kg connected through a compressed spring with a head cone of mass 10 kg. During terrestrial tests, the spring imparts a velocity of 5.1 m/s to the cone relative to the carrier rocket. Determine the velocities of the cone and the rocket if they are separated while orbiting the Earth at a velocity of 8 km/s.

Given: $m_1 = 5 \times 10^2$ kg, $m_2 = 10$ kg, $v_0 = 5.1$ m/s, $v = 8$ km/s $= 8 \times 10^3$ m/s.

u_1 — ? u_2 — ?

Solution. Writing the momentum conservation law for the third stage of the rocket in projections on the X-axis coinciding with the direction of orbital motion of the rocket, we obtain

$$(m_1 + m_2)\, v = m_1 u_1 + m_2 u_2, \qquad (1)$$

where m_1 is the mass of the carrier rocket, m_2 the mass of the cone, u_1 the velocity of the carrier rocket relative to the Earth after its separation from the cone, and u_2 the velocity of the cone relative to the Earth after its separation from the rocket.

Since the velocity of the satellite relative to the carrier rocket after the separation is the same as under terrestrial conditions, $u_2 - u_1 = v_0$, or

$$u_2 = u_1 + v_0, \quad u_1 = u_2 - v_0. \qquad (2)$$

Substituting consecutively each equation of (2) into Eq. (1), we obtain

$$u_1 = v - \frac{m_2}{m_1 + m_2} v_0,$$

$$u_1 = 8 \times 10^3 - \frac{10 \times 5.1}{5 \times 10^2 + 10}\ \frac{\text{m}}{\text{s}} \simeq 8 \times 10^3 \text{ m/s},$$

$$u_2 = v + \frac{m_1}{m_1 + m_2} v_0,$$

$$u_2 = 8 \times 10^3 + \frac{5 \times 10^2 \times 5.1}{5 \times 10^2 + 10}\ \frac{\text{m}}{\text{s}} \simeq 8005 \text{ m/s}.$$

151. Why is the blow of a hammer against an anvil on the chest of a circus performer harmless, while the same blow made directly against his chest would be lethal?

Answer. Since the mass of the anvil is larger than the mass of the hammer, the velocity acquired by it upon an inelastic collision with the hammer is small, and hence the force exerted on the chest is not very strong. Besides, the area of contact of the anvil with the performer's body is larger than that of the hammer. Therefore, the additional pressure exerted by the anvil on the body during the impact is lower than the pressure produced by the hammer. For these reasons, the blow is harmless.

EXERCISES

152. A meteorite and a rocket move at right angles relative to each other. The rocket hits the meteorite and gets stuck in it. The mass of the meteorite is m, the mass of the rocket is $m/2$, the velocity of the meteorite and the rocket being v and $2v$ respectively. Determine the momenta of the meteorite and the rocket after the collision.

153. A shell of mass 20 kg, flying horizontally at a velocity of 500 m/s, hits a flatcar with sand of mass 10 t and gets stuck in it. Determine the velocity acquired by the flatcar as a result of the impact.

154. What velocity will be acquired by a stationary boat having a mass of 200 kg with a load as a result of a shot fired by a passenger in the horizontal direction? The mass of the bullet is 10 g and its velocity is 800 m/s.

155. A shell of mass 50 kg, flying along a railway track at a velocity of 600 m/s, hits a flatcar with sand of mass 10 t and gets stuck in it. The velocity vector of the shell forms an angle of 45° with the horizontal at the moment of impact. Determine the velocity of the flatcar after the collision if the flatcar moves towards the shell at a velocity of 10 m/s.

156. A rocket whose mass without propellant is 400 g rises to an altitude of 125 m as a result of combustion of 50 g of fuel in it. Determine the velocity of the gas ejected from the rocket, assuming that the fuel burns instantaneously.

157. Two balls of mass 6 and 4 kg move along the same straight line at a velocity of 8 and 3 m/s respectively. What will their velocity be after a perfectly inelastic collision if (1) the first ball catches up with the second, (2) the balls move towards each other?

158. A boat of mass 150 kg and length 2 m, which is at rest on the surface of a lake, faces the shore at a distance of 0.7 m. A man whose weight is 70 kg goes over from the bow to the stern. Will the boat berth?

QUESTIONS FOR REVISION

1. Define force and mass and name their SI units of measurement. 2. What is density? 3. Formulate Newton's laws. 4. What is inertia? 5. List the types of force in mechanics. 6. Define the momentum of a body. 7. Formulate the momentum conservation law. 8. Formulate the law of universal gravitation. What is the physical meaning of the gravitational constant? 9. Why does the force of gravity vary with the altitude of a body above the Earth's surface? Does the mass of the body change in the process? 10. Define the orbital and escape velocities.

1.3. Work, Power, and Energy. Energy Conservation Law

The work done by a constant force is defined as

$$A = Fs\cos\alpha,$$

where F is the force acting on a body, s the displacement of the body under the action of the force, and α the angle between the directions of the force and the displacement. If $\alpha < \pi/2$, then $A > 0$, if $\alpha = \pi/2$, then $A = 0$, and if $\alpha > \pi/2$, then $A < 0$.

If a varying force acts on a body, then

$$A = \langle F\cos\alpha\rangle\, s,$$

where $\langle F\cos\alpha\rangle$ is the mean value of the projection of the force on the displacement. A more rigorous definition of the work done by a varying force is

$$A = \int_{s_1}^{s_2} F_s\, ds,$$

where $F_s = F \cos \alpha$ is the projection of the force on the displacement.

Power is defined as

$$N = A/t,$$

where t is the time during which the work is done.

For a uniform motion,

$$N = Fv,$$

where v is the velocity of the motion.

For a nonuniform motion, the concepts of **instantaneous** and **average power** are introduced:

$$N = \frac{dA}{dt} = Fv, \quad \langle N \rangle = F \langle v \rangle = A/t,$$

where v is the instantaneous velocity of the motion and $\langle v \rangle$ the average velocity of the varying motion.

The **efficiency** of a mechanism is defined as

$$\eta = A_u/A_d, \quad \eta = N_u/N_d, \quad \eta = W_u/W_d,$$

where A_u (N_u, W_u) is the useful work (power, energy) of the mechanism and A_d (N_d, W_d) is the work (power, energy) done by the mechanism.

There are two types of mechanical energy: **kinetic energy** and **potential energy**.

A moving body has a kinetic energy

$$W_k = mv^2/2,$$

where m is the mass of the moving body and v its velocity.

A body located above the ground has a potential energy

$$W_p = mgh,$$

where h is the height of the body above the ground.

A compressed or stretched spring has a potential energy

$$W_p = kx^2/2,$$

where k is the spring constant and x the compression (extension) of the spring.

The **total mechanical energy** W of a body is the sum of its kinetic and potential energies:

$$W = W_k + W_p.$$

A system of bodies is **closed** if bodies which do not constitute the system do not act on it. For a closed system of bodies (in which only conservative forces are acting), the total mechanical energy remains constant (the **law of conservation of mechanical energy**):

$$\sum_{i=1}^{n} W_i = \text{const},$$

where W_i is the total mechanical energy of an individual body of the system and n the number of bodies in the system.

For a body moving at a velocity v at a height h above the ground, the law of conservation of mechanical energy has the form

$$mv^2/2 + mgh = \text{const}.$$

When external forces are acting on a body (system of bodies), the total mechanical energy of the body (system of bodies) changes. The following relation holds in this case:

$$A = \Delta W = W - W_0,$$

where A is the work done by the external forces, ΔW the change in the total mechanical energy of the body (system of bodies), and W and W_0 the final and initial values of the mechanical energy of the body (system of bodies). The same relation is valid when nonconservative forces (like friction) are acting in a closed system of bodies. In this case, A is the work done by the nonconservative forces.

* * *

159. A bullet flying at a velocity of 400 m/s hits a barrage and comes to a halt over a distance of 0.5 m. Determine the resistance offered to the motion of the bullet if its mass is 24 g.

Given: $v_0 = 400$ m/s, $s = 0.5$ m, $m = 24$ g $= 2.4 \times 10^{-2}$ kg.

F — ?

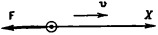


Fig. 71

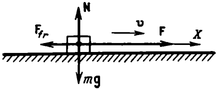


Fig. 72

Solution. We direct the X-axis along the motion of the bullet (Fig. 71). Since $\alpha = \pi$, $\cos\alpha = -1$, and

$$A = -Fs. \qquad (1)$$

On the other hand,

$$A = W - W_0. \qquad (2)$$

By hypothesis, $W = 0$ (the bullet has stopped), and

$$W_0 = mv_0^2/2. \qquad (3)$$

Substituting Eqs. (1) and (3) into (2), we obtain $-Fs = -mv_0^2/2$, whence

$$F = \frac{mv_0^2}{2s},$$

$$F = \frac{2.4 \times 10^{-2} \times 400^2}{2 \times 0.5}\ \mathrm{N} \simeq 3.8\ \mathrm{kN}.$$

160. A train of mass 600 t acquires a velocity of 60 km/h over a distance of 2.5 km from the previous station. What is the average power developed by the locomotive if the coefficient of friction is 0.005?

Given: $m = 600\ \mathrm{t} = 6 \times 10^5\ \mathrm{kg}$, $s = 2.5\ \mathrm{km} = 2.5 \times 10^3\ \mathrm{m}$, $v = 60\ \mathrm{km/h} = 16.7\ \mathrm{m/s}$, $\mu = 5 \times 10^{-3}$.

$\langle N \rangle$ — ?

Solution. We direct the X-axis along the motion of the train. The train is acted upon by the force of gravity $m\mathbf{g}$, the normal reaction $\mathbf{N}$ of the rails, the driving force $\mathbf{F}$, and the friction $\mathbf{F}_{fr}$ (Fig. 72). Since the change in the kinetic energy must be equal to the work done by external forces, $\Delta W = A$, or

$$W - W_0 = A_1 + A_2, \qquad (1)$$

where $W = mv^2/2$, $W_0 = 0$, $A_1 = Fs$ is the work done by the driving force, and $A_2 = -F_{fr}s$ the work done by friction. The work done by the force of gravity and by the normal reaction is zero since these forces are perpendicular to the displacement. For a horizontal surface, friction is defined as $F_{fr} = \mu N = \mu mg$. Substituting the expressions for W, W_0, A_1, and A_2 into Eq. (1), we obtain $mv^2/2 = Fs - \mu mgs$. Then the driving force is

$$F = \frac{mv^2}{2s} + \mu mg = m\left(\frac{v^2}{2s} + \mu g\right). \qquad (2)$$

By definition, the average velocity of a uniformly varying motion is

$$\langle v \rangle = (v + v_0)/2 = v/2 \qquad (3)$$

since $v_0 = 0$. By definition, the average power is

$$\langle N \rangle = F\langle v \rangle. \qquad (4)$$

Substituting Eqs. (2) and (3) into (4), we obtain

$$\langle N \rangle = \frac{mv}{2}\left(\frac{v^2}{2s} + \mu g\right),$$

$$\langle N \rangle = \frac{6 \times 10^6 \times 16.7}{2}\left(\frac{16.7^2}{2 \times 2.5 \times 10^3} + 5 \times 10^{-3} \times 9.8\right) \text{ W}$$
$$\simeq 0.52 \text{ MW}.$$

161. A load of mass 2 kg falling from a height of 5 m penetrates a soft soil to a depth of 5 cm. Determine the average resistance of the soil.

Given: $m = 2$ kg, $h = 5$ m, $h_1 = 5$ cm $= 0.05$ m.

$\langle F \rangle$ — ?

Solution. We direct the Y-axis vertically upwards and take the origin at a depth h_1 from the surface (Fig. 73). An external force (resistance of the soil) is acting on segment CO, and hence

$$\Delta W = A, \quad \text{or} \quad W - W_0 = A, \qquad (1)$$

where $W_0 = mgh + mgh_1$ is the mechanical energy of the load at point B and W the mechanical energy of the load at the depth h_1 from the surface. Since $y = 0$ at point O and $v = 0$, $W = 0$ as well.

The work done by external forces on segment CO is $A = -\langle F\rangle h_1$. Substituting the expressions for W_0, W, and A into Eq. (1), we obtain $0 - mgh - mgh_1 = -\langle F\rangle h_1$, whence

$$\langle F\rangle = mg\left(\frac{h}{h_1} + 1\right),$$

$$\langle F\rangle = 2 \times 9.8 \times \left(\frac{5}{0.05} + 1\right)\ \mathrm{N} \simeq 1.98\ \mathrm{kN}.$$

162. A block slides first down an inclined plane of length 42 cm and height 7 cm and then over a horizontal plane. Having covered a distance of 142 cm along the horizontal, it stops. Determine the coefficient of friction, assuming that it is the same everywhere.

Given: $l_1 = 42\ \mathrm{cm} = 0.42\ \mathrm{m}$, $l_2 = 142\ \mathrm{cm} = 1.42\ \mathrm{m}$,
$h = 7\ \mathrm{cm} = 0.07\ \mathrm{m}$.

$\mu - ?$

Solution. Let us consider the motion of the block on two segments: the inclined and horizontal planes (Fig. 74). During the motion down the inclined plane, the block experiences the action of the force of gravity mg, the normal reaction $\mathbf{N}_1$ of the inclined plane, and the friction

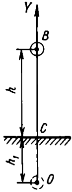


Fig. 73

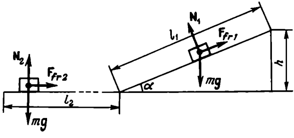


Fig. 74

$\mathbf{F}_{fr1}$ which is defined as

$$F_{fr1} = \mu N_1 = \mu mg \cos\alpha. \qquad (1)$$

During the motion over the horizontal plane, the block is acted upon by the force of gravity mg, the normal reaction $\mathbf{N}_2$ of the horizontal plane, and the friction $\mathbf{F}_{fr2}$ given by

$$F_{fr2} = \mu N_2 = \mu mg. \qquad (2)$$

The change in the total mechanical energy of the block is equal to the work done by the frictional forces F_{fr1} and F_{fr2}

$$\Delta W = A, \qquad (3)$$

where

$$\begin{aligned} \Delta W &= W - W_0 = 0 - mgh = -mgh, \\ A &= A_1 + A_2 = -F_{fr1}l_1 - F_{fr2}l_2. \end{aligned} \qquad (4)$$

Substituting expressions (4) into (3), we obtain $-mgh = -F_{fr1}l_1 - F_{fr2}l_2$. Taking Eqs. (1) and (2) into account, we obtain $mgh = \mu mg\,(l_1 \cos\alpha + l_2)$, whence $\mu = h/(l_1 \cos\alpha + l_2)$. The figure shows that $\cos\alpha = \sqrt{l_1^2 - h^2}/l_1$. Then

$$\mu = \frac{h}{\sqrt{l_1^2 - h^2} + l_2},$$

$$\mu = \frac{0.07}{\sqrt{0.42^2 - 0.07^2} + 1.42} \simeq 0.04.$$

163°. Determine the work done in order to compress by 20 cm a spring whose constant is 29.4 N/cm, assuming that deformations are elastic.

Given: $x_1 = 0$, $x_2 = 20$ cm $= 0.2$ m, $k = 29.4$ N/cm $= 2.94 \times 10^3$ N/m.

$A - ?$

Solution. Method 1. The work done to compress the spring is equal to the change in its potential energy:

$$A = \Delta W = \frac{kx_2^2}{2},$$

$$A = \frac{2.94 \times 10^3 \times 0.2^2}{2}\ \mathrm{J} = 58.8\ \mathrm{J}.$$

Method 2. By definition, the work done to compress the spring is

$$A=\int_{x_1}^{x_2} F\cos\alpha\, dx$$

$$=\int_{x_1}^{x_2}(-kx)(-1)\, dx=\frac{kx^2}{2}\Big|_{x_1}^{x_2}=\frac{kx_2^2}{2}=58.8\ \text{J}.$$

164°. A sledge moving over the horizontal surface of ice at a velocity of 5 m/s drives out on a road and comes

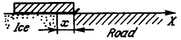


Fig. 75

to a halt. Determine the distance covered by the sledge on the road if the sledge is 1 m long and the coefficient of friction between the runners and the road is 0.5 (Fig. 75). The friction between the runners and the ice should be neglected.

Given: $\mu=0.5$, $v_0=5$ m/s, $l=1$ m.

L — ?

Solution. We divide the distance covered by the sledge over the rough surface of the road into two segments. On the first segment equal to the length l of the runners, the friction is varying since the coefficient of friction increases from zero to its value for the road. On the second segment l_1, when the runners leave the ice completely, the friction is constant. Let us calculate the work done by the friction on these two segments: $A_{fr}=A_{fr1}+A_{fr2}$.

We suppose that the sledge has covered a part x of the first segment (see Fig. 75). Since the friction increases in proportion to the covered distance, the friction acting on the sledge can be written as $F_{fr}=\mu mgx/l$. Then the work done by the friction on the first and second segments is

$$A_{fr1}=\int_0^l F_{fr}\cos\alpha\,dx=-\int_0^l \frac{\mu mg}{l}x\,dx$$

$$=-\frac{\mu mg}{l}\frac{x^2}{2}\Big|_0^l=-\frac{\mu mgl}{2},$$

$$A_{fr2}=-\mu mgl_1.$$

The total work done by the friction on these segments is

$$A_{fr}=-\mu mg\left(\frac{l}{2}+l_1\right). \quad (1)$$

The work done by the friction on the first segment can also be calculated as follows. Since friction is proportional to displacement, i.e. $F_{fr}=\mu mgx/l$, and increases from 0 to $F_{max}=\mu mg$, the average friction is

$$\langle F\rangle=(\mu mg+0)/2=\mu mg/2,$$

whence the work done by the friction on this segment is

$$A_1=-\langle F\rangle\, l=-\mu mgl/2.$$

On the other hand, the work done by friction is equal to the change in the kinetic energy of the sledge: $A_{fr}=\Delta W=W_2-W_1$. By hypothesis, $W_2=0$, $W_1=mv_0^2/2$, and hence

$$A_{fr}=-W_1=-mv_0^2/2. \quad (2)$$

Combining Eqs. (1) and (2), we get $-\mu mg\,(l/2+l_1)=-mv_0^2/2$, whence

$$l_1=(v_0^2-\mu gl)/(2\mu g).$$

The distance covered by the sledge is

$$L=l+l_1=l+\frac{v_0^2-\mu gl}{2\mu g}=\frac{v_0^2+\mu gl}{2\mu g},$$

$$L=\frac{5^2+0.5\times 9.8\times 1}{2\times 0.5\times 9.8}\ \text{m}\simeq 3.1\ \text{m}.$$

165°. A load of mass 5 kg falls freely from a certain height and reaches the ground in 2.5 s. Determine the work done by the force of gravity.

Given: $m=5$ kg, $t=2.5$ s.

A—?

Solution. The total work done by the force of gravity is

$$A=\int_H^0 F\cos\alpha\,\mathrm{d}h, \tag{1}$$

where h is the height from which the load falls, $F = mg$ is the force of gravity acting on the load, and $\cos\alpha = -1$. We transform expression (1) as follows:

$$A=-\int_H^0 mg\,\mathrm{d}h=-mgh\Big|_H^0=mgH.$$

Since $H=gt^2/2$, we have

$$A=mg\frac{gt^2}{2}=\frac{mg^2t^2}{2},$$

$$A=\frac{5\times 9.8^2\times 2.5^2}{2}\ \mathrm{J}\simeq 1.5\ \mathrm{kJ}.$$

166. A load of mass 0.5 kg falls from a certain height on a slab of mass 1 kg, fixed to a spring whose constant is $k = 9.8 \times 10^2$ N/m. Determine the maximum compression of the spring if the velocity of the load at the moment of impact is 5 m/s and the impact is inelastic.

Given: $m_1 = 0.5$ kg, $m_2 = 1$ kg, $k = 9.8\times 10^2$ N/m, $v = 5$ m/s.

---

x — ?

Solution. According to the energy conservation law, the total mechanical energy of the load and the slab after the impact is equal to the potential energy of the compressed spring:

$$(m_1+m_2)u^2/2+(m_1+m_2)gx=kx^2/2, \tag{1}$$

where m_2 is the mass of the slab and u the velocity of the load and the slab after the impact, which can be determined from the momentum conservation law for an inelastic collision: $m_1v = (m_1 + m_2)u$, whence

$$u=m_1v/(m_1+m_2). \tag{2}$$

Substituting Eq. (2) into (1), we obtain

$$\frac{m_1+m_2}{2}\,\frac{m_1^2v^2}{(m_1+m_2)^2}+(m_1+m_2)\,gx=\frac{kx^2}{2},$$

or

$$kx^2-2g\,(m_1+m_2)\,x-\frac{m_1^2v^2}{m_1+m_2}=0,$$

whence

$$x=\frac{g\,(m_1+m_2)+\sqrt{g^2\,(m_1+m_2)^2+km_1^2v^2/(m_1+m_2)}}{k},$$

$$x=\frac{9.8\,(0.5+1)+\sqrt{9.8^2\,(0.5+1)^2+9.8\times10^2\times0.5^2\times5^2/(0.5+1)}}{9.8\times10^2}$$

$$\simeq 8.2\times10^{-2}\ \text{m}.$$

(The negative root does not satisfy the conditions of the problem.)

167. The slope of a highway is 1 m per 20 m of the road. A motor car with its engine cut out moves down the slope at a constant velocity of 60 km/h. Determine the power of a car moving up the road at the same velocity if the mass of the car is 1.5 t.

Given: $\sin\alpha=h/l=1/20=0.05$, $v=60$ km/h $\simeq 17$ m/s, $m\simeq1.5$ t $=1.5\times10^3$ kg.

N—?

Solution. Since the motion of the car moving upwards is uniform,

$$N=Fv, \tag{1}$$

where F is the driving force of the engine during the ascent and v the velocity of the car. The car moving upwards is acted upon by the force of gravity mg, the normal reaction $\mathbf{N}$ of the highway, the driving force $\mathbf{F}$, and the friction $\mathbf{F}_{fr}$ (Fig. 76*a*). Writing Newton's second law for the car in projections on the X- and Y-axes (see Fig. 76*a*), we find that $F-F_{fr}-mg\ \sin\alpha=0$ ($a=0$ since $v=$ const), whence

$$F=F_{fr}+mg\sin\alpha. \tag{2}$$

The car moving downwards experiences the action of the force of gravity mg, the normal reaction N, and the friction F_{fr} (Fig. 76b). In analogy with the previous case, we can write $mg \sin\alpha - F_{fr} = 0$, whence

$$F_{fr} = mg \sin\alpha. \quad (3)$$

Substituting Eq. (3) into (2), we obtain $F = 2mg \sin\alpha$. Then Eq. (1) becomes

$$N = 2mg \sin\alpha \cdot v,$$

$$N = 2 \times 1.5 \times 10^3 \times 9.8 \times 0.05 \times 17 \text{ W} \simeq 25 \text{ kW}.$$

168. Determine the work done for lifting a load on an inclined plane, the average power, and the efficiency of the lifting mechanism if the mass of the load is 100 kg, the length of the inclined plane is 2 m, the slope of the plane is 30°, the coefficient of friction is 0.1, and the acceleration during the ascent is 1 m/s². The load was initially at rest at the foot of the inclined plane.

Given: $m = 100$ kg, $l = 2$ m, $\alpha = 30° \simeq 0.52$ rad, $\mu = 0.1$, $a = 1$ m/s².

A—? $\langle N\rangle$—? η—?

Solution. The change in the total mechanical energy is caused by the driving force and the friction acting on the load:

$$A + A_{fr} = \Delta W = W - W_0 = W. \quad (1)$$

Here A is the work done by the driving force and A_{fr} the force done by friction ($W_0 = 0$ by hypothesis). The

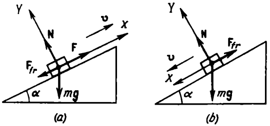


Fig. 76

final value of the total mechanical energy is

$$W = mv^2/2 + mgh = mal + mgl \sin \alpha$$

since $v^2 = 2al$.

The force of friction on an inclined plane is defined as $F_{tr} = \mu N = \mu mg \cos \alpha$, and hence $A_{tr} = -\mu mgl \cos \alpha$

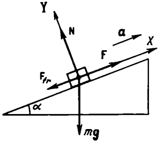


Fig. 77

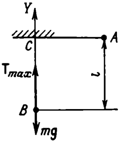


Fig. 78

(Fig. 77). Substituting these expressions into Eq. (1), we obtain

$$A = ml (a + g \sin \alpha + \mu g \cos \alpha),$$

$$A = 100 \times 2 \times (1 + 9.8 \times 0.5 + 0.1 \times 9.8 \times 0.87) \text{ J} = 1.35 \text{ kJ}.$$

The average power of the lifting mechanism is

$$\langle N \rangle = A/t,$$

where t is the time of lifting the load, which can be obtained from the equation for a uniformly accelerated motion:

$$l = at^2/2 \quad (v_0 = 0). \qquad (2)$$

From Eq. (2), $t = \sqrt{2l/a}$. Consequently,

$$\langle N \rangle = \frac{A}{\sqrt{2l/a}},$$

$$\langle N \rangle = \frac{1.35 \times 10^3}{\sqrt{2 \times 2/1}} \text{ W} = 675 \text{ W}.$$

By definition, the efficiency of the lifting mechanism is

$$\eta = A_u/A_d,$$

where $A_d = A = 1.35$ kJ.

The useful action of the lifting mechanism is the displacement of the load to the height $h = l \sin\alpha$. Therefore,

$$A_u = mgl \sin\alpha,$$
$$A_u = 100 \times 9.8 \times 2 \times 0.5 \text{ J} = 980 \text{ J},$$

whence

$$\eta = 980/(1.35 \times 10^3) \simeq 0.73.$$

169. A ball of mass m suspended on a string of length l is deflected through an angle of 90° from the vertical and released. Determine the maximum tension of the string.

Given: m, l, $\alpha = 90° \simeq 1.57$ rad.

T_{max} — ?

Solution. The tension of the string attains its maximum when the ball passes through point B. At this point, the ball experiences the action of the force of gravity mg and the tension T_{max} of the string (Fig. 78). Writing Newton's second law for the ball in projections on the Y-axis, we obtain

$$T_{max} - mg = ma_y, \tag{1}$$

where $a_y = v^2/R$ and $R = l$. Using these expressions, we can write Eq. (1) in the form $T_{max} - mg = mv^2/l$, whence

$$T_{max} = mg + mv^2/l. \tag{2}$$

In order to determine the velocity of the ball at point B, we apply the energy conservation law for points A and B:

$$W_A = W_B. \tag{3}$$

Since $W_A = mgl$ and $W_B = mv^2/2$, Eq. (3) becomes $mgl = mv^2/2$, whence $v^2 = 2gl$. Substituting the expression for v^2 into Eq. (2), we obtain

$$T_{max} = mg + m \cdot 2gl/l = 3mg.$$

170. A satellite of mass 12 t moves in a circular orbit around the Earth so that its kinetic energy is 54 GJ. What is the velocity of the satellite and its altitude?

Given: $m = 12\ \text{t} = 1.2 \times 10^4$ kg, $W_k = 54$ GJ $=$ 5.4×10^{10} J.

$v - ?\ h - ?$

Solution. The kinetic energy of the satellite is $W_k = mv^2/2$, whence

$$v^2 = 2W_k/m. \tag{1}$$

Then

$$v = \sqrt{\frac{2W_k}{m}},$$

$$v = \sqrt{\frac{2 \times 5.4 \times 10^{10}}{1.2 \times 10^4}}\ \frac{\text{m}}{\text{s}} = 3 \times 10^3\ \text{m/s}.$$

The altitude h of the satellite and its velocity v are connected through the following relation:

$$v^2 = g_0 R^2/(R + h) \tag{2}$$

(see Problem 134, formula (5)). Equating the right-hand sides of Eqs. (1) and (2), we obtain $2W_k/m = g_0R^2/(R + h)$, whence

$$h = mg_0 \frac{R^2}{2W_k} - R = R\left(\frac{mg_0R}{2W_k} - 1\right),$$

$$h = 6.37 \times 10^6 \times \left(\frac{1.2 \times 10^4 \times 9.8 \times 6.37 \times 10^6}{2 \times 5.4 \times 10^{10}} - 1\right)\ \text{m}$$

$$\simeq 3.8 \times 10^7\ \text{m}.$$

171. What is the altitude attained by a rocket launched vertically at a velocity of 9 km/s? Air resistance should be neglected.

Given: $v = 9$ km/s $= 9 \times 10^3$ m/s.

$h - ?$

Solution. A rocket of mass m is acted upon by a varying gravitational force

$$F = GmM/r^2, \tag{1}$$

where M is the Earth's mass and r the separation between the rocket and the centre of the Earth.

As the rocket attains an altitude h, this force will do the work

$$A = \int_{R}^{R+h} F \cos\alpha \, dr, \tag{2}$$

where $\alpha = \pi$ rad is the angle between the directions of the force and displacement and R the Earth's radius. Substituting Eq. (1) into (2), we obtain

$$A = -\int_{R}^{R+h} \frac{GmM}{r^2} dr = \frac{GmM}{r}\Big|_{R}^{R+h}$$

$$= GmM\left(\frac{1}{R+h} - \frac{1}{R}\right) = -\frac{GmMh}{R(R+h)}. \tag{3}$$

On the other hand, the work is equal to the change in the kinetic energy of the rocket: $A = \Delta W = W_2 - W_1$. At the altitude h, $W_2 = 0$, and hence

$$A = -W_1 = -mv^2/2. \tag{4}$$

Equating the right-hand sides of expressions (3) and (4), we obtain

$$GmMh/[R(R+h)] = mv^2/2. \tag{5}$$

Multiplying and dividing the left-hand side of Eq. (5) by R, we obtain

$$RhGM/[(R+h)R^2] = v^2/2. \tag{6}$$

Since $GM/R^2 = g_0$ is the free-fall acceleration at the Earth's surface, Eq. (6) can be written in the form $Rhg_0(R+h) = v^2/2$, whence

$$h = \frac{v^2 R}{2Rg_0 - v^2},$$

$$h = \frac{(9\times10^3)^2 \times 6.37\times10^6}{2\times6.37\times10^6\times9.8-(9\times10^3)^2}\ \mathrm{m} \simeq 1.17\times10^7\ \mathrm{m}.$$

172°. Calculate the escape velocity which should be imparted to a rocket at the Earth's surface.

Given: $g_0 = 9.8\ \mathrm{m/s^2}$, $R_1 = 6.37\times10^6\ \mathrm{m}$, $R_2 = \infty$.

v — ?

Solution. The rocket of mass m is acted upon by the varying gravitational force

$$F = GmM/r^2. \quad (1)$$

The work done by this force is

$$A = \int_{R_1}^{R_2} F \cos\alpha \, dr. \quad (2)$$

Considering that $\alpha = \pi$ rad and substituting Eq. (1) into (2), we obtain

$$A = -\int_{R_1}^{R_2} \frac{GmM}{r^2} dr = \frac{GmM}{r}\Big|_{R_1}^{\infty} = -\frac{GmM}{R_1}. \quad (3)$$

On the other hand, $A = \Delta W = W_2 - W_1$. Since $W_2 = 0$, we have

$$A = -W_1 = -mv^2/2. \quad (4)$$

Equating the right-hand sides of Eqs. (3) and (4), we obtain $GmM/R_1 = mv^2/2$, or (after transformations similar to those carried out in the previous problem), $g_0R_1 = v^2/2$, whence

$$v = \sqrt{2g_0R_1},$$

$$v = \sqrt{2 \times 9.8 \times 6.37 \times 10^6} \text{ m/s} \simeq 11.2 \times 10^3 \text{ m/s}.$$

173. A bullet flying along the horizontal hits a ball suspended on a light rigid rod and gets stuck in it. The mass of the bullet is equal to 0.001 times the mass of the ball. The distance between the point of suspension of the rod and the centre of the ball is 1 m. Determine the velocity of the bullet if the rod with the ball is known to be deflected through 10° as a result of impact.

Given: $m_2 = 1000m_1$, $l = 1$ m, $\alpha = 10° \simeq 0.17$ rad.

v—?

Solution. Writing the momentum conservation law for the inelastic impact in projections on the X-axis

(Fig. 79), we obtain $m_1 v = (m_1 + m_2) u$, whence

$$v = \frac{m_1 + m_2}{m_1} u. \quad (1)$$

Here v is the velocity of the bullet before the collision and u the velocity of the ball and the bullet after the collision. In Eq. (1), there is one more unknown velocity u in addition to v, which can be determined from the

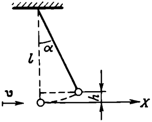


Fig. 79

energy conservation law. Let the centre of mass of the ball be raised to a height h as a result of the collision with the bullet. Then by the energy conservation law, $(m_1 + m_2) u^2/2 = (m_1 + m_2) gh$, whence

$$u^2 = 2gh. \quad (2)$$

It can be seen from the figure that $h = l - l \cos \alpha = l (1 - \cos \alpha)$. Substituting this expression for h into Eq. (2), we obtain $u^2 = 2gl (1 - \cos \alpha)$, whence $u = \sqrt{2gl (1 - \cos \alpha)}$. Then Eq. (1) can be reduced to the form

$$v = \frac{m_1 + m_2}{m_1} \sqrt{2gl (1 - \cos \alpha)}. \quad (3)$$

Using the trigonometric relation $\sin (\alpha/2) = \sqrt{(1 - \cos \alpha)/2}$, we transform expression (3) as follows:

$$v = 2 \frac{m_1 + m_2}{m_1} \sin \frac{\alpha}{2} \sqrt{gl},$$

$$v = 2 \frac{m_1 + 1000 m_1}{m_1} 0.09 \sqrt{9.8 \times 1} \frac{\text{m}}{\text{s}} \simeq 570 \text{ m/s}.$$

174. A man standing on a trolley pushes another trolley so that they are set in motion and stop after some time due to friction. Determine the ratio of the stopping distances of the trolleys if the mass of the first trolley with the man is thrice the mass of the second trolley.

Given: $m_1 = 3m_2$.

s_1/s_2 — ?

Solution. Writing the momentum conservation law for an elastic interaction between the trolleys in projections on the X-axis directed along the trajectory of the first trolley, we obtain

$$0 = m_1 u_1 - m_2 u_2, \tag{1}$$

where u_1 and u_2 are the velocities of the first and second trolleys immediately after the interaction.

Equation (1) implies that

$$m_1 u_1 = m_2 u_2. \tag{2}$$

Transforming Eq. (2), we get by hypothesis $3m_2 u_1 = m_2 u_2$, whence

$$u_1 = u_2/3. \tag{3}$$

The change in the kinetic energy of each trolley is equal to the work done by friction: $\Delta W = A$. Considering that $\Delta W = 0 - mu^2/2$ and $A = -F_{fr}s = -\mu mgs$, we obtain $-mu^2/2 = -\mu mgs$, or

$$u^2/2 = \mu gs. \tag{4}$$

Then relation (4) for the first and second trolleys can be written in the form

$$u_1^2/2 = \mu g s_1, \quad u_2^2/2 = \mu g s_2. \tag{5}$$

Dividing the first equation in (5) by the second termwise, we obtain $s_1/s_2 = u_1^2/u_2^2$ or, taking into account relation (3),

$$s_1/s_2 = u_2^2/(9u_2^2) = 1/9.$$

175. The beetle head of a pile driver has a mass of 400 kg and falls on a pipe of mass 100 kg which is driven into the ground. Determine the average resistance of the ground and the efficiency of the pile driver if the pile is known to sink into the ground by 5 cm upon each blow, the height from which the beetle head falls is 1.5 m, and the impact is inelastic.

Given: $m_1 = 400\ \mathrm{kg} = 4 \times 10^2\ \mathrm{kg}$, $m_2 = 100\ \mathrm{kg} = 10^2\ \mathrm{kg}$, $s = 5\ \mathrm{cm} = 5 \times 10^{-2}\ \mathrm{m}$, $h = 1.5\ \mathrm{m}$.

$\langle F \rangle$ — ? η — ?

Solution. The work done by the resistance of the ground is equal to the change in the total mechanical energy of the system of bodies (the pile and the beetle head):

$$A = \Delta W = W - W_0 = -W_0 \tag{1}$$

($W = 0$ by hypothesis).

By definition, the work done by the resistance force is

$$A = -\langle F \rangle s. \tag{2}$$

The initial mechanical energy of the system is equal to the sum of the kinetic and potential energies of the pile and the beetle head after their collision, i.e. $W_0 = W_p + W_k$. The kinetic energy can be presented in the form $W_k = p^2/[2(m_1 + m_2)]$, where p is the momentum of the pile and the beetle head after the collision. Since $W_p = (m_1 + m_2) gs$, we have

$$W_0 = (m_1 + m_2) gs + p^2/[2(m_1 + m_2)]. \tag{3}$$

According to the momentum conservation law for an inelastic collision, we have $p = p_1$, where p_1 is the momentum of the beetle head before it strikes against the pile. The mechanical energy conservation law for the beetle head falling from a height h (Fig. 80) implies that $p_1^2/(2m_1) = m_1 gh$, and therefore

$$p^2 = p_1^2 = 2m_1^2 gh. \tag{4}$$

Substituting relations (2), (3), and (4) into Eq. (1), we obtain

$$\langle F \rangle = (m_1 + m_2) g + \frac{m_1^2 gh}{(m_1 + m_2) s},$$

$$\langle F \rangle = 9.8 \times (4 \times 10^2 + 10^2) + \frac{(4 \times 10^2)^2 \times 9.8 \times 1.5}{(4 \times 10^2 + 10^2) \times 5 \times 10^{-2}}\ \mathrm{N}$$

$$\simeq 9.9 \times 10^4\ \mathrm{N}.$$

The efficiency of the mechanism is defined as

$$\eta = A_u/A_d. \tag{5}$$

The work done is $A_d = mgh = p_1^2/(2m)$, while the useful work $A_u = W_k$, where W_k is the kinetic energy of the

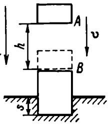


Fig. 80

pile and the beetle head after the collision, $W_k = p^2/[2(m_1 + m_2)]$. Since $p = p_1$, substituting these expressions into Eq. (5), we obtain

$$\eta = \frac{m_1}{m_1 + m_2},$$

$$\eta = \frac{400}{400+100} = 0.8.$$

176. A washer of mass 10 g rests at the top of a smooth hemisphere of radius 0.5 m. The washer starts to slide over the hemisphere under the action of a horizontal short impulse of force of 2×10^{-2} N·s. At what height from the base of the hemisphere will the washer be separated from its surface?

Given: $R = 0.5$ m, $m = 10$ g $= 10^{-2}$ kg, $F\Delta t = 2 \times 10^{-2}$ N·s.

H — ?

Solution. Before the moment of separation, the washer moves in a circle under the action of the force of gravity mg and the normal reaction **N** of the spherical surface. Writing Newton's second law for the washer in projections on the Y-axis directed along the radius towards the centre of the circle (Fig. 81), we obtain

$$mg\cos\alpha - N = ma_c,$$

where $a_c = v^2/R$ and $\cos\alpha = H/R$. At the moment of

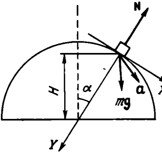


Fig. 81

separation, the normal reaction vanishes. Therefore,

$$mgH/R = mv^2/R, \quad \text{or} \quad mgH = mv^2. \qquad (1)$$

According to the law of conservation of mechanical energy, we have

$$mv^2/2 + mgH = (F\,\Delta t)^2/(2m) + mgR, \qquad (2)$$

where $F\,\Delta t$ is the impulse of the washer at the top of the hemisphere. From Eq. (2), we obtain

$$mv^2 = (F\,\Delta t)^2/m + 2mg\,(R - H).$$

Then

$$mgH = (F\,\Delta t)^2/m + 2mgR - 2mgH,$$

$$H = \frac{2}{3}R + \left(\frac{F\,\Delta t}{m}\right)^2 \frac{1}{3g},$$

$$H = \frac{2\times 0.5}{3} + \frac{2\times 10^{-2}}{10^{-2}}\,\frac{1}{3\times 9.8}\ \text{m} \simeq 0.47\ \text{m}.$$

177. A body of mass m slides down a hill of height h. What work should be done to lift the body to the top of the hill if the force is directed along the displacement?

Given: h, m.

$A_2 - ?$

Solution. Since the body sliding down the hill comes to a halt, its total mechanical energy has changed. The change in the total mechanical energy of the body during sliding must be equal to the work done by friction: $\Delta W_1 = A_1$. Since

$$\Delta W_1 = W_1 - W_{01} = 0 - mgh = -mgh, \qquad (1)$$

we obtain $A_1 = -mgh$.

The change in the total mechanical energy of the body during the ascent is

$$\Delta W_2 = A_2 + A_3. \qquad (2)$$

Here $\Delta W_2 = W_2 - W_{02} = mgh - 0 = mgh$, A_2 is the work done by the driving force, and A_3 the work done by friction during the ascent. Since the driving force is directed along the displacement, the normal reaction of the support (and hence the friction) will be the same as for the body sliding down the hill. Therefore, $A_3 = A_1$, and taking Eq. (1) into account, we obtain

$$A_3 = -mgh. \qquad (3)$$

Finally, from Eqs. (2) and (3) we get

$$A_2 = \Delta W_2 - A_3 = mgh - (-mgh) = 2mgh.$$

178. A load is lifted to a height h and then uniformly moved over a horizontal surface through a distance h. In which case will the work done be larger? The coefficient of friction between the load and the surface is μ, and the air resistance should be neglected.

Answer. The work done for lifting the load of mass m to a height h is mgh. The work done during the uniform displacement of the load over the horizontal surface through a distance h is μmgh. Since the coefficient of sliding friction is known to be always smaller than unity, the work done in the latter case will be smaller.

179. How should a ball be thrown to the floor from a height h for it to be bounced to a height H larger than h?

Answer. The ball should be thrown at a nonzero initial velocity. Then the initial mechanical energy of the ball is $mgh + mv^2/2$. If the impact of the ball against the floor is elastic, it will jump to the height H which can be determined from the energy conservation law $mgh + mv^2/2 = mgH$. This equation shows that $H > h$.

EXERCISES

180. A train of mass 500 t moves up the hill with a slope of 10 m per kilometre of the railroad at a velocity of 30 km/h. The coefficient of friction is 0.002. Determine the power developed by the locomotive of the train.

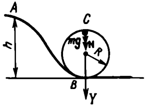


Fig. 82

181. A stone sliding over a horizontal surface of ice stops after covering a distance of 48 m. Determine the initial velocity of the stone if the coefficient of friction is 0.06.

182. A bullet of mass 10 g, flying at a velocity of 400 m/s, pierces a 5-cm thick board, after which its velocity decreases to half the initial value. Determine the average resistance offered by the board to the motion of the bullet.

183. A tank whose mass is 15 t and power is 368 kW moves up the hill with a slope of 30°. What is the maximum velocity of the tank?

184. A cyclist must make a loop of radius 8 m in a vertical plane. From what minimum height must the cyclist start to make the loop without falling (Fig. 82)? Friction should be neglected.

185. The beetle head of a pile driver of mass 500 kg falls on a pile of mass 100 kg at a velocity of 4 m/s. Determine the efficiency of the pile driver for elastic and inelastic collisions.

186. A chandelier of mass 100 kg is suspended from a ceiling on a metal chain whose length is 5 m. What is the maximum height to which the chandelier can be deflected without breaking the chain during swings if the rupture of the chain is known to occur at a tension of 2 kN?

187. An ice hockey player whose weight is 70 kg throws a puck of mass 0.3 kg in the horizontal direction at a velocity of 10 m/s. By what distance will the player move backwards if the coefficient of friction between the skates and ice is 0.02?

188. The spring of a toy pistol has a constant of 10 N/cm and a length of 15 cm. To what height will a ball of mass 10 g shot from the pistol in the vertical direction rise if the spring has been compressed to 5 cm? Air resistance should be neglected.

189. A box with sand whose mass is 10 kg is held by a spring constant of 30 N/cm. A bullet of mass 10 g moving at a velocity of 500 m/s hits the box from below and gets stuck in the sand. Determine the contraction of the spring.

190. Two loads whose masses are in the ratio 1:4 are connected through a compressed spring and rest on the horizontal surface of a table. When the spring is released, the load of the smaller mass acquires a kinetic energy of 40 J. Determine the potential energy of the compressed spring neglecting friction.

191. A ball falling from a height of 3 m is bounced to 2.5 m after the impact against the floor. How can this be put in accord with the law of conservation of mechanical energy?

QUESTIONS FOR REVISION

1. Define energy. 2. What is the kinetic energy of a body? 3. What is the difference between conservative and nonconservative forces? 4. What is the potential energy of a body? 5. Write the expressions for the potential energy of a body lifted above the ground and for the potential energy of a compressed spring. 6. Formulate the law of conservation of the total mechanical energy. Indicate the conditions under which this law is obeyed. 7. Define the mechanical work. 8. What is the relation between work and energy? 9. Write the formulas for calculating the work done by a constant and a varying force. 10. Define the mechanical power. 11. How are the average and instantaneous powers in varying motion defined? 12. In what units are energy, work, and power measured in SI? 13. Define the efficiency of a mechanism.

1.4. Statics

Statics deals with the equilibrium conditions for a body under the action of applied forces. **Equilibrium** is a state of rest or of uniform rectilinear motion or rotation. Equilibrium can be **stable**, **unstable** or **neutral**. In equilibrium, the potential energy has an extremal value. Analytically this is written in the form

$$\frac{dW_p}{dx} = 0.$$

The condition for stable equilibrium is that the potential energy must be minimum, which is equivalent to the following mathematical condition:

$$\frac{d^2W_p}{dx^2} > 0.$$

The necessary equilibrium condition for a point mass is the equality to zero of the sum of all the forces applied to it:

$$\sum_{i=1}^{n} \mathbf{F}_i = 0.$$

If we project all the forces acting on the point mass onto the X- and Y-axes, the equilibrium condition assumes the form

$$\sum_{i=1}^{n} F_{ix} = 0, \quad \sum_{i=1}^{n} F_{iy} = 0.$$

The equilibrium of a rigid body depends not only on the magnitude and direction of forces acting on it but also on the point of application of these forces.

The **moment of force (torque)** M about an axis is defined as the product of the force F by the arm l (i.e. the perpendicular dropped from the rotational axis on the line of action of the force):

$$M = Fl.$$

The moment of force tending to rotate a body counterclockwise about an axis is assumed to be **positive** and the one rotating a body clockwise **negative**.

The following two conditions are necessary for the equilibrium of a body:

1. The vector sum of all the forces applied to the body must be zero:

$$\sum_{i=1}^{n} \mathbf{F}_i = 0,$$

where n is the number of forces.

2. The algebraic sum of the torques about any axis must be zero:

$$\sum_{i=1}^{n} M_i = 0,$$

where n is the number of torques.

* * *

192. A load of mass 10 kg is balanced by two loads, the mass of the second load being 18 kg. The string holding the third load is directed from point A along the horizontal (Fig. 83*a*). Determine the mass of the third load and angle α.

Given: $m_1 = 10$ kg, $m_2 = 18$ kg.

m_3—? α—?

Solution. Point A is acted upon by the tensions $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3$ of the strings (Fig. 83*b*). Since the system of

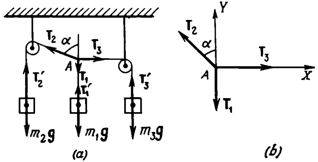


Fig. 83

bodies is stationary, we can write for each load

$$T_1' = m_1 g, \quad T_2' = m_2 g, \quad T_3' = m_3 g. \qquad (1)$$

The equilibrium condition for point A has the form

$$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 = 0, \qquad (2)$$

where $T_1 = T_1'$, $T_2 = T_2'$, and $T_3 = T_3'$. Choosing the directions of the X- and Y-axes and projecting Eq. (2) onto them, we obtain

$$-T_2 \sin\alpha + T_3 = 0, \qquad (3)$$

$$T_2 \cos\alpha - T_1 = 0. \qquad (4)$$

From Eqs. (4) and (1) we get $\cos\alpha = T_1/T_2 = m_1/m_2$, whence

$$\alpha = \arccos(m_1/m_2),$$

$$\alpha = \arccos(10/18) \simeq 0.98 \text{ rad}.$$

Equation (3) gives $T_3 = T_2 \sin\alpha$, or $m_3 g = m_2 g \sin\alpha$, whence

$$m_3 = m_2 \sin\alpha,$$

$$m_3 = 18 \times 0.832 \text{ kg} \simeq 14.9 \text{ kg}.$$

193. A wooden block lies on an inclined plane. What force must press the block against the inclined plane for it to remain at rest? The mass of the block is 2 kg, the length of the inclined plane is 1 m and its height is 60 cm. The coefficient of friction between the block and the plane is 0.4.

Given: $m = 2$ kg, $l = 1$ m, $h = 60$ cm $= 0.6$ m, $\mu = 0.4$.

$F - ?$

Solution. The block experiences the action of the force of gravity $m\mathbf{g}$, the normal reaction $\mathbf{N}$ of the inclined plane, the friction $\mathbf{F}_{fr}$, and the force $\mathbf{F}$ pressing the block against the inclined plane (Fig. 84). Since the block cannot rotate, the first equilibrium condition is sufficient for it to remain at rest:

$$m\mathbf{g} + \mathbf{N} + \mathbf{F} + \mathbf{F}_{fr} = 0. \qquad (1)$$

Choosing the directions of the X- and Y-axes and projecting Eq. (1) onto them, we obtain

$$mg \sin \alpha - F_{fr} = 0, \quad N - mg \cos \alpha - F = 0.$$

Considering that $F_{fr} = \mu N$, $\sin \alpha = h/l$, $\cos \alpha = \sqrt{l^2 - h^2}/l$ and solving the obtained system of equations, we find that

$$F = \frac{mg}{l}\left(\frac{h}{\mu} - \sqrt{l^2 - h^2}\right),$$

$$F = \frac{2 \times 9.8}{1} \times \left(\frac{0.6}{0.4} - \sqrt{1 - (0.6)^2}\right) \text{ N} \simeq 13.7 \text{ N}.$$

194. One end of a thin homogeneous beam AB of mass 100 kg rests on the smooth horizontal floor and the other

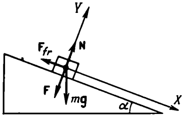


Fig. 84

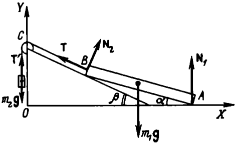


Fig. 85

end rests on a smooth plane inclined at an angle of 30° to the horizontal. The end B is supported by a string with a load passing over a pulley C (Fig. 85). Determine the mass of the load and the normal reactions of the floor and the inclined plane, neglecting friction.

Given: $m_1 = 100$ kg, $\beta = 30° \simeq 0.52$ rad.

m_2 — ? N_1 — ? N_2 — ?

Solution. Let us consider the forces acting on the beam: $m_1\mathbf{g}$ is the force of gravity, $\mathbf{N}_1$ and $\mathbf{N}_2$ are the normal reactions of the floor and the inclined plane, and $\mathbf{T}$ is the tension of the string. Under the action of these forces, the beam is in equilibrium. Writing the first equilibrium condition for the beam, we obtain

$$m_1\mathbf{g} + \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{T} = 0. \quad (1)$$

Directing the X- and Y-axes as shown in the figure and projecting Eq. (1) onto them, we get

$$N_2 \sin\beta - T\cos\beta = 0, \quad (2)$$

$$-m_1g + N_1 + N_2\cos\beta + T\sin\beta = 0. \quad (3)$$

Writing the second equilibrium condition for the beam about an axis passing through point B, we obtain

$$M_1 - M_2 = 0. \quad (4)$$

Here $M_1 = N_1 l_1$ and $M_2 = m_1 g l_2$ are the moments of forces N_1 and $m_1 g$ about the chosen axis, where $l_1 = L\cos\alpha$ and $l_2 = (L/2)\cos\alpha$ are the arms of the forces N_1 and m_1g, and $L = |AB|$ is the beam length. Substituting the expressions for M_1 and M_2 into Eq. (4), we obtain $N_1 L\cos\alpha - m_1 g\,(L/2)\cos\alpha = 0$, whence

$$N_1 = \frac{m_1 g}{2},$$

$$N_1 = \frac{100\times 9.8}{2}\ \text{N} = 490\ \text{N}.$$

The tension can be determined from Eqs. (2) and (3):

$$T = \frac{1}{2} m_1 g \sin\beta. \quad (5)$$

Using the equilibrium condition for the load, we get $T' = m_2 g$, where $T' = T$, whence

$$m_2 = \frac{T}{g} = \frac{m_1}{2}\sin\beta,$$

$$m_2 = \frac{100}{2\times 2}\ \text{kg} = 25\ \text{kg}.$$

Solving Eqs. (2) and (5) together, we obtain

$$N_2 = \frac{T \cos\beta}{\sin\beta} = \frac{m_1 g}{2} \cos\beta,$$

$$N_2 = \frac{100 \times 9.8 \times 0.87}{2} \text{ N} = 426 \text{ N}.$$

195. A thin homogeneous rod OB rests on two supports D and C separated by a distance a (Fig. 86). The coefficient of friction between the rod and the support is μ,

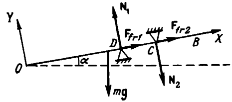


Fig. 86

the slope of the rod is α, and the length of region $|CB|$ is b. What must be the length L of the rod for it to be in equilibrium?

Given: $|OB_1| = L$, $|DC| = a$, $|CB| = b$, μ, α.

L—?

Solution. The rod OB experiences the action of the force of gravity mg, the normal reactions $\mathbf{N}_1$ and $\mathbf{N}_2$ of the supports D and C, and the frictional forces $\mathbf{F}_{fr1}$ and $\mathbf{F}_{fr2}$. The frictional forces are directed upwards along the rod since the rod has a tendency to slide down. Writing the first equilibrium condition for the rod, we obtain

$$m\mathbf{g} + \mathbf{N}_1 + \mathbf{F}_{fr1} + \mathbf{F}_{fr2} + \mathbf{N}_2 = 0. \quad (1)$$

Directing the X- and Y-axes as shown in the figure and projecting Eq. (1) onto them, we get

$$F_{fr1} + F_{fr2} - mg \sin\alpha = 0, \quad (2)$$

$$N_1 - N_2 - mg \cos\alpha = 0. \quad (3)$$

Considering that $F_{fr} \leqslant \mu N$, we can write Eqs. (2) and (3) in the form

$$\mu N_1 + \mu N_2 - mg \sin \alpha \geqslant 0, \tag{4}$$

$$N_1 - N_2 - mg \cos \alpha = 0. \tag{5}$$

Writing the second equilibrium condition for the rod about an axis passing through point C, we obtain

$$M_1 - M_2 = 0. \tag{6}$$

Here $M_1 = mgl_1$ and $M_2 = N_1 l_2$ are the moments of the forces mg and N_1 about the chosen axis, where $l_1 = (L/2 - b) \cos \alpha$ and $l_2 = a$ are the arms of the forces mg and N_1. Substituting the expressions for M_1 and M_2 into Eq. (6), we get

$$mg (L/2 - b) \cos \alpha - N_1 a = 0. \tag{7}$$

From Eqs. (5) and (7), we find $N_1 = mg \cos \alpha \, (L/2 - b)/a$ and $N_2 = mg \cos \alpha \, (L/2 - b - a)/a$. Taking this into account, we can calculate the length of the rod from Eq. (4):

$$L \geqslant 2b + a \, (1 + \tan \alpha/\mu).$$

196. A 4-m long ladder leans against a perfectly smooth wall at an angle of 60° to the horizontal. The coefficient of friction between the ladder and the floor is 0.33. To what distance along the ladder can a man climb before the ladder starts to slide down? The mass of the ladder should be neglected.

Given: $l = 4$ m, $\alpha = 60° \simeq 1.05$ rad, $\mu = 0.33$.

S — ?

Solution. The ladder is acted upon by the force of pressure $\mathbf{F}$ exerted by the man, the normal reactions $\mathbf{N}_1$ and $\mathbf{N}_2$ of the wall and the floor, and the friction $\mathbf{F}_{fr}$ (Fig. 87). The sliding of the ladder can be treated as a combination of two motions: rotation (about point O) and translation (against the X-axis). Writing the first equilibrium condition for the ladder, we get

$$\mathbf{F} + \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{F}_{fr} = 0. \tag{1}$$

Projecting Eq. (1) onto the X- and Y-axes, we obtain

$$F_{fr} - N_1 = 0, \quad N_2 - F = 0. \qquad (2)$$

Writing the second equilibrium condition for the ladder about point O, we get

$$M_1 - M_2 = 0. \qquad (3)$$

Here $M_1 = N_1 l_1$ and $M_2 = F l_2$ are the moments of the forces N_1 and F about the chosen axis, where $l_1 =$

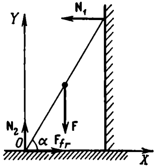


Fig. 87

$l \sin\alpha$ and $l_2 = S\cos\alpha$ are the arms of the forces N_1 and F. Using these relations, we can write Eq. (3) in the form $N_1 l \sin\alpha - FS\cos\alpha = 0$, whence

$$S = \frac{N_1 l \sin\alpha}{F\cos\alpha} = \frac{N_1 l}{F}\tan\alpha.$$

Using Eqs. (2) and the expression $F_{fr} = \mu N_2$ for friction, we obtain $N_1 = \mu F$, and therefore

$$S = \mu l \tan\alpha,$$

$$S = 0.33 \times 4 \times \sqrt{3}\ \mathrm{m} \simeq 2.3\ \mathrm{m}.$$

197. A sphere of radius 5 cm resting on a vertical wall is suspended on a 20-cm long string. The string touches the sphere at point C. Determine the coefficient of friction between the sphere and the wall.

Given: $l = 20\ \mathrm{cm} = 0.2\ \mathrm{m}$, $R = 5\ \mathrm{cm} = 0.05\ \mathrm{m}$.

μ — ?

Solution. The sphere is acted upon by the force of gravity mg, the normal reaction $\mathbf{N}$ of the wall, the tension

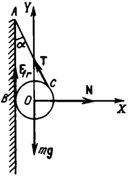


Fig. 88

$\mathbf{T}$ of the string, and the friction $\mathbf{F}_{fr}$ (Fig. 88). Writing the first equilibrium condition for the sphere, we obtain

$$m\mathbf{g} + \mathbf{N} + \mathbf{T} + \mathbf{F}_{fr} = 0. \quad (1)$$

Directing the X- and Y-axes as shown in the figure and projecting Eq. (1) onto them, we get

$$N - T \sin \alpha = 0, \quad -mg + T \cos \alpha + F_{fr} = 0. \quad (2)$$

Writing the second equilibrium condition for the sphere about an axis passing through point O, we obtain

$$-M_1 + M_2 = 0. \quad (3)$$

Here $M_1 = F_{fr} l_1$ and $M_2 = T l_2$ are the moments of the forces F_{fr} and T about the chosen axis, where $l_1 = l_2 = R$ are the arms of the forces. Taking this into account, we write Eq. (3) in the form

$$-F_{fr} R + TR = 0, \quad \text{or} \quad F_{fr} = T. \quad (4)$$

Solving the system of equations (2) and (4) and considering that $F_{fr} \leqslant \mu N$, we obtain

$$\mu \geqslant 1/\sin \alpha. \quad (5)$$

Since AC and AB are tangents to the circle drawn from the same point, $\angle OAC = \alpha/2$ and $\tan \alpha/2 = R/l$. Determining the sine of the angle from the formula

$$\sin\alpha = \frac{2\tan(\alpha/2)}{1+\tan^2(\alpha/2)},$$

we finally get

$$\mu \geqslant \frac{R^2+l^2}{2Rl},$$

$$\mu \geqslant \frac{0.2^2+0.05^2}{2\times 0.05\times 0.2} \simeq 2.13.$$

198. Two spheres of mass 3 and 5 kg are connected by a rod whose mass is 2 kg. Determine the position of the common centre of mass if the radii of the first and the second sphere are 5 and 7 cm respectively and the length of the rod is 30 cm.

Given: $m_1 = 3$ kg, $m_2 = 5$ kg, $m_3 = 2$ kg, $R_1 = 5$ cm $=$ 0.05 m, $R_2 = 7$ cm $= 0.07$ m, $l = 30$ cm $= 0.3$ m.

x—?

Solution. Since the centre of mass of the system coincides with its centre of gravity, we can determine its position from the equilibrium condition for the system in the gravitational field. For this purpose, we fix the system at an axis passing through the centre of gravity C (Fig. 89). Then it is sufficient to consider only one equilibrium condition for the system about this axis:

$$M_1 - M_2 + M_3 = 0. \tag{1}$$

Here $M_1 = m_1gl_1$, $M_2 = m_2gl_2$, and $M_3 = m_3gl_3$ are the moments of the forces m_1g, m_2g, and m_3g about the chosen axis, where $l_1 = l/2 + R_1 + x$, $l_2 = R_2 + l/2 - x$, and $l_3 = x$ are the arms of the forces m_1g, m_2g, and m_3g, x being the distance from the midpoint of the rod to the centre of gravity (see Fig. 89). Taking this into account, we write Eq. (1) as follows: $m_1g\ (l/2 + R_1 + x) - m_2g\ (R_2 + l/2 - x) + m_3gx = 0$, whence

$$x = \frac{m_2R_2 - m_1R_1 - l\,(m_2-m_1)/2}{m_1+m_2+m_3},$$

$$x = \frac{5\times 0.07 - 3\times 0.05 + 0.3\times(5-3)/2}{3+5+2}\ \text{m} \simeq 0.05\ \text{m}.$$

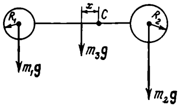


Fig. 89

199. A plane homogeneous plate has the shape of a circle with a circular hole having a radius equal to half the radius of the large circle and a common tangent with it (Fig. 90*a*). Determine the position of the centre of mass of the plate.

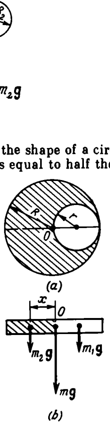


Fig. 90

Given: R.

x — ?

Solution. Since the centre of mass of the plate coincides with its centre of gravity, we shall determine the position of the centre of gravity of the plate. If we fill the hole, the force of gravity mg of the body can be presented as the resultant of two forces (Fig. 90*b*): the force of gravity m_1g of the hole portion and the force of gravity m_2g of the remaining portion (the circle with the hole). The plate will be in equilibrium about an axis passing through point O. Writing the second equilibrium condition for the system about the chosen axis, we obtain

$$-M_1 + M_2 = 0. \qquad (1)$$

Here $M_1 = m_1gr$ and $M_2 = m_2gx$ are the moments of the forces of gravity m_1g and m_2g about point O, where r and x are the arms of the forces of gravity m_1g and m_2g. Taking this into account, we can write Eq. (1) in the form

$-m_1gr + m_2gx = 0$, whence

$$x = m_1r/m_2. \tag{2}$$

The masses of homogeneous plates of the same thickness are given by

$$m = \rho Sh = \rho\pi R^2h, \quad m_1 = \rho S_1h = \rho\pi r^2h,$$
$$m_2 = m - m_1 = \rho\pi h\,(R^2 - r^2),$$

where ρ is the density of the material of the plate, S the area of the entire plate, S_1 the area of the hole, and h the plate thickness. Hence, we can write Eq. (2) in the form

$$x = \frac{\rho\pi hr^2r}{\rho\pi h\,(R^2-r^2)} = \frac{r^3}{R^2-r^2}.$$

Using the condition of the problem ($r = R/2$), we obtain

$$x = \frac{R^3}{8\,(R^2-R^2/4)} = \frac{R}{6}.$$

Consequently, the centre of mass lies at a distance of $R/6$ from the centre of the plate.

200°. The dependence of the potential energy of a system on the x-coordinate is given by $W(x) = -5x^2 + 4x - 3$. Determine the coordinate of a point corresponding to the equilibrium position of the system and indicate the type of equilibrium.

Given: $W(x) = -5x^2 + 4x - 3$.

x — ?

Solution. By definition, the system is in equilibrium if

$$\frac{dW}{dx} = 0.$$

Consequently, evaluating the first derivative $\frac{d}{dx}(-5x^2 + 4x - 3) = -10x + 4$ of the expression for the potential energy with respect to x and equating it to zero, we obtain the equation $-10x + 4 = 0$. Therefore, the coordinate of the point corresponding to the equilibrium position of the system is $x = 0.4$ m. In order to determine the type of equilibrium, we test the sign of the second derivative:

$$\frac{d^2W}{dx^2} = \frac{d}{dx}(10x+4) - 10 < 0.$$

Consequently, the equilibrium of the system is unstable.

201°. The potential energy of a body of mass 0.5 kg varies according to the law $W(x) = 6x^2 + 4x - 2$. Determine the acceleration of the body at the moment when it passes through the equilibrium position.

Given: $m = 0.5$ kg.

a — ?

Solution. By definition, the work dA is equal to the change dW in the energy of the body:

$$dA = dW. \tag{1}$$

On the other hand,

$$dA = F\,dx. \tag{2}$$

Equating the right-hand sides of Eqs. (1) and (2), we obtain $dW = F\,dx$, whence

$$F = \frac{dW}{dx}. \tag{3}$$

According to Newton's second law, $a = F/m$, or, taking Eq. (3) into account, $a = \frac{1}{m}\frac{dW}{dx}$. Differentiating the expression for the potential energy, we obtain

$$a = \frac{1}{m}\frac{d}{dx}(6x^2 + 4x - 2) = \frac{1}{m}(12x + 4). \tag{4}$$

The coordinate of the body in the equilibrium position is determined from the condition $\frac{dW}{dx} = 0$, or $\frac{d}{dx}(6x^2 + 4x - 2) = 12x + 4 = 0$, whence $x = -1/3$ m. Substituting x and m into Eq. (4), we get

$$a = \frac{1}{0.5}\left[12 - \left(-\frac{1}{3}\right) + 4\right] = 0.$$

202. Why is it easier to unscrew a nut with a long spanner than with a short spanner?

Answer. In order to turn a nut with a spanner, we must apply to it a force whose torque must be equal to or larger than the moment of the friction acting between the nut and the spanner. Since the torque is equal to the product of the force and its arm, the longer the spanner, the smaller the force that should be applied to it.

203. One truck is carrying wood and another hay. Which of the two will overturn more easily if their masses are equal?

Answer. The truck with hay will overturn more easily since its centre of mass is higher than that of the truck with wood. The stability of a body is the higher, the lower its centre of mass.

EXERCISES

204. A wall-mounted jib crane has a jibstay BC of length 4 m and a tie-rod AC of length 3 m, the distance AB between them being 1.5 m. Determine the forces acting on AC and BC if the mass of the load being hoisted is 2 t. What action is exerted by these forces on the jibstay and the tie-rod (Fig. 91)?

205. An electric lamp of mass 2 kg is suspended from the ceiling on a cord AB and is pulled towards the wall by string BC. Determine the tensions of the cord and the string if $\alpha = 60°$ and $\beta = 135°$ (Fig. 92).

206. A cylindrical tank of diameter 20 cm is filled with water and suspended from a hinge A on a weightless rod rigidly fixed to the tank (Fig. 93). Water flows at a veloc-

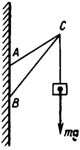


Fig. 91

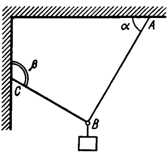


Fig. 92

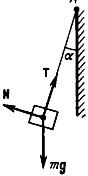


Fig. 93

ity of 4.43 m/s from a side hole of area 2 cm^2 drilled in the tank wall near its bottom. Through what angle from the vertical will the tank be deflected? The tank height is 1 m. The mass of the tank and the drop in the water level due to leakage should be neglected and the angle of deflection should be regarded as small.

207. A homogeneous rod AB is hinged to a vertical wall at point A and is held at an angle of 60° to the vertical with the help of a string BC forming an angle of 30° with it (Fig. 94). Determine the normal reaction of the hinge if the mass of the rod is 2 kg.

208. A 10-m long beam rests on two supports. A load of mass 5 t is put at a distance of 2 m from the left end of the beam. Determine the forces of pressure exerted by the beam on the supports if the beam mass is 10 t.

209. The mass of a roller is 100 kg and its radius is 0.5 m. What minimum force must be applied to the roller to roll it over a beam of height 10 cm?

210. A rod is subjected to the action of two parallel and opposite forces of 10 and 25 N. The points of application of the forces are at 1.5 m from each other. Determine the resultant of the forces and the point of its application.

211. Determine the position of the centre of mass of a homogeneous plate whose size and shape are indicated in Fig. 95.

212. A sphere of mass 10 kg rests on two identical inclined planes forming angles of 45° with the horizontal. Determine the force of pressure exerted by the sphere on the planes.

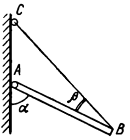


Fig. 94

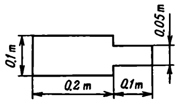


Fig. 95

QUESTIONS FOR REVISION

1. Define the equilibrium of a body. 2. Name all the types of equilibrium. 3. Formulate the equilibrium condition for a point mass. 4. What is torque? 5. How is the direction of the moment of force vector determined? 6. Define the arm of a force. 7. Formulate the equilibrium condition for a body relative to a fixed rotational axis. 8. Formulate the equilibrium conditions for a body in the general case. 9. Define the centre of mass of a system of bodies and indicate a method determining its position in space. 10. What is the centre of mass of a body? 11. How can the position of the centre of mass of a body of an arbitrary shape be determined?

1.5. Hydro- and Aerostatics

Pressure is defined as

$$p = F/S,$$

where F is the force acting on a surface of area S perpendicular to this force.

The pressure exerted by a column of a homogeneous liquid at a depth h **(hydrostatic pressure)** is defined as

$$p = \rho g h,$$

where ρ is the density of the liquid.

A fluid (i.e. liquid or gas) transmits a pressure exerted on its surface uniformly in all directions **(Pascal's law)**.

Pascal's law implies that if a homogeneous liquid is in equilibrium in communicating vessels, the pressure acting on the surfaces of the same level in these vessels will be the same.

A body immersed in a fluid is acted upon by a buoyant force equal to the weight of the fluid it displaces and applied at the centre of gravity of the displaced volume (the **Archimedean principle**). This principle leads to the **floatation condition:** *if the buoyancy is equal to the force of gravity acting on a body immersed in a fluid, the body floats in the fluid.*

* * *

213. Equal masses of mercury and water are poured in a cylindrical vessel. The total height of the liquid column is 29.2 cm. Determine the pressure exerted by the column on the bottom of the vessel (Fig. 96).

Given: $h = 29.2$ cm $= 0.292$ m, $m_1 = m_2$.

p — ?

Solution. The total pressure exerted by the liquids on the bottom of the vessel is

$$p = p_1 + p_2. \quad (1)$$

Here $p_1 = \rho_1 g h_1$ and $p_2 = \rho_2 g h_2$ are the pressures exerted by mercury and water on the bottom of the vessel, where ρ_1 and ρ_2 are the densities of mercury and water. Substituting these expressions into Eq. (1), we obtain

$$p = g(\rho_1 h_1 + \rho_2 h_2), \quad (2)$$

and

$$h = h_1 + h_2. \quad (3)$$

By hypothesis, the masses of the liquid columns are equal: $m_1 = m_2$, or $\rho_1 h_1 S = \rho_2 h_2 S$, whence

$$\rho_1 h_1 = \rho_2 h_2, \quad (4)$$

where S is the area of the vessel bottom. Using Eqs. (3) and (4), we obtain

$$h_1 = h\rho_2/(\rho_1 + \rho_2), \quad h_2 = h\rho_1/(\rho_1 + \rho_2). \quad (5)$$

Substituting Eqs. (5) into (2), we get

$$p = g\left(\frac{\rho_1\rho_2 h}{\rho_1+\rho_2} + \frac{\rho_1\rho_2 h}{\rho_1+\rho_2}\right) = \frac{2\rho_1\rho_2 g h}{\rho_1+\rho_2},$$

$$p = \frac{2\times 13.6\times 10^3\times 10^3\times 9.8\times 0.292}{13.6\times 10^3 + 10^3}\ \text{Pa} \simeq 5.3\ \text{kPa}.$$

214. The limbs of a U-tube are filled with water and oil separated by mercury (Fig. 97). The interfaces between

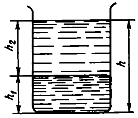


Fig. 96

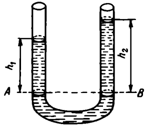


Fig. 97

mercury and the liquids in the limbs are at the same level. Determine the height of the water column if the height of the oil column is 20 cm.

Given: $h_2 = 20\ \text{cm} = 0.2\ \text{m}$.

$h_1 - ?$

Solution. According to Pascal's law, the pressure in the limbs of the U-tube is the same at the level AB (at the interfaces between mercury and the liquids):

$$p_1 = p_2. \tag{1}$$

Here $p_1 = \rho_1 g h_1$ and $p_2 = \rho_2 g h_2$ are the pressures in the left and right limbs respectively at the level AB, where

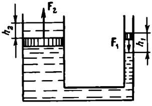


Fig. 98

ρ_1 and ρ_2 are the densities of water and oil. Substituting these expressions into Eq. (1), we obtain $\rho_1 g h_1 = \rho_2 g h_2$, whence

$$h_1 = \frac{\rho_2 h_2}{\rho_1},$$

$$h_1 = \frac{0.9 \times 10^3 \times 0.2}{10^3}\ \text{m} = 0.18\ \text{m}.$$

215. A force of 196 N is applied to the smaller piston of a hydraulic press. Under the action of the force, the piston is lowered by 25 cm during one stroke, and, as a result, the larger piston is raised by 5 cm (Fig. 98). What is the force of pressure transmitted to the larger piston?

Given: $h_1 = 25\ \text{cm} = 0.25\ \text{m}$, $h_2 = 5\ \text{cm} = 0.05\ \text{m}$, $F_1 = 196\ \text{N}$.

$F_2 - ?$

Solution. According to Pascal's law, $p_1 = p_2$, where $p_1 = F_1/S_1$ is the pressure exerted by the smaller piston of area S_1 on the liquid and $p_2 = F_2/S_2$ the pressure exerted by the liquid on the larger piston of area S_2. Writing Pascal's law in the form $F_1/S_1 = F_2/S_2$, we obtain

$$F_2 = F_1S_2/S_1. \quad (1)$$

Since the liquid is incompressible, the volume of the liquid flowing from the smaller to the larger cylinder is the same: $V_1 = V_2$, or $S_1h_1 = S_2h_2$, whence

$$S_2/S_1 = h_1/h_2. \quad (2)$$

Substituting ratio (2) into expression (1), we obtain

$$F_2 = F_1\frac{h_1}{h_2},$$

$$F_2 = 196\,\frac{0.25}{0.05}\ \mathrm{N} = 980\ \mathrm{N}.$$

216. What must be the height of a cylindrical vessel of radius 5 cm filled with water for the force of pressure exerted by water on the bottom to be equal to the force of pressure on the lateral surface?

Given: $R = 5\ \mathrm{cm} = 0.05\ \mathrm{m}$.

$h - ?$

Solution. By definition, the force of pressure on the bottom of the vessel is

$$F = pS. \quad (1)$$

Since the pressure on the bottom is $p = \rho gh$, where ρ is the density of water, h the vessel height, and $S = \pi R^2$ the area of the vessel bottom, expression (1) becomes $F = \rho gh\pi R^2$.

Similarly, the force of pressure on the lateral surface of the vessel is

$$F_{\mathrm{lat}} = \langle p\rangle\, S_{\mathrm{lat}},$$

where $\langle p\rangle = p/2$ is the mean pressure exerted by water on the lateral surface of the vessel and S_{lat} the area of the lateral surface. Considering that $p = \rho gh$ and $S_{\mathrm{lat}} =$

$2\pi Rh$, we obtain

$$F_{lat} = \rho g \pi R h^2.$$

Since $F = F_{lat}$, or $\rho g h \pi R^2 = \rho g \pi R h^2$, we obtain

$$h = R, \quad h = 0.05 \text{ m}.$$

217. A spring balance with a piece of copper-silver alloy suspended to it indicates 2.41 N in air and 2.17 N

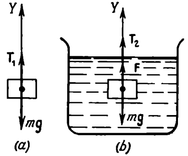


Fig. 99

in water. Determine the masses of copper and silver in the alloy, neglecting the buoyancy in air.

Given: $T_1 = 2.41$ N, $T_2 = 2.17$ N.

m_c — ? m_s — ?

Solution. During weighing in air, the piece of alloy is acted upon by the force of gravity mg and the tension $\mathbf{T}_1$ of the spring (Fig. 99*a*). Writing the equilibrium condition for the body in projections on the Y-axis, we obtain

$$-mg + T_1 = 0,$$

whence $mg = T_1$ and $m = T_1/g$. During weighing in water, the force of gravity mg, the tension $\mathbf{T}_2$ of the spring, and the buoyant force $\mathbf{F}$ act on the piece of alloy (Fig. 99*b*). Writing the equilibrium condition for the body in water, we get

$$m\mathbf{g} + \mathbf{T}_2 + \mathbf{F} = 0.$$

In projections on the Y-axis, this equation has the form

$$-mg + T_2 + F = 0. \qquad (1)$$

Considering that $mg = T_1$ and $F = \rho g V$, where V is the volume of the body and ρ the density of water, we transform Eq. (1) as follows:

$$T_2 - T_1 + \rho g V = 0. \qquad (2)$$

In other words, the mass of the body in air and its volume are

$$m = m_c + m_s, \qquad (3)$$

$$V = V_c + V_s, \quad \text{or} \quad V = m_c/\rho_c + m_s/\rho_s, \qquad (4)$$

where V_c and V_s are the volumes of copper and silver in the piece of alloy, and ρ_c and ρ_s the densities of copper and silver. Solving Eqs. (2)-(4) together, we obtain

$$m_s = \frac{\rho_s \rho_c}{(\rho_s - \rho_c)\, g} \left(\frac{T_2 - T_1}{\rho} + \frac{T_1}{\rho_c} \right),$$

$$m_s = \frac{10.5 \times 10^3 \times 8.9 \times 10^3}{(10.5 \times 10^3 - 8.9 \times 10^3) \times 9.8} \times \left(\frac{2.17 - 2.41}{10^3} + \frac{2.41}{8.9 \times 10^3} \right) \text{kg} \simeq 0.210 \text{ kg}.$$

Considering that $m = T_1/g$, we get from Eq. (3)

$$m_c = m - m_s = T_1/g - m_s,$$

$$m_c = (2.41/9.8 - 0.210) \text{ kg} = 0.0356 \text{ kg}.$$

218. A hollow iron sphere is weighed in air and kerosene. The readings of the dynamometer are 2.59 and 2.16 N respectively. Determine the volume of the cavity of the sphere, neglecting the buoyancy of air.

Given: $T_1 = 2.59$ N, $T_2 = 2.16$ N.

V_{cav} — ?

Solution. During weighing in air, the sphere is acted upon by the force of gravity mg and the tension $\mathbf{T}_1$ of the spring (see Fig. 99*a*). Writing the equilibrium condition for the sphere in projections on the Y-axis, we obtain

$$-mg + T_1 = 0,$$

whence $mg = T_1$, or $m = T_1/g$. During weighing in kerosene, the sphere experiences the action of the force of gravity mg, the tension $\mathbf{T}_2$ of the spring, and the buoyant force $\mathbf{F}$ exerted by kerosene (see Fig. 99*b*). Writing the equilibrium condition for the sphere in kerosene in projections on the Y-axis, we get

$$-mg + T_2 + F = 0. \quad (1)$$

Considering that $mg = T_1$ and $F = \rho_k gV$, where V is the volume of the sphere and ρ_k the density of kerosene, we transform Eq. (1) as follows:

$$T_2 - T_1 + \rho_k gV = 0. \quad (2)$$

The volume of the cavity is $V_{cav} = V - V_1$, where $V_1 = m/\rho_1$ is the volume occupied by iron. Consequently, $V_{cav} = V - m/\rho_1$. Since $m = T_1/g$, we have

$$V_{cav} = V - T_1/(\rho_1 g). \quad (3)$$

Solving Eqs. (2) and (3) together, we obtain

$$V_{cav} = \frac{T_1 - T_2}{\rho_k g} - \frac{T_1}{\rho_1 g},$$

$$V_{cav} = \left(\frac{2.59 - 2.16}{0.8 \times 10^3 \times 9.8} - \frac{2.59}{7.8 \times 10^3 \times 9.8}\right) \text{m}^3$$

$$\simeq 2.1 \times 10^{-5} \text{ m}^3.$$

219. A homogeneous body floats on the surface of kerosene so that the volume of the submerged part constitutes 0.92 of the entire volume of the body. Determine the volume of the submerged part of the body floating on the surface of water.

Given: $V_{sub} = 0.92V$.

V'_{sub} — ?

Solution. We denote the entire volume of the body by V, the volume of the part submerged in kerosene by V_{sub}, and the volume of the part submerged in water by V'_{sub}. The body in kerosene experiences the action of the force of gravity mg and the buoyant force $\mathbf{F} = -\rho_k V_{sub}\mathbf{g}$ exerted by kerosene (Fig. 100). The floatation condition implies that $mg = F$, or

$$mg = \rho_k V_{sub} g = \rho_k \cdot 0.92Vg, \quad (1)$$

where ρ_k is the density of kerosene.

