# V. S. Vladimirov

# Generalized Functions in Mathematical Physics

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В. С. Владимиров

# ОБОБЩЕННЫЕ ФУНКЦИИ В МАТЕМАТИЧЕСКОЙ ФИЗИКЕ

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# Generalized Functions in Mathematical Physics

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# Preface

As physics advances, its theoretical statements require ever "higher" mathematics. In this connection it is well worth quoting what the eminent English theoretical physicist Paul Dirac said in 1930 (Dirac [1]) when he gave a theoretical prediction of the existence of antiparticles:

> It seems likely that this process of increasing abstraction will continue in the future and that advance in physics is to be associated with a continual modification and generalisation of the axioms at the base of the mathematics rather than with a logical development of any one mathematical scheme on a fixed foundation.

Subsequent development of theoretical physics, particularly of quantum field theory, fully corroborated this view. Again in this connection we quote the apt words of N.N. Bogoliubov. In 1963 he said: "The basic concepts and methods of quantum field theory are becoming more and more mathematical."

The construction and investigation of mathematical models of physical phenomena constitute the subject of mathematical physics.

Since the time of Newton the search for and study of mathematical models of physical phenomena—the problems of mathematical physics—have made it necessary to resort to a wide range of mathematical tools and have thus stimulated the development of various areas of mathematics. Traditional (classical) mathematical physics had to do with the problems of classical physics: mechanics, hydrodynamics, acoustics, diffusion, heat conduction, potential theory, electrodynamics, optics and so forth. These problems all reduced to boundary-value problems for differential equations (the equations of mathematical physics). The basic mathematical tool for investigating such problems is the theory of differential equations and allied fields of mathematics: integral equations, the calculus of variations, approximate and numerical methods. With the advent of quantum physics, the range of mathematical tools expanded considerably through the use of the theory of operators, the theory of generalized functions, the theory of functions of complex variables, topological and algebraic methods, computational mathematics and computers. All these theories were pressed into service in addition to the traditional tools of mathematics. This intensive interaction of

theoretical physics and mathematics gradually brought to the fore a new domain, that of modern mathematical physics.

To summarize, then: modern mathematical physics makes extensive use of the latest attainments of mathematics, one of which is the theory of generalized functions. The present monograph is devoted to a brief exposition of the fundamentals of that theory and of certain of its applications to mathematical physics.

At the end of the 1920's Paul Dirac (see Dirac [3]) introduced for the first time in his quantum mechanical studies the so-called delta function ($\delta$ function), which has the following properties:

$$\delta(x) = 0, \quad x \neq 0; \qquad \int \delta(x)\,\varphi(x)\,dx = \varphi(0), \quad \varphi \in C. \qquad (*)$$

It was soon pointed out by mathematicians that from the purely mathematical point of view the definition is meaningless. It was of course clear to Dirac himself that the $\delta$ function is not a function in the classical meaning and, what is important, it operates as an operator (more precisely, as a functional) that relates, via formula $(*)$, to each continuous function $\varphi$ a number $\varphi(0)$, which is its value at the point 0. It required quite a few years and the efforts of many mathematicians§ in order to find a mathematically proper definition of the delta function, of its derivatives and, generally, of a generalized function.

The foundations of the mathematical theory of generalized functions were laid by the Soviet mathematician S.L. Sobolev in 1936 (see Sobolev [1]) when he successfully applied generalized functions to a study of the Cauchy problem for hyperbolic equations. After World War II, the French mathematician L. Schwartz attempted, on the basis of an earlier created theory of linear locally convex topological spaces§§, a systematic construction of a theory of generalized functions and explained it in his well-known monograph entitled *Théorie des distributions* [1] (1950-51). From then on the theory of generalized functions was developed intensively by many mathematicians. This precipitate development of the theory of generalized functions received its main stimulus from the requirements of mathematical and theoretical physics, in particular, the theory of differential equations and quantum physics. At the present time, the theory of generalized functions has advanced substantially and has found numerous applications in physics and mathematics, and more and more is becoming a workaday tool of the physicist, mathematician and engineer§§§. Generalized functions possess a number of remark-

§ See the pioneering works of Bochner [1] and Hadamard [1].

§§ See Dieudonné and Schwartz [1].

§§§ See Arsac [1], Bogoliubov, Logunov and Todorov [1], Bogoliubov, Medvedev and Polivanov [1], Bogoliubov and Shirkov [1], Bremermann [1], Ehrenpreis [1], Garding [1], Garsoux [1], Gelfand and Shilov [1], Gelfand

able properties that extend the capabilities of classical mathematical analysis; for example, any generalized function turns out to be infinitely differentiable (in the generalized meaning), convergent series of generalized functions may be differentiated termwise an infinite number of times, there always exists the Fourier transform of a generalized function, and so on. For this reason, the use of generalized function techniques substantially expands the range of problems that can be tackled and leads to appreciable simplifications that make elementary operations automatic.

The present monograph is an expanded version of a course of lectures that the author has been delivering to students, postgraduates, and associates of the Moscow Physics and Technology Institute and the Steklov Mathematical Institute.

I take this opportunity to thank all my associates for their constructive criticism that has helped to improve the presentation. In particular I wish to thank my pupils Yu. N. Drozhzhinov, V.V. Zharinov and R. Kh. Galeev.

The first Russian edition of this book was sold out in a short time. In preparing the second edition, I have taken into account a number of remarks, and part of the material has been reworked and supplemented. Inaccuracies and misprints have been corrected. Significant changes have been introduced into the portion devoted to the theory of holomorphic functions with nonnegative imaginary part in tubular regions over acute cones (Secs. 16-18). This part embodies new results.

*V. S. Vladimirov*

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and Vilenkin [1], Hörmander [1], Jost [1], Malgrange [1], Palamodov [1], Reed and Simon [1], Schwartz [1, 2], Sobolev [1, 2], Streater and Wightman [1], Treves [1], Vladimirov [1, 2], Zemanian [1], and others.

# Contents

# Symbols and Definitions

**0.1** We denote the *points* of an $n$-dimensional real space $\mathbb{R}^n$ by $x, y, \xi, \ldots; x = (x_1, x_2, \ldots, x_n)$. The points of an $n$-dimensional complex space $\mathbb{C}^n$ are given as $z, \zeta, \ldots; z = (z_1, z_2, \ldots, z_n) = x + iy$; $x = \mathrm{Re}\, z$ is the real part of $z$ and $y = \mathrm{Im}\, z$ is the imaginary part of $z$, $\bar{z} = x - iy$ is the complex conjugate of $z$. In the usual manner we introduce in $\mathbb{R}^n$ and $\mathbb{C}^n$ the *scalar products*

$$(x, \xi) = x_1\xi_1 + \ldots + x_n\xi_n, \quad \langle z, \zeta\rangle = z_1\bar{\zeta}_1 + \ldots + z_n\bar{\zeta}_n$$

and the *norms* (*lengths*)

$$|x| = \sqrt{(x, x)} = (x_1^2 + \ldots + x_n^2)^{1/2},$$
$$|z| = \sqrt{\langle z_1, z\rangle} = (|z_1|^2 + \ldots + |z_n|^2)^{1/2}.$$

**0.2** *Open sets* in $\mathbb{R}^n$ are denoted by $\mathcal{O}, \mathcal{O}', \ldots$; $\partial\mathcal{O}$ is the boundary of $\mathcal{O}$, or $\partial\mathcal{O} = \bar{\mathcal{O}} \setminus \mathcal{O}$. We will say that the set $A$ *is compact in the open set* $\mathcal{O}$ (or *is strictly contained in* $\mathcal{O}$) if $A$ is bounded and its closure $\bar{A}$ lies in $\mathcal{O}$; we then write $A \Subset \mathcal{O}$.

The following designations are used: $U(x_0; R)$ is an *open ball* of radius $R$ with centre at the point $x_0$; $S(x_0; R) = \partial U(x_0; R)$ is a *sphere* of radius $R$ with centre at the point $x_0$; $U_R = U(0; R)$, $S_R = S(0; R)$.

We use $\Delta(A, B)$ to denote the *distance* between the sets $A$ and $B$ in $\mathbb{R}^n$, that is,

$$\Delta(A, B) = \inf_{x \in A,\, y \in B} |x - y|.$$

We use $A^\varepsilon$ to denote the *$\varepsilon$-neighbourhood* of a set $A$, $A^\varepsilon = A + U_\varepsilon$ (Fig. 1a). If $\mathcal{O}$ is an open set, then $\mathcal{O}_\varepsilon$ designates the set of those points of $\mathcal{O}$ which are separated from $\partial\mathcal{O}$ by a distance greater than $\varepsilon$ (Fig. 1b):

$$\mathcal{O}_\varepsilon = [x : x \in \mathcal{O}, \Delta(x, \partial\mathcal{O}) > \varepsilon].$$

We use int $A$ to denote the set of interior points of the set $A$.

The *characteristic function of a set* $A$ is the function $\theta_A(x)$ which is equal to 1 when $x \in A$ and is equal to 0 when $x \notin A$. The characteristic function $\theta_{[0,\infty)}(x)$ of the semiaxis $x \geqslant 0$ is called the *Heaviside unit function* and is denoted $\theta(x)$ (Fig. 2):

$$\theta(x) = 0, \quad x < 0, \qquad \theta(x) = 1, \quad x \geqslant 0.$$

We write $\theta_n(x) = \theta(x_1) \ldots \theta(x_n)$.

The set $A$ is said to be *convex* if for any points $x'$ and $x''$ in $A$ the line segment joining them, $tx' + (1-t)\,x''$, $0 \leqslant t \leqslant 1$, lies entirely in $A$.

We will use ch $A$ to denote the *convex hull* of a set $A$.

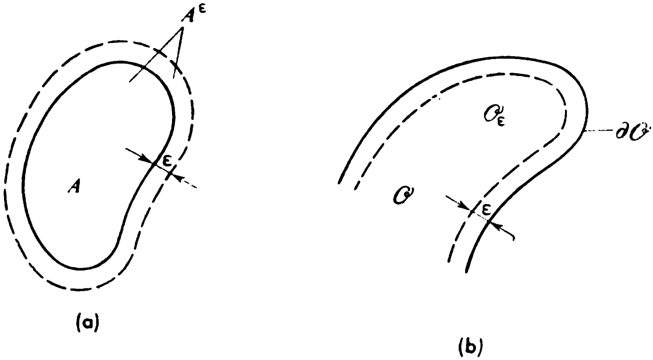



Figure 1

A real function $f(x) < +\infty$ is said to be *convex* on the set $A$ if for any points $x'$ and $x''$ in $A$ such that the line segment $tx' + (1-t)\,x''$ joining them lies entirely in $A$ the following inequality holds (Fig. 2b):

$$f(tx' + (1-t)\,x'') \leqslant tf(x') + (1-t)\,f(x'')$$

The function $f(x)$ is said to be *concave* if the function $-f(x)$ is convex.

**0.3** The Lebesgue integral of a function $f$ over an open set $\mathcal{O}$ is given as

$$\int_{\mathcal{O}} f(x)\,dx, \qquad \int_{\mathbb{R}^n} f(x)\,dx = \int f(x)\,dx.$$

The collection of all (complex-valued, measurable) functions $f$ specified on $\mathcal{O}$ for which the norm

$$\|f\|_{\mathcal{L}^p(\mathcal{O})} = \begin{cases} \left[\int_{\mathcal{O}} |f(x)|^p\,dx\right]^{1/p}, & 1 \leqslant p < \infty, \\ \operatorname{vrai\,sup}_{x\in\mathcal{O}} |f(x)|, & p = \infty, \end{cases}$$

is finite will be denoted as $\mathcal{L}^p(\mathcal{O})$, $1 \leqslant p \leqslant \infty$; we write $\|\ \| = \|\ \|_{\mathcal{L}^2(\mathbb{R}^n)}$, $\mathcal{L}^p(\mathbb{R}^n) = \mathcal{L}^p$.

If $f \in \mathcal{L}^p(\mathcal{O}')$ for any $\mathcal{O}' \Subset \mathcal{O}$, then $f$ is said to be *p-locally summable* in $\mathcal{O}$ (for $p = 1$, we say it is *locally summable in* $\mathcal{O}$).

The collection of $p$-locally summable functions in $\mathcal{O}$ is denoted $\mathcal{L}^p_{\text{loc}}(\mathcal{O})$, $\mathcal{L}^p_{\text{loc}}(\mathbb{R}^n) = \mathcal{L}^p_{\text{loc}}$.

A measurable function is said to be *finite* in $\mathcal{O}$ if it vanishes almost everywhere outside a certain $\mathcal{O}' \Subset \mathcal{O}$. The set of all functions in $\mathcal{L}^p(\mathcal{O})$ that are finite in $\mathcal{O}$ is denoted $\mathcal{L}^p_0(\mathcal{O})$.

(a)

(b)

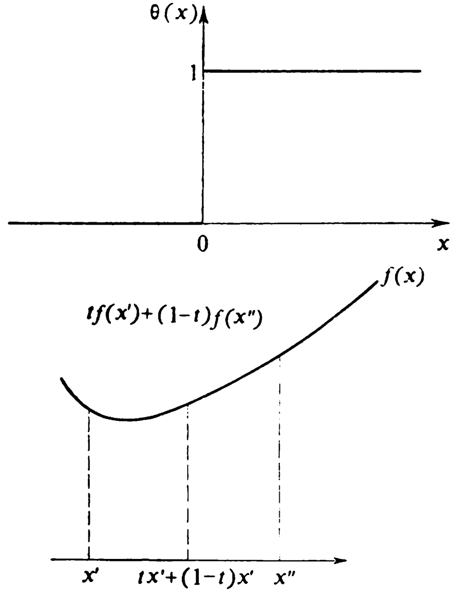



Figure 2

**0.4** Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a multi-index, that is to say its components $\alpha_j$ are nonnegative integers. We have the following symbolism:

$$\alpha! = \alpha_1!\,\alpha_2! \ldots \alpha_n!, \qquad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n},$$

$$\binom{\beta}{\alpha} = \binom{\beta_1}{\alpha_1}\binom{\beta_2}{\alpha_2}\cdots\binom{\beta_n}{\alpha_n} = \frac{\alpha!}{\beta!\,(\alpha-\beta)!},$$

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots \alpha_n.$$

Let $D = (D_1, D_2, \ldots, D_n)$, $D_j = \dfrac{\partial}{\partial x_j}$, $j = 1, 2, \ldots, n$. Then

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}.$$

It may sometimes happen that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ will be used to denote a multi-index with components of any sign: $\alpha_j \gtreqless 0$.

**0.5** We use $C^k(\mathcal{O})$ to denote the set of all functions $f(x)$ that are continuous in $\mathcal{O}$ together with all derivatives $D^\alpha f(x)$, $|\alpha| \leqslant k$; $C^\infty(\mathcal{O})$ is the collection of all functions infinitely differentiable in $\mathcal{O}$. The set of all functions $f(x)$ in $C^k(\mathcal{O})$ for which all derivatives $D^\alpha f(x)$, $|\alpha| \leqslant k$, admit continuous extension onto $\overline{\mathcal{O}}$ will be denoted by $C^k(\overline{\mathcal{O}})$. We introduce the norm in $C^k(\overline{\mathcal{O}})$ for $k < \infty$ via the formula

$$\|f\|_{C^k(\overline{\mathcal{O}})} = \sup_{\substack{x \in \overline{\mathcal{O}} \\ |\alpha| \leqslant k}} |D^\alpha f(x)|.$$

We also write $C(\mathcal{O}) = C^0(\mathcal{O})$, $C(\overline{\mathcal{O}}) = C^0(\overline{\mathcal{O}})$.

The *support* of a function $f(x)$ continuous in $\mathcal{O}$ is the closure, in $\mathcal{O}$, of those points where $f(x) \neq 0$; the support of $f$ is denoted by supp $f$. If supp $f \Subset \mathcal{O}$, then $f$ is finite in $\mathcal{O}$ (compare with Sec. 0.3).

We denote the collection of functions, finite in $\mathcal{O}$, of the class $C^k(\mathcal{O})$ by $C_0^k(\mathcal{O})$; $C_0(\mathcal{O}) = C_0^0(\mathcal{O})$. Finally, the set of all functions of the class $C^k(\overline{\mathcal{O}})$ that vanish on $\partial\mathcal{O}$ together with all derivatives up to order $k$ inclusive will be denoted by $C_0^k(\overline{\mathcal{O}})$; $C_0(\overline{\mathcal{O}}) = C_0^0(\overline{\mathcal{O}})$. We write $C^k(\mathbb{R}^n) = C^k$; $C_0^k(\mathbb{R}^n) = C_0^k$; $C_0^k(\overline{\mathbb{R}}^n) = \overline{C}_0^k$, $\overline{C}_0 = \overline{C}_0^0$ ($\overline{C}_0^k$ is the set of functions in $C^k$ that vanish at infinity together with all their derivatives up to order $k$ inclusive).

**0.6** Symbolism: $(a, b)$ is a bilinear form (linear in $a$ and $b$ separately); $\langle a, b\rangle$ is a linear-antilinear form (linear in $a$ and antilinear in $b$):

$$\langle \alpha a_1 + b a_2, \lambda b_1 + \mu b_2\rangle = \alpha\bar{\lambda}\langle a_1, b_1\rangle + \alpha\bar{\mu}\langle a_1, b_2\rangle + \beta\bar{\lambda}\langle a_2, b_1\rangle + \beta\bar{\mu}\langle a_2, b_2\rangle;$$

$\sigma_n = \int\limits_{|s|=1} ds = \dfrac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of a unit sphere in $\mathbb{R}^n$;

$A^T$ is the transpose of the matrix $A$.

We denote the uniform convergence of a sequence of functions, $\{\varphi_n(x)\}$, to a function $\varphi(x)$ on a set $A$ thus:

$$\varphi_n(x) \overset{x \in A}{\Longrightarrow} \varphi(x), \qquad n \to \infty;$$

if $A = \mathbb{R}^n$, then instead of $\overset{x \in \mathbb{R}^n}{\Longrightarrow}$ we write $\overset{x}{\Longrightarrow}$.

The sections are numbered in a single sequence. Each section is made up of subsections, the numbers of which are included in any reference to a section. Formulas are numbered separately in each subsection; they contain the number of the formula and of the subsection. When reference is made to a formula in a different section, the number of that section is also given.

Chapter I

# Generalized Functions and Their Properties

The exposition of the theory of generalized functions given in this chapter is tailored to the needs of theoretical and mathematical physics.

## 1 Basic and Generalized Functions

**1.1 Introduction** A generalized function is a generalization of the classical notion of a function. On the one hand, this generalization permits expressing in a mathematically proper form such idealized concepts as the density of a material point, the density of a point charge or dipole, the spatial density of a simple or double layer, the intensity of an instantaneous point source, the magnitude of an instantaneous force applied to a point, and so forth. On the other hand, the notion of a generalized function can reflect the fact that in reality one cannot measure the value of a physical quantity at a point but can only measure the mean values within sufficiently small neighbourhoods of the point and then proclaim the limit of the sequence of those mean values as the value of the physical quantity at the given point.

This can be explained by attempting to determine the density set up by a material point of mass 1. Assume that the point is the origin of coordinates. In order to determine the density, we distribute (or, as one often says, smear) the unit mass uniformly inside a sphere of radius $\varepsilon$ centered at 0. We then obtain the mean density $f_\varepsilon(x)$ that is equal (see Fig. 3) to

$$f_\varepsilon(x) = \begin{cases} \dfrac{1}{\frac{4}{3}\pi\varepsilon^3} & \text{if } |x| < \varepsilon, \\ 0 & \text{if } |x| > \varepsilon. \end{cases}$$

We are interested in the density at $\varepsilon = +0$. To begin with, for the desired density (which we denote by $\delta(x)$) we take the point limit of the sequence of mean densities $f_\varepsilon(x)$ as $\varepsilon \to +0$, that is, the function

$$\delta(x) = \begin{cases} +\infty & \text{if } x = 0, \\ 0 & \text{if } x \neq 0. \end{cases} \tag{1.1}$$

Of the density it is natural to require that the integral of the density over the entire space yield the total mass of substance, or

$$\int \delta(x)\,dx = 1. \tag{1.2}$$

But for the function $\delta(x)$ defined by (1.1), $\int \delta(x)\,dx = 0$. This means that the function does not restore the mass (it does not satisfy the requirement (1.2)) and therefore cannot be taken as

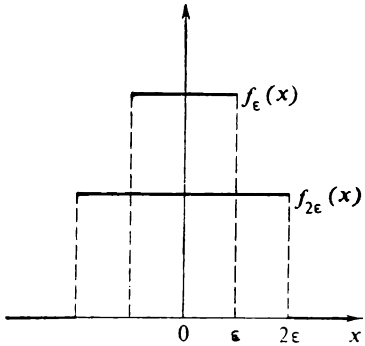



Figure 3

the desired mass. Thus the point limit of a sequence of mean densities $f_\varepsilon(x)$ is unsuitable for our purposes. What is the way out?

Let us now find a somewhat different limit of the sequence of mean densities $f_\varepsilon(x)$, the so-called *weak limit*. It will readily be seen that for any continuous function $\varphi(x)$

$$\lim_{\varepsilon \to +0} \int f_\varepsilon(x)\,\varphi(x)\,dx = \varphi(0). \tag{1.3}$$

Formula (1.3) states that the weak limit of a sequence of functions $f_\varepsilon(x)$, $\varepsilon \to +0$, is a functional $\varphi(0)$ (and not a function!) that relates to every continuous function $\varphi(x)$ a number $\varphi(0)$, which is its value at the point $x = 0$. It is this functional that we take for our sought-for density $\delta(x)$. And this is the famous *delta function* of Dirac. So now we can write

$$f_\varepsilon(x) \xrightarrow{\text{weak}} \delta(x), \qquad \varepsilon \to +0,$$

and we understand by this the limiting relation (1.3). The value of the functional $\delta$ on the function $\varphi$ (the number $\varphi(0)$) will be denoted thus:

$$(\delta, \varphi) = \varphi(0). \tag{1.4}$$

This equality yields the exact meaning of formula (*) (see Preface). The role of the "integral" $\int \delta(x)\,\varphi(x)\,dx$ is played here

by the quantity $(\delta, \varphi)$, which is the value of the functional $\delta$ on the function $\varphi$.

Let us now check to see that the functional $\delta$ restores the total mass. Indeed, as we have just said, the role of the "integral" $\int \delta(x)\,dx$ is handled by the quantity $(\delta, 1)$, which, by virtue of (1.4), is equal to the value of the function identically equal to 1 at the point $x = 0$, that is, $(\delta, 1) = 1$.

Also, generally, if masses $\mu_k$ are concentrated at distinct points $x_k$, $k = 1, 2, \ldots, N$, then the density that corresponds to such a mass distribution should be regarded as equal to

$$\sum_{1 \leqslant k \leqslant N} \mu_k \delta(x - x_k). \tag{1.5}$$

The expression (1.5) is a linear functional that associates with each continuous function $\varphi(x)$ a number

$$\sum_{1 \leqslant k \leqslant N} \mu_k \varphi(x_k).$$

Thus, the density corresponding to a point distribution of masses cannot be described within the framework of the classical concept of a function; to describe it requires resorting to entities of a more general mathematical nature, linear (continuous) functionals.

**1.2 The space of basic functions $\mathcal{D}(\mathcal{O})$** In the case of the delta function we have already seen that it is determined by means of continuous functions as a linear (continuous) functional on those functions. Continuous functions are said to be *basic functions* for the delta function. It is this viewpoint that serves as the basis for defining an arbitrary generalized function as a continuous linear functional on a collection of sufficiently "good" so-called basic functions. Clearly, the smaller the set of basic functions, the more continuous linear functionals there are on it. On the other hand, the supply of basic functions should be sufficiently large. In this subsection we introduce the important space of basic functions $\mathcal{D}(\mathcal{O})$ for any open set $\mathcal{O} \subset \mathbb{R}^n$.

Included in the set of basic functions $\mathcal{D}(\mathcal{O})$ are all finite functions infinitely differentiable in $\mathcal{O}$; $\mathcal{D}(\mathcal{O}) = C_0^\infty(\mathcal{O})$ (see Sec. 0.5). We define *convergence* in $\mathcal{D}(\mathcal{O})$ as follows. A sequence of functions $\varphi_1, \varphi_2, \ldots$ in $\mathcal{D}(\mathcal{O})$ converges to the function $\varphi$ (in $\mathcal{D}(\mathcal{O})$) if there exists a set $\mathcal{O}' \Subset \mathcal{O}$ such that $\operatorname{supp} \varphi_k \subset \mathcal{O}'$ and for every $\alpha$

$$D^\alpha \varphi_k(x) \overset{x \in \mathcal{O}}{\rightrightarrows} \mathcal{D}^\alpha \varphi(x), \qquad k \to \infty.$$

We then write: $\varphi_k \to \varphi$, $k \to \infty$ in $\mathcal{D}(\mathcal{O})$.

2*

A linear set $\mathcal{D}(\mathcal{O})$ equipped with convergence is called the *space of basic functions* $\mathcal{D}(\mathcal{O})$, and we have the following symbolism: $\mathcal{D} = \mathcal{D}(\mathbb{R}^n)$, $\mathcal{D}(a, b) = \mathcal{D}((a, b))$.

Clearly, if $\mathcal{O}_1 \subset \mathcal{O}_2$, then also $\mathcal{D}(\mathcal{O}_1) \subset \mathcal{D}(\mathcal{O}_2)$, and from the convergence in $\mathcal{D}(\mathcal{O}_1)$ there follows convergence in $\mathcal{D}(\mathcal{O}_2)$.

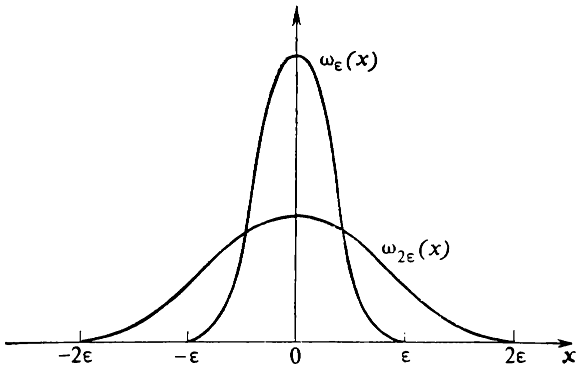



Figure 4

An instance of a nonzero basic function is the "cap" in Fig. 4:

$$\omega_\varepsilon(x) = \begin{cases} C_\varepsilon e^{-\frac{\varepsilon^2}{\varepsilon^2 - |x|^2}}, & |x| \leqslant \varepsilon, \\ 0, & |x| > \varepsilon. \end{cases}$$

In what follows, the function $\omega_\varepsilon$ will play the part of an averaging function; and so we shall regard the constant $C_\varepsilon$ as such that

$$\int \omega_\varepsilon(x)\, dx = 1, \quad \text{that is,} \quad C_\varepsilon \varepsilon^n \int\limits_{|\xi|<1} e^{-\frac{1}{1-|\xi|^2}}\, d\xi = 1.$$

The following lemma yields other instances of basic functions.

Lemma *For any set $A$ and any number $\varepsilon > 0$ there is a function $\eta_\varepsilon \in C^\infty$ such that*

$$\eta_\varepsilon(x) = 1, \quad x \in A^\varepsilon; \qquad \eta_\varepsilon(x) = 0, \quad x \overline{\in} A^{3\varepsilon};$$
$$0 \leqslant \eta_\varepsilon(x) \leqslant 1, \qquad |\mathcal{D}^\alpha \eta_\varepsilon(x)| \leqslant K_\alpha \varepsilon^{-|\alpha|}.$$

*Proof*. Let $\theta_{A^{2\varepsilon}}$ be a characteristic function of the set $A^{2\varepsilon}$ (see Sec. 0.2). Then the function

$$\eta_\varepsilon(x) = \int \theta_{A^{2\varepsilon}}(y)\, \omega_\varepsilon(x-y)\, dy = \int\limits_{A^{2\varepsilon}} \omega_\varepsilon(x-y)\, dy,$$

where $\omega_\varepsilon$ is the "cap", has the required properties. The proof is complete.

Corollary *Let $\mathcal{O}$ be an open set. Then for any $\mathcal{O}' \Subset \mathcal{O}$ there is a function $\eta \in \mathcal{D}$ such that $\eta(x) = 1$, $x \in \mathcal{O}'$, $0 \leqslant \eta(x) \leqslant 1$.*

This follows from the lemma when $A = \mathcal{O}'$ and $\varepsilon = \frac{1}{4}\Delta(\mathcal{O}', \partial\mathcal{O}) > 0$.

Let $\mathcal{O}_k$, $k = 1, 2, \ldots$, be a countable system of open sets. We say that this system forms a *locally finite cover* of the open set $\mathcal{O}$ if $\mathcal{O} = \bigcup_{k\geqslant 1} \mathcal{O}_k$, $\mathcal{O}_k \Subset \mathcal{O}$, and any compact $K \Subset \mathcal{O}$ intersects only a finite number of sets $\{\mathcal{O}_k\}$.

Theorem I (decomposition of unity) *Let $\{\mathcal{O}_k\}$ be a locally finite cover of $\mathcal{O}$. Then there is a system of functions $\{e_k\}$ such that*

$$e_k \in \mathcal{D}(\mathcal{O}_k), \quad 0 \leqslant e_k(x) \leqslant 1, \quad \sum_{k\geqslant 1} e_k(x) = 1, \quad x \in \mathcal{O}.$$

*Remark.* For each $x \in \mathcal{O}$ that has a nonzero sum there is only a finite number of summands $e_k(x)$; the set of functions $\{e_k\}$ is termed the *decomposition of unity corresponding to the given locally finite cover $\{\mathcal{O}_k\}$ of the open set $\mathcal{O}$.*

*Proof.* We will prove that there is another locally finite cover $\{\mathcal{O}'_k\}$ of the set $\mathcal{O}$ such that $\mathcal{O}'_k \Subset \mathcal{O}_k$. Construct $\mathcal{O}'_k$ and set

$$K_1 = \mathcal{O} \setminus \bigcup_{k\geqslant 2} \mathcal{O}_k.$$

Then $K_1 \subset \mathcal{O}_1 \Subset \mathcal{O}$ and $K_1$ is closed in $\mathcal{O}$. Hence $K_1 \Subset \mathcal{O}_1$, for $\mathcal{O}'_1$ we take an open set such that $K_1 \Subset \mathcal{O}'_1 \Subset \mathcal{O}_1$. Then the sets $\mathcal{O}'_1, \mathcal{O}_2, \ldots$ form a locally finite cover of $\mathcal{O}$. In similar fashion we construct an open set $\mathcal{O}'_2 \Subset \mathcal{O}_2$, etc. We thus obtain the required cover $\{\mathcal{O}'_k\}$.

By the corollary to the lemma, there exist functions such that

$$\eta_k(x) = 1, \qquad x \in \mathcal{O}'_k, \qquad 0 \leqslant \eta_k(x) \leqslant 1.$$

Putting

$$e_k(x) = \frac{\eta_k(x)}{\sum_{k\geqslant 1} \eta_k(x)} \quad \left(\sum_{k\geqslant 1} \eta_k(x) \geqslant 1\right)$$

we obtain the required decomposition of unity. This completes the proof.

We have thus seen that there are various functions in $\mathcal{D}(\mathcal{O})$. We will now see that there are a sufficiently large number of such functions.

Let $f$ be a locally summable function in $\mathcal{O}$, $f \in \mathcal{L}^1_{\mathrm{loc}}(\mathcal{O})$. The convolution of $f$ and the "cap", $\omega_\varepsilon$,

$$f_\varepsilon(x) = \int f(y)\,\omega_\varepsilon(x-y)\,dy = \int \omega_\varepsilon(y)\,f(x-y)\,dy$$

(wherever it is defined) is called the *mean function* of $f$ (or the *regularization* of $f$).

Let $f \in \mathcal{L}^p(\mathcal{O})$, $1 \leqslant p \leqslant \infty$ ($f(x)$ is regarded as zero outside $\mathcal{O}$). Then $f_\varepsilon \in C^\infty$ and the following inequality holds:

$$\| f_\varepsilon \|_{\mathcal{L}^p(\mathcal{O})} \leqslant \| f \|_{\mathcal{L}^p(\mathcal{O})}. \tag{2.1}$$

Indeed, the fact that $f_\varepsilon \in C^\infty$ follows from the properties of the function $f$ and from the definition of a mean function. When $1 \leqslant p < \infty$ the inequality (2.1) follows from the Hölder inequality:

$$\begin{aligned} \| f_\varepsilon \|^p_{\mathcal{L}^p(\mathcal{O})} &= \int_{\mathcal{O}} | f_\varepsilon(x)|^p \, dx = \int_{\mathcal{O}} \left| \int f(y)\, \omega_\varepsilon(x-y)\, dy \right|^p dx \\ &\leqslant \int_{\mathcal{O}} \int_{\mathcal{O}} | f(y)|^p \, \omega_\varepsilon(x-y)\, dy \left[ \int \omega_\varepsilon(x-y)\, dy \right]^{p-1} dx \\ &= \int_{\mathcal{O}} \int_{\mathcal{O}} | f(y)|^p \, \omega_\varepsilon(x-y)\, dy\, dx \\ &\leqslant \int_{\mathcal{O}} | f(y)|^p \, dy = \| f \|^p_{\mathcal{L}^p(\mathcal{O})}. \end{aligned}$$

The case of $p = \infty$ is considered in a similar fashion.

Theorem II *Let $f \in \mathcal{L}^1_0(\mathcal{O})$ and $f(x) = 0$ almost everywhere outside $K \Subset \mathcal{O}$. Then for all $\varepsilon < \Delta(K, \partial\mathcal{O})$ the mean function $f_\varepsilon \in \mathcal{D}(\mathcal{O})$ and (see Secs. 0.3 and 0.5 for the symbolism)*

$$f_\varepsilon \to f, \ \varepsilon \to +0 \begin{cases} \text{in } C(\overline{\mathcal{O}}) \text{ if } f \in C_0(\mathcal{O}), \\ \text{in } \mathcal{L}^p(\mathcal{O}) \ (1 \leqslant p < \infty) \text{ if } f \in \mathcal{L}^p_0(\mathcal{O}), \\ \text{almost everywhere in } \mathcal{O} \text{ if } f \in \mathcal{L}^\infty_0(\mathcal{O}). \end{cases}$$

*Proof*. If $\varepsilon < \Delta(K, \partial\mathcal{O})$, then $f_\varepsilon(x)$ is finite in $\mathcal{O}$, and since $f_\varepsilon \in C^\infty(\mathcal{O})$, it follows that $f_\varepsilon \in \mathcal{D}(\mathcal{O})$.

Let $f \in C_0(\mathcal{O})$. Then from the estimate

$$\begin{aligned} | f_\varepsilon(x) - f(x) | &= \left| \int [f(y) - f(x)]\, \omega_\varepsilon(x-y)\, dy \right| \\ &\leqslant \max_{|x-y| \leqslant \varepsilon} |f(x) - f(y)| \int \omega_\varepsilon(x-y)\, dy \\ &= \max_{|x-y| \leqslant \varepsilon} |f(x) - f(y)|, \qquad x \in \mathcal{O}, \end{aligned}$$

and from the uniform continuity of the function $f$ follows the uniform convergence, on $\mathcal{O}$, of $f_\varepsilon(x)$ to $f(x)$ as $\varepsilon \to +0$.

Let $f \in \mathcal{L}_0^p(\mathcal{O})$, $1 \leqslant p < \infty$. Take an arbitrary $\delta > 0$. There is a function $g \in C_0(\mathcal{O})$ such that

$$\|f-g\|_{\mathcal{L}^p(\mathcal{O})} < \frac{\delta}{3}.$$

From what has been proved, there will be an $\varepsilon_0$ such that

$$\|g-g_\varepsilon\|_{\mathcal{L}^p(\mathcal{O})} < \frac{\delta}{3} \quad \text{for all } \varepsilon < \varepsilon_0.$$

From this, using the inequality (2.1), we obtain, for all $\varepsilon < \varepsilon_0$,

$$\begin{aligned}\|f-f_\varepsilon\|_{\mathcal{L}^p(\mathcal{O})} &\leqslant \|f-g\|_{\mathcal{L}^p(\mathcal{O})} + \|g-g_\varepsilon\|_{\mathcal{L}^p(\mathcal{O})} + \|(g-f)_\varepsilon\|_{\mathcal{L}^p(\mathcal{O})} \\ &\leqslant 2\|f-g\|_{\mathcal{L}^p(\mathcal{O})} + \|g-g_\varepsilon\|_{\mathcal{L}^p(\mathcal{O})} < \frac{2\delta}{3} + \frac{\delta}{3} = \delta.\end{aligned}$$

And this means that $f_\varepsilon \to f$, $\varepsilon \to +0$ in $\mathcal{L}^p(\mathcal{O})$.

Now if $f \in \mathcal{L}_0^\infty(\mathcal{O})$, we can indicate a sequence of functions taken from $C_0(\mathcal{O})$ that converges to $f(x)$ almost everywhere in $\mathcal{O}$. From this fact and from what has been proved, it follows that $f_\varepsilon(x) \to f(x)$, $\varepsilon \to +0$, almost everywhere in $\mathcal{O}$. The theorem is proved.

Corollary 1 *$\mathcal{D}(\mathcal{O})$ is dense in $\mathcal{L}^p(\mathcal{O})$, $1 \leqslant p < \infty$.*

Corollary 2 *$\mathcal{D}(\mathcal{O})$ is dense in $C_0^k(\overline{\mathcal{O}})$ (in the norm $C^k(\overline{\mathcal{O}})$) if $\mathcal{O}$ is bounded or $\mathcal{O} = \mathbb{R}^n$.*

**1.3 The space of generalized functions $\mathcal{D}'(\mathcal{O})$** A generalized function specified on an open set $\mathcal{O}$ is any continuous linear functional on the space of basic functions $\mathcal{D}(\mathcal{O})$.

We will write the value of the functional (generalized function) $f$ on the basic function $\varphi$ as $(f, \varphi)$. By analogy with ordinary functions, we sometimes formally write $f(x)$ instead of $f$, and regard $x$ as the argument of the basic functions on which the functional $f$ operates.

We now give an explanation of the definition of a generalized function.

(1) A generalized function $f$ is a *functional* on $\mathcal{D}(\mathcal{O})$, that is, with each $\varphi \in \mathcal{D}(\mathcal{O})$ there is associated a (complex-valued) number $(f, \varphi)$.

(2) A generalized function $f$ is a *linear functional* on $\mathcal{D}(\mathcal{O})$, that is, if $\varphi$ and $\psi$ belong to $\mathcal{D}(\mathcal{O})$ and $\lambda$ and $\mu$ are complex numbers, then

$$(f, \lambda\varphi + \mu\psi) = \lambda(f, \varphi) + \mu(f, \psi).$$

(3) A generalized function $f$ is a *continuous functional* on $\mathcal{D}(\mathcal{O})$, that is, if $\varphi_k \to \varphi$, $k \to \infty$ in $\mathcal{D}(\mathcal{O})$, then

$$(f, \varphi_k) \to (f, \varphi), \qquad k \to \infty.$$

The generalized functions $f$ and $g$ specified in $\mathcal{O}$ are said to be *equal* in $\mathcal{O}$ if they are equal as functionals on $\mathcal{D}(\mathcal{O})$, that is, if for any $\varphi$ in $\mathcal{D}(\mathcal{O})$, $(f, \varphi) = (g, \varphi)$. We will then write: $f = g$ in $\mathcal{O}$ or $f(x) = g(x)$, $x \in \mathcal{O}$.

Denote by $\mathcal{D}'(\mathcal{O})$ the set of all generalized functions specified in $\mathcal{O}$. The set $\mathcal{D}'(\mathcal{O})$ is a linear set if we define the linear combination $\lambda f + \mu g$ of the generalized functions $f$ and $g$ in $\mathcal{D}'(\mathcal{O})$ as a functional acting via the formula [thus, the form $(f, \varphi)$ is a bilinear form (see Sec. 0.6)]

$$(\lambda f + \mu g, \varphi) = \lambda (f, \varphi) + \mu (g, \varphi), \qquad \varphi \in \mathcal{D}(\mathcal{O}).$$

Suppose $f \in \mathcal{D}'(\mathcal{O})$. We define a generalized function $\bar{f}$ in $\mathcal{D}'(\mathcal{O})$, which is the complex conjugate of $f$, as follows:

$$(\bar{f}, \varphi) = \overline{(f, \bar{\varphi})}, \qquad \varphi \in \mathcal{D}(\mathcal{O}).$$

The generalized functions

$$\operatorname{Re} f = \frac{f + \bar{f}}{2}, \qquad \operatorname{Im} f = \frac{f - \bar{f}}{2i}$$

are respectively the *real part* and the *imaginary part* of $f$ so that

$$f = \operatorname{Re} f + i \operatorname{Im} f, \qquad \bar{f} = \operatorname{Re} f - i \operatorname{Im} f.$$

If $\operatorname{Im} f = 0$, then $f$ is said to be a *real* generalized function.

*Example.* The delta function is real.

Let us define *convergence* in $\mathcal{D}'(\mathcal{O})$. A sequence of generalized functions $f_1, f_2, \ldots$ in $\mathcal{D}'(\mathcal{O})$ converges to a generalized function $f \in \mathcal{D}'(\mathcal{O})$ if for any basic function $\varphi \in \mathcal{D}(\mathcal{O})$ $(f_k, \varphi) \to$ $\to (f, \varphi)$, $k \to \infty$. We then write

$$f_k \to f, \qquad k \to \infty \quad \text{in} \quad \mathcal{D}'(\mathcal{O}).$$

This convergence is termed a *weak convergence*. The linear set $\mathcal{D}'(\mathcal{O})$ together with the convergence with which it is equipped is called the *space* $\mathcal{D}'(\mathcal{O})$ *of generalized functions*. In symbols: $\mathcal{D}' = \mathcal{D}'(\mathbb{R}^n)$, $\mathcal{D}'(a, b) = \mathcal{D}'((a, b))$.

Quite obviously, if $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathcal{D}'(\mathcal{O}_2) \subset \mathcal{D}'(\mathcal{O}_1)$, and from convergence in $\mathcal{D}'(\mathcal{O}_2)$ follows convergence in $\mathcal{D}'(\mathcal{O}_1)$.

For this reason, for any generalized function $f$ in $\mathcal{D}'(\mathcal{O})$ there is a (unique) restriction to any open set $\mathcal{O}' \subset \mathcal{O}$ such that $f \in$ $\in \mathcal{D}'(\mathcal{O}')$.

*Remark*. Linear functionals on $\mathcal{D}(\mathcal{O})$ need not necessarily be continuous on $\mathcal{D}(\mathcal{O})$. However, not a single discontinuous linear function has been constructed explicitly on $\mathcal{D}(\mathcal{O})$; one can only prove their existence theoretically by using the axiom of choice.

Theorem *For a linear functional $f$ on $\mathcal{D}(\mathcal{O})$ to belong to $\mathcal{D}'(\mathcal{O})$, that is, for it to be a generalized function in $\mathcal{O}$, it is necessary and sufficient that for any open set $\mathcal{O}' \Subset \mathcal{O}$ there exist numbers $K = K(\mathcal{O}')$ and $m = m(\mathcal{O}')$ such that*

$$|(f, \varphi)| \leqslant K \|\varphi\|_{C^m(\overline{\mathcal{O}}')}, \qquad \varphi \in \mathcal{D}(\mathcal{O}'). \tag{3.1}$$

*Proof*. Sufficiency is obvious. We prove necessity. Suppose $f \in \mathcal{D}'(\mathcal{O})$ and $\mathcal{O}' \Subset \mathcal{O}$. If the inequality (3.1) does not hold, there will be a sequence $\varphi_k$, $k = 1, 2, \ldots$, of functions in $\mathcal{D}(\mathcal{O}')$ such that

$$|(f, \varphi_k)| \geqslant k \|\varphi_k\|_{C^k(\overline{\mathcal{O}}')}. \tag{3.2}$$

But the sequence

$$\psi_k = \frac{\varphi_k}{\sqrt{k}\, \|\varphi_k\|_{C^k(\overline{\mathcal{O}}')}} \to 0, \qquad k \to \infty \quad \text{in} \quad \mathcal{D}(\mathcal{O}),$$

since $\operatorname{supp} \psi_k \subset \mathcal{O}' \Subset \mathcal{O}$, and for $k \geqslant |\beta|$ we have

$$|D^\beta \psi_k(x)| = \left| D^\beta \frac{\varphi_k(x)}{\sqrt{k}\, \|\varphi_k\|_{C^k(\overline{\mathcal{O}}')}} \right| \leqslant \frac{1}{\sqrt{k}} \to 0, \qquad k \to \infty.$$

Therefore, $(f, \psi_k) \to 0$, $k \to \infty$. On the other hand, by virtue of (3.2) we get

$$|(f, \psi_k)| = \frac{|(f, \varphi_k)|}{\sqrt{k}\, \|\varphi_k\|_{C^k(\overline{\mathcal{O}}')}} \geqslant \sqrt{k} \to \infty, \qquad k \to \infty.$$

This contradiction proves the theorem.

Suppose $f \in \mathcal{D}'(\mathcal{O})$. If it is possible in the inequality (3.1) to choose the integer $m$ as independent of $\mathcal{O}'$, then we say that the generalized function $f$ is of *finite order*; the smallest such $m$ is termed the *order* of $f$ in $\mathcal{O}$. For example, the order of the delta function is 0; the order of the generalized function

$$(f, \varphi) = \sum_{k \geqslant 1} \varphi^{(k)}(k)$$

in $(0, \infty)$ is infinite.

*Remark.* The theorem just proved signifies that if we introduce in the space $\mathscr{D}(\mathcal{O})$ a topology of an inductive limit (union) of an increasing sequence of countable-normed spaces $C_0^\infty(\overline{\mathcal{O}}_k)$, where $\mathcal{O}_1 \Subset \mathcal{O}_2 \Subset, \ldots, \bigcup\limits_{k\geq 1} \mathcal{O}_k = \mathcal{O}$, with norms

$$\|\varphi\|_{C^\nu(\overline{\mathcal{O}}_k)}, \qquad \nu = 0, 1, \ldots, \varphi \in C_0^\infty(\overline{\mathcal{O}}_k),$$

then $\mathscr{D}'(\mathcal{O})$ becomes the conjugate space of $\mathscr{D}(\mathcal{O})$ (see Bourbaki [1], and Dieudonné and Schwartz [1]). Here, the inequality (3.1) is preserved for all functions $\varphi$ in $C_0^m(\overline{\mathcal{O}}')$ (see Corollary 2 to Theorem II of Sec. 1.2).

**1.4 The completeness of the space of generalized functions $\mathscr{D}'(\mathcal{O})$** The property of the completeness of the space $\mathscr{D}'(\mathcal{O})$ is extremely important.

Theorem *Let there be a sequence of generalized functions $f_1$, $f_2, \ldots$ in $\mathscr{D}'(\mathcal{O})$ such that for every function $\varphi \in \mathscr{D}(\mathcal{O})$, the numerical sequence $(f_k, \varphi)$ converges as $k \to \infty$. Then the functional $f$ on $\mathscr{D}(\mathcal{O})$ defined by $(f, \varphi) = \lim\limits_{k\to\infty}(f_k, \varphi)$ is also linear and continuous on $\mathscr{D}(\mathcal{O})$, or $f \in \mathscr{D}'(\mathcal{O})$.*

*Proof.* The linearity of the limiting functional $f$ is obvious. Let us prove its continuity on $\mathscr{D}(\mathcal{O})$. Let $\varphi_\nu \to 0$, $\nu \to \infty$ in $\mathscr{D}(\mathcal{O})$; we have to prove that $(f, \varphi_\nu) \to 0$, $\nu \to \infty$. Assuming the contrary and passing, if necessary, to a sequence, we may assume that for all $\nu = 1, 2, \ldots$ the inequality $|(f, \varphi_\nu)| \geqslant 2a$ holds true for some $a > 0$. Since $(f, \varphi_\nu) = \lim\limits_{k\to\infty}(f_k, \varphi_\nu)$, it follows that for every $\nu = 1, 2, \ldots$ there is a number $k_\nu$ such that $|(f_{k_\nu}, \varphi_\nu)| \geqslant a$. But this is impossible due to the lemma which follows. This contradiction completes the proof of the continuity of $f$. The proof of the theorem is complete.

Lemma *Given a sequence of functionals $f_1, f_2, \ldots$ taken from a weakly bounded set $M' \subset \mathscr{D}'(\mathcal{O})$, that is, $|(f, \varphi)| < C_\varphi$, $f \in M'$ for all $\varphi$ in $\mathscr{D}(\mathcal{O})$, and suppose the sequence of basic functions $\varphi_1, \varphi_2, \ldots$ in $\mathscr{D}(\mathcal{O})$ tends to 0 in $\mathscr{D}(\mathcal{O})$. Then $(f_k, \varphi_k) \to 0$ as $k \to \infty$.*

*Proof.* Assume the lemma is not true. Then, passing, if necessary, to a subsequence, we can say that $|(f_k, \varphi_k)| \geqslant c > 0$. The convergence of $\varphi_k$ to 0 in $\mathscr{D}(\mathcal{O})$ means that supp $\varphi_k \subset \mathcal{O}' \Subset \mathcal{O}$ and for every $\alpha$

$$D^\alpha \varphi_k(x) \overset{x\in\mathcal{O}}{\Longrightarrow} 0, \qquad k \to \infty.$$

Therefore, by again passing, if necessary, to a subsequence, we can assume that

$$|D^\alpha \varphi_k(x)| \leqslant \frac{1}{4^k}, \qquad |\alpha| \leqslant k = 0, 1, \ldots .$$

Set $\psi_k = 2^k \varphi_k$; then

$$\operatorname{supp} \psi_k \subset \mathcal{O}' \Subset \mathcal{O} \quad \text{and} \quad |D^\alpha \psi_k(x)| \leqslant \frac{1}{2^k}, \tag{4.1}$$
$$|\alpha| \leqslant k = 0, 1, \ldots ,$$

so that

$$|(f_k, \psi_k)| = 2^k |(f_k, \varphi_k)| \geqslant 2^k c \to \infty, \qquad k \to \infty. \tag{4.2}$$

Now construct the subsequences $\{f_{k_\nu}\}$ and $\{\psi_{k_\nu}\}$ in the following manner. Choose $f_{k_1}$ and $\psi_{k_1}$ so that $|(f_{k_1}, \psi_{k_1})| \geqslant 2$. Suppose $f_{k_j}$ and $\psi_{k_j}$, $j = 1, \ldots, \nu - 1$, have already been constructed; construct $f_{k_\nu}$ and $\psi_{k_\nu}$. Since $\psi_k \to 0$, $k \to \infty$ in $\mathcal{D}(\mathcal{O})$, it follows that $(f_{k_j}, \psi_k) \to 0$, $k \to \infty$, $j = 1, \ldots, \nu - 1$, and therefore there is a number $N$ such that for all $k \geqslant N$,

$$|(f_{k_j}, \psi_k)| \leqslant \frac{1}{2^{\nu - j}}, \qquad j = 1, \ldots, \nu - 1. \tag{4.3}$$

Now note that $|(f_k, \psi_{k_j})| \leqslant c_{k_j}$, $j = 1, \ldots, \nu - 1$. Finally, by virtue of (4.2) choose a number $k_\nu \geqslant N_1$ such that

$$|(f_{k_\nu}, \psi_{k_\nu})| \geqslant \sum_{1 \leqslant j \leqslant \nu - 1} c_{k_j} + \nu + 1. \tag{4.4}$$

Thus, by (4.3)-(4.4), the (generalized) functions $f_{k_\nu}$ and $\psi_{k_\nu}$ thus constructed are such that

$$|(f_{k_j}, \psi_{k_\nu})| \leqslant \frac{1}{2^{\nu - j}}, \qquad j = 1, \ldots, \nu - 1, \tag{4.5}$$

$$|(f_{k_\nu}, \psi_{k_\nu})| \geqslant \sum_{1 \leqslant j \leqslant \nu - 1} |(f_{k_\nu}, \psi_{k_j})| + \nu + 1. \tag{4.6}$$

Set $\psi = \sum\limits_{j \geqslant 1} \psi_{k_j}$. By virtue of (4.1) this series converges in $\mathcal{D}(\mathcal{O})$ and, hence, $\psi \in \mathcal{D}(\mathcal{O})$ and

$$(f_{k_\nu}, \psi) = (f_{k_\nu}, \psi_{k_\nu}) + \sum_{\substack{j=1 \\ j \neq \nu}} (f_{k_\nu}, \psi_{k_j}). \tag{4.7}$$

Whence, taking into account the inequalities (4.5) and (4.6), we get

$$|(f_{k_\nu}, \psi)|$$
$$\geqslant |(f_{k_\nu}, \psi_{k_\nu})| - \sum_{1\leqslant j\leqslant \nu-1} |(f_{k_\nu}, \psi_{k_j})| - \sum_{j\geqslant \nu+1}^{\infty} |(f_{k_\nu}, \psi_{k_j})|$$
$$\geqslant \nu+1-\sum_{j\geqslant\nu+1}\frac{1}{2^{j-\nu}}=\nu.$$

That is, $(f_{k_\nu}, \psi) \to \infty$, $\nu \to \infty$. But this contradicts the boundedness of the sequence $(f_k, \psi)$, $k \to \infty$ $(f_k \in M')$. The proof of the lemma is complete.

Corollary *If a set $M' \subset \mathscr{D}'(\mathcal{O})$ is weakly bounded, then for any $\mathcal{O}' \Subset \mathcal{O}$ there exist numbers $K$ and $m$ such that the inequality* (3.1) *is preserved for all $\varphi \in \mathscr{D}(\mathcal{O}')$ and $f \in M'$.*

The proof is analogous to the proof of the theorem of Sec. 1.3 with the lemma invoked as well.

**1.5 The support of a generalized function** Generally speaking, generalized functions do not have values at separate points. Nevertheless, one may speak of the vanishing of a generalized function in an open set. We say that a generalized function $f$ in $\mathscr{D}'(\mathcal{O})$ *vanishes* in an open set $\mathcal{O}' \subset \mathcal{O}$ if its restriction to $\mathcal{O}'$ (see Sec. 1.3) is a zero functional in $\mathscr{D}'(\mathcal{O}')$, that is, $(f, \varphi) = 0$ for all $\varphi \in \mathscr{D}(\mathcal{O}')$. We then write: $f(x) = 0$, $x \in \mathcal{O}'$.

Suppose a generalized function $f$ in $\mathscr{D}'(\mathcal{O})$ vanishes in $\mathcal{O}$. It then obviously vanishes also in the neighbourhood of every point of the set $\mathcal{O}$.

Conversely, let $f$ in $\mathscr{D}'(\mathcal{O})$ vanish in some neighbourhood $U(y) \subset \mathcal{O}$ of each point $y$ of the open set $\mathcal{O}$.

Using the cover $\{U(y), y \in \mathcal{O}\}$ of the set $\mathcal{O}$, let us construct its locally finite cover $\{\mathcal{O}_k\}$ so that each $\mathcal{O}_k$ is contained in some $U(y)$. Let $\mathcal{O}'_1 \Subset \mathcal{O}'_2 \Subset \ldots$, $\bigcup_{\nu\geqslant 1} \mathcal{O}'_\nu = \mathcal{O}$. By the Heine-Borel lemma, the compact $\overline{\mathcal{O}}'_2$ is covered by a finite number of neighbourhoods $U(y)$: $U(y_1), \ldots, U(y_{N_1})$; the compact $\overline{\mathcal{O}}'_3 \setminus \mathcal{O}'_1$ is covered likewise by a finite number of such neighbourhoods: $U(y_{N_1+1}), \ldots, U(y_{N_1+N_2})$; and so forth. Setting $\mathcal{O}_k = U(y_k) \cap \mathcal{O}'_2$, $k = 1, \ldots, N_1$, $\mathcal{O}_k = U(y_k) \cap (\mathcal{O}'_3 \setminus \overline{\mathcal{O}}'_1)$, $k = N_1 + 1, \ldots, N_1 + N_2$, and so on, we obtain the required cover $\{\mathcal{O}_k\}$.

Let $\{e_k\}$ be the decomposition of unity that corresponds to the constructed cover $\{\mathcal{O}_k\}$ of the set $\mathcal{O}$ (see Sec. 1.2). Then for any $\varphi$

in $\mathcal{D}(\mathcal{O})$, supp $(\varphi e_k) \subset U(y)$ for some $y$ and for that reason $(f, \varphi e_k) = 0$; consequently

$$(f, \varphi) = (f, \sum_{k \geqslant 1} e_k \varphi) = \sum_{k \geqslant 1} (f, \varphi e_k) = 0.$$

We are thus convinced that the following lemma holds true.

Lemma *If a generalized function in $\mathcal{D}'(\mathcal{O})$ vanishes in some neighbourhood of every point of the open set $\mathcal{O}$, then it also vanishes in the whole set $\mathcal{O}$.*

Suppose $f \in \mathcal{D}'(\mathcal{O})$. The union of all neighbourhoods where $f = 0$ forms an open set $\mathcal{O}_f$, which is called the *zero set* of the generalized function $f$. By the lemma, $f = 0$ in $\mathcal{O}_f$; furthermore $\mathcal{O}_f$ is the largest open set in which $f$ vanishes.

The *support* of a generalized function $f$ is the completion of $\mathcal{O}_f$ to $\mathcal{O}$; the support of $f$ is symbolized as supp $f$, so that supp $f = \mathcal{O} \setminus \mathcal{O}_f$; supp $f$ is a closed set in $\mathcal{O}$. If supp $f \Subset \mathcal{O}$, then $f$ is said to be *finite* in $\mathcal{O}$.

From the foregoing follow the assertions:

(a) *If the supports of $f \in \mathcal{D}'(\mathcal{O})$ and $\varphi \in \mathcal{D}(\mathcal{O})$ do not have any points in common, then* $(f, \varphi) = 0$.

(b) *To have $x \in$ supp $f$, it is necessary and sufficient that $f$ should not vanish in any neighbourhood of the point $x$.*

Let $A$ be a closed set in $\mathcal{O}$. We denote by $\mathcal{D}'(\mathcal{O}, A)$ the collection of generalized functions in $\mathcal{D}'(\mathcal{O})$, whose supports are contained in $A$, and with convergence: $f_k \to 0$, $k \to \infty$ in $\mathcal{D}'(\mathcal{O}, A)$ if $f_k \to 0$, $k \to \infty$ in $\mathcal{D}'(\mathcal{O})$ and supp $f_k \subset A$. We write $\mathcal{D}'(A) = \mathcal{D}'(\mathbb{R}^n, A)$.

A similar meaning is also attributed to other spaces of generalized functions, for example, $\mathcal{S}'(A)$, $\mathcal{L}^2_s(A)$ and so on (see Secs. 5 and 7 below).

The lemma that was proved in this subsection admits of a generalization. In Sec. 1.3 we saw that any generalized function $f$ in $\mathcal{D}'(\mathcal{O})$ induces, in each $\mathcal{O}' \subset \mathcal{O}$, its own local element $f \in \mathcal{D}'(\mathcal{O}')$. The converse is also true: it is possible to "join together" a single generalized function out of any collection of coordinated local elements. To be more precise, the following theorem holds.

Theorem of piecewise joining *Suppose that for each point $y \in \mathcal{O}$ there exists a neighbourhood $U(y) \Subset \mathcal{O}$ and in it there is specified a generalized function $f_y$, wherein $f_{y_1}(x) = f_{y_2}(x)$ if $x \in U(y_1) \cap U(y_2) \neq \varnothing$. Then there is a unique generalized function $f$ in $\mathcal{D}'(\mathcal{O})$ that coincides with $f_y$ in $U(y)$ for all $y \in \mathcal{O}$.*

*Proof*. Again, as in the proof of the lemma, we construct, with respect to the cover $\{U(y), y \in \mathcal{O}\}$, a locally finite cover $\{\mathcal{O}_k\}$,

$\mathcal{O}_k \subset U(y_k)$, of the set $\mathcal{O}$ and the appropriate decomposition of unity $\{e_k\}$. Set

$$(f, \varphi) = \sum_{k \geqslant 1} (f_{y_k}, \varphi e_k), \qquad \varphi \in \mathcal{D}(\mathcal{O}). \tag{5.1}$$

Since the number of summands in the right member of (5.1) is always finite and does not depend on $\varphi \in \mathcal{D}(\mathcal{O}')$ for any $\mathcal{O}' \Subset \mathcal{O}$, it follows that functional $f$ defined by it is linear and continuous on $\mathcal{D}(\mathcal{O})$, or $f \in \mathcal{D}'(\mathcal{O})$. Furthermore, if $\varphi \in \mathcal{D}(U(y))$, then $\varphi e_k \in \mathcal{D}(U(y_k))$ and, hence, $(f_y, \varphi e_k) = (f_{y_k}, \varphi e_k)$ so that by virtue of (5.1)

$$(f, \varphi) = \sum_{k \geqslant 1} (f_{y_k}, \varphi e_k) = \Big(f_y, \varphi \sum_{k \geqslant 1} e_k\Big) = (f_y, \varphi).$$

That is, $f = f_y$ in $U(y)$. The uniqueness of the generalized function $f$ thus constructed follows from the lemma. The proof of the theorem is complete.

**1.6 Regular generalized functions** The simplest example of a generalized function is a functional generated by a function $f(x)$ locally summable in $\mathcal{O}$:

$$(f, \varphi) = \int f(x)\, \varphi(x)\, dx, \qquad \varphi \in \mathcal{D}(\mathcal{O}). \tag{6.1}$$

From the properties of the linearity of an integral and from the theorem on passage to the limit under the integral sign it follows that the functional in the right member of (6.1) is linear and continuous on $\mathcal{D}(\mathcal{O})$, that is, $f \in \mathcal{D}'(\mathcal{O})$.

Generalized functions determined, via (6.1), by functions locally summable in $\mathcal{O}$ are termed *regular* generalized functions. All other generalized functions are said to be *singular*.

Lemma (Du Bois Reymond) *For a function $f(x)$ that is locally summable in $\mathcal{O}$ to vanish almost everywhere in $\mathcal{O}$, it is necessary and sufficient that the regular generalized function $f$ generated by it vanish in $\mathcal{O}$.*

*Proof*. Necessity is obvious. We prove sufficiency. Let

$$\int f(x)\, \varphi(x)\, dx = 0, \qquad \varphi \in \mathcal{D}(\mathcal{O}). \tag{6.2}$$

Take an arbitrary $\mathcal{O}' \Subset \mathcal{O}$; let $\chi_{\mathcal{O}'}$ be a characteristic function of $\mathcal{O}'$. By Theorem II of Sec. 1.2, there exists a sequence of functions $\varphi_k(x)$, $k = 1, 2, \ldots$, from $\mathcal{D}(\mathcal{O})$, which sequence con-

verges to the function $e^{-i \arg f(x)}\chi_{\mathcal{O}'}(x)$ almost everywhere in $\mathcal{O}$, and $|\varphi_k(x)| \leqslant 1$ almost everywhere in $\mathcal{O}$. From this, using the Lebesgue theorem on passage to the limit under the sign of the Lebesgue integral, we conclude, taking into account (6.2), that

$$\begin{aligned}\int_{\mathcal{O}'} |f(x)|\, dx &= \int f(x)\, e^{-i \arg f(x)}\chi_{\mathcal{O}'}(x)\, dx \\ &= \lim_{k\to\infty}\left\{\int f(x)\,\varphi_k(x)\, dx \right. \\ &\qquad \left. + \int f(x)\,[e^{-i \arg f(x)}\chi_{\mathcal{O}'}(x) - \varphi_k(x)]\, dx\right\} \\ &= \int_{\mathcal{O}'} \lim_{k\to\infty} f(x)\,[e^{-i \arg f(x)} - \varphi_k(x)]\, dx = 0,\end{aligned}$$

so that $f(x) = 0$ almost everywhere in $\mathcal{O}'$. Due to the arbitrary nature of the set $\mathcal{O}' \Subset \mathcal{O}$, we conclude that $f(x) = 0$ almost everywhere in $\mathcal{O}$. The proof of the lemma is complete.

From the lemma just proved it follows that any regular generalized function in $\mathcal{O}$ is defined by a unique function (unique up to the values on a set of measure zero) that is locally summable in $\mathcal{O}$. Consequently there is a one-to-one correspondence between functions locally summable in $\mathcal{O}$ and regular generalized functions in $\mathcal{O}$. For this reason we will henceforth identify a function $f(x)$ locally summable in $\mathcal{O}$ with the generalized function from $\mathcal{D}'(\mathcal{O})$ that is generated by it via (6.1). In this sense, "ordinary" functions (that is, functions locally summable in $\mathcal{O}$) are (regular) generalized functions taken from $\mathcal{D}'(\mathcal{O})$.

From the Du Bois Reymond lemma it follows likewise that both definitions of the support of a continuous function in $\mathcal{O}$ that were given in Sec. 0.5 and Sec. 1.5 coincide.

Also note that if the sequence $f_k(x)$, $k = 1, 2, \ldots$, of functions locally summable in $\mathcal{O}$ converges uniformly to the function $f(x)$ on each compact $K \Subset \mathcal{O}$, then, also, $f_k \to f$, $k \to \infty$ in $\mathcal{D}'(\mathcal{O})$.

Suppose $f \in \mathcal{D}'(\mathcal{O})$ and $\mathcal{O}_1 \subset \mathcal{O}$. We will say that a generalized function $f$ *belongs to a class* $C^k(\mathcal{O}_1)$ if, in $\mathcal{O}_1$, it coincides with the function $f_1$ of the class $C^k(\mathcal{O}_1)$, that is, for any $\varphi \in \mathcal{D}(\mathcal{O}_1)$,

$$(f, \varphi) = \int f_1(x)\,\varphi(x)\, dx.$$

If, besides, $f_1 \in C^k(\overline{\mathcal{O}}_1)$, then we will say that $f$ belongs to the class $C^k(\overline{\mathcal{O}}_1)$.

**1.7 Measures** A more general class of generalized functions that contains the regular generalized functions is generated by measures. A *measure* on a Borel set $A$ is a completely additive (complex-valued) function of the set

$$\mu(E) = \int_E \mu(dx),$$

which function is specified and finite on all bounded Borel subsets $E$ of the set $A$, $|\mu(E)| < \infty$.

For details of measure theory and integration see Kolmogorov and Fomin [1].

The measure $\mu(E)$ on $A$ can be uniquely represented in terms of 4 nonnegative measures $\mu_j(E) \geqslant 0$ on $A$ via the formula $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$; here,

$$\int_E \mu(dx) = \int_E \mu_1(dx) - \int_E \mu_2(dx) + i\int_E \mu_3(dx) - i\int_E \mu_4(dx). \tag{7.1}$$

The measure $\mu(E)$ on an open set $\mathcal{O}$ determines a generalized function $\mu$ in $\mathcal{O}$ by the formula

$$(\mu, \varphi) = \int \varphi(x)\,\mu(dx), \qquad \varphi \in \mathscr{D}(\mathcal{O}), \tag{7.2}$$

where the integral is the Lebesgue-Stjieltes integral. From the properties of this integral it follows that we do indeed have $\mu \in \mathscr{D}'(\mathcal{O})$.

*Remark*. For measures $\mu$ on $\mathcal{O}$ that are absolutely continuous with respect to the Lebesgue measure, that is, $\mu(dx) = f(x)(dx)$, where $f \in \mathscr{L}^1_{\mathrm{loc}}(\mathcal{O})$, formula (7.2) defines regular generalized functions $f$ (see Sec. 1.6).

An assertion similar to the lemma of Du Bois Reymond holds (see Sec. 1.6).

Lemma *For a measure $\mu(E)$ on $\mathcal{O}$ to be a zero measure, it is necessary and sufficient for the generalized function $\mu$ defined by it to vanish in $\mathcal{O}$.*

*Proof*. The proof is based on the following assertion: for the measure $\mu(E)$ on $\mathcal{O}$ to be a zero measure, it is necessary and sufficient that

$$\int_{\mathcal{O}} \varphi(x)\,\mu(dx) = 0, \qquad \varphi \in C_0(\mathcal{O}) \tag{7.3}$$

whence immediately follows necessity. We now prove sufficiency. Assuming that (7.3) holds for all $\varphi$ in $\mathscr{D}(\mathcal{O})$, we will prove that

it holds also for an arbitrary $\varphi$ in $C_0(\mathcal{O})$. Suppose supp $\varphi \subset \subset \mathcal{O}' \Subset \mathcal{O}$. By Theorem II, Sec. 1.2, there exists a sequence of functions $\varphi_k$, $k = 1, 2, \ldots$, in $\mathcal{D}(\mathcal{O})$ such that supp $\varphi_k \subset \subset \mathcal{O}' \Subset \mathcal{O}$ and $\varphi_k \to \varphi$, $k \to \infty$ in $C(\overline{\mathcal{O}'})$. Therefore

$$\int_{\mathcal{O}} \varphi(x)\mu(dx) = \int_{\mathcal{O}'} \lim_{k\to\infty} \varphi_k(x)\,\mu(dx) = \lim_{k\to\infty} \int_{\mathcal{O}} \varphi_k(x)\,\mu(dx) = 0,$$

which is what we set out to prove. The lemma is proved.

From the lemma it follows that there exists a one-to-one correspondence between measures on $\mathcal{O}$ and the generalized functions generated by them via formula (7.2). For this reason, we will in future identify the measure $\mu(E)$ on $\mathcal{O}$ and the generalized function $\mu$ in $\mathcal{D}'(\mathcal{O})$ generated by that measure.

Theorem I *For a generalized function $f$ in $\mathcal{D}'(\mathcal{O})$ to be a measure on $\mathcal{O}$, it is necessary and sufficient that its order in $\mathcal{O}$ be equal to* 0.

*Proof. Necessity.* Let $f \in \mathcal{D}'(\mathcal{O})$ be a measure $\mu$ on $\mathcal{O}$. Then for any $\mathcal{O}' \Subset \mathcal{O}$ and any $\varphi \Subset \mathcal{D}(\mathcal{O}')$ we have

$$|(f, \varphi)| = \left| \int \varphi(x)\,\mu(dx) \right| \leqslant \int_{\mathcal{O}'} |\mu(dx)| \max_{x\in\overline{\mathcal{O}'}} |\varphi(x)|,$$

whence we conclude that the order of $f$ in $\mathcal{O}$ is 0 (see Sec. 1.3).

*Sufficiency.* Let the order of $f \in \mathcal{D}'(\mathcal{O})$ in $\mathcal{O}$ be 0, that is, for all $\mathcal{O}' \Subset \mathcal{O}$

$$|(f,\varphi)| \leqslant K(\mathcal{O}')\,\|\varphi\|_{C(\overline{\mathcal{O}'})}, \qquad \varphi \in \mathcal{D}(\mathcal{O}'). \tag{7.4}$$

Let $\mathcal{O}_k$, $k = 1, 2, \ldots$, be a strictly increasing sequence of open sets that exhausts $\mathcal{O}$: $\mathcal{O}_k \Subset \mathcal{O}_{k+1}$, $\bigcup_k \mathcal{O}_k = \mathcal{O}$. Since the set $\mathcal{D}(\mathcal{O}_k)$ is dense in $C_0(\overline{\mathcal{O}}_k)$ in the norm $C(\overline{\mathcal{O}}_k)$ (see Corollary 2 to Theorem II of Sec. 1.2), it follows from inequality (7.4) that the functional $f$ admits of a (linear) continuous extension onto $C_0(\overline{\mathcal{O}}_k)$. By the Riesz-Radon theorem, there is a measure $\mu_k$ on $\overline{\mathcal{O}}_k$ such that

$$(f, \varphi) = \int \varphi(x)\,\mu_k(dx), \qquad \varphi \in C_0(\overline{\mathcal{O}}_k).$$

From this it follows that the measures $\mu_k$ and $\mu_{k+1}$ coincide on $\mathcal{O}_k$; therefore there exists a single measure $\mu$ on $\mathcal{O}$ that coincides with the measure $\mu_k$ in $\mathcal{O}_k$ and with the generalized function $f$ in $\mathcal{O}$. The proof of the theorem is complete.

A generalized function $f$ in $\mathcal{D}'(\mathcal{O})$ is said to be *nonnegative* in $\mathcal{O}$ if $(f, \varphi) \geqslant 0$ for all $\varphi \in \mathcal{D}(\mathcal{O})$, $\varphi(x) \geqslant 0$, $x \in \mathcal{O}$.

Theorem II *For a generalized function in* $\mathscr{D}'(\mathcal{O})$ *to be a nonnegative measure on* $\mathcal{O}$, *it is necessary and sufficient that it be nonnegative in* $\mathcal{O}$.

*Proof*. Necessity is obvious. We prove sufficiency. Suppose $f \in \mathscr{D}'(\mathcal{O})$ is nonnegative in $\mathcal{O}$. Suppose $\varphi \in \mathscr{D}(\mathcal{O}')$, $\mathcal{O}' \Subset \mathcal{O}$. By the corollary to the lemma of Sec. 1.2 there is a function $\eta \in \mathscr{D}(\mathcal{O})$, $\eta(x) = 1$, $x \in \mathcal{O}'$. For this reason,

$$-\|\varphi\|_{C(\overline{\mathcal{O}}')}\eta(x) \leqslant \varphi(x) \leqslant \|\varphi\|_{C(\overline{\mathcal{O}}')}\eta(x), \qquad x \in \mathcal{O}.$$

Whence, using the nonnegative nature of the functional $f$ in $\mathcal{O}$, we get

$$-(f, \eta)\|\varphi\|_{C(\overline{\mathcal{O}}')} \leqslant (f, \varphi) \leqslant (f, \eta)\|\varphi\|_{C(\overline{\mathcal{O}}')}$$

or

$$|(f, \varphi)| \leqslant (f, \eta)\|\varphi\|_{C(\overline{\mathcal{O}}')}, \qquad \varphi \in \mathscr{D}(\mathcal{O}').$$

This inequality shows that the order of the generalized function $f$ is 0. By Theorem I, $f$ is a (nonnegative) measure on $\mathcal{O}$, and the proof is complete.

The simplest instance of a measure (and, what is more, a measure of a singular generalized function) is the delta function of Dirac (see Sec. 1.1), which operates via the rule

$$(\delta, \varphi) = \varphi(0), \qquad \varphi \in \mathscr{D}.$$

Clearly, $\delta \in \mathscr{D}'$, $\delta(x) = 0$, $x \neq 0$ so that $\operatorname{supp} \delta = \{0\}$.

We will now prove that $\delta(x)$ is a *singular* generalized function. Suppose, on the contrary, that there is a function $f \in \mathscr{L}^1_{\text{loc}}(\mathbb{R}^n)$ such that for any $\varphi \in \mathscr{D}$

$$\int f(x)\varphi(x)\,dx = (\delta, \varphi) = \varphi(0). \tag{7.5}$$

Since $|x|^2\varphi \in \mathscr{D}$, it follows from (7.5) that

$$\int f(x)|x|^2\varphi(x)\,dx = |x|^2\varphi(x)|_{x=0} = 0 = (|x|^2 f, \varphi)$$

for all $\varphi \in \mathscr{D}$. Thus the function $|x|^2 f(x)$ that is locally summable in $\mathbb{R}^n$ is equal to zero in the sense of generalized functions. By the Du Bois Reymond lemma (see Sec. 1.6), $|x|^2 f(x) = 0$ almost everywhere and, hence, $f(x) = 0$ almost everywhere in $\mathbb{R}^n$. But this contradicts the equation (7.5). The contradiction proves the singularity of the delta function.

Suppose $\omega_\varepsilon(x)$ is a "cap" (see Sec. 1.2). We will prove that

$$\omega_\varepsilon(x) \to \delta(x), \qquad \varepsilon \to +0 \quad \text{in} \quad \mathcal{D}'. \tag{7.6}$$

The sequence $\omega_\varepsilon(x)$, $\varepsilon \to +0$ is depicted in Fig. 4.

Indeed, by the definition of convergence in $\mathcal{D}'$ the relation (7.6) is equivalent to

$$\lim_{\varepsilon\to+0} \int \omega_\varepsilon(x)\,\varphi(x)\,dx = \varphi(0), \qquad \varphi \in \mathcal{D}.$$

This equation follows from the estimate

$$\left|\int \omega_\varepsilon(x)\,\varphi(x)\,dx - \varphi(0)\right| \leqslant \int \omega_\varepsilon(x)\,|\varphi(x)-\varphi(0)|\,dx$$

$$\leqslant \max_{|x|\leqslant\varepsilon} |\varphi(x)-\varphi(0)| \int \omega_\varepsilon(x)\,dx = \max_{|x|\leqslant\varepsilon} |\varphi(x)-\varphi(0)|$$

and from the continuity of the function $\varphi$.

The surface $\delta$ function is a generalization of the point $\delta$ function. Let $S$ be a piecewise smooth surface in $\mathbb{R}^n$ and let $\mu$ be a continuous function on $S$. We introduce the generalized function $\mu\delta_S$ that operates via the rule

$$(\mu\delta_S, \varphi) = \int_S \mu(x)\,\varphi(x)\,dS, \qquad \varphi \in \mathcal{D}.$$

Clearly $\mu\delta \in \mathcal{D}'$; $\mu\delta_S(x) = 0$, $x \notin S$ so that $\operatorname{supp} \mu\delta_S \subset S$; $\mu\delta_S$ is a singular measure if $\mu \not\equiv 0$.

The generalized function $\mu\delta_S(x)$ is termed a *simple layer* on the surface $S$. It describes the spatial density of masses or charges concentrated on the surface $S$ with surface density $\mu$. (Here, the density of the simple layer is defined as a weak limit of the densities that correspond to a discrete distribution on the surface $S$,

$$\sum_k \mu(x_k)\,\Delta S_k \delta(x-x_k), \qquad x_k \in S$$

when the surface $S$ is subdivided without limit; compare Sec. 1.1.)

*Remark.* Locally summable functions and $\delta$ functions describe the density distribution of masses, charges, forces and the like (see Sec. 1.1). For this reason, generalized functions are also termed *distributions* (see Schwartz [1, 2]). If, for example, a generalized function $f$ is the density of masses or charges, then the expression $(f, 1)$ is the total mass or total charge (on the assumption that $f$ is meaningful on a function identically equal to 1 because 1 is not finite!). In particular, $(\delta, 1) = 1$; $(f, 1) = \int f(x)\,dx$ if $f \in \mathcal{L}^1$.

3*

**1.8 Sochozki formulas** We now introduce another important singular generalized function $\mathscr{P}\frac{1}{x}$ that operates in accordance with the formula

$$\left(\mathscr{P}\frac{1}{x},\ \varphi\right)=\mathrm{PV}\int\frac{\varphi(x)}{x}\,dx=\lim_{\varepsilon\to+0}\left(\int_{-\infty}^{-\varepsilon}+\int_{\varepsilon}^{\infty}\right)\frac{\varphi(x)}{x}\,dx.$$

The functional $\mathscr{P}\frac{1}{x}$ is a linear functional. Its continuity on $\mathscr{D}=\mathscr{D}(\mathbb{R}^1)$ follows from the equation

$$\left|\left(\mathscr{P}\frac{1}{x},\varphi\right)\right|=\left|\mathrm{PV}\int\frac{\varphi(x)}{x}\,dx\right|=\left|\mathrm{PV}\int_{-R}^{R}\frac{\varphi(0)+x\varphi'(x')}{x}\,dx\right|$$

$$\leqslant\int_{-R}^{R}|\varphi'(x')|\,dx\leqslant 2R\max_{x}|\varphi'(x)|,\quad \varphi\in\mathscr{D}(-R,R). \tag{8.1}$$

Here, $x'$ is some point in the interval $(-R,\ R)$. Thus $\mathscr{P}\frac{1}{x}\in\mathscr{D}'$.

The generalized function $\mathscr{P}\frac{1}{x}$ coincides with the function $\frac{1}{x}$ for $x\neq0$ (in the meaning of Sec. 1.6). It is called the *finite part* or *principal value* of the integral of the function $\frac{1}{x}$. Let us now set up the equality

$$\lim_{\varepsilon\to+0}\int\frac{\varphi(x)}{x+i\varepsilon}\,dx=-i\pi\varphi(0)+\mathrm{PV}\int\frac{\varphi(x)}{x}\,dx, \tag{8.2}$$
$$\varphi\in\mathscr{D}.$$

Indeed, if $\varphi(x)=0$ for $|x|>R$, then

$$\lim_{\varepsilon\to+0}\int\frac{\varphi(x)}{x+i\varepsilon}\,dx=\lim_{\varepsilon\to+0}\int_{-R}^{R}\frac{x-i\varepsilon}{x^2+\varepsilon^2}\varphi(x)\,dx$$

$$=\varphi(0)\lim_{\varepsilon\to+0}\int_{-R}^{R}\frac{x-i\varepsilon}{x^2+\varepsilon^2}\,dx+\lim_{\varepsilon\to+0}\int_{-R}^{R}\frac{x-i\varepsilon}{x^2+\varepsilon^2}[\varphi(x)-\varphi(0)]\,dx$$

$$=-2i\varphi(0)\lim_{\varepsilon\to+0}\arctan\frac{R}{\varepsilon}+\int_{-R}^{R}\frac{\varphi(x)-\varphi(0)}{x}\,dx$$

$$=-i\pi\varphi(0)+\mathrm{PV}\int\frac{\varphi(x)}{x}\,dx.$$

The relation (8.2) means that there is a limit to the sequence $\frac{1}{x+i\varepsilon}$ as $\varepsilon\to+0$ in $\mathscr{D}'$, which limit we denote by $\frac{1}{x+i0}$; and

this limit **is equal** tc $-i\pi\delta(x)+\mathcal{P}\frac{1}{x}$. Thus

$$\frac{1}{x+i0}=-i\pi\delta(x)+\mathcal{P}\frac{1}{x}. \qquad (8.3)$$

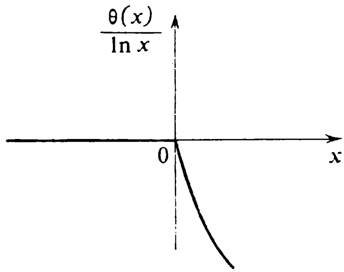
Similarly

$$\frac{1}{x-i0}=i\pi\delta(x)\mathcal{P}\frac{1}{x}. \qquad (8.3')$$

The formulas (8.3) and (8.3′) were actually first obtained in "integral" form of the type (8.2) in 1873 by the Russian mathematician Julian Sochozki (see Sochozki [1]). At the present time these formulas are widely used in quantum physics.

We will now prove that the order of $\mathcal{P}\frac{1}{x}$ in $\mathbb{R}^1$ is equal to 1. Indeed, from (8.1) it follows that its order in $\mathbb{R}^1$ does not exceed 1.

Figure 5




If its order in $\mathbb{R}^1$ were equal to 0, then by Theorem I of Sec. 1.7, $\mathcal{P}\frac{1}{x}$ would be a measure on $\mathbb{R}^1$. But then the integral $\mathrm{PV}\int\frac{\varphi(x)}{x}\,dx$ would be defined on all continuous functions that are finite in $\mathbb{R}^1$, which, as we know, is not true (for example, it is not defined on functions equal to $\frac{\theta(x)}{\ln x}$ in the neighbourhood of 0; Fig. 5).

We note in passing that the order of $\mathcal{P}\frac{1}{x}$ in $\{x\neq 0\}$ is equal to 0 because $\mathcal{P}\frac{1}{x}$ coincides with the locally summable function $\frac{1}{x}$ when $x\neq 0$.

The generalized function $\mathcal{P}\frac{1}{x}$ is a continuation of the regular generalized function $\frac{1}{x}$ from the set $\{x\neq 0\}$ onto the whole axis $\mathbb{R}^1$. The question now is: does any locally summable function in $\mathcal{O}\neq\mathbb{R}^n$ admit of continuation onto the whole space $\mathbb{R}^n$ as

a generalized function from $\mathcal{D}'(\mathbb{R}^n)$? The answer is negative, as will be seen from the following example:

$$e^{1/x} \in \mathcal{D}' \qquad (x \neq 0).$$

If there existed a function $f \in \mathcal{D}'(\mathbb{R}^1)$ that coincided with $e^{1/x}$ for $x \neq 0$, we would have

$$(f, \varphi) = \int e^{\frac{1}{x}} \varphi(x)\, dx, \qquad \varphi \in \mathcal{D} \quad (x \neq 0). \tag{8.4}$$

Let $\varphi_0 \in \mathcal{D}$, $\varphi_0(x) = 0$ for $x < 1$ and $x > 2$, $\varphi_0(x) \geqslant 0$,

$$\int \varphi_0(x) dx = 1.$$

Then

$$\varphi_k(x) = e^{-\frac{k}{2}} k\varphi_0(kx) \to 0, \qquad k \to \infty \quad \text{in} \quad \mathcal{D},$$

$$\int e^{\frac{1}{x}} \varphi_k(x)\, dx = \int e^{\frac{1}{x} - \frac{k}{2}} k\varphi_0(kx)\, dx =$$

$$= \int_1^2 e^{k\left(\frac{1}{y} - \frac{1}{2}\right)} \varphi_0(y)\, dy \geqslant \int_1^2 \varphi_0(y)\, dy = 1,$$

$$\varphi_k \in \mathcal{D} \qquad (x \neq 0),$$

but this contradicts (8.4):

$$1 \leqslant \int e^{\frac{1}{x}} \varphi_k(x)\, dx = (f, \varphi_k) \to 0, \qquad k \to \infty.$$

**1.9 Change of variables in generalized functions** Let $f \in \mathcal{L}^1_{\text{loc}}(\mathcal{O})$ and $x = Ay + b$ be a nonsingular linear transformation of $\mathcal{O}$ onto $\mathcal{O}_1$. Then for any $\varphi \in \mathcal{D}(\mathcal{O}_1)$ we have

$$\int_{\mathcal{O}_1} f(Ay + b) \varphi(y)\, dy = \frac{1}{|\det A|} \int_{\mathcal{O}} f(x)\, \varphi[A^{-1}(x - b)]\, dx.$$

This equality is taken for the definition of the generalized function $f(Ay + b)$ for any $f(x)$ in $\mathcal{D}'(\mathcal{O})$:

$$\big(f(Ay + b), \varphi(y)\big) = \left(f(x), \frac{\varphi[A^{-1}(x - b)]}{|\det A|}\right), \tag{9.1}$$

$$\varphi \in \mathcal{D}(\mathcal{O}_1).$$

Since the operation $\varphi(x) \to \varphi[A^{-1}(x - b)]$ is linear and continuous from $\mathcal{D}(\mathcal{O}_1)$ into $\mathcal{D}(\mathcal{O})$, the functional $f(Ay + b)$ defined by the right-hand side of (9.1) belongs to $\mathcal{D}'(\mathcal{O}_1)$.

In particular, if $A$ is a rotation, that is $A^T = A^{-1}$ and $b = 0$, then $(f(Ay), \varphi) = (f, \varphi(A^T x))$; if $A$ is a similarity (with reflection), that is, $A = cI$, $c \neq 0$ and $b = 0$, then

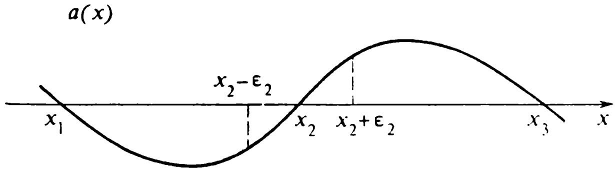
$$(f(cy), \varphi) = \frac{1}{|c|^n}\left(f, \varphi\left(\frac{x}{c}\right)\right).$$

If $A = I$, then (a shift equal to $b$)

$$(f(y+b), \varphi) = (f, \varphi(x-b)).$$

The foregoing enables us to define translation-invariant, spherically symmetrical, centrally symmetrical, homogeneous, periodic, Lorentz-invariant, and other generalized functions.

Figure 6




*Examples*: (a) $\delta(-x) = \delta(x)$; (b) $(\delta(x - x_0), \varphi) = \varphi(x_0)$.

Let $a \in C^1$. We define the generalized function $\delta(a(x))$ via the formula

$$\delta(a(x)) = \lim_{\varepsilon \to +0} \omega_\varepsilon(a(x)) \quad \text{in} \quad \mathscr{D}'(c, d) \tag{9.2}$$

where $\omega_\varepsilon$ is the "cap".

Suppose the function $a(x)$ has isolated and simple zeros which we denote by $x_k$, $k = 1, 2, \ldots$ (Fig. 6). Then $\delta(a(x))$ exists in $\mathscr{D}'(\mathbb{R}^1)$ and is given by the sum

$$\delta(a(x)) = \sum_k \frac{\delta(x - x_k)}{|a'(x_k)|}. \tag{9.3}$$

By virtue of the theorem of piecewise joining (see Sec. 1.5), it suffices to prove formula (9.3) locally, in a sufficiently small neighbourhood of each point. Let $\varphi \in \mathscr{D}(x_k - \varepsilon_k, x_k + \varepsilon_k)$ and let the number $\varepsilon_k$ be so small that in the interval $(x_k - \varepsilon_k, x_k + \varepsilon_k)$ the function $a(x)$ is monotonic. Making use of the limiting relation (7.6), we have the following chain of equalities:

$$(\delta(a(x)), \varphi) = \lim_{\varepsilon \to +0} \sum \int_{x_k - \varepsilon_k}^{x_k + \varepsilon_k} \omega_\varepsilon[a(x)]\varphi(x)\, dx$$

$$= \lim_{\varepsilon \to +0} \int_{a(x_k - \varepsilon_k)}^{a(x_k + \varepsilon_k)} \omega_\varepsilon(y)\, \varphi[a^{-1}(y)] \frac{dy}{a'[a^{-1}(y)]}$$

$$= \frac{\varphi(x_k)}{|a'(x_k)|} = \left(\frac{\delta(x - x_k)}{|a'(x_k)|}, \varphi\right).$$

Now if $\varphi \in \mathscr{D}(\alpha, \beta)$, where the interval $(\alpha, \beta)$ does not contain a single zero of $x_k$, then

$$(\delta(a(x)), \varphi) = \lim_{\varepsilon \to +0} \int_{\alpha}^{\beta} \omega_\varepsilon[a(x)]\, \varphi(x)\, dx = 0.$$

The local elements $\frac{\delta(x-x_k)}{|a'(x_k)|}$ in $(x_k - \varepsilon_k, x_k + \varepsilon_k)$ and 0 in $(\alpha, \beta)$ are clearly in agreement. The proof of (9.3) is complete.

*Examples*: (a) $\delta(x^2 - a^2) = \frac{1}{2a}[\delta(x-a) + \delta(x+a)]$;

(b) $\delta(\sin x) = \sum_{k=-\infty}^{\infty} \delta(x - k\pi)$.

**1.10 Multiplication of generalized functions** Suppose $f \in \mathscr{L}^1_{\mathrm{loc}}(\mathscr{O})$ and $a \in C^\infty(\mathscr{O})$. Then for any $\varphi$ in $\mathscr{D}(\mathscr{O})$ we have the equality

$$(af, \varphi) = \int a(x) f(x) \varphi(x)\, dx = (f, a\varphi).$$

This equality is then taken for the definition of the product of the generalized function $f$ in $\mathscr{D}'(\mathscr{O})$ by the function $a$ that is infinitely differentiable in $\mathscr{O}$:

$$(af, \varphi) = (f, a\varphi), \qquad \varphi \in \mathscr{D}(\mathscr{O}). \tag{10.1}$$

Since the operation $\varphi \to a\varphi$, $a \in C^\infty(\mathscr{O})$, is linear and continuous from $\mathscr{D}(\mathscr{O})$ into $\mathscr{D}(\mathscr{O})$, it follows that the functional $af$ defined by the right-hand side of (10.1) is a generalized function in $\mathscr{D}'(\mathscr{O})$.

The following inclusion holds true:

$$\operatorname{supp}(af) \subset \operatorname{supp} a \cap \operatorname{supp} f$$

because $\mathscr{O}_{af} \supset \mathscr{O}_a \cup \mathscr{O}_f$ (see Sec. 1.5) and

$$\begin{aligned} \operatorname{supp}(af) &= \mathscr{O} \setminus \mathscr{O}_{af} \subset \mathscr{O} \setminus (\mathscr{O}_a \cup \mathscr{O}_f) \\ &= (\mathscr{O} \setminus \mathscr{O}_a) \cap (\mathscr{O} \setminus \mathscr{O}_f) = \operatorname{supp} a \cap \operatorname{supp} f. \end{aligned}$$

If $f \in \mathscr{D}'(\mathscr{O})$, then we have the equality

$$f = \eta f \tag{10.2}$$

where $\eta$ is any function of the class $C^\infty$, that function being equal to 1 in the neighbourhood of the support of $f$.

Indeed, for any $\varphi \in \mathcal{D}(\mathcal{O})$, the supports of $f$ and $(1-\eta)\varphi$ have no points in common, and for this reason (see Sec. 1.5)

$$(f, (1-\eta)\varphi) = 0 = (f(1-\eta), \varphi),$$

which is equivalent to (10.2).

*Examples*: (a) $a(x)\delta(x) = a(0)\delta(x)$ since for all $\varphi \in \mathcal{D}$

$$(a\delta, \varphi) = (\delta, a\varphi) = a(0)\varphi(0) = (a(0)\delta, \varphi);$$

(b) $x\mathcal{P}\frac{1}{x} = 1$ since

$$\left(x\mathcal{P}\frac{1}{x}, \varphi\right) = \left(\mathcal{P}\frac{1}{x}, x\varphi\right) = \int \varphi(x)\,dx = (1, \varphi).$$

The following question arises: Is it not possible, in the class of generalized functions, to define the multiplication of any generalized functions and so that the multiplication is associative and commutative and agrees with the above-defined multiplication by an infinitely differentiable function? L. Schwartz demonstrated that no such multiplication can be defined. Indeed, if it existed, then, using examples (a) and (b), we would have the following contradictory chain of equalities:

$$0 = 0\,\mathcal{P}\frac{1}{x} = (x\delta(x))\,\mathcal{P}\frac{1}{x} = (\delta(x)\,x)\,\mathcal{P}\frac{1}{x} = \delta(x)\left(x\mathcal{P}\frac{1}{x}\right)$$
$$= \delta(x).$$

In order to define a product of two generalized functions $f$ and $g$, it is necessary that they have (to put it crudely) the following properties: $f$ must be just as "irregular" in the neighbourhood of an (arbitrary) point as $g$ is "regular" in that neighbourhood, and conversely.

# 2 Differentiation of Generalized Functions

**2.1 Derivatives of generalized functions** Let $f \in C^k(\mathcal{O})$. Then for all $\alpha$, $|\alpha| \leqslant k$, and $\varphi \in \mathcal{D}(\mathcal{O})$ we have the following integration-by-parts formula:

$$(D^\alpha f, \varphi) = \int D^\alpha f(x)\,\varphi(x)\,dx = (-1)^{|\alpha|}\int f(x)\,D^\alpha\varphi(x)\,dx$$
$$= (-1)^{|\alpha|}(f, D^\alpha\varphi).$$

We take this equation for the definition of a (generalized) derivative $D^\alpha f$ of the generalized function $f$ in $\mathcal{D}'(\mathcal{O})$:

$$(D^\alpha f, \varphi) = (-1)^{|\alpha|}(f, D^\alpha\varphi), \qquad \varphi \in \mathcal{D}(\mathcal{O}) \tag{1.1}$$

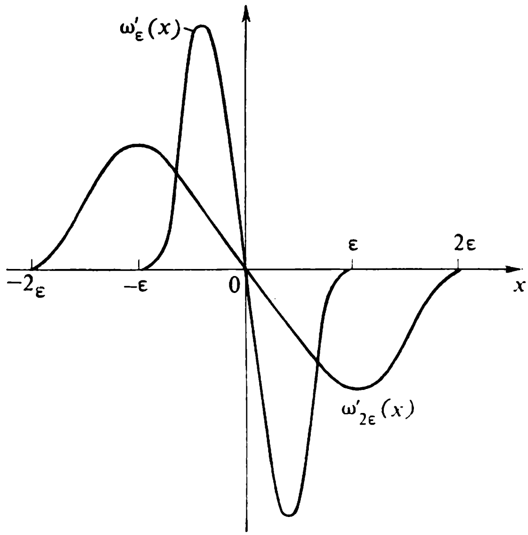
Since the operation $\varphi \to (-1)^{|\alpha|} D^\alpha \varphi$ is linear and continuous from $\mathcal{D}(\mathcal{O})$ into $\mathcal{D}(\mathcal{O})$, the functional $D^\alpha f$ defined by the right-hand side of (1.1) is a generalized function in $\mathcal{D}'(\mathcal{O})$.

In particular, when $f = \delta$, then (1.1) takes the form

$$(D^\alpha \delta, \varphi) = (-1)^{|\alpha|} D^\alpha \varphi(0), \qquad \varphi \in \mathcal{D}.$$

It follows from this definition that if a generalized function $f$ in $\mathcal{D}'(\mathcal{O})$ belongs to the class $C^k(\mathcal{O}_1)$ in $\mathcal{O}_1 \subset \mathcal{O}$ (see Sec. 1.6),

Figure 7




then its classical and generalized derivatives $D^\alpha f$, $|\alpha| \leqslant k$, coincide in $\mathcal{O}_1$.

The following properties of the operation of differentiation of generalized functions hold true:

(a) The operation of differentiation $f \to D^\alpha f$ is linear and continuous from $\mathcal{D}'(\mathcal{O})$ into $\mathcal{D}'(\mathcal{O})$.

Linearity is obvious. We will prove continuity. Suppose $f_k \to 0$, $k \to \infty$ in $\mathcal{D}'(\mathcal{O})$. Then for all $\varphi \in \mathcal{D}(\mathcal{O})$ we have

$$(D^\alpha f_k, \varphi) = (-1)^{|\alpha|} (f_k, D^\alpha \varphi) \to 0, \qquad k \to \infty$$

and this signifies that $D^\alpha f_k \to 0$, $k \to \infty$ in $\mathcal{D}'(\mathcal{O})$. For example,

$$D^\alpha \omega_\varepsilon(x) \to D^\alpha \delta(x), \qquad \varepsilon \to +0 \quad \text{in} \quad \mathcal{D}'. \tag{1.2}$$

The relation (1.2) follows from the relation (7.6) of Sec. 1. The sequence $\omega'_\varepsilon(x)$, $\varepsilon \to +0$, is depicted in Fig. 7.

In particular, if the series

$$\sum_{k\geqslant 1} u_k(x)=S(x), \qquad u_k\in\mathscr{L}^1_{\mathrm{loc}}(\mathcal{O})$$

converges uniformly on every compact $K\Subset\mathcal{O}$, then it may be differentiated term by term any number of times, and the resulting series will converge in $\mathscr{D}'(\mathcal{O})$,

$$\sum_{k\geqslant 1} D^\alpha u_k(x)=D^\alpha S(x).$$

True enough, the sequence of the partial sums of this series converges to $S(x)$ in $\mathscr{D}'(\mathcal{O})$ (see Sec. 1.6).

(b) Any generalized function $f\in\mathscr{D}'(\mathcal{O})$ (in particular, any function locally summable in $\mathcal{O}$) is infinitely differentiable (in the generalized sense).

Indeed, since $f\in\mathscr{D}'(\mathcal{O})$, it follows that $\frac{\partial f}{\partial x_j}\in\mathscr{D}'(\mathcal{O})$; in turn, $\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)\in\mathscr{D}'(\mathcal{O})$ and so forth.

(c) The result of differentiation does not depend on the order of differentiation:

$$D^{\alpha+\beta}f=D^\alpha(D^\beta f)=D^\beta(D^\alpha f). \tag{1.3}$$

Indeed,

$$\begin{aligned}(D^{\alpha+\beta}f,\varphi)&=(-1)^{|\alpha|+|\beta|}(f,D^{\alpha+\beta}\varphi)=(-1)^{|\alpha|}(D^\beta f,D^\alpha\varphi)\\&=(D^\alpha(D^\beta f),\varphi)=(-1)^{|\beta|}(D^\alpha f,D^\beta\varphi)\\&=(D^\beta(D^\alpha f),\varphi),\end{aligned}$$

whence follow the equalities (1.3).

(d) If $f\in\mathscr{D}'(\mathcal{O})$ and $a\in C^\infty(\mathcal{O})$, then the Leibniz formula holds true for the differentiation of a product $af$,

$$D^\alpha(af)=\sum_{\beta\leqslant\alpha}\binom{\beta}{\alpha}D^\beta a D^{\alpha-\beta}f. \tag{1.4}$$

Indeed, if $\varphi\in\mathscr{D}(\mathcal{O})$, then

$$\begin{aligned}\left(\frac{\partial(af)}{\partial x_1},\varphi\right)&=-\left(af,\frac{\partial\varphi}{\partial x_1}\right)=-\left(f,a\frac{\partial\varphi}{\partial x_1}\right)\\&=-\left(f,\frac{\partial(a\varphi)}{\partial x_1}-\frac{\partial a}{\partial x_1}\varphi\right)\\&=-\left(f,\frac{\partial(a\varphi)}{\partial x_1}\right)+\left(f,\frac{\partial a}{\partial x_1}\varphi\right)\\&=\left(\frac{\partial f}{\partial x_1},a\varphi\right)+\left(\frac{\partial a}{\partial x_1}f,\varphi\right)\\&=\left(a\frac{\partial f}{\partial x_1}+\frac{\partial a}{\partial x_1}f,\varphi\right),\end{aligned}$$

whence follows (1.4) for $\alpha = (1, 0, \ldots, 0)$.

(e) $$\operatorname{supp} D^{\alpha} f \subset \operatorname{supp} f. \tag{1.5}$$

Indeed, if $f \in \mathcal{D}'(\mathcal{O})$, then for all $\varphi \in \mathcal{D}(\mathcal{O}_f)$ we have $D^{\alpha}\varphi \in \mathcal{D}(\mathcal{O}_f)$ and

$$(D^{\alpha} f, \varphi) = (-1)^{|\alpha|} (f, D^{\alpha}\varphi) = 0$$

so that $\mathcal{O}_{D^{\alpha} f} \supset \mathcal{O}_f$, whence follows the inclusion (1.5).

**2.2 The antiderivative of a generalized function** Every function $f(x)$ continuous in an interval $(a, b)$ has in $(a, b)$ a unique (up to an additive constant) antiderivative $f^{(-1)}(x)$,

$$f^{(-1)}(x) = \int^{x} f(\xi)\, d\xi + C, \qquad f^{(-1)'}(x) = f(x).$$

The last equality is what we will start with to define the antiderivative of an arbitrary generalized function $f$ (of one variable).

Suppose $f \in \mathcal{D}'(a, b)$. The generalized function $f^{(-1)}$ in $\mathcal{D}'(a, b)$ is termed the *antiderivative* of the generalized function $f$ in $(a, b)$ if $f^{(-1)'} = f$, that is,

$$(f^{(-1)}, \varphi') = -(f, \varphi), \qquad \varphi \in \mathcal{D}(a, b). \tag{2.1}$$

The equality (2.1) shows that the function $f^{(-1)}$ is not specified on all basic functions taken from $\mathcal{D}(a, b)$, but only on their first derivatives. Our problem is to extend that functional onto the whole space $\mathcal{D}(a, b)$, and in a manner so that the extended functional $f^{(-1)}$ is linear and continuous on $\mathcal{D}(a, b)$, and to determine the degree of arbitrariness in such an extension.

First assume that $f^{(-1)}$ (the antiderivative of $f$) exists in $\mathcal{D}'(a, b)$. Construct it. Let $\varphi \in \mathcal{D}(a, b)$. (We assume the function $\varphi$ to be continued by means of zero onto the entire axis $\mathbb{R}^1$.) We fix an arbitrary point $x_0 \in (a, b)$. Then

$$\varphi(x) = \psi'(x) + \omega_{\varepsilon}(x - x_0) \int \varphi(\xi)\, d\xi \tag{2.2}$$

where $\omega_{\varepsilon}$ is the "cap" when $\varepsilon < \min(x_0 - a, b - x_0)$ (see Sec. 1.2) and

$$\psi(x) = \int_{-\infty}^{x} \left[\varphi(x') - \omega_{\varepsilon}(x' - x_0) \int \varphi(\xi)\, d\xi\right] dx'. \tag{2.3}$$

We will prove that $\psi \in \mathcal{D}(a, b)$. Indeed, $\psi \in C^{\infty}$ and $\psi(x) = 0$ for $x < a'' = \min(a', x_0 - \varepsilon) > a$ if $\operatorname{supp} \varphi \subset [a', b'] \subset (a, b)$.

Furthermore, for $x > b'' = \max(b', x_0 + \varepsilon) < b$,

$$\psi(x) = \int_{-\infty}^{\infty} \varphi(x')\,dx' - \int_{-\infty}^{\infty} \omega_\varepsilon(x' - x_0)\,dx' \int_{-\infty}^{\infty} \varphi(\xi)\,d\xi = 0.$$

Thus, $\operatorname{supp}\psi \subset [a'', b''] \subset (a, b)$ (Fig. 8). Hence $\psi \in \mathscr{D}(a, b)$.

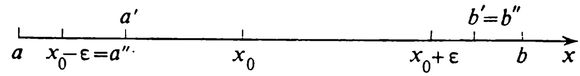
Applying the functional $f^{(-1)}$ to (2.2), we obtain

$$(f^{(-1)}, \varphi) = (f^{(-1)}, \psi') + (f^{(-1)}, \omega_\varepsilon(x - x_0)) \int \varphi(\xi)\,d\xi.$$

That is to say, taking into account (2.1),

$$(f^{(-1)}, \varphi) = -(f, \psi) + C \int \varphi(\xi)\,d\xi \tag{2.4}$$

where $C = (f^{(-1)}, \omega_\varepsilon(x - x_0))$. Thus, if $f^{(-1)}$ exists, then it is expressed by (2.4), where $\psi$ is defined by (2.3)

Figure 8 

Let us now prove the converse: given an arbitrary constant $C$, the functional $f^{(-1)}$ defined by equalities (2.4) and (2.3) defines the antiderivative of $f$ in $(a, b)$.

Indeed, the functional $f^{(-1)}$ is clearly linear. Let us prove that it is continuous on $\mathscr{D}(a, b)$. Let $\varphi_k \to 0$, $k \to \infty$ in $\mathscr{D}(a, b)$, that is, $\operatorname{supp}\varphi_k \subset [a', b'] \subset (a, b)$ and $\varphi_k^{(\alpha)}(x) \overset{x}{\rightrightarrows} 0$, $k \to \infty$. Then, by what has already been proved,

$$\psi_k(x) = \int_{-\infty}^{x} \left[\varphi_k(x') - \omega_\varepsilon(x' - x_0) \int \varphi_k(\xi)\,d\xi\right] dx' = 0$$

outside $[a'', b''] \subset [a, b]$

and, obviously, $\psi_k^{(\alpha)}(x) \overset{x}{\rightrightarrows} 0$, $k \to \infty$, that is, $\psi_k \to 0$, $k \to \infty$ in $\mathscr{D}(a, b)$. Therefore, by virtue of the continuity of $f$ on $\mathscr{D}(a, b)$, we have

$$(f^{(-1)}, \varphi_k) = -(f, \psi_k) + C \int \varphi_k(\xi)\,d\xi \to 0, \qquad k \to \infty,$$

which is what was affirmed. Consequently, $f^{(-1)} \in \mathscr{D}'(a, b)$. It remains to verify that $f^{(-1)}$ is the antiderivative of $f$ in $(a, b)$. Indeed, substituting $\varphi'$ for $\varphi$ in (2.3) and noting that $\int \varphi'(\xi)\,d\xi = 0$, we get $\psi = \varphi$, and then from (2.4) there follows the equality (2.1), which is what we set out to prove. We have thus proved the following theorem.

Theorem *Every generalized function $f$ in $\mathscr{D}'(a, b)$ has in $(a, b)$ an antiderivative $f^{(-1)}$, and every antiderivative of it is expressed by the formula (2.4), where $\psi$ is defined by (2.3) and $C$ is an arbitrary constant.*

This theorem states that a solution of the differential equation

$$u' = f, \qquad f \in \mathscr{D}'(a, b) \tag{2.5}$$

exists in $\mathscr{D}'(a, b)$ and its general solution is of the form $u = f^{(-1)} + C$, where $f^{(-1)}$ is some antiderivative of $f$ in $(a, b)$ and $C$ is an arbitrary constant. In particular, if $f \in C(a, b)$, then any solution in $\mathscr{D}'(a, b)$ of the equation (2.5) is a classical solution. For example, the general solution of the equation $u' = 0$ in $\mathscr{D}'(a, b)$ is the arbitrary constant.

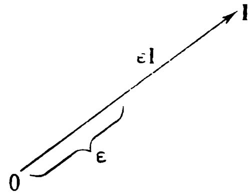
The definition of the antiderivative $f^{(-n)}$ of order $n$ in $(a, b)$ of the generalized function $f \in \mathscr{D}'(a, b)$, $f^{(-n)(n)} = f$, is similar. Applying this theorem to a recurrent chain for $f^{(-k)}$ (the antiderivatives of $f$ of order $k$),

$$f^{(-1)\prime} = f, \; f^{(-2)\prime} = f^{(-1)}, \; \ldots, \; f^{(-n)\prime} = f^{(-n+1)},$$

we conclude that $f^{(-n)}$ exists in $\mathscr{D}'(a, b)$ and is unique up to an arbitrary additive polynomial of degree $n - 1$.

**2.3 Examples** (a) Let us compute the density of charges corresponding to the dipole of the moment $+1$ located at the point $x = 0$ and oriented in a given direction $\mathbf{l} = (l_1, \ldots, l_n)$, $|\mathbf{l}| = 1$ (Fig. 9).

Figure 9




Approximately corresponding to this dipole is the charge density (see Secs. 1.1 and 1.7)

$$\frac{1}{\varepsilon}\,\delta(x - \varepsilon\mathbf{l}) - \frac{1}{\varepsilon}\,\delta(x), \qquad \varepsilon > 0,$$

Passing to the limit here as $\varepsilon \to +0$ in $\mathscr{D}'(\mathbb{R}^n)$,

$$\left(\frac{1}{\varepsilon}\,\delta(x - \varepsilon\mathbf{l}) - \frac{1}{\varepsilon}\,\delta(x), \varphi\right) = \frac{1}{\varepsilon}\,[\varphi(\varepsilon\mathbf{l}) - \varphi(0)]$$

$$\to \frac{\partial\varphi(0)}{\partial\mathbf{l}} = \left(\delta, \frac{\partial\varphi}{\partial\mathbf{l}}\right) = -\left(\frac{\partial\delta}{\partial\mathbf{l}}, \varphi\right),$$

we conclude that the desired density is equal to

$$-\frac{\partial \delta(x)}{\partial \mathbf{l}} = -(\mathbf{l}, D\delta(x)).$$

Let us now verify that the total charge of the dipole is 0:

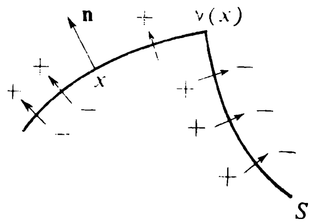
$$\left(-\frac{\partial \delta}{\partial \mathbf{l}}, 1\right) = \left(\delta, \frac{\partial 1}{\partial \mathbf{l}}\right) = (\delta, 0) = 0,$$

and that its moment is equal to 1:

$$\left(-\frac{\partial \delta}{\partial \mathbf{l}}, (x, \mathbf{l})\right) = \left(\delta, \frac{\partial (x, \mathbf{l})}{\partial \mathbf{l}}\right) = (\delta, |\mathbf{l}|) = (\delta, 1) = 1.$$

(b) A generalization of $-\frac{\partial \delta(x)}{\partial \mathbf{l}}$ is a double layer on a surface. Let $S$ be a piecewise smooth two-sided surface, $\mathbf{n}$ a normal to $S$

Figure 10




(Fig. 10) and $\nu$ a continuous function on $S$. We introduce the generalized function $-\frac{\partial}{\partial \mathbf{n}}(\nu \delta_S)$, which operates via the rule

$$\left(-\frac{\partial}{\partial \mathbf{n}}(\nu \delta_S), \varphi\right) = \int_S \nu(x) \frac{\partial \varphi(x)}{\partial \mathbf{n}}\, dS, \qquad \varphi \in \mathscr{D}.$$

Clearly

$$-\frac{\partial}{\partial \mathbf{n}}(\nu \delta_S) \in \mathscr{D}', \qquad \operatorname{supp}\left[-\frac{\partial}{\partial \mathbf{n}}(\nu \delta_S)\right] \subset S.$$

The generalized function $-\frac{\partial}{\partial \mathbf{n}}(\nu \delta_S)$ is called a *double layer* on the surface $S$. It describes the spatial density of charges corresponding to the distribution of dipoles on the surface $S$ with surface moment density $\nu(x)$, the charges oriented in the given direction of the normal $\mathbf{n}$ on $S$. (Here, the density of the double layer is defined as the weak limit of the densities corresponding to the discrete arrangement of dipoles on the surface $S$,

$$-\sum_k \frac{\partial}{\partial \mathbf{n}_k}[\nu(x_k)\, \Delta S_k \delta(x - x_k)], \qquad x_k \in S,$$

in the case of an unbounded partitioning of the surface $S$; compare Sec. 1.7.)

(c) Let a function $f(x)$ be piecewise continuously differentiable in $(a, b)$ and let $\{x_k\}$ be points in $(a, b)$ at which it or its derivative has discontinuities of the first kind (Fig. 11). Then

$$f' = f'_{\text{cl}}(x) + \sum_k [f]_{x_k} \delta(x - x_k) \tag{3.1}$$

where $f'_{\text{cl}}(x)$ is the classical derivative of the function $f(x)$, equal to $f'(x)$ when $x \neq x_k$, and is not defined at the points

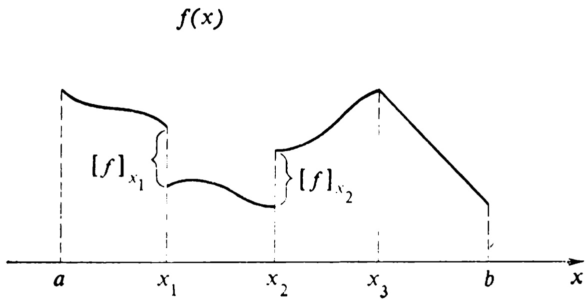



Figure 11

$\{x_k\}$; $[f]_{x_k}$ is a jump of the function $f(x)$ at the point $x_k$,

$$[f]_{x_k} = f(x_k + 0) - f(x_k - 0).$$

Indeed, for any $\varphi \in \mathcal{D}(a, b)$ we have

$$(f', \varphi) = -(f, \varphi') = -\sum_k \int_{x_k}^{x_{k+1}} f(x)\varphi'(x)\,dx$$

$$= \sum_k \int_{x_k}^{x_{k+1}} f'_{\text{cl}}(x)\varphi(x)\,dx$$

$$-\sum_k [f(x_{k+1} - 0)\varphi(x_{k+1}) - f(x_k + 0)\varphi(x_k)]$$

$$= \int f'_{\text{cl}}(x)\varphi(x)\,dx + \sum_k [f(x_k + 0) - f(x_k - 0)]\varphi(x_k)$$

$$= (f'_{\text{cl}}, \varphi) + \sum_k [f]_{x_k} (\delta(x - x_k), \varphi),$$

which completes the proof of (3.1).

In particular, if $\theta$ is the Heaviside unit function (see Sec. 0.2), then

$$\theta'(x) = \delta(x) \tag{3.2}$$

In the theory of electric circuits, the Heaviside unit function is called the *unit-step function*, and the delta function is called the *unit-impulse function*. Formula (3.2) states that the unit-impulse function is a derivative of the unit-step function.

(d) The following formulas hold true:

$$x^m \delta^{(k)}(x) = \begin{cases} 0, & k = 0, 1, \ldots, m-1, \\ (-1)^m m! \binom{m}{k} \delta^{(k-m)}(x), & k \geqslant m \end{cases} \tag{3.3}$$

Indeed,

$$\begin{aligned} (x^m \delta^{(k)}, \varphi) &= (-1)^k (x^m \varphi)^{(k)} |_{x=0} \\ &= (-1)^k \sum_{0 \leqslant j \leqslant k} \binom{j}{k} (x^m)^{(k)} \varphi^{(j-k)}(x) |_{x=0} \\ &= \begin{cases} 0, & k = 0, 1, \ldots, m-1, \\ (-1)^k m! \binom{m}{k} \varphi^{(m-k)}(0) \end{cases} \\ &= (-1)^m m! \binom{m}{k} (\delta^{(m-k)}, \varphi), \quad k \geqslant m. \end{aligned}$$

(e) The trigonometric series

$$\sum_{k=-\infty}^{\infty} a_k e^{ikx}, \qquad |a_k| \leqslant A(1 + |k|)^m \tag{3.4}$$

converges in $\mathscr{D}'$.

True enough, the series

$$\frac{a_0 x^{m+2}}{(m+2)!} + \sum_{k \neq 0} \frac{a_k}{(ik)^{m+2}} e^{ikx}$$

converges uniformly in $\mathbb{R}^1$; hence, the series which is a derivative of it of order $m + 2$ converges in $\mathscr{D}'$ and its sum determines the sum of the series (3.4) (see Sec. 2.1(a)).

(f) Let us prove the formula

$$\frac{1}{2\pi} \sum_k e^{ikx} = \sum_k \delta(x - 2k\pi) \tag{3.5}$$

To do this we expand the $2\pi$-periodic function (Fig. 12)

$$f_0(x)=\frac{x}{2}-\frac{x^2}{4\pi}, \qquad 0\leqslant x<2\pi,$$

into a Fourier series that converges uniformly in $\mathbb{R}^1$:

$$f_0(x)=\frac{\pi}{6}-\frac{1}{2\pi}\sum_{k\neq 0}\frac{1}{k^2}e^{ikx}. \tag{3.6}$$

By virtue of (e), the series (3.5) can be differentiated termwise in $\mathcal{D}'$ any number of times. As a result, we get

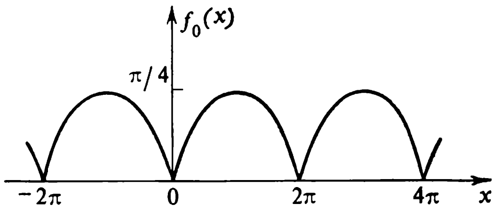
$$f'(x)=\frac{1}{2}-\frac{x}{2\pi}=-\frac{i}{2\pi}\sum_{k\neq 0}\frac{1}{k}e^{ikx}, \qquad 0\leqslant x<2\pi,$$

$$f_0''(x)=-\frac{1}{2\pi}+\sum_{k}\delta(x-2k\pi)=\frac{1}{2\pi}\sum_{k\neq 0}e^{ikx},$$

whence follows formula (3.5). In differentiating the function $f_0'(x)$ (Fig. 13), we made use of the formula (3.1).

Note that the left-hand side of (3.5) is nothing other than the Fourier series of the $2\pi$-periodic generalized function $\sum_k \delta(x-2k\pi)$, the graph of which is symbolically depicted in Fig. 14

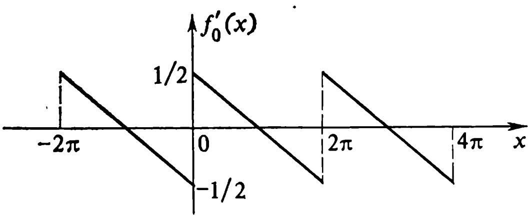
Figure 12




(see Sec. 7.2 for more details).

(g) Let $G$ be a region in $\mathbb{R}^n$ with a piecewise smooth boundary $S$ and let $\mathbf{n}=\mathbf{n}_x$ be an outer normal to $S$ at the point $x\in S$ (Fig. 15).

Figure 13




Suppose $f\in C^2(G)\cap C^1(\overline{G})$ and $f(x)=0$ outside $\overline{G}$. Then for any $\varphi\in\mathcal{D}$ we have the following Green's formula:

$$\int_G (f\nabla^2\varphi-\varphi\nabla^2 f)\,dx=\int_S\left(f\frac{\partial\varphi}{\partial\mathbf{n}}-\varphi\frac{\partial f}{\partial\mathbf{n}}\right)dS. \tag{3.7}$$

We can rewrite Green's formula (3.7) as follows in terms of the generalized functions (of a simple layer and a double layer) that were introduced in Sec. 1.7 and Sec. 2.3(b):

$$\nabla^2 f = \nabla^2_{\text{cl}} f - \frac{\partial f}{\partial \mathbf{n}} \delta_S - \frac{\partial}{\partial \mathbf{n}} (f \delta_S) \tag{3.7'}$$

where $\nabla^2_{\text{cl}} f$ is the classical Laplacian of $f$:

$$\nabla^2_{\text{cl}} f(x) = \begin{cases} \nabla^2 f(x), & x \in G, \\ 0, & x \,\overline{\in}\, \overline{G}, \\ \text{not defined}, & x \in S. \end{cases}$$

By $f$ and $\frac{\partial f}{\partial \mathbf{n}}$ on $S$ we mean the boundary values of $f$ and $\frac{\partial f}{\partial \mathbf{n}}$ on $S$ from within the domain $G$.

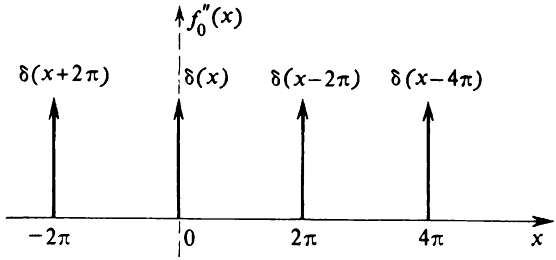



Figure 14

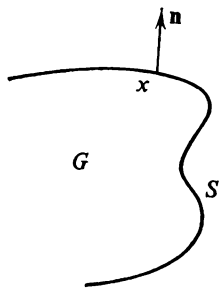



Figure 15

(h) Let us verify that the function $\frac{1}{|x|}$ in $\mathbb{R}^3$ satisfies the Poisson equation

$$\nabla^2 \frac{1}{|x|} = -4\pi\delta(x). \tag{3.8}$$

True enough, the function $\frac{1}{|x|}$ is locally integrable in $\mathbb{R}^3$ and

$$\nabla^2 \frac{1}{|x|} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\frac{1}{r}\right) = 0, \qquad x \neq 0. \tag{3.9}$$

Let $\varphi \in \mathscr{D}$, $\operatorname{supp} \varphi \subset U_R$. Then

$$\left(\nabla^2 \frac{1}{|x|}, \varphi\right) = \left(\frac{1}{|x|}, \nabla^2\varphi\right) = \int\limits_{U_R} \frac{1}{|x|} \nabla^2\varphi(x)\,dx$$
$$= \lim_{\varepsilon \to +0} \int\limits_{\varepsilon<|x|<R} \frac{1}{|x|} \nabla^2\varphi(x)\,dx.$$

Applying Green's formula (3.7) for $f = \frac{1}{|x|}$ and $G = [x\colon \varepsilon < < |x| < R]$ (Fig. 16) and taking into account (3.9), we obtain

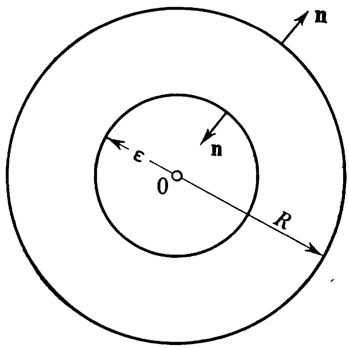


Figure 16

the formula (3.8):

$$\left(\nabla^2 \frac{1}{|x|}, \varphi\right) = \lim_{\varepsilon \to +0}\Big[ \int\limits_{\varepsilon<|x|<R} \nabla^2 \frac{1}{|x|}\,\varphi(x)\,dx$$
$$+ \Big(\int\limits_{S_\varepsilon} + \int\limits_{S_R}\Big)\left(\frac{1}{|x|}\frac{\partial\varphi}{\partial \mathbf{n}} - \varphi\frac{\partial}{\partial \mathbf{n}}\frac{1}{|x|}\right) dS\Big]$$
$$= \lim_{\varepsilon \to +0} \int\limits_{S_\varepsilon} \left(-\frac{1}{|x|}\frac{\partial\varphi}{\partial|x|} - \varphi\frac{1}{|x|^2}\right) dS$$
$$= \lim_{\varepsilon \to +0} \frac{-1}{\varepsilon^2}\int\limits_{S_\varepsilon} \varphi\,dS$$
$$= \lim_{\varepsilon \to +0} \left\{\frac{1}{\varepsilon^2}\int\limits_{S_\varepsilon} [\varphi(0) - \varphi(x)]\,dS - 4\pi\varphi(0)\right\}$$
$$= -4\pi(\delta, \varphi).$$

The equation (3.8) may be interpreted as follows: the function $\frac{1}{|x|}$ is the Newtonian (Coulomb) potential generated by the charge $+1$ at the point $x = 0$.

Similarly,

$$\nabla^2 \ln|x| = 2\pi\delta(x), \qquad n = 2,$$

$$\nabla^2 \frac{1}{|x|^{n-2}} = -(n-2)\,\sigma_n\delta(x), \qquad n \geqslant 3 \tag{3.10}$$

where $\sigma_n$ is the surface area of a unit sphere in $\mathbb{R}^n$ (see Sec. 0.6).

The function $\mathscr{E}_n(x)$, which is equal to

$$-\frac{1}{(n-2)\sigma_n|x|^{n-2}}\ (n\geqslant 3), \quad \frac{1}{2\pi}\ln|x|\ (n=2), \quad \frac{1}{2}|x|\ (n=1),$$

is termed the *fundamental solution* of the Laplace operator.

**2.4 The local structure of generalized functions** We will now prove that the local space $\mathscr{D}'(\mathcal{O})$ is a (smallest) extension of the space $\mathscr{L}^\infty(\mathcal{O})$ such that, in it, differentiation is always possible.

Theorem *Let $f \in \mathscr{D}'(\mathcal{O})$ and let the open set $\mathcal{O}' \Subset \mathcal{O}$. Then there exist a function $g \in \mathscr{L}^\infty(\mathcal{O}')$ and an integer $m \geqslant 0$ such that*§

$$f(x) = D_1^m \ldots D_n^m g(x), \qquad x \in \mathcal{O}'. \tag{4.1}$$

*Proof*. According to the theorem of Sec. 1.3, there exist numbers $K$ and $k$ such that the following inequality holds:

$$|(f, \varphi)| \leqslant K\|\varphi\|_{C^k(\overline{\mathcal{O}'})}, \qquad \varphi \in \mathscr{D}(\mathcal{O}'). \tag{4.2}$$

Since, for $\psi \in \mathscr{D}(\mathcal{O}')$, $\psi(x) = \int\limits_{-\infty}^{x} D_j\psi\, dx_j'$, it follows that

$$\max_{x\in\overline{\mathcal{O}'}} |\psi(x)| \leqslant d \max_{x\in\overline{\mathcal{O}'}} |D_j\psi(x)|$$

where $d$ is the diameter of $\mathcal{O}'$. Therefore, applying this inequality a sufficient number of times, we obtain from (4.2) the inequality

$$|(f, \varphi)| \leqslant C \max_{x\in\overline{\mathcal{O}'}} |D_1^k \ldots D_n^k \varphi(x)|, \qquad \varphi \in \mathscr{D}(\mathcal{O}'). \tag{4.3}$$

Furthermore, for $\psi \in \mathscr{D}(\mathcal{O}')$

$$\psi(x) = \int\limits_{-\infty}^{x_1} \ldots \int\limits_{-\infty}^{x_n} \frac{\partial^n \psi(y)}{\partial y_1 \ldots \partial y_n}\, dy_1 \ldots dy_n$$

§ The derivative in (4.1) is to be understood in the sense of generalized functions.

and therefore

$$|\psi(x)| \leqslant \int_{\mathcal{O}'} |D_1 \ldots D_n \psi(y)| \, dy.$$

From this fact and from (4.3) there follows the inequality (for $m = k + 1$)

$$|(f, \varphi)| \leqslant C \int_{\mathcal{O}'} |D_1^m \ldots D_n^m \varphi(x)| \, dx, \qquad \varphi \in \mathcal{D}(\mathcal{O}'). \tag{4.4}$$

From the Hahn-Banach theorem it follows that the continuous linear functional $f^*$:

$$\chi(x) = (-1)^{mn} D_1^m \ldots D_n^m \varphi(x) \to (f^*, \chi) = (f, \varphi), \qquad \varphi \in \mathcal{D}(\mathcal{O}') \tag{4.5}$$

admits of an extension to a continuous linear functional on $\mathcal{L}^1(\mathcal{O}')$ with norm $\leqslant C$ by virtue of the inequality (4.4):

$$|(f^*, \chi)| = |(f, \varphi)| \leqslant C \, \|\chi\|_{\mathcal{L}^1(\mathcal{O}')}.$$

By a theorem of F. Riesz, there exists a function $g \in \mathcal{L}^\infty(\mathcal{O}')$ with norm $\|g\|_{\mathcal{L}^\infty(\mathcal{O}')} \leqslant C$ such that

$$(f^*, \chi) = (-1)^{mn} \int_{\mathcal{O}'} g(x) \chi(x) \, dx.$$

From this and from (4.5) we derive, for all $\varphi \in \mathcal{D}(\mathcal{O}')$,

$$(f, \varphi) = (-1)^{mn} \int_{\mathcal{O}'} g(x) D_1^m \ldots D_n^m \varphi(x) \, dx$$

$$= (D_1^m \ldots D_n^m g, \varphi),$$

which is equivalent to (4.1). This completes the proof of the theorem.

Corollary *Under the conditions of the theorem,*

$$f(x) = D_1^{m+1} \ldots D_n^{m+1} g_1(x) \quad \textit{in} \quad \mathcal{O}', \qquad g_1 \in C(\overline{\mathcal{O}'}). \tag{4.6}$$

The representation (4.6) follows from the representation (4.1) if the function $g(x)$ is continued via zero onto the whole of $\mathbb{R}^n$ and if we put

$$g_1(x) = \int_{-\infty}^{x_1} \ldots \int_{-\infty}^{x_n} g(y) \, dy_1 \ldots dy_n.$$

**2.5 Generalized functions with compact support** We introduce convergence on a set of functions $C^\infty(\mathcal{O})$:

$$\varphi_k \to 0, \quad k \to \infty \quad \text{in} \quad C^\infty(\mathcal{O})$$

$$\text{if } D^\alpha \varphi_k(x) \underset{x \in \mathcal{O}'}{\Longrightarrow} 0, \quad k \to \infty \text{ for all } \alpha \text{ and } \mathcal{O}' \Subset \mathcal{O}.$$

From this definition it follows that convergence in $\mathcal{D}(\mathcal{O})$ implies convergence in $C^\infty(\mathcal{O})$, but not vice versa.

Suppose a generalized function $f$ in $\mathcal{D}'(\mathcal{O})$ has compact support in $\mathcal{O}$, $\operatorname{supp} f = K \Subset \mathcal{O}$. Suppose $\eta \in \mathcal{D}(\mathcal{O})$, $\eta(x) = 1$ in the neighbourhood of $K$ (see Sec. 1.2). We will construct a function $\tilde{f}$ on $C^\infty(\mathcal{O})$ via the rule

$$(\tilde{f}, \varphi) = (f, \eta\varphi), \qquad \varphi \in C^\infty(\mathcal{O}). \tag{5.1}$$

Clearly, $\tilde{f}$ is a linear functional on $C^\infty(\mathcal{O})$. Furthermore, since the operation $\varphi \to \eta\varphi$ is continuous from $C^\infty(\mathcal{O})$ into $\mathcal{D}(\mathcal{O})$, it follows that $\tilde{f}$ is a continuous functional on $C^\infty(\mathcal{O})$. The functional $f$ is an extension of the function $f$ from $\mathcal{D}(\mathcal{O})$ onto $C^\infty(\mathcal{O})$, since for $\varphi \in \mathcal{D}(\mathcal{O})$

$$(\tilde{f}, \varphi) = (f, \eta\varphi) = (\eta f, \varphi) = (f, \varphi)$$

by virtue of the equality (10.2) of Sec. 1.

We will show that there is a unique linear and continuous extension of $f$ onto $C^\infty(\mathcal{O})$ (in particular, the extension (5.1) does not depend on the auxiliary function $\eta$). Let $\tilde{\tilde{f}}$ be another such extension of $f$. We introduce a sequence of functions $\{\eta_k\}$ in $\mathcal{D}(\mathcal{O})$ such that $\eta_k(x) = 1$, $x \in \mathcal{O}_k$ ($\mathcal{O}_1 \Subset \mathcal{O}_2 \Subset \ldots$, $\mathcal{O} = \bigcup_k \mathcal{O}_k$), so that $\eta_k \to 1$, $k \to \infty$ in $C^\infty(\mathcal{O})$. Therefore, for any $\varphi \in C^\infty(\mathcal{O})$ we will have $\eta_k\varphi \to \varphi$, $k \to \infty$ in $C^\infty(\mathcal{O})$. Hence,

$$(\tilde{f}, \varphi) = (\tilde{f}, \lim_{k\to\infty} \eta_k\varphi) = \lim_{k\to\infty} (\tilde{f}, \eta_k\varphi) = \lim_{k\to\infty} (f, \eta_k\varphi)$$

$$= \lim_{k\to\infty} (\tilde{\tilde{f}}, \eta_k\varphi) = (\tilde{\tilde{f}}, \lim_{k\to\infty} \eta_k\varphi) = (\tilde{\tilde{f}}, \varphi), \quad \varphi \in C^\infty(\mathcal{O}),$$

so that $f = \tilde{\tilde{f}}$.

We have thus proved the necessity of the conditions in the following theorem.

Theorem *For a generalized function $f$ in $\mathcal{D}'(\mathcal{O})$ to have compact support in $\mathcal{O}$, it is necessary and sufficient that it admit of a linear and continuous extension onto $C^\infty(\mathcal{O})$.*

*Proof of sufficiency.* Suppose $f \in \mathcal{D}'(\mathcal{O})$ admits of a linear and continuous extension of $\tilde{f}$ onto $C^\infty(\mathcal{O})$. If $f$ did not possess com

pact support in $\mathscr{D}$, then it would be possible to indicate a sequence of functions $\{\varphi_k\}$ in $\mathscr{D}(\mathcal{O})$ such that $\operatorname{supp}\varphi_k \subset \mathcal{O}\setminus\overline{\mathcal{O}}_k$ ($\mathcal{O}_1 \Subset \mathcal{O}_2 \Subset \ldots$, $\bigcup_k \mathcal{O}_k = \mathcal{O}$) and $(f, \varphi_k) = 1$. On the other hand $\varphi_k \to 0$, $k \to \infty$ in $C^\infty(\mathcal{O})$, and therefore $(\tilde{f}, \varphi_k) \to 0$, $k \to \infty$. But $(\tilde{f}, \varphi_k) = (f, \varphi_k) = 1$, which is contradictory. The proof is complete.

Let $f$ be a generalized function with compact support in $\mathcal{O}$. Then, by virtue of (5.1), we have

$$(f, \varphi) = (f, \eta\varphi), \qquad \varphi \in \mathscr{D}(\mathcal{O}).$$

Since $\eta \in \mathscr{D}(\mathcal{O})$ and $\operatorname{supp}\eta \Subset \mathcal{O}$, it follows that $\eta \in \mathscr{D}(\mathcal{O}')$ for some $\mathcal{O}' \Subset \mathcal{O}$. Therefore $\eta\varphi \in \mathscr{D}(\mathcal{O}')$ for all $\varphi \in \mathscr{D}(\mathcal{O})$. By the theorem of Sec. 1.3 there exist numbers $K = K(\mathcal{O}')$ and $m = m(\mathcal{O}')$ such that the following inequality holds:

$$|(f, \varphi)| = |(f, \eta\varphi)| \leqslant K \|\eta\varphi\|_{C^m(\overline{\mathcal{O}'})}, \quad \varphi \in \mathscr{D}(\mathcal{O}),$$

whence immediately follows the inequality

$$|(f, \varphi)| \leqslant C \|\varphi\|_{C^m(\overline{\mathcal{O}'})}, \qquad \varphi \in \mathscr{D}(\mathcal{O}). \tag{5.2}$$

Inequality (5.2) implies the following assertion: any generalized function with compact support in $\mathcal{O}$ has a finite order in $\mathcal{O}$ (see Sec. 1.3).

We denote by $\mathscr{E}'$ the collection of generalized functions with compact support in $\mathbb{R}^n$. It has thus been proved that $\mathscr{E}' = C^\infty(\mathbb{R}^n)'$.

**2.6 Generalized functions with point support** Generalized functions whose supports consist of isolated points admit of explicit description. This is given by the following theorem.

Theorem *If the support of a generalized function $f \in \mathscr{D}'$ consists of a unique point $x = 0$, then it is uniquely representable in the form*

$$f(x) = \sum_{|\alpha| \leqslant N} c_\alpha D^\alpha \delta(x) \tag{6.1}$$

*where $N$ is the order of $f$, and $c_\alpha$ are certain constants.*

*Proof.* Suppose $\eta \in \mathscr{D}(U_1)$, $\eta(x) = 1$, $|x| \leqslant 1/2$. Then for any $\varepsilon > 0$ we have $f = \eta\left(\frac{x}{\varepsilon}\right) f$ and, hence, for any $\varphi \in \mathscr{D}$

$$\begin{aligned}(f, \varphi) &= \left(\eta\left(\frac{x}{\varepsilon}\right) f, \varphi\right) \\ &= \left(f, \eta\left(\frac{x}{\varepsilon}\right)(\varphi - \varphi_N)\right) + \left(f, \eta\left(\frac{x}{\varepsilon}\right)\varphi_N\right)\end{aligned} \tag{6.2}$$

where

$$\varphi_N(x) = \sum_{|\alpha| \leqslant N} \frac{D^\alpha \varphi(0)}{\alpha!} x^\alpha.$$

Since $\eta\left(\frac{x}{\varepsilon}\right)(\varphi - \varphi_N) \in \mathscr{D}(U_\varepsilon)$, by applying the inequality (5.2) we get

$$\left|\left(f, \eta\left(\frac{x}{\varepsilon}\right)(\varphi-\varphi_N)\right)\right|$$

$$\leqslant C \left\| \eta\left(\frac{x}{\varepsilon}\right)(\varphi - \varphi_N) \right\|_{C^N(\bar{U}_\varepsilon)}$$

$$= C \max_{\substack{|x| \leqslant \varepsilon \\ |\alpha| \leqslant N}} \left| D^\alpha \left\{ \eta\left(\frac{x}{\varepsilon}\right) [\varphi(x) - \varphi_N(x)] \right\} \right|$$

$$\leqslant C \max_{\substack{|x| \leqslant \varepsilon \\ |\alpha| \leqslant N}} \sum_{\beta \leqslant \alpha} \binom{\beta}{\alpha} \left| D^\beta \eta\left(\frac{x}{\varepsilon}\right) D^{\alpha-\beta}[\varphi(x) - \varphi_N(x)] \right|$$

$$\leqslant C' \max_{|\alpha| \leqslant N} \sum_{\beta \leqslant \alpha} \varepsilon^{-|\beta|} \varepsilon^{N-|\alpha-\beta|} \varepsilon \leqslant C' \max_{|\alpha| \leqslant N} \varepsilon^{N-|\alpha|+1} = C''\varepsilon.$$

In the right-hand member of (6.2), let $\varepsilon \to +0$. By virtue of the resulting estimate, the first term will tend to zero. But the second term does not at all depend on $\varepsilon$ and is equal to $(\tilde{f}, \varphi_N)$, where $\tilde{f}$ is the extension of $f$ onto $C^\infty(\mathbb{C})$ (see Sec. 2.5). Therefore the equation (6.2) takes the form

$$(f, \varphi) = (\tilde{f}, \varphi_N) = \sum_{|\alpha| \leqslant N} \frac{D^\alpha \varphi(0)}{\alpha!} (\tilde{f}, x^\alpha).$$

Now set

$$c_\alpha = \frac{(-1)^{|\alpha|}}{\alpha!} (\tilde{f}, x^\alpha)$$

and we get the representation (6.1):

$$(f, \varphi) = \sum_{|\alpha| \leqslant N} (-1)^{|\alpha|} c_\alpha D^\alpha \varphi(0) = \sum_{|\alpha| \leqslant N} c_\alpha (D^\alpha \delta, \varphi),$$

$$\varphi \in \mathscr{D}.$$

We now prove the uniqueness of the representation (6.1). If there is another such representation

$$f(x) = \sum_{|\alpha| \leqslant N} c'_\alpha D^\alpha \delta(x),$$

then by subtracting we obtain

$$0=\sum_{|\alpha|\leqslant N}(c'_\alpha-c_\alpha)\,D^\alpha\delta(x),$$

whence

$$0=\sum_{|\alpha|\leqslant N}(c'_\alpha-c_\alpha)\,(D^\alpha\,\delta,\;x^k)$$

$$=\sum_{|\alpha|\leqslant N}(c'_\alpha-c_\alpha)\,(-1)^{|\alpha|}D^\alpha x^k|_{x=0}=(-1)^{|k|}\,k!(c'_k-c_k),$$

that is, $c'_k=c_k$, and the proof of the theorem is complete.

*Example.* The general solution of the equation

$$x^m u\,(x)=0 \tag{6.3}$$

in the class $\mathscr{D}'\,(\mathbb{R}^1)$ is given by the formula

$$u(x)=\sum_{0\leqslant k\leqslant m-1}c_k\,\delta^{(k)}(x) \tag{6.4}$$

where $c_k$ are arbitrary constants.

Indeed, if $u\in\mathscr{D}'$ is a solution of the equation (6.3), then either $u=0$ or supp $u$ coincides with the point $x=0$. By the theorem that has just been proved,

$$u\,(x)=\sum_{0\leqslant k\leqslant N}c_k\delta^{(k)}\,(x) \tag{6.5}$$

for certain numbers $c_k$ and integer $N\geqslant 0$. Taking into account (3.3) and substituting (6.5) into (6.3), we have

$$0=(-1)^m m!\sum_{m\leqslant k\leqslant N}\binom{m}{k}c_k\delta^{(m-k)}\,(x),$$

whence it follows that $c_k=0$, $k\geqslant m$. Thus, in the representation (6.5) we can assume $N=m-1$, and the formula (6.4) is proved. It remains to note that the right-hand side of (6.4) satisfies equation (6.3) for arbitrary constants $c_k$, $k=0,1,\ldots,m-1$.

## 3 Direct Product of Generalized Functions

**3.1 The definition of a direct product** Let $f(x)$ and $g\,(y)$ be locally summable functions in the open sets $\mathcal{O}_1\subset\mathbb{R}^n$ and $\mathcal{O}_2\subset\mathbb{R}^m$ respectively. The function $f\,(x)\,g\,(y)$ will also be locally summable in $\mathcal{O}_1\times\mathcal{O}_2$. It defines a (regular) generalized function

$f(x)\, g(y) = g(y)\, f(x)$ in $\mathcal{D}'(\mathcal{O}_1 \times \mathcal{O}_2)$ operating on the basic functions $\varphi(x, y)$ in $\mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2)$ via the formulas

$$\begin{aligned}(f(x)\, g(y), \varphi) &= \int\limits_{\mathcal{O}_1 \times \mathcal{O}_2} f(x)\, g(y)\, \varphi(x, y)\, dx\, dy \\ &= \int\limits_{\mathcal{O}_1} f(x) \int\limits_{\mathcal{O}_2} g(y)\, \varphi(x, y)\, dy\, dx \\ &= \int\limits_{\mathcal{O}_1 \times \mathcal{O}_2} g(y)\, f(x)\, \varphi(x, y)\, dx\, dy \\ &= \int\limits_{\mathcal{O}_2} g(y) \int\limits_{\mathcal{O}_1} f(x)\, \varphi(x, y)\, dx\, dy,\end{aligned}$$

that is,

$$(f(x)\, g(y), \varphi) = (f(x), (g(y), \varphi(x, y))), \tag{1.1}$$

$$(g(y)\, f(x), \varphi) = (g(y), (f(x), \varphi(x, y))). \tag{1.1'}$$

These equations express the Fubini theorem on the coincidence of iterated integrals and a multiple integral.

We take (1.1) and (1.1′) as the starting equalities for defining the *direct products* $f(x) \times g(y)$ and $g(y) \times f(x)$ of the generalized functions $f \in \mathcal{D}'(\mathcal{O}_1)$ and $g \in \mathcal{D}'(\mathcal{O}_2)$:

$$(f(x) \times g(y), \varphi) = (f(x), (g(y), \varphi(x, y))), \tag{1.2}$$

$$(g(y) \times f(x), \varphi) = (g(y), (f(x), \varphi(x, y))), \tag{1.2'}$$

where $\varphi \in \mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2)$.

Now let us verify that this definition is proper, that is, that the right-hand side of (1.2) defines a continuous linear functional on $\mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2)$.

Since for every $x \in \mathcal{O}_1$ the function $\varphi(x, y) \in \mathcal{D}(\mathcal{O}_2)$ while $g \in \mathcal{D}'(\mathcal{O}_2)$, it follows that the function

$$\psi(x) = (g(y), \varphi(x, y)), \qquad \varphi \in \mathcal{D}'(\mathcal{O}_1 \times \mathcal{O}_2), \tag{1.3}$$

is defined in $\mathcal{O}_1$. We now prove the following lemma.

Lemma *Let* $\mathcal{O}' \Subset \mathcal{O}_1 \times \mathcal{O}_2$ *and* $g \in \mathcal{D}'(\mathcal{O}_2)$. *Then there exist an open set* $\widetilde{\mathcal{O}}_1 = \widetilde{\mathcal{O}}_1(\mathcal{O}') \Subset \mathcal{O}_1$ *and numbers* $C = C(\mathcal{O}', g) \geqslant 0$, *with*

*integer* $m = m(\mathcal{O}', g) \geqslant 0$, *such that*

$$\psi \in \mathcal{D}(\tilde{\mathcal{O}}_1) \quad \text{if} \quad \varphi \in \mathcal{D}(\mathcal{O}'); \tag{1.4}$$

$$D^\alpha \psi(x) = (g(y),\ D_x^\alpha \varphi(x,y)), \tag{1.5}$$

$$\varphi \in \mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2);$$

$$|D^\alpha \psi(x)| \leqslant C \max_{\substack{(x,y) \in \bar{\mathcal{O}}' \\ |\beta| \leqslant m}} |D_x^\alpha D_y^\beta \varphi(x, y)|, \tag{1.6}$$

$$\varphi \in \mathcal{D}(\mathcal{O}'), \qquad x \in \mathcal{O}_1.$$

*Proof.* We will prove that the function $\psi(x)$ defined by (1.3) is finite in $\mathcal{O}_1$. Since $\operatorname{supp} \varphi \subset \mathcal{O}' \Subset \mathcal{O}_1 \times \mathcal{O}_2$, it follows that

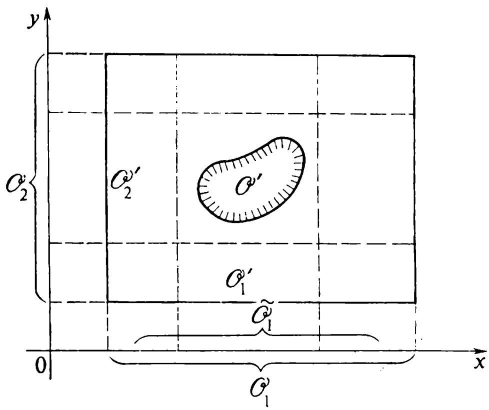



Figure 17

there are open sets $\mathcal{O}_1' \Subset \mathcal{O}_1$ and $\mathcal{O}_2' \Subset \mathcal{O}_2$ such that $\mathcal{O}' \Subset \Subset \mathcal{O}_1' \times \mathcal{O}_2'$ (Fig. 17). Therefore, if $x \in \mathcal{O}_1 \setminus \mathcal{O}_1'$, then $\varphi(x, y) = 0$ for all $y \in \mathcal{O}_2$ and for this reason $\psi(x) = (g, 0) = 0$, so that $\psi(x) = 0$ outside $\mathcal{O}_1'$. Choosing an open set $\tilde{\mathcal{O}}_1$ such that $\mathcal{O}_1' \Subset \tilde{\mathcal{O}}_1 \Subset \mathcal{O}_1$, we conclude that $\operatorname{supp} \psi \subset \tilde{\mathcal{O}}_1$.

Now let us prove that $\psi$ is continuous in $\mathcal{O}_1$. We fix an arbitrary point $x \in \mathcal{O}_1$ and let $x_k \to x$, $x_k \in \mathcal{O}_1$. Then

$$\varphi(x_k, y) \to \varphi(x, y), \qquad x_k \to x \quad \text{in} \quad \mathcal{D}(\mathcal{O}_2). \tag{1.7}$$

Indeed, $\operatorname{supp} \varphi(x_k, y) \subset \mathcal{O}_2^{\bullet} \Subset \mathcal{O}_2$ and

$$D_y^\alpha(x_k, y) \overset{y \in \mathcal{O}_2}{\Longrightarrow} D_y^\alpha \varphi(x, y), \qquad x_k \to x.$$

Taking advantage of the continuity of the functional $g$, we obtain from (1.3) and (1.7), as $x_k \to x$,

$$\psi(x_k) = (g(y), \varphi(x_k, y)) \to (g(y), \varphi(x, y)) = \psi(x),$$

which is to say that the function $\psi$ is continuous at an arbitrary point $x$. Thus $\psi \in C(\mathcal{O}_1)$.

Now we will prove that $\psi \in C^\infty(\mathcal{O}_1)$ and that the differentiation formula (1.5) holds true. Let $e_1 = (1, 0, \ldots, 0)$. Then for every $x \in \mathcal{O}_1$

$$\chi_h(y) = \frac{1}{h}[\varphi(x + he_1, y) - \varphi(x, y)] \to \frac{\partial \varphi(x, y)}{\partial x_1}, \qquad (1.8)$$
$$h \to 0 \quad in \quad \mathcal{D}(\mathcal{O}_2).$$

Indeed, supp $\chi_h \subset \mathcal{O}_2' \Subset \mathcal{O}_2$ for sufficiently small $h$ and

$$D^\alpha \chi_h(y) \overset{y \in \mathcal{O}_2}{\rightrightarrows} D_y^\alpha \frac{\partial \varphi(x, y)}{\partial x_1}, \qquad h \to 0.$$

Since $g \in \mathcal{D}'(\mathcal{O}_2)$, then using (1.3) and (1.8), we get

$$\frac{\psi(x + he_1) - \varphi(x)}{h} = \frac{1}{h}[g(y), \varphi(x + he_1, y)) - (g(y), \varphi(x, y))]$$
$$= \left(g(y), \frac{\varphi(x + he_1, y) - \varphi(x, y)}{h}\right)$$
$$= (g, \chi_h) \to \left(g(y), \frac{\partial \varphi(x, y)}{\partial x_1}\right),$$

whence follows the truth of formula (1.5) for $\alpha = (1, 0, \ldots, 0)$ and, hence, for all first derivatives

$$\frac{\partial \psi(x)}{\partial x_j} = \left(g(y), \frac{\partial \varphi(x, y)}{\partial x_j}\right), \qquad j = 1, 2, \ldots, n.$$

Again applying the same reasoning to this formula, we see that (1.5) holds true for all second derivatives, and so forth; hence for all derivatives And since the function $D^\alpha \varphi(x, y)$ also belongs to $\mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2)$, we conclude from (1.5) (by what has been proved) that $D^\alpha \psi(x)$ is a continuous function in $\mathcal{O}_1$ for all $\alpha$ so that $\psi \in C^\infty(\mathcal{O}_1)$.

From this and also from the fact that supp $\psi \subset \tilde{\mathcal{O}}_1$ we conclude that $\psi \in \mathcal{D}(\mathcal{O}_1)$ and (1.4) is proved.

Let us prove the inequality (1.6). Let $x \in \mathcal{O}_1$. Then, by what has already been proved, $D_x^\alpha \varphi(x, y) \in \mathcal{D}(\mathcal{O}_2')$, $\mathcal{O}_2' \Subset \mathcal{O}_2$. By

the theorem of Sec. 1.3 there exist a number $C \geqslant 0$ and an integer $m \geqslant 0$ that depend solely on $g$ and $\mathcal{O}_2'$, such that

$$|D^{\alpha}\psi(x)| = |(g(y), D_x^{\alpha}\varphi(x, y))| \leqslant C \max_{\substack{y \in \overline{\mathcal{O}}_2' \\ |\beta| \leqslant m}} |D_y^{\beta} D_x^{\alpha}\varphi(x, y)|,$$

$x \in \mathcal{O}_1$, whence follows inequality (1.6). The proof of the lemma is complete.

Corollary *The operation*

$$\varphi(x, y) \to \psi(x) = (g(y), \varphi(x, y))$$

*is linear and continuous from* $\mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2)$ *into* $\mathcal{D}(\mathcal{O}_1)$.

Indeed, the linearity of the operation is obvious. Furthermore, if $\varphi \in \mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2)$, then, by the lemma, $\psi \in \mathcal{D}(\mathcal{O}_1)$ so that this operation carries $\mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2)$ into $\mathcal{D}(\mathcal{O}_1)$. Let us prove that it is continuous. Let $\varphi_k \to 0$, $k \to \infty$ in $\mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2)$. Then

$$\operatorname{supp} \varphi_k \subset \mathcal{O}' \Subset \mathcal{O}_1 \times \mathcal{O}_2, \quad D_x^{\alpha} D_y^{\beta} \varphi_k(x, y) \overset{(x, y)}{\rightrightarrows} 0, \quad k \to \infty.$$

From this and from (1.4) and (1.6) we derive the following for the sequence $\varphi_k(x) = (g(y), \varphi_k(x, y))$, $k = 1, 2, \ldots$:

$$\operatorname{supp} \psi_k \subset \widetilde{\mathcal{O}}_1 \Subset \mathcal{O}_1, \qquad D^{\alpha}\psi_k(x) \overset{x}{\rightrightarrows} 0, \quad k \to \infty,$$

so that $\psi_k \to 0$, $k \to \infty$ in $\mathcal{D}'(\mathcal{O}_1)$.

Let us return to formula (1.2), the definition of the direct product $f(x) \times g(y)$. By the corollary to the lemma that was just proved, the operation $\varphi \to \psi$ is linear and continuous from $\mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2)$ to $\mathcal{D}(\mathcal{O}_1)$ and, hence, the right-hand side of (1.2), which is equal to $(f, \psi)$, defines a linear and continuous functional on $\mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2)$ so that $f(x) \times g(y) \in \mathcal{D}'(\mathcal{O}_1 \times \mathcal{O}_2)$.

Similarly, using (1.2′), we can prove that $g(y) \times f(x) \in \mathcal{D}'(\mathcal{O}_1 \times \mathcal{O}_2)$.

### 3.2 The properties of a direct product

(a) *Commutativity of a direct product.*

$$f(x) \times g(y) = g(y) \times f(x), \qquad (2.1)$$
$$f \in \mathcal{D}'(\mathcal{O}_1), \qquad g \in \mathcal{D}'(\mathcal{O}_2).$$

Indeed, on the basic functions $\varphi \in \mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2)$ of the form

$$\varphi(x, y) = \sum_{1 \leqslant i \leqslant N} u_i(x) v_i(y), \qquad (2.2)$$
$$u_i \in \mathcal{D}(\mathcal{O}_1), \qquad v_i \in \mathcal{D}(\mathcal{O}_2),$$

the equality (2.1) follows from the definitions (1.2) and (1.2'):

$$(f(x)\times g(y),\varphi)=\sum_{1\leqslant i\leqslant N}(f,u_i)(g,v_i)=(g(y)\times f(x),\varphi).$$

In order to extend (2.1) to any basic functions in $\mathscr{D}(\mathcal{O}_1\times\mathcal{O}_2)$ let us prove a lemma that states that the set of basic functions of the form (2.2) is dense in $\mathscr{D}(\mathcal{O}_1\times\mathcal{O}_2)$.

Lemma *For any $\varphi\in\mathscr{D}(\mathcal{O}_1\times\mathcal{O}_2)$ there exists a sequence of basic functions in $\mathscr{D}(\mathcal{O}_1\times\mathcal{O}_2)$,*

$$\varphi_k(x,y)=\sum_{1\leqslant i\leqslant N_k}u_{ik}(x)v_{ik}(y),\quad u_{ik}\in\mathscr{D}(\mathcal{O}_1),\quad v_{ik}\in\mathscr{D}(\mathcal{O}_2),\quad k=1,2,\ldots,$$

*that converges to $\varphi$ in $\mathscr{D}(\mathcal{O}_1\times\mathcal{O}_2)$.*

*Proof*. Suppose supp $\varphi\Subset\tilde{\mathcal{O}}_1\times\tilde{\mathcal{O}}_2\Subset\mathcal{O}_1'\times\mathcal{O}_2'\Subset\mathcal{O}_1\times\mathcal{O}_2$. By the Weierstrass theorem, there exists a sequence of polynomials $P_k(x,y)$, $k=1,2,\ldots$, such that

$$|D^\alpha\varphi(x,y)-D^\alpha P_k(x,y)|<\frac{1}{k},\qquad|\alpha|\leqslant k,$$
$$(x,y)\in\bar{\mathcal{O}}_1'\times\bar{\mathcal{O}}_2'.\tag{2.3}$$

Suppose $\xi(x)\in\mathscr{D}(\mathcal{O}_1')$, $\xi(x)=1$, $x\in\tilde{\mathcal{O}}_1$; $\eta(y)\in\mathscr{D}(\mathcal{O}_2')$, $\eta(y)=1$, $y\in\tilde{\mathcal{O}}_2$. Then the sequence of functions

$$\varphi_k(x,y)=\xi(x)\eta(y)P_k(x,y),\quad k=1,2,\ldots,$$

is the required sequence. Indeed, supp $\varphi_k\subset\mathcal{O}_1'\times\mathcal{O}_2'\Subset\mathcal{O}_1\times\mathcal{O}_2$ and for all $k\geqslant|\alpha|$, by virtue of (2.3), we have

$$|D^\alpha\varphi(x,y)-D^\alpha\varphi_k(x,y)|\leqslant\begin{cases}\frac{1}{k}\text{ if }(x,y)\in\tilde{\mathcal{O}}_1\times\tilde{\mathcal{O}}_2,\\ \frac{c_\alpha}{k}\text{ if }(x,y)\in\mathcal{O}_1'\times\mathcal{O}_2'\setminus(\tilde{\mathcal{O}}_1\times\tilde{\mathcal{O}}_2),\end{cases}$$

for certain $c_\alpha$ estimated in terms of max $|D^\beta\xi|$ and max $|D^\beta\eta|$, $\beta\leqslant\alpha$. And that means that $\varphi_k\to\varphi$, $k\to\infty$ in $\mathscr{D}(\mathcal{O}_1\times\mathcal{O}_2)$. The proof of the lemma is complete.

Let $\varphi$ be an arbitrary basic function in $\mathscr{D}(\mathcal{O}_1\times\mathcal{O}_2)$. By the lemma, there exists a sequence $\{\varphi_k\}$ of functions of the form (2.2) that converges to $k$ in $\mathscr{D}(\mathcal{O}_1\times\mathcal{O}_2)$. From this, taking advantage of the continuity, on $\mathscr{D}(\mathcal{O}_1\times\mathcal{O}_2)$, of the functionals $f(x)\times g(y)$ and $g(y)\times f(x)$ (see Sec. 3.1) and also taking advantage of the

above-proved equation (2.1) on functions of the form (2.2), we obtain (2.1) in the general case:

$$(f(x) \times g(y), \varphi) = \lim_{k\to\infty} (f(x) \times g(y), \varphi_k)$$
$$= \lim_{k\to\infty} (g(y) \times f(x), \varphi_k) = (g(y) \times f(x), \varphi)$$

(b) *Associativity of a direct product.* If $f \in \mathscr{D}'(\mathscr{O}_1)$, $g \in \mathscr{D}'(\mathscr{O}_2)$ and $h \in \mathscr{D}'(\mathscr{O}_3)$, then

$$[f(x) \times g(y)] \times h(z) = f(x) \times [g(y) \times h(z)]. \tag{2.4}$$

Indeed, if $\varphi \in \mathscr{D}(\mathscr{O}_1 \times \mathscr{O}_2 \times \mathscr{O}_3)$, then

$$([f(x) \times g(y)] \times h(z), \varphi) = (f(x) \times g(y), (h(z), \varphi(x, y, z)))$$
$$= (f(x), (g(y), (h(z), \varphi(x, y, z))))$$
$$= (f(x), (g(y) \times h(z), \varphi(x, y, z)))$$
$$= (f(x) \times [g(y) \times h(z)], \varphi).$$

Henceforth, taking into account the commutativity and associativity of the operation of a direct product, we will write

$$(f \times g) \times h = f \times g \times h.$$

*Example.* $\delta(x) = \delta(x_1) \times \delta(x_2) \times \ldots \times \delta(x_n)$.

(c) If $g \in \mathscr{D}'(\mathscr{O}_2)$, the operation $f \to f \times g$ is linear and continuous from $\mathscr{D}'(\mathscr{O}_1)$ into $\mathscr{D}'(\mathscr{O}_1 \times \mathscr{O}_2)$.

The linearity of the operation is obvious. Let us prove continuity. Suppose $f_k \to 0$, $k \to \infty$ in $\mathscr{D}(\mathscr{O}_1)$. Then for all $\varphi \in \mathscr{D}(\mathscr{O}_1 \times \mathscr{O}_2)$ we have

$$(f_k(x) \times g(y), \varphi) = (f_k(x), (g(y), \varphi(x, y))) =$$
$$= (f_k, \psi) \to 0, \qquad k \to \infty.$$

That is, $f_k(x) \times g(y) \to 0$, $k \to \infty$ in $\mathscr{D}'(\mathscr{O}_1 \times \mathscr{O}_2)$. Here we made use of the fact that $\psi \in \mathscr{D}(\mathscr{O}_1)$ by virtue of the lemma of Sec. 3.1.

(d) The following formula holds:

$$\operatorname{supp}(f \times g) = \operatorname{supp} f \times \operatorname{supp} g, \tag{2.5}$$
$$f \in \mathscr{D}'(\mathscr{O}_1), \qquad g \in \mathscr{D}'(\mathscr{O}_2).$$

Corollary *If $f(x) \times 1(y) = 0$ in $\mathscr{O}_1 \times \mathscr{O}_2$, then $f(x) = 0$ in $\mathscr{O}_1$.*

Indeed, suppose $(x_0, y_0) \in \operatorname{supp} f \times \operatorname{supp} g$ and $U(x_0, y_0)$ is the neighbourhood of the point $(x_0, y_0)$ lying in $\mathscr{O}_1 \times \mathscr{O}_2$. There exist neighbourhoods $U_1$ and $U_2$ of the points $x_0$ and $y_0$

respectively such that $U_1 \times U_2 \subset U(x_0, y_0)$. From the definition of the support of a generalized function (see Sec. 1.5(b)) it follows that there are functions $\varphi_1 \in \mathcal{D}(U_1)$ and $\varphi_2 \in \mathcal{D}(U_2)$ such that $(f, \varphi_1) \neq 0$ and $(g, \varphi_2) \neq 0$. And so $(f \times g, \varphi_1\varphi_2) =$ $= (f, \varphi_1)(g, \varphi_2) \neq 0$. From this fact, due to the arbitrariness

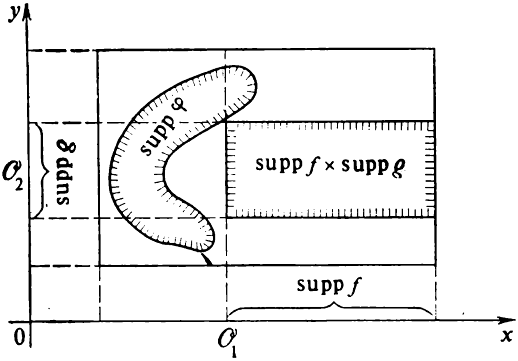



Figure 18

of the neighbourhood $U(x_0, y_0)$, it follows that $(x_0, y_0) \in$ $\in \operatorname{supp}(f \times g)$ so that

$$\operatorname{supp} f \times \operatorname{supp} g \subset \operatorname{supp}(f \times g). \tag{2.6}$$

Let us now prove the converse inclusion. Take a basic function $\varphi$ in $\mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2)$ such that

$$\operatorname{supp} \varphi \subset (\mathcal{O}_1 \times \mathcal{O}_2) \setminus (\operatorname{supp} f \times \operatorname{supp} g)$$

(Fig. 18). Then there is a neighbourhood $U$ of the set $\operatorname{supp} f$ such that for every $x \in U$, $\operatorname{supp} \varphi(x, y) \subset \mathcal{O}_2 \setminus \operatorname{supp} g$. Therefore (see Sec. 1.5(a))

$$\psi(x) = (g(y), \varphi(x, y)) = 0, \qquad x \in U,$$

and, hence, $\operatorname{supp} \psi \cap \operatorname{supp} f = \varnothing$, and so

$$(f \times g, \varphi) = (f, \psi) = 0.$$

Thus the zero set $\mathcal{O}_{f \times g}$ contains $(\mathcal{O}_1 \times \mathcal{O}_2) \setminus (\operatorname{supp} f \times \operatorname{supp} g)$ and, hence, the following inclusion holds true:

$$\operatorname{supp}(f \times g) \subset \operatorname{supp} f \times \operatorname{supp} g.$$

This, together with the converse inclusion (2.6), proves the equality (2.5).

(e) The following formulas, which are readily verifiable, hold true: if $f \in \mathscr{D}'(\mathcal{O}_1)$ and $g \in \mathscr{D}'(\mathcal{O}_2)$, then

$$D_x^\alpha D_y^\beta\,[f(x) \times g(y)] = D^\alpha f(x) \times D^\beta g(y), \tag{2.7}$$

$$a(x)\,b(y)\,[f(x) \times g(y)] = [a(x)\,f(x)] \times [b(y)\,g(y)], \tag{2.8}$$

$$(f \times g)(x + x_0,\ y + y_0) = f(x + x_0) \times g(y + y_0). \tag{2.9}$$

**3.3 Some applications** We will say that a generalized function $F(x, y)$ in $\mathscr{D}'(\mathcal{O}_1 \times \mathcal{O}_2)$ does not depend on the variables $y$ if it can be represented in the form

$$F(x, y) = f(x) \times 1(y), \qquad f \in \mathscr{D}'(\mathcal{O}_1) \tag{3.1}$$

(and then $F \in \mathscr{D}'(\mathcal{O}_1 \times \mathbb{R}^m)$). The generalized function $f(x) \times 1(y) = 1(y) \times f(x)$ acts on the basic functions $\varphi$ in $\mathscr{D}(\mathcal{O}_1 \times \mathbb{R}^m)$ via the rule

$$(f(x) \times 1(y), \varphi) = \left(f(x), \int \varphi(x, y)\,dy\right)$$
$$= (1(y) \times f(x), \varphi) = \int (f(x), \varphi(x, y))\,dy.$$

We have thus obtained the equality

$$\left(f(x), \int \varphi(x, y)\,dy\right) = \int (f(x), \varphi(x, y))\,dy \tag{3.2}$$

which holds for all $f \in \mathscr{D}'(\mathcal{O}_1)$ and $\psi \in \mathscr{D}(\mathcal{O}_1 \times \mathbb{R}^m)$.

The formula (3.2) may be regarded as a peculiar kind of generalization of the Fubini theorem.

Suppose $F \in \mathscr{D}'(\mathcal{O} \times (a, b))$. The following three statements are equivalent:

(1) $F(x, y)$ does not depend on the variable $y$;

(2) $F(x, y)$ is invariant in $\mathcal{O} \times (a, b)$ with respect to translations along $y$, that is,

$$F(x, y + h) = F(x, y); \qquad a < y, \qquad y + h < b; \tag{3.3}$$

(3) $F(x, y)$ satisfies the following equation in $\mathcal{O} \times (a, b)$:

$$\frac{\partial F(x,y)}{\partial y} = 0 \tag{3.4}$$

Corollary *If* $f \in \mathscr{D}'(\mathcal{O})$ *and* $\frac{\partial f}{\partial x_j} = 0$, $j = 1, \ldots, n$, *in* $\mathcal{O}$, *then* $f =$ *constant in* $\mathcal{O}$; *if* $f$ *is invariant with respect to translations along all arguments in* $\mathcal{O}$, *then* $f =$ *constant in* $\mathcal{O}$.

*Proof*. (1) $\rightarrow$ (2). It follows from (3.1) by virtue of (2.9).