Similarly, we can write the floatation condition for water:

$$mg = F_w, \quad \text{or} \quad mg = \rho_w V'_{sub} g, \tag{2}$$

where ρ_w is the density of water.

Solving Eqs. (1) and (2) together, we obtain $\rho_k \times 0.92 V g = \rho_w V'_{sub} g$, whence

$$V'_{sub} = \frac{0.92 \rho_k}{\rho_w} V,$$

$$V'_{sub} = \frac{0.92 \times 0.8 \times 10^3}{10^3} V \simeq 0.74 V.$$

220. One end of a 20-cm long thin wooden stick is hinged and the free end is immersed in water. What part of the stick will be submerged in water in equilibrium?

Given: $L = 20 \text{ cm} = 0.2 \text{ m}$.

l — ?

Solution. A stick submerged in water is acted upon by the force of gravity mg, the buoyant force **F** of water, and the normal reaction **N** of the hinge (Fig. 101). Writing the equilibrium condition for a body having a rotational axis, we obtain for the stick

$$M_1 - M_2 = 0. \tag{1}$$

Here $M_1 = Fl_1$ and $M_2 = mgl_2$ are the moments of the forces F and mg about the axis passing through point O,

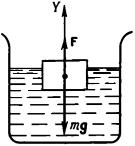


Fig. 100

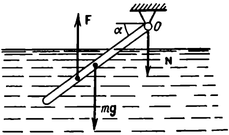


Fig. 101

$l_1 = (L - l/2) \cos \alpha$ and $l_2 = (L/2) \cos \alpha$ being the arms of the forces F and mg.

Substituting the expressions for M_1 and M_2 into Eq. (1), we get

$$F (L - l/2) \cos \alpha - mg (L/2) \cos \alpha = 0. \quad (2)$$

Considering that $F = \rho_w gSl$ and $mg = \rho_{wood} gSL$, where S is the cross-sectional area of the stick, we can write Eq. (2) in the form

$$\rho_w gSl (L - l/2) - \rho_{wood} gSL^2/2 = 0,$$

whence

$$l^2 - 2Ll + \rho_{wood} L^2/\rho_w = 0. \quad (3)$$

Solving Eq. (3), we find that

$$l = L \pm \sqrt{L^2 - \rho_{wood} L^2/\rho_w} = L (1 \pm \sqrt{1 - \rho_{wood}/\rho_w}).$$

Since the length of the submerged part cannot exceed its length, we obtain

$$l = L (1 - \sqrt{1 - \rho_{wood}/\rho_w}),$$

$$l = 0.2 (1 - \sqrt{1 - 0.8 10^3/10^3}) \text{ m} \simeq 0.11 \text{ m}.$$

221. What is heavier in water: a brick or a piece of iron of the same mass?

Answer. The piece of iron will be heavier.

222. Is the buoyant force acting on a body at different depths the same?

Answer. Since liquids are practically incompressible, they have nearly the same density at different depths. Consequently, the buoyant forces will be the same at different depths.

223. Where is the draught of a ship larger: in river or at sea?

Answer. The draught of the ship in sea water is smaller than in river since the density of sea water is higher than that of fresh water.

EXERCISES

224. What force of pressure can be developed by a hydraulic press if the force applied to the longer arm of the lever transmitting pressure to the smaller piston is

100 N, the ratio of the lever arms is 1:9, and the areas of the pistons are 5 and 500 cm^2? The press efficiency is 0.8.

225. A barometer tube is inclined at an angle of 30° to the horizontal. What is the length of the mercury column under the normal atmospheric pressure?

226. At what depth is the pressure in fresh water thrice higher than the atmospheric pressure equal to 1.017×10^5 Pa?

227. Two communicating tubes with different cross-sectional areas are filled with mercury. Then 272 g of water are poured in a wider tube of 8-cm^2 cross-sectional area. What will be the difference in the mercury levels in the tubes?

228. A bar made of gold and silver is weighed in air and then in water. The readings of the spring balance are 3 and 2.756 N respectively. Determine the mass of the gold and silver in the bar.

229. A piece of wood floats in water so that three-fourths of its volume are submerged. What is the density of the wood?

230. A hollow copper sphere of volume 44.5 cm^3 floats in water so that it is submerged to half. Determine the volume of the cavity.

231. A homogeneous stick is hinged at the upper end, and its lower end is immersed in water. The stick is in equilibrium when half of it is submerged. Determine the density of the material of which the stick is made.

QUESTIONS FOR REVISION

1. Define pressure and the force of pressure. What is the SI unit of pressure? 2. Formulate Pascal's law. 3. What is the pressure in a liquid at a certain depth h? 4. How can the force of pressure exerted on the bottom and the walls of a vessel be calculated? 5. Formulate the laws of communicating vessels for homogeneous and heterogeneous liquids. 6. Explain the principle of operation of a hydraulic press. 7. What is the buoyant force? 8. Formulate the Archimedean principle. 9. Formulate the floatation conditions for a body. 10. Define the lifting force.

Chapter 2
MOLECULAR PHYSICS AND THERMODYNAMICS

2.1. Basic Concepts of the Molecular Theory

HEAT AND WORK

All substances consist of atoms and molecules. The amount of substance is measured in special units called **moles**. A mole of any substance contains the same number of molecules $N_A = 6.02 \times 10^{23}$ mol^{-1} **(Avogadro's constant)**.

All molecules are in increasing random motion known as **thermal motion**.

The sum of the kinetic energy W_k of the random motion of all molecules of a body and the potential energy W_p of their interaction is called the **internal energy** U of the body:

$$U = W_k + W_p.$$

For an ideal gas, the potential energy of molecules is neglected, and hence its internal energy is completely determined by its temperature:

$$U = cmT.$$

Here c is the specific heat of the gas at constant volume, m the mass of the gas, and T its thermodynamic temperature (in K): $T = 273 + t$, where t is the gas temperature in degrees centigrade.

Gas molecules possess the mean kinetic energy of translational motion, defined as

$$W_k = (1/2)m_0 \langle v^2 \rangle,$$

where m_0 is the mass of a molecule and $v_{r.m.s} = \sqrt{\langle v^2 \rangle}$ the **root-mean-square velocity** of translational motion of molecules.

According to the molecular theory, this energy is connected with thermodynamic temperature through the following relation:

$$W_k = (3/2)kT,$$

where $k = R/N_A$ is the **Boltzmann constant.**

The internal energy of a body may change as a result of two processes, viz. the heat exchange and the conversion of mechanical energy into the internal energy of the body.

In all processes occurring with a body, the energy conservation law (**the first law of thermodynamics**) is observed: *the amount of heat supplied to the body is spent on increasing the internal energy of the body and on doing work by the body on the surroundings*:

$$Q = \Delta U + A. \quad (1)$$

Here, it is assumed that
$Q > 0$ if the body receives a certain amount of heat,
$Q < 0$ if the body gives away a certain amount of heat,
$A > 0$ if the body does a work, and
$A < 0$ if the work is done on a body.

Let us consider two special cases of the change in internal energy.

1. The change in internal energy during heat exchange (without doing a mechanical work). In this case, Eq. (1) becomes

$$Q = \Delta U. \quad (2)$$

The change ΔU in internal energy can be calculated from the following relations:

$$\Delta U = cm\,\Delta T \quad (3)$$

for heating and cooling,

$$\Delta U = \lambda m \quad (4)$$

for melting and crystallization,

$$\Delta U = rm \quad (5)$$

for vaporization and condensation, and

$$\Delta U = qm \quad (6)$$

for combustion of a substance. Here c is the specific heat of a body, λ the latent heat of fusion, r the latent heat of vaporization, q the heat of combustion of the substance, m the mass of the body, and ΔT the change in temperature, $\Delta T = \Theta - T_{\text{in}}$, where Θ is the final and T_{in} the initial temperature of the body. For crystallization and condensation, we must assume in Eqs. (4) and (5) that $\Delta U < 0$. In the general case, Eq. (2) has the form

$$Q = \sum_{i=1}^{n} \Delta U_i. \tag{7}$$

If several bodies take part in heat exchange, the algebraic sum of the amounts of heat given away by the bodies whose internal energy decreases and the amounts of heat received by the bodies whose internal energy increases is zero for a closed system of bodies. This statement is known as the **heat-balance equation**:

$$\sum_{i=1}^{n} Q_{i\,\text{giv}} + \sum_{j=1}^{n} Q_{j\,\text{rec}} = 0, \tag{8}$$

where Q_{giv} and Q_{rec} are calculated by Eq. (7).

2. The change in internal energy as a result of doing mechanical work by a body (without heat exchange with the surroundings). In this case, Eq. (1) becomes

$$0 = \Delta U + A,$$

whence

$$\Delta U = -A. \tag{9}$$

The mechanical work A can be calculated from the familiar relations.

In the presence of heat losses, the distinction should be made between the amount of heat Q_{u} spent to change the internal energy of a body (useful heat) and the total amount of heat Q_{l} (taking into account heat losses). In this case, the concept of efficiency η is introduced:

$$\eta = \frac{Q_{\text{u}}}{Q_{\text{l}}} 100\%.$$

* * *

232. Determine the mass of a hydrogen molecule.

Given: $M = 2 \times 10^{-3}$ kg/mol.

$m_0 - ?$

Solution. If we take a mole of hydrogen whose molecular mass is $M = 2 \times 10^{-3}$ kg/mol, it contains, by definition, $N_A = 6.02 \times 10^{23}$ mol^{-1} molecules. Consequently, the mass of a hydrogen molecule is

$$m_0 = \frac{M}{N_A},$$

$$m_0 = \frac{2 \times 10^{-3}}{6.02 \times 10^{23}} \text{ kg} \simeq 3.32 \times 10^{-27} \text{ kg}.$$

233. How many water molecules are contained in a drop whose mass is 0.2 g?

Given: $m = 0.2 \text{ g} = 0.2 \times 10^{-3}$ kg, $M = 18 \times 10^{-3}$ kg/mol.

$N - ?$

Solution. The number of moles in the drop of mass m is

$$\nu = m/M.$$

Since a mole contains N_A molecules, ν moles of water contain νN_A molecules. Consequently,

$$N = \frac{m}{M} N_A,$$

$$N = \frac{0.2 \times 10^{-3}}{18 \times 10^{-3}} \times 6.02 \times 10^{23} \simeq 6.7 \times 10^{21}.$$

234. Determine the root-mean-square velocity of oxygen molecules at 20 °C. At what temperature is this velocity equal to 500 m/s?

Given: $T_1 = 293$ K, $v_{2\text{r.m.s}} = 500$ m/s.

$v_{1\text{r.m.s}} - ?$ $T_2 - ?$

Solution. By definition, the mean kinetic energy of a gas molecule is

$$W_{k1} = (1/2) m_0 \langle v_1^2 \rangle, \tag{1}$$

where m_0 is the mass of a molecule and $v_{1\text{r.m.s}} = \sqrt{\langle v_1^2 \rangle}$ the root-mean-square velocity. Alternately, according to the molecular theory, the mean kinetic energy of translational motion of a gas molecule is given by

$$W_{k1} = (3/2)kT_1. \quad (2)$$

Equating the right-hand sides of Eqs. (1) and (2), we obtain $(1/2)m_0\langle v_1^2 \rangle = (3/2)kT_1$, whence

$$v_{1\text{r.m.s}} = \sqrt{3kT_1/m_0}. \quad (3)$$

Since $k = R/N_A$, expression (3) can be written in the form $v_{1\text{r.m.s}} = \sqrt{3RT_1/(m_0N_A)}$ or, considering that $m_0N_A = M$, we obtain

$$v_{1\text{r.m.s}} = \sqrt{3RT_1/M},$$

$$v_{1\text{r.m.s}} = \sqrt{\frac{3 \times 8.3 \times 293}{32 \times 10^{-3}}} \frac{\text{m}}{\text{s}} \simeq 480 \text{ m/s}.$$

Similarly, $v_{2\text{r.m.s}} = \sqrt{3RT_2/M}$, whence

$$T_2 = \frac{v_{2\text{r.m.s}}^2 M}{3R},$$

$$T_2 = \frac{500^2 \times 32 \times 10^{-3}}{3 \times 8.3} \text{ K} \simeq 320 \text{ K}.$$

235. What amount of heat must be supplied to 2 kg of ice taken at -10 °C to melt it, to heat the obtained water to 100 °C, and to vaporize it?

Given: $m = 2$ kg, $T_1 = 263$ K, $T_2 = 373$ K.

Q—?

Solution. The change in the internal energy of ice is

$$\Delta U_1 = c_{ice}m(T_m - T) \quad (1)$$

for its heating to the melting point,

$$\Delta U_2 = \lambda m \quad (2)$$

for melting the ice,

$$\Delta U_3 = c_w m (T_b - T_m) \quad (3)$$

for heating the obtained water to the boiling point, and

$$\Delta U_4 = rm \tag{4}$$

for vaporizing the obtained water.

Then the total amount of heat that must be supplied to the body is

$$Q = \Delta U_1 + \Delta U_2 + \Delta U_3 + \Delta U_4. \tag{5}$$

Substituting Eqs. (1)-(4) into (5), we obtain

$$Q = c_{\text{ice}} m\,(T_m - T_1) + \lambda m + c_w m\,(T_b - T_m) + rm$$
$$= m\,[c_{\text{ice}}\,(T_m - T_1) + \lambda + c_w\,(T_b - T_m) + r],$$
$$Q = 2 \times [2.1 \times 10^3 \times (273 - 263) + 3.35 \times 10^5$$
$$+4.19 \times 10^3 \times (373 - 273) + 22.6 \times 10^5]\ \text{J}$$
$$\simeq 61\ \text{MJ}.$$

236. A metal body of mass 192 g heated to 100 °C is immersed in a brass calorimeter of mass 128 g containing 240 g of water at 8.4 °C. The final temperature which sets in the calorimeter is 21.5 °C. Determine the specific heat of the tested body.

Given: $m_1 = 128\ \text{g} = 0.128\ \text{kg}$, $m_2 = 240\ \text{g} = 0.24\ \text{kg}$, $T_1 = T_2 = 281.4\ \text{K}$, $m_3 = 192\ \text{g} = 0.192\ \text{kg}$, $T_3 = 373\ \text{K}$, $\Theta = 294.5\ \text{K}$.

c_3—?

Solution. Three bodies participate in heat exchange: the calorimeter, the water in it, and the metal which gives away heat, while water and the calorimeter receive heat. Let us write the heat-balance equation:

$$Q_1 + Q_2 + Q_3 = 0, \tag{1}$$

where $Q_1 = \Delta U_1 = c_1 m_1 (\Theta - T_1)$ and $Q_2 = \Delta U_2 = c_2 m_2 (\Theta - T_1)$ are the amounts of heat received by the calorimeter and water respectively and $Q_3 = \Delta U_3 = c_3 m_3 (\Theta - T_3)$ the amount of heat given away by the body. Substituting these relations into Eq. (1), we obtain $c_1 m_1 (\Theta - T_1) + c_2 m_2 (\Theta - T_1) + c_3 m_3 (\Theta - T_3) = 0,$

whence

$$c_3 = \frac{(c_1 m_1 + c_2 m_2)(\Theta - T_1)}{m_3 (T_3 - \Theta)},$$

$$c_3 = \frac{(0.38 \times 10^3 \times 0.128 + 4.19 \times 10^3 \times 0.24)(294.5 - 281.4)}{0.192 \times (373 - 294.5)} \frac{\text{J}}{\text{kg} \cdot \text{K}}$$

$$\simeq 9.2 \times 10^2 \text{ J/(kg} \cdot \text{K)}.$$

237. A brass calorimeter of mass 100 g contains 5 g of ice at -10 °C. Molten lead of mass 30 g at the melting point is poured into the calorimeter. What substances will be in the calorimeter after the heat exchange and what will be the temperature after the stabilization? Heat losses for vaporization should be neglected.

Given: $m_1 = 100 \text{ g} = 0.1 \text{ kg}$, $m_2 = 5 \times 10^{-3}$ kg,
$T_2 = 263$ K, $m_3 = 30 \text{ g} = 3 \times 10^{-2}$ kg.

m—? Θ—?

Solution. The heat-balance equation cannot be written straightaway for problems of this type since the result of the heat exchange is unknown. In order to determine the result, we must analyze the process.

The heat exchange in the system involves three bodies: the calorimeter and the ice receive while the lead gives away the amounts of heat Q_1, Q_2, and Q_3 respectively. Naturally, the heat-balance equation

$$Q_1 + Q_2 + Q_3 = 0 \qquad (1)$$

is valid in this case. At the end of the process, a certain temperature will be established in the system. Its value Θ lies in the temperature interval $T_2 < \Theta < T_{m3}$, where T_2 is the initial temperature of ice and the calorimeter and T_{m3} the initial temperature (melting point) of lead. Such an indeterminacy in the final temperature is due to the fact that phase transitions of ice and lead may occur in the process of heat exchange. For this reason, we shall proceed as follows: we shall choose arbitrarily the value Θ of the final temperature and then calculate the values of Q_1, Q_2, and Q_3 for this temperature, after which the fulfilment of the heat-balance equation (1) should be verified. If the equation does not hold, we shall use its left-hand side to make final assumptions concerning the result of the heat-exchange process. Thus, we assume that $\Theta = T_{m2}$, where T_{m2} is the melting point of ice. In other words, we assume that the heat exchange proceeds as follows: the lead is crystallized and cooled, while the calorimeter and the icc are heated to the temperature $T_{m2} = 273$ K. We also assume that the ice does not melt. We can now determine Q_1, Q_2, and Q_3:

$$Q_1 = c_1 m_1 (T_{m2} - T_2),$$

$$Q_1 = 0.38 \times 10^3 \times 0.1 \times (273-263) \text{ J} = 380 \text{ J},$$

$$Q_2 = c_2 m_2 (T_{m2} - T_2),$$

$$Q_2 = 2.1 \times 10^3 \times 5 \times 10^{-3} \times (273-263) \text{ J} = 105 \text{ J},$$

$$Q_3 = -\lambda_3 m_3 + c_3 m_3 (T_{m2} - T_{m3}),$$

$$Q_3 = -3 \times 10^{-2} \times [0.25 \times 10^5 + 0.13 \times 10^3 \times (600-273)] \text{ J} \simeq -2025 \text{ J}.$$

Let us determine the sum $Q_1 + Q_2 + Q_3 = (380 + 105 - 2025)$ J $= -1540$ J. It can be seen that under the assumptions made by us, the heat-balance equation does not hold. Since its left-hand side is negative, we assume that the final temperature $\Theta = T_{m2}$ as before, but a part of the ice of mass m has melted, i.e. the ice receives the amount of heat $Q_2 + \lambda m$. Let us substitute the refined value of the amount of heat received by the ice into Eq. (1) and determine the mass of water (melted ice) from it:

$$m = 1540/(3.35 \times 10^5) \text{ kg} = 4.6 \times 10^{-3} \text{ kg}.$$

Since the mass of the obtained water does not exceed the initial mass of the ice, our last assumptions concerning the result of heat exchange are valid.

Thus, the established temperature is 273 K, and the calorimeter contains ice of mass 4×10^{-4} kg, water of mass 4.6×10^{-3} kg, and solid lead of mass 3×10^{-3} kg.

238. Determine the mass of water converted into steam as a result of pouring 10 kg of molten lead at the melting point into a vessel containing 1 kg of water at 20 °C. The vessel is made of brass and has a mass of 0.5 kg. Heat losses should be neglected.

Given: $m_1 = 1$ kg, $T_1 = T_2 = 293$ K, $m_3 = 10$ kg, $m_2 = 0.5$ kg.

m — ?

Solution. Although the result of the heat-exchange process is not known to us, we assume that the final temperature of bodies is equal to the boiling point of water: $\Theta = T_{b1}$ (see the solution of Problem 237), i.e. the heat exchange proceeds as follows: the water and the calorimeter, having received the amount of heat Q_1 and Q_2 respectively, are heated, while the lead, having given away the amount of heat Q_3, solidifies and cools to $\Theta = 373$ K. We also assume that a part of the water of mass m is converted into steam. Under these assumptions, the following condition holds for the value of the mass of the steam:

$$0 \leqslant m \leqslant m_1. \tag{1}$$

Writing the heat-balance equation for such a result of the heat-exchange process, we obtain

$$Q_1 + Q_2 + Q_3 = 0, \tag{2}$$

where

$$Q_1 = c_1 m_1 (\Theta - T_1) + r_1 m, \tag{3}$$

$$Q_2 = c_2 m_2 (\Theta - T_1), \tag{4}$$

$$Q_3 = -\lambda_3 m_3 + c_3 m_3 (\Theta - T_{m3}). \tag{5}$$

From the system of equations (2)-(5), we can determine the mass of vaporized water:

$$m = \frac{m_3 [\lambda_3 + c_3 (T_{m3} - \Theta)] - (m_1 c_1 + m_2 c_2)(\Theta - T_1)}{r_1},$$

$$m = \frac{10 \times [0.25 \times 10^5 + 0.13 \times 10^3 \times (600 - 373)]}{22.6 \times 10^5}\ \text{kg}$$

$$- \frac{(1 \times 4.19 \times 10^3 + 0.5 \times 0.38 \times 10^3)(373 - 293)}{22.6 \times 10^5}\ \text{kg}$$

$$\simeq 0.086\ \text{kg}.$$

Since the obtained value of the mass satisfies condition (1), our assumptions about the result of heat exchange are correct.

239. A vessel whose heat capacity is 0.63 kJ/K contains 0.5 l of water and 250 g of ice at 0 °C. What temperature will set in after the addition of 90 g of steam at 100 °C?

Given: $C_1 = 0.63$ kJ/K $= 0.63 \times 10^3$ J/K, $V_2 = 0.5$ l $= 5 \times 10^{-4}$ m³, $m_3 = 250$ g $= 0.25$ kg, $m_4 = 90$ g $= 0.09$ kg, $T_{m3} = 273$ K, $T_{b3} = 373$ K.

Θ — ?

Solution. As in the previous problems (237 and 238), the result of the heat-exchange process is unknown to us. Therefore, we assume that all the bodies will have as a result of heat exchange a temperature Θ which is higher than the melting point of ice but lower than the boiling point of water. In this process, the calorimeter is heated, having received the amount of heat

$$Q_1 = C_1 (\Theta - T_{m3}), \tag{1}$$

the water of mass $m_2 = \rho_2 V_2$ is heated, having received the amount of heat

$$Q_2 = c_2 m_2 (\Theta - T_{m3}) = c_2 \rho_2 V_2 (\Theta - T_{m3}), \tag{2}$$

the ice is melted and the obtained water is heated, having received the amount of heat

$$Q_3 = \lambda_3 m_3 + c_2 m_3 (\Theta - T_{m3}), \tag{3}$$

the steam is condensed and the formed water is cooled, having given away the amount of heat

$$Q_4 = -r_4 m_4 + c_2 m_4 (\Theta - T_{b3}). \tag{4}$$

Writing the heat-balance equation, we obtain

$$Q_1 + Q_2 + Q_3 + Q_4 = 0. \tag{5}$$

Substituting the expressions for Q_1 through Q_4 into Eq. (5), we get

$$C_1 (\Theta - T_{m3}) + c_2 \rho_2 V_2 (\Theta - T_{m3}) + \lambda_3 m_3$$
$$+ c_2 m_3 (\Theta - T_{m3}) - r_4 m_4 + c_2 m_4 (\Theta - T_{b3}) = 0,$$

whence the steady-state temperature is

$$\Theta=\frac{m_4r_4+C_1T_{m3}-m_3\lambda_3+c_2\,[m_4T_{b3}+(\rho_2V_2+m_3)\,T_{m3}]}{C_1+c_2\,(\rho_2V_2+m_4+m_3)},$$

$$\Theta=\frac{0.09\times 22.6\times 10^5+0.63\times 10^3\times 273-0.25\times 3.35\times 10^5}{0.63\times 10^3+4.19\times 10^3\times (10^3\times 5\times 10^{-4}+0.09+0.25)}\ \text{K}$$

$$+\frac{4.19\times 10^3\times [0.09\times 373+(10^3\times 5\times 10^{-4}+0.25)\times 273]}{0.63\times 10^3+4.19\times 10^3\times (10^3\times 5\times 10^{-4}+0.09+0.25)}\ \text{K}$$

$$\simeq 311\ \text{K}.$$

Since $273\ \text{K} < 311\ \text{K} < 373\ \text{K}$, our assumptions about the result of heat exchange are correct.

240. A vessel from which air is rapidly pumped out contains water at 0 °C. As a result of intense evaporation, the water gradually freezes. What fraction of the initial amount of water can be frozen in this way?

Given: $T = 273$ K.

m_1/m — ?

Solution. We denote by m the initial mass of water and by m_1 and m_2 the masses of water converted into ice and steam respectively. The amount of heat required for evaporation can only be obtained at the expense of the heat released during the freezing of water. In this case, the heat-balance equation has the form

$$Q_1 + Q_2 = 0.$$

Here $Q_1 = \Delta U_1 = -m_1\lambda$ is the amount of heat liberated during the freezing of water and $Q_2 = \Delta U_2 = m_2r$ the amount of heat spent for the evaporation of water, where λ is the latent heat of melting for ice and r the latent heat of vaporization for water. Substituting these expressions into the heat-balance equation, we obtain $m_1\lambda = m_2r$, whence $m_2 = m_1\lambda/r$. Since the mass of water remains unchanged during the heat exchange, we have $m = m_1 + m_2$, or $m = m_1 + m_1\lambda/r$, whence

$$\frac{m_1}{m}=\frac{r}{\lambda+r},$$

$$\frac{m_1}{m}=\frac{22.6\times 10^5}{3.35\times 10^5+22.6\times 10^5}=0.87.$$

241. During the preparation of ice in a domestic refrigerator, the temperature of water drops from 16 to 12 °C during 5 min, and the water is converted into ice in 1 h 55 min. Determine the latent heat of fusion for ice.

Given: $\Delta t_1 = 5\ \text{min} = 0.3 \times 10^3$ s, $T_1 = 289$ K,
$T_2 = 285$ K, $\tau = 1$ h 55 min $= 6.9 \times 10^3$ s.

λ — ?

Solution. The refrigerator receives heat from water during its cooling and crystallization and transfers it to the ambient with the help of a special apparatus. For this reason, the temperature in the refrigerator chamber remains constant. Therefore, we can assume that the rate of change in the internal energy of water is also constant:

$$\Delta U/t = k = \text{const}. \quad (1)$$

Since the internal energy of water has changed by $\Delta U = cm\,(T_{\text{m}} - T_1) - \lambda m$ during the time $t = \Delta t_1 + \tau$, using these expressions in Eq. (1), we obtain

$$m\,[c\,(T_{\text{m}} - T_1) - \lambda]/(\Delta t_1 + \tau) = k. \quad (2)$$

Since the temperature of water decreases during the time Δt_1, the internal energy has changed by $\Delta U_1 = cm\,(T_2 - T_1)$. Taking this into account, we obtain from Eq. (1)

$$cm\,(T_2 - T_1)/\Delta t_1 = k. \quad (3)$$

Substituting Eq. (3) into (2), we finally get

$$\lambda = c\left[T_{\text{m}} - T_1 - (T_2 - T_1)\,\frac{\Delta t_1 + \tau}{\Delta t_1}\right],$$

$$\lambda = 4.19 \times 10^3 \times \left[273 - 289 - (285 - 289) \times \left(\frac{0.3 \times 10^3 + 6.9 \times 10^3}{0.3 \times 10^3}\right)\right]\ \text{J/kg}$$

$$\simeq 3.35 \times 10^5\ \text{J/kg}.$$

242. A red-hot aluminium cube put on ice whose temperature is −20 °C completely sinks in it. Determine the initial temperature of the cube, neglecting the change in the cube volume as a result of cooling.

Given: $T_1 = 253$ K.

$T_2 - ?$

Solution. For the aluminium cube to be completely sunk in ice, the ice in the volume of the cube should be melted. Therefore, we shall assume that the cube and ice in the same volume take part in heat exchange. Writing the heat-balance equation, we obtain

$$Q_1 + Q_2 + Q_3 = 0, \qquad (1)$$

where Q_1 is the amount of heat received by ice during its heating to the melting point, Q_2 the amount of heat received by ice during its melting, and Q_3 the amount of heat given away by the cube during its cooling to the melting point of ice. Considering that $Q_1 = c_1 m_1 (T_m - T_1)$, $Q_2 = \lambda m_1$, and $Q_3 = c_2 m_2 (T_m - T_2)$, we transform Eq. (1) as follows:

$$c_1 m_1 (T_m - T_1) + \lambda m_1 = c_2 m_2 (T_2 - T_m). \qquad (2)$$

Since $m_1 = \rho_1 V$ and $m_2 = \rho_2 V$, where V is the cube volume, Eq. (2) becomes

$$c_1 \rho_1 V (T_m - T_1) + \lambda \rho_1 V = c_2 \rho_2 V (T_2 - T_m),$$

whence

$$T_2 = \frac{c_1 \rho_1 (T_m - T_1) + \lambda \rho_1}{c_2 \rho_2} + T_m,$$

$$T_2 = \frac{2.1 \times 10^3 \times 0.9 \times 10^3 \times (273 - 253) + 3.3 \times 10^5 \times 0.9 \times 10^3}{0.88 \times 10^3 \times 2.7 \times 10^3} \text{ K}$$

$$+ 273 \text{ K} \simeq 414 \text{ K}.$$

243. A lead bullet fired vertically upwards reaches a height of 1200 m. Falling to the ground, it is heated as a result of impact. Assuming that 50% of the mechanical energy of the bullet are spent on its heating, determine the increase in its temperature. Air resistance should be neglected.

Given: $h = 1.2 \times 10^3$ m, $\eta = 50\%$, or $\eta = 0.5$.

$\Delta T - ?$

Solution. Using the relation

$$\Delta U = -\eta A,$$

where $\Delta U = cm\ \Delta T$ is the change in the internal energy of the bullet as a result of heating by ΔT and $A = W - W_0 = -mgh$ the work done by forces during the inelastic impact against the ground, we obtain $cm\ \Delta T = \eta mgh$, whence

$$\Delta T = \frac{\eta gh}{c},$$

$$\Delta T = \frac{0.5 \times 9.8 \times 1.2 \times 10^3}{0.13 \times 10^3}\ \mathrm{K} \simeq 45\ \mathrm{K}.$$

244. At what velocity does a lead bullet striking a wall melt? The temperature of the bullet before the impact is 100 °C, and 60 % of its mechanical energy are converted into the internal energy.

Given: $T = 373$ K, $\eta = 60\%$, or $\eta = 0.6$.

v—?

Solution. We shall use the relation

$$\sum \Delta U = -\eta A. \qquad (1)$$

Here $\sum \Delta U = m\lambda + cm\,(T_m - T)$ is the change in the internal energy of the bullet during its melting and heating to the melting point and $A = W - W_0 = -mv^2/2$ the work done by forces during the inelastic collision of the bullet with the wall, where T_m is the melting point of lead. Taking these relations into account, we write Eq. (1) in the form $cm\,(T_m - T) + m\lambda = \eta mv^2/2$, whence

$$v = \sqrt{\frac{2\,[c\,(T_m - T) + \lambda]}{\eta}},$$

$$v = \sqrt{\frac{2 \times [0.13 \times 10^3 \times (600 - 373) + 0.25 \times 10^5]}{0.6}}\ \frac{\mathrm{m}}{\mathrm{s}} \simeq 426\ \mathrm{m/s}.$$

245. A sledge of mass 6 kg slides down a hill with a slope of 30°. Having traversed a distance of 50 m along the hill, the sledge attains a velocity of 4.5 m/s. Determine the amount of heat liberated as a result of friction of the runners against snow (Fig. 102).

Given: $m = 6$ kg, $\alpha = 30° \simeq 0.52$ rad, $l = 50$ m, $v = 4.5$ m/s.

Q—?

Solution. Let us consider a closed system of bodies consisting of the runners and the ice surface of the hill. We shall use the relation

$$\Delta U = -A, \qquad (1)$$

where $A = W - W_0$ is the work done by the force of friction between the runners and ice, $W = mv^2/2 + mgh$

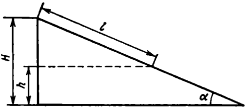


Fig. 102

the total mechanical energy of the sledge at the end of motion, and $W_0 = mgH$ the total mechanical energy of the sledge at the beginning of motion. Taking into account the expressions for A, W, and W_0, we write Eq. (1) in the form

$$\Delta U = -(mv^2/2 + mgh - mgH)$$
$$= mg\,(H - h) - mv^2/2 = m\,(gl \sin\alpha - v^2/2).$$

Then

$$\Delta U = 6 \times (9.7 \times 50 \times 0.5 - 4.5^2)\ \text{J} \simeq 1.4\ \text{kJ}.$$

Since the change in the internal energy of the interacting bodies is measured by the amount of heat, the amount of heat liberated as a result of friction of the runners against snow must be equal to the change in the internal energy of these bodies: $Q = \Delta U \simeq 1.4$ kJ.

246. While heating 300 g of water from 18 to 68 °C on a spirit lamp, 7 g of spirit were burnt. Determine the efficiency of the spirit lamp.

Given: $m_1 = 300\ \text{g} = 0.3\ \text{kg}$, $T_1 = 291\ \text{K}$, $T_2 = 341\ \text{K}$, $m_2 = 7\ \text{g} = 7 \times 10^{-3}\ \text{kg}$.

η — ?

Solution. By definition,

$$\eta = \frac{Q_u}{Q_l} 100\%. \qquad (1)$$

Here $Q_u = \sum \Delta U_1 = c_1 m_1 (T_2 - T_1)$ is the amount of heat spent to change the internal energy of water and $Q_l = \Delta U_2 = q m_2$ the amount of heat liberated as a result of combustion of spirit, where c_1 is the specific heat of water and q the latent heat of combustion of spirit. Substituting the expressions for Q_u and Q_l into Eq. (1), we obtain

$$\eta = \frac{c_1 m_1 (T_2 - T_1)}{q m_2} 100\%,$$

$$\eta = \frac{4.19 \times 10^3 \times 0.3 \times (341 - 291)}{2.93 \times 10^7 \times 7 \times 10^{-3}} 100\% \simeq 31\%.$$

247. The efficiency of a locomotive is 30%. Determine the consumption of fuel in the locomotive per hour if its power is 7.36×10^2 W.

Given: $\eta = 30\%$, $N = 7.36 \times 10^2$ W, $t = 1$ h $= 3.6 \times 10^3$ s.

m — ?

Solution. The efficiency of the locomotive is

$$\eta = \frac{Q_u}{Q_l} 100\%. \qquad (1)$$

Here $Q_u = \Delta U_1 = A = Nt$ is the amount of heat spent to change the internal energy of the fuel gas, as a result of which a mechanical work is done, and $Q_l = \Delta U_2 = qm$ the amount of heat liberated during the combustion of the fuel, where q is the latent heat of combustion of the fuel. Substituting the expressions for Q_u and Q_l into Eq. (1), we obtain

$$\eta = \frac{Nt \cdot 100\%}{qm},$$

whence

$$m = \frac{Nt \cdot 100\%}{\eta q},$$

$$m = \frac{7.36 \times 10^2 \times 3.6 \times 10^3 \times 100}{30 \times 4.61 \times 10^7} \text{ kg} \simeq 0.19 \text{ kg}.$$

248. An engine consumes 25 kg of petrol per hour and is cooled with water. The temperature difference of water at the inlet and outlet of the radiator is 15 K. Determine the water flow rate per second if 30% of the energy liberated as a result of combustion of petrol are spent for heating water.

Given: $m = 25$ kg, $\tau = 1$ h $= 3.6 \times 10^3$ s, $\Delta T = 15$ K, $\eta = 30\%$, or $\eta = 0.3$.

m_s — ?

Solution. By hypothesis,

$$\eta = Q_u/Q_l,$$

where Q_u is the amount of heat required for heating water and Q_l the amount of heat liberated as a result of combustion of the fuel. Considering that $Q_u = cm_1 \Delta T$ and $Q_l = qm$, we obtain

$$\eta = cm_1 \Delta T/(qm),$$

whence the mass of water is

$$m_1 = qm\eta/(c \Delta T).$$

Consequently, the water consumption per second is

$$m_s = \frac{m_1}{\tau} = \frac{qm\eta}{c \Delta T \tau},$$

$$m_s = \frac{4.61 \times 10^7 \times 25 \times 0.3}{4.19 \times 10^3 \times 15 \times 3.6 \times 10^3} \frac{\text{kg}}{\text{s}} \simeq 1.53 \text{ kg/s}.$$

249. Under what conditions can ice be a heater?

Answer. Ice can be a heater in contact with the bodies whose temperature is lower than the temperature of ice. Since the temperature of crystallization of water under normal conditions is 273 K, the temperature of the bodies for which ice is a heater must be lower than this temperature.

250. Why don't damp match sticks burn?

Answer. A match stick burns when its temperature attains the value at which the substance of its head is ignited. When a damp match stick is rubbed against a matchbox, the match stick receives an energy the major part of which is spent to evaporate the moisture contained in it. Therefore, the temperature of the head cannot attain the value at which it is ignited.

251. Why is salt sprinkled over pavements in winter to remove ice from them?

Answer. The mixture of ice and salt has a lower melting point than that of pure ice. Since the temperature of atmospheric air is higher than this melting point, the mixture rapidly melts.

252. Two bodies (copper and iron) of the same mass fall from the same height. Which of the bodies will be heated to a higher temperature as a result of the impact?

Answer. We shall assume that the initial temperatures of the bodies are equal and that the same fraction of their mechanical energy is converted into the internal energy as a result of the inelastic impact. Since the masses of the bodies, the initial heights, and the final velocities at the moment of impact are equal, we have

$$\Delta U_1 = \Delta U_2,$$

or

$$c_1 m\,(T_1 - T) = c_2 m\,(T_2 - T), \tag{1}$$

where T_1 and T_2 are the temperatures of the copper and iron bodies at the moment of the impact and T is the initial temperature of the bodies. Since $c_1 < c_2$, it follows from Eq. (1) that $(T_1 - T) > (T_2 - T)$, i.e. as a result of the impact, the copper body will be heated to a higher temperature than the iron body.

EXERCISES

253. Calculate the masses of an ozone O_3, a carbon dioxide CO_2, and a methane CH_4 molecule.

254. How many molecules are contained in 1 kg of (a) hydrogen H_2 and (b) oxygen O_2 under normal conditions?

255. The mean kinetic energy of translational motion of gas molecules is 1.7×10^{-23} J at 5000 °C. What are the values of this energy at -273 and 1000 °C?

256. A mixture consists of 24 l of water at 12 °C and 40 l of water at 80 °C. Determine the steady-state temperature if the thermal losses during mixing are 420 kJ.

257. A piece of aluminium having a mass of 200 g and heated to 100 °C is placed into a calorimeter containing 270 g of water at 12 °C. The temperature at thermal equilibrium is 23 °C. Determine the specific heat of aluminium if the heat capacity of the calorimeter is 42 J/K.

258. A piece of ice of 20 kg at −20 °C is immersed in 20 l of water at 70 °C. Will the entire mass of ice be melted?

259. A lead bullet flies at a velocity of 200 m/s. What will be the change in the bullet temperature if its entire energy is spent for its heating?

260. A steam hammer of mass 10 t falls freely from a height of 2.5 m on an iron ingot of mass 200 kg so that 30% of the amount of heat liberated during the impact are spent for heating the ingot. How many strokes has the hammer made if the temperature of the ingot has increased by 20 °C?

261. A locomotive having a mass of 213.5 t and a velocity of 72 km/h comes to a halt. What amount of heat is liberated during braking?

262. What power is developed by a bicycle engine if the petrol consumption over 100 km of the path is 1.7 l at a velocity of 25 km/h? The efficiency of the engine is 20%.

263. What will happen with the level of water in a glass with a piece of ice after its melting?

QUESTIONS FOR REVISION

1. Formulate the basic concepts of the molecular theory. 2. Describe experiments confirming the fundamentals of the molecular theory. 3. What is the internal energy of a substance? 4. Name two methods by which the internal energy of bodies can be changed. 5. Formulate the first law of thermodynamics. 6. Name the states of aggregation of matter. 7. What is melting (crystallization) of a substance? 8. Does the temperature of a substance change during its melting (crystallization)? What is the term applied to this temperature and what factors determine it? 9. Define vaporization (condensation) of a substance. 10. Describe two methods of vaporization. 11. Does the temperature of a substance change during vaporization (condensation)? What is the term applied to this temperature and what factors determine it? 12. Write formulas for calculating the change in the internal energy of a substance during melting (crystallization) and during vaporization (condensation)

of a substance. 13. What is the change in the internal energy of a substance during its heating or cooling if the substance remains in the same state of aggregation? 14. Define specific heat, latent heat of fusion, and latent heat of vaporization. 15. What is the change in the internal energy of a substance during its combustion? 16. What is the latent heat of combustion of fuel? 17. Write the heat-balance equation.

2.2. Properties of Gases

The state of any gas can be characterized by its mass m, the volume V occupied by it, the pressure p exerted by the gas on the vessel walls, the temperature T, and the molar mass M. Equations relating these quantities are known as the **equations of state for the gas**.

Boyle's law. For $m = \text{const}$ and $T = \text{const}$, we have

$$pV = \text{const},$$

or for two states of a gas,

$$p_1V_1 = p_2V_2.$$

This law describes **isothermal processes**.

Charles' law. For $m = \text{const}$ and $p = \text{const}$, we have

$$V/T = \text{const},$$

or for two states of a gas,

$$V_1/T_1 = V_2/T_2.$$

This law describes **isobaric processes**.

Gay-Lussac's law. For $m = \text{const}$ and $V = \text{const}$, we have

$$p/T = \text{const},$$

or for two states of a gas,

$$p_1/T_1 = p_2/T_2.$$

This law describes **isochoric processes**.

Ideal gas law. For $m = \text{const}$,

$$pV/T = \text{const},$$

or for two states of a gas,

$$p_1V_1/T_1 = p_2V_2/T_2.$$

Consequently, any state of a gas can be compared with its state under normal conditions: $p_0 = 1.01 \times 10^5$ Pa, $T_0 = 273$ K (0 °C), and $V_0 = 22.4 \times 10^{-3}$ m^3 (the volume of a mole of any gas under normal conditions).

Clapeyron's equation of state (for an ideal gas):

$$pV = (m/M)\, RT,$$

where m is the mass of the gas, M its molar mass, and $R = 8.3$ J/(mol·K) a constant which is the same for all gases and is known as the **molar gas constant**.

Dalton's law. *In a vessel containing a mixture of several gases which do not enter into chemical reactions with one another, the pressure is equal to the sum of partial pressures (viz. the pressures produced by each gas in the absence of other gases in the vessel)*:

$$p = p_1 + p_2 + \ldots + p_n.$$

Any problem on gas laws can be solved by using Clapeyron's equation of state or the ideal gas law (depending on the conditions of the problem). Boyle's, Charles', and Gay-Lussac's laws can be treated as special cases of these equations.

* * *

264. The volume of an air bubble increases threefold as it rises from the bottom of a lake to the surface. What is the depth of the lake?

Given: $V_2 = 3V_1$.

h — ?

Solution. We assume that the temperature of water in the lake is the same at any depth. Then the temperature of air in the bubble is constant, and according to Boyle's law,

$$p_1 V_1 = p_2 V_2,$$

where p_1 and p_2 are the pressures of air in the bubble at the bottom and at the surface of the lake respectively, and V_1 and V_2 the volumes of the bubble at the bottom and at the surface.

The air pressure in the bubble at the surface is obviously equal to the atmospheric pressure p_0, i.e. $p_2 = p_0$. Then $p_1 V_1 = 3p_0 V_1$, whence $p_1 = 3p_0$.

Consequently, the increase in pressure at the bottom is $\Delta p = p_1 - p_0 = 3p_0 - p_0 = 2p_0$. It is known from hydrostatics that $\Delta p = \rho g h$, where ρ is the density of the water and h the depth of the lake. Equating the right-hand sides of the last equations, we obtain $2p_0 = \rho g h$, whence

$$h = \frac{2p_0}{\rho g},$$

$$h = \frac{2 \times 1.01 \times 10^5}{10^3 \times 9.8}\ \text{m} \simeq 20.6\ \text{m}.$$

265. A vessel with air whose pressure is 97 kPa is connected with a piston pump. After five strokes of the piston, the air pressure in the vessel becomes 29 kPa. Determine the ratio of the volumes of the vessel and the cylinder of the pump.

Given: $p_0 = 97\ \text{kPa} = 9.7 \times 10^4\ \text{Pa}$,
$p_5 = 29\ \text{kPa} = 2.9 \times 10^4\ \text{Pa}$, $n = 5$.

$V_1/V_2 - ?$

Solution. Let V_1 and V_2 be the volumes of the vessel and the cylinder of the pump. After the first connection of the cylinder with the vessel, we have, according to Boyle's law, $p_0 V_1 = p_1 (V_1 + V_2)$, whence $p_1 = p_0 V_1/(V_1 + V_2)$. After the second connection, $p_1 V_1 = p_2 (V_1 + V_2)$, whence $p_2 = p_1 V_1/(V_1 + V_2)$, or $p_2 = p_0 V_1^2/(V_1 + V_2)^2$. Similarly, after the fifth connection, $p_5 = p_0 V_1^5/(V_1 + V_2)^5$. Let us transform the obtained equation:

$$\frac{p_0}{p_5} = \left(\frac{V_1 + V_2}{V_1}\right)^5, \quad \log \frac{p_0}{p_5} = 5 \log \frac{V_1 + V_2}{V_1},$$

whence

$$\log \frac{V_1 + V_2}{V_1} = \frac{1}{5} \log \frac{p_0}{p_5},$$

$$\log \frac{V_1 + V_2}{V_1} = \frac{1}{5} \log \frac{9.7 \times 10^4}{2.9 \times 10^4} \simeq \frac{1}{5} \log 3.35$$

$$= \frac{1}{5}\, 0.5250 = 0.1050.$$

From the table of antilogarithms, we find that $(V_1 + V_2)/V_1 \simeq 1.274$, whence $V_1/V_2 \simeq 3.65$.

266. A closed vessel whose volume is 1 l contains 12 kg of oxygen. Determine the pressure of the oxygen at 15 °C.

Given: $V_1 = 1\ \text{l} = 10^{-3}\ \text{m}^3$, $m = 12$ kg, $T_1 = 288$ K.

$p_1 - ?$

Solution. Let us compare the state of the given mass of oxygen (p_1, V_1, T_1) with its state under normal conditions (p_0, V_0, T_0). According to the ideal gas law,

$$p_1 V_1/T_1 = p_0 V_0/T_0.$$

Since $V_0 = m/\rho_0$, where ρ_0 is the density of oxygen under normal conditions, $p_1 V_1/T_1 = p_0 m/(T_0 \rho_0)$, whence

$$p_1 = \frac{p_0 m T_1}{V_1 T_0 \rho_0},$$

$$p_1 = \frac{1.01 \times 10^5 \times 12 \times 288}{10^{-3} \times 273 \times 1.43}\ \text{Pa} \simeq 8.94 \times 10^8\ \text{Pa}.$$

267. Calculate the pressure of a mole of a gas occupying a volume of 1 l at a temperature of 300 K.

Given: $T_1 = 300$ K, $V_1 = 1\ \text{l} = 10^{-3}\ \text{m}^3$.

$p_1 - ?$

Solution. We compare the state of a given mass of the gas (p_1, V_1, T_1) with its state under normal conditions (p_0, V_0, T_0). According to the ideal gas law, we have

$$p_1 V_1/T_1 = p_0 V_0/T_0,$$

whence

$$p_1 = \frac{p_0 V_0 T_1}{V_1 T_0},$$

$$p_1 = \frac{1.01 \times 10^5 \times 22.4 \times 10^{-3} \times 300}{10^{-3} \times 273}\ \text{Pa} \simeq 2.49\ \text{MPa}.$$

268. How many molecules of air are contained in a room of volume 240 m³ at a temperature of 15 °C under a pressure of 10^5 Pa?

Given: $V_1 = 240\ \text{m}^3$, $T_1 = 288$ K, $p_1 = 10^5$ Pa.

$N - ?$

Solution. In order to use the ideal gas law, we shall compare the states of a given mass of the air under given (p_1, V_1, T_1) and normal (p_0, V_0, T_0) conditions. This gives

$$p_1V_1/T_1 = p_0V_0/T_0. \quad (1)$$

Since the volume of the air under normal conditions is $V_0 = m/\rho_0$, where ρ_0 is the density of the air under normal conditions, Eq. (1) assumes the form $p_1V_1/T_1 = p_0m/(T_0\rho_0)$, whence

$$m = p_1V_1T_0\rho_0/(p_0T_1).$$

Let us calculate the number ν of moles contained in the given mass of the air:

$$\nu = m/M = p_1V_1T_0\rho_0/(p_0T_1M).$$

Since the number of molecules in a mole of any gas (including air) is equal to Avogadro's constant N_A, we have

$$N = N_A\nu = N_A \frac{p_1V_1T_0\rho_0}{p_0T_1M},$$

$$N = 6.02 \times 10^{23} \times \frac{10^5 \times 240 \times 273 \times 1.29}{1.01 \times 10^5 \times 288 \times 29 \times 10^{-3}} \simeq 6 \times 10^{27}.$$

269. The volume of air in a room is 100 m³. What is the mass of the air leaving the room as a result of an increase in temperature from 10 to 25 °C if the atmospheric pressure is 102 kPa?

Given: $V_1 = 100$ m³, $T_1 = 283$ K, $T_2 = 298$ K, $p_1 = 102$ kPa $= 1.02 \times 10^5$ Pa.

Δm — ?

Solution. Let us compare the states of the mass m_1 of the air in the room under the given (p_1, V_1, T_1) and normal (p_0, V_0', T_0) conditions:

$$p_1V_1/T_1 = p_0V_0'/T_0 = p_0m_1/(T_0\rho_0),$$

where $V_0' = m_1/\rho_0$. Then

$$m_1 = p_1V_1T_0\rho_0/(T_1p_0).$$

As the temperature increases, the volume and pressure will not increase, but the mass of the air in the room will be different (m_2). Similarly, we shall compare the states of the mass m_2 of air at the given (p_1, V_1, T_2) and normal (p_0, V_0'', T_0) conditions. This gives $p_1V_1/T_2 = p_0V_0''/T_0 = p_0m_2/(T_0\rho_0)$, where $V_0'' = m_2/\rho_0$. Then

$$m_2 = p_1V_1T_0\rho_0/(T_2p_0).$$

Thus, the mass of the escaped air is

$$\Delta m = m_1 - m_2 = \frac{p_1V_1T_0\rho_0}{p_0T_1} - \frac{p_1V_1T_0\rho_0}{p_0T_2} = \frac{p_1V_1T_0\rho_0}{p_0}\,\frac{T_2-T_1}{T_1T_2},$$

$$\Delta m = \frac{1.02\times 10^5\times 100\times 273\times 1.29}{1.01\times 10^5}\,\frac{298-283}{298\times 283}\text{ kg} \simeq 6.33\text{ kg}.$$

270. A cylinder with the area of the base of 100 cm² contains air. A piston is at 50 cm from the cylinder bottom. When loaded by a weight of 50 kg, the piston is lowered by 10 cm. Determine the temperature of air after the loading if its pressure before the loading was 101 kPa and the temperature was 12 °C.

Given: $S = 100\text{ cm}^2 = 10^{-2}\text{ m}^2$, $h_1 = 50\text{ cm} = 0.5\text{ m}$, $m = 50\text{ kg}$, $l = 10\text{ cm} = 0.1\text{ m}$, $p_0 = 101\text{ kPa} = 1.01\times 10^5\text{ Pa}$, $T_1 = 285\text{ K}$.

T_2—?

Solution. Let us consider two states of air under the piston: before the loading and after it. The state of air before the loading is characterized by the parameters p_1, V_1, and T_1, and after the loading by p_2, V_2, and T_2, where $V_1 = Sh_1$, $p_1 = p_0$, $p_2 = p_0 + p'$, $p' = mg/S$, and $V_2 = Sh_2$, or since $h_2 = h_1 - l$, $V_2 = S(h_1 - l)$. Let us apply the ideal gas law to these two states: $p_1V_1/T_1 = p_2V_2/T_2$, whence

$$T_2 = p_2V_2T_1/(p_1V_1). \qquad (1)$$

Substituting the expressions for p_1, V_1, p_2, and V_2 into Eq. (1), we obtain

$$T_2 = \frac{(p_0+mg/S)\,S\,(h_1-l)\,T_1}{p_0Sh_1} = \frac{(p_0+mg/S)\,(h_1-l)\,T_1}{p_0h_1},$$

$$T_2 = \frac{(1.01\times 10^5 + 50\times 9.8\times 10^2)\,(0.5-0.1)\times 285}{1.01\times 10^5\times 0.5}\text{ K} \simeq 339\text{ K}.$$

271. Determine the density of hydrogen at a temperature of 15 °C under a pressure of 98 kPa.

Given: $T = 288$ K, $p = 98$ kPa $= 9.8 \times 10^4$ Pa.

$\rho - ?$

Solution. Using Clapeyron's equation of state for an ideal gas, we obtain $pV = mRT/M$. Since $V = m/\rho$, the above equation becomes $pm/\rho = m/(MRT)$, whence

$$\rho = \frac{pM}{RT},$$

$$\rho = \frac{9.8 \times 10^4 \times 2 \times 10^{-3}}{8.32 \times 288} \frac{\text{kg}}{\text{m}^3} \simeq 0.082 \text{ kg/m}^3.$$

272. Ten grams of oxygen are under a pressure of 0.303 MPa at a temperature of 10 °C. After heating at constant pressure, the oxygen occupies a volume of 10 l. Determine the initial volume and the final temperature of the gas.

Given: $m = 10$ g $= 10^{-2}$ kg, $p = 0.303$ MPa $=$ 3.03×10^5 Pa, $T_1 = 283$ K, $V_2 = 10$ l $= 10^{-2}$ m^3.

$V_1 - ?$ $T_2 - ?$

Solution. Using Clapeyron's equation of state $pV_1 = mRT_1/M$, we can determine the volume of oxygen before heating:

$$V_1 = \frac{mRT_1}{Mp},$$

$$V_1 = \frac{10^{-2} \times 8.32 \times 283}{32 \times 10^{-3} \times 3.03 \times 10^5} \text{ m}^3 \simeq 2.4 \times 10^{-3} \text{ m}^3.$$

By hypothesis, the oxygen is heated at constant pressure, and hence its temperature after heating can be determined from Charles' law $V_1/T_1 = V_2/T_2$, whence

$$T_2 = \frac{V_2 T_1}{V_1},$$

$$T_2 = \frac{10^{-2} \times 283}{2.4 \times 10^{-3}} \text{ K} \simeq 1.18 \times 10^3 \text{ K}.$$

273. Hydrogen leaks from a cylinder of volume 10 l because of a faulty valve. At 7 °C, the manometer indicates 5 MPa. At 17 °C, the reading of the manometer

remains unchanged. Determine the mass of the gas that has leaked from the cylinder.

Given: $V = 10\ \text{l} = 10^{-2}\ \text{m}^3$, $T_1 = 280$ K, $p = 5$ MPa $= 5 \times 10^6$ Pa, $T_2 = 290$ K.

Δm — ?

Solution. Using Clapeyron's equation of state $pV = m_1RT_1/M$, we can determine the initial mass of hydrogen:

$$m_1 = pVM/(RT_1).$$

Similarly, we can find the mass m_2 of hydrogen after the leakage:

$$m_2 = pVM/(RT_2).$$

Consequently, the mass of the leaked gas is

$$\Delta m = m_1 - m_2 = \frac{pVM}{RT_1} - \frac{pVM}{RT_2} = \frac{pVM}{R}\,\frac{T_2 - T_1}{T_1T_2},$$

$$\Delta m = \frac{5 \times 10^6 \times 10^{-2} \times 2 \times 10^{-3}\,(290 - 280)}{8.32 \times 290 \times 280}\ \text{kg} \simeq 1.5 \times 10^{-3}\ \text{kg}.$$

274. Determine the density of a mixture consisting of 4 g of hydrogen and 32 g of oxygen at a temperature of 7 °C under a pressure of 93 kPa.

Given: $m_1 = 4\ \text{g} = 4 \times 10^{-3}$ kg, $m_2 = 32\ \text{g} = 32 \times 10^{-3}$ kg, $p = 93$ kPa $= 9.3 \times 10^4$ Pa, $T = 280$ K.

ρ — ?

Solution. According to Dalton's law, the pressure of the mixture is

$$p = p_1 + p_2, \tag{1}$$

where p_1 and p_2 are the partial pressures of hydrogen and oxygen under the given conditions. Writing Clapeyron's equation of state for each gas separately, we obtain

$$p_1V_1 = m_1RT_1/M_1, \quad p_2V_2 = m_2RT_2/M_2.$$

By hypothesis, $V_1 = V_2 = V$ and $T_1 = T_2 = T$. Then for hydrogen we have $p_1V = m_1RT/M_1$, whence

$$p_1 = m_1RT/(M_1V). \tag{2}$$

Similarly, for oxygen we get

$$p_2 = m_2RT/(M_2V). \tag{3}$$

Substituting expressions (2) and (3) into (1), we obtain

$$p = \frac{m_1}{M_1}\frac{RT}{V} + \frac{m_2}{M_2}\frac{RT}{V} = \frac{RT}{V}\left(\frac{m_1}{M_1} + \frac{m_2}{M_2}\right),$$

whence

$$V = \frac{RT}{p}\left(\frac{m_1}{M_1} + \frac{m_2}{M_2}\right). \tag{4}$$

By definition, the density of the gas mixture is

$$\rho = m/V, \tag{5}$$

where $m = m_1 + m_2$ is the mass of the gas mixture.
Substituting Eq. (4) into (5), we get

$$\rho = \frac{m_1+m_2}{m_1/M_1+m_2/M_2}\frac{p}{RT},$$

$$\rho = \frac{4\times10^{-3}+32\times10^{-3}}{4\times10^{-3}/(2\times10^{-3})+32\times10^{-3}/(32\times10^{-3})}\frac{9.3\times10^4}{8.32\times280}\frac{\text{kg}}{\text{m}^3}$$

$$\simeq 0.48\ \text{kg/m}^3.$$

275. The volumes of two vessels with a gas are 3 and 4 l respectively. The gas is under a pressure of 202 kPa in the first vessel and under 101 kPa in the second vessel. What will be the gas pressure after the connection of the vessels? Assume that the temperature in the vessels is the same and maintained constant.

Given: $V_1 = 3\ \text{l} = 3\times10^{-3}\ \text{m}^3$, $V_2 = 4\ \text{l} = 4\times10^{-3}\ \text{m}^3$, $p_1 = 202\ \text{kPa} = 2.02\times10^5\ \text{Pa}$, $p_2 = 101\ \text{kPa} = 1.01\times10^5\ \text{Pa}$.

p — ?

Solution. According to Dalton's law,

$$p = p_3 + p_4. \tag{1}$$

Since the process is isothermal, the partial pressure of the gas in each vessel can be determined from Boyle's law: $p_1V_1 = p_3V$ and $p_2V_2 = p_4V$, where $V = V_1 + V_2$. Then the partial pressure of the gas in each vessel after the connection is

$$p_3 = p_1 V_1/V, \quad p_4 = p_2 V_2/V. \tag{2}$$

Substituting expressions (2) into (1), we obtain

$$p = \frac{p_1 V_1}{V} + \frac{p_2 V_2}{V} = \frac{p_1 V_1 + p_2 V_2}{V},$$

$$p = \frac{2.02\times10^5\times3\times10^{-3} + 1.01\times10^5\times4\times10^{-3}}{3\times10^{-3}+4\times10^{-3}}\ \text{Pa} \simeq 144\ \text{kPa}.$$

276. Three grams of water are introduced into a 10-l vessel containing dry air under normal conditions and heated to 100 °C. Determine the pressure of damp air in the vessel at this temperature. The pressure dependence of the boiling point for the water should be neglected.

Given: $m = 3\ \text{g} = 3\times10^{-3}\ \text{kg}$, $V = 10\ \text{l} = 10^{-2}\ \text{m}^3$, $T = 373\ \text{K}$.

p — ?

Solution. According to Dalton's law, $p = p_1 + p_2$, where p_1 and p_2 are the pressures of dry air and water vapour respectively. Since the heating occurs at constant volume, the pressure of dry air at 373 K can be determined from Gay-Lussac's law: $p_0/T_0 = p_1/T$, whence $p_1 = p_0 T/T_0$. Let us determine the pressure of water vapour, treating it as an ideal gas. The number ν of moles of water vapour in the vessel is

$$\nu = m/M,$$

where M is the molar mass of the vapour. Then the volume occupied by vapour under normal conditions is

$$V_0' = V_0 \nu = V_0 m/M. \tag{1}$$

From the ideal gas law $p_0 V_0'/T_0 = p_2 V/T$, we can find the vapour pressure: $p_2 = p_0 V_0' T/(T_0 V)$, or, taking into account Eq. (1), $p_2 = p_0 T V_0 m/(T_0 V M)$. Consequently, the pressure of damp air is

$$p = \frac{p_0 T}{T_0} + \frac{p_0 T V_0 M}{T_0 V M} = \frac{p_0 T}{T_0}\left(1 + \frac{V_0}{V}\,\frac{m}{M}\right),$$

$$p = \frac{1.01\times10^5\times373}{273}\left(1 + \frac{22.4\times10^{-3}\times3\times10^{-3}}{10^{-2}\times18\times10^{-3}}\right)\ \text{Pa}$$

$$\simeq 190\ \text{kPa}.$$

277. How many air molecules are contained in a cubic centimetre of a vessel at a temperature of 10 °C if the air in the vessel is rarefied to a pressure of 1.33 μPa?

Given: $V = 1\ \text{cm}^3 = 10^{-6}\ \text{m}^3$, $T = 283\ \text{K}$,
$p = 1.33\ \mu\text{Pa} = 1.33 \times 10^{-6}\ \text{Pa}$.

N — ?

Solution. The number of molecules in a cubic centimetre of air can be calculated from Avogadro's constant N_A and the number ν of moles contained in a cubic centimetre of air by the formula

$$N = \nu N_A.$$

The number ν of moles contained in a cubic centimetre of air can be defined as the ratio m/M, where m is the mass of air contained in a cubic centimetre and M the molar mass of air. The ratio m/M can be determined from Clapeyron's equation of state:

$$m/M = pV/(RT).$$

Thus, the required number of molecules is

$$N = \frac{pV}{RT} N_A,$$

$$N = \frac{1.33 \times 10^{-6} \times 10^{-6}}{8.32 \times 283} \times 6.02 \times 10^{23} = 3.4 \times 10^{8}.$$

278. A sphere of mass 40 g filled with air is in equilibrium in a lake at a depth of 100 m and at a temperature of 8 °C. Determine the mass of the air in the sphere if the atmospheric pressure is 99.7 kPa. The walls of the sphere are made of thin rubber.

Given: $h = 100\ \text{m}$, $T = 281\ \text{K}$, $m = 40\ \text{g} = 4 \times 10^{-2}\ \text{kg}$,
$p_{atm} = 99.7\ \text{kPa} = 9.97 \times 10^{4}\ \text{Pa}$.

m_1 — ?

Solution. The sphere immersed in water is acted upon by the force of gravity mg and the buoyant force $\mathbf{F} = \rho V g$ exerted by the water, where ρ is the density of water (Fig. 103). By hypothesis, $mg = F$, or $mg = \rho V g$, whence $V = m/\rho$. The mass of air in the sphere can be determined from Clapeyron's equation of state:

$$m_1 = MpV/(RT). \quad (1)$$

Here $p = p_{atm} + p_h$, where $p_h = \rho hg$ is the hydrostatic pressure of water at a depth h. Substituting the expressions for p and V into Eq. (1), we obtain

$$m_1 = \frac{M(p_{atm} + \rho hg)\, m/\rho}{RT},$$

$$m_1 = \frac{29 \times 10^{-3} \times (9.97 \times 10^4 + 10^3 \times 10^2 \times 9.8) \times 4 \times 10^{-3} \times 10^{-3}}{8.32 \times 281}\ \text{kg}$$

$$\simeq 5.36 \times 10^{-4}\ \text{kg}.$$

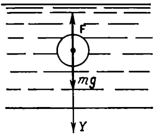


Fig. 103

279. Why does the body of a deep-water fish swell when it is brought to the surface?

Answer. At a large depth, the pressure inside the fish is balanced by the external pressure which is much higher than the atmospheric pressure. When the fish is brought to the surface, the external pressure becomes equal to the atmospheric pressure, while the pressure inside the fish remains unchanged. As a result, the volume of the fish sharply increases.

280. Electric bulbs are filled with krypton at a low pressure. Why?

Answer. During the operation of a bulb, the gas in it is strongly heated, which considerably increases the gas pressure. If the initial pressure were not low, it would lead to an explosion of the bulb.

EXERCISES

281. A narrow cylindrical tube sealed at one end is in the horizontal position. Air in the tube (whose volume is 240 mm^3) is separated from the atmospheric air by a mercury column of length 15 cm. If the tube is turned to the vertical position with the open end facing up, the air in the tube will occupy a volume of 200 mm^3. Determine the atmospheric pressure.

282. A 2-l cylinder contains a gas under a pressure of 0.33×10^5 Pa, while a 6-l cylinder contains the same gas under a pressure of 0.66×10^5 Pa. The cylinders are connected through a tube with a valve. What pressure will be established in the cylinders after the opening of the valve?

283. A cylinder contains a gas under a pressure of 131.3×10^5 Pa at a temperature of 30 °C. As a result of leakage, the pressure in the cylinder drops to 2.02×10^5 Pa and the temperature to —25 °C. What fraction of the gas remains in the cylinder?

284. A narrow cylindrical tube of length 100 cm open at two ends is immersed in mercury to half its length. The upper end is closed and the tube is taken out of mercury. Determine the height of the mercury column that remains in the tube, assuming that the atmospheric pressure is normal.

285. Determine the difference in the masses of air filling a room of volume 50 m^3 in summer and winter if the temperature in summer reaches 40 °C and in winter drops to zero. The atmospheric pressure should be regarded as normal.

286. Determine the mass of a mole of a mixture consisting of 25 g of oxygen and 75 g of nitrogen.

287. To what temperature should a flask containing air at 20 °C be heated for its density to decrease by a factor of 1.5?

288. Determine the mass of oxygen containing in a cylinder having a volume of 1 l under a pressure of 0.93×10^5 Pa at a temperature of 17 °C.

289. What is the volume occupied by 3 g of carbon dioxide under a pressure of 133 kPa at a temperature of 27 °C?

290. Determine the density of oxygen contained in a cylinder under a pressure of 3×10^5 Pa at a temperature of 17 °C.

291. Plot the p-V graphs for isothermal, isobaric, and isochoric processes.

QUESTIONS FOR REVISION

1. Define an isothermal process. 2. Formulate Boyle's law. 3. What curve is called an isotherm? Plot an isotherm in the p-V coordinates. 4. What process can be referred to as isobaric? 5. Formulate Charles' law. 6. What curve is called an isobar? Plot an isobar in the V-T coordinates. 7. What process can be referred to as isochoric? 8. Formulate Gay-Lussac's law. 9. What curve is called an isochor? Plot an isochor in the p-V coordinates. 10. Formulate the ideal gas law. 11. Write Clapeyron's equation of state for an ideal gas. 12. Formulate Dalton's law. 13. Clarify the physical meaning of thermodynamic temperature. 14. Name the parameters characterizing the state of a gas and give their SI units.

2.3. Work Done by a Gas. Heat Engines

An **elementary work** done by a gas contained in a cylinder under a piston during expansion is

$$dA = F \cos \alpha \, dl.$$

For $\alpha = 0$, we obtain

$$dA = F \, dl = pS \, dl = p \, dV, \qquad (1)$$

where $p = F/S$ is the gas pressure, S the area of the piston, and $dV = S \, dl$ the change in the gas volume.

The **total expansion work** of a gas from volume V_1 to V_2 is determined by integration:

$$A = \int_{V_1}^{V_2} p \, dV.$$

A **heat engine** is a device in which the internal energy of a fuel is converted into mechanical work.

Any heat engine includes a heater, a working substance, and a cooler (Fig. 104). The heater supplies an amount of heat Q_1 to the working substance. The cooler receives an

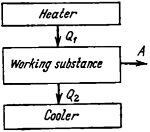


Fig. 104

amount of heat Q_2 from the working substance. The **mechanical work** done by the working substance is

$$A = Q_1 - Q_2.$$

The operation of a heat engine is characterized by the efficiency defined by the formula

$$\eta = \frac{Q_1 - Q_2}{Q_1} 100\%.$$

The efficiency of an ideal Carno heat engine is

$$\eta = \frac{T_1 - T_2}{T_1} 100\%,$$

where T_1 and T_2 are the temperatures of the heater and the cooler respectively. The efficiency of a heat engine is always less than unity (**the second law of thermodynamics**). An ideal heat engine is reversible (refrigerator).

* * *

292°. Ten grams of carbon dioxide are heated from 20 to 30 °C at constant pressure. Determine the expansion work of the gas and the change in its internal energy.

Given: $m = 10$ g $= 10^{-2}$ kg, $T_1 = 293$ K, $T_2 = 303$ K.

$A - ?$ $\Delta U - ?$

Solution. The change in the internal energy of carbon dioxide during its heating from T_1 to T_2 can be determined by the formula

$$\Delta U = c_V m\,(T_2 - T_1),$$

where c_V is the specific heat of carbon dioxide at constant volume. [1] Consequently,

$$\Delta U = 0.83 \times 10^3 \times 10^{-2} \times (303 - 293)\ \mathrm{J} = 83\ \mathrm{J}.$$

The expansion work of a gas is

$$A = \int\limits_{V_1}^{V_2} p\,\mathrm{d}V. \tag{1}$$

By hypothesis, the pressure p of carbon dioxide is constant and hence can be taken outside the integral sign. Expression (1) then becomes

$$A = p\int\limits_{V_1}^{V_2} \mathrm{d}V = p\,(V_2 - V_1). \tag{2}$$

From Clapeyron's equation of state

$$pV = mRT/M$$

we determine the volume of carbon dioxide at a temperature T: $V = mRT/(Mp)$. Consequently, at the temperatures T_1 and T_2 carbon dioxide will occupy the volumes

$$V_1 = mRT_1/(Mp), \quad V_2 = mRT_2/(Mp). \tag{3}$$

Substituting expressions (3) into (2), we obtain

$$A = p\left(\frac{mRT_2}{Mp} - \frac{mRT_1}{Mp}\right) = \frac{mR}{M}(T_2 - T_1),$$

$$A = \frac{10^{-2} \times 8.32}{44 \times 10^{-3}} \times (303 - 293)\ \mathrm{J} \simeq 18.9\ \mathrm{J}.$$

293°. Six grams of oxygen (O_2) taken at a temperature of 30 °C expand at constant pressure so that the volume

[1] This formula can be applied since the internal energy of an ideal gas depends only on its temperature (we neglect the potential energy of interaction between gas molecules).

increases twofold due to a heat supplied from outside. Determine the expansion work of the gas, the change in the internal energy of oxygen, and the amount of heat supplied to it.

Given: $m = 6$ g $= 6 \times 10^{-3}$ kg, $T_1 = 303$ K, $V_2 = 2V_1$.

A—? ΔU—? Q—?

Solution. The change in the internal energy of oxygen as a result of its heating by ΔT is

$$\Delta U = c_V m \, \Delta T, \qquad (1)$$

where c_V is the specific heat of oxygen at constant volume (see footnote 1 to Problem 292°) and $\Delta T = T_2 - T_1$ the change in its temperature, T_1 and T_2 being the initial and final temperatures of oxygen.

Since oxygen expands at a constant pressure p, we can use Charles' law $V_1/T_1 = V_2/T_2$ to determine the final temperature of oxygen:

$$T_2 = V_2 T_1/V_1 = 2V_1 T_1/V_1 = 2T_1. \qquad (2)$$

Substituting expression (2) into (1), we get

$$\Delta U = c_V m \, (T_2 - T_1) = c_V m \, (2T_1 - T_1) = c_V m T_1,$$
$$\Delta U = 0.92 \times 10^3 \times 6 \times 10^{-3} \times 303 \text{ J} \simeq 1.67 \text{ kJ}.$$

Let us determine the expansion work of the gas:

$$A = \int_{V_1}^{V_2} p \, dV,$$

or, considering that $p = \text{const}$,

$$A = p \int_{V_1}^{V_2} dV = p\,(V_2 - V_1) = p\,(2V_1 - V_1) = pV_1. \qquad (3)$$

Using Clapeyron's equation of state $pV_1 = mRT_1/M$, we can determine the volume of oxygen at the temperature T_1:

$$V_1 = mRT_1/(Mp). \qquad (4)$$

Substituting expression (4) into (3), we get

$$A = p\frac{mRT_1}{Mp} = \frac{mRT_1}{M},$$

$$A = \frac{6\times10^{-3}\times8.32\times303}{32\times10^{-3}}\ \mathrm{J} \simeq 473\ \mathrm{J}.$$

The amount of heat supplied to the gas can be determined from the first law of thermodynamics:

$$Q = \Delta U + A,$$

$$Q = (1670 + 473)\ \mathrm{J} \simeq 2.14\ \ \mathrm{kJ}.$$

294°. Ten grams of nitrogen (N_2) expand isothermally at a temperature of -20 °C so that the gas pressure drops from 202 to 101 kPa. Determine the expansion work of the gas, the change in the internal energy of nitrogen, and the amount of heat supplied to it.

Given: $m = 10\ \mathrm{g} = 10^{-2}\ \mathrm{kg}$, $T = 253\ \mathrm{K}$, $p_1 = 202\ \mathrm{kPa} = 2.02\times10^5\ \mathrm{Pa}$, $p_2 = 101\ \mathrm{kPa} = 1.01\times10^5\ \mathrm{Pa}$.

A—? ΔU—? Q—?

Solution. Since the change in the internal energy of a gas is

$$\Delta U = cm\,\Delta T$$

and the gas temperature remains constant during expansion, the change in temperature $\Delta T = 0$, and hence $\Delta U = 0$. Let us determine the expansion work of the gas:

$$A = \int_{V_1}^{V_2} p\,\mathrm{d}V. \qquad (1)$$

Using Clapeyron's equation of state, we can determine the gas pressure:

$$p = mRT/(MV). \qquad (2)$$

Substituting expression (2) into the integrand of (1), we obtain

$$A = \int_{V_1}^{V_2} \frac{mRT}{MV}\,\mathrm{d}V = \frac{mRT}{M}\int_{V_1}^{V_2}\frac{\mathrm{d}V}{V}$$

$$= \frac{mRT}{M}\ln V\Big|_{V_1}^{V_2} = \frac{mRT}{M}\ln\frac{V_2}{V_1}. \qquad (3)$$

Having determined the volume ratio $V_2/V_1 = p_1/p_2$ by Boyle's law and substituting it into Eq. (3), we obtain

$$A = \frac{mRT}{M} \ln \frac{p_1}{p_2},$$

$$A = \frac{10^{-3} \times 8.32 \times 253}{28 \times 10^{-3}} \ln \frac{2.02 \times 10^5}{1.01 \times 10^5} \simeq 521 \text{ J}.$$

The amount of heat supplied to the gas can be determined from the first law of thermodynamics:

$$Q = \Delta U + A,$$

or, since $\Delta U = 0$,

$$Q = A \simeq 521 \text{ J}.$$

295°. Air of mass 0.2 g is contained in a cylinder under a weightless piston having an area of 15 cm² at a temperature of 20 °C. Determine the work done during a slow uniform ascent of the piston from a height of 10 to 20 cm. The atmospheric pressure is normal.

Given: $S = 15\ \text{cm}^2 = 1.5 \times 10^{-3}\ \text{m}^2$, $m = 0.2\ \text{g} = 2 \times 10^{-4}\ \text{kg}$, $T = 293\ \text{K}$, $h_1 = 10\ \text{cm} = 0.1\ \text{m}$, $h_2 = 20\ \text{cm} = 0.2\ \text{m}$.

$A - ?$

Solution. During the uniform ascent of the piston, it is acted upon by the external force $\mathbf{F}_{ext}$, the force of pressure $\mathbf{F}$ exerted by the air under the piston, and the force of pressure $\mathbf{F}_{atm}$ of the atmospheric air (Fig. 105). Let us determine the work done by the external force to lift the piston to the height h_2:

$$A = \int_{h_1}^{h_2} F_{ext}\, dh. \qquad (1)$$

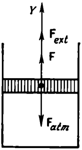


Fig. 105

In order to determine the external force, we write the equilibrium condition for the piston in projections on the Y-axis: $F_{ext} + F - F_{atm} = 0$, whence

$$F_{ext} = F_{atm} - F.$$

By definition, $F_{atm} = p_0 S$ and $F = pS$. Here p_0 is the atmospheric pressure and $p = mRT/(MV)$ the pressure of the air under the piston, where $V = Sh$ is the volume of the air under the piston when the latter is at an arbitrary height h. Then the expression for the force of pressure exerted by the air under the piston has the form $F = mRT/(Mh)$. Substituting the expressions for F_{atm} and F into Eq. (1) and integrating, we obtain

$$A = \int_{h_1}^{h_2} (F_{atm} - F)\,dh = \int_{h_1}^{h_2} F_{atm}\,dh - \int_{h_1}^{h_2} F\,dh$$

$$= \int_{h_1}^{h_2} p_0 S\,dh - \int_{h_1}^{h_2} \frac{mRT}{Mh}\,dh = p_0 S \int_{h_1}^{h_2} dh - \frac{mRT}{M} \int_{h_1}^{h_2} \frac{dh}{h}$$

$$= p_0 S (h_2 - h_1) - \frac{mRT}{M} \ln \frac{h_2}{h_1},$$

$$A = \left[1.01 \times 10^5 \times 1.5 \times 10^{-3} \times (0.2 - 0.1) - \frac{2 \cdot 10^{-4} \times 8.32 \times 293}{29 \times 10^{-3}} \ln \frac{0.2}{0.1}\right] \text{J} \simeq 3.5 \text{ J}.$$

Since the piston moves slowly, the temperature T can be assumed to be constant and taken outside the integral sign.

296. An ideal heat engine received 3360 J of heat during a cycle from a heater whose temperature is 500 K. Determine the amount of heat given away during a cycle to a cooler whose temperature is 400 K. Calculate the work done by the engine during a cycle.

Given: $T_1 = 500$ K, $Q_1 = 3360$ J, $T_2 = 400$ K.

Q_2 — ? A — ?

Solution. By definition, the efficiency of an ideal heat engine is $\eta = (Q_1 - Q_2)/Q_1 = (T_1 - T_2)/T_1$, whence

$$Q_2 = \frac{Q_1 T_2}{T_1},$$

$$Q_2 = \frac{3360 \times 400}{500} \text{ J} = 2688 \text{ J}.$$

The work done by the engine during a cycle is

$$A = Q_1 - Q_2,$$

$$A = (3360 - 2688) \text{ J} = 672 \text{ J}.$$

297. An ideal Carnot heat engine with a reverse cycle (refrigerator) used water at 0 °C as a cooler and water at 100 °C as a heater. What amount of water should be frozen in the cooler to vaporize 500 g of the water in the heater (boiler)?

Given: $t_1 = 100\,°C$, $T_1 = 373$ K, $t_2 = 0\,°C$, $T_2 = 273$ K, $m_1 = 500$ g $= 0.5$ kg.

m_2 — ?

Solution. The amount of heat liberated during the cooling of the water of mass m_2 is

$$Q_2 = \lambda m_2, \qquad (1)$$

where λ is the latent heat of fusion for ice.

The amount of heat required to vaporize the water of mass m_1 is

$$Q_1 = r m_1, \qquad (2)$$

where r is the latent heat of vaporization for the water.

Using the expression $\eta = (Q_1 - Q_2)/Q_1 = (T_1 - T_2)/T_1$ for the efficiency of the ideal Carnot heat engine, we find that

$$Q_2 = T_2 Q_1/T_1. \qquad (3)$$

Substituting expressions (1) and (2) into (3), we obtain $\lambda m_2 = T_2 m_1 r/T_1$, whence

$$m_2 = \frac{T_2 m_1 r}{T_1 \lambda},$$

$$m_2 = \frac{273 \times 0.5 \times 22.6 \times 10^5}{373 \times 3.35 \times 10^5}\ \text{kg} \simeq 2.47\ \text{kg}.$$

298. Determine the work done by a heat engine during a cycle shown in Fig. 106.

Given: p_1, p_2, V_1, V_2.

A — ?

Solution. The expansion work of a gas is numerically equal to the area of the figure bounded by the graph $p = p(V)$, the abscissa axis, and the ordinates of the initial and the final point of the graph. The work done on segments *1-2* and *3-4* is zero since the gas volume does not

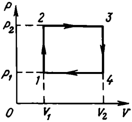


Fig. 106

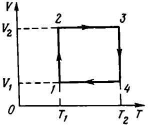


Fig. 107

change on these segments. The work done by the engine during a cycle is numerically equal to the difference in the areas of rectangles $V_1 23 V_2$ and $V_1 14 V_2$, i.e. to the area of rectangle *1234*. Using the figure, we can determine this area:

$$A = p_2 (V_2 - V_1) - p_1 (V_2 - V_1) = (p_2 - p_1)(V_2 - V_1).$$

EXERCISES

299. Two hundred grams of nitrogen (N_2) are heated by 100 K first isobarically and then isochorically. What amounts of heat are required in these cases?