(2) $\rightarrow$ (3). Passing to the limit in $\frac{F(x, y+h) - F(x, y)}{h} = 0$ as $h \rightarrow 0$ in $\mathscr{D}'(\mathcal{O} \times (a, b))$, we conclude that $F$ satisfies the equation (3.4).

(3) $\rightarrow$ (1). Let $F$ satisfy the equation (3.4) in $\mathcal{O} \times (a, b)$. Then, proceeding as in Sec. 2.2, for any $\varphi$ in $\mathscr{D}(\mathcal{O} \times (a, b))$ we obtain the representation

$$\varphi(x, y) = \frac{\partial \psi(x, y)}{\partial y} + \omega_\varepsilon(y - y_0) \int \varphi(x, \xi)\, d\xi, \tag{3.5}$$

where $y_0 \in (a, b)$, $\varepsilon < \min(y_0 - a, b - y_0)$ and

$$\psi(x, y) = \int_{-\infty}^{y} \left[ \varphi(x, y') - \omega_\varepsilon(y' - y_0) \int \varphi(x, \xi)\, d\xi \right] dy'$$

$$\in \mathscr{D}(\mathcal{O} \times (a, b)).$$

By introducing the generalized function $f(x)$ taken from $\mathscr{D}'(\mathcal{O})$, which function acts on the basic functions $\chi$ taken from $\mathscr{D}(\mathcal{O})$ via the rule

$$(f, \chi) = (F(x, y), \omega_\varepsilon(y - y_0)\, \chi(x)),$$

and by taking into account (3.4), we obtain from (3.5)

$$\begin{aligned}(F, \varphi) &= \left(F(x, y), \frac{\partial \psi(x, y)}{\partial y} + \omega_\varepsilon(y - y_0) \int \varphi(x, \xi)\, d\xi\right)\\ &= \left(f(x), \int \varphi(x, \xi)\, d\xi\right).\end{aligned}$$

That is, $F(x, y) = f(x) \times 1(y)$, which is what we set out to prove.

The proof is similar for the following assertion (compare Sec. 2.6). Suppose $F \in \mathscr{D}'(\mathcal{O} \times \mathbb{R}^1)$. The equation

$$yu(x, y) = F(x, y) \tag{3.6}$$

is always solvable and its general solution is of the form

$$(u, \varphi) = (F, \psi) + (f(x) \times \delta(y), \varphi), \quad \varphi \in \mathscr{D}(\mathcal{O} \times \mathbb{R}^1) \tag{3.7}$$

where $f$ is an arbitrary generalized function in $\mathcal{D}'(\mathcal{O})$,

$$\psi(x, y) = \frac{1}{y}[\varphi(x, y) - \eta(y)\varphi(x, 0)], \tag{3.8}$$

$\eta(y)$ is an arbitrary function in $\mathcal{D}(\mathbb{R}^1)$ equal to 1 in the neighbourhood of $y = 0$.

Indeed, since the operation $\varphi \to \psi$ given by (3.8) is linear and continuous from $\mathcal{D}(\mathcal{O} \times \mathbb{R}^1)$ to $\mathcal{D}(\mathcal{O} \times \mathbb{R}^1)$, the right-hand side of (3.7) is a generalized function in $\mathcal{D}'(\mathcal{O} \times \mathbb{R}^1)$ and the first term $(F, \psi)$ satisfies equation (3.6) while the second term, the generalized function $f(x) \times \delta(y)$, satisfies the homogeneous equation

$$yu(x, y) = 0 \tag{3.9}$$

which corresponds to the equation (3.6).

It remains to prove that $f(x) \times \delta(y)$, $f \in \mathcal{D}'(\mathcal{O})$, is the general solution of the equation (3.9) in $\mathcal{D}'(\mathcal{O} \times \mathbb{R}^1)$. Suppose $u(x, y)$ is a solution of the equation (3.9) in $\mathcal{D}'(\mathcal{O} \times \mathbb{R}^1)$. Then, by virtue of (3.8),

$$\varphi(x, y) = y\psi(x, y) + \eta(y)\varphi(x, 0), \qquad \psi \in \mathcal{D}(\mathcal{O} \times \mathbb{R}^1)$$

and therefore

$$\begin{aligned}(u, \varphi) &= (u, y\psi) + (u, \eta(y)\varphi(x, 0)) \\ &= (u, \eta(y)\varphi(x, 0)).\end{aligned} \tag{3.10}$$

By introducing the generalized function $f(x)$, taken from $\mathcal{D}'(\mathcal{O})$, that acts on the basic functions $\chi$ in $\mathcal{D}(\mathcal{O})$ via the rule

$$(f, \chi) = (u(x, y), \eta(y)\chi(x)),$$

we obtain from (3.10)

$$(u, \varphi) = (f, \varphi(x, 0)) = (f(x) \times \delta(y), \varphi), \qquad \varphi \in \mathcal{D}(\mathcal{O} \times \mathbb{R}^1),$$

that is, $u(x, y) = f(x) \times \delta(y)$, which is what we set out to prove.

**3.4 Generalized functions that are smooth with respect to some of the variables** Suppose $f(x, y)$ is a generalized function in $\mathcal{D}'(\mathcal{O}_1 \times \mathcal{O}_2)$ and $\varphi(x)$ is a basic function in $\mathcal{D}(\mathcal{O}_1)$. We introduce the generalized function $f_\varphi(y)$ in $\mathcal{D}'(\mathcal{O}_2)$ via the

formula

$$(f_\varphi, \psi) = (f, \varphi(x)\psi(y)), \qquad \psi \in \mathscr{D}(\mathcal{O}_2).$$

From this definition there follows the differentiation formula

$$D^\alpha f_\varphi(y) = (D_y^\alpha f)_\varphi(y). \tag{4.1}$$

Indeed

$$((D_y^\alpha f)_\varphi, \psi) = (D_y^\alpha f, \psi\varphi) = (-1)^{|\alpha|}(f, \varphi D^\alpha \psi)$$
$$= (-1)^{|\alpha|}(f_\varphi, D^\alpha\psi) = (D^\alpha f_\varphi, \psi),$$
$$\psi \in \mathscr{D}(\mathcal{O}_2).$$

We will say that the generalized function $f(x, y)$ taken from $\mathscr{D}'(\mathcal{O}_1 \times \mathcal{O}_2)$ belongs to the class $C^p(\mathcal{O}_2)$ with respect to $y$, $p = 0, 1, \ldots$, if for any $\varphi \in \mathscr{D}(\mathcal{O}_1)$ the generalized function $f_\varphi \in C^p(\mathcal{O}_2)$; but if $f_\varphi \in C^p(\overline{\mathcal{O}}_2)$, we then say that $f \in C^p(\overline{\mathcal{O}}_2)$ with respect to $y$ (compare Sec. 1.6).

Suppose $f \in C(\mathcal{O}_2)$ with respect to $y$. It then follows that for every $y \in \mathcal{O}_2$ there exists a restriction $f_y(x)$ in $\mathscr{D}'(\mathcal{O}_1)$ of the generalized function $f(x, y)$, and

$$f_\varphi(y) = (f_y, \varphi), \qquad \varphi \in \mathscr{D}(\mathcal{O}_1), \quad y \in \mathcal{O}_2. \tag{4.2}$$

Indeed, for a fixed $y_0 \in \mathcal{O}_2$, $\varphi \to f_\varphi(y_0)$ is a linear functional on $\mathscr{D}(\mathcal{O}_1)$. Let us now prove that it is continuous. To do this, note that for all sufficiently large $k \geqslant N(y_0)$, the functional

$$\varphi \to (f_\varphi(y), \omega_{1/k}(y - y_0)),$$

where $\omega_{1/k}$ is the "cap" (see Sec. 1.2), belongs to $\mathscr{D}'(\mathcal{O}_1)$. But by virtue of (7.6), Sec. 1,

$$(f_\varphi(y), \omega_{1/k}(y - y_0)) \to (f_\varphi(y), \delta(y - y_0)) = f_\varphi(y_0),$$
$$k \to \infty.$$

By the theorem on the completeness of the space $\mathscr{D}'(\mathcal{O}_1)$ we conclude therefrom that the functional $f_\varphi(y_0)$ belongs to $\mathscr{D}'(\mathcal{O}_1)$. Denoting it by $f_{y_0}$, we obtain (4.2).

Taking into account formula (4.1), we obtain, from (4.2),

$$D^\alpha(f_y, \varphi) = ((D_y^\alpha f)_y, \varphi), \qquad \varphi \in \mathscr{D}(\mathcal{O}_1), \quad y \in \mathcal{O}_2. \tag{4.3}$$

*Example.* Suppose a generalized function $F$ does not depend on the variable $y$, $F(x, y) = f(x) \times 1(y)$, $f \in \mathscr{D}'(\mathcal{O}_1)$ (see Sec. 3.3). Then $F \in C^\infty(\mathbb{R}^m)$ with respect to $y$ and $F_y(x) = f(x)$.

Recall that the class $C_0(\mathcal{O})$, which is defined in Sec. 0.5, consists of functions continuous and finite in $\mathcal{O}$. We introduce convergence thus: $f_k \to 0$, $k \to \infty$ in $C_0(\mathcal{O})$ if

$$\operatorname{supp} f_k \subset \mathcal{O}' \Subset \mathcal{O} \quad \text{and} \quad f_k(x) \overset{x\in\mathcal{O}}{\rightrightarrows} 0, \quad k \to \infty.$$

Lemma *If $f \in C(\mathcal{O}_2)$ with respect to $y$, then the operation*

$$\chi \to (f_y, \chi(x, y))$$

*is continuous from $\mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2)$ into $C_0(\mathcal{O}_2)$.*

*Proof*. Let $\chi \in \mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2)$. Then $\operatorname{supp} \chi \subset \mathcal{O}_1' \times \mathcal{O}_2'$, where $\mathcal{O}_1' \Subset \mathcal{O}_1$ and $\mathcal{O}_2' \Subset \mathcal{O}_2$. Put $\psi(y) = (f_y, \chi(x, y))$. We have $\operatorname{supp} \psi \subset \subset \mathcal{O}_2' \Subset \mathcal{O}_2$. We now prove that $\psi \in C(\mathcal{O}_2)$. Let $y_0$ be an arbitrary point in $\mathcal{O}_2$ and let $y_k \to y_0$, $k \to \infty$.
Then

$$|\psi(y_k) - \psi(y_0)| \leqslant |(f_{y_k}, \chi(x, y_0)) - (f_{y_0}, \chi(x, y_0))| + |(f_{y_k}, \chi(x, y_k) - \chi(x, y_0))|. \tag{4.4}$$

The first summand on the right of (4.4) tends to 0 as $k \to \infty$ by virtue of the continuity of the function $(f_y, \chi(x, y_0))$, and the second summand, by the weak boundedness of the set $\{f_{y_k}\} \subset \mathcal{D}'(\mathcal{O}_1)$ and by virtue of the fact that

$$\chi(x, y_k) - \chi(x, y_0) \to 0, \quad k \to \infty \quad \text{in} \quad \mathcal{D}(\mathcal{O}_1)$$

via the lemma of Sec. 1.4. Thus $\psi \in C(\mathcal{O}_2)$ and for this reason $\psi \in C_0(\mathcal{O}_2)$.

Let $\chi_k \to 0$, $k \to \infty$ in $\mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2)$. Then $\operatorname{supp} \chi_k \subset \mathcal{O}_1' \times \mathcal{O}_2'$ where $\mathcal{O}_1' \Subset \mathcal{O}_1$ and $\mathcal{O}_2' \Subset \mathcal{O}_2$. Putting $\psi_k(y) = (f_y, \chi_k(x, y))$, we have $\operatorname{supp} \psi_k \subset \mathcal{O}_2' \Subset \mathcal{O}_2$. Furthermore, the set of generalized functions $\{f_y, y \in \mathcal{O}_2'\}$ in $\mathcal{D}'(\mathcal{O}_1)$ is weakly bounded. For this reason, by applying the inequality (3.1) of Sec. 1 (see also the corollary to the lemma of Sec. 1.4), for certain $K > 0$, $m \geqslant 0$, and for all $y \in \mathcal{O}_2$ we obtain

$$|\psi_k(y)| = |(f_y, \chi_k(x, y))| \leqslant K \|\chi_k(x, y)\|_{C^m(\overline{\mathcal{O}}_1')}$$
$$\leqslant K \|\chi_k\|_{C^m(\overline{\mathcal{O}}_1' \times \overline{\mathcal{O}}_2')} \to 0, \quad k \to \infty.$$

This completes the proof of the lemma.

Let us now prove the following formula: if $f \in C(\mathcal{O}_2)$ with respect to $y$, then

$$(f, \chi) = \int (f_y, \chi(x, y))\, dy, \qquad \chi \in \mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2) \tag{4.5}$$

or $f(x, y) = f_y(x)$.

Indeed, by virtue of (4.2) the equality (4.5) holds true on the basic functions $\chi$ of the form $\sum \varphi(x) \psi(y)$, where $\varphi \in \mathcal{D}(\mathcal{O}_1)$ and $\psi \in \mathcal{D}(\mathcal{O}_2)$:

$$\begin{aligned}\left(f, \sum \varphi(x) \psi(y)\right) &= \sum \int f_\varphi(y) \psi(y)\, dy \\ &= \sum \int (f_y, \varphi) \psi(y)\, dy \\ &= \int \left(f_y, \sum \varphi(x) \psi(y)\right) dy. \end{aligned} \tag{4.6}$$

But the set of such functions is dense in $\mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2)$ (see Sec. 3.2 (a)) and, besides, by the lemma,

$$\left(f_y, \sum \varphi(x) \psi(y)\right) \to (f_y, \chi(x, y)) \quad \text{in} \quad C_0(\mathcal{O}_2)$$

if $\sum \varphi(x) \psi(y) \to \chi(x, y)$ in $\mathcal{D}(\mathcal{O}_1 \times \mathcal{O}_2)$. It is from this and from (4.6) that the formula (4.5) follows.

## 4 The Convolution of Generalized Functions

**4.1 The definition of a convolution.** Let $f$ and $g$ be locally summable functions in $\mathbb{R}^n$. If the integral $\int f(y) g(x-y)\, dy$ exists for almost all $x \in \mathbb{R}^n$ and defines a locally summable function in $\mathbb{R}^n$, then it is called the *convolution* of the functions $f$ and $g$ and is symbolized as $f * g$ so that

$$\begin{aligned}(f * g)(x) &= \int f(y) g(x-y)\, dy \\ &= \int g(y) f(x-y)\, dy = (g * f)(x). \end{aligned} \tag{1.1}$$

We note two cases where the convolution $f * g$ definitely exists.

(a) Let $f \in \mathscr{L}^1_{\text{loc}}$, $g \in \mathscr{L}^1_{\text{loc}}$, $\operatorname{supp} f \subset A$, $\operatorname{supp} g \subset B$, and the sets $A$ and $B$ are such that for any $R > 0$ the set

$$T_R = [(x, y)\colon x \in A,\ y \in B,\ |x + y| \leqslant R]$$

is bounded in $\mathbb{R}^{2n}$ (Fig. 19). Then $f * g \in \mathscr{L}^1_{\text{loc}}$.

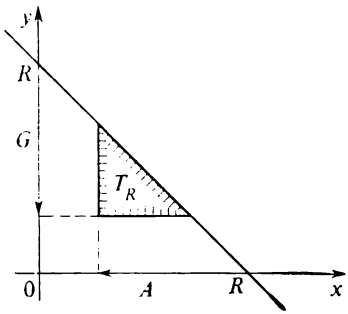



Figure 19

Indeed, using the Fubini theorem, we have, for all $R > 0$,

$$\int\limits_{|x|<R} |f*g|\, dx \leqslant \int\limits_{|x|<R}\int |f(y)|\,|g(x-y)|\, dy\, dx$$

$$\leqslant \int\limits_{T_R} |f(y)|\,|g(\xi)|\, dy\, d\xi < \infty.$$

In particular, if $f$ or $g$ is finite, then $T_R$ is bounded.

(b) Let $f \in \mathscr{L}^p$ and $g \in \mathscr{L}^q$ if $\frac{1}{p} + \frac{1}{q} \geqslant 1$. Then $f * g \in \mathscr{L}^r$, where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$.

Indeed, choosing the numbers $\alpha \geqslant 0$, $\beta \geqslant 0$, $s \geqslant 1$ and $t \geqslant 1$ such that

$$\frac{1}{r} + \frac{1}{s} + \frac{1}{t} = 1, \quad \alpha r = p = (1-\alpha)\, s, \quad \beta r = q = (1-\beta)\, t,$$

and then

$$p + \frac{pr}{s} = r = q + \frac{qr}{t},$$

and making use of the Hölder inequality and the Fubini theorem, we obtain the required estimate

$$\|f*g\|^r_{\mathscr{L}^r} = \int \left| \int f(y)\, g(x-y)\, dy \right|^r dx$$

$$\leqslant \int \left[ \int |f(y)|^\alpha |g(x-y)|^\beta\, |f(y)|^{1-\alpha} |g(x-y)|^{1-\beta}\, dy \right]^r dx$$

$$\leqslant \int \int |f(y)|^{\alpha r} |g(x-y)|^{\beta r}\, dy \left[ \int |f(y)|^{(1-\alpha)s}\, dy \right]^{\frac{r}{s}}$$

$$\times \left[ \int |g(x-y)|^{(1-\beta)t}\, dy \right]^{\frac{r}{t}} dx$$

$$\leqslant \|f\|^r_{\mathscr{L}^p} \|q\|^r_{\mathscr{L}^q}.$$

The convolution $f*g$ defines a regular functional on $\mathscr{D}(\mathbb{R}^n)$ via the rule

$$\begin{aligned}(f*g, \varphi) &= \int (f*g)(x)\, \varphi(x)\, dx \\ &= \int \varphi(x) \int f(y)\, g(x-y)\, dy\, dx \\ &= \int f(y) \int g(x-y)\, \varphi(x)\, dx\, dy \\ &= \int f(y) \int g(\xi)\, \varphi(y+\xi)\, d\xi\, dy.\end{aligned}$$

That is

$$(f*g, \varphi) = \int f(x)\, g(y)\, \varphi(x+y)\, dx\, dy, \qquad \varphi \in \mathscr{D}. \tag{1.2}$$

(In deriving (1.2) we made constant use of the Fubini theorem.)

We will say that the sequence $\{\eta_k\}$ of functions taken from $\mathscr{D}(\mathbb{R}^n)$ converges to 1 in $\mathbb{R}^n$ if (a) for any compact $K$ there is a number $N = N(K)$ such that $\eta_k(x) = 1$, $x \in K$, $k \geqslant N$, and (b) the functions $\{\eta_k\}$ are uniformly bounded together with all derivatives, $|D^\alpha \eta_k(x)| < c_\alpha$, $x \in \mathbb{R}^n$, $k = 1, 2, \ldots$.

Note that there always exist such sequences, for example:

$$\eta_k(x) = \eta\left(\frac{x}{k}\right), \quad \text{where} \quad \eta \in \mathscr{D},\ \eta(x) = 1, \quad |x| < 1.$$

Let us now prove that the equality (1.2) can be rewritten as

$$(f*g, \varphi) = \lim_{k\to\infty} (f(x) \times g(y), \eta_k(x; y)\, \varphi(x+y)), \qquad \varphi \in \mathscr{D}, \tag{1.3}$$

where $\{\eta_k\}$ is any sequence of functions taken from $\mathcal{D}(\mathbb{R}^{2n})$ that converges to 1 in $\mathbb{R}^{2n}$.

Indeed, the function $c_0 \mid f(x) g(y) \varphi(x+y) \mid$ is summable on $\mathbb{R}^{2n}$ and dominates the sequence of functions $f(x) g(y) \times \times \eta_k(x; y) \varphi(x+y)$, $k = 1, 2, \ldots$, that converges almost everywhere in $\mathbb{R}^{2n}$ to the function $f(x) g(y) \varphi(x+y)$. From this, making use of the Lebesgue theorem, we obtain

$$\int f(x) g(y) \varphi(x+y)\, dx\, dy = \lim_{k\to\infty} \int f(x) g(y) \eta_k(x; y) \varphi(x+y)\, dx\, dy$$

which is equivalent to (1.3) by virtue of (1.2).

Proceeding from the equalities (1.3) and (1.2), we define a convolution of generalized functions as follows. Suppose $f$ and $g$ taken from $\mathcal{D}'(\mathbb{R}^n)$ are such that their direct product $f(x) \times g(y)$ admits of an extension $(f(x) \times g(y), \varphi(x+y))$ to functions of the form $\varphi(x+y)$, where $\varphi$ is any function in $\mathcal{D}(\mathbb{R}^n)$, in the following sense: no matter what sequence $\{\eta_k\}$ there is of functions from $\mathcal{D}(\mathbb{R}^{2n})$, which sequence converges to 1 in $\mathbb{R}^{2n}$, there exists a limit to the numerical sequence,

$$\lim_{k\to\infty} (f(x) \times g(y), \eta_k(x, y) \varphi(x+y)) = (f(x) \times g(y), \varphi(x+y)),$$

and that limit does not depend on the sequence $\{\eta_k\}$. Note that for every $k$ the function $\eta_k(x; y) \varphi(x+y) \in \mathcal{D}(\mathbb{R}^{2n})$ and so our numerical sequence is defined.

The *convolution* $f * g$ is the functional

$$\begin{aligned}(f * g, \varphi) &= (f(x) \times g(y), \varphi(x+y)) \\ &= \lim_{k\to\infty} (f(x) \times g(y), \eta_k(x; y) \varphi(x+y)), \qquad (1.4) \\ &\qquad \varphi \in \mathcal{D}(\mathbb{R}^n).\end{aligned}$$

Let us prove that the functional $f*g$ belongs to $\mathcal{D}'(\mathbb{R}^n)$, that is, it is a generalized function. For this purpose, it suffices, by virtue of the theorem on the completeness of the space $\mathcal{D}'$ (see Sec. 1.4), to establish the continuity of the linear functionals

$$(f(x) \times g(y), \eta_k(x; y) \varphi(x+y)), \qquad k = 1, 2, \ldots, \qquad (1.5)$$

on $\mathcal{D}(\mathbb{R}^n)$. Let $\varphi_\nu \to 0$, $\nu \to \infty$ in $\mathcal{D}(\mathbb{R}^n)$. Then

$$\eta_k(x; y) \varphi_\nu(x+y) \to 0, \qquad \nu \to \infty \quad \text{in} \quad \mathcal{D}(\mathbb{R}^{2n})$$

since $\eta_k \in \mathscr{D}(\mathbb{R}^{2n})$. From this, since the functional $f(x) \times g(y)$ on $\mathscr{D}(\mathbb{R}^{2n})$ (see Sec. 3.1) is continuous, we obtain

$$(f(x) \times g(y),\ \eta_k(x;\ y)\,\varphi_\nu(x+y)) \to 0, \qquad \nu \to \infty$$

and this completes the proof of the continuity of the functionals (1.5) on $\mathscr{D}(\mathbb{R}^n)$.

Note that since $\varphi(x+y)$ does not belong to $\mathscr{D}(\mathbb{R}^{2n})$ (it is not finite in $\mathbb{R}^{2n}$), the right-hand side of (1.4) does not exist for simply any pairs of generalized functions $f$ and $g$ and, thus, the convolution does not always exist.

The convolution of any number of generalized functions is defined in similar fashion. For example, let $f$, $g$ and $h$ be generalized functions taken from $\mathscr{D}'(\mathbb{R}^n)$ and let $\{\eta_k\}$ be the sequence of functions from $\mathscr{D}(\mathbb{R}^{3n})$ that converges to 1 in $\mathbb{R}^{3n}$. The convolution $f * g * h$ is the functional

$$\begin{aligned}(f*g*h,\ \varphi) &= (f(x) \times g(y) \times h(z),\ \varphi(x+y+z))\\ &= \lim_{k\to\infty} (f(x) \times g(y) \times h(z),\ \eta_k(x;\ y;\ z)\,\varphi(x+y+z)), \qquad (1.6)\\ &\qquad\qquad \varphi \in \mathscr{D}(\mathbb{R}^n),\end{aligned}$$

if that functional exists.

## 4.2 The properties of a convolution

(a) *Commutativity of a convolution.* If the convolution $f*g$ exists, then so also does the convolution $g*f$, and they are equal:

$$f*g = g*f.$$

This statement follows from the definition of a convolution (see Sec. 4.1) and from the commutativity of a direct product (see Sec. 3.2(a)):

$$\begin{aligned}(f*g,\varphi) &= \lim_{k\to\infty} (f(x)\times g(y),\ \eta_k(x;y)\,\varphi(x+y))\\ &= \lim_{k\to\infty} (g(y)\times f(x),\ \eta_k(x;y)\,\varphi(x+y))\\ &= (g*f,\varphi),\\ &\qquad \varphi\in\mathscr{D}.\end{aligned}$$

Similarly, from the definition (1.6) we obtain

$$f*g*h = f*h*g = h*f*g = \ \ldots \text{ and so forth.}$$

(b) *Convolution with the delta function.* The convolution of any generalized function $f$ in $\mathscr{D}'$ with the $\delta$ function exists and is equal to $f$:

$$f * \delta = \delta * f = f. \tag{2.1}$$

True enough, let $\varphi \in \mathscr{D}(\mathbb{R}^n)$ and let $\{\eta_k\}$ be a sequence of functions taken from $\mathscr{D}(\mathbb{R}^{2n})$ that converges to 1 in $\mathbb{R}^{2n}$. Then

$$\eta_k(x; 0)\,\varphi(x) \to \varphi, \qquad k \to \infty \quad \text{in} \quad \mathscr{D}(\mathbb{R}^n)$$

and so

$$\begin{aligned}(f * \delta, \varphi) &= \lim_{k\to\infty} (f(x) \times \delta(y), \eta_k(x; y)\,\varphi(x+y)) \\ &= \lim_{k\to\infty} (f(x), \eta_k(x; 0)\,\varphi(x)) \\ &= (f, \varphi),\end{aligned}$$

which is what we set out to prove.

*Remark.* The meaning of the formula $f = f * \delta$ is that any generalized function $f$ may be expanded in terms of $\delta$ functions, which, formally, is often written thus:

$$f(x) = \int f(\xi)\,\delta(x-\xi)\,d\xi.$$

It is precisely this formula which one has in mind when we say that every material body consists of mass points, every source consists of source points, and so on (compare Sec. 1.1).

(c) *The shift of a convolution.* If the convolution $f * g$ exists, then so also does the convolution $f(x+h) * g(x)$ for all $h \in \mathbb{R}^n$, and

$$f(x+h) * g(x) = (f * g)(x+h). \tag{2.2}$$

That is, the operations of shift and convolution commute; in other words, the convolution operator

$$f \to f * g$$

is a translation invariant operator.

Indeed, let $\{\eta_k\}$ be a sequence of functions in $\mathscr{D}(\mathbb{R}^{2n})$ that converges to 1 in $\mathbb{R}^{2n}$. Then for any $h \in \mathbb{R}^n$,

$$\eta_k(x-h; y) \to 1, \qquad k \to \infty \quad \text{in} \quad \mathbb{R}^{2n}.$$

Now, using the definition of a shift (see Sec. 1.9) and of a convolution (see Sec. 4.1), we obtain, for all $\varphi \in \mathscr{D}(\mathbb{R}^n)$,

$$
\begin{aligned}
((f * g)(x+h), \varphi) &= (f * g, \varphi(x-h)) \\
&= \lim_{k \to \infty} (f(x) \times g(y), \eta_k(x-h; y)\,\varphi(x-h+y)) \\
&= \lim_{k \to \infty} (f(x+h) \times g(y), \eta_k(x; y)\,\varphi(x+y)) \\
&= (f(x+h) * g, \varphi),
\end{aligned}
$$

which is what we set out to prove. Here we made use of formula (2.9) of Sec. 3 for the shift of a direct product.

(d) *The reflection of a convolution.* If the convolution $f * g$ exists, then so also does the convolution $f(-x) * g(-x)$, and

$$f(-x) * g(-x) = (f * g)(-x). \tag{2.3}$$

The proof is similar to that of (c).

(e) *Differentiating a convolution.* If the convolution $f * g$ exists, then there exist the convolutions $D^\alpha f * g$ and $f * D^\alpha g$, and we have

$$D^\alpha f * g = D^\alpha (f * g) = f * D^\alpha g. \tag{2.4}$$

It will suffice to prove this assertion for the first derivatives $D_j$, $j = 1, \ldots, n$. Let $\varphi \in \mathscr{D}(\mathbb{R}^n)$ and let $\{\eta_k\}$ be a sequence of functions taken from $\mathscr{D}(\mathbb{R}^{2n})$ that converges to 1 in $\mathbb{R}^{2n}$. Then the sequence $\{\eta_k + D_j\eta_k\}$ also converges to 1 in $\mathbb{R}^{2n}$. From this fact, taking advantage of the existence of the convolution $f * g$ (see Sec. 4.1), we obtain the following chain of equalities (with $\eta_k \equiv \eta_k(x; y)$):

$$
\begin{aligned}
(D_j(f * g), \varphi) &= -(f * g, D_j\varphi) \\
&= -\lim_{k \to \infty} \left(f(x) \times g(y), \eta_k \frac{\partial \varphi(x+y)}{\partial x_j}\right) \\
&= -\lim_{k \to \infty} \left(f(x) \times g(y), \frac{\partial}{\partial x_j}[\eta_k \varphi(x+y)] - \varphi(x+y)\frac{\partial \eta_k}{\partial x_j}\right) \\
&= \lim_{k \to \infty} \left(\frac{\partial}{\partial x_j}[f(x) \times g(y)], \eta_k \varphi(x+y)\right) \\
&\quad + \lim_{k \to \infty} \left(f(x) \times g(y), \left[\eta_k + \frac{\partial \eta_k}{\partial x_j}\right]\varphi(x+y)\right) \\
&\quad - \lim_{k \to \infty} (f(x) \times g(y), \eta_k \varphi(x+y)) \\
&= \lim_{k \to \infty} (D_j f(x) \times g(y), \eta_k \varphi(x+y)) + (f * g, \varphi) - (f * g, \varphi) \\
&= (D_j f * g, \varphi),
\end{aligned}
$$

whence follows the first equality (2.4) for $D_j$. The second one of (2.4) follows from the first and from the commutativity of a convolution:

$$D_j(f*g) = D_j(g*f) = D_jg*f = f*D_jg.$$

From (2.1) and (2.4) there follow the equalities

$$D^\alpha f = D^\alpha\delta*f = \delta*D^\alpha f, \qquad f\in\mathscr{D}'.$$

Note that the existence of the convolutions $D^\alpha f*g$ and $f*D^\alpha g$ for $|\alpha|\geqslant 1$ is not yet enough for the existence of the convolution $f*g$ and for the truth of (2.4). For example,

$$\theta'*1=\delta*1=1, \quad \text{but} \quad \theta*1'=\theta*0=0.$$

(f) The operation $f\to f*g$ is linear on the set of those generalized functions $f$ for which the convolution with $g$ exists.

This property of a convolution follows directly from the definition of a convolution (1.4) and from the linearity of the operation $f\to f\times g$ (see Sec. 3.2(c)).

In passing we may note that the operation $f\to f*g$ is not, generally speaking, continuous from $\mathscr{D}'$ into $\mathscr{D}'$, as the following example shows:

$$\delta(x-k)\to 0, \; k\to\infty \text{ in } \mathscr{D}'; \text{ however, } 1*\delta(x-k)=1.$$

(g) If the convolution $f*g$ exists, then

$$\operatorname{supp}(f*g)\subset\overline{\operatorname{supp} f+\operatorname{supp} g}. \tag{2.5}$$

Indeed, suppose $\{\eta_k\}$ is a sequence of functions taken from $\mathscr{D}(\mathbb{R}^{2n})$ that converges to 1 in $\mathbb{R}^{2n}$ and $\varphi\in\mathscr{D}(\mathbb{R}^n)$ is such that

$$\operatorname{supp}\varphi\cap\overline{\operatorname{supp} f+\operatorname{supp} g}=\varnothing. \tag{2.6}$$

Since $\operatorname{supp}(f\times g)=\operatorname{supp} f\times\operatorname{supp} g$ (see Sec. 3.2(d)), we conclude that

$$\operatorname{supp}[f(x)\times g(y)]\cap\operatorname{supp}[\eta_k(x;y)\varphi(x+y)]$$
$$\subset[\operatorname{supp} f\times\operatorname{supp} g]\cap[(x,y): x+y\subset\operatorname{supp}\varphi]=\varnothing.$$

And so, due to Sec. 1.5(a), we have

$$(f*g,\varphi)=\lim_{k\to\infty}(f(x)\times g(y), \eta_k(x;y)\varphi(x+y))=0$$

for all basic functions $\varphi$ in $\mathcal{D}(\mathbb{R}^n)$ that satisfy the condition (2.6). And this means that the inclusion (2.5) holds.

*Remark.* The set $\operatorname{supp} f + \operatorname{supp} g$ may also be open. Generally speaking, there is no equality in the inclusion (2.5). For example, for the convolution $\delta' * \theta = \delta$ it takes the form $\{0\} \subset \{x \geqslant 0\}$.

(h) *Associativity of a convolution.* Generally, the operation of convolution is not associative; for example,

$$(1 * \delta') * \theta = 1' * \theta = 0 * \theta = 0, \quad \text{but} \quad 1 * (\delta' * \theta) = 1 * \delta = 1.$$

However, this unpleasantness does not arise if the convolution $f * g * h$ exists. To be more precise, the following assertion holds true.

If the convolutions $f * g$ and $f * g * h$ exist, then so also does the convolution $(f * g) * h$, and we have

$$(f * g) * h = f * g * h. \tag{2.7}$$

Indeed, suppose $\{\eta_k\}$ and $\{\xi_k\}$ are sequences of functions from $\mathcal{D}(\mathbb{R}^{2n})$ that converge to 1 in $\mathbb{R}^{2n}$. Then the sequence

$$\eta_i(x;\, y)\, \xi_k(x + y;\, z), \qquad i \to \infty, \quad k \to \infty,$$

of functions taken from $\mathcal{D}(\mathbb{R}^{3n})$ converges to 1 in $\mathbb{R}^{3n}$. From this fact and from the existence of the convolution $f * g * h$ (see Sec. 4.1) there follows the existence of a double limit:

$$\lim_{\substack{i \to \infty \\ k \to \infty}} (f(x) \times g(y) \times h(z),\, \eta_i(x;\, y)\, \xi_k(x + y;\, z)\, \varphi(x + y + z))$$
$$= (f * g * h,\, \varphi), \qquad \varphi \in \mathcal{D}(\mathbb{R}^n),$$

and, consequently, of the repeated limit

$$\begin{aligned}
&(f * g * h, \varphi)\\
&= \lim_{k \to \infty} \lim_{i \to \infty} (f(x) \times g(y) \times h(z),\, \eta_i \xi_k (x + y;\, z)\, \varphi(x + y + z))\\
&= \lim_{k \to \infty} \lim_{i \to \infty} (f(x) \times g(y),\, \eta_i (h(z),\, \xi_k(x + y;\, z)\, \varphi(x + y + z)))\\
&= \lim_{k \to \infty} ((f * g)(t),\, (h(z),\, \xi_k(t;\, z)\, \varphi(t + z)))\\
&= \lim_{k \to \infty} ((f * g)(t) \times h(z),\, \xi_k(t;\, z)\, \varphi(t + z))\\
&= ((f * g) * h,\, \varphi),
\end{aligned}$$

which proves (2.7) and the existence of the convolution $(f * g) * h$ (see Sec. 4.1)

Corollary *If there exist convolutions* $f*g*h$, $f*g$, $g*h$ *and* $f*h$, *then there exist convolutions* $(f*g)*h$, $f*(g*h)$ *and* $(f*h)*g$ *and we have*

$$f*g*h = (f*g)*h = f*(g*h) = (f*h)*g,$$

*which in this case means the convolution is associative.*

**4.3 The existence of a convolution** Let us establish certain sufficient conditions (besides those given in Sec. 4.1), under which a convolution definitely exists in $\mathcal{D}'$. Recall (see Fig. 19) that

$$T_R = [(x, y): x \in A, \quad y \in B, \quad |x+y| \leqslant R].$$

For the definition of the space $\mathcal{D}'(A)$ see Sec. 1.5.

Theorem *Let* $f \in \mathcal{D}'(A)$, $g \in \mathcal{D}'(B)$ *and suppose that for any* $R > 0$ *the set* $T_R$ *is bounded in* $\mathbb{R}^{2n}$. *Then the convolution* $f*g$ *exists in* $\mathcal{D}'(\overline{A+B})$ *and may be represented as*

$$(f*g, \varphi) = (f(x) \times g(y), \xi(x)\eta(y)\varphi(x+y)), \qquad (3.1)$$
$$\varphi \in \mathcal{D}$$

*where* $\xi$ *and* $\eta$ *are any functions in* $C^\infty$ *that are equal to* 1 *in* $A^\varepsilon$ *and* $B^\varepsilon$ *and are equal to* 0 *outside* $A^{2\varepsilon}$ *and* $B^{2\varepsilon}$ *respectively* ($\varepsilon$ *is any number greater than* 0). *Here the operation* $f \to f*g$ *is continuous from* $\mathcal{D}'(A)$ *into* $\mathcal{D}'(\overline{A+B})$.

*Proof*. Let $\varphi \in \mathcal{D}(U_R)$ and let $\{\eta_k\}$ be a sequence of functions in $\mathcal{D}(\mathbb{R}^{2n})$ that converges to 1 in $\mathbb{R}^{2n}$. Since

$$\operatorname{supp}(f \times g) = \operatorname{supp} f \times \operatorname{supp} g \subset A \times B$$

(see Sec. 3.2(d)), it follows that

$$\operatorname{supp}\{[f(x) \times g(y)]\,\varphi(x+y)\}$$
$$\subset [(x, y): x \in A, \ y \in B, \ |x+y| \leqslant R] = T_R.$$

And since $T_R$ is a bounded set, there is a number $N = N(R)$ such that $\eta_k(x; y) = 1$ in the neighbourhood of $T_R$ for all $k \geqslant N$. For this reason,

$$(f*g, \varphi) = \lim_{k\to\infty} (f(x) \times g(y), \eta_k(x; y)\,\varphi(x+y))$$
$$= \lim_{k\to\infty} ([f(x) \times g(y)]\,\varphi(x+y), \eta_k(x; y))$$
$$= (f(x) \times g(y), \eta_N(x; y)\,\varphi(x+y))$$

and the representation

$$(f * g,\ \varphi) = (f(x) \times g(y),\ \eta_N(x;\ y)\,\varphi(x+y)),\qquad \varphi \in \mathcal{D}(U_R), \tag{3.2}$$

is proved. Clearly, the representation (3.2) does not depend on the auxiliary function $\eta_N(x;\ y)$. It can be replaced by the function $\xi(x)\,\eta(y)$. Indeed, the function $\xi(x)\,\eta(y)\,\varphi(x+y) \in \mathcal{D}(\mathbb{R}^{2n})$, since the set

$$T_{R,\varepsilon} = [(x,\ y){:}\ x \in A^2,\quad y \in B^{2\varepsilon},\quad |x+y| \leqslant R] \subset T_R^{4\varepsilon}$$

for any $R > 0$ and $\varepsilon > 0$ bounded in $\mathbb{R}^{2n}$; furthermore, the function $[\eta_N(x;\ y) - \xi(x)\,\eta(y)]\,\varphi(x+y)$, $\varphi \in \mathcal{D}(U_R)$, vanishes in the neighbourhood of $T_R$. This completes the proof of the representation (3.1).

From (2.5) it follows that

$$\operatorname{supp}(f * g) \subset \overline{A+B}$$

so that the operation $f \to f * g$ carries $\mathcal{D}'(A)$ into $\mathcal{D}'(\overline{A+B})$. Its continuity follows from the continuity of the direct product $f \times g$ with respect to $f$ (see Sec. 3.2(c)) and from the representation (3.1): if $f_k \to 0$, $k \to \infty$ in $\mathcal{D}'(A)$, then

$$(f_k * g,\ \varphi) = (f_k(x) \times g(y),\ \xi(x)\,\eta(y)\,\varphi(x+y)) \to 0$$

for all $\varphi \in \mathcal{D}$, that is, $f_k * g \to 0$, $k \to \infty$ in $\mathcal{D}'(\overline{A+B})$. The proof of the theorem is complete.

Note that the continuity of the convolution $f * g$ relative to the collection of $f$ and $g$ may not occur, as the following simple example illustrates:

$$\delta(x+k) \to 0,\quad k \to +\infty,\quad \delta(x-k) \to 0,\quad k \to +\infty.$$

However,

$$\delta(x+k) * \delta(x-k) = \delta * \delta = \delta.$$

We note here an important special case of this theorem.

*If $f \in \mathcal{D}'$ and $g \in \mathcal{E}'$, then the convolution $f * g$ exists and can be represented in the form*

$$(f * g,\ \varphi) = (f(x) \times g(y),\ \eta(y)\,\varphi(x+y)),\quad \varphi \in \mathcal{D}, \tag{3.3}$$

*where $\eta$ is any basic function taken from $\mathcal{D}$ that is equal to 1 in the neighbourhood of the support of $g$.*

Indeed, in this case the boundedness condition of the set $T_R$ is fulfilled for all $R > 0$ (Fig. 20): if $\operatorname{supp} g \subset \overline{U}_{R'}$, then

$$T_R = [(x, y): x \in R^n, \ y \in \operatorname{supp} g, \ |x + y| \leqslant R] \subset \overline{U}_{R+R'} \times \overline{U}_{R'}.$$

Similarly, *if* $f \in \mathscr{D}'$ *and* $g_1, \ldots, g_m \in \mathscr{E}'$, *then there exists a convolution* $f * g_1 * \ldots * g_m$ *(see Sec. 4.1) that is associative*

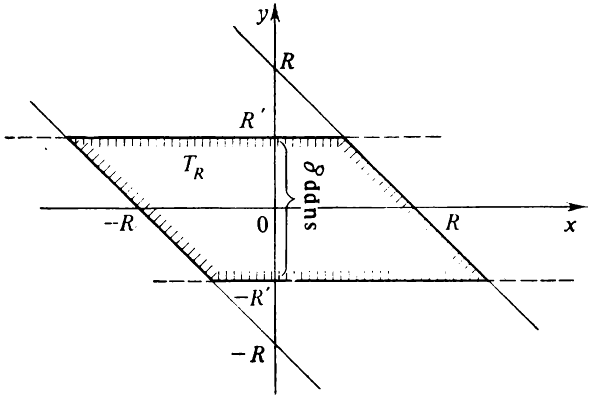



Figure 20

*and commutative (see Sec. 4.2(a), (h)) and the formula (3.3) is generalized thus:*

$$\begin{aligned}(f * g_1 * \ldots * g_m, \varphi) &= (f(x) \times g_1(y) \times \ldots \times g_m(z), \\ &\qquad \eta_1(y) \ldots \eta_m(z) \times \varphi(x + y + \ldots + z)), \\ &\qquad \varphi \in \mathscr{D}.\end{aligned} \tag{3.4}$$

*But if* $f \in C^\infty$ *and* $g \in \mathscr{E}'$, *then the convolution* $f * g \in C^\infty$, *and the formula (3.3) takes on the form*

$$(f * g)(x) = (\tilde{g}(y), f(x - y)), \tag{3.5}$$

*where* $\tilde{g}$ *is the extension of* $g$ *onto* $C^\infty = C^\infty(\mathbb{R}^n)$ *(see Sec. 2.5).*

True enough, as in the proof of the lemma of Sec. 3.1, it is established that the function

$$(\tilde{g}(y), f(x - y)) = (g(y), \eta(y) f(x - y)) \in C^\infty.$$

Then, from the representation (3.3) we have, for all $\varphi \in \mathscr{D}$,

$$(f * g, \varphi) = \left(g(y), \eta(y) \int f(\xi) \varphi(\xi + y)\, d\xi\right)$$
$$= \left(g(y), \int \eta(y) f(x - y) \varphi(x)\, dx\right).$$

Noticing now that $\eta(y) f(x - y) \varphi(x) \in \mathscr{S}(\mathbb{R}^{2n})$ and using the formula (3.2) of Sec. 3, we obtain

$$(f * g, \varphi) = \int (g(y), \eta(y) f(x - y)) \varphi(x)\, dx,$$

whence follows formula (3.5).

In similar fashion, *if $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and $g \in \mathscr{E}'$, then the convolution $f * g$ in $\mathbb{R}^n \setminus$ supp $g$ is expressed by the formula (3.5); in particular, $f * g \in C^\infty(\mathbb{R}^n \setminus \text{supp}\, g)$.*

**4.4 Cones in $\mathbb{R}^n$** A *cone* in $\mathbb{R}^n$ (with vertex at 0) is a set $\Gamma$ with the property that if $x \in \Gamma$, then $\lambda x$ too belongs to $\Gamma$ for

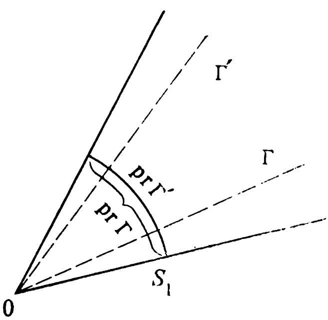



**Figure 21**

all $\lambda > 0$. Denote by pr $\Gamma$ the intersection of $\Gamma$ and the unit sphere (Fig. 21). The cone $\Gamma'$ is said to be *compact* in the cone $\Gamma$ if $\text{pr}\, \overline{\Gamma'} \subset \text{pr}\, \Gamma$ (Fig. 21); we then write $\Gamma' \Subset \Gamma$.

The cone

$$\Gamma^* = [\xi: (\xi, x) \geqslant 0, \ \forall x \in \Gamma]$$

is said to be *conjugate* to the cone $\Gamma$. Clearly, $\Gamma^*$ is a closed convex cone with vertex at 0 (Fig. 22) and $(\Gamma^*)^* = \overline{\text{ch}\, \Gamma}$; here ch $\Gamma$ is the convex hull of $\Gamma$ (see Sec. 0.2).

A cone $\Gamma$ is said to be *acute* if there exists a plane of support for $\overline{\text{ch}\,\Gamma}$ that has a unique common point 0 with $\overline{\text{ch}\,\Gamma}$ (Fig. 22).

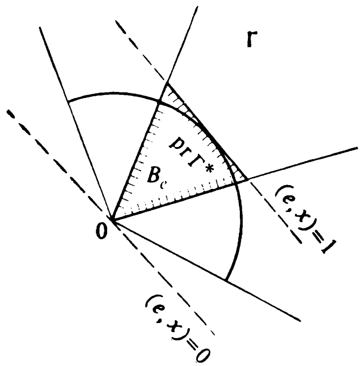



Figure 22

*Examples of convex acute cones.* (a) an $n$-hedral cone in $\mathbb{R}^n$:

$$C = [x : (e_1, x) > 0, \ldots, (e_n, x) > 0]$$

is acute (convex and open) if and only if the vectors $e_1, \ldots, e_n$ form a basis in $\mathbb{R}^n$. Then

$$C^* = [\xi : \xi = \sum_{1 \leqslant k \leqslant n} \lambda_k e_k, \lambda_k \geqslant 0].$$

In particular, the positive quadrantal angle

$$\mathbb{R}^n_+ = [x : x_1 > 0, \ldots, x_n > 0], \qquad (\mathbb{R}^n_+)^* = \overline{\mathbb{R}^n_+}.$$

(b) The future light cone in $\mathbb{R}^{n+1}$.

$$V^+ = [x : (x_0, \mathbf{x}) : x_0 > |\mathbf{x}|], \qquad (V^+)^* = \bar{V}^+.$$

(c) The origin of coordinates $\{0\}$, $\{0\}^* = \mathbb{R}^n$.

Note, however, that the cone $\mathbb{R}^1_+ \times \mathbb{R}^{n-1} = [x : x_1 > 0]$ is not acute.

(d) The cone $P_n \subset \mathbb{R}^{n^2}$ of positive (Hermitian) $n \times n$ matrices $X = (x_{pq})$, $P_n^* = \overline{P}_n$, where $\overline{P}_n$ is the cone of nonnegative matrices. This follows from the assertion that, for $X \in P_n$, it is necessary and sufficient that for all $\Xi \in \overline{P}_n$, $\Xi \neq 0$,

$$(X, \Xi) = \operatorname{Tr}(X\Xi) = \sum_{p,q} x_{pq}\xi_{qp} > 0.$$

Lemma 1 *The following statements are equivalent:*
(1) *the cone* $\Gamma$ *is acute;*
(2) *the cone* $\overline{\operatorname{ch}\Gamma}$ *does not contain an integral straight line;*
(3) $\operatorname{int}\Gamma^* \neq \varnothing$;
(4) *for any* $C' \Subset \operatorname{int}\Gamma^*$ *there exists a number* $\sigma = \sigma(C') > 0$ *such that*

$$(\xi, x) \geqslant \sigma |\xi| |x|, \qquad \xi \in C', \quad x \in \overline{\operatorname{ch}\Gamma}; \tag{4.1}$$

(5) *for any* $e \in \operatorname{pr}\operatorname{int}\Gamma^*$ *the set*

$$B_e = [x : 0 \leqslant (e, x) \leqslant 1, x \in \overline{\operatorname{ch}\Gamma}]$$

*is bounded (Fig. 22).*

*Proof.* (1) $\to$ (2). If the cone $\overline{\operatorname{ch}\Gamma}$ contains an integral straight line $x = x^0 + te$, $-\infty < t < \infty$ ($|e| = 1$), then it also contains the straight line $x = te$, $-\infty < t < \infty$. Consequently, any plane of support for $\overline{\operatorname{ch}\Gamma}$ must contain that straight line, but this contradicts (1).

(2) $\to$ (3). If $\operatorname{int}\Gamma^* \neq \varnothing$, then, since $\Gamma^*$ is a convex cone with vertex at 0, it lies in some $(n-1)$-dimensional plane $(e, x) = 0$ ($|e| = 1$). For this reason, $\pm e \in \Gamma^{**} = \overline{\operatorname{ch}\Gamma}$. But then the integral straight line $y = te$, $-\infty < t < \infty$ too lies in $\overline{\operatorname{ch}\Gamma}$, but this contradicts (2).

(3) $\to$ (4). Since all points of the cone $C'$ different from 0 are interior points relative to $\Gamma^*$, it follows that $(\xi, x) > 0$ for all $\xi \in C'$ and $x \in \overline{\operatorname{ch}\Gamma}$. From this fact and also from the continuity and the homogeneity of the form $(\xi, x)$ follows the existence of a number $\sigma > 0$ for which the inequality (4.1) holds true.

(4) $\to$ (5). Let us take an arbitrary $e \in \operatorname{pr}\operatorname{int}\Gamma^*$. Then, by applying the inequality (4.1), $(e, x) \geqslant \sigma |x|$, $x \in \overline{\operatorname{ch}\Gamma}$, we conclude that the set $B_e$ is bounded: $|x| \leqslant \frac{(e, x)}{\sigma} \leqslant \frac{1}{\sigma}$.

(5) $\to$ (1). If for some $e \in \operatorname{pr}\operatorname{int}\Gamma^*$ the set $B_e$ is bounded, then the plane $(e, x) = 0$ cannot have any other points in common with $\overline{\operatorname{ch}\Gamma}$, with the exception of 0.

The proof of the lemma is complete.

Lemma 2 *Let* $\Gamma$ *be a convex cone. Then* $\Gamma = \Gamma + \Gamma$.

*Proof.* The inclusion $\Gamma \subset \Gamma + \Gamma$ is obvious. Let $x \in \Gamma + \Gamma$ so that $x = y + z$, where $y \in \Gamma$ and $z \in \Gamma$. Then for all $\lambda \in (0, 1)$ we have $x = \lambda \frac{y}{\lambda} + (1 - \lambda) \frac{z}{\lambda - \lambda} \in \Gamma$ and for this reason $\Gamma + \Gamma \subset$ $\subset \Gamma$, thus completing the proof of the lemma.

The *indicator* of the cone $\Gamma$ is the function

$$\mu_\Gamma(\xi) = -\inf_{x \in \mathrm{pr}\,\Gamma} (\xi, x).$$

From the definition of the indicator it follows that $\mu_\Gamma(\xi)$ is a convex (see Sec. 0.2) and, hence, continuous (see, for example, Vladimirov [1], Chapter II) and homogeneous first-degree function defined on the whole of $\mathbb{R}^n$. Besides,

$$\mu_\Gamma(\xi) \leqslant \mu_{\mathrm{ch}\,\Gamma}(\xi),$$
$$\mu_\Gamma(\xi) = -\Delta(\xi, \partial\Gamma^*), \qquad \xi \in \Gamma^*$$

and $\mu_\Gamma(\xi) > 0$ for $\xi \,\bar{\in}\, \Gamma^*$. Thus

$$\Gamma^* = [\xi\colon \mu_\Gamma(\xi) \leqslant 0]$$

so that the indicator of a cone fully defines only the closure of its convex hull, by virtue of $\overline{\mathrm{ch}\,\Gamma} = \Gamma^{**}$.

*Example.*

$$\mu_{V^+}(\xi) = \begin{cases} \dfrac{1}{\sqrt{2}}(|\xi| - \xi_0), & \xi \,\bar{\in}\, -V^+, \\ |\xi|, & \xi \in -V^+. \end{cases}$$

Lemma 3 *If $\Gamma$ is a convex cone, then for any $a \geqslant 0$*

$$[\xi\colon \mu_\Gamma(\xi) \leqslant a] = \Gamma^* + \bar{U}_a. \tag{4.2}$$

*Proof.* The inclusion

$$\Gamma^* + \bar{U}_a \subset [\xi\colon \mu_\Gamma(\xi) \leqslant a] \tag{4.3}$$

is trivial: if $\xi = \xi_1 + \xi_2$, $\xi_1 \in \Gamma^*$, $|\xi_2| \leqslant a$, then

$$\mu_\Gamma(\xi) = -\inf_{x \in \mathrm{pr}\,\Gamma} (\xi, x) = -\inf_{x \in \mathrm{pr}\,\Gamma} [(\xi_1, x) + (\xi_2, x)]$$
$$\leqslant -\inf_{x \in \mathrm{pr}\,\Gamma} (\xi_2, x) \leqslant a,$$

since $(\xi_1, x) \geqslant 0$, $x \in \Gamma$.

Now let us prove the inverse inclusion of (4.3). Let the point $\xi_0$ be such that $\mu_\Gamma(\xi_0) \leqslant a$. If $\xi_0 \in \Gamma^*$ or $|\xi_0| \leqslant a$, then $\xi_0 \in \Gamma^* + U_a$. Now let $\xi_0 \,\bar{\in}\, \Gamma^*$ and $|\xi_0| > a$. Let the point $\xi_1 \in \Gamma^*$ realize the distance from $\xi_0$ to $\Gamma^*$, $\Delta(\xi_0, \Gamma^*) = |\xi_0 - \xi_1|$.

Then, since $\Gamma^*$ is a convex cone (Fig. 23), it follows that

(a) $$(\xi_1 - \xi_0, \xi) \geqslant 0, \qquad \xi \in \Gamma^*;$$

(b) $$(\xi_1 - \xi_0, \xi_1) = 0.$$

From the inequality (a) it follows that $\xi_1 - \xi_0 \in \Gamma^{**} = \bar{\Gamma}$ and therefore

$$a \geqslant \mu_\Gamma(\xi_0) = -\inf_{x \in \mathrm{pr}\,\bar{\Gamma}} (\xi_0, x) \geqslant -\left(\xi_0, \frac{\xi_1 - \xi_0}{|\xi_1 - \xi_0|}\right).$$

Now the latter is equivalent, by virtue of (b), to the inequality $|\xi_1 - \xi_0| \leqslant a$. Thus, the point $\xi_0 = \xi_1 + (\xi_0 - \xi_1)$ is repre-

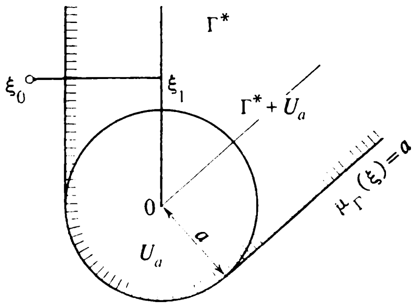



Figure 23

sented in the form of a sum of two terms $\xi_1 \in \Gamma^*$ and $\xi_0 - \xi_1 \in \bar{U}_a$ that is, $\xi_0 \in \Gamma^* + \bar{U}_a$. This completes the proof of the inverse inclusion of (4.3) and also the equality (4.2). The proof of Lemma 3 is complete.

Suppose $\Gamma$ is a closed convex acute cone. Set $C = \operatorname{int}\Gamma^*$ (via Lemma 1, $C \neq \varnothing$). The smooth $(n-1)$-dimensional surface $S$ without an edge is said to be $C$-like if each straight line $x = x_0 + te$, $-\infty < t < \infty$, $e \in \mathrm{pr}\,\Gamma$, intersects it in a unique point; in other words: for any $x \in S$ the cone $\Gamma + x$ intersects $S$ in a unique point $x$ (Fig. 24). Thus, the $C$-like surface $S$ cuts $\mathbb{R}^n$ into two infinite regions $S_+$ and $S_-$: $S_+$ lies above $S$ and $S_-$ lies below $S$; $S_+ \cup S_- \cup S = \mathbb{R}^n$. At every point $x$ of the surface $S$, the normal $\mathbf{n}_x$ is contained in the cone $\Gamma^* + x$ (Fig. 24).

*Example*. The surface $S$ in $\mathbb{R}^{n+1}$, which surface is specified by the equation

$$x_0 = f(\mathbf{x}), \quad |\operatorname{grad} f(\mathbf{x})| \leqslant \sigma < 1, \quad \mathbf{x} \in \mathbb{R}^n, \quad f \in C^1,$$

is $V^+$-like (space-like).

Lemma 4 *If $S$ is a C-like surface, then*

$$\bar{S}_+ = S + \Gamma. \tag{4.4}$$

*Proof.* Suppose $x_0 \in \bar{S}_+$. The straight line $x = x_0 + te$, $|t| < \infty$, $e \in \mathrm{pr}\,\Gamma$, intersects $S$ at some point $x_1 = x_0 - t_1 e$, $t_1 \geqslant 0$

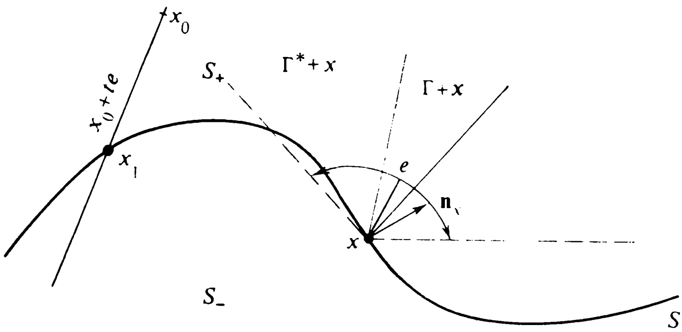



Figure 24

(Fig. 24) so that $x_0 = x_1 + t_1 e$, $x_1 \in S$, $t_1 e \in \Gamma$, and the inclusion $\bar{S}_+ \subset S + \Gamma$ is proved. Clearly, the inclusion $S + \Gamma \subset \bar{S}_+$ together with the inverse inclusion leads to the equation (4.4), thus completing the proof of the lemma.

Lemma 5 *Let $S$ be a C-like surface. Then for any $R > 0$ there is a number $R'(R) > 0$ such that the set*

$$T_R = [(x, y): x \in S,\ y \in \Gamma,\ |x + y| \leqslant R]$$

*is contained in the sphere $U_{R'} \subset \mathbb{R}^{2n}$.*

*Proof.* Since $S$ is a $C$-like surface, it follows that any point $x \in S$ that can be represented as $\xi - y$, $y \in \Gamma$, $|\xi| \leqslant R$, is of the form $x = \xi - eT$, $e \in \mathrm{pr}\,\Gamma$, where the number $T = T(e, \xi)$ is uniquely determined by $e$ and $\xi$ and constitutes a continuous function of the argument $(e, \xi)$ on the compact $e \in \mathrm{pr}\,\Gamma$, $|\xi| \leqslant R$. Hence the set $[(y, \xi): y = eT(e, \xi),\ e \in \mathrm{pr}\,\Gamma,\ |\xi| \leqslant R]$ is bounded and so also is the set $T_R$. The lemma is proved.

We will say that a $C$-like surface $S$ is a *strictly C-like* surface if, under the conditions of Lemma 5,

$$R'(R) \leqslant a(1 + R)^{\nu}, \qquad \nu \geqslant 1, \quad a > 0. \tag{4.5}$$

*Example.* The plane $(e, x) = 0$, $e \in \mathrm{pr}\,C$, is strictly $C$-like with $\nu = 1$ (by virtue of Lemma 1).

**4.5 Convolution algebras $\mathscr{D}'(\Gamma+)$ and $\mathscr{D}'(\Gamma)$** We will say that a set $A$ is *bounded on the side of the cone* $\Gamma$ if $A \subset \Gamma + K$, where $K$ is a certain compact (Fig. 25). It is clear that the sets bounded on the side of the cone $\{0\}$ are compacts in $\mathbb{R}^n$.

Suppose $\Gamma$ is a closed cone in $\mathbb{R}^n$. The collection of generalized functions in $\mathscr{D}'$ whose supports are bounded on the side of the

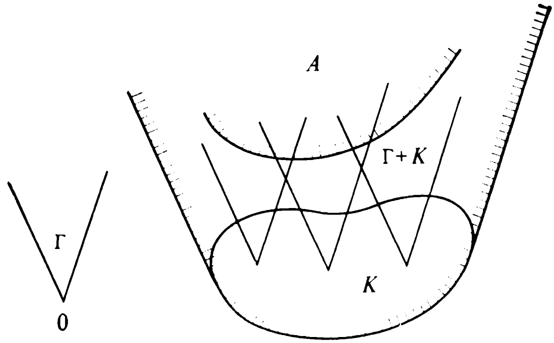



**Figure 25**

cone $\Gamma$ will be denoted by $\mathscr{D}'(\Gamma+)$. We define *convergence* in $\mathscr{D}'(\Gamma+)$ in the following manner: $f_k \to 0$, $k \to \infty$ in $\mathscr{D}'(\Gamma+)$, if $f_k \to 0$, $k \to \infty$ in $\mathscr{D}'$, and $\operatorname{supp} f_k \subset \Gamma + K$, where the compact $K$ does not depend on $k$§. Set $\mathscr{D}'(\{0\}+) = \mathscr{E}'$; $\mathscr{E}'$ is the space of generalized functions with compact supports (compare Sec. 2.5).

Let $\Gamma$ be a closed convex acute cone, $C = \operatorname{int} \Gamma^*$, $S$ a $C$-like surface, and $S_+$ the region lying above $S$ (see Sec. 4.4).

*If $f \in \mathscr{D}'(\Gamma+)$ and $g \in \mathscr{D}'(\overline{S}_+)$, then the convolution $f * g$ exists in $\mathscr{D}'$ and can be represented as*

$$(f * g, \varphi) = (f(x) \times g(y), \xi(x)\eta(y)\varphi(x+y)), \qquad (5.1)$$
$$\varphi \in \mathscr{D},$$

*where $\xi$ and $\eta$ are any functions in $C^\infty$ that are equal to 1 in* $(\operatorname{supp} f)^\varepsilon$ *and* $(\operatorname{supp} g)^\varepsilon$ *and are equal to 0 outside* $(\operatorname{supp} f)^{2\varepsilon}$ *and* $(\operatorname{supp} g)^{2\varepsilon}$, *respectively ($\varepsilon$ is any number $>0$). Here, if* $\operatorname{supp} f \subset \Gamma + K$, *where $K$ is a compact, then* $\operatorname{supp}(f * g) \subset \overline{S}_+ + K$ *and the respective operations $f \to f * g$ and $g \to f * g$ are continuous.*

§ A similar meaning will be attached to other spaces of generalized functions as well; for example, $\mathscr{S}'(\Gamma+)$, $\mathscr{L}_s^2(\Gamma+)$ and so forth (see Secs. 5 and 7 below).

This assertion follows from the theorem of Sec. 4.3 for $A = \Gamma + K$ and $B = \bar{S}_+$ if we note that by Lemmas 2, 4 and 5 of Sec. 4.4 the set

$$T_R = [(x, y): x \in \Gamma + K, y \in \bar{S}_+, |x + y| \leqslant R]$$
$$= [(x, y): x \in \Gamma + K, y \in S + \Gamma, |x + y| \leqslant R]$$

is bounded for all $R > 0$ and

$$\overline{\Gamma + K + \bar{S}_+} = \overline{\Gamma + K + \Gamma + S} = \overline{\Gamma + S + K} = \bar{S}_+ + K.$$

We now note an important special case of the last criterion for the existence of a convolution.

Theorem *Let* $\Gamma$ *be a closed convex acute cone. If* $f \in \mathscr{D}'(\Gamma+)$ *and* $g \in \mathscr{D}'(\Gamma+)$, *then the convolution* $f * g$ *exists in* $\mathscr{D}'(\Gamma+)$ *and can be represented as (5.1); here, the operation* $f \to f * g$ *is continuous from* $\mathscr{D}'(\Gamma+)$ *into* $\mathscr{D}'(\Gamma+)$.

*Proof*. Since $\Gamma + K$, where $K$ is a compact, is contained in $\bar{S}_+$ for some $C$-like surface $S$ (which depends on $K$), it follows, by the preceding criterion, that the convolution $f * g$ exists in $\mathscr{D}'$ and can be represented by the formula (5.1). Let us prove that $f * g \in \mathscr{D}'(\Gamma +)$. Suppose supp $f \subset \Gamma + K_1$ and supp $g \subset \Gamma + K_2$, where $K_1$ and $K_2$ are certain compacts in $\mathbb{R}^n$. Then, using the inclusion (2.5) and Lemma 2 of Sec. 4.4, we obtain

$$\operatorname{supp}(f * g) \subset \overline{\Gamma + K_1 + \Gamma + K_2} = \Gamma + K_1 + K_2$$

so that $f * g \in \mathscr{D}'(\Gamma +)$. The continuity of the operation $f \to f * g$ from $\mathscr{D}'(\Gamma +)$ into $\mathscr{D}'(\Gamma +)$ also follows from this inclusion. The proof of the theorem is complete.

In similar fashion we can prove that the convolution of any number of generalized functions taken from $\mathscr{D}'(\Gamma +)$ (see Sec. 4.1) exists in $\mathscr{D}'(\Gamma +)$ and can be expressed by a formula similar to (5.1).

From this and from the results of Sec. 4.2(h) it follows that the convolution of generalized functions taken from $\mathscr{D}'(\Gamma +)$ is *associative*.

A linear set is termed an *algebra* if the operation of multiplication is defined on it, and the operation is linear with respect to every factor separately. An algebra is said to be *associative* if $x(yz) = (xy)z$ and *commutative* if $xy = yx$.

The results established in this subsection enable us to assert that the set of generalized functions $\mathscr{D}'(\Gamma +)$ forms an algebra that is associative and commutative if for the operation of mul-

tiplication we take the convolution operation $*$. Such algebras are called *convolution algebras*; the unit element here is the $\delta$ function (see Sec. 4.2(b)).

Finally, note that the set of generalized functions $\mathscr{D}'(\Gamma)$ also forms a convolution algebra, a subalgebra of the algebra $\mathscr{D}'(\Gamma+)$. Indeed, if $f \in \mathscr{D}'(\Gamma)$ and $g \in \mathscr{D}'(\Gamma)$, then $\operatorname{supp}(f*g) \subset \subset \operatorname{supp} f + \operatorname{supp} g \subset \Gamma + \Gamma = \Gamma$ so that $f*g \in \mathscr{D}'(\Gamma)$. (Here, we again took advantage of the inclusion (2.5) and Lemma 2 of Sec. 4.4.)

**4.6 Regularization of generalized functions** Let us extend the concept of a convolution $f*g$ when $f$ and $g$ are generalized

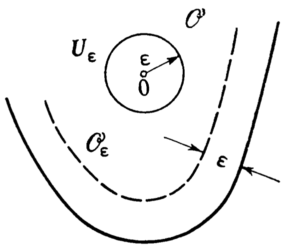



Figure 26

functions taken from $\mathscr{D}'(\mathcal{O})$ and $g$ is with a compact and sufficiently small support in $\mathcal{O}$: $\operatorname{supp} g \subset U_\varepsilon$ and $\mathcal{O}_\varepsilon \neq \varnothing$ (see Sec. 0.2 and Fig. 26). In accordance with formula (3.3) we set, by definition,

$$(f*g, \varphi) = (f(x) \times g(y), \eta(y)\varphi(x+y)), \qquad (6.1)$$
$$\varphi \in \mathscr{D}(\mathcal{O}_\varepsilon),$$

where $\eta \in \mathscr{E}(\mathcal{O}_\varepsilon)$, $\eta(y) = 1$ in the neighbourhood of $\operatorname{supp} g$.

By construction, the operation $\varphi \to \eta(y)\varphi(x+y)$ is linear and continuous from $\mathscr{D}(\mathcal{O}_\varepsilon)$ into $\mathscr{D}(\mathcal{O} \times \mathcal{O})$. From this it follows that the right-hand side of (6.1) defines a continuous linear functional on $\mathscr{D}(\mathcal{O}_\varepsilon)$ so that $f*g \in \mathscr{D}'(\mathcal{O}_\varepsilon)$. Furthermore, it is easy to see (compare Sec. 4.3) that the right-hand side of (6.1) is not dependent on the auxiliary function $\eta$. Finally, as in Sec. 4.2, it can be established that the convolution $f*g$ is commutative and continuous with respect to $f$ and $g$ separately, and $f*\delta = f$.

In particular, if $\alpha \in \mathscr{D}(U_\varepsilon)$, then, using the representation (6.1) and acting in a manner similar to that of Sec. 4.3 when deriving (3.5) we obtain a representation for the convolution $f*\alpha$:

$$(f*\alpha)(x) = (f(y), \alpha(x-y)), \qquad x \in \mathcal{O}_\varepsilon, \qquad (6.2)$$

whence follows $f * \alpha \in C^\infty(\mathcal{O}_\varepsilon)$ and

$$(f * \alpha)(0) = (f(y), \alpha(-y)) = (\delta, f * \alpha). \tag{6.3}$$

By virtue of (6.2), the formula (6.1) takes the form

$$(f * g, \varphi) = (f, g(-y) * \varphi), \qquad \varphi \in \mathcal{D}(\mathcal{O}_\varepsilon). \tag{6.4}$$

Note that when $\mathcal{O} = \mathbb{R}^n$, formula (6.4) also follows from (3.3) and (6.2).

Let $\omega_\varepsilon(x)$ be the "cap" (see Sec. 1.2) and let $f$ be a generalized function in $\mathcal{D}'(\mathcal{O})$. The convolution

$$f_\varepsilon(x) = (f * \omega_\varepsilon)(x) = (f(y), \omega_\varepsilon(x - y))$$

is termed a *regularization of $f$* (compare this with the definition of a mean function for the case of $f \in \mathcal{L}^1_{\mathrm{loc}}(\mathcal{O})$ see Sec. 1.2). By what has been proved, the regularization $f_\varepsilon \in C^\infty(\mathcal{O}_\varepsilon)$.

Now let us prove that

$$f_\varepsilon \to f, \qquad \varepsilon \to +0 \text{ in } \mathcal{D}'(\mathcal{O}). \tag{6.5}$$

True enough, the limiting relation (6.5) follows from the relation $\omega_\varepsilon(x) \to \delta(x)$, $\varepsilon \to +0$ in $\mathcal{D}'$, (see Sec. 1.7) and from the continuity of a convolution, by virtue of

$$f_\varepsilon = f * \omega_\varepsilon \to f * \delta = f, \qquad \varepsilon \to +0 \quad \text{in} \quad \mathcal{D}'(\mathcal{O}).$$

To summarize: every generalized function taken from $\mathcal{D}'(\mathcal{O})$ is a weak limit of its regularizations. Let us use this statement and establish a stronger result.

Theorem *Every generalized function $f$ in $\mathcal{D}'(\mathcal{O})$ is a weak limit of the basic functions in $\mathcal{D}(\mathcal{O})$, that is, $\mathcal{D}(\mathcal{O})$ is dense in $\mathcal{D}'(\mathcal{O})$.*

*Proof.* Let $f_\varepsilon(x)$ be a regularization of $f$. Furthermore, let $\mathcal{O}_1 \Subset \mathcal{O}_2 \Subset \ldots, \bigcup_k \mathcal{O}_k = \mathcal{O}$, $\varepsilon_k = \Delta(\mathcal{O}_k, \partial\mathcal{O}) > 0$ and $\eta_k \in \mathcal{D}(\mathcal{O}_k)$, $\eta_k(x) = 1$, $x \in \mathcal{O}_{k-1}$. We will prove that the sequence $\eta_k(x) f_{\varepsilon_k}(x)$, $k = 1, 2, \ldots$, of basic functions taken from $\mathcal{D}(\mathcal{O})$ converges to $f$ in $\mathcal{D}'(\mathcal{O})$. Indeed, $\varepsilon_k \to 0$, as $k \to \infty$ and by (6.5) for all $\varphi \in \mathcal{D}(\mathcal{O})$ we have

$$\lim_{k\to\infty} (\eta_k f_{\varepsilon_k}, \varphi) = \lim_{k\to\infty} (f_{\varepsilon_k}, \eta_k\varphi) = \lim_{k\to\infty} (f_{\varepsilon_k}, \varphi) = (f, \varphi),$$

which completes the proof.

*Remark*. From the completeness of the space $\mathcal{D}'(\mathcal{O})$ (see Sec. 1.4) there follows a converse statement to the theorem that has just been proved: any weak limit of locally summable functions in $\mathcal{O}$ is a generalized function in $\mathcal{D}'(\mathcal{O})$. Therefore, it is possible to construct a theory of generalized functions by proceeding from weakly convergent sequences of ordinary, locally summable functions. With regard to this approach, see Antosik, Mikusinski and Sikorski [1].

It is appropriate at this point to mention the following analogy. The relation of generalized functions to basic functions is reminiscent, in a certain sense, of the relation of irrational numbers to rational numbers: by augmenting the set of rational numbers by means of all possible limits of sequences of rational numbers, we obtain real numbers; by augmenting the set of basic functions by all weak limits of sequences of basic functions, we obtain generalized functions.

**4.7 Convolution as a continuous linear translation-invariant operator** An operator $L$ acting from $\mathcal{D}'$ to $\mathcal{D}'$ is said to be translation-invariant if $Lf(x+h) = (Lf)(x+h)$ for all $f \in \mathcal{D}'$ and for all translations $h \in \mathbb{R}^n$.

Recall that the definition of convergence in the space $C^\infty = C^\infty(\mathbb{R}^n)$ is given in Sec. 2.5 and in the space $\mathcal{E}'$ in Sec. 4.5; $\mathcal{E}'$ is a collection of continuous linear functionals on $C^\infty$ (see Sec. 2.5).

Theorem *For an operator L to be linear, continuous and translation-invariant from $\mathcal{E}'$ to $\mathcal{D}'$, it is necessary and sufficient that it be a convolution operator, that is to say, that it be representable in the form $L = f_0 *$, where $f_0$ is some generalized function taken from $\mathcal{D}'$; then $f_0$, the kernel of the operator L, is unique and is expressed by the formula $f_0 = L\delta$.*

*Proof*. Sufficiency follows from the results of Sec. 4.3 and Sec. 4.2, according to which the convolution operator $f \to f_0 * f$, $f_0 \in \mathcal{D}'$, is linear, continuous and translation-invariant from $\mathcal{E}'$ to $\mathcal{D}'$, and $f_0 * \delta = f_0$. To prove necessity let us first establish the following lemma.

Lemma *For an operator L to be linear, continuous and translation-invariant from $\mathcal{D}$ to $C^\infty$, it is necessary and sufficient that it be a convolution operator $L = f_0 *$, $f_0 \in \mathcal{D}'$; here, the kernel $f_0$ is unique.*

*Proof*. To prove sufficiency, it remains to establish the continuity of the operation

$$\varphi \to f_0 * \varphi = (f_0(y), \varphi(x-y))$$

(see (6.2)) from $\mathscr{D}$ to $C^\infty$. But this follows from the inequality (see the theorem of Sec. 1.3)

$$|D^\alpha (f_0 * \varphi)(x)| = |(f_0(y), D^\alpha \varphi(x-y))| \leqslant K \|\varphi\|_{C^{m+|\alpha|}}, \tag{7.1}$$

which holds true for all $\varphi \in \mathscr{D}(U_R)$ and $|x| \leqslant R_1$ (the numbers $K$ and $m$ in (7.1) depend on $R$ and $R_1$).

*Necessity*. From the assumed conditions it follows that the functional $(L\varphi)(0)$ is linear and continuous on $\mathscr{D}$. For this reason there exists an (obviously) unique generalized function $f_0 \in \mathscr{D}'$ such that $(L\varphi)(0) = (f_0(-x), \varphi)$. From this, by the property of translational invariance of the operator $L$, for all $x_0 \in \mathbb{R}^n$ we derive

$$(L\varphi(x+x_0))(0) = (L\varphi)(x_0) = (f_0(-x), \varphi(x+x_0)) = (f_0(x), \varphi(x_0 - x)) = (f_0 * \varphi)(x_0),$$

thus completing the proof of the lemma.

*Proof of necessity of the hypothesis of the theorem*. The operator $L_1 = L - L\delta *$ is linear, continuous and translation-invariant from $\mathscr{E}'$ to $\mathscr{D}'$ (see proof of sufficiency). Besides, for all $x_0 \in \mathbb{R}^n$ we have

$$L_1\delta(x+x_0) = (L\delta - L\delta * \delta)(x+x_0) = (L\delta - L\delta)(x+x_0) = 0$$

so that $L_1$ vanishes on all translations of the $\delta$ function. Now let $\varphi$ be an arbitrary basic function in $\mathscr{D}$. Then

$$\frac{1}{N^n} \sum_{0 \leqslant |k| \leqslant N} \varphi\left(\frac{k}{N}\right) \delta\left(x - \frac{k}{N}\right) \to \varphi(x), \quad N \to \infty \text{ in } \mathscr{E}',$$

because for any $\varphi \in C^\infty$

$$\frac{1}{N^n} \sum_{0 \leqslant |k| \leqslant N} \varphi\left(\frac{k}{N}\right) \psi\left(\frac{k}{N}\right) \to \int \varphi(x)\psi(x)\,dx, \quad N \to \infty.$$

Therefore, by virtue of the linearity and the continuity from $\mathscr{E}'$ to $\mathscr{D}'$ of the operator $L_1$,

$$L_1\varphi = \lim_{N\to\infty} L_1 \left[\frac{1}{N^n} \sum_{0 \leqslant |k| \leqslant N} \varphi\left(\frac{k}{N}\right) \delta\left(x - \frac{k}{N}\right)\right] = 0, \quad \varphi \in \mathscr{D}.$$

Now let $f$ be any generalized function in $\mathscr{E}'$. There exists a sequence $\{f_k\}$ of functions in $\mathscr{D}$ that converges to $f$ in $\mathscr{E}'$ (see Sec. 4.6). From this fact and from the continuity from $\mathscr{E}'$ to $\mathscr{D}'$ of the operator $L_1$ we conclude that $L_1 f = \lim_{k\to\infty} L_1 f_k = 0$ for all $f \in \mathscr{E}'$ so that $L_1 = 0$ and, hence, $L = L\delta * = f_0 *$.

The uniqueness of the kernel $f_0$ of the operator $L$ stems from the following reasoning: if $f_1 \in \mathscr{D}'$ is such that $f_1 * f = 0$ for all $f \in \mathscr{E}'$ and, hence, for all $f \in \mathscr{D}'$, then, by the above-proved lemma, $f_1 = 0$. This completes the proof of the theorem.

**4.8 Some applications** (a) *Newtonian potential.* Let $f \in \mathscr{D}'$. The convolutions

$$V_n = \frac{1}{|x|^{n-2}} * f, \quad n \geqslant 3; \qquad V_2 = \ln\frac{1}{|x|} * f, \quad n = 2$$

(if they exist) are called the *Newtonian* (for $n = 2$, the *logarithmic*) *potential* with density $f$.

The potential $V_n$ satifies the Poisson equation

$$\nabla^2 V_n = -(n-2)\,\sigma_n f, \quad n \geqslant 3; \qquad \nabla^2 V_2 = -2\pi f.$$

Indeed, using the formulas (3.10) of Sec. 2 and (2.4), we obtain, for $n \geqslant 3$,

$$\nabla^2 V_n = \nabla^2\left(\frac{1}{|x|^{n-2}} * f\right) = \nabla^2 \frac{1}{|x|^{n-2}} * f$$
$$= -(n-2)\,\sigma_n \delta * f = -(n-2)\,\sigma_n f.$$

We proceed in similar fashion in the case of $n = 2$ as well.

If $f = \rho(x)$ is a finite function summable on $\mathbb{R}^n$, $n \geqslant 3$, then the corresponding Newtonian potential $V_n$ is called the *volume potential.* In this case, $V_n$ is a locally summable function in $\mathbb{R}^n$ and is given by the integral

$$V_n(x) = \int \frac{\rho(y)\,dy}{|x-y|^{n-2}} \tag{8.1}$$

in accordance with formula (1.1) for the convolution of a finite function $\rho(x)$ summable in $\mathbb{R}^n$ with the function $|x|^{-n+2}$ locally summable in $\mathbb{R}^n$.

Let $f = \mu\delta_S$ and $f = -\frac{\partial}{\partial \mathbf{n}}(\nu\delta_S)$ be a simple layer and a double layer on a piecewise-smooth surface $S \subset \mathbb{R}^n$, $n \geqslant 3$, with surface densities $\mu$ and $\nu$ (see Secs. 1.7 and 2.3). The corresponding

Newtonian potentials

$$V_n^{(0)} = \frac{1}{|x|^{n-2}} * \mu\delta_S, \qquad V_n^{(1)} = -\frac{1}{|x|^{n-2}} * \frac{\partial}{\partial \mathbf{n}} (\nu\delta_S)$$

are, respectively, the *surface potentials of a simple layer* and *a double layer* with densities $\mu$ and $\nu$.

If $S$ is a bounded surface, then the surface potentials $V_n^{(0)}$ and $V_n^{(1)}$ are locally summable functions in $\mathbb{R}^n$ and can be represented by the integrals

$$V_n^{(0)}(x) = \int_S \frac{\mu(y)}{|x-y|^{n-2}} \, dS_y,$$

$$V_n^{(1)}(x) = \int_S \nu(y) \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|x-y|^{n-2}} \, dS_y. \tag{8.2}$$

For the sake of definiteness, let us prove the representation (8.2) for the potential $V_n^{(1)}$. Using the representation (3.3) and the definition of a double layer (see Sec. 2.3), for all $\varphi \in \mathcal{D}$ we obtain a chain of equalities (the function $\eta \in \mathcal{D}$ and $\eta(x) \equiv 1$ in the neighbourhood of $S$):

$$\begin{aligned}
(V_n^{(1)}, \varphi) &= -\left(\frac{1}{|x|^{n-2}} * \frac{\partial}{\partial \mathbf{n}} (\nu\delta_S), \varphi\right) \\
&= -\left(\frac{1}{|\xi|^{n-2}} \times \frac{\partial}{\partial \mathbf{n}} (\nu\delta_S)(y), \eta(y)\varphi(y+\xi)\right) \\
&= -\left(\frac{\partial}{\partial \mathbf{n}} (\nu\delta_S), \eta(y) \int \frac{\varphi(y+\xi)}{|\xi|^{n-2}} d\xi\right) \\
&= \int \nu(y) \frac{\partial}{\partial \mathbf{n}} \left[\eta(y) \int \frac{\varphi(y+\xi)}{|\xi|^{n-2}} d\xi\right] dS_y \\
&= \int_S \nu(y) \frac{\partial}{\partial \mathbf{n}} \int \frac{\varphi(x)}{|x-y|^{n-2}} \, dx \, dS_y \\
&= \int_S \nu(y) \int \varphi(x) \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|x-y|^{n-2}} \, dx \, dS_y \\
&= \int \varphi(x) \int_S \nu(y) \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|x-y|^{n-2}} \, dS_y \, dx,
\end{aligned}$$

whence follows the required formula (8.2) for $V_n^{(1)}$. The change in the order of integration is ensured by the Fubini theorem, by

virtue of the existence of the iterated integral

$$\int_S |\nu(y)| \int |\varphi(x)| \left| \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|x-y|^{n-2}} \right| dx\, dS_y.$$

(b) *Green's formula.* Let the region $G \subset \mathbb{R}^n$, $n \geqslant 3$, be bounded by a piecewise-smooth boundary $S$ and let the function $u \in C^2(G) \cap C^1(\bar{G})$. Then it can be represented in the form of a sum of three Newtonian potentials via *Green's formula* ($\mathbf{n}$ is an outer normal to $S$):

$$-\frac{1}{(n-2)\sigma_n} \left\{ \int_G \frac{\nabla^2 u(y)}{|x-y|^{n-2}} dy \right.$$

$$\left. - \int_S \left[ \frac{1}{|x-y|^{n-2}} \frac{\partial u(y)}{\partial \mathbf{n}} - u(y) \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|x-y|^{n-2}} \right] dS_y \right\}$$

$$= \begin{cases} u(x), & x \in G, \\ 0, & x \,\bar{\in}\, \bar{G}. \end{cases} \tag{8.3}$$

Indeed, assuming that the function $u(x)$ has been continued by zero for $x \,\bar{\in}\, \bar{G}$ and taking advantage of the formulas (3.7′) and (3.10) of Sec. 2, we conclude that

$$u = \delta * u = -\frac{1}{(n-2)\sigma_n} \nabla^2 \frac{1}{|x|^{n-2}} * u$$

$$= -\frac{1}{(n-2)\sigma_n |x|^{n-2}} * \nabla^2 u$$

$$= -\frac{1}{(n-2)\sigma_n |x|^{n-2}} * \left[ \nabla^2_{\mathrm{cl}} u - \frac{\partial u}{\partial n} \delta_S - \frac{\partial}{\partial n} (u \delta_S) \right].$$

Whence, using (8.1) and (8.2), we convince ourselves that the representation (8.3) holds true.

In particular, if the function $u(x)$ is harmonic in the region $G$, then the representation (8.3) transforms into the *Green's formula for harmonic functions*:

$$\frac{1}{(n-2)\sigma_n} \int_S \left[ \frac{1}{|x-y|^{n-2}} \frac{\partial u(y)}{\partial \mathbf{n}} - u(y) \frac{\partial}{\partial \mathbf{n}_y} \frac{1}{|x-y|^{n-2}} \right] dS_y$$

$$= \begin{cases} u(x), & x \in G, \\ 0, & x \,\bar{\in}\, \bar{G}. \end{cases} \tag{8.4}$$

Formulas similar to (8.3) and (8.4) occur in the case of $n = 2$ as well. Then, the fundamental solution $-\frac{1}{(n-2)\sigma_n |x|^{n-2}}$ must be replaced by $\frac{1}{2\pi} \ln |x|$.

*Remark.* Green's formula (8.4) expresses the values of the harmonic function in the region in terms of its values and the values of its normal derivative on the boundary of that region. In that sense, it is similar to Cauchy's formula for analytic functions.

(c) A *convolution equation* has the form

$$a * u = f \tag{8.5}$$

where $a$ and $f$ are specified generalized functions in $\mathcal{D}'$ and $u$ is an unknown generalized function in $\mathcal{D}'$. Convolution equations involve all linear partial differential equations with constant coefficients:

$$a(x) = \sum_{|\alpha| \leqslant m} a_\alpha D^\alpha \delta(x), \qquad a * u = \sum_{|\alpha| \leqslant m} a_\alpha D^\alpha u(x);$$

linear difference equations:

$$a(x) = \sum_\alpha a_\alpha \delta(x - x_\alpha), \qquad a * u = \sum_\alpha a_\alpha u(x - x_\alpha);$$

linear integral equations of the first kind:

$$a \in \mathcal{L}^1_{\text{loc}}, \qquad a * u = \int u(y)\, a(x - y)\, dy;$$

linear integral equations of the second kind:

$$a = \delta + \mathcal{K}, \qquad \mathcal{K} \in \mathcal{L}^1_{\text{loc}},$$

$$a * u = u(x) + \int u(y)\, \mathcal{K}(x - y)\, dy;$$

linear integro-differential equations; and so forth.

The *fundamental solution* of the convolution operator $a *$ is a generalized function $\mathcal{E} \in \mathcal{D}'$ that satisfies the equation (8.5) for $f = \delta$,

$$a * \mathcal{E} = \delta \tag{8.6}$$

Generally speaking, the fundamental solution $\mathcal{E}$ is not unique; it is determined up to the summand $\mathcal{E}_0$, which is an arbitrary solu-

tion in $\mathscr{D}'$ of the homogeneous equation $a * \mathscr{E}_0 = 0$. Indeed,

$$a * (\mathscr{E} + \mathscr{E}_0) = a * \mathscr{E} + a * \mathscr{E}_0 = \delta.$$

*Examples.* (1) The function $\mathscr{E}_n(x)$ defined in Sec. 2.3(h) is a fundamental solution of the Laplace operator: $\nabla^2 \mathscr{E}_n = \delta$.

(2) The formula $\theta(x) + C$ yields the general form of the fundamental solution in $\mathscr{D}'$ of the operator $\frac{d}{dx} = \delta' *$ (see Sec. 2.2 and Sec. 2.3(c)).

Let the fundamental solution $\mathscr{E}$ of the operator $a *$ in $\mathscr{D}'$ exist. We denote by $A(a, \mathscr{E})$ the collection of those generalized functions $f$ taken from $\mathscr{D}'$ for which the convolutions $\mathscr{E} * f$ and $a * \mathscr{E} * f$ exist in $\mathscr{D}'$.

The following theorem holds.

Theorem *Suppose $f \in A(a, \mathscr{E})$. Then the solution $u$ of the equation (8.5) exists and can be expressed by the formula*

$$u = \mathscr{E} * f. \tag{8.7}$$

*The solution of (8.5) is unique in the class $A(a, \mathscr{E})$.*

*Proof.* The generalized function $u = \mathscr{E} * f$ satisfies (8.5) since, by virtue of the commutativity and the associativity of a convolution (see Sec. 4.2(h)) the convolutions $\mathscr{E} * f$ and $a * \mathscr{E} = \delta$ exist:

$$a * u = a * (\mathscr{E} * f) = a * \mathscr{E} * f = (a * \mathscr{E}) * f = \delta * f = f.$$

Uniqueness: if $a * u = 0$ and $u \in A(a, \mathscr{E})$, then

$$u = u * \delta = u * (a * \mathscr{E}) = u * a * \mathscr{E} = (u * a) * \mathscr{E}$$
$$= 0 * \mathscr{E} = 0,$$

which is what we set out to prove. The proof is complete.

*Remark.* We can give the solution $u = \mathscr{E} * f$, (8.7), the following physical interpretation. Let us represent the source $f(x)$ in the form of a "sum" of point sources $f(\xi)\,\delta(x-\xi)$ (see Sec. 4.2(b)),

$$f(x) = f * \delta = \int f(\xi)\,\delta(x-\xi)\,d\xi.$$

The fundamental solution $\mathscr{E}(x)$ is the perturbation due to the point source $\delta(x)$. Whence, by virtue of the linearity and translational invariance of the convolution operator $a *$ (see Sec. 4.7) it follows that each point source $f(\xi)\,\delta(x-\xi)$ generates a per-

turbation $f(\xi)\,\mathscr{E}(x-\xi)$. It is therefore natural to expect tha the "sum" (superposition) of these perturbations

$$\int f(\xi)\,\mathscr{E}(x-\xi)\,d\xi=\mathscr{E}*f$$

will yield a total perturbation due to the source $f$, that is, the solution $u$ of the equation (8.5). This nonrigorous reasoning is brought into shape by the theorem proved above.

(d) *Equations in convolution algebras.* Let $A$ be a convolution algebra, for example $\mathscr{D}'(\Gamma+)$, $\mathscr{D}'(\Gamma)$ (see Sec. 4.5). Let us consider the equation (8.5) in the algebra $A$, that is, we will assume that $a\in A$ and $f\in A$; the solution $u$ will also be sought in $A$. In the algebra $A$, the above theorem takes the following form: *if the fundamental solution $\mathscr{E}$ of the operator $a*$ exists in $A$, then the solution $u$ of equation (8.5) is unique in $A$, exists for any $f$ taken from $A$, and can be expressed by the formula $u=\mathscr{E}*f$.*

The fundamental solution $\mathscr{E}$ of the operator $a*$ in the algebra $A$ is conveniently denoted as $a^{-1}$ so that, by (8.6),

$$a^{-1}*a=\delta \tag{8.8}$$

In other words, $a^{-1}$ is the *inverse element* of $a$ in the algebra $A$.

The following proposition is very useful when constructing fundamental solutions in the $A$ algebra:

*if $a_1^{-1}$ and $a_2^{-1}$ exist in $A$, then*

$$(a_1*a_2)^{-1}=a_1^{-1}*a_2^{-1}. \tag{8.9}$$

Indeed,

$$\begin{aligned}(a_1*a_2)*(a_1^{-1}*a_2^{-1})&=(a_2*a_1)*(a_1^{-1}*a_2^{-1})\\&=a_2*((a_1*a_1^{-1})*a_2^{-1})\\&=a_2*(\delta*a_2^{-1})=a_2*a_2^{-1}=\delta.\end{aligned}$$

Formula (8.9) forms the basis of operational calculus.

(e) *Fractional differentiation and integration.* Denote by $\mathscr{D}'_+$ the algebra $\mathscr{D}'(\overline{\mathbb{R}^1_+})$.

We introduce the generalized function $f_\alpha$, taken from $\mathscr{D}'_+$, that depends on a real parameter $\alpha$, $-\infty<\alpha<\infty$, via the formula

$$f_\alpha(x)=\begin{cases}\dfrac{\theta(x)\,x^{\alpha-1}}{\Gamma(\alpha)}, & \alpha>0,\\ f'_{\alpha+1}, & \alpha\leqslant 0.\end{cases}$$

Let us verify that

$$f_\alpha * f_\beta = f_{\alpha+\beta}. \tag{8.10}$$

Indeed, if $\alpha > 0$ and $\beta > 0$, then (see Sec. 4.1)

$$\begin{aligned} f_\alpha * f_\beta &= \frac{\theta(x)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^x y^{\alpha-1}(x-y)^{\beta-1}\,dy \\ &= \frac{\theta(x)\,x^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt \\ &= \frac{\theta(x)\,x^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} = f_{\alpha+\beta}. \end{aligned}$$

Now if $\alpha \leqslant 0$ or $\beta \leqslant 0$, then, by choosing integers $m > -\alpha$ and $n > -\beta$, we obtain

$$\begin{aligned} f_\alpha * f_\beta &= f_{\alpha+m}^{(m)} * f_{\beta+n}^{(n)} = (f_{\alpha+m} * f_{\beta+n})^{(m+n)} \\ &= f_{\alpha+\beta+m+n}^{(m+n)} = f_{\alpha+\beta}, \end{aligned}$$

which is what we set out to prove.

Let us consider the convolution operator $f_\alpha *$ in the algebra $\mathscr{D}'_+$. Since $f_0 = \theta' = \delta$, it follows from (8.10) that the fundamental solution $f_\alpha^{-1}$ of the operator $f_\alpha *$ exists and is equal to $f_{-\alpha}$: $f_\alpha^{-1} = f_{-\alpha}$. Furthermore, for integral $n < 0$, $f_n = \delta^{(n)}$, and for this reason $f_n * u = \delta^{(n)} * u = u^{(n)}$, which means the operator $f_n *$ is the operator of $n$-fold differentiation. Finally, for integral $n > 0$,

$$(f_n * u)^{(n)} = f_{-n} * (f_n * u) = (f_{-n} * f_n) * u = \delta * u = u,$$

which is to say that $f_n * u$ is an antiderivative of order $n$ of the generalized function $u$ (see Sec. 2.2).

By virtue of what has been said, the operator $f_\alpha *$ is termed the *operator of fractional differentiation of order* $\alpha$ for $\alpha < 0$ and the *operator of fractional integration of order* $\alpha$ for $\alpha > 0$ (it is also called the *Riemann-Liouville* operator).

*Example.* Let $f \in \mathscr{D}'_+$. Then

$$D^{1/2} f = D(f_{1/2} * f) = \frac{1}{\sqrt{\pi}} \frac{d}{dx} \int_0^x \frac{f(y)\,dy}{\sqrt{x-y}}.$$

(f) *Heaviside's operational calculus* is nothing but analysis in the convolution algebra $\mathscr{D}'_+$. To illustrate, let us calculate in the

algebra $\mathcal{D}'_+$ the fundamental solution $\mathcal{E}(t)$ of the differential operator

$$P\left(\frac{d}{dt}\right)=\frac{d^m}{dt^m}+a_1\frac{d^{m-1}}{dt^{m-1}}+\ldots+a_m,$$

where $a_j$ are constants. In the $\mathcal{D}'_+$ algebra the corresponding equation takes the form

$$P(\delta)*\mathcal{E}=\delta,$$
$$P(\delta)(t)=\delta^{(m)}(t)+a_1\delta^{(m-1)}(t)+\ldots+a_m\delta(t).$$

If, in the $D'_+$ algebra, we factor $P(\delta)$§,

$$P(\delta)=*\prod_j(\delta'-\lambda_j\delta)^{k_j},$$

and take advantage of (8.9), we obtain

$$P^{-1}(\delta)(t)=\mathcal{E}(t)=\left[*\prod_j(\delta'-\lambda_j\delta)^{k_j}\right]^{-1}$$
$$=*\prod_j(\delta'-\lambda_j\delta)^{-k_j}. \tag{8.11}$$

But it is easy to verify that

$$*(\delta'-\lambda\delta)^{-k}=*[(\delta'-\lambda\delta)^{-1}]^k=\frac{\theta(t)\,t^{k-1}}{(k-1)!}e^{\lambda t}, \tag{8.12}$$

whence, by continuing the equalities (8.11), we derive

$$\mathcal{E}(t)=*\prod_j\frac{\theta(t)\,t^{k_j-1}}{(k_j-1)!}e^{\lambda_j t}. \tag{8.13}$$

The convolution (8.13) admits of explicit calculation. By decomposing the right side of (8.11) into partial fractions in the $\mathcal{D}'_+$ algebra, we obtain

$$\mathcal{E}(t)=*\prod_j(\delta'-\lambda_j\delta)^{-k_j}$$
$$=\sum_j[c_{j,k_j}*(\delta'-\lambda_j\delta)^{-k_j}+\ldots+c_{j,1}*(\delta'-\lambda_j\delta)^{-1}],$$

§The symbol $*\prod\limits_{1\leqslant j\leqslant l}a_j$ stands for $a_1*a_2*\ldots*a_l$.

whence, using (8.12), we finally derive

$$\mathscr{E}(t)=\theta(t)\sum_j\left[c_{j,k_j}\frac{t^{k_j-1}}{(k_j-1)!}+\ldots+c_{j,1}\right]e^{\lambda_j t}.$$

We thus have the following rule for finding the fundamental solution of the operator $P\left(\frac{d}{dt}\right)$: substitute $p$ for $\frac{d}{dt}$, set up the polynomial $P(p)$, decompose the expression $\frac{1}{P(p)}$ into partial fractions:

$$\frac{1}{P(p)}=\prod_j(p-\lambda_j)^{-k_j}=\sum_j[c_{j,k_j}(p-\lambda_j)^{-k_j}+\ldots+c_{j,1}(p-\lambda_j)^{-1}],$$

and with each partial fraction $(p-\lambda)^{-k}$ associate the right side of formula (8.12).

*Example.* Fing $\mathscr{E}$ if $\mathscr{E}''+\omega^2\mathscr{E}=\delta$.

We have

$$\frac{1}{p^2+\omega^2}=\frac{1}{2\omega i}\left(\frac{1}{p-i\omega}+\frac{1}{p+i\omega}\right)\leftrightarrow\frac{\theta(t)}{2\omega i}(e^{i\omega t}-e^{-i\omega t})$$
$$=\theta(t)\frac{\sin\omega t}{\omega}=\mathscr{E}(t).$$

# 5 Generalized Functions of Slow Growth

**5.1 The space $\mathscr{S}$ of basic (rapidly diminishing) functions** We refer to the space of basic functions $\mathscr{S}=\mathscr{S}(\mathbb{R}^n)$ all functions infinitely differentiable in $\mathbb{R}^n$ that decrease together with all their derivatives, as $|x|\to\infty$, faster than any power of $|x|^{-1}$. We introduce in $\mathscr{S}$ a countable number of norms via the formula

$$\|\varphi\|_p=\sup_{x,\ |\alpha|\leqslant p}(1+|x|^2)^{\frac{p}{2}}|D^\alpha\varphi(x)|,\qquad \varphi\in\mathscr{S},\quad p=0,1,\ldots.$$

Clearly,

$$\|\varphi\|_0\leqslant\|\varphi\|_1\leqslant\|\varphi\|_2\leqslant\ldots,\qquad \varphi\in\mathscr{S}. \tag{1.1}$$

We define convergence in $\mathscr{S}$ as follows: the sequence of functions $\varphi_1,\varphi_2,\ldots$ in $\mathscr{S}$ converges to 0, $\varphi_k\to 0$, $k\to\infty$ in $\mathscr{S}$, if for all $p=0,1,\ldots$ $\|\varphi_k\|_p\to 0$, $k\to\infty$. In other words, $\varphi_k\to 0$,

$k \to \infty$ in $\mathscr{S}$, if for all $\alpha$ and $\beta$

$$x^\alpha D^\beta \varphi_k(x) \overset{x \in R^n}{\rightrightarrows} 0, \qquad k \to \infty.$$

It is clear that $\mathscr{D} \subset \mathscr{S}$, and if $\varphi_k \to 0$, $k \to \infty$ in $\mathscr{D}$, then $\varphi_k \to 0$, $k \to \infty$ in $\mathscr{S}$.

However, $\mathscr{S}$ does not coincide with $\mathscr{D}$; for example, the function $e^{-|x|^2}$ belongs to $\mathscr{S}$ but does not belong to $\mathscr{D}$ (it is not finite).

Yet $\mathscr{D}$ is dense in $\mathscr{S}$, that is, for any $\varphi \in \mathscr{S}$ there is a sequence $\{\varphi_k\}$ of functions in $\mathscr{D}$ such that $\varphi_k \to \varphi$, $k \to \infty$ in $\mathscr{S}$.

Indeed, the sequence of functions, in $\mathscr{D}$,

$$\varphi_k(x) = \varphi(x)\, \eta\left(\frac{x}{k}\right), \qquad k = 1, 2, \ldots,$$

where $\eta \in \mathscr{D}$, $\eta(x) = 1$, $|x| < 1$, converges to $\varphi$ in $\mathscr{S}$.

Let us denote by $\mathscr{S}_p$ the adjoining of $\mathscr{S}$ in the $p$th norm; $\mathscr{S}_p$ is a Banach space.

The following inclusions hold:

$$\mathscr{S}_0 \supset \mathscr{S}_1 \supset \mathscr{S}_2 \supset \ldots \tag{1.2}$$

Each inclusion

$$\mathscr{S}_{p+1} \subset \mathscr{S}_p, \qquad p = 0, 1, \ldots,$$

is continuous, by (1.1). We will now prove that this inclusion is totally continuous (compact), that is, it is possible, from each infinite bounded set in $\mathscr{S}_{p+1}$, to choose a sequence that converges in $\mathscr{S}_p$.

Indeed, let $M$ be an infinite set bounded in $\mathscr{S}_{p+1}$, $\|\varphi\|_{p+1} < C$, $\varphi \in M$. From this, for all $\varphi \in M$ and $|\alpha| \leqslant p$, we obtain

$$\left|\frac{\partial}{\partial x_j} D^\alpha \varphi(x)\right| < C, \qquad j = 1, \ldots, n;$$

$$(1 + |x|^2)^{p/2} D^\alpha \varphi(x) \to 0, \qquad |x| \to \infty.$$

Suppose $R_k$, $k = 1, 2, \ldots$, is an increasing sequence of positive numbers such that

$$(1 + |x|^2)^{p/2} |D^\alpha \varphi(x)| < \frac{1}{k}, \quad |x| > R_k, \quad |\alpha| \leqslant p. \tag{1.3}$$

By Ascoli's lemma there is a sequence $\{\varphi_j^{(1)}\}$ of functions in $M$ that converges in $C^p(\bar{U}_{R_1})$; furthermore, by the same lemma there

is a subsequence $\{\varphi_j^{(2)}\}$ of the sequence $\{\varphi_j^{(1)}\}$ that converges in $C^p(\overline{U}_{R_2})$, and so on. It remains to remark that by virtue of (1.3) the diagonal sequence $\{\varphi_k^{(k)}\}$ converges in $\mathscr{S}_p$.

The following lemma gives an exact characteristic of functions taken from the space $\mathscr{S}_p$.

Lemma *So that $\varphi \in \mathscr{S}_p$, it is necessary and sufficient that $\varphi \in C^p$ and $|x|^p D^\alpha \varphi(x) \to 0$ for $|x| \to \infty$ and $|\alpha| \leqslant p$, so that $\varphi \in \overline{C}_0^p$.*

*Proof*. Necessity is obvious. Let us prove sufficiency. Suppose $\varphi \in \overline{C}_0^p$ and $\varphi_\varepsilon = \varphi * \omega_\varepsilon$ is a regularization of $\varphi$ (see Sec. 4.6). Furthermore, let $\{\eta_k\}$ be a sequence of functions taken from $\mathscr{D}$ that converges to 1 in $\mathbb{R}^n$ (see Sec. 4.1). Then the sequence $\{\varphi_{1/k}\eta_k\}$ of functions taken from $\mathscr{D} \subset \mathscr{S}$ converges to $\varphi$ in $\mathscr{S}_p$. Indeed, let $\varepsilon > 0$; there exists a number $R = R(\varepsilon)$ such that

$$(1+|x|^2)^{p/2}|D^\alpha \varphi(x)| < \varepsilon, \quad |x| > R, \quad |\alpha| \leqslant p. \tag{1.4}$$

Let $N_1$ be a number such that $\eta_k(x) = 1$, $|x| \leqslant R+1$, $k \geqslant N_1$. Finally, from Theorem II of Sec. 1.2 follows the existence of a number $N \geqslant N_1$ such that for all $k \geqslant N$, $|x| \leqslant R+1$, and $|\alpha| \leqslant p$, the following inequality holds true:

$$(1+|x|^2)^{p/2}|D^\alpha \varphi(x) - D^\alpha \varphi_{1/k}(x)| < \varepsilon. \tag{1.5}$$

Now, using (1.4) and (1.5) for $k \geqslant N$, we obtain

$$\|\varphi - \varphi_{1/k}\eta_k\|_p = \sup_{\substack{x \\ |\alpha| \leqslant p}} (1+|x|^2)^{p/2} |D^\alpha[\varphi(x) + \varphi_{1/k}(x)\eta_k(x)]|$$

$$\leqslant \varepsilon + \sup_{\substack{|x| > R+1 \\ |\alpha| \leqslant p}} (1+|x|^2)^{p/2} \Big[ |D^\alpha \varphi(x)|$$

$$+ \sum_{\beta \leqslant \alpha} \binom{\beta}{\alpha} |D^\beta \varphi_{1/k}(x) D^{\alpha-\beta}\eta_k(x)| \Big]$$

$$\leqslant 2\varepsilon + C_p' \sup_{\substack{|x| > R+1 \\ |\beta| \leqslant p}} (1+|x|^2)^{p/2} |D^\beta \int \omega_{1/k}(y)\varphi(x-y)\,dy|$$

$$\leqslant 2\varepsilon + C_p' \sup_{\substack{|x| > R+1 \\ |\beta| \leqslant p}} \int \omega_{1/k}(y)(1+|x|^2)^{p/2}|D^\beta \varphi(x-y)|\,dy$$

$$\leqslant 2\varepsilon + C_p \sup_{\substack{|x| > R+1 \\ |\beta| \leqslant p}} \int \omega_{1/k}(y)[(1+|x-y|^2)^{p/2}$$

$$+ |y|^p]|D^\beta \varphi(x-y)|\,dy$$

$$\leqslant 2\varepsilon + C_p\varepsilon + C_p\varepsilon \int \omega_{1/k}(y)(1+|y|^2)\,dy \leqslant (2+3C_p)\varepsilon,$$

which is what we set out to prove. The proof of the lemma is complete.

It follows from the lemma that $\mathscr{S}$ is a complete space and

$$\mathscr{S} = \bigcap_{p\geqslant 0} \mathscr{S}_{p}. \qquad (1.6)$$

The operations of differentiation $\varphi \to D^{\alpha}\varphi$ and of the nonsingular linear change of variables $\varphi(x) \to \varphi(Ax + b)$ are linear and continuous from $\mathscr{S}$ to $\mathscr{S}$. This follows directly from the definition of convergence in the space $\mathscr{S}$.

On the other hand, multiplication by an infinitely differentiable function may take one outside the domain of $\mathscr{S}$, for example, $e^{-|x|^2}e^{|x|^2} = 1 \overline{\in} \mathscr{S}$.

Suppose the function $a \in C^{\infty}$ grows at infinity together with all its derivatives not faster than the polynomial

$$|D^{\alpha}a(x)| \leqslant C_{\alpha}(1 + |x|)^{m_{\alpha}}. \qquad (1.7)$$

We denote by $\theta_M$ the set of all such functions. This is called the set of multipliers in $\mathscr{S}$.

The operation $\varphi \to a\varphi$, where $a \in \theta_M$, is continuous (and, obviously, linear) from $\mathscr{S}$ to $\mathscr{S}$.

Indeed, if $\varphi \in \mathscr{S}$, then $a\varphi \in C^{\infty}$ and, by virtue of (1.7),

$$\begin{aligned}\| a\varphi \|_p &= \sup_{x,\ |\alpha|\leqslant p} (1 + |x|^2)^{p/2} |D^{\alpha}(a\varphi)| \\ &\leqslant \sup_{x,\ |\alpha|\leqslant p} (1 + |x|^2)^{p/2} \sum_{\beta\leqslant\alpha} \binom{\beta}{\alpha} |D^{\beta}\varphi(x)\, D^{\alpha-\beta}a(x)| \\ &\leqslant K_p \sup_{x,\ |\alpha|\leqslant p} (1 + |x|^2)^{p/2+N_p/2} |D^{\alpha}\varphi(x)| \\ &= K_p \| \varphi \|_{p+N_p}, \qquad p = 0, 1, \ldots,\end{aligned}$$

where $N_p$ is the smallest integer not less than $\max\limits_{|\alpha|\leqslant p} m_{\alpha}$. These inequalities signify that $a\varphi \in \mathscr{S}$ and the operation $\varphi \to a\varphi$ is continuous from $\mathscr{S}$ to $\mathscr{S}$.

### 5.2 The space $\mathscr{S}'$ of generalized functions of slow growth

A *generalized function of slow growth* is any continuous linear functional on the space $\mathscr{S}$ of basic functions. We denote by $\mathscr{S}' = \mathscr{S}'(\mathbb{R}^n)$ the set of all generalized functions of slow growth. Clearly, $\mathscr{S}'$ is a linear set and $\mathscr{S}' \subset \mathscr{D}'$.

We define convergence in $\mathscr{S}'$ as weak convergence of a sequence of functionals: a sequence of generalized functions $f_1, f_2, \ldots$

taken from $\mathcal{S}'$ converges to the generalized function $f \in \mathcal{S}'$, $f_k \to f$, $k \to \infty$ in $\mathcal{S}'$, if for any $\varphi \in \mathcal{S}$, $(f_k, \varphi) \to (f, \varphi)$ $k \to \infty$. The linear set $\mathcal{S}'$ equipped with convergence is termed the *space* $\mathcal{S}'$ *of generalized functions of slow growth.*

From this definition it follows that convergence in $\mathcal{S}'$ implies convergence in $\mathcal{D}'$.

Theorem (L. Schwartz) *Let* $M'$ *be a weakly bounded set of functionals from* $\mathcal{S}'$, *that is,* $|(f, \varphi)| < C_\varphi$ *for all* $f \in M'$ *and* $\varphi \in \mathcal{S}$. *Then there are numbers* $K \geqslant 0$ *and* $m \geqslant 0$ *such that*

$$|(f, \varphi)| \leqslant K \| \varphi \|_m, \qquad f \in M', \quad \varphi \in \mathcal{S}. \tag{2.1}$$

*Proof.* If the inequality (2.1) does not hold, then there will be sequences $\{f_k\}$ of functionals from $M'$ and sequences $\{\varphi_k\}$ of functions taken from $\mathcal{S}$ such that

$$|(f_k, \varphi_k)| \geqslant k \| \varphi_k \|_k, \qquad k = 1, 2, \ldots \tag{2.2}$$

The sequence of functions

$$\psi_k(x) = \frac{\varphi_k(x)}{\sqrt{k} \| \varphi_k \|_k}, \qquad k = 1, 2, \ldots,$$

tends to 0 in $\mathcal{S}$ because for $k \geqslant p$

$$\| \psi_k \|_p = \frac{\| \varphi_k \|_p}{\sqrt{k} \| \varphi_k \|_k} \leqslant \frac{1}{\sqrt{k}}.$$

The sequence of functionals $\{f_k\}$ is bounded on every basic function $\varphi$ taken from $\mathcal{S}$. For this reason, we have an analogue of the lemma of Sec. 1.4 according to which $(f_k, \psi_k) \to 0$, $k \to \infty$. On the other hand, the inequality (2.2) yields

$$|(f, \psi_k)| = \frac{1}{\sqrt{k} \| \varphi_k \|_k} |(f, \varphi_k)| \geqslant \sqrt{k}.$$

The resulting contradiction proves the theorem.

From the Schwartz theorem we have just proved there follow a number of corollaries.

Corollary 1 *Every generalized function of slow growth has a finite order* (compare Sec. 1.3), that is to say, it admits of an extension as a continuous linear functional from some (least) conjugate space $\mathcal{S}'_m$; then, for $f$, the inequality (2.1) takes the form

$$|(f, \varphi)| \leqslant \| f \|_{-m} \| \varphi \|_m, \qquad \varphi \in \mathcal{S}, \tag{2.3}$$

where $\|f\|_{-m}$ is the norm of the functional $f$ in $\mathscr{S}'_m$ and $m$ is the order of $f$.

Thus, the following relations hold true:

$$\mathscr{S}'_0 \subset \mathscr{S}'_1 \subset \mathscr{S}'_2 \subset \ldots, \qquad \mathscr{S}' = \bigcup_{p \geqslant 0} \mathscr{S}'_p \tag{2.4}$$

They are duals of (1.2) and (1.6).

Also note that every inclusion

$$\mathscr{S}'_1 \subset \mathscr{S}'_{p+1}, \qquad p = 0, 1, \ldots,$$

is fully continuous (see Sec. 5.1); in particular, every weakly convergent sequence of functionals taken from $\mathscr{S}'_p$ converges in norm in $\mathscr{S}'_{p+1}$.

Corollary 2 *Every (weakly) convergent sequence of generalized functions of slow growth converges weakly in some space $\mathscr{S}'_p$ and, hence, converges in norm in $\mathscr{S}'_{p+1}$.*

This follows from the Schwartz theorem since every (weakly) convergent sequence of functionals taken from $\mathscr{S}'$ is a weakly bounded set in $\mathscr{S}'$; it also follows from the remark referring to Corollary 1.

Corollary 3 *The space of generalized functions of slow growth is complete.*

This follows from the weak completeness of the conjugate spaces $\mathscr{S}'_p$ and from Corollary 2.

**5.3 Examples of generalized functions of slow growth and elementary operations in $\mathscr{S}'$** If $f(x)$ is a function of slow growth in $\mathbb{R}^n$, that is, for some $m \geqslant 0$

$$\int |f(x)| (1 + |x|)^{-m} \, dx < \infty,$$

then it determines a regular functional $f$ in $\mathscr{S}'$ via the formula (6.1) of Sec. 1,

$$(f, \varphi) = \int f(x) \varphi(x) \, dx, \qquad \varphi \in \mathscr{S}.$$

Not every locally summable function defines a generalized function of slow growth, for example, $e^x \overline{\in} \mathscr{S}'$. On the other hand, not every locally summable function taken from $\mathscr{S}'$ is of slow growth. For example, the function $(\cos e^x)' = -e^x \sin e^x$ is not a function of slow growth, yet it defines a generalized function

from $\mathscr{S}'$ via the formula

$$((\cos e^x)', \ \varphi) = -\int \cos e^x \varphi'(x)\, dx, \qquad \varphi \in \mathscr{S}.$$

However, there can be no such unpleasantness as regards nonnegative functions (and even measures), as we shall now see.

A measure $\mu$ specified on $\mathbb{R}^n$ (see Sec. 1.7) is said to be a *measure of slow growth* if for some $m \geqslant 0$

$$\int (1 + |x|)^{-m} \mu(dx) < \infty.$$

It defines a generalized function in $\mathscr{S}'$ via formula (7.2) of Sec. 1,

$$(\mu, \ \varphi) = \int \varphi(x)\, \mu(dx), \qquad \varphi \in \mathscr{S}.$$

If a nonnegative measure $\mu$ defines a generalized function in $\mathscr{S}'$ then $\mu$ is of slow growth.

Indeed, since $\mu \in \mathscr{S}'$, it follows from the Schwartz theorem that it is of finite order $m$ so that

$$\left| \int \varphi(x)\, \mu(dx) \right| \leqslant K \| \varphi \|_m, \qquad \varphi \in \mathscr{S}. \tag{3.1}$$

Let $\{\eta_k\}$ be a sequence of nonnegative functions in $\mathscr{D}$ that tend to 1 in $\mathbb{R}^n$ (see Sec. 4.1). Substituting into (3.1)

$$\varphi(x) = \eta_k(x)\,(1 + |x|^2)^{-m/2}$$

and making use of the nonnegativity of the measure $\mu$, we obtain

$$\int \eta_k(x)\,(1 + |x|^2)^{-m/2} \mu(dx) \leqslant C$$

where $C$ does not depend on $k$. From this, by virtue of the Fatou lemma, it follows that the measure $\mu$ is of slow growth.

If $f \in \mathscr{E}'$, then $f \in \mathscr{S}'$, and

$$(f, \ \varphi) = (f, \ \eta\varphi), \qquad \varphi \in \mathscr{S}, \tag{3.2}$$

where $\eta \in \mathscr{D}$ and $\eta = 1$ in the neighbourhood of the support of $f$ (compare (10.2) of Sec. 1).

Indeed, since the operation $\varphi \to \eta\varphi$ is linear and continuous from $\mathscr{S}$ to $\mathscr{D}$, the functional $(f, \eta\varphi)$ on the right-hand side of (3.2)

is linear and continuous on $\mathscr{S}$ so that $f \in \mathscr{S}'$. The uniqueness of the extension follows from the density of $\mathscr{D}$ in $\mathscr{S}$ (see Sec. 5.1); in particular, it is independent of the auxiliary function $\eta$.

If $f \in \mathscr{S}'$, then every derivative $D^\alpha f \in \mathscr{S}'$ as well; here, the operation $f \to D^\alpha f$ is continuous (and linear) from $\mathscr{S}'$ to $\mathscr{S}'$.

Indeed, since the operation $\varphi \to D^\alpha \varphi$ is linear and continuous from $\mathscr{S}$ to $\mathscr{S}$ (see Sec. 5.1), it follows that the right-hand side of

$$(D^\alpha f, \varphi) = (-1)^{|\alpha|} (f, D^\alpha \varphi), \qquad \varphi \in \mathscr{S},$$

is a continuous linear functional on $\mathscr{S}$ (compare Sec. 2.1).

If $f \in \mathscr{S}'$ and $\det A \neq 0$, then $f(Ax+b) \in \mathscr{S}'$, and the operation $f(x) \to f(Ax+b)$ is continuous (and linear) from $\mathscr{S}'$ to $\mathscr{S}'$.

True enough, since the operation $\varphi(x) \to \varphi[A^{-1}(x-b)]$ is linear and continuous from $\mathscr{S}$ to $\mathscr{S}$ (see Sec. 5.1), the right-hand side of

$$(f(Ay+b), \varphi) = \left(f, \frac{\varphi[A^{-1}(x-b)]}{|\det A|}\right), \qquad \varphi \in \mathscr{S},$$

is a continuous linear functional on $\mathscr{S}$ (compare Sec. 1.9).

If $f \in \mathscr{S}'$ and $a \in \theta_M$, then $af \in \mathscr{S}'$, and the operation $f \to af$ is continuous (and linear) from $\mathscr{S}'$ to $\mathscr{S}'$.

Indeed, since the operation $\varphi \to a\varphi$ is linear and continuous from $\mathscr{S}$ to $\mathscr{S}$ (see Sec. 1.5), it follows that the right-hand side of the equality

$$(af, \varphi) = (f, a\varphi), \qquad \varphi \in S,$$

is a continuous linear functional on $\mathscr{S}$ (compare Sec. 1.10).

Thus, the set $\theta_M$ contains all multipliers in $\mathscr{S}'$ (actually, it consists of them; prove it).

*Example.* If $|a_k| \leqslant C(1+|k|)^N$, then

$$\sum_k a_k \delta(x-k) \in \mathscr{S}'.$$

## 5.4 The structure of generalized functions of slow growth

We will now prove that the space $\mathscr{S}'$ is a (smallest) extension of the collection of functions of slow growth in $\mathbb{R}^n$ such that in it differentiation is always possible (compare Sec. 2.4). Hence the name of $\mathscr{S}'$ as the space of generalized functions of slow growth.

Theorem *If $f \in \mathscr{S}'$, then there exist a continuous function $g$ of slow growth in $\mathbb{R}^n$ and an integer $m \geqslant 0$ such that*

$$f(x) = D_1^m \ldots D_n^m g(x). \tag{4.1}$$

*Proof*. Let $f \in \mathscr{S}'$. By the theorem of L. Schwartz (see Sec. 5.2) there exist numbers $K$ and $p$ such that for all $\varphi \in \mathscr{S}$

$$|(f, \varphi)| \leqslant K \|\varphi\|_p \leqslant K \max_{|\alpha| \leqslant p} \int |D_1 \ldots D_n [(1+|x|^2)^{p/2} D^\alpha \varphi(x)]|\, dx,$$

that is,

$$|(f, \varphi)| \leqslant K \max_{|\alpha| \leqslant p} \| D_1 \ldots D_n [(1+|x|^2)^{p/2} D^\alpha \varphi] \|_{\mathscr{L}^1}, \quad (4.2)$$

With every function $\varphi$ of $\mathscr{S}$ we associate a vector function $\{\psi_\alpha\}$ with components

$$\psi_\alpha(x) = D_1 \ldots D_n [(1 + |x|^2)^{p/2} D^\alpha \varphi(x)], \qquad |\alpha| \leqslant p. \quad (4.3)$$

In this way we define a one-to-one mapping $\varphi \to \{\psi_\alpha\}$ of the space $\mathscr{S}$ into the direct sum $\bigoplus_{|\alpha| \leqslant p} \mathscr{L}^1$ with norm

$$\|\{f_\alpha\}\| = \max_{|\alpha| \leqslant p} \|f_\alpha\|_{\mathscr{L}^1}.$$

On the linear subset $[\{\psi_\alpha\}, \varphi \in \mathscr{S}]$ of the space $\bigoplus_{|\alpha| \leqslant p} \mathscr{L}^1$, in which the components $\psi_\alpha$ are defined by (4.3), we introduce a linear functional $f^*$:

$$(f^*, \{\psi_\alpha\}) = (f, \varphi). \quad (4.4)$$

By virtue of the estimate (4.2),

$$|(f^*, \{\psi_\alpha\})| = |(f, \varphi)| \leqslant K \max_{|\alpha| \leqslant p} \|\psi_\alpha\|_{\mathscr{L}^1} = K \|\{\psi_\alpha\}\|,$$

the functional $f^*$ is continuous. By the Hahn-Banach and F. Riesz theorems there exists a vector function $\{\chi_\alpha\} \in \bigoplus_{|\alpha| \leqslant p} \mathscr{L}^\infty$ such that

$$(f^*, \{\psi_\alpha\}) = \sum_{|\alpha| \leqslant p} \int \chi_\alpha(x)\, \psi_\alpha(x)\, dx.$$

That is to say, by virtue of (4.3) and (4.4), we have

$$(f, \varphi) = \sum_{\alpha| \leqslant p} \int \chi_\alpha(x)\, D_1 \ldots D_n [(1 + |x|^2)^{p/2} D^\alpha \varphi(x)]\, dx, \qquad \varphi \in \mathscr{S}.$$

Integrating the right-hand side of this equation by parts, we are convinced of the existence of continuous functions $g_\alpha$, $|\alpha| \leqslant \leqslant p+2$, of slow growth such that

$$(f, \varphi) = (-1)^{pn} \int \sum_{|\alpha| \leqslant (p+2)n} g_\alpha(x) D_1^{p+2} \ldots D_n^{p+2} \varphi(x)\, dx,$$

whence follows the representation (4.1) for $m = p+2$. The proof of the theorem is complete.

Corollary *If $f \in \mathscr{S}'$, then there exists an integer $p \geqslant 0$ such that for any $\varepsilon > 0$ there are functions $g_{\alpha, \varepsilon}$, $|\alpha| \leqslant p$, which are continuous, of slow growth in $\mathbb{R}^n$ and such that vanish outside the $\varepsilon$-neighbourhood of the support of $f$, so that*

$$f(x) = \sum_{|\alpha| \leqslant p} D^\alpha g_{\alpha, \varepsilon}(x). \tag{4.5}$$

Indeed, suppose $\varepsilon > 0$ and $\eta \in \theta_M$, $\eta(x) = 1$, $x \in (\operatorname{supp} f)^{\varepsilon/3}$, and $\eta(x) = 0$, $x \bar{\in} (\operatorname{supp} f)^\varepsilon$. (By the lemma of Sec. 1.2, such functions exist.) Then, taking into account the representation (4.1) and using the Leibniz formula (see Sec. 2.1), we have

$$\begin{aligned} f(x) &= \eta(x) f(x) \\ &= \eta(x) D_1^m \ldots D_n^m g(x) \\ &= D_1^m \ldots D_n^m [\eta(x) g(x)] + \sum_{|\alpha| \leqslant mn-1} \eta_\alpha(x) D^\alpha g(x), \end{aligned}$$

where $\eta_\alpha \in \theta_M$ and $\eta_\alpha(x) = 0$, $x \bar{\in} (\operatorname{supp} f)^\varepsilon$. Each term in the last sum is again transformed in that fashion, and so on. Then, in a finite number of steps, we arrive at the representation (4.5) with $p = mn$ and $g_{\alpha, \varepsilon} = \chi_\alpha g$, where $\chi_\alpha$ are certain functions taken from $\theta_M$ with support in $(\operatorname{supp} f)^\varepsilon$.

### 5.5 The direct product of generalized functions of slow growth

Let $f(x) \in \mathscr{S}'(\mathbb{R}^n)$ and $g(y) \in \mathscr{S}'(\mathbb{R}^m)$. Since $\mathscr{S}' \subset \mathscr{D}'$, the direct product $f(x) \times g(y)$ is a generalized functon in $\mathscr{D}'(\mathbb{R}^{n+m})$ (see Sec. 3.1). We will prove that $f(x) \times g(y) \in \mathscr{S}'(\mathbb{R}^{n+m})$.

By the definition of the functional $f(x) \times g(y)$ (see Sec. 3.1),

$$(f(x) \times g(y), \varphi) = (f(x), (g(y), \varphi(x, y))). \tag{5.1}$$

We will now prove that the right-hand side of (5.1) is a continuous linear functional on $\mathscr{S}(\mathbb{R}^{n+m})$.

To do this, we set up the following lemma that is similar to the lemma of Sec. 3.1.

Lemma *If* $g \in \mathcal{S}'$, *then for all* $\alpha$

$$D^{\alpha}\psi(x) = (g(y),\ D_x^{\alpha}\varphi(x,\ y)), \qquad \varphi \in \mathcal{S}(\mathbb{R}^{n+m}), \tag{5.2}$$

*and there is an integer* $q \geqslant 0$ *such that*

$$\| \psi \|_p \leqslant \| g \|_{-q} \, \| \varphi \|_{p+q}, \qquad p = 0,\ 1,\ \ldots, \tag{5.3}$$

*so that the operation* $\varphi \to \psi = (g(y),\ \varphi(x,\ y))$ *is continuous (and linear) from* $\mathcal{S}(\mathbb{R}^{n+m})$ *to* $\mathcal{S}(\mathbb{R}^n)$.

*Proof*. As in the proof of the lemma of Sec. 3.1, we establish the truth of the equality (5.2) for all $\alpha$ and the continuity of the right-hand side. Consequently, $\psi \in C^{\infty}$. Let $q$ be the order of $g$. Applying (2.3) to the right side of (5.2), we obtain for all $x \in \mathbb{R}^n$ the estimate

$$|D^{\alpha}\psi(x)| \leqslant \| g \|_{-q} \sup_{y,\ |\beta| \leqslant q} (1 + |y|^2)^{q/2}\, |D_x^{\alpha} D_y^{\beta}\varphi(x,\ y)|,$$

whence follows (5.3):

$$\begin{aligned} \| \psi \|_p &= \sup_{x,\ |\alpha| \leqslant p} (1 + |x|^2)^{p/2}\, |D^{\alpha}\psi(x)| \\ &\leqslant \|g\|_{-q} \sup_{(x,\ y),\ |\alpha| \leqslant p,\ |\beta| \leqslant q} (1 + |x|^2)^{p/2} (1 + |y|^2)^{q/2}\, |D_x^{\alpha} D_y^{\beta}\varphi(x,\ y)| \\ &\leqslant \| g \|_{-q} \| \varphi \|_{p+q}, \qquad p = 0,\ 1,\ \ldots\ . \end{aligned}$$

The proof of the lemma is complete.

From this lemma it follows that the right-hand side of (5.1), which is equal to $(f,\ \psi)$, is a continuous and linear functional on $\mathcal{S}(\mathbb{R}^{n+m})$ so that $f(x) \times g(y) \in \mathcal{S}'(\mathbb{R}^{n+m})$ (compare Sec. 3.1).

All the properties of a direct product that are listed in Sec. 3.2 for the space $\mathcal{D}'$ hold true also for the space $\mathcal{S}'$. This assertion follows from the density of $\mathcal{D}$ in $\mathcal{S}$ (see Sec. 5.1). In particular, the operation $f(x) \to f(x) \times g(y)$ is continuous from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^{n+m})$.

Finally, the formula (3.2) of Sec. 3 holds true also for $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^{n+m})$:

$$\left(f(x),\ \int \varphi(x,\ y)\, dy\right) = \int (f(x),\ \varphi(x,\ y))\, dy. \tag{5.4}$$

**5.6 The convolution of generalized functions of slow growth** Let $f \in \mathscr{S}'$, $g \in \mathscr{S}'$ and let their convolution $f * g$ exist in $\mathscr{D}'$. Now: When does $f * g \in \mathscr{S}'$ and when is the operation $f \to f * g$ continuous from $\mathscr{S}'$ to $\mathscr{S}'$? We state three sufficient criteria for the existence of a convolution in $\mathscr{S}'$.

(a) Let $f \in \mathscr{S}'$ and $g \in \mathscr{E}'$. Then the convolution $f * g$ belongs to $\mathscr{S}'$ and can be represented as

$$(f * g, \varphi) = (f(x) \times g(a), \eta(y) \varphi(x+y)), \quad \varphi \in \mathscr{S}, \tag{6.1}$$

where $\eta$ is any function from $\mathscr{D}$ equal to 1 in the neighbourhood of the support of $g$; here, the operation $f \to f * g$ is continuous from $\mathscr{S}'$ to $\mathscr{S}'$, and the operation $g \to f * g$ is continuous from $\mathscr{E}'$ to $\mathscr{S}'$.

Indeed, the convolution $f * g \in \mathscr{D}'$ and the representation (3.3) of Sec. 4 holds true on the basic functions in $\mathscr{D}$. Since $f(x) \times g(y) \in \mathscr{S}'(\mathbb{R}^{2n})$ (see Sec. 5.5), and the operation $\varphi \to \eta(y) \times \varphi(x+y)$ is linear and continuous from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^{2n})$:

$$\|\eta(y)\varphi(x+y)\|_p \leqslant \sup_{(x,y), |\alpha| \leqslant p} (1+|x|^2+|y|^2)^{p/2} |D^\alpha[\eta(y)\varphi(x+y)]|$$

$$\leqslant C_p \sup_{(x,y), |\alpha| \leqslant p} (1+|x+y|^2)^{p/2} |D^\alpha \varphi(x+y)| = C_p \|\varphi\|_p,$$

it follows that the right-hand side of (6.1) defines a continuous linear functional on $\mathscr{S}$ so that $f * g \in \mathscr{S}'$.

(b) Let $\Gamma$ be a closed convex acute cone in $\mathbb{R}^n$ with vertex at 0, $C = \operatorname{int} \Gamma^*$, $S$ a strictly $C$-like surface, and $S_+$ the domain lying above $S$ (see Sec. 4.4).

If $f \in \mathscr{S}'(\Gamma +)$ and $g \in \mathscr{S}'(\bar{S}+)$, then the convolution $f * g$ exists in $\mathscr{S}'$ and can be represented as

$$(f * g, \varphi) = (f(x) \times g(y), \xi(x)\eta(y)\varphi(x+y)), \tag{6.2}$$
$$\varphi \in \mathscr{S}$$

where $\xi$ and $\eta$ are any $C^\infty$-functions, $|D^\alpha \xi(x)| \leqslant c_\alpha$, $|D^\alpha \eta(y)| \leqslant c_\alpha$, equal to 1 in $(\operatorname{supp} f)^\varepsilon$ and $(\operatorname{supp} g)^\varepsilon$ and equal to 0 outside $(\operatorname{supp} f)^{2\varepsilon}$ and $(\operatorname{supp} g)^{2\varepsilon}$ respectively ($\varepsilon$ is any number $> 0$).[§] Here, if $\operatorname{supp} f \subset \Gamma + K$, where $K$ is a compact, then the operation $f \to f * g$ is continuous from $\mathscr{S}'(\Gamma + K)$ to $\mathscr{S}'(\bar{S}_+ + K)$.

To prove this assertion, it remains—by using the representation (5.1) of Sec. 4 and by reasoning as in Sec. 5.6(a)—to establish

[§] According to the lemma of Sec. 1.2, such functions exist.

the continuity of the operation $\varphi \to \chi = \xi(x)\,\eta(y)\,\varphi(x+y)$ from $\mathscr{S}(\mathbb{R}^n)$ to $\mathscr{S}(\mathbb{R}^{2n})$. For all $\varphi \in \mathscr{S}$ we have

$$\|\chi\|_p = \sup_{(x,\,y)\atop |\alpha|\leqslant p} (1+|x|^2+|y|^2)^{p/2}\,|D^\alpha_{(x,\,y)}[\xi(x)\,\eta(y)\,\varphi(x+y)]|$$

$$\leqslant C'_p \sup_{x\in\Gamma+K+\overline{U}_{2\varepsilon},\; y\in\overline{S}_+} (1+|x|^2+|y|^2)^{p/2}\,|D^\alpha_{(x,\,y)}\varphi(x+y)|$$

$$\leqslant 2^p C'_p \sup_{x\in T(\xi)\atop |\alpha|\leqslant p} (1+|x|^2+|\xi|^2)^{p/2}\,|D^\alpha\varphi(\xi)|,$$

where

$$T(\xi) = [x : x\in\Gamma+K+\overline{U}_{2\varepsilon},\; x=\xi-y,\; y\in\overline{S}_+].$$

Since $S$ is assumed to be a strictly $C$-like surface, it follows that the set $T(\xi)$ is contained in a sphere of radius $a(1+|\xi|)^\nu$, $\nu \geqslant 1$ (see Sec. 4.4). Therefore, continuing our estimates, we obtain

$$\|\chi\|_p \leqslant C''_p \sup_{\xi\atop |\alpha|\leqslant p} [1+|\xi|^2+a^2(1+|\xi|)^{2\nu}]^{p/2}\,|D^\alpha\varphi(\xi)|$$

$$\leqslant C_p \|\varphi\|_{p(|\nu|+1)}, \qquad p = 0, 1, \ldots,$$

which is what we set out to do.

From the criterion obtained it follows, in particular, that the set of generalized functions $\mathscr{S}'(\Gamma+)$ forms a convolution algebra, a subalgebra of the algebra $\mathscr{D}'(\Gamma+)$; in the same way, $\mathscr{S}'(\Gamma)$ also forms a convolution algebra, a subalgebra of the algebra $\mathscr{S}'(\Gamma+)$.

(c) Let $f\in\mathscr{S}'$ and $\eta\in\mathscr{S}$. Then the convolution $f*\eta$ exists in $\theta_M$ and can be represented in the form [compare (6.2) of Sec. 4]

$$(f*\eta,\ \varphi) = (f,\ \eta*\varphi(-x)), \tag{6.3}$$

$$\varphi\in\mathscr{S},$$

$$f*\eta = (f(y),\ \eta(x-y)). \tag{6.3'}$$

Here, there is an integer $m \geqslant 0$ (of order $f$) such that

$$|D^\alpha(f*\eta)(x)| = \|f\|_{-m}(1+|x|^2)^{m/2}\,\|\eta\|_{m+|\alpha|}, \tag{6.4}$$

$$x\in\mathbb{R}^n.$$

Indeed, suppose $\{\eta_k(x;\ y)\}$ is a sequence of functions taken from $\mathscr{D}(\mathbb{R}^{2n})$ that converges to 1 in $\mathbb{R}^{2n}$, and $\varphi \in \mathscr{S}(\mathbb{R}^n)$. Then

$$\int \eta(y)\,\eta_k(x;\ y)\,\varphi(x+y)\,dy \to \int \eta(y)\,\varphi(x+y)\,dy,$$

$$k \to \infty \text{ in } \mathscr{S}.$$

From this, if we make use of the definitions of a convolution (see Sec. 4.1) and of a direct product (see Sec. 3.1), we obtain, for all $\varphi \in \mathscr{S}$, the representation (6.3):

$$\begin{aligned}(f*\eta,\ \varphi) &= \lim_{k\to\infty}(f(x)\times\eta(y),\ \eta_k(x;\ y)\,\varphi(x+y))\\ &= \lim_{k\to\infty}\Big(f(x),\ \int \eta(y)\,\eta_k(x;\ y)\,\varphi(x+y)\,dy\Big)\\ &= \Big(f(x),\ \int \eta(y)\,\varphi(x+y)\,dy\Big)\\ &= \Big(f(x),\ \int \varphi(\xi)\,\eta(\xi-x)\,d\xi\Big) = (f,\ \eta*\varphi(-x)).\end{aligned}$$

Noting that $\varphi(\xi)\,\eta(\xi-x) \in \mathscr{S}(\mathbb{R}^{2n})$ and taking advantage of (5.4), we continue our chain of equalities

$$(f*\eta,\ \varphi) = \int (f(x),\ \eta(\xi-x))\,\varphi(\xi)\,d\xi,$$

whence follows the representation (6.3′).

As in the proof of the lemma of Sec. 3.1, we conclude from the representation (6.3) that $f*\eta \in C^\infty$ and the following formula holds true:

$$D^\alpha(f*\eta)(x) = (f(y),\ D_x^\alpha\eta(x-y)). \tag{6.5}$$

Let $m$ be the order of $f$. Applying the inequality (2.3) to the right-hand side of (6.5), we obtain the inequality (6.4):

$$\begin{aligned}|D^\alpha(f*\eta)(x)| &\leqslant \|f\|_{-m}\,|D_x^\alpha\eta(x-y)|_m\\ &= \|f\|_{-m}\sup_{y,\ |\beta|\leqslant m}(1+|y|^2)^{m/2}\,|D_x^\alpha D_y^\beta\eta(x-y)|\\ &= \|f\|_{-m}\sup_{\xi,\ |\beta|\leqslant m}(1+|x-\xi|^2)^{m/2}\,|D^{\alpha+\beta}\eta(\xi)|\end{aligned}$$

$$\leqslant \|f\|_{-m} (1+|x|^2)^{m/2} \sup_{\substack{\xi \\ |\beta|\leqslant m}} (1+|\xi|^2)^{m/2} |D^{\alpha+\beta}\eta(\xi)|$$

$$\leqslant \|f\|_{-m} (1+|x|^2)^{m/2} \|\eta\|_{m+|\alpha|}.$$

Corollary *$\mathscr{S}$ is dense in $\mathscr{S}'$.*

That is true. From what has been proved, if $f \in \mathscr{S}'$, then its regularization $f_\varepsilon = f * \omega_\varepsilon \in \theta_M$ and $f_\varepsilon \to f$, $\varepsilon \to +0$ in $\mathscr{S}'$ (see Sec. 5.6(a)). Therefore $\theta_M$ is dense in $\mathscr{S}'$. But $\mathscr{S}$ is dense in $\theta_M$ because if $a \in \theta_M$, then

$$\mathscr{S} \ni e^{-\varepsilon|x|^2}a \to a, \qquad \varepsilon \to +0 \quad \text{in} \quad \mathscr{S}'.$$

Chapter II

# Integral Transformations of Generalized Functions

One of the most powerful tools of investigation of problems in mathematical physics is the method of integral transformations. In this chapter we consider the theories of the Fourier transformation and the Laplace transformation that is closely linked with it; we also consider the transformations of Cauchy-Bochner, Hilbert and Poisson for the class of generalized functions of slow growth.

## 6 The Fourier Transform of Generalized Functions of Slow Growth

A remarkable property of the class of generalized functions of slow growth is that the operation of the Fourier transform does not take one outside that class.

**6.1 The Fourier transform of basic functions in $\mathcal{S}$** Since the basic functions $\varphi(x)$ in $\mathcal{S}$ are summable on $\mathbb{R}^n$, the (classical) operation of the Fourier transform $F[\varphi]$ is defined on them:

$$F[\varphi](\xi) = \int \varphi(x)\, e^{i(\xi, x)}\, dx, \qquad \varphi \in \mathcal{S}.$$

In this case, the function $F[\varphi](\xi)$, which is the *Fourier* transform of the function $\varphi(x)$, is bounded and continuous in $\mathbb{R}^n$. The basic function $\varphi(x)$ decreases at infinity faster than any power of $|x|^{-1}$. Therefore, its Fourier transform may be differentiated under the integral sign any number of times:

$$D^{\alpha} F[\varphi](\xi) = \int (ix)^{\alpha}\, \varphi(x)\, e^{i(\xi, x)}\, dx$$

$$= F[(ix)^{\alpha}\varphi](\xi), \tag{1.1}$$

whence it follows that $F[\varphi] \in C^\infty$. Furthermore, every derivative $D^\alpha \varphi(x)$ has the same properties and so

$$F[D^\alpha \varphi](\xi) = \int D^\alpha \varphi(x) e^{i(\xi, x)} dx$$
$$= (-i\xi)^\alpha F[\varphi](\xi). \tag{1.2}$$

From (1.2) it follows, for one thing, that $F[\varphi](\xi)$ is a summable function on $\mathbb{R}^n$.

From the general theory of the Fourier transformation it follows that the function $\varphi(x)$ is expressed in terms of its Fourier transform $F[\varphi](\xi)$ with the aid of the inverse Fourier transform, $F^{-1}$:

$$\varphi = F^{-1}[F[\varphi]] = F[F^{-1}[\varphi]] \tag{1.3}$$

where

$$F^{-1}[\psi](x) = \frac{1}{(2\pi)^n} \int \psi(\xi) e^{-i(x, \xi)} d\xi$$
$$= \frac{1}{(2\pi)^n} F[\psi](-x)$$
$$= \frac{1}{(2\pi)^n} \int \psi(-\xi) e^{i(x, \xi)} d\xi$$
$$= \frac{1}{(2\pi)^n} F[\psi(-\xi)]. \tag{1.4}$$

Lemma *The operation of the Fourier transform $F$ carries $\mathcal{S}$ into itself in reciprocal one-to-one fashion and reciprocal continuous fashion.*[§]

*Proof.* Let $\varphi \in \mathcal{S}$. Then, using (1.1) and (1.2), for all $p = 0, 1, \ldots$ and all $\alpha$ we obtain

$$(1+|\xi|^2)^{p/2} |D^\alpha F[\varphi](\xi)| \leqslant (1+|\xi|^2)^{\left[\frac{p+1}{2}\right]} |D^\alpha F[\varphi](\xi)|$$
$$\leqslant \left| \int (1-\nabla^2)^{\left[\frac{p+1}{2}\right]} [(ix)^\alpha \varphi(x)] e^{i(\xi, x)} dx \right|$$
$$\leqslant C \sup_x (1+|x|^2)^{\frac{n+1}{2}} |(1-\nabla^2)^{\left[\frac{p+1}{2}\right]} [x^\alpha \varphi(x)]|,$$

whence we derive the estimates (see Sec. 5.1)

$$\| F[\varphi] \|_p \leqslant C_p \| \varphi \|_{p+n+1}, \quad p = 0, 1, \ldots, \tag{1.5}$$

[§] We say that the mapping $F$ is a (linear) *isomorphism of $\mathcal{S}$ onto $\mathcal{S}$.*

for certain $C_p$ that do not depend on $\varphi$. (Here, $[x]$ is the integral part of the number $x \geqslant 0$.) The estimate (1.5) shows that the operation $\varphi \to F[\varphi]$ transforms $\mathcal{S}$ to $\mathcal{S}$ and is continuous. Furthermore, from (1.3) and (1.4) it follows that any function of $\varphi$ taken from $\mathcal{S}$ is a Fourier transform of the function $\psi = F^{-1}[\varphi]$ taken from $\mathcal{S}$, $\varphi = F[\psi]$, and if $F[\varphi] = 0$, then $\varphi = 0$ as well. This means that the mapping $\varphi \to F[\varphi]$ carries $\mathcal{S}$ to $\mathcal{S}$ in a reciprocal one-to-one fashion. The properties are similar for the operation of the inverse Fourier transform, $F^{-1}$. This completes the proof of the lemma.

**6.2 The Fourier transform of generalized functions in $\mathcal{S}'$** First let $f(x)$ be a summable function on $\mathbb{R}^n$. Then its Fourier transform,

$$F[f](\xi) = \int f(x)\, e^{i(\xi, x)}\, dx, \quad |F[f](\xi)| \leqslant \int |f(x)|\, dx < \infty,$$

is a (continuous) bounded function in $\mathbb{R}^n$ and, hence, determines a regular generalized function of slow growth via the formula (see Sec. 5.3)

$$(F[f], \varphi) = \int F[f](\xi)\, \varphi(\xi)\, d\xi, \qquad \varphi \in \mathcal{S}.$$

Using the Fubini theorem on changing the order of integration, we transform the last integral:

$$\begin{aligned}\int F[f](\xi)\, \varphi(\xi)\, d\xi &= \int \left[\int f(x)\, e^{i(\xi, x)}\, dx\right] \varphi(\xi)\, d\xi \\ &= \int f(x) \int \varphi(\xi)\, e^{i(x, \xi)}\, d\xi\, dx \\ &= \int f(x)\, F[\varphi](x)\, dx,\end{aligned}$$

that is,

$$(F[f], \varphi) = (f, F[\varphi]), \qquad \varphi \in \mathcal{S}.$$

It is this equation that we take for the definition of the *Fourier transform $F[f]$ of any generalized function $f$ of slow growth*:

$$(F[f], \varphi) = (f, F[\varphi]), \qquad f \in \mathcal{S}', \quad \varphi \in \mathcal{S}. \tag{2.1}$$

Since by the lemma of Sec. 6.1 the operation $\varphi \to F[\varphi]$ is linear and continuous from $\mathcal{S}$ to $\mathcal{S}$, the functional $F[f]$ defined by the right-hand side of (2.1) is a generalized function taken from $\mathcal{S}'$ and, what is more, the operation $f \to F[f]$ is linear and continuous from *$\mathcal{S}'$ to $\mathcal{S}'$*.

Let us now introduce into $\mathscr{S}'$ yet another Fourier transform operation, which we will denote by $F^{-1}$, via the formula [compare (1.4)]

$$F^{-1}[f] = \frac{1}{(2\pi)^n} F[f(-x)], \qquad f \in \mathscr{S}', \tag{2.2}$$

where $f(-x)$ is a reflection of $f(x)$ (see Sec. 1.9). Clearly, $F^{-1}$ is a linear and continuous operation from $\mathscr{S}'$ to $\mathscr{S}'$.

Now we will prove that $F^{-1}$ is the inverse of $F$, that is,

$$F^{-1}[F[f]] = f, \quad F[F^{-1}[f]] = f, \quad f \in \mathscr{S}'. \tag{2.3}$$

Indeed, by virtue of (1.3) and (1.4), the formulas (2.3) hold on the set $\mathscr{S}$, which is dense in $\mathscr{S}'$ (see Sec. 5.6); the operations $F$ and $F^{-1}$ are continuous from $\mathscr{S}'$ to $\mathscr{S}'$. Hence, the formulas (2.3) hold true for all $f$ in $\mathscr{S}'$ as well.

From (2.3) it follows that any $f$ in $\mathscr{S}'$ is a Fourier transform of some $g = F^{-1}[f]$ in $\mathscr{S}'$, $f = F[g]$, and if $F[f] = 0$, then $f = 0$. Thus, we have proved that the operation $f \to F[f]$ transforms $\mathscr{S}'$ to $\mathscr{S}'$ in a reciprocal one-to-one fashion and a reciprocal continuous fashion, that is, we have a (linear) isomorphism of $\mathscr{S}'$ onto $\mathscr{S}'$.

Suppose $f(x, y) \in \mathscr{S}'(\mathbb{R}^{n+m})$, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. We introduce the Fourier transform $F_x[f]$ with respect to the variables $x = (x_1, \ldots, x_n)$ by putting, for any basic function $\varphi(\xi, y)$ in $\mathscr{S}(\mathbb{R}^{n+m})$,

$$(F_x[f], \varphi) = (f, F_\xi[\varphi]). \tag{2.4}$$

As in Sec. 6.1, we establish that the operation

$$\varphi(\xi, y) \to F_\xi[\varphi](x, y) = \int \varphi(\xi, y) e^{i(x, \xi)} d\xi$$

accomplishes a (linear) isomorphism of $\mathscr{S}(\mathbb{R}^{n+m})$ onto $\mathscr{S}(\mathbb{R}^{n+m})$ so that the formula (2.4) does indeed define a generalized function $F_x[f](\xi, y)$ in $\mathscr{S}'(\mathbb{R}^{n+m})$. The operation of Fourier inversion is defined in a manner similar to that of (2.2):

$$F_\xi^{-1}[g] = \frac{1}{(2\pi)^n} F_\xi[g(-\xi, y)](x, y), \quad g \in \mathscr{S}'(\mathbb{R}^{n+m}). \tag{2.5}$$

The operation $f \to F_x[f]$ is a (linear) isomorphism of $\mathscr{S}'(\mathbb{R}^{n+m})$ onto $\mathscr{S}'(\mathbb{R}^{n+m})$.

*Example.*

$$F[\delta(x - x_0)] = e^{i(\xi, x_0)}. \tag{2.6}$$

Indeed,

$$(F[\delta(x - x_0)], \varphi) = (\delta(x - x_0), F[\varphi]) = F[\varphi](x_0)$$

$$= \int \varphi(\xi) e^{i(x_0, \xi)} d\xi = (e^{i(x_0, \xi)}, \varphi), \quad \varphi \in \mathscr{S}.$$

Putting $x_0 = 0$ in (2.6), we get

$$F[\delta] = 1 \tag{2.7}$$

whence, by (2.2), we derive

$$\delta = F^{-1}[1] = \frac{1}{(2\pi)^n} F[1]$$

so that

$$F[1] = (2\pi)^n \delta(\xi). \tag{2.8}$$

**6.3 Properties of the Fourier transform** The formulas for the Fourier transform given in this subsection hold true on the basic functions in $\mathscr{S}$. But $\mathscr{S}$ is dense in $\mathscr{S}'$. Therefore, these formulas remain true also for all generalized functions in $\mathscr{S}'$.

(a) *Differentiating a Fourier transform*:

$$D^\alpha F[f] = F[(ix)^\alpha f]. \tag{3.1}$$

In particular, putting $f = 1$ in (3.1) and using (2.8), we obtain

$$F[x^\alpha] = (-i)^{|\alpha|} D^\alpha F[1] = (2\pi)^n (-i)^{|\alpha|} D^\alpha \delta(\xi). \tag{3.2}$$

(b) *The Fourier transform of a derivative*:

$$F[D^\alpha f] = (-i\xi)^\alpha F[f]. \tag{3.3}$$

Putting $f = \delta$ in (3.3) and using (2.7), we obtain

$$F[D^\alpha \delta] = (-i\xi)^\alpha F[\delta] = (-i\xi)^\alpha. \tag{3.4}$$

(c) *The Fourier transform of a translation*:

$$F[f(x - x_0)] = e^{i(\xi, x_0)} F[f]. \tag{3.5}$$

(d) *The translation of a Fourier transform*:

$$F[f](\xi + \xi_0) = F[e^{i(\xi_0, x)} f](\xi). \tag{3.6}$$

(e) *The Fourier transform under a linear transformation of the argument* (see Sec. 5.3):

$$F[f(Ax)](\xi) = \frac{1}{|\det A|} F[f](A^{-1T}\xi), \qquad \det A \neq 0. \tag{3.7}$$

Here, $A \to A^T$ denotes the transpose operation of the matrix $A$.

(f) *The Fourier transform of a direct product:*

$$F[f(x)\times g(y)] = F_x[f(x)\times F[g](\eta)]$$
$$= F_y[F[f](\xi)\times g(y)]$$
$$= F[f](\xi)\times F[g](\eta). \qquad (3.8)$$

(g) Analogous formulas hold true also for the Fourier transform $F_x$ (see Sec. 6.2), for example:

$$D_x^\alpha D_y^\beta F_x[f] = F_x[(ix)^\alpha D_y^\beta f],$$
$$F_x[D_x^\alpha D_y^\beta f] = (-i\xi)^\alpha D_y^\beta F_x[f]. \qquad (3.9)$$

**6.4 The Fourier transform of generalized functions with compact support** If $f$ is a generalized function with compact support, $f\in\mathscr{E}'$, then it is of slow growth, $f\in\mathscr{S}'$ (see Sec. 5.3), and therefore its Fourier transform exists. What is more, the following theorem holds true.

Theorem *If $f\in\mathscr{E}'$, then the Fourier transform $F[f]$ exists in $\theta_M$ and can be represented as*

$$F[f](\xi) = (f(x),\ \eta(x)\,e^{i(\xi,x)}), \qquad (4.1)$$

*where $\eta$ is any function in $\mathscr{D}$ equal to 1 in the neighbourhood of the support of $f$. And there exist numbers $C_\alpha\geqslant 0$ and $m\geqslant 0$ such that*

$$|D^\alpha F[f](\xi)|\leqslant \|f\|_{-m}C_\alpha(1+|\xi|^2)^{m/2},\quad \xi\in\mathbb{R}^n. \qquad (4.2)$$

*Proof*. Taking into account the equalities (3.2) of Sec. 5 and (3.3), we obtain, for all $\varphi\in\mathscr{S}$,

$$(D^\alpha F[f],\varphi) = (-1)^{|\alpha|}(F[f],\ D^\alpha\varphi) = (-1)^{|\alpha|}(f,\ F[D^\alpha\varphi])$$
$$= (-1)^{|\alpha|}(f,\ \eta(x)(-ix)^\alpha F[\varphi])$$
$$= \left(f(x),\ \int \eta(x)(ix)^\alpha\varphi(\xi)\,e^{i(x,\xi)}\,d\xi\right).$$

Now, noting that

$$\eta(x)(ix)^\alpha\varphi(\xi)\,e^{i(x,\xi)}\in\mathscr{S}(\mathbb{R}^{2n})$$

and using (5.4) of Sec. 5,

$$\left(f(x),\ \int \eta(x)(ix)^\alpha\varphi(\xi)\,e^{i(x,\xi)}\,d\xi\right)$$
$$= \int (f(x),\ \eta(x)(ix)^\alpha e^{i(x,\xi)})\,\varphi(\xi)\,d\xi,$$

we derive the following equation from the preceding ones:

$$(D^\alpha F[f], \varphi) = \int (f(x), \eta(x)(ix)^\alpha e^{i(x, \xi)}) \varphi(\xi)\, d\xi.$$

It follows from this equation that

$$D^\alpha F[f](\xi) = (f(x), \eta(x)(ix)^\alpha e^{i(x, \xi)}). \tag{4.3}$$

And from (4.3), for $\alpha = 0$, follows the formula (4.1).

From the representation (4.3), as in the proof of the lemma of Sec. 5.5, we derive that $F[f] \in C^\infty$. Let $m$ be the order of $f$. Applying to the right-hand side of (4.3) the inequality (2.3) of Sec. 5, we obtain, for all $\xi \in \mathbb{R}^n$, the estimate (4.2):

$$|D^\alpha F[f](\xi)| = |(f(x), \eta(x)(ix)^\alpha e^{i(x, \xi)})|$$

$$\leqslant \|f\|_{-m} \|\eta(x)(ix)^\alpha e^{i(x, \xi)}\|_m$$

$$= \|f\|_{-m} \sup_{x,\ |\beta| \leqslant m} (1 + |x|^2)^{m/2} |D_x^\beta [\eta(x) x^\alpha e^{i(x, \xi)}]|$$

$$\leqslant \|f\|_{-m} C_\alpha (1 + |\xi|^2)^{m/2}$$

for certain $C_\alpha \geqslant 0$. Thus, $F[f] \in \theta_M$, and the proof of the theorem is complete.

*Remark.* As may be seen from the proof of the theorem, the numbers $C_\alpha$ that appear in the inequality (4.2) may be chosen as being independent of the family of generalized functions $f$ if all supports of that family are uniformly bounded.

**6.5 The Fourier transform of a convolution** Let $f \in \mathcal{S}'$ and $g \in \mathcal{E}'$. Then their convolution $f * g \in \mathcal{S}'$ (see Sec. 5.6 (a)) and its Fourier transform can be calculated from the formula

$$F[f * g] = F[f]\, F[g]. \tag{5.1}$$

True enough, by virtue of (6.1) of Sec. 5, the convolution $f * g \in \mathcal{S}'$ can be represented in the form

$$(f * g, \varphi) = (f(x), (g(y), \eta(y) \varphi(x + y))), \quad \varphi \in \mathcal{S},$$

where $\eta \in \mathcal{D}$, $\eta(y) = 1$ in the neighbourhood of supp $g$. Taking this representation into account and making use of the definition of the Fourier transform (see Sec. 6.2), we obtain

$$(F[f * g], \varphi) = (f * g, F[\varphi])$$

$$= \left(f(x), \left(g(y), \eta(y) \int \varphi(\xi) e^{i(x+y, \xi)}\, d\xi\right)\right).$$

Using the formulas (5.4) of Sec. 5 and (4.1) and taking into account that $F[g] \in \theta_M$, we transform the resulting equation:

$$\begin{aligned}(F[f*g],\ \varphi) &= \left(f(x),\ \int (g(y),\ \eta(y)\, e^{i(\xi,\, y)})\, e^{i(x,\,\xi)}\varphi(\xi)\, d\xi\right)\\ &= \left(f(x),\ \int F[g](\xi)\, e^{i(x,\,\xi)}\varphi(\xi)\, d\xi\right)\\ &= (f,\ F[F[g]\varphi])\\ &= (F[f],\ F[g]\varphi) = (F[g]F[f],\ \varphi),\end{aligned}$$

whence follows (5.1).

Some other cases follow in which (5.1) holds true:

(a) Let $f \in \mathcal{S}'$, $g \in \mathcal{S}$. Then $f * g \in \theta_M$. This follows from Sec. 5.6 (c).

(b) Let $f$ and $g \in \mathcal{L}^2$. Then $f * g \in C$ and $(f * g)(x) = o(1)$, $|x| \to \infty$, that is, $f * g \in \bar{C}_0$ (see Sec. 0.5).

Indeed, in this case, $F[f]$ and $F[g] \in \mathcal{L}^2$ and, hence, $F[f]\,F[g] \in \mathcal{L}^1$. Besides, $f(y)\, g(x-y)\, \varphi(x) \in \mathcal{L}^1(\mathbb{R}^{2n})$ for all $\varphi \in \mathcal{S}$ by virtue of the Cauchy-Bunyakovsky inequality:

$$\begin{aligned}&\left[\int |f(y)\, g(x-y)|\ |\varphi(x)|\, dx\, dy\right]^2\\ &\qquad \leqslant \left[\int |f(y)|^2\, |\varphi(x)|\, dx\, dy\right]\left[\int |g(x-y)|^2\, |\varphi(x)|\, dx\, dy\right]\\ &\qquad \leqslant \|f\|^2\, \|g\|^2 \left[\int |\varphi(x)|\, dx\right]^2 < \infty.\end{aligned}$$

Therefore, using (1.1) for the convolution $f * g$ (see Sec. 4.1(b)), we obtain the following equalities with the aid of the Fubini theorem for all $\varphi \in \mathcal{S}$:

$$\begin{aligned}(F[f*g],\ \varphi) &= (f*g,\ F[\varphi])\\ &= \int F[\varphi](x) \int f(y)\, g(x-y)\, dy\, dx\\ &= \int f(y) \int g(x-y)\, F[\varphi](x)\, dx\, dy\\ &= \int f(y) \int F[g(x-y)](\xi)\, \varphi(\xi)\, d\xi\, dy\\ &= \int f(y) \int F[g](\xi)\, \varphi(\xi)\, e^{i(y,\,\xi)}\, d\xi\, dy\\ &= \int F[g]\, F[f]\, \varphi\, d\xi.\end{aligned}$$

From these follows formula (5.1). Therefore

$$f * g = F^{-1}[F[f]\,F[g]] \in C$$

and by the Riemann-Lebesgue theorem $(f * g)(x) \to 0$, $|x| \to \infty$.

*Remark.* If it is known that the convolution $f * g$ exists in $\mathscr{S}'$ [for example, for $f \in \mathscr{S}'(\Gamma+)$ and $g \in \mathscr{S}'(\bar{S}_+)$ (see Sec. 5.6(b))], then (5.1) may serve as a definition of the product of the generalized functions $F[f]$ and $F[g]$ (compare Sec. 1.10).

### 6.6 Examples

(a)
$$F[e^{-\alpha^2x^2}] = \frac{\sqrt{\pi}}{\alpha}\, e^{-\frac{\xi^2}{4\alpha^2}}, \qquad n = 1. \tag{6.1}$$

True enough, the function $e^{-\alpha^2x^2}$ is summable on $\mathbb{R}^1$ and therefore

$$F(e^{-\alpha^2x^2}) = \int e^{-\alpha^2x^2+i\xi x}\,dx = \frac{1}{\alpha}\int e^{-\sigma^2+i\frac{\xi}{\alpha}\sigma}\,d\sigma$$

$$= \frac{1}{\alpha}\, e^{-\frac{\xi^2}{4\alpha^2}} \int e^{-\left(\sigma+\frac{i\xi}{2\alpha}\right)^2}\,d\sigma$$

$$= \frac{1}{\alpha}\, e^{-\frac{\xi^2}{4\alpha^2}} \int\limits_{\operatorname{Im}\zeta=\frac{\xi}{2\alpha}} e^{-\zeta^2}\,d\zeta.$$

In the last integral, the line of integration may be shifted onto the real axis and therefore

$$F[e^{-\alpha^2x^2}] = \frac{1}{\alpha}\, e^{-\frac{\xi^2}{4\alpha^2}} \int_{\infty}^{\infty} e^{-\sigma^2}\,d\sigma = \frac{\sqrt{\pi}}{\alpha}\, e^{-\frac{\xi^2}{4\alpha^2}}$$

(b) A multidimensional analogue of formula (6.1) is

$$F[e^{-(Ax,\,x)}] = \frac{\pi^{n/2}}{\sqrt{\det A}}\, e^{-\frac{1}{4}(A^{-1}\xi,\,\xi)}, \tag{6.2}$$

where $A$ is a real positive-definite matrix.

To obtain (6.2) with the aid of a nonsingular real linear transformation $x = By$, let us reduce the quadratic form $(Ax, x)$ to a sum of squares:

$$(Ax, x) = (ABy, By) = (B^T ABy, y) = |y|^2.$$

Note that

$$A^{-1} = BB^T, \qquad \det A \mid \det B \mid^2 = 1.$$

From this, using the formula (6.1), we obtain

$$F[e^{-(Ax, x)}] = \int e^{-(Ax, x)+i(\xi, x)}\, dx$$

$$= |\det B| \int e^{-(ABy, By)+i(\xi, By)}\, dy$$

$$= \frac{1}{\sqrt{\det A}} \int e^{-|y|^2+i(B^T\xi, y)}\, dy$$

$$= \frac{1}{\sqrt{\det A}} \prod_{1\leqslant j\leqslant n} \int e^{-y_j^2+i(B^T\xi)_j y_j}\, dy_j$$

$$= \frac{\pi^{n/2}}{\sqrt{\det A}} e^{-\frac{1}{4}|B^T\xi|^2} = \frac{\pi^{n/2}}{\sqrt{\det A}} e^{-\frac{1}{4}(\xi, BB^T\xi)}$$

$$= \frac{\pi^{n/2}}{\sqrt{\det A}} e^{-\frac{1}{4}(\xi, A^{-1}\xi)}.$$

(c) Let the function $f(x)$ be of slow growth in $\mathbb{R}^n$ (see Sec. 5.3). Then

$$F[f](\xi) = \lim_{R\to\infty} \int_{|x|<R} f(x)\, e^{i(\xi, x)}\, dx \quad \text{in} \quad \mathscr{S}'. \tag{6.3}$$

Indeed,

$$\theta(R - |x|)\, f(x) \to f(x), \; R \to \infty \text{ in } \mathscr{S}',$$

whence by virtue of the continuity, in $\mathscr{S}'$, of the Fourier transform operation $F$, follows the equation (6.3).

In particular, for $f \in \mathscr{L}^2$ the following theorem of Plancherel holds true: *The Fourier transform $F[f]$ is expressed by the equation*

$$F[f](\xi) = \lim_{R\to\infty} \int_{|x|<R} f(x)\, e^{i(\xi, x)}\, dx \quad \text{in} \quad \mathscr{L}^2.$$

*It maps $\mathscr{L}^2$ onto $\mathscr{L}^2$ reciprocally in one-to-one fashion and reciprocally continuously*; *the Parseval equation*

$$(2\pi)^n \langle f, \varphi\rangle = \langle F[f], F[\varphi]\rangle, \; f, \varphi \in \mathscr{L}^2$$

*holds true so that*

$$(2\pi)^n \, \| f \|^2 = \| F[f] \|^2, \qquad f \in \mathcal{L}^2$$

(the scalar product $\langle \cdot, \cdot \rangle$ is defined in Sec. 0.5).

(d) Let $f$ be an arbitrary generalized function of slow growth. By the theorem of Sec. 5.4 there exists a function $g(x)$, which is continuous and of slow growth in $\mathbb{R}^n$, and an integer $m \geqslant 0$ such that

$$f(x) = D_1^m \ldots D_n^m g(x).$$

From this, using (3.3), we get

$$F[f] = (-i)^{mn} \xi_1^m \ldots \xi_n^m F[g], \tag{6.4}$$

and the Fourier transform $F[g]$ may be computed via (6.3).

(e)
$$F[e^{ix^2}] = \sqrt{\pi}\, e^{-\frac{1}{4}(\xi^2 - \pi)}, \qquad n = 1. \tag{6.5}$$

True enough, from the convergence of the improper integral (Fresnel's integral)

$$\int_{-\infty}^{\infty} e^{iy^2}\, dy = \sqrt{\pi}\, e^{\frac{i\pi}{4}}$$

it follows that the sequence of Fourier transforms

$$\int_{-R}^{R} e^{ix^2 + i\xi x}\, dx = e^{-\frac{1}{4}\xi^2} \int_{-R+\frac{\xi}{2}}^{R+\frac{\xi}{2}} e^{iy^2}\, dy, \qquad R \to \infty,$$

converges uniformly with respect to $\xi$ on every finite interval to the function

$$e^{-\frac{i}{4}\xi^2} \int_{-\infty}^{\infty} e^{iy^2}\, dy = \sqrt{\pi}\, e^{-\frac{1}{4}(\xi^2 - \pi)}.$$

From this, by virtue of (c), we conclude that (6.5) holds true on all basic functions in $\mathcal{D}$. But $\mathcal{D}$ is dense in $\mathcal{S}$ (see Sec. 6.1) and so (6.5) holds true in $\mathcal{S}'$.