300. A vessel contains 20 g of nitrogen (N_2) and 32 g of oxygen (O_2). Determine the change in the internal energy of the gas mixture during its cooling by 28 °C.

301. Nitrogen (N_2) under an initial pressure of 1.01×10^5 Pa having a volume of 10 l expands isothermally so that its volume is doubled. Determine the work done by the gas.

302. The pressure of nitrogen in a vessel of volume 3 l increases by 2.2 MPa as a result of heating. Determine the amount of heat supplied to the gas.

303. During the expansion of a monatomic gas from 0.2 to 0.5 m^3, its pressure increases from 404 to 808 kPa. Determine the work done by the gas, the amount of heat supplied to it, and the change in its internal energy.

304. The amount of heat received by a working substance from the heater in an ideal heat engine is 6.3 kJ. Determine the efficiency of the engine and the work done by it during a cycle if 80% of the heat are transferred to the cooler.

305. Determine the work done by the heat engine shown in Fig. 107 during a cycle.

306. Two kilograms of air are contained in a vessel at a temperature of 16 °C. What work will be done by the gas during its isobaric heating to 100 °C?

307. A cylinder contains nitrogen at a temperature of 20 °C. A light piston under a load of 100 kg is at a height of 50 cm from the bottom of the cylinder. Determine the work done by the gas during its isobaric heating to 60 °C. The area of the base of the cylinder is 100 cm^2. The atmospheric pressure is normal.

QUESTIONS FOR REVISION

1. Formulate the first law of thermodynamics. 2. Write formulas for calculating the elementary and the total expansion work of a gas. 3. Formulate the second law of thermodynamics. 4. What is a heat engine? a refrigerator? 5. How is the efficiency of a heat engine determined? 6. How is the efficiency of an ideal Carnot heat engine determined? 7. Explain how the work done by a heat engine is determined graphically.

2.4. Saturated and Unsaturated Vapour. Humidity

The surface of a liquid is always under its vapour due to evaporation. A vapour can be saturated or unsaturated.

Saturated vapour is in a dynamic equilibrium with its liquid (the number of molecules escaping from the surface of the liquid per second is equal to the number of molecules returning to the liquid).

The density and pressure of **unsaturated vapour** are lower than the corresponding values for saturated vapour at the same temperature.

In its properties, unsaturated vapour does not differ from an ordinary gas. Therefore, Clapeyron's equation of state

$$pV = mRT/M$$

is valid for it.

Air containing water vapour is referred to as **humid (moist) air**, while air containing no water vapour is called **dry**.

Humid air is characterized by absolute and relative humidity.

The **absolute humidity** ρ is defined as the mass m of water vapour contained in a cubic metre of air at a given temperature. In other words, it is equal to the density of water vapour at a given temperature.

The **relative humidity** B is the ratio of the density ρ of water vapour at a given temperature to the density ρ_{sat} of saturated vapour at the same temperature:

$$B = \frac{\rho}{\rho_{sat}}\ 100\ \%.$$

Relative humidity can be determined with the help of the psychrometric table.

Unsaturated vapour can be converted into saturated vapour as a result of isochoric cooling.

The temperature at which a vapour becomes saturated as a result of isochoric cooling is known as the **dew point** T_{dew}. Vapour cooled below the dew point is partially condensed to liquid.

* * *

308. Determine the absolute humidity of air if the partial pressure of vapour in it is 14 kPa and the temperature is 60 °C.

Given: $p = 14\ \text{kPa} = 1.4 \times 10^4\ \text{Pa},\ t = 60\ °\text{C},\ T = 333\ \text{K}.$

$\rho - ?$

Solution. Using Clapeyron's equation of state $pV = mRT/M$, we obtain

$$m/V = Mp/(RT).$$

Consequently, the absolute humidity of air is

$$\rho = \frac{Mp}{RT},$$

$$\rho = \frac{18 \times 10^{-3} \times 1.4 \times 10^4}{8.32 \times 333}\ \frac{\text{kg}}{\text{m}^3} \simeq 9.1 \times 10^{-2}\ \text{kg/m}^3.$$

309. Determine the absolute humidity of air if its temperature is 15 °C and the relative humidity is 80%.

Given: $t = 15\,°C$, $T = 288$ K, $B = 80\% = 0.8$.

$\rho - ?$

Solution. The relative humidity is defined as

$$B = \rho/\rho_{sat},$$

whence $\rho = B\rho_{sat}$. The saturated vapour density at 288 K can be determined from Table 5: $\rho_{sat} = 12.8 \times 10^{-3}$ kg/m^3. Therefore,

$$\rho = 0.8 \times 12.8 \times 10^{-3}\ \text{kg/m}^3 \simeq 1.02 \times 10^{-2}\ \text{kg/m}^3.$$

310. A litre of moist air at 50 °C has a mass of 1.04 g under normal atmospheric pressure. Determine the absolute humidity of air.

Given: $V = 1\ l = 10^{-3}$ m^3, $t = 50\,°C$, $T = 323$ K, $m = 1.04$ g $= 1.04 \times 10^{-3}$ kg, $p = 1.01 \times 10^5$ Pa.

$\rho - ?$

Solution. The absolute humidity is defined as

$$\rho = m_v/V, \tag{1}$$

where m_v is the mass of water vapour in a given volume of air. By hypothesis,

$$m = m_{air} + m_v, \tag{2}$$

$$p = p_{air} + p_v, \tag{3}$$

where m_{air} is the mass of dry air in the given volume, and p_{air} and p_v are the partial pressures of dry air and water vapour. The partial pressures p_{air} and p_v can be determined from Clapeyron's equation of state, written for dry air and water vapour

$$p_{air} = m_{air}RT/(M_{air}V), \tag{4}$$

$$p_v = m_vRT/(M_vV), \tag{5}$$

where M_{air} and M_v are the molar masses of dry air and water vapour respectively.

Substituting expressions (4) and (5) into (3), we obtain $p = \left(\frac{m_v}{M_v} + \frac{m_{air}}{M_{air}}\right)\frac{RT}{V}$, or, taking into account relation (2), $p=\left(\frac{m_v}{M_v}+\frac{m-m_v}{M_{air}}\right)\frac{RT}{V}$, whence

$$m_v=\frac{M_v}{M_{air}-M_v}\left(\frac{pVM_{air}}{RT}-m\right). \qquad (6)$$

Substituting expression (6) into (1), we obtain

$$\rho=\frac{M_v}{M_{air}-M_v}\left(\frac{pM_{air}}{RT}-\frac{m}{V}\right),$$

$$\rho=\frac{18\times10^{-3}}{29\times10^{-3}-18\times10^{-3}}\times\left(\frac{1.01\times10^5\times29\times10^{-3}}{8.32\times323}-\frac{1.04\times10^{-3}}{10^{-3}}\right)\frac{\text{kg}}{\text{m}^3}$$

$$\simeq 0.082\ \text{kg/m}^3.$$

311. A vessel contains air whose relative humidity at 10 °C is 60%. Determine the relative humidity of air after the reduction of its volume to one-third of the initial value and heating to 100 °C.

Given: $t_1=10\ °\text{C}$, $T_1=283\ \text{K}$, $B_1=60\%=0.6$, $n=3$, $t_2=100\ °\text{C}$, $T_2=373\ \text{K}$.

$B_2-?$

Solution. The absolute humidity of air before its compression and heating is

$$\rho_1 = B_1\rho_{sat1},$$

where ρ_{sat1} is the saturated vapour density at T_1. After the reduction of the air volume by a factor of n, its density also increases by a factor of n, i.e.

$$\rho_2 = n\rho_1 = nB_1\rho_{sat1}. \qquad (1)$$

It is well known that the saturated vapour pressure at the boiling point of a liquid is equal to the atmospheric pressure p_0. Consequently, when the moist air is heated to 373 K, the pressure of saturated vapour contained in the air becomes equal to the normal atmospheric pressure p_0. The saturated vapour density ρ_{sat3} under these conditions can be determined from Clapeyron's equation of state:

$$\rho_{sat3} = M_v p_0/(RT_2). \qquad (2)$$

Then the relative humidity of air after its compression and heating is

$$B_2 = \frac{\rho_2}{\rho_{sat2}} 100\,\%. \qquad (3)$$

Substituting expressions (1) and (2) into (3), we find that

$$B_2 = \frac{nB_1\rho_{sat1}RT_2}{M_v p_2} 100\%,$$

$$B_2 = \frac{3\times 0.6\times 9.4\times 10^{-3}\times 8.32\times 373}{18\times 10^{-3}\times 1.01\times 10^5} 100\% \simeq 2.9\%.$$

312. Determine the absolute and relative humidities of moist air if its temperature is 18 °C and the dew point is 8 °C.

Given: $t = 18\ °C$, $T = 291\ K$, $t_{dew} = 8\ °C$, $T_{dew} = 281\ K$.

$\rho - ?\ B - ?$

Solution. Knowing the dew point, we can determine, by using Table 17, the mass of saturated vapour contained in a cubic metre of air. This quantity will determine the absolute humidity ρ of air at any temperature equal to or higher than the dew point. Consequently, $\rho = 8.3 \times 10^{-3}$ kg/m^3. Let us determine the relative humidity of air:

$$B = \frac{\rho}{\rho_{sat}} 100\%,$$

where ρ_{sat} is the saturated vapour density at 291 K. Therefore,

$$B = \frac{8.3\times 10^{-3}}{15.4\times 10^{-3}} 100\% \simeq 54\%.$$

313. The relative humidity of air in a room is 63% and the temperature is 18 °C. What must be the temperature drop outdoors for the window-pane in the room to become misty?

Given: $B = 63\% = 0.63$, $t = 18\ °C$, $T = 291\ K$.

$\Delta T - ?$

Solution. The absolute humidity of air is defined as

$$\rho = B\rho_{sat},$$

where ρ_{sat} is the saturated vapour density at 291 K. Then

$$\rho = 0.63 \times 15.4 \times 10^{-3} \text{ kg/m}^3 = 9.7 \times 10^{-3} \text{ kg/m}^3.$$

As the air temperature outdoors drops to the dew point, vapour near the window-pane becomes saturated and starts to condense—the glass becomes misty. The mass of vapour contained in a cubic metre does not change up to the dew point $T_{dew} = 283.5$ K. Consequently, the window-pane becomes misty when the outdoor temperature drops by

$$\Delta T = (291 - 283.5) \text{ K} = 7.5 \text{ K}.$$

314. Two vessels of volumes 5 and 3 m^3 contain air at temperatures of 15 and 28 °C and relative humidities of 22 and 46% respectively. Determine the relative humidity of air after the connection of these vessels.

Given: $V_1 = 5$ m^3, $B_1 = 22\% = 0.22$, $t_1 = 15$ °C, $T_1 = 288$ K, $V_2 = 3$ m^3, $B_2 = 46\% = 0.46$, $t_2 = 28$ °C, $T_2 = 301$ K.

B—?

Solution. The mass of water vapour in each vessel before the connection is

$$m_1 = \rho_1 V_1, \quad m_2 = \rho_2 V_2, \tag{1}$$

where ρ_1 and ρ_2 are the densities of water vapour in the vessels. The relative humidity of air in each vessel before the connection is

$$B_1 = \rho_1/\rho_{sat1}, \quad B_2 = \rho_2/\rho_{sat2},$$

whence $\rho_1 = B_1\rho_{sat1}$ and $\rho_2 = B_2\rho_{sat2}$. Then (see Eqs. (1))

$$m_1 = B_1\rho_{sat1}V_1, \quad m_2 = B_2\rho_{sat2}V_2. \tag{2}$$

The absolute humidity of air after the connection of the vessels becomes $\rho = (m_1 + m_2)/(V_1 + V_2)$, or, taking Eqs. (2) into account,

$$\rho = \frac{B_1\rho_{sat1}V_1 + B_2\rho_{sat2}V_2}{V_1 + V_2}.$$

The relative humidity of air after the connection of the vessels is

$$B=\frac{\rho}{\rho_{sat}}\,100\%=\frac{B_1\rho_{sat1}V_1+B_2\rho_{sat2}V_2}{\rho_{sat}\,(V_1+V_2)}\,100\%. \qquad (3)$$

Neglecting the heat capacity of the vessels and the mass of water vapour in view of the smallness of the vessels, we obtain from the heat-balance equations

$$m_{v1}c_v\,(\Theta-T_1)=m_{v2}c_v\,(T_2-\Theta), \qquad (4)$$

where Θ is the temperature of air in the vessels after their connection. Since $m_{v1}=\rho_v V_1$ and $m_{v2}=\rho_v V_2$, Eq. (4) can be written in the form $\rho_v V_1 c_v\,(\Theta-T_1)=\rho_v V_2 c_v\,(T_2-\Theta)$, whence

$$\Theta=\frac{V_1T_1+V_2T_2}{V_1+V_2},$$

$$\Theta=\frac{5\times288+3\times301}{5+3}\ \text{K}=293\ \text{K}.$$

Knowing T_1, T_2, and Θ, from Table 17 we obtain $\rho_{sat1}=12.8\times10^{-3}\ \text{kg/m}^3$, $\rho_{sat2}=27.2\times10^{-3}\ \text{kg/m}^3$, and $\rho_{sat}=17.3\times10^{-3}\ \text{kg/m}^3$. Substituting these values into Eq. (3), we get

$$B=\frac{0.22\times12.8\times10^{-3}\times5+0.46\times27.2\times10^{-3}\times3}{17.3\times10^{-3}\times(5+3)}\,100\%=37\%.$$

315. Explain why mist appears in low-lying areas at night after a hot summer day.

Answer. At night, after a hot summer day the air temperature drops by 10-15 °C. As a result, the water vapour contained in air becomes saturated and is partially condensed in the form of mist.

316. Why is mist seen in the exhaled air in winter and is not seen in summer?

Answer. The temperature of the exhaled air is about 36 °C. The water vapour contained in the air is cooled in winter below the dew point, becomes saturated, and partially condenses in the form of mist. On the other hand, the temperature of the ambient air in summer is above the dew point, and hence no mist is formed.

EXERCISES

317. The absolute humidity of air at 60 °C is 5×10^{-3} kg/m^3. Determine the absolute humidity of the air after its temperature drops to 20 °C.

318. Determine the absolute humidity of air in a room if its relative humidity is 80% and the temperature is 15 °C.

319. Determine the absolute humidity of air if the partial pressure of vapour contained in it is 1.4×10^4 Pa and the temperature is 60 °C.

320. The pressure of air at a temperature of 26 °C and a relative humidity of 70% is 1.017×10^5 Pa. Determine the air pressure after the temperature drops to −5 °C and the relative humidity becomes 80%, other conditions being the same.

321. The relative humidity of air in the evening is 60% at 16 °C. At night, the air temperature drops to 4 °C and dew is precipitated. What amount of water vapour is condensed from a cubic metre of the air?

322. What was the relative humidity of air at 20 °C if under a pressure of 6 MPa the dew point is 100 °C?

323. How many water molecules are contained in a room of volume 100 m^3 under normal conditions and at a relative humidity of 20%?

324. 3.5 g of water and 2.9 g of water vapour are contained in a cylinder under a piston at 40 °C. The gas in the cylinder expands isothermally. At what volume will the water in the cylinder be evaporated completely?

325. 20 000 m^3 of air at 18 °C and a relative humidity of 50% are supplied to a room. The air is pumped from outdoors where the temperature is 10 °C and the relative humidity is 60%. What amount of water must be evaporated for the relative humidity of the air in the room to remain unchanged?

QUESTIONS FOR REVISION

1. What is evaporation? 2. What is the dynamic equilibrium between a vapour and its liquid? 3. Define saturated vapour. 4. Define unsaturated vapour. What equation describes the behaviour of such a vapour? 5. What is the difference between moist and dry

air? 6. What is meant by absolute humidity? 7. What is meant by relative humidity? 8. Define the dew point. 9. How can unsaturated vapour be converted into saturated vapour? 10. What instruments are used to measure air humidity?

2.5. Properties of Solids and Liquids

THERMAL EXPANSION OF SOLIDS AND LIQUIDS

When a solid is heated, its linear dimensions l vary according to the law

$$l = l_0 (1 + \alpha \, \Delta T), \quad (1)$$

where l_0 is the linear dimensions of the body at 273 K, $\Delta T = T - T_0$ the change in the temperature of the body during its heating from $T_0 = 273$ K to T, and α the coefficient of linear expansion.

The volume of solids and liquids changes as a result of heating according to the law

$$V = V_0 (1 + \beta \, \Delta T), \quad (2)$$

where V_0 is the volume of a body at 273 K and β the coefficient of volume expansion, connected with the coefficient α of linear expansion through the approximate relation $\beta \approx 3\alpha$. The coefficients β of volume expansion for liquids are given in Table 16.

A change in the volume of solids and liquids as a result of heating leads to a change in their density according to the law

$$\rho = \rho_0/(1 + \beta \, \Delta T), \quad (3)$$

where ρ_0 is the density of a body at 273 K.

Since $\alpha \, \Delta T \ll 1$ and $\beta \, \Delta T \ll 1$, the following approximate formulas should be used in calculations:

$$1/(1 + x) \approx 1 - x, \quad (1 + x)(1 + y) \approx 1 + x + y,$$

where $x \ll 1$ and $y \ll 1$.

* * *

326. An iron ruler has a length of 1 m at 15 °C. What will be the change in its length as a result of its cooling to −35 °C?

Given: $t_1 = 15\,°C$, $T_1 = 288$ K, $l_1 = 1$ m, $t_2 = -35\,°C$, $T_2 = 238$ K.

Δl — ?

Solution. According to the law of linear expansion,

$$l_1 = l_0\,[1 + \alpha\,(T_1 - T_0)], \quad (1)$$

$$l_2 = l_0\,[1 + \alpha\,(T_2 - T_0)], \quad (2)$$

where l_2 is the length of the ruler after its cooling to T_2 and α the coefficient of linear expansion for iron. Then the change in the ruler length is

$$\Delta l = l_1 - l_2 = l_0\,[1 + \alpha\,(T_1 - T_0)] - l_0\,[1 + \alpha\,(T_2 - T_0)] = l_0\alpha\,(T_1 - T_2). \quad (3)$$

Expressing l_0 from Eq. (1) and substituting it into (3), we get

$$\Delta l = \frac{\alpha\,(T_1 - T_2)\,l_1}{1 + \alpha\,(T_1 - T_0)}. \quad (4)$$

Considering that $\alpha\,(T_1 - T_0) \ll 1$, expression (4) can be approximately written as

$$\Delta l \approx \alpha l_1\,(T_1 - T_2)\,[1 - \alpha\,(T_1 - T_0)] \approx l_1\alpha\,(T_1 - T_2),$$

$$\Delta l \simeq 1.2 \times 10^{-8} \times 1 \times (288 - 238)\ \text{m} \simeq 6 \times 10^{-4}\ \text{m}.$$

327. At any temperature, the difference in the lengths of aluminium and copper rods is 15 cm. What are their lengths at 0 °C?

Given: $\Delta l = 15$ cm $= 0.15$ m, $t_0 = 0\,°C$, $T_0 = 273$ K.

l_{al} — ? l_c — ?

Solution. According to the law of linear expansion, we have

$$l_{al} = l_{0\,al}\,(1 + \alpha_{al}\,\Delta T), \quad (1)$$

$$l_c = l_{0\,c}\,(1 + \alpha_c\,\Delta T), \quad (2)$$

where α_{al} and α_c are the coefficients of linear expansion for aluminium and copper. By hypothesis,

$$l_c - l_{al} = \Delta l, \quad l_{0\,c} - l_{0\,al} = \Delta l. \quad (3)$$

Subtracting Eq. (2) from (1) termwise, we obtain

$$l_{al} - l_c = l_{0\,al}(1+\alpha_{al}\Delta T) - l_{0\,c}(1+\alpha_c\Delta T)$$
$$= l_{0\,al} - l_{0\,c} + l_{0\,al}\alpha_{al}\Delta T - l_{0\,c}\alpha_c\Delta T. \qquad (4)$$

We transform expression (4), taking into account formulas (3): $-\Delta l = -\Delta l + l_{0\,al}\alpha_{al}\,\Delta T - l_{0\,c}\,\alpha_c\,\Delta T$, whence

$$l_{0\,al}\alpha_{al} - l_{0\,c}\alpha_c = 0. \qquad (5)$$

Having expressed $l_{0\,al}$ from the second of Eqs. (3) and substituting it into (5), we obtain $l_{0\,c}\alpha_{al} - \alpha_{al}\,\Delta l - l_{0\,c}\alpha_c = 0$, whence

$$l_{0\,c} = \Delta l\alpha_{al}/(\alpha_{al} - \alpha_c).$$

Similarly, we find that

$$l_{0\,al} = \Delta l\alpha_c/(\alpha_{al} - \alpha_c).$$

Then

$$l_{0\,c} = \frac{0.15\times 2.4\times 10^{-5}}{2.4\times 10^{-5} - 1.7\times 10^{-5}}\ \mathrm{m} \simeq 0.51\ \mathrm{m},$$

$$l_{0\,al} = \frac{0.15\times 1.7\times 10^{-5}}{2.4\times 10^{-5} - 1.7\times 10^{-5}}\ \mathrm{m} \simeq 0.36\ \mathrm{m}.$$

328. The amount of heat spent in heating an iron bar is 1.68 MJ. What is the change in the volume of the bar?

Given: $Q = 1.68\ \mathrm{MJ} = 1.68\times 10^6\ \mathrm{J}$.

ΔV -- ?

Solution. According to the law of volume expansion, $V = V_0(1 + \beta\,\Delta T)$, whence

$$\Delta V = V - V_0 = V_0\beta\,\Delta T \approx 3\alpha V_0\,\Delta T.$$

Here $\beta = 3\alpha$ is the coefficient of volume expansion for iron and α the coefficient of linear expansion for this substance.

The amount of heat required to heat the bar by ΔT is

$$Q = cm\,\Delta T,$$

whence $\Delta T = Q/(cm)$. Since the mass of the bar is $m = \rho V_0$, we can write

$$\Delta T = Q/(c\rho V_0).$$

The change in the volume of the bar is

$$\Delta V = \frac{V_0 3\alpha Q}{c\rho V_0} = \frac{3\alpha Q}{c\rho},$$

$$\Delta V = \frac{3 \times 1.2 \times 10^{-5} \times 1.68 \times 10^{6}}{0.46 \times 10^{3} \times 7.8 \times 10^{3}}\ \text{m}^3 \simeq 1.69 \times 10^{-5}\ \text{m}^3.$$

329. Water and kerosene have the same volume of 4 l at 0 °C. Determine the difference in their volumes at 50 °C.

Given: $T_0 = 273$ K, $V_{0\,w} = V_{0\,k} = 4$ l $= 4 \times 10^{-3}$ m^3, $T = 323$ K.

ΔV — ?

Solution. According to the law of volume expansion, we have

$$V_w = V_{0\,w}(1 + \beta_w \Delta T), \quad V_k = V_{0\,k}(1 + \beta_k \Delta T),$$

where $\Delta T = T - T_0$, and β_w and β_k are the coefficients of volume expansion for water and kerosene. Then the difference in the volumes of water and kerosene is

$$\Delta V = V_k - V_w = V_{0\,k}(1 + \beta_k \Delta T) - V_{0\,w}(1 + \beta_w \Delta T) = V_0 (T - T_0)(\beta_k - \beta_w),$$

where $V_0 = V_{0\,k} = V_{0\,w}$. Consequently,

$$\Delta V = 4 \times 10^{-3} \times (323 - 273) \times (10 \times 10^{-4} - 1.8 \times 10^{-4})\ \text{m}^3 \simeq 1.64 \times 10^{-4}\ \text{m}^3.$$

330. An iron tank has a capacity of 50 l of kerosene at 0 °C. What amount of kerosene will flow out of the full tank if it is brought in a room where the temperature is 20 °C?

Given: $V_0 = 50$ l $= 5 \times 10^{-2}$ m^3, $t_0 = 0$ °C, $T_0 = 273$ K, $t = 20$ °C, $T = 293$ K.

Δm — ?

Solution. The increase in the volume of kerosene as a result of heating by 20 °C will be larger than the increase in the volume of the tank, and a part of the kerosene will flow out. We denote by m_0 the mass of kerosene in the tank at 273 K and by m the mass of kerosene at room temperature. Then the mass of kerosene flowing out of the tank is

$$\Delta m = m_0 - m, \quad \text{or}$$
$$\Delta m = \rho_0 V_0 - \rho V, \qquad (1)$$

where ρ_0 is the density of kerosene at 273 K, and ρ and V are the density of kerosene and the volume of the tank at room temperature.

The volume of the iron tank after its heating is

$$V = V_0 (1 + \beta_1 \Delta T), \qquad (2)$$

where $\beta_1 \approx 3\alpha_1$. The density of kerosene after the heating of the tank is

$$\rho = \rho_0/(1 + \beta_k \Delta T). \qquad (3)$$

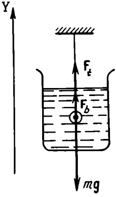


Fig. 108

Substituting expressions (2) and (3) into (1), we obtain

$$\Delta m = \rho_0 V_0 - \frac{\rho_0}{1+\beta_k \Delta T} V_0 (1 + 3\alpha_1 \Delta T)$$
$$= \frac{\rho_0 V_0 (\beta_k - 3\alpha_1)(T - T_0)}{1 + \beta_k (T - T_0)} \approx \rho_0 V_0 (\beta_k - 3\alpha_1)(T - T_0),$$
$$\Delta m = 0.8 \times 10^3 \times 5 \times 10^{-2} \times (10 \times 10^{-4} - 3 \times 1.2 \times 10^{-5})$$
$$\times (293 - 273) \text{ kg} \simeq 0.8 \text{ kg}.$$

331. A steel ball of mass 100 g is tied to a string and immersed in kerosene. What will be the change in the tension of the string if the entire system is heated from 20 to 50 °C (Fig. 108)?

Given: $m = 100 \text{ g} = 0.1 \text{ kg}$, $t_1 = 20\ °\text{C}$, $T_1 = 293$ K, $t_2 = 50\ °\text{C}$, $T_2 = 323$ K.

ΔF_t — ?

Solution. At temperature T_1, the ball immersed in kerosene experiences the action of the force of gravity mg, the tension $\mathbf{F}_{t1}$ of the string, and the buoyant force $\mathbf{F}_{b1}$. We write the equilibrium condition for the ball in projections on the Y-axis:

$$F_{t1} + F_{b1} - mg = 0,$$

whence

$$F_{t1} = mg - F_{b1}. \tag{1}$$

Here $F_{b1} = \rho_k V_s g$, where ρ_k is the density of kerosene. Since $\rho_k = \rho_{0\,k}/(1 + \beta_k \Delta T_1)$, $V_s = V_{0\,s}(1 + \beta_s \Delta T_1) = V_{0\,s}(1 + 3\alpha_s \Delta T_1)$, $V_{0\,s} = m/\rho_{0\,s}$, and $\Delta T_1 = T_1 - T_0$, the buoyant force is

$$F_{b1} = \frac{\rho_{s\,k} m [1+3\alpha_s (T_1 - T_0)] g}{[1+\beta_k (T_1 - T_0)] \rho_{0\,s}}, \tag{2}$$

where α_s is the coefficient of linear expansion for steel. Taking Eq. (2) into account, we can write Eq. (1) as

$$F_{t1} = mg - \frac{\rho_{0\,k} m [1+3\alpha_s (T_1 - T_0)] g}{[1+\beta_k (T_1 - T_c)] \rho_{0\,s}}$$

$$\approx mg - \frac{m\rho_{0\,k}}{\rho_{0\,s}} [1 - (\beta_k - 3\alpha_s)(T_1 - T_0)]\, g.$$

Similarly, the tension of the string at temperature T_2 is

$$F_{t2} = mg - \frac{\mu_{0\,k} m [1+3\alpha_s (T_2 - T_s)] g}{[1+\beta_k (T_2 - T_0)] \rho_{0\,s}}$$

$$\approx mg - \frac{m\rho_{0\,k}}{\rho_{0\,s}} [1 - (\beta_k - 3\alpha_0)(T_2 - T_0)]\, g.$$

Therefore

$$\Delta F_t = F_{t2} - F_{t1} \approx \frac{mg\rho_{0\,k}}{\rho_{0\,s}} (\beta_k - 3\alpha_s)(T_2 - T_1),$$

$$\Delta F_t = 0.1 \times 9.8 \times 0.8 \times 10^3 \times (10 \times 10^{-4} - 3 \times 1.2 \times 10^{-5}) \times \left(\frac{323 - 293}{7.8 \times 10^{-3}}\right) \text{ N} \simeq 3 \text{ mN}.$$

332. The mass of alcohol taken at 0 °C in a volume of 500 cm³ is 400 g. Determine the density of alcohol at 15 °C.

Given: $t_0 = 0$ °C, $T_0 = 273$ K, $V_0 = 500\ \text{cm}^3 = 5 \times 10^{-4}\ \text{m}^3$, $t = 15$ °C, $T = 288$ K, $m = 400\ \text{g} = 0.4$ kg.

$\rho - ?$

Solution. The density of alcohol is

$$\rho = \frac{\rho_0}{1+\beta\,\Delta T}. \tag{1}$$

Since $\rho_0 = m/V_0$, we have (see Eq. (1))

$$\rho = \frac{m}{V_0(1+\beta\,\Delta T)}.$$

Using the formula for approximate calculation, we find that

$$\rho \approx \frac{m}{V_0}(1-\beta\,\Delta T) = \frac{m}{V_0}[1-\beta(T-T_0)],$$

$$\rho \simeq \frac{0.4}{5\times 10^{-4}} \times [1-11\times 10^{-4}\times(288-273)]\,\frac{\text{kg}}{\text{m}^3}$$

$$\simeq 0.79\times 10^3\ \text{kg/m}^3.$$

333. Heating changes not only the volume of a cube cut out of a single crystal but also its shape. Why?

Answer. Due to anisotropy, the coefficient of linear expansion in a single crystal is different in different directions, and for this reason the shape of the cube changes.

334. Why are the most sensitive measuring instruments made of a special alloy like invar?

Answer. Invar has a very small coefficient of linear expansion. For this reason, random temperature fluctuations do not affect the accuracy of a measuring instrument made of invar.

EXERCISES

335. The length of an aluminium ruler is 79.5 cm and that of an iron ruler is 80 cm at 0 °C. At what temperature will the lengths of the rulers become equal?

336. The length of a copper and an iron wire is 500 m at 0 °C. Determine the difference in their lengths at 30 °C.

337. A lead ball has a volume of 1.8 dm^3 at 20 °C. Determine the increase in its volume as a result of heating to 100 °C.

338. At 20 °C, kerosene and sulphuric acid have the same volume of 500 cm^3. What will be the difference in the volumes of the liquids at 0 °C?

339. The temperature of one arm of a U-tube is 10 °C and that of the other arm is 80 °C. The level of kerosene in one arm is 280 mm and that in the other arm is 300 mm. Determine the coefficient of volume expansion for kerosene.

340. The volume of a brass vessel increases as a result of heating by 0.6%. By how many degrees was the vessel heated?

341. 60 g of water at 75 °C are poured in an indentation made in ice at a temperature of 0 °C. What will be the volume of the indentation after the cooling of water?

342. Petroleum is contained in an iron cylindrical tank whose height is 6 m and the diameter of the base is 5 m. At 0 °C, the level of petroleum is 20 cm below the brim. At what temperature will petroleum start to overflow from the tank? The thermal expansion of the tank should be also taken into account.

SURFACE TENSION IN LIQUIDS. CAPILLARY PHENOMENA

The surface layer of a liquid is in a stressed state and possesses a potential energy. **Surface tension** is defined as

$$\sigma = F_{s.t}/l \quad \text{or} \quad \sigma = W/S,$$

where $F_{s\,t}$ is the force of surface tension, l the length of the contour bounding the surface of the liquid, W the potential energy of the surface layer, and S the area of the surface layer.

A curved surface layer exerts an excess pressure on a liquid in comparison with the pressure of a liquid with a flat surface layer. A convex surface layer compresses the lower layers of a liquid, while a concave layer stretches them.

The **excess pressure** p_{ex} exerted on a liquid by a spherical surface of radius R is

$$p_{ex} = \pm 2\sigma/R,$$

where the plus sign corresponds to the convex meniscus and the minus sign to the concave meniscus.

In narrow tubes (capillaries), the curvature of the surface of a liquid (meniscus) becomes considerable due to wetting or nonwetting of the tube walls for the liquid. The excess pressure causes a noticeable rise or fall in the level of the liquid.

The height to which a liquid perfectly wetting the capillary walls rises in the capillary is

$$h = 2\sigma/\rho g R,$$

where ρ is the density of the liquid, g the free-fall acceleration, and R the radius of the capillary.

* * *

343. A thin aluminium ring of radius 7.8 cm is in contact with a soap solution. What force must be applied to separate the ring from the surface of the solution? Assume that the solution is at room temperature and that the mass of the ring is 7 g.

Given: $R = 7.8\,\text{cm} = 7.8 \times 10^{-2}\,\text{m}$, $m = 7\,\text{g} = 7 \times 10^{-3}$ kg.

F — ?

Solution. The ring is acted upon by the force of gravity mg, the force of surface tension $\mathbf{F}_{s\,t}$, and an external force $\mathbf{F}$ (Fig. 109). Since both the external and internal surfaces of the ring touch the solution, the force of surface tension

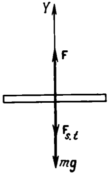


Fig. 109

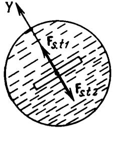


Fig. 110

is

$$F_{s.t} = 2\sigma l,$$

where $l = 2\pi R$. The condition of separation of the ring from the solution in projections on the Y-axis has the form $F = mg + F_{s.t}$, or

$$F = mg + 2\sigma l = mg + 4\pi\sigma R.$$

Then

$$F = 7 \times 10^{-3} \times 9.8 + 4 \times 3.14 \times 4 \times 10^{-2} \times 7.8 \times 10^{-2}\ \text{N} \simeq 0.11\ \text{N}.$$

344. A wooden stick of length 4 cm floats on the surface of water. A soap solution is carefully poured at one side of the stick. With what acceleration will the stick start to move if its mass is 1 g? The water resistance should be disregarded.

Given: $l = 4\ \text{cm} = 4 \times 10^{-2}\ \text{m}$, $m = 1\ \text{g} = 10^{-3}\ \text{kg}$.

a — ?

Solution. The stick in the horizontal plane is acted upon by the forces of surface tension exerted by water ($F_{s\ t1}$) and by the soap solution ($F_{s.t2}$) (Fig. 110). Writing Newton's second law for the stick in projections on the Y-axis, we obtain

$$F_{s\ t1} - F_{s.t2} = ma,$$

whence

$$a = (F_{s\ t1} - F_{s.t2})/m.$$

Since $F_{s\ t1} = \sigma_1 l$ and $F_{s\ t2} = \sigma_2 l$, where σ_1 and σ_2 are the surface tensions for water and the soap solution, we have

$$a = \frac{l(\sigma_1 - \sigma_2)}{m},$$

$$a = \frac{4 \times 10^{-2} \times (7.4 \times 10^{-2} - 4 \times 10^{-2})}{10^{-3}}\ \frac{\text{m}}{\text{s}^2} = 1.36\ \text{m/s}^2.$$

345. What energy is liberated as a result of merging of small water drops of radius 2×10^{-3} mm into a single drop of radius 2 mm?

Given: $r = 2 \times 10^{-3}\ \text{mm} = 2 \times 10^{-6}\ \text{m}$, $R = 2\ \text{mm} = 2 \times 10^{-2}\ \text{m}$.

W — ?

Solution. The change in the potential energy of the surface layer of the drops as a result of a decrease in the surface area by ΔS during their mergence is

$$\Delta W = \sigma \, \Delta S = \sigma \, (S_1 - S_2), \qquad (1)$$

where S_1 is the surface area of all the small drops, S_2 the surface area of the large drop, and σ the surface tension for water.

Obviously, $S_1 = 4\pi r^2 n$ and $S_2 = 4\pi R^2$. The mass of water remains constant, and hence

$$nm = M, \qquad (2)$$

where n is the number of small drops, m the mass of a small drop, and M the mass of the large drop. Since $m = \rho V_1 = (4/3)\,\rho\pi r^3$ and $M = \rho V_2 = (4/3)\,\rho\pi R^3$, we obtain from Eq. (2)

$$n = R^3/r^3.$$

Consequently, the surface area of all the small drops is $S_1 = 4\pi r^2 R^3/r^3 = 4\pi R^3/r$. Substituting the expressions for S_1 and S_2 into Eq. (1), we find that

$$\Delta W = \sigma \left(\frac{4\pi R^3}{r} - 4\pi R^2 \right) = 4\pi R^2 \sigma \left(\frac{R}{r} - 1 \right),$$

$$\Delta W = 4 \times 3.14 \times (2 \times 10^{-3})^2 \times 7.4 \times 10^{-2} \times \left[\left(\frac{2 \times 10^{-3}}{2 \times 10^{-6}} \right) - 1 \right] \text{J} = 3.7 \text{ mJ}.$$

346. What is the air pressure in a bubble of radius 5×10^{-3} mm under the water surface?

Given: $R = 5 \times 10^{-3}\text{ mm} = 5 \times 10^{-6}\text{ m}.$

p—?

Solution. The air pressure in the bubble is

$$p = p_0 + p_{ex},$$

where p_0 is the atmospheric pressure and p_{ex} the excess pressure. Since $p_{ex} = 2\sigma/R$, we have $p = p_0 + 2\sigma/R$, where σ is the surface tension for water. Then

$$p = 1.01 \times 10^5 \text{ Pa} + \frac{2 \times 7.4 \times 10^{-2}}{5 \times 10^{-6}} \text{ Pa} = 130 \text{ kPa}.$$

347. Two soap bubbles of radius 10 and 5 cm are blown at different ends of the same tube. Determine the pressure difference in the bubbles. Will their size change if they are left to themselves?

Given: $R_1 = 10\,\text{cm} = 0.1\,\text{m}$, $R_2 = 5\,\text{cm} = 0.05\,\text{m}$.

$\Delta p - ?$

Solution. The pressure p in a soap bubble is

$$p = p_0 + p_{ex}, \tag{1}$$

where p_0 is the atmospheric pressure and p_{ex} the excess pressure produced by the curved surface layer of the soap solution. Obviously, $p_{ex} = 2\cdot 2\sigma/R$, where σ is the surface tension for the soap solution. The factor 2 is due to the fact that the soap film has two surfaces: outer and inner. Then Eq. (1) assumes the form

$$p = p_0 + 4\sigma/R. \tag{2}$$

Equation (2) can be written for the first and the second bubble in the form $p_1 = p_0 + 4\sigma/R_1$ and $p_2 = p_0 + 4\sigma/R_2$. Then the pressure difference in the bubbles is

$$\Delta p = p_2 - p_1 = \frac{4\sigma (R_1 - R_2)}{R_1 R_2},$$

$$\Delta p = \frac{4\times 4\times 10^{-2}\times (0.1-0.05)}{0.1\times 0.05}\ \text{Pa} = 1.6\ \text{Pa}.$$

Calculations show that the pressure in the smaller bubble is higher than that in the larger bubble. Therefore, air flows from the smaller bubble to the larger bubble. The volume of the smaller bubble will decrease and that of the larger bubble will increase until the pressures become equal.

348. A long capillary of radius 1 mm open at both ends is filled with water and turned to the vertical position. Determine the height of the water column remaining in the capillary, neglecting the thickness of its wall.

Given: $R = 1\,\text{mm} = 10^{-3}\,\text{m}$.

$h - ?$

Solution. The water column in the capillary is acted upon by the force of gravity mg and the forces of surface tension F_{st} in the upper and lower meniscuses (Fig. 111). Writing the equilibrium condition for the water column in projections of forces on the Y-axis, we obtain

$$2F_{st} - mg = 0.$$

Considering that $F_{st} = \sigma 2\pi R$ and $mg = \rho g V = \rho g \pi R^2 h$, where σ is the surface tension and ρ the density of water, we obtain $2\sigma \cdot 2\pi R - \rho g \pi R^2 h = 0$, whence

$$h = \frac{4\sigma}{\rho g R},$$

$$h = \frac{4 \times 7.4 \times 10^{-2}}{10^3 \times 9.8 \times 10^{-3}} \text{ m} \simeq 3 \times 10^{-2} \text{ m}.$$

349. The difference in the levels of a wetting liquid in the arms of a U-tube is 23 mm (Fig. 112). The diameters of the channels in the arms of the tube are 2 and 0.4 mm. The density of the liquid is 0.8 g/cm³. Determine the surface tension for the liquid.

Given: $h = 23\text{ mm} = 2.3 \times 10^{-2}\text{ m}$, $D_1 = 2\text{ mm} = 2 \times 10^{-3}\text{ m}$, $D_2 = 0.4\text{ mm} = 0.4 \times 10^{-3}\text{ m}$, $\rho = 0.8\text{ g/cm}^3 = 0.8 \times 10^3\text{ kg/m}^3$.

σ — ?

Solution. The equilibrium condition for the liquid in the U-tube is $p_A = p_B$, where p_A and p_B are the pressures

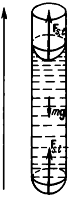


Fig. 111

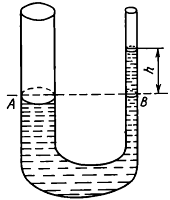


Fig. 112

in the left and right arms on the level AB. Considering that $p_A = p_0 - p_{ex1}$ and $p_B = p_0 - p_{ex2} + p_h$, where p_0 is the atmospheric pressure, $p_{ex1} = 2\sigma/R_1 = 4\sigma/D_1$, $p_{ex2} = 2\sigma/R_2 = 4\sigma/D_2$, and $p_h = \rho gh$, the equilibrium condition for the liquid assumes the form $p_0 - 4\sigma/D_1 = p_0 - 4\sigma/D_2 + \rho gh$, whence

$$\sigma = \frac{\rho g h D_1 D_2}{4(D_1 - D_2)},$$

$$\sigma = \frac{0.8 \times 10^3 \times 9.8 \times 2.3 \times 10^{-2} \times 2 \times 10^{-3} \times 0.4 \times 10^{-3}}{4 \times (2 \times 10^{-3} - 0.4 \times 10^{-3})} \frac{\text{N}}{\text{m}}$$

$$\simeq 2.25 \times 10^{-2}\ \text{N/m}.$$

350. Why do the hair of a brush stick together when the brush is taken out of water?

Answer. If the brush is taken out of water, its hair is covered by a water film and sticks together under the action of surface tension.

351. What liquids can be poured above the brim in a glass?

Answer. Nonwetting liquids, since the forces of interaction between the liquid molecules are stronger than the forces of interaction between the liquid and glass molecules. The resultant of all these forces is directed into the liquid and keeps the liquid molecules which are above the brim.

EXERCISES

352. Determine the surface tension for oil whose density is 0.91 g/cm^3 if 304 drops are formed by passing 4 cm^3 of the oil through a pipette. The diameter of the pipette nozzle is 1.2 mm.

353. What is the mass of a drop of water flowing out of a glass tube of diameter 1 mm? Assume that the diameter of a drop is equal to the diameter of the tube.

354. An air bubble of diameter 0.002 mm is in water near the surface. Determine the density of air in the bubble.

355. What is the excess of pressure in a soap bubble over the atmospheric pressure if the diameter of the bubble is 5 mm?

356. What is the ratio of the heights of water and kerosene columns in capillaries, other conditions being equal?

357. The difference in the mercury levels in communicating capillaries of radius 0.5 and 2 mm is 10.5 mm. Determine the surface tension for mercury.

358. A liquid rises in a capillary to 80 cm. Determine the height of the liquid column which will be in the capillary if it is filled with the liquid in the horizontal position and then turned to the vertical position.

359. A water drop of mass 0.2 g is between two glass plates separated by a distance of 0.1 mm. Determine the force of attraction between the plates.

DEFORMATION OF SOLIDS. HOOKE'S LAW

Deformation is a change in the shape and size of bodies under the action of applied forces. Deformations can be elastic and inelastic.

A deformation is **elastic** if it disappears after the forces cease to act on a body. An **inelastic** deformation is partially retained after the forces are removed.

Elastic deformations obey **Hooke's law**:

$$\Delta l = (1/E)\,\sigma l_0,$$

where $\Delta l = l - l_0$ is the **absolute elongation** (deformation) of a body, l the length of the deformed body, l_0 its initial length, E the **modulus of elongation (Young's modulus)**, $\sigma = F/S$ the **stress**, F the force acting on the body, and S the cross-sectional area of the body.

The stress σ_u at which a body begins to be ruptured is called the **ultimate strength**.

* * *

360. Determine the elongation of a brass rod of length 4 m and cross-sectional area 0.4 cm² under the action of a force of 1 kN.

Given: $l_0 = 4$ m, $S = 0.4\ \text{cm}^2 = 4 \times 10^{-5}\ \text{m}^2$, $F = 1\ \text{kN} = 10^3$ N.

Δl — ?

Solution. According to Hooke's law, the absolute elongation is

$$\Delta l = (1/E)\,\sigma l_0 = Fl_0/ES,$$

where E and σ are Young's modulus and the stress for brass. This gives

$$\Delta l = \frac{10^3 \times 4}{0.9 \times 10^{11} \times 4 \times 10^{-5}}\ \mathrm{m} \simeq 1.1 \times 10^{-3}\ \mathrm{m}.$$

361. At what limiting load will a steel rope of diameter 1 cm rupture if the ultimate strength for steel is 1 GPa?

Given: $D = 1\ \mathrm{cm} = 10^{-2}\ \mathrm{m}$, $\sigma_u = 1\ \mathrm{GPa} = 10^9\ \mathrm{Pa}$.

F_{lim} — ?

Solution. The ultimate strength is

$$\sigma_u = F_{\mathrm{lim}}/S,$$

where $S = \pi D^2/4$ is the cross-sectional area of the rope. Consequently, the limiting load, i.e. the force acting on the rope, is

$$F_{\mathrm{lim}} = \sigma_u S = \frac{\sigma_u \pi D^2}{4},$$

$$F_{\mathrm{lim}} = \frac{10^9 \times 3.14 \times (10^{-2})^2}{4}\ \mathrm{N} \simeq 78.5\ \mathrm{kN}.$$

362. A steel bar tightly fits the gap between two stationary walls at 0 °C. Determine the stress in the bar material at 20 °C.

Given: $t_0 = 0\ ^\circ\mathrm{C}$, $T_0 = 273\ \mathrm{K}$, $t = 20\ ^\circ\mathrm{C}$, $T = 293\ \mathrm{K}$.

σ — ?

Solution. If the free bar were heated by ΔT, its length would be $l = l_0(1 + \alpha\,\Delta T)$, whence

$$\Delta l = l - l_0 = \alpha l_0\,\Delta T, \qquad (1)$$

where α is the coefficient of linear expansion for steel and l_0 the length of the bar at T_0.

The distance between the walls does not change, and hence Δl is the absolute compression of the bar emerging during its heating. It follows from Hooke's law that the stress of the bar material at 293 K is

$$\sigma = E\,\Delta l/l_0,$$

where E is Young's modulus for steel, or, taking into account expression (1),

$$\sigma = E\alpha l_0 \, \Delta T/l_0 = E\alpha \, (T - T_0),$$

$$\sigma = 2.2 \times 10^{11} \times 1.1 \times 10^{-5} \times (293 - 273) \text{ Pa}$$
$$= 48.4 \text{ MPa}.$$

363. A steel wire of length 1 m is fixed at one end so that it can oscillate in the vertical plane. A load of mass 50 kg is tied to the free end of the wire, which is deflected to the horizontal position and then released. Determine the absolute elongation of the wire at the lowest point of the trajectory of the load. The cross-sectional area of the wire is 0.8 mm^2, and its mass can be neglected.

Given: $l_0 = 1$ m, $m = 50$ kg, $S = 0.8 \text{ mm}^2 = 8 \times 10^{-7} \text{ m}^2$.

Δl — ?

Solution. According to Hooke's law, the absolute elongation of the wire is

$$\Delta l = F l_0/(ES), \qquad (1)$$

where F is the tensile force acting on the wire at point A and E Young's modulus for steel. Let us consider the forces acting on the load at the moment it passes through point A: the force of gravity mg, and the tension $\mathbf{T}$ of the wire (Fig. 113). Writing Newton's second law for the load suspended on the wire in projections on the Y-axis, we obtain

$$T - mg = ma_y.$$

Here $a_y = a_c = v^2/l$, where l is the length of the stretched wire at point A. Hence

$$T = mg + mv^2/l = m\,(g + v^2/l). \qquad (2)$$

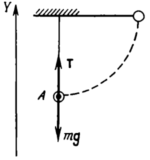


Fig. 113

In order to find the velocity v of the load at point A, we apply the energy conservation law for two positions of the load at points A and B: $W_A = W_B$, or $mv^2/2 = mgl$, whence $v^2 = 2gl$. Substituting this expression into Eq. (2), we obtain

$$T = m\,(g + 2gl/l) = 3mg.$$

According to Newton's third law, the force F stretching the wire is equal in magnitude to the tensile force T: $F = T = 3mg$. Using this expression, we write Eq. (1) in the form

$$\Delta l = \frac{3mgl_0}{ES},$$

$$\Delta l = \frac{3 \times 50 \times 9.8 \times 1}{2.3 \times 10^{11} \times 8 \times 10^{-7}}\ \text{m} \simeq 8.4 \times 10^{-3}\ \text{m}.$$

364. Why are cutting tools not made of glass whose hardness is the same as that of instrument steel?

Answer. Glass has a low tensile (and flexural) strength at room temperature in comparison with steel.

EXERCISES

365. What force must be applied to a steel wire of length 2 m to stretch it by 1 mm? The cross-sectional area of the wire is 0.5 mm^2.

366. A copper wire of diameter 1 mm breaks under a load of 188.4 N. Determine the ultimate tensile strength for copper.

367. What is the minimum length of a lead wire at which it is ruptured under the force of gravity if the rupture takes place near the point of suspension?

368. An iron wire is stretched in the horizontal position between stationary supports at 30 °C. What force will be exerted by the wire at the points where it is fixed upon a temperature drop to —10 °C? The cross-sectional area of the wire is 2 mm^2.

369. A copper rod of length 1 m uniformly rotates about a vertical axis passing through its end. At what angular velocity will the rod break? The ultimate strength for copper is $\sigma_u = 235$ MPa.

370. Determine the work done in stretching a steel wire of length 1 m and radius 1 mm, from which a load of mass 100 kg is suspended.

QUESTIONS FOR REVISION

1. What is the coefficient of linear expansion? 2. What is the coefficient of volume expansion? 3. What is the relation between the coefficients of linear and volume expansion? 4. What is the physical meaning of the surface tension for a liquid? 5. Write a formula for calculating the excess pressure under the curved surface of a liquid. 6. Write a formula for calculating the height of a liquid column in a capillary. 7. What is the deformation of a solid? 8. Name the types of deformation of solids. 9. Define the ultimate strength for a body. 10. How is stress defined? 11. Formulate Hooke's law for elastic deformations. 12. What is the physical meaning of Young's modulus? 13. Give SI units of Young's modulus and stress.

Chapter 3
ELECTRICITY

3.1. Electrostatics

COULOMB'S LAW. ELECTROSTATIC FIELD STRENGTH

According to **Coulomb's law**, the force of interaction between two stationary point electric charges is

$$F = q_1 q_2/(4\pi\varepsilon_0 \varepsilon r^2),$$

where q_1 and q_2 are the magnitudes of the charges, r is the distance between them, ε the **relative permittivity of the medium** in which the charges are located, and ε_0 the electric constant, $\varepsilon_0 = 1/(4\pi 9 \times 10^9)$ F/m $\simeq 8.85 \times 10^{-12}$ F/m.

The following concepts are introduced for calculating the interaction between nonpoint charges.

1. **Linear charge density** for a uniformly charged filament

$$\tau = \frac{dq}{dl},$$

where dq and dl are the charge of the filament and its element of length.

2. **Surface charge density** for a uniformly charged surface

$$\sigma = \frac{dq}{dS},$$

where dq and dS are the charge of the element and its surface area.

Below we give the basic formulas for calculating the strength of an electrostatic field.

The **field strength** is defined as

$$\mathbf{E} = \mathbf{F}/q.$$

Hence the force acting on a charge q in a magnetic field $\mathbf{E}$ is

$$\mathbf{F} = q\mathbf{E}.$$

The strength of the field produced by a point charge or a charged sphere is

$$E = Q/(4\pi\varepsilon_0\varepsilon r^2),$$

where Q is the charge producing the field and r the distance from the charge (or the centre of the charged sphere) to a given point of the field.

The field strength produced by an infinite uniformly charged plane **(uniform field)** is

$$E = \sigma/(2\varepsilon_0\varepsilon),$$

where σ is the surface charge density and ε the permittivity of the medium. The vector $\mathbf{E}$ is normal to the plane.

The field strength produced by two unlikely charged parallel planes **(the field of a parallel-plate capacitor)** is given by

$$E = \sigma/(\varepsilon_0\varepsilon).$$

If an electric field is produced by several point charges q_i, the electric field strength vector $\mathbf{E}$ at a given point is

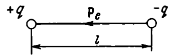


Fig. 114

equal to the vector sum of the field strengths $\mathbf{E}_i$ produced at this point by each charge separately **(the principle of superposition of electric fields)**:

$$\mathbf{E} = \sum_{i=1}^{n} \mathbf{E}_i.$$

Two unlike point charges of the same magnitude, separated by a certain distance l, form an **electric dipole** (Fig. 114). The points at which the charges are located are known as the **poles of the dipole**. The arm **l** of the dipole is a vector directed from the negative to the positive pole, whose length is equal to the separation between the poles. The quantity $\mathbf{p}_e = q\mathbf{l}$ is known as the **electric moment of the dipole** (or **electric dipole moment**).

* * *

371. Two identical small balls of mass 0.1 g each are suspended on strings of length 25 cm. After identical charges have been imparted to the balls, they diverge to 5 cm. Determine the charges of the balls.

Given: $m_1 = m_2 = m = 0.1\ \text{g} = 10^{-4}\ \text{kg}$, $l_1 = l_2 = l = 25\ \text{cm} = 25 \times 10^{-2}\ \text{m}$, $r = 5\ \text{cm} = 5 \times 10^{-2}\ \text{m}$.

q—?

Solution. Each ball experiences the action of the force of gravity $m\mathbf{g}$, the tension $\mathbf{T}$ of the string, and the electric force $\mathbf{F}$ of interaction between the balls (Fig. 115). Writing the equilibrium condition for a ball under the action of the applied forces, we obtain

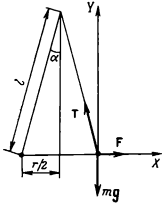


Fig. 115

$$m\mathbf{g} + \mathbf{T} + \mathbf{F} = 0. \quad (1)$$

In projections on the X- and Y-axes, Eq. (1) can be written as

$$\begin{aligned} -T \sin \alpha + F &= 0, \\ T \cos \alpha - mg &= 0. \end{aligned} \quad (2)$$

Considering that $F = q^2/(4\pi\varepsilon_0\varepsilon r^2)$, we write Eqs. (2) in the form

$$\begin{aligned} T \sin \alpha &= q^2/(4\pi\varepsilon_0\varepsilon r^2), \\ T \cos \alpha &= mg. \end{aligned} \quad (3)$$

Dividing the first equation from (3) by the second termwise, we find that $\tan \alpha = q^2/(4\pi\varepsilon_0\varepsilon r^2 mg)$. Since the angle α is small, we can write $\tan \alpha \approx \sin \alpha = r/(2l)$. Then

$r/(2l) = q^2/(4\pi\varepsilon_0\varepsilon r^2 mg)$, whence

$$q = r\sqrt{\frac{2\pi\varepsilon_0\varepsilon rmg}{l}},$$

$$q = 5\times10^{-2}\sqrt{\frac{2\times3.14\times8.85\times10^{-12}\times1\times5\times10^{-2}\times10^{-4}\times9.8}{25\times10^{-2}}}\ \mathrm{C}$$

$\simeq 5.2\,\mathrm{nC}$.

372. Two positively charged bodies bearing charges of 1.67 and 3.33 nC are situated at a distance of 20 cm from each other. At what point on the line connecting the bodies should the third body bearing a charge of -0.67 nC be placed for it to be in equilibrium? The masses of the bodies should be neglected.

Given: $q_1 = 1.67$ nC $= 1.67\times10^{-9}$ C, $q_2 = 3.33$ nC $= 3.33\times10^{-9}$ C, $r = 20$ cm $= 0.2$ m, $|q_3| = 0.67$ nC $= 0.67\times10^{-9}$ C.

x — ?

Solution. The body C with the charge q_3 is acted upon by the forces $\mathbf{F}_1$ and $\mathbf{F}_2$, viz. the forces of interaction with

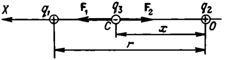


Fig. 116

the charges q_1 and q_2 respectively (Fig. 116). Writing the equilibrium condition for the body C in projections on the X-axis, we obtain

$$F_1 - F_2 = 0,$$

whence $F_1 = F_2$. Considering that

$$F_1 = \frac{q_1\,|q_3|}{4\pi\varepsilon_0\varepsilon\,(r-x)^2},\quad F_2 = \frac{q_2\,|q_3|}{4\pi\varepsilon_0\varepsilon x^2},$$

where x is the distance between the charges q_2 and q_3, we get

$$\frac{q_1\,|q_3|}{4\pi\varepsilon_0\varepsilon\,(r-x)^2} = \frac{q_2\,|q_3|}{4\pi\varepsilon_0\varepsilon x^2},$$

whence

$$x=\frac{\sqrt{q_2}}{\sqrt{q_1}+\sqrt{q_2}}\,r,$$

$$x=\frac{\sqrt{3.33\times10^{-9}}}{\sqrt{1.67\times10^{-9}}+\sqrt{3.33\times10^{-9}}}\,0.2\,\mathrm{m}\simeq 0.12\,\mathrm{m}.$$

373. According to the hypothesis put forth by N. Bohr, the electron in a hydrogen atom moves in a circular orbit. Calculate the velocity of the electron if the radius of its orbit is 0.5×10^{-8} cm (Fig. 117).

Given: $R=0.5\times10^{-8}\,\mathrm{cm}=5\times10^{-11}\,\mathrm{m}.$

v — ?

Solution. The electron moving in a circular orbit is under the action of the electrostatic force of its interaction with the nucleus:

$$F=|e|\,q/(4\pi\varepsilon_0 \varepsilon R^2),$$

where $|e|$ is the magnitude of the electron charge, q the nuclear charge of the hydrogen atom, and R the radius of the electron orbit. Neglecting the force of gravitational interaction between the electron and the nucleus, we write Newton's second law in projections on the Y-axis (see Fig. 117):

$$F=ma_y,$$

where m is the electron mass and $a_y=a_c=v^2/R$. This gives

$$\frac{|e|\,q}{4\pi\varepsilon_0\varepsilon R^2}=\frac{mv^2}{R},$$

whence

$$v=\sqrt{\frac{|e|\,q}{4\pi\varepsilon_0\varepsilon mR}},$$

$$v=\sqrt{\frac{1.6\times10^{-19}\times1.6\times10^{-19}}{4\times3.14\times8.85\times10^{-12}\times1\times9.1\times10^{-31}\times5\times10^{-11}}}\,\frac{\mathrm{m}}{\mathrm{s}}$$

$$\simeq 2.25\times10^{6}\,\mathrm{m/s}.$$

374. Two point charges of 6.7 and -13.3 nC are situated at a distance of 5 cm from each other. Determine the electric field strength at a point located at a distance of 3 cm from the positive charge and 4 cm from the negative charge.

Given: $q_1 = 6.7\ \mathrm{nC} = 6.7 \times 10^{-9}\ \mathrm{C}$, $q_2 = -13.3\ \mathrm{nC} = -13.3 \times 10^{-9}\ \mathrm{C}$, $r = 5\ \mathrm{cm} = 5 \times 10^{-2}\ \mathrm{m}$, $r_1 = 3\ \mathrm{cm} = 3 \times 10^{-2}\ \mathrm{m}$, $r_2 = 4\ \mathrm{cm} = 4 \times 10^{-2}\ \mathrm{m}$.

F — ?

Solution. According to the superposition principle, the field strength at point A is $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$, where $\mathbf{E}_1$ is the electric field strength at point A due to the charge q_1 and $\mathbf{E}_2$ is that due to the charge q_2 (Fig. 118).

Since q_1 and q_2 are point charges, we have $E_1 = q_1/(4\pi\varepsilon_0\varepsilon r_1^2)$ and $E_2 = q_2/(4\pi\varepsilon_0\varepsilon r_2^2)$. By hypothesis, the angle between the vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ is 90°, and the field strength can be determined from the Pythagorean theorem: $E = \sqrt{E_1^2 + E_2^2}$, or, using the expressions for E_1 and E_2,

$$E = \sqrt{\frac{q_1^2}{(4\pi\varepsilon_0\varepsilon r_1^2)^2} + \frac{q_2^2}{(4\pi\varepsilon_0\varepsilon r_2^2)^2}}$$

$$= \frac{1}{4\pi\varepsilon_0\varepsilon}\sqrt{\frac{q_1^2}{r_1^4} + \frac{q_2^2}{r_2^4}},$$

$$E = \frac{1}{4 \times 3.14 \times 8.85 \times 10^{-12}}$$

$$\times \sqrt{\frac{(6.7 \times 10^{-9})^2}{(3 \times 10^{-2})^4} + \frac{(13.3 \times 10^{-9})^2}{(4 \times 10^{-2})^4}}\ \frac{\mathrm{V}}{\mathrm{m}}$$

$$\simeq 101\ \mathrm{kV/m}.$$

375°. A thin rod of length 20 cm is uniformly charged with a linear density of 1 nC/cm. Determine the electric

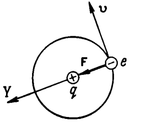


Fig. 117

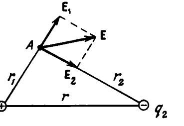


Fig. 118

field strength produced by the rod at point A (Fig. 119) lying on the continuation of its axis at 10 cm from the

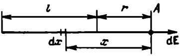


Fig. 119

nearest end and the force of interaction between the rod and a charge of 10^{-8} C placed at point A.

Given: $l = 20$ cm $= 0.2$ m, $\tau = 1$ nC/cm $= 10^{-7}$ C/m, $r = 10$ cm $= 0.1$ m, $q_0 = 10^{-8}$ C.

$E - ?$ $F - ?$

Solution. We divide the rod into small elements dx of length and assume that the charge dq of an element dx is a point charge. Let us determine the field strength dE produced at point A by the element dx of the rod separated from A by a distance x:

$$dE = \frac{dq}{4\pi\varepsilon_0\varepsilon x^2}.$$

Since $\tau = dq/dx$, we have $dq = \tau\, dx$ and

$$dE = \frac{\tau\, dx}{4\pi\varepsilon_0\varepsilon x^2}.$$

The electric field strength vectors due to other elements of length of the rod have the same direction as the vector **dE**. The electric field strength at point A is

$$E = \int_r^{r+l} \frac{\tau\, dx}{4\pi\varepsilon_0\varepsilon x^2} = \frac{\tau}{4\pi\varepsilon_0\varepsilon} \int_r^{r+l} \frac{dx}{x^2}$$

$$= \frac{\tau}{4\pi\varepsilon_0\varepsilon} \left(\frac{1}{x}\right)\Big|_r^{r+l} = \frac{\tau}{4\pi\varepsilon_0\varepsilon} \frac{l}{r(r+l)},$$

$$E = \frac{10^{-7}}{4 \times 3.14 \times 8.85 \times 10^{-12}} \frac{0.2}{0.1 \times (0.1 + 0.2)} \frac{\mathrm{V}}{\mathrm{m}} = 6\ \mathrm{kV/m}.$$

Then the force of interaction between the rod and the charge is

$$F = Eq_0,$$

$$F = 6 \times 10^3 \times 10^{-8}\ \mathrm{N} = 6 \times 10^{-5}\ \mathrm{N}.$$

376. Three identical point charges of 5 nC each are at the vertices of a square with a side of 40 cm. Determine the electric field strength at the fourth vertex of the square (Fig. 120).

Given: $a = 40\ \mathrm{cm} = 0.4\ \mathrm{m}$, $q = 5\ \mathrm{nC} = 5 \times 10^{-9}\ \mathrm{C}$.

$E - ?$

Solution. According to the superposition principle, the electric field at point A is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3, \quad (1)$$

where $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{E}_3$ are the field strengths produced by the charges q_1, q_2, and q_3 respectively at point A. Writing Eq. (1) in projections on the chosen directions of the X- and Y-axes, we obtain

$$\begin{aligned} E_x &= E_1 \sin \alpha + E_2 + E_3 \cos \alpha, \\ E_y &= -E_1 \cos \alpha + E_3 \sin \alpha. \end{aligned} \quad (2)$$

Considering that $E_1 = E_3 = q/(4\pi\varepsilon_0\varepsilon a^2)$, $E_2 = q/(4\pi\varepsilon_0\varepsilon r^2)$, $r^2 = 2a^2$, $\sin \alpha = \cos \alpha = \sqrt{2}/2$ and substituting these

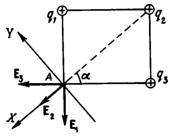


Fig. 120

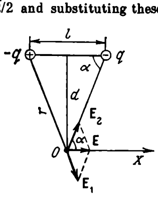


Fig. 121

expressions into Eqs. (2), we find that

$$E_y = 0,$$

$$E_x = E = \frac{q\sqrt{2}/2}{4\pi\varepsilon_0 \varepsilon a^2} + \frac{q}{4\pi\varepsilon_0 \varepsilon 2a^2} + \frac{q\sqrt{2}/2}{4\pi\varepsilon_0 \varepsilon a^2}$$

$$\approx 1.9 \frac{q}{4\pi\varepsilon_0 \varepsilon a^2},$$

$$E = 1.9 \frac{5\times10^{-9}}{4\times3.14\times8.85\times10^{-12}\times1\times0.4^2} \frac{\text{V}}{\text{m}} \simeq 590 \text{ V/m}.$$

377. Determine the electric field strength produced by a dipole at a point lying on the perpendicular to the dipole arm at a distance of 50 cm from its midpoint if the dipole charges are 10^{-8} and -10^{-8} C and the dipole arm is 5 cm.

Given: $d = 50$ cm $= 0.5$ m, $q_1 = 10^{-8}$ C, $q_2 = -10^{-8}$ C, $l = 5$ cm $= 5\times10^{-2}$ m.

E — ?

Solution. According to the superposition principle, the field strength produced by the dipole at point O is equal to the sum of the field strengths $\mathbf{E}_1$ and $\mathbf{E}_2$ produced by the charges q_1 and q_2 respectively (Fig. 121):

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2,$$

or in projections on the X-axis,

$$E = E_1 \cos\alpha + E_2 \cos\alpha,$$

where α is the angle between the vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ and the X-axis. Since the dipole is formed by point charges, we can write

$$E_1 = E_2 = q/(4\pi\varepsilon_0 \varepsilon r^2), \quad E = 2E_1 \cos\alpha.$$

Analyzing the figure, we find that $r = \sqrt{d^2 + l^2/4}$ and $\cos\alpha = l/(2\sqrt{d^2 + l^2/4})$. Substituting the expressions for E_1, r, and $\cos\alpha$ into the expression for the electric field strength of the dipole, we obtain

$$E = \frac{2q}{4\pi\varepsilon_0 \varepsilon (d^2 + l^2/4)} \frac{l}{2\sqrt{d^2 + l^2/4}} = \frac{2ql}{\pi\varepsilon_0 \varepsilon (4d^2 + l^2)^{3/2}}. \quad (1)$$

Since in the denominator of expression (1) $4d^2 \gg l^2$, we can write

$$E \approx \frac{2ql}{\pi\varepsilon_0\varepsilon (4d^2)^{3/2}} = \frac{ql}{4\pi\varepsilon_0\varepsilon d^3},$$
$$E = \frac{10^{-8}\times 5\times 10^{-2}}{4\times 3.14\times 8.85\times 10^{-12}\times 0.5^3}\ \frac{\text{V}}{\text{m}} = 36\ \text{V/m}. \quad (2)$$

Equation (2) shows that the electric field strength produced by the dipole decreases in inverse proportion to d^3, i.e. faster than the electric field strength of a point charge which decreases in inverse proportion to d^2.

378. Two metal concentric spheres of radius 5 and 10 cm bear charges of 2×10^{-8} and -10^{-8} C. Determine the electric field strength produced by the spheres at points separated from the centre of the spheres by 3, 8, and 14 cm. Plot the graph of the dependence of the electric field strength on the distance from the centre of the spheres.

Given: $R_1 = 5\ \text{cm} = 5\times 10^{-2}\ \text{m}$, $R_2 = 10\ \text{cm} = 0.1\ \text{m}$, $q_1 = 2\times 10^{-8}\ \text{C}$, $q_2 = -10^{-8}\ \text{C}$, $r_1 = 3\ \text{cm} = 3\times 10^{-2}\ \text{m}$, $r_2 = 8\ \text{cm} = 8\times 10^{-2}\ \text{m}$, $r_3 = 14\ \text{cm} = 14\times 10^{-2}\ \text{m}$.

E_1 — ? E_2 — ? E_3 — ? $E = E(r)$ — ?

Solution. According to the superposition principle, the electric field strength at any point is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2,$$

where $\mathbf{E}_1$ is the electric field strength at any point produced by the sphere with the charge q_1, and $\mathbf{E}_2$ the electric field strength at the same point produced by the sphere with the charge q_2. Using the superposition principle, we find the field strengths at points A, B, and C in projections on the r-axis.

For point A (Fig. 122), we have

$$E_A = E_{1A} + E_{2A}.$$

Since $E_{1A} = E_{2A} = 0$ (the field strength inside a conductor is zero), $E_A = 0$ (point A lies inside the spheres).

For point B, we have

$$E_B = E_{1B} - E_{2B},$$

or, considering that $E_{1B} = q_1/(4\pi\varepsilon_0\varepsilon r_2^2)$ and $E_{2B} = 0$, we get

$$E_B = \frac{q_1}{4\pi\varepsilon_0\varepsilon r_2^2},$$

$$E_B = \frac{2\times 10^{-8}}{4\times 3.14\times 8.85\times 10^{-12}\times (8\times 10^{-2})^2}\,\frac{\text{V}}{\text{m}} \simeq 28\ \text{kV/m}.$$

For point C, we have

$$E_C = E_{1C} - E_{2C},$$

or, considering that $E_{1C} = q_1/(4\pi\varepsilon_0\varepsilon r_3^2)$ and $E_{2C} = |q_2|/(4\pi\varepsilon_0\varepsilon r_3^2)$, we get

$$E_C = \frac{q_1}{4\pi\varepsilon_0\varepsilon r_3^2} - \frac{|q_2|}{4\pi\varepsilon_0\varepsilon r_3^2} = \frac{q_1 - |q_2|}{4\pi\varepsilon_0\varepsilon r_3^2},$$

$$E_C = \frac{2\times 10^{-8} - 10^{-8}}{4\times 3.14\times 8.85\times 10^{-12}\times (14\times 10^{-2})^2}\,\frac{\text{V}}{\text{m}} \simeq 4.6\ \text{kV/m}.$$

In order to plot the graph, we shall use the obtained values of E_A, E_B, and E_C and calculate the electric field strengths at points M and N lying on the surfaces of the spheres. For all the points inside the smaller sphere, we

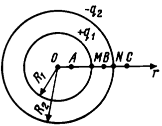


Fig. 122

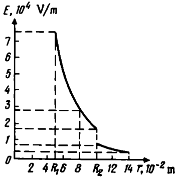


Fig. 123

have

$$E_1' = E_A = 0.$$

The field strength E_1'' for a point on the outer surface of the smaller sphere is

$$E_1'' = \frac{q_1}{4\pi\varepsilon_0\varepsilon R_1^2},$$

$$E_1'' = \frac{2\times10^{-8}}{4\times3.14\times8.85\times10^{-12}\times(5\times10^{-2})^2}\,\frac{\text{V}}{\text{m}} = 72\ \text{kV/m}.$$

The field strength for a point on the inner surface of the larger sphere is

$$E_2' = \frac{q_1}{4\pi\varepsilon_0\varepsilon R_2^2},$$

$$E_2' = \frac{2\times10^{-8}}{4\times3.14\times8.85\times10^{-12}\times0.1^2}\,\frac{\text{V}}{\text{m}} = 18\ \text{kV/m}.$$

The field strength for a point on the outer surface of the larger sphere is

$$E_2'' = \frac{q_1-|q_2|}{4\pi\varepsilon_0\varepsilon R_2^2},$$

$$E_2'' = \frac{2\times10^{-8}-10^{-8}}{4\times3.14\times8.85\times10^{-12}\times0.1^2}\,\frac{\text{V}}{\text{m}} = 9\ \text{kV/m}.$$

Using the obtained values, we plot the E vs r graph (Fig. 123). The figure shows that the E vs r dependence is quite complicated. For $r < R_1$, the curve coincides with the abscissa axis ($E = 0$), for $r = R_1$ and $r = R_2$, the curve suffers discontinuities.

379°. Two identical positive charges are separated by a distance of 20 cm. Determine a point on the straight line perpendicular to the line connecting the charges and passing through its midpoint at which the electric field strength has the maximum value.

Given: $q_1 = q_2$, $l = 20\ \text{cm} = 0.2\ \text{m}$.

x — ?

Solution. We take an arbitrary point A on the X-axis, separated from point O by a distance x (Fig. 124). According to the superposition principle, the electric field strength at this point is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2, \tag{1}$$

where $\mathbf{E}_1$ and $\mathbf{E}_2$ are the field strengths produced at point A by the charges q_1 and q_2 respectively.

Projecting Eq. (1) onto the X-axis, we obtain

$$E = E_1 \cos\alpha + E_2 \cos\alpha, \tag{2}$$

where α is the angle between the vectors $\mathbf{E}_1$ and $\mathbf{E}_2$.

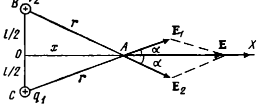


Fig. 124

Since $E_1 = E_2$, expression (2) can be written in the form

$$E = 2E_1 \cos\alpha. \tag{3}$$

From $\triangle OAC$, we find that $\cos\alpha = \frac{x}{\sqrt{x^2+l^2/4}} = \frac{2x}{\sqrt{4x^2+l^2}}$.

Considering that $E_1 = q/(4\pi\varepsilon_0 \varepsilon r^2)$, where $r = \sqrt{x^2+l^2/4} = \sqrt{4x^2+l^2}/2$, we write Eq. (3) as

$$E = \frac{2q\cdot 2x\cdot 4}{4\pi\varepsilon_0\varepsilon\,(4x^2+l^2)\sqrt{4x^2+l^2}}$$

$$= \frac{4qx}{\pi\varepsilon_0\varepsilon\,(4x^2+l^2)^{3/2}}. \tag{4}$$

If the field strength at a given point has the maximum value,

$$\frac{dE}{dx} = 0. \tag{5}$$

Substituting expression (4) into (5) and differentiating, we obtain

$$\frac{d}{dx}\left[\frac{4qx}{\pi\varepsilon_0\varepsilon\,(4x^2+l^2)^{3/2}}\right]$$

$$= \frac{4q}{\pi\varepsilon_0\varepsilon}\;\frac{\frac{dx}{dx}(4x^2+l^2)^{3/2} - x\frac{d}{dx}(4x^2+l^2)^{3/2}}{[(4x^2+l^2)^{3/2}]^2} = 0,$$

whence

$$\frac{dx}{dx}(4x^2+l^2)^{3/2}-x\frac{d}{dx}(4x^2+l^2)^{3/2}=0,$$

$$(4x^2+l^2)^{3/2}-x\frac{3}{2}(4x^2+l^2)^{1/2}4\cdot 2x=0,$$

$$(4x^2+l^2)^{1/2}[(4x^2+l^2)-12x^2]=0,\ 4x^2+l^2-12x^2=0,$$

$$x=\pm\sqrt{l^2/8}=\pm l/(2\sqrt{2}),$$

$$x=\pm 0.2/(2\sqrt{2})\ \mathrm{m}\simeq\pm 0.071\ \mathrm{m}.$$

380. A field is produced by an infinite vertical plane with a surface charge density of 4 nC/cm². A ball of mass 1 g bearing a charge of 1 nC is suspended on a string in the field. Determine the angle between the string and the plane.

Given: $\sigma = 4\ \mathrm{nC/cm^2} = 4\times 10^{-5}\ \mathrm{C/m^2}$, $m = 1\ \mathrm{g} = 10^{-3}\ \mathrm{kg}$, $q = 1\ \mathrm{nC} = 10^{-9}\ \mathrm{C}$.

α — ?

Solution. The charged ball suspended on the string in the electric field is acted upon by the force of gravity mg, the electric force **F** with which the field of the plane acts on the charged ball, and the tension **T** of the string (Fig. 125). Writing the equilibrium conditions for the ball in projections on the X- and Y-axes, we obtain

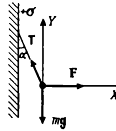


Fig. 125

$F - T\sin\alpha = 0,\quad$ or $\quad F = T\sin\alpha,$

$$-mg + T\cos\alpha = 0, \quad \text{or } mg = T\cos\alpha. \qquad (1)$$

Dividing Eqs. (1) termwise, we get

$$\tan\alpha = F/(mg). \qquad (2)$$

Considering that $F = qE$, $E = \sigma/(2\varepsilon_0\varepsilon)$ and substituting these expressions into Eq. (2), we obtain $\tan\alpha =$

$q\sigma/(2\varepsilon_0\varepsilon mg)$, whence

$$\alpha = \arctan \frac{q\sigma}{2\varepsilon_0\varepsilon mg},$$

$$\alpha = \arctan \frac{10^{-9}\times 4\times 10^{-6}}{2\times 8.85\times 10^{-12}\times 10^{-3}\times 9.8} \simeq \arctan 0.23$$
$$\simeq 0.225\ \text{rad}.$$

381. An electric field is produced by two infinite, parallel planes with surface charge densities of 2 and

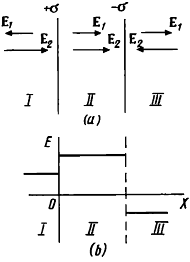


Fig. 126

—4 nC/m² (Fig. 126*a*). Determine the field strength between the planes and beyond them. Plot the graphs of the electric field strengths for regions *I-III*.

Given: $\sigma_1 = 2\ \text{nC/m}^2 = 2\times 10^{-9}\ \text{C/m}^2$, $\sigma_2 = -4\ \text{nC/m}^2 = -4\times 10^{-9}\ \text{C/m}^2$.

E_I—? E_{II}—? E_{III}—?

Solution. Let us consider the regions *I*, *II*, and *III* of the field (see Fig. 126*a*).

Region I. According to the superposition principle, the field strength in this region is

$$\mathbf{E}_I = \mathbf{E}_1 + \mathbf{E}_2, \qquad (1)$$

or in projections on the X-axis,

$$E_I = E_2 - E_1,$$

where $E_2 = |\sigma_2|/(2\varepsilon_0\varepsilon)$ and $E_1 = \sigma_1/(2\varepsilon_0\varepsilon)$. Using these expressions, we write Eq. (1) in the form

$$E_I = \frac{|\sigma_2|}{2\varepsilon_0\varepsilon} - \frac{\sigma_1}{2\varepsilon_0\varepsilon} = \frac{|\sigma_2| - \sigma_1}{2\varepsilon_0\varepsilon},$$

$$E_I = \frac{4\times 10^{-9} - 2\times 10^{-9}}{2\times 8.85\times 10^{-12}\times 1}\,\frac{\mathrm{V}}{\mathrm{m}} \simeq 113\ \mathrm{V/m}.$$

Similarly, we determine the values of the field strength for the other regions.

Region II:

$$E_{II} = E_2 + E_1 = \frac{|\sigma_2|}{2\varepsilon_0\varepsilon} + \frac{\sigma_1}{2\varepsilon_0\varepsilon} = \frac{|\sigma_2| + \sigma_1}{2\varepsilon_0\varepsilon},$$

$$E_{II} = \frac{4\times 10^{-9} + 2\times 10^{-9}}{2\times 8.85\times 10^{-12}\times 1}\,\frac{\mathrm{V}}{\mathrm{m}} \simeq 339\ \mathrm{V/m}.$$

Region III:

$$E_{III} = E_1 - E_2 = \frac{\sigma_1}{2\varepsilon_0\varepsilon} - \frac{|\sigma_2|}{2\varepsilon_0\varepsilon} = \frac{\sigma_1 - |\sigma_2|}{2\varepsilon_0\varepsilon},$$

$$E_{III} = \frac{2\times 10^{-9} - 4\times 10^{-9}}{2\times 8.85\times 10^{-12}\times 1}\,\frac{\mathrm{V}}{\mathrm{m}} \simeq -113\ \mathrm{V/m}.$$

Using the obtained values, we plot the graph of change of the field strength along the X-axis (Fig. 126*b*).

382. As a result of a collision between an air molecule and a cosmic particle, an electron is formed in the vicinity of the negative plate of a parallel-plate capacitor. At what velocity will the electron reach the positive plate if the plate charge is 1 nC, the plate area is 60 cm², and the separation between the plates is 5 mm?

Given: $q = 1\ \mathrm{nC} = 10^{-9}\ \mathrm{C}$, $S = 60\ \mathrm{cm}^2 = 6\times 10^{-3}\ \mathrm{m}^2$, $d = 5\ \mathrm{mm} = 5\times 10^{-2}\ \mathrm{m}$.

---

v — ?

Solution. The electron formed as a result of the collision starts to move in the uniform field of the parallel-plate capacitor with a constant acceleration at zero initial velocity. From the equation $d = at^2/2$ of a uniformly accelerated motion, we determine the time of motion of the electron from one plate to another: $t = \sqrt{2d/a}$. Substituting this expression into the formula $v = at$ for velocity, we obtain

$$v = a\sqrt{2d/a} = \sqrt{2ad}. \quad (1)$$

According to Newton's second law, the acceleration of the electron is

$$a = F/m,$$

where m is the electron mass, $F = |e|E$ the electric force exerted on the electron by the capacitor field, e the electron charge, $E = \sigma/(\varepsilon_0\varepsilon)$ the electric field strength within the capacitor, and $\sigma = q/S$ the surface charge density on the capacitor plates. Using these expressions in Eq. (1), we obtain

$$v = \sqrt{\frac{2|e|qd}{S\varepsilon_0\varepsilon m}},$$

$$v = \sqrt{\frac{2\times 1.6\times 10^{-19}\times 10^{-9}\times 5\times 10^{-3}}{6\times 10^{-3}\times 8.85\times 10^{-12}\times 1\times 9.1\times 10^{-31}}}\ \frac{\text{m}}{\text{s}}$$

$$\simeq 5.76\times 10^{6}\ \text{m/s}.$$

383. Explain why a light pith ball is first attracted to an electrostatically charged rod and is then repelled from it?

Answer. When the charged rod is brought to the ball, charges of opposite signs are induced on the ball. A charge opposite to that on the rod is induced on the side facing the rod, while the charge induced on the opposite side of the ball is of the same sign as that on the rod. The force of attraction between unlike charges is stronger than the force of repulsion of like charges since the unlike charges on the ball and the rod are closer than the like charges. When the ball and the rod come in contact, the charge on the ball opposite to that on the rod is neutralized. The ball acquires a charge of the same sign as that on the rod and is repelled from it.

EXERCISES

384. What charge must be imparted to two balls of mass 1 g each for the force of mutual repulsion of the charges to balance the gravitational force of mutual attraction of the balls? The balls are in air.

385. Two identically charged balls of mass 0.5 g each and suspended on strings of length 1 m are deflected 4 cm apart. Determine the charge of each ball.

386. A steel ball of radius 0.5 cm is immersed in kerosene and is in a uniform electric field of strength 35 kV/cm, which is directed vertically upwards. Determine the charge on the ball if it is in equilibrium.

387. A uniformly charged wire ring of radius 10 cm carries a charge of —5 nC. Determine the electric field strength on the axis of the ring at points separated from its centre by 0, 5, 8, 10, and 15 cm. Plot the graph of dependence of the field strength on the distance from the centre of the ring.

388. A small ball of mass 100 mg and charge 16.7 nC is suspended on a string. To what distance must an identical charge be brought to it from below for the tensile force to be reduced by half?

389. What is the ratio of the gravitational force acting between two protons to the force of their Coulomb repulsion?

390. Three negative charges of 9 nC each are arranged at the vertices of an equilateral triangle. What charge must be placed at the centre of the triangle for the system to be in equilibrium?

391. A ball having a mass of 0.4 g and a charge of 4.9 nC is suspended on a string in the field of a parallel-plate air capacitor with a charge of 4.43 nC and a plate area of 50 cm^2. By what angle will the string with the ball be deflected from the vertical?

392. Two like-charged balls suspended on identical strings deflect through a certain angle. What is the density of the material of which the balls are made if the angle of deflection of the strings remains unchanged after their immersion in kerosene?

ELECTROSTATIC FIELD POTENTIAL. WORK DONE IN MOVING A CHARGE IN AN ELECTROSTATIC FIELD

The **electrostatic field potential** is defined as

$$\varphi = -A/q,$$

where A is the work done by the field forces in moving a charge from infinity to a given point in the field.

The **potential of the field produced by a point charge** or **a charged sphere** is

$$\varphi = Q/(4\pi\varepsilon_0 \varepsilon r).$$

If a field is produced by several point charges, the potential of the field at a given point is equal to the algebraic sum of the potentials of the fields produced at this point by each charge separately:

$$\varphi = \sum_{i=1}^{n} \varphi_i.$$

The relation between the strength E of an electrostatic field and the **potential difference** U across the plates of a parallel-plate capacitor (uniform field) is as follows:

$$E = (\varphi_1 - \varphi_2)/d = U/d,$$

where d is the separation between the plates of the parallel-plate capacitor.

The **work done by an electric force in moving a charge from one point of an electrostatic field to another** is

$$A = q(\varphi_1 - \varphi_2) = qU.$$

* * *

393. A dust particle of mass 10^{-8} g is suspended between the plates of a parallel-plate air capacitor to which a voltage of 5 kV is applied. The separation between the plates is 5 cm. What is the charge of the particle?

Given: $m = 10^{-8}$ g $= 10^{-11}$ kg, $U = 5$ kV $= 5 \times 10^3$ V, $d = 5 \times 10^{-2}$ m.

q — ?

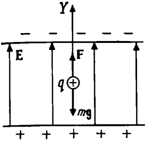


Fig. 127

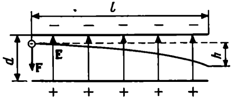


Fig. 128

Solution. In the electrostatic field, the dust particle experiences the action of the force of gravity mg and the electric force **F** exerted by the field (Fig. 127). Writing the equilibrium condition for the particle in projections on the Y-axis, we obtain

$$F - mg = 0,$$

whence $F = mg$.

Considering that $F = qE$ and $E = U/d$, we get $qU/d = mg$, whence

$$q = \frac{mgd}{U},$$

$$q = \frac{10^{-11} \times 9.8 \times 5 \times 10^{-2}}{5 \times 10^{3}}\ \mathrm{C} = 9.8 \times 10^{-16}\ \mathrm{C}.$$

394. An electron flies into a parallel-plate air capacitor at a velocity of 6×10^7 m/s parallel to the plates. The separation between the plates is 1 cm and the potential difference is 600 V. Determine the deflection of the electron caused by the field of the capacitor if the length of its plate is 5 cm.

Given: $v = 6 \times 10^7$ m/s, $d = 1$ cm $= 10^{-2}$ m, $U = 600$ V, $l = 5$ cm $= 5 \times 10^{-2}$ m.

h — ?

Solution. The force exerted by the electric field on the electron is

$$F = |e|E,$$

where $E = U/d$.

Since the electric field strength is directed upwards (Fig. 128), the force acting on the electron is directed downwards. We can represent the motion of the electron as a superposition of two independent motions in the horizontal and vertical directions. In the horizontal direction, the electron, as before, will move uniformly since no force is acting on it in this direction. At the same time, it will be displaced with a uniform downward acceleration due to the electric force. The trajectory of its motion will be a parabola. The motion of the electron in the capacitor is similar to the motion of the body thrown along the horizontal. During the time of motion, the electron will traverse along the horizontal a distance

$$l = vt, \qquad (1)$$

and will be displaced along the vertical by a distance

$$h = at^2/2, \qquad (2)$$

where a is the electron acceleration.

Solving Eqs. (1) and (2) together, we find that

$$h = al^2/(2v^2). \qquad (3)$$

In order to calculate the acceleration, we apply Newton's second law. Since only one force F acts on the electron in the vertical direction (we neglect the force of gravity acting on the electron), we have $F = ma$, whence $a = F/m$, or

$$a = |e| U/(dm). \qquad (4)$$

Substituting expression (4) into (3), we obtain

$$h = \frac{l^2 |e| U}{2v^2 dm},$$

$$h = \frac{(5\times10^{-2})^2\times1.6\times10^{-19}\times600}{2\times(6\times10^7)^2\times10^{-2}\times9.1\times10^{-31}}\ \text{m}$$

$$\simeq 3.65\times10^{-3}\ \text{m}.$$

395. Two balls bearing charges of 6.7 and 13.3 nC are at a distance of 40 cm from each other. What work should be done to bring the separation down to 25 cm?

Given: $q_1 = 6.7\ \text{nC} = 6.7 \times 10^{-9}\ \text{C}$, $q_2 = 13.3\ \text{nC} = 13.3 \times 10^{-9}\ \text{C}$, $r_1 = 40\ \text{cm} = 0.4\ \text{m}$, $r_2 = 25\ \text{cm} = 0.25\ \text{m}$.

A_{ext} — ?

Solution. In this type of problems, it is convenient to assume that one ball is stationary and the other ball moves in the field produced by the first ball. Let the ball bearing the charge q_1 produce a field, while the ball bearing the charge q_2 move in the field from the point at a distance r_1 from the charge q_1 to the point at a distance r_2 from it. The work done by an external force is

$$A_{ext} = -A = q_2 (\varphi_2 - \varphi_1), \tag{1}$$

where φ_1 and φ_2 are the potentials of the initial and final points of the field. Since the field is produced by a point charge, we can write

$$\varphi_1 = \frac{q_1}{4\pi\varepsilon_0 \varepsilon r_1}, \quad \varphi_2 = \frac{q_1}{4\pi\varepsilon_0 \varepsilon r_2}. \tag{2}$$

Substituting expressions (2) into (1), we obtain

$$A_{ext} = q_2 \left(\frac{q_1}{4\pi\varepsilon_0 \varepsilon r_2} - \frac{q_1}{4\pi\varepsilon_0 \varepsilon r_1} \right) = \frac{q_1 q_2 (r_1 - r_2)}{4\pi\varepsilon_0 \varepsilon r_1 r_2},$$

$$A_{ext} = \frac{6.7 \times 10^{-9} \times 13.3 \times 10^{-9} \times (0.4 - 0.25)}{4 \times 3.14 \times 8.85 \times 10^{-12} \times 1 \times 0.4 \times 0.25}\ \text{J} \simeq 1.2\ \mu\text{J}.$$

396. A point charge of 10^{-8} C is at a distance of 50 cm from the surface of a sphere of radius 9 cm, charged to a potential of 25 kV. What work must be done to decrease the separation between the sphere and the charge to 20 cm?

Given: $l_1 = 50\ \text{cm} = 0.5\ \text{m}$, $R = 9\ \text{cm} = 0.09\ \text{m}$, $\varphi_{sph} = 25\ \text{kV} = 25 \times 10^3\ \text{V}$, $q = 10^{-8}\ \text{C}$, $l_2 = 20\ \text{cm} = 0.2\ \text{m}$.

A — ?

Solution. Let the charge q move from point B to point C in the electrostatic field produced by the charged sphere (Fig. 129). The work done in moving the charge in the electrostatic field is

$$A_{el} = q (\varphi_B - \varphi_C). \tag{1}$$

Considering that the potentials of the field produced by the charged sphere at points B and C are $\varphi_B = q_{\mathrm{sph}}/[4\pi\varepsilon_0\varepsilon(R+l_1)]$ and $\varphi_C = q_{\mathrm{sph}}/[4\pi\varepsilon_0\varepsilon(R+l_2)]$, we write Eq. (1) in the form

$$A_{\mathrm{el}} = q\left[\frac{q_{\mathrm{sph}}}{4\pi\varepsilon_0\varepsilon(R+l_1)} - \frac{q_{\mathrm{sph}}}{4\pi\varepsilon_0\varepsilon(R+l_2)}\right]$$

$$= \frac{qq_{\mathrm{sph}}(l_2-l_1)}{4\pi\varepsilon_0\varepsilon(R+l_1)(R+l_2)}. \qquad (2)$$

The potential of the field on the surface of the sphere is

$$\varphi_{\mathrm{sph}} = \frac{q_{\mathrm{sph}}}{4\pi\varepsilon_0\varepsilon R},$$

whence the charge on the sphere is

$$q_{\mathrm{sph}} = \varphi_{\mathrm{sph}}\cdot 4\pi\varepsilon_0\varepsilon R. \qquad (3)$$

Substituting Eq. (3) into (2), we find that

$$A_{\mathrm{el}} = \frac{q\varphi_{\mathrm{sph}}\cdot 4\pi\varepsilon_0\varepsilon R(l_2-l_1)}{4\pi\varepsilon_0\varepsilon(R+l_1)(R+l_2)}$$

$$= qR\varphi_{\mathrm{sph}}\frac{l_2-l_1}{(R+l_1)(R+l_2)},$$

$$A_{\mathrm{el}} = 10^{-8}\times 9\times 10^{-2}\times 25\times 10^{3}\times\frac{0.2-0.5}{(0.09+0.5)(0.09+0.2)}\ \mathrm{J}$$

$$\simeq -39.5\,\mu\mathrm{J}.$$

The minus sign indicates that the electric force prevents the charge from moving, i.e. is directed against the motion. In order to displace the charge, an external force must be applied to it in the direction of motion. The work

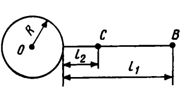


Fig. 129

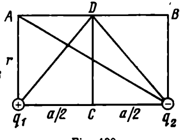


Fig. 130

done by this force will be $A = -A_{el}$. Consequently, $A = 39.5$ μJ.

397. Determine the work done by electric forces in moving a charge of 1 nC from point A to B and from point C to D if $r = 6$ cm, $a = 8$ cm, $q_1 = 3.33$ nC, and $q_2 = -3.33$ nC (Fig. 130).

Given: $q = 1\,\text{nC} = 10^{-9}\,\text{C}$, $r = 6\,\text{cm} = 6 \times 10^{-2}\,\text{m}$, $a = 8\,\text{cm} = 8 \times 10^{-2}\,\text{m}$, $q_1 = 3.33\,\text{nC} = 3.33 \times 10^{-9}\,\text{C}$, $q_2 = -3.33\,\text{nC} = -3.33 \times 10^{-9}\,\text{C}$.

$A - ?$

Solution. 1. The work done by an electric force in moving the charge q from point A to B is

$$A_{AB} = q\,(\varphi_A - \varphi_B). \tag{1}$$

Here $\varphi_A = \varphi_{A_1} + \varphi_{A_2} = q_1/(4\pi\varepsilon_0\varepsilon r_1) + q_2/(4\pi\varepsilon_0\varepsilon r_2)$ and $\varphi_B = \varphi_{B_1} + \varphi_{B_2} = q_1/(4\pi\varepsilon_0\varepsilon r_2) + q_2/(4\pi\varepsilon_0\varepsilon r_1)$ are the potentials of the field produced by the charges q_1 and q_2 at points A and B, where $r_1 = r$ and $r_2 = \sqrt{r^2 + a^2}$. Substituting the expressions for φ_A, φ_B, r_1, and r_2 into Eq. (1), we obtain

$$A_{AB} = q\left[\left(\frac{q_1}{4\pi\varepsilon_0\varepsilon r} + \frac{q_2}{4\pi\varepsilon_0\varepsilon\sqrt{r^2+a^2}}\right) - \left(\frac{q_1}{4\pi\varepsilon_0\varepsilon\sqrt{r^2+a^2}} + \frac{q_2}{4\pi\varepsilon_0\varepsilon r}\right)\right]$$

$$= \frac{q\,(q_1 - q_2)\,(\sqrt{r^2+a^2} - r)}{4\pi\varepsilon_0\varepsilon r\sqrt{r^2+a^2}},$$

$$A_{AB} = \frac{10^{-9} \times (3.33 \times 10^{-9} + 3.33 \times 10^{-9})}{4 \times 3.14 \times 8.85 \times 10^{-12} \times 1 \times 6 \times 10^{-2}} \times \frac{(\sqrt{(6 \times 10^{-2})^2 + (8 \times 10^{-2})^2} - 6 \times 10^{-2})}{\sqrt{(6 \times 10^{-2})^2 + (8 \times 10^{-2})^2}}\ \text{J}$$

$$\simeq 8 \times 10^{-7}\ \text{J}.$$

2. The work done by an electric force in moving the charge q from point C to D is

$$A_{CD} = q\,(\varphi_C - \varphi_D). \tag{2}$$

Here $\varphi_C = \varphi_{C_1} + \varphi_{C_2} = q_1/(4\pi\varepsilon_0\varepsilon r_3) + q_2/(4\pi\varepsilon_0\varepsilon r_3)$
and $\varphi_D = \varphi_{D_1} + \varphi_{D_2} = q_1/(4\pi\varepsilon_0\varepsilon r_4) + q_2/(4\pi\varepsilon_0\varepsilon r_4)$

are the potentials of the field produced by the charges q_1 and q_2 at points C and D, where $r_3 = a/2$ and $r_4 = \sqrt{r^2 + a^2/4}$. Substituting the expressions for φ_C, φ_D, r_3, and r_4 into Eq. (2), we obtain

$$A_{CD} = q\left[\left(\frac{q_1}{4\pi\varepsilon_0\varepsilon a/2} + \frac{q_2}{4\pi\varepsilon_0\varepsilon a/2}\right) - \left(\frac{q_1}{4\pi\varepsilon_0\varepsilon\sqrt{r^2+a^2/4}} + \frac{q_2}{4\pi\varepsilon_0\varepsilon\sqrt{r^2+a^2/4}}\right)\right]$$
$$= \frac{2q\,(q_1+q_2)\,(\sqrt{r^2+a^2/4} - a/2)}{4\pi\varepsilon_0\varepsilon a\sqrt{r^2+a^2/4}}.$$

Since $q_1 = -q_2$, we have $(q_1 + q_2) = 0$ and $A_{CD} = 0$.

398. What is the velocity of an electron that has crossed an accelerating potential difference of 200 V (Fig. 131)?

Given: $U = 200$ V.

v — ?

Solution. In the electric field, the electron experiences the action of the force $F = |e|E$ which makes it fly with

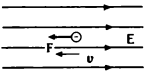


Fig. 131

a uniform acceleration against the electric field vector. The electric force does thereby a work on the electron, which is equal to the change in its kinetic energy:

$$A = \Delta W_k = W_k - W_{k_0}.$$

Considering that $A = |e|U$, $W_k = mv^2/2$, and $W_{k_0} = 0$, we obtain $|e|U = mv^2/2$, whence

$$v = \sqrt{\frac{2|e|U}{m}},$$

$$v = \sqrt{\frac{2\times1.6\times10^{-19}\times200}{9.1\times10^{-31}}}\ \frac{\text{m}}{\text{s}} \simeq 8.4\times10^6\ \text{m/s}.$$

399. A ball of mass 1 g moves from point A whose potential is 600 V to point B whose potential is zero. Determine the velocity of the ball at point A if its velocity at point B is 20 cm/s. The charge of the ball is 10 nC.

Given: $m = 1\ \text{g} = 10^{-3}$ kg, $\varphi_1 = 600$ V, $\varphi_2 = 0$, $v = 20$ cm/s $= 0.2$ m/s, $q = 10$ nC $= 10^{-8}$ C.

v_0 — ?

Solution. The ball moves under the action of the electric force exerted by the field. The change in the kinetic energy of the ball is equal to the work done by the electric force:

$$\Delta W_k = A. \tag{1}$$

Since $\Delta W_k = mv^2/2 - mv_0^2/2$ and $A = q(\varphi_1 - \varphi_2)$, Eq. (1) can be written in the form $mv^2/2 - mv_0^2/2 = q(\varphi_1 - \varphi_2)$, whence

$$v_0 = \sqrt{v^2 - 2q(\varphi_1 - \varphi_2)/m},$$

$$v_0 = \sqrt{0.2^2 - 2 \times 10^{-8} \times 600 \times 10^3}\ \text{m/s} \simeq 0.17\ \text{m/s}.$$

400. A ball of mass 40 mg, having a charge of 1 nC, moves from infinity at a velocity of 10 cm/s. To what minimum distance can the ball approach a point charge equal to 1.33 nC?

Given: $m = 40\ \text{mg} = 4 \times 10^{-5}$ kg, $q_1 = 1$ nC $= 10^{-9}$ C, $v = 10$ cm/s $= 0.1$ m/s, $q_2 = 1.33$ nC $= 1.33 \times 10^{-9}$ C.

r — ?

Solution. The charged ball experiences the action of the force exerted by the electric field. The work done by the force is equal to the change in the kinetic energy of the ball:

$$\Delta W_k = A.$$

Since $\Delta W_k = W_k - W_{k_0} = 0 - mv^2/2 = -mv^2/2$, $A = q_1(\varphi_\infty - \varphi_1) = q_1[0 - q_2/(4\pi\varepsilon_0 \varepsilon r)] = -q_1 q_2/(4\pi\varepsilon_0 \varepsilon r)$, we have $-mv^2/2 = -q_1 q_2/(4\pi\varepsilon_0 \varepsilon r)$,

whence

$$r = \frac{q_1 q_2}{2\pi\varepsilon_0 e m v^2},$$

$$r = \frac{10^{-9} \times 1.33 \times 10^{-9}}{2 \times 3.14 \times 8.85 \times 10^{-12} \times 4 \times 10^{-5} \times 0.1^2} \text{ m} \simeq 6 \times 10^{-2} \text{ m}.$$

401°. An electron moves in the field produced by a charged sphere of radius 10 cm along the radius connecting the points separated by 12 and 15 cm from the centre of the sphere. The velocity of the electron changes thereby from 2×10^5 to 2×10^6 m/s. Determine the surface charge density of the sphere.

Given: $R = 10\text{ cm} = 0.1\text{ m}$, $r_1 = 12\text{ cm} = 0.12\text{ m}$, $r_2 = 15\text{ cm} = 0.15\text{ m}$, $v_1 = 2 \times 10^5\text{ m/s}$, $v_2 = 2 \times 10^6\text{ m/s}$.

σ — ?

Solution. During the motion of the electron, the electric field does a work equal to the change in the electron kinetic energy:

$$A = \Delta W = W_2 - W_1 = \frac{mv_2^2}{2} - \frac{mv_1^2}{2}$$

$$= \frac{m}{2}(v_2^2 - v_1^2). \qquad (1)$$

On the other hand, the force exerted by the electric field on the moving electron is

$$F = |e| E,$$

where $E = q/(4\pi\varepsilon_0 e r^2)$ is the electric field strength produced by the charged sphere. Hence it is clear that the electric force depends on the distance r and varies during the motion of the electron in the field. Therefore, the work done by the force is

$$A = \int_{r_1}^{r_2} F \cos\alpha \, dr,$$

or, taking into account the expressions for F and E and considering that $\cos\alpha = 1$ and the charge on the sphere is $q = \sigma 4\pi R^2$, we have

$$A = \int_{r_1}^{r_2} \frac{\sigma 4\pi R^2 \mid e \mid dr}{4\pi\varepsilon_0 \varepsilon r^2}.$$

Taking the constant quantities outside the integral sign and integrating, we obtain

$$A = \frac{\sigma R^2 \mid e \mid}{\varepsilon_0 \varepsilon} \int_{r_1}^{r_2} \frac{dr}{r^2} = \frac{\sigma R^2 \mid e \mid}{\varepsilon_0 \varepsilon} \left(-\frac{1}{r}\right)\Bigg|_{r_1}^{r_2}$$

$$= \frac{\sigma R^2 \mid e \mid (r_2 - r_1)}{\varepsilon_0 \varepsilon r_1 r_2}. \qquad (2)$$

Equating the right-hand sides of Eqs. (1) and (2), we find that

$$\frac{m(v_2^2 - v_1^2)}{2} = \frac{\sigma R^2 \mid e \mid (r_2 - r_1)}{\varepsilon_0 \varepsilon r_1 r_2},$$

whence

$$\sigma = \frac{\varepsilon_0 \varepsilon r_1 r_2 m (v_2^2 - v_1^2)}{2R^2 \mid e \mid (r_2 - r_1)},$$

$$\sigma = \frac{8.85 \times 10^{-12} \times 0.12 \times 0.15 \times 9.1 \times 10^{-31} \times [(2 \times 10^6)^2 - (2 \times 10^5)^2]}{2 \times 0.1^2 \times 1.6 \times 10^{-19} \times (0.15 - 0.12)} \frac{\text{C}}{\text{m}^2}$$

$\simeq 5.96$ nC/m^2.

402. Identical charges are imparted to two metal spheres of different radii. Will the charges flow from one sphere to another after their connection with a conductor?

Answer. Since the potentials of the spheres are different $[\varphi = Q/(4\pi\varepsilon_0 \varepsilon R)]$, after the connection the charge will flow from the sphere with a higher potential (smaller radius) to the sphere with a lower potential (larger radius) until the potentials of the spheres assume the same value.

EXERCISES

403. Two parallel plane plates separated by 10 cm are charged to a potential difference of 1 kV. What force will act on a charge of 10^{-4} C placed between the plates?

404. Two charges of 1 C and of -6.67 nC are at a distance of 10 cm from each other. What work should be done to transfer the second charge to a point separated from the first charge by 1 m?

405. A point mass bearing a charge of 0.67 nC and moving in an accelerating electric field acquires a kinetic energy of 10^7 eV. Determine the potential difference between the initial and final points of the trajectory of the particle in the field if its initial kinetic energy is zero.

406. As a result of a radioactive decay of a polonium atom, an alpha-particle is emitted at a velocity of 1.6×10^7 m/s. What potential difference should be applied to impart the same velocity to the alpha-particle?

407. Determine the force of mutual repulsion between two balls in air if each ball is charged to a potential of 600 V. The diameter of each ball is 1 cm and the distance between their centres is 20 cm.

408. A field is produced by a thin rod which is bent to form a half-ring and is uniformly charged with a linear density of 20 nC/m. A point charge of -1 nC is placed at the centre of the half-ring. Determine the work that must be done to remove the charge from the centre of the half-ring to infinity.

409. A small ball bearing a charge of 6.67 nC is at the centre of a hollow metal sphere of radius 1 m carrying a charge of 3.34 nC. Determine the potentials of the field at points separated from the centre of the sphere by 0.5, 1, and 10 m.

410. Point charges of 1.33, -0.66, 0.99, and -1.32 nC are arranged at the vertices of a square. Determine the potential of the field at the centre of the square if its diagonal is 20 cm.

411. Determine the potential of the point of a field produced by a metal sphere with a surface charge density of 10^{-11} C/cm^2 and having a radius of 1 cm if the distance from this point to the surface of the sphere is 9 cm.

Capacitance. The Energy of a Charged Conductor, Capacitor, and of an Electrostatic Field

The **capacitance of a conductor** is

$$C = q/\varphi.$$

The **capacitance of a sphere** is

$$C = 4\pi\varepsilon\varepsilon_0 R,$$

where R is the radius of the sphere.

The **capacitance of a parallel-plate capacitor** is

$$C = \varepsilon\varepsilon_0 S/d,$$

where S is the area of a capacitor plate and d the separation between the plates.

The **capacitance of a capacitor bank** consisting of parallel-connected capacitors is

$$C = \sum_{i=1}^{n} C_i$$

and that of series-connected capacitors is

$$\frac{1}{C} = \sum_{i=1}^{n} \frac{1}{C_i},$$

where C_i is the capacitance of a capacitor and n the number of capacitors in a bank.

The **electric energy of a charged conductor** is

$$W = (1/2)\,q\varphi = (1/2)\,C\varphi^2.$$

The **electric energy of a charged capacitor** is

$$W = (1/2)\,qU = (1/2)\,CU^2.$$

The **energy of a (uniform) electrostatic field** is

$$W = (1/2)\,\varepsilon_0\varepsilon E^2 V,$$

where V is the volume occupied by the field.

The **volume energy density of a field** is

$$w = (1/2)\,\varepsilon_0\varepsilon E^2.$$

* * *

412. A metal sphere of radius 5 cm is charged to a potential of 150 V. Determine the potential and field strength at point A separated from the surface of the sphere by 10 cm.

Given: $R = 5\ \text{cm} = 5 \times 10^{-2}\ \text{m}$, $\varphi_{sph} = 150\ \text{V}$,
$d = 10\ \text{cm} = 0.1\ \text{m}$.

φ—? E—?

Solution. The potential of the field produced by the charged sphere at point A (Fig. 132) is

$$\varphi = q/(4\pi\varepsilon_0\varepsilon r), \tag{1}$$

where q is the charge on the sphere and $r = R + d$ the distance from the centre O of the sphere to point A.

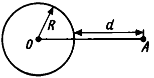


Fig. 132

The capacitance of the sphere is

$$C_{sph} = 4\pi\varepsilon_0\varepsilon R. \tag{2}$$

On the other hand, $C_{sph} = q/\varphi_{sph}$, whence $q = C_{sph}\varphi_{sph}$, or, taking into account Eq. (2),

$$q = 4\pi\varepsilon_0\varepsilon R\varphi_{sph}. \tag{3}$$

Substituting expression (3) into (1), we find that

$$\varphi = \frac{4\pi\varepsilon_0\varepsilon R\varphi_{sph}}{4\pi\varepsilon_0\varepsilon r} = \frac{R\varphi_{sph}}{r} = \frac{R\varphi_{sph}}{R+d},$$

$$\varphi = \frac{5\times10^{-2}\times150}{5\times10^{-2}+0.1}\ \text{V} = 50\ \text{V}.$$

The strength of the field produced by the charged sphere at point A is

$$E = q/(4\pi\varepsilon_0\varepsilon r^2),$$

or, taking into account Eq. (3),

$$E = \frac{4\pi\varepsilon_0\varepsilon R\varphi_{sph}}{4\pi\varepsilon_0\varepsilon r^2} = \frac{R\varphi_{sph}}{r^2} = \frac{R\varphi_{sph}}{(R+d)^2},$$

$$E = \frac{5\times10^{-2}\times150}{(5\times10^{-2}+0.1)^2}\ \frac{\text{V}}{\text{m}} \simeq 3.3\times10^2\ \text{V/m}.$$

413. Two spheres of radius 5 and 8 cm are charged to a potential of 120 and 50 V respectively and connected by a wire. Determine the potentials of the spheres after they have been connected, and the charge that has passed from one sphere to the other.

Given: $R_1 = 5\ \text{cm} = 5\times10^{-2}\ \text{m}$, $R_2 = 8\ \text{cm} = 8\times10^{-2}\ \text{m}$, $\varphi_1 = 120\ \text{V}$, $\varphi_2 = 50\ \text{V}$.

φ — ? Δq — ?

Solution. Knowing the radii of the spheres, we can write the expressions for their capacitances: $C_1 = 4\pi\varepsilon_0\varepsilon R_1$ and $C_2 = 4\pi\varepsilon_0\varepsilon R_2$. Then the charge on each sphere before the connection is

$$q_1 = C_1\varphi_1 = 4\pi\varepsilon_0\varepsilon R_1\varphi_1, \qquad q_2 = C_2\varphi_2 = 4\pi\varepsilon_0\varepsilon R_2\varphi_2 \qquad (1)$$

respectively. The total charge on the spheres before the connection is

$$\begin{aligned} q_1 + q_2 &= 4\pi\varepsilon_0\varepsilon R_1\varphi_1 + 4\pi\varepsilon_0\varepsilon R_2\varphi_2 \\ &= 4\pi\varepsilon_0\varepsilon\,(R_1\varphi_1 + R_2\varphi_2). \qquad (2) \end{aligned}$$

After the connection, the charges on the spheres will be redistributed: a part of the charge from the sphere with a higher potential will pass to the sphere with a lower potential. As a result, the potentials of the spheres equalize and become equal to φ. Consequently, expressions (1) and (2) after the connection become

$$q_1' = 4\pi\varepsilon_0\varepsilon R_1\varphi, \quad q_2' = 4\pi\varepsilon_0\varepsilon R_2\varphi,$$

$$\begin{aligned} q_1' + q_2' &= 4\ \pi\varepsilon_0\varepsilon R_1\varphi + 4\pi\varepsilon_0\varepsilon R_2\varphi \\ &= 4\pi\varepsilon_0\varepsilon\,(R_1 + R_2)\,\varphi. \qquad (3) \end{aligned}$$

However, the total charge on the spheres does not change as a result of the connection. Consequently, $q_1 + q_2 =$

$b_1' + q_2'$, or, taking Eqs. (2) and (3) into account,

$$4\pi\varepsilon_0\varepsilon(R_1\varphi_1 + R_2\varphi_2) = 4\pi\varepsilon_0\varepsilon(R_1 + R_2)\varphi,$$

whence

$$\varphi = \frac{\varphi_1 R_1 + \varphi_2 R_2}{R_1 + R_2},$$

$$\varphi = \frac{120\times 5\times 10^{-2} + 50\times 8\times 10^{-2}}{5\times 10^{-2} + 8\times 10^{-2}}\ \text{V} \simeq 77\ \text{V}.$$

Knowing φ, we can determine the charge that has passed from one sphere to the other:

$$\Delta q = q_1 - q_1' = 4\pi\varepsilon_0\varepsilon R_1\varphi_1 - 4\pi\varepsilon_0\varepsilon R_1\varphi$$
$$= 4\pi\varepsilon_0\varepsilon R_1(\varphi_1 - \varphi),$$
$$\Delta q = 4\times 3.14\times 8.85\times 10^{-12}\times 1\times 5\times 10^{-2}\times(120-77)\ \text{C}$$
$$= 2.4\times 10^{-10}\ \text{C}.$$

414. Three charged water drops of radius 1 mm each merge into one large drop. Determine the potential of the large drop if each small drop carries a charge of 10^{-10} C.

Given: $n = 3$, $r = 1\ \text{mm} = 10^{-3}\ \text{m}$, $q = 10^{-10}\ \text{C}$.

φ — ?

Solution. The potential of the large drop is

$$\varphi = Q/C, \tag{1}$$

where $Q = nq$ and $C = 4\pi\varepsilon_0\varepsilon R$.

The radius R of the large drop can be determined from the mass conservation law:

$$M = nm, \tag{2}$$

where $m = \rho V = (4/3)\rho\pi r^3$ is the mass of a small drop and $M = \rho V_1 = (4/3)\rho\pi R^3$ the mass of the large drop. Using these expressions, we reduce Eq. (2) to the form $R^3 = nr^3$, whence $R = r\sqrt[3]{n}$. Substituting the expressions for Q, C, and R into Eq. (1), we obtain

$$\varphi = \frac{nq}{4\pi\varepsilon_0\varepsilon r\sqrt[3]{n}},$$

$$\varphi = \frac{3\times 10^{-10}}{4\times 3.14\times 8.85\times 10^{-12}\times 1\times 10^{-3}\sqrt[3]{3}}\ \text{V} \simeq 1.87\ \text{kV}.$$

415. The area of a plate of a parallel-plate air capacitor is 60 cm^2, the capacitor charge is 1 nC, and the potential difference across its plates is 90 V. Determine the distance between the capacitor plates.

Given: $S = 60\ cm^2 = 6 \times 10^{-3}\ m^2$, $q = 1\ nC = 10^{-9}\ C$, $U = 90\ V$.

$d - ?$

Solution. The capacitance of a parallel-plate capacitor is

$$C = \varepsilon_0 \varepsilon S/d, \quad \text{or} \quad C = q/U.$$

Equating the right-hand sides of these expressions, we obtain $\varepsilon_0 \varepsilon S/d = q/U$, whence

$$d = \frac{US\varepsilon_0\varepsilon}{q},$$

$$d = \frac{90 \times 6 \times 10^{-3} \times 8.85 \times 10^{-12} \times 1}{10^{-9}}\ m \simeq 4.8 \times 10^{-3}\ m.$$

416. The plates of a parallel-plate capacitor are insulated from each other by an insulator layer. The capacitor is charged to a potential difference of 1 kV and disconnected from the voltage source. Determine the permittivity of the insulator if the potential difference across the capacitor plates increases to 3 kV after its removal.

Given: $U_1 = 1\ kV = 10^3\ V$, $U_2 = 3\ kV = 3 \times 10^3\ V$.

$\varepsilon - ?$

Solution. Since the capacitor is disconnected from the voltage source, the charge on its plates will be the same in both cases: $q_1 = q_2$. Considering that $q_1 = U_1C_1$ and $q_2 = U_2C_2$, we obtain $U_1C_1 = U_2C_2$, whence

$$U_2/U_1 = C_1/C_2. \qquad (1)$$

Using the expressions for the capacitance of a parallel-plate capacitor $C_1 = \varepsilon_1\varepsilon_0 S/d$ and $C_2 = \varepsilon_2\varepsilon_0 S/d$, we get

$$C_1/C_2 = \varepsilon_1/\varepsilon_2. \qquad (2)$$

Comparing Eqs. (1) and (2), we find that $U_2/U_1 = \varepsilon_1/\varepsilon_2$, whence

$$\varepsilon_1 = \frac{U_2}{U_1}\,\varepsilon_2,$$

$$\varepsilon_1 = \frac{3\times 10^3 \times 1}{10^3} = 3.$$

417. A parallel-plate air capacitor with a plate separation of 5 cm is charged to 200 V and disconnected from a voltage source. What will be the voltage across the capacitor after its plates have been moved apart to 10 cm?

Given: $d_1 = 5\ \text{cm} = 5\times 10^{-2}\ \text{m}$, $U_1 = 200\ \text{V}$,
$d_2 = 10\ \text{cm} = 0.1\ \text{m}$.

$U_2 - ?$

Solution. The capacitance of the capacitor and the voltage across it are

$$C_1 = \varepsilon_0\varepsilon S/d_1, \quad U_1 = Q/C_1 = Qd_1/(\varepsilon_0\varepsilon S) \qquad (1)$$

before moving the plates apart and

$$C_2 = \varepsilon_0\varepsilon S/d_2, \quad U_2 = Q/C_2 = Qd_2/(\varepsilon_0\varepsilon S) \qquad (2)$$

after that.

We assume that the charge on the capacitor does not change as a result of moving the plates apart since the capacitor is disconnected from the voltage source. Dividing expressions (1) by (2) termwise, we obtain $U_1/U_2 = d_1/d_2$, whence

$$U_2 = \frac{d_2U_1}{d_1},$$

$$U_2 = \frac{0.1\times 200}{5\times 10^{-2}}\ \text{V} = 400\ \text{V}.$$

418. The distance between the plates of a parallel-plate air capacitor connected to a voltage source of emf 180 V is 5 mm and the area of the plates is 175 cm^2. Determine the work done in moving the plates apart to 12 mm if (1) the capacitor is disconnected from the voltage source; (2) the capacitor is connected to the voltage source all the time.

Given: $\mathscr{E} = 180$ V, $d_1 = 5$ mm $= 5 \times 10^{-3}$ m, $d_2 = 12$ mm $=$ 12×10^{-3} m, $S = 175$ cm$^2 = 1.75 \times 10^{-2}$ m^2.

A_1—? A_2—?

Solution. 1. The work done in this case is measured by the change in the energy of the capacitor, i.e.

$$A_1 = \Delta W = W_2 - W_1, \quad (1)$$

where $W_1 = q^2/(2C_1)$ and $C_1 = \varepsilon_0 \varepsilon S/d_1$ are the energy and the capacitance of the capacitor before moving the plates apart, and $W_2 = q^2/(2C_2)$ and $C_2 = \varepsilon_0 \varepsilon S/d_2$ the energy and the capacitance after moving the plates apart. If the capacitor is disconnected from the source, the charge q on the plates remains unchanged when they are placed apart,

$$q = C_1 U_1 = \varepsilon_0 \varepsilon S U_1/d_1,$$

or, since $U_1 = \mathscr{E}$ before moving the plates apart,

$$q = \varepsilon_0 \varepsilon S \mathscr{E}/d_1.$$

Using the expressions for W_1, W_2, C_1, C_2, and q, we write Eq. (1) as follows:

$$A_1 = \frac{\varepsilon_0^2 \varepsilon^2 S^2 \mathscr{E}^2 d_2}{2d_1^2 \varepsilon_0 \varepsilon S} - \frac{\varepsilon_0^2 \varepsilon^2 S^2 \mathscr{E}^2 d_1}{2d_1^2 \varepsilon_0 \varepsilon S}$$

$$= \frac{\varepsilon_0 \varepsilon S \mathscr{E}^2}{2d_1^2}(d_2 - d_1),$$

$$A_1 = \frac{8.85 \times 10^{-12} \times 1 \times 1.75 \times 10^{-2} \times 180^2 \times (12 \times 10^{-3} - 5 \times 10^{-3})}{2 \times (5 \times 10^{-3})^2} \text{ J}$$

$$= 705 \text{ nJ}.$$

2. If the capacitor is connected to the source, the potential difference across its plates remains unchanged. The total work done in moving the plates apart is

$$A = A_2 - A_s, \quad (2)$$

where A_2 is the work done by an external force and $A_s = \mathscr{E}\Delta q = \mathscr{E}(q_1 - q_2)$ is the work done by the source on moving the charge Δq, where $q_1 = C_1 \mathscr{E}$ is the capacitor charge before moving the plates apart and $q_2 = C_2 \mathscr{E}$ the charge after moving the plates apart. The work done by the source A_s is taken with the minus sign since the source does a negative work when a charge is moved from the positive to the negative plate.

On the other hand, the total work is equal to the change in the energy of the capacitor:

$$A = \Delta W = W_2 - W_1. \qquad (3)$$

Equating expressions (2) and (3), we find that $A_2 - A_s = W_2 - W_1$, whence

$$A_2 = W_2 - W_1 + A_s. \qquad (4)$$

Here $W_1 = C_1\mathscr{E}^2/2$ and $W_2 = C_2\mathscr{E}^2/2$ is the energy of the capacitor before and after moving the plates apart. Substituting the expressions for A_s, W_1, and W_2 into Eq. (4), we obtain

$$A_2 = C_2\mathscr{E}^2/2 - C_1\mathscr{E}^2/2 + \mathscr{E}\,(C_1\mathscr{E} - C_2\mathscr{E})$$
$$= (C_1 - C_2)\,\mathscr{E}^2/2,$$

or, considering that $C_1 = \varepsilon_0\varepsilon S/d_1$ and $C_2 = \varepsilon_0\varepsilon S/d_2$,

$$A_2 = \frac{\mathscr{E}^2}{2}\left(\frac{\varepsilon_0\varepsilon S}{d_1} - \frac{\varepsilon_0\varepsilon S}{d_2}\right) = \frac{\varepsilon_0\varepsilon S\mathscr{E}^2\,(d_2 - d_1)}{2d_1d_2},$$

$$A_2 = \frac{8.85\times10^{-2}\times1\times1.75\times10^{-12}\times180^2\times(12\times10^{-3}-5\times10^{-3})}{2\times5\times10^{-3}\times12\times10^{-3}}\ \mathrm{J}$$

$$= 293\ \mathrm{nJ}.$$

419. Three capacitors of capacitance 1, 2, and 3 μF are connected in series and to a voltage source with a potential difference of 220 V. What are the charge and voltage of each capacitor?

Given: $C_1 = 1\ \mu\mathrm{F} = 10^{-6}\ \mathrm{F}$, $C_2 = 2\ \mu\mathrm{F} = 2\times10^{-6}\ \mathrm{F}$, $C_3 = 3\ \mu\mathrm{F} = 3\times10^{-6}\ \mathrm{F}$, $U = 220\ \mathrm{V}$.

q_1—? q_2—? q_3—? U_1—? U_2—? U_3—?

Solution. We denote by φ_1 the potential of plate *1*, φ_2 the potentials of plates *2* and *3* (the potentials are equal since the plates are connected), φ_3 the potentials of plates *4* and *5*, and φ_4 the potential of plate *6* (Fig. 133). Plate *1* receives a charge $+q$ from the source. The charge $-q$ is induced on plate *2* and the charge $+q$ on plate *3*.

Similarly, the charge $-q$ is induced on plate *4* and the charge $+q$ on plate *5*, and $-q$ on plate *6*. Therefore, the

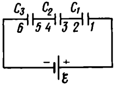


Fig. 133

charges on series-connected capacitors are equal. We denote these charges by $q_1 = q_2 = q_3 = q$. By definition,

$$\varphi_1 - \varphi_2 = q/C_1 = U_1,$$
$$\varphi_2 - \varphi_3 = q/C_2 = U_2,$$
$$\varphi_3 - \varphi_4 = q/C_3 = U_3.$$

Summing up these equations termwise, we obtain

$$\varphi_1 - \varphi_4 = q\,(1/C_1 + 1/C_2 + 1/C_3).$$

Considering that $\varphi_1 - \varphi_4 = U$ is the voltage across the source, we find that

$$q = \frac{U}{1/C_1 + 1/C_2 + 1/C_3},$$
$$q = \frac{220}{1/10^{-6} + 1/(2\times 10^{-6}) + 1/(3\times 10^{-6})}\ \mathrm{C} = 12\times 10^{-6}\,\mathrm{C}$$

Then the voltages across the capacitors are

$$U_1 = \frac{q}{C_1},\quad U_1 = \frac{12\times 10^{-3}}{10^{-6}}\ \mathrm{V} = 120\ \mathrm{V},$$
$$U_2 = \frac{q}{C_2},\quad U_2 = \frac{12\times 10^{-3}}{2\times 10^{-6}}\ \mathrm{V} = 60\ \mathrm{V},$$
$$U_3 = \frac{q}{C_3},\quad U_3 = \frac{12\times 10^{-3}}{3\times 10^{-6}}\ \mathrm{V} = 40\ \mathrm{V}.$$

Thus, in series connection, the capacitors of different capacitances are under different voltages. The lower the capacitance, the higher the voltage across the capacitor plates.

420. Seven capacitors of capacitance 2 and 1 μF are connected between terminals A and B as shown in Fig. 134*a*. Calculate the capacitance of the system.

Given: $C_1 = 2\ \mu F = 2 \times 10^{-6}$ F, $C_2 = 1\ \mu F = 10^{-6}$ F.

C — ?

Solution. Let us consider the subcircuit between points D and E (the right-hand side of Fig. 134*a*) separately.

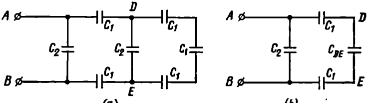


Fig. 134

It consists of two parallel branches one of which includes a capacitor of capacitance C_2 and the other includes three capacitors of capacitance C_1, connected in series. The capacitance of this subcircuit is $C_{DE} = C_2 + C'$, where C' is determined from the condition $1/C' = 1/C_1 + 1/C_1 + 1/C_1 = 3/C_1$, whence $C' = C_1/3$. Then

$$C_{DE} = C_2 + C_1/3 = (3C_2 + C_1)/3. \quad (1)$$

We replace the subcircuit between points D and E by an equivalent capacitance C_{DE} (Fig. 134*b*). The required capacitance C is now the sum of the capacitances of two parallel branches one of which includes a capacitor of capacitance C_2 and the other consists of three series-connected capacitors of capacitance C_1, C_{DE}, and C_1. Therefore, $C = C_2 + C''$, where C'' can be determined from the condition

$$1/C'' = 1/C_1 + 1/C_{DE} + 1/C_1 = 2/C_1 + 1/C_{DE} \quad (2)$$

We transform expression (2) by using Eq. (1):

$$\frac{1}{C''}=\frac{2}{C_1}+\frac{3}{3C_2+C_1}=\frac{6C_2+5C_1}{C_1(3C_2+C_1)},$$

whence $C''=\dfrac{C_1(3C_2+C_1)}{6C_2+5C_1}$.

Consequently, the total capacitance of the system is

$$C=C_2+\frac{C_1(3C_2+C_1)}{6C_2+5C_1},$$

$$C=10^{-6}+\frac{2\times10^{-6}\times(3\times10^{-6}+2\times10^{-6})}{6\times10^{-6}+5\times2\times10^{-6}}\ \text{F}=1.62\ \mu\text{F}.$$

421. A Leyden jar of capacitance 3.3 nF is charged to a potential difference of 20 kV. Assuming that 10% of the jar energy is dissipated during a discharge in the form of acoustic and electromagnetic waves, determine the amount of heat liberated during the discharge.

Given: $C=3.3\ \text{nF}=3.3\times10^{-9}\ \text{F}$, $U=20\ \text{kV}=2\times10^4\ \text{V}$, $W'=0.1W$.

Q—?

Solution. The electric energy of the charged Leyden jar is

$$W=CU^2/2. \tag{1}$$

The energy $W'=0.1W$ is lost by radiation. Consequently, the amount of liberated heat is

$$Q=W-0.1W=0.9W,$$

or, taking into account Eq. (1),

$$Q=0.9\,\frac{CU^2}{2},$$

$$Q=\frac{0.9\times3.3\times10^{-9}\times(2\times10^4)^2}{2}\ \text{J}\simeq0.6\ \text{J}.$$

422. A capacitor of capacitance 1 mF at a voltage of 1200 V is used for a shot butt welding of copper wire. Determine the mean useful power of a discharge if its duration is 10^{-6} s. The efficiency of the set-up is 4%.

Given: $C=1\ \text{mF}=10^{-3}\ \text{F}$, $U=1.2\times10^3\ \text{V}$, $t=10^{-6}\ \text{s}$, $\eta=0.04$.

$\langle N_u\rangle$—?

Solution. The mean power of the discharge is

$$\langle N \rangle = A/t,$$

where $A = \Delta W = W_{in} - W_{fin} = CU^2/2 - 0 = CU^2/2$. Consequently,

$$\langle N \rangle = CU^2/(2t).$$

The mean useful power is

$$\langle N_u \rangle = \eta \langle N \rangle = \eta \frac{CU^2}{2t},$$

$$\langle N_u \rangle = \frac{0.04 \times 10^{-3} \times (1.2 \times 10^3)^2}{2 \times 10^{-6}} \text{ W} = 28.8 \text{ MW}.$$

423. A metal sphere of radius 3 cm bears a charge of 2×10^{-8} C. The sphere is immersed in kerosene so that it does not touch the vessel walls. Determine the volume energy density of the field at points lying at 2 and 4 cm from the centre of the sphere.

Given: $R = 3 \text{ cm} = 3 \times 10^{-2}$ m, $q = 2 \times 10^{-8}$ C, $r_1 = 2 \times 10^{-2}$ m, $r_2 = 4 \times 10^{-2}$ m, $\varepsilon = 2$.

w_1—? w_2—?

Solution. The volume energy density is defined as

$$w_1 = \varepsilon_0 \varepsilon E_1^2/2.$$

A point lying at a distance r_1 from the centre of the charged sphere is inside the sphere where the electric field strength E_1 is zero. Therefore, $w_1 = 0$ at this point. For a point separated by r_2 from the centre of the sphere, the volume energy density is

$$w_2 = \varepsilon_0 \varepsilon E_2^2/2,$$

where E_2 is the electric field strength at this point. Since $E_2 = q/(4\pi\varepsilon_0\varepsilon r_2^2)$, we have

$$w_2 = \frac{\varepsilon_0 \varepsilon}{2} \left(\frac{q}{4\pi\varepsilon_0 \varepsilon r_2^2} \right)^2 = \frac{q^2}{32\pi^2 \varepsilon_0 \varepsilon r_2^4},$$

$$w_2 = \frac{(2 \times 10^{-8})^2}{32 \times 3.14^2 \times 8.85 \times 10^{-12} \times 2 \times (4 \times 10^{-2})^4} \frac{\text{J}}{\text{m}^3}$$

$$\simeq 0.056 \text{ J/m}^3.$$

424. The plates of a parallel-plate air capacitor are attracted to each other with a force F. Will this force change if an insulator plate is introduced into the air gap between the capacitor plates?

Answer. Introducing the plate made of insulator, we reduce the electric field strength in the space occupied by the insulator but do not change the field strength in the gaps between the insulator and the capacitor plates. Therefore, the attractive force F between the plates remains unchanged.

EXERCISES

425. A capacitor consists of three tin foils of area $10\ cm^2$ each, separated by mica layers of thickness 0.5 mm. The extreme foils are connected to each other. Determine the capacitance of the capacitor.

426. An air capacitor consists of two circular plates of radius 10 cm. The separation between the plates is 1 cm and the potential difference is 120 V. Determine the charge on the capacitor.

427. A capacitor charged to a voltage of 100 V is connected in parallel to a capacitor of the same capacitance, charged to a voltage of 200 V. What voltage will appear across the plates?

428. Three capacitors of capacitance 2, 4, and 6 pF are connected in parallel and to a source of voltage of 1 kV. Determine the charges on the capacitors.

429. A bank consisting of two series-connected Leyden jars of capacitance 300 and 500 pF is charged to a voltage of 12 kV. Determine the voltage and the charge on the plates of the first and second jars.

430. A sphere of radius 25 cm is charged to a potential of 600 V. What amount of heat will be liberated in a conductor connecting the sphere to the ground?

431. Determine the volume energy density of the electrostatic field at a point located at 2 cm from the surface of a charged sphere of radius 1 cm. The surface charge density of the sphere is $16.5\ \mu C/m^2$ and the permittivity of the medium is 2.

QUESTIONS FOR REVISION

1. Formulate Coulomb's law. 2. Explain the physical meaning of relative permittivity. 3. Define electric field strength. 4. Write formulas for calculating the electric field strength produced by a point charge, charged sphere, infinitely charged plane, and a parallel-plate capacitor. 5. Formulate the superposition principle for electric fields. 6. What is potential difference? 7. What is the work done in moving a charge in an electrostatic field equal to? 8. Write formulas for calculating the potential of the field produced by a point charge and a charged sphere. 9. What is the relation between the field strength and potential difference for a uniform electrostatic field? 10. Define capacitance. 11. Write formulas for calculating the capacitance of a sphere and a parallel-plate capacitor. 12. Calculate the capacitance of a capacitor bank for series and parallel connection of capacitors in it. 13. Write formulas for calculating the energy of a charged capacitor. 14. Give an expression for the volume energy density of an electrostatic field. 15. Name the units of charge, electric field strength, and potential.

3.2. Direct Current

CURRENT IN METALS

Current in metals is due to the motion of free electrons in a conductor. The amount of electricity (charge) q passing through the cross-sectional area of the conductor per second is called the **current**:

$$I = q/t.$$

The **current density** is defined by

$$j = I/S,$$

where S is the cross-sectional area of a conductor, normal to the direction of the current.

Ohm's law for a conductor has the form

$$I = U/R,$$

where U is the voltage across the conductor and R the **resistance of the conductor.**

For a homogeneous conductor,

$$R = \rho l/S,$$

where ρ is the **resistivity of the conducting material,** l the conductor length, and S the cross-sectional area of the conductor.

The temperature dependence of the resistivity of a conducting material is given by

$$\rho = \rho_0 (1 + \alpha \Delta T),$$

where ρ_0 is the resistivity of the material at 273 K, α the temperature resistance coefficient, and $\Delta T = T - T_0$ is the change in the temperature of the conductor.

For series-connected conductors, the total resistance is equal to the sum of the resistances of individual conductors:

$$R = \sum_{i=1}^{n} R_i.$$

For parallel-connected conductors, the reciprocal of the total resistance is equal to the sum of the reciprocals of the resistances of individual conductors:

$$\frac{1}{R} = \sum_{i=1}^{n} \frac{1}{R_i},$$

where R_i is the resistance of an individual conductor and n the number of conductors in a given subcircuit.

Ohm's law for a closed circuit is

$$I = \mathscr{E}/(R + r),$$

where $\mathscr{E}$ is the electromotive force (emf) of the current source, R the resistance of the external subcircuit, and r the resistance of the internal subcircuit.

The voltage U across a subcircuit containing an emf source is

$$U = \mathscr{E} \pm IR_t,$$

where R_t is the total resistance of the subcircuit. The minus sign is taken when the current in the source is directed from the negative to the positive pole (Fig. 135*a*), while the plus sign is taken when the current in the source is directed from the positive to the negative pole (Fig. 135*b*).

When several sources are connected in series, their total emf and the total internal resistance can be determined from the following relations:

$$\mathscr{E}=\sum_{i=1}^{n}\mathscr{E}_i,\quad r=\sum_{i=1}^{n}r_i.$$

When the sources are connected in parallel, these quantities can be found from the relation

$$\frac{\mathscr{E}}{r}=\sum_{i=1}^{n}\frac{\mathscr{E}_i}{r_i},$$

where $\mathscr{E}_i$ is the emf of an individual source and r_i is the internal resistance of the source.

For calculating branched circuits, Kirchhoff's laws should be used.

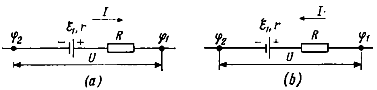


Fig. 135

Kirchhoff's first law. *The algebraic sum of the currents converging at a junction is zero*:

$$\sum_{i=1}^{n}I_i=0,$$

where n is the number of currents.

The directions of currents are chosen arbitrarily so that the algebraic sum contains both positive and negative currents. The current arriving at a junction is usually assumed to be positive and that flowing away from the junction is assumed to be negative.

If a circuit diagram contains N junctions, using Kirchhoff's first law, we can write $N-1$ independent equations.

Kirchhoff's second law. *In any arbitrarily chosen closed contour of a branched circuit, the algebraic sum of the voltage drops across individual elements of the contour is equal to the algebraic sum of the emf's encountered during the circum-*

vention of the contour:

$$\sum_{i=1}^{n} I_i R_i = \sum_{i=1}^{n} \mathscr{E}_i.$$

The direction of circumvention of the contour is chosen clockwise or counterclockwise. In this case, the product $I_i R_i$ is assumed to be positive if the direction of the current I_i coincides with the chosen direction of circumvention. An emf is assumed to be positive if it increases the potential in the direction of circumvention of the contour.

According to Kirchhoff's second law, we can write $n - (N - 1)$ equations, where n is the number of unknown currents and N the number of junctions in the circuit diagram.

* * *

432. Determine the resistance between points A and D (Fig. 136*a*) if each of the three resistances is 1 Ω (the resistance of the leads should be neglected).

Given: $R = 1\ \Omega$, $n = 3$.

R_{AD}—?

Solution. Since points A and C, as well as points B and D, are connected by conductors whose resistance is

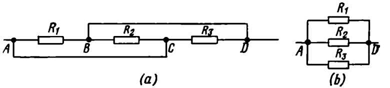


Fig. 136

disregarded, diagram (*a*) can be replaced by an equivalent diagram (*b*). It can be seen that the resistance between points A and D can be calculated from the formula for parallel connection of conductors:

$$1/R_{AD} = 1/R_1 + 1/R_2 + 1/R_3 = n/R,$$

whence

$$R_{AD} = R/n,$$

$$R_{AD} = 1/3\ \Omega \simeq 0.33\ \Omega.$$

433. Determine the total resistance of the electric circuit (Fig. 137) if the internal resistance of the source is 1 Ω and the resistances of other resistors are 4, 3, 12, and 6 Ω respectively.

Given: $r = 1\ \Omega$, $R_1 = 4\ \Omega$, $R_2 = 3\ \Omega$, $R_3 = 12\ \Omega$, $R_4 = 6\ \Omega$.

R—?

Solution. In the circuit diagram (Fig. 137*a*), the resistors R_1 and R_3 are connected in parallel, and hence the diagram can be drawn in a different form by making points A and B coincide (Fig. 137*b*). It is now easier to calculate the total resistance of the circuit. Denoting the total resistance of the resistors R_1 and R_3 by R', we can write

$$1/R' = 1/R_1 + 1/R_3,$$

whence

$$R' = R_1R_3/(R_1 + R_3).$$

The resistors R' and R_2 are connected in series. Denoting their total resistance by R'', we can write

$$R'' = R' + R_2 = R_1R_3/(R_1 + R_3) + R_2. \qquad (1)$$

We can now determine the total resistance R''' of the parallel subcircuits with resistances R'' and R_4: $1/R''' =$

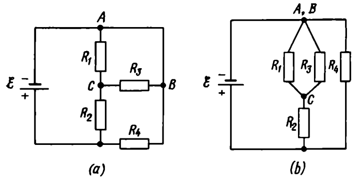


Fig. 137

$1/R'' + 1/R_4$, or, taking into account Eq. (1), $\frac{1}{R'''} = \frac{1}{R_1R_3/(R_1+R_3)+R_2} + \frac{1}{R_4}$, whence

$$R''' = \frac{R_4(R_1R_3+R_1R_2+R_2R_3)}{R_1R_4+R_3R_4+R_1R_3+R_1R_2+R_2R_3}. \quad (2)$$

Finally, the total resistance of the entire circuit is $R = r + R'''$, or, taking into account Eq. (2),

$$R = r + \frac{R_4(R_1R_3+R_1R_2+R_2R_3)}{R_1R_4+R_3R_4+R_1R_3+R_1R_2+R_2R_3},$$

$$R = 1 + \frac{6\times(4\times12+4\times3+3\times12)}{4\times6+12\times6+4\times12+4\times3+3\times12}\ \Omega = 4\ \Omega.$$

434. Determine the current obtained from a battery of emf 6 V if the resistances of different resistors are 2, 6, 3,

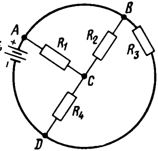


Fig. 138

and 1.5 Ω respectively. The internal resistance of the battery should be neglected (Fig. 138).

Given: $\mathscr{E} = 6$ V, $R_1 = 2\ \Omega$, $R_2 = 6\ \Omega$, $R_3 = 3\ \Omega$, $R_4 = 1.5\ \Omega$.

I—?

Solution. The resistors R_1 and R_2 are connected in parallel. Denoting their total resistance by R, we obtain

$$1/R = 1/R_1 + 1/R_2,$$

whence

$$R = R_1R_2/(R_1 + R_2). \quad (1)$$

The subcircuit ABC is connected in series with the resistor R_4. Consequently, the total resistance of the subcircuit is $R' = R + R_4$, or, taking into account Eq. (1),

$$R' = \frac{R_1R_2}{R_1+R_2} + R_4 = \frac{R_1R_2+R_4(R_1+R_2)}{R_1+R_2}. \qquad (2)$$

The subcircuit $ABCD$ is connected in parallel with the resistor R_3. Denoting their total resistance by R'', we determine it from the condition $1/R'' = 1/R' + 1/R_3$, or, taking into account Eq. (2), $\frac{1}{R''} = \frac{R_1+R_2}{R_1R_2+R_4(R_1+R_2)} + \frac{1}{R_3}$,

whence

$$R'' = \frac{R_3(R_1R_2+R_1R_4+R_2R_4)}{R_1R_3+R_2R_3+R_1R_2+R_4R_1+R_4R_2}. \qquad (3)$$

The current consumed from the battery can be determined from Ohm's law for a closed circuit:

$$I = \mathscr{E}/R'',$$

or, taking into account Eq. (3),

$$I = \frac{\mathscr{E}(R_1R_3+R_2R_3+R_1R_2+R_4R_1+R_4R_2)}{R_3(R_1R_2+R_1R_4+R_2R_4)},$$

$$I = \frac{6\times(2\times3+6\times3+2\times6+1.5\times2+1.5\times6)}{3\times(2\times6+2\times1.5+6\times1.5)}\ \text{A} = 4\ \text{A}.$$

This problem can be solved by using Kirchhoff's laws. We leave it to the reader to verify this.

435°. What must be a direct current for which a charge of 50 C passes through the cross section of a conductor during the time interval from 5 to 10 s after the moment of switching on the current? What charge will pass through the cross section of the conductor during the same time if the current in the conductor changes with time according to the law $I = 6 + 3t$?

Given: $\Delta q = 50$ C, $t_1 = 5$ s, $t_2 = 10$ s.

I—? q_2—?

Solution. For a direct current, we can write

$$I = \Delta q_1/\Delta t,$$

where $\Delta t = t_2 - t_1$. Then

$$I = \frac{50}{10-5}\ \text{A} = 10\ \text{A}.$$

If the current changes with time, the charge passing through the cross section of a conductor during this time interval is

$$q_2 = \int_{t_1}^{t_2} I\,dt = \int_{t_1}^{t_2} (6+3t)\,dt = \int_{t_1}^{t_2} 6\,dt + \int_{t_1}^{t_2} 3t\,dt$$

$$= \left(6t + \frac{3t^2}{2}\right)\Big|_{t_1}^{t_2} = 6t_2 + \frac{3}{2}t_2^2 - 6t_1 - \frac{3}{2}t_1^2,$$

$$q_2 = \left(6\times 10 + \frac{3}{2}\times 10^2 - 6\times 5 - \frac{3}{2}\times 5^2\right)\ \text{C} = 142.5\ \text{C}.$$

436. A current passes through a conductor having a cross-sectional area of 50 mm^2. The mean drift velocity

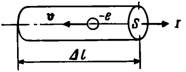


Fig. 139

of free electrons is 0.282 mm/s and the electron number density is 7.9×10^{27} m^{-3}. Determine the current and the current density in the conductor.

Given: $S = 50\ \text{mm}^2 = 5\times 10^{-5}\ \text{m}^2$, $\langle v\rangle = 0.282\ \text{mm/s} = 2.82\times 10^{-4}\ \text{m/s}$, $n = 7.9\times 10^{27}\ \text{m}^{-3}$.

I—? j—?

Solution. We isolate in the conductor a region of length Δl and of cross-sectional area S (Fig. 139). The charge passing through the cross section of the conductor during the time interval Δt is

$$\Delta q = ne\,\Delta V,$$

where n is the number density of free electrons, e the electron charge, $\Delta V = S\Delta l$ the volume of the isolated region of the conductor. By definition, $I = \Delta q/\Delta t$, or, using the expressions for Δq and ΔV, we have

$$I = neS\ \Delta l/\Delta t. \qquad (1)$$

Noting that, by definition, $\Delta l/\Delta t = \langle v \rangle$ is the mean drift velocity of free electrons, we can write Eq. (1) in the form

$$I = neS\ \langle v \rangle,$$

$$I = 7.9 \times 10^{27} \times 1.6 \times 10^{-19} \times 5 \times 10^{-5} \times 2.82 \times 10^{-4}\,\text{A} \simeq 17.8\ \text{A}.$$

Then the current density is

$$j = \frac{I}{S},$$

$$j = \frac{17.8}{5 \times 10^{-5}}\ \frac{\text{A}}{\text{m}^2} = 3.56 \times 10^5\ \text{A/m}^2.$$

437. A circuit having a resistance of 100 Ω is fed by a d.c. source. An ammeter having an internal resistance of 1 Ω and connected to the circuit indicated 5 A. What was the current in the circuit before the connection of the ammeter?

Given: $R = 100\ \Omega$, $R_0 = 1\ \Omega$, $I = 5$ A.

I_0—?

Solution. Before the connection of the ammeter, the current in a subcircuit was, according to Ohm's law,

$$I_0 = U/R. \qquad (1)$$

After the connection of the ammeter, the current becomes

$$I = U/(R + R_0). \qquad (2)$$

Solving Eqs. (1) and (2) together, we obtain

$$I_0 = \frac{R + R_0}{R}\, I,$$

$$I_0 = \frac{(100 + 1) \times 5}{100}\ \text{A} = 5.05\ \text{A}.$$

438. A cell having an emf of 2.1 V and an internal resistance of 0.2 Ω is connected to a potentiometer.

Determine the current in the circuit and the resistance of the potentiometer if the voltage across the cell terminals is 2 V. What must be the length of the iron wire

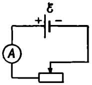


Fig. 140

used for preparing the potentiometer if its cross-sectional area is 0.75 mm²?

Given: $\mathscr{E}=2.1$ V, $r=0.2\ \Omega$, $U=2$ V, $S=0.75\ \text{mm}^2=7.5\times10^{-7}\ \text{m}^2$.

I—? R—? l—?

Solution. According to Ohm's law for a closed circuit, the current is

$$I=\mathscr{E}/(R+r). \tag{1}$$

On the other hand, according to Ohm's law for a conductor, viz. potentiometer (Fig. 140), the same current is given by

$$I=U/R. \tag{2}$$

Solving Eqs. (1) and (2) together, we obtain

$$R=\frac{Ur}{\mathscr{E}-U},$$

$$R=\frac{2\times0.2}{0.1}\ \Omega=4\ \Omega,$$

$$I=\frac{U}{R},$$

$$I=\frac{2}{4}\ \text{A}=0.5\ \text{A}.$$

Since $R=\rho l/S$, the length of the iron wire is

$$l=\frac{RS}{\rho},$$

$$l=\frac{4\times7.5\times10^{-7}}{1.2\times10^{-7}}\ \text{m}=25\ \text{m}.$$

439. A galvanic cell of an emf 1.5 V and an internal resistance of 1 Ω is connected to an external resistance of 4 Ω. Determine the current in the circuit, the voltage drop in the internal subcircuit, and the voltage across the terminals of the cell.

Given: $\mathscr{E} = 1.5$ V, $r = 1$ Ω, $R = 4$ Ω.

I—? U_1—? U_2—?

Solution. The current can be determined from Ohm's law for a closed circuit:

$$I = \frac{\mathscr{E}}{R+r},$$

$$I = \frac{1.5}{4+1}\ \text{A} = 0.3\ \text{A}.$$

The voltage drop in the internal subcircuit is

$$U_1 = Ir,$$

$$U_1 = 0.3 \times 1\ \text{V} = 0.3\ \text{V}.$$

The voltage across the terminals of the cell is smaller than the emf by the voltage drop in the internal subcircuit. Therefore,

$$U_2 = \mathscr{E} - Ir,$$

$$U_2 = (1.5 - 0.3)\ \text{V} = 1.2\ \text{V}.$$

440. The internal resistance r of a cell is smaller by a factor of k than the external resistance R by which the cell of emf $\mathscr{E}$ is closed. Determine the ratio of the voltage U across the terminals of the cell to the emf $\mathscr{E}$.

Given: $R/r = k$, $\mathscr{E}$.

$U/\mathscr{E}$—?

Solution. According to Ohm's law for a closed circuit,

$$I = \mathscr{E}/(R + r). \qquad (1)$$

Since the voltage across the terminals of the cell is smaller than the emf by the magnitude of the voltage drop in the internal subcircuit, $U = \mathscr{E} - Ir$, or, taking into account Eq. (1),

$$U = \mathscr{E} - \frac{\mathscr{E}r}{R+r} = \mathscr{E}\frac{R}{R+r} = \frac{\mathscr{E}}{1+r/R}. \qquad (2)$$

By hypothesis, $R/r = k$, or $r/R = 1/k$. Then Eq. (2) can be reduced to the form $U = \mathscr{E}/(1 + 1/k) = k\mathscr{E}/(k + 1)$, whence

$$U/\mathscr{E} = k/(k + 1).$$

Equation (2) shows that if the external resistance is large in comparison with the internal resistance, the ratio r/R becomes small in comparison with unity, and the value of the voltage approaches the value of the emf.

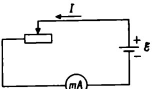


Fig. 141

441. A potentiometer made of an iron wire, a milliammeter, and an emf source are connected in series. At 0 °C, the resistance of the potentiometer is 200 Ω and the resistance of the milliammeter is 20 Ω. The milliammeter indicates 30 mA. What will be the reading of the milliammeter when the potentiometer is heated to 50 °C? The internal resistance of the source should be neglected (Fig. 141).

Given: $T_0 = 273$ K, $R_0 = 200\ \Omega$, $R_a = 20\ \Omega$, $I_0 = 30$ mA $= 3 \times 10^{-2}$ A, $T = 323$ K.

I—?

Solution. According to Ohm's law for a closed circuit, before the heating of the potentiometer, the current was

$$I_0 = \mathscr{E}/(R_0 + R_a). \tag{1}$$

After the heating, the current becomes

$$I = \mathscr{E}/(R + R_a). \tag{2}$$

Here $R = R_0(1 + \alpha\, \Delta T)$, $\Delta T = T - T_0$, where α is the temperature resistance coefficient of the iron. The change in the resistance of the milliammeter is neglected. Solving Eqs. (1) and (2) together, we obtain

$$I = \frac{I_0(R_0 + R_a)}{R_0[1 + \alpha(T - T_0)] + R_a},$$

$$I = \frac{3 \times 10^{-2} \times (200 + 20)}{200 \times [1 + 6 \times 10^{-3} \times (323 - 273)] + 20}\ \mathrm{A} \simeq 2.36 \times 10^{-2}\ \mathrm{A}.$$

442. Calculate the emf and the internal resistance of a battery consisting of three emf sources (Fig. 142) if the emf's of the sources are 10, 20, and 30 V and their internal resistances are 1 Ω each.

Given: $\mathscr{E}_1 = 10$ V, $\mathscr{E}_2 = 20$ V, $\mathscr{E}_3 = 30$ V, $r_1 = r_2 = r_3 = r = 1\ \Omega$.

$\mathscr{E}$—? r—?

Solution. The emf sources are connected in the circuit in series and in parallel. If we assume that the emf's $\mathscr{E}_2$

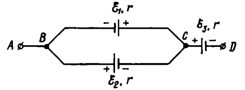


Fig. 142

and $\mathscr{E}_3$ are positive, the emf $\mathscr{E}_1$ will be negative. Let us determine the total resistance r' and the total emf for the subcircuit BC consisting of parallel-connected sources $\mathscr{E}_1$ and $\mathscr{E}_2$:

$$1/r' = 1/r_1 + 1/r_2 = 2/r,$$

$$\mathscr{E}'/r' = \mathscr{E}_1/r_1 + \mathscr{E}_2/r_2 = (\mathscr{E}_1 + \mathscr{E}_2)/r,$$

whence

$$r' = \frac{r}{2},$$

$$\mathscr{E}' = \frac{r'(\mathscr{E}_1 + \mathscr{E}_2)}{r} = \frac{\mathscr{E}_1 + \mathscr{E}_2}{2}.$$

Let us now calculate the total emf $\mathscr{E}$ and the total resistance r of the entire battery connected between points A and D:

$$\mathscr{E} = \mathscr{E}' + \mathscr{E}_3 = \frac{\mathscr{E}_1 + \mathscr{E}_2}{2} + \mathscr{E}_3,$$

$$\mathscr{E} = \frac{-10 + 20}{2} + 30\ \text{V} = 35\ \text{V},$$

$$r = r' + r_3 = \frac{r}{2} + r_3 = \frac{r}{2} + r = \frac{3}{2} r,$$

$$r = 1.5\ \Omega.$$

443. Determine the emf and the internal resistance of a current source shunted by a resistance of 6 Ω if the emf of the source without a shunt is 12 V and its internal resistance is 3 Ω.

Given: $R = 6\ \Omega$, $\mathscr{E} = 12$ V, $r = 3\ \Omega$.

$\mathscr{E}'$—? r'—?

Solution. Considering the shunt as a source whose internal resistance is R and the emf is zero, we obtain a battery consisting of two parallel-connected sources. Then the internal resistance r' and the emf $\mathscr{E}'$ of such a battery can be found from the relations

$$1/r' = 1/r + 1/R, \quad \mathscr{E}'/r' = \mathscr{E}/r + 0/R,$$

whence $r' = rR/(r + R)$ and $\mathscr{E}' = r'\mathscr{E}/r$. This gives

$$r' = \frac{3 \times 6}{3+6}\ \Omega = 2\ \Omega, \quad \mathscr{E}' = \frac{2 \times 12}{3}\ \text{V} = 8\ \text{V}.$$

444. A voltmeter with an internal resistance of 2500 Ω indicates a voltage of 125 V in the circuit. What is the series resistance to be connected to the voltmeter so that it

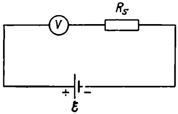


Fig. 143

indicates 100 V (Fig. 143)?

Given: $R = 2.5 \times 10^3\ \Omega$, $U_1 = 125$ V, $U_2 = 100$ V.

R_s—?

Solution. The voltage across the series resistance is $U_1 - U_2$. Using Ohm's law for an element of the circuit, we can write

$$R_s = (U_1 - U_2)/I. \quad (1)$$

With a series connection, the current in different elements of a circuit is the same. Therefore, knowing the internal resistance R of the voltmeter and the voltage drop U_2 across it, we have from Ohm's law for this element of the circuit

$$I = U_2/R. \qquad (2)$$

Substituting expression (2) into (1), we find that

$$R_s = \frac{(U_1 - U_2) R}{U_2},$$

$$R_s = \frac{(125-100) \times 2.5 \times 10^3}{100}\ \Omega = 625\ \Omega.$$

445. A milliammeter is intended for measuring currents below 10 mA. What should be done to use this

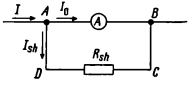


Fig. 144

instrument for measuring current up to 1 A if its internal resistance is 9.9 Ω?

Given: $I_0 = 10$ mA $= 10^{-2}$ A, $R_0 = 9.9\ \Omega$, $I = 1$ A.

R_{sh}—?

Solution. The milliammeter is connected in series to a circuit. If the current I in it is stronger than the maximum rated value I_0 of the instrument, a resistor R_{sh} known as a shunt should be connected in parallel to it (Fig. 144). It is chosen so that the current through the instrument is below I_0.

Applying Kirchhoff's first law to the junction A, we obtain

$$I - I_0 - I_{sh} = 0. \qquad (1)$$

Choosing the direction of circumvention of the closed contour $ABCDA$ clockwise and applying Kirchhoff's second law to the contour, we get

$$I_0R_0 - I_{sh}R_{sh} = 0. \qquad (2)$$

Solving Eqs. (1) and (2) together, we obtain

$$R_{sh} = \frac{I_0R_0}{I - I_0},$$

$$R_{sh} = \frac{10^{-2}\times 9.9}{1-10^{-2}}\ \Omega = 0.1\ \Omega.$$

446. Four resistors of 1 kΩ each and two current sources of emf 1.5 and 1.8 V are connected to form an

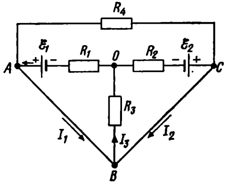


Fig. 145

electric circuit. Determine the current in all the resistors, neglecting the internal resistances of the sources (Fig. 145).

Given: $R = 1\ \text{k}\Omega = 10^3\ \Omega$, $\mathscr{E}_1 = 1.5$ V, $\mathscr{E}_2 = 1.8$ V.

I_1, I_2, I_3, I_4—?

Solution. It can be seen from the figure that the resistor R_4 is short-circuited by the conductor ABC whose resistance is neglected. Consequently, the current does not flow through R_4, i.e. $I_4 = 0$. The remaining part of the circuit contains two junctions at points O and B and three closed contours. Consequently, we can write one equation by using Kirchhoff's first law and two equations by using Kirchhoff's second law. Applying Kirchhoff's first law to the junction B, we find that

$$I_1 + I_2 - I_3 = 0.$$

Applying Kirchhoff's second law to the contour $AOBA$ (circumvented counterclockwise), we obtain

$$I_3R + I_1R = \mathscr{E}_1.$$

Similarly, applying Kirchhoff's second law to the contour $OCBO$ (circumvented counterclockwise), we get

$$I_3R + I_2R = \mathscr{E}_2.$$

Solving the obtained system of equations, we determine the required currents:

$$I_3 = \frac{\mathscr{E}_1 + \mathscr{E}_2}{3R},$$

$$I_3 = \frac{1.5 + 1.8}{3 \times 10^3} \text{ A} = 1.1 \text{ mA},$$

$$I_2 = \frac{\mathscr{E}_2 - \mathscr{E}_1 + I_3R}{2R},$$

$$I_2 = \frac{1.8 - 1.5 + 1.1 \times 10^{-3} \times 10^3}{2 \times 10^3} \text{ A} = 0.7 \text{ mA},$$

$$I_1 = I_3 - I_2,$$

$$I_1 = (1.1 \times 10^{-3} - 0.7 \times 10^{-3}) \text{ A} = 0.4 \text{ mA}.$$

447. Two cells of emf 1.9 and 1.1 V having internal resistances of 0.8 and 0.1 Ω are connected in parallel to an external resistance of 10 Ω (Fig. 146). Determine the current in the external circuit.

Given: $\mathscr{E}_1 = 1.9$ V, $\mathscr{E}_2 = 1.1$ V, $r_1 = 0.8$ Ω, $r_2 = 0.1$ Ω, $R = 10$ Ω.

I—?

Solution. The electric circuit contains two junctions at points K and C and three unknown currents. Consequently, we can write one equation by using Kirchhoff's first law and two equations by using Kirchhoff's second law. Applying Kirchhoff's first law to the junction K, we obtain

$$I_1 + I_2 - I = 0.$$

Applying Kirchhoff's second law to the contour $ABDKA$ (circumvented counterclockwise), we find that

$$I_1r_1 + IR = \mathscr{E}_1.$$

Similarly, applying Kirchhoff's second law to the contour $ABCKA$ (circumvented counterclockwise), we obtain

$$I_1r_1 - I_2r_2 = \mathscr{E}_1 - \mathscr{E}_2.$$

Solving the obtained system of equations, we get

$$I = \frac{\mathscr{E}_1r_2 + \mathscr{E}_2r_1}{R(r_1+r_2)+r_1r_2},$$

$$I = \frac{1.9\times0.1+1.1\times0.8}{10\times(0.8+0.1)+0.8\times0.1}\ \mathrm{A} \simeq 0.12\ \mathrm{A}.$$

448. The resistances of elements AB, BC, and AD of a circuit are 1000, 500, and 200 Ω respectively. A galvanic cell whose poles are connected to points A and C has an emf of 1.8 V. A galvanometer detects a current of 0.5 mA in the direction indicated by the arrow. Determine the emf of the other galvanic cell, neglecting the internal resistances of the cells and of the galvanometer (Fig. 147).