(f) A multidimensional analogue of (6.5) is the equation [compare (b)]

$$F\left[e^{i(Ax,\,x)}\right]=\frac{\pi^{n/2}}{\sqrt{\det A}}\,e^{i\frac{\pi n}{4}-\frac{i}{4}(A^{-1}\xi,\,\xi)} \tag{6.6}$$

where $A$ is a real positive-definite matrix.

(g)
$$F\left[\frac{1}{|x|^2}\right]=\frac{2\pi^2}{|\xi|},\qquad n=3. \tag{6.7}$$

We have

$$\int\limits_{|x|<R}\frac{e^{i(\xi,\,x)}}{|x|^2}\,dx=\int\limits_0^R\int\limits_0^\pi\int\limits_0^{2\pi}\frac{e^{i|\xi|\rho\cos\theta}}{\rho^2}\,\rho^2\,d\psi\sin\theta\,d\theta\,d\rho$$
$$=2\pi\int\limits_0^R\int\limits_{-1}^1 e^{i|\xi|\rho\mu}\,d\mu\,d\rho=\frac{4\pi}{|\xi|}\int\limits_0^R\frac{\sin(|\xi|\rho)}{\rho}\,d\rho.$$

Since

$$\left|\int\limits_R^\infty\frac{\sin(|\xi|\rho)}{\rho}\,d\rho\right|=\left|\frac{\cos(|\xi|R)}{|\xi|R}-\frac{1}{|\xi|}\int\limits_R^\infty\frac{\cos(|\xi|\rho)}{\rho^2}\,d\rho\right|\leqslant\frac{2}{|\xi|R},$$

$$\int\limits_0^\infty\frac{\sin(|\xi|\rho)}{\rho}\,d\rho=\frac{\pi}{2},\qquad |\xi|\neq 0,$$

it follows that

$$\frac{4\pi}{|\xi|}\int\limits_0^R\frac{\sin(|\xi|\rho)}{\rho}\,d\rho\to\frac{2\pi^2}{|\xi|},\qquad R\to\infty\quad\text{in}\quad\mathscr{S}'$$

and, by virtue of (c), the equation (6.7) holds true.

(h) Let $n=2$. We introduce the generalized function $\mathrm{Pf}\,\frac{1}{|x|^2}$ taken from $\mathscr{S}'$, which function operates via the rule

$$\left(\mathrm{Pf}\,\frac{1}{|x|^2},\,\varphi\right)=\int\limits_{|x|<1}\frac{\varphi(x)-\varphi(0)}{|x|^2}\,dx+\int\limits_{|x|>1}\frac{\varphi(x)}{|x|^2}\,dx.$$

Obviously, $\mathrm{Pf}\,\frac{1}{|x|^2}=\frac{1}{|x|^2}$ for $x\neq 0$. Let us prove the formula

$$F\left[\mathrm{Pf}\,\frac{1}{|x|^2}\right]=-2\pi\ln|\xi|-2\pi c_0, \tag{6.8}$$

where

$$c_0 = \int_0^1 \frac{1 - J_0(u)}{u}\, du - \int_1^\infty \frac{J_0(u)}{u}\, du.$$

Indeed, for all $\varphi \in \mathcal{S}$ the following chain of equalities holds true:

$$\left(F\left[\mathrm{Pf}\,\frac{1}{|x|^2}\right], \varphi\right) = \left(\mathrm{Pf}\,\frac{1}{|x|^2}, F[\varphi]\right)$$

$$= \int\limits_{|x|<1} \frac{F[\varphi](x) - F[\varphi](0)}{|x|^2}\, dx + \int\limits_{|x|>1} \frac{F[\varphi](x)}{|x|^2}\, dx$$

$$= \int\limits_{|x|<1} \frac{1}{|x|^2} \int \varphi(\xi)\, [e^{i(x,\xi)} - 1]\, d\xi\, dx$$

$$+ \int\limits_{|x|>1} \frac{1}{|x|^2} \int \varphi(\xi)\, e^{i(x,\xi)}\, d\xi\, dx$$

$$= \int_0^1 \frac{1}{r} \int \varphi(\xi) \int_0^{2\pi} (e^{ir|\xi|\cos\theta} - 1)\, d\theta\, d\xi\, dr$$

$$+ \int_1^\infty \frac{1}{r} \int \varphi(\xi) \int_0^{2\pi} e^{ir|\xi|\cos\theta}\, d\theta\, d\xi\, dr$$

$$= 2\pi \int_0^1 \frac{1}{r} \int \varphi(\xi)\, [J_0(r|\xi|) - 1]\, d\xi\, dr$$

$$+ 2\pi \int_1^\infty \frac{1}{r} \int \varphi(\xi)\, J_0(r|\xi|)\, d\xi\, dr$$

$$= 2\pi \int \varphi(\xi) \left[\int_0^{1} \frac{J_0(r|\xi|) - 1}{r}\, dr + \int_1^\infty \frac{J_0(r|\xi|)}{r}\, dr\right] d\xi$$

$$= 2\pi \int \varphi(\xi) \left[\int_0^{|\xi|} \frac{J_0(u) - 1}{u}\, du + \int_{|\xi|}^\infty \frac{J_0(u)}{u}\, du\right] d\xi$$

$$= -2\pi \int \varphi(\xi)\, (\ln|\xi| + c_0)\, d\xi.$$

And formula (6.8) follows from this.

(i) Let $\Gamma$ be a closed convex acute cone in $\mathbb{R}^n$ (with vertex at 0) and let $f \in \mathcal{S}'(\Gamma+)$ (see Sec. 5.6(b)). Then the following for-

mula holds true in the sense of convergence in $\mathscr{S}'$:

$$F[f](\xi) = \lim_{\substack{\xi' \to 0 \\ \xi' \Subset \operatorname{int} \Gamma^*}} (f(x), \eta(x) e^{i(x, \xi) - (x, \xi')}), \tag{6.9}$$

where $\eta$ is any $C^\infty$-function with the following properties:

$$|D^\alpha_\eta(x)| \leqslant C_\alpha; \qquad \eta(x) = 1, \quad x \in (\operatorname{supp} f)^\varepsilon;$$
$$\eta(x) = 0, \quad x \bar{\in} (\operatorname{supp} f)^{2\varepsilon}$$

($\varepsilon$ is any number $> 0$).

To prove (6.9), we first note that

$$\eta(x) e^{-(x, \xi')} \in \mathscr{S} \quad \text{for all} \quad \xi' \in \operatorname{int} \Gamma^*, \tag{6.10}$$

$$\eta(x) f(x) e^{-(x, \xi')} \to f(x), \ \xi' \to 0, \ \xi' \Subset \operatorname{int} \Gamma^* \text{ in } \mathscr{S}'. \tag{6.11}$$

Indeed, if $x \bar{\in} (\operatorname{supp} f)^{2\varepsilon}$, then $\eta(x) = 0$; but if $x \in (\operatorname{supp} f)^{2\varepsilon}$, then $x = x' + x''$, where $x' \in \Gamma$, $|x''| \leqslant R$ for some $R > 0$. Let $\xi' \in C' \Subset \operatorname{int} \Gamma^*$. Then by Lemma 1 of Sec. 4.4 there is a number $\sigma = \sigma(C') > 0$ such that $(x', \xi') \geqslant \sigma |x'| |\xi'|$ and therefore

$$\begin{aligned} -(x, \xi') &= -(x', \xi') - (x'', \xi') \\ &\leqslant -\sigma |x'| |\xi'| + R |\xi'| \\ &\leqslant (-\sigma |x| + \sigma R + R) |\xi'|. \end{aligned}$$

The relations (6.10) and (6.11) follow from the resulting estimate and from the properties of the function $\eta(x)$.

Now, for all $\varphi \in \mathscr{S}$, we have a chain of equalities:

$$(F[f], \varphi) = (f, F[\varphi]) = \lim_{\substack{\xi' \to 0 \\ \xi' \Subset \operatorname{int} \Gamma^*}} (\eta(x) f(x) e^{-(x, \xi')}),$$

$$\int \varphi(\xi) e^{i(x, \xi)} d\xi = \lim_{\substack{\xi' \to 0 \\ \xi' \Subset \operatorname{int} \Gamma^*}} \int (f(x), \eta(x) e^{i(x, \xi) - (x, \xi')}) \varphi(\xi) d\xi,$$

whence follows formula (6.9). Here we made use of (5.4) of Sec. 5, since

$$\eta(x) \varphi(\xi) e^{i(x, \xi) - (x, \xi')} \in \mathscr{S}(\mathbb{R}^{2n}) \text{ for all } \xi' \in \operatorname{int} \Gamma^*.$$

(j) $$F[\theta(x)] = \frac{i}{\xi + i0} = \pi\delta(\xi) + i\mathscr{P}\frac{1}{\xi}, \tag{6.12}$$

$$F[\theta(-x)] = \frac{-i}{\xi - i0} = \pi\delta(\xi) - i\mathscr{P}\frac{1}{\xi}. \tag{6.12'}$$

These formulas follow from (6.9) and from the Sochozki formulas (8.3) and (8.3′) of Sec. 1, for example:

$$F[\theta]=\lim_{\xi'\to+0}\int_0^\infty e^{ix(\xi+i\xi')}\,dx=\lim_{\xi'\to+0}\frac{i}{\xi+i\xi'}=\frac{i}{\xi+i0}.$$

(k) $$F[\operatorname{sign} x]=F[\theta(x)]-F[\theta(-x)]=2i\mathscr{P}\frac{1}{\xi}. \qquad (6.13)$$

(l) $$F\left[\mathscr{P}\frac{1}{x}\right]=-2\pi F^{-1}\left[\mathscr{P}\frac{1}{x}\right]=\pi i\operatorname{sign}\xi. \qquad (6.14)$$

(m) Let $V^+$ be a future light cone in $\mathbb{R}^{n+1}$ (see Sec. 4.4) and let $\theta_{V^+}(x)$ be its characteristic function. Then, by virtue of (i),

$$F[\theta_{V^+}]=\lim_{\xi_0'\to+0}\int_{V^+}e^{i(x,\xi)-x_0\xi_0'}\,dx$$

$$=2^n\pi^{\frac{n-1}{2}}\Gamma\left(\frac{n+1}{2}\right)[-(\xi_0+i0)^2+|\xi|^2]^{-\frac{n+1}{2}} \qquad (6.15)$$

(for a simple method of computing this integral see Sec. 10.2).

(n) *Hermite polynomials and functions.* Definitions:

$$H_n(x)=(-1)^n(n!)^{-1/2}\pi^{-1/4}2^{-n/2}e^{x^2}\frac{d^ne^{-x^2}}{dx^n},\qquad n=0,1,\dots.$$

are *Hermite polynomials*, and

$$\mathscr{H}_n(x)=e^{-x^2/2}H_n(x),\qquad n=0,1,\dots$$

are *Hermite functions* (the wave functions of a harmonic oscillator).

Differential equations:

$$L^-\mathscr{H}_n=\sqrt{n}\,\mathscr{H}_{n-1},\quad L^+\mathscr{H}_n=\sqrt{n+1}\,\mathscr{H}_{n+1},\quad L^+L^-\mathscr{H}_n=n\mathscr{H}_n \qquad (6.16)$$

$$\frac{dH_n(x)}{dx}=\sqrt{2n}\,H_{n-1}(x),\quad n=0,1,\dots\ (\mathscr{H}_{-1}=H_{-1}=0), \qquad (6.17)$$

where

$$L^\pm=\frac{1}{\sqrt{2}}\left(x\mp\frac{d}{dx}\right),\qquad L^-L^+-L^+L^-=1. \qquad (6.18)$$

Recurrence relation:

$$\sqrt{n+1}\, H_{n+1}(x) = \sqrt{2}xH_n(x) - \sqrt{n}\, H_{n-1}(x), \qquad (6.19)$$
$$n = 1, 2, \ldots .$$

Orthonormality:

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \int_{-\infty}^{\infty} \mathscr{H}_n(x) \mathscr{H}_m(x)\, dx = \delta_{nm}. \qquad (6.20)$$

The Fourier transform:

$$F[\mathscr{H}_n] = \sqrt{2\pi}\, i^n \mathscr{H}_n(\xi). \qquad (6.21)$$

We now prove (6.21), which follows from

$$H_n\left(\frac{d}{i\,d\xi}\right) e^{-\xi^2/2} = i^n \mathscr{H}_n(\xi) \qquad (6.22)$$

by virtue of (see (6.1))

$$\begin{aligned} F[e^{-x^2/2} H_n(x)] &= H_n\left(\frac{d}{i\,d\xi}\right) F[e^{-x^2/2}] \\ &= \sqrt{2\pi}\, H_n\left(\frac{d}{i\,d\xi}\right) e^{-\xi^2/2} \\ &= \sqrt{2\pi}\, i^n e^{-\xi^2/2} H_n(\xi) \\ &= \sqrt{2\pi}\, i^n \mathscr{H}_n(\xi). \end{aligned}$$

The equality (6.22) holds true for $n = 0$. Its truth, for $n > 0$, follows from the recurrence relations (6.19) and (6.17):

$$\begin{aligned} H_n\left(\frac{d}{i\,d\xi}\right) e^{-\xi^2/2} &= \sqrt{\frac{2}{n}} \frac{d}{i\,d\xi} H_{n-1}\left(\frac{d}{i\,d\xi}\right) e^{-\xi^2/2} \\ &\quad - \sqrt{\frac{n-1}{n}}\, H_{n-2}\left(\frac{d}{i\,d\xi}\right) e^{-\xi^2/2} \\ &= \sqrt{\frac{2}{n}} \frac{d}{i\,d\xi} [i^{n-1} H_{n-1}(\xi) e^{-\xi^2/2}] \\ &\quad - \sqrt{\frac{n-1}{n}}\, i^{n-2} H_{n-2}(\xi) e^{-\xi^2/2} \\ &= \frac{i^{n-2}}{\sqrt{n}} e^{-\xi^2/2} [-\sqrt{2}\,\xi H_{n-1}(\xi) + \sqrt{2}\, H'_{n-1}(\xi) \\ &\qquad - \sqrt{n-1}\, H_{n-2}(\xi)] \end{aligned}$$

$$= \frac{i^n}{\sqrt{n}} e^{-\xi^2/2} [\sqrt{2}\,\xi H_{n-1}(\xi) - 2\sqrt{n-1}\,H_{n-2}(\xi) + \sqrt{n-1}\,H_{n-2}(\xi)]$$

$$= i^n e^{-\xi^2/2} H_n(\xi) = i^n \mathscr{H}_n(\xi).$$

Smoothness: $\mathscr{H}_n \in \mathscr{S}$, and

$$\| \mathscr{H}_n \|_p \leqslant c_p (1+n)^{p+2}, \quad p = 0, 1, \ldots, n = 0, 1, \ldots \tag{6.23}$$

The estimate (6.23) follows from the equations (6.16) and from the formulas (6.20) and (6.21). Regarding $p$ as even, we have

$$\| \mathscr{H}_n \|_p = \sup_{\substack{x \\ 0 \leqslant \alpha \leqslant p}} | (1+x^2)^{p/2} \mathscr{H}_n^{(\alpha)}(x) |$$

$$= \frac{1}{\sqrt{2\pi}} \sup_{\substack{x \\ 0 \leqslant \alpha \leqslant p}} \left| \int \left(1 - \frac{d^2}{d\xi^2}\right)^{p/2} [\xi^\alpha \mathscr{H}_n(\xi)] (1+\xi^2) e^{ix\xi} \frac{d\xi}{1+\xi^2} \right|$$

$$\leqslant \frac{1}{\sqrt{2\pi}} \sup_{\substack{x \\ 0 \leqslant \alpha \leqslant p}} \left| \sum_{\ldots + \alpha_s + \ldots + \alpha_m + \ldots \leqslant 2p+2} \int c_{\ldots \alpha_s \ldots \alpha_m \ldots} \ldots (L^+)^{\alpha_s} \ldots (L^-)^{\alpha_m} \ldots \mathscr{H}_n(\xi) \frac{e^{ix\xi} d\varepsilon}{1+\xi^2} \right.$$

$$\leqslant c_p' (1+n)^{p+1} \sum_{0 \leqslant k \leqslant 2p+2} \int | \mathscr{H}_k(\xi) | \frac{d\xi}{1+\xi^2}$$

$$\leqslant c_p' (1+n)^{p+1} \sum_{0 \leqslant k \leqslant 2p+2} \| \mathscr{H}_k \| \sqrt{\int \frac{d\xi}{1+\xi^2}}$$

$$\leqslant c_p (1+n)^{p+2}$$

For odd $p$, the estimate (6.23) follows from (1.1) of Sec. 5.

Let $f \in \mathscr{S}'$. The numbers

$$a_n(f) = (f, \mathscr{H}_n), \qquad n = 0, 1, \ldots \tag{6.24}$$

will be called *Fourier coefficients*. The formal series

$$\sum_{0 \leqslant n < \infty} a_n(f) \mathscr{H}_n(x) \tag{6.25}$$

will be called the *Fourier series* of the generalized function $f$ with respect to the orthonormal system of Hermite functions $\{\mathscr{H}_n\}$.

Completeness in $\mathscr{L}^2$: if $f \in \mathscr{L}^2$, then its Fourier series (6.25) is unique, converges in $\mathscr{L}^2$ to $f$, and the following Parseval-Steklov

equation holds true:

$$\|f\|^2=\sum_{0\leqslant n<\infty}|a_n(f)|^2. \tag{6.26}$$

For the function $\varphi$ to belong to $\mathcal{S}$, it is necessary and sufficient that its Fourier coefficients satisfy the condition

$$\|(L^-L^+)^m\varphi\|^2=\sum_{0\leqslant n<\infty}|a_n(\varphi)|^2 n^{2m}<\infty, \tag{6.27}$$

$$m=0, 1, \ldots.$$

Then the Fourier series of $\varphi$ converges to $\varphi$ in $\mathcal{S}$.

Indeed, if $\varphi\in\mathcal{S}$, then $(L^-L^+)^m\varphi\in\mathcal{L}^2$ for all $m\geqslant 0$. Therefore, by virtue of (6.16) and (6.24),

$$\begin{aligned} a_n((L^-L^+)^m\varphi) &= \int (L^-L^+)^m\varphi(x)\,\mathcal{H}_n(x)\,dx \\ &= \int \varphi(x)(L^+L^-)^m\,\mathcal{H}_n(x)\,dx \\ &= n^m\int \varphi(x)\,\mathcal{H}_n(x)\,dx = n^m a_n(\varphi), \end{aligned}$$

whence, by the Parseval-Steklov equation (6.26), follows (6.27).

Conversely, if the coefficients $\{a_n\}$ satisfy the condition $\sum\limits_{0\leqslant n<\infty}|a_n|^2 n^{2m}<\infty$ for all $m\geqslant 0$, then by (6.23) the series $\sum\limits_{0\leqslant n<\infty} a_n\mathcal{H}_n(x)$ converges in $\mathcal{S}$ to some $\varphi\in\mathcal{S}$ such that $a_n=a_n(\varphi)$.

For $f$ to belong to $\mathcal{S}'$, it is necessary and sufficient that its Fourier coefficients satisfy the following condition: there exist $p\geqslant 0$ and $C$ such that

$$|a_n(f)|\leqslant C(1+n)^p, \quad n=0, 1, \ldots. \tag{6.28}$$

Here, the Fourier series of $f$ is unique, converges to $f$ in $\mathcal{S}'$, and the Parseval-Steklov equation holds:

$$(f, \varphi)=\sum_{0\leqslant n<\infty} a_n(f)\,a_n(\varphi), \qquad \varphi\in\mathcal{S}. \tag{6.29}$$

Indeed, if $f\in\mathcal{S}'$ and $m$ is the order of $f$ (see Sec. 5.2), then by (6.24) and (6.23) the estimate (6.28) holds:

$$|a_n(f)|=|(f, \mathcal{H}_n)|\leqslant\|f\|_{-m}\,\|\mathcal{H}_n\|_m\leqslant c_m\|f\|_{-m}(1+n)^{m+2}.$$

Conversely, if the coefficients $\{a_n\}$ satisfy the condition (6.28), $|a_n| \leqslant C(1+n)^p$, $n=0, 1, \ldots$, then by virtue of (6.27) the series $\sum\limits_{0\leqslant n<\infty} a_n\mathscr{H}_n(x)$ converges in $\mathscr{S}'$ to some $f \in \mathscr{S}'$, and the following equation holds:

$$(f, \varphi) = \sum_{0\leqslant n<\infty} a_n a_n(\varphi), \qquad \varphi \in \mathscr{S}, \tag{6.30}$$

since

$$\Big(\sum_{0\leqslant n\leqslant N} a_n\mathscr{H}_n, \varphi\Big) = \sum_{0\leqslant n\leqslant N} a_n a_n(\varphi) \to (f, \varphi), \qquad N\to\infty,$$

by virtue of the completeness of the space $\mathscr{S}'$ (see Sec. 5.2). Putting $\varphi = \mathscr{H}_m$ in (6.30) and taking into account that by (6.20), $a_n(\mathscr{H}_m) = \delta_{nm}$, we get $a_n = a_n(f)$.

It remains to prove the uniqueness of the Fourier series: if $f \in \mathscr{S}'$ and $a_n(f) = 0$, $n = 0, 1, \ldots$, then $f = 0$. But this follows from (6.30).

*Remark*. Let us introduce two sequence spaces: we define convergence in them in a natural manner in accord with the estimates (6.27) and (6.28). The results that have been proved signify that the operation $f \to \{a_n(f), n = 0, 1, \ldots\}$ is a linear isomorphism of $\mathscr{S}$ and $\mathscr{S}'$ onto the sequence of spaces that satisfy the conditions (6.27) and (6.28) respectively. [The continuity of this operation follows from (6.27) and (6.29).]

(o) *An integral representation of the Bessel function*:

$$J_\nu(x) = \frac{1}{\sqrt{\pi}\,\Gamma(\nu+1/2)}\left(\frac{x}{2}\right)^\nu \int_{-1}^{1} e^{ix\xi}(1-\xi^2)^{\nu-1/2}\,d\xi, \tag{6.31}$$

$$\operatorname{Re}\nu > -\frac{1}{2}.$$

The Bessel function

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}$$

is a unique (up to a factor) solution, bounded in zero, of the Bessel equation

$$(xu')' + \left(x - \frac{\nu^2}{x}\right)u = 0.$$

By virtue of the equation

$$\frac{1}{\sqrt{\pi}\,\Gamma(\nu+1/2)}\int_{-1}^{1}(1-\xi^2)^{\nu-1/2}\,d\xi$$

$$=\frac{1}{\sqrt{\pi}\,\Gamma(\nu+1/2)}\int_{0}^{1}(1-\mu)^{-1/2}\mu^{\nu-1/2}d\mu$$

$$=\frac{B(1/2,\nu+1/2)}{\sqrt{\pi}\,\Gamma(\nu+1/2)}=\frac{\Gamma(1/2)\,\Gamma(\nu+1/2)}{\sqrt{\pi}\,\Gamma(\nu+1/2)\,\Gamma(\nu+1)}$$

$$=\frac{1}{\Gamma(\nu+1)}$$

the asymptotic behaviour, as $x\to+0$, of both sides of (6.31) is the same. And so to prove (6.31) it remains to prove that the right-hand side of (6.31) satisfies the Bessel equation. But this is established by direct verification:

$$[xJ_\nu'(x)]'+\left(x+\frac{\nu^2}{x}\right)J_\nu(x)$$

$$=x^{\nu+1}\int_{-1}^{1}(1-\xi^2)^{\nu+1/2}e^{ix\xi}d\xi+(2\nu+1)\,ix^{\nu}\int_{-1}^{1}(1-\xi^2)^{\nu-1/2}e^{ix\xi}\xi\,d\xi$$

$$=0.$$

(p) *The Hankel transform.* Let $f(|x|)\in\mathcal{L}^2$, that is, $'\|f\|^2=\int_0^\infty|f(r)|^2r^{n-1}\,dr<\infty$. The function

$$g(\rho)=\frac{(2\pi)^{n/2}}{\rho^{\frac{n-2}{2}}}\int_0^\infty f(r)\,r^{n/2}J_{\frac{n-2}{2}}(r\rho)\,dr \tag{6.32}$$

is termed a *Hankel transform of order* $(n-2)/2$ of the function $f(r)$; the integral here converges in the norm $'\|\ \ \|$.

The following inversion formula holds:

$$f(r)=\frac{(2\pi)^{-n/2}}{r^{\frac{n-2}{2}}}\int_0^\infty g(\rho)\,\rho^{n/2}J_{\frac{n-2}{2}}(r\rho)\,d\rho, \tag{6.32'}$$

and the Parseval equation holds:

$$(2\pi)^{n'} \| f \|^2 = ' \| g \|^2.$$

*Special cases:*

$$n=1, \quad g(\rho)=2\int_0^{\infty} f(r)\cos r\rho\, dr,$$

$$n=2, \quad g(\rho)=\frac{1}{2\pi}\int_0^{\infty} f(r)\, rJ_0(r\rho)\, dr,$$

$$n=3, \quad g(\rho)=\frac{4\pi}{\rho}\int_0^{\infty} f(r)\, r\sin r\rho\, dr.$$

To prove the inversion formulas (6.32) and (6.32′) and the Parseval equation, it is sufficient to demonstrate, by the Plancherel theorem (see Sec. 6.6(c)), that the right-hand sides of (6.32) and (6.32′) are the direct and inverse Fourier transforms of the functions $f(|x|)$ and $g(|\xi|)$, respectively.

Indeed, using (6.31) we have

$$\begin{aligned} \mathcal{F}[f(|x|)] &= \int f(|x|)\, e^{i(\xi,x)}dx \\ &= \int_0^{\infty} f(r)\, r^{n-1} \int_{|s|=1} e^{ir(\xi,s)}ds\, dr \\ &= \sigma_{n-1}\int_0^{\infty} f(r)\, r^{n-1}\int_0^{\pi} e^{ir\rho\cos\theta}\sin^{n-2}\theta\, d\theta\, dr \\ &= \sigma_{n-1}\int_0^{\infty} f(r)\, r^{n-1}\int_{-1}^{1} e^{ir\rho\mu}(1-\mu^2)^{\frac{n-3}{2}}\, d\mu\, dr \\ &= \frac{(2\pi)^{n/2}}{\rho^{\frac{n-2}{2}}}\int_0^{\infty} f(r)\, r^{n/2}J_{\frac{n-2}{2}}(r\rho)\, dr, \end{aligned}$$

which is what we wanted; here, $\sigma_{n-1}$ is the surface area of a unit sphere in $\mathbb{R}^{n-1}$ (see Sec. 0.6).

# 7 Fourier Series of Periodic Generalized Functions

**7.1 The definition and elementary properties of periodic generalized functions** A generalized function $f(x)$ taken from $\mathscr{D}'(\mathbb{R}^n)$ is said to be *periodic with an n-period* $T = (T_1, T_2, \ldots, T_n)$,

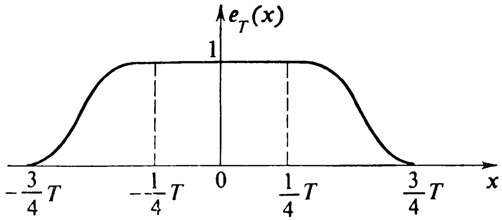



Figure 27

$T_j > 0$, if it is periodic with respect to each argument $x_j$ with period $T_j$, that is, if it satisfies the condition (see Sec. 1.9)

$$f(x_1, \ldots, x_j + T_j, \ldots, x_n) = f(x), \quad j = 1, \ldots, n.$$

We use $\mathscr{D}'_T$ to denote the collection of all periodic generalized functions of an $n$-period $T$.

We now prove that for every $n$-period $T$ there exists a special expansion of unity in $\mathbb{R}^n$ (see Sec. 1.2):

$$\sum_{|k|\geqslant 0} e_T(x+kT) = 1, \qquad e_T \geqslant 0, \quad e_T \in \mathscr{D},$$

$$\operatorname{supp} e_T \subset \left(-\frac{3}{4}T_1, \frac{3}{4}T_1\right) \times \ldots \times \left(-\frac{3}{4}T_n, \frac{3}{4}T_n\right); \quad (1.1)$$

$e_T(x)$ is an even function with respect to each variable; here we set $kT = (k_1T_1, \ldots, k_nT_n)$.

Let $T > 0$. We denote by $e_T(x)$ an even function taken from $\mathscr{D}(\mathbb{R}^1)$ with the properties: $\operatorname{supp} e_T \subset \left(-\frac{3}{4}T, \frac{3}{4}T\right)$, $e_T(x) = 1$ in the neighbourhood of the interval $\left[-\frac{1}{4}T, \frac{1}{4}T\right]$, and $e_T(x) = 1 - e_T(x+T)$, $x \in \left[-\frac{3}{4}T, -\frac{1}{4}T\right]$ (Fig. 27). It is easy to see that such functions exist. Clearly, the function $e_T$ satisfies the equation

$$\sum_{|k|\geqslant 0} e_T(x+kT) = 1, \qquad x \in \mathbb{R}^1. \quad (1.2)$$

Setting

$$e_T(x) = e_{T_1}(x) \ldots e_{T_n}(x), \tag{1.3}$$

we are convinced of the existence of the required expansion of unity.

We now introduce the generalized function

$$\delta_T(x) = \sum_{|k| \geqslant 0} \delta(x - kT).$$

Quite obviously, $\delta_T \in \mathscr{D}'_T \cap \mathscr{S}'$ (see Sec. 5.3).

Let us now prove the following representation: if $f \in \mathscr{D}'_T$, then

$$f = (e_T f) * \delta'_T. \tag{1.4}$$

Indeed, using (1.1) and the periodicity of $f(x)$, we have

$$\begin{aligned} f(x) &= f(x) \sum_{|k| \geqslant 0} e_T(x + kT) \\ &= \sum_{|k| \geqslant 0} f(x) e_T(x + kT) \\ &= \sum_{|k| \geqslant 0} f(x + kT) e_T(x + kT) \\ &= \sum_{|k| \geqslant 0} (e_T f) * \delta(x + kT), \end{aligned}$$

whence, taking advantage of the continuity in $\mathscr{S}'$ of the convolution (see Sec. 5.6(a)), we obtain the representation (1.4).

From (1.4) it follows, in particular, that $\mathscr{D}'_T \subset \mathscr{S}'$. Besides, setting $f = \delta_T$ in (1.4), we obtain

$$\delta_T = (e_T \delta_T) * \delta_T. \tag{1.5}$$

Let $f \in \mathscr{D}'_T$ and $\varphi \in C^\infty \cap \mathscr{D}'_T$. We introduce the scalar product $(f, \varphi)_T$ by the rule

$$(f, \varphi)_T = (f, e_T \varphi).$$

For this definition to be proper, it is necessary to demonstrate that the right-hand side of the equation is independent of the choice of the auxiliary function $e_T(x)$ with the properties (1.1).

Indeed, let $e'_T(x)$ be another such function. Then, using the representation (1.4) and formula (6.1) of Sec. 5, we obtain

$$
\begin{aligned}
(f, e'_T\varphi) &= ((e_T f) * \delta_T, e'_T\varphi) \\
&= (e_T(x) f(x) \times \delta_T(y), e'_T(x+y)\varphi(x+y)) \\
&= (f(x), (\delta_T(y), e_T(x) e'_T(x+y)\varphi(x+y))) \\
&= \Big(f(x), e_T(x) \sum_{|k|\geqslant 0} e'_T(x-kT)\varphi(x-kT)\Big) \\
&= (f, e_T\varphi),
\end{aligned}
$$

which is what we set out to prove.

If $f \in \mathscr{L}^1_{\text{loc}} \cap \mathscr{D}'_T$, then

$$
(f, \varphi)_T = \int_0^{T_1} \dots \int_0^{T_n} f(x)\varphi(x)\,dx. \tag{1.6}
$$

True, since the scalar product $(\ ,\ )_T$ is independent of the choice of the function $e_T$, it suffices to compute it for the concrete function (1.3):

$$
\begin{aligned}
(f, \varphi)_T &= \int e_T(x) f(x)\varphi(x)\,dx \\
&= \left[\int_{-\frac{3}{4}T_1}^{-\frac{1}{4}T_1} e_{T_1}(x_1) + \int_{-\frac{1}{4}T_1}^{\frac{1}{4}T_1} e_{T_1}(x_1) + \int_{\frac{1}{4}T_1}^{\frac{3}{4}T_1} e_{T_1}(x)\right] \\
&\qquad \times \int e_{T_2}(x_2)\dots e_{T_n}(x_n) f(x)\varphi(x)\,dx_2\dots dx_n\,dx_1 \\
&= \int_{-\frac{1}{2}T_1}^{\frac{1}{2}T_1} \int e_{T_2}(x_2)\dots e_{T_n}(x_n) f(x)\varphi(x)\,dx_2\dots dx_n\,dx_1
\end{aligned}
$$

(*continued on page 130*)

$$+\left[\int_{\frac{1}{4}T_1}^{\frac{3}{4}T_1} e_{T_1}(x_1) - \int_{-\frac{3}{4}T_1}^{-\frac{1}{4}T_1} e_{T_1}(x_1+T_1)\right]$$

$$\times \int e_{T_2}(x_2) \dots e_{T_n}(x_n) f(x) \varphi(x)\, dx_2 \dots dx_n\, dx_1$$

$$= \int_{-\frac{1}{2}T_1}^{\frac{1}{2}T_1} \int_{-\frac{1}{2}T_2}^{\frac{1}{2}T_2} \int e_{T_3}(x_3) \dots e_{T_n}(x_n) f(x) \varphi(x)\, dx_3 \dots dx_n\, dx_2\, dx_1$$

$$\dots = \int_{-\frac{1}{2}T_1}^{\frac{1}{2}T_1} \int_{-\frac{1}{2}T_2}^{\frac{1}{2}T_2} \dots \int_{-\frac{1}{2}T_n}^{\frac{1}{2}T_n} f(x) \varphi(x)\, dx.$$

In particular, the trigonometric functions

$$e^{i(k\omega,\, x)}, \qquad \omega = \left(\frac{2\pi}{T_1}, \dots, \frac{2\pi}{T_n}\right)$$

are periodic with $n$-period $T$ and for them

$$(e^{i(k\omega,x)},\ e^{-i(k'\omega,x)})_T = \delta_{kk'}\, T_1 \dots T_n. \tag{1.7}$$

**7.2 Fourier series of periodic generalized functions** Let $f \in \mathscr{D}'_T$. The formal series

$$f(x) \sim \sum_{|k| \geqslant 0} c_k(f)\, e^{i(k\omega,\, x)}, \qquad c_k(f) = \frac{(f, e^{-i(k\omega,\, x)})_T}{T_1 \dots T_n}, \tag{2.1}$$

is termed a *Fourier series* and the numbers $c_k(f)$ are called *Fourier coefficients* of the generalized function $f$.

*Example 1.* If $f \in \mathscr{L}^1_{\text{loc}} \cap \mathscr{D}'_T$, then its Fourier series (2.1) turns into the classical Fourier series by virtue of (1.6).

*Example 2.* The following equation in $\mathscr{S}'$ holds true:

$$\sum_{|k| \geqslant 0} \delta(x + kT) = \frac{1}{T_1 \dots T_n} \sum_{|k| \geqslant 0} e^{i(k\omega,\, x)}. \tag{2.2}$$

It follows from the one-dimensional formula (3.5) of Sec. 2 and from the continuity, in $\mathscr{S}'$, of the direct product (see Sec. 5.5).

Fix $m \geqslant 0$. Let $f \in \mathcal{D}'_T$ and let $m$ be the order of $f$. Then there is a number $C_m \geqslant 0$, not dependent on $f$ and $k$, such that

$$| c_k (f) | \leqslant C_m \| f \|_{-m} (1 + | k |)^m. \tag{2.3}$$

Indeed, using the definition of a scalar product $(,)_T$ and fixing the auxiliary function $e_T (x)$, we obtain the estimate (2.3):

$$\begin{aligned} |c_k (f)| &= \frac{1}{T_1 \ldots T_n} |(f, e^{-i (k\omega, x)})_T| \\ &= \frac{1}{T_1 \ldots T_n} |(f, e_T e^{-i (k\omega, x)})| \\ &\leqslant \frac{\| f \|_{-m}}{T_1 \ldots T_n} \sup_{\substack{x \\ |\alpha| \leqslant m}} (1 + |x|^2)^{m/2} |D^\alpha [e_T (x) e^{-i (k\omega, x)}]| \\ &\leqslant C' \| f \|_{-m} \sup_{\substack{x \\ |\alpha| \leqslant m}} \sum_{\beta \leqslant \alpha} \binom{\beta}{\alpha} |D^{\alpha-\beta} e_T (x)| |(-ik\omega)^\beta| \\ &\leqslant C \| f \|_{-m} (1 + |k|)^m. \end{aligned}$$

Theorem *The Fourier series of any generalized function $f$ in $\mathcal{D}'_T$ converges to $f$ in $\mathcal{S}'$:*

$$f (x) = \sum_{|k| \geqslant 0} c_k (f) e^{i (k\omega, x)}. \tag{2.4}$$

*Proof*. Substituting (2.2) into the right-hand side of (1.4),

$$\begin{aligned} f &= (e_T f) * \sum_{|k| \geqslant 0} \frac{1}{T_1 \ldots T_n} e^{i (k\omega, x)} \\ &= \sum_{|k| \geqslant 0} \frac{1}{T_1 \ldots T_n} (e_T f) * e^{i (k\omega, x)}, \end{aligned}$$

and using (3.3) of Sec. 4 for the convolution,

$$\begin{aligned} (e_T f) * e^{i (k\omega, x)} &= (f (y), e_T (y) e^{i (k\omega, x-y)}) \\ &= T_1 \ldots T_n c_k (f) e^{i (k\omega, x)}, \end{aligned}$$

we obtain the expansion of $f$ in the form of the Fourier series (2.4) converging in $\mathcal{S}'$. The proof is complete.

Corollary 1 *A generalized function $f$ in $\mathcal{D}'_T$ is completely determined by the set of its Fourier coefficients $\{c_k (f)\}$.*

Corollary 2 *If $f \in \mathcal{D}'_T$ and $\varphi \in C^\infty \cap \mathcal{D}'_T$, then the following generalized Parseval-Steklov equation holds true:*

$$(f, \varphi)_T = \sum_{|k| \geqslant 0} c_k(f)\, c_k(\varphi). \tag{2.5}$$

Corollary 3 *The Fourier series of a generalized function $f$ taken from $\mathcal{D}'_T$ may be differentiated termwise an infinite number of times:*

$$D^\alpha f(x) = \sum_{|k| \geqslant 0} c_k(f)\, (ik\omega)^\alpha e^{i(k\omega, x)}, \tag{2.6}$$

*so that*

$$c_k(D^\alpha f) = (ik\omega)^\alpha c_k(f). \tag{2.7}$$

**7.3 The convolution algebra $\mathcal{D}'_T$** We introduce, on the set $\mathcal{D}'_T$, the convolution operation $\circledast$ via the rule

$$f \circledast g = (e_T f) * g, \qquad f, g \in \mathcal{D}'_T. \tag{3.1}$$

The convolution $f \circledast g$ does not depend on the auxiliary function $e_T$ and is commutative.

This assertion stems from the equality

$$(e_T f) * g = (e'_T g) * f \tag{3.2}$$

that follows from the identity (1.4) and from the properties of continuity, associativity, and commutativity of the convolution $*$ (see Sec. 4.2):

$$\begin{aligned}(e_T f) * g &= (e_T f) * ((e'_T g) * \delta_T) = ((e_T f) * \delta_T) * (e'_T g)\\ &= f * (e'_T g) = (e'_T g) * f.\end{aligned}$$

The operation $f \to f \circledast g$ is linear and continuous from $\mathcal{D}'_T$ to $\mathcal{S}'$ (see Sec. 5.6(a)).

Finally, $f \circledast g \in \mathcal{D}'_T$.

This follows from the property of translation of a convolution (see Sec. 4.2(c)):

$$\begin{aligned}(f \circledast g)(x + kT) &= (e_T f) * g(x + kT)\\ &= (e_T f) \circledast g = (f \circledast g)(x).\end{aligned}$$

The convolution of any number of generalized functions $f_1, f_2, \ldots, f_m$ taken from $\mathcal{D}'_T$ is determined in similar fashion via

the rule

$$f_1 \circledast f_2 \circledast \ldots \circledast f_m = (e'_T f_1) * (e''_T f_2) * \ldots * (e_T^{(m)} f_m). \qquad (3.3)$$

This convolution is associative and commutative (see Sec. 4.2).

*Example 1.* If $f$ and $g \in \mathscr{L}'_{\text{loc}} \cap \mathscr{D}'_T$, then

$$\begin{aligned}(f \circledast g)(x) &= \int_0^{T_1} \ldots \int_0^{T_n} f(x-y)\, g(y)\, dy \\ &= \int_0^{T_1} \ldots \int_0^{T_n} f(y)\, g(x-y)\, dy = (g \circledast f)(x).\end{aligned} \qquad (3.4)$$

*Example 2.*

$$f \circledast e^{i(k\omega, x)} = T_1 \ldots T_n c_k(f)\, e^{i(k\omega, x)}. \qquad (3.5)$$

Indeed,

$$\begin{aligned}(f \circledast e^{i(k\omega, x)}, \varphi) &= ((e_T f) * e^{i(k\omega, x)}), \varphi) \\ &= (e_T(\xi)\, f(\xi) \times e^{i(k\omega, y)}, \varphi(\xi + y)) \\ &= \left(f(\xi), e_T(\xi) \int e^{i(k\omega, y)} \varphi(\xi + y)\, dy\right) \\ &= \left(f(\xi), e_T(\xi)\, e^{-i(k\omega, \xi)} \int e^{i(k\omega, x)} \varphi(y)\, dx\right) \\ &= T_1 \ldots T_n c_k(f) \int e^{i(k\omega, x)} \varphi(x)\, dx.\end{aligned}$$

In particular, by virtue of (1.7),

$$e^{i(k\omega, x)} \circledast e^{i(k'\omega, x)} = T_1 \ldots T_n \delta_{kk'} e^{i(k\omega, x)}. \qquad (3.6)$$

*Example 3.* The formula (1.4) takes the form

$$f = f \circledast \delta_T, \qquad D^\alpha f = f \circledast D^\alpha \delta_T, \qquad f \in \mathscr{D}'_T \qquad (3.7)$$

and, generally, if $f$ and $g \in \mathscr{D}'_T$, then

$$c_k(f \circledast g) = T_1 \ldots T_n c_k(f)\, c_k(g). \qquad (3.8)$$

Indeed, using (3.5) we have

$$c_k(f \circledast g) = \frac{1}{T_1 \ldots T_n} e^{-i(k\omega, x)} (f \circledast g) \circledast e^{i(k\omega, x)}$$

$$= \frac{1}{T_1 \ldots T_n} e^{-i(k\omega, x)} (f \circledast (g \circledast e^{i(k\omega, x)}))$$

$$= c_k(g) e^{-i(k\omega, x)} (f \circledast e^{i(k\omega, x)})$$

$$= T_1 \ldots T_n c_k(f) c_k(g).$$

From the foregoing it follows that $\mathscr{D}'_T$ forms a convolution algebra relative to the convolution operation $\circledast$ (see Sec. 4.5). The algebra $\mathscr{D}'_T$ is associative and commutative; its unit element is $\delta_T$ (see (3.7)); it contains zero divisors [see (3.6)].

What has been said in Sec. 4.8(d) holds true for equations $a \circledast u = f$ in the convolution algebra $\mathscr{D}'_T$. But here we have more precise statements.

Theorem *For the operator* $a \circledast$, $a \in \mathscr{D}'_T$, *to have an inverse* $\mathscr{E} \circledast$ *in* $\mathscr{D}'_T$, *it is necessary and sufficient that for certain* $L > 0$ *and* $m$ *the following inequality hold*:

$$|c_k(a)| \geqslant L(1 + |k|)^{-m}. \tag{3.9}$$

*Here, the fundamental solution* $\mathscr{E}$ *is unique and is expressible as a Fourier series*:

$$\mathscr{E}(x) = \frac{1}{T_1^2 \ldots T_n^2} \sum_{|k| \geqslant 0} \frac{1}{c_k(a)} e^{i(k\omega, x)}. \tag{3.10}$$

*Proof. Sufficiency*. By (3.9), the series (3.10) converges in $\mathscr{S}'$ and its sum $\mathscr{E} \in \mathscr{D}'_T$. We will prove that $\mathscr{E}$ satisfies the equation $a \circledast \mathscr{E} = \delta_T$. By the theorem of Sec. 7.2, it suffices to prove the equality of the Fourier coefficients:

$$c_k(a \circledast \mathscr{E}) = c_k(\delta_T) = \frac{1}{T_1 \ldots T_n}.$$

But this is fulfilled by virtue of (3.8) and (3.10).

*Necessity*. Suppose, in $\mathscr{D}'_T$, there is a fundamental solution $\mathscr{E}$ of the operator $a \circledast$, $a \circledast \mathscr{E} = \delta_T$. Then it is unique (see Sec. 4.8(d)) and from the equalities

$$c_k(a \circledast \mathscr{E}) = c_k(a) c_k(\mathscr{E}) T_1 \ldots T_n = c_k(\delta_T) = \frac{1}{T_1 \ldots T_n}$$

we derive

$$c_k(\mathscr{E}) = \frac{1}{T_1^2 \ldots T_n^2 c_k(a)}. \tag{3.11}$$

Therefore the expansion (3.10) holds true. Furthermore, denoting by $m$ the order of $\mathscr{E}$ and applying the estimates (2.3) and (3.11), we obtain the estimate (3.9):

$$|c_k(\mathscr{E})| = \frac{1}{T_1^2 \ldots T_n^2} \frac{1}{|c_k(a)|} \leqslant C \| \mathscr{E} \|_{-m} (1+|k|)^m.$$

The proof of the theorem is complete.

**7.4 Examples** a) Solve the "quadratic" equation in $\mathscr{D}'_T$:

$$u \circledast u = \delta_T. \tag{4.1}$$

We have

$$c_k^2(u) = \frac{1}{T_1^2 \ldots T_n^2}, \qquad c_k(u) = \pm \frac{1}{T_1 \ldots T_n}.$$

Therefore equation (4.1) has a continuum of solutions:

$$u(x) = \frac{1}{T_1 \ldots T_n} \sum_{|k| \geqslant 0} \varepsilon_k e^{i(k\omega, x)}, \qquad \varepsilon_k = \pm 1. \tag{4.2}$$

(b) $$\left(\frac{d}{dx} - \lambda\right) \mathscr{E} = \delta_T, \qquad n = 1, \quad \lambda \neq ik\omega, \quad k = 0, \pm 1, \ldots.$$

Rewriting

$$(\delta'_T - \lambda \delta_T) \circledast \mathscr{E} = \delta_T,$$

we obtain

$$T \frac{ik\omega - \lambda}{T} c_k(\mathscr{E}) = \frac{1}{T}$$

so that

$$\mathscr{E}(x) = \frac{1}{T} \sum_{|k| \geqslant 0} \frac{1}{ik\omega - \lambda} e^{ik\omega x}. \tag{4.3}$$

(c) Let us consider the eigenvalue problem:

$$\delta'_T \circledast u = \lambda u, \qquad u \in \mathscr{D}'_T,$$

$$\lambda_k = ik\omega, \qquad u_k(x) = e^{ik\omega x}, \qquad k = 0, \pm 1, \ldots, \tag{4.4}$$

are eigenvalues and the associated eigenfunctions of the operator $\delta'_T \circledast$.

(d) Let $f \in \mathscr{D}'_T$, $n = 1$. We consider the problem of finding the antiderivative $f^{(-1)}$ in $\mathscr{D}'_T$ (see Sec. 2.2):

$$\frac{df^{(-1)}}{dx} = f, \qquad \delta'_T \circledast f^{(-1)} = f.$$

From the equation

$$\frac{ik\omega}{T} c_k (f^{(-1)}) = c_k (f)$$

it follows that the antiderivative $f^{(-1)}$ exists in $\mathscr{D}'_T$ if and only if $c_0 (f) = 0$ and can be expressed by the Fourier series

$$f^{(-1)} (x) = \sum_{|k|>0} \frac{c_k (f)}{ik\omega} e^{ik\omega x} + C \tag{4.5}$$

where $C$ is an arbitrary constant.

(e) *Bernoulli polynomials.* Set $f_0 = T\delta_T - 1$. Since $c_0 (f_0) = Tc_0 (\delta)_T - c_0 (1) = 0$, it follows that $f_0^{(-1)}$ exists in $\mathscr{D}'_T$; we choose $c_0 (f_0^{(-1)}) = 0$, and so forth. As a result we obtain a sequence of antiderivatives $f_0^{(-m)} (x)$ in $\mathscr{D}'_T$, $m = 1, 2, \ldots$, which are polynomials on the basic period $(0, T)$. They are called *Bernoulli polynomials.* Let us find their Fourier-series expansion. We have

$$(f_0^{(-m)})^{(m)} = \delta_T^{(m)} \circledast f_0^{(-m)} = f_0 = T\delta_T - 1$$

and therefore

$$c_k (\delta_T^{(m)} \circledast f_0^{(-m)}) = (ik\omega)^m c_k (f_0^{(-m)}) = 1, \qquad k \neq 0.$$

Consequently,

$$f_0^{(-m)} (x) = \sum_{|k|>0} \frac{1}{(ik\omega)^m} e^{ik\omega x}. \tag{4.6}$$

For example,

$$f_0^{(-1)} (x) = \frac{T}{2} - x, \qquad 0 < x < T.$$

# 8 Positive Definite Generalized Functions

**8.1 The definition and elementary properties of positive definite generalized functions** Suppose $f \in \mathscr{D}'(\mathbb{R}^n)$; a generalized function $f^* (x) = \overline{f} (-x)$ in $\mathscr{D}' (\mathbb{R}^n)$ is said to be the $*$-*Hermite conjugate of* $f$; if $f = f^*$, then $f$ is said to be a $*$-*Hermite* (generalized) function.

The function $f(x)$, which is continuous in $\mathbb{R}^n$, is a *positive definite* function, $f \gg 0$, if for any points $x_1, \ldots, x_l$ in $\mathbb{R}^n$ and for the complex numbers $z_1, \ldots, z_l$ the following inequality holds true:

$$\sum_{1 \leqslant j,\, k \leqslant l} f(x_j - x_k)\, z_j \bar{z}_k \geqslant 0.$$

From this definition follow certain properties: setting $l = 1$, we obtain $f(0) \geqslant 0$; also, setting $l = 2$, $x_1 = x$, $x_2 = 0$, we have

$$f(0)(|z_1|^2 + |z_2|^2) + f(x) z_1 \bar{z}_2 + f(-x) \bar{z}_1 z_2 \geqslant 0$$

whence it follows that $f$ is a $*$-Hermite bounded function:

$$f = f^*, \quad |f(x)| \leqslant f(0). \tag{1.1}$$

Finally, replacing the integral by the limit of the sequence of Riemann sums, we obtain the inequality

$$\int f(x - \xi)\, \varphi(x)\, \bar{\varphi}(\xi)\, dx\, d\xi \geqslant 0, \qquad \varphi \in \mathscr{D},$$

that is,

$$(f, \varphi * \varphi^*) \geqslant 0, \quad \varphi \in \mathscr{D}. \tag{1.2}$$

We take property (1.2) as the basis for defining positive definite generalized functions. A generalized function $f$ is said to be *positive definite*, $f \gg 0$, if it satisfies the condition

$$(f, \varphi * \varphi^*) \geqslant 0, \qquad \varphi \in \mathscr{D}. \tag{1.3}$$

This definition immediately implies that if $f \gg 0$, then $f(-x) \gg 0$ and $\bar{f} \gg 0$ as well.

Furthermore, for the generalized function $f$ to be positive definite, it is necessary and sufficient that

$$f * \alpha * \alpha^* \gg 0, \quad \alpha \in \mathscr{E}'. \tag{1.4}$$

Indeed, if $f \gg 0$, then, using (6.4) of Sec. 4 and the properties of commutativity and associativity of a convolution (see Sec. 4.2), we have, for all $\alpha \in \mathscr{E}'$ and $\varphi \in \mathscr{D}$,

$$\begin{aligned}(f * \alpha * \alpha^*, \varphi * \varphi^*) &= (f, (\alpha * \alpha^*)(-x) * (\varphi * \varphi^*)) \\ &= (f, [\alpha(-x) * \varphi] * [\alpha^*(-x) * \varphi^*]) \\ &= (f, [\alpha(-x) * \varphi] * [\alpha(-x) * \varphi]^*) \geqslant 0,\end{aligned}$$

since $\alpha * \varphi \in \mathscr{D}$ if $\alpha \in \mathscr{E}'$ and $\varphi \in \mathscr{D}$ (see Sec. 4.2(g) and Sec. 4.6). Thus, condition (1.4) is fulfilled. Conversely, suppose the condition (1.4) is fulfilled, so that if $\alpha \in \mathscr{D}$, then $f * \alpha * \alpha^*$ is a continuous positive definite function and therefore, by (1.1), $(f * \alpha * \alpha^*)(0) \geqslant 0$. Now, taking advantage of formula (6.3) of Sec. 4, we have, for all $\alpha \in \mathscr{D}$,

$$(f(-y), \alpha * \alpha^*) = (f, (\alpha * \alpha^*)(-y)) = (f * \alpha * \alpha^*)(0) \geqslant 0$$

so that $f(-x) \gg 0$ and therefore $f \gg 0$.

If $f \gg 0$, then $f = f^*$.

Indeed, from what has been proved, for all $\alpha \in \mathscr{D}$,

$$(f * \alpha * \alpha^*)^* = f^* * (\alpha * \alpha^*) = f * (\alpha * \alpha^*).$$

If in the last equation we let $\alpha \to \delta$ in $\mathscr{E}'$ [and then $\alpha^* \to \delta$ in $\mathscr{E}'$, and from] formula (5.1) of Sec. 4 it follows that $\alpha * \alpha^* \to \delta$ in $\mathscr{E}'$ as well] and use the continuity property of a convolution (see Sec. 4.3), we obtain $f = f^*$.

For what follows we will need the following lemma.

Lemma *For every integer $p \geqslant 0$ there is a function $\omega(x)$ with the properties*:

$$\omega \in C^{2p}; \qquad \omega(x) = 0, \quad |x| > 1;$$

$$F[\omega](\xi) \geqslant \frac{A}{(1+|\xi|^2)^{p+n+1}}. \tag{1.5}$$

*Proof*. Let $\chi \in \mathscr{D}$, $\chi(x) = 0$ for $|x| > 1/2$ and

$$\gamma(x) = \frac{1}{(2\pi)^n} \int \frac{e^{-i(x,\xi)}\, d\xi}{(1+|\xi|^2)^{p+n+1}} = F^{-1}\left[\frac{1}{(1+|\xi|^2)^{p+n+1}}\right].$$

Let us verify that the function $\omega = \gamma(\chi * \chi^*)$ has the properties (1.5). Since $\gamma \in C^{2p}$ and $\chi * \chi^* \in C^\infty$, it follows that $\omega \in C^{2p}$. Furthermore, by virtue of Sec. 4.2(g), $\operatorname{supp} \omega \subset \operatorname{supp} \chi + \operatorname{supp} \chi^* \subset U_{1/2} + U_{1/2} = U_1$. Finally, using the formula of the Fourier transform of a convolution (see Sec. 6.5), we have

$$\begin{aligned} F[\omega](\xi) &= F[\gamma(\chi * \chi^*)] \\ &= F\left[F^{-1}\left[\frac{1}{(1+|\xi|^2)^{p+n+1}}\right] F^{-1}[F[\chi]F[\chi^*]]\right] \\ &= \frac{1}{(2\pi)^n (1+|\xi|^2)^{p+n+1}} * |F[\chi]|^2 \\ &= \frac{1}{(2\pi)^n} \int \frac{|F[\chi]|^2(y)\, dy}{(1+|\xi-y|^2)^{p+n+1}} \end{aligned}$$

and therefore

$$F[\omega](\xi) \geqslant \frac{1}{(2\pi)^n} \int\limits_{|y|<1} \frac{|F[\chi]|^2(y)\,dy}{(1+|\xi-y|^2)^{p+n+1}}$$

$$\geqslant \frac{1}{(2\pi)^n\,[1+(|\xi|+1)^2]^{p+n+1}} \int\limits_{|y|<1} |F[\chi]|^2(y)\,dy$$

$$\geqslant \frac{A}{(1+|\xi|^2)^{p+n+1}}.$$

The proof of the lemma is complete.

**8.2 The Bochner-Schwartz theorem** Suppose $f \in \mathscr{D}'$ and $f \gg 0$. Then, in the sphere $U_3 = [x : |x| < 3]$, $f$ has a finite order $m \geqslant 0$ (see Sec. 1.3),

$$|(f, \varphi)| \leqslant K \|\varphi\|_{C^m(\overline{U}_3)}, \qquad \varphi \in C_0^m(U_3).$$

Take an integer $p \geqslant 0$ such that $2p \geqslant m$, and let $\omega$ be a function with the properties (1.5). Then the function $\omega * \omega^* \in C_0^m(U_2)$ and, consequently, the generalized function $g = f * \omega * \omega^* = f * (\omega * \omega^*) \gg 0$, is continuous in the sphere $\overline{U}_1$.

We will now prove that $g$ is bounded in $\mathbb{R}^n$.

Let $\alpha_n \in \mathscr{D}$, supp $\alpha_n \subset U_{1/n}$, $\alpha_n \geqslant 0$, $\int \alpha_n\,dx = 1$, $\alpha_n \to \delta$, $n \to \infty$. The sequence of functions $g * \alpha_n * \alpha_n^*$, $n = 2, 3, \ldots$, is uniformly bounded,

$$|(g * \alpha_n * \alpha_n^*)(x)| \leqslant \max_{|x|\leqslant 2/n} |g(x)|\,\|\alpha_n * \alpha_n^*\|_{\mathscr{L}^1} \leqslant \max_{|x|\leqslant 1} |g(x)|$$

(see Sec. 4.1), and converges weakly on the set $\mathscr{D}$, which is dense in $\mathscr{L}'$ (see Sec. 1.2). In that case, the limiting generalized function $g$ may be identified with the function $g(x)$ taken from $\mathscr{L}^\infty$.

We now prove that $g(x)$ is the Fourier transform of a nonnegative measure finite on $\mathbb{R}^n$. Since $g$ is bounded and $\mathscr{D}$ is dense in $\mathscr{S}$ (see Sec. 5.1), it follows that the inequality

$$(g, \varphi * \varphi^*) \geqslant 0$$

holds true for all $\varphi \in \mathscr{S}$ and, hence,

$$(F^{-1}[g], F[\varphi * \varphi^*]) = (F^{-1}[g], |F[\varphi]|^2) \geqslant 0, \quad \varphi \in \mathscr{S}. \qquad (2.1)$$

But the operation $F$ is an isomorphism of $\mathscr{S}$ onto $\mathscr{S}$ (see Sec. 6.1). Therefore, the inequality (2.1) is equivalent to the inequality

$$(F^{-1}[g], |\psi|^2) \geqslant 0, \quad \psi \in \mathscr{S}. \qquad (2.2)$$

Now let $\varphi$ be any nonnegative basic function in $\mathcal{S}$ and let $\{\eta_k\}$ be a sequence of nonnegative functions taken from $\mathcal{D}$ that tend to 1 in $\mathbb{R}^n$ (see Sec. 4.1). Then

$$|\eta_k \sqrt{\varphi+1/k}\,|^2 = \eta_k^2(\varphi+1/k) \to \varphi, \qquad k\to\infty \quad \text{in} \quad \mathcal{S}$$

and, consequently, by virtue of (2.2),

$$(F^{-1}[g], \varphi) = \lim_{k\to\infty}(F^{-1}[g], |\eta_k \sqrt{\varphi+1/k}|^2) \geqslant 0, \quad \varphi \in \mathcal{S}.$$

By Theorem II of Sec. 1.7, $F^{-1}[g] = \nu$ is a nonnegative measure on $\mathbb{R}^n$ and $g = F[\nu]$. But $\nu = F^{-1}[g] \in \mathcal{S}'$. For this reason, $\nu$ is a measure of slow growth (see Sec. 5.3) so that for all $\varphi \in \mathcal{S}$ we have

$$(F^{-1}[g], \varphi) = \int \varphi(\xi)\,\nu(d\xi) = (g, F^{-1}[\varphi]). \tag{2.3}$$

Let $\{\eta_k\}$ be a nondecreasing sequence of nonnegative functions taken from $\mathcal{D}$ that tend to 1 in $\mathbb{R}^n$. Then $F^{-1}[\eta_k] \to \delta$, $k\to\infty$, on all functions bounded in $\mathbb{R}^n$ and continuous in the neighbourhood of zero. Setting $\varphi = \eta_k$ in (2.3),

$$\int \eta_k(\xi)\,\nu(d\xi) = (g, F^{-1}[\eta_k]),$$

passing to the limit as $k\to\infty$, and making use of the Levi theorem, we obtain

$$\int \nu(d\xi) = g(0)$$

which is precisely the assertion.

Now let us prove that the equation

$$u * \omega * \omega^* = g \tag{2.4}$$

has a unique solution in the class of positive definite generalized functions from $\mathcal{D}'$ and that that solution is given by the formula

$$u = F^{-1}\left[\frac{1}{|F[\omega]|^2}\right]. \tag{2.5}$$

Indeed, by virtue of the inequality (1.5), the generalized function $u$ given by (2.5) is actually the sole solution of equation (2.4) in the class $\mathcal{S}'$ by virtue of the theorem on the Fourier trans-

form of a convolution (see Sec. 6.5):

$$F[u]\,|F[\omega]|^2 = F[g] = \nu. \tag{2.6}$$

It remains to prove that the solution of the homogeneous equation $u * \omega * \omega^* = 0$ in the class of generalized functions $u$, which can be represented in the form of a difference $u_1 - u_2$, where $u_j \gg 0$, $u_j \in \mathcal{D}'$, is trivial: $u = 0$. Suppose such a $u$ satisfies that equation. Then for all $\alpha \in \mathcal{D}$ the function $u * \alpha * \alpha^*$ also satisfies that equation:

$$(u * \omega * \omega^*) * \alpha * \alpha^* = 0 = (u * \alpha * \alpha^*) * \omega * \omega^*.$$

But the function $u * \alpha * \alpha^* = u_1 * \alpha * \alpha^* - u_2 * \alpha * \alpha^*$ is a difference of continuous positive definite functions and, hence, is bounded in $\mathbb{R}^n$ (and all the more so in $\mathcal{S}'$). From what has been proved, $u * \alpha * \alpha^* = 0$. Passing to the limit in this equation as $\alpha \to \delta$ in $\mathcal{E}'$ and using the continuity of a convolution, we obtain $u = 0$, which is what we set out to prove.

The generalized function $f \gg 0$ also satisfies equation (2.4). By virtue of the uniqueness of the solution of that equation, $f$ coincides with the generalized function $u$ given by (2.5). Hence, $f$ is the inverse Fourier transform of the measure $\mu = \nu\,|F[\omega]|^{-2}$ of slow growth, by the inequality (1.5).

We have thus proved the necessity of the conditions of the following theorem.

Theorem (Bochner-Schwartz) *For a generalized function $f$ taken from $\mathcal{D}'$ to be positive definite, it is necessary and sufficient that it be a Fourier transform of a nonnegative measure of slow growth,* $f = F[\mu]$, $\mu \in \mathcal{S}'$, $\mu \geqslant 0$.

*Sufficiency.* If $f = F[\mu]$, $\mu \geqslant 0$, $\mu \in \mathcal{S}'$, then $(\mu, |\varphi|^2) \geqslant 0$ for all $\varphi \in \mathcal{S}$, whence [compare (2.1)]

$$(\mu, |F[\varphi]|^2) = (F[\mu], F^{-1}[|F[\varphi]|^2) = (f, \varphi * \varphi^*) \geqslant 0,$$
$$\varphi \in \mathcal{D},$$

so that $f \gg 0$, and the theorem is proved.

Corollary 1 *If $f \in \mathcal{D}'$, $f \gg 0$, then $f \in \mathcal{S}'$.*

Corollary 2 (Bochner) *For a generalized function $f$ that is continuous in the neighbourhood of zero to be positive definite, it is necessary and sufficient that it be a Fourier transform of positive measure $\nu$ finite on $\mathbb{R}^n$:*

$$f(x) = \int e^{i(x,\xi)}\nu(d\xi), \quad \nu \geqslant 0, \quad \int \nu(d\xi) = f(0); \tag{2.7}$$

*here, $f(x)$ is a continuous function on $\mathbb{R}^n$.*

Corollary 3 *For a generalized function $f$ to be positive definite, it is necessary and sufficient that it be (uniquely) represented, for some integer $m \geqslant 0$, in the form*

$$f(x) = (1 - \Delta)^m f_0(x)$$

*where $f_0(x)$ is a continuous positive definite function.*

This is a consequence of the following chain of equalities:

$$f = F[\mu] = F\left[(1+|\xi|^2)^m \frac{\mu}{(1+|\xi|^2)^m}\right]$$
$$= (1-\Delta)^m F[\nu]$$

where the measure $\nu = \mu(1 + |\xi|^2)^{-m} \geqslant 0$, for sufficiently large $m$, may be made finite on $\mathbb{R}^n$.

**8.3 Examples** (a) Let the polynomial $P(\xi) \geqslant 0$. Then

$$P(-iD)\,\delta \gg 0$$

In particular, $\delta \gg 0$.

(b) If $f \gg 0$ and $g \in \mathscr{E}'$, $g \gg 0$, then $f * g \gg 0$.

Indeed, the measure $F[g] \geqslant 0$, $F[g] \in \theta_M$ (see Sec. 6.4) and therefore the measure $F[f * g] = F[f]\,F[g] \geqslant 0$ is of slow growth.

(c) If $f \gg 0$ and $F[g] \in \mathscr{E}'$, $g \gg 0$, then $gf \gg 0$.

Indeed, $g \in \theta_M$, $gf \in \mathscr{S}'$ and

$$F^{-1}[gf] = F^{-1}[g] * F^{-1}[f]$$

is a nonnegative measure taken from $\mathscr{S}'$ and, hence, is of slow growth [see Sec. 5.3].

(d) $e^{-(Ax,\, x)} \gg 0$, where $A$ is a real positive definite matrix (see Sec. 6.6(b)).

(e) $\frac{1}{|x|} \gg 0$, $n = 3$ (see Sec. 6.6(g)).

(f) $\pi\delta(x) \pm i\mathscr{P}\frac{1}{x} \gg 0$, $n = 1$ (see Sec. 6.6(j)).

(g) For $f \in \mathscr{D}'_T$ to be positive definite, it is necessary and sufficient that its Fourier coefficients $c_k(f)$ be nonnegative. Then for all $\varphi \in C^\infty \cap \mathscr{D}'_T$ the following inequality holds:

$$(f, \varphi \circledast \varphi^*)_T \geqslant 0 \tag{3.1}$$

This follows from the theorem of Sec. 7.2, by which theorem

$$F^{-1}[f] = \sum_{|k| \geqslant 0} c_k(f)\,\delta(\xi - k\omega), \tag{3.2}$$

and from the Bochner-Schwartz theorem. To prove the inequality (3.1), let us take advantage of the machinery developed in Sec. 7. Using (3.1) of Sec. 7 and (3.3) of Sec. 7, we have the chain of equalities

$$
\begin{aligned}
(f, \varphi \circledast \varphi^*)_T &= (e_T f, \varphi \circledast \varphi^*) = (f * \delta, e_T (\varphi \circledast \varphi^*)) \\
&= (\delta, f(-x) * e_T (\varphi \circledast \varphi^*)) \\
&= (\delta, f(-x) \circledast (\varphi \circledast \varphi^*)) \\
&= (\delta, f(-x) \circledast \varphi \circledast \varphi^*) \\
&= (\delta, f(-x) * (e_T\varphi) * (e_T\varphi^*)) \\
&= (\delta, f(-x) * [(e_T\varphi) * (e_T\varphi)^*]) \\
&= (f, (e_T\varphi) * (e_T\varphi)^*),
\end{aligned}
$$

so that

$$(f, \varphi \circledast \varphi^*)_T = (f, (e_T\varphi) * (e_T\varphi)^*), \tag{3.3}$$

whence follows inequality (3.1).

## 9 The Laplace Transform of Generalized Functions of Slow Growth

The fundamentals of the general theory of the Laplace transform of generalized functions were developed by Schwartz [3] and Lions [1]. However, this theory has been developed into its most complete form for the case—so important in applications of mathematical physics—of generalized functions of slow growth.

**9.1 The Laplace transform defined** Let $\Gamma$ be a closed convex acute cone in $\mathbb{R}^n$ with vertex at 0 (see Sec. 4.4); we put $C = \operatorname{int} \Gamma^*$ (by Lemma 1 of Sec. 4.4 the cone $C \neq \varnothing$; $C$ is an open and convex cone). Denote by $T^C$ a tubular region in $\mathbb{C}^n$ with base $C$:

$$T^C = \mathbb{R}^n + iC = [z = x + iy\colon x \in \mathbb{R}^n,\ y \in C].$$

Suppose $g \in \mathscr{S}'(\Gamma+)$ (see Sec. 4.5 and Sec. 5.6). We will use the term *Laplace transform* $L[g]$ of the generalized function $g$ for the expression

$$L[g] = F[g(\xi)\, e^{-(y, \xi)}](x), \tag{1.1}$$

where $F$ is the Fourier transform operation.

*Example.*

$$L[\delta(\xi-\xi_0)] = e^{i(z,\xi_0)}. \tag{1.2}$$

This follows from (2.6) of Sec. 6.

Let us now prove that for all $y \in C$

$$g(\xi)\, e^{-(y,\xi)} \in \mathscr{S}'$$

so that the Laplace transform $L[g]$ is a generalized function of slow growth with respect to $x$ for all $y \in C$.

True enough, suppose $\eta$ is any function of the class $C^\infty$ with the following properties:

$$|D^\alpha \eta(\xi)| \leqslant c_\alpha; \qquad \eta(\xi) = 1, \qquad \xi \in (\operatorname{supp} g)^\varepsilon;$$
$$\eta(\xi) = 0, \qquad \xi \bar{\in} (\operatorname{supp} g)^{2\varepsilon},$$

where $\varepsilon$ is any number $> 0$. Then, by what was proved in Sec. 6.6(i),

$$\eta(\xi)\, e^{-(y,\xi)} \in \mathscr{S}(\mathbb{R}^n) \quad \text{for all} \quad y \in C \tag{1.3}$$

and therefore, by (10.2) of Sec. 1,

$$g(\xi)\, e^{-(y,\xi)} = g(\xi)\, \eta(\xi)\, e^{-(y,\xi)} \in \mathscr{S}',$$

which is what was to be proved.

The Laplace transform $L$ is a linear and one-to-one operation. This follows from the appropriate properties of the Fourier transform (see Sec. 6.2).

Let us now prove the representation

$$L[g] = (g(\xi), \eta(\xi)\, e^{i(z,\xi)}), \qquad z \in T^C. \tag{1.4}$$

This representation does not depend on the auxiliary function $\eta$ with the above-indicated properties.

Indeed, let $y \in C$ and $\varphi \in \mathscr{S}$. Then (compare Sec. 6.6(i))

$$(L[g], \varphi) = (F[g(\xi)\, e^{-(y,\xi)}], \varphi) = (g(\xi)\, e^{-(y,\xi)}, F[\varphi])$$
$$= \left(g(\xi), \eta(\xi)\, e^{-(y,\xi)} \int \varphi(x)\, e^{i(x,\xi)}\, dx\right)$$
$$= \int \varphi(x)\, (g(\xi), \eta(\xi)\, e^{i(z,\xi)})\, dx,$$

whence follows (1.4). Here, we made use of formula (5.4) of Sec. 5, since, by virtue of (1.3),

$$\eta(\xi)\, e^{-(y,\,\xi)}\varphi(x)\, e^{i(x,\,\xi)} \in \mathscr{S}(\mathbb{R}^{2n}).$$

We set $f(z) = L[g]$ and will prove that the function $f(z)$ is holomorphic in $T^C$ and the following differentiation formula holds true:

$$D^{\alpha} f(z) = ((i\xi)^{\alpha} g(\xi),\, \eta(\xi)\, e^{i(z,\,\xi)}). \tag{1.5}$$

The proof is analogous to that of the lemma of Sec. 3.1. The continuity of the function $f(x)$ in $T^C$ follows from the representation (1.4), from the continuity of the function $\eta(\xi)\, e^{i(z,\,\xi)}$ with respect to $z$ in $T^C$ in the sense of convergence in $\mathscr{S}$,

$$\eta(\xi)\, e^{i(z,\,\xi)} \to \eta(\xi)\, e^{i(z_0,\,\xi)}, \qquad z \to z_0 \quad \text{in} \quad \mathscr{S},$$
$$z \in T^C, \qquad z_0 \in T^C$$

(see the estimates in Sec. 6.6(i)), and from the continuity of the functional $g$ on $\mathscr{S}$,

$$f(z) = (g(\xi),\, \eta(\xi)\, e^{i(z,\,\xi)}) \to (g(\xi),\, \eta(\xi)\, e^{i(z_0,\,\xi)}) = f(z_0),$$
$$z \to z_0.$$

To prove the holomorphicity of the function $f(z)$ in $T^C$ it suffices to establish, by virtue of the familiar Hartogs theorem, the existence of all first derivatives $\frac{\partial f}{\partial z_j}$, $j = 1, \ldots, n$. Suppose $e_1 = (1, 0, \ldots, 0)$. Then for each $z \in T^C$

$$\chi_h(\xi) = \frac{1}{h}\left[\eta(\xi)\, e^{i(z+he_1,\,\xi)} - \eta(\xi)\, e^{i(z,\,\xi)}\right] \to$$
$$\to \eta(\xi)\, i\xi_1 e^{i(z,\,\xi)}, \qquad h \to 0 \quad \text{in} \quad \mathscr{S}.$$

Therefore, from the representation (1.4) and from the linearity and the continuity of the functional $g$ on $\mathscr{S}$ as $h \to 0$ we have

$$\frac{f(z+he_1)-f(z)}{h}$$
$$= \frac{1}{h}\left[(g(\xi),\, \eta(\xi)\, e^{i(z+he_1,\,\xi)}) - (g(\xi),\, \eta(\xi)\, e^{i(z,\,\xi)})\right]$$
$$= (g(\xi),\, \chi_h(\xi)) \to$$
$$\to (g(\xi),\, \eta(\xi)\, i\xi_1 e^{i(z,\,\xi)}) = (i\xi_1 g(\xi),\, \eta(\xi)\, e^{i(z,\,\xi)}),$$

so that the derivative $\frac{\partial f}{\partial z_1}$ exists and the differentiation formula (1.5) holds for $\alpha = (1, 0, \ldots, 0)$ and, hence, also for all first derivatives

$$\frac{\partial f}{\partial z_j} = (i\xi_j g(\xi), \eta(\xi) e^{i(z, \xi)}), \qquad j = 1, \ldots, n. \tag{1.6}$$

Applying this reasoning to (1.6), we see that the formulas (1.5) hold for all second derivatives, and so forth. This completes the proof.

Our task now is to give a full description of holomorphic functions that are the Laplace transforms of generalized functions taken from the algebras $\mathscr{S}'(\Gamma+)$ and $\mathscr{S}'(\Gamma)$. We will refer to the generalized function $g(\xi)$ of $\mathscr{S}'(\Gamma+)$, for which $f = L[g]$, as the *spectral function* of the function $f(z)$. The spectral function is unique (if it exists) and, by virtue of (1.1),

$$g(\xi) = e^{(y, \xi)} F_x^{-1} [f(x + iy)](\xi) \tag{1.7}$$

and the right-hand side of (1.7) is independent of $y \in C = \operatorname{int} \Gamma^*$.

**9.2 Properties of the Laplace transform** These properties follow from the appropriate properties of the Fourier transform (see Sec. 6.3).

(a) *Differentiation of the Laplace transform:*

$$D^\alpha L[g] = L[(i\xi)^\alpha g], \tag{2.1}$$

which is precisely formula (1.5).

(b) *The Laplace transform of a derivative*:

$$L[D^\alpha g] = (-iz)^\alpha L[g]. \tag{2.2}$$

It suffices to prove (2.2) for the first derivatives. We have

$$\begin{aligned} L\left[\frac{\partial g}{\partial \xi_j}\right] &= F\left[\frac{\partial g(\xi)}{\partial \xi_j} e^{-(y, \xi)}\right] \\ &= F\left[\frac{\partial}{\partial \xi_j}(g(\xi) e^{-(y, \xi)})\right] + y_j F[g(\xi) e^{-(y, \xi)}] \\ &= (-ix_j + y_j) F[g(\xi) e^{-(y, \xi)}] = -iz_j L[g]. \end{aligned}$$

In particular, setting $g = \delta(\xi - \xi_0)$ in (2.2) and using (1.2), we obtain

$$L[D^\alpha \delta(\xi - \xi_0)] = (-iz)^\alpha e^{i(z, \xi_0)}. \tag{2.3}$$

(c) *The translation of a Laplace transform.* If $\operatorname{Im} a \in C$, then

$$L[g(\xi)\, e^{i(a,\xi)}] = L[g](z+a). \tag{2.4}$$

Indeed,

$$\begin{aligned} L[g(\xi)\, e^{i(a,\xi)}] &= F[g(\xi)\, e^{i(\operatorname{Re} a,\xi)}\, e^{-(y+\operatorname{Im} a,\xi)}] \\ &= L[g](x+\operatorname{Re} a + iy + i \operatorname{Im} a) \\ &= L[g](z+a). \end{aligned}$$

(d) *The Laplace transform of a translation*:

$$L[g(\xi-\xi_0)] = e^{i(z,\xi_0)}\, L[g](z). \tag{2.5}$$

(e) *The Laplace transform under a linear transformation of the argument*:

$$L[g(A\xi)] = \frac{1}{|\det A|}\, L[g](A^{-1T}z), \qquad z \in T^{A^T C}. \tag{2.6}$$

$$(z, \zeta) \in T^{C_1 \times C_2}.$$

(f) *The Laplace transform of a direct product.* If $g_1(\xi) \in \mathscr{S}'(\Gamma_+^1)$ and $g_2(\eta) \in \mathscr{S}'(\Gamma_2 +)$, then

$$L[g_1 \times g_2](z, \zeta) = L[g_1](z)\, L[g_2](\zeta), \tag{2.7}$$

(g) *The Laplace transform of a convolution.* If $g \in \mathscr{S}'(\Gamma +)$ and $g_1 \in \mathscr{S}'(\Gamma +)$, then $g * g_1 \in \mathscr{S}'(\Gamma +)$ (see Sec. 5.6(b)) and

$$L[g * g_1] = L[g]\, L[g_1]. \tag{2.8}$$

Let us first prove the following formula: if $g \in \mathscr{D}'(\Gamma +)$ and $g_1 \in \mathscr{D}'(\Gamma +)$, then for all $y \in C$

$$(ge^{-(y,\xi)}) * (g_1 e^{-(y,\xi)}) = (g * g_1)\, e^{-(y,\xi)}. \tag{2.9}$$

Indeed, using the formula (5.1) of Sec. 4, we have, for all $\varphi \in \mathscr{D}$,

$$\begin{aligned} &((ge^{-(y,\xi)}) * (g_1 e^{-(y,\xi)}), \varphi) \\ &\quad = (g(\xi)\, e^{-(y,\xi)} \times g_1(\xi')\, e^{-(y,\xi')}, \eta_1(\xi)\, \eta_2(\xi')\, \varphi(\xi+\xi')) \\ &\quad = (g(\xi) \times g_1(\xi'), \eta_1(\xi)\, \eta_2(\xi')\, e^{-(y,\xi+\xi')}\varphi(\xi+\xi')) \\ &\quad = (g * g_1, \varphi e^{-(y,\xi)}) \\ &\quad = (e^{-(y,\xi)}(g * g_1), \varphi), \end{aligned}$$

which is what we set out to prove. From (2.9), for $g$ and $g_1 \in \in \mathscr{S}'(\Gamma+)$, and from the formula for the Fourier transform of a convolution [see (5.1) of Sec. 6] there follows immediately the formula (2.8):

$$\begin{aligned} L[g * g_1] &= F[(g * g_1)\, e^{-(y,\xi)}] = F[(ge^{-(y,\xi)}) * (g_1 e^{-(y,\xi)})] \\ &= F[ge^{-(y,\xi)}]\, F[g_1 e^{-(y,\xi)}] \\ &= L[g]\, L[g_1]. \end{aligned}$$

*Remark*. In the case of a single variable, the Laplace transform is defined differently in the operational calculus of Heaviside: if the original $g \in \mathscr{S}'([0, \infty)+)$, then its image (the Laplace transform) is the function

$$F[g(t)\, e^{-\sigma t}](-\omega],$$

which is holomorphic in the right half-plane $\sigma > 0$ of the complex plane $p = \sigma + i\omega$. In particular,

$$\theta(t) \leftrightarrow \int_0^\infty e^{-pt}\, dt = \frac{1}{p}.$$

However, we will adhere to the definition (1.1) in the case of $n = 1$ as well.

## 9.3 Examples

(a) $$L[f_\alpha] = \frac{1}{(-iz)^\alpha}, \qquad y > 0, \tag{3.1}$$

where $f_\alpha$ is a generalized function from $\mathscr{S}'_+ = \mathscr{S}' \cap \mathscr{D}'_+$ that was introduced in Sec. 4.8(e). The branch of the function $(-iz)^\alpha$ in the half-plane $y > 0$ is chosen so that it is positive for $z = iy$, $y > 0$.

Let $\alpha > 0$. Then

$$L[f_\alpha](iy) = \int_0^\infty \frac{\xi^{\alpha-1}}{\Gamma(\alpha)}\, e^{-y\xi} d\xi = \frac{1}{y^2\Gamma(\alpha)} \int_0^\infty u^{\alpha-1} e^{-u} du = \frac{1}{y^\alpha}$$

so that the functions $L[f_\alpha](z)$ and $(-iz)^{-\alpha}$, which are holomorphic in the upper half-plane, coincide on the line $z = iy$, $y > 0$. By virtue of the principle of analytic extension, (3.1) holds for

$\alpha > 0$. But if $\alpha \leqslant 0$, then $\alpha + m > 0$ for some integer $m$. Therefore, $f_\alpha = f_{\alpha+m}^{(m)}$ and, by what has been proved,

$$L[f_\alpha] = L[f_{\alpha+m}^{(m)}] = (-iz)^m L[f_{\alpha+m}]$$
$$= (-iz)^m (-iz)^{-\alpha-m} = (-iz)^{-\alpha}.$$

(b)
$$L[\theta(\xi)\sin\omega\xi] = \frac{\omega}{\omega^2 - z^2}$$
$$L[\theta(\xi)\cos\omega\xi] = \frac{-iz}{\omega^2 - z^2} \qquad (3.2)$$

These follow from the equations [see (a) for $\alpha = 1$]

$$L[\theta(\xi)e^{i\omega\xi}] \quad \frac{i}{z+\omega}, \qquad L[\theta(\xi)e^{-i\omega\xi}] = \frac{i}{z-\omega}$$

(c) Let us prove the equation

$$L\left[\frac{\theta(\xi)\sqrt{\pi}}{\Gamma(\nu+1/2)}\left(\frac{\xi}{2}\right)^\nu J_\nu(\xi)\right] = (1-z^2)^{-\nu-1/2}, \qquad (3.3)$$
$$\nu > -1/2.$$

By (a) we have the equations

$$L[e^{i\xi}f_{\nu-1/2}(\xi) = \left(\frac{i}{z+1}\right)^{\nu+1/2}$$
$$L[e^{-i\xi}f_{\nu-1/2}(\xi)] = \left(\frac{i}{z-1}\right)^{\nu+1/2}$$

Therefore, using the formula for the Laplace transform of a convolution, we have

$$L[(e^{i\xi}f_{\nu-1/2}) * (e^{-i\xi}f_{\nu-1/2})] = (1-z^2) \quad ^{\nu-1/2}.$$

But

$$(e^{i\xi}f_{\nu-1/2}) * (e^{-i\xi}f_{\nu-1/2})$$
$$= \frac{\theta(\xi)}{\Gamma^2(\nu+1/2)}\int_0^\xi e^{i(\xi-t)}\, e^{-it}\,(\xi-t)^{\nu-1/2}\,t^{\nu-1/2}\,dt$$
$$= \frac{\theta(\xi)\,e^{i\xi}\xi^{2\nu}}{\Gamma^2(\nu+1/2)}\int_0^1 e^{-2i\xi u}\,[(1-u)\,u]^{\nu-1/2}\,du$$

$$= \frac{\theta(\xi)\,\xi^{2\nu}}{\Gamma^2(\nu+1/2)} \int_{-1}^{1} e^{-i\xi v} \left(\frac{1-v^2}{4}\right)^{\nu-1/2} \frac{dv}{2}$$

$$= \frac{\theta(\xi)\sqrt{\pi}}{\Gamma(\nu+1/2)} \left(\frac{\xi}{2}\right)^{\nu} J_\nu(\xi)$$

where $u = (v+1)/2$, and (3.3) is proved. Here, we made use of (6.20) of Sec. 6.