Given: $R_1 = 10^3\ \Omega$, $R_2 = 5\times10^2\ \Omega$, $R_3 = 2\times10^2\ \Omega$, $\mathscr{E}_1 = 1.8$ V, $I_2 = 0.5$ mA $= 5\times10^{-4}$ A.

$\mathscr{E}_2$—?

Solution. The electric circuit has two junctions at points A and B and three unknown currents. Consequently, we can write one equation by using Kirchhoff's first law

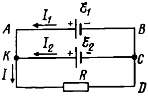


Fig. 146

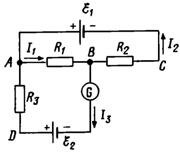


Fig. 147

and two equations by using Kirchhoff's second law. Applying Kirchhoff's first law to the junction B, we obtain

$$I_1 - I_2 - I_3 = 0.$$

Applying Kirchhoff's second law to the contour $ABC\mathscr{E}_1A$ (circumvented counterclockwise), we find that

$$I_1R_1 + I_2R_2 = \mathscr{E}_1.$$

Similarly, applying Kirchhoff's second law to the contour $AB\mathscr{E}_2DA$ (circumvented clockwise), we obtain

$$I_1R_1 + I_3R_3 = \mathscr{E}_2.$$

Solving the obtained system of equations, we get

$$\mathscr{E}_2 = \frac{\mathscr{E}_1R_1 + I_3(R_1R_2 + R_1R_3 + R_2R_3)}{R_1 + R_2},$$

$$\mathscr{E}_2 = \frac{1.8\times10^3 + 5\times10^{-4}\times(10^3\times5\times10^3 + 10^3\times2\times10^3 + 5\times10^3\times2\times10^3)}{10^3 + 5\times10^3}$$

$= 1.47$ V.

449. A wire and a sliding contact are connected to point A of a homogeneous wire ring and to a diametrically opposite point B respectively. What will be the change in the readings of a voltmeter during the motion of the sliding contact (Fig. 148)?

Answer. The subcircuit between points A and B can be regarded as a parallel connection of two conductors of the same resistance. When the sliding contact moves upwards or downwards, the resistance of one of these conductors

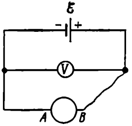


Fig. 148

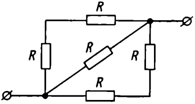


Fig. 149

will decrease. This leads to a decrease in the resistance of the subcircuit AB since the resistance R of a subcircuit consisting of two parallel-connected conductors of resistances R_1 and R_2 is always smaller than the resistance R_1 if $R_1 < R_2$. Therefore, the voltage drop across AB, and hence the readings of the voltmeter, will decrease.

450. Why is the voltage across the terminals of a current source close to zero during short-circuiting, although the current in the circuit has the maximum value?

Answer. During short-circuiting, the resistance of the external subcircuit is small in comparison with the internal resistance of the source. Therefore, although the current in the circuit is large, the voltage drop in the external subcircuit, which is equal to the voltage across the terminals of the source, is small.

EXERCISES

451. Calculate the total resistance of a subcircuit if the resistance of each side and a diagonal of a square is 8 Ω. The resistance of the leads should be neglected (Fig. 149).

452. Eight conductors of resistance 20 Ω each are connected pairwise into four parallel branches. Determine the total resistance of the circuit.

453. A copper and an iron wires of the same length are connected in parallel to form a circuit. The diameter of the iron wire is twice that of the copper wire. The current in the copper wire is 60 mA. What is the current in the iron wire?

454. The resistance of the filament of a vacuum tube is 40 Ω, the resistance of the engaged part of a potentiometer is 20 Ω, and the current in the circuit is 0.2 A. When the same cell is connected to a resistance of 10 Ω, the current is 0.1 A. Determine the emf of the cell and its internal resistance.

455. When a cell is connected to a resistance of 4.5 Ω, the current in the circuit is 0.2 A. When the same cell is connected to a resistance of 10 Ω, the current is 0.1 A. Determine the emf of the cell and its internal resistance.

456. A cell is first connected to an external resistance of 5 Ω and produces a current of 0.25 A. Then it is connected to an external resistance of 9 Ω, which produces a current of 0.15 A. Determine the current through the cell during short-circuiting.

457. Two cells of emf's 1.8 and 2 V and internal resistances 0.3 and 0.2 Ω are connected in a battery so that a current of 4 A flows in an external circuit of resistance 0.2 Ω. How are the cells connected?

458. Six cells having an emf of 1.5 V and an internal resistance of 0.4 Ω each are connected in a battery so that a current of 6 A flows in an external circuit of resistance 0.2 Ω. How are the cells connected?

459. The resistance of a voltmeter is 200 Ω. A conductor of resistance 1 kΩ is connected in series to it. By what factor is the value of the division of the instrument changed?

460. Determine the resistance of a shunt of a galvanometer rated for 1 A if the internal resistance of the galvanometer is 20 Ω and the full scale corresponds to a current of 5 mA.

461. Determine the change in the temperature of a copper conductor whose resistance has increased twofold.

462. The external resistance of a circuit is 1.4 Ω, the emf's of the sources are 2 V each, and their internal resistances are 1 and 1.5 Ω. Determine the currents in each source and in the entire circuit (Fig. 150).

463. Determine the current indicated by an ammeter if the voltage across the terminals of a source is 2.1 V and the resistances are 5.6 and 3 Ω. The resistance of the ammeter and the internal resistance of the source should be neglected (Fig. 151).

464. Determine the current through a resistor R_2 and the voltage drop in it if the resistances of the circuit

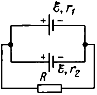


Fig. 150

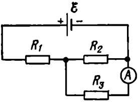


Fig. 151

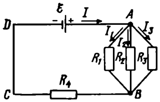


Fig. 152

elements are $R_1 = R_3 = 40\ \Omega$, $R_2 = 80\ \Omega$, and $R_4 = 34\ \Omega$, and the emf of the generator is 100 V. The internal resistance of the generator should be neglected (Fig. 152).

WORK AND POWER OF CURRENT. THERMAL EFFECT OF CURRENT

The **work** done by an electric field in moving a charge q over a subcircuit is

$$A = qU = IUt,$$

where I is the current in the subcircuit, U the voltage across the subcircuit, and t the time during which the current flows in it.

According to the energy conservation law,

$$A = \Delta W,$$

where ΔW is the change in the energy of a conductor and surrounding bodies.

The **power** developed by a current passing through a conductor of resistance R is given by

$$N = \Delta W/t = A/t = IU = I^2R.$$

The amount of heat liberated upon the passage of a current through a conductor is determined by **Joule's law:**

$$Q = I^2Rt.$$

The **efficiency** is the quantity defined as

$$\eta = \frac{A_u}{A_d}\,100\%, \quad \eta = \frac{W_u}{W_s}\,100\%, \quad \eta = \frac{N_u}{N_l}\,100\%,$$

where A_u (W_u, N_u) is the useful work (energy, power) and A_d (W_s, N_l) the work done (spent energy, liberated power).

* * *

465. Determine the emf and the internal resistance of an accumulator supplying a power of 9.5 W to an external surface at a current of 5 A and 14.4 W at 8 A.

Given: $I_1 = 5$ A, $N_1 = 9.5$ W, $I_2 = 8$ A, $N_2 = 14.4$ W.

$\mathscr{E}$—? r—?

Solution. According to Ohm's law for a closed circuit,

$$I_1 = \mathscr{E}/(R_1 + r), \quad I_2 = \mathscr{E}/(R_2 + r). \qquad (1)$$

The resistances R_1 and R_2 can be determined from the relations $N_1 = I_1^2 R_1$ and $N_2 = I_2^2 R_2$, whence

$$R_1 = N_1/I_1^2, \quad R_2 = N_2/I_2^2. \qquad (2)$$

Substituting expressions (2) into (1), we obtain

$$I_1 = \mathscr{E}/(N_1/I_1^2 + r), \quad I_2 = \mathscr{E}/(N_2/I_2^2 + r). \qquad (3)$$

Solving Eqs. (3) together, we find that

$$\mathscr{E} = \frac{N_1 I_2^2 - N_2 I_1^2}{I_1 I_2 (I_2 - I_1)}, \quad r = \frac{N_1 I_2 - N_2 I_1}{I_1 I_2 (I_2 - I_1)}.$$

Then

$$\mathscr{E} = \frac{9.5 \times 8^2 - 14.4 \times 5^2}{5 \times 8 \times (8-5)} \text{ V} \simeq 2.1 \text{ V},$$

$$r = \frac{9.5 \times 8 - 14.4 \times 5}{5 \times 8 \times (8-5)} \ \Omega \simeq 0.03 \ \Omega.$$

466. In an electric circuit, the same power is developed at external resistances of 2 and 0.1 Ω. Determine the internal resistance of the source.

Given: $R_1 = 2\ \Omega$, $R_2 = 0.1\ \Omega$, $N_1 = N_2$.

r—?

Solution. According to Ohm's law for a closed circuit, for two values of the external resistance we have

$$\begin{aligned} I_1 &= \mathscr{E}/(R_1 + r), \quad I_2 = \mathscr{E}/(R_2 + r), \\ \mathscr{E} &= I_1 (R_1 + r), \quad \mathscr{E} = I_2 (R_2 + r). \end{aligned} \qquad (1)$$

The current is connected to power through the relation $N = I^2R$, whence $I = \sqrt{N/R}$. Substituting $I_1 = \sqrt{N/R_1}$ and $I_2 = \sqrt{N/R_2}$ into Eqs. (1), we obtain

$$\mathscr{E} = \sqrt{N/R_1}(R_1 + r), \quad \mathscr{E} = \sqrt{N/R_2}(R_2 + r). \qquad (2)$$

Equating the right-hand sides of expressions (2) and transforming the obtained equality, we can determine r:

$$\sqrt{N/R_1}(R_1+r) = \sqrt{N/R_2}(R_2+r),$$
$$\sqrt{R_2}(R_1+r) = \sqrt{R_1}(R_2+r),$$
$$\sqrt{R_2}R_1 + \sqrt{R_2}r = \sqrt{R_1}R_2 + \sqrt{R_1}r,$$
$$r(\sqrt{R_2} - \sqrt{R_1}) = \sqrt{R_1}R_2 - \sqrt{R_2}R_1$$
$$= \sqrt{R_1R_2}(\sqrt{R_2} - \sqrt{R_1}),$$
$$r = \sqrt{R_1R_2},$$
$$r = \sqrt{2 \times 0.1}\ \Omega \simeq 0.45\ \Omega.$$

467. A battery is formed by a few parallel-connected cells. When the current in an external circuit is 2 A, the useful power is 7 W. Determine the number of cells in the battery if the emf of a cell is 5.5 V and the internal resistance is 5 Ω.

Given: $I = 2$ A, $N = 7$ W, $\mathscr{E} = 5.5$ V, $r = 5\ \Omega$.

n—?

Solution. The useful power developed in the resistance R is

$$N = I^2R. \qquad (1)$$

According to Ohm's law for a closed circuit, we can write for the current $I = \mathscr{E}_b/(R + r_b)$, whence $R = (\mathscr{E}_b - Ir_b)/I$, where $\mathscr{E}_b = \mathscr{E}$ is the emf of the battery consisting of n identical parallel-connected cells, and $r_b = r/n$ is the internal resistance of the battery. Substituting the expressions for r_b, $\mathscr{E}_b$, and R into Eq. (1), we obtain $N = I(\mathscr{E} - Ir/n)$, whence

$$n = \frac{I^2r}{\mathscr{E}I - N},$$

$$n = \frac{2^2 \times 5}{5.5 \times 2 - 7} = 5.$$

468. A cell having an internal resistance of 4 Ω and an emf of 12 V is closed by a conductor of resistance 8 Ω. What amount of heat will be liberated in the external circuit per second?

Given: $r = 4\ \Omega$, $\mathscr{E} = 12$ V, $R = 8\ \Omega$, $t = 1$ s.

Q/t—?

Solution. The amount of heat liberated in the external circuit per second is

$$Q/t = I^2/R. \tag{1}$$

According to Ohm's law for a closed circuit, we have

$$I = \mathscr{E}/(R + r). \tag{2}$$

Substituting expression (2) into (1), we obtain

$$\frac{Q}{t} = \frac{\mathscr{E}^2 R}{(R+r)^2},$$

$$\frac{Q}{t} = \frac{12^2 \times 8}{(8+4)^2}\ \frac{\text{J}}{\text{s}} = 8\ \text{J/s}.$$

469. A kettle filled with 1 l of water at 16 °C is put on a hot plate of power 0.5 kW. The water in the kettle boils in 20 min after switching on the hot plate. What amount of heat is lost for heating the kettle and for radiation?

Given: $N = 5 \times 10^2$ W, $V = 1\ \text{l} = 10^{-3}\ \text{m}^3$, $T_1 = 289$ K, $t_1 = 20\ \text{min} = 1.2 \times 10^3$ s.

W'—?

Solution. The total energy spent for heating water and the kettle and for radiation is

$$W_s = Nt.$$

The useful energy required to heat water is

$$W_u = \Delta U = cm\,(T_2 - T_1).$$

Here ΔU is the change in the internal energy of water as a result of heating, c the specific heat for water, and $m = \rho V$ is the mass of water, where ρ is its density. Then the energy losses for heating the kettle, for radiation, etc. are

$$W' = W_s - W_u = Nt - c\rho V\,(T_2 - T_1),$$

$$W' = [5 \times 10^2 \times 1.2 \times 10^3 - 4.19 \times 10^3 \times 10^3 \times 10^{-3} \times (373-289)]\ \text{J} \simeq 250\ \text{kJ}.$$

470. A homogeneous iron conductor of length 100 m is connected to a d.c. source of 100 V for 10 s. What will be the change in the conductor temperature? The change in the conductor resistance upon heating should be neglected.

Given: $l = 10^2$ m, $U = 10^2$ V, $t = 10$ s.

ΔT—?

Solution. The amount of heat required for heating the iron conductor is

$$Q_1 = \Delta U = cm\,\Delta T. \qquad (1)$$

Here $m = DSl$ is the conductor mass, S the cross-sectional area, c the specific heat for iron, and D the density of iron.

According to Joule's law, the total amount of heat liberated in the conductor is

$$Q_2 = U^2t/R. \qquad (2)$$

Here $R = \rho l/S$, where ρ is the resistivity of iron. Neglecting heat losses, we can assume that $Q_1 = Q_2$, or, taking into account expressions (1) and (2), $cDSl\,\Delta T = U^2St/(\rho l)$, whence

$$\Delta T = \frac{U^2t}{cDl^2\rho},$$

$$\Delta T = \frac{(10^2)^2 \times 10}{0.46 \times 10^3 \times 7.8 \times 10^3 \times (10^2)^2 \times 1.2 \times 10^{-7}}\ \text{K} \simeq 23.3\ \text{K}.$$

471. Determine the resistance of the leads to a source of voltage 120 V if fuses made of a lead wire, having a cross-sectional area of 1 mm² and a length of 2 cm, melt during short-circuiting in 0.03 s. The initial temperature of a fuse is 27 °C.

Given: $U = 120$ V, $S = 1\ \text{mm}^2 = 10^{-6}\ \text{m}^2$, $l = 2\ \text{cm} = 2 \times 10^{-2}$ m, $t = 3 \times 10^{-2}$ s, $T = 300$ K.

R—?

Solution. The amount of heat required for heating the lead to the melting point and its fusion at this temperature is

$$Q_1 = \Delta U_1 + \Delta U_2.$$

Since $\Delta U_1 = cm\, \Delta T$, $\Delta U_2 = \lambda m$, $m = DlS$, and $\Delta T = T_m - T$, we have

$$Q_1 = DlS\,[c\,(T_m - T) + \lambda], \tag{1}$$

where D is the density of lead, c the specific heat for lead, and T_m its melting point.

During short-circuiting, the resistance of the circuit is $R + R_f$, where $R_f = \rho l/S$ is the resistance of the fuse.

The short-circuit current is

$$I = U/(R + R_f).$$

According to Joule's law, the amount of heat liberated in the fuse during a time t is $Q_2 = I^2 R_f t$. Therefore,

$$Q_2 = \frac{U^2 \rho l t}{(R + \rho l/S)^2 S}. \tag{2}$$

Assuming that the entire amount of heat liberated in the fuse is spent for its heating and melting, we can write $Q_1 = Q_2$, or, taking into account Eqs. (1) and (2),

$$DlS\,[c\,(T_m - T) + \lambda] = \frac{U^2 \rho l t}{(R + \rho l/S)^2 S},$$

whence

$$R = \frac{1}{S}\left\{U \sqrt{\frac{\rho t}{D\,[c\,(T_m - T) + \lambda]}} - \rho t\right\},$$

$$R = \frac{1}{10^{-6}}\left[120 \times \left(\frac{2.1 \times 10^{-7} \times 3 \times 10^{-2}}{11.3 \times 10^3 \times [0.13 \times 10^3 \times (600 - 300) + 0.25 \times 10^5]}\right)^{1/2} - 2.1 \times 10^{-7} \times 2 \times 10^{-2}\right] \Omega \simeq 0.34\ \Omega.$$

472. Air contained in a closed vessel of volume 1 l under normal conditions is heated by an electric heater rated for a current of 0.2 A and a voltage of 10 V. In what time will the pressure in the vessel rise to 1 MPa? The efficiency of the heater is 50%.

Given: $V = 1\ \text{l} = 10^{-3}\ \text{m}^3$, $I = 0.2$ A, $U = 10$ V, $p_2 = 1$ MPa $= 10^6$ Pa, $\eta = 50\% = 0.5$.

t—?

Solution. The amount of heat liberated by the current during a time t can be determined from Joule's law:

$$Q_1 = I^2Rt = IUt. \qquad (1)$$

The amount of heat required to heat the air from $T_1 =$ 273 K to a temperature T_2 is

$$Q_2 = cm\,(T_2 - T_1), \qquad (2)$$

where c is the specific heat for air and m its mass. By hypothesis, $Q_2 = 0.5Q_1$, or, taking into account expressions (1) and (2), $cm\,(T_2 - T_1) = 0.5IUT$, whence

$$t = cm\,(T_2 - T_1)/(0.5IU). \qquad (3)$$

Here $m = \rho V$, where ρ is the density of air.

Since the volume V occupied by the gas does not change upon heating, by applying Gay-Lussac's law $p_1/T_1 = p_2/T_2$, we can determine the final temperature of the gas: $T_2 = T_1p_2/p_1$. Substituting the expressions for m and T_2 into Eq. (3), we obtain

$$t = \frac{c\rho V\,(T_1p_2/p_1 - T_1)}{0.5IU} = \frac{c\rho VT_1\,(p_2/p_1 - 1)}{0.5IU},$$

$$t = \frac{1.005\times10^3\times1.29\times10^{-3}\times10^{-3}\times273\times[10^6/(1.01\times10^5)-1]}{0.5\times0.2\times10}\ \text{s}$$

$$\simeq 3.18\times10^3\ \text{s}.$$

473. Determine the efficiency of a current source with an internal resistance of 0.1 Ω loaded by a resistance of 1.5 Ω.

Given: $r = 0.1$ Ω, $R = 1.5$ Ω.

η—?

Solution. By definition, the efficiency is

$$\eta = \frac{N_u}{N_l}\ 100\%,$$

where N_u is the useful power liberated in the external subcircuit and N_l is the power liberated in the entire closed circuit. The powers N_u and N_l can be determined from the relations $N_u = I^2R$ and $N_l = I^2(R + r)$. This gives

$$\eta = \frac{I^2R \cdot 100\%}{I^2(R+r)} = \frac{R}{R+r}\,100\%,$$

$$\eta = \frac{1.5}{1.5+0.1}\,100\% \simeq 94\%.$$

474°. The useful power liberated in an external subcircuit attains its maximum value of 5 W at a current of 5 A. Determine the internal resistance and the emf of the current source.

Given: $N_{max} = 5$ W, $I_{max} = 5$ A.

r—? $\mathscr{E}$—?

Solution. According to Joule's law, the useful power liberated in the external subcircuit of resistance R is

$$N = I^2R. \tag{1}$$

According to Ohm's law for a closed circuit,

$$I = \mathscr{E}/(R + r).$$

Therefore (see Eq. (1)), we can write

$$N = \mathscr{E}^2R/(R + r)^2. \tag{2}$$

The power liberated in the external subcircuit has the maximum value ($N = N_{max}$) when the following condition is satisfied:

$$\frac{dN}{dR} = 0. \tag{3}$$

Differentiating Eq. (3) with respect to R, we can find the external resistance R_{max} at which the power has the maximum value:

$$\frac{d}{dR}\left[\frac{\mathscr{E}^2R}{(R+r)^2}\right] = \frac{\mathscr{E}^2(R+r)^2 - \mathscr{E}^2R\cdot 2(R+r)}{(R+r)^4} = 0,$$

$$(R+r)^2 - 2R(R+r) = 0, \quad (R+r)(R+r-2R) = 0,$$

$$R + r - 2R = 0, \quad R = R_{max} = r.$$

Substituting R_{max} into Eqs. (1) and (2), we obtain $N_{max} = I^2_{max}r$, whence

$$r = N_{max}/I^2_{max},$$
$$r = (5/5^2)\ \Omega = 0.2\ \Omega.$$

Therefore, $N_{max} = \mathscr{E}^2r/(4r^2) = \mathscr{E}^2/(4r)$, whence

$$\mathscr{E} = \sqrt{4rN_{max}} = \sqrt{\frac{4N_{max}}{I^2_{max}}} = 2\,\frac{N_{max}}{I_{max}},$$

$$\mathscr{E} = 2\,\frac{5}{5}\ \text{V} = 2\ \text{V}.$$

475. Is the work done by a current source in the internal subcircuit a constant quantity for a given source?

Answer. No, it is not.

476. Why is the filament of a bulb strongly heated, while the leads remain cold?

Answer. The amount of heat liberated in conductors connected in series is directly proportional to their resistance (Joule's law). The resistance of the filament is very high, while the resistance of the leads is very low, and hence the filament is heated strongly, while the leads are heated only slightly.

EXERCISES

477. Three conductors of resistance 3, 6, and 8 Ω respectively are connected in parallel. The amount of heat liberated in the first conductor is 21 kJ. Determine the amount of heat liberated in the second and third conductors during the same time.

478. Two conductors of resistance 10 and 6 Ω are connected first in series and then in parallel between two points with a potential difference of 20 V. Determine the amount of heat liberated in each conductor per second.

479. A current flows through a copper conductor of length 2 m and a cross-sectional area of 0.4 mm^2. The amount of heat liberated per second is 0.35 J. How many electrons pass through the cross section of the conductor per second?

480. What is the length of a Nichrome conductor of diameter 0.5 mm used for making an electric heater operating at a voltage of 120 V and producing 1 MJ of heat per hour?

481. Two conductors of the same resistance R are connected to a circuit at a voltage U first in parallel and then in series. In which case is the power consumed from the circuit higher?

482. A pot containing 1 l of water and 0.5 kg of ice at 0 °C is put on a hot plate of power 600 W. In what time will the temperature of the water in the pot rise to 60 °C if the efficiency of the hot plate is 80%?

483. A potential difference of 10 V is applied at the ends of a lead wire of length 1 m. What time will elapse from the beginning of the passage of current to the moment at which the lead starts melting? The initial temperature of the lead is 27 °C.

484. Two identical refractory vessels containing tin and lead respectively are heated on a hot plate. The masses of the tin and the lead are equal, and their initial temperature is 20 °C. What will be the ratio of the durations of melting of the metals?

485. Determine the current in a circuit of a lead accumulator if its emf is 2.2 V, the external resistance is 0.5 Ω, and the efficiency is 65%.

CURRENT IN ELECTROLYTES AND GASES

Solutions conducting electric current are known as **electrolytes**. The current in them is due to the motion of positive and negative ions. The passage of the current through an electrolyte is accompanied by the deposition of the constituents of the dissolved material **(electrolysis)**.

The mass of a substance deposited during an electrolysis can be calculated by using **Faraday's generalized law of electrolysis:**

$$m = AIt/(Fn),$$

where m is the mass of the deposited substance, A its molar mass, I the current through the electrolyte, t the time during which the current flows, F the **Faraday constant,** and n the valence.

Gases become conductors when they are ionized, i.e.

contain free electrons and positive and negative ions.

Extrinsic conduction is due to the ionization of a gas caused by an external effect (flame, ultraviolet or X-radiation, etc.).

Intrinsic conduction is due to charged particles accelerated by an electric field and ionizing neutral molecules as a result of collisions.

The minimum potential difference required to ionize an atom or a molecule of a substance by an electron accelerated by the potential is known as the **ionization potential** U_i for the given atom or molecule.

The energy acquired by an electron having passed through this potential difference is called the **ionization energy**:

$$W_i = eU_i,$$

where e is the electron charge.

The **glow discharge**, the **arc discharge**, the **spark discharge**, and the **corona discharge** are different types of self-sustained gas discharge.

* * *

486. During nickel plating of a plate, its surface is covered by a nickel layer of thickness 0.05 mm. Determine the mean current density if the nickel plating lasts for 2.5 h.

Given: $h = 0.05\ \text{mm} = 5 \times 10^{-5}\ \text{m}$, $t = 2.5\ \text{h} = 9 \times 10^3\ \text{s}$.

j—?

Solution. The current density is defined as

$$j = I/S,$$

where S is the cross-sectional area of the conducting part of the electrolyte, equal to the area of the plate.

In order to determine the current, we shall apply Faraday's generalized law

$$m = AIt/(Fn). \tag{1}$$

Here

$$m = \rho V = \rho h S, \tag{2}$$

where ρ is the density of nickel.

Substituting Eq. (2) into (1), we obtain the current:

$$I = \rho hSFn/(At).$$

Then the current density is

$$j = \frac{\rho hSFn}{ASt} = \frac{\rho hFn}{At},$$

$$j = \frac{8.8\times10^3\times5\times10^{-5}\times9.65\times10^4\times2}{58.7\times10^{-3}\times9\times10^3}\ \frac{\text{A}}{\text{m}^2} = 160\ \text{A/m}^2.$$

487. During an electrolysis of silver nitrate solution lasting for an hour, 9.4 g of silver are deposited at the cathode. Determine the emf of polarization if the voltage across the bath terminals is 4.2 V and the resistance of the solution is 1.5 Ω.

Given: $t = 1\ \text{h} = 3.6\times10^3$ s, $m = 9.4\ \text{g} = 9.4\times10^{-3}$ kg, $U = 4.2$ V, $R = 1.5\ \Omega$.

$\mathscr{E}_p$—?

Solution. During the electrolysis of silver nitrate solution, the symmetry of the electrodes made of the same material is violated, and electrodes are polarized. In this case, the emf of polarization appears, and Ohm's law for a subcircuit containing an emf of polarization becomes $U = IR + \mathscr{E}_p$, whence

$$\mathscr{E}_p = U - IR. \tag{1}$$

Since $I = q/t$, and according to Faraday's law, $q = mFn/A$, Eq. (1) can be transformed as follows:

$$\mathscr{E}_p = U - \frac{mFnR}{At},$$

$$\mathscr{E}_p = 4.2 - \frac{9.4\times10^{-3}\times9.65\times10^4\times1\times1.5}{107.9\times10^{-3}\times3.6\times10^3}\ \text{V} \simeq 0.7\ \text{V}.$$

488. How many atoms of a bivalent metal will be deposited on a square centimetre of the electrode surface in 5 min at a current density of 0.1 A/dm²?

Given: $S = 1\ \text{cm}^2 = 10^{-4}\ \text{m}^2$, $t = 5\ \text{min} = 300$ s, $j = 0.1\ \text{A/dm}^2 = 10\ \text{A/m}^2$, $n = 2$.

n_1—?

Solution. According to Faraday's law, we have

$$m = AIt/(Fn),$$

or, considering that $I = jS$ and $m = AjSt/(Fn)$, we obtain

$$m/A = jSt/(Fn). \quad (1)$$

Here m/A is the number of moles. Multiplying this quantity by Avogadro's constant, we obtain the number n_1 of atoms deposited on the electrode surface:

$$n_1 = mN_A/A.$$

Using Eq. (1), we get

$$n_1 = \frac{jStN_A}{Fn},$$

$$n_1 = \frac{10 \times 10^{-4} \times 300 \times 6.02 \times 10^{23}}{9.65 \times 10^4 \times 2} = 9.4 \times 10^{18}.$$

489. A copper plate containing 12% of impurities is the anode in a copper sulphate solution. During an electrolysis, copper is dissolved and deposited in pure form at the cathode. What is the cost of the refining of 1 kg of such a copper if the voltage across a bath is maintained at 6 V, and the cost of 1 kWh of energy is 4 kopecks?

Given: $M = 1$ kg, $U = 6$ V.

x—?

Solution. The mass of the pure copper deposited at the cathode is

$$m = M - 0.12M = 0.88M.$$

According to Faraday's law,

$$m = AIt/(Fn) = Aq/(Fn),$$

whence

$$q = mnF/A = 0.88MnF/A. \quad (1)$$

The energy spent during the electrolysis is $W = qU$, or, taking into account expression (1),

$$W = \frac{0.88MnFU}{A},$$

$$W = \frac{0.88 \times 1 \times 2 \times 9.65 \times 10^4 \times 6}{64 \times 10^{-3}} \text{ J} = 4.5 \text{ kWh}.$$

Consequently, the cost of the refining of 1 kg of such a copper is $x = 18$ kop.

490°. Determine the mass of copper deposited at the cathode as a result of the passage of a current which uniformly increases from 0 to 4 A through a copper sulphate solution for 10 s.

Given: $t_1 = 0$, $t_2 = 10$ s, $I_1 = 0$, $I_2 = 4$ A.

m—?

Solution. According to Faraday's law, the mass of the substance deposited at the cathode is

$$m = Aq/(Fn), \tag{1}$$

where the charge passing through the copper sulphate solution during the time t_2 is

$$q = \int_0^{t_2} I\,dt. \tag{2}$$

By hypothesis,

$$I = kt, \tag{3}$$

where k is the proportionality factor. For the instant of time t_2, $I_2 = kt_2$, whence $k = I_2/t_2$. This gives

$$I = I_2 t/t_2. \tag{4}$$

Substituting Eq. (4) into (2) and integrating, we obtain

$$q = \int_0^{t_2} \frac{I_2}{t_2} t\,dt = \frac{I_2}{t_2} \int_0^{t_2} t\,dt = \frac{I_2}{t_2} \frac{t_2^2}{2} = \frac{I_2 t_2}{2}. \tag{5}$$

Using expression (5) in (1), we get

$$m = \frac{AI_2t_2}{2Fn},$$

$$m = \frac{64 \times 10^{-3} \times 4 \times 10}{2 \times 9.65 \times 10^4 \times 2}\ \text{kg} = 6.65 \times 10^{-6}\ \text{kg}.$$

491. Determine the mass of oxygen evolved as a result of the passage of a charge of 16 C through an aqueous solution of sulphuric acid. The mass of an oxygen atom is 2.6×10^{-26} kg.

Given: $q = 16$ C, $m = 2.6 \times 10^{-26}$ kg.

M—?

Solution. The molecules of sulphuric acid H_2SO_4 dissociate in water into positive (H^+) and negative (SO_4^{--}) ions. This reaction can be written in the form

$$H_2SO_4 \rightarrow 2H^+ + SO_4^{--}.$$

The group SO_4^{--} deposited at the anode enters into the reaction with water:

$$SO_4 + H_2O = H_2SO_4 + O.$$

As a result, gaseous oxygen is evolved at the anode. The last equation shows that as a group SO_4^{--} bearing a charge of $2e$ passes through the solution, an oxygen atom is evolved at the anode. Consequently, when a charge q passes through the solution, $q/(2e)$ atoms of oxygen will be evolved at the anode. Denoting by m the mass of an oxygen atom, we find that the mass of the evolved oxygen is

$$M = \frac{mq}{2e},$$

$$M = \frac{2.6 \times 10^{-26} \times 16}{2 \times 1.6 \times 10^{-19}} \text{ kg} = 1.3 \times 10^{-6} \text{ kg}.$$

492. Undiluted sulphuric acid is stored in iron containers, while diluted acid is stored in glass vessels. Why?

Answer. Undiluted sulphuric acid is not an electrolyte, while diluted acid is. If diluted acid were stored in an iron container, an electrolysis would take place.

493. The direction of current in modern galvanic baths is being altered. Why?

Answer. A metal is deposited at an article during a passage of the forward current. During a passage of a short-time reverse current, the deposited metal is partially dissolved (mainly at microscopic projections of the cathode). This improves the quality of plating.

494. Determine the ionization energy for a helium atom if its ionization potential is 24.5 V.

Given: $U_i = 24.5$ V.

W_i—?

Solution. By definition, the ionization energy is

$$W_1 = eU_1,$$

where e is the electron charge. Therefore,

$$W_1 = 1.6 \times 10^{-19} \times 24.5 \text{ J} = 39.2 \times 10^{-19} \text{ J}.$$

495. What must be the minimum velocity required for an electron to ionize hydrogen atoms? The ionization potential for a hydrogen atom is 13.5 V.

Given: $U_1 = 13.5$ V.

v—?

Solution. In order to ionize a hydrogen atom, the electron must have a kinetic energy equal to the ionization energy:

$$W_k = W_1, \quad \text{or} \quad mv^2/2 = eU_1,$$

where e and m are the electron charge and mass. Then

$$v = \sqrt{\frac{2eU_1}{m}},$$

$$v = \sqrt{\frac{2 \times 1.6 \times 10^{-19} \times 13.5}{9.1 \times 10^{-31}}} \frac{\text{m}}{\text{s}} = 2.2 \times 10^6 \text{ m/s}.$$

496. Determine the saturation current density in a gas-discharge tube with an interelectrode distance of 10 cm if 10 pairs of monovalent ions are formed in a cubic centimetre of the tube per second under the action of cosmic radiation.

Given: $l = 10$ cm $= 0.1$ m, $n_{pt} = 10$ cm$^{-3} \cdot$ s^{-1} $=$ 10^7 m$^{-3} \cdot$ s^{-1}.

j_s—?

Solution. The saturation current density is

$$j_s = I_s/S, \qquad (1)$$

where I_s is the saturation current and S the cross-sectional area of the tube.

Considering that $I_s = q/t$, $q = enV$, and $V = lS$, where q is the charge passing through the tube during a time t, e the charge of a monovalent ion, V the volume of the tube, and $n = 2n_p$ is the ion concentration (n_p the number of ion pairs), and using these expressions in Eq. (1), we obtain

$$j_s = 2en_p lS/(tS) = 2en_p l/t. \qquad (2)$$

Denoting by $n_p/t = n_{pt}$ the number of ion pairs formed in a cubic metre of the tube per second, we can write expression (2) in the form

$$j_s = 2en_{pt}l,$$
$$j_s = 2 \times 1.6 \times 10^{-19} \times 10^7 \times 0.1 \text{ A/m}^2$$
$$= 3.2 \times 10^{-13} \text{ A/m}^2.$$

497. Determine the mean velocity of the directional motion of monovalent ions in an ionization chamber if their concentration is 10^3 cm^{-3} and the saturation current density is 10^{-12} A/m^2.

Given: $n = 10^3 \text{ cm}^{-3} = 10^9 \text{ m}^{-3}$, $j_s = 10^{-12} \text{ A/m}^2$.

$\langle v \rangle$—?

Solution. By definition,

$$\langle v \rangle = l/t, \qquad (1)$$

where l is the length of the chamber.

Using the solution of Problem 496, we can write

$$j_s = 2en_{pt}l = en_t l, \qquad (2)$$

where n_t is the number of ions formed in a cubic metre per second. It follows from expression (2) that $l = j_s/(en_t)$, or since $n_t = n/t$,

$$l = j_s t/(en). \qquad (3)$$

Substituting expression (3) into (1), we obtain

$$\langle v \rangle = \frac{j_s t}{ent} = \frac{j_s}{en},$$

$$\langle v \rangle = \frac{10^{-12}}{1.6 \times 10^{-19} \times 10^9} \frac{\text{m}}{\text{s}} = 6.2 \times 16^{-3} \text{ m/s}.$$

498. Why does a charged electroscope always get discharged under the room conditions?

Answer. The electric field produced between the charged leaves of the electroscope always contains a certain number of ions formed as a result of action of external ionizers (like cosmic rays). These ions move to the electroscope leaves and neutralize their charge.

499. Why are high-tension transmission lines always supplied with two additional wires which are not insulated from the steel supports and are arranged above the main wires?

Answer. These additional wires are intended for protecting high-tension transmission lines from lightning discharges.

EXERCISES

500. The amount of copper deposited at the cathode per hour during an electrolysis of copper sulphate is 0.5 kg. The surface area of the electrodes immersed in the electrolyte is 7.5 m^2. Determine the current density.

501. What amount of electric energy should be supplied for depositing 500 mg of silver during an electrolysis of a silver nitrate solution? The potential difference between the electrodes is 4 V.

502. The amount of a bivalent metal deposited at the cathode in an electrolytic bath during 10 min at a current of 5 A is 1.017 g. Determine the atomic mass of the metal.

503. Determine the thickness of the copper layer deposited at the cathode during a 5-h electrolysis of copper sulphate if the current density is 0.8 A/dm^2.

504. What amount of copper will be deposited at the cathode during an electrolysis if the amount of supplied electric energy is 5 kWh? The voltage across the bath terminals is 10 V and the efficiency of the set-up is 75%.

505. A power of 37 kW is supplied during an electrolysis of a sulphuric acid solution. Determine the resistance of the electrolyte if 0.3 g of hydrogen are evolved at the cathode during 50 min.

506. Determine the mass of chlorine evolved as a result of the passage of a charge of 16 C through a hydrochloric acid solution.

507. An electron flying at a velocity of 2.2×10^6 m/s ionizes a gas. Determine the ionization potential for the gas.

508. Determine the saturation current in an ionization chamber with a surface area of the electrodes of 100 cm^2 and an interelectrode separation of 6.2 cm. An ionizer generates 10^9 monovalent ions of each sign in a cubic centimetre of the chamber per second.

QUESTIONS FOR REVISION

1. What is electric current? 2. Define current and current density, name their units. 3. What is the resistance of a conductor? How does it depend on the geometrical size, material, and temperature of the conductor? 4. Define voltage, emf, and potential difference. What is the relation between them? 5. Formulate Ohm's law for a homogeneous and a heterogeneous conductor and for a closed circuit. 6. Write expressions for the resistance of a subcircuit containing series- and parallel-connected conductors. 7. What are the duties of a shunt and a series resistance? What are the ways of their connection to measuring instruments? 8. Formulate Kirchhoff's laws. 9. Write formulas for calculating the work and the power of a current. 10. Formulate Joule's law. 11. What process is called electrolysis? 12. Formulate Faraday's laws of electrolysis. 13. Explain the physical meaning of electrochemical and chemical equivalents. 14. What is the physical meaning of the Faraday constant? 15. What is gas discharge? 16. Name the types of gas discharge. 17. Define ionization potential.

3.3. Electromagnetism

MAGNETIC FIELD OF A CURRENT. FORCES ACTING IN A MAGNETIC FIELD ON A CURRENT-CARRYING CONDUCTOR, A MOVING ELECTRIC CHARGE, AND A CURRENT LOOP

Magnetic induction is a vector physical quantity whose magnitude is given by

$$B = \mu\mu_0 I/(2\pi r)$$

for an infinitely long straight conductor,

$$B = \mu\mu_0 I/(2R)$$

at the centre of a circular loop, and

$$B = \mu\mu_0 In$$

on the axis of a solenoid (inside it), where r is the shortest distance from the conductor to a point where the induction is determined, R the radius of the loop, and n the number of turns per unit length of the solenoid. In all cases, the direction of the vector $\mathbf{B}$ is determined by the right-hand screw rule.

If a magnetic field is produced by several currents, the magnetic induction at a given point is the vector sum of the magnetic inductions $\mathbf{B}_i$ of the fields produced at the point by each current separately (the **principle of superposition of magnetic fields**):

$$\mathbf{B} = \sum_{i=1}^{n} \mathbf{B}_i.$$

Magnetic induction and magnetic field strength are connected through the relation

$$B = \mu\mu_0 H,$$

where H is the **magnetic field strength**, μ the **permeability of a medium**, and μ_0 the **magnetic constant**.

A **current-carrying conductor in a uniform magnetic field** is acted upon by the **Ampère force**

$$F = IlB \sin \alpha,$$

where I is the current in the conductor, l its length, and α the angle between the directions of the magnetic induction vector and the current. The direction of the Ampère force can be determined by the left-hand rule.

If a magnetic field is nonuniform, the Ampère force acting on an element of length $\mathrm{d}l$ of the conductor is

$$\mathrm{d}F = IB \sin \alpha \, \mathrm{d}l.$$

The force acting in this case on the entire conductor is

$$F = \int_l IB \sin \alpha \, \mathrm{d}l.$$

An electric charge moving in a magnetic field experiences the action of the **Lorentz force**

$$F = qvB \sin \alpha,$$

where q is the charge, v the velocity of its motion, and α the angle between the directions of the magnetic induction and the velocity vectors. The direction of the Lorentz force can be determined by the left-hand rule (for a positive charge).

A current loop in a magnetic field is under the action of a force couple with a torque

$$M = p_m B \sin \alpha,$$

where $p_m = IS$ is the magnetic moment of the current loop, S its area, and α the angle between the directions of the magnetic induction vector and the normal to the plane of the loop.

* * *

509. The currents $I_1 = I_2$ and $I_3 = I_1 + I_2$ pass through three long straight conductors arranged in the same plane parallel to one another at a distance of 3 cm

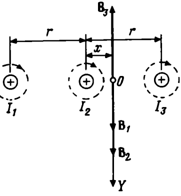


Fig. 153

(Fig. 153). Determine the position of a straight line such that the magnetic induction of the field produced by the currents is zero at each of its points.

Given: $r = 3$ cm $= 3 \times 10^{-2}$ m, $I_1 = I_2$, $I_3 = I_1 + I_2$.

x—?

Solution. Let us suppose that the currents I_1, I_2, and I_3 flow in the plane perpendicular to the plane of the figure away from us (the directions of the currents are indicated by crosses). The magnetic induction vectors produced by the currents are directed, according to the right-hand screw rule, along the tangent at any point of the magnetic field line (the field lines are indicated by dashed circles).

The sought straight line on which the magnetic induction vector is zero can be arranged between the currents I_2 and I_3 at a distance x from I_2. Indeed, the induction vectors $\mathbf{B}_1$ and $\mathbf{B}_2$ of the fields produced by the currents I_1 and I_2 at point O are directed downwards, while the induction vector $\mathbf{B}_3$ of the field produced by the current I_3 at this point is directed upwards. According to the superposition principle, we can write

$$\mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3 = 0,$$

or in projections on the Y-axis,

$$B_1 + B_2 - B_3 = 0. \quad (1)$$

The magnetic induction of the field produced by an infinitely long straight current-carrying conductor is

$$B = \mu\mu_0 I/(2\pi r).$$

This gives

$$B_1 = \frac{\mu\mu_0 I_1}{2\pi (r+x)},$$
$$B_2 = \frac{\mu\mu_0 I_2}{2\pi x}, \quad (2)$$
$$B_3 = \frac{\mu\mu_0 (I_1+I_2)}{2\pi (r-x)}.$$

Substituting expressions (2) into (1), we obtain

$$\frac{\mu\mu_0 I_1}{2\pi (r+x)} + \frac{\mu\mu_0 I_2}{2\pi x} - \frac{\mu\mu_0 (I_1+I_2)}{2\pi (r-x)} = 0,$$

or after the transformation $4x^2 + rx - r^2 = 0$, we get

$$x = \frac{-r \pm \sqrt{r^2+16r^2}}{8} = \frac{-3\times 10^{-2} \pm 12.4\times 10^{-2}}{8}\ \mathrm{m}.$$

Consequently, $x_1 \simeq 1.2 \times 10^{-2}$ m. The second root of the quadratic equation, $x_2 = -1.9 \times 10^{-2}$ m, corresponds to a point lying between the currents I_1 and I_2, which is also possible. Therefore, the problem has two solutions.

510. Currents of 10 A flow in the same direction along two long straight conductors separated by 5 cm. Determine the magnetic induction at a point lying at a distance of 3 cm from each conductor.

Given: $l = 5\ \text{cm} = 5\times10^{-2}\ \text{m}$, $I_1 = I_2 = I = 10\ \text{A}$, $r = 3\ \text{cm} = 3\times10^{-2}\ \text{m}$.

B—?

Solution. The magnetic induction vector **B** of the field at point A is the vector sum of the magnetic inductions $\mathbf{B}_1$ and $\mathbf{B}_2$ of the fields produced by each current separately at this point (Fig. 154). The directions of the vectors $\mathbf{B}_1$ and $\mathbf{B}_2$ can be determined from the right-hand screw rule.

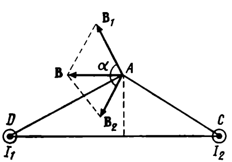


Fig. 154

The numerical value of the magnetic induction at point A can be found by using the cosine law:

$$B = \sqrt{B_1^2 + B_2^2 + 2B_1B_2\cos\alpha}. \qquad (1)$$

The magnetic inductions of the fields produced by each current at point A are

$$B_1 = \mu\mu_0 I/(2\pi r_1), \qquad B_2 = \mu\mu_0 I/(2\pi r_2).$$

Since $r_1 = r_2 = r$ and $B_1 = B_2$, we have (see Eq. (1))

$$B = \sqrt{2B_1^2 + 2B_1^2\cos\alpha} = B_1\sqrt{2 + 2\cos\alpha}. \qquad (2)$$

Applying the cosine law to $\triangle ADC$, we find that $l^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos\alpha$, whence $\cos\alpha = (r_1^2 + r_2^2 - l^2)/(2r_1r_2) = (2r^2 - l^2)/(2r^2)$. Substituting the expressions for B_1 and $\cos\alpha$ into Eq. (2), we obtain

$$B = \frac{\mu\mu_0 I}{2\pi r}\sqrt{2 + \frac{2(2r^2 - l^2)}{2r^2}} = \frac{\mu\mu_0 I}{2\pi r^2}\sqrt{4r^2 - l^2},$$

$$B = \frac{4\times 3.14\times 10^{-7}\times 10}{2\times 3.14\times(3\times 10^{-2})^2}\sqrt{4\times(3\times 10^{-2})^2 - (5\times 10^{-2})^2}\ \mathrm{T}$$
$$\simeq 66.6\ \mu\mathrm{T}.$$

511. A current of 10 A passes through a ring made of a copper wire and having a cross-sectional area of 1 mm^2. A potential difference of 0.15 V is applied across the ends of the ring. Determine the magnetic induction of the field at the centre of the ring.

Given: $S = 1\ \mathrm{mm}^2 = 10^{-6}\ \mathrm{m}^2$, $I = 10$ A, $U = 0.15$ V.

B—?

Solution. The magnetic induction of the field at the centre of a circular current is

$$B = \mu\mu_0 I/(2r), \tag{1}$$

where r is the radius of the ring. According to Ohm's law for a conductor, the potential difference across the ends of the ring is

$$U = IR. \tag{2}$$

Considering that $R = \rho l/S$ and $l = 2\pi r$, where R is the resistance of the ring and l its length, we write Eq. (2) as $U = I\rho 2\pi r/S$, whence

$$r = US/(I2\pi\rho). \tag{3}$$

Substituting expression (3) into (1), we obtain

$$B = \frac{\mu\mu_0\pi\rho I^2}{US},$$

$$B = \frac{4\times 3.14\times 10^{-7}\times 1\times 3.14\times 0.17\times 10^{-7}\times 10^2}{0.15\times 10^{-6}}\ \mathrm{T} = 44\ \mu\mathrm{T}.$$

512. Identical currents of 2 A flow in two identical circular loops of radius 5 cm whose planes are mutually perpendicular and whose centres coincide. Determine the magnetic induction of the field at the centre of the loops.

Given: $R = 5$ cm $= 5 \times 10^{-2}$ m, $I = 2$ A.

B—?

Solution. The magnetic induction of the field produced by each circular current at the centre of the corresponding loop is

$$B_1 = B_2 = \mu\mu_0 I/(2R). \tag{1}$$

According to the right-hand screw rule for the chosen directions of the currents in the loops, the vector $\mathbf{B}_1$ is directed away from us and the vector $\mathbf{B}_2$ to the right (Fig. 155). According to the superposition principle, the induction of the resultant magnetic field is

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2.$$

Since $\mathbf{B}_1$ and $\mathbf{B}_2$ are mutually perpendicular, we can apply the Pythagorean theorem to the corresponding right triangle and find that

$$B = \sqrt{B_1^2 + B_2^2} = B_1\sqrt{2}. \tag{2}$$

Substituting Eq. (1) into (2), we obtain

$$B = \frac{\mu\mu_0\sqrt{2}}{2R},$$

$$B = \frac{4 \times 3.14 \times 10^{-7} \times 1 \times 2 \times 1.41}{2 \times 5 \times 10^{-2}}\ \mathrm{T} \simeq 35.4\ \mu\mathrm{T}.$$

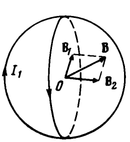


Fig. 155

513. A solenoid of length 20 cm and diameter 5 cm should produce a magnetic induction of 1.26 mT on its axis.

Determine the potential difference that must be applied across the ends of the solenoid winding if a copper wire of diameter 0.5 mm is used for it.

Given: $L = 20$ cm $= 0.2$ m, $D = 5$ cm $= 5 \times 10^{-2}$ m, $B = 1.26$ mT $= 1.26 \times 10^{-3}$ T, $d = 0.5$ mm $=$ 5×10^{-4} m.

U—?

Solution. The potential difference across the ends of the solenoid winding is

$$U = IR. \tag{1}$$

Using the expression $B = \mu\mu_0 In$, we can determine the current through the solenoid:

$$I = B/(\mu\mu_0 n), \tag{2}$$

where $n = N/L$ is the number of turns per unit length of the solenoid. Since the total number of turns is $N = L/d$, expression (2) can be written in the form

$$I = Bd/(\mu\mu_0 L). \tag{3}$$

The resistance of the winding is

$$R = \rho l/S.$$

Since ρ is the resistivity of copper, $l = \pi DN = \pi DL/d$, and $S = \pi d^2/4$, we have

$$R = 4\rho DL/d^3. \tag{4}$$

Transforming Eq. (1) by (3) and (4), we obtain

$$U = \frac{4\rho BLD}{\mu\mu_0 d^2},$$

$$U = \frac{4 \times 0.17 \times 10^{-7} \times 1.26 \times 10^{-3} \times 0.2 \times 5 \times 10^{-2}}{4 \times 3.14 \times 10^{-7} \times (5 \times 10^{-4})^2} \text{ V} \simeq 2.7 \text{ V}.$$

514. Two parallel conductors carrying the same current are separated by a distance of 8.7 cm and attract each other with a force of 2.5×10^{-2} N. Determine the current in the conductors if the length of each conductor is 320 cm and the currents in them have the same direction.

Given: $r = 8.7\ \text{cm} = 8.7\times10^{-2}\ \text{m}$, $F = 2.5\times10^{-2}\ \text{N}$, $l = 320\ \text{cm} = 3.2\ \text{m}$.

I—?

Solution. Since, by hypothesis, $r \ll l$, we can assume that the wires are infinitely long. Therefore, the magnetic induction of the field produced by the conductor I_1 (Fig. 156) is

$$B_1 = \mu\mu_0 I_1/(2\pi r). \tag{1}$$

According to Ampère's law, the force acting on the conductor I_2 is $F_2 = I_2 B_1 l$, or, taking into account Eq. (1), $F_2 = I_2\mu\mu_0 I_1 l/(2\pi r)$. Since $I_1 = I_2 = I$, we have $F_2 = \mu\mu_0 I^2 l/(2\pi r)$, whence

$$I = \sqrt{\frac{2\pi r F_2}{\mu\mu_0 l}},$$

$$I = \sqrt{\frac{2\times3.14\times8.7\times10^{-2}\times2.5\times10^{-2}}{4\times3.14\times10^{-7}\times3.2}}\ \text{A} \simeq 58\ \text{A}.$$

515°. A metal rod of length 15 cm is perpendicular to an infinitely long straight wire carrying a current of 2 A. Determine the force exerted by the magnetic field produced by the wire on the rod if the current in the rod is 0.5 A and the distance between the wire and the nearest end of the rod is 5 cm.

Given: $l = 15\ \text{cm} = 15\times10^{-2}\ \text{m}$, $I_2 = 2\ \text{A}$, $r = 5\ \text{cm} = 5\times10^{-2}\ \text{m}$, $I_1 = 0.5\ \text{A}$.

F_1—?

Solution. We divide the rod CD into small elements of length dx and consider such an element at an arbitrary distance x from the vertical wire (Fig. 157). According to Ampère's law, the force acting on each element dx is equal to

$$dF_1 = I_1 B_2\, dx$$

and is directed vertically upwards, in accordance with the left-hand rule for the chosen directions of the currents. Here

$$B_2 = \mu\mu_0 I_2/(2\pi x) \tag{1}$$

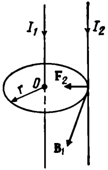


Fig. 156

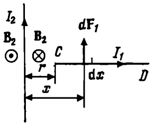


Fig. 157

is the magnetic induction of the field produced by the current I_2 at a point lying at the distance x from the vertical wire. Since the forces acting on other elements of the rod are also directed vertically upwards, the total force acting on the rod is

$$F_1 = \int_r^{r+l} dF_1 = \int_r^{r+l} I_1 B_2 \, dx. \qquad (2)$$

Substituting Eq. (1) into (2) and integrating the obtained expression, we find that

$$F_1 = \int_r^{r+l} \frac{\mu\mu_0 I_2 I_1 \, dx}{2\pi x} = \frac{\mu\mu_0 I_2 I_1}{2\pi} \int_r^{r+l} \frac{dx}{x}$$

$$= \frac{\mu\mu_0 I_2 I_1}{2\pi} \ln x \Big|_r^{r+l}$$

$$= \frac{\mu\mu_0 I_2 I_1}{2\pi} \ln \frac{r+l}{r},$$

$$F_1 = \frac{4 \times 3.14 \times 10^{-7} \times 1 \times 2 \times 0.5}{2 \times 3.14}$$

$$\times \ln \frac{5 \times 10^{-2} + 15 \times 10^{-2}}{5 \times 10^{-2}} \text{ N}$$

$$\simeq 2.8 \times 10^{-7} \text{ N}.$$

516°. Two long straight parallel conductors are separated by a distance of 5 cm and carry currents of 10 and

20 A. What work per unit length of a conductor must be done to increase the separation between the conductors to 10 cm if the currents flow in the same direction?

Given: $r_1 = 5$ cm $= 5 \times 10^{-2}$ m, $I_1 = 10$ A, $I_2 = 20$ A, $r_2 = 10$ cm $= 10 \times 10^{-2}$ m.

A/l—?

Solution. For the sake of definiteness, we assume that the conductor carrying the current I_1 is stationary, and the distance r changes due to the displacement of the conductor carrying the current I_2. The second conductor is in the magnetic field B_1 produced by the conductor with the current I_1. Therefore, it is acted upon by the Ampère force

$$F_2 = I_2 B_1 l \quad (\sin \alpha = 1), \tag{1}$$

which is directed, according to the left-hand rule, towards the first conductor (Fig. 158). This force cannot increase the distance between the conductors. Therefore, in order to displace the second conductor, we must apply to it an external force $\mathbf{F}_{ext}$ equal in magnitude and opposite to the force $\mathbf{F}_2$, i.e. $\mathbf{F}_{ext} = -\mathbf{F}_2$, or, in scalar form, $F_{ext} = F_2$. Since we assume that the first conductor is long, the magnetic induction of the field produced by it is

$$B_1 = \mu\mu_0 I_1/(2\pi r). \tag{2}$$

Taking Eq. (2) into account, we can write Eq. (1) in the form

$$F_{ext} = \mu\mu_0 I_1 I_2 l/(2\pi r).$$

Then the external force per unit length of the second conductor is

$$f_{ext} = F_{ext}/l = \mu\mu_0 I_1 I_2/(2\pi r). \tag{3}$$

Hence it follows that the force decreases with the displacement of the conductor, and the work done by such a varying force is

$$\frac{A}{l} = \int_{r_1}^{r_2} f_{ext}\, dr.$$

Substituting Eq. (3) into the integrand and integrating, we obtain

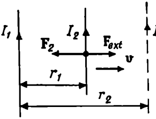


Fig. 158

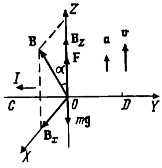


Fig. 159

$$\frac{A}{l}=\int_{r_1}^{r_2}\frac{\mu\mu_0 I_1 I_2}{2\pi}\frac{dr}{r}=\frac{\mu\mu_0 I_1 I_2}{2\pi}\int_{r_1}^{r_2}\frac{dr}{r}$$

$$=\frac{\mu\mu_0 I_1 I_2}{2\pi}\ln\frac{r_2}{r_1},$$

$$\frac{A}{l}=\frac{4\times3.14\times10^{-7}\times10\times20}{2\times3.14}\ln\frac{10\times10^{-2}}{5\times10^{-2}}\ \text{J/m}$$

$$=27.6\ \mu\text{J/m}.$$

517. A straight conductor of mass 2 kg, carrying a current of 4 A, moves vertically upwards in a uniform magnetic field of induction 2 T directed at an angle of 30° to the vertical. Three seconds after the beginning of motion, the velocity of the conductor is 10 m/s. Determine the conductor length.

Given: $B=2$ T, $\alpha=30°\simeq0.52$ rad, $m=2$ kg, $I=4$ A, $t=3$ s, $v=10$ m/s.

l—?

Solution. The conductor *CD* moving in the magnetic field is acted upon by the force of gravity mg and the Ampère force $\mathbf{F}$ (Fig. 159). Since the conductor moves at an angle α to the direction of $\mathbf{B}$, the Ampère force is

$$F=IB_xl,$$

where $B_x = B \sin\alpha$ is the x-component of the vector **B**. Consequently,

$$F = IBl \sin\alpha. \qquad (1)$$

Since the motion of the conductor is uniformly accelerated, we can write Newton's second law in projections of forces and accelerations on the Z-axis:

$$F - mg = ma,$$

or, taking into account expression (1),

$$IBl \sin\alpha - mg = ma. \qquad (2)$$

Using the formula $v = at$, we find that $a = v/t$ and write Eq. (2) in the form

$$IBl \sin\alpha - mg = mv/t,$$

whence

$$l = \frac{m(g + v/t)}{IB \sin\alpha},$$

$$l = \frac{2 \times (9.8 + 10/3)}{4 \times 2 \times 0.5} \text{ m} \simeq 6.57 \text{ m}.$$

518. A cyclotron is intended for accelerating protons to an energy of 5 MeV. Determine the maximum radius of the proton orbit if the magnetic induction of the field is 1 T.

Given: $W_k = 5 \text{ MeV} = 8 \times 10^{-13}$ J, $B = 1$ T.

R—?

Solution. The proton moves in the cyclotron in a helical orbit consisting of semicircles with gradually increasing radii. The Lorentz force $F = qvB$ is acting on the proton. Writing Newton's second law for this particle in projections on the Y-axis directed to the centre of the circle, we get

$$F = ma_y, \qquad (1)$$

where $a_y = a_c = v^2/R$. Then Eq. (1) can be written in the form $qvB = mv^2/R$, whence

$$R = mv/(qB). \qquad (2)$$

Transforming expression (2), we obtain

$$mv = \sqrt{2m}\sqrt{mv^2/2} = \sqrt{2mW_k}, \qquad (3)$$

where W_k is the kinetic energy of the proton.

Substituting Eq. (3) into (2), we get

$$R = \frac{\sqrt{2mW_k}}{qB},$$

$$R = \frac{\sqrt{2 \times 1.67 \times 10^{-27} \times 8 \times 10^{-13}}}{1.6 \times 10^{-19} \times 1} \text{ m} = 0.32 \text{ m}.$$

519. An electron accelerated by a potential difference of 300 V moves parallel to a straight conductor at a distance of 4 mm from it (Fig. 160). What force will act on the electron if a current of 5 A is passed through the conductor?

Given: $r = 4 \text{ mm} = 4 \times 10^{-3}$ m, $I = 5$ A, $U = 300$ V.

F—?

Solution. When the current is passed through the conductor, the electron is acted upon by the Lorentz force

$$F = eBv. \quad (1)$$

Since the electron has been preliminarily accelerated in an electric field, we can write $A = \Delta W_k$, or $eU = mv^2/2$, where m is the electron mass. Therefore,

$$v = \sqrt{2eU/m}. \quad (2)$$

The magnetic induction of the field produced by a long straight current-carrying conductor is

$$B = \mu\mu_0 I/(2\pi r). \quad (3)$$

Substituting expressions (2) and (3) into (1), we obtain

$$F = \mu\mu_0 e \frac{I}{2\pi r} \sqrt{\frac{2eU}{m}},$$

$$F = \frac{4 \times 3.14 \times 10^{-7} \times 1 \times 1.6 \times 10^{-19} \times 5}{2 \times 3.14 \times 4 \times 10^{-3}} \times \sqrt{\frac{2 \times 1.6 \times 10^{-19} \times 300}{9.1 \times 10^{-31}}} \text{ N} \simeq 4 \times 10^{-16} \text{ N}.$$

520. An electron moves in a magnetic field of induction 2 mT in a helix of radius 2 cm and pitch 5 cm. Determine the electron velocity.

Given: $R = 2$ cm $= 2 \times 10^{-2}$ m, $h = 5$ cm $= 5 \times 10^{-2}$ m,
$B = 2$ mT $= 2 \times 10^{-3}$ T.

v—?

Solution. The motion of the electron in a helix can be represented as a motion in a circle at a velocity v_y under

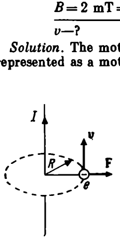


Fig. 160

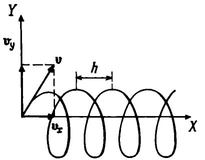


Fig. 161

the action of the Lorentz force in a plane perpendicular to the magnetic induction, and a uniform motion along the field at a velocity v_x (Fig. 161). Then the resultant velocity of the electron is

$$v = \sqrt{v_x^2 + v_y^2}. \tag{1}$$

Let us determine v_x and v_y separately. Writing Newton's second law for the electron in projections on the Y-axis, we obtain

$$F = ma_y,$$

where $F = eBv_y$ and $a_y = v_y^2/R$. Then $eBv_y = mv_y^2/R$, whence

$$v_y = eBR/m. \tag{2}$$

The velocity v_x can be determined from the relation $h = v_xT$:

$$v_x = h/T, \tag{3}$$

where T is the time during which the electron is translated in the horizontal direction by a pitch of the helix. On the other hand, T is equal to the time during which the electron traverses (at a velocity v_y) the distance equal to the length of the circumference, i.e. $T = 2\pi R/v_y$, or, taking into account Eq. (2),

$$T = 2\pi Rm/(eBR) = 2\pi m/(eB). \qquad (4)$$

Using relations (3) and (4), we get

$$v_x = h/T = heB/(2\pi m). \qquad (5)$$

Substituting Eqs. (2) and (5) into (1), we obtain

$$v = \sqrt{\frac{h^2e^2B^2}{4\pi^2m^2} + \frac{e^2B^2R^2}{m^2}} = \frac{eB}{2\pi m}\sqrt{h^2 + 4\pi^2R^2},$$

$$v = \frac{1.6\times10^{-19}\times2\times10^{-3}}{2\times3.14\times9.1\times10^{-31}}$$

$$\times\sqrt{(5\times10^{-2})^2 + 4\times3.14^2\times(2\times10^{-2})^2}\ \text{m/s}$$

$$\simeq 7.6\times10^6\ \text{m/s}.$$

521. An electron beam is directed into a space containing a uniform electric field of strength 1 kV/m and a uniform magnetic field of induction 1 mT, which is perpendicular to the electric field. The velocity of the electrons is constant and directed at right angles to vectors **E** and **B** (Fig. 162*a*). Determine this velocity. What will

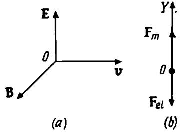


Fig. 162

be the motion of the electrons in the absence of an electric field? What will be the radius of curvature of the trajectory of the electrons in this case?

Given: $E = 10^3$ V/m, $B = 1$ mT $= 10^{-3}$ T.

v—? R—?

Solution. The forces acting on an electron in the electric and magnetic fields are

$$F_{el} = eE, \quad F_m = F_L = evB \ (\sin\alpha = 1). \qquad (1)$$

Since the electron moves uniformly in a straight line, the electric and magnetic forces are mutually balanced. Writing the equilibrium condition for the electron in projections on the Y-axis (Fig. 162*b*), we obtain

$$F_m - F_{el} = 0,$$

or, taking into account expressions (1), $evB - eE = 0$, whence

$$v = E/B, \quad v = (10^3/10^{-3}) \text{ m/s} = 10^6 \text{ m/s}.$$

If we switch off the electric field, the electron will be acted upon only by the magnetic field exerting on it the force

$$F_m = F_L = evB.$$

Since the electron velocity is perpendicular to the Lorentz force, the electron will move in a circle. According to Newton's second law,

$$F_L = ma_c.$$

Taking into account expression (1) and the fact that $a_c = v^2/R$, we obtain $evB = mv^2/R$, whence

$$R = \frac{mv}{eB},$$

$$R = \frac{9.1 \times 10^{-31} \times 10^6}{1.6 \times 10^{-19} \times 10^{-3}} \text{ m} = 5.7 \times 10^{-3} \text{ m}.$$

522. Explain why two conductors carrying currents in the same direction attract each other.

Answer. The first conductor produces in the surrounding space a magnetic field whose direction can be determined by the right-hand screw rule (Fig. 163). The second conductor will be in this field. Since it carries a current, it is acted upon by the Ampère force F_2 whose direction can be determined by the left-hand rule. Similarly, we can show

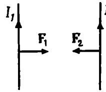


Fig. 163

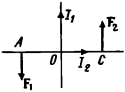


Fig. 164

that the first conductor is acted upon by the Ampère force F_1 equal to F_2.

523. What will be the behaviour of two current-carrying conductors arranged at right angles (Fig. 164)?

Answer. A conductor carrying a current I_1 produces in the surrounding space a magnetic field of induction B_1. According to the left-hand rule, the Ampère force F_1 acting on the segment AO of the second conductor is directed downwards, and the Ampère force F_2 acting on the segment OC is directed upwards. This couple of forces rotates the conductor AC about point O counterclockwise. Similarly, we can show that at the same time the first conductor starts to rotate about point O clockwise.

EXERCISES

524. Determine the magnetic induction of the field produced by an infinitely long straight wire carrying a current of 5 A at a point lying at 2 cm from it.

525. Determine the magnetic induction of the field at the centre of a circular wire loop of radius 1 cm, carrying a current of 1 A.

526. Assuming that the electron in a hydrogen atom moves in a circular orbit of radius 0.53×10^{-8} cm, determine the magnetic induction of the field at the centre of the orbit. Consider that a circular current equivalent to the moving electron is 0.01 mA.

527. Calculate the magnetic induction in a solenoid with an iron core if 400 turns of wire are wound over 40 cm of its length. The current in the solenoid is 8 A and the permeability of iron is 183.

528. Currents of 30 A flow in opposite directions through two long parallel wires separated by a distance of

16 cm. Determine the magnetic induction of the field at a point located at 10 cm from each wire.

529. A force of 0.15 N is acting on a straight conductor of length 0.5 m which is normal to lines of a magnetic field of induction 2×10^{-2} T. Determine the current through the conductor.

530. An electron is directed into a uniform magnetic field whose induction is 20 mT at right angles to the field lines at a velocity of 10^8 cm/s. Calculate the radius of the circle in which the electron will move.

531. A proton moves at a velocity of 10^8 cm/s at right angles to a uniform magnetic field of induction 1 T. Determine the force acting on the proton and the radius of the circle in which it moves.

532. An electron describes a circle of radius 4 mm in a magnetic field. The electron velocity is 3.6×10^6 m/s. Determine the magnetic induction of the field.

533. What is the direction of the force exerted by the magnetic field of the Earth on a horizontal current-carrying conductor in the northern hemisphere if the conductor lies in the magnetic meridian plane and the current flows through it from north to south?

WORK DONE IN MOVING A CURRENT-CARRYING CONDUCTOR IN A MAGNETIC FIELD. ELECTROMAGNETIC INDUCTION. MAGNETIC FIELD ENERGY

The **magnetic flux** through a plane surface of area S in a uniform magnetic field of induction B is

$$\Phi = BS \cos \alpha,$$

where α is the angle between the directions of the magnetic induction vector **B** and the normal n to the plane surface S.

The **elementary work done in moving a current-carrying conductor** in a magnetic field is

$$dA = I \, d\Phi,$$

where I is the current in the conductor and $d\Phi$ the change in the magnetic flux.

The **total work done in moving a current-carrying conductor** is

$$A = \int_{\Phi_1}^{\Phi_2} I\, d\Phi = I(\Phi_2 - \Phi_1) = I\,\Delta\Phi.$$

The **emf induced in a solenoid** is connected with the rate of change of the magnetic flux through the relation

$$\mathscr{E} = -N\frac{\Delta\Phi}{\Delta t},$$

where N is the number of turns in the solenoid, $\Delta\Phi = \Phi_2 - \Phi_1$ the change in the magnetic flux, and $\Delta t = t_2 - t_1$ the time during which this change occurs.

The direction of the current induced in a closed conductor can be determined by using **Lenz's law**. The emf induced in a solenoid as a result of change of its own magnetic flux is

$$\mathscr{E} = -L\frac{\Delta I}{\Delta t},$$

where L is the inductance of the solenoid, $\Delta I = I_2 - I_1$ the change in the current in the solenoid, and $\Delta t = t_2 - t_1$ the time during which this change takes place.

The **inductance of a solenoid** is given by

$$L = \mu\mu_0 n^2 lS,$$

where μ is the permeability of the solenoid core, μ_0 the magnetic constant, $n = N/l$ the number of turns per unit length of the solenoid, N the total number of turns in it, l the solenoid length, and S the cross-sectional area of the solenoid.

The **energy of the magnetic field of a solenoid** is

$$W = LI^2/2.$$

* * *

534°. A straight conductor having a length of 20 cm and carrying a current of 2 A moves uniformly in a uniform magnetic field of induction 1 T. The velocity of the conductor is 15 cm/s and perpendicular to the magnetic induction vector. Determine the work done in moving the conductor for 5 s.

Given: $B = 1$ T, $l = 20$ cm $= 0.2$ m, $I = 2$ A, $v = 15$ cm/s $= 0.15$ m/s, $t = 5$ s.

A—?

Solution. By definition, the total work done in moving a conductor in a magnetic field is

$$A = \int_{\Phi_1}^{\Phi_2} I\,\mathrm{d}\Phi, \tag{1}$$

where $\mathrm{d}\Phi = B\,\mathrm{d}S$ is the magnetic flux through the surface element $\mathrm{d}S$ swept by the conductor during its motion for a time interval $\mathrm{d}t$ (Fig. 165).

It can be seen from the figure that $\mathrm{d}S$ is the area of the parallelogram $abb'a'$ with sides $ab = l$ and $aa' = v\,\mathrm{d}t$. Consequently, $\mathrm{d}S = lv\,\mathrm{d}t$, which gives

$$\mathrm{d}\Phi = Blv\,\mathrm{d}t. \tag{2}$$

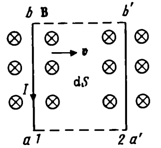


Fig. 165

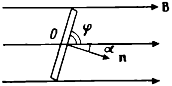


Fig. 166

Substituting expression (2) into the integrand of (1) and integrating, we obtain

$$A = \int_0^t IBlv\,\mathrm{d}t = IBlv \int_0^t \mathrm{d}t = IBlvt,$$

$$A = 2 \times 1 \times 0.2 \times 0.15 \times 5 \text{ J} = 0.3 \text{ J}.$$

535. A flat coil having a radius of 25 cm and formed by 75 turns is in a uniform magnetic field of induction 0.25 T. The plane of the coil forms an angle of 60° with the direction of the magnetic induction vector. Determine the torque acting on the coil in the magnetic field if the current through the coil is 3 A. What work must be done to remove the coil from the magnetic field?

Given: $B = 0.25$ T, $R = 25$ cm $= 0.25$ m, $N = 75$, $\varphi = 60° = \pi/3$ rad, $I = 3$ A.

M—? A_{ext} — ?

Solution. The magnetic field exerts on the coil containing N turns a torque

$$M = N p_m B \sin \alpha. \tag{1}$$

Considering that $p_m = IS$, $S = \pi R^2$, and $\alpha = \pi/2 - \varphi$ (Fig. 166), we transform Eq. (1) as follows:

$$M = NI\pi R^2 B \sin(\pi/2 - \varphi) = NI\pi R^2 B \cos\varphi,$$

$$M = 75 \times 3 \times 3.14 \times 0.25^2 \cos(\pi/3) = 5.5 \text{ N}\cdot\text{m}.$$

The work done by the magnetic field on the removal of the coil from it is

$$A = I(\Phi_2 - \Phi_1).$$

On the other hand, in order to remove the coil from the field, we must apply to it an external (say, mechanical) force which will do the work

$$A_{ext} = -A = I(\Phi_1 - \Phi_2). \tag{2}$$

Here $\Phi_1 = NBS\cos\alpha$ and $\Phi_2 = 0$ since $B_2 = 0$. Substituting these expressions into Eq. (2) and considering that $S = \pi R^2$, we obtain

$$A_{ext} = INB\pi R^2 \cos(\pi/2 - \varphi) = INB\pi R^2 \sin\varphi,$$

$$A_{ext} = 3 \times 75 \times 0.25 \times 3.14 \times 0.25^2 \sin(\pi/3) = 8.6 \text{ J}.$$

536. A circular loop having a radius of 5 cm and carrying a current of 1 A is in a uniform magnetic field of induction 4×10^{-2} T. The loop is arranged so that its plane is perpendicular to the magnetic induction. What work must be done to turn the loop through 90° about an axis coinciding with its diameter (Fig. 167)?

Given: $R = 5$ cm $= 5 \times 10^{-2}$ m, $I = 1$ A, $\varphi = 90° = \pi/2$ rad, $B = 4 \times 10^{-2}$ T.

A_{ext}—?

Solution. The initial position of a current loop is the position of stable equilibrium since the torque in this case is zero. Therefore, the magnetic field does a negative work during the rotation of the loop, while the work done by an external force is positive. By definition, the work done by the magnetic field to rotate the loop is

$$A = I\,(\Phi_2 - \Phi_1), \qquad (1)$$

where $\Phi_1 = BS\cos\alpha_1$ and $\Phi_2 = BS\cos\alpha_2$ are the magnetic fluxes through the plane of the loop in the initial and final positions and $S = \pi R^2$ is the area of the surface bounded by the current loop. Since $\alpha_1 = 0$ and $\alpha_2 = \pi/2$, we can write

$$\Phi_1 = B\pi R^2 \cos\alpha_1, \qquad \Phi_2 = B\pi R^2 \cos\alpha_2.$$

Then expression (1) assumes the form

$$A = IB\pi R^2 (\cos\alpha_2 - \cos\alpha_1).$$

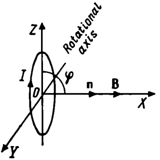


Fig. 167

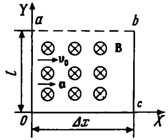


Fig. 168

Since $A_{ext} = -A$, the work done by the external force rotating the current loop is

$$A_{ext} = IB\pi R^2 (\cos\alpha_1 - \cos\alpha_2),$$
$$A_{ext} = 1 \times 4 \times 10^{-2} \times 3.14 \times (5\times10^{-2})^2 \times (1-0)\ \mathrm{J}$$
$$= 3.14 \times 10^{-4}\ \mathrm{J}.$$

537. A straight conductor having a length of 1.5 m and moving with a uniform acceleration of 10 m/s² in a uniform magnetic field at an initial velocity of 3 m/s covers a distance of 0.5 m. Determine the mean emf induced in the conductor. The magnetic induction of the field is equal to 0.2 T and perpendicular to the velocity of the conductor. Determine the instantaneous value of the emf induced in the conductor at the end of the path.

Given: $l = 1.5$ m, $v_0 = 3$ m/s, $a = 10$ m/s², $\Delta x = 0.5$ m, $B = 0.2$ T.

$\langle\mathscr{E}\rangle$—? $\mathscr{E}$—?

Solution. According to Faraday's law, the emf induced in the conductor moving at a certain velocity in a magnetic field is

$$\mathscr{E} = -\frac{\Delta\Phi}{\Delta t}, \tag{1}$$

where $\Delta\Phi = B\,\Delta S$ is the magnetic flux crossed by the conductor during its motion in the magnetic field for a time interval Δt (Fig. 168). The figure shows that the area ΔS pierced by the magnetic flux $\Delta\Phi$ can be determined as the area of the rectangle $Oabc$, i.e. $\Delta S = l\,\Delta x$. Then $\Delta\Phi = Bl\,\Delta x$ and (see Eq. (1))

$$\mathscr{E} = -\frac{Bl\,\Delta x}{\Delta t} = -Blv. \tag{2}$$

If the motion of the conductor is nonuniform in the magnetic field, taking the mean velocity of the conductor for the velocity v and using Eq. (2), we calculate the mean value of the induced emf:

$$\langle\mathscr{E}\rangle = -Bl\langle v\rangle.$$

The mean velocity of a uniformly accelerated motion is defined as

$$\langle v\rangle = (v_0 + v)/2, \quad v^2 - v_0^2 = 2a\,\Delta x,$$

whence $v = \sqrt{v_0^2 + 2a\,\Delta x}$ is the final velocity of such a motion. Consequently, $\langle v\rangle = (v_0 + \sqrt{v_0^2 + 2a\,\Delta x})/2$. Therefore,

$$\langle \mathscr{E}\rangle = -Bl\,\frac{v_0 + \sqrt{v_0^2 + 2a\,\Delta x}}{2},$$

$$\langle \mathscr{E}\rangle = -0.2\times 1.5\,\frac{3+\sqrt{3^2+2\times 10\times 0.5}}{2}\ \mathrm{V} = -0.99\ \mathrm{V}.$$

Similarly, if v is the instantaneous velocity of the conductor, the instantaneous induced emf is

$$\mathscr{E} = -Blv, \qquad (3)$$

where $v = \sqrt{v_0^2 + 2a\,\Delta x}$ is the instantaneous velocity at the end of the path. Then Eq. (3) becomes

$$\mathscr{E} = -Bl\sqrt{v_0^2 + 2a\,\Delta x},$$

$$\mathscr{E} = -0.2\times 1.5\times\sqrt{3^2+2\times 10\times 0.5}\ \mathrm{V} = -1.08\ \mathrm{V}.$$

538°. A current loop rotates uniformly in a uniform magnetic field of induction 4 mT at a frequency of 20 s^{-1}. The area of the loop is 20 cm^2. The rotational axis of the loop lies in its plane and is perpendicular to the magnetic induction vector. Determine the maximum magnetic flux through the plane of the loop and the emf induced in the loop during its rotation. Plot the graphs for the time dependences of the magnetic flux and the induced emf.

Given: $B = 4\ \mathrm{mT} = 4\times 10^{-3}\ \mathrm{T}$, $\nu = 20\ \mathrm{s}^{-1}$, $S = 20\ \mathrm{cm}^2 = 2\times 10^{-3}\ \mathrm{m}^2$.

Φ_{max}—? $\mathscr{E}_{max}$—?

Solution. The magnetic flux through the plane of the loop is

$$\Phi = BS\cos\alpha,$$

where α is the angle between the normal to the plane of the loop and the direction of the magnetic induction vector. If the loop rotates uniformly, the angle α changes with time according to the law $\alpha = \omega t$. Therefore,

$$\Phi = BS \cos \omega t. \qquad (1)$$

According to Faraday's law of electromagnetic induction, the induced emf is

$$\mathscr{E} = -\frac{d\Phi}{dt},$$

or, taking into account Eq. (1),

$$\mathscr{E} = -\frac{d}{dt}(BS \cos \omega t) = BS\omega \sin \omega t. \qquad (2)$$

Expression (1) shows that the magnetic flux through the rotating loop has the maximum value when $\cos \omega t = 1$. In this case,

$$\Phi = \Phi_{max} = BS,$$

$$\Phi_{max} = 4 \times 10^{-3} \times 2 \times 10^{-3}\ \text{Wb} = 8\ \mu\text{Wb}.$$

Similarly, from expression (2) we can determine the maximum emf induced in the loop: $\mathscr{E}_{max} = BS\omega$ or, since $\omega = 2\pi\nu$,

$$\mathscr{E}_{max} = BS2\pi\nu,$$

$$\mathscr{E}_{max} = 4 \times 10^{-3} \times 2 \times 10^{-3} \times 2 \times 3.14 \times 20\ \text{V} \simeq 1\ \text{mV}.$$

Using Eqs. (1) and (2), we can plot the graphs for the time dependences of Φ and $\mathscr{E}$ (Fig. 169). Since $\omega = 2\pi/T$, it is convenient to consider the instants of time t corresponding to $(1/4)T$, $(1/2)T$, $(3/4)T$, and T.

539°. A rectangular coil of area 500 cm², consisting of 200 turns of a wire, rotates uniformly in a uniform magnetic field about an axis passing through its centre and parallel to one of the sides (Fig. 170) at a frequency of 10 s⁻¹. The maximum value of the emf induced in the coil is 150 V. Determine the magnetic induction of the field.

Given: $S = 500\ \text{cm}^2 = 5 \times 10^{-2}\ \text{m}^2$, $N = 200$, $\nu = 10\ \text{s}^{-1}$, $\mathscr{E}_{max} = 150\ \text{V}$.

B—?

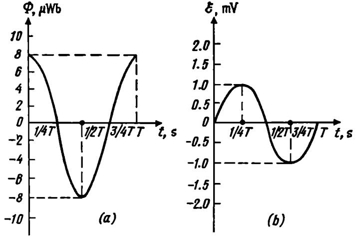


Fig. 169

Solution. According to Faraday's law, the emf induced in a coil consisting of N turns and rotating in a magnetic field is

$$\mathscr{E} = -N \frac{d\Phi}{dt}, \tag{1}$$

where $\Phi = BS \cos \omega t$ and $\omega = 2\pi\nu$. Substituting these expressions into Eq. (1) and differentiating, we obtain

$$\mathscr{E} = -N \frac{d}{dt} (BS \cos 2\pi\nu) = NBS 2\pi\nu \sin 2\pi\nu t.$$

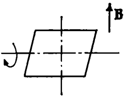


Fig. 170

The induced emf assumes the maximum value at the moments of time when $\sin 2\pi\nu t = 1$:

$$\mathscr{E} = \mathscr{E}_{max} = NBS2\pi\nu,$$

whence

$$B = \frac{\mathscr{E}_{max}}{NS2\pi\nu},$$

$$B = \frac{150}{200 \times 5 \times 10^{-2} \times 2 \times 3.14 \times 10}\ \mathrm{T} \simeq 0.24\ \mathrm{T}.$$

540. An aluminium ring is placed in a uniform magnetic field so that its plane is perpendicular to the magnetic induction vector. The diameter of the ring is 25 cm and the thickness of the wire of which it is made is 2 mm. Determine the rate of change of the magnetic induction of the field if the current induced in the ring is 12 A.

Given: $D = 25\ \mathrm{cm} = 0.25\ \mathrm{m}$, $d = 2\ \mathrm{mm} = 2 \times 10^{-3}\ \mathrm{m}$, $I = 12\ \mathrm{A}$.

$\Delta B/\Delta t$—?

Solution. According to Faraday's law,

$$\mathscr{E} = -\frac{\Delta\Phi}{\Delta t}.$$

Here $\Phi = BS$ is the magnetic flux through the surface bounded by the ring. Since the area S of the surface is constant, we can write

$$\Delta\Phi = \Delta(BS) = S\ \Delta B, \quad \mathscr{E} = -S\frac{\Delta B}{\Delta t}, \tag{1}$$

whence $|\Delta B/\Delta t| = \mathscr{E}/S$. The surface bounded by the ring is a circle, and hence its area is

$$S = \pi D^2/4. \tag{2}$$

According to Ohm's law for a closed circuit,

$$\mathscr{E} = IR,$$

where $R = \rho l/S_w$ is the resistance of the ring. Since the length of the ring is $l = \pi D$ and the cross-sectional area of the wire is $S_w = \pi d^2/4$, the expression for the ring resistance can be written in the form $R = 4\rho D/d^2$. Therefore,

$$\mathscr{E} = 4\rho DI/d^2. \quad (3)$$

Substituting expressions (2) and (3) into (1), we obtain

$$\left|\frac{\Delta B}{\Delta t}\right| = \frac{16I\rho}{\pi D d^2},$$

$$\left|\frac{\Delta B}{\Delta t}\right| = \frac{16 \times 12 \times 0.26 \times 10^{-7}}{3.14 \times 0.25 \times (2 \times 10^{-3})^2}\ \frac{\mathrm{T}}{\mathrm{s}} \simeq 1.6\ \mathrm{T/s}.$$

541. A solenoid consisting of 80 turns and having a diameter of 8 cm is in a uniform magnetic field of induction 60.3 mT. The solenoid is rotated through an angle of 180° for 0.2 s. Determine the mean value of the emf induced in the solenoid if its axis is directed along the field before and after the rotation.

Given: $N = 80$, $d = 8\ \mathrm{cm} = 8 \times 10^{-2}\ \mathrm{m}$, $\alpha_1 = 0$ rad, $\alpha_2 = 180° \simeq 3.14$ rad, $\Delta t = 0.2$ s, $B = 60.3\ \mathrm{mT} = 60.3 \times 10^{-3}\ \mathrm{T}$.

$\mathscr{E}$—?

Solution. The emf induced in the solenoid is

$$\mathscr{E} = -N\frac{\Delta\Phi}{\Delta t}, \quad (1)$$

where $\Delta\Phi = \Phi_2 - \Phi_1 = BS(\cos\alpha_2 - \cos\alpha_1)$ is the change in the magnetic flux piercing the solenoid during its rotation through an angle of 180°. Since $S = \pi d^2/4$, $\cos\alpha_2 = -1$, and $\cos\alpha_1 = +1$, we have

$$\Delta\Phi = -2B\pi d^2/4 = -B\pi d^2/2. \quad (2)$$

Substituting Eq. (2) into (1), we find that

$$\mathscr{E} = \frac{NB\pi d^2}{2\Delta t},$$

$$\mathscr{E} = \frac{80 \times 60.3 \times 10^{-3} \times 3.14 \times (8 \times 10^{-2})^2}{2 \times 0.2}\ \mathrm{V} \simeq 0.24\ \mathrm{V}.$$

542. A coil having a negligibly low resistance and an inductance of 3 H is connected to a current source with an emf of 15 V and a negligibly low internal resistance. In

what time will the current induced in the coil attain a value of 50 A?

Given: $\mathscr{E}_1 = 15\text{ V},\ I = 50\text{ A},\ L = 3\text{ H}.$

t—?

Solution. According to Ohm's law for a closed circuit,

$$\mathscr{E} = I\,(R + r),$$

where $\mathscr{E}$ is the total emf in the circuit, which in the case under consideration is the sum of the emf $\mathscr{E}_1$ of the source and the emf $\mathscr{E}_2$ of self-induction emerging in the coil after its connection to the source. Consequently,

$$\mathscr{E} = \mathscr{E}_1 + \mathscr{E}_2 = \mathscr{E}_1 - L\frac{\Delta I}{\Delta t} = I\,(R + r), \qquad (1)$$

where $\mathscr{E}_2 = -L\frac{\Delta I}{\Delta t}$ is the emf of self-induction.

By hypothesis, the resistances R and r are negligibly low, and hence Eq. (1) can be written in the form $\mathscr{E}_1 - L\frac{\Delta I}{\Delta t} \simeq 0$, whence the rate of change of the current is

$$\frac{\Delta I}{\Delta t} = \frac{\mathscr{E}_1}{L}\,. \qquad (2)$$

Since the rate of change of the current is constant, $I = \frac{\Delta I}{\Delta t}t$, whence $t = \frac{I\,\Delta t}{\Delta I}$, or, taking into account Eq. (2),

$$t = \frac{IL}{\mathscr{E}_1}\,,$$

$$t = \frac{50 \times 3}{15}\text{ s} = 10\text{ s}.$$

543. A solenoid of length 60 cm and diameter 10 cm contains 1000 turns. The current in it increases uniformly by 0.2 A per second. A ring made of copper wire of cross-sectional area 2 mm² is put on the solenoid. Determine the current induced in the ring.

Given: $l = 60$ cm $\Rightarrow 0.6$ m, $D = 10$ cm $= 0.1$ m, $N = 10^3$, $\Delta I/\Delta t = 0.2$ A/s, $S_w = 2$ mm$^2 = 2 \times 10^{-6}$ m^2.

I_r—?

Solution. The change in the current through the solenoid leads to the emergence of an emf of self-induction

$$\mathscr{E} = -L\frac{\Delta I}{\Delta t},$$

where $L = \mu\mu_0 n^2 lS$ is the inductance of the solenoid. Since $n = N/l$ and $S = \pi D^2/4$, the inductance of the solenoid is

$$L = \mu\mu_0 \pi D^2 N^2/(4l).$$

Therefore,

$$\mathscr{E} = -\frac{\mu\mu_0\pi D^2 N^2}{4l}\frac{\Delta I}{\Delta t}.$$

The emf $\mathscr{E}_r$ induced in the ring is smaller than the obtained value of the emf of self-induction in the solenoid consisting of N turns by a factor of N, i.e.

$$\mathscr{E}_r = \frac{\mathscr{E}}{N} = -\frac{\mu\mu_0\pi D^2 N}{4l}\frac{\Delta I}{\Delta t}. \qquad (1)$$

According to Ohm's law for a closed circuit, the current induced in the ring is

$$I_r = |\mathscr{E}_r|/R_r,$$

where $R_r = \rho l_r/S_w$ is the resistance of the ring. Since $l_r = \pi D$, we have $R_r = \rho\pi D/S_w$. Therefore,

$$I_r = |\mathscr{E}_r| S_w/(\pi\rho D),$$

and using expression (1), we obtain

$$I_r = \frac{\mu\mu_0 NDS_w}{4l\rho}\frac{\Delta I}{\Delta t},$$

$$I_r = \frac{4\times 3.14\times 10^{-7}\times 10^3\times 0.1\times 2\times 10^{-6}}{4\times 0.6\times 1.7\times 10^{-8}}\times 0.2\text{ A} \simeq 1.23\text{ mA}.$$

544. A solenoid of length 50 cm and diameter 0.8 cm consists of 20 000 turns of copper wire and is under a constant voltage. Determine the time during which the amount of heat liberated in the solenoid winding is equal to the magnetic field energy of the solenoid.

Given: $l = 50$ cm $= 0.5$ m, $D = 0.8$ cm $= 0.8 \times 10^{-2}$ m, $N = 2 \times 10^4$.

t—?

Solution. According to Joule's law, the amount of heat liberated in the solenoid winding carrying a direct current during a time t is

$$Q = I^2Rt,$$

where $R = \rho l/S_w$ is the resistance of the copper wire used for the winding. The length of the wire wound on the solenoid is equal to the product of the length of a turn by the number of turns in the solenoid: $l = \pi DN$. The cross-sectional area of the wire is $S_w = \pi d_w^2/4$. Since the turns are wound without gaps, the diameter of the wire is $d_w = l/N$. Then $S_w = \pi l^2/(4N^2)$ and $R = 4\rho DN^3/l^2$. Consequently,

$$Q = 4I^2\rho DN^3t/l^2. \quad (1)$$

The magnetic field energy of the solenoid is

$$W = LI^2/2, \quad (2)$$

where $L = \mu\mu_0 n^2 lS$ is the inductance of the solenoid. Since the number of turns per unit length and the cross-sectional area of the solenoid are $n = N/l$ and $S = \pi D^2/4$, the inductance of the solenoid is

$$L = \mu\mu_0\pi D^2N^2/(4l). \quad (3)$$

Using Eq. (3), we can write Eq. (2) as follows:

$$W = \mu\mu_0\pi D^2N^2I^2/(8l). \quad (4)$$

Equating expressions (1) and (4) by hypothesis, we obtain $I^24\rho DN^3t/l^2 = \mu\mu_0\pi D^2N^2I^2/(8l)$, whence

$$t = \frac{\mu\mu_0\pi Dl}{32\rho N},$$

$$t = \frac{4 \times 3.14 \times 10^{-7} \times 1 \times 3.14 \times 0.8 \times 10^{-2} \times 0.5}{32 \times 1.7 \times 10^{-8} \times 2 \times 10^4} \text{ s}$$

$$= 1.45 \times 10^{-6} \text{ s}.$$

545. In what direction will the current flow through an

ammeter (Fig. 171) at the moment of disconnection of the circuit by the key K?

Answer. Before the circuit is disconnected, the currents I_1 and I_2 flow through the solenoid and the ammeter re-

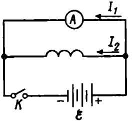


Fig. 171

spectively. After the disconnection, the emf of self-induction will appear in the solenoid and will tend to maintain the current in the solenoid at the previous level. This induced current will pass through the ammeter, and the pointer will be diflected in the derection opposite to the initial direction.

EXERCISES

546. What work must be done to move by 25 cm a conductor having a length of 40 cm and carrying a current of 21 A in a uniform magnetic field of induction 1.2 T?

547. A rectangular coil of area 40 cm^2 is in a uniform magnetic field of induction 0.06 T. The coil consists of 200 turns and can be rotated about an axis perpendicular to the magnetic field lines. When a current of 0.5 A is passed through the coil, it is turned so that its plane becomes perpendicular to the magnetic field lines. What work must be done to rotate the coil from this position by 1/4 of a turn; by 1/2 of a turn; by a complete turn?

548. A flat loop of area 10 cm^2 is in a uniform magnetic field perpendicular to its plane. Determine the current in the loop if the field decreases at a constant rate of 0.1 T/s and the resistance of the loop is 10 Ω.

549. The current in a coil having a length of 50 cm, a diameter of 10 cm, and consisting of 1000 turns increases

uniformly by 0.1 A per second. A ring made of copper wire of cross-sectional area 2 mm^2 is put on the coil. Determine the current in the ring, assuming that the magnetic flux through the solenoid and the ring is the same.

550. A square loop made of copper wire and having an area of 25 cm^2 is placed in a magnetic field of induction 0.1 T. The normal to the loop is parallel to the magnetic induction vector. The cross-sectional area of the wire is 1 mm^2. What charge will pass through the loop after switching off the field?

551. A coil having 30 turns rotates about a horizontal axis lying in its plane and perpendicular to the magnetic meridian plane at a frequency of 10 s^{-1}. The magnetic field strength of the Earth is 40 A/m. The maximum emf induced in the coil is 0.001 V. Determine the area of the coil.

552. A field of induction 0.7 T is produced between the poles of a dynamo (d.c. generator). The armature of the generator consists of 100 turns having an area of 500 cm^2 each. Determine the rotational frequency of the armature if the maximum emf induced in it is 200 V.

553. A current of 2 A is passed through a 20-cm long coil having a diameter of 3 cm and consisting of 400 turns. Determine the inductance of the coil and the magnetic flux piercing it.

554. If a current passing through a solenoid changes by 50 A per second, an emf of self-induction of 0.08 V appears across the ends of the winding. Determine the inductance of the solenoid.

555. A coil consisting of 100 turns is short-circuited and placed in a magnetic field of strength 9.6 kA/m. The area of a turn is 5 cm^2 and the planes of the turns are perpendicular to the magnetic field lines in the coil. What charge will pass through the coil as a result of its removal from the field? The coil resistance is 2 Ω.

556. The winding of an electromagnet has an inductance of 0.5 H, a resistance of 15 Ω, and is under a constant voltage. Determine the time during which the amount of heat liberated in the winding is equal to the magnetic field energy of the core of the electromagnet.

557. A closed solenoid with an iron core of length

150 cm and cross-sectional area 20 cm^2 contains 1200 turns. Determine the magnetic field energy of the solenoid if the current through it is 1 A and the permeability of iron is 1400.

QUESTIONS FOR REVISION

1. What is magnetic induction vector? 2. What is the relation between magnetic field strength and magnetic induction? 3. What is the permeability of a substance? 4. Name the units of magnetic field strength and magnetic induction. 5. What is the magnitude of the Ampère force acting on a current-carrying conductor in a magnetic field? What is the direction of the force? 6. What is the value of the Lorentz force acting on a charge moving in a magnetic field? What is the direction of the force? 7. Write an expression for the torque acting on a current loop in a magnetic field. 8. Define the magnetic flux through a surface element. What is the unit of measurement for magnetic flux? 9. Write a formula for calculating the work done in moving a current-carrying conductor in a magnetic field. 10. Characterize the phenomenon of electromagnetic induction. 11. Formulate Lenz's law. 12. Formulate Faraday's law of electromagnetic induction. 13. What is self-induction? 14. Write an expression for the emf of the self-induction. 15. Define inductance. 16. Write an expression for the inductance of a solenoid. 17. In what units is inductance measured? 18. What is the magnetic field energy of a solenoid?

Chapter 4
OPTICS

4.1. Basic Quantities and Laws in Photometry

Luminous flux is defined as the **radiant energy** W emitted by a point source per unit time:

$$\Phi = W/t.$$

Luminous intensity is defined as the luminous flux Φ emitted per unit solid angle ω by a point source in a given direction:

$$I = \Phi/\omega.$$

Solid angle is a spatial angle bounded by a conical surface whose vertex coincides with a point source and the area S of the base is a part of a spherical surface of radius R:

$$\omega = S/R^2.$$

Illuminance is defined as a luminous flux incident on a given surface S per unit area of the surface:

$$E = \Phi_{\text{in}}/S.$$

The illuminance produced by a point source at a distance r from it is

$$E = I \cos \alpha/r^2,$$

where α is the angle between the incident ray and the normal to the surface at the point of incidence.

Luminous emittance of a source is defined as the luminous flux Φ emitted by a bright surface S per unit area of the surface:

$$R = \Phi/S.$$

Luminance (brightness) of a source in a certain direction φ is defined as the luminous intensity I of the source in this direction per unit area of the surface S of the source:

$$B = I/(S \cos \varphi),$$

where φ is the angle between the normal to the surface and the direction of observation.

The luminous emittance and the luminance of a source are connected through the relation

$$R = \pi B.$$

* * *

558. Calculate the luminous flux incident on an area element of 10 cm² placed at a distance of 2 m from a source of luminous intensity 200 cd.

Given $S = 10\ \text{cm}^2 = 10^{-3}\ \text{m}^2$, $r = 2$ m, $I = 2 \times 10^2$ cd.

Φ—?

Solution. We assume that the source is at the centre of the sphere of radius r. The area element S is a part of the spherical surface. Then the illuminance of the area element is

$$E = I/r^2 \qquad (1)$$

since $\alpha = 0$. On the other hand,

$$E = \Phi/S. \qquad (2)$$

Equating the right-hand sides of relations (1) and (2), we obtain $I/r^2 = \Phi/S$, whence

$$\Phi = \frac{IS}{r^2},$$

$$\Phi = \frac{2 \times 10^2 \times 10^{-3}}{2^2}\ \text{lm} = 0.05\ \text{lm}.$$

559. Two electric lamps placed close to each other illuminate a screen located at a distance of 1 m from the lamps. One of the lamps is switched off. To what distance should the other lamp be brought closer to the screen for its illuminance to remain unchanged?

Given: $r_1 = 1$ m.

Δr—?

Solution. The illuminance of the screen produced by the two lamps is

$$E_{\text{I}} = E_1 + E_2 = 2I/r_1^2,$$

where E_1 and E_2 are the illuminances of the screen due to each lamp separately.

After one lamp has been switched off and the other has been brought closer to the screen, its illuminance becomes

$$E_{\text{II}} = I/r_2^2,$$

where r_2 is a new distance from the lamp to the screen.

By hypothesis, $E_{\text{I}} = E_{\text{II}}$, and hence $2I/r_1^2 = I/r_2^2$, from which we get $r_2 = \sqrt{r_1^2/2} = r_1/\sqrt{2}$. Consequently,

$$\Delta r = r_1 - r_2 = r_1 - r_1/\sqrt{2} = r_1 (1 - 1/\sqrt{2}),$$

$$\Delta r = 1 \ (1 - 1/1.41) \text{ m} \simeq 0.3 \text{ m}.$$

560. A screen and a plane mirror whose surfaces are parallel are placed at the same distance of 1 m on both

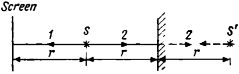


Fig. 172

sides of a point source of 2 cd (Fig. 172). What is the illuminance at the centre of the screen?

Given: $r = 1$ m, $I = 2$ cd.

E—?

Solution. The illuminance of the screen is produced by the rays directly emitted by the source S (e.g. ray *1*) and the rays incident on the screen after the reflection by the mirror (e.g. ray *2*). The rays incident on the screen after the reflection by the mirror can be regarded as the rays emitted by a source S', viz. the virtual image of the source S formed by the mirror and located at a distance r behind it. Since the solid angle in which the rays propagate does not change as a result of reflection by a plane mirror, the luminous intensity of the source S' is the same as that of the source S. Then the illuminance of the screen is

$$E = E_1 + E_2 = I/r_1^2 + I/r_2^2,$$

where $r_1 = r$ and $r_2 = 3r$. Consequently,

$$E = \frac{I}{r^2} + \frac{I}{9r^2} = \frac{10I}{9r^2},$$

$$E = \frac{10 \times 2}{9 \times 1}\ \text{lx} \simeq 2.2\ \text{lx}.$$

561. Through what angle should an area element be turned for its illuminance to decrease by half in comparison with the value corresponding to the normal incidence of the rays?

Given: $E_2 = 0.5E_1$.

α—?

Solution. The illuminance of the area element for the normal incidence of the rays is

$$E_1 = I/r^2. \tag{1}$$

The illuminance of the same area element for an oblique incidence of the rays is

$$E_2 = I \cos \alpha / r^2. \tag{2}$$

By hypothesis, $E_2 = 0.5E_1$. Using expressions (1) and (2), we find that $I \cos \alpha/r^2 = 0.5I/r^2$, whence

$$\cos \alpha = 0.5,\ \alpha \simeq 1.05\ \text{rad}.$$

562. A lamp suspended at a height of 5 m illuminates an area element on the ground. At what distance from the centre of the area element is the illuminance of the ground surface smaller than that at the centre by a factor of two (Fig. 173)?

Given: $h = 5$ m, $E_0 = 2E$.

l—?

Solution. The illuminance of the ground surface at the centre of the area element is

$$E_0 = I/h^2.$$

The illuminance of the ground surface at a distance l from the centre of the area element is

$$E = I \cos \alpha / r^2.$$

The figure shows that $\cos \alpha = h/r$ and $r = \sqrt{h^2 + l^2}$. Then $E = Ih/(\sqrt{h^2 + l^2})^3$. Considering that $E_0 = 2E$, we obtain $I/h^2 = 2Ih/(\sqrt{h^2 + l^2})^3$, whence

$$l = h\sqrt{\sqrt[3]{4} - 1},$$

$$l = 5\sqrt{\sqrt[3]{4} - 1}\ \text{m} \simeq 3.83\ \text{m}.$$

563. A dome in the form of a hemisphere of radius 1 m is illuminated by two identical lamps suspended at a height of 2 m above the ground and separated from each other by the same distance (Fig. 174). Determine the illuminance at the points of the hemisphere located at the minimum distance from the sources if the total luminous flux produced by each lamp is 10^3 lm.

Given: $R = 1$ m, $h = 2$ m, $l = 2$ m, $\Phi = 10^3$ lm.

E—?

Solution. Point C lying on the straight line connecting a source A with the centre O of the hemisphere is at the minimum distance from the source A (see Fig. 174). The

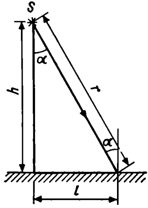


Fig. 173

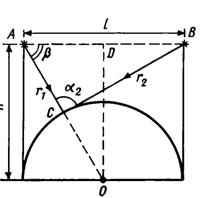


Fig. 174

illuminance at point C is produced by the sources located at points A and B and separated from it by the distances $|AC| = r_1$ and $|BC| = r_2$. Consequently, the illuminance at point C is

$$E = I \cos \alpha_1/r_1^2 + I \cos \alpha_2/r_2^2, \tag{1}$$

where

$$I = \Phi/(4\pi), \ \alpha_1 = 0 \tag{2}$$

since the ray AC propagates along the radius of the hemisphere and coincides with the normal to its surface at the point of incidence. The angle α_2 can be determined from $\triangle ABC$ by using the cosine law:

$$\cos \alpha_2 = (r_1^2 + r_2^2 - l^2)/(2r_1r_2). \tag{3}$$

It follows from the figure that

$$r_1 = |AO| - |OC|, \tag{4}$$

where $|OC| = R$. From $\triangle AOD$, we find that

$$|AO| = \sqrt{|AD|^2 + |OD|^2} = \sqrt{l^2/4 + h^2}.$$

Since, by hypothesis,

$$l = h = 2R, \tag{5}$$

we have $|AO| = R\sqrt{5} \approx 2.24R$. Substituting the expressions for $|AO|$ and $|OC|$ into Eq. (4), we obtain

$$r_1 = 2.24R - R \approx 1.24R. \tag{6}$$

Applying the cosine law to $\triangle ABC$, we find that

$$r_2^2 = r_1^2 + l^2 - 2r_1 l \cos \beta. \tag{7}$$

The value of $\cos \beta$ can be determined from $\triangle AOD$ by considering that $|AD| = l/2 = R$. This gives

$$\cos \beta = |AD|/|AO| = R/(2.24R) \simeq 0.45. \tag{8}$$

Substituting expressions (6), (5), and (8) into (7), we obtain

$$r_2^2 = (1.24R)^2 + 4R^2 - 4 \times 1.24R^2 \times 0.45 \approx 3R^2,$$

whence

$$r_2 \approx 1.73R. \tag{9}$$

Substituting expressions (5), (6), and (9) into (3), we get

$$\cos\alpha_2 = \frac{(1.24R)^2+(1.73R)^2-4R^2}{2\times 1.24R\times 1.73R} \simeq 0.12. \qquad (10)$$

Substituting expressions (2), (6), (9), and (10) into (1), we finally obtain

$$E=\frac{\Phi}{4\pi}\left[\frac{1}{(1.24R)^2}+\frac{0.12}{(1.73R)^2}\right]\approx\frac{0.69\Phi}{4\pi R^2}\approx 0.055\,\frac{\Phi}{R^2},$$

$$E=\frac{0.055\times 10^3}{1}\ \text{lx}=55\ \text{lx}.$$

564. Determine the illuminance of the Earth's surface produced by solar rays incident along the normal to the surface. The Sun's luminance is $1.2\times 10^9\ \text{cd/m}^2$. The distance between the Earth and the Sun is 1.5×10^8 km and the Sun's radius is 7×10^5 km.

Given: $B=1.2\times 10^9\ \text{cd/m}^2$, $r=1.5\times 10^8\ \text{km} = 1.5\times 10^{11}\ \text{m}$, $R=7\times 10^5\ \text{km}=7\times 10^8\ \text{m}$.

E—?

Solution. Since the distance from the Earth to the Sun is very long, we assume that the rays emitted by the Sun are incident on the Earth in a parallel beam. Assuming that the Sun is a plane luminous disc, we find that its luminance is

$$B = 2I/S,$$

where $S=\pi R^2$. The coefficient 2 is introduced since the flat disc emits in two directions. Then $B = 2I/(\pi R^2)$, whence

$$I = \pi BR^2/2. \qquad (1)$$

By hypothesis, $\cos\alpha = 1$. Then the illuminance of the Earth's surface is

$$E = I/r^2. \qquad (2)$$

Substituting expression (1) into (2), we obtain

$$E=\frac{\pi BR^2}{2r^2},$$

$$E=\frac{3.14\times 1.2\times 10^9\times(7\times 10^8)^2}{2\times(1.5\times 10^{11})^2}\ \text{lx}\simeq 8\times 10^4\ \text{lx}.$$

565. An electric bulb whose luminous intensity is 100 cd is enclosed in a frosted spherical dome of diameter 5 cm. Determine the luminous emittance and the luminance of the bulb, neglecting the light absorption by the glass of which the dome is made.

Given: $I = 100$ cd, $D = 5$ cm $= 5 \times 10^{-2}$ m.

R—? B—?

Solution. The luminous emittance of a source is defined as

$$R = \Phi/S,$$

where $\Phi = I\omega$ is the emitted luminous flux, $\omega = 4\pi$ the total solid angle, and $S = \pi D^2$. Therefore,

$$R = \frac{4\pi I}{\pi D^2} = \frac{4I}{D^2},$$

$$R = \frac{4 \times 10^2}{(5 \times 10^{-2})^2} \frac{\text{lm}}{\text{m}^2} = 1.6 \times 10^5 \text{ lm/m}^2.$$

The luminance of the bulb is

$$B = \frac{R}{\pi},$$

$$B = \frac{1.6 \times 10^5}{3.14} \frac{\text{cd}}{\text{m}^2} = 5.1 \times 10^4 \text{ cd/m}^2.$$

566. A lamp of luminous intensity 60 cd is used for printing photographs. If the lamp is placed at a distance of 1.5 m from a photograph, the exposure time is 2.5 s. Determine the exposure time for a lamp of luminous intensity 40 cd located at a distance of 2 m from the photograph.

Given: $I_1 = 60$ cd, $r_1 = 1.5$ m, $t_1 = 2.5$ s, $I_2 = 40$ cd, $r_2 = 2$ m.

t_2—?

Solution. The radiant energy received by a photographic paper illuminated for the time t is equal to the product of the luminous flux Φ and the exposure time t:

$$W = \Phi t = ESt.$$

Consequently, for the two cases we can write

$$W_1 = E_1St_1,\ W_2 = E_2St_2. \tag{1}$$

The quality of the photographs is the same if the same radiant energy $W_1 = W_2$ falls on the photographic paper, or, taking into account Eq. (1), $E_1St_1 = E_2St_2$, whence

$$t_2 = E_1t_1/E_2.$$

According to the law of illuminance, $E_1 = I_1/r_1^2$ and $E_2 = I_2/r_2^2$. Therefore,

$$t_2 = \frac{I_1r_2^2t_1}{r_1^2I_2},$$

$$t_2 = \frac{60\times 2^2\times 2.5}{1.5^2\times 40}\ \text{s} = 6.64\ \text{s}.$$

567. Why is it difficult to see in the daytime the interior of a room through a window glass from outside without approaching the face close to the glass?

Answer. Two luminous fluxes propagate from the window outside: the first is produced by the rays reflected by the window glass and the second by the rays passing through the window from the room. Since the second flux is produced by the rays multiply reflected by the room walls, its intensity is much lower than that of the first flux due to absorption and scattering of light upon reflection. For this reason, the observer sees the glittering glass rather than the interior of the room.

568. The snow on the sloping surfaces of roofs usually melts sooner than on the ground. Why?

Answer. In medium latitudes, the Sun at noon is not at the zenith, and hence the solar rays are incident on a horizontal surface at a certain angle. If a roof has a slope, the angle of incidence of the rays on it is smaller than the angle of incidence on the horizontal surface of the ground. Since the illuminance of a surface is defined as $E = I\cos\alpha/r^2$, the illuminance of the roof is larger than that of the ground, which causes a rapid melting of snow on the roofs.

EXERCISES

569. A lamp of luminous intensity 400 cd is suspended on a pole of height 6 m. Calculate the illuminance of the ground at 8 m from the foot of the pole.

570. A source of light should be placed between two screens so that the left screen must be illuminated twice stronger than the right screen. At what distance from the left screen must the source of light be placed if the separation between the screens is 100 cm?

571. The light from an electric bulb of luminous intensity 200 cd is incident on a small horizontal area element at an angle of 45°, producing an illuminance of 141 lx. Determine the distance r between the lamp and the area element and the height h at which the lamp is suspended (see Fig. 173).

572. Two lamps of luminous intensity 200 cd each are suspended on a pole at 3 and 4 m above the ground. Determine the illuminance of the ground at 2 m from the foot of the pole.

573. A lamp is suspended from the ceiling and is between a vertical picture and a plane mirror. Determine the luminous flux incident on the picture of area 0.5 m^2 if the distance from the lamp to the picture and to the mirror is 4 and 2 m respectively. The luminous intensity of the lamp is 96 cd.

574. Two point sources are 2 m apart. A small area element is arranged at an angle α on the perpendicular passing through the midpoint of the line segment connecting the sources at 1 m from the line. At an angle $\alpha = 15°$, the illuminance of both sides of the area element is the same and equal to 20 lx. Determine the luminous intensities of the sources.

575. A lamp in which light is emitted by an incandescent sphere of diameter 3 mm has a luminous intensity of 85 cd. Determine the luminance of the lamp if its spherical bulb of diameter 6 cm is made of a transparent glass.

576. Determine the illuminance of the edge of a round table of diameter 1 m if it is illuminated by a lamp suspended at a height of 1 m from the centre of the table. The total luminous flux of the lamp is 600 lx.

577. A lamp of luminous intensity 1000 cd is suspended at 8 m from the surface of the ground. Determine the area of the surface whose illuminance is not lower than 1 lx.

QUESTIONS FOR REVISION

1. What is luminous flux? In what units is it measured? 2. What is luminous intensity? In what units is it measured? 3. Define solid angle. 4. What is illuminance? 5. Write a formula for calculating the illuminance produced by a point source. 6. Define luminous emittance. In what units is it expressed? 7. What is the luminance of a source? In what units is it measured? 8. Write a formula connecting the luminous emittance and the luminance of a source.

4.2. Geometrical Optics

REFLECTION OF LIGHT. MIRRORS

The **reflection of light** at the interface between two media obeys the following law:

The incident ray AO, the reflected ray OB, and the normal OC to the reflecting surface at the point of incidence of the ray lie in the same plane. The angle of incidence i is equal to the angle of reflection i_1 (Fig. 175).

If the rays incident on the interface between two media are parallel, they remain parallel after reflection. In

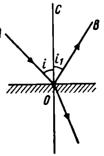


Fig. 175

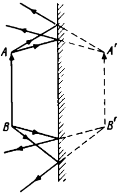


Fig. 176

this case, we have a **specular reflection**, and the reflecting surface is a **plane mirror**.

When constructing the image of an object formed by a plane mirror, one should remember that all rays emerging from a point of the object (say, point A in Fig. 176), propagate after reflection so that their continuations will intersect behind the mirror at point A' which is a virtual image of point A. Similarly, we construct an image

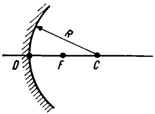


Fig. 177

B' of point B. Therefore, the image of the object is erect, virtual, and equal in size to the object which is symmetrical to it about the plane of the mirror.

If a reflecting surface is spherical, we have a **spherical mirror**. The centre of the spherical surface (point C) is the **optical centre of the mirror** (Fig. 177). The vertex O of the spherical segment is known as the **pole of the mirror**. The straight line passing through the optical centre and the pole of the mirror is the **principal optical axis of the mirror**. Any straight line passing through the optical centre of the mirror is called an **auxiliary optical axis**. Spherical mirrors can be concave or convex.

A parallel bundle of rays incident on a **concave mirror** converges at a single point, viz. the **focus of the mirror**, after reflection. The focus F lying on the principal optical axis is called the **principal focus of the mirror** (Fig. 178a). The geometrical locus of all foci forms a **focal plane.**

A parallel bundle of rays is reflected by a **convex mirror** so that the continuations of the reflected rays intersect behind the mirror at a single point F known as the **virtual focus of the mirror** (Fig. 178b).

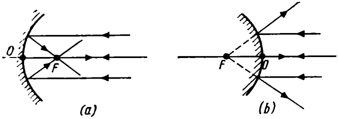


Fig. 178

While constructing images formed by spherical mirrors, it is convenient to make use of the following rays incident on a mirror:

(a) a ray parallel to the principal optical axis of the mirror, which after reflection passes through the principal focus;

(b) a ray passing through the principal focus, which after reflection propagates parallel to the principal optical axis;

(c) a ray passing through the optical centre of the mirror, which after reflection propagates along the same line in the backward direction.

The **focal length** $|OF| = F$ **(focus) of a mirror** having a **radius** R **of curvature** is

$$F = R/2.$$

If an object is at a distance d from the mirror and its image is at a distance f from it, we have

$$\pm 1/d \pm 1/f = \pm 1/F.$$

In this formula, the distances from the mirror to real points are taken with the plus sign and the distances from the mirror to virtual points with the minus sign.

The ratio of the linear size $A'B'$ of the image to the linear size AB of an object is called the **magnification of a mirror:**

$$\Gamma = |A'B'|/|AB| = f/d.$$

* * *

578. A plane mirror AB can rotate about a horizontal axis O. A light ray is incident on the mirror at an angle α. By what angle will the reflected ray be rotated upon the rotation of the mirror through an angle β (Fig. 179*a*)?

Given: α, β.

γ—?

Solution. As the mirror is rotated through the angle β, the normal to the mirror will also be rotated through

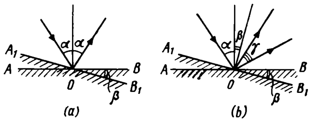


Fig. 179

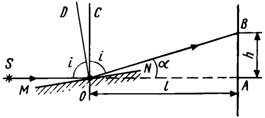


Fig. 180

the angle β, and hence after the rotation of the mirror, the angle of incidence is $\alpha + \beta$, while the angle between the incident and reflected rays is $2(\alpha + \beta)$. Before the rotation of the mirror, the angle between the incident and reflected rays was 2α. Consequently, the reflected ray will be rotated through the angle (Fig. 179*b*)

$$\gamma = 2(\alpha + \beta) - 2\alpha = 2\beta.$$

579. A light ray propagating along the horizontal is incident on a vertical screen. If a plane mirror is placed on the path of the ray, the light spot on the screen is displaced upwards by 3.5 cm (Fig. 180). Determine the angle of incidence of the ray on the mirror if the distance between the mirror and the screen is 50 cm.

Given: $h = 3.5\,\text{cm} = 3.5 \times 10^{-2}\,\text{m}$, $l = 50\,\text{cm} = 0.5\,\text{m}$.

i—?

Solution. Let us consider the path of the ray SO incident at point O of the plane mirror MN. According to the construction, DO is the perpendicular dropped on the mirror, CO the perpendicular to the horizontal line SA, and OB the reflected ray. The figure shows that $\angle DOC = \pi/2 - i$, $\angle COB = \pi/2 - \alpha$, and $\angle DOB = i$. Then $(\pi/2 - i) + (\pi/2 - \alpha) = i$, whence

$$i = (\pi - \alpha)/2. \qquad (1)$$

On the other hand, we can write

$$\tan \alpha = |AB|/|OA| = h/l \approx \alpha. \qquad (2)$$

Substituting expression (2) into (1), we find that

$$i = \frac{\pi - h/l}{2},$$

$$i = \frac{3.14 - 3.5 \times 10^{-2}/0.5}{2} = 1.535\,\text{rad}.$$

580. How many images are formed of a bright point located between two plane mirrors arranged at 45° to each other?

Given: $\alpha = 45° \simeq 0.785\,\text{rad}$.

n—?

Solution. If the bright point A_0 is placed between mirrors *1* and *2* (Fig. 181), the rays emerging from it will be incident on the mirrors and after undergoing multiple reflection will form virtual images of the sources on their continuations. Constructing consecutively the images of point A_0 formed by mirrors *1* and *2*, we find that A_1 is the image of point A_0 formed by mirror *1*, A_2 and A_3 are the images of points A_0 and A_1 formed by mirror *2*, A_4 and A_5 are the images of points A_2 and A_3 formed by mirror *1*, and finally, A_7 and A_6 are the images of points A_4 and A_5 formed by mirror *2*. All the subsequent images will coincide with the ones obtained earlier. Thus, seven images will be formed by the mirrors arranged at 45°.

581. The radius of curvature of a concave mirror is 80 cm. At what distance from the mirror must an object be placed for its real image to be twice as large as the object?

Given: $R = 80\,\text{cm} = 0.8\,\text{m}$, $|A'B'| = 2|AB|$.

d—?

Solution. We place the object AB between the focus and the optical centre of the mirror and construct its image (Fig. 182). For this purpose, we consider the rays emerging from point A of the object AB. Of all these rays, incident on the mirror, we choose the ray parallel to the principal optical axis. After reflection by the mirror, it will pass through the principal focus F. For the second ray, we take the one propagating from point A through the focus F. After reflection, the ray will propagate parallel to the principal optical axis. The point A' of intersection of the reflected rays will be a real image of point A. The images of all other points of the object AB will lie on the perpendicular dropped from point A' onto the principal optical axis. Consequently, $A'B'$ will be a real, magnified, and inverted image of the object AB.

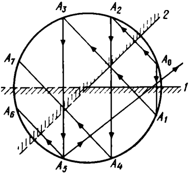


Fig. 181

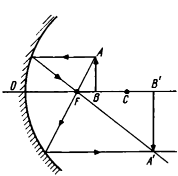


Fig. 182

In order to determine the distance from the object to the mirror, we can use the formula for a concave mirror:

$$1/d + 1/f = 1/F.$$

Considering that $F = R/2$ and $|A'B'|/|AB| = f/d = 2$, i.e. $f = 2d$, we find that $1/d + 1/(2d) = 2/R$, whence

$$d = 3R/4,$$
$$d = (3 \times 0.8)/4 \text{ m} = 0.6 \text{ m}.$$

582. Determine the principal focal length of a mirror if a bright spot and its image lie on the principal optical axis of the concave mirror at 16 and 100 cm respectively from the principal focus.

Given: $a = 16\,\text{cm} = 0.16\,\text{m}$, $b = 100\,\text{cm} = 1\,\text{m}$.

F—?

Solution. Let us construct the image of the bright spot S lying between the centre of the mirror and its principal focus. For this purpose, we consider two rays, viz. the ray SO propagating along the principal optical axis and an arbitrary ray SA (Fig. 183). After reflection by the mirror, the ray SO will propagate in the opposite direction along the principal optical axis. In order to find the path of the ray SA after reflection by the mirror, we draw the focal plane MN and an auxiliary optical axis CD parallel to the ray SA, which will intersect the focal plane at

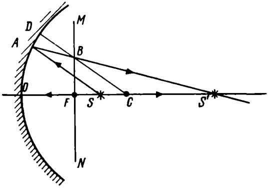


Fig. 183

point B. The ray SA must pass through this point after reflection by the mirror (according to the property of the bundle of rays parallel to an auxiliary optical axis). The image S' of the bright spot S will lie on the principal optical axis at the point of intersection of the continuations of the reflected rays.

In order to determine the principal focal length F, we can use the formula for a concave mirror:

$$1/d + 1/f = 1/F,$$

where $d = |OF| + |FS| = F + a$, and $f = |OF| + |FS'| = F + b$. This gives

$$1/(F + a) + 1/(F + b) = 1/F,$$

whence

$$F = \sqrt{ab},$$

$$F = \sqrt{0.16 \times 1}\ \text{m} = 0.4\ \text{m}.$$

We leave it to the reader to prove independently that the answer will be the same if the bright spot lies between the pole and the principal focus of the mirror.

583. A concave mirror is placed on the path of a converging bundle of rays so that the point of intersection of the rays is behind the mirror at 20 cm from its pole (Fig. 184). The rays reflected by the mirror converge at a point at a distance equal to half the focal length of the mirror. Determine the radius of curvature of the mirror.

Given: $|AO| = 20\,\text{cm} = 0.2\,\text{m}$, $|OA_1| = F/5$.

R—?

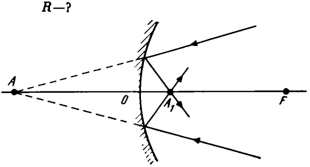


Fig. 184

Solution. If we place a point source of light at point A_1, its virtual image will be at point A (according to the principle of reversibility of paths). Consequently, if we assume that point A_1 is an image, point A will be the virtual source corresponding to the image. In this case, the formula for a concave mirror has the form

$$-1/d + 1/f = 1/F,$$

where $d = |AO|$, $f = |A_1O| = F/5$, and $F = R/2$. Therefore,

$$-1/d + 1/[R/(5 \times 2)] = 1/(R/2),$$

whence

$$R = 8d,$$
$$R = 8 \times 0.2 \text{ m} = 1.6 \text{ m}.$$

584. A screen MN is placed in front of a convex mirror at 5 cm from its pole (Fig. 185). An object AB of height

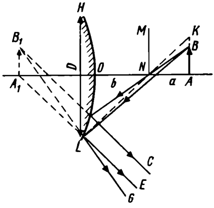


Fig. 185

3 cm is situated at 5 cm from the screen. At what positions of the eye will an observer see the image of the entire object? What is the maximum size of the object (for a given arrangement of the object, mirror, and screen) for which the mirror will form the image of the entire object? The diameter of the mirror is 10 cm.

Given: $b = 5\,\text{cm} = 5 \times 10^{-2}\,\text{m}$, $a = 5\,\text{cm} = 5 \times 10^{-2}\,\text{m}$, $|AB| = 3\,\text{cm} = 3 \times 10^{-2}\,\text{m}$, $D = 10\,\text{cm} = 0.1\,\text{m}$.

$|AK|$—?

Solution. Let A_1B_1 be the virtual image of the object AB. Obviously, B_1 will be the point of intersection of the continuations of the rays reflected by the mirror and lying within the cone CB_1G. Similarly, A_1 will be the point of intersection of the continuations of the rays reflected by the mirror and contained in the cone AA_1E. In order to see the image of the entire object AB, the observer's eye must be in the space in front of the mirror between the rays LC and A_1E. The maximum size of the object is determined from the similarity of triangles LON and NAK (the curvature of the segment OL can be neglected): $|AK|/|LO| = a/b$, whence

$$|AK| = \frac{a}{b}\,|LO| = \frac{a}{b}\,\frac{D}{2},$$

$$|AK| = \frac{5 \times 10^{-2} \times 0.1}{5 \times 10^{-2} \times 2}\,\text{m} = 5 \times 10^{-2}\,\text{m}.$$

585. At what distance from a convex spherical mirror must an object be located for its image to be closer to the mirror than the object by a factor of 1.5? The radius of curvature of the mirror is 1.6 m. Construct the image of the object.

Given: $f = d/1.5$, $R = 1.6\,\text{m}$.

d—?

Solution. Using the formula for a convex mirror, we obtain

$$1/d - 1/f = -1/F.$$

Considering that $f = d/1.5$ and $F = R/2$, we find that

$$1/d - 1/(d/1.5) = -1/(R/2),$$

whence

$$d = 0.5R/2,$$

$$d = (0.5 \times 1.6)/2\ \text{m} = 0.4\ \text{m}.$$

In order to construct the image of point B, we shall use two rays (Fig. 186): the ray propagating along an auxiliary optical axis BC, which after reflection will pass in the direction of point B, and the ray propagating parallel to the principal optical axis, which after reflection will pass

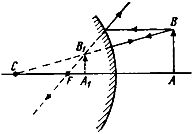


Fig. 186

so that its continuation behind the mirror will pass through the principal focus F. The point of intersection of the continuations of the reflected rays will give the image B_1 of point B. Dropping a perpendicular from point B_1 onto the principal optical axis, we obtain the image A_1B_1 of the object AB. The obtained image is diminished, virtual, and erect.

586. A bright point S is on the principal optical axis of a concave mirror of radius 40 cm at 30 cm from its pole (Fig. 187). At what distance in front of the concave mirror should a plane mirror be placed for the rays reflected by the mirrors to return to point S?

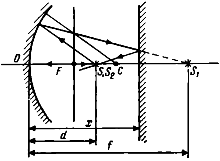


Fig. 187

Given: $R = 40\,\text{cm} = 0.4\,\text{m}$, $d = 30\,\text{cm} = 0.3\,\text{m}$.

x—?

Solution. In the absence of the plane mirror, point S_1 would be the image of the bright point S formed by the concave mirror (a similar construction of the image was carried out in Problem 582). If the plane mirror is placed on the path of the rays reflected by the concave mirror and propagating to point S_1, the rays will be incident on it in a converging bundle and after reflection will form the final image at point S_2. Point S_1 can be regarded as a virtual source for the plane mirror (see Problem 583). The image S_2 of the virtual source formed by this mirror will be real and symmetric to the source S_1 about the plane of the mirror.

Since the plane mirror divides the distance between the source S_1 and its image S_2 by half, and, by hypothesis, point S_2 must coincide with point S, the plane mirror must be placed at equal distances from points S and S_1. Let us determine the distance from the concave mirror to the image of the bright point formed by it using the formula for a concave mirror:

$$1/d + 1/f = 1/F.$$

Considering that $F = R/2$, we find that $1/d + 1/f = 2R$, whence

$$f = Rd/(2d - R). \qquad (1)$$

The figure shows that the distance between the plane and concave mirrors is

$$x = d + (f - d)/2 = (d + f)/2. \qquad (2)$$

Substituting expression (1) into (2), we obtain

$$x = \frac{d^2}{2d - R},$$

$$x = \frac{0.3^2}{2 \times 0.3 - 0.4}\,\text{m} = 0.45\,\text{m}.$$

587. Two identical concave mirrors are arranged opposite to each other so that their foci coincide. A point source of light is placed at 50 cm from the first mirror on the

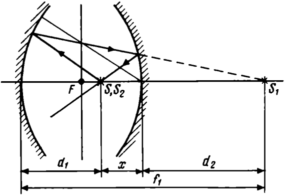


Fig. 188

common axis of the mirrors (Fig. 188). Where will the image be formed after the reflection of the rays by the mirrors? The radius of curvature of each mirror is 80 cm.

Given: $d_1 = 50\,\text{cm} = 0.5\,\text{m}$, $R = 80\,\text{cm} = 0.8\,\text{m}$.

x—?

Solution. In the absence of the second mirror, point S_1 would be the image of a bright point S formed by the first mirror. If we place the second mirror on the path of the rays reflected by the first mirror, the rays will be incident on it in a converging bundle and after reflection will form the final image at point S_2. Point S_1 can be regarded as a virtual source for the second mirror. The image S_2 of the virtual source formed by the second mirror is real and is at a distance x from the second mirror. Applying the formula for a concave mirror to the first mirror, we can determine the position of point S_1:

$$1/d_1 + 1/f_1 = 1/F, \text{ or } 1/d_1 + 1/f_1 = 2/R,$$

whence

$$f_1 = Rd_1/(2d_1 - R). \qquad (1)$$

Similarly, applying the same formula to the second mirror, we can find the position of point S_2:

$$-1/d_2 + 1/f_2 = 1/F,$$

or, considering that $F = R/2$, and $f_2 = x$, we obtain $-1/d_2 + 1/x = 2/R$, whence

$$x = Rd_2/(2d_2 + R). \tag{2}$$

The figure shows that $d_2 = f_1 - 2F$. Therefore (see Eq. (1)),

$$d_2 = \frac{Rd_1}{2d_1 - R} - \frac{2R}{2} = \frac{R(R - d_1)}{2d_1 - R}. \tag{3}$$

Using Eq. (3), we can reduce Eq. (2) to the form

$$x = \frac{R^2(R-d_1)/(2d_1-R)}{2R(R-d_1)/(2d_1-R)+R} = R - d_1,$$

$$x = (0.8 - 0.5)\ \text{m} = 0.3\ \text{m}.$$

It follows from the solution that the positions of points S and S_2 coincide.

588. A point source of light of luminous intensity 75 cd lies on the principal optical axis of a concave mirror of radius 50 cm at 35 cm from its pole. Determine the maximum illuminance of a screen placed at 2.5 m from the mirror and perpendicular to the principal optical axis.

Given: $R = 50\ \text{cm} = 0.5\ \text{m}$, $d = 35\ \text{cm} = 0.35\ \text{m}$, $I = 75\ \text{cd}$, $L = 2.5\ \text{m}$.

E—?

Solution. The illuminance is maximum at the centre of the screen and equal to the sum of the illuminances from the sources S and S' (its image formed by the concave mirror, Fig. 189):

$$E_0 = E + E_1 = I/r^2 + I_1/r_1^2, \tag{1}$$

where $r = L - d$ is the distance from the source S, $r_1 = L - f$ the distance from the source S' to the screen, I the luminous intensity of the source S, and I_1 the luminous intensity of the source S'. In contrast to Problem 560, the luminous intensity of the source S' is not equal to that of the source S since the solid angle in which the rays reflected by the concave mirror propagate is not equal to the solid angle for the rays incident on the mirror. In order to calculate the luminous intensity of the source

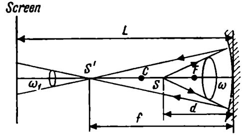


Fig. 189

S', let us consider the luminous flux $\Phi = I\omega$ incident on the mirror from the source. If we neglect the energy losses during reflection, the same flux will propagate within the solid angle ω_1: $\Phi = I_1\omega_1$. On the other hand, $\omega d^2 = \omega_1 f^2$ (see Fig. 189). From these relations, we obtain

$$I_1 = I\,(f/d)^2. \qquad (2)$$

Substituting expression (2) into (1) and taking into account the expressions for r and r_1, we obtain

$$E_0 = I\left[\frac{1}{(L-d)^2} + \left(\frac{f}{d}\right)^2 \frac{1}{(L-f)^2}\right]. \qquad (3)$$

The distance from the image S' of the source to the mirror is

$$1/d + 1/f = 1/F,$$

where $F = R/2$. Therefore,

$$f = \frac{Rd}{2d-R},$$

$$f = \frac{0.5\times 0.35}{2\times 0.35 - 0.5}\ \mathrm{m} = 0.875\ \mathrm{m}.$$

Finally, we can write

$$E_0 = 75\left[\frac{1}{(2.5-0.35)^2} + \left(\frac{0.875}{0.35}\right)^2 \frac{1}{(2.5-0.875)^2}\right] \mathrm{lx} \simeq 194\ \mathrm{lx}.$$

589. At what angle must a ray be incident on a plane mirror for the reflected ray to be perpendicular to the incident ray?

Answer. The ray must be incident at an angle of $\pi/4$ rad. In this case, according to the law of reflection, the angle of reflection is $\pi/4$ rad, and hence the angle between the incident and reflected rays is $\pi/2$ rad, i.e. the rays are mutually perpendicular.

590. If the surface of water vibrates, the images of objects acquire intricate forms. Why?

Answer. The vibrating surface of water can be regarded as a combination of concave and convex mirrors of different radii, which form different images.

EXERCISES

591. A person stands in front of a plane mirror and then moves away from it by 1 m. What is the increase in the distance between the person and his image?

592. An object is between two parallel plane mirrors. How many images of the object are formed?

593. A light ray reflected by the mirror of a galvanometer is incident on the central division of the scale separated by 1.5 m from the mirror and perpendicular to the incident ray. When a current is passed through the galvanometer, the mirror turns so that the light spot on the scale is displaced by 2 m. Determine the angle of rotation of the mirror.

594. A concave mirror forms a $3\times$ inverted image of an object. The distance from the object to its image is 28 cm. Determine the principal focal length of the mirror.

595. The image formed by a concave mirror is smaller than the object by a factor of four. If the object is moved closer to the mirror by 5 cm, its image will have a size half that of the object. Determine the principal focal length of the mirror.

596. The focal length of a concave spherical mirror is 1 m. At what distance from the mirror must a point source of light be placed for its image to coincide with the source?

597. At what distance from the face should a convex mirror of diameter 5 cm be held to see the face completely if the focal length of the mirror is 7.5 cm and the length of the face is 20 cm?

598. A bright spot is at 1 m from the pole of a convex mirror, and its image bisects the segment of the optical axis between the pole of the mirror and its focus. Determine the radius of curvature of the mirror.

599. Concave and convex spherical mirrors of the same radius of curvature equal to 60 cm are arranged so that their principal optical axes coincide. Where should an object be placed at right angles to the principal optical axis for its images in the mirrors to be identical? The distance between the poles of the mirrors is 150 cm.

600. Construct the image of an object formed by a convex mirror. Does the type of the image depend on the distance from the object to the mirror?

601. Construct the image of an object formed by a concave mirror if the object is (1) behind the optical centre of the mirror; (2) between the focus and the optical centre; (3) between the focus and the pole of the mirror.

REFRACTION OF LIGHT. THIN LENSES. OPTICAL INSTRUMENTS

The **refraction of light** at the interface between two media obeys the following law:

The incident ray AO, the refracted ray OB, and the normal OC to the refracting surface at the point of incidence of the ray lie in the same plane (Fig. 190). The ratio of the sine of the angle of incidence i to the sine of the an-

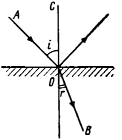


Fig. 190

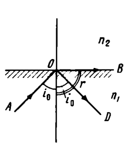


Fig. 191

gle of refraction r is a constant equal to the relative index of the second medium with respect to the first:

$$\sin i/\sin r = n_{21}.$$

The **relative refractive index** n_{21} is connected with **absolute refractive indices** n_2 and n_1 and with the velocities v_1 and v_2 of propagation of light in two contacting media through the following relations:

$$n_{21} = n_2/n_1 = v_1/v_2.$$

The **velocity of propagation of light in vacuum** is $c = 3 \times 10^8$ m/s. The **velocity of propagation of light in air** is $v \approx c$.

The **total reflection of light** is observed when light propagates from an optically more dense medium to an optically less dense medium and consists in the complete reflection of light from the interface between two media if the angle of incidence of all the rays is larger than the critical angle.

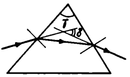


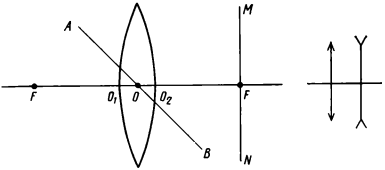
Fig. 192

The **critical angle** i_0 (Fig. 191) is the angle of incidence for which the refracted ray passes along the interface between two media. The value of the angle can be found from the relation

$$\sin i_0 = n_2/n_1.$$

When light passes through a prism, it is deflected towards its base (Fig. 192). The angle γ is known as the **prism angle** and δ the **angle of deflection** between the incident ray and the ray emerging from the prism. In order to rotate a ray through $\pi/2$ and π rad, a right-angle isosceles prism known as a **totally reflecting prism** is used.

A transparent body bounded by two spherical surfaces or a spherical surface and a plane is called a **lens**. The point through which any ray propagates without changing its direction is called the **principal optical centre** O **of a lens** (Fig. 193a). The straight line FF passing through the vertices O_1 and O_2 of the spherical surfaces is known as the



(a) (b)

Fig. 193

principal optical axis of a lens. Any straight line AB passing through the optical centre of a lens is called an **auxiliary optical axis.**

Lenses converting a parallel bundle of rays incident on them into a converging bundle of rays are called **converging lenses.**

Lenses converting a parallel bundle of rays incident on them into a diverging bundle of rays are known as **diverging lenses.**

Schematic diagrams of converging and diverging lenses are presented in Fig. 193*b*.

A bundle of rays parallel to the principal optical axis converges at the **principal focus** F after refraction. For a converging lens, the principal focus is **real**, while for a diverging lens, it is **virtual**. Each lens has two foci, viz. **front** and **rear foci**. The plane MN drawn through the principal focus at right angles to the principal optical axis is called the **focal plane** (see Fig. 193*a*). Each ray parallel to the optical axis and refracted by the lens passes through the same point lying in the focal plane of the lens.

In order to construct images formed by lenses, it is convenient to use the following rays from the entire bundle incident on a lens:

(a) a ray parallel to the principal optical axis of the lens, which after refraction passes through the principal focus;

(b) a ray passing through the principal focus, which after refraction propagates parallel to the principal optical axis;

(c) a ray passing through the optical centre of the lens, which after refraction does not change its direction.

The formula for a lens and the sign convention are the same as those for spherical mirrors, i.e.

$$\pm 1/d \pm 1/f = \pm 1/F.$$

The **focal length** F of a lens, the **radii** R_1 and R_2 of curvature of the spherical surfaces, and the absolute refractive indices n_1 and n_2 of the substance of the lens and of the surrounding medium are connected through the following relation:

$$\frac{1}{F} = \left(\frac{n_1}{n_2} - 1\right)\left(\pm\frac{1}{R_1} \pm \frac{1}{R_2}\right),$$

where the signs in front of the terms containing R_1 and R_2 are positive for convex surfaces and negative for concave surfaces.

The quantity reciprocal to the focal length is known as the **focal (lens) power**:

$$D = 1/F.$$

The focal power of a system of lenses in contact with one another is equal to the sum of the focal powers of the lenses constituting the system:

$$D = D_1 + D_2 + D_3 + \ldots .$$

The **linear magnification** of a lens is the ratio of the linear size A_1B_1 of the image to the linear size AB of an object:

$$\Gamma = |A_1B_1|/|AB| = f/d.$$

The **total magnification of an optical system of lenses** is equal to the product of the magnifications produced by each lens separately:

$$\Gamma = \Gamma_1\Gamma_2\Gamma_3 \ldots .$$

* * *

602. A light ray is incident on a glass plate whose refractive index is 1.5. Determine the angle of incidence of the ray if the angle between the reflected and refracted rays is 90° (Fig. 194).

Given: $n = 1.5$, $\gamma = 90° \simeq 1.57$ rad.

i—?

Solution. It can be seen from the figure that $i + \gamma + r = \pi$, whence $r = \pi - \gamma - i$. Substituting the value of γ, we obtain

$$r = \pi/2 - i. \qquad (1)$$

On the other hand, according to Snell's law,

$$\sin i/\sin r = n. \qquad (2)$$

From expression (1), we find that $\sin r = \sin(\pi/2 - i) = \cos i$. Then Eq. (2) can be written in the form $\sin i/\cos i = \tan i = n$, whence

$$i = \arctan n,$$

$$i = \arctan 1.5 \simeq 0.98 \text{ rad}.$$

603. The absolute refractive indices of diamond and glass are 2.42 and 1.5. What is the ratio of the thicknesses of these materials if the time of propagation of light in them is the same?

Given: $n_1 = 2.42$, $n_2 = 1.5$.

l_2/l_1—?

Solution. The absolute refractive indices n_1 and n_2 of diamond and glass are connected with the velocities v_1 and v_2 of propagation of light in them through the following relation:

$$n_1/n_2 = v_2/v_1. \qquad (1)$$

Since light propagates in a homogeneous medium at a constant velocity, we can write

$$v_1 = l_1/t,\ v_2 = l_2/t, \qquad (2)$$

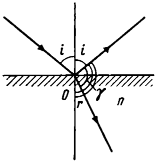


Fig. 194

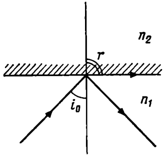


Fig. 195

where t is the time of propagation of light through a substance, and l_1 and l_2 are the thicknesses of the diamond and glass.

Dividing expressions (2) termwise, we obtain

$$v_2/v_1 = l_2/l_1. \tag{3}$$

Comparing Eqs. (1) and (3), we find that

$$l_2/l_1 = n_1/n_2,$$
$$l_2/l_1 = 2.42/1.5 = 1.61.$$

604. Determine the critical angle of incidence of a ray on the interface between glass and water (Fig. 195).

Given: $n_1 = 1.5$, $n_2 = 1.33$.

$i_0 — ?$

Solution. The critical angle of incidence at which total internal reflection is observed can be determined from the condition $\sin i_0 = n_2/n_1$, whence

$$i_0 = \arcsin (n_2/n_1),$$
$$i_0 = \arcsin (1.33/1.5) \simeq 1.08 \text{ rad}.$$

605. Light rays emerge from turpentine into air. The critical angle for these rays is 42°53′. Determine the velocity of propagation of light in the turpentine.

Given: $i_0 = 42°53' \simeq 0.64$ rad.

$v — ?$

Solution. The refractive indices n_1 and n_2 of turpentine and air are connected with the velocities of propagation of light in these media through the relation

$$n_2/n_1 = v/c. \tag{1}$$

On the other hand, the critical angle of incidence at which total internal reflection is observed can be found from the condition

$$\sin i_0 = n_2/n_1. \tag{2}$$

Comparing Eqs. (1) and (2), we find that $\sin i_0 = v/c$, whence

$$v = c \sin i_0,$$

$$v = 3 \times 10^8 \times 0.68 \text{ m/s} \simeq 2.01 \times 10^8 \text{ m/s}.$$

606. A scratch is made on the lower face of a plane-parallel glass plate. An observer looking from above sees the scratch at 4 cm from the upper face of the plate. What is the thickness of the plate?

Given: $h = 4\ \text{cm} = 4 \times 10^{-2}$ m.

H—?

Solution. Let the scratch be at point A on the lower surface of the glass plate. We construct the image of point A which is seen by the observer (Fig. 196). For this purpose, we consider two rays, viz. the ray AC incident at right angles on the upper surface of the plate and the ray AD incident on the upper surface at a small angle i. The fig-

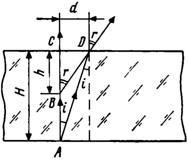


Fig. 196

ure shows that point B is a virtual image of point A. In order to determine the thickness H of the plate, we consider $\triangle ACD$: $|AC| = |CD|/\tan i$, or, since $|AC| = H$ and $|CD| = d$,

$$H = d/\tan i.$$

The segment d can be found from $\triangle BCD$: $|CD| = |CB| \tan r$, or, considering that $|CD| = d$ and $|CB| = h$, we obtain $d = h \tan r$. Then $H = h \tan r/\tan i$. Since the angles r and i are small, we can replace the ratio of their tangents by the ratio of their sines, i.e. $\tan r/\tan i \approx \sin r/\sin i$. Consequently,

$$H = h \sin r/\sin i.$$

But according to the law of refraction, $\sin r/\sin i = n_g/n_{air} = n_g$ since $n_{air} = 1$. Therefore,

$$H = hn_g,$$

$$H = 4 \times 10^{-2} \times 1.5 \text{ m} = 6 \times 10^{-2} \text{ m}.$$

607. Determine the angle of deflection of a ray by a glass prism with a prism angle of 3° if the angle of incidence of the ray on the front face of the prism is zero.

Given: $\gamma = 3° \simeq 0.052$ rad, $i_1 = 0$.

δ — ?

Solution. Figure 197 shows that $\angle AKM = \angle BCD = \angle NKE = \gamma$. We denote $\angle NKF = \alpha$ and $\angle EKF = \delta$. Then

$$\delta = \alpha - \gamma. \qquad (1)$$

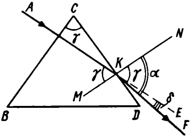


Fig. 197

According to the law of refraction, for the face CD we have $\sin \gamma/\sin \alpha = 1/n$. Since the angles α and γ are small, we can write $\sin \gamma/\sin \alpha \approx \gamma/\alpha$. Then $\gamma/\alpha = 1/n$, whence

$$\alpha = \gamma n. \qquad (2)$$

Substituting relation (2) into (1), we obtain

$$\delta = \gamma n - \gamma = \gamma (n - 1),$$

$$\delta = 0.052\,(1.5 - 1)\text{ rad} = 0.026\text{ rad}.$$

608. A light ray is incident on a triangular glass prism with a prism angle of 45° and emerges from it at an angle of 30°. Determine the angle of incidence of the ray.

Given: $\gamma = 45° \simeq 0.79$ rad, $r_2 = 30° \simeq 0.52$ rad.

i_1—?

Solution. Let us consider $\triangle DKE$ formed by the ray DE and the perpendiculars KD and KE to the faces AC

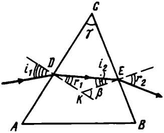


Fig. 198

and BC of the prism (Fig. 198). According to the well-known theorem in geometry, the exterior angle β of this triangle is equal to the sum of the interior angles r_1 and i_2 which are not adjacent to it, i.e. $\beta = r_1 + i_2$. But $\beta = \gamma$ as angles with mutually perpendicular sides. Consequently, $\gamma = r_1 + i_2$, whence

$$r_1 = \gamma - i_2. \qquad (1)$$

According to the law of refraction, for the face CB we have $\sin i_2/\sin r_2 = 1/n$, whence $\sin i_2 = \sin r_2/n = 0.5/1.5 = 0.333$, $i_2 = 19.5°$. Therefore, according to Eq. (1), $r_1 = 45° - 19.5° = 25.5°$.

According to the law of refraction, for the face AC we have $\sin i_1/\sin r_1 = n$, whence $\sin i_1 = n \sin r_1$;

$$\sin i_1 = 1.5 \times 0.43 = 0.645,$$
$$i_1 = 40°12' \simeq 0.698 \text{ rad}.$$

609. A converging lens forms a real twofold magnified image of an object. Determine the focal length of the lens if the distance between the lens and the image of the object is 24 cm. Construct the image formed by the lens.

Given: $\Gamma = 2$, $f = 24\ \text{cm} = 0.24\ \text{m}$.

F—?

Solution. In order to construct the image of the upper point A of the object AB, we consider the paths of two rays (Fig. 199). After refraction, the first ray propagating

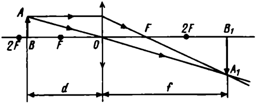


Fig. 199

parallel to the principal optical axis will pass through the principal focus of the lens. The second ray passing through the principal optical centre of the lens will not change its direction. The point A_1 of intersection of the rays will be a real image of point A. Dropping a perpendicular from point A_1 onto the principal optical axis, we shall obtain a real, magnified, and inverted image A_1B_1 of the object AB. In order to determine the focal length, we shall use the formula for a thin lens $1/d + 1/f = 1/F$, whence

$$F = df/(d + f). \tag{1}$$

The linear magnification of the lens is $\Gamma = |A_1B_1|/|AB| = f/d$, whence

$$d = f/\Gamma. \tag{2}$$

Substituting expression (2) into (1), we obtain

$$F = \frac{f/\Gamma \cdot f}{f/\Gamma + f} = \frac{f}{\Gamma + 1},$$

$$F = \frac{0.24}{2+1}\ \mathrm{m} = 8 \times 10^{-2}\ \mathrm{m}.$$

610. Determine the focal length of a double-convex glass lens immersed in water if its focal length in air is known to be 20 cm.

Given: $F_1 = 20\ \mathrm{cm} = 0.2\ \mathrm{m}$.

F_2—?

Solution. The focal length of a double-convex lens is connected with the absolute refractive indices n_1 and n_2 of the lens material and of the surrounding medium through the following relation:

$$\frac{1}{F} = \left(\frac{n_1}{n_2} - 1\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

For the lens in air, we have

$$\frac{1}{F_1} = (n_1 - 1)\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{1}$$

Similarly, for the lens in water, we have

$$\frac{1}{F_2} = \left(\frac{n_1}{n_2} - 1\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$= \left(\frac{n_1 - n_2}{n_2}\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right). \tag{2}$$

Dividing relation (1) by (2) termwise, we obtain $F_2/F_1 = (n_1 - 1)\, n_2/(n_1 - n_2)$, whence

$$F_2 = \frac{F_1 n_2 (n_1 - 1)}{n_1 - n_2},$$

$$F_2 = \frac{0.2 \times 1.33 \times (1.5 - 1)}{1.5 - 1.33}\ \mathrm{m} \simeq 0.8\ \mathrm{m}.$$

611. Determine the principal focal length of a plano-convex glass lens in turpentine if the r^dius of curvature of its convex surface is 25 cm.

Given: $R_2 = 25\ \mathrm{cm} = 0.25\ \mathrm{m}$.

F — ?

Solution. The focal length of the lens can be determined from the formula

$$\frac{1}{F}=\left(\frac{n_1}{n_2}-1\right)\left(\frac{1}{R_1}+\frac{1}{R_2}\right). \quad (1)$$

Since for the plane surface of the lens we have $1/R_1 = 0$, Eq. (1) can be reduced to the form $1/F = (n_1/n_2 - 1)/R_2$, whence

$$F=\frac{R_2}{n_1/n_2-1}=\frac{R_2 n_2}{n_1-n_2},$$

$$F=\frac{0.25\times 1.47}{1.5-1.47}\ \text{m}=12.25\ \text{m}.$$

612. Construct the image of a bright spot lying on the principal optical axis of a thin converging lens at the midpoint between the lens and its focus. Characterize the image.

Given: $d = 0.5F$.

f—?

Solution. Using the formula for a converging lens $1/d - 1/f = 1/F$, we obtain

$$f = Fd/(F - d),$$

or, considering that $d = 0.5F$,

$$f = F\cdot 0.5F/(F - 0.5F) = F.$$

Consequently, the image of the bright spot will be virtual and located at a distance F from the lens. In order to construct the image of the bright spot S, we shall use two rays, viz. the ray SO propagating along the principal optical axis and an arbitrary ray SA (Fig. 200). The ray SO passes through the lens without being refracted. In order to determine the path of the ray SA after its refraction in the lens, we draw the focal plane MN and an auxiliary optical axis DO parallel to the ray SA, which will intersect the focal plane at point B. The ray SA must pass through this point after refraction in the lens (in accordance with the property of the bundle of rays parallel to an auxiliary optical axis). The image S' of the bright spot S will lie

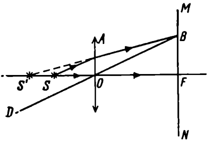


Fig. 200

on the principal optical axis at the point of intersection of the continuations of the refracted rays.

613. What is the magnification of a magnifier whose lens power is 16 D? Construct the image of an object formed by the magnifier.

Given: $D = 16$ D.

Γ—?

Solution. The magnification produced by the magnifier is

$$\Gamma = L/F = LD,$$

where F is the focal length of the magnifier and L the distance of normal vision (which is equal to 0.25 m for a normal eye). Consequently $\Gamma = 0.25 \times 16 = 4$.

The object to be examined through a magnifier is placed between the magnifier and its focus (Fig. 201). In order to construct the image of point A of the object, we shall use two rays emerging from it, viz. the ray parallel to the principal optical axis, which after refraction will

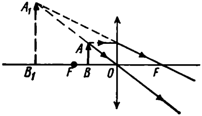


Fig. 201

pass through the focus, and the ray passing through the principal optical centre of the lens, which will not change its direction. The image A_1 of point A will be formed at the point of intersection of the continuations of the refracted rays. Similarly, we can obtain the image B_1 of point B. Consequently, the image A_1B_1 of the object AB is virtual, magnified, and erect.

614. A converging bundle of rays is incident on a diverging lens. After passing through the lens, the rays intersect at a point lying at 15 cm from the lens. If we remove

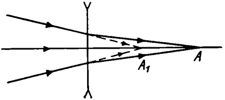


Fig. 202

the lens, the point of intersection will be shifted by 5 cm towards the previous position of the lens. Determine the focal power of the lens.

Given: $d = 15\ \text{cm} = 0.15\ \text{m}$, $l = 5\ \text{cm} = 0.05\ \text{m}$.

$D - ?$

Solution. If we place a point source of light at point A, point A_1 will be its virtual image (Fig. 202). Using the formula for a diverging lens $1/d - 1/f = -1/F$, we can write

$$F = fd/(d - f), \qquad (1)$$

where $f = d - l$. By definition, the focal power of a diverging lens is $D = -1/F$ or, using Eq. (1),

$$D = -\frac{d-f}{fd} = -\frac{l}{d(d-l)},$$

$$D = -\frac{0.05}{0.15 \times (0.15 - 0.05)}\ \text{D} \simeq -3.3\ \text{D}.$$

615. A diverging lens with a focal length of 12 cm is placed between two point sources so that it is twice as

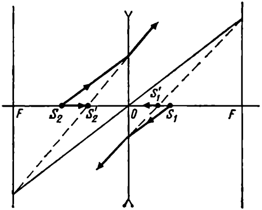


Fig. 203

close to one of them than to the other. The distance between the images of the sources is found to be 7.8 cm. Determine the distance between the sources.

Given: $F = 12$ cm $= 0.12$ m, $d_2 = 2d_1$, $l = 7.8$ cm $=$ 0.078 m.

L—?

Solution. Let us construct the images S_1' and S_2' of the bright spots S_1 and S_2 (Fig. 203) (a detailed construction is described in Problem 612). Using the formula for a diverging lens to each source separately, we obtain

$$1/d_1 - 1/f_1 = -1/F, \; 1/d_2 - 1/f_2 = -1/F.$$

By hypothesis, $d_2 = 2d_1$, $l = f_1 + f_2$, and $L = d_1 + d_2$. Therefore,

$$L = \frac{-9F(F-l) + 3F\sqrt{8F^2 + (F-l)^2}}{4(2F-l)},$$

$$L = \frac{-9 \times 0.12\,(0.12 - 0.078) + 3 \times 0.12\sqrt{8\,(0.12)^2 + (0.12 - 0.078)^2}}{4\,(2 \times 0.12 - 0.078)}$$

$$= \frac{-0.045 + 0.123}{0.648},$$

whence $L \simeq 0.12$ m. (The negative root does not satisfy the hypothesis.)

616. A bundle of rays parallel to the principal optical axis is incident on a double-convex lens whose focal length is 12 cm. Another double-convex lens having a focal length of 2 cm is at 14 cm from the first lens. The principal optical axes of the lenses coincide. Where will the image be formed?

Given: $F_1 = 12$ cm $= 0.12$ m, $l = 14$ cm $= 0.14$ m, $F_2 = 2$ cm $= 0.02$ m.

f_2—?

Solution. We shall construct the paths of the rays in the given optical system (Fig. 204). By hypothesis, the foci F_1 and F_2 of the lenses coincide ($l = F_1 + F_2$). Consequently, the bundle of rays emerging from the second lens will be parallel to the principal optical axis. No image will be formed in this case (the image is at infinity).

617. The objective of a photographic camera has a focal length of 50 mm. What must be the exposure time for a motor car moving uniformly at a velocity of 72 km/h at 2 km from the camera at right angles to its optical axis for its image to be displaced on the picture by 0.005 mm during this time? Construct the image.

Given: $F = 50\text{ mm} = 5 \times 10^{-2}\text{ m}$, $d = 2\text{ km} = 2 \times 10^{3}\text{ m}$, $v = 72\text{ km/h} = 20\text{ m/s}$, $s_1 = 0.005\text{ mm} = 5 \times 10^{-6}\text{ m}$.

t—?

Solution. In order to construct the image of the car (object AB), we consider two rays (Fig. 205*a*), viz. the ray parallel to the principal optical axis, which after re-

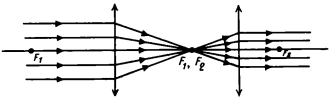


Fig. 204

fraction will pass through the focus F of the objective, and the ray passing through the optical centre O of the objective, which will not change its direction. The point A_1 of intersection of the refracted rays will be a real image of point A. Dropping a perpendicular from point A_1 onto the principal optical axis, we shall obtain a real, diminished, and inverted image A_1B_1 of the object AB.

During the exposure time t, the car is displaced by a distance $s = vt$, whence

$$t = s/v. \tag{1}$$

It follows from Fig. 205*b* that $s/s_1 = d/f$, whence

$$s = s_1 d/f. \tag{2}$$

Using the formula for a converging lens $1/d + 1/f = 1/F$, we find that

$$d/f = (d - F)/F. \tag{3}$$

Substituting expression (3) into (2), we obtain

$$s = s_1 (d - F)/F.$$

Then from Eq. (1) we get

$$t = \frac{s_1 (d - F)}{Fv} \approx \frac{s_1 d}{Fv},$$

$$t = \frac{5 \times 10^{-6} \times 2 \times 10^{3}}{5 \times 10^{-3} \times 20} \text{ s} = 10^{-2} \text{ s}.$$

618. The height of the image of an object, formed on the opaque glass of a camera, is 30 mm when the object

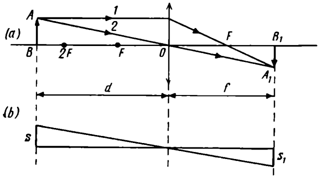


Fig. 205

is at 15 m and 51 mm when the object is at 9 m. Determine the focal length of the objective.

Given: $d_1 = 15$ m, $h_1 = 30$ mm $= 3 \times 10^{-2}$ m, $d_2 = 9$ m, $h_2 = 51$ mm $= 5.1 \times 10^{-2}$ m.

F—?

Solution. Using the formula for a converging lens for the distances d_1 and d_2, we obtain

$$1/d_1 + 1/f_1 = 1/F,\ 1/d_2 + 1/f_2 = 1/F. \qquad (1)$$

Using the formula for the magnification of the lens for the same distances, we find that $h/h_1 = d_1/f_1$ and $h/h_2 = d_2/f_2$, whence

$$f_1 = h_1 d_1/h,\ f_2 = h_2 d_2/h, \qquad (2)$$

where h is the height of the object.

Substituting Eq. (2) into (1), we obtain

$$1/d_1 + h/h_1 d_1 = 1/F,\ 1/d_2 + h/h_2 d_2 = 1/F. \qquad (3)$$

Solving Eqs. (3) together, we get

$$F = \frac{d_2 h_2 - d_1 h_1}{h_2 - h_1},$$

$$F = \frac{9 \times 5.1 \times 10^{-2} - 15 \times 3 \times 10^{-2}}{5.1 \times 10^{-2} - 3 \times 10^{-2}}\ \text{m} \simeq 0.43\ \text{m}.$$

619. Using a telescope with a focal length of 50 cm of the objective, an observer can clearly see objects at 50 m from the objective. In what direction and by what distance should the eyepiece be shifted to accommodate the telescope to infinity? Construct the image.

Given: $F = 50$ cm $= 0.5$ m, $d = 50$ m.

l—?

Solution. Using the formula for a converging lens $1/d + 1/f_1 = 1/F$, we can determine the distance between the objective and the image formed by it

$$f_1 = \frac{dF}{d - F},$$

$$f_1 = \frac{50 \times 0.5}{50 - 0.5}\ \text{m} = 0.505\ \text{m}.$$

When the telescope is adjusted to infinity, the image must be formed by the objective in its focal plane, i.e. $f_2 = 0.5$ m. Consequently, the eyepiece must be shifted towards the objective by a distance

$$l = f_1 - f_2,$$

$$l = (0.505 - 0.5)\ \text{m} = 0.005\ \text{m}.$$

Let us construct the image of an object formed by the telescope (Fig. 206). For this purpose, we shall take two rays emerging from point A of the object AB.

The first ray passing through the focus F_1 of the objective will propagate parallel to the principal optical axis

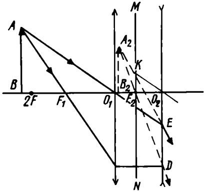


Fig. 206

after refraction until it is incident on the eyepiece at point D. After refraction in the eyepiece, it will propagate so that its continuation will intersect the optical axis at the focus F_2.