(d) We now prove the formula

$$\sin\xi = \int_0 J_0(\xi-t)\,J_0(t)\,dt. \tag{3.4}$$

Since the right and left members of (3.4) are odd, it suffices to prove the formula for $\xi > 0$. It is therefore sufficient to prove the convolution equation

$$\theta \sin\xi = (\theta J_0) * (\theta J_0),$$

which is equivalent, by virtue of (b) and (c), to the trivial equality

$$\frac{1}{1-z^2} = \frac{1}{\sqrt{1-z^2}}\,\frac{1}{\sqrt{1-z^2}}, \qquad y > 0.$$

(e) Let us find the fundamental solution $\mathscr{E}$ of the operator $(\theta J_0)\,*$ in the algebra $\mathscr{S}'_+$. By (c), we have

$$L[\theta J_0] = \frac{1}{\sqrt{1-z^2}} \neq 0, \qquad y > 0.$$

Consequently,

$$L[\mathscr{E}] = \sqrt{1-z^2} = \frac{1-z^2}{\sqrt{1-z^2}}$$

whence

$$\mathscr{E}(\xi) = \theta(\xi)\,J_0(\xi) + [\theta(\xi)\,J_0(\xi)]''$$

$$= \theta(\xi)\,J_0(\xi) + \delta'(\xi)\,J_0(\xi) + 2\delta(\xi)\,J_0'(\xi) + \theta(\xi)\,J_0''(\xi)$$

$$= -\frac{\theta(\xi)\,J_0'(\xi)}{\xi} + \delta'(\xi),$$

that is

$$\mathscr{E}(\xi) = \theta(\xi)\,\frac{J_1(\xi)}{\xi} + \delta'(\xi). \tag{3.5}$$

(f) Let $f \in \mathscr{L}^1_{\text{loc}} \cap \mathscr{D}'_T$ (see Sec. 7), $n = 1$. Then

$$L[\theta f] = \frac{1}{1 - e^{izT}} \int_0^T f(\xi) e^{iz\xi} d\xi. \tag{3.6}$$

Indeed,

$$L[\theta f](z) = \int_0^\infty f(\xi) e^{iz\xi} d\xi$$

$$= \int_0^\infty e^{iz(t+T)} f(t+T)\, dt + \int_0^T e^{iz\xi} f(\xi)\, d\xi$$

$$= e^{izT} L[\theta f] + \int_0^T e^{iz\xi} f(\xi)\, d\xi,$$

whence follows (3.6).

## 10 The Cauchy Kernel and the Transforms of Cauchy-Bochner and Hilbert

**10.1 The space $\mathscr{H}_s$** We denote by $\mathscr{L}^2_s$ the Hilbert space consisting of all functions $g(\xi)$ with finite norm

$$\| g \|_{(s)} = \left[ \int |g(\xi)|^2 (1 + |\xi|^2)^s d\xi \right]^{1/2} = \| g(\xi)(1 + |\xi|^2)^{s/2} \|.$$

We denote by $\mathscr{H}_s$ the collection of all (generalized) functions $f(x)$ that are Fourier transforms of functions in $\mathscr{L}^2_s$, $f = F[g]$, with norm

$$\| f \|_s = \| F^{-1}[f] \|_{(s)} = \| g \|_{(s)} \,. \tag{1.1}$$

The parameter $s$ can assume any real values.

Clearly, $\mathscr{H}_0 = \mathscr{L}^2 = \mathscr{L}^2_0$ and

$$\| g \|_{(0)} = \| g \| = (2\pi)^{-n/2} \| f \| = \| f \|_0$$

by virtue of the Parseval equation (see Sec. 6.6(c)).

From the definition of the space $\mathscr{H}_s$ we find that for $f \in \mathscr{H}_s$ it is necessary that the function be representable as

$$f(x) = (1 - \Delta)^m f_1(x), \qquad f_1 \in \mathscr{L}^2, \quad m = 0, \quad s \geqslant 0;$$

$$m = 1 + \left[ -\frac{s}{2} \right], \qquad s < 0. \tag{1.2}$$

The space $\mathscr{H}_s$ is a Hilbert space isomorphic to $\mathscr{L}^2_s$. And

$$\mathscr{S} \subset \mathscr{H}_s \subset \mathscr{H}_{s'} \subset \mathscr{S}', \qquad s' < s,$$

where inclusion is to be understood as embedding together with the appropriate topology, $\| f \|_{s'} \leqslant \| f \|_s$, $f \in \mathscr{H}_s$.

Let us now prove that $\mathscr{S}$ is dense in $\mathscr{H}_s$.

By virtue of (1.1) it suffices to prove that $\mathscr{D}$ is dense in $\mathscr{L}^2_s$. Let $g \in \mathscr{L}^2_s$ and $\varepsilon > 0$. Then

$$\psi(\xi) = g(\xi)(1 + |\xi|^2)^{s/2} \in \mathscr{L}^2.$$

But $\mathscr{D}$ is dense in $\mathscr{L}^2$ (see Sec. 1.2, Corollary 1 to Theorem II). Therefore there is a function $\psi_1$ in $\mathscr{D}$ such that $\| \psi - \psi_1 \| < \varepsilon$. Putting

$$g_1(\xi) = \psi_1(\xi)(1 + |\xi|^2)^{-s/2} \in \mathscr{D}$$

we obtain

$$\|g - g_1\|^2_{(s)} = \int |g(\xi) - g_1(\xi)|^2 (1 + |\xi|^2)^s \, d\xi$$

$$= \| \psi - \psi_1 \|^2 < \varepsilon^2,$$

which is what we affirmed. Let us now prove that

$$\mathscr{H}_s \subset \overline{C}^l_0 \quad \text{if } l \text{ is an integer,} \quad l < s - n/2.$$

*Remark*. This assertion is a simple special case of the Sobolev embedding theorems [2].

To prove this, note that if $f \in \mathscr{H}_s$, then for all $|\alpha| \leqslant l < s - n/2$

$$\xi^\alpha F^{-1}[f] \in \mathscr{L}^1$$

by the Cauchy-Bunyakovsky inequality

$$\| \xi^\alpha F^{-1}[f] \|_{\mathscr{L}^1}$$

$$= \int |\xi^\alpha (1 + |\xi|^2)^{-s/2} (1 + |\xi|^2)^{s/2} F^{-1}[f](\xi)| \, d\xi$$

$$\leqslant \left[ \int |\xi|^{2|\alpha|} (1 + |\xi|^2)^{-s} d\xi \right]^{1/2} \| (1 + |\xi|^2)^{s/2} F^{-1}[f] \|$$

$$= K \| F^{-1}[f] \|_{(s)} = K \| f \|_s < \infty.$$

Hence,

$$D^\alpha f(x) = D^\alpha F[F^{-1}[f]] = \int (-i\xi)^\alpha F^{-1}[f](\xi) e^{i(x, \xi)} \, d\xi \in C$$

and by virtue of the Riemann-Lebesgue theorem $D^\alpha f(x) = o(1)$, $|x| \to \infty$ for all $|\alpha| \leqslant l$. Which means that $f \in \overline{C}_0^l$ (concerning notation see Sec. 0.5).

Now let $s$ be an integer $\geqslant 0$. In that case the space $\mathcal{L}_s^2$ consists of those and only those functions $g(\xi)$ for which $\xi^\alpha g \in \mathcal{L}^2$ for all $|\alpha| \leqslant s$. Therefore the space $\mathcal{H}_s$ consists of those and only those functions $f$ for which the generalized derivatives $D^\alpha f \in \mathcal{L}^2$ for all $|\alpha| \leqslant s$ (by the Plancherel theorem). Furthermore, as follows from the readily verifiable identity

$$\|f\|_s^2 = \|f\|_{s-1}^2 + \sum_{1 \leqslant j \leqslant n} \|D_j f\|_{s-1}^2$$

the space $\mathcal{H}_s$ consists of those functions $f$ in $\mathcal{H}_{s-1}$ for which $D_j f \in \mathcal{H}_{s-1}$, $j = 1, \ldots, n$.

Now let us describe the conjugate of the $\mathcal{H}_s$ space. Since $\mathcal{S}$ is dense in $\mathcal{H}_s$, it follows that every continuous linear form on $\mathcal{H}$ is uniquely defined by its restriction on $\mathcal{S}$.

Lemma *If $L(f)$ is a continuous linear form on $\mathcal{H}_s$, then for some $f_0$ in $\mathcal{H}_{-s}$,*

$$L(f) = (f_0, f), \qquad f \in \mathcal{S}, \tag{1.3}$$

*and the norm of that form is $\|f_0\|_{-s}$. Thus, $\mathcal{H}_{-s}$ is the conjugate space of $\mathcal{H}_s$.*

*Proof*. By hypothesis, the linear form

$$L_1(\chi) = L(F^{-1}[\chi(\xi)(1 + |\xi|^2)^{-s/2}])$$

is continuous on $\mathcal{L}^2$ and coincides with the form $L(f)$ for

$$\chi(\xi) = (1 + |\xi|^2)^{s/2} F[f]. \tag{1.4}$$

Now, since the mapping $f \to \chi$ given by (1.4) is biunique and reciprocally continuous from $\mathcal{H}_s$ to $\mathcal{L}^2$, it follows that $\|L_1\| = \|L\|$. By the F. Riesz theorem there is a function $g_1 \in \mathcal{L}^2$ such that

$$L_1(\chi) = \int g_1(\xi)\chi(\xi)\,d\xi, \qquad \|g_1\| = \|L_1\|.$$

From this, if we introduce the function

$$g(\xi) = g_1(\xi)(1 + |\xi|^2)^{s/2}$$

taken from $\mathscr{L}^2_{-s}$ and then set $f_0=F\ g]$, we obtain, for all $f$ in $\mathscr{S}$, the representation (1.3):

$$L(f)=L_1((1+|\xi|^2)^{s/2}F[f])$$
$$=\int g_1(\xi)(1+|\xi|^2)^{s/2}F[f](\xi)\,d\xi$$
$$=\int g(\xi)F[f](\xi)\,d\xi=(F[g],f)=(f_0,f)$$

and, besides,

$$\|L\|=\|g_1\|=\|g(1+|\xi|^2)^{-s/2}\|=\|f_0\|_{-s},$$

which is what we set out to establish. The proof of the lemma is complete.

For $f\in\mathscr{H}_s$, it is necessary and sufficient that

$$f=g*L_s,\qquad g\in\mathscr{L}^2, \tag{1.5}$$

where the kernel $L_s$ is given by the formula

$$L_s(x)=F^{-1}[(1+|\xi|^2)^{-s/2}]. \tag{1.6}$$

Here, the mapping $g\to f=g*L_s$ is biunique and reciprocally continuous from $\mathscr{L}^2$ onto $\mathscr{H}_s$.

Indeed, (1.5) is equivalent to (see Sec. 6.5, Remark)

$$F[f]=F[g](1+|\xi|^2)^{-s/2},$$

which is what sets up the reciprocally one-to-one and reciprocally continuous correspondence between $\mathscr{L}^2$ and $\mathscr{H}_s$.

The kernel $L_s$ has the obvious property [by virtue of (1.6)]

$$L_s*L_\sigma=L_{s+\sigma}.$$

Explicitly, the expression is

$$L_s(x)=\begin{cases}(1-\Delta)^{-s/2}\delta(x) & \text{if}-s \text{ is even and } \geqslant 0,\\ \dfrac{2|x|^{s/4-n/2}}{(2\pi)^{n/2}2^{s/4}\Gamma(s/4)}K_{n/2-s/4}(|x|) & \text{if } s>n,\end{cases}$$

where $K_\nu$ is the Bessel function of an imaginary argument.

*Remark.* The convolution (1.5) is called a *Bessel potential.*

Let $f\in\mathscr{H}_s$ and $f_0\in\mathscr{H}_\sigma$. Then the convolution $f*f_0$ exists in $\mathscr{S}'$, is expressed by the formula

$$f*f_0=F^{-1}[F[f]\,F[f_0]], \tag{1.7}$$

and is continuous with respect to $f$ and $f_0$ together: if $f \to 0$ in $\mathscr{H}_s$ and $f_0 \to 0$ in $\mathscr{H}_\sigma$, then $f * f_0 \to 0$ in $\mathscr{S}'$.

Indeed, represent $f$ and $f_0$ in the form (1.2):

$$f = (1-\Delta)^m f_1, \quad f_1 \in \mathscr{L}^2, \quad f_0 = (1-\Delta)^l f_{01}, \quad f_{01} \in \mathscr{L}^2.$$

By virtue of the formula (1.7), just proved, for the convolution $f_1 * f_{01}$ (see Sec. 6.5(b)), also of the rule of differentiating a convolution (see Sec. 4.2(e)), and of the properties of the Fourier transform (see Sec. 6.3), we are convinced that (1.7) holds true:

$$\begin{aligned}(1-\Delta)^{m+l}(f_1 * f_{01}) &= (1-\Delta)^m f_1 * (1-\Delta)^l f_{01} = f * f_0 \\ &= (1-\Delta)^{m+l} F^{-1}[F[f]F[f_0]] \\ &= F^{-1}[(1+|\xi|^2)^{m+l} F[f_1] F[f_{01}]] \\ &= F^{-1}[F[(1-\Delta)^m f_1] F[(1-\Delta)^l f_{01}]] \\ &= F^{-1}[F[f] F[f_0]].\end{aligned}$$

From the representation (1.7) follows the continuity of the convolution $f * f_0$ with respect to $f$ and $f_0$ together.

*Example.* $\mathscr{P}\frac{1}{x} * \mathscr{P}\frac{1}{x} = -\pi^2\delta(x)$.

This follows from (6.14) of Sec. 6:

$$\mathscr{P}\frac{1}{x} * \mathscr{P}\frac{1}{x} = F^{-1}[(\pi i \operatorname{sign}\xi)^2] = -\pi^2 F^{-1}[1] = -\pi^2\delta(x).$$

Generalizing, we obtain the following: if $f \in \mathscr{H}_s$, $f_0 \in \mathscr{H}_\sigma$, and $F[f_1], \ldots, F[f_m]$ belong to $\mathscr{L}^\infty$, then their convolution exists in $\mathscr{S}'$ and can be represented as

$$f * f_0 * f_1 * \ldots * f_m = F^{-1}[F[f] F[f_0] F[f_1] \ldots F[f_m]]. \qquad (1.8)$$

Analogously, if $f \in \mathscr{H}_s$ and $F[f_1], \ldots, F[f_m]$ belong to $\mathscr{L}^\infty$, then their convolution exists in $\mathscr{H}_s$, can be represented as

$$f * f_1 * \ldots * f_m = F^{-1}[F[f] F[f_1] \ldots F[f_m]], \qquad (1.9)$$

and is continuous in $f$ from $\mathscr{H}_s$ to $\mathscr{H}_s$.

If $f_0 \in \mathscr{H}_{-s}$ and $f \in \mathscr{H}_s$, $s \geqslant 0$, then the following formula holds:

$$f_0 * f = (f_0(x'), f(x-x')) \qquad (1.10)$$

True enough, by virtue of (1.7), for all $\varphi$ in $\mathscr{S}$ we have

$$\begin{aligned}(f_0 * f, \varphi) &= (F^{-1}[F[f_0]F[f]], \varphi)\\ &= \left[F[f_0]F[f], \frac{1}{(2\pi)^n}\int \varphi(x) e^{-i(x,\xi)}\,dx\right)\\ &= \frac{1}{(2\pi)^n}\int \varphi(x)(F[f_0]F[f], e^{-i(x,\xi)})\,dx,\end{aligned}$$

since $F[f_0]F[f] \in \mathscr{L}^1$. Therefore

$$\begin{aligned}(f_0 * f)(x) &= \frac{1}{(2\pi)^n}(F[f_0]F[f], e^{-i(x,\xi)})\\ &= \frac{1}{(2\pi)^n}(F[f_0], F[f]e^{-i(x,\xi)})\\ &= \frac{1}{(2\pi)^n}(f_0, F[F[f]e^{-i(x,\xi)}])\\ &= \frac{1}{(2\pi)^n}\left(f_0(x'), \int F[f](\xi)e^{-i(x-x',\xi)}\,d\xi\right)\\ &= (f_0(x'), F^{-1}[F[f]](x-x'))\\ &= (f_0(x'), f(x-x')).\end{aligned}$$

We denote by $\mathscr{D}'_{\mathscr{L}^2}$ the inductive limit (union) of the increasing sequence of spaces $\mathscr{H}_{-s}$, $s = 0, 1, \ldots$,

$$\mathscr{D}'_{\mathscr{L}^2} = \bigcup_{s\geqslant 0}\mathscr{H}_{-s}.$$

By virtue of the lemma, $\mathscr{D}'_{\mathscr{L}^2}$ is a collection of continuous linear functionals on the countable-normed space $\mathscr{D}_{\mathscr{L}^2}$, which is a projective limit (intersection) of the decreasing sequence of spaces $\mathscr{H}_s$, $s = 0, 1, \ldots$,

$$\mathscr{D}_{\mathscr{L}^2} = \bigcap_{s=0}^{\infty}\mathscr{H}_s.$$

The space $\mathscr{D}_{\mathscr{L}^2}$ is an algebra with respect to the operation of ordinary multiplication (associative, commutative without unity, see Sec. 4.5); and, for all $f$ and $g$ in $\mathscr{D}_{\mathscr{L}^2}$,

$$\|fg\|_s \leqslant c_{p-s}\|f\|_p\|g\|_s, \quad s \geqslant 0, \quad p > s + n/2. \tag{1.11}$$

Indeed, since $f \in \mathscr{H}_s$ and $g \in \mathscr{H}_s$ for all $s$, it follows, in particular, that $f \in \mathscr{L}^2$ and $g \in \mathscr{L}^2$. We put $\tilde{f} = F^{-1}[f]$ and $\tilde{g} =$

$= F^{-1}[g]$. Using the definition (1.1) of a norm in $\mathscr{H}_s$, the formula of the Fourier transform of a convolution (see Sec. 6.5), the Fubini theorem, the Cauchy-Bunyakovsky inequalities, and

$$+ |\xi + \xi'|^2 \leqslant (1 + |\xi|^2)(1 + |\xi'|^2),$$

for all $s \geqslant 0$ and $p > s + n/2$, we obtain the inequality (1.11):

$$\|fg\|_s^2 = \int |F^{-1}[fg](\xi)|^2 (1 + |\xi|^2)^s \, d\xi,$$

$$= \int (1 + |\xi|^2)^s \left| \int \tilde{f}(\xi')(1 + |\xi'|^2)^{p/2} \frac{\tilde{g}(\xi - \xi')\, d\xi'}{(1 + |\xi'|^2)^{p/2}} \right|^2 d\xi$$

$$\leqslant \int |\tilde{f}(\xi')|^2 (1 + |\xi'|^2)^p \, d\xi'$$

$$\times \int (1 + |\xi|^2)^s \int |\tilde{g}(\xi - \xi')|^2 (1 + |\xi'|^2)^{-p} \, d\xi' \, d\xi$$

$$= \|f\|_p^2 \int |\tilde{g}(\eta)|^2 (1 + |\xi' + \eta|^2)^s (1 + |\xi'|^2)^{-p} \, d\eta \, d\xi'$$

$$\leqslant \|f\|_p^2 \int |\tilde{g}(\eta)|^2 (1 + |\eta|^2)^s \, d\eta \int (1 + |\xi'|^2)^{s-p} \, d\xi'$$

$$= \int \frac{d\xi'}{(1 + |\xi'|^2)^{p-s}} \|f\|_p^2 \|g\|_s^2.$$

We set

$$S'_0 = \bigcup_{s \geqslant 0} \mathscr{L}^2_{-s} = F[\mathscr{D}'_{\mathscr{L}^2}]$$

as the inductive limit (union) of the spaces $\mathscr{L}^2_{-s}$, $s = 0, 1, \ldots$.

For $f$ to belong to $f \in \mathscr{S}'$, it is necessary and sufficient that it be representable as

$$f(x) = x^\alpha f_0(x), \qquad f_0 \in \mathscr{D}'_{\mathscr{L}^2}. \tag{1.12}$$

Sufficiency is obvious and necessity follows from the representation

$$F[f] = (iD)^\alpha g(\xi),$$

where $g$ is a continuous function of slow growth in $\mathbb{R}^n$ (see Sec. 5.4), that is, $f_0 = F^{-1}[g] \in \mathscr{H}_s$ for some $s$, whence follows the required representation (1.12).

Let the generalized function $f_0$ from $\mathscr{S}'$ be continuously dependent, in $\mathscr{S}'$, on the parameter $\sigma$ on the compact $K$, that is, $(f_\sigma, \varphi) \in C(K)$ for any $\varphi \in \mathscr{S}$, and let $\mu$ be a finite measure on $K$. We introduce the generalized function $\int\limits_K f_\sigma \mu(d\sigma)$ taken from $\mathscr{S}'$

by means of the equation

$$\Big(\int_K f_\sigma \mu(d\sigma), \varphi\Big) = \int_K (f_\sigma, \varphi)\, \mu(d\sigma), \qquad \varphi \in \mathscr{S}.$$

It is easy to see that

$$F\Big[\int_K f_\sigma \mu(d\sigma)\Big] = \int_K F[f_\sigma]\, \mu(d\sigma). \tag{1.13}$$

**10.2 The Cauchy kernel $\mathscr{K}_C(z)$** Let $C$ be a connected open cone in $\mathbb{R}^n$ with vertex at 0 and let $C^*$ be the conjugate cone $C$ (see Sec. 4.4). The function

$$\mathscr{K}_C(z) = \int_{C^*} e^{i(z, \xi)}\, d\xi = L[\theta_{C^*}] = F[\theta_{C^*} e^{-(y, \xi)}] \tag{2.1}$$

is termed the *Cauchy kernel of a tubular region* $T^C$; here, $\theta_{C^*}(\xi)$ is the characteristic function of the cone $C^*$.

If the cone $C$ is not acute, then by virtue of Lemma 1 of Sec. 4.4, mes $C^* = 0$ and, hence, $\mathscr{K}_C(z) \equiv 0$. Furthermore, since $C^* = (\text{ch}\, C)^*$, it follows that $\mathscr{K}_C(z) \equiv \mathscr{K}_{\text{ch}\, C}(z)$. Therefore, without restricting generality, we may regard the cone $C$ as acute and convex.

By what has been proved (see Sec. 9.1), the kernel $\mathscr{K}_C(z)$ is a holomorphic function in $T^C$; and, moreover, the integral in (2.1) converges uniformly with respect to $z$ in any tubular region $T^K$, $K \Subset C$ ($\bar{K}$ is a compact).

We will now show that the kernel $\mathscr{K}_C(z)$ can be represented by the integral

$$\mathscr{K}_C(z) = i^n \Gamma(n) \int_{\text{pr}\, C^*} \frac{d\sigma}{(z, \sigma)^n}, \qquad z \in T^C. \tag{2.2}$$

Indeed, since $(y, \sigma) > 0$ for all $y \in C$, $\sigma \in \text{pr}\, C^*$, it follows that the denominator of the integrand on the right of (2.2) is equal to $[(x, \sigma) + i(y, \sigma)]^n$ and does not vanish in $T^C$, and, consequently, the right-hand side of (2.2) is a holomorphic function in $T^C$. Since the kernel $\mathscr{K}_C(z)$ is also a holomorphic function in $T^C$, it suffices to prove (2.2) on the manifold $z = iy$, $y \in C$. But when $x = 0$

the formula (2.2) follows readily from (2.1):

$$\mathcal{K}_C(iy) = \int_{C^*} e^{-(y,\xi)}\,d\xi = \int_{\mathrm{pr}\,C^*}\int_0^\infty e^{-\rho(y,\sigma)}\rho^{n-1}\,d\rho\,d\sigma$$

$$= \int_{\mathrm{pr}\,C^*} \frac{d\sigma}{(y,\sigma)^n}\int_0^\infty e^{-u}u^{n-1}\,du = i^n\Gamma(n)\int_{\mathrm{pr}\,C^*}\frac{d\sigma}{(iy,\sigma)^n}.$$

From the representation (2.2) it follows that the *kernel* $\mathcal{K}_C(z)$. and also the kernel $\mathcal{K}_{-C}$ are holomorphic in the region

$$D = \mathbb{C}^n \setminus \bigcup_{\sigma\in\mathrm{pr}\,C^*}[z:(z,\sigma)=0].$$

It is easy to see that the region $D$ contains the tubular domains $T^C$

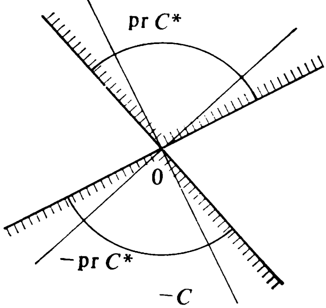



Figure 28

and $T^{-C}$ and also the real points of the cones $C$ and $-C$.

The kernels $\mathcal{K}_C$ and $\mathcal{K}_{-C}$ satisfy the relations

$$\mathcal{K}_{-C}(z) = (-1)^n\mathcal{K}_C(z) = \overline{\mathcal{K}_C(\bar{z})} = \mathcal{K}_C(-z),$$

$$\mathcal{K}_C(\lambda z) = \lambda^{-n}\mathcal{K}_C(z) \qquad \lambda\in\mathbb{C}^1\setminus\{0\},\quad z\in T^C\cup T^{-C}. \tag{2.3}$$

Let us now prove the estimate

$$|D^\alpha\mathcal{K}_C(z)| \leqslant M_\alpha\Delta^{-n-|\alpha|}(y), \qquad z\in T^C\cup T^{-C}, \tag{2.4}$$

where $\Delta(y) = \Delta(y, -\partial C\cup\partial C)$ is the distance from $y$ to the boundary of the cone $-C\cup C$: $\Delta(y) = \inf_{\sigma\in\mathrm{pr}\,C^*}(\sigma, y)$, $y\in C$ (see Sec. 0.2 and Fig. 28).

Indeed, using the representation (2.2), we have, for $z \in T^C$, the estimate (2.4):

$$|D^\alpha \mathscr{K}_C(z)| \leqslant M_\alpha \int_{\mathrm{pr}\, C^*} \frac{|\sigma^\alpha|\, d\sigma}{|(z, \sigma)|^{n+|\alpha|}}$$

$$\leqslant M_\alpha \sup_{\sigma \in \mathrm{pr}\, C^*} (y, \sigma)^{-n-|\alpha|} = M_\alpha \Delta^{-n-|\alpha|}(y).$$

The estimate (2.4) for $z \in T^{-C}$ follows from (2.4) that was proved for $z \in T^C$ and from the properties (2.3). More rigorous reasoning yields the estimate

$$|D^\alpha \mathscr{K}_C(z)| \leqslant M'_\alpha \Delta^{-n+1-|\alpha|}(y)\,[|x|^2 + \Delta^2(y)]^{-1/2} \qquad (2.4')$$
$$z \in T^C \cup T^{-C}.$$

We now prove the estimate for all $s \geqslant 0$,

$$\| D^\alpha \mathscr{K}_C(x+iy)\|_s \leqslant K_{s,\alpha}[1 + \Delta^{-s}(y)]\, \Delta^{-n/2-|\alpha|}(y),$$
$$y \in -C \cup C. \qquad (2.5)$$

Indeed, by (2.1) and (2.2) for $y \in C$ we have the estimate (2.5):

$$\|D^\alpha \mathscr{K}_C(x+iy\| = \| F^{-1}[D^\alpha \mathscr{K}_C(x+iy)]\|^2_{(s)}$$
$$= \| (-i\xi)^\alpha \theta_{C^*}(\xi)\, e^{-(y,\xi)} \|^2_{(s)}$$
$$= \int_{C^*} e^{-2(y,\xi)} (1 + |\xi|^2)^s |\xi^\alpha|^2\, d\xi$$
$$\leqslant \int_0^\infty (1+\rho^2)^s \rho^{n-1+2|\alpha|} \int_{\mathrm{pr}\, C^*} e^{-2\rho(y,\sigma)}\, d\sigma\, d\rho$$
$$\leqslant \frac{\sigma_n}{2} \int_0^\infty e^{-2\rho\Delta(y)} (1+\rho^2)^s \rho^{n-1+2|\alpha|}\, d\rho$$
$$= \frac{\sigma_n}{2^{n+1+2|\alpha|}\Delta^{n+2|\alpha|}(y)} \int_0^\infty e^{-u}\left[1 + \frac{u^2}{4\Delta^2(y)}\right]^s u^{n-1+2|\alpha|}\, du$$
$$\leqslant K^2_{s,\alpha}[1 + \Delta^{-s}(y)]^2 \Delta^{-n-2|\alpha|}(y).$$

Here, $\sigma_n$ is the surface area of a unit sphere in $\mathbb{R}^n$. The case $y \in -C$ can be considered with use of the relations (2.3).

The kernel $\mathscr{K}_C(z)$ assumes a boundary value equal to $(\pm 1)^n F[\theta_{\pm C^*}]$, respectively,

$$\mathscr{K}_C(x+iy) \to (\pm 1)^n \mathscr{K}_C(\pm x) = (\pm 1)^n F[\theta_{\pm C^*}], \tag{2.6}$$

as $y \to 0$, $y \in \pm C$ in norm in $\mathscr{H}_s$ for arbitrary $s < -n/2$.

Indeed, by what has been proved, $\mathscr{K}_C(x+iy) \in \mathscr{H}_s$ for $y \in -C \cup C$ and for any $s$, while the generalized functions $F[\theta_{\pm C^*}] \in \mathscr{H}_s$ for all $s < -n/2$ (since $\theta_{\pm C^*} \in \mathscr{L}_s^2$ for $s < -n/2$). Therefore, when $s < -n/2$ and $y \in C$, $y \to 0$, we have

$$\| \mathscr{K}_C(x+iy) - F[\theta_{C^*}] \|_s^2 = \| \theta_{C^*} e^{-(y,\xi)} - \theta_{C^*} \|_{(s)}^2$$
$$= \int_{C^*} [e^{-(y,\xi)} - 1]^2 (1+|\xi|^2)^s \, d\xi \to 0.$$

But if $y \in -C$, $y \to 0$, then

$$\mathscr{K}_C(x+iy) = (-1)^n \mathscr{K}_C(-x-iy) \to$$
$$\to (-1)^n \mathscr{K}_C(-x) = (-1)^n F[\theta_{-C^*}]$$

and the formula (2.6) is proved.

From the formulas (2.3) and (2.6) we have the following relations for the boundary values of the kernels $\mathscr{K}_C(z)$ and $\mathscr{K}_{-C}(z)$:

$$\left.\begin{aligned} &\mathscr{K}_{-C}(x) = \overline{\mathscr{K}_C(x)} = \mathscr{K}_C(-x), && x \in \mathbb{R}^n,\\ &\mathscr{K}_{-C}(x) = (-1)^n \mathscr{K}_C(x), && x \in C \cup (-C);\end{aligned}\right\} \tag{2.7}$$

$$\left.\begin{aligned} \operatorname{Re} \mathscr{K}_C(x) &= \frac{1}{2} F[\theta_{C^*} + \theta_{-C^*}] \\ &= \frac{1}{2}[\mathscr{K}_C(x) + \mathscr{K}_C(-x)], \\ \operatorname{Im} \mathscr{K}_C(x) &= \frac{1}{2i} F[\theta_{C^*} - \theta_{-C^*}] \\ &= \frac{1}{2i}[\mathscr{K}_C(x) - \mathscr{K}_C(-x)]. \end{aligned}\right\} \tag{2.8}$$

From this, taking into account the trivial equalities

$$(\theta_{C^*} - \theta_{-C^*})^2 = (\theta_{C^*} + \theta_{-C^*})^2 = \theta_{C^*} + \theta_{-C^*},$$

$$(\theta_{C^*} - \theta_{-C^*})(\theta_{C^*} + \theta_{-C^*}) = \theta_{C^*} - \theta_{-C^*},$$

and making use of (1.9) for the convolution, we obtain the following relations between the generalized functions $\operatorname{Re}\mathscr{K}_C(x)$ and $\operatorname{Im}\mathscr{K}_C(x)$:

$$-\operatorname{Im}\mathscr{K}_C * \operatorname{Im}\mathscr{K}_C = \operatorname{Re}\mathscr{K}_C * \operatorname{Re}\mathscr{K}_C = \frac{1}{2}(2\pi)^n \operatorname{Re}\mathscr{K}_C, \quad (2.9)$$

$$\operatorname{Im}\mathscr{K}_C * \operatorname{Re}\mathscr{K}_C = \frac{1}{2}(2\pi)^n \operatorname{Im}\mathscr{K}_C. \quad (2.10)$$

Let us now calculate the real and imaginary parts of the kernel $\mathscr{K}_C(x)$. To do this, we introduce, for $k = 0, 1, \ldots$, the generalized functions

$$\delta^{(k)}[(x, \delta)] \quad \text{and} \quad \mathscr{P}^{(k)}\frac{1}{(x, \sigma)}$$

that operate on the basic functions $\varphi$ in $\mathscr{S}$ via the rules

$$(\delta^{(k)}[(x, \sigma)], \varphi) = (-1)^k \int\limits_{(x, \sigma)=0} \left(\frac{\partial}{\partial\sigma}\right)^k \varphi(x)\, dS_x,$$

$$\left(\mathscr{P}^{(k)}\frac{1}{(x, \sigma)}, \varphi\right) = (-1)^k \mathrm{PV}\int\limits_{-\infty}^{\infty} \frac{1}{\lambda} \int\limits_{(x, \sigma)=0} \frac{\partial^k}{\partial\lambda^k}\varphi(x+\lambda\sigma)\, dS_x\, d\lambda.$$

The generalized functions that have just been introduced depend continuously, in $\mathscr{S}'$, on the parameter $\sigma$ on the unit sphere $S_1$ (in the sense of Sec. 10.1).

Let us prove the equation

$$\mathscr{K}_C(x) = \pi(-i)^{n-1} \int\limits_{\operatorname{pr} C^*} \delta^{(n-1)}[(x, \sigma)]\, d\sigma$$
$$-(-i)^n \int\limits_{\operatorname{pr} C^*} \mathscr{P}^{(n-1)}\frac{1}{(x, \sigma)}\, d\sigma. \quad (2.11)$$

Using the representation (2.2), we have, for all $y \in C$ and $\varphi \in \mathscr{S}$,

$$\int \mathscr{K}_C(x+iy)\varphi(x)\, dx = i^n\Gamma(n) \int \int\limits_{\operatorname{pr} C^*} \frac{d\sigma}{[(x, \sigma)+i(y, \sigma)]^n}\varphi(x)\, dx$$
$$= i^n\Gamma(n) \int \int \frac{\varphi(x)\, dx}{[(x, \sigma)+i(y, \sigma)]^n}\, d\sigma. \quad (2.12)$$

Now let $y \to 0$, $y \in C$. Then $0 < \varepsilon = (y, \sigma) \to 0$ for $\sigma \in \operatorname{pr} C^*$, and the integral

$$\int \frac{\varphi(x)\,dx}{[(x, \sigma)+i\varepsilon]^n}$$

$$= \int_{-\infty}^{\infty} \frac{1}{(\lambda+i\varepsilon)^n} \int_{(x,\sigma)=0} \varphi(x+\lambda\sigma)\,dS_x\,d\lambda$$

$$= \frac{1}{\Gamma(n)} \int_{-\infty}^{\infty} \frac{1}{\lambda+i\varepsilon} \frac{\partial^{n-1}}{\partial\lambda^{n-1}} \int_{(x,\sigma)=0} \varphi(x+\lambda\sigma)\,dS_x\,d\lambda$$

$$= \frac{1}{\Gamma(n)} \int_{-\infty}^{\infty} \ln(\lambda+i\varepsilon) \frac{\partial^{n}}{\partial\lambda^{n}} \int_{(x,\sigma)=0} \varphi(x+\lambda\sigma)\,dS_x\,d\lambda$$

is uniformly bounded with respect to $(\varepsilon, \sigma)$ for all $0 < \varepsilon \leqslant 1$ and $\sigma \in \operatorname{pr} C^*$; furthermore, by virtue of the Sochozki formula [see (8.3) of Sec. 1], that integral tends, as $\varepsilon \to +0$, to the limit

$$-\frac{i\pi}{\Gamma(n)} \int_{(x,\sigma)=0} \left(\frac{\partial}{\partial\sigma}\right)^{n-1} \varphi(x)\,dS_x$$

$$+\frac{1}{\Gamma(n)} \mathrm{PV} \int_{-\infty}^{\infty} \frac{1}{\lambda} \frac{\partial^{n-1}}{\partial\lambda^{n-1}} \int_{(x,\sigma)=0} \varphi(x+\lambda\sigma)\,dS_x\,d\lambda,$$

that is to say, if we make use of the notation that has been introduced, then it tends to the limit

$$\left(\frac{(-1)^n i\pi}{\Gamma(n)} \delta^{(n-1)}[(x,\sigma)] + \frac{(-1)^{n-1}}{\Gamma(n)} \mathscr{P}^{(n-1)} \frac{1}{(x,\sigma)}, \varphi\right)$$

$$= \left(\frac{1}{[(x,\sigma)+i0]^n}, \varphi\right).$$

Therefore, if in the integral (2.12) we pass to the limit as $y \to 0$, $y \in C$, and if we make use of the Lebesgue theorem and the limiting relation (2.6), we obtain (2.11). In passing we also obtained the equalities

$$\mathscr{K}_C(x) = i^n \Gamma(n) \int_{\operatorname{pr} C^*} \frac{d\sigma}{[(x,\sigma)+i0]^n}, \tag{2.13}$$

$$\frac{1}{[(x,\sigma)+i0]^n} = \frac{(-1)^n i\pi}{\Gamma(n)} \delta^{(n-1)}[(x,\sigma)] + \frac{(-1)^{n-1}}{\Gamma(n)} \mathscr{P}^{(n-1)} \frac{1}{(x,\sigma)},$$

$$\sigma \in \operatorname{pr} C^*.$$

Finally, separating the real and imaginary parts in (2.11), we obtain the following useful formulas:

$$\operatorname{Re} \mathcal{K}_C(x) = \begin{cases} \pi(-1)^{\frac{n-1}{2}} \int\limits_{\operatorname{pr} C^*} \delta^{(n-1)}[(x,\sigma)]\, d\sigma, & n \text{ odd}, \\ (-1)^{n/2-1} \int\limits_{\operatorname{pr} C^*} \mathcal{P}^{(n-1)} \frac{1}{(x,\sigma)}\, d\sigma, & n \text{ even}, \end{cases} \tag{2.14}$$

$$\operatorname{Im} \mathcal{K}_C(x) = \begin{cases} (-1)^{\frac{n-1}{2}} \int\limits_{\operatorname{pr} C^*} \mathcal{P}^{(n-1)} \frac{1}{(x,\sigma)}\, d\sigma, & n \text{ odd}, \\ \pi(-1)^{n/2} \int\limits_{\operatorname{pr} C^*} \delta^{(n-1)}[(x,\sigma)]\, d\sigma, & n \text{ even}. \end{cases} \tag{2.15}$$

*Example 1.*

$$\mathcal{K}_{\mathbb{R}_+^n}(z) = \frac{i^n}{z_1 \ldots z_n} = \mathcal{K}_n(z), \qquad z \in T^{\mathbb{R}_+^n} = T^n \tag{2.16}$$

$$\mathcal{K}_n(x) = \left[\pi\delta(x_1) + i\mathcal{P}\frac{1}{x_1}\right] \times \ldots \times \left[\pi\delta(x_n) + i\mathcal{P}\frac{1}{x_n}\right]$$

*Example 2.*

$$\mathcal{K}_{V^+}(z) = 2^n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right)(-z^2)^{-\frac{n+1}{2}}, \qquad z \in T^{V^+} \tag{2.17}$$

where $z^2 = z_0^2 - z_1^2 - \ldots - z_n^2$. Let us compute the Cauchy kernel $\mathcal{K}_{V^+}(z)$. As was mentioned above, it suffices to compute it for $x = 0$; since $(V^+)^* = \overline{V}^+$ (see Sec. 4.4), it follows that

$$\mathcal{K}_{V^+}(iy) = \int\limits_{V^+} e^{-(y,\xi)}\, d\xi, \qquad y \in V^+.$$

Furthermore, by virtue of the invariance of that integral relative to the restricted Lorentz group, $L_+^{\uparrow}$, it suffices to compute it for $y = (y_0, 0)$, $y_0 > 0$. We have

$$\mathcal{K}_{V^+}(iy_0, 0) = \int\limits_{V^+} e^{-y_0\xi_0}\, d\xi = \int\limits_0^\infty e^{-y_0\xi} \int\limits_{|\xi|<\xi_0} d\xi\, d\xi_0$$

$$= \frac{\sigma_n}{n} \int\limits_0^\infty e^{-y_0\xi_0} \xi_0^n\, d\xi_0 = \frac{\sigma_n}{n y_0^{n+1}} \int\limits_0^\infty e^{-u} u^n\, du$$

$$= \Gamma(n)\,\sigma_n y_0^{-n-1}$$

$$= 2^n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right) [-(iy_0)^2]^{-\frac{n+1}{2}}.$$

By extending the resulting equality to all $y \in V^+$ and further onto all $z \in T^{V^+}$, we obtain (2.17).

*Example 3.*

$$\mathcal{K}_{\mathcal{P}_n}(Z) = \pi^{\frac{n(n-1)}{2}} i^{n^2} \frac{1!\dots(n-1)!}{(\det Z)^n}, \qquad Z \in T^{\mathcal{P}_n}, \tag{2.18}$$

where $\mathcal{P}_n$ is a cone of positive $n$-by-$n$ matrices (see Sec. 4.4), and $T^{\mathcal{P}_n} = [Z = X + iY,\ Y = \operatorname{Im} Z > 0]$. In order to compute the Cauchy kernel $\mathcal{K}_{\mathcal{P}_n}(Z)$ note that $\mathcal{P}_n^* = \overline{\mathcal{P}}_n$ and, therefore, by (2.1),

$$\mathcal{K}_{\mathcal{P}_n}(iY) = \int_{\mathcal{P}_n} e^{-\operatorname{Tr}(Y\Xi)} d\Xi,$$

where $d\Xi$ is the Lebesgue measure in $\mathbb{R}^{n^2}$,

$$d\Xi = d\xi_{11}\dots d\xi_{nn} \prod_{p<q} d \operatorname{Re} \xi_{pq} d \operatorname{Im} \xi_{pq}.$$

By virtue of the invariance of the last integral with respect to the transformations $Y \to U^{-1} YU$, where $U$ is any unitary matrix, it suffices to compute that integral for diagonal matrices $Y$ of the form $Y_0 = [\lambda_1, \dots, \lambda_n]$, $\lambda_j > 0$, $j = 1, \dots, n$,

$$\mathcal{K}_{\mathcal{P}_n}(iY_0) = \int_{\mathcal{P}_n} e^{-\sum_{p=1}^{n} \lambda_p \xi_{pp}} d\Xi.$$

Then the transformation $\xi_{pq} \to \frac{\xi_{pq}}{\sqrt{\lambda_p \lambda_q}}$ carries $\mathcal{P}_n$ onto itself, and its Jacobian is equal to $(\lambda_1 \dots \lambda_n)^n = (\det Y_0)^n = (\det Y)^n$. Consequently,

$$\mathcal{K}_{\mathcal{P}_n}(iY_0) = \mathcal{K}_{\mathcal{P}_n}(iY) = (\det Y)^{-n} \int_{\mathcal{P}_n} e^{-\operatorname{Tr}\Xi} d\Xi.$$

The last constant has been computed (see, for example, Bochner [2])

$$\int_{\mathcal{P}_n} e^{-\operatorname{Tr}\Xi} d\Xi = \pi^{\frac{n(n-1)}{2}} 1! \ldots (n-1)!.$$

Therefore

$$\mathcal{K}_{\mathcal{P}_n}(iY) = i^{n^2} \pi^{\frac{n(n-1)}{2}} 1! \ldots (n-1)! \, [\det(iY)]^{-n},$$
$$Y \in \mathcal{P}_n.$$

Extending this equation to all $Z \in T^{\mathcal{P}_n}$, we obtain the formula (2.18).

**10.3 The Cauchy-Bochner transform** Suppose $f \in \mathcal{H}_s$. The function

$$f(z) = \frac{1}{(2\pi)^n} (f(x'), \mathcal{K}_C(z - x')), \qquad z \in T^C \cup T^{-C}, \tag{3.1}$$

is called the *Cauchy-Bochner transform* (*integral*). It is assumed here that the cone $C$ is convex and acute.

Since $\mathcal{K}_C(x + iy) \in \mathcal{H}_s$ for all $s$ and $y \in -C \cup C$ (see Sec. 10.2), it follows that by (1.10) the right-hand side of (3.1) may be rewritten in the form of a convolution:

$$f(z) = \frac{1}{(2\pi)^n} f(x') * \mathcal{K}_C(x' + iy), \qquad z \in T^C \cup T^{-C}. \tag{3.2}$$

*Example.* When $n = 1$, $C = (0, \infty)$ and $f \in \mathcal{L}^2$, the Cauchy-Bochner integral (3.1) turns into the classical Cauchy integral:

$$f(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(x')}{x' - z} dx'.$$

The function $f(z)$ is holomorphic in $T^C \cup T^{-C}$, and

$$D^\alpha f(z) = \frac{1}{(2\pi)^n} (f(x'), D^\alpha \mathcal{K}_C(z - x')), \tag{3.3}$$

$$|D^\alpha f(z)| \leqslant \frac{K_{-s,\alpha}}{(2\pi)^n} \|f\|_s [1 + \Delta^s(y)] \Delta^{-n/2-|\alpha|}(y), \tag{3.4}$$

where the numbers $K_{-s,\alpha}$ are the same as in the estimate (2.5).

The holomorphic nature of the function $f(z)$ in $T^C \cup T^{-C}$ and the differentiation formula (3.3) follow directly from the facts that the Cauchy kernel $\mathscr{K}_c(z)$ is a holomorphic function in $T^C \cup T^{-C}$ and $\mathscr{K}_C(x+iy) \in \mathscr{H}_s$ for all $s$ and $y \in -C \cup C$ (see Sec. 10.2). The estimate (3.4) follows from (3.3) and also from the lemma of Sec. 10.1 and from (2.5):

$$|D^\alpha f(z)| \leqslant \frac{1}{(2\pi)^n} \|f\|_s \|D^\alpha \mathscr{K}_C(z-x')\|_{-s}$$
$$= \frac{\|f\|_s}{(2\pi)^n} \|D^\alpha \mathscr{K}_C(x+iy)\|_{-s}$$
$$\leqslant \frac{K_{-s,\alpha}}{(2\pi)^n} \|f\|_s [1+\Delta^s(y)] \Delta^{-n/2-|\alpha|}(y).$$

Now let us prove, for all $s$ and $y \in -C \cup C$, the estimates

$$\|D^\alpha f(x+iy)\|_s \leqslant N_\alpha \|f\|_s \Delta^{-|\alpha|}(y), \tag{3.5}$$

$$\|D^\alpha f(x+iy)\|_{s-|\alpha|} \leqslant \|f\|_s. \tag{3.6}$$

Indeed, from the representation (3.2) and from the definition of the kernel $\mathscr{K}_C(z)$ it follows that

$$F^{-1}[D^\alpha f(x+iy)] = \frac{1}{(2\pi)^n} F^{-1}[f * D^\alpha \mathscr{K}_C]$$
$$= F^{-1}[f] F^{-1}[D^\alpha \mathscr{K}_C]$$
$$= F^{-1}[f] (i\xi)^\alpha F^{-1}[\mathscr{K}_C]$$
$$= (i\xi)^\alpha F^{-1}[f](\xi) e^{-(y,\xi)} \theta_{C^*}(\xi). \tag{3.7}$$

Therefore

$$\|D^\alpha f(x+iy)\|_s^2 = \int_{C^*} |F^{-1}[f](\xi)|^2 |\xi^\alpha|^2 e^{-2(y,\xi)} (1+|\xi|^2)^s \, d\xi$$
$$\leqslant \|f\|_s^2 \sup_{\xi \in C^*} |\xi|^{2|\alpha|} e^{2=(y,\xi)} = \|f\|_s^2 \sup_{\rho \geqslant 0} \rho^{2|\alpha|} \sup_{\sigma \in \operatorname{pr} C^*} e^{-2\rho(y,\sigma)}$$
$$= \|f\|_s^2 \sup_{\rho \geqslant 0} \rho^{2|\alpha|} e^{-2\rho\Delta(y)}$$
$$= \|f\|_s^2 \, 2^{-2|\alpha|} \Delta^{-2|\alpha|}(y) \sup_{u \geqslant 0} u^{2|\alpha|} e^{-u},$$

which is what yields the estimate (3.5). The estimate (3.6) is derived in similar fashion but more simply.

As $y \to 0$ for $y \in \pm C$, the function $f(z)$ assumes in $\mathscr{H}_s$, in norm, the boundary values $f_\pm(x)$, which are respectively equal to

$$f_+ = \frac{1}{(2\pi)^n} f * \mathscr{K}_C, \qquad f_- = \frac{(-1)^n}{(2\pi)^n} f * \overline{\mathscr{K}}_C. \tag{3.8}$$

Indeed, taking into account (3.7), we have, for $y \in C$,

$$F^{-1}[f(x+iy) - f_+] = F^{-1}[f](\xi)\,[e^{-(y,\xi)} - 1]\,\theta_{C^*}(\xi).$$

Therefore, when $y \to 0$, $y \in C$, we obtain

$$\|f(x+iy) - f_+(x)\|_s^2$$
$$= \int_{C^*} |F^{-1}[f](\xi)|^2\,[e^{-(y,\xi)} - 1]^2\,(1 + |\xi|^2)^s\, d\xi \to 0,$$

which is what we set out to prove. The case of $y \to 0$, $y \in -C$, is considered in similar fashion with use made of the formulas (2.6) and (2.7).

**10.4 The Hilbert transform** Suppose $f \in \mathscr{H}_s$ for some $s$. The *Hilbert transform* $f_1$ of the generalized function $f$ is the convolution

$$f_1 = -\frac{2}{(2\pi)^n} f * \operatorname{Im} \mathscr{K}_C. \tag{4.1}$$

Applying the Fourier transform to (4.1) and using (2.8), we obtain

$$F[f_1] = -i\,(\theta_{C^*} - \theta_{-C^*})\,F[f],$$

whence it follows that

$$f_1 \in \mathscr{H}_s \quad \text{and} \quad \operatorname{supp} F[f_1] \subset -C^* \cup C^*. \tag{4.3}$$

If $f \in \mathscr{H}_s$, then the conditions

$$(1) \qquad \operatorname{supp} F[f] \subset -C^* \cup C^*, \tag{4.4}$$

$$(2) \qquad f = \frac{2}{(2\pi)^n} f_1 * \operatorname{Im} \mathscr{K}_C \tag{4.5}$$

$$(3) \qquad f = \frac{2}{(2\pi)^n} f * \operatorname{Re} \mathscr{K}_C \tag{4.6}$$

are equivalent.

Indeed, from (1) $\to$ (2), by virtue of (4.2) and (2.8),

$$F[f] = i(\theta_{C^*} - \theta_{-C^*}) F[f_1].$$

From (2) $\to$ (3), by virtue of (4.1) and (2.9),

$$f = \frac{2}{(2\pi)^n} f_1 * \operatorname{Im} \mathscr{K}_C = -\frac{4}{(2\pi)^{2n}} (f * \operatorname{Im} \mathscr{K}_C) * \operatorname{Im} \mathscr{K}_C$$

$$= -\frac{4}{(2\pi)^{2n}} f * (\operatorname{Im} \mathscr{K}_C * \operatorname{Im} \mathscr{K}_C) = \frac{2}{(2\pi)^n} f * \operatorname{Re} \mathscr{K}_C$$

and from the associativity of the convolution (see Sec. 10.1). Finally, from (3) $\to$ (1), by (2.8),

$$F[f] = (\theta_{C^*} + \theta_{-C^*}) F[f].$$

We will say that the generalized functions $f$ and $f_1$ in $\mathscr{H}_s$ form a *pair* of Hilbert transforms if they satisfy the relations (4.1) and (4.5):

$$f_1 = -\frac{2}{(2\pi)^n} f * \operatorname{Im} \mathscr{K}_C, \qquad f = \frac{2}{(2\pi)^n} f_1 * \operatorname{Im} \mathscr{K}_C. \tag{4.7}$$

*Example*. When $n = 1$ (see Sec. 10.2),

$$\mathscr{K}_C(z) = \frac{i}{z}, \quad \operatorname{Re} \mathscr{K}_C(x) = \pi\delta(x), \quad \operatorname{Im} \mathscr{K}_C(x) = \mathscr{P}\frac{1}{x},$$

the formulas (4.7) take the form

$$f_1 = -\frac{1}{\pi} f * \mathscr{P}\frac{1}{x}, \qquad f = \frac{1}{\pi} f_1 * \mathscr{P}\frac{1}{x}, \tag{4.8}$$

and the relation (4.6) turns into the identity $f = f$. When $f \in \mathscr{L}^2$, the formulas (4.8) turn into the classical Hilbert transform formulas.

*Remark 1*. We note here the difference between the cases $n = 1$ and $n \geqslant 2$: for $n = 1$, the condition (4.4) is absent because $-C^* \cup C^* = \mathbb{R}^1$, whereas for $n \geqslant 2$ that condition is essential.

*Remark 2*. The results of this subsection were obtained by Beltrami and Wohlers [1] ($n = 1$) and Vladimirov [3] ($n \geqslant 2$).

**10.5 Holomorphic functions of the class $H_a^{(s)}(C)$** Suppose $C$ is a convex acute open cone, $a \geqslant 0$, and let $s$ be a real number. Denote by $H_a^{(s)}(C)$ the Banach space consisting of functions $f(z)$

holomorphic in $T^C$ with norm

$$\| f \|_a^{(s)} = \sup_{y\in C} e^{-a|y|} \| f(x+iy) \|_s. \tag{5.1}$$

We write $H_0^{(s)}(C) = H^{(s)}(C)$ and $\| \ \|_0^{(s)} = \| \ \|^{(s)}$.

Lemma *Let the function $f(z)$ be holomorphic in $T^C$ and let it satisfy the following condition of growth: for any $\varepsilon > 0$ there is a number $M(\varepsilon)$ such that*

$$\| f(x+iy) \|_s \leqslant M(\varepsilon)\, e^{(a+\varepsilon)|y|}\, [1 + \Delta^{-\gamma}(y)], \quad y \in C, \tag{5.2}$$

*for certain $s$, $a \geqslant 0$ and $\gamma \geqslant 0$ (all dependent only on $f$). Then $f(z)$ is the Laplace transform of the function $g$ in $\mathcal{L}_{s'}^2(C^* + \bar{U}_a)$, where $s' = s$ if $\gamma = 0$, $s' < s - \gamma$ if $\gamma > 0$; here the following estimate holds true:*

$$\| g \|_{(s')} \leqslant \sqrt{\frac{1+\gamma}{s-s'-\gamma}}\, e^{2+a} \inf_{0<\varepsilon\leqslant 1} M(\varepsilon) \inf_{\sigma\in \operatorname{pr} C} [1+\Delta^{-\gamma}(\sigma)]. \tag{5.3}$$

*Proof.* We introduce the generalized function $g_y(\xi)$, taken from $\mathcal{D}'(\mathbb{R}^n \times C)$, via the formula

$$g_y(\xi) = e^{(y,\xi)} F_x^{-1} [f(x+iy)](\xi, y). \tag{5.4}$$

Here, $F_x^{-1}$ signifies the Fourier transform with respect to the variables $x$ (see Sec. 6.2). We will prove that $g_y(\xi)$ does not depend on $y \in C$. Indeed, differentiating (5.4) with respect to $y_j$ and using the Cauchy-Riemann conditions, we have

$$\begin{aligned}\frac{\partial g_y(\xi)}{\partial y_j} &= \xi_j e^{(y,\xi)} F_x^{-1}[f(x+iy)] + e^{(y,\xi)} F_x^{-1}\left[\frac{\partial f(x+iy)}{\partial y_j}\right] \\ &= e^{(\xi,y)} \left\{\xi_j F_x^{-1}[f(x+iy)] + iF_x^{-1}\left[\frac{\partial}{\partial x_j} f(x+iy)\right]\right\} \\ &= e^{(y,\xi)} F_x^{-1}[f(x+iy)](\xi_j + i^2\xi_j) = 0, \\ & \qquad j = 1, \ldots, n,\end{aligned}$$

whence, by virtue of the criterion of Sec. 3.3, we conclude that $g_y(\xi)$ does not depend on $y \in C$, $g_y(\xi) = g(\xi)$. And then from (5.4) it follows that $g(\xi)\, e^{-(y,\xi)} \in \mathcal{S}'$ for all $y \in C$ and

$$f(x+iy) = F[g(\xi)\, e^{-(y,\xi)}], \qquad z \in T^C. \tag{5.5}$$

Furthermore, by the hypothesis $f(x+iy) \in \mathcal{H}_s$ for every $y \in C$ so that, by (5.5), $g(\xi)\, e^{-(y,\xi)}$ is a function in $\mathcal{L}_s^2$ and

$$\| g(\xi)\, e^{-(y,\xi)} \|_{(s)}^2 = \| f(x+iy) \|_s^2, \qquad y \in C.$$

From this, by (5.2), for all $\varepsilon > 0$ we derive the inequality

$$\int | g(\xi) |^2 e^{-2(y,\xi)} (1+|\xi|^2)^s \, d\xi \leqslant M^2(\varepsilon)\, e^{2(a+\varepsilon)|y|} [1+\Delta^{-\gamma}(y)]^2, \qquad y \in C. \quad (5.6)$$

We will now prove that $g(\xi) = 0$ almost everywhere outside $C^* + \overline{U}_a$. Let $\xi_0 \overline{\in} C^* + \overline{U}_a$. By lemma 3 of Sec. 4.4

$$C^* + \overline{U}_a = [\xi : \mu_C(\xi) \leqslant a]$$

so that

$$\mu_C(\xi_0) = - \inf_{y \in \mathrm{pr}\, C} (\xi_0, y) > a.$$

Therefore there is a point $y_0 \in \mathrm{pr}\, C$ such that $(\xi_0, y_0) < -a - \varkappa$ for certain $\varkappa > 0$ (see Fig. 23). This inequality also holds, in continuity, in a sufficiently small neighbourhood $|\xi - \xi_0| < \delta$. Therefore, putting $y = ty_0$ in (5.6), we obtain, for all $t > 0$, the inequality

$$e^{2t(a+\varkappa)} \int_{|\xi-\xi_0|<\delta} | g(\xi) |^2 (1+|\xi|^2)^s \, d\xi \leqslant \int_{|\xi-\xi_0|<\delta} | g(\xi) |^2 e^{-2(y,\xi)} (1+|\xi|^2)^s \, d\xi \leqslant M^2(\varepsilon)\, e^{2t(a+\varepsilon)} [1+t^{-\gamma}\Delta^{-\gamma}(y_0)]^2,$$

which is possible (assuming $\varepsilon < \varkappa$) only if $g(\xi) = 0$ almost everywhere in $|\xi - \xi_0| < \delta$. Since $\xi_0$ is an arbitrary point outside $C^* + \overline{U}_a$, it follows that $g(\xi) = 0$ almost everywhere outside $C^* + \overline{U}_a$ so that $\operatorname{supp} g \subset C^* + \overline{U}_a$.

In (5.6), set $y = t\sigma$, $t > 0$. We then get

$$t^{2\gamma} \int_{C^*+\overline{U}_a} | g(\xi) |^2 e^{-2t(\sigma,\xi)} (1+|\xi|^2)^s \, d\xi \leqslant M^2(\varepsilon)\, e^{2t(a+\varepsilon)} \left[ t^{\gamma} + \frac{1}{\Delta^{\gamma}(\sigma)} \right]^2, \qquad t > 0. \quad (5.7)$$

Let $\gamma = 0$. Passing to the limit in (5.7) as $t \to +0$ and using the Fatou lemma, we obtain

$$\int\limits_{C^*+\bar{U}_a} |g(\xi)|^2 (1+|\xi|^2)^s \, d\xi \leqslant 4M^2(\xi), \qquad \varepsilon > 0,$$

whence follows the inequality (5.3).

Now let $\gamma > 0$. Take into account the inequality $(\sigma, \xi) \leqslant |\xi|$, divide the inequality (5.7) through by $t^{1-2\delta}$, where $\delta$ is an arbitrary number $0 < \delta \leqslant 1$, integrate the resulting inequality with respect to $t$ on $(0,1)$, and take advantage of the Fubini theorem. Assuming $0 < \varepsilon \leqslant 1$, we then obtain the inequality

$$\int\limits_{C^*+\bar{U}_a} |g(\xi)|^2 (1+|\xi|^2)^s \int\limits_0^1 t^{2(\gamma+\delta)-1} e^{-2t|\xi|} \, dt \, d\xi$$
$$\leqslant \frac{1}{2\delta} e^{2(a+1)} M^2(\varepsilon) [1+\Delta^{-\gamma}(\sigma)]^2. \quad (5.8$$

Now, taking into account the estimate

$$\int\limits_0^1 t^{2(\gamma+\delta)-1} e^{-2t|\xi|} \, dt \geqslant \min(1, |\xi|^{-2(\gamma+\delta)}) \int\limits_0^1 u^{2\gamma+1} e^{-2u} \, du$$
$$\geqslant \frac{e^{-2}}{2(\gamma+1)} (1+|\xi|^2)^{-\gamma-\delta},$$

we derive from (5.8) the estimate

$$\int\limits_{C^*+\bar{U}_a} |g(\xi)|^2 (1+|\xi|^2)^{s-\gamma-\delta} \, d\xi$$
$$\leqslant \frac{e^{2(a+2)}}{\delta} (\gamma+1) M^2(\varepsilon) [1+\Delta^{-\gamma}(\sigma)]^2,$$

whence it follows that $g \in \mathcal{L}^2_{s'}(C^* + \bar{U}_a)$ for all $s' = s - \gamma - \delta < s - \gamma$ and the estimate (5.3) holds true. Finally, from (5.4) it follows that $f = L[g]$. The proof of the lemma is complete.

Theorem *For the function $f(z)$ to belong to the class $H_a^{(s)}(C)$, it is necessary and sufficient that its spectral function $g(\xi)$ belong to the class $\mathcal{L}^2_s(C^* + \bar{U}_a)$.*

*Here, the following equalities hold:*

$$\|f\|_a^{(s)} = \|f_+\|_s = \|g\|_{(s)}, \tag{5.9}$$

*where $f_+(x)$ is a boundary value in $\mathcal{H}_s$ of the function $f(z)$ as $y \to 0$, $y \in C$, and $f_+ = F[g]$.*

*Proof. Necessity.* Let $f \in H_a^{(s)}(C)$. Then from the lemma [for $\gamma = 0$ and $M(\varepsilon)$ independent of $\varepsilon$] it follows that $f(z) = L[g]$, where $g \in \mathscr{L}_s^2(C^* + \bar{U}_a)$.

*Sufficiency.* Let $f(z) = L[g]$, where $g \in \mathscr{L}_s^2(C^* + \bar{U}_a)$. By what has been proved (see Sec. 9.1), the function $f(z)$ is holomorphic in $T^{\operatorname{int} C^{**}} = T^C$ and is given by the integral

$$f(z) = \int\limits_{C^* + \bar{U}_a} g(\xi)\, e^{i(z, \xi)}\, d\xi = F[g(\xi)\, e^{-(y, \xi)}], \qquad z \in T^C. \tag{5.10}$$

Let us prove that $f \in H_a^{(s)}(C)$. Using the relations of the norms in the spaces $H_a^{(s)}(C)$, $\mathscr{H}_s$ and $\mathscr{L}_s^2$, we obtain from (5.10)

$$\begin{aligned}
\| f \|_a^{(s)2} &= \sup_{y \in C} e^{-2a|y|} \| f(x + iy) \|_s^2 \\
&= \sup_{y \in C} e^{-2a|y|} \| g(\xi)\, e^{-(y, \xi)} \|_{(s)}^2 \\
&= \sup_{y \in C} e^{-2a|y|} \int\limits_{C^* + \bar{U}_a} | g(\xi) |^2 e^{-2(y, \xi)} (1 + | \xi |^2)^s\, d\xi \\
&= \int | g(\xi) |^2 (1 + | \xi |^2)^s\, d\xi = \| g \|_{(s)}^2.
\end{aligned}$$

That is,

$$\| f \|_a^{(s)} = \| g \|_{(s)}. \tag{5.11}$$

We now prove that the function $f(z)$ assumes, when $y \to 0$, $y \in C$, a (unique) boundary value in $\mathscr{H}_s$ equal to $f_+ = F[g]$. This follows from the limiting relation

$$\begin{aligned}
\| f(x + iy) - F[g] \|_s^2 &= \| L[g](x + iy) - F[g](x) \|_s^2 \\
&= \int\limits_{C^* + \bar{U}_a} | g(\xi) |^2 [e^{-(y, \xi)} - 1]^2 (1 + | \xi |^2)^s\, d\xi \to 0, \\
&\qquad y \to 0, \quad y \in C.
\end{aligned}$$

Thus, $\| g \|_{(s)} = \| f_+ \|_s$, which together with (5.11) yields (5.9). The proof is complete.

Corollary 1 *The spaces* $H_a^{(s)}(C)$ and $\mathcal{L}_s^2(C^* + \bar{U}_a)$ *are (linearly) isomorphic and isometric, and the isomorphism is realized via the Laplace transformation* $g \to L[g] = f$.

Corollary 2 *Any function* $f(z)$ *in* $H_a^{(s)}(C)$ *has, for* $y \to 0$, $y \in C$, *a (unique) boundary value* $f_+(x)$ *in* $\mathcal{H}_s$ *and the correspondence* $f \to f_+$ *is isometric.*

*Remark.* The theorem on the existence of boundary values in $\mathcal{D}'_{\mathcal{L}^p}$ was proved by a different method by Tillmann [2] and Luszczki and Zieleźny [1] ($n = 1$).

Corollary 3 (an analogue of Liouville's theorem) *If the cone C is not acute and* $f \in H_0^{(s)}(C)$, *then* $f(z) \equiv 0$.

True enough, by Lemma 1 of Sec. 4.4, mes $C^* = 0$. In that case, as follows from the proof of the lemma, the function $g(\xi) = 0$ almost everywhere in $\mathbb{R}^n$ so that $f(z) = L[g] \equiv 0$.

**10.6 The generalized Cauchy-Bochner representation** Here we continue the investigation started in Subsection 5 when $a = 0$.

Theorem I *For a function* $f(z)$ *to belong to* $H^{(s)}(C)$, *it is necessary and sufficient that it possess the generalized Cauchy-Bochner integral representation*

$$\frac{1}{(2\pi)^n}(f_+(x'), \mathcal{K}_C(z-x')) = \begin{cases} f(z), & z \in T^C, \\ 0, & z \in T^{-C}, \end{cases} \tag{6.1}$$

*where* $f_+(x)$ *is a boundary value in* $\mathcal{H}_s$ *of the function* $f(z)$ *as* $y \to 0$, $y \in C$.

*Proof. Necessity.* Let $f \in H^{(s)}(C)$. By the theorem of Sec. 10.5, $f(z)$ is the Laplace transform of the function $g$ in $\mathcal{L}_s^2(C^*)$ so that

$$f(z) = F[g(\xi)e^{-(y,\xi)}\theta_{C^*}(\xi)], \qquad z \in T^C,$$

$$0 = F[g(\xi)e^{-(y,\xi)}\theta_{-C^*}(\xi)], \qquad z \in T^{-C}.$$

From this fact, using the definition of the kernel $\mathcal{K}_C(z)$ [see (2.1)] and using (1.7) and (1.9) for the convolution, we obtain the representation (6.1):

$$f(z) = \frac{1}{(2\pi)^n}F[g] * \mathcal{K}_C = \frac{1}{(2\pi)^n}(f_+(x'), \mathcal{K}_C(z-x')), \qquad z \in T^C,$$

$$0 = \frac{1}{(2\pi)^n}F[g] * \mathcal{K}_{-C} = \frac{(-1)^n}{(2\pi)^n}(f_+(x'), \mathcal{K}_C(z-x')), \qquad z \in T^{-C}.$$

Here we made use of one of the equalities of (2.3): $\mathcal{K}_{-C}(z) = (-1)^n \mathcal{K}_C(z)$, and also used the relation $f_+ = F[g]$.

*Sufficiency*. Suppose $f(z)$ has the representation (6.1). Then $f \in H^{(s)}(C)$ (see Sec. 10.3). Theorem I is proved.

*Remark*. For $s = 0$, Theorem I becomes the Bochner theorem [2]; for $n = 1$, Theorem I was obtained by Beltrami and Wohlers [1]; for arbitrary $n$ and $s$ see Vladimirov [3].

Theorem II *The following statements are equivalent*:

(1) $f_+$ *is a boundary value in* $\mathcal{H}_s$ *of some function taken from* $H^{(s)}(C)$;

(2) $f_+$ *belongs to* $\mathcal{H}_s$ *and satisfies the relations*

$$\operatorname{Re} f_+ = -\frac{2}{(2\pi)^n} \operatorname{Im} f_+ * \operatorname{Im} \mathcal{K}_C, \qquad \operatorname{Im} f_+ = \frac{2}{(2\pi)^n} \operatorname{Re} f_+ * \operatorname{Im} \mathcal{K}_C. \tag{6.2}$$

*That is*, $\operatorname{Re} f_+$ *and* $\operatorname{Im} f_+$ *form a pair of Hilbert transforms*;

(3) $f_+$ *belongs to* $\mathcal{H}_s$ *and* supp $F^{-1}[f_+] \subset C^*$.

*Proof*. (1) $\to$ (2). Let $f_+(x)$ be a boundary value in $\mathcal{H}_s$ of the function $f(z)$ taken from $H^{(s)}(C)$. Then, by Theorem I, $f_+ \in \mathcal{H}_s$ and for $f(z)$ the generalized Cauchy-Bochner representation (6.1) holds true, from which follow, by (3.8), the relations

$$f_+ = \frac{1}{(2\pi)^n} f_+ * \mathcal{K}_C, \qquad 0 = \frac{1}{(2\pi)^n} f_+ * \overline{\mathcal{K}}_C. \tag{6.3}$$

From this we obtain relations (6.2) by separating the real and imaginary parts.

(2) $\to$ (3). Let $f_+$ in $\mathcal{H}_s$ satisfy the relation (6.2). Then it will also satisfy (6.3). Applying the inverse Fourier transform to the first of the relations (6.3) and making use of (2.6), we obtain

$$\begin{aligned} F^{-1}[f_+] &= F^{-1}[f_+]\, F^{-1}[\mathcal{K}_C] \\ &= F^{-1}[f_+]\, \theta_{C^*}(\xi), \end{aligned}$$

whence it follows that supp $F^{-1}[f_+] \subset C^*$.

(3) $\to$ (1). If $f_+ \in \mathcal{H}_s$ and supp $F^{-1}[f_+] \subset C^*$, then $F^{-1}[f_+] \in \mathcal{L}_s^2(C^*)$. By the theorem of Sec. 10.5, the function $f(z) = L[g] \in H^{(s)}(C)$ and the boundary value of it in $\mathcal{H}_s$ is equal to $F[g] = f_+$.

Theorem II is proved.

*Remark*. For $n = 1$, $C = (0, \infty) = \mathbb{R}_+^1$, the formulas (6.2) take on the following form:

$$\operatorname{Re} f_+ = -\frac{1}{\pi} \operatorname{Im} f_+ * \mathcal{P}\frac{1}{x}, \qquad \operatorname{Im} f_+ = \frac{1}{\pi} \operatorname{Re} f_+ * \mathcal{P}\frac{1}{x}. \tag{6.4}$$

In physics, the formulas (6.4) are called dispersion relations (without subtraction). It is natural to regard (6.2) as a generalization of the dispersion relations to the multidimensional case with causality with respect to an arbitrary convex acute closed cone $C^*$.

## 11 Poisson Kernel and Poisson Transform

**11.1 The definition and properties of the Poisson kernel** Let $C$ be a convex acute open cone in $\mathbb{R}^n$ (with vertex at 0). The function

$$\mathcal{P}_C(x, y) = \frac{|\mathcal{K}_C(x+iy)|^2}{(2\pi)^n \mathcal{K}_C(2iy)}, \qquad (x, y) \in T^C \tag{1.1}$$

is termed the *Poisson kernel* of the tubular region $T^C$. Here, $\mathcal{K}_C$ is the Cauchy kernel (see Sec. 10.2).

*Example 1* [see (2.16) of Sec. 10].

$$\mathcal{P}_{\mathbb{R}^n_+}(x, y) = \frac{y_1 \cdots y_n}{\pi^n |z_1|^2 \cdots |z_n|^2} \equiv \mathcal{P}_n(x, y). \tag{1.2}$$

*Example 2* [see (2.17) of Sec. 10].

$$\mathcal{P}_{V^+}(x, y) = \frac{2^n \Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+3}{2}}} \frac{(y^2)^{\frac{n+1}{2}}}{|(x+iy)^2|^{n+1}}. \tag{1.3}$$

The following is a list of the properties of the Poisson kernel $\mathcal{P}_C$ that follow from the corresponding properties of the Cauchy kernel $\mathcal{K}_C$ (see Sec. 10.2):

(a)
$$0 \leqslant \mathcal{P}_C(x, y) = \mathcal{P}_C(-x, y) \in C^\infty(T^C) \tag{1.4}$$

follows from the holomorphicity of the kernel $\mathcal{K}_C(z)$ in $T^C$ and from the fact that $\mathcal{K}_C(2iy) > 0$, $y \in C$.

(b)
$$\int \mathcal{P}_C(x, y)\, dx = 1, \qquad y \in C, \tag{1.5}$$

follows from the Parseval equation applied to (2.1) of Sec. 10:

$$\int |\mathcal{K}_C(x+iy)|^2\, dx = (2\pi)^n \int_{C^*} e^{-2(y, \xi)}\, d\xi = (2\pi)^n K_C(2iy), \qquad y \in C;$$

(c)
$$\mathcal{P}_C(x, y) \leqslant \frac{\mathcal{K}_C^2(iy)}{(2\pi)^n \mathcal{K}_C(2iy)}, \qquad (x, y) \in T^C \tag{1.6}$$

follows from the estimate

$$|\mathcal{K}_C(x, iy)| \leqslant \int_{C^*} e^{-(y,\xi)}\,d\xi = \mathcal{K}_C(iy);$$

(d) $\|\mathcal{P}_C(x, y)\|_{\mathcal{L}^p}$

$$\leqslant \frac{\mathcal{K}_C^{2-\frac{2}{p}}(iy)}{(2\pi)^{n\left(1-\frac{1}{p}\right)}\mathcal{K}_C^{1-\frac{1}{p}}(2iy)}, \qquad y \in C, \quad 1 \leqslant p \leqslant \infty, \qquad (1.7)$$

follows from (1.4)-(1.6) by virtue of

$$\|\mathcal{P}_C(x, y)\|_{\mathcal{L}^p}^p = \int \mathcal{P}_C^p(x, y)\,dx \leqslant \sup_x \mathcal{P}_C^{p-1}(x, y)\int \mathcal{P}_C(x, y)\,dx$$

$$\leqslant \frac{\mathcal{K}_C^{2p-2}(iy)}{(2\pi)^{n(p-1)}\mathcal{K}_C^{p-1}(2iy)};$$

(e) $0 < F_x[\mathcal{P}_C(x, y)](\xi)$

$$= \frac{[\theta_{C^*}(\xi)\,e^{-(y,\xi)}] * [\theta_{C^*}(-\xi)\,e^{(y,\xi)}]}{\mathcal{K}_C(2iy)} \leqslant 1 \qquad (1.8)$$

$$\xi \in \mathbb{R}^n, \qquad y \in C,$$

follows from the Fourier transform formula of a convolution and from the fact that

$$F_x^{-1}[\mathcal{K}_C(z)] = \theta_{C^*}(\xi)\,e^{-(y,\xi)} \in \mathcal{L}^1,$$

$$F_x^{-1}[\overline{\mathcal{K}_C}(z)] = \theta_{C^*}(-\xi)\,e^{(y,\xi)} \in \mathcal{L}^1;$$

(f) $$\mathcal{P}_C(x, y) \gg 0$$

is a continuous positive definite function for all $y \in C$ (see Sec. 8.2); it follows from (1.8) and from the fact that

$$[\theta_{C^*}(\xi)e^{-(y,\xi)}] * [\theta_{C^*}(-\xi)\,e^{(y,\xi)}] \in \mathcal{L}^1, \qquad y \in C;$$

(g) $$[\theta_{C^*}(\xi)\,e^{-(y,\xi)}] * [\theta_{C^*}(-\xi)\,e^{(y,\xi)}] \gg 0 \qquad (1.9)$$

is a continuous positive definite function for all $y \in C$; it follows from (1.4) and (1.8);

(h) $F_x[\mathcal{P}_C(x, y)](\xi) = e^{-|(y, \xi)|}, \quad \xi \in -C^* \cup C^*, \quad y \in C, \qquad (1.10)$

follows from (1.8) by virtue of the following manipulations for $\xi \in C^*$:

$$F_x[\mathcal{P}_C(x, y)](\xi) = \frac{1}{\mathcal{K}_C(2iy)} \int\limits_{\substack{-\xi' \in C^* \\ \xi-\xi' \in C^*}} e^{-(y, \xi-\xi')+(y, \xi')}\, d\xi'$$

$$= \frac{1}{\mathcal{K}_C(2iy)} e^{-(y, \xi)} \int\limits_{-C^*} e^{2(y, \xi')}\, d\xi' = e^{-(y, \xi)}.$$

But if $\xi \in -C^*$, then the equation being proved remains true because of the evenness of the kernel $\mathcal{P}_C(x, y)$ with respect to $x$;

(i) $|D_x^\alpha \mathcal{P}_C(x, y)| \leqslant M'_\alpha |y|^n \Delta^{-2n-|\alpha|}(y), \quad (x, y) \in T^C; \qquad (1.11)$

it follows from the estimates (2.4) of Sec. 10 and from the estimate

$$\mathcal{K}_C(2iy) = \int\limits_{C^*} e^{-2(y, \xi)}\, d\xi \geqslant \int\limits_{C^*} e^{-2|y||\xi|}\, d\xi = \frac{\varkappa}{|y|^n}, \qquad \varkappa > 0 \qquad (1.12)$$

(j) $\| D_x^\alpha \mathcal{P}_C(x, y) \|_s$

$$\leqslant K_{s,\alpha,p}[1+\Delta^{-s}(y)][1+\Delta^{-p}(y)]\Delta^{-n-|\alpha|}(y)|y|^n,$$

$$y \in C, \quad s \geqslant 0, \quad p > s + n/2, \qquad (1.13)$$

so that $\mathcal{P}_C(x, y) \in \mathcal{H}_s$ for all $s$ and $y \in C$.

This follows from the inequalities (1.11) and (2.5) of Sec. 10 and from the estimate (1.12):

$$\|D_x^\alpha \mathcal{P}_C(x, y)\|_s = \frac{1}{(2\pi)^n \mathcal{K}_C(2iy)} \| D^\alpha \mathcal{K}_C(x+iy) \overline{\mathcal{K}_C(x+iy)} \|_s$$

$$\leqslant \frac{|y|^n}{(2\pi)^n \varkappa} \left\| \sum_\beta \binom{\alpha}{\beta} D^\beta \mathcal{K}_C(x+iy) \overline{D^{\alpha-\beta}\mathcal{K}_C(x+iy)} \right\|_s$$

$$\leqslant \frac{|y|^n}{(2\pi)^n \varkappa} c_{p-s} \sum_\beta \binom{\alpha}{\beta} \| D^\beta \mathcal{K}_C(x+iy\|_s \| D^{\alpha-\beta}\mathcal{K}_C(x+iy)\|_p$$

$$\leqslant \frac{c_{p-s}}{(2\pi)^n \varkappa} \sum_\beta \binom{\alpha}{\beta} K_{s,\beta} K_{(p,\alpha-\beta)} [1+\Delta^{-s}(y)][1+\Delta^{-p}(y)]$$

$$\times \Delta^{-n-|\alpha|}(y)|y|^n.$$

**11.2 The Poisson transform and Poisson representation** Let $f \in \mathscr{H}_s$, $-\infty < s < \infty$. We call the convolution [see (1.10) of Sec. 10]

$$\mathscr{F}(x, y) = f(x) * \mathscr{P}_C(x, y)$$
$$= (f(x'), \mathscr{P}_C(x - x', y)), \qquad (x, y) \in T^C, \qquad (2.1)$$

the *Poisson transform* (*or integral*).

By virtue of (j) of Sec. 11.1, the Poisson integral exists for every $y \in C$ and is a continuous operation from $\mathscr{H}_s$ to $\mathscr{H}_s$.

*Example.* If $f \in \mathscr{L}^2 = \mathscr{H}_0$, then the Poisson integral becomes the classical Poisson integral:

$$\mathscr{F}(x, y) = \int f(x') \mathscr{P}_C(x - x', y)\, dx'.$$

The following is a partial list of the properties of the Poisson integral.

(a) $$\mathscr{F}(x, y) \in C^\infty(T^C). \qquad (2.2)$$

This follows from (1.4) and from (1.13).

(b) $$\| \mathscr{F}(x, y) \|_s^2 \leqslant \| f \|_s^2, \qquad y \in C. \qquad (2.3)$$

This follows from (1.8) by virtue of the following manipulations:

$$\| \mathscr{F}(x, y) \|_s^2 = \| F_x^{-1}[\mathscr{F}(x, y)] \|_{(s)}^2 = \| F[f] F_x^{-1}[\mathscr{P}_C(x, y)] \|_{(s)}^2 \leqslant \| f \|_s^2.$$

(c) Theorem (generalized Poisson representation) *For* $f(z)$ *to belong to* $H^{(s)}(C)$, *it is necessary and sufficient that it be uniquely represented as the Poisson integral*

$$f(z) = (\chi(x'), \mathscr{P}_C(x - x', y)), \qquad z \in T^C, \qquad (2.4)$$

*where* $\chi \in \mathscr{H}_s$ *and* supp $F^{-1}[\chi] \subset C^*$; *here*, $\chi = f_+$ *where* $f_+(x)$ *is the boundary value in* $\mathscr{H}_s$ *of the function* $f(z)$ *as* $y \to 0$, $y \in C$.

*Proof. Necessity.* Since $f \in H^{(s)}(C)$, it follows, by the theorem of Sec. 10.5, that there is a function $g \in \mathscr{L}_s^2(C^*)$ such that $f_+ = F[g] \in \mathscr{H}_s$ and

$$f(z) = F[g(\xi) e^{-(y, \xi)}](x), \qquad z \in T^C. \qquad (2.5)$$

From this, using (1.10), we obtain for the function $f(z)$ the generalized Poisson representation (2.4):

$$f(z) = F[g(\xi) F_x [\mathcal{P}_C(x, y)](\xi)] = F[g](x) * \mathcal{P}_C(x, y)$$
$$= f_+(x) * \mathcal{P}_C(x, y) = (f_+(x'), \mathcal{P}_C(x - x', y)), \quad z \in T^C.$$

The generalized Poisson representation (2.4) is unique since, by (1.8), $F_x^{-1}[\mathcal{P}_C(x, y)](\xi) \neq 0$, $\xi \in \mathbb{R}^n$, $y \in C$.