The second ray passing through the optical centre O_1 of the objective will not change its direction and will be incident on the eyepiece at point E. In order to determine the path of the ray after refraction in the eyepiece, we draw the focal plane MN through the focus F_2 and an auxiliary optical axis parallel to the ray and intersecting the focal plane at point K. The second ray is refracted so that its continuation will pass through point K.

The point A_2 of intersection of the continuations of the first and second rays emerging from the eyepiece is a virtual image of point A. Dropping a perpendicular from point A_2 onto the principal optical axis, we shall obtain a virtual, diminished, and erect image A_2B_2 of the object AB.

620. A microscope consists of an objective and an eyepiece arranged so that the distance between their principal foci is 18 cm. Determine the magnification produced by the microscope if the focal lengths of the objective and the eyepiece are 2 and 40 mm respectively. Construct the image of an object formed by the microscope.

Given: $l = 18\ \text{cm} = 0.18\ \text{m}$, $F_1 = 2\ \text{mm} = 2 \times 10^{-3}\ \text{m}$, $F_2 = 40\ \text{mm} = 4 \times 10^{-2}\ \text{m}$.

$\Gamma - ?$

Solution. We shall construct the image of an object AB which is normally placed near the focal plane of the objective. For this purpose, we take two rays emerging from point A of the object AB (Fig. 207).

The first ray passing through the focus F_1 of the objective will propagate parallel to the principal optical axis after refraction until it is incident on the eyepiece at point D. After refraction in the eyepiece, the ray will pass through its focus F_2.

The second ray passing through the optical centre O_1 of the objective will not change its direction and will be incident on the eyepiece at point E. In order to determine

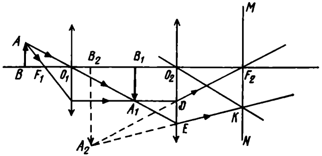


Fig. 207

the path of the ray after refraction in the eyepiece, we draw the focal plane MN through the focus F_2 and an auxiliary optical axis parallel to the ray and intersecting the focal plane at point K. The second ray must pass through this point after refraction in the eyepiece.

The point A_2 of intersection of the continuations of the first and second rays emerging from the eyepiece is a virtual image of point A. Dropping a perpendicular from point A_2 onto the principal optical axis, we shall obtain a virtual, magnified, and inverted image A_2B_2 of the object AB.

Since the microscope consists of two lenses (objective and eyepiece), its magnification is given by

$$\Gamma = \Gamma_1\Gamma_2, \tag{1}$$

where Γ_1 and Γ_2 are the magnifications of the objective and the eyepiece respectively. By definition, the magnification of the objective is

$$\Gamma_1 = f_1/d_1. \tag{2}$$

Since $f_1 \approx l$ and $d_1 \approx F_1$, we have $\Gamma_1 \approx l/F_1$. The eyepiece operates as a magnifier, and hence

$$\Gamma_2 = L/F_2, \tag{3}$$

where $L = 0.25$ m is the distance of normal vision. Substituting expressions (2) and (3) into (1), we obtain

$$\Gamma = \frac{lL}{F_1F_2},$$

$$\Gamma = \frac{0.18 \times 0.25}{2\times 10^{-3} \times 4 \times 10^{-2}} = 562.$$

621. A bundle of rays is incident on a diverging lens of focal length F_1 parallel to its principal optical axis. At what distance from the centre of the diverging lens must a converging lens be placed for the rays emerging from it to propagate, as before, parallel to the principal optical axis? The focal length F_2 of the converging lens is twice as large as that of the diverging lens (Fig. 208).

Given: F_1, $F_2 = 2F_1$.

l—?

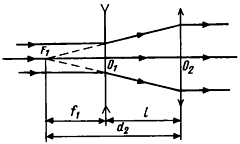


Fig. 208

Solution. For the diverging lens, we can write

$$-1/f_1 + 1/d_1 = -1/F_1. \quad (1)$$

The source of a parallel bundle of rays is assumed to be at infinity so that $1/d_1 = 0$, and from Eq. (1) we obtain

$$1/F_1 = 1/f_1,$$

whence $F_1 = f_1$. Consequently, the virtual image of the source formed by the diverging lens is at its virtual focus and is a source for the converging lens. For the converging lens, we have

$$1/d_2 + 1/f_2 = 1/F_2. \quad (2)$$

The figure shows that $d_2 = f_1 + l$. By hypothesis, the rays emerging from the converging lens must propagate in a parallel bundle, i.e. $1/f_2 = 0$. Substituting the expressions for F_2, d_2, and f_2 into Eq. (2), we obtain

$$1/(2F_1) = 1/(F_1 + l),$$

whence $l = F_1$, i.e. the required distance is equal to the focal length of the diverging lens.

622. Water is poured into a concave mirror whose radius of curvature is 50 cm. The focal power of the obtained system is 5.3 D. Calculate the focal length of the water lens.

Given: $R = 50$ cm $= 0.5$ m, $D = 5.3$ D.

$F - ?$

Solution. The optical system consists of the concave mirror and the plano-convex water lens brought in contact. Therefore, its focal power is

$$D = D_1 + 2D_2, \qquad (1)$$

where $D_1 = 1/F_1 = 2/R$ is the focal power of the concave mirror and D_2 the focal power of the water lens. The coefficient 2 is introduced into Eq. (1) since the rays pass twice through the water lens. It follows from this equation that $D_2 = (D - D_1)/2$, or, taking into account the expression for D_1,

$$D_2 = (D - 2/R)/2. \qquad (2)$$

By definition, the focal length is $F_2 = 1/D_2$, or, taking into account expression (2),

$$F_2 = \frac{2}{D-2/R},$$

$$F_2 = \frac{2}{5.3-2/0.5} \text{ m} \simeq 1.54 \text{ m}.$$

623. A double-convex lens is obtained from two identical thin watch glasses the space between which is filled

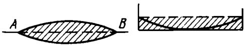


Fig. 209

with water. The focal power of such a lens is 4 D. Determine the focal power of a plano-concave lens consisting of a thin watch glass touching the bottom of a thin-walled cylindrical glass vessel if the space between the watch glass and the bottom of the vessel is filled with water.

Given: $D_0 = 4$ D.
$D_2 - ?$

Solution. In order to determine the focal power of the plano-concave lens, we shall fill the watch glass placed at the bottom of the cylindrical vessel with water (Fig. 209). We shall obtain a thin plane-parallel layer of water which can be treated as a system consisting of two water lenses: plano-convex (upper lens) and plano-concave (lower lens). The focal power of the plane-parallel water layer is

$$D = 0. \tag{1}$$

On the other hand,

$$D = D_1 + D_2, \tag{2}$$

where D_1 and D_2 are the focal powers of the plano-convex and plano-concave lenses. The plano-convex lens is half the double-convex lens cut along the line AB. Therefore, its focal power is

$$D_1 = D_0/2 = 2 \text{ D}. \tag{3}$$

Substituting expressions (1) and (3) into (2), we find that $0 = 2 \text{ D} + D_2$, whence $D_2 = -2 \text{ D}$.

624. Determine the focal power of a pair of spectacles for a long-sighted person whose distance of normal vision is 40 cm.

Given: $L_{l-s} = 40 \text{ cm} = 0.4 \text{ m}$, $L_n = 25 \text{ cm} = 0.25 \text{ m}$.
D—?

Solution. Without the spectacles, the image S' of point S is formed on the retina R of the eye if point S is at the distance SO equal to the distance L_{l-s} of normal vision for a long-sighted eye from the eye lens EL (Fig. 210). For the long-sighted eye to be able to examine

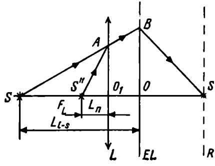


Fig. 210

closer objects, the lens L of the spectacles should be used. Then the image S' of point S'' lying at a distance L_n of normal vision for a normal eye from the lens of the spectacles will, as usual, be formed on the retina R of the eye, and the person sees not the point S'' itself but its virtual image S lying at the distance L_{l-s} of normal vision for the long-sighted eye from the eye lens EL. The focal power of the lens-eye optical system (we neglect the distance between the eye and the lens) is

$$D = D_L + D_{EL}, \tag{1}$$

where D_L and D_{EL} are the focal powers of the lens of the spectacles and the eye lens. On the other hand,

$$1/d + 1/f = 1/F = D, \tag{2}$$

where f is the distance from the retina to the eye lens.

Since the lens of the spectacles is at a small distance O_1O from the eye lens, we can neglect this distance and write $d = L_n$. The focal power of the eye lens can be determined by applying the formula for a thin lens without the spectacles:

$$1/d' + 1/f = 1/F_{EL} = D_{EL}. \tag{3}$$

Here $d' = L_{l-s}$. Taking into account expressions (2) and (3), we can write Eq. (1) in the form

$$1/L_n + 1/f = D_L + 1/L_{l-s} + 1/f,$$

whence

$$D_L = \frac{L_{l-s} - L_n}{L_{l-s} L_n},$$

$$D_L = \frac{0.4 - 0.25}{0.4 \times 0.25}\ \mathrm{D} = 1.5\ \mathrm{D}.$$

625. A short-sighted person distinguishes small objects at a distance not longer than 15 cm. Determine this distance of normal vision in spectacles with a focal power of -3 D.

Given: $d_1 = 15$ cm $= 0.15$ m, $D_2 = -3$ D.
$d_2 - ?$

Solution. The focal power of the eye without spectacles in the case of myopia is

$$D_1 = 1/d_1 + 1/f, \qquad (1)$$

where $d_1 = |SO|$ is the distance between an object and the eye lens and $f = |OS'|$ the distance between the eye lens and the retina (Fig. 211).

The focal power of a short-sighted eye with spectacles is

$$D = D_1 + D_2 = 1/d_2 + 1/f, \qquad (2)$$

where D_2 is the focal power of the spectacles and $d_2 \approx |S''O|$ the distance between the object and the eye lens

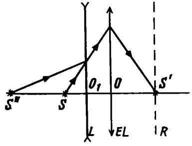


Fig. 211

(we neglect the distance between the lens of the spectacles and the eye lens).

Solving Eqs. (1) and (2) together, we obtain $1/d_1 + D_2 = 1/d_2$, whence

$$d_2 = \frac{d_1}{1 + D_2 d_1},$$

$$d_2 = \frac{0.15}{1 - 3 \times 0.15} \text{ m} \simeq 0.27 \text{ m}.$$

The short-sighted person with spectacles sees the virtual image S of the object S'' located at a longer distance from the eye than without the spectacles.

626. Why does the outline of objects above heated soil seem vibrating on a hot day?

Answer. On a hot day, heated air currents from the soil circulate in the vertical direction. The air density changes continuously and randomly due to nonuniform heating, and so does the refractive index which is related to density. For this reason, the outline of objects seems vibrating.

627. How can you explain the lustre of precious stones?

Answer. The lustre of precious stones is explained by total internal reflection which is attained by special facetting.

628. Why are plants not watered on a hot sunny day?

Answer. Drops of water left on the leaves of plants after watering play the role of small lenses which may focus solar rays and burn the leaves.

EXERCISES

629. What is the true depth of a river if it seems to be 2 m along the vertical according to a rough visual estimate?

630. A light ray propagates from glass into water. The angle of incidence of the ray at the interface between the two media is 40°. Determine the angle of refraction and the critical angle.

631. A light ray is incident on a trihedral prism made of quartz glass at an angle of 36°. The prism angle is 40°. At what angle will the ray emerge from the prism and what is the angle of its deflection from the initial direction?

632. Where will the image of an object placed at 30 cm from a converging lens with a focal length of 60 cm be formed? What is the type of the image?

633. The distance between a lamp and a screen on an optical bench is 1 m. A converging lens placed between them forms a diminished image of the lamp on the screen. If the lens is shifted towards the lamp by 60 cm, a magnified image is formed on the screen. Determine the focal length of the lens.

634. Construct the image of an object formed by a converging lens if the object is (1) beyond twice the focal length; (2) between the focus and a point at twice the focal length from the lens; (3) between the focus and the lens.

635. The radii of curvature of the surfaces of a double-convex glass lens placed in water are 50 cm each. Determine the focal power of the lens.

636. Construct the image of an object produced by a diverging lens. Does the type of the image depend on the distance between the object and the lens?

637. Determine the focal length of a diverging lens if a converging bundle of rays incident on it intersects the principal optical axis after refraction at a point lying at 25 cm from the lens. In the absence of the lens, the point of intersection of the rays will be at 7 cm from the point of convergence through the lens.

638. A terrain must be photographed from an aeroplane flying at an altitude of 4 km to obtain pictures on the scale 1:5000. Determine the focal power of the objective.

639. In order to obtain good-quality photographs, of size 12×9 cm, photographic paper must be exposed for 8 s. In what proportion should the exposure time be increased for obtaining a 48×36 cm photograph?

640. A telescope with a focal length of 50 cm is adjusted to infinity. After the eyepiece has been shifted by a certain distance, the objects at 50 m from the objective become focussed. By what distance has the eyepiece been shifted during focussing?

641. The focal lengths of the objective and the eyepiece of a microscope are 3 and 50 mm respectively. The distance between the objective and the eyepiece is 135 mm and the distance between the object and the objective is 3.1 mm. Determine the magnification of the instrument.

642. One surface of a double-concave lens is silver plated. The radius of curvature of the lens surfaces is 20 cm. An object of height 5 cm is at 50 cm from the lens. Determine the height of the image formed by the optical system.

QUESTIONS FOR REVISION

1. What is meant by absolute and relative refractive indices? 2. Formulate the law of reflection of light. 3. Define the pole, the optical centre, and the focus of a mirror. 4. Write a formula for calculating the focal length of a spherical mirror. 5. Formulate the law of refraction of light. 6. Describe total internal reflection. Under what conditions can it be observed? 7. What types of lens do you know? 8. What is the focus and the optical centre of a lens? 9. What is the focal power of a lens? In what units is it expressed? 10. Write a formula for calculating the focal length of a lens. 11. Explain how to construct an image formed by a mirror. 12. Explain the principle of construction of an image formed by a lens.

4.3. Quantum Properties of Light

Some phenomena like photoelectric effect or light pressure lead to a new concept of light according to which a light flow is a flow of elementary particles, viz. **photons (quanta of light)**. One of the characteristics of a photon is its energy. A monochromatic light flow consists of photons with the same energy

$$\varepsilon = h\nu = hc/\lambda,$$

where h is **Planck's constant**, ν the frequency of light, c the velocity of light in vacuum, and λ the wavelength.

The phenomenon consisting in the emission of electrons by metal bodies exposed to light radiation is known as **extrinsic photoelectric effect (photoeffect)**.

Photoeffect obeys **Einstein's law**

$$\varepsilon = h\nu = A + mv^2/2,$$

where ε is the energy of a photon, A the **work function** for a given metal, $mv^2/2$ the maximum kinetic energy of an emitted electron, and m the electron mass.

Photoelectric effect is observed only when a metal is irradiated by light at a frequency larger than or equal to the critical frequency ν_0 known as **photoelectric threshold**. In other words, the photoelectric threshold corresponds to the photon energy equal to the work function for a given metal:

$$h\nu_0 = A.$$

In this case, the velocity v of electrons, and hence their kinetic energy $mv^2/2$, are zero.

Light incident on bodies exerts a pressure on them, which is determined by the luminous intensity and reflectivity of the surface of a body. **Light pressure** is given by

$$p = I\,(1 + \rho)/c,$$

where p is the light pressure, I the luminous intensity, i.e. the radiant energy incident on a unit area of the body surface per second, c the velocity of light in vacuum, and ρ the **reflection coefficient**.

If a body reflects light rays specularly, then $\rho = 1$ and $p = 2I/c$. If a body absorbs light completely (blackbody), then $\rho = 0$ and $p = I/c$.

* * *

643. Determine the maximum wavelength of light for which the photoeffect can be observed in platinum.

Given: $A = 8.5 \times 10^{-19}$ J.

λ_{max} — ?

Solution. Using the formula $A = h\nu_0$, we can determine the photoelectric threshold for platinum:

$$\nu_0 = A/h. \qquad (1)$$

This frequency corresponds to the required maximum wavelength

$$\lambda_{max} = c/\nu_0,$$

or, taking into account Eq. (1),

$$\lambda_{max} = \frac{ch}{A},$$

$$\lambda_{max} = \frac{3 \times 10^8 \times 6.62 \times 10^{-34}}{8.5 \times 10^{-19}}\ \text{m} \simeq 2.34 \times 10^{-7}\ \text{m}.$$

644. Determine the maximum velocity of an electron emitted by cesium exposed to light of wavelength 400 nm.

Given: $\lambda = 400\ \text{nm} = 4 \times 10^{-7}$ m.

v_{max} — ?

Solution. From Einstein's law for photoelectric effect $h\nu = A + mv^2/2$, we have $v_{max} = \sqrt{2(h\nu - A)/m}$. Considering that $\nu = c/\lambda$, we obtain

$$v_{max} = \sqrt{\frac{2}{m}\left(\frac{hc}{\lambda} - A\right)},$$

$$v_{max} = \sqrt{\frac{2}{9.1\times10^{-31}} \times \left(\frac{6.62\times10^{-34}\times3\times10^{8}}{4\times10^{-7}} - 3.2\times10^{-19}\right)} \text{ m/s}$$

$$= 6.5 \times 10^5 \text{ m/s}.$$

645. The maximum wavelength of light at which the photoeffect for potassium is observed is 6.2×10^{-5} cm. Determine the work function for potassium.

Given: $\lambda_{max} = 6.2 \times 10^{-5}$ cm $= 6.2 \times 10^{-7}$ m.

$A - ?$

Solution. The maximum wavelength at which the photoeffect is observed for a metal is connected with the photoelectric threshold for the metal through the relation

$$\nu_0 = c/\lambda_{max}. \tag{1}$$

The work function for the metal is $A = h\nu_0$, or, using Eq. (1),

$$A = \frac{hc}{\lambda_{max}},$$

$$A = \frac{6.62\times10^{-34}\times3\times10^{8}}{6.2\times10^{-7}} \text{ J} \simeq 3.2\times10^{-19} \text{ J}.$$

646. The maximum wavelength of light at which the photoeffect for tungsten is observed is 0.275 μm. Determine the work function for tungsten, the maximum kinetic energy of emitted electrons, and the maximum velocity of the electrons knocked out by light of wavelength 0.18 μm from the metal.

Given: $\lambda_{max} = 0.275$ μm $= 0.275 \times 10^{-6}$ m,
$\lambda = 0.18$ μm $= 0.18 \times 10^{-6}$ m.

$A - ?$ $W_{max} - ?$ $v_{max} - ?$

Solution. The work function for tungsten is

$$A = \frac{hc}{\lambda_{max}},$$

$$A = \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{0.275 \times 10^{-6}} \text{ J} \simeq 7.2 \times 10^{-19} \text{ J}.$$

Using Einstein's law for photoelectric effect and considering that $\nu = c/\lambda$, we obtain $hc/\lambda = A + mv_{max}^2/2$, whence the maximum kinetic energy is

$$W_{max} = \frac{mv_{max}^2}{2} = \frac{hc}{\lambda} - A,$$

$$W_{max} = \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{0.18 \times 10^{-6}} - 7.2 \times 10^{-19} \text{ J}$$

$$\simeq 3.8 \times 10^{-19} \text{ J}.$$

Knowing the maximum energy of the emitted electrons, we can determine the maximum velocity corresponding to it:

$$v_{max} = \sqrt{\frac{2W_{max}}{m}},$$

$$v_{max} = \sqrt{\frac{2 \times 3.8 \times 10^{-19}}{9.1 \times 10^{-31}}} \frac{\text{m}}{\text{s}} \simeq 9.1 \times 10^5 \text{ m/s}.$$

647. The energy of a photon is equal to the kinetic energy of an electron having an initial velocity of 10^6 m/s and accelerated by a potential difference of 4 V. Determine the wavelength of the photon.

Given: $\varepsilon = W_k$, $v_0 = 10^6$ m/s, $U = 4$ V.

λ — ?

Solution. The energy of the photon is $\varepsilon = h\nu = hc/\lambda$, whence

$$\lambda = hc/\varepsilon. \quad (1)$$

By hypothesis,

$$\varepsilon = W_k = mv^2/2, \quad (2)$$

where m is the mass of the electron and v its final velocity as a result of acceleration by the electric field.

The work done by the electric field is equal to the change in the kinetic energy of the electron: $mv^2/2 - mv_0^2/2 = A$, whence $mv^2/2 = mv_0^2/2 + A$, or, since $A = eU$,

$$mv^2/2 = mv_0^2/2 + eU. \quad (3)$$

Here $mv_0^2/2$ is the initial kinetic energy of the electron and e its charge. Solving Eqs. (2) and (3) together, we find that

$$\varepsilon = mv_0^2/2 + eU. \qquad (4)$$

Substituting expression (4) into (1), we obtain

$$\lambda = \frac{hc}{mv_0^2/2 + eU},$$

$$\lambda = \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{9.1 \times 10^{-31} \times (10^6)^2/2 + 1.6 \times 10^{-19} \times 4} \text{ m} \simeq 1.8 \times 10^{-7} \text{ m}.$$

648. The electrons knocked out of the surface of a metal during photoelectric effect by a radiation of frequency 2×10^{15} Hz are completely retarded by a deceleration field with a potential difference of 7 V. When the radiation frequency is 4×10^{15} Hz, the potential difference required to stop the electrons is 15 V. Using these data, calculate Planck's constant.

Given: $\nu_1 = 2 \times 10^{15}$ Hz, $U_1 = 7$ V, $\nu_2 = 4 \times 10^{15}$ Hz, $U_2 = 15$ V.

h — ?

Solution. Let us write Einstein's law for the two cases of photoeffect considered in the problem:

$$h\nu_1 = A + mv_1^2/2, \quad h\nu_2 = A + mv_2^2/2. \qquad (1)$$

Since the electrons escaping from the surface of the metal are completely retarded by the decelerating electric field, the change in their kinetic energy is equal to the work done by the electric field:

$$mv^2/2 = eU. \qquad (2)$$

Using Eq. (2), we can write (1) in the form

$$h\nu_1 = A + eU_1, \quad h\nu_2 = A + eU_2.$$

Solving this system of equations, we find that

$$h = \frac{e(U_2 - U_1)}{\nu_2 - \nu_1},$$

$$h = \frac{1.6 \times 10^{-19} \times (15 - 7)}{4 \times 10^{15} - 2 \times 10^{15}} \text{ J} \cdot \text{s} = 6.4 \times 10^{-34} \text{ J} \cdot \text{s}.$$

649. What is the number of photons incident per second on a human eye if the eye perceives light of wavelength 0.5 μm at a luminous power of 2×10^{-17} W?

Given: $t = 1$ s, $\lambda = 0.5\ \mu m = 5 \times 10^{-7}$ m, $N = 2 \times 10^{-17}$ W.

$n - ?$

Solution. The total radiant energy incident on the eye is $W = Nt$. The energy of a photon is $\varepsilon = hc/\lambda$. Then the number of photons incident on the eye is

$$n = \frac{W}{\varepsilon} = \frac{Nt\lambda}{hc},$$

$$n = \frac{2 \times 10^{-17} \times 1 \times 5 \times 10^{-7}}{6.62 \times 10^{-34} \times 3 \times 10^{8}} = 50.$$

650. A water drop of volume 0.2 ml is heated by light of wavelength 0.75 μm, absorbing 10^{10} photons per second. Determine the rate of heating of the drop.

Given: $V = 0.2$ ml $= 0.2 \times 10^{-6}$ m^3, $\lambda = 0.75\ \mu m = 7.5 \times 10^{-7}$ m, $n = 10^{10}$ s^{-1}.

$\Delta T/\Delta t - ?$

Solution. The amount of heat received by the drop is

$$Q = c_w m\, \Delta T, \tag{1}$$

where m is the mass of the drop, c_w the specific heat for water, and ΔT the change in the temperature of the water during its heating. The energy given away by light during the time Δt is

$$W = n\varepsilon\, \Delta t, \tag{2}$$

where ε is the energy of a photon. Neglecting all possible energy losses, we assume that the entire energy received by the drop is spent on its heating, i.e. $W = Q$, or, taking into account expressions (1) and (2), $n\varepsilon\, \Delta t = mc_w\, \Delta T$, whence we can determine the rate of heating:

$$\Delta T/\Delta t = n\varepsilon/(mc_w). \tag{3}$$

Observing that $m = \rho V$, where ρ is the density of water and $\varepsilon = hc/\lambda$, we can write expression (3) as follows:

$$\frac{\Delta T}{\Delta t} = \frac{nhc}{\lambda\rho V c_w},$$

$$\frac{\Delta T}{\Delta t}=\frac{10^{10}\times 6.62\times 10^{-34}\times 3\times 10^{8}}{7.5\times 10^{-7}\times 10^{3}\times 0.2\times 10^{-9}\times 4.2\times 10^{3}}\ \frac{\text{K}}{\text{s}}$$
$$=3.15\times 10^{-9}\ \text{K/s}.$$

651. Determine the light pressure exerted on the walls of an electric bulb of power 100 W. The bulb is made in the form of a sphere of radius 5 cm, and its walls reflect 10% of the incident light. Assume that the entire power consumed by the bulb is converted into radiation.

Given: $N=10^2$ W, $R=5$ cm $=5\times 10^{-2}$ m, $\rho=0.1$.

p—?

Solution. The light pressure is given by

$$p = I\,(1+\rho)/c. \qquad (1)$$

Since $I = N/S$, where $S = 4\pi R^2$ is the surface area of the bulb, Eq. (1) can be transformed as follows:

$$p=\frac{N}{4\pi R^2 c}\,(1+\rho),$$
$$p=\frac{10^2}{4\times 3.14\times (5\times 10^{-2})^2\times 3\times 10^{8}}\,(1+0.1)\ \text{Pa}=12\ \mu\text{Pa}.$$

652. A light beam of wavelength 0.49 μm is incident at right angles to a surface and exerts a pressure of 5 μPa on it. How many photons are incident on a square metre of the surface per second. The reflection coefficient of light from a given surface is 0.25.

Given: $\lambda=0.49$ μm $=4.9\times 10^{-7}$ m, $p=5$ μPa $=5\times 10^{-6}$ Pa, $\rho=0.25$.

n—?

Solution. Using the formula $p = I\,(1+\rho)/c$ for light pressure, we can determine the energy of all the photons incident on a square metre of the surface per second:

$$I = pc/(1+\rho).$$

The energy of a photon is

$$\varepsilon = h\nu = hc/\lambda.$$

Consequently, the number of photons incident on a square metre of the surface per second is

$$n=\frac{I}{\varepsilon}=\frac{p\lambda}{h(1+\rho)},$$

$$n=\frac{5\times10^{-6}\times4.9\times10^{-7}}{6.62\times10^{-34}\times(1+0.25)}\ \mathrm{m^{-2}\cdot s^{-1}}=2.9\times10^{21}\ \mathrm{m^{-2}\cdot s^{-1}}.$$

653. A radiant energy flow of 63 J is incident on a surface of area 100 cm^2 per minute. Determine the light pressure for the cases when the surface completely reflects and completely absorbs the entire radiant flux.

Given: $S=100\ \mathrm{cm^2}=10^{-2}\ \mathrm{m^2}$, $t=1\ \mathrm{min}=60\ \mathrm{s}$, $W=63\ \mathrm{J}$.

p—?

Solution. 1. Since $\rho = 1$, the light pressure is

$$p = 2I/c.$$

By definition, $I = W/(St)$, and hence

$$p=\frac{2W}{cSt},$$

$$p=\frac{2\times63}{3\times10^{8}\times10^{-2}\times60}\ \mathrm{Pa}=0.7\ \mu\mathrm{Pa}.$$

2. Since $\rho=0$, $p=I/c$, or

$$p=\frac{W}{cSt},$$

$$p=\frac{63}{3\times10^{8}\times10^{-2}\times60}\ \mathrm{Pa}=0.35\ \mu\mathrm{Pa}.$$

654. Why are photographic films developed in red light?

Answer. Red light does not act on the film in view of the small energy of photons at this frequency, which is insufficient to initiate a chemical reaction in the emulsion layer.

655. On which surface (black or white) does light exert a higher pressure?

Answer. The light incident on a white surface produces on it a pressure $p = 2I/c$. Since the light is practically absorbed by a black surface, its pressure is $p = I/c$ (see Problem 653). Hence it follows that a higher pressure is exerted on the white surface.

EXERCISES

656. Determine the energy of a photon if the wavelength corresponding to it is 17×10^{-13} m.

657. Determine the work function for zinc if the wavelength corresponding to its photoelectric threshold is 300 nm.

658. Determine the photoelectric threshold for lithium.

659. The photoelectric threshold for potassium corresponds to a wavelength of 0.577 μm. What must be the potential difference between the electrodes to stop the electron emission from the surface of the potassium if the cathode is exposed to light with a wavelength of 0.4 μm?

660. Determine the photoelectric threshold for cesium if the retarding potential for a radiation of wavelength 0.35 μm is 1.47 V.

661. What fraction of the energy of a photon causing a photoeffect is spent for the work function if the maximum velocity of electrons knocked out from the surface of zinc is 10^6 m/s? The photoelectric threshold for zinc corresponds to a wavelength of 290 nm.

662. A radiant flux of power 5 μW and wavelength of 0.36 μm is incident on the surface of a metal. Determine the photoelectron saturation current if 5% of the incident photons knock out electrons from the metal.

663. A photon flux of intensity 10^{18} photons/s is incident at an angle of 45° on a mirror surface of area 10 cm². The wavelength of incident light is 400 nm. Determine the light pressure exerted on the surface if the reflection coefficient for the surface is 0.75.

664. A radiant flux of power 1 μW is incident along the normal to a surface of area 1 cm². Determine the light pressure if the reflection coefficient is 0.8.

665. Determine the light pressure of solar radiation exerted on a square metre of the Earth's surface normal to the direction of the radiation if the solar constant is 8.38 kJ/(cm²·min). The coefficient of reflection of light from the Earth's surface should be neglected.

666. A parallel bundle of rays of wavelength 0.5 μm is incident along the normal on a blackened surface pro-

ducing a pressure of 10^{-9} N/cm^2 on it. Determine the number of photons contained in a cubic metre of incident light flow.

QUESTIONS FOR REVISION

1. What are the values of the mass, velocity, and energy of a photon? 2. What is photoelectric effect? 3. Name the types of photoelectric effect. 4. Explain the working principle of a photocell. 5. Formulate the laws of photoeffect. 6. Write Einstein's law for photoeffect. 7. What is photoelectric threshold? 8. What is retarding potential? 9. Write a formula for calculating light pressure. 10. What is the value of light pressure exerted on the surface of a blackbody? on a mirror surface?

Chapter 5
OSCILLATIONS AND WAVES

5.1. Mechanical Vibrations and Waves

Harmonic oscillations occur under the action of a force F which is proportional to the displacement x of a body and directed to the equilibrium position:

$$F = -kx,$$

where k is the **proportionality factor.**

The law of motion for harmonic oscillations is

$$x = A \sin(\omega t + \varphi_0),$$

where x is the displacement of a body from the equilibrium position at a given instant of time, A the **amplitude of oscillations**, $\omega t + \varphi_0$ the **phase of oscillations**, φ_0 the **initial phase**, and ω the cyclic frequency.

Cyclic frequency ω is connected with the **frequency** ν **of oscillations** and the period T through the relations

$$\omega = 2\pi\nu = 2\pi/T.$$

The **period of natural oscillations of a simple pendulum** is

$$T = 2\pi\sqrt{l/g},$$

where l is the length of the pendulum and g the free-fall acceleration.

The **period of natural vibrations of a simple oscillator** is

$$T = 2\pi\sqrt{m/k},$$

where m is the mass of a vibrating body and k the spring constant.

The **instantaneous velocity** of a **harmonically oscillating body** is

$$v = \frac{dx}{dt} = A\omega \cos(\omega t + \varphi_0) = v_{max} \cos(\omega t + \varphi_0),$$

where $A\omega = v_{max}$ is the **velocity amplitude**.

The **acceleration** of a **harmonically oscillating body** at a given instant is

$$a = \frac{dv}{dt} = \frac{d^2x}{dt} = -A\omega^2 \sin(\omega t + \varphi_0)$$
$$= -a_{max} \sin(\omega t + \varphi_0),$$

where $A\omega^2 = a_{max}$ is the **acceleration amplitude**.

The **force causing harmonic oscillations**

$$F = ma = -mA\omega^2 \sin(\omega t + \varphi_0) = -m\omega^2 x,$$

where $mA\omega^2 = F_{max}$ is the **force amplitude** and m the mass of an oscillating body. Since $F = -kx$, we have $k = m\omega^2$.

The **total energy of a harmonically oscillating body** is

$$W = mA^2\omega^2/2 = kA^2/2.$$

Besides natural oscillations, a body can perform **forced oscillations** under the action of an external force. If the frequency of the external force coincides with the frequency of natural oscillations, **resonance** is observed.

A body can simultaneously participate in several vibrations. In this case, vibrations are added. Let us consider two particular cases of addition of vibrations.

1. Two vibrations occurring along the same straight line in the same direction with the same period but with different amplitudes and initial phases are added. The equations for component vibrations have the form

$$x_1 = A_1 \sin(\omega t + \varphi_{01}), \qquad x_2 = A_2 \sin(\omega t - \varphi_{02}).$$

The resultant vibration is expressed through the equation

$$x = x_1 + x_2 = A \sin(\omega t + \varphi_0),$$

where $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2 \cos(\varphi_{02} - \varphi_{01})}$ is the amplitude of the resultant vibration and $\varphi_0 = \arctan \frac{A_1 \sin \varphi_{01} + A_2 \sin \varphi_{02}}{A_1 \cos \varphi_{01} + A_2 \cos \varphi_{02}}$ is its initial phase.

2. Two mutually perpendicular vibrations of the same period but with different amplitudes and initial phases are added. The trajectory of the resultant vibration is given by

$$\frac{x_1^2}{A_1^2} + \frac{x_2^2}{A_2^2} - 2\frac{x_1x_2}{A_1A_2}\cos(\varphi_{02} - \varphi_{01}) = \sin^2(\varphi_{02} - \varphi_{01}).$$

Depending on the phase difference and the ratio of amplitudes, this will be either a straight line, or an ellipse, or a circle.

The process of propagation of vibrations in an elastic medium is known as a wave. If the direction of vibrations coincides with the direction of propagation of a wave, the wave is referred to as **longitudinal** (e.g. an acoustic wave in air). If the direction of vibrations is perpendicular to the direction of propagation of a wave, we have a **transverse wave**. The **wavelength** is defined as

$$\lambda = vT,$$

where v is the velocity of propagation of a wave and T the period of vibration.

The **equation for a plane wave** has the form

$$x = A \sin \omega[t - (r/v)] = A \sin(\omega t - kr),$$

where $k = 2\pi/\lambda$ is the **wave number** and r the distance covered by the wave from the source of vibrations to the point under consideration (Fig. 212).

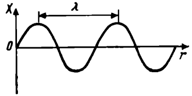


Fig. 212

The **phase difference for two vibrating points** located at distances r_1 and r_2 from the source of vibrations is

$$\Delta\varphi = \varphi_2 - \varphi_1 = 2\pi (r_2 - r_1)/\lambda.$$

When a plane wave is incident on the interface between two media, a **reflected** wave is formed. Being superim-

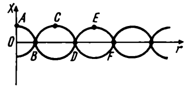


Fig. 213

posed on the incident wave, it forms a **standing wave**.

The **equation for a standing wave** is

$$x = 2A \cos kr \sin \omega t,$$

where $A(r) = 2A \cos kr$ is the amplitude of the standing wave.

The amplitude of a standing wave has its maximum value at the points satisfying the condition

$$r = 2n\lambda/4$$

and known as the **antinodes of the standing wave**. Here $n = 0, 1, 2, \ldots$ (Fig. 213, points $A, C, E, \ldots$).

The amplitude of a standing wave has its minimum value at the points satisfying the condition

$$r = (2n + 1)\lambda/4$$

and called the **nodes of the standing wave**. Here $n = 0, 1, 2, \ldots$ (Fig. 213, points $B, D, F, \ldots$).

* * *

667. A material particle of mass 10 g vibrates according to the law $x = 0.05 \sin (0.6t + 0.8)$. Determine the maximum force acting on the particle and its total energy.

Given: $m = 10 \text{ g} = 10^{-2} \text{ kg}$, $x = 0.05 \sin (0.6t + 0.8)$.

$F_0 - ?$ $W - ?$

Solution. Comparing the general form of equation for harmonic oscillations $x = A \sin(\omega t + \varphi_0)$ with the equation $x = 0.05 \sin(0.6t + 0.8)$ given in the problem, we find that $A = 5 \times 10^{-2}$ m, $\omega = 0.6$ rad/s, and $\varphi_0 = 0.8$ rad. Using the expression $F = -mA\omega^2 \sin(\omega t + \varphi_0)$ for the force causing harmonic oscillations, we obtain

$$F_0 = mA\omega^2,$$

$$F_0 = 10^{-2} \times 5 \times 10^{-2} \times (0.6)^2 \text{ N} = 1.8 \times 10^{-4} \text{ N}.$$

The total energy of the vibrating particle is

$$W = \frac{mA^2\omega^2}{2},$$

$$W = \frac{10^{-2} \times (5 \times 10^{-2})^2 \times (0.6)^2}{2} \text{ J} = 4.5 \ \mu\text{J}.$$

668°. Write the equation for a harmonic oscillation whose amplitude is 10 cm, the period is 10 s, and the initial phase is zero. Determine the displacement, velocity, and acceleration of an oscillating body 12 s after the beginning of the oscillation.

Given: $A = 10$ cm $= 0.1$ m, $T = 10$ s, $\varphi_0 = 0$, $t_1 = 12$ s.

x — ? v_1 — ? a_1 — ?

Solution. Writing the equation for harmonic oscillations, we obtain

$$x = A \sin(\omega t + \varphi_0) = A \sin(2\pi t/T + \varphi_0).$$

Substituting the given quantities into this equation, we get

$$x = 0.1 \sin\left(\frac{2 \times 3.14}{10} t\right) = 0.1 \sin 0.628t.$$

For the instant t_1, we can write

$$x = 0.1 \sin(0.628 \times 12) \text{ m} \simeq 0.095 \text{ m}.$$

The velocity of the oscillating body is

$$v = \frac{dx}{dt} = A\omega \cos(\omega t + \varphi_0),$$

or, substituting the values of the given quantities, we obtain

$$v = 0.1 \times 0.628 \cos 0.628t = 0.0628 \cos 0.628t.$$

For the instant t_1, we have

$$v_1 = 0.0628 \cos (0.628 \times 12) \text{ m/s} \simeq 1.95 \times 10^{-2} \text{ m/s}.$$

The acceleration of the oscillating body is

$$a = \frac{dv}{dt} = -A\omega^2 \sin(\omega t + \varphi_0),$$

or, substituting the values of the given quantities, we obtain

$$a = -0.1 \times 0.628^2 \sin 0.628t = -0.0393 \sin 0.628t.$$

For the instant t_1, we have

$$a_1 = -0.0393 \sin (0.628 \times 12) \text{ m/s}^2$$
$$= -3.73 \times 10^{-2} \text{ m/s}^2.$$

669. In what minimum time after the beginning of oscillations will the displacement of a point from the equilibrium position be equal to half the amplitude if the period of oscillations is 24 s and the initial phase is zero?

Given: $x_1 = A/2$, $T = 24$ s, $\varphi_0 = 0$.

t_1 — ?

Solution. Writing the equation for harmonic oscillations, we obtain

$$x = A \sin (\omega t + \varphi_0) = A \sin (2\pi t/T + \varphi_0).$$

By hypothesis, $x_1 = A \sin (2\pi t_1/T) = (1/2)A$. Cancelling out A, we get

$$\sin (2\pi t_1/T) = 1/2.$$

The value of the sine is 1/2 when its argument is $\pi/6$, i.e. $2\pi t/T = \pi/6$, whence

$$t_1 = \frac{T}{12},$$

$$t_1 = \frac{24}{12} \text{ s} = 2 \text{ s}.$$

670. Write the equation for a harmonic oscillation if its amplitude is 5 cm, the period is 4 s, and the initial phase is $\pi/4$ rad. Plot the time dependences of displacement, velocity, and acceleration.

Given: $A = 5$ cm $= 0.05$ m, $T = 4$ s, $\varphi_0 = \pi/4$ rad.

$x = x(t)$ — ? $v = v(t)$ — ? $a = a(t)$ — ?

Solution. Substituting the given values into the equation $x = A \sin(2\pi t/T + \varphi_0)$, we obtain

$$x = 0.05 \sin\left(\frac{2\pi}{4} t + \frac{\pi}{4}\right). \qquad (1)$$

In order to plot the time dependence of the displacement x, we compile the table for the values of the displacement x at various instants t. For the initial instant $t_0 = 0$, from Eq. (1) we obtain

$$x_0 = 0.05 \sin(\pi/4) \text{ m} \simeq 0.035 \text{ m}.$$

Using Eq. (1), we can determine the time t_{max} corresponding to the maximum displacement $x_{max} = A$:

$$0.05 = 0.05 \sin\left(\frac{2\pi}{4} t_{max} + \frac{\pi}{4}\right),$$

whence $\sin\left(\frac{2\pi}{4} t_{max} + \frac{\pi}{4}\right) = 1$, $\frac{2\pi}{4} t_{max} + \frac{\pi}{4} = \frac{\pi}{2}$,

$t_{max} = 0.5$ s. We can determine the values of x at instants $t_1 = t_{max} + T/4$, $t_2 = t_1 + T/4$, etc. The obtained values are used for compiling the following table:

t, s	0	0.5	1.5	2.5	3.5	4.5
x, m	0.035	0.05	0	−0.05	0	0.05

Using the table, we plot the time dependence of the displacement $x = x(t)$ (Fig. 214*a*). In order to plot the velocity graph, we make use of the equation

$$v = A\omega \cos(2\pi t/T + \varphi_0),$$

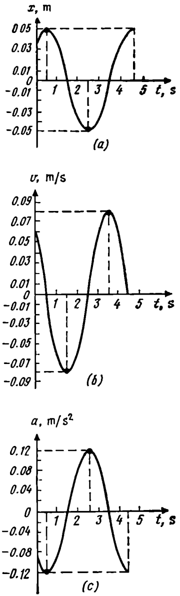

x, m
0.05
0.03
0.01
0
-0.01
-0.03
-0.05
1
2
3
4
5
t, s
(a)
v, m/s
0.09
0.07
0.05
0.03
0.01
0
-0.01
-0.03
-0.05
-0.07
-0.09
1
2
3
4
5
t, s
(b)
a, m/s²
0.12
0.08
0.04
0
-0.04
-0.08
-0.12
1
2
3
4
5
t, s
(c)

Fig. 214

or, substituting the given values of the quantities,

$$v = 0.05\,\frac{2\pi}{4}\cos\left(\frac{2\pi}{4}t+\frac{\pi}{4}\right) \simeq 0.08\cos\left(\frac{2\pi}{4}t+\frac{\pi}{4}\right).$$

In analogy with the previous case, we compile the following table:

t, s	0	0.5	1.5	2.5	3.5	4.5
v, m/s	0.06	0	−0.08	0	0.08	0

Using the table, we plot the time dependence of the velocity $v = v(t)$ (Fig. 214*b*). In order to plot the acceleration graph, we make use of the equation

$$a = -A\omega^2 \sin(2\pi t/T + \varphi_0),$$

or, substituting the given values of the quantities,

$$a = -0.05\left(\frac{2\pi}{4}\right)^2 \sin\left(\frac{2\pi}{4}t+\frac{\pi}{4}\right)$$

$$\simeq -0.12\sin\left(\frac{2\pi}{4}t+\frac{\pi}{4}\right).$$

In analogy with the previous cases, we compile the following table:

t, s	0	0.5	1.5	2.5	3.5	4.5
a, m/s^2	−0.08	−0.12	0	0.12	0	−0.12

Using the table, we plot the time dependence of the acceleration $a = a(t)$ (Fig. 214*c*).

671. Write the equation for a harmonic oscillation of a body if its total energy is 3×10^{-5} J and the maximum force acting on the body is 1.5 mN. The period of oscillations is 2 s and the initial phase is 60°.

Given: $W = 3 \times 10^{-5}$ J, $F_{max} = 1.5$ mN $= 1.5 \times 10^{-3}$ N, $T = 2$ s, $\varphi_0 = 60° = \pi/3$ rad.

$x = x(t)$ — ?

Solution. The displacement of an oscillating body is given by

$$x = A \sin (2\pi t/T + \varphi_0). \qquad (1)$$

In order to determine the amplitude, we shall write the expressions for the total energy and the maximum force:

$$W = mA^2\omega^2/2, \quad F_{max} = mA\omega^2. \qquad (2)$$

Dividing Eqs. (2) termwise, we obtain

$$A = \frac{2W}{F_{max}},$$

$$A = \frac{2 \times 3 \times 10^{-5}}{1.5 \times 10^{-3}} \text{ m} = 4 \times 10^{-2} \text{ m}.$$

Then Eq. (1) becomes

$$x = 4 \times 10^{-2} \sin (\pi t + \pi/3).$$

672. A pendulum consists of a heavy ball of mass 100 g suspended on a string of length 50 cm. Determine the period of oscillations of the pendulum and its energy if the maximum angle by which it is deflected from the equilibrium position is 15°.

Given: $m = 100$ g $= 0.1$ kg, $l = 50$ cm $= 0.5$ m, $\alpha = 15° \simeq 0.26$ rad.

T—? W—?

Solution. Assuming that the ball is a simple pendulum and considering that it oscillates harmonically, we can determine the period of its oscillations:

$$T = 2\pi \sqrt{l/g},$$

$$T = 2 \times 3.14 \sqrt{0.5/9.8} \text{ s} \simeq 1.42 \text{ s}.$$

The total energy of the pendulum deflected through an angle α is the potential energy

$$W_t = mgh. \qquad (1)$$

It follows from Fig. 215 that $h = |AO| - |OB|$. Since $|OB| = l \cos \alpha$, we have

$$h = l - l \cos \alpha = l (1 - \cos \alpha). \qquad (2)$$

Substituting expression (2) into (1), we obtain

$$W_t = mgl\,(1 - \cos\alpha),$$

$$W_t = 0.1 \times 9.8 \times 0.5 \times (1 - 0.97)\ \text{J} \simeq 15\ \text{mJ}.$$

673. A ball is suspended on a long string. It is first lifted along the vertical to the point of suspension, and then deflected through a small angle. In what case will the ball return to the initial position sooner after having been released?

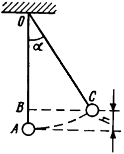


Fig. 215

Given: l.

$\overline{t_1/t_2 - ?}$

Solution. Let us consider the first case. From the equation $l = gt_1^2/2$, we can determine the time t_1 of free fall of the ball from a height equal to the length l of the string:

$$t_1 = \sqrt{2l/g}. \qquad (1)$$

In the second case, the time t_2 of motion of the ball from the deflected position can be determined from the equation $x = A \sin(\omega t + \varphi_0)$ for harmonic oscillations. Since at the initial instant the pendulum suffers the maximum deflection from the equilibrium position, its initial phase is $\varphi_0 = \pi/2$. Since $x = 0$ in equilibrium, we can write $0 = A \sin(\omega t_2 + \pi/2)$. Hence, $\sin(\omega t_2 + \pi/2) = 0$ and $\omega t_2 + \pi/2 = \pi$, whence

$$t_2 = \pi/(2\omega) = T/4. \qquad (2)$$

The ball is a simple pendulum, and hence its period of oscillations can be determined from the formula $T = 2\pi\sqrt{l/g}$. Substituting this expression into (2), we obtain

$$t_2 = T/4 = (\pi/2)\sqrt{l/g}. \qquad (3)$$

Dividing Eq. (1) by (3) termwise, we get

$$\frac{t_1}{t_2} = \frac{\sqrt{2l/g}}{\pi\sqrt{l/g}/2} = \frac{2\sqrt{2}}{\pi} \simeq 0.9.$$

Consequently, the ball returns to the initial position sooner in the first case.

674. A small ball is suspended on a string of length 1 m from the ceiling of a carriage. At what velocity of the carriage will the ball oscillate with the maximum amplitude under the action of impacts of the wheels against the joints of the rails? The length of a rail is 12.5 m.

Given: $l = 1$ m, $s = 12.5$ m.

v — ?

Solution. The ball performs forced oscillations at a frequency ν equal to the frequency of impacts of the wheels against the joints:

$$\nu = v/s. \tag{1}$$

If the size of the ball is small as compared with the length of the string, it can be treated as a simple pendulum whose period of oscillations is $T_0 = 2\pi \sqrt{l/g}$. Then the natural frequency of oscillations is

$$\nu_0 = 1/T_0 = (1/2\pi) \sqrt{g/l}. \tag{2}$$

The amplitude of undamped forced oscillations attains its maximum value in resonance, i.e. when $\nu \approx \nu_0$. Substituting this condition into Eqs. (1) and (2), we find that $v/s = (1/2\pi)\sqrt{g/l}$, whence

$$v = \frac{s}{2\pi} \sqrt{\frac{g}{l}},$$

$$v = \frac{12.5}{2 \times 3.14} \sqrt{\frac{9.8}{1} \frac{\mathrm{m}}{\mathrm{s}}} \simeq 6.2 \text{ m/s}.$$

675. A copper ball suspended on a spring performs vertical vibrations. What will be the change in the period of vibrations if an aluminium ball of the same radius is suspended on the spring?

Given: $\rho_1 = 8.9 \times 10^3$ kg/m³, $\rho_2 = 2.7 \times 10^3$ kg/m³.

T_1/T_2 — ?

Solution. Since the balls suspended on the spring are simple oscillators, the periods of their vibrations are

$$T_1 = 2\pi \sqrt{m_1/k}, \quad T_2 = 2\pi \sqrt{m_2/k},$$

where $m_1 = (4/3)\pi R^3\rho_1$ and $m_2 = (4/3)\pi R^3\rho_2$ are the masses of the copper and the aluminium ball. Therefore,

$$\frac{T_1}{T_2} = \frac{2\pi\sqrt{m_1/k}}{2\pi\sqrt{m_2/k}} = \sqrt{\frac{\rho_1}{\rho_2}},$$

$$\frac{T_1}{T_2} = \sqrt{\frac{8.9\times 10^3}{2.7\times 10^3}} \simeq 1.8,$$

i.e. the period of vibrations will decrease.

676. Two vibrations of the same direction and frequency have amplitudes of 20 and 50 cm. The second vibration leads in phase the first vibration by 30°. Determine the amplitude and the initial phase of the sum of the vibrations if the initial phase of the first vibration is zero.

Given: $A_1 = 20$ cm $= 0.2$ m, $A_2 = 50$ cm $= 0.5$ m, $\varphi_{01} = 0$, $\varphi_{02} - \varphi_{01} = 30° \simeq 0.52$ rad.

A — ? φ_0 — ?

Solution. When two vibrations occurring in the same direction are added, the amplitude of the resultant vibration is

$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\varphi_{02} - \varphi_{01})},$$

$$A = \sqrt{(0.2)^2 + (0.5)^2 + 2\times 0.2\times 0.5\times 0.87}\ \text{m} \simeq 0.68\ \text{m}.$$

From the expression $\tan\varphi_0 = \dfrac{A_1\sin\varphi_{01} + A_2\sin\varphi_{02}}{A_1\cos\varphi_{01} + A_2\cos\varphi_{02}}$, we can find the initial phase of the resultant vibration:

$$\varphi_0 = \arctan\frac{A_1\sin\varphi_{01} + A_2\sin\varphi_{02}}{A_1\cos\varphi_{01} + A_2\cos\varphi_{02}},$$

$$\varphi_0 = \arctan\frac{0.2\times 0 + 0.5\times 0.5}{0.2\times 1 + 0.5\times 0.87} = \arctan 0.394 \simeq 0.38\ \text{rad}.$$

677. A point takes part simultaneously in two mutually perpendicular vibrations with multiple periods, equal amplitudes, and zero initial phases. Plot the trajectory of the point if the period of vibrations along the Y-axis is twice the period of vibrations along the X-axis.

Given: $A_1 = A_2 = A$, $\varphi_{01} = \varphi_{02} = 0$, $T_2 = 2T_1$.

$y(x)$ — ?

Solution. The equations for harmonic vibrations along the X- and Y-axes have the form

$$x = A \sin (2\pi t/T_1), \qquad (1)$$

$$y = A \sin (2\pi t/T_2). \qquad (2)$$

Since, by hypothesis, $T_2 = 2T_1$, Eq. (2) can be reduced to the form

$$y = A \sin (2\pi t/2T_1) = A \sin (\pi t/T_1). \qquad (3)$$

Since the equation for the resultant trajectory in this case is transcendental, we shall solve this problem nu-

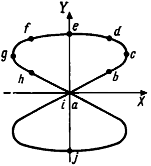


Fig. 216

merically. Using Eqs. (1) and (3), we compile the table of the values of x and y at various instants t expressed in fractions of the period T_1:

t	x	y	point on trajectory	t	x	y	point on trajectory
0	0	0	*a*	$5T_1/8$	$-0.7A$	$0.9A$	*f*
$T_1/8$	$0.7A$	$0.4A$	*b*	$3T_1/4$	$-A$	$0.7A$	*g*
$T_1/4$	A	$0.7A$	*c*	$7T_1/8$	$-0.7A$	$0.4A$	*h*
$3T_1/8$	$0.7A$	$0.9A$	*d*	T_1	0	0	*i*
$T_1/2$	0	A	*e*				

Using the table, we plot the trajectory *abcdefghi* of the point (Fig. 216). In view of symmetry, we can supplement the trajectory with the region aji. We leave it to the reader to verify the correctness of this assumption.

678. The displacement from the equilibrium position of a point lying at 4 cm from the source of vibrations is equal to half the amplitude in a time $T/6$. Determine the wavelength.

Given: $r=4$ cm $=0.04$ m, $t=T/6$, $x=A/2$.

$\lambda-?$

Solution. We transform the wave equation

$$x=A\sin\omega(t-r/v),$$

considering that $\omega=2\pi/T$ and $\lambda=vT$:

$$x=A\sin\left(\frac{2\pi t}{T}-\frac{2\pi r}{vT}\right)=A\sin\left(\frac{2\pi t}{T}-\frac{2\pi r}{\lambda}\right).$$

Then $\frac{A}{2}=A\sin\left(\frac{2\pi T}{6T}-\frac{2\pi\cdot 0.04}{\lambda}\right)$, and after transformations we obtain $\frac{1}{2}=\sin\left(\frac{\pi}{3}-\frac{0.08\pi}{\lambda}\right)$, which shows that the argument of the sine is $\pi/6$, i.e. $\frac{\pi}{3}-\frac{0.08\pi}{\lambda}=\frac{\pi}{6}$. Consequently, $\lambda=0.48$ m.

679. A transverse wave propagates along an elastic cord at a velocity of 15 m/s. The period of vibration of points on the cord is 1.2 s and the amplitude of vibration is 2 cm. Determine the wavelength, the phase, and the displacement of a point separated by 45 m from the source of vibration in 4 s.

Given: $v=15$ m/s, $T=1.2$ s, $A=2$ cm $=2\times10^{-2}$ m, $r=45$ m, $t=4$ s.

$\lambda-?$ $\varphi-?$ $x-?$

Solution. The wavelength is

$$\lambda=vT,$$

$$\lambda=15\times1.2\text{ m}=18\text{ m}.$$

The phase and the displacement of any point can be determined from the wave equation

$$x=A\sin\omega(t-r/v).$$

The phase of vibration is equal to the argument of the sine in the wave equation:

$$\varphi=\omega\left(t-\frac{r}{v}\right)=\frac{2\pi}{T}\left(t-\frac{r}{v}\right),$$

$$\varphi=\frac{2\pi}{1.2}\left(4-\frac{45}{15}\right)\text{ rad}\simeq 5.24 \text{ rad}.$$

The displacement of the point is

$$x = 2 \times 10^{-2}\cdot\sin 5.24 \text{ m} \simeq -1.73 \times 10^{-2}\text{ m}.$$

680. Vibrations of period 0.25 s propagate along a straight line at a velocity of 48 m/s. Ten seconds after the emergence of vibrations at the initial points, the displacement of a point at 43 m from it is found to be 3 cm. Determine the phase and the displacement of a point separated by 45 m from the source of vibrations at the same instant.

Given: $T=0.25$ s, $v=48$ m/s, $t=10$ s, $r_1=43$ m, $r_2=45$ m, $x_1=3$ cm $=3\times 10^{-2}$ m.

φ_2 — ? x_2 — ?

Solution. Writing the equations describing the vibrations of the points separated by the distances r_1 and r_2 from the source, we obtain

$$x_1=A\sin\omega(t-r_1/v),\quad x_2=A\sin\omega(t-r_2/v).$$

Considering that $\omega=2\pi/T$, we have

$$x_1=A\sin\frac{2\pi}{T}\left(t-\frac{r_1}{v}\right),\quad x_2=A\sin\frac{2\pi}{T}\left(t-\frac{r_2}{v}\right).$$

The amplitude of vibrations can be determined from the equation for x_1:

$$A=\frac{x_1}{\sin[(2\pi/T)(t-r_1/v)]},$$

$$A=\frac{3\times 10^{-2}}{\sin[(2\pi/0.25)(10-43/48)]}\text{ m}\simeq 6\times 10^{-2}\text{ m}.$$

Using the equation for x_2, we can determine the phase at the point separated by the distance r_2 from the source of vibrations:

$$\varphi_2=\frac{2\pi}{T}\left(t-\frac{r_2}{v}\right),$$

$$\varphi_2=\frac{2\pi}{0.25}\left(10-\frac{45}{48}\right)\text{ rad}=145\,\frac{\pi}{2}\text{ rad}\simeq 227.65\text{ rad}.$$

The displacement at the point separated by the distance r_2 from the source of vibrations at the instant t is

$$x_2=A\sin\varphi_2,$$

$$x_2=6\times10^{-2}\sin\left(145\,\frac{\pi}{2}\right)\text{ m}=6\times10^{-2}\sin\frac{\pi}{2}$$
$$=6\times10^{-2}\text{ m}.$$

681. Two points are 6 and 12 m away from a source of vibrations. Determine the phase difference for the points if the period of vibrations is 0.04 s and the velocity of propagation of vibrations is 300 m/s.

Given: $r_1=6$ m, $r_2=12$ m, $T=0.04$ s, $v=300$ m/s.

$\Delta\varphi$ — ?

Solution. The equations for vibrations of the points are

$$x_1=A\sin\frac{2\pi}{T}\left(t-\frac{r_1}{v}\right),\quad x_2=A\sin\frac{2\pi}{T}\left(t-\frac{r_2}{v}\right),$$

whence the phases of vibrations of the points are

$$\varphi_1=\frac{2\pi}{T}\left(t-\frac{r_1}{v}\right),\quad \varphi_2=\frac{2\pi}{T}\left(t-\frac{r_2}{v}\right). \qquad (1)$$

Taking into account expressions (1), the phase difference $\Delta\varphi=\varphi_1-\varphi_2$ is

$$\Delta\varphi=\frac{2\pi}{T}\left(t-\frac{r_1}{v}\right)-\frac{2\pi}{T}\left(t-\frac{r_2}{v}\right)=\frac{2\pi}{Tv}(r_2-r_1),$$

$$\Delta\varphi=\frac{2\pi}{0.04\times300}(12-6)\text{ rad}=\pi\text{ rad}.$$

The points vibrate in antiphase.

682. The distance between the second and sixth antinodes of a standing wave is 20 cm. Determine its wavelength.

Given: $\Delta r_{6,2}=20$ cm $=0.2$ m.

λ — ?

Solution. By hypothesis, $\Delta r_{6,2} = r_6 - r_2$, where r_6 is the distance from the source of vibrations to the sixth antinode of the standing wave and r_2 the distance from the source to the second antinode. But the distance r from the source to an antinode is connected with the wavelength through the relation $r_n = 2n\lambda/4$, where n is the number of the antinode. Therefore,

$$r_6 = 2\cdot 6\lambda/4 = 3\lambda, \quad r_2 = 2\cdot 2\lambda/4 = \lambda, \quad \Delta r_{6,2} = 3\lambda - \lambda = 2\lambda,$$

whence

$$\lambda = \frac{\Delta r_{6,2}}{2},$$

$$\lambda = \frac{0.2}{2}\ \text{m} = 0.1\ \text{m}.$$

683. Standing waves are excited in a cord of 3 m whose one end is fastened to a wall and the other end

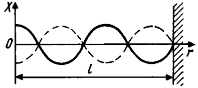


Fig. 217

vibrates at a frequency of 5 Hz. Six nodes are formed between the source and the wall. Determine the velocity of propagation of the wave in the cord.

Given: $l = 3$ m, $\nu = 5$ Hz.

v — ?

Solution. The velocity of propagation of a wave is $v = \lambda/T = \lambda\nu$. Figure 217 shows that $l = 11\lambda/4$, whence $\lambda = 4l/11$. Therefore,

$$v = \frac{4l\nu}{11},$$

$$v = \frac{4\times 3\times 5}{11}\ \text{m/s} = 5.45\ \text{m/s}.$$

684. Will the period of oscillations of a swing change if a load is placed on the board?

Answer. The swing should be regarded as a simple pendulum. Therefore, the period of oscillations will not change since it does not depend on the mass of the load.

685. If a person carries a load tied to one end of a string, the amplitude of the load becomes very large at a certain pace. Explain the phenomenon.

Answer. If the frequency of periodic jerks during walking coincides with the natural frequency of the load, resonance will take place, which leads to a considerable increase in the amplitude of the load.

EXERCISES

686. The period of vibrations of a material particle is 2.4 s, the amplitude is 5 cm, and the initial phase of vibrations is $\pi/2$. What are the displacement, velocity, and acceleration of the vibrating particle 0.4 s after the beginning of the vibrations?

687. A body performs harmonic oscillations according to the law $x = 50 \sin (\pi/3)t$ cm. Determine the amplitude of the force and the total energy of the body if its mass is 2 kg.

688. In what time does a body vibrating harmonically according to the law $x = A \sin \omega t$ cover (1) the entire trajectory from the equilibrium position to the extreme position; (2) the first half of the trajectory; (3) its second half?

689. Plot on the same graph two harmonic oscillations of the same amplitude 3 cm and with a period of 8 s, having the phases $\pi/4$ and $3\pi/2$.

690. Determine the maximum velocity and the maximum acceleration of a vibrating point if its amplitude is 5 cm and the period is 4 s.

691. A material particle vibrates harmonically with a period of 2 s, an amplitude of 50 mm, and a zero initial phase. Determine the velocity of the particle at the instant when its displacement from the equilibrium position is 25 mm.

692. The amplitude of harmonic vibrations of a material particle is 2 cm and its total energy of vibrations is

3×10^{-7} J. At what displacement from the equilibrium position does a force of 2.25×10^{-5} N act on the particle?

693. A pendulum clock is accurate when the length of the pendulum is 55.8 cm. What will be the lag of the clock in 24 h if the pendulum length is increased by 0.5 cm? Assume that the pendulum is simple.

694. Determine the period of vibrations of a simple oscillator if its mass is 196 g and the spring constant is 2×10^2 N/m.

695. Two harmonic oscillations of the same direction, the same period of 8 s, and the same amplitude of 0.02 m are added. The phase difference in oscillations is $\pi/4$ and the initial phase of one of them is zero. Write the equation for the resultant oscillation.

696. A point participates in two mutually perpendicular vibrations simultaneously. The equations for the vibrations are $x = 2 \sin \omega t$ and $y = 2 \cos \omega t$. Determine the trajectory of the point.

697. A wave propagates from a source of vibrations along a straight line. The displacement of a point at the moment of time $0.5T$ is 5 cm. The point is separated from the source by $\lambda/3$. Determine the amplitude of vibrations.

698. A source performs undamped vibrations according to the law $x = 0.05 \sin 500\pi t$ m. Determine the displacement of a point 60 cm from the source 0.01 s after the beginning of the vibrations. The velocity of propagation of the vibrations is 300 m/s.

699. Determine the phase difference for two points separated by 20 cm for a wave propagating at a velocity of 2.4 m/s at a frequency of 3 Hz.

700. A tuning fork (a source of sound waves) is placed in front of an observer's ear and another identical tuning fork is placed at a distance of 47.5 cm from the first. The observer does not hear any sound with such an arrangement. Determine the frequency of vibrations of the tuning fork.

QUESTIONS FOR REVISION

1. What vibrations are called undamped harmonic vibrations? 2. Write expressions for the displacement, the velocity, and the acceleration of a vibrating point. 3. Define the amplitude, the pe-

riod, and the phase of an oscillation. 4. Write formulas for the periods of oscillations of a simple pendulum and a simple oscillator. 5. Write an expression for the total energy of an undamped harmonic oscillation. Why is it constant? 6. What oscillations are referred to as damped? forced? 7. Define resonance. 8. What process can be called a wave process? 9. Name the main types of mechanical waves. 10. What is wavelength? 11. Write a formula connecting the wavelength, the velocity of propagation of a wave, and its period. 12. Write an equation for a travelling wave. 13. Explain how a standing wave is formed. 14. What are the nodes and antinodes of a standing wave? How can their positions be determined relative to the source of vibrations?

5.2. Electromagnetic Oscillations and Waves

When a plane frame of area S, consisting of N turns, rotates uniformly in a magnetic field of induction B at an angular velocity ω, the emf induced in the frame is

$$\mathscr{E} = \mathscr{E}_0 \sin(\omega t + \varphi_0),$$

where $\mathscr{E}_0 = NSB\omega$ is the **emf amplitude**.

If the voltage supplied to a circuit without inductance and capacitance varies according to the law

$$U = U_0 \sin \omega t,$$

the current I varies according to a similar law

$$I = I_0 \sin \omega t.$$

The resistance of such a circuit is known as **(ohmic) resistance** and is given by

$$R = U_0/I_0.$$

The direct current which liberates in a circuit with an ohmic resistance the same energy per unit time as an alternating current is known as the **effective value** I_{eff} **of the alternating current**. The d.c. voltage corresponding to the effective current is called the **effective voltage** U_{eff}. The effective voltage and current are connected with the amplitude values of the alternating current through the following relations:

$$U_{eff} = U_0/\sqrt{2}, \quad I_{eff} = I_0/\sqrt{2}.$$

The parameters of an a.c. circuit with an ohmic resistance are calculated in the same way as for a d.c. circuit by using effective currents and voltages.

Let us consider a circuit consisting of a small ohmic resistance R and a capacitor of capacitance C (Fig. 218*a*). If the voltage across the capacitor varies according to the law

$$U = U_0 \sin \omega t,$$

we can neglect the ohmic resistance R and write

$$q = q_0 \sin \omega t, \quad I = I_0 \cos \omega t,$$

where $q_0 = CU_0$ and $I_0 = q_0\omega = CU_0\omega$.

The reactance of such a circuit (**capacitive reactance**) can be determined from Ohm's law:

$$R_C = U_0/I_0 = 1/\omega C.$$

Let us consider a circuit consisting of a small ohmic resistance R and a coil of inductance L (Fig. 218*b*). If the voltage applied to the circuit varies according to the law

$$U = U_0 \sin \omega t,$$

we can neglect the ohmic resistance R and write

$$I = -I_0 \cos \omega t,$$

where $I_0 = U_0/(\omega L)$.

The reactance of such a circuit (**inductive reactance**) can be determined from Ohm's law:

$$R_L = U_0/I_0 = \omega L.$$

The **impedance of an a.c. circuit** containing an ohmic resistance and capacitive and inductive reactances

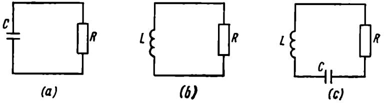


Fig. 218

(Fig. 218c) is

$$Z=\sqrt{R^2+\left(\omega L-\frac{1}{\omega C}\right)^2}.$$

A system consisting of two windings coupled through a core is called a **transformer**. If the primary winding contains N_1 turns and the secondary winding has N_2 turns, the **transformation ratio** is

$$k = N_1/N_2 = \mathscr{E}_1/\mathscr{E}_2,$$

where $\mathscr{E}_1$ and $\mathscr{E}_2$ are the emf's induced in the primary and secondary windings.

If the voltage drop in the ohmic resistance of the primary winding of a transformer is negligibly small, and if the secondary winding is disconnected, then $\mathscr{E}_1 = U_1$ and $\mathscr{E}_2 = U_2$, which gives

$$N_1/N_2 = U_1/U_2.$$

The **efficiency of a transformer** is the ratio of the power P_2 developed by the secondary winding to the power P_1 applied to the primary winding:

$$\eta=\frac{P_2}{P_1}100\%.$$

In a circuit consisting of an inductance L and a capacitance C, there can be induced **natural electromagnetic oscillations** whose period is determined by **Thomson's formula**

$$T=2\pi\sqrt{LC}.$$

Such a circuit is a source of **electromagnetic waves** which involve the propagation of oscillations of electric and magnetic fields.

The **equations for a plane electromagnetic wave** have the form

$$E = E_0 \sin \omega (t - r/c), \quad B = B_0 \sin \omega (t - r/c),$$

where $c = 3 \times 10^8$ m/s is the **velocity of electromagnetic waves in vacuum**.

* * *

701. An a.c. circuit with an effective voltage of 110 V contains a series-connected capacitor of capacitance 50 μF and a coil of inductance 200 mH and ohmic resistance 4 Ω. Determine the amplitude of the current in the circuit if the frequency of the alternating current is 100 Hz. Find the a.c. frequency at which the voltage resonance is observed in the circuit.

Given: $U_{eff} = 110$ V, $C = 50\ \mu\text{F} = 5 \times 10^{-5}$ F,
$L = 200\ \text{mH} = 0.2$ H, $R = 4\ \Omega$, $\nu = 100$ Hz.

I_0 — ? ν_r — ?

Solution. According to Ohm's law, the amplitude of the current is

$$I_0 = \frac{U_0}{\sqrt{R^2 + [\omega L - 1/(\omega C)]^2}}. \qquad (1)$$

Considering that $U_0 = U_{eff}\sqrt{2}$ and $\omega = 2\pi\nu$, we can write Eq. (1) in the form

$$I_0 = \frac{U_{eff}\sqrt{2}}{\sqrt{R^2 + [2\pi\nu L - 1/(2\pi\nu C)]^2}},$$

$$I_0 = \frac{110 \times 1.41}{\sqrt{4^2 + [2 \times 3.14 \times 100 \times 0.2 - 1/(2 \times 3.14 \times 100 \times 5 \times 10^{-5})]^2}}$$
$$\simeq 1.65 \text{ A}.$$

At the voltage resonance, the amplitudes of voltage across the capacitor and the coil are equal: $U_{0C} = U_{0L}$. Since $U_{0C} = I_0/(2\pi\nu_r C)$ and $U_{0L} = 2\pi\nu_r L I_0$, we obtain

$$\nu_r = \frac{1}{2\pi\sqrt{LC}},$$

$$\nu_r = \frac{1}{2 \times 3.14\sqrt{0.2 \times 5 \times 10^{-5}}} \text{ Hz} \simeq 50 \text{ Hz}.$$

702°. An electric circuit with a low resistance contains a capacitor of capacitance 0.2 μF and a coil of inductance 1 mH. During the resonance, the current in the circuit varies according to the law $I = 0.02 \sin \omega t$ (Fig. 219).

Determine the instantaneous value of current and the instantaneous values of voltages across the capacitor and the coil one-third of the period after the beginning of oscilla-

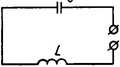


Fig. 219

tions. Plot the time dependences for the current and the voltages.

Given: $C = 0.2\ \mu F = 2 \times 10^{-7}$ F, $L = 1$ mH $= 10^{-3}$ H, $t = T/3$.

$I - ?\ \ U_C - ?\ \ U_L - ?$

Solution. Since the resistance $R \simeq 0$, the cyclic frequency $\omega = 2\pi/T$, or, using Thomson's formula,

$$\omega = \frac{2\pi}{2\pi \sqrt{LC}} = \frac{1}{\sqrt{LC}},$$

$$\omega = \frac{1}{\sqrt{10^{-3} \times 2 \times 10^{-7}}} \frac{\text{rad}}{\text{s}} = 7.1 \times 10^{4} \text{ rad/s}.$$

According to the law of current variation, we can write

$$I = 0.02 \sin (2\pi t/T). \tag{1}$$

For $t = T/3$, the instantaneous value of the current is

$$I = 0.02 \sin\left(\frac{2\pi}{T}\,\frac{1}{3}\,T\right) \text{A} = 1.73 \times 10^{-2} \text{ A}.$$

The voltage across the capacitor is $U_C = q/C$, where q is the charge on the capacitor. By definition, the current is $I = dq/dt$, whence $q = \int\limits_0^t I\, dt$. Substituting Eq. (1) into the integrand of this formula and integrating, we obtain

$$q = \int\limits_0^t 0.02 \sin \omega t\, dt = -\frac{0.02}{\omega} \cos \omega t.$$

Then the voltage across the capacitor is

$$U_C = -\frac{0.02}{\omega C}\cos\omega t = \frac{0.02}{\omega C}\sin\left(\omega t - \frac{\pi}{2}\right)$$
$$= \frac{0.02}{\omega C}\sin\left(\frac{2\pi}{T}t - \frac{\pi}{2}\right).$$

For $t = T/3$, we get

$$U_C = \frac{0.02}{7.1\times 10^4 \times 2\times 10^{-7}}\sin\left(\frac{2\pi}{T}\frac{T}{3} - \frac{\pi}{2}\right)\ \text{V} = 0.7\ \text{V}.$$

The voltage across the coil is

$$U_L = L\frac{dI}{dt} = L\frac{d}{dt}(0.02\sin\omega t) = 0.02\omega L\cos\omega t$$
$$= 0.02\omega L\sin\left(\omega t + \frac{\pi}{2}\right).$$

For $t = T/3$, we obtain

$$U_L = 0.02\times 7.1\times 10^4\times 10^{-3}\sin\left(\frac{2\pi}{T}\frac{T}{3} + \frac{\pi}{2}\right)\ \text{V}$$
$$\simeq -0.7\ \text{V}.$$

The time dependences for the current and the voltages across the capacitor and the coil are shown in Fig. 220*a* and *b*.

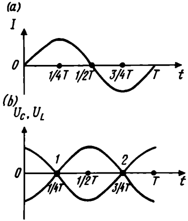


Fig. 220

703°. The voltage across the capacitor in the electric circuit shown in Fig. 219 varies according to the law $U_C = 0.01 \sin \omega t$. Determine the instantaneous values of the current and the voltages across the capacitor and the coil after one-sixth of the period if the capacitance of the capacitor is 0.02 μF and the inductance of the coil is 10 mH. Plot the time dependences for the current and the voltages across the capacitor and the coil.

Given: $C = 0.02\ \mu\text{F} = 2 \times 10^{-8}$ F, $L = 10$ mH $= 10^{-2}$ H, $t = T/6$.

U_C—? I—? U_L—?

Solution. In analogy with Problem 702, for $R = 0$, we have

$$\omega = \frac{1}{\sqrt{LC}},$$

$$\omega = \frac{1}{\sqrt{10^{-2} \times 2 \times 10^{-8}}}\ \frac{\text{rad}}{\text{s}} \simeq 7.1 \times 10^4\ \text{rad/s}.$$

The instantaneous value of the voltage across the capacitor is

$$U_C = 0.01 \sin \omega t. \quad (1)$$

For $t = T/6$, we have

$$U_C = 0.01 \sin\left(\frac{2\pi}{T} \frac{T}{6}\right)\ \text{V} \simeq 8.6\ \text{mV}.$$

The charge on the capacitor is $q = CU_C$, or, taking into account Eq. (1), $q = 0.01C \sin \omega t$. The current is

$$I = \frac{dq}{dt} = \frac{d}{dt}(0.01C \sin \omega t) = 0.01C\omega \cos \omega t$$

$$= 0.01C\omega \sin\left(\omega t + \frac{\pi}{2}\right). \quad (2)$$

For $t = T/6$, the instantaneous value of the current is

$$I = 0.01 \times 2 \times 10^{-8} \times 7.1 \times 10^4 \sin\left(\frac{2\pi}{6} + \frac{\pi}{2}\right)\ \text{A} = 7.1\ \mu\text{A}.$$

The voltage across the coil is $U_L = L\frac{dI}{dt}$, or, taking into account Eq. (2),

$$U_L = L \frac{d}{dt}(0.01C\omega \cos \omega t) = -0.01LC\omega^2 \sin \omega t.$$

Since $\omega^2 = 1/(LC)$, we have

$$U_L = -0.01 \sin \omega t = 0.01 \sin (\omega t + \pi).$$

For $t = T/6$, the instantaneous value of the voltage across the coil is

$$U_L = 0.01 \sin \left(\frac{2\pi}{T} \frac{T}{6} + \pi\right) \text{ V} \simeq -8.6 \text{ mV}.$$

The time dependences for the current and the voltages can be constructed in analogy with the previous problem (we leave this to the reader).

704. Determine the amplitude of the emf induced during the rotation of a rectangular frame at a frequency of 50 Hz in a uniform magnetic field of induction 0.2 T if the area of the frame is 100 cm² and the induction vector is perpendicular to the rotational axis of the frame. The initial phase is zero.

Given: $\nu = 50$ Hz, $S = 100$ cm$^2 = 10^{-2}$ m^2, $B = 0.2$ T, $\varphi_0 = 0$.

$\mathscr{E}_0$ — ?

Solution. According to Faraday's law, the emf induced in the frame is

$$\mathscr{E} = -\frac{\Delta\Phi}{\Delta t}. \tag{1}$$

Here $\Delta\Phi = \Phi_2 - \Phi_1$ is the change in the magnetic flux during the time Δt, where $\Phi_1 = BS \cos \omega t$ and $\Phi_2 = BS \cos [\omega (t + \Delta t)]$ are the magnetic fluxes piercing the frame at the instants t and $t + \Delta t$. Therefore,

$$\Delta\Phi = \Phi_2 - \Phi_1 = BS \{\cos [\omega (t + \Delta t)] - \cos \omega t\}. \tag{2}$$

Using the trigonometric formula $\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$, we transform expression (2) as follows:

$$\Delta\Phi = BS (\cos \omega t \cos \omega \Delta t - \sin \omega t \sin \omega \Delta t - \cos \omega t).$$

Since $\omega \Delta t$ is small, $\cos \omega \Delta t \simeq 1$ and $\sin \omega \Delta t \simeq \omega \Delta t$. Consequently,

$$\begin{aligned}\Delta\Phi &= BS (\cos \omega t - \omega \Delta t \sin \omega t - \cos \omega t)\\ &= -BS \omega \Delta t \sin \omega t.\end{aligned} \tag{3}$$

Substituting expression (3) into (1), we obtain $\mathscr{E} = BS\omega \sin \omega t$, which shows that the emf amplitude is $\mathscr{E}_0 = BS\omega$, or, since $\omega = 2\pi\nu$,

$$\mathscr{E}_0 = 2\pi\nu BS,$$

$$\mathscr{E}_0 = 2 \times 3.14 \times 50 \times 0.2 \times 10^{-2} \text{ V} = 628 \text{ mV}.$$

The emf can be determined by differentiation if we write Faraday's law in the form

$$\mathscr{E} = -\frac{d\Phi}{dt}.$$

705. The voltage across a subcircuit carrying an alternating current varies with time according to the law $U = U_0 \sin(\omega t + \pi/6)$. At the moment $t = T/12$, the instantaneous voltage is 10 V. Determine the voltage amplitude.

Given: $t = T/12$, $U_1 = 10$ V.

U_0—?

Solution. Substituting the values of U_1 and t into the equation

$$U = U_0 \sin(\omega t + \pi/6)$$

and considering that $\omega = 2\pi/T$, we obtain

$$10 = U_0 \sin\left(\frac{2\pi}{T}\frac{T}{12} + \frac{\pi}{6}\right), \quad \text{or} \quad 10 = U_0 \sin\frac{\pi}{3},$$

whence

$$U_0 = \frac{10}{\sin(\pi/3)} = \frac{10}{0.87} \text{ V} \simeq 11.5 \text{ V}.$$

706. An electric furnace whose resistance is 22 Ω is fed by an a.c. generator. Determine the amount of heat liberated by the furnace in 1 h if the current amplitude is 10 A.

Given: $R = 22$ Ω, $t = 1$ h $= 3.6 \times 10^4$ s, $I_0 = 10$ A.

Q—?

Solution. According to Joule's law, the amount of heat liberated in an a.c. circuit is

$$Q = I_{\text{eff}}^2 Rt = (I_0/\sqrt{2})^2 Rt,$$

$$Q = (10/\sqrt{2})^2 \times 22 \times 3.6 \times 10^4 \text{ J} = 39.6 \text{ MJ}.$$

707. The current in the primary winding of a transformer is 0.5 A and the voltage across its ends is 220 V. The current in the secondary winding is 11 A and the voltage across its ends is 9.5 V. Determine the efficiency of the transformer.

Given: $I_1 = 0.5$ A, $U_1 = 220$ V, $I_2 = 11$ A, $U_2 = 9.5$ V.

η — ?

Solution. The efficiency of the transformer is defined as

$$\eta = \frac{P_2}{P_1} 100\%.$$

Since $P_2 = I_2U_2$ and $P_1 = I_1U_1$, we can write

$$\eta = \frac{I_2U_2}{I_1U_1} 100\%,$$

$$\eta = \frac{11 \times 9.5}{0.5 \times 220} 100\% = 95\%.$$

708. During what time will a neon lamp glow if it is connected for 1 min to an a.c. circuit with an effective voltage of 120 V and a frequency of 50 Hz? The lamp starts and stops glowing at 84 V.

Given: $t_0 = 1$ min $= 60$ s, $U_{\text{eff}} = 120$ V, $\nu = 50$ Hz, $U_g = 84$ V.

t_x — ?

Solution. When the lamp is connected to the circuit, the voltage across its electrodes varies according to the law

$$U = U_0 \sin 2\pi\nu t, \tag{1}$$

where

$$U_0 = \sqrt{2} U_{\text{eff}}. \tag{2}$$

The neon lamp starts and stops glowing at $U_g < U_0$. Consequently, the duration of its glow during half a period is

$$\Delta t = t_2 - t_1, \tag{3}$$

where t_1 and t_2 are the instants when the lamp starts and stops glowing. Since the interval t_0 contains $2t_0\nu$ time intervals Δt, the lamp glows during the time

$$t_x = 2t_0\nu\, \Delta t. \tag{4}$$

From Eqs. (1) and (2), we find that $U = \sqrt{2}U_{eff} \sin 2\pi\nu t$. If $t = t_1$ and $t = t_2$, we have $U = U_g$, and hence

$$U_g = \sqrt{2}U_{eff} \sin 2\pi\nu t_1, \; U_g = \sqrt{2}U_{eff} \sin 2\pi\nu t_2,$$

whence

$$\sin 2\pi\nu t_1 = \frac{U_g}{\sqrt{2}U_{eff}},$$

$$\sin (2\pi\cdot 50 t_1) = \frac{84}{1.41 \times 120} \simeq \frac{1}{2}.$$

Consequently, $100\pi t_1 = \pi/6$, whence $t_1 = 1/600$ s. Similarly $\sin(2\pi\cdot 50t_2) \simeq 1/2$, i.e. $100\pi t_2 = 5\pi/6$, whence $t_2 = 5/600$ s. Substituting the values of t_1 and t_2 into Eq. (3), we obtain

$$\Delta t = (5/600 - 1/600)\text{ s} = 1/150 \text{ s}.$$

The time during which the lamp glows can be determined by substituting the values of Δt, t_0, and ν into Eq. (4):

$$t_x = (2 \times 60 \times 50 \times 1/150)\text{ s} = 40 \text{ s}.$$

709. The potential difference across the plates of a capacitor in an oscillatory circuit varies according to the law $U = 50 \cos(10^4\pi t)$. The capacitance of the capacitor is 0.9 μF. Determine the inductance of the circuit, the law of time variation of the current in the circuit, and the wavelength corresponding to the circuit.

Given: $C = 0.9\ \mu\text{F} = 9 \times 10^{-7}$ F.

L—? λ—? $I(t)$—?

Solution. Using Thomson's formula

$$T = 2\pi\sqrt{LC}, \qquad (1)$$

we can determine $L = T^2/(4\pi^2 C)$, or, considering that $T = 2\pi/\omega$, $L = 1/(C\omega^2)$. It follows from the equation $U = 50 \cos 10^4\pi t$ that $\omega = 10^4\pi$ rad/s. Consequently, the inductance of the circuit is

$$L = \frac{1}{9 \times 10^{-7} \times (10^4 \times 3.14)^2}\text{ H} \simeq 1.12 \text{ mH}.$$

The wavelength is $\lambda = cT$, or, taking into account expression (1),

$$\lambda = \frac{2\pi c}{\omega},$$

$$\lambda = \frac{2 \times 3.14 \times 3 \times 10^8}{10^4 \times 3.14}\ \text{m} = 6 \times 10^4\ \text{m}.$$

By definition, the current is

$$I = -I_0 \sin \omega t. \tag{2}$$

Since $I_0 = U_0/(\omega L)$, we have (see Eq. (2))

$$I = -\frac{U_0}{\omega L} \sin \omega t.$$

The values of U_0 and ω can be determined from the equation $U = 50 \cos 10^4 \pi t$: $U_0 = 50$ V and $\omega = 10^4 \pi$ rad/s. Consequently,

$$I = -\frac{50}{10^4 \times 3.14 \times 1.12 \times 10^{-3}} \sin 10^4 \pi t = -1.42 \sin 10^4 \pi t.$$

710. The capacitance of the variable capacitor of the oscillatory circuit of a receiver varies between C_1 and $C_2 = 9C_1$. Determine the waveband of the receiver if the capacitance C_1 of the capacitor corresponds to a wavelength of 3 m.

Given: $C_2 = 9C_1$, $\lambda_1 = 3$ m.

λ_2 — ?

Solution. We denote by λ_1 and λ_2 the wavelengths limiting the waveband. Then

$$\lambda_1 = cT_1 = 2\pi c \sqrt{LC_1}, \tag{1}$$

$$\lambda_2 = cT_2 = 2\pi c \sqrt{LC_2}, \tag{2}$$

where T_1 and T_2 are the minimum and maximum periods of oscillations of the circuit. Considering that $C_2 = 9C_1$, we can write Eq. (2) in the form

$$\lambda_2 = 6\pi c \sqrt{LC_1}. \tag{3}$$

Comparing Eqs. (1) and (3), we find that $\lambda_2 = 3\lambda_1$, i.e. $\lambda_2 = 3 \times 3 = 9$ m. Consequently, the waveband of the oscillatory circuit is limited by the wavelengths 3 and 9 m.

711. A direct and an alternating currents under the same voltage are passed in turn through the filament of an electric lamp. Will the amount of heat liberated by the filament be the same in the two cases?

Answer. Considering that the filament has purely ohmic resistance, we can say that it is heated in the two cases to the same extent since, by hypothesis, the d.c. voltage is equal to the effective voltage U_{eff} of the alternating current. According to Joule's law, for direct and alternating currents we have

$$Q_1 = U^2t/R, \quad Q_2 = U_{eff}^2t/R$$

respectively. Comparing these expressions, we find that $Q_1 = Q_2$.

712. How is energy transferred from the primary winding of a transformer to the secondary winding, although the windings are not connected to each other through a conductor?

Answer. When an electric current passes through the primary winding of the transformer, it produces a magnetic flux which entirely passes through the core and pierces the turns of the secondary winding. Since we are dealing with an alternating current in the primary wining, the magnetic flux varies with time and an emf is induced in the secondary winding.

713. Why are skip (silent) zones observed in radiocommunication on short waves?

Answer. In view of nonhomogeneity of the atmosphere for electromagnetic waves of this range, they are refracted, which leads to the emergence of skip zones.

EXERCISES

714. A resistor of resistance 15 Ω and a coil of inductance 50 mH are connected in series to an a.c. circuit under a voltage of 120 V. Determine the frequency of the current if the amplitude of the current in the circuit is 7 A.

715. An oscillatory circuit has an inductance of 1.6 mH and a capacitance of 0.04 μF. The maximum voltage across the terminals is 200 V. Determine the maximum current in the circuit, neglecting the resistance of the circuit.

716. Determine the instantaneous and effective values of the emf in an a.c. circuit 0.002 s after the beginning of oscillations if the amplitude value of the emf is 127 V. The frequency of the alternating current is 50 Hz and its initial phase is zero.

717. The primary winding of a transformer with a transformation ratio of 8 is connected to a circuit under a voltage of 220 V. The resistance of the secondary winding is 2 Ω and the current in it is 3 A. Determine the voltage across the terminals of the secondary, neglecting the losses in the primary.

718. A coil of length 50 cm and cross-sectional area 3 cm^2 contains 1000 turns and is connected in parallel to an air capacitor. The capacitor consists of two plates of area 75 cm^2 each, separated by 5 mm. Determine the period of oscillations of the obtained oscillatory circuit.

719. Determine the period of oscillations of an oscillatory circuit emitting an electromagnetic wave of wavelength 3 km.

720. Determine the period of oscillations in an oscillatory circuit containing a capacitor of capacitance 500 pF and a coil of inductance 1 mH.

721. The inductance of the coil in an oscillatory circuit varies between 50 and 500 H and the capacitance of the capacitor varies between 10 and 1000 pF. What frequency waveband can be obtained by tuning such a circuit?

722. Determine the natural frequency of electrical oscillations in a circuit containing a coil of inductance 3 mH and a capacitor of capacitance 2 μF.

723. Electromagnetic waves propagate in a homogeneous medium at a velocity of 2×10^8 m/s. What is the wavelength of electromagnetic oscillations in the medium if their frequency in vacuum is 1 MHz?

QUESTIONS FOR REVISION

1. Describe the construction of a simple oscillatory circuit. 2. What are the natural frequency and period of oscillations of a circuit? 3. Write an expression for the emf induced in a frame rotating uniformly at a certain angular velocity in a magnetic field.

4. What are the effective values of voltage and current? 5. What are the values of resistance and capacitive and inductive reactances? 6. Write an expression for the impedance of an a.c. circuit. 7. What device is called a transformer? 8. What is transformation ratio? 9. Define the efficiency of a transformer. 10. Write an equation for a plane electromagnetic wave. 11. What is the velocity of an electromagnetic wave in vacuum?

5.3. Wave Properties of Light

Phenomena like interference. diffraction, and dispersion of light determine its wave properties.

Interference of light is a superposition of coherent light waves as a result of which the maxima and minima in light intensity are formed in certain regions of space. This phenomenon is associated with the redistribution of radiant energy in space. Light waves are called **coherent** if their phase difference remains constant in time.

The **condition for the maximum intensity of light** is

$$\Delta = 2k\ (\lambda/2),$$

where $\Delta = n_2 r_2 - n_1 r_1$ is the **optical path difference** for waves, $k = 1, 2, 3, \ldots$ is an integer, and λ the wavelength.

The **condition for the minimum intensity** has the form

$$\Delta = (2k + 1)\ (\lambda/2).$$

When interference is observed in thin plane-parallel plates in reflected light, the conditions for the maximum and minimum intensities have the form

$$\Delta = 2dn \cos r - (\lambda/2) = 2k\ (\lambda/2),$$

$$\Delta = 2dn \cos r - (\lambda/2) = (2k + 1)\ (\lambda/2),$$

where d is the thickness of a thin plate, n the refractive index of the material of the plate, and r the angle of refraction of light in it.

For interference in the transmitted light, the conditions for maximum and minimum change places.

Diffraction of light is a phenomenon observed when light propagates in a medium with clearly manifested inhomogeneities and consisting in the light propagation to the region of umbra. Diffraction of light can be observed by using a **diffraction grating.**

The condition for the principal maxima for a normal incidence of light on a diffraction grating has the form

$$d \sin \varphi = k\lambda,$$

where d is the grating constant, φ the angle of deflection of a ray from the initial direction, $k = 1, 2, 3, \ldots$ is an integer (spectrum order), and λ the wavelength of light incident on the grating.

Dispersion of light consists in the dependence of the refractive index on the wavelength. An experimental evidence of dispersion is the decomposition of white light into spectrum by a trihedral prism. The spectrum of white light contains seven primary colours which are continuously transformed into one another. Consequently, white light is compound. Red-hot solids and liquids give continuous spectra, while incandescent gases produce line spectra typical of a given element.

In optics, a concept of light ray is often used. A **ray** is a line along which light transfers an energy.

* * *

724. A light wave of wavelength 700 nm propagates in air. What is its wavelength in water?

Given: $\lambda_1 = 700\ \text{nm} = 7 \times 10^{-7}\ \text{m}$.

$\lambda_2 - ?$

Solution. The wavelengths λ_1 and λ_2 of the light waves in air and in water are connected with the velocities v_1 and v_2 of propagation of the waves in air and in water through the following relations:

$$\lambda_1 = v_1/\nu, \quad \lambda_2 = v_2/\nu, \tag{1}$$

where ν is the frequency of light oscillations, which does not change upon a transition of light from one medium to another. Dividing Eqs. (1) termwise, we obtain

$$\lambda_1/\lambda_2 = v_1/v_2. \tag{2}$$

The velocities of propagation of light in air and in water are connected with the absolute refractive indices n_1 and n_2 of the media through the relation

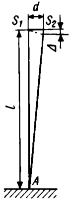


Fig. 221

$$v_1/v_2 = n_2/n_1. \qquad (3)$$

Comparing expressions (2) and (3), we find that $\lambda_1/\lambda_2 = n_2/n_1$, whence

$$\lambda_2 = \frac{\lambda_1 n_1}{n_2},$$

$$\lambda_2 = \frac{7\times 10^{-7}\times 1}{1.33}\ \mathrm{m} \simeq 5.26\times 10^{-7}\ \mathrm{m}.$$

725. Two coherent sources S_1 and S_2 emitting at a wavelength of 0.5 μm are separated by 2 mm. A screen is placed parallel to the line connecting the sources at 2 m from them. What will be observed at point A of the screen (Fig. 221)?

Given: $\lambda = 0.5\ \mu\mathrm{m} = 5\times 10^{-7}$ m, $d = 2$ mm $=$ 2×10^{-3} m, $l = 2$ m.

Δ — ?

Solution. At point A of the screen, there will be the intensity maximum if the path difference for the rays emanating from the sources S_1 and S_2 is equal to an integral number of wavelengths, and the intensity minimum if the difference is equal to an odd number of half-waves. Let us calculate the path difference (see Fig. 221):

$$\Delta = |S_2A| - |S_1A|,$$

where $|S_2A| = \sqrt{l^2+d^2}$ and $|S_1A| = l$. Consequently,

$$\Delta = \sqrt{l^2+d^2} - l = l\sqrt{1+(d/l)^2} - l.$$

Since $d/l \ll 1$, using the formula of approximate calculations, we obtain

$$\Delta \approx l\left[1+\frac{1}{2}\left(\frac{d}{l}\right)^2\right] - l = \frac{d^2}{2l},$$

$$\Delta \simeq \frac{(2\times 10^{-3})^2}{2\times 2}\ \mathrm{m} = 10^{-6}\ \mathrm{m}.$$

Comparing the values of Δ and λ, we find that the path difference Δ is equal to an integral number of wavelengths (two). Consequently, at point A there will be the intensity maximum.

726. A vertical soap film is first observed in reflected light through a red glass ($\lambda_1 = 6.3 \times 10^{-7}$ m). The distance between two neighbouring red lines is 3 mm in this case. Then the film is observed through the blue glass ($\lambda_2 = 4 \times 10^{-7}$ m). Determine the separation between two neighbouring blue lines, assuming that the shape of the film does not change during the observations.

Given: $\lambda_1 = 6.3 \times 10^{-7}$ m, $x_1 = 3$ mm $= 3 \times 10^{-3}$ m, $\lambda_2 = 4 \times 10^{-7}$ m.

x_2 — ?

Solution. The rays which get into an observer's eye are reflected from a thin wedge at right angles to its sur-

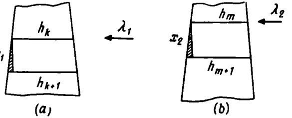


Fig. 222

face. The optical path differences for the kth and $(k+1)$th red lines are respectively

$$\Delta_k = 2h_k n - \lambda_1/2 = k\lambda_1,$$
$$\Delta_{k+1} = 2h_{k+1} n - \lambda_1/2 = (k+1)\lambda_1 \tag{1}$$

($\cos r = 1$ in both cases). Here h_k and h_{k+1} are the thicknesses of the vertical soap film whose section is a wedge (Fig. 222*a*), corresponding to the lines. From Eqs. (1), we obtain

$$\Delta_{k+1} - \Delta_k = 2h_{k+1}n - \frac{\lambda_1}{2} - \left(2h_k n - \frac{\lambda_1}{2}\right)$$
$$= (k+1)\lambda_1 - k\lambda_1,$$

whence

$$2n\ (h_{k+1} - h_k) = \lambda_1. \qquad (2)$$

Similarly, for the blue lines (Fig. 222*b*)

$$2n\ (h_{m+1} - h_m) = \lambda_2. \qquad (3)$$

Dividing expression (2) by (3) termwise, we find that

$$\frac{h_{k+1}-h_k}{h_{m+1}-h_m} = \frac{\lambda_1}{\lambda_2}. \qquad (4)$$

On the other hand, it follows from the similarity of the hatched triangles (see Fig. 222) that

$$\frac{h_{k+1}-h_k}{h_{m+1}-h_m} = \frac{x_1}{x_2}. \qquad (5)$$

Equating the right-hand sides of Eqs. (4) and (5), we obtain $\lambda_1/\lambda_2 = x_1/x_2$, whence

$$x_2 = x_1 \frac{\lambda_2}{\lambda_1},$$

$$x_2 = 3 \times 10^{-3} \times \frac{4 \times 10^{-7}}{6.3 \times 10^{-7}}\ \text{m} = 1.9 \times 10^{-3}\ \text{m}.$$

727. Determine the radius of curvature of a lens used for observing Newton's rings if the distance between the

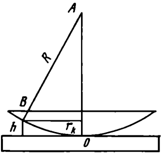


Fig. 223

second and third bright rings is 0.5 mm. The instrument is illuminated by light of wavelength 5.5×10^{-7} m, and observations are carried out in reflected light.

Given: $\Delta_{3,2} = 0.5\ \text{mm} = 0.5 \times 10^{-3}\ \text{m}$, $\lambda = 5.5 \times 10^{-7}\ \text{m}$, $n = 1$.

$R - ?$

Solution. From $\triangle OAB$ (Fig. 223), we have $|BA|^2 = |BO|^2 + |AO|^2$, or $R^2 = r_k^2 + (R-h)^2$, whence $r_k^2 - 2Rh + h^2 = 0$. Neglecting h^2 which is small in comparison with the other terms, we obtain $r_k = \sqrt{R2h}$. On the other hand, the path difference for the kth bright ring in reflected light is

$$\Delta_k = 2hn - \lambda/2 = 2k\ (\lambda/2),$$

whence $2h = (2k+1)/(\lambda/2n)$. Therefore,

$$r_k = \sqrt{(2k+1)\frac{\lambda}{2}\frac{R}{n}}.$$

For $k = 2$ and $k = 3$, we have

$$r_2 = \sqrt{(2\times 2+1)\frac{\lambda R}{2n}} = \sqrt{\frac{5\lambda R}{2n}},$$

$$r_3 = \sqrt{(2\times 3+1)\frac{\lambda R}{2n}} = \sqrt{\frac{7\lambda R}{2n}}.$$

This gives

$$\Delta r_{3,2} = r_3 - r_2 = \sqrt{\frac{7\lambda R}{2n}} - \sqrt{\frac{5\lambda R}{2n}}$$

$$= \sqrt{\frac{\lambda R}{2n}}\,(\sqrt{7} - \sqrt{5}) = 0.4\sqrt{\frac{\lambda R}{2n}},$$

whence

$$R = \frac{\Delta r_3^2 n}{0.08\lambda},$$

$$R = \frac{(0.5\times 10^{-3})^2 \times 1}{0.08 \times 5.5 \times 10^{-7}}\ \text{m} = 5.7\ \text{m}.$$

728. Determine the maximum spectrum order for the yellow line of sodium with a wavelength of 5.89×10^{-7} m if the diffraction grating constant is 2 μm.

Given: $\lambda = 5.89 \times 10^{-7}$ m, $d = 2$ μm $= 2 \times 10^{-6}$ m.

k_{max} — ?

Solution. Using the formula $d \sin \varphi = k\lambda$ for a diffraction grating, we obtain

$$k = d \sin \varphi/\lambda. \qquad (1)$$

Expression (1) shows that for given d and λ, the order k of the spectrum will be maximum when $\sin \varphi = 1$, i.e. at an angle of deflection $\varphi = 1.57$ rad. Consequently,

$$k_{max} = \frac{d}{\lambda},$$

$$k_{max} = \frac{2 \times 10^{-6}}{5.89 \times 10^{-7}} \simeq 3.$$

729. At what distance from a diffraction grating should a screen be placed to obtain a distance of 50 mm

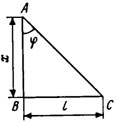


Fig. 224

between the zero-order maximum and the fourth-order spectrum for light with a wavelength of 500 nm? The diffraction grating constant is 0.02 mm.

Given: $k = 4$, $l = 50$ mm $= 5 \times 10^{-2}$ m, $\lambda = 500$ nm $=$ 5×10^{-7} m, $d = 0.02$ mm $= 2 \times 10^{-5}$ m.

x — ?

Solution. Using the formula $d \sin \varphi = k\lambda$ for a diffraction grating, we obtain

$$\sin \varphi = k\lambda/d. \qquad (1)$$

On the other hand, $\sin\varphi$ can be determined from $\triangle ABC$ (Fig. 224) in which the side BC is a part of the screen located at a distance $x = |AB|$ from the diffraction grating. At point B we have the zero-order maximum (undeflected image), while at point C the image of the fourth-order spectrum is observed:

$$\sin\varphi = |BC|/|AC| = l/\sqrt{l^2+x^2}. \qquad (2)$$

Comparing Eqs. (1) and (2), we find that $k\lambda/d = l/\sqrt{l^2+x^2}$, whence

$$x = \sqrt{\frac{d^2l^2 - k^2\lambda^2l^2}{k^2\lambda^2}} = \frac{l}{k\lambda}\sqrt{d^2-k^2\lambda^2},$$

$$x = \frac{5\times10^{-2}}{4\times5\times10^{-7}}\sqrt{(2\times10^{-5})^2-4^2\times(5\times10^{-7})^2}\,\mathrm{m} \simeq 0.5\,\mathrm{m}.$$

730. Determine the diffraction angle for the second-order spectrum for light emitted by sodium with a wavelength of 589 nm if 1 mm of the diffraction grating contains five lines.

Given: $k=2$, $\lambda = 589\ \mathrm{nm} = 5.89\times10^{-7}\ \mathrm{m}$,
$N_0 = 5\ \mathrm{mm}^{-1} = 5\times10^{3}\ \mathrm{m}^{-1}$.

φ — ?

Solution. Using the formula $d\sin\varphi = k\lambda$ for a diffraction grating, we find that

$$\sin\varphi = k\lambda/d. \qquad (1)$$

Since the number of lines per unit length of the grating is connected with the grating constant through the relation $N_0 = 1/d$, Eq. (1) can be written in the form $\sin\varphi = k\lambda N_0$, whence

$$\varphi = \arcsin k\lambda N_0,$$

$$\varphi = \arcsin 2\times5.89\times10^{-7}\times5\times10^{3}\ \mathrm{rad} \simeq 5.8\times10^{-3}\ \mathrm{rad}.$$

731. Determine the maximum order of the spectrum that can be formed by a diffraction grating with 500 lines in 1 mm if the wavelength of incident light is 590 nm. What is the maximum wavelength that can be observed in the spectrum of the grating?

Given: $N_0 = 500\ \text{mm}^{-1} = 5 \times 10^5\ \text{m}^{-1}$, $\lambda = 590\ \text{nm} = 5.9 \times 10^{-7}\ \text{m}$.

k_{max} — ? λ_{max} — ?

Solution. Using the formula $d \sin \varphi = k\lambda$ for a diffraction grating, we find that

$$k = d \sin \varphi / \lambda. \tag{1}$$

Considering that $d = 1/N$, we transform Eq. (1) as follows:

$$k = \sin \varphi / (\lambda N_0). \tag{2}$$

Expression (2) shows that for given λ and N_0, the maximum order k_{max} of the spectrum can be observed for the maximum value $\sin \varphi_{max} = 1$, i.e.

$$k_{max} = \frac{\sin \varphi_{max}}{\lambda N_0} = \frac{1}{\lambda N_0},$$

$$k_{max} = \frac{1}{5.9 \times 10^{-7} \times 5 \times 10^5} \simeq 3.$$

The maximum wavelength that can be observed by using this grating is

$$\lambda_{max} = \frac{d \sin \varphi_{max}}{k_{max}} = \frac{1}{k_{max} N_0},$$

$$\lambda_{max} = \frac{1}{3 \times 5 \times 10^5}\ \text{m} = 6.67 \times 10^{-7}\ \text{m}.$$

732. Water is illuminated by red light. What colour does a diver see under the water surface?

Answer. The diver sees red light since the colour perceived by the eye is determined by the frequency of oscillations that does not change upon a transition of light to water and not by the wavelength that changes.

733. While observing a soap film formed in a plane vertical frame, one can notice that interference fringes are displaced with time downwards. The upper part of the film gradually becomes black, and the film is ruptured. Explain the phenomenon.

Answer. Water in the film gradually flows down, as a result, the lower part of the film becomes thicker and the upper part thinner. The conditions for interference are different on the regions with different thicknesses, which leads to the appearance of dark and bright fringes on the surface. As the water flows down, the thickness of various regions changes, bringing about the changes in the conditions for interference, which is manifested in a downward shift of the fringes. When the thickness of the upper part becomes very small ($d \simeq 0$), the path difference will be $\lambda/2$. The waves at various parts of the film suppress one another, and the film becomes black.

734. A pinhole camera forms the image of an object with the help of a small orifice. As the diameter of the orifice decreases, the sharpness of the image is first improved and then deteriorated. Why?

Answer. As long as the diameter of the orifice is much larger than the wavelength of the light incident on it, diffraction can be disregarded, and the image is sharp. It can be constructed by using the laws of geometrical optics. But when the diameter of the orifice becomes commensurate with the wavelength, diffraction cannot be neglected any longer (i.e. the light will propagate in the region of umbra). The image is blurred.

735. Are the spectra of the Sun, the Moon, planets, and stars identical?

Answer. The spectra of the Moon and planets are the same as the spectrum of the Sun since they glow due to reflected solar rays. The spectra of stars may differ.

736. Why is red light used for stop signals in traffic?

Answer. Red light has a larger wavelength in comparison with light of the other colours constituting the spectrum. The rays with a larger wavelength are scattered by the atmosphere (dust particles and water vapour contained in it) to a smaller extent and hence are seen better from a long distance.

EXERCISES

737. The wavelength of some rays in water is 435 nm. What is the wavelength of these rays in air?

738. In the experiment with Fresnel's mirrors, the distance between the virtual images of a light source is 0.5 mm, and the distance to the screen is 5 m. The interference fringes formed in green light are separated by 5 mm. Determine the wavelength of green light.

739. Light from a projection lantern passes through a small hole covered by a blue glass, falls on a screen with two small holes separated by 1 mm, and then falls on another screen placed 1.7 m behind the first. The distance between the interference fringes on the second screen is found to be 0.8 mm. Determine the wavelength of blue light.

740. White light falls on a soap film with a refractive index of 1.33 at an angle of 45°. What must be the minimum thickness of the film for the reflected rays to be yellow and have a wavelength of 6×10^{-5} cm?

741. Newton's rings are formed between a plane glass and a lens with a radius of curvature of 8.6 m. Monochromatic light is incident along the normal to the surface. It was found as a result of measurements that the diameter of the fourth dark ring is 9 mm. Determine the wavelength of incident light.

742. Violet light of wavelength 0.45 μm is incident along the normal to a diffraction grating with a grating constant of 2 μm. What is the maximum order of the spectrum that can be observed by using the grating?

743. A diffraction grating has 500 lines in 1 mm. At what distance from the central nondecomposed line will the beginning and end of the visible first-order spectrum be on a screen placed at 2 m from the grating if the grating is parallel to the screen and illuminated by light incident along the normal to its surface?

QUESTIONS FOR REVISION

1. What phenomenon is called interference of light? 2. What waves are referred to as coherent? 3. Write the conditions for the maximum and minimum intensity of light during interference. 4. Give the ways of obtaining coherent sources of waves. 5. What phenomenon is called diffraction of light? 6. Under what conditions can diffraction of light be observed? 7. Describe the construction of a diffraction grating. 8. Write a formula for a diffraction grating. 9. What phenomenon is called dispersion of light? 10. Explain the reason behind the formation of spectra. 11. Name the types of spectra you know. 12. Describe the construction of a spectroscope.

Chapter 6

STRUCTURE OF ATOMS AND ATOMIC NUCLEI

6.1. Structure of Atoms

An **atom** of any element consists of a positively charged nucleus and electrons moving around it. The total charge of all the electrons constituting the atom is equal to the charge of its nucleus.

The simplest atom as regards the structure is the hydrogen atom consisting of a nucleus and an electron moving around it. According to **Bohr's first postulate**, it moves in one of n circular orbits without emitting any energy. As the electron goes over from one orbit to another, the hydrogen atom emits or absorbs an energy quantum **(Bohr's second postulate)**:

$$h\nu = W_2 - W_1,$$

where h is Planck's constant, ν the frequency of the emitted light, and W_1 and W_2 the total energy of the electron in the atom on the corresponding orbit, which can be calculated by the formula

$$W_n = -\frac{e^4 m}{8h^2\varepsilon_0^2 n^2},$$

where e is the electron charge, m the electron mass, ε_0 the **electric constant**, and $n = 1, 2, 3, \ldots$ the number of an electron orbit.

The wavelength λ of light emitted by the hydrogen atom as a result of an electron transition from one orbit to another can be determined from the formula

$$\frac{1}{\lambda} = \mathrm{R}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right),$$

where R is the **Rydberg constant**.

For visible light, $n_1 = 2$ is the number of the orbit to which the electron jumps and $n_2 = 3, 4, 5, 6$ is the number of the orbit from which the electron jumps.

* * *

745. Determine the energy emitted by the hydrogen atom as a result of an electron transition from the third to the first orbit.

Given: $n_1 = 1,\ n_2 = 3.$

$\varepsilon - ?$

Solution. According to Bohr's second postulate, the energy emitted by the atom as a result of an electron transition from the third to the first orbit is

$$\varepsilon = W_{n2} - W_{n1}. \qquad (1)$$

Here $W_{n2} = -e^4m/(8h^2\varepsilon_0^2n_2^2)$ and $W_{n1} = -e^4m/(8h^2\varepsilon_0^2n_1^2)$ are the energies of the electron on the third and first orbits. Substituting the expressions for W_{n2} and W_{n1} into Eq. (1), we obtain

$$\varepsilon = -\frac{e^4m}{8h^2\varepsilon_0^2n_2^2} + \frac{e^4m}{8h^2\varepsilon_0^2n_1^2} = \frac{e^4m}{8h^2\varepsilon_0^2}\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right),$$

$$\varepsilon = \frac{(1.6\times10^{-19})^4\times9.1\times10^{-31}}{8\times(6.62\times10^{-34})^2\times(8.87\times10^{-12})^2}\left(1 - \frac{1}{3^2}\right)\ \mathrm{J}$$

$$= 1.94\times10^{-18}\ \mathrm{J}.$$

746. By what amount has the electron energy in the hydrogen atom changed upon the emission of a photon with a wavelength of 4.86×10^{-7} m?

Given: $\lambda = 4.86 \times 10^{-7}$ m.

$\Delta W - ?$

Solution. The change in the energy of the atom upon the emission of the photon is $\Delta W = h\nu$, or, considering that $\nu = c/\lambda$,

$$\Delta W = \frac{hc}{\lambda},$$

$$\Delta W = \frac{6.62\times10^{-34}\times3\times10^{8}}{4.86\times10^{-7}}\ \mathrm{J} = 4.4\times10^{-19}\ \mathrm{J}.$$

747. Calculate the total energy of the electron on the second orbit of the hydrogen atom.

Given: $n = 2$.

$W - ?$

Solution. The energy of the electron on the second orbit of the hydrogen atom is

$$W = -\frac{e^4 m}{8h^2\varepsilon_0^2 n^2},$$

$$W = -\frac{(1.6\times10^{-19})^4\times9.1\times10^{-31}}{8\times(6.62\times10^{-34})^2\times(8.87\times10^{-12})^2\times2^2}\ \text{J}$$
$$= 5.44\times10^{-19}\ \text{J}.$$

748. Determine the maximum and minimum wavelengths in the visible region of the emission spectrum of the hydrogen atom.

Given: $n_2 = 2$.

$\lambda_{max} - ?\ \ \lambda_{min} - ?$

Solution. The wavelength of light emitted by the hydrogen atom as a result of an electron transition from one orbit to another is determined from the formula

$$\frac{1}{\lambda} = R\left(\frac{1}{n_2^2} - \frac{1}{n_3^2}\right). \tag{1}$$

The minimum energy is emitted by the atom upon a transition of the electron to the second orbit from the third orbit ($n_3 = 3$), which is the closest to the second orbit. This corresponds to the emission of light with the maximum wavelength λ_{max}. Consequently,

$$\lambda_{max} = \frac{1}{1.097\times10^7\times(1/2^2 - 1/3^2)}\ \text{m} = 6.56\times10^{-7}\ \text{m}.$$

The maximum energy is emitted by the atom when the electron goes over to the second orbit from an infinitely remote orbit ($n_\infty = \infty$), which corresponds to the emission of light with the minimum wavelength λ_{min}. Consequently,

$$\lambda_{min} = \frac{1}{1.097\times10^7\times(1/2^2 - 1/\infty)}\ \text{m} = 3.65\times10^{-7}\ \text{m}.$$

However, this wavelength does not belong to the visible light region ($\lambda_{min} < 400$ nm). Therefore, using the condition $\lambda_{min} \geqslant 400$ nm and Eq. (1), we can choose n and find that $\lambda_{min} = 4.1 \times 10^{-7}$ m for $n = 6$. Thus, only four lines can be seen in the visible region of the emission spectrum of the hydrogen atom.

749. As an electron goes over from some orbit to the second orbit, the hydrogen atom emits light with a wavelength of 4.34×10^{-7} m. Determine the number of the unknown orbit.

Given: $\lambda = 4.34 \times 10^{-7}$ m, $n_2 = 2$.

n_k — ?

Solution. We shall use the formula for the wavelength of light emitted by the hydrogen atom as a result of an electron transition from one orbit to another:

$$\frac{1}{\lambda} = R\left(\frac{1}{n_2^2} - \frac{1}{n_k^2}\right),$$

whence

$$n_k = \frac{1}{\sqrt{1/n_2^2 - 1/(\lambda R)}},$$

$$n_k = \frac{1}{\sqrt{1/2^2 - 1/(4.34 \times 10^{-7} \times 1.097 \times 10^7)}} = 5.$$

750. A beam of light emitted by a gas-discharge tube filled with atomic hydrogen is incident along the normal to a diffraction grating with a grating constant of 5×10^{-4} cm. From what orbit must the electron come to the second orbit for the spectral line in the fifth-order spectrum to be observed at an angle of 41°?

Given: $d = 5 \times 10^{-4}$ cm $= 5 \times 10^{-6}$ m, $k = 5$, $n_2 = 2$, $\varphi = 41° \simeq 0.72$ rad.

n_k — ?

Solution. Using the formula $d \sin \varphi = k\lambda$ for a diffraction grating, we can determine the wavelength of the emitted light, corresponding to the given spectral line:

$$\lambda = d \sin \varphi / k.$$

From the formula $\frac{1}{\lambda}=R\left(\frac{1}{n_i^2}-\frac{1}{n_k^2}\right)$, we can determine the number of the unknown orbit:

$$n_k=\frac{1}{\sqrt{1/n_i^2-1/(\lambda R)}}, \qquad (1)$$

or, using expression (1),

$$n_k=\frac{1}{\sqrt{1/n_i^2-k/(d\sin\varphi R)}},$$

$$n_k=\frac{1}{\sqrt{1/2^2-5/(5\times10^{-6}\times0.656\times1.097\times10^{7})}}\simeq3.$$

751. The structure of an atom (nucleus + electrons) resembles the structure of the solar system (the Sun + planets). What is the difference between them?

Answer. In the atom, electric forces of attraction act between the electrons and the nucleus, while in the solar system, gravitational forces of attraction act between the Sun and the planets.

Besides, the motion of an electron in the atom does not obey the principles and laws of classical mechanics and is governed by the laws of quantum mechanics.

752. What is the difference between an atom in the ground state and an atom in an excited state?

Answer. The main difference between these atoms consists in that the electrons in the atom in the excited state move in orbits which are further from the nucleus and possess a higher energy in comparison with the electrons in the atom in the ground state.

EXERCISES

753. When a photon is emitted by the hydrogen atom, the total energy of the atom changes by 2.56 eV. Determine the wavelength of the emitted light.

754. What must be the maximum energy of electrons bombarding hydrogen atoms for the emission spectrum of the hydrogen atoms excited by these electrons to contain only one spectral line?

755. Calculate the total energy of the electron on the third orbit of the hydrogen atom.

756. Determine the wavelength corresponding to the third spectral line in the visible region of the spectrum emitted by the hydrogen atom.

757. As the electron in the hydrogen atom goes over from the fourth stationary orbit to the second orbit, the green line is emitted in the hydrogen spectrum. Determine the wavelength corresponding to the line.

758. The electron in the hydrogen atom can move in circular orbits of radii 0.5×10^{-8} and 2×10^{-10} m. What is the ratio of the angular velocities of the electron moving in these orbits?

759. The radius of the electron orbit in the hydrogen atom is 2×10^{-10} m. What is the wavelength of photons that may cause the ionization of the atom?

760. What is the frequency of rotation of the electron in the hydrogen atom if it moves in a circular orbit of radius 5×10^{-11} m?

761. When mercury vapour is bombarded by electrons, the energy of a mercury atom increases by 4.9 eV. What is the wavelength emitted by the atom as a result of the transition to the ground state?

QUESTIONS FOR REVISION

1. Describe the planetary model of an atom. 2. What is the drawback of the planetary model? 3. Formulate Bohr's postulates. 4. Write a formula for the total energy of the hydrogen atom. 5. Write a formula for the angular velocity of an electron moving in a stationary orbit. 6. Derive a formula for the radius of a stationary orbit. 7. What is a quantum number? 8. Write a formula for calculating the wavelength in the visible region of the emission spectrum of the hydrogen atom.

6.2. Structure of Atomic Nuclei

An **atomic nucleus** consists of nucleons, viz. protons and neutrons. The number of **nucleons in** a nucleus is equal to the mass number A (the atomic mass of an element in atomic mass units (amu), rounded to an integer). The number Z of **protons** in a nucleus is equal to the atomic number of the element in Mendeleev's Periodic

Table and corresponds to the charge of the nucleus in the units of elementary charge (electron charge). Consequently, the number of **neutrons** in a nucleus is $N = A - Z$.

For any element and elementary particle we have a notation: ${}_Z^AX$, where ${}_1^1p$ stands for a proton, ${}_0^1n$ for a neutron, ${}_{-1}^{0}e$ for an electron, ${}_1^0e$ for a positron, and ${}_2^4\alpha$ for an alpha-particle.

Atoms of the same element may have different numbers of neutrons in a nucleus. Such atoms are known as **isotopes** of a given element.

The **rest mass of a nucleus** is smaller than the rest mass of a neutral atom by the mass of the electrons constituting the electron shell of the atom:

$$M_{\text{nuc}} = M_a - Zm_e,$$

where m_e is the electron mass.

The **mass defect of a nucleus** is the difference between the sum of the rest masses of the nucleons and the rest mass of the nucleus:

$$\Delta m = Zm_p + (A - Z)\, m_n - M_{\text{nuc}},$$

where m_p is the rest mass of a free proton (which is not bound to the nucleus) and m_n the rest mass of a free neutron.

The **binding energy of a nucleus** is defined as the work that must be done to split the nucleus of an atom into separate nucleons and to remove them at distances beyond the range of nuclear forces without imparting a kinetic energy to them. The binding energy of a nucleus is expressed by

$$W_b = \Delta mc^2,$$

where Δm is the mass defect of the nucleus and c the velocity of light in vacuum.

When atomic nuclei are bombarded by elementary particles, atomic nuclei of one element may be transformed into atomic nuclei of another element. The reactions of transformation of elements **(nuclear reactions)** are schematically written in the form

$${}_Z^AX + {}_2^4\alpha \rightarrow {}_{Z+2}^{A+4}Y, \quad {}_Z^AX + {}_{-1}^{0}e \rightarrow {}_{Z-1}^{A}Y.$$

In these reactions, the law of conservation of electric charge and of the number of nucleons must be observed.

The investigation of the properties of atoms led to the discovery of **radioactivity**. Radioactive substances can emit three types of radiation: α-radiation, β-radiation, and γ-radiation.

Alpha-radiation is the flow of positively charged particles, viz. doubly ionized helium atoms, **beta-radiation** is an electron flow, while **gamma-radiation** is formed by electromagnetic waves of very short wavelength.

The number of radioactive (nondecayed) atoms decreases with time during a radioactive decay according to the law

$$N = N_0 e^{-\lambda t},$$

where N_0 is the number of radioactive atoms at the initial instant, λ the **radioactive decay constant**, and t the decay period.

The **half-life** of a radioactive isotope is defined as the period of time during which half the radioactive substance existing at the initial moment undergoes a decay:

$$T_{1/2} = \ln 2/\lambda = 0.693/\lambda.$$

* * *

762. What is the structure of the nucleus of the lithium isotope $^7_3\mathrm{Li}$?

Given: $^7_3\mathrm{Li}$.

Z—? N—?

Solution. It follows from the notation for the lithium isotope $^7_3\mathrm{Li}$ that its nucleus consists of seven nucleons ($A = 7$): three protons ($Z = 3$) and four neutrons ($N = 7 - 3 = 4$).

763. What is the difference between the nuclei of the nitrogen isotopes $^{14}_7\mathrm{N}$ and $^{15}_7\mathrm{N}$?

Given: $^{14}_7\mathrm{N}$, $^{15}_7\mathrm{N}$.

Z—? N—?

Solution. The notation for the nitrogen isotopes $^{14}_7\mathrm{N}$ and $^{15}_7\mathrm{N}$ shows that the number of protons in their nuclei is the same and equal to seven ($Z = 7$), and the number of neutrons is respectively $N_1 = 14 - 7 = 7$ and $N_2 = 15 - 7 = 8$. Consequently, the nuclei of the isotopes differ in the number of neutrons in them.

764. Calculate the mass defect of the nucleus of the isotope $^{20}_{10}\mathrm{Ne}$.

Given: $^{20}_{10}\mathrm{Ne}$, $m_p = 1.6724 \times 10^{-27}$ kg, $m_n = 1.6748 \times 10^{-27}$ kg, $M_{\mathrm{nuc}} = 33.1888 \times 10^{-27}$ kg.

Δm — ?

Solution. The mass defect of a nucleus is

$$\Delta m = Zm_p + (A - Z)m_n - M_{\mathrm{nuc}}. \qquad (1)$$

The notation of the element $^{20}_{10}\mathrm{Ne}$ shows that $A = 20$ and $Z = 10$. Then Eq. (1) can be written in the form

$$\Delta m = 10m_p + (20 - 10)m_n - M_{\mathrm{nuc}}$$
$$= 10\,(m_p + m_n) - M_{\mathrm{nuc}}.$$
$$\Delta m = [10 \times (1.6724 \times 10^{-27} + 1.6748 \times 10^{-27}) - 33.1888 \times 10^{-27}]\ \mathrm{kg} = 2.882 \times 10^{-28}\ \mathrm{kg}.$$

765. Determine the binding energy of the nucleus of the lithium isotope $^{7}_{3}\mathrm{Li}$.

Given: $^{7}_{3}\mathrm{Li}$, $m_p = 1.6724 \times 10^{-27}$ kg, $m_n = 1.6748 \times 10^{-27}$ kg, $M_{\mathrm{nuc}} = 11.6475 \times 10^{-27}$ kg.

W_{b} — ?

Solution. The binding energy of a nucleus is given by

$$W_{\mathrm{b}} = \Delta mc^2. \qquad (1)$$

Since $\Delta m = Zm_p + (A - Z)\,m_n - M_{\mathrm{nuc}}$, Eq. (1) can be written in the form

$$W_{\mathrm{b}} = [Zm_p + (A - Z)\,m_n - M_{\mathrm{nuc}}]\,c^2.$$

It follows from the notation for the lithium isotope $^{7}_{3}\mathrm{Li}$ that $A = 7$ and $Z = 3$. Substituting the values of A and Z into Eq. (1), we obtain

$$W_{\mathrm{b}} = [3m_p + 4m_n - M_{\mathrm{nuc}}]\,c^2,$$
$$W_{\mathrm{b}} = [3 \times 1.6724 \times 10^{-27} + 4 \times 1.6748 \times 10^{-27} - 11.6475 \times 10^{-27}]\,(3 \times 10^8)^2\ \mathrm{J} = 6.201 \times 10^{-12}\ \mathrm{J}.$$

766. An unknown element and a proton are formed as a result of the capture of an α-particle by the nucleus of the nitrogen isotope ${}^{14}_{7}N$. Write the nuclear reaction equation and identify the unknown element.

Given: ${}^{4}_{2}\alpha$, ${}^{14}_{7}N$, ${}^{1}_{1}p$.

${}^{A}_{Z}X - ?$

Solution. We write the nuclear reaction equation

$$ {}^{14}_{7}N + {}^{4}_{2}\alpha \rightarrow {}^{1}_{1}p + {}^{A}_{Z}X. \quad (1)$$

Since the sums of the mass numbers and charges on the right- and left-hand sides of Eq. (1) must be equal, we have $14 + 4 = 1 + A$ and $7 + 2 = 1 + Z$, whence $A = 17$ and $Z = 8$. Consequently, the obtained element can be presented as ${}^{17}_{8}X$. From Mendeleev's Periodic Table, we find that it is the oxygen isotope ${}^{17}_{8}O$.

767. As a result of the capture of a neutron by the nucleus of the isotope ${}^{14}_{7}N$, an unknown element and an α-particle are formed. Write the nuclear reaction equation and identify the unknown element.

Given: ${}^{1}_{0}n$, ${}^{14}_{7}N$, ${}^{4}_{2}\alpha$.

${}^{A}_{Z}X - ?$

Solution. We write the nuclear reaction equation

$$ {}^{14}_{7}N + {}^{1}_{0}n \rightarrow {}^{A}_{Z}X + {}^{4}_{2}\alpha.$$

According to the law of conservation of mass number and charge, we have $14 + 1 = A + 4$ and $7 + 0 = Z + 2$, whence $A = 11$ and $Z = 5$. Consequently, we can write the unknown element in the form ${}^{11}_{5}X$. Using Mendeleev's Periodic Table, we find that it is the boron isotope ${}^{11}_{5}B$.

768. Radioactive sodium ${}^{24}_{11}Na$ decays with the emission of β-particles. The half-life of sodium is 14.8 h. Calculate the number of atoms decomposed in 1 mg of the given radioactive sample during 10 h.

Given: ${}^{24}_{11}Na$, $T_{1/2} = 14.8\ h \simeq 5.33 \times 10^4\ s$, $t = 10\ h = 3.6 \times 10^4\ s$, $m = 1\ mg = 10^{-6}\ kg$.

$\Delta N - ?$

Solution. The number of atoms that have decayed during the time t is

$$\Delta N = N_0 - N, \tag{1}$$

where N_0 is the number of undecayed atoms at the initial instant, equal to the number of all atoms in 1 mg of $^{24}_{11}Na$, and N the number of undecayed atoms during the time t. Since $N = N_0 e^{-\lambda t}$, Eq. (1) can be written in the form

$$\Delta N = N_0 - N_0 e^{-\lambda t} = N_0 (1 - e^{-\lambda t}). \tag{2}$$

Considering that $\lambda = \ln 2/T_{1/2}$, we transform expression (2) as follows:

$$\Delta N = N_0 (1 - e^{-\ln 2t/T_{1/2}}) = N_0 [1 - (e^{\ln 2})^{-t/T_{1/2}}]$$

$$= N_0 (1 - 2^{-t/T_{1/2}}). \tag{3}$$

Since the number of atoms contained in a mole of $^{24}_{11}Na$ is equal to Avogadro's constant N_A, the given mass m contains the number N_0 of atoms equal to the product of the number m/M of moles and Avogadro's constant:

$$N_0 = mN_A/M, \tag{4}$$

where M is the molar mass of sodium. Substituting expression (4) into (3), we obtain

$$\Delta N = \frac{m}{M} N_A (1 - 2^{-t/T_{1/2}}),$$

$$\Delta N = \frac{10^{-6}}{24 \times 10^{-3}} \times 6.02 \times 10^{23} \times (1 - 2^{-3.6 \times 10^4/(5.33 \times 10^4)})$$

$$\simeq 9.3 \times 10^{18}.$$

769. Determine the half-life of radon if from 10^6 atoms 175 000 atoms decay during 24 hours.

Given: $t = 1$ day $= 8.64 \times 10^4$ s, $N_0 = 10^6$, $\Delta N = 1.75 \times 10^5$.

$T_{1/2}$ — ?

Solution. The half-life of radon is

$$T_{1/2} = 0.693/\lambda. \tag{1}$$

The radioactive decay constant λ can be determined from the relation $\Delta N = N_0 (1 - e^{-\lambda t})$ (see Problem 768),

whence

$$\lambda = \frac{1}{t \log e} \log \frac{N_0}{N_0 - \Delta N}. \qquad (2)$$

Substituting expression (2) into (1), we obtain

$$T_{1/2} = \frac{0.693 (\log e)\, t}{\log [N_0/(N_0 - \Delta N)]},$$

$$T_{1/2} = \frac{0.693 \times 0.43 \times 8.64 \times 10^4}{\log [10^6/(10^6 - 1.75 \times 10^5)]}\ \text{s} \simeq 3.3 \times 10^5\ \text{s}.$$

770. Why cannot α-particles emitted by radioactive samples initiate nuclear reactions in heavy elements?

Answer. The energy of such α-particles is insufficient to overcome the repulsive electric forces exerted by the nucleus of a heavy element to approach it to a distance of 10^{-15} m starting from which the forces of interaction between nucleons considerably exceed the force of electrostatic interaction.

771. What is the difference in the composition of atomic nuclei of radioactive elements and ordinary atomic nuclei?

Answer. The number of neutrons in atomic nuclei of radioactive elements is considerably larger than the number of protons in ordinary atomic nuclei. For example, the nucleus of the uranium isotope $^{238}_{92}U$ contains 146 neutrons and 92 protons.

EXERCISES

772. Describe the structure of the nucleus of the potassium isotope $^{38}_{19}K$.

773. What is the difference between the nuclei of the oxygen isotopes $^{16}_{8}O$, $^{17}_{8}O$, and $^{18}_{8}O$?

774. What minimum energy must be spent to split the nucleus of the helium isotope $^{4}_{2}He$?

775. Determine the mass defect of the nucleus of the hydrogen isotope $^{2}_{1}H$.

776. How are the mass number and the atomic number of an element change upon a proton radioactive decay?

777. Determine the reaction product for the bombardment of the nuclei of the magnesium isotope $^{24}_{12}Mg$ by

α-particles if neutrons are known to be liberated in the nuclear reaction.

778. Write the nuclear reaction equation and identify the unknown element formed as a result of the bombardment of nuclei of the aluminium isotope $^{27}_{13}Al$ by α-particles if one of the reaction products is neutron.

779. How many nuclei of the iodine isotope $^{131}_{53}I$ from 10^9 nuclei decay per second?

780. The initial mass of a radioactive isotope decreases in 8 h to one-third. In what proportion will it decrease in 24 h from the initial instant?

781. Explain why electrons are emitted by an atomic nucleus as a result of β-decay.

QUESTIONS FOR REVISION

1. What particles constitute an atomic nucleus? 2. How can the number of the protons and the neutrons constituting a nucleus be determined? 3. Define an isotope. 4. What is the mass defect of a nucleus? 5. Give an expression for the binding energy of a nucleus. 6. What laws must be observed in writing nuclear reaction equations? 7. What three types of radiation are emitted by radioactive substances? 8. Formulate the law of radioactive decay. 9. Define half-life.

APPENDICES

1. Derivatives of Some Functions

Function	Derivative	Function	Derivative
$c = \text{const}$	0	$\sin x$	$\cos x$
x^n	nx^{n-1}	$\cos x$	$-\sin x$
a^x	$a^x \ln a$	$\tan x$	$\dfrac{1}{\cos^2 x}$
e^x	e^x		
$\ln x$	$\dfrac{1}{x}$	$\cot x$	$-\dfrac{1}{\sin^2 x}$

2. Indefinite Integrals of Some Functions

Function	Integral	Function	Integral
x^n	$\dfrac{x^{n+1}}{n+1}+C$	$\sin x$	$-\cos x + C$
		$\cos x$	$\sin x + C$
$\dfrac{1}{x}$	$\ln \lvert x \rvert + C$	$\tan x$	$-\ln \lvert \cos x \rvert + C$
e^x	$e^x + C$	$\cot x$	$\ln \lvert \sin x \rvert + C$

3. SI Units of Physical Quantities

Quantity	Unit		
	name	notation	dimensions
	Base units		
Length	metre	m	m
Mass	kilogram	kg	kg

(Table 3 continued)

Quantity	Unit name	Unit notation	Unit dimensions
Time	second	s	s
Current	ampere	A	A
Temperature	kelvin	K	K
Luminous intensity	candela	cd	cd
Amount of substance	mole	mol	mol
	Supplementary units		
Plane angle	radian	rad	dimensionless
Solid angle	steradian	sr	dimensionless
	Derived units		
Area	square metre	m^2	m^2
Volume	cubic metre	m^3	m^3
Period	second	s	s
Frequency of periodic process	hertz	Hz	s^{-1}
Rotational frequency	second inverse	s^{-1}	s^{-1}
Density	kilogram per cubic metre	kg/m^3	$m^{-3} \cdot kg$
Velocity	metre per second	m/s	$m \cdot s^{-1}$
Acceleration	metre per second squared	m/s^2	$m \cdot s^{-2}$
Force	newton	N	$m \cdot kg \cdot s^{-2}$
Pressure	pascal	Pa	$m^{-1} \cdot kg \cdot s^{-2}$
Work, energy, amount of heat	joule	J	$m^2 \cdot kg \cdot s^{-2}$
Power	watt	W	$m^2 \cdot kg \cdot s^{-3}$
Electric charge (quantity of electricity)	coulomb	C	$s \cdot A$
Electric voltage, potential difference, emf	volt	V	$m^2 \cdot kg \cdot s^{-3} \cdot A^{-1}$
Electric field strength	volt per metre	V/m	$m \cdot kg \cdot s^{-3} \cdot A^{-1}$
Electric capacitance	farad	F	$m^{-2} \cdot kg^{-1} \cdot s^4 \cdot A^2$
Electric resistance	ohm	Ω	$m^2 \cdot kg \cdot s^{-3} \cdot A^{-2}$
Magnetic flux	weber	Wb	$m^2 \cdot kg \cdot s^{-2} \cdot A^{-1}$
Magnetic induction	tesla	T	$kg \cdot s^{-2} \cdot A^{-1}$
Magnetic field strength	ampere per metre	A/m	$m^{-1} \cdot A$
Inductance	henry	H	$m^{-2} \cdot kg \cdot s^{-2} \cdot A^{-2}$

Table 3 (continued)

Quantity	Unit		
	name	notation	dimensions
Luminous flux	lumen	lm	cd·sr
Illuminance	lux	lx	m^{-2}·cd·sr
Lens power	diopter	D	m^{-1}

4. Approximate Values of Some Fundamental Physical Constants

Physical constant	Notation	Numerical value
Earth's radius	R	6.37×10^{6} m
Earth's mass	M	5.97×10^{24} kg
Normal free-fall acceleration	g	9.8 m/s^2
Gravitational constant	G	6.67×10^{-11} m^3/(kg·s^2)
Avogadro's constant	N_A	6.02×10^{23} mol^{-1}
Molar gas constant	R	8.32 J/(mol·K)
The Boltzmann constant	k	1.38×10^{-23} J/K
Volume of a mole of a gas under normal conditions	V_0	22.41 m^3/mol
Elementary charge	e	1.6×10^{-19} C
Electron mass	m_e	9.11×10^{-31} kg
Faraday's constant	F	9.65×10^{4} C/mol
Speed of light in vacuum	c	3×10^{8} m/s
Planck's constant	h	6.62×10^{-34} J·s
The Rydberg constant	R	1.097×10^{7} m^{-1}
Proton mass	m_p	1.6724×10^{-27} kg
Neutron mass	m_n	1.6748×10^{-27} kg

5. Density of Some Substances, 10^3 kg/m^3

($p = 1.01 \times 10^5$ Pa, $T = 273$ K)

Solids		*Liquids*	
Aluminium	2.7	Alcohol	0.8
Brick	1.8	Kerosene	0.8
Copper	8.9	Mercury	13.6
Ice	0.9	Oil	0.9
Iron	7.8	Petrol	0.7
Lead	11.3	Water	
Nickel	8.8	fresh	1.0
		sea	1.03

Solids		*Gases*	
Silver	10.5	Air	1.29×10^{-3}
Steel	7.8	Helium	0.18×10^{-3}
Wood	0.8	Hydrogen	0.089×10^{-3}
		Oxygen	1.43×10^{-3}

6. Longitudinal Elastic Modulus, 10^{11} Pa

Aluminium	0.7	Iron	2.1
Brass	0.9	Lead	0.17
Copper	1.2	Steel	2.2

7. Surface Tension of Some Liquids at Room Temperature, 10^{-2} N/m

Alcohol	2.2	Mercury	47.1
Aniline	4.3	Soap solution	4.0
Kerosene	3.6	Water	7.4

8. Specific Heat, 10^3 J/(kg·K)

Air	1.005	Iron	0.46
Alcohol	2.42	Lead	0.13
Aluminium	0.88	Nitrogen	1.05
Brass	0.38	Oxygen	0.92
Carbon dioxide	0.83	Steel	0.46
Copper	0.38	Tin	0.23
Hydrogen	14.20	Water	4.19
Ice	2.10		

The specific heats for gases are given at constant volume.

9. Molar Mass of Some Gases, 10^{-3} kg/mol

Air	29	Oxygen O_2	32
Carbon dioxide CO_2	44	Nitrogen N_2	28
Helium He_2	4	Water vapour H_2O	18
Hydrogen H_2	2		

10. Melting Point for Some Solids, K

Aluminium	933	Iron	1803
Brass	1173	Lead	600
Copper	1356	Silver	1233
Ice	273	Tin	505

11. Latent Heat of Melting, 10^5 J/kg

Aluminium	3.90	Lead	0.25
Copper	1.80	Silver	1.01
Ice	3.35	Tin	0.58

12. Temperature of Vaporization, K

($p = 1.01 \times 10^5$ Pa)

Alcohol	351	Mercury	630
Ether	308	Water	373

13. Latent Heat of Vaporization, 10^5 J/kg

Alcohol	9.05	Mercury	2.82
Ether	3.68	Water	22.60

14. Heat of Combustion, 10^7 J/kg

Alcohol	2.93	Petrol	4.61
Coal	2.93	Petroleum	4.61
Kerosene	4.61	Wood	1.26

15. Coefficient of Linear Expansion for Some Solids, 10^{-5} K^{-1}

Aluminium	2.40	Invar	0.15
Brass	1.90	Iron	1.20
Copper	1.70	Lead	2.90
Glass	0.90	Steel	1.10

16. Coefficient of Volume Expansion for Some Liquids, 10^{-4} K^{-1}

Alcohol	11.0	Petroleum	10.0
Kerosene	10.0	Sulphuric acid	5.6
Mercury	1.8	Water	1.8

17. Density of Saturated Water Vapour at Various Temperatures

T, K	ρ, 10^{-3} kg/m^3	T, K	ρ, 10^{-3} kg/m^3
263	2.14	283	9.40
264	2.33	284	10.00
265	2.54	285	10.70
266	2.76	286	11.40
267	2.99	287	12.10
268	3.24	288	12.80
269	3.51	289	13.60
270	3.81	290	14.50
271	4.13	291	15.40
272	4.47	292	16.30
273	4.84	293	17.30
274	5.20	294	18.30
275	5.60	295	19.40
276	6.00	296	20.60
277	6.40	297	21.80
278	6.80	298	23.00
279	7.30	299	24.40
280	7.80	300	25.80
281	8.30	301	27.20
282	8.80	302	28.70

18. Relative Permittivity

Ebonite	3.0	Mica	7.0
Glass	7.0	Paraffine	2.0
Glycerine	39.1	Water	81.0
Kerosene	2.0		

19. Resistivity and Temperature Coefficient of Resistance for Some Conductors

Substances	ρ, 10^{-7} $\Omega\cdot$m	α, 10^{-3} K^{-1}
Aluminium	0.26	3.6
Copper	0.17	4.2
Iron	1.20	6.0

Table 19 (continued)

Substances	ρ, 10^{-7} $\Omega \cdot m$	α, 10^{-3} K^{-1}
Lead	2.10	4.3
Nichrome	11.0	0.4
Silver	0.16	3.6
Tungsten	0.55	5.2

20. Absolute Refractive Index

Air	1.00029	Ice	1.31
Alcohol	1.36	Quartz	1.54
Diamond	2.42	Turpentine	1.47
Glass	1.50	Water	1.33

21. Work Function for Some Metals, 10^{-19} J

Cesium	3.2	Potassium	3.2
Lithium	3.8	Tungsten	7.2
Platinum	8.5	Zinc	6.6

22. Basic Properties of Some Elementary Particles

Particle	Symbol	Charge, 10^{-19} C	Mass, 10^{-27} kg
Alpha-particle	${}^4_2\alpha$	3.2	6.6446
Electron	${}^{0}_{-1}e$	—1.6	0.000911
Neutron	${}^1_0n$	0	1.6748
Positron	${}^0_1e$	1.6	0.000911
Proton	${}^1_0p$	1.6	1.6724

23. Mass of the Nuclei of Some Light Isotopes, 10^{-27} kg

${}^1_1H$	1.6726	${}^{11}_5B$	18.2767
${}^3_2He$	3.3436	${}^{14}_7N$	23.2461
${}^4_2He$	6.6446	${}^{16}_8O$	26.5527
${}^6_3Li$	9.9855	${}^{17}_8O$	28.2202
${}^7_3Li$	11.6475	${}^{20}_{10}Ne$	33.1888

24. Approximate Calculations

While solving problems in physics, we often deal with approximate numerical values, including many constants such as $g = 9.8$ m/s^2.

Modern students can use various calculators which give a large number of significant digits. A student has to decide about the number of digits that must be retained in calculations and disregard the remaining digits. The rules for approximate calculations are given below.

1. In addition and subtraction, the result is rounded so that it does not contain significant digits that are not present at least in one of the given quantities.

Example. $3.351 + 2.45 + 1.2534 \simeq 7.05$.

2. In multiplication, the factors are rounded so that each of them contains the number of significant digits equal to that in the factor with the minimum number of significant digits.

Example. $2.51 \times 1.2 \times 5.245 \simeq 2.5 \times 1.2 \times 5.2$.

The final result must contain the same number of significant digits as that in the rounded factors.

3. In division, the same rule as in multiplication should be observed.

Example. $6.24 \div 2.124 \simeq 6.24 \div 2.12$.

4. While raising to the second (or third) power, the result must contain the number of significant digits equal to that in the base.

Example. $1.25^2 \simeq 1.56$.

5. While extracting a square (or cube) root, the result must contain the same number of significant digits as that in the radicand.

Example. $\sqrt{6.82} \simeq 2.61$.

These rules must also be applied while calculating complex expressions.

ANSWERS TO PROBLEMS

4. $v_2 = 8$ m/s. 5. $x = -1.5 \times 10^3$ m, $y = 4.5 \times 10^3$ m. 6. $x = 19 \times 10^3$ m, $y = 2 \times 10^4$ m. 7. $S = 5$ m, $S_x = 4$ m, $S_y = -3$ m. 30. $a \simeq 0.05$ m/s^2, $t \simeq 53$ s. 31. $t = 20$ s. 32. $a = 0.8$ m/s^2, $S_{10} = 7.6$ m. 33. $a = 1.6$ m/s^2, $v_0 = 5$ m/s. 34. $v_1 = 19.8$ m/s, $t = 3$ s, $v_2 = 9.9$ m/s. 35. $t \simeq 0.5$ s. 36. $\langle v \rangle \simeq 8.2$ m/s. 37. $\langle v \rangle \simeq 16.7$ m/s. 38. $S = -200$ m, $l = 2{,}2 \times 10^2$ m. 39. $a = 2$ m/s^2, $\langle v_1 \rangle = 20$ m/s, $\langle v_2 \rangle = 28.6$ m/s, $\langle v \rangle = 23.5$ m/s. 40. $h = 20$ m, $t = 1$ s, $v_1 = 10.2$ m/s, $v_2 = 10.6$ m/s. 54. $\varphi \simeq 0.068$ rad, $v \simeq 221.7$ m/s. 55. $v \simeq 21.7$ m/s. 56. $h \simeq 20$ m. 57. Increase twofold. 58. $s \simeq 8.7 \times 10^4$ m, $t \simeq 10^2$ s. 59. $v = \sqrt{2gh + v_0^2}$. 60. $s \simeq 5.1$ m. 61. $\alpha = \arctan\left(\frac{1}{4} - \frac{a}{g}\right)$. 62. $s \simeq 6.3$ m. 63. $v \simeq 11.5$ m/s. 64. $t \simeq 17.3 \times 10^3$ s. 75. $l = 0.37$ m. 76. $v \simeq 7.6 \times 10^3$ m/s. 77. $\omega \simeq 7.2 \times 10^{-5}$ rad/s, $v \simeq 3.25 \times 10^2$ m/s. 78. $v \simeq 32$ m/s. 79. $N = 90$ rev, $\varepsilon \simeq 0.14$ rad/s^2. 80. $\varepsilon \simeq 3.2$ rad/s^2. 81. $t = 10$ s. 82. $R \simeq 6.1$ m.

* * *

101. $F \simeq 0.12$ N. 102. $F = 10^3$ N. 103. $s \simeq 24$ m. 104. $T \simeq 5$ N, $a \simeq 2.1$ m/s^2. 105. $a = 0.3$ m/s^2, $s = 27$ m, $T \simeq 24$ kN. 106. $a = g/3$, $T = mg/3$. 107. $a = 2.45$ m/s^2, $h = 1.8$ m, $v \simeq 3$ m/s. 108. $T = 70$ N. 109. $a \simeq 2.7$ m/s^2, $T \simeq 43$ N. 110. $\mu \simeq 0.3$, $T_1 = 8.2$ N, $T_2 = 2$ N. 111. $a \simeq 3.3$ m/s^2, $T = 13$ N, $F_{pr} = 26$ N. 121. $v \simeq 16$ m/s. 122. $\nu = 0.98$ s^{-1}. 123. $\alpha \simeq 0.018$ rad. 124. $T \simeq 632$ N. 125. $F \simeq 61.5$ kN. 126. $\alpha \simeq 1.3$ rad. 127. $\mu \geqslant 0.2$. 128. $\nu \simeq 2.1$ s^{-1}. 137. $F = 2.70$ mN. 138. $\rho = 3\pi/(GT^2)$. 139. $T = 5400$ s, $h = 2.7 \times 10^5$ m, $v \simeq 7.7 \times 10^3$ m/s. 140. $\Delta T \simeq 10^2$ s, $\Delta h \simeq 7.8 \times 10^4$ m. 141. $T_2 = T_1$. 142. $a \simeq 10g$. 143. $h = 6.37 \times 10^6$ m. 144. $T = 7.54 \times 10^3$ s, $V = 1.6 \times 10^3$ m/s. 145. $T = \sqrt{3\pi/(G\rho)}$. 152. $P = \sqrt{2}\, mv$. 153. $v \simeq 1$ m/s. 154. $v = 4 \times 10^{-2}$ m/s. 155. $v_2 = -7.8$ m/s. 156. $v = 400$ m/s. 157. $u_{I} = 6$ m/s, $u_{II} = 3.6$ m/s. 158. The boat will not reach the bank.

* * *

180. $N = 123$ kW. 181. $v_0 \simeq 7.5$ m/s. 182. $\langle F \rangle = 12$ kN. 183. $v_{max} = 5$ m/s. 184. $h = 20$ m. 185. $\eta_1 \simeq 55.6\%$, $\eta_2 \simeq 83.3\%$. 186. $h = 2.6$ m. 187. $s \simeq 0.07$ m. 188. $h \simeq 50$ m. 189. $\Delta l = 0.03$ m. 190. $W_n = 50$ J.

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204. $F_1 \simeq 52$ kN, $F_2 \simeq 39$ kN. 205. $T_1 \simeq 14.3$ N, $T_2 \simeq 10.1$ N. 206. $\alpha = 1.4 \times 10^{-2}$ rad. 207. $N \simeq 9.8$ N. 208. $F_1 \simeq 60$ kN, $F_2 \simeq 90$ kN. 209. $F = 580$ N. 210. $F = 15$ N, $x \simeq 1$ m. 211. $x = 0.13$ m from the left end of the plate. 212. $F_{\text{pr}} \simeq 69$ N.

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224. $F_{\text{pr}} = 72$ kN. 225. $l = 1.52$ m. 226. $H \simeq 20.8$ m. 227. $l = 2.5 \times 10^{-2}$ m. 228. $m_{\text{E}} \simeq 9.6 \times 10^{-2}$ kg, $m_{\text{S}} \simeq 20.4 \times 10^{-2}$ kg. 229. $\rho = 0.75 \times 10^3$ kg/m³. 230. $V_{\text{c}} = 4.2 \times 10^{-5}$ m³. 231. $\rho = 0.75 \times 10^3$ kg/m³.

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253. $m_{O_2} = 8 \times 10^{-26}$ kg, $m_{CO_2} \simeq 7.3 \times 10^{-26}$ kg, $m_{CH_4} \simeq 2.7 \times 10^{-26}$ kg. 254. $N_{H_2} = 3 \times 10^{26}$, $N_{O_2} = 1.9 \times 10^{25}$. 255. $W_{\text{b}1} = 0$, $W_{\text{b}2} = 2.6 \times 10^{-20}$ J. 256. $\Theta \simeq 326$ K. 257. $c \simeq 840$ kJ/(kg·K). 258. 15 kg of ice will melt. 259. $\Delta T = 154$ K. 260. 25 times. 261. $Q = 42.7$ MJ. 262. $N \simeq 736$ W. 263. It will not change.

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281. $p_{\text{atm}} = 100$ kPa. 282. $p \simeq 58$ kPa. 283. $m_2 \simeq 0.019 m_1$. 284. $l = 25$ cm. 285. $\Delta m = 8.2$ kg. 286. $M \simeq 29 \times 10^{-3}$ kg/mol. 287. $T \simeq 440$ K. 288. $m \simeq 1.2 \times 10^{-3}$ kg. 289. $V \simeq 0.128$ m³. 290. $\rho \simeq 4$ kg/m³.

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299. $Q_1 = 27$ kJ, $Q_2 = 21$ kJ. 300. $\Delta U \simeq -1.4$ kJ. 301. $A \simeq 0.7$ kJ. 302. $Q \simeq 23$ kJ. 303. $A = 182$ kJ, $Q = 667$ kJ, $\Delta U \simeq 485$ kJ. 304. $\eta = 20\%$, $A = 1.26$ kJ. 305. $A = \frac{m}{M} R\,(T_1 T_2) \ln \frac{V_2}{V_1}$. 306. $A = 48$ kJ. 307. $A = 136$ J.

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317. $\rho = 5 \times 10^{-3}$ kg/m³. 318. $\rho = 10.24 \times 10^{-3}$ kg/m³. 319. $m \simeq 2.8 \times 10^{-3}$ kg. 320. $p \simeq 99.7$ kPa. 321. $m = 1.76 \times 10^{-3}$ kg. 322. $B \simeq 58.7\%$. 323. $N \simeq 3.3 \times 10^{24}$. 324. $V = 0.1$ m³. 325. $m = 41.2$ kg.

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335. $T = 803$ K. 336. $\Delta l = 7.5 \times 10^{-2}$ m. 337. $\Delta V = 1.25 \times 10^{-5}$ m³. 338. $\Delta V = 4.4 \times 10^{-6}$ m³. 339. $\beta \simeq 10^{-3}$ K^{-1}. 340. $\Delta T \simeq 100$ K. 341. $V \simeq 1.06 \times 10^{-4}$ m³. 342. $T = 309$ K. 352. $\sigma \simeq 3.14 \times 10^{-2}$ N/m. 353. $m = 2.25 \times 10^{-5}$ kg. 354. $\rho = 3.2$ kg/m³. 355. $\Delta p = 64$ Pa. 356. $h_{\text{w}}/h_{\text{k}} \simeq 1.64$. 357. $\sigma \simeq 0.47$ N/m. 358. $h = 1.6$ m. 359. $F = 2.96$ N. 365. $F = 55$ N. 366. $\sigma_{\text{u}} = 235$ MPa. 367. $l = 177$ m. 368. $F \simeq 202$ N. 369. $\omega = 238$ rad/s. 370. $A = 0.72$ mJ.

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384. $q \simeq 8.6 \times 10^{-14}$ C. 385. $q_1 = q_2 \simeq 4.2$ nC. 386. $q = 10$ nC 387. $E = 0$, 1600, 1710, 1600, 1150 V/m. 388. $r = 7.2 \times 10^{-2}$ m. 389. By a factor of 1.25×10^{36}. 390. $q \simeq 5.2$ nC. 391. $\alpha \simeq$ 0.122 rad. 392. $\rho = 1.6 \times 10^3$ kg/m^3. 403. $F = 1$ N. 404. $A =$ 540 J. 405. $U = 2.4$ mV. 406. $U = 2.66$ MV. 407. $F =$ 25 nN. 408. $A \simeq 565$ nJ. 409. $\varphi_1 = \varphi_2 = 90$ V, $\varphi_3 = 9$ V. 410. $\varphi \simeq 15$ V. 411. $\varphi \simeq 12.6$ V. 425. $C \simeq 248$ pF. 426. $q \simeq 3.3$ nC. 427. $U = 150$ V. 428. $q_1 = 2$ nC, $q_2 = 4$ nC, $q_3 = 6$ nC. 429. $U_1 = 7.5$ kV, $U_2 = 4.5$ kV, $q_1 = q_2 = 2.25$ μC. 430. $Q =$ 0.5 mJ. 431. $w = 97$ mJ/m^3.

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451. $R_e = 4$ Ω. 452. $R_e = 10$ Ω. 453. $I = 34$ mA. 454. $I =$ 33.5 mA. 455. $\mathscr{E} = 1.1$ V, $r = 1$ Ω. 456. $I_{\text{sh.c}} = 1.5$ A. 457. In series. 458. In three parallel groups of two series-connected elements. 459. Sixfold. 460. $R = 0.1$ Ω. 461. $\Delta T = 238$ K. 462. $I = 1$ A, $I_1 = 0.6$ A, $I_2 = 0.4$ A. 463. $I = 0.2$ A. 464. $I_2 = 0.4$ A, $U_2 = 32$ V. 477. $Q_2 = 10.5$ kJ, $Q_3 \simeq 7.9$ kJ. 478. $Q_1 \simeq 15.6$ J, $Q_2 \simeq 9.4$ J, $Q_3 = 40$ J, $Q_4 \simeq 6.7$ J. 479. $n \simeq$ 1.27×10^{19}. 480. $l \simeq 9.3$ m. 481. $N_1 = 4N_2$. 482. $t \simeq 1.14 \times$ 10^3 s. 483. $t \simeq 1.6$ s. 484. Lead melts sooner by a factor of 1.67. 485. $I = 2.86$ A. 500. $j \simeq 56$ A/m^2. 501. $W = 1.8$ kJ. 502. $A = 65.4$ g/mol. 503. $h \simeq 5.4 \times 10^{-5}$ m. 504. $m =$ 0.445 kg. 505. $R = 0.4$ Ω. 506. $m \simeq 5.9 \times 10^{-6}$ kg. 507. $U_1 =$ 13.5 V. 508. $I_{\text{sat}} = 10^{-7}$ A.

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524. $B = 50$ μT. 525. $B = 62.8$ μT. 526. $B \simeq 119$ mT. 527. $B = 1.84$ T. 528. $B = 96$ μT. 529. $I = 15$ A. 530. $R \simeq$ 2.8×10^{-4} m. 531. $F = 1.6 \times 10^{-13}$ N, $R \simeq 1.04 \times 10^{-2}$ m. 532. $B = 5$ mT. 546. $A = 2.5$ J. 547. $A_1 = 0.024$ J, $A_2 = 0.048$ J, $A_3 = 0$. 548. $I = 10^{-5}$ A. 549. $I = 0.74$ A. 550. $q = 0.074$ C. 551. $S = 1.07 \times 10^{-2}$ m^2. 552. $\nu = 8$ s^{-1}. 553. $L = 7.1 \times 10^{-4}$ H, $\Phi = 3.55$ μWb. 554. $L = 1.6$ mH. 555. $q = 2.4$ mC. 556. $t =$ 0.017 s. 557. $W = 1.69$ J.

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569. $E = 2.4$ lx. 570. $r = 0.4$ m. 571. $r = 1$ m, $h = 0.71$ m. 572. $E = 14.26$ lx. 573. $\Phi = 3.75$ lm. 574. $I_1 = 80$ cd, $I_2 = 46$ cd. 575. $B = 1.2 \times 10^7$ cd/m^2. 576. $E = 34.2$ lx. 577. $S = 1.055 \times$ 10^3 m^2.

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591. $x = 2$ m. 592. Infinitely large number. 593. $\varphi = 2.9 \times$ 10^{-4} rad. 594. $F \simeq 0.1$ m. 595. $F = 0.25$ m. 596. $x = 1.26$ m. 597. $x = 0.45$ m. 598. $R = 2$ m. 599. $x = 1.05$ m (from the concave mirror). 629. $h = 2.66$ m. 630. $r = 0.82$ rad, $i_e = 1.13$ rad. 631. $r_2 \simeq 0.41$ rad, $\delta \simeq 0.34$ rad. 632. $f = 0.6$ m, virtual image.

633. $F = 0.16$ m. 635. $D = 0.42$ D. 637. $F = 6.4$ m. 638. $D = 1.25$ D. 639. $t = 50$ s. 640. $l = 5 \times 10^{-3}$ m. 641. $\Gamma = 187.5$. 642. $|A_1B_1| = 9 \times 10^{-3}$ m.

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656. $\mathscr{E} = 1.15 \times 10^{-18}$ J. 657. $A = 6.55 \times 10^{-19}$ J. 658. $\nu_0 = 5.8 \times 10^{14}$ Hz. 659. $U = 0.95$ V. 660. $\nu = 4.76 \times 10^{14}$ Hz. 661. $k = 60\%$. 662. $I_{sat} = 7.27 \times 10^{-19}$ A. 663. $p = 2.03$ μPa. 664. $p = 6 \times 10^{-11}$ Pa. 665. $p = 4.6$ μPa. 666. $n = 2.52 \times 10^{13}$ m^{-3}.

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686. $x = 2.5 \times 10^{-2}$ m, $v = 0.11$ m/s, $a = -0.17$ m/s^2. 687. $F_0 = 109$ N, $W \simeq 2.7$ kJ. 688. $t_1 = T/4$, $t_2 = T/12$, $t_3 = T/6$. 690. $v_{max} = 7.85 \times 10^{-2}$ m/s, $a_{max} = 12.3 \times 10^{-2}$ m/s^2. 691. $v = 0.136$ m/s. 692. $x = -1.5 \times 10^{-2}$ m. 693. $\Delta t = 3.46 \times 10^2$ s. 694. $T \simeq 0.2$ s. 695. $x = 3.7 \times 10^{-2} \sin(0.785t + 0.392)$. 696. $x^2/4 + y^2/4 = 1$. 697. $A \simeq 5.8$ cm. 698. $x = 0$. 699. $\Delta\varphi = 1.57$ rad. 700. $\nu = 348$ Hz.

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714. $\nu = 61$ Hz. 715. $I_{max} = 1$ A. 716. $\mathscr{E}_{eff} = 50$ V, $\mathscr{E} = 70$ V. 717. $U = 21.5$ V. 718. $T = 6.28 \times 10^{-7}$ s. 719. $T = 10^{-5}$ s. 720. $T \simeq 4.7 \times 10^{-6}$ s. 721. From 2.3 to 7.1 kHz. 722. $\nu = 71$ kHz. 723. $\lambda = 200$ m.

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737. $\lambda = 5.79 \times 10^{-7}$ m. 738. $\lambda = 5 \times 10^{-7}$ m. 739. $\lambda = 4.7 \times 10^{-7}$ m. 740. $d_{min} = 1.3 \times 10^{-7}$ m. 741. $\lambda = 5.89 \times 10^{-7}$ m. 742. $k_{max} = 4$. 743. $l_1 = 0.4$ m, $l_2 = 0.8$ m.

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753. $\lambda = 4.86 \times 10^{-7}$ m. 754. $W = 1.94 \times 10^{-19}$ J. 755. $W = -2.42 \times 10^{-19}$ J. 756. $\lambda = 4.34 \times 10^{-7}$ m. 757. $\lambda = 4.88 \times 10^{-7}$ m. 758. $\omega_1/\omega_2 = 8$. 759. $\lambda = 3.45 \times 10^{-7}$ m. 760. $\nu = 3.6 \times 10^{16}$ Hz. 761. $\lambda = 25.3 \times 10^{-7}$ m.

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772. $A = 39$, $Z = 19$. 773. In the number of neutrons. 774. $W \simeq 4.53 \times 10^{-12}$ J. 775. $\Delta m \simeq 2.44 \times 10^{-29}$ kg. 776. $\Delta A = 1$, $\Delta Z = 1$. 777. ${}^{27}_{14}Si$. 778. ${}^{30}_{15}P$. 779. $\Delta N = 10^3$. 780. $n = 27$.

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