*Sufficiency.* Suppose a generalized function $\chi$ is such that $g = F^{-1}[\chi] \in \mathcal{L}_s^2(C^*)$. Then by the theorem of Sec. 10.5 the function $f(z)$ defined by (2.5) belongs to $H^{(s)}(C)$ and, by what has been proved, can be represented by the integral (2.4) with $\chi = F[g] = f_+$. This completes the proof of the theorem.

Corollary 1 *Under the hypothesis of the theorem, we get*

$$\operatorname{Re} f(z) = (\operatorname{Re} f_+(x'), \quad \mathcal{P}_C(x - x', y)),$$
$$\operatorname{Im} f(z) = (\operatorname{Im} f_+(x'), \quad \mathcal{P}_C(x - x', y)). \qquad (2.6)$$

Corollary 2 *If $f(x)$ is a real generalized function in $\mathcal{H}_s$ and* $\operatorname{supp} F[f] \subset -C^* \cup C^*$, *then the function*

$$u(x, y) = (f(x'), \mathcal{P}_C(x - x', y)) \qquad (2.7)$$

*is a real part of some function of the class $H^{(s)}(C)$ and assumes, in the sense of $\mathcal{H}_s$ as $y \to 0$, $y \in C$, the value of $f(x)$.*

Indeed, putting

$$f_+(x) = F[\theta_{C^*}(\xi) F^{-1}[f](\xi)](x),$$

we obtain

$$f_+ \in \mathcal{H}_s, \qquad \operatorname{supp} F^{-1}[f_+] \subset C^* \quad \text{and} \quad f = 2\operatorname{Re} f_+$$

so that

$$u(x, y) = 2\operatorname{Re}(f_+(x'), \mathcal{P}_C(x - x', y)).$$

From this and from the theorem follow the required assertions.

*Example.* The function $\mathcal{K}_C(z + iy')$ belongs to the class $H^{(s)}(C)$ for all $y' \in C$ and $s$ [see estimate (2.5) of Sec. 10 in which $\Delta(y + y') \geqslant \Delta(y')$, $y \in C$]. Suppose $C'$ is an arbitrary (convex open) subcone of the cone $C$, $C' \subset C$. Applying (2.4) to the function $\mathcal{K}_C(z + iy')$ of the class $H^{(s)}(C')$, we obtain

$$\mathcal{K}_C(z + iy') = \int \mathcal{K}_C(x' + iy') \mathcal{P}_{C'}(x - x', y)\, dx',$$
$$(x, y) \in T^{C'}, \quad y' \subset C. \qquad (2.8)$$

From this, using the Cauchy-Bunyakovsky inequality and (1.5), we obtain the following inequality:

$$|\mathcal{K}_C(z+iy')|^2 \leqslant \int |\mathcal{K}_C(x'+iy')|^2 \mathcal{P}_{C'}(x-x', y)\,dx' \times \int \mathcal{P}_{C'}(x-x', y)\,dx' = \int \mathcal{P}_{C'}(x-x', y)|\mathcal{K}_C(x'+iy')|^2\,dx'. \tag{2.9}$$

In terms of the Poisson kernel (1.1), the inequality (2.9) takes the form

$$\mathcal{P}_C(x, y+y') \leqslant \frac{\mathcal{K}_C(iy')}{\mathcal{K}_C(iy+iy')} \int \mathcal{P}_{C'}(x-x', y)\,\mathcal{P}_C(x', y')\,dx', \quad (x, y) \in T^{C'}, \quad y' \in C. \tag{2.10}$$

In particular, for $C' = C$, $y' = y$ the formula (2.10) assumes the form

$$\mathcal{P}_C(x, 2y) \leqslant 2^n \int \mathcal{P}_C(x-x', y)\,\mathcal{P}_C(x', y)\,dx', \qquad (x, y) \in T^C. \tag{2.11}$$

Here we made use of the property of homogeneity (of degree $-n$) of the kernel $\mathcal{K}_C$ [see (2.3) of Sec. 10].

### 11.3 Boundary values of the Poisson integral

(a) $$\int \mathcal{P}_C(x, y)\,\varphi(x)\,dx \to \varphi(0), \quad y \to 0, \quad y \in C, \tag{3.1}$$

for any function $\varphi \in \mathcal{L}^\infty$ continuous in 0.

By virtue of (1.5), it suffices, when proving this assertion, to establish the following limiting relation: for any $\delta > 0$

$$\int\limits_{|x|>\delta} \mathcal{P}_C(x, y)\,dx \to 0, \qquad y \to 0, \quad y \in C. \tag{3.2}$$

Let us construct an auxiliary function $\omega(x)$ with the properties:

(1) $\omega$ is a real continuous function in $\mathbb{R}^n$, $\omega(x) \to 0$, $|x| \to \infty$;

(2) $\omega(0) = 1$, $|\omega(x)| < 1$, $x \neq 0$;

(3) $\int \mathcal{P}_C(x, y)\,\omega(x)\,dx \to 1$, $y \to 0$, $y \in C$.

Suppose $\eta \in \mathcal{D}$ (int $C^*$), $\eta \geqslant 0$, $\int \eta(\xi)\, d\xi = 1$ (int $C^* \neq \varnothing$ since $C$ is an acute cone; see Sec. 4.4). The function

$$\omega(x) = \operatorname{Re} \int \eta(\xi)\, e^{i(x,\xi)} d\xi = \int \eta(\xi) \cos(x, \xi)\, d\xi$$

possesses the required properties (1) to (3). Indeed, property (1) follows from the Riemann-Lebesgue theorem. Let us prove property (2). It is clear that $\omega(0) = 1$ and $|\omega(x)| \leqslant 1$. Suppose $\omega(x_0) = \pm 1$, $x_0 \neq 0$, that is, $1 = \pm \int \eta(\xi) \cos(x_0, \xi)\, d\xi$; but this contradicts the hypothesis $\int \eta(\xi)\, d\xi = 1$. Property (3) follows from the corollary to the theorem of Sec. 11.2 by virtue of which

$$\operatorname{Re} \int \eta(\xi)\, e^{i(z,\xi)}\, d\xi = \int \mathcal{P}_C(x - x', y)\, \omega(x')\, dx', \qquad z \in T^C.$$

Putting $x = 0$ here, then taking into account (1.4) and passing to the limit as $y \to 0$, $y \in C$, we obtain relation (3).

Suppose $\delta > 0$. By the properties (1) and (2) there exists a number $\varepsilon > 0$ such that $|\omega(x)| \leqslant 1 - \varepsilon$, $|x| \geqslant \delta$. From this fact, taking into account property (3), we obtain

$$1 = \lim_{y \to 0,\, y \in C} \left[ \int_{|x| \leqslant \delta} \mathcal{P}_C(x, y)\, \omega(x)\, dx + \int_{|x| > \delta} \mathcal{P}_C(x, y)\, \omega(x)\, dx \right]$$

$$\leqslant \lim_{y \to 0,\, y \in C} \left[ \int_{|x| \leqslant \delta} \mathcal{P}_C(x, y)\, dx + (1 - \varepsilon) \int_{|x| > \delta} \mathcal{P}_C(x, y)\, dx \right]$$

$$\leqslant \lim_{y \to 0,\, y \in C} \left[ 1 - \varepsilon \int_{|x| > \delta} \mathcal{P}_C(x, y)\, dx \right],$$

which completes the proof of relation (3.2).

(b) If $f \in \mathcal{H}_s$, then its Poisson integral

$$\mathcal{F}(x, y) \to f(x), \quad y \to 0, \quad y \in C \text{ in } \mathcal{H}_s. \tag{3.3}$$

Indeed, by (1.8) and (3.1),

$$|F_x[\mathcal{P}_C(x, y)](\xi) - 1|^2 \leqslant 4, \quad \xi \in \mathbb{R}^n, \quad y \in C;$$

$$|F_x[\mathcal{P}_C(x, y)](\xi) - 1|^2$$

$$= \left| \int \mathcal{P}_C(x, y)\, e^{-i(x,\xi)} dx - 1 \right|^2 \to 0, \quad y \to 0, \quad y \in C.$$

For this reason [compare (2.3)], by the Lebesque theorem, as $y \to 0$, $y \in C$, we have

$$\| \mathcal{F}(x, y) - f(x) \|_s^2 = \| F_x^{-1} [\mathcal{F}(x, y) - f(x)] \|_{(s)}^2$$
$$= \int |F_x [\mathcal{P}_C (x, y)](\xi) - 1|^2 |F^{-1}[f](\xi)|^2 (1 + |\xi|^2)^s \, d\xi \to 0,$$

which is what we set out to prove.

(c) $\mathcal{P}_C(x, y) \to \delta(x)$, $y \to 0$, $y \in C$ in $\mathcal{H}_s$, $s < -n/2$; this follows from (3.3) since $\delta \in \mathcal{H}_s$ for all $s < -n/2$ and

$$\mathcal{P}_C(x, y) = \delta(x) * \mathcal{P}_C(x, y) \to \delta(x), \quad y \to 0, \quad y \in C \quad \text{in } \mathcal{H}_s.$$

(d) In the case of an $n$-hedral cone (see Sec. 4.4)

$$C = [y: (y, e_1) > 0, \ldots, (y, e_n) > 0]$$

the limiting relation (3.1) admits of extension to a more general class of functions $\varphi(x)$, namely: if $\varphi(x)$ is continuous in 0 and such that the integral

$$\int \mathcal{P}_C(x, y) |\varphi(x)| \, dx \leqslant K, \qquad y \in C, \quad |y| < a \tag{3.4}$$

is bounded, then

$$\int \mathcal{P}_C(x, y) \varphi(x) \, dx \to \varphi(0), \qquad y \to 0, \quad y \in C. \tag{3.5}$$

Indeed, since the (nonsingular) linear mapping

$$z \to Tz = [(z, e_1), \ldots, (z, e_n)]$$

carries the region $T^C$ onto the region $T^n$ [see (2.16) of Sec. 10], it follows that it suffices to prove the assertion for the cone $\mathbb{R}_+^n$ and the kernel

$$\mathcal{P}_n(x, y) = \prod_{\leqslant j \leqslant n} \frac{y_j}{\pi} \frac{1}{x_j^2 + y_j^2} \text{ [see (1.2)].}$$

By virtue of (1.5), we only need to prove that for all $\delta > 0$ ($\delta < \sqrt{n}$)

$$\int\limits_{|x| > \delta} \mathcal{P}_n(x, y) |\varphi(x)| \, dx \to 0, \qquad y \to 0, \quad y \in \mathbb{R}_+^n. \tag{3.6}$$

But the limiting relation (3.6) follows from the estimate

$$\frac{1}{x_k^2 y + y_k^2} \leqslant \frac{2}{x_k^2 + (\delta^2 a^2)/n}, \qquad |x_k| > \frac{\delta a}{\sqrt{n}}$$

and from the estimate (3.4) by virtue of the following chain of inequalities for $|y| < a\sqrt{1 - \delta^2/n}$, $y \in \mathbb{R}_+^n$:

$$\int\limits_{|x|>\delta} \mathcal{P}_n(x, y)|\varphi(x)|\,dx \leqslant \sum_{1\leqslant k\leqslant n} \frac{y_1 \cdots y_n}{\pi^n} \int\limits_{|x_k| > \frac{\delta a}{\sqrt{n}}} \frac{|\varphi(x)|\,dx}{(x_1^2 + y_1^2) \cdots (x_n^2 + y_n^2)}$$

$$\leqslant 2 \sum_{1\leqslant k\leqslant n} \frac{y_1 \cdots y_n}{\pi^n} \int \frac{|\varphi(x)|\,dx}{(x_1^2 + y_1^2) \cdots (x_k^2 + \delta^2 a^2/n) \cdots (x_n^2 + y_n^2)}$$

$$\leqslant 2\frac{\sqrt{n}}{\delta a} \sum_{1\leqslant k\leqslant n} y_k \int \mathcal{P}_n(x, y_1, \ldots, \delta a/\sqrt{n}, \ldots, y_n)|\varphi(x)|\,dx$$

$$\leqslant 2\frac{\sqrt{n}}{\delta a} K \sum_{1\leqslant k\leqslant n} y_k.$$

(e) The following holds for an $n$-hedral cone $C$ [see (d)]: if $P(x)$ is a polynomial and the integral

$$\int |P(x)|\,\mathcal{P}_C(x, y)\,dx < \infty$$

for some $y \in C$, then $P(x) = \text{constant}$.

Indeed, as in (d), it suffices to prove this assertion for the cone $\mathbb{R}_+^n$ and the kernel $\mathcal{P}_n(x, y)$. For $n = 1$ it is readily demonstrated by induction on the degree of the polynomial $P$. Then we apply induction on $n$: let

$$P(x) = x_n^m P_m(\tilde{x}) + \ldots + x_n P_1(\tilde{x}) + P_0(\tilde{x}),$$
$$\tilde{x} = (x_1, \ldots, x_{n-1}).$$

Then by the Fubini theorem the integral

$$\int\limits_{-\infty}^{\infty} |x_n^m P_m(\tilde{x}) + \ldots + P_0(\tilde{x})|\,\mathcal{P}_1(x_n, y_n)\,dx_n < \infty$$

for almost all $\tilde{x} \in \mathbb{R}^{n-1}$ and therefore, by what has been proved, $P(x) = P_0(\tilde{x})$, and so forth.

*Remark 1.* The question arises as to whether the assertions of (d) and (e) hold true for an arbitrary acute (convex) cone $C$. The assertion (e) has been proved for the cone $V^+$ ($n = 4$) by Vladimirov [10(II)]

*Remark 2.* For $s = 0$, that is, in $\mathcal{H}_0 = \mathcal{L}^2$ (and in $\mathcal{L}^p$, $1 \leqslant p \leqslant \infty$), the theory is given (by use of a different method) in Stein and Weiss [1].

## 12 Algebras of Holomorphic Functions

In this section we give an internal description of the Laplace transform of generalized functions from the algebras $\mathcal{S}'(C^* +)$ and $\mathcal{S}'(C^*)$ in a manner similar to that in Sec. 10.5 for functions from $\mathcal{L}_s^2(C^* + \overline{U}_a)$.

**12.1 The definition of the $H_+(C)$ and $H(C)$ algebras** Let $C$ be a connected open cone with vertex at 0. Denote by $H_a^{(\alpha,\beta)}(C)$, $\alpha \geqslant 0, \beta \geqslant 0$, $a \geqslant 0$, the set of all functions $f(z)$ that are holomorphic in $T^C$ and that satisfy the following growth condition:

$$|f(z)| \leqslant Me^{a|y|}(1 + |z|^2)^{\alpha/2}[1 + \Delta^{-\beta}(y)], \quad z \in T^C. \tag{1.1}$$

We introduce the convergence (topology) in $H_a^{(\alpha,\beta)}(C)$ in accordance with the estimate (1.1) by means of the norm

$$\|f\|_a^{(\alpha,\beta)} = \sup_{z\in T^C} \frac{|f(z)|e^{-a|y|}}{(1+|z|^2)^{\alpha/2}[1+\Delta^{-\beta}(y)]}.$$

The spaces $H_a^{(\alpha,\beta)}(C)$ are Banach spaces and

$$H_a^{(\alpha,\beta)}(C) \subset H_{a'}^{(\alpha',\beta')}(C), \qquad \alpha' \geqslant \alpha, \quad \beta' \geqslant \beta, \quad a' \geqslant a, \tag{1.2}$$

with the inclusion (1.2) to be understood together with the appropriate topology, by virtue of the obvious inequality

$$\|f\|_a^{(\alpha,\beta)} \leqslant \|f\|_{a'}^{(\alpha',\beta')}. \tag{1.3}$$

We set

$$H_a(C) = \bigcup_{\alpha\geqslant 0,\ \beta\geqslant 0} H_a^{(\alpha,\beta)}(C), \qquad H_+(C) = \bigcup_{a\geqslant 0} H_a(C),$$

$$H_a^0(C) = \bigcup_s H_a^{(s)}(C), \qquad H_+^0(C) = \bigcup_{a\geqslant 0} H_a^0(C).$$

The set $H_+(C)$ forms an algebra of functions that are holomorphic in $T^C$ and that satisfy the estimate (1.1) for certain $a \geqslant 0$, $\alpha \geqslant 0$ and $\beta \geqslant 0$ relative to the operation of ordinary multiplication. This algebra is associative, commutative, contains a unit element but does not contain divisors of zero. Furthermore, $H_0(C) = H(C)$ is a subalgebra of the algebra $H_+(C)$ and contains the unit element. We endow the spaces $H_a(C)$, $H_+(C)$, $H_a^0(C)$ and $H_+^0(C)$ with a topology of the inductive limit (union) of the increasing sequence of spaces $H_a^{(\alpha,\beta)}(C)$, $H_a(C)$, $H_a^{(s)}(C)$ and $H_a^0(C)$, respectively (see Dieudonné and Schwartz [1], Bourbaki [1]). In what follows, we will drop the index 0 for $a = 0$.

**12.2 Isomorphism of the algebras $\mathscr{S}'(C^*+) \sim H_+(C)$ and $\mathscr{S}'(C^*) \sim H(C)$** Let $C$ be an acute convex cone. By Lemma 1 of Sec. 4.4 int $C^* \neq \varnothing$. We choose an (arbitrary) basis $e_1, \ldots, e_n$ in $\mathbb{R}^n$ such that $e_j \in \operatorname{pr} \operatorname{int} C^*$, $j = 1, \ldots, n$. Then we construct the polynomial

$$l(z) = (e_1, z) \ldots (e_n, z).$$

We will say that $l(z)$ is an *admissible polynomial for the cone $C$*. Since $(e_j, y) > 0$ for all $y \in C$, it follows that

$$l(z) = [(e_1, x) + i(e_1, y)] \ldots [(e_n, x) + i(e_n, y)] \neq 0, \quad z \in T^C. \tag{2.1}$$

We now convince ourselves that the following lemma holds true:

Lemma *Let the function $f(z)$ be holomorphic in $T^C$ and let it satisfy the following growth condition: for any number $\varepsilon > 0$ there is a number $M(\varepsilon)$ such that*

$$|f(x+iy)| \leqslant M(\varepsilon) e^{(a+\varepsilon)|y|}(1+|z|^2)^{\alpha/2}[1+\Delta^{-\beta}(y)], \quad z \in T^C \tag{2.2}$$

*for certain $a \geqslant 0$, $\alpha \geqslant 0$ and $\beta \geqslant 0$ (that depend solely on $f$). Then $f(z)$ can, for $\delta > \alpha + n/2$, be represented in the form*

$$f(z) = l^\delta(z) f_\delta(z), \quad f_\delta \in H_a^{(s)}(C), \quad s < -\beta - n(\delta - 1/2), \tag{2.3}$$

$$\|f_\delta\|_a^{(s)} \leqslant K_{s,\delta} \inf_{0<\varepsilon\leqslant 1} M(\varepsilon) \inf_{\sigma \in \operatorname{pr} C} [1 + \Delta^{-\beta-n(\delta-1/2)}(\sigma)]. \tag{2.4}$$

*Here, $l(z)$ is any admissible polynomial for the cone $C$.*

*Proof*. By (2.1) the function

$$f_\delta(z) = f(z)\, l^{-\delta}(z)$$

is holomorphic in $T^C$. Since $(e_j, y) \geqslant \sigma \mid y \mid$, $j = 1, \ldots, n$, $y \in C$, for some $\sigma > 0$ [see (4.1) of Sec. 4], it follows that

$$\mid (l, z) \mid^2 = [(e_1, x)^2 + (e_1, y^2)] \ldots [(e_n, x)^2 + (e_n, y^2)]$$

$$\geqslant [(e_1, x)^2 + \sigma^2 \mid y \mid^2] \ldots [(e_n, x)^2 + \sigma^2 \mid y \mid^2]$$

$$\geqslant (\sigma \mid y \mid)^{2n-2} [(e_1, x)^2 + \ldots + (e_n, x)^2 + \sigma^2 \mid y \mid^2], \quad z \in T^C. \tag{2.5}$$

Since the vectors $e_1, \ldots, e_n$ are linearly independent, there is a number $b > 0$ such that

$$(e_1, x)^2 + \ldots + (e_n, x)^2 \geqslant b^2 \mid x \mid^2.$$

From this, continuing the estimates (2.5), we obtain

$$\mid l (x + iy) \mid^2 \geqslant (\sigma \mid y \mid)^{2n-2} [b^2 \mid x \mid^2 + \sigma^2 \mid y \mid^2], \qquad z \in T^C.$$

Taking into account the estimate thus obtained and the estimate (2.2), we have, for all $z \in T^C$,

$$\mid f_\delta (x + iy) \mid^2 = \mid f (x + iy) \mid^2 \mid l (x + iy) \mid^{-2\delta}$$

$$\leqslant M^2 (\varepsilon) e^{2 (a+\varepsilon) \mid y \mid} \frac{(1 + \mid x + iy \mid^2)^\alpha [1 + \Delta^{-\beta} (y)]^2}{(\sigma \mid y \mid)^{2\delta (n-1)} [b^2 \mid x \mid^2 + \sigma^2 \mid y \mid^2]^\delta}$$

$$\leqslant K_1^2 M^2 (\varepsilon) e^{2 (a+\varepsilon) \mid y \mid} \frac{[1 + \Delta^\beta (y)]^2 (1 + \mid x \mid^2 + \mid y \mid^2)^\alpha}{\Delta^{2\beta}(y) \mid y \mid^{2\delta (n-1)} ( \mid x \mid^2 + \mid y \mid^2 )^\delta}$$

$$\leqslant K_1^2 M^2 (\varepsilon) e^{2 (a+\varepsilon) \mid y \mid} \frac{[1 + \Delta^\beta (y)]^2 [1 + \mid x \mid^2 + \Delta^2 (y)]^\alpha}{\Delta^{2\beta + 2\delta (n-1)} (y) [ \mid x \mid^2 + \Delta^2 (y)]^\delta}.$$

Here, we also took into account that $\Delta (y) \leqslant \mid y \mid$ and $\delta > \alpha$. Therefore

$$\| f_\delta (x + iy) \|^2$$

$$\leqslant K_1^2 M^2 (\varepsilon) e^{2 (a+\varepsilon) \mid y \mid} \frac{[1 + \Delta^\beta (y)]^2}{\Delta^{2\beta + 2\delta (n-1)} (y)} \int \frac{[1 + \mid x \mid^2 + \Delta^2 (y)]^\alpha}{[ \mid x \mid^2 + \Delta^2 (y)]^\delta} dx$$

$$\leqslant K_1^2 M^2 (\varepsilon) e^{2 (a+\varepsilon) \mid y \mid} \frac{[1 + \Delta^\beta (y)]^2}{\Delta^{2\beta + n (2\delta - 1)} (y)} \int \frac{[1 + \Delta^2 (y) (1 + \mid \xi \mid^2)]^\alpha}{(1 + \mid \xi \mid^2)^\delta} d\xi .$$

By virtue of the choice of the number $\delta$, $2\delta - 2\alpha > n$, but then $2\alpha \leqslant n(2\delta - 1)$ and, continuing our estimates, we obtain

$$\| f_\delta (x+iy) \|_0^2 \leqslant K_2^2 M^2 (\varepsilon) e^{2(a+\varepsilon)|y|} [1 + \Delta^{-\beta - n(\delta - 1/2)}(y)]^2, \quad y \in C, \tag{2.6}$$

where the number $K_2$ depends solely on $\alpha$, $\beta$, $\delta$ and on the admissible polynomial $l$. The estimate (2.6) shows that the function $f_\delta$ satisfies the conditions of the lemma of Sec. 10.5 for $s = 0$ and $\gamma = \beta + n(\delta - 1/2)$. Therefore $f_\delta = L[g_\delta]$, where $g_\delta \in \mathscr{L}_s^2 (C^* + \bar{U}_a)$, $s < -\beta - n(\delta - 1/2)$, and satisfies the estimate

$$\| g_\delta \|_{(s)} \leqslant K_2' \sqrt{\frac{\beta + n(\delta - 1/2) + 1}{-s - \beta - n(\delta - 1/2)}} \inf_{0 < \varepsilon \leqslant 1} M(\varepsilon) \inf_{\sigma \in \operatorname{pr} C} [1 + \Delta^{-\gamma}(\sigma)].$$

By the theorem of Sec. 10.5, $f_\delta = L[g_\delta] \in H_a^{(s)}(C)$ and satisfies the estimate (2.4) with a certain $K_{s,\delta}$. The proof of the lemma is complete.

Theorem *The following statements are equivalent:*
(1) *$f$ belongs to $H_a(C)$;*
(2) *$f$ can be represented in the form*

$$f(z) = l^\delta (z) f_\delta (z), \qquad f_\delta = L[g_\delta] \in H_a^{(s)}(C) \tag{2.7}$$

*for all admissible polynomials $l(z)$ for the cone $C$ for all $s < s_0$ and for all $\delta > \delta_0 \geqslant n/2$ ($s_0$ and $\delta_0$ depend solely on $f$);*

(3) *$f$ possesses the spectral function $g$ taken from $\mathscr{S}'(C^* + \bar{U}_a)$. Here the following operations are continuous: $f \to f_\delta \to g_\delta \to g \to f$.*

*Proof*. (1) $\to$ (2). Let $f \in H_a^{(\alpha, \beta)}(C)$. From the lemma, for $M(\varepsilon) = \| f \|_a^{(\alpha, \beta)}$, it follows that for $\delta > \alpha + n/2$, $f(z)$ can be represented as (2.7), and the function $f_\delta \in H_a^{(s)}(C)$, $s < -\beta - n(\delta - 1/2)$, and satisfies the estimate

$$\| f_\delta \|_a^{(s)} \leqslant K_{s,\delta} \| f \|_a^{(\alpha, \beta)} \inf_{\sigma \in \operatorname{pr} C} [1 + \Delta^{-\beta - n(\delta - 1/2)}(\sigma)]. \tag{2.8}$$

with some $K_{s\,\delta}$. The estimate (2.8) is what signifies that the operation $f \to f_\delta$ is continuous from $H_a^{(\alpha, \beta)}(C)$ to $H_a^{(s)}(C)$.

(2) $\to$ (3). Suppose $f(z)$ can be represented as (2.7). Assuming $\delta > \delta_0 \geqslant n/2$ to be integral and using the theorem of Sec. 10.5 and property (b) of Sec. 9.2, we conclude that the spectral func-

tion $g$ of the function $f$ can be represented as

$$g(\xi) = l^{\delta}(-iD)\, g_{\delta}(\xi), \qquad g_{\delta} \in \mathcal{L}_s^2(C^* + \overline{U}_a), \tag{2.9}$$

that is, $g \in \mathcal{S}'(C^* + \overline{U}_a)$.

(3) → (1). Let $f = L[g]$, where $g \in \mathcal{S}'(C^* + \overline{U}_a)$. Then $f(z)$ is a holomorphic function in $T^{\operatorname{int} C^{**}} = T^C$ and can be represented in the form (see Sec. 9.1)

$$f(z) = (g(\xi), \eta(\xi)\, e^{i(z, \xi)}), \qquad z \in T^C,$$

where $\eta \in C^{\infty}$; $\eta(\xi) = 1$, $\xi \in (C^* + \overline{U}_a)^{\varepsilon/2}$; $\eta(\xi) = 0$, $\xi \overline{\in} (C^* + \overline{U}_a)^{\varepsilon}$; $|D^{\alpha}\eta(\xi)| \leqslant c_{\alpha}(\varepsilon)$; $\varepsilon$ is an arbitrary number, $0 < \varepsilon \leqslant 1$. Since $g \in \mathcal{S}'$, it follows, by the Schwartz theorem (see Sec. 5.2), that it is of finite order $m$. Furthermore, by what was proved in Sec. 9.1, $\eta(\xi)\, e^{i(z, \xi)} \in \mathcal{S}$ for all $z \in T^C$. Hence, for all $z \in T^C$ the following estimates hold true:

$$|f(z)| \leqslant \|g\|_{-m} \|\eta(\varepsilon)\, e^{i(z, \xi)}\|_m$$

$$= \|g\|_{-m} \sup_{\substack{\xi \\ |\alpha| \leqslant m}} (1 + |\xi|^2)^{m/2} |D^{\alpha}[\eta(\xi)\, e^{i(z, \xi)}]|$$

$$\leqslant \|g\|_{-m} \sup_{\substack{\xi \\ |\alpha| \leqslant m}} (1 + |\xi|^2)^{m/2} \sum_{\beta \leqslant \alpha} \binom{\beta}{\alpha} e^{-(y, \xi)} |z^{\beta}|\, |D^{\alpha-\beta}\eta(\xi)|$$

$$\leqslant K'_m(\varepsilon) \|g\|_{-m} (1 + |z|^2)^{m/2} \sup_{\xi \in (C^* + \overline{U}_a)^{\varepsilon}} (1 + |\xi|^2)^{m/2} e^{-(y, \xi)}$$

$$\leqslant K'_m(\varepsilon) \|g\|_{-m} (1 + |z|^2)^{m/2}$$

$$\times \sup_{\xi_1 \in C^*, |\xi_2| \leqslant a + \varepsilon} (1 + |\xi_1 + \xi_2|^2)^{m/2} e^{-(y, \xi_1) - (y, \xi_2)}$$

$$\leqslant K''_m(\varepsilon) \|g\|_{-m} e^{(a+\varepsilon)|y|} (1 + |z|^2)^{m/2} \sup_{\xi_1 \in C^*} (1 + |\xi_1|^2)^{m/2} e^{-(y, \xi_1)}$$

$$\leqslant K''_m(\varepsilon) \|g\|_{-m} e^{(a+\varepsilon)|y|} (1 + |z|^2)^{m/2} \sup_{\rho \geqslant 0} (1 + \rho^2)^{m/2} e^{-\Delta(y)\rho}$$

$$\leqslant K''_m(\varepsilon) \|g\|_{-m} e^{(a+\varepsilon)|y|} (1 + |z|^2)^{m/2} \sup_{t \geqslant 0} \left[1 + \frac{t^2}{\Delta^2(y)}\right]^{m/2} e^{-t},$$

that is,

$$|f(z)| \leqslant K_m(\varepsilon) \|g\|_{-m} e^{(a+\varepsilon)|y|} (1 + |z|^2)^{m/2} [1 + \Delta^{-m}(y)], \qquad z \in T^C.$$

Thus, the function $f(z)$ satisfies the conditions of the lemma with $\alpha = \beta = m$ and $M(\varepsilon) = K_m(\varepsilon) \| g \|_{-m}$. In this case, when $\delta > m + n/2$, it can be represented as (2.7), where $f_\delta \in H_a^{(s)}(C)$ for $s < -m - n(\delta - 1/2) < 0$, and it satisfies the estimate

$$\| f_\delta \|_a^{(s)} \leqslant K'_{s,\delta} \| g \|_{-m} \inf_{0<\varepsilon\leqslant 1} K_m(\varepsilon) \inf_{\sigma \in \operatorname{pr} C} [1 + \Delta^{-m-n(\delta-1/2)}(\sigma)] = K_{s,\delta} \| g \|_{-m}. \tag{2.10}$$

By the theorem of Sec. 10.5, the function $f_\delta(z)$ is the Laplace transform of the function $g_\delta$ taken from $\mathscr{L}_s^2(C^* + \overline{U}_a)$,

$$f_\delta(z) = \int_{C^*+U_a} g_\delta(\xi) e^{i(z,\xi)}\, d\xi, \qquad z \in T^C, \tag{2.11}$$

which, by virtue of (2.10), satisfies the inequality

$$\| g_\delta \|_{(s)} = \| f \|_a^{(s)} \leqslant K_{s,\delta} \| g \|_{-m}. \tag{2.12}$$

Applying to the integral (2.11) the Cauchy-Bunyakovsky inequality and using the definition of a norm in the space $\mathscr{L}_s^2$, we get, for all $z \in T^C$,

$$|f_\delta(z)| \leqslant \int_{C^*+U_a} |g_\delta(\xi)| (1+|\xi|^2)^{s/2} e^{-(y,\xi)} (1+|\xi|^2)^{-s/2}\, d\xi \leqslant \| g_\delta \|_{(s)} \left[ \int_{C^*+U_a} (1+|\xi|^2)^{-s} e^{-2(y,\xi)}\, d\xi \right]^{1/2} \tag{2.13}$$

Before continuing the estimate (2.13), we take note of the inequality

$$-(y,\xi) \leqslant -\Delta(y)(|\xi| - a)\,\theta(|\xi| - a) + a|y|, \qquad y \in C, \quad \xi \in C^* + \overline{U}_a. \tag{2.14}$$

Indeed, if $\xi = \xi_1 + \xi_2$, $\xi_1 \in C^*$, $|\xi_2| \leqslant a$, then

$$-(y,\xi) = -(y,\xi_1) - (y,\xi_2) \leqslant -\Delta(y)|\xi_1| + a|y| \leqslant \begin{cases} a|y| & \text{if} \quad |\xi| < a, \\ -\Delta(y)(|\xi| - a) + a|y| & \text{if} \quad |\xi| \geqslant a. \end{cases}$$

Taking into account the inequality (2.14), we continue the estimate (2.13):

$$|f_\delta(z)|^2 \leqslant \|g_\delta\|_s^2 \int\limits_{C^*+U_a} (1+|\xi|^2)^{-s} e^{-2\Delta(y)(|\xi|-a)\theta(|\xi|-a)+2a|y|}\, d\xi$$

$$\leqslant \|g_\delta\|_s^2 e^{2a|y|}\left[\int\limits_{|\xi|<a} (1+|\xi|^2)^{-s} d\xi + \int\limits_{|\xi|>a} (1+|\xi|^2)^{-s} e^{-2\Delta(y)(|\xi|-a)}\, d\xi\right]$$

$$= \|g_\delta\|_s^2 e^{2a|y|}\Big\{M_s'(a)+\sigma_n \int\limits_0^\infty [1+(r+a)^2]^{-s}$$

$$\times (r+a)^{n-1} e^{-2\Delta y(r)}\, dr\Big\}$$

$$= \|g_\delta\|_{(s)}^2 e^{2a|y|}\left\{M_s'(a)+\sigma_n \int\limits_0^\infty \left[1+\left(\frac{u}{\Delta(y)}+a\right)^2\right]^{-s}\right.$$

$$\left.\times \left(\frac{u}{\Delta(y)}+a\right)^{n-1} e^{-2u}\frac{du}{\Delta(y)}\right\}.$$

That is to say, for some $M_s(a)$,

$$|f_\delta(z)| \leqslant M_s(a)\, \|g_\delta\|_{(s)}\, e^{a|y|}\,[1+\Delta^{s-n/2}(y)], \qquad z \in T^C.$$

Whence, taking into account the estimate (2.12), we obtain from the representation (2.7)

$$|f(z)| = |l^\delta(z)|\,|f_\delta(z)| \leqslant M_s(a)\, |z|^{n\delta}\, \|g_\delta\|_{(s)}\, e^{a|y|}\,[1+\Delta^{s-n/2}(y)]$$

$$\leqslant M_s(a) K_{s,\delta}\, \|g\|_{-m}\, e^{a|y|}\,(1+|z|^2)^{\frac{n\delta}{2}}\,[1+\Delta^{s-n/2}(y)], \quad z \in T^C,$$

so that $f \in H_a^{(n\delta,\, s-n/2)}(C)$ and the operation $g \to f$ is continuous from $\mathscr{S}'(C^*+\bar{U}_a)$ to $H_a(C)$.

It remains to note that the operation $f_\delta \to g_\delta$ is continuous from $H_a^0(C)$ to $S_0'(C^*+\bar{U}_a)$ (by the theorem of Sec. (10.5), and the operation $g_\delta \to g$ is continuous from $S_0'(C^*+\bar{U}_a)$ to $\mathscr{S}'(C^*+\bar{U}_a)$ [by (2.9)]. The proof of the theorem is complete.

Corollary 1 *The algebras $H_+(C)$ and $\mathscr{S}'(C^*+)$ and also their subalgebras $H(C)$ and $\mathscr{S}'(C^*)$ are isomorphic, and that isomorphism is accomplished via the Laplace transformation.*

Corollary 2 *For $g \in \mathscr{S}'(C^*+\bar{U}_a)$, it is necessary and sufficient that for any admissible polynomial for the cone $C$ and for any*

*integer* $\delta \geqslant \delta_0(g)$ *it be representable in the form*

$$g(\xi) = l^{\delta}(-iD)\, g_\delta(\xi) \qquad g_\delta \in S_0'(C^* + \overline{U}_a), \qquad (2.15)$$

*the operation* $g \to g_\delta$ *being continuous from* $\mathscr{S}'(C^* + \overline{U}_a)$ *to* $S_0'(C^* + \overline{U}_a)$.

Corollary 3 *The operation* $f \to D^\alpha f$ *is continuous in* $H_a(C)$. This follows from the continuity of the operations

$$f \to g \to (i\xi)^\alpha g \to D^\alpha f.$$

*Remark.* These results have been proved by Vladimirov [4] by a different method.

Corollary 4 *Any function* $f(z)$ *in* $H_+(C)$ *has a (unique) boundary value* $f_+(x)$ *as* $y \to 0$, $y \in C$ *in* $\mathscr{S}'$, *which value is equal to* $F[g] = f_+$, *and the operation* $f \to f_+$ *is continuous from* $H_+(C)$ *to* $\mathscr{S}'$.

*Remark.* The theorem on the existence, in $\mathscr{S}'$, of boundary values of functions taken from the algebra $H(C)$ has been proved by Vladimirov [5, 8] and Tillmann [1]. The proof given here is taken from Vladimirov [6]. More general conceptions of boundary values of holomorphic functions have been considered in the works of Köthe [1] ($\mathscr{D}'$), Sato [1] (hyperfunctions), Komatsu [1] (ultradistributions), and Martineau [1].

## 12.3 The Paley-Wiener-Schwartz theorem and its generalizations

Theorem (Paley-Wiener-Schwartz) *For a function* $f(z)$ *to be integral and to satisfy the conditions of growth: for any* $\varepsilon > 0$ *there is a number* $M(\varepsilon)$ *such that*

$$|f(z)| \leqslant M(\varepsilon)\, e^{(a+\varepsilon)|y|}(1+|z|^2)^{\alpha/2}, \qquad z \in \mathbb{C}^n, \qquad (3.1)$$

*for certain* $a \geqslant 0$ *and* $\alpha \geqslant 0$ *(that are dependent on* $f$*), it is necessary and sufficient that its spectral function* $g$ *belong to* $\mathscr{E}'(\overline{U}_a)$. *Here,* $f(z)$ *satisfies the growth condition*

$$|f(z)| \leqslant M e^{a|y|}(1+|z|^2)^{\alpha'/2}, \qquad z \in \mathbb{C}^n, \qquad (3.2)$$

*for certain* $M$ *and* $\alpha' \geqslant \alpha$.

*Proof. Necessity.* Suppose $f(z)$ is an integral function satisfying the growth condition (3.1). Then by the theorem of Sec. 12.2, $f(z)$ is the Laplace transform of the generalized function $g$ in

$\mathcal{S}'(C^* + \bar{U}_a)$ for any convex acute cone $C$. Hence

$$\operatorname{supp} g \subset \bigcap_{C} (C^* + \bar{U}_a) = \bar{U}_a,$$

so that $g \in \mathcal{E}'(\bar{U}_a)$.

*Sufficiency*. Suppose $f(z)$ is the Laplace transform of the generalized function $g$ taken from $\mathcal{E}'(\bar{U}_a)$. Let us cover $\mathbb{R}^n \setminus \{0\}$ with a finite number of convex acute open cones $C_j$, $j = 1, \ldots, N$. Then $g \in \mathcal{S}'(C_j^* + \bar{U}_a)$, $j = 1, \ldots, N$. By the theorem of Sec. 12.2 in each $T^{C_j}$, $f(z)$ satisfies an estimate of the type (3.2),

$$|f(z)| \leqslant M_j e^{a|y|} (1 + |z|^2)^{\alpha_j/2}, \quad z \in T^{C_j}, \; j = 1, \ldots, N. \tag{3.3}$$

Setting $M = \max\limits_j M_j$ and $\alpha' = \max\limits_j \alpha_j$, we obtain from (3.3) the estimate (3.2) in $\mathbb{C}^n$. The theorem is proved.

Corollary *For a function $f(z)$ to be integral and to satisfy the growth condition,*

$$|f(z)| \leqslant K_N e^{a|y|} (1 + |z|^2)^{-N}, \qquad z \in \mathbb{C}^n, \tag{3.4}$$

*it is necessary and sufficient, for all $N \geqslant 0$, that its spectral function $\varphi$ belong to $\mathcal{D}$ and* $\operatorname{supp} \varphi \subset \bar{U}_a$. *Here,*

$$K_N = \int_{|\xi|<a} |(1-\Delta)^N \varphi(\xi)| \, d\xi. \tag{3.5}$$

This follows from the Paley-Wiener-Schwartz theorem and from the estimate

$$|L[\varphi](z)| = (1 + |z|^2)^{-N} \left| \int_{|\xi|<a} e^{i(z,\xi)} (1-\Delta)^N \varphi(\xi) \, d\xi \right|$$

$$\leqslant (1 + |z|^2)^{-N} e^{a|y|} \int_{|\xi|<a} |(1-\Delta)^N \varphi(\xi)| \, d\xi, \quad \varphi \in \mathcal{D}, \quad \operatorname{supp} \varphi \subset \bar{U}_a,$$

where $N$ is an integer $\geqslant 0$.

**12.4 The space $H_a(C)$ is the projective limit of the spaces $H_{a'}(C')$** Suppose $C'$ is a convex open cone, compact in the cone $C$, and $a' > a \geqslant 0$. We denote by $\Delta'(y)$ the distance from the point $y$ to the boundary of the cone $C'$. Then $\Delta'(y) < \Delta(y)$, $y \in C'$,

and therefore the following inequality holds:

$$\sup_{z\in T^{C'}} e^{-(a+\varepsilon)|y|} \frac{|f(z)|}{(1+|z|^2)^{\alpha/2}[1+\Delta'^{-\beta}(y)]}$$
$$\leqslant \sup_{z\in T^{C}} e^{-a|y|} \frac{|f(z)|}{(1+|z|^2)^{\alpha/2}[1+\Delta^{-\beta}(y)]}.$$

That is, (the norm $\|\ \|'$ corresponds to the cone $C'$)

$$\|f\|'^{(\alpha,\beta)}_{a+\varepsilon} \leqslant \|f\|^{(\alpha,\beta)}_{a}, \tag{4.1}$$

whence we conclude that

$$H_a(C) \subset H_{a'}(C'), \tag{4.2}$$

and this inclusion is continuous.

We introduce the intersection of spaces

$$\bigcap_{C' \Subset C,\ a'>a} H_{a'}(C')$$

with convergence $f_k \to 0$, $k \to \infty$, if $f_k \to 0$, $k \to \infty$, in each of the $H_{a'}(C')$. In other words, we equip this intersection with a topology of the projective limit (of the intersection) of a decreasing sequence of the spaces $H_{a'}(C')$, $a' \to a+0$, $C' \to C$, $C' \Subset C$. We have the equality

$$H_a(C) = \bigcap_{C' \Subset C,\ a'>a} H_{a'}(C') \tag{4.3}$$

which holds true together with the corresponding topology.

Indeed, the truth of the inclusion

$$H_a(C) \subset \bigcap_{C' \Subset C,\ a'>a} H_{a'}(C') \tag{4.4}$$

has already been proved by (4.2). We now prove the inverse inclusion

$$\bigcap_{C' \Subset C,\ a'>a} H_{a'}(C') \subset H_a(C). \tag{4.5}$$

Let $f \in H_{a'}(C')$ for all $C' \Subset C$ and $a' > a$. By the theorem of Sec. 12.2 $f = L[g]$, where $g \in \mathscr{S}'(C'^* + \bar{U}_{a'})$. Noting that

$$\bigcap_{C' \Subset C,\, a' > a} (C'^* + \bar{U}_{a'}) = C^* + \bar{U}_a,$$

we conclude that $g \in \mathscr{S}'(C^* + \bar{U}_a)$ and, hence, $f(z) \in H_a(C)$. Furthermore, the operation $f \to g$ is continuous from $H_{a'}(C')$ to $\mathscr{S}'(C'^* + \bar{U}_{a'})$. But

$$\bigcap_{C' \Subset C,\, a' > a} \mathscr{S}'(C'^* + \bar{U}_{a'}) = \mathscr{S}'(C^* + \bar{U}_a)$$

and this equality is continuous in both directions. Finally, the operation $g \to f$ is continuous from $\mathscr{S}'(C^* + U_a)$ to $H_a(C)$. And this means that the inclusion $f \to f$ is continuous from $\bigcap_{C' \Subset C,\, a' > a} H_{a'}(C')$ to $H_a(C)$. The inclusion (4.5) together with the inclusion (4.4) is what yields (4.3), which is what we set out to prove.

The equality (4.3) gives a different definition of functions of the class $H_a(C)$, which definition is convenient for applications.

For a function $f(z)$ that is holomorphic in $T^C$ to belong to $H_a(C)$, it is necessary and sufficient that, for any arbitrary cone $C' \Subset C$ and an arbitrary number $\varepsilon > 0$, there exist numbers $\alpha' \geqslant 0$, $\beta' \geqslant 0$, and $M' > 0$ such that

$$|f(z)| \leqslant M' e^{(a+\varepsilon)|y|} \frac{(1+|z|^2)^{\alpha'/2}}{|y|^{\beta'}}, \qquad z \in T^{C'}. \tag{4.6}$$

Indeed, if $f \in H_a(C)$, then $f \in H_a^{(\alpha,\beta)}(C)$ for certain $\alpha \geqslant 0$ and $\beta \geqslant 0$ such that $f$ satisfies the inequality (1.1). Let $C' \Subset C$. By Lemma 1 of Sec. 4.4 there is a number $\varkappa > 0$ such that

$$\Delta(y) = \inf_{\sigma \in \operatorname{pr} C^*} (\sigma, y) \geqslant \varkappa |y|, \qquad y \in C'.$$

From this and from the inequality (1.1) follows the inequality (4.6) for $\varepsilon = 0$, $\beta' = \beta$ and for certain $\alpha' \geqslant \alpha$ and $M'(C') \geqslant M$.

Conversely, if $f(z)$ is holomorphic in $T^C$ and, for arbitrary $C' \Subset C$ and $\varepsilon > 0$, satisfies the estimate (4.6), then, taking into account that $\Delta'(y) \leqslant |y|$, where $\Delta'(y)$ is the distance from $y$ to $\partial C'$, we obtain $f \in H_{a+\varepsilon}(C)'$, whence, by (4.3), it follows that $f \in H_a(C)$.

**12.5 The Schwartz representation** Suppose an acute (convex open) cone $C$ is such that the Cauchy kernel $\mathscr{K}_C(z)$ is the divisor of unity in the algebra $H(C)$, that is, $1/\mathscr{K}_C(z) \in H(C)$. Such cones $C$ will be called *regular*[§]. For example, the cones $\mathbb{R}^n_+$ and $V^+$ are regular (see Sec. 13.5 below).

The Schwartz kernel of the region $T^C$, where $C$ is a regular cone, relative to the point $z^0 = x^0 + iy^0 \in T^C$ is the function

$$\mathscr{S}_C(z;\, z^0) = \frac{2\mathscr{K}_C(z)\,\mathscr{K}_C(-\bar{z}^0)}{(2\pi)^n\,\mathscr{K}_C(z-\bar{z}^0)} - \mathscr{P}_C(x^0,\, y^0), \qquad z \in T^C. \quad (5.1)$$

We note some properties of the Schwartz kernel.

(a) $$\mathscr{S}_C(z;\, z) = \mathscr{P}_C(x,\, y), \qquad z \in T^C. \quad (5.2)$$

This property follows from (5.1) when $z^0 = z$, from the definition of the Poisson kernel (1.1) of Sec. 11, and from the property (2.3) of Sec. 10 of the Cauchy kernel.

(b) $$\int \mathscr{S}_C(z-x';\, z^0-x')\,dx' = 1, \qquad z \in T^C, \quad z^0 \in T^C. \quad (5.3)$$

This property follows from the Parseval equation applied to (2.1) of Sec. 10,

$$\int \mathscr{K}_C(z-x')\,\mathscr{K}_C(-\bar{z}^0+x')\,dx' = \int \mathscr{K}_C(z-x')\,\overline{\mathscr{K}_C(z^0-x')}\,dx'$$

$$= (2\pi)^n \int_{C^*} e^{i(z-\bar{z}^0,\, \xi)}\,d\xi = (2\pi)^n\,\mathscr{K}_C(z-\bar{z}^0),$$

and from the property (1.5), Sec. 11, of the Poisson kernel.

(c) $$|\mathscr{S}_C(z;\, z^0)| \leqslant \frac{\mathscr{K}_C(2iy)}{|\mathscr{K}_C(z-\bar{z}^0)|}\,\mathscr{P}_C(x,\, y) + \left[\frac{\mathscr{K}_C(2iy^0)}{|\mathscr{K}_C(z-\bar{z}^0)|} + 1\right]\mathscr{P}_C(x^0,\, y^0), \quad (5.4)$$

$$z \in T^C, \qquad z^0 \in T^C.$$

This property follows from the definitions of the Schwartz and the Poisson kernel and from the estimate $2\,|ab| \leqslant |a|^2 + |b|^2$.

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[§]Apparently, all acute cones are regular; for homogeneous cones of positivity it has been proved that $\mathscr{K}_C(z) \neq 0$, $z \in T^C$ (Rothaus [1]).

*Example 1* [see (2.16) of Sec. 10 and (1.2) of Sec. 11]:

$$\mathcal{S}_{\mathbf{R}^n_+}(z;\ z^0) = \mathcal{S}_n(z;\ z^0)\Big|$$

$$= \frac{2i^n}{(2\pi)^n}\left(\frac{1}{z_1}-\frac{1}{\bar{z}_1^0}\right)\ldots\left(\frac{1}{z_n}-\frac{1}{\bar{z}_n^0}\right) - \mathcal{P}_n(x^0,\ y^0).$$

In particular, for $n = 1$, $C = (0,\ \infty)$,

$$\mathcal{S}_1(z;\ z^0) = \frac{i}{\pi}\left(\frac{1}{z}-\frac{x^0}{|z^0|^2}\right), \qquad \mathrm{Re}\,\mathcal{S}_1(z;\ z^0) = \mathcal{P}_1(x,\ y).$$

*Example 2* [see (2.17) of Sec. 10 and (1.3) of Sec. 11]:

$$\mathcal{S}_{V^+}(z;\ z^0) = \frac{\Gamma\left(\frac{n+1}{2}\right)\left[-(z-\bar{z}^0)^2\right]^{\frac{n+1}{2}}}{\pi^{\frac{n+3}{2}}\,(-z^2)^{\frac{n+1}{2}}\,\left[-(\bar{z}^0)^2\right]^{\frac{n+1}{2}}} - \mathcal{P}_{V^+}(x^0,\ y^0).$$

Let the boundary value $f_+(x)$ of a function $f(z)$ of the class $H(C)$ (see Sec. 12.2) satisfy the condition

$$f_+(x)\,\mathcal{K}_C(x-\bar{z}^0) \in \mathcal{H}_s \tag{5.5}$$

for some $s$ and for all $z^0 \in T^C$. Then the generalized function (5.5) is the boundary value in $\mathcal{S}'$ of the function $f(z)\,\mathcal{K}_C(z-\bar{z}^0)$ of the class $H(C)$ and therefore the support of its inverse Fourier transform is contained in the cone $C^*$. By Theorem II of Sec. 10.6, the function $f(z)\,\mathcal{K}_C(z-\bar{z}^0)$ belongs to the class $H^{(s)}(C)$ and its boundary value in $\mathcal{H}_s$ is equal to $f_+(x)\,\mathcal{K}_C(x-\bar{z}^0)$ since $\mathcal{K}_C(x+iy) \in \theta_M$ for all $y \in C$ [see Sec. 5.3 and Sec. 10.2, estimate (2.4)]. Applying Theorem I of Sec. 10.6 to the function $f(z)\,\mathcal{K}_C(z-\bar{z}^0)$, we obtain

$$f(z)\,\mathcal{K}_C(z-\bar{z}^0)$$

$$= \frac{1}{(2\pi)^n}\left(f_+(x')\,\mathcal{K}_C(x'-\bar{z}^0),\ \mathcal{K}_C(z-x')\right), \qquad z \in T^C,\ z^0 \in T^C. \tag{5.6}$$

Putting $z^0 = z$ in (5.6) and taking into account (5.2) for the function $f(z)$, we derive the generalized Poisson representation

$$f(z) = \left(f_+(x'),\ \mathcal{P}_C(x-x',\ y)\right), \qquad z \in T^C. \tag{5.7}$$

Then, interchanging $z$ and $z^0$ in (5.6), we obtain

$$f(z^0)\,\mathcal{K}_C(z^0-\bar{z})=\frac{1}{(2\pi)^n}\,(f_+(x')\,\mathcal{K}_C(x'-\bar{z}),\ \mathcal{K}_C(z^0-x')),$$

whence, passing to the complex conjugate, we derive

$$\bar{f}(z^0)\,\mathcal{K}_C(z-\bar{z}^0)=\frac{1}{(2\pi)^n}\,(\bar{f}_+(x')\,\mathcal{K}_C(z-x'),\ \mathcal{K}_C(x'-\bar{z}^0)). \quad (5.8)$$

Subtracting (5.7) from (5.6), we get the relation

$$\mathcal{K}_C(z-\bar{z}^0)\,[f(z)-\bar{f}(z^0)]$$
$$=\frac{2i}{(2\pi)^n}\,(\operatorname{Im} f_+(x'),\ \mathcal{K}_C(z-x')\,\mathcal{K}_C(x'-\bar{z}^0)),\quad z\in T^C,\quad z^0\in T^C. \quad (5.9)$$

Suppose $C$ is a regular cone so that $\mathcal{K}_C(z)\neq 0$, $z\in T^C$. Divide (5.9) by $\mathcal{K}_C(z-\bar{z}^0)$ and, in (5.9), in accordance with formula (5.7), make the substitution

$$\operatorname{Im} f(z^0) = (\operatorname{Im} f_+(x'),\ \mathcal{P}_C(x^0-x',\ y^0)),\qquad z^0\in T^C. \quad (5.10)$$

As a result we obtain the representation

$$f(z)=i\left(\operatorname{Im} f_+(x'),\ \frac{2}{(2\pi)^n}\,\mathcal{K}_C(z-x')\,\mathcal{K}_C(x'-\bar{z}^0)\right.$$
$$\left.-\mathcal{P}_C(x^0-x',\ y^0)\right)+\operatorname{Re} f(z^0)$$

or, using the definition (5.1) of the Schwartz kernel,

$$f(z) = i\,(\operatorname{Im} f_+(x'),\ \mathcal{S}_C(z-x';\ z^0-x'))+\operatorname{Re} f(z^0),$$
$$z\in T^C,\quad z^0\in T^C. \quad (5.11)$$

Formula (5.11) is called the *generalized Schwartz representation*.

This completes the proof of the following theorem.

Theorem *If $C$ is an acute cone, then any function $f(z)$ of the class $H(C)$ that satisfies the condition (5.5) can be represented in terms of its boundary value $f_+$ by the Poisson integral (5.7) and can also be represented in terms of the imaginary part of its boundary value by the formula (5.9). And if, besides, the cone $C$ is regular, then for any such function $f(z)$ the generalized Schwartz representation (5.11) holds true.*

**12.6 A generalization of the Phragmén-Lindelöf theorem** The Phragmén-Lindelöf theorem in the theory of holomorphic functions is defined as any generalization of the maximum principle to the case of unbounded regions or to more general (than continuous) boundary values. Here we give one such generalization of the maximum principle that will be used later on in Sec. 20.1.

Theorem *If the boundary value $f_+(x)$ of a function $f(z)$ of the class $H(C)$, where $C$ is an acute cone, is bounded: $|f_+(x)| \leqslant M$, $x \in \mathbb{R}^n$, then we also have $|f(z)| \leqslant M$, $z \in T^C$; what is more, for $f(z)$ we have the generalized Poisson representation*

$$f(z) = \int \mathscr{P}_C(x-x', y)\, f_+(x')\, dx', \qquad z \in T^C. \tag{6.1}$$

*Remark*. For $n = 1$ this theorem was proved by Nevanlinna [1].

*Proof*. Since $\mathscr{K}_C(x+iy) \in \mathscr{L}^2$ for all $y \in C$ (see Sec. 10.2), it follows that $f_+(x)\, \mathscr{K}_C(x-\bar{z}^0) \in \mathscr{L}^2$ for all $z^0 \in T^C$ and, hence, the condition (5.5) is fulfilled for $s = 0$. By the theorem of Sec. 12.5, for the function $f(z)$ the Poisson representation (6.1) holds; from this and from the property (1.5), Sec. 11.1, of the kernel $\mathscr{P}_C$ follows the estimate

$$|f(z)| \leqslant M \int \mathscr{P}_C(x-x', y)\, dx' = M, \qquad z \in T^C,$$

which completes the proof of the theorem.

## 13 Equations in Convolution Algebras

Let $\Gamma$ be a closed convex acute solid cone in $\mathbb{R}^n$ (with vertex at 0). Then the sets of generalized functions of slow growth $\mathscr{S}'(\Gamma+)$ and $\mathscr{S}'(\Gamma)$ form convolution algebras [$\mathscr{S}'(\Gamma)$ is a subalgebra of $\mathscr{S}'(\Gamma+)$] (see Sec. 5.6(b)) that are isomorphic to the algebras of the holomorphic functions $H_+(C)$ and $H(C)$, respectively, where $C = \operatorname{int} \Gamma^*$, and the isomorphism is accomplished by the operation of the Laplace transform (see Sec. 12.2).

**13.1 Divisors of unity in the $H_+(C)$ and $H(C)$ algebras** As was shown in Sec. 4.8(d) the solvability of the equation

$$a * u = f, \quad a \text{ and } f \in \mathscr{S}'(\Gamma+),$$

in the convolution algebra $\mathscr{S}'(\Gamma+)$ reduces to the existence of a fundamental solution $\mathscr{E}$ (the kernel of the inverse operator

$a^{-1}*$) of the convolution operator $a*$,

$$a * \mathscr{E} = \delta, \tag{1.1}$$

in the same algebra $\mathscr{S}'(\Gamma+)$. The equation (1.1) is equivalent to the algebraic equation

$$L[a] f = 1. \tag{1.2}$$

in the algebra $H_+(C)$ with respect to the unknown function $f(z) = L[\mathscr{E}]$. Therefore the question of the existence of a fundamental solution of the operator $a*$ in the algebra $\mathscr{S}'(\Gamma+)$ reduces to the question of the possibility of dividing unity by the function $f_0(z) = L[a]$ in the $H_+(C)$ algebra. In other words, the question reduces to studying the divisors of unity in the $H_+(C)$ algebra: if $f \in H_+(C)$, then we want to know under what conditions $\frac{1}{f} \in H_+(C)$.

The necessary condition for this, $f(z) \neq 0$, $z \in T^C$, is not a sufficient condition, as will be seen by the following simple example: $f(z) = e^{-i/z} \in H(0, \infty)$ since $|f(z)| = e^{-y/|z|^2} \leqslant 1$. However, $\frac{1}{f} \bar{\in} H_+(0, \infty)$ since

$$\left|\frac{1}{f(z)}\right| = e^{\frac{y}{|z|^2}} > e^{\frac{1}{y}\left(1-\frac{x^2}{y^2}\right)}.$$

We first note that the study of divisors of unity in the $H_+(C)$ algebra reduces to studying the divisors of unity in its subalgebra, the $H(C)$ algebra. Indeed, any function $f(z)$ in $H_+(C)$ [that is, $f \in H_a^{(\alpha, \beta)}(C)$ for certain $a \geqslant 0$, $\alpha \geqslant 0$ and $\beta \geqslant 0$] can be represented in the form

$$f(z) = e^{-i(z, e)} f_e(z), \qquad f_e \in H(C), \tag{1.3}$$

where $e$ is an arbitrary point in int $\Gamma$ such that $(y, e) \geqslant a|y|$ for all $y \in C$ (by Lemma 1 of Sec. 4.4, such points exist). Indeed,

$$|f_e(z)| = |e^{i(z, e)} f(z)| \leqslant \|f\|_a^{(\alpha, \beta)} (1 + |z|^2)^\alpha [1 + \Delta^{-\beta}(y)], \quad z \in T^C,$$

so that $f_e \in H^{(\alpha, \beta)}(C)$.

From the results of Sec. 12.4 we have the following theorem.

Theorem *For $f \in H(C)$ to be a divisor of unity in the $H(C)$ algebra, it is necessary and sufficient that, for any cone $C' \Subset C$ and any number $\varepsilon > 0$, there exist numbers $\alpha' \geqslant 0$, $\beta' \geqslant 0$ and*

$M' > 0$ *such that*

$$|f(z)| \geqslant M' e^{-\varepsilon|y|} (1 + |z|^2)^{-\alpha'/2} |y|^{\beta'}, \qquad z \in T^{C'}. \qquad (1.4)$$

The condition (1.4) is hard to verify. We now point to several sufficient criteria for the divisibility of unity in the $H(C)$ algebra that follow from the theorem that has just been proved.

## 13.2 On division by a polynomial in the $H(C)$ algebra

Theorem *Suppose $P(z) \not\equiv 0$ is a polynomial, and a function $f(z)$ is holomorphic in $T^C$ and $Pf \in H(C)$. Then $f \in H(C)$ and the operation $f \to Pf$ has a continuous inverse in $H(C)$.*

Corollary *If the polynomial $P(z)$ does not vanish in $T^C$, then $\frac{1}{P} \in H(C)$.*

Indeed, in that case, $\frac{1}{P(z)}$ is a holomorphic function in $T^C$ and $P\frac{1}{P} = 1 \in H(C)$.

To prove this theorem we take advantage of the following result obtained by Hörmander [see inequality (2.3) of Sec. 14 when $p = 0$]:

For a given polynomial $P(z) \not\equiv 0$ there are numbers $m \geqslant 0$ (an integer) and $K > 0$ such that for any $\varphi \in C^m(\mathbb{R}^{2n})$ the following holds true:

$$|\varphi(x, y)| \leqslant K \sup_{\substack{(x, y) \\ |\gamma| \leqslant m}} (1 + |z|^2)^{m/2} |D^\gamma_{(x, y)} [P(z)\varphi(x, y)]|. \qquad (2.1)$$

By hypothesis $Pf \in H(C)$. By Corollary 3 (see Sec. 12.2), $D^\gamma(Pf) \in H(C)$. Therefore there will be numbers $\alpha_0 \geqslant 0$ and $\beta_0 \geqslant 1$ such that $D^\gamma(Pf) \in H^{(\alpha, \beta)}(C)$, $|\gamma| \leqslant m$, $\alpha \geqslant \alpha_0$, $\beta \geqslant \beta_0$ ($\beta$ even).

Let $C'$ be an open convex cone, $C' \Subset C$. Then there will be an (open convex) cone $C''$ such that $C' \Subset C'' \Subset C$. Let us construct a function $\eta(\sigma)$ of the class $C^\infty(S_1)$ that is equal to 1 on pr $C'$ and equal to 0 outside pr $C''$. (From the lemma of Sec. 1.2 it follows that such functions do exist.) If in the inequality (2.1) we put

$$\varphi(x, y) = \frac{f(z)\,\eta\left(\frac{y}{|y|}\right)|y|^m}{(1 + |z|^2)^{m+\alpha/2}(1 + |y|^{-\beta})} \in C^m(\mathbb{R}^{2n}),$$

for all $z \in T^{C'}$, we obtain

$$\frac{|f(z)|\,|y|^m}{(1+|z|^2)^{m+\alpha/2}(1+|y|^{-\beta})}$$

$$\leqslant K \sup_{\substack{(x,y)\\|\gamma|\leqslant m}} (1+|z|^2)^{m/2}\left|D^\gamma_{(x,y)}\left[\frac{P(z)f(z)\eta\left(\frac{y}{|y|}\right)|y|^{\beta+m}}{(1+|z|^2)^{m+\alpha/2}(1+|y|^\beta)}\right]\right|$$

$$\leqslant K_1(C') \sup_{\substack{(x,y)\in T^{C''}\\|\gamma|\leqslant m}} \frac{(|y|^\beta+|y|^{\beta+m})\,|D^\gamma[P(z)f(z)]|}{(1+|z|^2)^{\frac{m+\alpha}{2}}(1+|y|^\beta)}$$

$$\leqslant K_2(C') \sup_{\substack{(x,y)\in T^{C''}\\|\gamma|\leqslant m}} \frac{|D^\gamma[P(z)f(z)]|}{(1+|z|^2)^{\alpha/2}(1+|y|^{-\beta})}.$$

Now, taking into account that

$$\Delta'(y) \leqslant |y|, \quad y \in C'; \quad \Delta(y) \geqslant \sigma|y|, \quad y \in C'', \tag{2.2}$$

where $\Delta(y)$ and $\Delta'(y)$ are the distances from the point $y$ to $\partial C$ and $\partial C'$, respectively, let us continue our estimates

$$\frac{|f(z)|}{(1+|z|^2)^{m+\alpha/2}[1+(\Delta')^{-\beta-m}(y)]}$$

$$\leqslant K_3(C') \sup_{\substack{(x,y)\in T^C\\|\gamma|\leqslant m}} \left|\frac{D^\gamma[P(z)f(z)]}{(1+|z|^2)^{\alpha/2}[1+\Delta^{-\beta}(y)]}\right|,$$

whence, by (2.2) (the norm $\|\;\;\|'$ corresponds to the cone $C'$)

$$\|f\|'^{(2m+\alpha,\, m+\beta)} \leqslant K_3(C') \max_{|\gamma|\leqslant m} \|D^\gamma(Pf)\|^{(\alpha,\beta)}. \tag{2.3}$$

Now, by Corollary 3 (see Sec. 12.2), the operation $Pf \to D^\alpha(Pf)$ is continuous in $H(C)$ so that for certain $M_1 > 0$, $\alpha' \geqslant 0$, and $\beta' \geqslant 0$ the following estimates hold:

$$\|D^\gamma(Pf)\|^{(\alpha,\beta)} \leqslant M_1\|Pf\|^{(\alpha',\beta')}, \qquad |\gamma| \leqslant m.$$

Taking into account these estimates, let us rewrite the inequality (2.3) as

$$\|f\|'^{(2m+\alpha,\, m+\beta)} \leqslant M(C')\|Pf\|^{(\alpha',\beta')}. \tag{2.4}$$

The estimate (2.4) shows (see Sec. 12.4) that $f \in H(C)$ and the operation $f \to Pf$ has a continuous inverse in $H(C)$. The proof is complete.

*Remark.* This theorem was proved in Bogoliubov and Vladimirov [1]. It resembles the theorem of Hörmander [2] on the division of a generalized function of slow growth by a polynomial.

## 13.3 Estimates for holomorphic functions with nonnegative imaginary part in $T^C$

Theorem *Suppose a function $f(z)$ is holomorphic in $T^C$ and $\operatorname{Im} f(z) \geqslant 0$, $z \in T^C$. Then it satisfies the following estimate: for any cone $C' \Subset C$ there is a number $M(C')$ such that*

$$|f(z)| \leqslant M(C') \frac{1+|z|^2}{|y|}, \qquad z \in T^{C'}. \tag{3.1}$$

*That is, $f \in H^{(2,1)}(C')$ for all $C' \Subset C$ so that $f \in H(C)$.*

Corollary *If under the hypothesis of the theorem $f(z) \not\equiv 0$ in $T^C$, then $\frac{1}{f} \in H(C)$.*

We now prove the corollary. If $\operatorname{Im} f(z) > 0$ in $T^C$, then the function $\frac{1}{f(z)}$ is holomorphic in $T^C$ and

$$\operatorname{Im} \frac{1}{f(z)} = \frac{-\operatorname{Im} f(z)}{|f|^2} < 0.$$

By the theorem, $\frac{1}{f} \in H(G)$. But if $\operatorname{Im} f(z) \geqslant 0$ vanishes at some point in the region $T^C$, then, by virtue of the maximum principle for harmonic functions, $\operatorname{Im} f(z) \equiv 0$ in $T^C$ so that $f(z) =$ constant $\neq 0$ and therefore, trivially, $\frac{1}{f} \in H(C)$.

*Remark.* The estimate (3.1) was obtained in Vladimirov [7].

To prove the theorem, let us first prove three lemmas.

Lemma 1 *If a function $f(z)$ is holomorphic and $\operatorname{Im} f(z) \geqslant 0$ in the unit polycircle $S^n = [z : |z_1| < 1, \ldots, |z_n| < 1]$, then*

$$\operatorname{Im} f(0) \frac{1 - \max\limits_{1 \leqslant j \leqslant n} |z_j|}{1 + \max\limits_{1 \leqslant j \leqslant n} |z_j|}$$

$$\leqslant |f(z) - \operatorname{Re} f(0)| \leqslant \operatorname{Im} f(0) \frac{1 + \max\limits_{1 \leqslant j \leqslant n} |z_j|}{1 - \max\limits_{1 \leqslant j \leqslant n} |z_j|}, \qquad z \in S^n. \tag{3.2}$$

*In particular,*

$$|f(z)| \leqslant \frac{2|f(0)|}{1-\max\limits_{1\leqslant j\leqslant n}|z_j|}, \qquad z\in S^n. \tag{3.3}$$

*Proof.* If $\operatorname{Im} f(z) \equiv 0$, then, as we have seen, $f(z) =$ constant and the estimate (3.2) is trivially fulfilled. We can therefore assume that $\operatorname{Im} f(z) > 0$, $z \in S^n$. Let us fix an arbitrary point $z \in S^n$ and put $\rho = \max\limits_{1\leqslant j\leqslant n} |z_j|$ so that $0 \leqslant \rho < 1$. We consider the function $\varphi(\lambda) = f\left(\lambda\frac{z}{\rho}\right)$ that is holomorphic and $\operatorname{Im} \varphi(\lambda) \geqslant 0$ in the circle $|\lambda| < 1$. The function

$$\psi(\lambda)=\frac{\varphi(\lambda)-\operatorname{Re}\varphi(0)-i\operatorname{Im}\varphi(0)}{\varphi(\lambda)-\operatorname{Re}\varphi(0)+i\operatorname{Im}\varphi(0)}$$

is holomorphic and $|\psi(\lambda)| < 1$ in the circle $|\lambda| < 1$; what is more, $\psi(0) = 0$. By the Schwartz lemma, $|\psi(\lambda)| \leqslant |\lambda|$ and therefore

$$\operatorname{Im}\varphi(0)\frac{1-|\lambda|}{1+|\lambda|} \leqslant |\varphi(\lambda)-\operatorname{Re}\varphi(0)| = \left|i\operatorname{Im}\varphi(0)\,\frac{1+\psi(\lambda)}{1-\psi(\lambda)}\right|$$

$$\leqslant \operatorname{Im}\varphi(0)\frac{1+|\lambda|}{1-|\lambda|}, \qquad |\lambda|<1, \tag{3.4}$$

and so

$$|\varphi(\lambda)| \leqslant |\operatorname{Re}\varphi(0)| + \operatorname{Im}\varphi(0)\frac{1+|\lambda|}{1-|\lambda|} \leqslant \frac{2|\varphi(0)|}{1-|\lambda|}, \quad |\lambda|<1. \tag{3.5}$$

If in the estimates (3.4) and (3.5) we put $\lambda = \rho < 1$, we obtain the estimates (3.2) and (3.3). And that is the end of the proof of Lemma 1.

Lemma 2 *If a function $f(z)$ is holomorphic and* $\operatorname{Im} f(z) \geqslant 0$ *in $T^n = [z : y_1 > 0, \ldots, y_n > 0]$, then*

$$|f(z)| \leqslant \sqrt{2}\,|f(\mathbf{i})| \max_{1\leqslant j\leqslant n} \frac{1+|z_j|^2}{y_j}, \qquad z\in T^{\mathbb{R}^n_+} \tag{3.6}$$

*where $\mathbf{i} = (i, i, \ldots, i)$.*

*Proof.* The biholomorphic mapping

$$w_j=\frac{z_j-i}{z_j+i}, \qquad z_j=i\,\frac{1+w_j}{1-w_j}, \qquad j=1,\ldots,n$$

transforms the tubular region $T^n$ on $S^n$, and the function $f(z)$ into the function

$$\varphi(w)=f\left(i\,\frac{1+w_1}{1-w_1},\ \ldots,\ i\,\frac{1+w_n}{1-w_n}\right)$$

which is holomorphic, and $\operatorname{Im} \varphi(w) \geqslant 0$ in $S^n$. Applying (3.3) to $\varphi(w)$, we obtain

$$|\varphi(w)| \leqslant \frac{2|\varphi(0)|}{1-\max\limits_{1 \leqslant j \leqslant n} |w_j|}, \qquad w \in S^n.$$

From this, if we pass to the old variables, we obtain the estimate

$$|f(z)| \leqslant \frac{2|f(\mathbf{i})|}{1-\max\limits_{1 \leqslant j \leqslant n} \left|\frac{z_j-i}{z_j+i}\right|}, \qquad z \in T^n. \tag{3.7}$$

Let us prove the inequality

$$\left|\frac{z-i}{z+i}\right| < 1-\frac{\sqrt{2}\, y}{1+|z|^2}, \qquad y > 0. \tag{3.8}$$

Putting

$$\alpha = \frac{1+x^2}{y} + y \geqslant 2,$$

we reduce inequality (3.8) to the equivalent inequality

$$2\alpha^2 - (\alpha + 2)(\sqrt{2}\alpha - 1) > 0, \qquad \alpha \geqslant 2.$$

Now the latter inequality does indeed occur: it holds for $\alpha = 2$, and the derivative of the left-hand side is greater than zero:

$$(4 - 2\sqrt{2})\,\alpha + 1 - 2\sqrt{2} > 0, \qquad \alpha \geqslant 2.$$

If we take into account the inequality (3.8), we obtain from (3.7) the inequality (3.6). Lemma 2 is proved.

Lemma 3 *Suppose the function $f(z)$ is holomorphic and $\operatorname{Im} f(z) \geqslant 0$ in $T^C$, where $C$ is an $n$-hedral acute cone,*

$$C = [y\colon (e_j, y) > 0,\ j = 1, \ldots, n] \qquad (|e_j| = 1).$$

*Then $f(z)$ satisfies the estimate*

$$|f(z)| \leqslant \sqrt{2}\left|f(T^{-1}\mathbf{i})\right| \frac{1+|z|^2}{\Delta(y)}, \qquad z \in T^C, \tag{3.9}$$

*where $T$ is a linear transformation,*

$$y \to Ty = [(e_1, y), \ldots, (e_n, y)].$$

*Proof*. Since the cone $C$ is not empty, the vectors $e_1, \ldots e_n$ are linearly independent and, hence, the matrix $T^{-1}$ exists. The biholomorphic mapping

$$w = Tz, \qquad z = T^{-1}w$$

transforms the region $T^C$ into the region $T^n$, and the function $f(z)$ into the function $f(T^{-1}w)$, which is holomorphic, and $\operatorname{Im} f(T^{-1}w) \geqslant 0$ in $T^n$. Applying the estimate (3.6) to that function, we obtain

$$|f(T^{-1}w)| \leqslant \sqrt{2}\,|f(T^{-1}\mathbf{i})| \max_{1\leqslant j\leqslant n} \frac{1+|w_j|^2}{\operatorname{Im} w_j}, \qquad w \in T^n.$$

From this, passing to the variables $z$, we derive the estimate

$$|f(z)| \leqslant \sqrt{2}\,|f(T^{-1}\mathbf{i})| \max_{1\leqslant j\leqslant n} \frac{1+|(e_j, z)|^2}{(e_j, y)}, \qquad z \in T^C,$$

from which, and also from the relations

$$|(e_j, z)|^2 \leqslant |e_j|^2 |z|^2 = |z|^2; \qquad \Delta(y) = \max_{1\leqslant j\leqslant n} (e_j, y), \quad y \in C,$$

follows the inequality (3.9). The proof of Lemma 3 is complete.

*Proof of the theorem*. Let $C' \Subset C$. Cover the cone $\overline{C}'$ with a finite number of $n$-hedral open cones $C_k \Subset C$, $k = 1, \ldots, N$, and in each cone $C_k$ choose a cone $C'_k \Subset C_k$ so that the cones $C'_1, \ldots, C'_N$ still cover the cone $C'$. In each region $T^{C_k}$ the estimate (3.9) holds true:

$$|f(z)| \leqslant \sqrt{2}\,|f(T_k^{-1}\mathbf{i})| \frac{1+|z|^2}{\Delta_k(y)}, \qquad k = 1, \ldots, N, \quad (3.10)$$

where $\Delta_k(y)$ and $T_k$ have the same meaning relative to the cone $C_k$ as $\Delta(y)$ and $T$ do relative to the cone $C$. Furthermore, since $C'_k \Subset C_k$, it follows that there exist numbers $\sigma_k$ such that $\Delta_k(y) \geqslant \sigma_k |y|$ for all $y \in C'_k$ (see Lemma 1 of Sec. 4.4). Taking into consideration this inequality, we obtain from (3.10) the estimates

$$|f(z)| \leqslant \frac{\sqrt{2}}{\sigma_k} |f(T_k^{-1}\mathbf{i})| \frac{1+|z|^2}{|y|}, \qquad z \in T^{C'_k}, \quad k = 1, \ldots, N,$$

whence follows the estimate (3.1) in $T^{C'}$ for

$$M(C') = \max_{1\leqslant k\leqslant N} \frac{\sqrt{2}}{\sigma_k} |f(T_k^{-1}\mathbf{i})|.$$

The theorem is proved.

**13.4 Divisors of unity in the algebra $W(C)$** Denote by $\dot{\mathbb{R}}^n$ the fact that the point at infinity is adjoined to $\mathbb{R}^n$. Denote by $W(C)$ the Banach algebra consisting of functions holomorphic in $T^C$ that are Laplace transforms of generalized functions of the form $\lambda\delta(\xi) + g(\xi)$, where $\lambda$ is an arbitrary number and $g$ is an arbitrary function in $\mathscr{L}^1(C^*)$. (The set of such generalized functions forms a convolution algebra; see Sec. 4.1.) The algebra $W(C)$ is called a *Wiener algebra*. Thus, any element $f \in W(C)$ can be represented as

$$f(z) = \lambda + \int_{C^*} g(\xi)\, e^{i(z,\,\xi)}\, d\xi,$$

$$\|f\|_{W(C)} = |\lambda| + \int_{C^*} |g(\xi)|\, d\xi.$$

Here, the function $f(z)$ is continuous in $\dot{\bar{T}}^C$, the closure of $T^C$ in $\dot{\mathbb{R}}^{2n}$, and the following inequality holds:

$$\|f\|^{(0,\,0)} = \frac{1}{4} \sup_{z \in T^C} |f(z)| \leqslant \frac{1}{4} \|f\|_{W(C)},$$

so that the $W(C)$ algebra is a subalgebra of the $H(C)$ algebra and the embedding of $W(C)$ in $H(C)$ is continuous.

If $f \in W(C)$ and $f(z) \neq 0$ in $T^C \cup \dot{\mathbb{R}}^n$, then $\frac{1}{f} \in W(C)$. Indeed, in that case

$$f^+(x) = \lambda + \int_{C^*} g(\xi)\, e^{i(x,\,\xi)}\, d\xi \neq 0, \qquad x \in \mathbb{R}^n, \quad \lambda \neq 0, \quad g \in \mathscr{L}^1.$$

By Wiener's theorem (see Wiener [1])

$$\frac{1}{f^+(x)} = \frac{1}{\lambda} + \int g_1(\xi)\, e^{i(x,\,\xi)}\, d\xi, \qquad g_1 \in \mathscr{L}^1. \tag{4.1}$$

Furthermore, since $f(z) \neq 0$, $z \in T^C \cup \dot{\mathbb{R}}^n$, it follows that for all $C' \Subset C$

$$\inf_{z \in T^{C'}} |f(z)| > 0.$$

By the theorem of Sec. 13.1 (for $\alpha' = \beta' = \varepsilon = 0$), $\frac{1}{f} \in H(C)$ so that $\frac{1}{f} = L[g]$, $g \in \mathscr{S}'(C^*)$. Therefore $\frac{1}{f^+} = F[g]$. Com-

paring that equality with (4.1), we obtain

$$g(\xi) = \frac{1}{\lambda}\delta(\xi) + g_1(\xi)$$

so that $\operatorname{supp} g_1 \subset C^*$ and therefore $g_1 \in \mathscr{L}^1(C^*)$. But then, by (4.1),

$$\frac{1}{f(z)} = \frac{1}{\lambda} + \int\limits_{C^*} g_1(\xi) e^{i(z,\xi)} d\xi \in W(C).$$

**13.5 Example** The Cauchy kernel $\mathscr{K}_{V^+}(z)$ [see (2.17) of Sec. 10] is a divisor of unity in the algebra $H(V^+)$,

$$\frac{1}{\mathscr{K}_{V^+}(z)} = \frac{1}{2^n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right)} (-z^2)^{\frac{n+1}{2}} \in H(V^+).$$

This follows from the corollary to the theorem of Sec. 13.2 and from the fact that the polynomial

$$z^2 = (x+iy)^2 = x^2 - y^2 + 2i(x, y) \neq 0, \qquad z \in T^{V^+}.$$

Indeed, we would otherwise have

$$x^2 = y^2, \qquad x_0 y_0 = (\mathbf{x}, \mathbf{y}), \qquad y^2 > 0, \quad y_0 > 0,$$

that is,

$$0 < y^2 = x_0^2 - \mathbf{x}^2 = \frac{(\mathbf{x}, \mathbf{y})^2}{y_0^2} - |\mathbf{x}|^2 \leqslant \frac{|\mathbf{x}|^2}{y_0^2}(|\mathbf{y}|^2 - y_0^2) = -\frac{|\mathbf{x}|^2}{y_0^2} y^2 < 0,$$

which is a contradiction.

Thus, in the algebra $\mathscr{S}'(\overline{V}^+)$ there is an inverse operator of $\theta_{\overline{V}^+}*$. What is more, it is possible to define arbitrary real powers $\theta_{\overline{V}^+}^{\alpha}*$ of that convolution operator by putting

$$L[\theta_{\overline{V}^+}^{\alpha}] = \mathscr{K}_{V^+}^{\alpha}(z), \quad z \in T^{V^+}. \tag{5.1}$$

For the sake of definiteness, we choose that branch of the holomorphic function $\mathscr{K}_{V^+}^{\alpha}(z)$ that is positive for $z = iy$ [(see (2.2) of Sec. 10]. From (5.1) it follows that

$$\theta_{\overline{V}^+}^{\alpha} * \theta_{\overline{V}^+}^{\beta} = \theta_{\overline{V}^+}^{\alpha+\beta}, \text{ where } \alpha, \beta \text{ are arbitrary.} \tag{5.2}$$

The powers $\Box^\alpha$ of the d'Alembert operator are defined in similar fashion:

$$\Box = \Box\delta * = \frac{\partial^2}{\partial\xi_0^2} - \frac{\partial^2}{\partial\xi_1^2} - \ldots - \frac{\partial^2}{\partial\xi_n^2} = \frac{\partial^2}{\partial\xi_0^2} - \nabla^2.$$

We have

$$L[\Box] = -z^2 = c_n \mathscr{K}_{V+}^{-\frac{2}{n+1}}(z), \qquad c_n = 2^{\frac{2n}{n+1}} \pi^{\frac{n-1}{n+1}} \Gamma^{\frac{2}{n+1}}\left(\frac{n+1}{2}\right).$$

Therefore, setting

$$L[\Box^\alpha] = c_n^\alpha \mathscr{K}_{V+}^{-\frac{2\alpha}{n+1}}(z),$$

we obtain

$$\Box^\alpha = c_n^\alpha \theta_{\overline{V}+}^{-\frac{2\alpha}{n+1}} *, \qquad \Box = c_n \theta_{\overline{V}+}^{-\frac{2}{n+1}} * \tag{5.3}$$

by virtue of (5.1). By (5.2) the following relation holds true:

$$\Box^\alpha \Box^\beta = \Box^{\alpha+\beta}, \text{ where } \alpha, \beta \text{ are arbitrary.} \tag{5.4}$$

In particular, for $\alpha = -1$, we have, from (5.3),

$$\Box^{-1} = \frac{1}{c_n} \theta_{\overline{V}+}^{\frac{2}{n+1}} * = \frac{1}{c_n}\left(\theta_{\overline{V}+}^{-\frac{n-1}{n+1}} * \theta_{\overline{V}+}\right) * = \frac{1}{c_n^{\frac{n+1}{2}}} \Box^{\frac{n-1}{2}} \theta_{\overline{V}+} *.$$

That is,

$$\mathscr{E}(\xi) = \frac{1}{2^n \pi^{\frac{n-1}{2}} \Gamma\left(\frac{n+1}{2}\right)} \Box^{\frac{n-1}{2}} \theta_{\overline{V}+}(\xi). \tag{5.5}$$

From this, for $n = 3$, we obtain a familiar result:

$$\mathscr{E}(\xi) = \frac{1}{8\pi} \Box \theta_{\overline{V}+}(\xi) \tag{5.6}$$

for the fundamental solution of a three-dimensional wave operator.

*Remark 1.* Fractional powers of the operator $\Box$ were introduced in a different manner by Riesz [1].

*Remark 2.* In similar fashion we can introduce fractional and negative powers of the operator $\theta_{C^*} *$ in the $H(\mathscr{C})$ algebra for any regular cone $C$ (see Sec. 13.5):

$$L[\theta_{C^*}^\alpha *] = \mathscr{K}_C^\alpha(z), \qquad z \in T^C, \quad -\infty < \alpha < \infty.$$

Chapter III

# Some Applications in Mathematical Physics

## 14 Differential Operators With Constant Coefficients

The theory of generalized functions has exerted a strong influence on the development of the theory of linear differential equations. First to be mentioned here are the fundamental works of L. Gårding, L. Hörmander, B. Malgrange, I. M. Gelfand, L. Ehrenpreis of the 1950s devoted to the general theory of linear partial differential equations irrespective of their type. The results of these studies are summarized in Hörmander's *Linear Partial Differential Operators* [1], which appeared in 1963. In recent years, big advances have been made in the theory of the so-called pseudodifferential operators [a generalization of differential and integral (singular) operators]. [§]

**14.1 Fundamental solutions in $\mathscr{D}'$** One of the basic and most profound results is the proof of the existence of a fundamental solution $\mathscr{E}(x)$ in $\mathscr{D}'$ of any linear differential operator $P(D) \not\equiv 0$ with constant coefficients (see Sec. 4.8(c)), that is,

$$P(D)\mathscr{E}(x) = \delta(x) \tag{1.1}$$

where

$$P(D) = \sum_{|\alpha| \leqslant m} a_\alpha D^\alpha, \qquad \sum_{|\alpha|=m} |a_\alpha| \neq 0, \tag{1.2}$$

is a differential operator of the $m$th order.

This result was first obtained independently by L. Ehrenpreis [2] (1954) and B. Malgrange [2] (1953).

Before proceeding to the proof of the existence of a fundamental solution, we will first prove two lemmas on polynomials.

Lemma 1 *If*

$$P(\xi) = \sum_{|\alpha| \leqslant m} a_\alpha \xi^\alpha, \qquad \sum_{|\alpha|=m} |a_\alpha| \neq 0,$$

*is an arbitrary polynomial of degree $m \geqslant 1$, then there exists a nonsingular linear real transformation of coordinates*

$$\xi = C\xi', \qquad \det C \neq 0, \qquad C = (c_{ki}),$$

[§] See Hörmander [3-5] and Kohn and Nirenberg [1, 2].

*that transforms the polynomial P to the form*

$$\widetilde{P}(\xi') = d\xi_1'^m + \sum_{0 \leqslant k \leqslant m-1} \widetilde{P}_k(\xi_2', \ldots, \xi_n')\, \xi_1'^k, \qquad a \neq 0.$$

*Proof*. The coefficient of $\xi_1'^m$ in the polynomial $\widetilde{P}(\xi') = P(C\xi')$ is equal to

$$\sum_{|\alpha|=m} a_\alpha c_{11}^{\alpha_1} c_{21}^{\alpha_2} \ldots c_{n1}^{\alpha_n}. \tag{1.3}$$

Since $\sum_{|\alpha|=m} |a_\alpha| \neq 0$, we can choose $n$ real numbers $c_{11}, c_{21}, \ldots, c_{n1}$ so that the expression (1.3) is not zero; we then have $\sum_{1 \leqslant k \leqslant n} |c_{k1}| \neq 0$. The remaining numbers $c_{ki}$ are chosen to be arbitrary real numbers so that det $C \neq 0$. The proof of Lemma 1 is complete.

Lemma 2 *Suppose*

$$P(\xi) = a\xi_1^m + \sum_{0 \leqslant k \leqslant m-1} P_k(\xi_2, \ldots, \xi_n)\, \xi_1^k, \qquad a \neq 0, \tag{1.4}$$

*is a polynomial. Then there is a constant $\varkappa$, dependent solely on $m$, such that for every point $\xi \in \mathbb{R}^n$ there is an integer $k$, $0 \leqslant k \leqslant m$, such that the folloning inequality holds*:

$$\left| P\left(\xi_1 + i\tau \frac{k}{m}, \ \xi_2, \ldots, \xi_n\right) \right| \geqslant a\varkappa, \qquad |\tau| = 1. \tag{1.5}$$

*Proof*. Fix $\xi \in \mathbb{R}^n$. Factor the polynomial $P(z, \xi_2, \ldots, \xi_n)$ into factors involving $z$:

$$P(z, \xi_2, \ldots, \xi_n) = a(z - \lambda_1) \ldots (z - \lambda_m)$$

so that $\lambda_j = \lambda_j(\xi_2, \ldots, \xi_n)$, $j = 1, 2, \ldots, m$, and

$$P\left(\xi_1 + i\tau \frac{k}{m}, \ \xi_2, \ldots, \xi_n\right) = a\left(\xi_1 - \lambda_1 + i\tau \frac{k}{m}\right) \ldots \left(\xi_1 - \lambda_m + i\tau \frac{k}{m}\right). \tag{1.6}$$

Using the "box" principle, we conclude that among the $m + 1$ circles $|\tau'| = \frac{j}{m}$, $j = 0, 1, \ldots, m$, there is at least one, $|\tau| = \frac{k}{m}$, distant from $m$ points $\lambda_1 - \xi_1, \ldots, \lambda_m - \xi_1$

at least by $\frac{1}{2m}$ (Fig. 29). From this and from (1.6) follows the inequality (1.5) for $\varkappa = \left(\frac{1}{2m}\right)^m$, which completes the proof of Lemma 2.

Theorem (Malgrange-Ehrenpreis) *Every differential operator with constant coefficients $P(D) \not\equiv 0$ has a fundamental solution in $\mathcal{D}'$.*

*Proof*. Since a nonsingular linear real transformation carries $\mathcal{D}'$ onto $\mathcal{D}'$ (see Sec. 1.10), then by virtue of Lemma 1 it suffices

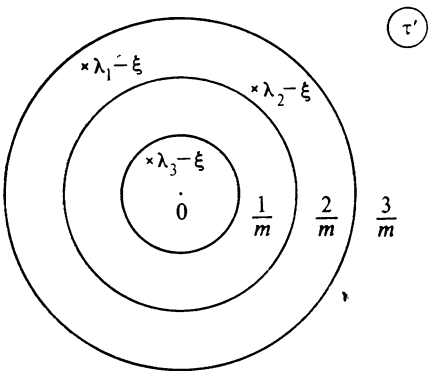



Figure 29

to prove the theorem for the case where the polynomial $P(i\xi)$ is of the form (1.4).

Suppose $f_0, f_1, \ldots, f_m$ are measurable nonnegative functions specified on $\mathbb{R}^n$ and such that $\sum\limits_{0 \leqslant k \leqslant m} f_k(\xi) \equiv 1$, $\xi \in \mathbb{R}^n$, and $f_k(\xi) = 0$ for those $\xi$ for which

$$\min_{|\tau|=1} \left| P\left(i\xi_1 - \tau \frac{k}{m},\ i\xi_2, \ \ldots,\ i\xi_n\right) \right| < a\varkappa \tag{1.7}$$

(by Lemma 2, such functions and an $\varkappa > 0$ exist).

We now determine the generalized function $\mathscr{E}$ by putting, for all $\varphi \in \mathcal{D}$,

$$(\mathscr{E}, \varphi) = \frac{1}{(2\pi)^n} \sum_{0 \leqslant k \leqslant m} \int f_k(\xi)$$

$$\times \frac{1}{2\pi i} \int\limits_{|\tau|=1} \frac{L[\varphi]\left(\xi_1 + i\tau \frac{k}{m},\ \xi_2,\ \ldots,\ \xi_n\right)}{P\left(i\xi_1 - \tau \frac{k}{m},\ i\xi_2,\ \ldots,\ i\xi_n\right)} \frac{d\tau}{\tau}\, d\xi, \tag{1.8}$$

where $L[\varphi]$ is the Laplace transform of the function $\varphi$ (see Sec. 9.1). We will prove that the expression on the right of (1.8) exists and determines a linear and continuous functional on $\mathcal{D}$, that is, $\mathcal{E} \in \mathcal{D}'$. But this assertion follows from the following estimates:

$$|(\mathcal{E}, \varphi)|$$

$$\leqslant \frac{1}{(2\pi)^n} \sum_{0 \leqslant k \leqslant m} \int f_k(\xi) \frac{\max\limits_{|\tau|=1} \left| L[\varphi]\left(\xi_1 + i\tau \frac{k}{m}, \xi_2, \ldots, \xi_n\right)\right|}{\min\limits_{|\tau|=1} \left| P\left(i\xi_1 - \tau \frac{k}{m}, i\xi_2, \ldots, i\xi_n\right)\right|} d\xi$$

$$\leqslant \frac{1}{(2\pi)^n a\varkappa} \sum_{0 \leqslant k \leqslant m} \max e^{|\operatorname{Re}\tau| \frac{k}{m}} \int \frac{d\xi}{\left(1 + \left|\xi_1 + \frac{i\tau k}{m}\right|^2 + \xi_2^2 + \ldots + \xi_n^2\right)^N}$$

$$\times \int\limits_{|x|<R} |(1-\Delta)^N \varphi(x)|\, dx.$$

That is to say, from the estimate

$$|(\mathcal{E}, \varphi)| \leqslant K_N \int\limits_{|x|<R} |(1-\Delta)^N \varphi(x)|\, dx, \tag{1.9}$$

which holds for all integral $N > n/2$ and for all $\varphi \in \mathcal{D}(U_R)$. In deriving the estimate (1.9), we made use of the estimates (3.4) to (3.5) of Sec. 12 for the integral function $L[\varphi](\zeta)$ and also the estimate (1.7) and the properties of the functions $\{f_k\}$.

It remains to verify that the constructed generalized function $\mathcal{E}$ in $\mathcal{D}'$ satisfies the equation (1.1). Using (1.8), for all $\varphi \in \mathcal{D}$ we have

$$(P(D)\mathcal{E}, \varphi) = (\mathcal{E}, P(-D)\varphi)$$

$$= \frac{1}{(2\pi)^n} \sum_{0 \leqslant k \leqslant m} \int f_k(\xi) \frac{1}{2\pi i}$$

$$\times \int\limits_{|\tau|=1} \frac{L[P(-D)\varphi]\left(\xi_1 + i\tau \frac{k}{m}, \xi_2, \ldots, \xi_n\right)}{P\left(i\xi_1 - \tau \frac{k}{m}, i\xi_2, \ldots, i\xi_n\right)} \frac{d\tau}{\tau} d\xi$$

$$= \frac{1}{(2\pi)^n} \sum_{0 \leqslant k \leqslant m} \int f_k(\xi) \frac{1}{2\pi i} \int\limits_{|\tau|=1} L[\varphi]\left(\xi_1 + i\tau \frac{k}{m}, \xi_2, \ldots, \xi_n\right) \frac{d\tau}{\tau} d\xi$$

$$= \frac{1}{(2\pi)^n} \sum_{0 \leqslant k \leqslant m} \int f_k(\xi) F[\varphi](\xi)\, d\xi$$

$$= \frac{1}{(2\pi)^n} \int F[\varphi](\xi)\, d\xi = \varphi(0) = (\delta, \varphi),$$

which is what was required, and the theorem is proved.

Having the fundamental solution $\mathscr{E}$ of the operator $P(D)$, we can construct a solution $u$ in $\mathscr{D}'$ of the equation

$$P(D)\, u = f, \qquad f \in \mathscr{D}' \tag{1.10}$$

in the form of a convolution

$$u = \mathscr{E} * f \tag{1.11}$$

for those $f$ in $\mathscr{D}'$ for which this convolution exists in $\mathscr{D}'$ (see Sec. 4.8(c)). Thus, by choosing various fundamental solutions, it is possible to obtain various classes of right-hand members for which the equation (1.10) is solvable in the form of the convolution (1.11).

**14.2 Fundamental solutions of slow growth** In Subsection 1 we established the fact that every nonzero differential operator with constant coefficients has at least one fundamental solution in $\mathscr{D}'$. The question arises—one that is important in applications—of how to find a fundamental solution with the required properties of growth, of support, of smoothness, and so forth. A convenient tool in this respect is the method of Fourier transforms. However, the Fourier transform technique that was developed in Sec. 6 is applicable to generalized functions of slow growth. For this reason, in constructing a fundamental solution by the method of Fourier transforms we confine ourselves from the very start to the class $\mathscr{S}'$.

The equation (1.1) in the class $\mathscr{S}'$ is equivalent to the algebraic equation (see Sec. 6.3(b))

$$P(-i\xi)\, \tilde{\mathscr{E}}(\xi) = 1 \tag{2.1}$$

with respect to the Fourier transform $F[\mathscr{E}] = \tilde{\mathscr{E}}$. Thus, the problem of seeking a fundamental solution of slow growth turns out to be a special case of the more general problem of "dividing" a generalized function of slow growth by a polynomial, that is, of the problem of finding a solution $u$ in $\mathscr{S}'$ of the equation

$$P(\xi)\, u = f \tag{2.2}$$

where $P \not\equiv 0$ is a polynomial and $f$ is a specified generalized function in $\mathscr{S}'$. The solvability of the problem of "division" was proved in 1958 independently by Hörmander [2] and Łojasiewicz [1].

The proof is based on the following lemma.

Lemma *The mapping*

$$\varphi \to P\varphi, \qquad \varphi \in \mathscr{S},$$

*has a continuous inverse in* $\mathscr{S}$; *in other words, for every integer* $p \geqslant 0$ *there are numbers* $K_p > 0$ *and an integer* $p' = p'(p) \geqslant 0$ *such that the following inequality holds*:

$$\| \varphi \|_p \leqslant K_p \| P\varphi \|_{p'}, \qquad \varphi \in C^{p'}(\mathbb{R}^n). \tag{2.3}$$

The existing proofs of this lemma are extremely complicated. We confine ourselves here to the proof of only the case $n = 1$.

First we will prove (2.3) for the case $P(\xi) = \xi$. Setting $\psi = \xi\varphi$, we have

$$\varphi = \frac{\psi}{\xi}, \qquad |\varphi(\xi)| \leqslant \begin{cases} \max\limits_{|\xi| \leqslant 1} |\psi'(\xi)|, & |\xi| \leqslant 1, \\ |\psi(\xi)|, & |\xi| > 1; \end{cases}$$

$$\varphi' = \frac{\psi'}{\xi} - \frac{\psi}{\xi^2}, \qquad |\varphi'(\xi)| \leqslant \begin{cases} \frac{3}{2} \max\limits_{|\xi| \leqslant 1} |\psi''(\xi)|, & |\xi| \leqslant 1, \\ |\psi'(\xi)| + |\psi(\xi)|, & |\xi| > 1; \end{cases}$$

and so forth. Consequently,

$$\| \varphi \|_p = \sup_{\substack{\xi \\ |\alpha| \leqslant p}} (1 + |\xi|^2)^{p/2} |\varphi^{(\alpha)}(\xi)| \\ \leqslant K_p \sup_{\substack{\xi \\ |\alpha| \leqslant p+1}} (1 + |\xi|^2)^{p/2} |\psi^{(\alpha)}(\xi)| \leqslant K_p \| \xi\varphi \|_{p+1},$$

which is what was required ($p' = p + 1$).

From the fact that the inequality (2.3) holds for $P = \xi$ follows that it also holds for $P = \xi - \xi_0$ and, hence, for all polynomials $P = a \prod\limits_{1 \leqslant k \leqslant m} (\xi - \xi_k)$:

$$\| \varphi \|_p \leqslant K_p^{(1)} \| (\xi - \xi_1) \varphi \|_{p+1} \leqslant K_p^{(2)} \| (\xi - \xi_1)(\xi - \xi_2) \varphi \|_{p+2} \leqslant \\ \ldots \leqslant K_p^{(m)} \| (\xi - \xi_1)(\xi - \xi_2) \ldots (\xi - \xi_m) \varphi \|_{p+m} = K_p \| P\varphi \|_{p+m}.$$

Using the Hörmander lemma, we will prove that the equation (2.2) is always solvable in $\mathscr{S}'$.

Indeed, consider the linear functional

$$P\varphi \to (f, \varphi)$$

defined on a linear subspace $[\psi : \psi = P\varphi, \ \varphi \in \mathcal{S}]$ of the space $\mathcal{S}$. By the Hörmander lemma, this functional is continuous: if $P\varphi_k \to 0$, $k \to \infty$ in $\mathcal{S}$, then $\varphi_k \to 0$, $k \to \infty$ in $\mathcal{S}$ and therefore $(f, \varphi_k) \to 0$, $k \to \infty$. By the Hahn-Banach theorem, there exists a (linear) continuous extension $u$, $\varphi \to (u, \varphi)$, of that functional on the whole of $\mathcal{S}$ so that $u \in \mathcal{S}'$ and $(u, P\varphi) = (f, \varphi)$. And this means that the functional $u$ satisfies (2.2).

Passing to the Fourier transform, we are convinced that the following theorem holds.

Theorem (Hörmander-Łojasiewicz) *The equation*

$$P(D)\, u = f, \tag{2.4}$$

*where* $P(D) \not\equiv 0$, *is solvable in* $\mathcal{S}'$ *for all* $f \in \mathcal{S}'$.

Corollary *Every nonzero linear differential operator with constant coefficients has a fundamental solution of slow growth.*

**14.3 A descent method** Let us consider the linear differential equation with constant coefficients in the space $\mathbb{R}^{n+1}$ of the variables $(x, t) = (x_1, \ldots, x_n, t)$,

$$P(D, D_0)\, u = f(x) \times \delta(t), \qquad f \in \mathcal{D}'(\mathbb{R}^n), \tag{3.1}$$

where $D_0 = \frac{\partial}{\partial t}$, $\quad P(D, D_0) = \sum\limits_{1 \leqslant q \leqslant p} P_q(D)\, D_0^q + P_0(D)$, and

$P_q(D)$ are differential operators with respect to the variables $x$.

We will say that the generalized function $u(x, t)$ taken from $\mathcal{D}'(\mathbb{R}^{n+1})$ admits of an extension to functions of the form $\varphi(x)\, 1(t)$, where $\varphi \in \mathcal{D}(\mathbb{R}^n)$, if no matter what the sequence of basic functions $\eta_k(x, t)$, $k = 1, 2, \ldots$, in $\mathcal{D}(\mathbb{R}^{n+1})$, which sequence converges to 1 in $\mathbb{R}^{n+1}$ (see Sec. 4.1), there exists a limit

$$\lim_{k\to\infty} (u, \varphi(x)\, \eta_k(x, t)) = (u, \varphi(x)\, 1(t)), \qquad \varphi \in \mathcal{D}(\mathbb{R}^n), \tag{3.2}$$

and that limit is independent of the sequence $\{\eta_k\}$.

We denote the functional (3.2) by $u_0$,

$$(u_0, \varphi) = (u, \varphi(x)\, 1(t)) = \lim_{k\to\infty} (u, \varphi(x)\, \eta_k(x, t)), \qquad \varphi \in \mathcal{D} \tag{3.3}$$

Clearly, for any $k$ the functional $(u, \varphi(x)\,\eta_k(x, t))$ is linear and continuous on $\mathscr{D}$, that is, it belongs to $\mathscr{D}'$. Therefore, by the theorem on the completeness of the space $\mathscr{D}'$ (see Sec. 1.4) the functional $u_0$ as well belongs to $\mathscr{D}'$: $u_0 \in \mathscr{D}'$.

We will call the generalized function $u_0(x)$ the *generalized integral with respect to* $t$ of the generalized function $u(x, t)$.

We now give a criterion of existence for a generalized integral with respect to $t$.

Theorem I *In order that for* $u$ *in* $\mathscr{D}'(\mathbb{R}^{n+1})$ *there exist* $u_0$—*a generalized integral with respect to* $t$—*it is necessary and sufficient that there exist a convolution* $u * [\delta(x) \times 1(t)]$. *Here, the following equation holds*:

$$u * [\delta(x) \times 1(t)] = u_0(x) \times 1(t). \tag{3.4}$$

We prove sufficiency. Suppose there exists a convolution $u * [\delta(x) \times 1(t)]$. Then there exists a generalized function $u_0$ in $\mathscr{D}'$ such that (3.4) holds (see Sec. 4.2(c) and Sec. 3.3).

We will prove that $u_0$ is a generalized integral with respect to $t$ for $u(x, t)$. Suppose $\{\xi_k(x)\}$, $\{\eta_k(x, t)\}$, and $\{\chi_i(t)\}$ are sequences of basic functions in $\mathscr{D}(\mathbb{R}^n)$, $\mathscr{D}(\mathbb{R}^{n+1})$, and $\mathscr{D}(\mathbb{R}^1)$, which sequences converge to 1 in $\mathbb{R}^n$, $\mathbb{R}^{n+1}$, and $\mathbb{R}^1$ respectively. Suppose $\varphi \in \mathscr{D}(\mathbb{R}^n)$. Then there is a number $N$ such that $\xi_k\varphi = \varphi$ for all $k \geqslant N$. Furthermore, for every $k$ there is a number $i_k$ such that

$$\eta_k(x, t) \int \chi_{i_k}(t')\,\omega_\varepsilon(t + t')\,dt' = \eta_k(x, t) \tag{3.5}$$

where $\omega_\varepsilon$ is the "cap" (see Sec. 1.2). Indeed, if $(x, t) \in \operatorname{supp} \eta_k \subset \subset [(x, t) : |t| < R_k]$, then, choosing the number $i_k$ so that $\chi_{i_k}(t) = 1$ for $|t| \leqslant R_k + \varepsilon$, we obtain

$$\int \chi_{i_k}(t')\,\omega_\varepsilon(t \quad t')\,dt' = \int_{|t'|<\varepsilon} \omega_\varepsilon(t')\,\chi_{i_k}(t' - t)\,dt'$$

$$= \int \omega_\varepsilon(t')\,dt' = 1.$$

Now, making use of (3.4) and (3.5) and also of the definitions of a generalized integral with respect to $t$ (3.3) and of a convolution (see Sec. 4.1), and also noting that the sequence

$$\xi_k(x)\,\xi_k(x')\,\eta_k(x, t)\,\chi_{i_k}(t') \qquad k = 1, 2, \ldots,$$

of basic functions in $\mathcal{D}(\mathbb{R}^{2n+2})$ converges to 1 in $\mathbb{R}^{2n+2}$, we have, for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\lim_{k\to\infty}(u, \varphi(x)\eta_k(x, t))$$

$$= \lim_{k\to\infty}\left(u(x, t), \xi_k(x)\varphi(x)\eta_k(x, t)\int \chi_{i_k}(t')\,\omega_\varepsilon(t+t')\,dt'\right)$$

$$= \lim_{k\to\infty}(u(x, t)\times\delta(x')\times 1(t'),$$

$$\xi_k(x)\,\xi_k(x')\,\eta_k(x, t)\chi_{i_k}(t')\,\varphi(x+x')\,\omega_\varepsilon(t+t'))$$

$$= (u * [\delta(x)\times 1(t)], \varphi(x)\,\omega_\varepsilon(t)) = (u_0(x)\times 1(t), \varphi(x)\,\omega_\varepsilon(t))$$

$$= \left(u_0(x), \varphi(x)\int \omega_\varepsilon(t)\,dt\right) = (u_0, \varphi),$$

which is what we set out to prove.

We now prove necessity. Suppose for $u$ there exists $u_0$, which is a generalized integral with respect to $t$. Suppose $\xi_k(x, t; x', t')$, $k = 1, 2, \ldots$, is a sequence of basic functions in $\mathcal{D}(\mathbb{R}^{2n+2})$ that converges to 1 in $\mathbb{R}^{2n+2}$. Let $\varphi \in \mathcal{D}(\mathbb{R}^{n+1})$. Then for every compact $K \subset \mathbb{R}^{n+1}$ there is a number $N$ such that for all $k \geqslant N$

$$\int \xi_k(x, t; 0, t')\,\varphi(x, t+t')\,dt' = \int \varphi(x, t+t')\,dt' = \int \varphi(x, t')\,dt',$$
$$(x, t) \in K.$$

Consequently, there exists a sequence $\eta_k(x, t)$ of functions taken from $\mathcal{D}(\mathbb{R}^{n+1})$, which sequence converges to 1 in $\mathbb{R}^{n+1}$ and is such that the sequence of functions

$$\chi_k(x, t) = \int \xi_k(x, t; 0, t')\,\varphi(x, t+t')\,dt'$$
$$- \eta_k(x, t)\int \varphi(x, t')\,dt' + \eta_k(x, t), \quad k = 1, 2, \ldots, \tag{3.6}$$

in $\mathcal{D}(\mathbb{R}^{n+1})$ converges to 1 in $\mathbb{R}^{n+1}$. Let $\varphi_0(x)$ be a function in $\mathcal{D}(\mathbb{R}^n)$ equal to 1 on supp $\varphi(x, t)$ so that $\varphi = \varphi_0\varphi$. Then, using (3.6) and the definitions of a generalized integral in $t$ and of a convolution, we have

$$(u_0(x)\times 1(t), \varphi)$$

$$= \left(u_0, \int \varphi(x, t)\,dt\right) = \left(u_0(x), \varphi_0(x)\int \varphi(x, t)\,dt\right)$$

$$= \lim_{k\to\infty} \left(u(x, t), \varphi_0(x) \eta_k(x, t) \int \varphi(x, t)\, dt\right)$$

$$= -\lim_{k\to\infty} (u(x, t), \varphi_0(x) \chi_k(x, t)) + \lim_{k\to\infty} (u(x, t), \varphi_0(x) \eta_k(x, t))$$

$$+ \lim_{k\to\infty} \left(u(x, t), \varphi_0(x) \int \xi_k(x, t; 0, t') \varphi(x, t+t')\, dt'\right)$$

$$= -(u_0, \varphi_0) + (u_0, \varphi_0)$$

$$+ \lim_{k\to\infty} (u(x, t) \times [\delta(x') \times 1(t')],$$

$$\xi_k(x, t; x', t') \varphi_0(x+x') \varphi(x+x', t+t'))$$

$$= (u * [\delta(x) \times 1(t)], \varphi_0\varphi) = (u * [\delta(x) \times 1(t)], \varphi),$$

which is what we set out to prove.

Corollary 1 *Suppose the function* $u(x, t)$ *is measurable and* $\int |u(x, t)|\, dt \in \mathscr{L}^1_{\text{loc}}$. *Then its generalized integral in* $t$ *exists in* $\mathscr{L}^1_{\text{loc}}$ *and can be represented by the classical integral*

$$u_0(x) = \int u(x, t)\, dt. \tag{3.7}$$

*Remark.* The formula (3.7) shows that a generalized integral in $t$ is an extension of the classical concept of an integral in $t$ to generalized functions.

Corollary 2 *If* $u = f(x) \times \delta(t)$, *where* $f \in \mathscr{D}'$, *then* $u_0 = f$.

Theorem II *If the solution* $u$ *in* $\mathscr{D}'(\mathbb{R}^{n+1})$ *of the equation (3.1) possesses* $u_0$ *(a generalized integral in* $t$*), then* $u_0$ *satisfies the equation*

$$P_0(D)\, u_0 = f(x). \tag{3.8}$$

*Proof.* Let $\eta_k(x, t)$, $k = 1, 2, \ldots,$ be a sequence of functions in $\mathscr{D}(\mathbb{R}^{n+1})$ that converges to 1 in $\mathbb{R}^{n+1}$. Then for $q = 1, 2, \ldots,$ the sequence of functions $\eta_k + D_0^q \eta_k$, $k = 1, 2, \ldots,$ also converges to 1 in $\mathbb{R}^{n+1}$ and, hence, for all $\varphi$ in $\mathscr{D}(\mathbb{R}^n)$,

$$\lim_{k\to\infty} (u, \varphi(x) D_0^q \eta_k(x, t))$$

$$= \lim_{k\to\infty} (u, \varphi(x) [\eta_k(x, t) + D_0^q \eta_k(x, t)]) - \lim_{k\to\infty} (u, \varphi(x) \eta_k(x, t))$$

$$= (u_0, \varphi) - (u_0, \varphi) = 0.$$

Taking into account the resulting equation, we verify that $u_0$ satisfies (3.8):

$$(P_0(D)u_0, \varphi)$$
$$=(u_0, P_0(-D)\varphi)=\lim_{k\to\infty}(u, P_0(-D)\varphi(x)\eta_k(x, t))$$
$$=\lim_{k\to\infty}\Big(u, P_0(-D)\varphi(x)\eta_k(x, t)+\sum_{1\leqslant q\leqslant p}(-1)^q P_q(-D)\varphi(x)D_0^q\eta_k(x, t)\Big)$$
$$=\lim_{k\to\infty}(u, P(-D, -D_0)\varphi(x)\eta_k(x, t))$$
$$=\lim_{k\to\infty}(P(D, D_0)u, \varphi(x)\eta_k(x, t))$$
$$=\lim_{k\to\infty}(f(x)\times\delta(t), \varphi(x)\eta_k(x, t))$$
$$=\lim_{k\to\infty}(f(x), \varphi(x)\eta_k(x, 0))=(f, \varphi).$$

The theorem is proved.

The foregoing method of obtaining a solution $u_0(x)$ of the equation (3.8) in $n$ variables in terms of the solution $u(x, t)$ of equation (3.1) in $n+1$ variables is termed the *method of descent with respect to the variable t.*

The descent method is particularly convenient for the construction of fundamental solutions. Applying Theorem II for $f = \delta(x)$, we obtain the following corollary.

Corollary *If a fundamental solution* $\mathscr{E}(x, t)$ *of the operator* $P(D, D_0)$ *possesses* $\mathscr{E}_0(x)$ *(a generalized integral in* $t$*), then* $\mathscr{E}_0$ *is a fundamental solution of the operator* $P_0(D)$.

The fundamental solution $\mathscr{E}_0$ satisfies the relation

$$\mathscr{E}_0(x)\times 1(t)=\mathscr{E}*[\delta(x)\times 1(t)]. \tag{3.9}$$

The physical meaning of (3.9) consists in the fact that $\mathscr{E}_0(x)$ is a perturbation (independent of $t$) of a source $\delta(x)\times 1(t)$ concentrated along the axis $t$ (compare Sec. 4.8(c)).

**14.4 Examples** (a) Particular solutions of the equation $\xi u = 1$ are the generalized functions

$$\frac{1}{\xi+i0}, \quad \frac{1}{\xi-i0}, \quad \mathscr{P}\frac{1}{\xi}$$

which, by virtue of the Sochozki formulas (8.3) and (8.3′) of Sec. 1, differ by the expression const $\times$ $\delta(\xi)$, which is the general solution of the homogeneous equation $\xi u = 0$ (see Sec. 2.6).

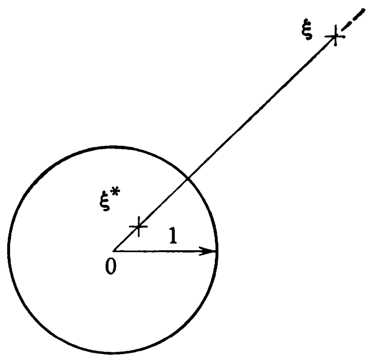



Figure 30

(b) If a polynomial $P(\xi)$ does not have real zeros, then the function $\frac{1}{P(\xi)}$ belongs to $\theta_M$ and is the sole solution of the equation $P(\xi)u = 1$.

This assertion follows from the following lemma.

Lemma *If a polynomial* $P(\xi) \neq 0$, $\xi \in \mathbb{R}^n$, *there are constants* $C > 0$ *and* $\nu$ *such that the following inequality holds true*:

$$|P(\xi)| \geqslant C(1 + |\xi|^2)^{-\nu}, \qquad \xi \in \mathbb{R}^n. \tag{4.1}$$

*Proof*. It suffices to prove the estimate (4.1) for $|\xi| > 1$. To do this, perform an inversion transformation (Fig. 30):

$$\xi^* = \frac{\xi}{|\xi|^2}, \qquad \xi = \frac{\xi^*}{|\xi^*|^2}, \qquad |\xi||\xi^*| = 1.$$

Suppose $m$ is the degree of $P$. The polynomial

$$P^*(\xi^*) = |\xi^*|^{2m} P\left(\frac{\xi^*}{|\xi^*|^2}\right) \tag{4.2}$$

has a unique zero in $\mathbb{R}^n$: $\xi^* = 0$. Therefore there exist numbers $C_1 > 0$ and $\mu > 0$ such that

$$|P^*(\xi^*)| \geqslant C_1 |\xi^*|^{\mu}, \qquad |\xi^*| < 1,$$

and so, by (4.2),

$$|P(\xi)| \geqslant C_1 |\xi^*|^{\mu-2m} = C_1 |\xi|^{2m-\mu}, \qquad |\xi| > 1.$$

The proof of the lemma is complete.

(c) The equation

$$\xi_1 u(\xi) = f(\xi)$$

is solvable for any $f$ in $\mathscr{S}'$ and its general solution is of the form

$$(u, \varphi) = (f, \psi) + (\delta(\xi_1) \times u_1(\xi_2, \ldots, \xi_n), \varphi), \qquad \varphi \in \mathscr{S}, \quad (4.3)$$

where $u_1$ is an arbitrary generalized function in $\mathscr{S}'(\mathbb{R}^{n-1})$,

$$\psi(\xi) = \frac{1}{\xi_1}[\varphi(\xi) - \eta(\xi_1)\varphi(0, \xi_2, \ldots, \xi_n)]$$

where $\eta(\xi_1)$ is an arbitrary function in $\mathscr{D}$ equal to 1 in the neighbourhood of 0.

The proof of this assertion is similar to that for the space $\mathscr{D}'$ (see Sec. 3.3).

(d) The function $\mathscr{E}(t) = \theta(t) Z(t)$, where $Z(t)$ is a solution of the homogeneous differential equation (compare Sec. 4.8(f))

$$P\left(\frac{d}{dt}\right) Z = Z^{(m)} + a_1 Z^{(m-1)} + \ldots + a_m Z = 0$$

that satisfies the conditions

$$Z(0) = Z'(0) = \ldots = Z^{(m-2)}(0) = 0, \qquad Z^{(m-1)}(0) = 1,$$

is a fundamental solution of the operator $P\left(\frac{d}{dt}\right)$.

Indeed, using (3.1) of Sec. 2, we obtain

$$\mathscr{E}'(t) = \theta(t) Z'(t), \ \ldots, \ \mathscr{E}^{(m-1)}(t) = \theta(t) Z^{(m-1)}(t),$$

$$\mathscr{E}^{(m)}(t) = \delta(t) + \theta(t) Z^{(m)}(t),$$

whence

$$P\left(\frac{d}{dt}\right) \mathscr{E}(t) = \theta(t) P\left(\frac{d}{dt}\right) Z(t) + \delta(t) = \delta(t),$$

which completes the proof.

In particular, the function

$$\mathscr{E}(t) = \theta(t)\, e^{-at} \tag{4.4}$$

is a fundamental solution of the operator $\frac{d}{dt} + a$.

(e) *A fundamental solution of the heat conduction operator,*

$$\frac{\partial \mathscr{E}}{\partial t} - a^2 \nabla^2 \mathscr{E} = \delta(x, t). \tag{4.5}$$

Applying the Fourier transform $F_x$ (see Sec. 6.2) to (4.5) and using the formulas (3.8) and (3.9) of Sec. 6,

$$F_x[\delta(x, t)] = F_x[\delta(x) \times \delta(t)] = F[\delta](\xi) \times \delta(t)$$

$$= 1(\xi) \times \delta(t),$$

$$F_x\left[\frac{\partial \mathscr{E}}{\partial t}\right] = \frac{\partial}{\partial t} F_x[\mathscr{E}], \qquad F_x[\nabla^2 \mathscr{E}] = -|\xi|^2 F_x[\mathscr{E}],$$

we obtain, for the generalized function $\tilde{\mathscr{E}}(\xi, t) = F_x[\mathscr{E}]$, the equation

$$\frac{\partial \tilde{\mathscr{E}}}{\partial t} + a^2 |\xi|^2 \tilde{\mathscr{E}} = 1(\xi) \times \delta(t). \tag{4.6}$$

Taking into account (4.4) with $a^2 |\xi|^2$ substituted for $a$, we conclude that the solution in $\mathscr{S}'$ of the equation (4.6) is the function

$$\tilde{\mathscr{E}}(\xi, t) = \theta(t)\, e^{-a^2|\xi|^2 t}.$$

From this, using the inverse Fourier transform $F_\xi^{-1}$ and (6.2) of Sec. 6, we obtain

$$F_\xi^{-1}[\tilde{\mathscr{E}}] = \frac{\theta(t)}{(2\pi)^n} \int e^{-a^2|\xi|^2 t - i(\xi, x)}\, d\xi = \frac{\theta(t)}{(2a\sqrt{\pi t})^n}\, e^{-\frac{|x|^2}{4a^2 t}}.$$

That is,

$$\mathscr{E}(x, t) = \frac{\theta(t)}{(2a\sqrt{\pi t})^n}\, e^{-\frac{|x|^2}{4a^2 t}}. \tag{4.7}$$

(f) *Fundamental solution of the wave operator* $\square$. It was demonstrated in Sec. 13.5 that the (generalized) function

$$\mathscr{E}_n(x) = \frac{1}{2\pi^{\frac{n-2}{2}}\, \Gamma\left(\frac{n+1}{2}\right)}\, \square^{\frac{n-1}{2}}\, \theta(x_0)\, \theta(x^2), \qquad x = (x_0, \mathbf{x}), \tag{4.8}$$

where $\theta(x_0)\,\theta(x^2) = \theta_{\bar{V}^+}(x)$ is the characteristic function of the light cone of the future $\bar{V}^+ = [x : x_0 \geqslant |\mathbf{x}|]$, is a fundamental solution of the wave operator $\square$. Putting $n = 1$ in (4.8), we have

$$\mathscr{E}_1(x) = \frac{1}{2}\,\theta(x_0)\,\theta(x^2). \tag{4.9}$$

We will prove, for $n \geqslant 2$, the equality

$$\square\theta(x_0)\,\theta(x^2) = 2(n-1)\,\theta(x_0)\,\delta(x^2), \tag{4.10}$$

where the generalized function $\theta(x_0)\,\delta(x^2)$ is given by

$$\begin{aligned}(\theta(x_0)\,\delta(x^2), \varphi) &= \frac{1}{2}\int_0^\infty \frac{1}{x_0}\int_{|\mathbf{x}|=x_0} \varphi\, dS_{\mathbf{x}}\, dx_0 \\ &= \frac{1}{2}\int_{\mathbb{R}^n} \frac{\varphi(\mathbf{x}, |\mathbf{x}|)}{|\mathbf{x}|}\, d\mathbf{x}, \qquad \varphi \in \mathscr{D}(\mathbb{R}^{n+1}).\end{aligned} \tag{4.11}$$

Using the technique of differential forms and the Stokes theorem (see, for example, Vladimirov [1]), for all $\varphi \in \mathscr{D}(\mathbb{R}^{n+1})$ we have

$$\begin{aligned}&(\square\theta(x_0)\,\theta(x^2), \varphi) \\ &= (\theta(x_0)\,\theta(x^2), \square\varphi) = \int_{V^+} \square\varphi(x)\, dx_0 \wedge dx_1 \wedge \ldots \wedge dx_n \\ &= \int_{V^+} d\Big(\frac{\partial\varphi}{\partial x_0}\, dx_1 \wedge \ldots \wedge dx_n + \frac{\partial\varphi}{\partial x_1}\, dx_0 \wedge dx_2 \wedge \ldots \wedge dx_n + \\ &\qquad + (-1)^{n-1}\frac{\partial\varphi}{\partial x_n}\, dx_0 \wedge dx_1 \wedge \ldots \wedge dx_{n-1}\Big) \\ &= \int_{\partial V^+} \frac{\partial\varphi}{\partial x_0}\, dx_1 \wedge \ldots \wedge dx_n + \frac{\partial\varphi_1}{\partial x_1}\, dx_0 \wedge dx_2 \wedge \ldots \wedge dx_n + \\ &\qquad \ldots + (-1)^{n-1}\frac{\partial\varphi}{\partial x_n}\, dx_0 \wedge dx_1 \wedge \ldots \wedge dx_{n-1}.\end{aligned}$$

But on $\partial V^+ \setminus \{0\}$ we have, by virtue of the equation $x_0^2 = |\mathbf{x}|^2$, the relation

$$x_0\, dx_0 = x_1\, dx_1 + \ldots + x_n\, dx_n.$$

Therefore, continuing our chain of equations, we obtain

$$(\square\theta(x_0)\theta(x^2), \varphi) = \int_{\partial V^+} d\left\{\frac{\varphi}{x_0}[x_1\, dx_2 \wedge \quad \wedge dx_n + \right.$$

$$\left. \ldots + (-1)^{n-1} x_n\, dx_1 \wedge \quad \wedge dx_{n-1}]\right\}$$

$$-2(n-1)\int_{\partial V^+} \varphi \frac{dx_1 \wedge \ldots \wedge dx_n}{2x_0} \qquad (4.12)$$

By the Stokes theorem and by virtue of the finiteness of $\varphi$, the first integral in the right-hand member of (4.12) is zero.

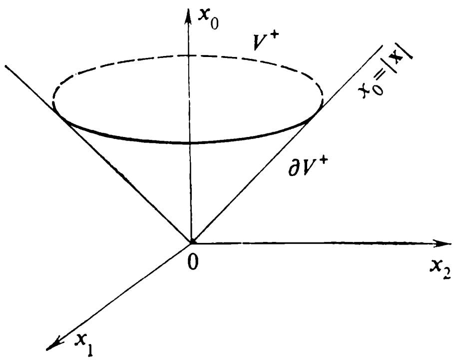



Figure 31

In the second integral, integration is performed along the outer side of the surface $\partial V^+$ (Fig. 31) so that $dx_1 \wedge \ldots \wedge dx_n = -d\mathbf{x}$ and therefore

$$(\square\theta(x_0)\theta(x^2), \varphi) = 2(n-1)\int_{R^n} \frac{\varphi(\mathbf{x}, |\mathbf{x}|)}{|\mathbf{x}|}\, d\mathbf{x},$$

whence, by (4.11), follows (4.10).

Putting $n = 2\nu + 1$ in (4.8) and using (4.10), we obtain

$$\mathscr{E}_{2\nu+1}(x) = \frac{1}{2^{2\nu-1}\pi^{\nu}\Gamma(\nu)} \square^{\nu-1}\theta(x_0)\,\delta(x^2). \qquad (4.13)$$

In particular, for $\nu = 1$ we derive the following from (4.13):

$$\mathscr{E}_3(x) = \frac{1}{2\pi}\theta(x_0)\,\delta(x^2). \qquad (4.14)$$

To find a formula similar to (4.13) for $n = 2\nu$, we take advantage of the descent method with respect to the variable $x_{n+1}$ (see

Sec. 14.3). To do this, it must be shown that $\mathscr{E}_{2\nu+1}(x, x_{2\nu+1})$, $x = (x_0, x_1, \ldots, x_{2\nu})$ possesses a generalized integral with respect to variable $x_{2\nu+1}$. Let the sequence $\eta_k(x, x_{2\nu+1})$,

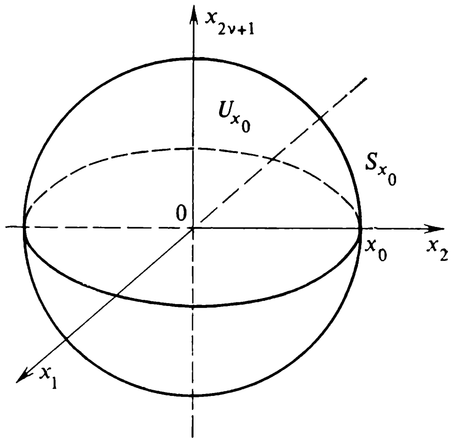



Figure 32

$k = 1, 2, \ldots$, of basic functions in $\mathscr{D}(\mathbb{R}^{n+2})$ converge to 1 in $\mathbb{R}^{n+2}$. Then, using (4.13) and (4.11), we have, for all $\varphi \in \mathscr{D}(\mathbb{R}^{n+2})$,

$$\lim_{k\to\infty} (\mathscr{E}_{2\nu+1}(x, x_{2\nu+1}), \varphi(x)\,\eta_k(x, x_{2\nu+1}))$$

$$= \frac{1}{2^{2\nu-1}\pi^\nu\Gamma(\nu)}$$

$$\times \lim_{k\to\infty} \frac{1}{2}\int_0^\infty \frac{1}{x_0} \int_{|x|^2+x_{2\nu+1}^2=x_0^2} \square^{\nu-1}[\varphi(x)\,\eta_k(x, x_{2\nu+1})]\,dS_{(x,\,x_{2\nu+1})}\,dx_0$$

$$= \frac{1}{2^{2\nu}\pi^\nu\Gamma(\nu)} \int_0^\infty \frac{1}{x_0} \int_{|x|^2+x_{2\nu+1}^2=x_0^2} \square^{\nu-1}\varphi(x)\,dS_{(x,\,x_{2\nu+1})}\,dx_0$$

Transform the last integral. Since $\square^{\nu-1}\varphi(x)$ does not depend on $x_{2\nu+1}$, then, by replacing the surface integral over the sphere $S_{x_0} = [(x, x_{2\nu+1}) : |\mathbf{x}|^2 + x_{2\nu+1}^2 = x_0^2]$ with twice the integral over the sphere $|\mathbf{x}| < x_0$ (Fig. 32), we get

$$(\mathscr{E}_{2\nu}, \varphi) = \frac{1}{2^{2\nu-1}\pi^\nu\Gamma(\nu)} \int_0^\infty \int_{|x|<x_0} \frac{\square^{\nu-1}\varphi(x)}{\sqrt{\overline{x^2}}}\,d\mathbf{x}\,dx_0.$$

That is,

$$\mathscr{E}_{2\nu}(x)=\frac{1}{2^{2\nu-1}\pi^{\nu}\Gamma(\nu)}\Box^{\nu-1}[\theta(x_0)x_+^2]^{-1/2}, \tag{4.15}$$

where

$$[\theta(x_0)x_+^2]^{-1/2}=\begin{cases}(x^2)^{-1/2}, & \text{if } x_0\geqslant|\mathbf{x}|,\\ 0, & \text{if } x_0<|\mathbf{x}|.\end{cases} \tag{4.16}$$

Putting $\nu=1$ in (4.15), we have

$$\mathscr{E}_2(x)=\frac{1}{2\pi\sqrt{\theta(x_0)x_+^2}}. \tag{4.17}$$

(g) *Fundamental solution of the Laplace operator* $\nabla^2$. In Sec. 2.3(h), it was demonstrated that the function

$$\mathscr{E}_n(x)=-\frac{1}{(n-2)\sigma_n|x|^{n-2}},\quad n\geqslant 3;\quad \mathscr{E}_2(x)=\frac{1}{2\pi}\ln|x| \tag{4.18}$$

is a fundamental solution of the Laplace operator.

Let us compute $\mathscr{E}_n$ by the method of Fourier transforms. We have

$$-|\xi|^2F[\mathscr{E}_n]=1.$$

Let $n=2$. The generalized function $-\mathscr{P}\frac{1}{|\xi|^2}$ defined in Sec. 6.6(h) satisfies that equation, and its Fourier transform is equal to $2\pi\ln|x|+2\pi C_0$, where $C_0$ is some constant. Therefore

$$F^{-1}\left[-\mathscr{P}\frac{1}{|\xi|^2}\right]=\frac{1}{4\pi^2}F\left[-\mathscr{P}\frac{1}{|\xi|^2}\right]=\frac{1}{2\pi}\ln|x|+\frac{C_0}{2\pi}.$$

Since the constant satisfies the homogeneous Laplace equation, then by dropping the term $\frac{C_0}{2\pi}$ we see that $\mathscr{E}_2$ may be chosen equal to $\frac{1}{2\pi}\ln|x|$. Now let $n=3$. In this case, the function $-|\xi|^{-2}$ is locally summable in $\mathbb{R}^n$ and is of slow growth, and therefore, in accordance with Sec. 14.2,

$$\mathscr{E}_n(x)=-F^{-1}\left[\frac{1}{|\xi|^2}\right]=-\frac{1}{(2\pi)^n}F\left[\frac{1}{|\xi|^2}\right],$$

whence, using (6.6) of Sec. 6, we obtain (4.18) for $n=3$. The computation of $\mathscr{E}_n(x)$ is similar for $n>3$ as well.

It is particularly simple to construct $\mathscr{E}_n(x)$ for $n\geqslant 3$ by the descent method with respect to the variable $t$ (see Sec. 14.3)

from the fundamental solutions of the heat conduction operator or the wave operator. For example, by using (3.7), we obtain from (4.7) for $a = 1$ the formula (4.18) for $n \geqslant 3$:

$$\mathscr{E}_n(x) = -\int \mathscr{E}(x, t)\, dt = -\int_0^\infty \frac{1}{(2\sqrt{\pi t})^n} e^{-\frac{|x|^2}{4t}}\, dt$$

$$= \frac{-|x|^{-n+2}}{4\pi^{n/2}} \int_0^\infty e^{-u} u^{n/2-2}\, du$$

$$= -\Gamma(n/2-1) \frac{|x|^{-n+2}}{4\pi^{n/2}} = -\frac{1}{(n-2)\sigma_n |x|^{n-2}}$$

Computation is analogous in the case of the fundamental solution $\mathscr{E}_{n,k}(x)$ of the iterated Laplace operator $(\nabla^2)^k$ for $2k < n$:

$$\mathscr{E}_{n,k}(x) = \frac{(-1)^k \Gamma(n/2-k)}{2^{2k}\pi^{n/2}(k-1)!} |x|^{2k\ n} \tag{4.19}$$

(h) *Fundamental solution of the Cauchy-Riemann operator,*

$$\frac{\partial}{\partial \bar{z}} \mathscr{E} = \delta(x, y), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{4.20}$$

Applying the operator $\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$ to the equation (4.20), we obtain

$$\frac{1}{2}\nabla^2 \mathscr{E} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\delta$$

whence, using formulas (1.11) and (4.18) we have, for $n = 2$,

$$\mathscr{E} = 2\mathscr{E}_2 * \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\delta = \frac{1}{\pi}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \ln\sqrt{x^2+y^2},$$

that is,

$$\mathscr{E}(x, y) = \frac{1}{\pi z}, \qquad z = x + iy \tag{4.21}$$

(i) *Fundamental solution of the transfer operator,*

$$\frac{1}{v}\frac{\partial \mathscr{E}_s}{\partial t} + (s, \operatorname{grad} \mathscr{E}_s) + \alpha \mathscr{E}_s = \delta(x, t), \quad |s| = 1, \quad v > 0, \quad \alpha \geqslant 0. \tag{4.22}$$

Applying the Fourier transform $F_x$ to (4.22), we obtain, for the generalized function $F_x[\mathscr{E}_s] = \tilde{\mathscr{E}}_s(\xi, t)$, the equation

$$\frac{1}{v}\frac{\partial \tilde{\mathscr{E}}_s}{\partial t} + [\alpha - i(s, \xi)]\tilde{\mathscr{E}}_s = 1(\xi) \times \delta(t). \tag{4.23}$$

From this, using (4.4), we conclude that the solution, in $\mathscr{S}'$, of the equation (4.23) is the function $\tilde{\mathscr{E}}_s(\xi, t) = v\theta(t)e^{[i(s, \xi)-\alpha]vt}$. Now applying the inverse Fourier transform $F_\xi^{-1}$ and using the formula (2.6) of Sec. 6, we obtain, for $x_0 = vts$, the fundamental solution of the transfer operator

$$\mathscr{E}_s(x, t) = v\theta(t)e^{-\alpha tv}\delta(x - vts). \tag{4.24}$$

To compute the fundamental solution $\mathscr{E}_s^0(x)$ of the stationary transfer operator

$$(s, \operatorname{grad} \mathscr{E}_s^0) + \alpha\mathscr{E}_s^0 = \delta(x) \tag{4.25}$$

let us take advantage of the descent method with respect to the variable $t$ (see Sec. 14.3):

$$(\mathscr{E}_s, \varphi(x)\,1(t)) = v\int_0^\infty e^{-\alpha vt}(\delta(x - vts), \varphi)\,dt$$

$$= v\int_0^\infty e^{-\alpha vt}\varphi(vts)\,dt = \int_0^\infty e^{-\alpha u}\varphi(us)\,du$$

$$= \left(\frac{e^{-\alpha|x|}}{|x|^2}\delta\left(s - \frac{x}{|x|}\right), \varphi\right),$$

so that

$$\mathscr{E}_s^0(x) = \frac{e^{-\alpha|x|}}{|x|^2}\delta\left(s - \frac{x}{|x|}\right). \tag{4.26}$$

(j) *Fundamental solution of the Schrödinger operator,*

$$i\frac{\partial \mathscr{E}}{\partial t} + \frac{1}{2m}\nabla^2\mathscr{E} = \delta(x, t). \tag{4.27}$$

Applying the Fourier transform $F_x$ to (4.27), we obtain, for the generalized function $F_x[\mathscr{E}] = \tilde{\mathscr{E}}(\xi, t)$, the equation [compare item (e)]

$$i\frac{\partial\tilde{\mathscr{E}}}{\partial t}-\frac{1}{2m}|\xi|^2\tilde{\mathscr{E}}=1(\xi)\times\delta(t),$$

whence, by (4.4), we have

$$\tilde{\mathscr{E}}(\xi, t)=-i\theta(t)\,e^{-\frac{i}{2m}|\xi|^2 t}$$

$$=-i\lim_{\varepsilon\to+0}\theta(t)\left(\frac{m}{m+i\varepsilon}\right)^{n/2}e^{-\frac{i}{2(m+i\varepsilon)}|\xi|^2 t}.$$

Using the continuity, in $\mathscr{S}'$, of the operator of the Fourier transform $F_\xi^{-1}$ and applying formula (4.7) for $a^2=\frac{i}{2(m+i\varepsilon)}$, we obtain

$$\mathscr{E}(x, t)=-i\theta(t)\left(\frac{m}{2\pi t}\right)^{n/2}e^{i\left[\frac{|x|^2}{2t}(m+i0)-\frac{\pi n}{4}\right]}. \tag{4.28}$$

In particular, for $n=1$ we have

$$\mathscr{E}(x, t)=-\frac{1+i}{\sqrt{2}}\theta(t)\sqrt{\frac{m}{2\pi t}}\,e^{i\frac{m}{2t}x^2}. \tag{4.29}$$

**14.5 A comparison of differential operators** Let $P(D)$ be a differential operator with constant coefficients of order $m$ defined by (1.2). We set

$$\tilde{P}^2(\xi)=\sum_{|\beta|\leqslant m}|P^{(\beta)}(i\xi)|^2.$$

$$P^{(\beta)}(\xi)=D^\beta P(\xi)=\beta!\sum_{\beta\leqslant\alpha}a_\alpha\binom{\beta}{\alpha}\xi^{\alpha-\beta}.$$

The Leibniz formula takes the form

$$P(D)(fg)=\sum_{|\beta|\leqslant m}\frac{D^\beta f}{\beta!}P^{(\beta)}(D)\,g. \tag{5.1}$$

It can be verified directly:

$$P(D)(fg)=\sum_{|\alpha|\leqslant m}a_\alpha D^\alpha(fg)=\sum_{|\alpha|\leqslant m}a_\alpha\sum_{\beta\leqslant\alpha}\binom{\beta}{\alpha}D^\beta fD^{\alpha-\beta}g$$

$$=\sum_{|\beta|\leqslant m}D^\beta f\sum_{-\beta\leqslant\alpha}a_\alpha\binom{\beta}{\alpha}D^{\alpha-\beta}g=\sum_{|\beta|\leqslant m}\frac{D^\beta f}{\beta!}P^{(\beta)}(D)\,g.$$

We now prove the *Hörmander inequality*: if $\mathcal{O}$ is a bounded open set, then for any $\alpha$ there is a number $C_\alpha = C_\alpha(P, \mathcal{O})$ such that

$$\| P^{(\alpha)}(D)_\varphi \| \leqslant C_\alpha \| P(D)\varphi \|, \qquad \varphi \in \mathcal{D}(\mathcal{O}). \tag{5.2}$$

Here, $\| \ \| = \| \ \|_{\mathcal{L}^2(\mathcal{O})}$ (see Sec. 0.3).

*Proof*. It suffices to prove the inequality (5.2) for $|\alpha| = 1$ and apply the recurrent process. We prove it for $\alpha = (1, 0, \ldots, 0)$. Suppose $\varphi \in \mathcal{D}(\mathcal{O})$. Then by (5.1)

$$P(D)(x_1\varphi) = x_1 P(D)\varphi + P^{(1)}(D)\varphi, \qquad P^{(1)}(\xi) = \frac{\partial P(\xi)}{\partial \xi_1}. \tag{5.3}$$

Forming the scalar product of (5.3) on the right by $P^{(1)*}(D) = \overline{P^{(1)}(-D)}$ (compare Sec. 8.1) and taking into account that the operators $P(D)$ and $P^{(1)}(D)$ commute, we obtain

$$\langle P^{(1)}(D)(x_1\varphi), P^*(D)\varphi\rangle = \langle x_1 P(D)\varphi, P^{(1)*}(D)\varphi\rangle + \| P^{(1)}(D)\varphi \|^2. \tag{5.4}$$

But, by (5.3),

$$P^{(1)}(D)(x_1\varphi) = x_1 P^{(1)}(D)\varphi + P^{(2)}(D)\varphi, \qquad P^{(2)}(\xi) = \frac{\partial P^{(1)}(\xi)}{\partial \xi_1}.$$

Substituting the resulting expression into (5.4), we have

$$\begin{aligned}\| P^{(1)}(D)\varphi \|^2 &= \langle x_1 P^{(1)}(D)\varphi, P^*(D)\varphi\rangle + \langle P^{(2)}(D)\varphi, P^*(D)\varphi\rangle \\ &\quad - \langle x_1 P(D)\varphi, P^{(1)*}(D)\varphi\rangle \\ &\leqslant \| x_1 P^{(1)}(D)\varphi \| \, \| P^*(D)\varphi \| + \| P^{(2)}(D)\varphi \| \, \| P^*(D)\varphi \| \\ &\quad + \| x_1 P(D)\varphi \| \quad \| P^{(1)*} \times (D)\varphi \|.\end{aligned}$$

From this, putting $R_1 = \sup_{x \in \mathcal{O}} [|x_1|, 1]$ and noting that

$$\| P(D)\varphi \| = \| P^*(D)\varphi \|$$

we obtain the inequality

$$\| P^{(1)}(D)\varphi \|^2 \leqslant 2R_1 [\| P^{(1)}(D)\varphi \| + \| P^{(2)}(D)\varphi \|] \, \| P(D)\varphi \|. \tag{5.5}$$

Suppose the inequality (5.2) is proved for all polynomials of degree $< m$. (If the degree of $P$ is 0, then it is trivial.) Then there is a number $C_1$ such that $\| P^{(2)}(D)\varphi \| \leqslant C_1 \| P^{(1)}(D)\varphi \|$,

$\varphi \in \mathscr{D}(\mathcal{O})$. Substituting this inequality into the inequality (5.5) and cancelling $\|P^{(1)}(D)\varphi\|$, we are convinced that (5.2) holds true for $\alpha = (1, 0, \ldots, 0)$ with $C_\alpha = 2R_1(1 + C_1)$.

From the Hörmander inequality follows the corollary: if $\mathcal{O}$ is an open bounded set and $P(D) \not\equiv 0$, then

$$P(D)\mathscr{L}^2(\mathcal{O}) \supset \mathscr{L}^2(\mathcal{O}). \tag{5.6}$$

To prove the inclusion (5.6) it is necessary to establish the existence, in $\mathscr{L}^2(\mathcal{O})$, of the solution of the equation

$$P(D)u = f, \qquad \varphi \in \mathscr{L}^2(\mathcal{O}),$$

that is, of the equation

$$\langle u, P^*(D)\varphi\rangle = \langle f, \varphi\rangle, \qquad \varphi \in \mathscr{D}(\mathcal{O}). \tag{5.7}$$

The equation (5.7) defines an antilinear form on the functions $P^*(D)\mathscr{D}(\mathcal{O})$, which form is continuous in the norm $\mathscr{L}^2(\mathcal{O})$, by virtue of the inequality (5.2),

$$\|\varphi\| \leqslant C\|P^*(D)\varphi\|, \qquad \varphi \in \mathscr{D}(\mathcal{O}).$$

This form, by the Hahn-Banach theorem, may be extended antilinearly and continuously onto all $\mathscr{L}^2(\mathcal{O})$, which form, by the Riesz theorem, is what determines the required solution $u(x)$ in $\mathscr{L}^2(\mathcal{O})$.

Note that we have again obtained the existence of a fundamental solution of the operator $P(D) \not\equiv 0$ in $\mathscr{D}'(\mathcal{O})$ for any open bounded set $\mathcal{O}$. To prove this, it suffices to represent $\delta = D^\alpha f$, $f \in \mathscr{L}^2(\mathcal{O})$, and take advantage of the inclusion (5.6).

We will say that the operator $P(D)$ is *stronger* than the operator $Q(D)$ in an open bounded set $\mathcal{O}$, and we write: $Q < P$ in $\mathcal{O}$ if there is a constant $K = K(P, Q, \mathcal{O})$ such that

$$\|Q(D)\varphi\| \leqslant K\|P(D)\varphi\|, \qquad \varphi \in \mathscr{D}(\mathcal{O}). \tag{5.8}$$

Theorem *The following statements are equivalent*:
(1) *the operator $P$ is stronger than the operator $Q$ in $\mathcal{O}$*;
(2) *there exists a constant $C$ such that*

$$|Q(i\xi)| \leqslant C\widetilde{P}(\xi), \qquad \xi \in \mathbb{R}^n. \tag{5.9}$$

*Proof*. (1) $\to$ (2). Let $\varphi \in \mathscr{D}(\mathcal{O})$, $\varphi \not\equiv 0$. Applying the inequality (5.8) with $e^{i(x,\xi)}\varphi(x)$ substituted for $\varphi(x)$, we obtain

$$\|Q(D)e^{i(x,\xi)}\varphi\|^2 \leqslant K^2\|P(D)e^{i(x,\xi)}\varphi\|^2, \qquad \xi \in \mathbb{R}^n, \tag{5.10}$$

where $K$ does not depend on $\xi$. Noting that

$$P(D)\, e^{i(x,\,\xi)} = P(i\xi)\, e^{i(x,\,\xi)}$$

and using the Leibniz formula (5.1), from (5.10) we obtain

$$\Big\| \sum_{\alpha} Q^{(\alpha)}(i\xi) \frac{D^{\alpha}\varphi}{\alpha!} \Big\|^2 \leqslant K^2 \Big\| \sum_{\alpha} P^{(\alpha)}(i\xi) \frac{D^{\alpha}\varphi}{\alpha!} \Big\|^2. \tag{5.11}$$

Suppose $m$ is the largest of the orders of the operators $Q$ and $P$. We will prove that the quadratic form

$$\Big\| \sum_{|\alpha| \leqslant m} \frac{D^{\alpha}\varphi}{\alpha!} \lambda_{\alpha} \Big\|^2$$

of the variables $\{\lambda_{\alpha}\}$ is positive definite. Indeed, if for $\{\lambda_{\alpha}\} \neq 0$ that form vanishes, then

$$\sum_{|\alpha| \leqslant m} \frac{\lambda_{\alpha}}{\alpha!} D^{\alpha}\varphi(x) = 0, \qquad x \in \mathbb{R}^n.$$

Applying the Laplace transform to the resulting equation,

$$\sum_{|\alpha| \leqslant m} \frac{\lambda_{\alpha}}{\alpha!} (-iz)^{\alpha} L[\varphi](z) = 0, \qquad z \in \mathbb{C}^n,$$

and taking into account that $L[\varphi](z)$ is an integral function (see Sec. 12.3), we conclude that $L[\varphi] \equiv 0$, whence $\varphi \equiv 0$, which cannot be.

Thus, there exists a constant $\sigma > 0$, not dependent on $\{\lambda_{\alpha}\}$, such that

$$\frac{1}{\sigma} \sum_{|\alpha| \leqslant m} |\lambda_{\alpha}|^2 \leqslant \Big\| \sum_{|\alpha| \leqslant m} \frac{D^{\alpha}\varphi}{\alpha!} \lambda_{\alpha} \Big\|^2 \leqslant \sigma \sum_{|\alpha| \leqslant m} |\lambda_{\alpha}|^2. \tag{5.12}$$

Applying the first of the inequalities of (5.12) to $\lambda_{\alpha} = Q^{(\alpha)}(i\xi)$, the second to $\lambda_{\alpha} = P^{(\alpha)}(i\xi)$, and taking into account the inequality (5.11), we obtain the inequality

$$\widetilde{Q}^2(\xi) \leqslant (\sigma K)^2 \widetilde{P}^2(\xi) \tag{5.13}$$

whence follows the inequality (5.9) for $C = \sigma K$.

(2) $\to$ (1). Let $\varphi \in \mathcal{D}(\mathcal{O})$. Multiplying inequality (5.9) by $|F^{-1}[\varphi](\xi)|^2$ and applying the Parseval equation (see Sec. 6.6(c)), we obtain

$$\|Q(i\xi)F^{-1}[\varphi](\xi)\|^2 = \|F^{-1}[Q(D)\varphi]\|^2 = \frac{1}{(2\pi)^n}\|Q(D)\varphi\|^2$$

$$\leqslant \frac{C}{(2\pi)^n}\sum_{|\alpha|\leqslant m}\|P^{(\alpha)}(D)\varphi\|^2. \quad (5.14)$$

Applying to the right-hand side of (5.14) the Hörmander inequality (5.2), we see that inequality (5.8) holds for a certain $K$ (dependent on $\mathcal{O}$). And this means that $Q < P$ in $\mathcal{O}$. The proof of the theorem is complete.

Corollary 1 *If $Q < P$ in some $\mathcal{O}$, then $Q < P$ in any open bounded set.*

Corollary 2 *The inequality (5.9) is equivalent to the inequality*

$$\widetilde{Q}(\xi) \leqslant C_1\widetilde{P}(\xi), \qquad \xi \in \mathbb{R}^n.$$

**14.6 Elliptic and hypoelliptic operators** An operator $P(D)$ is said to be *elliptic* (or, respectively, *hypoelliptic*) if it possesses a (real) analytic (respectively $C^\infty$) fundamental solution $\mathcal{E}(x)$ for $x \neq 0$. Every elliptic operator is hypoelliptic.

*Examples.* The Laplace and Cauchy-Riemann operators are elliptic (see Sec. 14.4(g) and (h)); the heat conduction operator is hypoelliptic (see Sec. 14.4(e)).

Theorem I *For the operator $P(D)$ to be hypoelliptic, it is necessary and sufficient that for any open set $\mathcal{O}$ every solution $u(x)$ in $\mathcal{D}'(\mathcal{O})$ of the equation $P(D)u = f$, where $f \in C^\infty(\mathcal{O})$, belong to $C^\infty(\mathcal{O})$.*

*Proof.* Sufficiency is obvious. We will prove necessity. Let $\mathcal{E}(x)$ be a fundamental solution of the class $C^\infty(\mathbb{R}^n \setminus \{0\})$ of the operator $P(D)$ and $u \in \mathcal{D}'(\mathcal{O})$ a solution in $\mathcal{O}$ of the equation $P(D)u = f$. Suppose $\mathcal{O}'$ is an arbitrary open set, compact in $\mathcal{O}$, and $\eta \in \mathcal{D}(\mathcal{O})$, $\eta(x) = 1$, $x \in \mathcal{O}'$. The generalized function $\eta u$ is finite and satisfies the equation [see (5.1)]

$$P(D)(\eta u) = \eta f + f_1,$$

where $\eta f \in C^\infty$, $\eta f$ is finite, $f_1 \in \mathcal{D}'$ and $\operatorname{supp} f_1 \subset \operatorname{supp}\eta \setminus \mathcal{O}'$. Therefore (see Sec. 14.1)

$$\eta u = \mathcal{E} * (\eta f) + \mathcal{E} * f_1$$

whence it follows that $u \in C^{\infty}(\mathcal{O}')$ (see Sec. 4.3). Since $\mathcal{O}' \Subset \mathcal{O}$ is arbitrary, it follows that $u \in C^{\infty}(\mathcal{O})$. Theorem I is proved.

The algebraic conditions for hypoellipticity may be indicated: for an operator $P(D)$ to be hypoelliptic, it is necessary and sufficient that for all $\alpha$, $|\alpha| \geqslant 1$,

$$\frac{P^{(\alpha)}(-i\xi)}{P(-i\xi)} \to 0, \qquad |\xi| \to \infty. \tag{6.1}$$

This result was obtained by Hörmander [1].

The proof of the following theorem is similar to that of Theorem I.

Theorem I′ *For an operator $P(D)$ to be elliptic, it is necessary and sufficient that, for any $\mathcal{O}$, every solution $u(x)$ in $\mathcal{D}'(\mathcal{O})$ of the equation $P(D)u = 0$ be (real) analytic in $\mathcal{O}$.*

The algebraic condition for ellipticity: for an operator $P(D)$ to be elliptic, it is necessary and sufficient that its principal part

$$P_m(D) = \sum_{|\alpha|=m} a_\alpha D^\alpha$$

satisfy the condition $P_m(\xi) \neq 0$, $\xi \neq 0$.

This result was established by Petrovskii [1] (1939), for classical solutions, by Weyl [1] (1940) for generalized solutions for the Laplace operator, and by Hörmander (1955) (see [1]) in the general case.

We now prove a theorem on wiping out isolated singularities of harmonic functions.

A generalized function $u(x)$ in $\mathcal{D}'(G)$ is said to be *harmonic* in a region $G$ if it satisfies the Laplace equation $\nabla^2 u = 0$ in $G$ (and then $u \in C^{\infty}$ via Theorem I).

Theorem II *If the function $u$ is harmonic in the region $G \setminus \{0\}$ and*

$$u(x) = o(|x|^{-n+2}),\ n \geqslant 3;\ u(x) = o(\ln|x|),\ n = 2, \qquad |x| \to 0, \tag{6.2}$$

*then $u$ is harmonic in $G$.*

*Proof*. Let $U_R \Subset G$. We introduce the function $\tilde{u}(x)$, which is equal to $u(x)$ in $\overline{U}_R$ and 0 outside $\overline{U}_R$. This function is summable on $\mathbb{R}^n$ and by (3.7′) of Sec. 2 the generalized function in $\mathcal{D}'(\mathbb{R}^n)$

$$\nabla^2\tilde{u} + \frac{\partial u}{\partial \mathbf{n}}\delta_{S_R} + \frac{\partial}{\partial \mathbf{n}}(u\delta_{S_R}) = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}.$$

From this, by the theorem of Sec. 2.6, we have

$$\nabla^2\tilde{u} = -\frac{\partial u}{\partial \mathbf{n}}\delta_{S_R} - \frac{\partial}{\partial \mathbf{n}}(u\delta_{S_R}) + \sum_{|\alpha|\leqslant m} c_\alpha D^\alpha\delta \quad \text{in} \quad \mathbb{R}^n. \tag{6.3}$$

Since $\tilde{u}$ is finite, it follows that, by using (1.11), we obtain, from (6.3),

$$\begin{aligned}\tilde{u} &= \mathcal{E}_n * \nabla^2\tilde{u}\\ &= -\mathcal{E}_n * \frac{\partial u}{\partial \mathbf{n}}\delta_{S_R} - \mathcal{E}_n * \frac{\partial}{\partial \mathbf{n}}(u\delta_{S_R}) + \sum_{|\alpha|\leqslant m} c_\alpha \mathcal{E}_n * D^\alpha\delta\\ &= V_n^{(0)} + V_n^{(1)} + \sum_{|\alpha|\leqslant m} c_\alpha D^\alpha \mathcal{E}_n\end{aligned} \tag{6.4}$$

where

$$\mathcal{E}_n(x) = k_n |x|^{-n+2}, \quad n \geqslant 3; \qquad \mathcal{E}_2(x) = \frac{1}{2\pi}\ln|x|$$

is a fundamental solution of the Laplace operator, and $V_n^{(0)}(x)$ and $V_n^{(1)}(x)$ are surface potentials of a simple and double layer on the sphere $S_R$. From the representation (6.4) for $|x| < R$ and from the condition (6.2) it follows that $c_\alpha = 0$, so that

$$u(x) = V_n^{(0)}(x) + V_n^{(1)}(x), \qquad |x| < R,$$

whence it follows that $u(x)$ is a harmonic function in the ball $U_R$. The proof of the theorem is complete.

**14.7 Hyperbolic operators** Let $C$ be a convex open cone in $\mathbb{R}^n$ with vertex at 0. The operator $P(D)$ is said to be *hyperbolic relative to the cone $C$* if it satisfies the condition: there is a point $y_0 \in \mathbb{R}^n$ such that

$$P(y_0 - iz) \neq 0 \quad \text{for all} \quad z \in T^C. \tag{7.1}$$

Theorem *For the operator $P(D)$ to be hyperbolic relative to a cone $C$, it is necessary and sufficient that it have a (unique) fundamental solution $\mathcal{E}(x)$ in the algebra $\mathcal{D}'(C^*)$, which solution can be represented as*

$$\mathcal{E}(x) = e^{(y_0, x)}\mathcal{E}_0(x), \qquad \mathcal{E}_0 \in \mathcal{S}'(C^*), \tag{7.2}$$

*where the point $y_0 \in \mathbb{R}^n$ is defined in (7.1).*

*Proof. Necessity.* If the operator $P(D)$ is hyperbolic relative to the cone $C$, then the polynomial $P(y_0 - iz)$ does not vanish in the tubular region $T^C$. Therefore $1/P(y_0 - iz) \in H(C)$ (see Sec. 13.2) so that

$$\frac{1}{P[-i(iy_0+z)]} = L[\mathscr{E}_0](z), \qquad \mathscr{E}_0 \in \mathscr{S}'(C^*). \tag{7.3}$$

Setting $\zeta = z + iy_0$, we obtain (see Sec. 9.2(c))

$$\frac{1}{P(-i\zeta)} = L[\mathscr{E}_0](\zeta - iy_0) = L[\mathscr{E}_0(x)\, e^{(y_0,\, x)}],$$

whence follows the representation (7.2).

*Sufficiency.* If the operator $P(D)$ has a fundamental solution of the form of (7.2), then the function $L[\mathscr{E}_0](z)$ is holomorphic in $T^C$ and, hence, by virtue of (7.3) the polynomial $P(y_0 - iz)$ does not vanish in $T^C$ since the operator $P(D)$ is hyperbolic relative to the cone $C$.

The uniqueness of a fundamental solution in the algebra $\mathscr{D}'(C^*)$ was proved in Sec. 4.8(d).

The theorem is proved.

*Example 1.* The wave operator $\Box$ is hyperbolic relative to the light cone of the future $V^+$, and (see Sec. 13.5)

$$\Box(-iz) = -z_0^2 + z_1^2 + \ldots + z_n^2 \neq 0 \text{ in } T^{V^+}.$$

*Example 2.* The differential operator $P\left(\frac{d}{dt}\right)$ (see Sec. 14.4(d)) is hyperbolic relative to the cone $(0, \infty)$.

*Remark.* The cone $C$ is connected component of the open set (see Hörmander [1])

$$[y : P_m(y) \neq 0].$$

# 15 Cauchy Problem

**15.1 The generalized Cauchy problem for a hyperbolic equation** Let $S$ be a $C$-like surface of the class $C^\infty$ and let $S_+$ be a region lying above $S$ (see Sec. 4.5); $P(D)$ is a hyperbolic operator relative to the cone $C$ of order $m$.

We consider the classical Cauchy problem

$$P(D)\, u = f(x), \qquad x \in S_+, \tag{1.1}$$

$$\left.\frac{\partial^k u}{\partial \mathbf{n}^k}\right|_S = u_k(x), \qquad k=0, 1, \ldots, m-1, \tag{1.2}$$

that is, the problem of finding a function $u \in C^m(S_+) \cap C^{m-1}(\bar{S}_+)$ that satisfies the equation (1.1) in $S_+$ and the conditions (1.2) on $S$. For solvability of the Cauchy problem (1.1)-(1.2) it is necessary that $f \in C(S_+)$ and $u_k \in C^{m-k-1}(S)$.

Suppose that the classical solution $u$ of the problem (1.1)-(1.2) exists and $f \in C(\bar{S}_+)$. We continue the functions $f$ and $u$ by zero onto $S_-$ and denote the extended functions by $\tilde{f}$ and $\tilde{u}$ respectively. Then the function $\tilde{u}(x)$ satisfies the following differential equation over the entire space $\mathbb{R}^n$:

$$P(D)\tilde{u} = \tilde{f}(x) + \sum_{0 \leqslant k \leqslant m-1} \frac{\partial^k}{\partial \mathbf{n}^k}(v_k \delta_S)(x), \tag{1.3}$$

where $v_k \delta_S$ is the density of a simple layer on $S$ with surface density $v_k$ (see Sec. 1.7), uniquely defined by the functions $\{u_j\}$, by the surface $S$, and by the operator $P(D)$. The generalized function $\frac{\partial^k}{\partial \mathbf{n}^k}(v_k \delta_S)$ acts on the basic functions $\varphi$ via the rule (compare Sec. 2.3(b))

$$\left(\frac{\partial^k}{\partial \mathbf{n}^k}(v_k \delta_S), \varphi\right) = (-1)^k \int_S v_k(x) \frac{\partial^k \varphi(x)}{\partial \mathbf{n}^k} dS.$$

Let us prove (1.3). Using equation (1.1) and the conditions (1.2), we have, for all $\varphi \in \mathcal{D}$,

$$\begin{aligned}
&(P(D)\tilde{u}, \varphi) \\
&\quad = (\tilde{u}, P(-D)\varphi) = \int_{S_+} u(x) P(-D)\varphi(x)\, dx \\
&\quad = \int_{S_+} P(D)u(x)\varphi(x)\, dx + \sum_{0 \leqslant k \leqslant m-1} (-1)^k \int_S v_k(x) \frac{\partial^k \varphi(x)}{\partial \mathbf{n}^k} dS \\
&\quad = \int \tilde{f}(x)\varphi(x)\, dx + \sum_{0 \leqslant k \leqslant m-1} \left(\frac{\partial^k}{\partial \mathbf{n}^k}(v_k \delta_S), \varphi\right),
\end{aligned}$$

which is equivalent to (1.3).

*Example 1.* For the Cauchy problem for the ordinary differential equation

$$P\left(\frac{d}{dt}\right)u \equiv u^{(m)} + a_1 u^{(m-1)} + \ldots + a_m u = f(t), \qquad (1.4)$$

$$u^{(k)}(0) = u_k, \qquad k = 0, 1, \ldots, m-1, \qquad (1.5)$$

the equation (1.3) takes the form (see Sec. 2.3(c))

$$P\left(\frac{d}{dt}\right)\tilde{u} = \tilde{f}(t) + \sum_{0 \leqslant k \leqslant m-1} v_k \delta^k(t) \qquad (1.6)$$

where

$$v_k = \sum_{0 \leqslant j \leqslant m-1-k} a_{m-1-j} u_j \qquad (a_0 = 1). \qquad (1.7)$$

Indeed, using formula (3.1) of Sec. 2 and the initial conditions (1.5), we have

$$\tilde{u}^{(k)} = u_{\mathrm{cl}}^{(k)}(t) + \sum_{0 \leqslant j \leqslant k-1} u_j \delta^{(k-j-1)}(t), \qquad k = 1, 2, \ldots, n.$$

From this and from (1.4) follows equation (1.6):

$$P\left(\frac{d}{dt}\right)\tilde{u} = P_{\mathrm{cl}}\left(\frac{d}{dt}\right)\tilde{u}(t) + u_0\delta^{(m-1)}(t) + (a_1u_0 + u_1)\,\delta^{(m-2)}(t) +$$
$$\ldots + (a_{m-1}u_0 + \ldots + a_1u_{m-2} + u_{m-1})\,\delta(t)$$
$$= \tilde{f}(t) + \sum_{0 \leqslant k \leqslant m-1} v_k \delta^{(k)}(t),$$

where the numbers $v_k$ are defined by equations (1.7).

*Example 2.* For the Cauchy problem for the wave equation

$$\Box\, u = f(x), \qquad x = (x_0, \mathbf{x}), \qquad (1.8)$$

$$u\Big|_{x_0=0} = u_0(\mathbf{x}), \qquad \frac{\partial u}{\partial x_0}\Big|_{x_0=0} = u_1(\mathbf{x}), \qquad (1.9)$$

the equation (1.3) takes the form

$$\Box\, \tilde{u} = \tilde{f}(x) + u_0(\mathbf{x}) \times \delta'(x_0) + u_1(\mathbf{x}) \times \delta(x_0). \qquad (1.10)$$

Thus, the classical solution $u(x)$ of the Cauchy problem (1.1)-(1.2), being a continuation onto $S_-$ via zero, satisfies equation (1.3) in $\mathbb{R}^n$. Here the initial conditions (1.2) play the role of

sources concentrated on the surface $S$ (as the sum of densities of layers of different orders). For example, for the wave equation, by virtue of (1.10), that is the sum of the densities of a simple layer and a double layer on the plane $x_0 = 0$, that is, an instantaneously operating source (for $x_0 = 0$).

Now we can generalize the classical Cauchy problem for a hyperbolic operator $P(D)$ in the following manner. Suppose the generalized function $F \in \mathscr{D}'(\bar{S}_+)$. We use the term *generalized Cauchy problem* for the operator $P(D)$ with the source $F$ to describe the problem of finding, in $\mathbb{R}^n$, a generalized solution $u$ in $\mathscr{D}'(\bar{S}_+)$ of the equation

$$P(D)\, u = F(x). \tag{1.11}$$

By the foregoing, all solutions of the classical Cauchy problem are contained among the solutions of the generalized Cauchy problem.

The following theorem holds true:

Theorem *A solution of the generalized Cauchy problem exists uniquely and is expressed by the formula*

$$u = \mathscr{E} * F, \tag{1.12}$$

*where $\mathscr{E}$ is a fundamental solution of the operator $P(D)$ in $\mathscr{D}'(C^*)$. This solution depends continuously on $F$ in the sense of convergence in the space $\mathscr{D}'(\bar{S}_+)$.*

To prove the theorem, it is necessary to take advantage of the results of Sec. 4.5, for $\Gamma = C^*$ and $K = \varnothing$, and also of Sec. 4.8(d). Here the operation $F \to \mathscr{E} * F$ is continuous from $\mathscr{D}'(\bar{S}_+)$ to $\mathscr{D}'(\bar{S}_+)$.

*Remark.* The foregoing is carried over in obvious fashion to hyperbolic systems. Let $A$ be an $N \times N$ matrix whose elements are differential operators with constant coefficients. The operator $A(D)$ is said to be hyperbolic with respect to the cone $C$ if the operator $\det A(D)$ is hyperbolic relative to $C$.

*Example 1'.* The formula (1.12) for solving the classical Cauchy problem (1.4)-(1.5) takes the form

$$u(t) = \int_0^t f(\tau)\, Z(t-\tau)\, d\tau + \sum_{0 \leqslant k \leqslant m-1} v_k Z^{(k)}(t), \tag{1.13}$$

where $Z(t)$ is a solution of the homogeneous equation $P\left(\frac{d}{dt}\right) Z = 0$ that satisfies the conditions $Z^{(k)}(0) = 0$, $0 \leqslant k \leqslant m-2$, $Z^{(m-1)}(0) = 1$, and the numbers $v_k$ are given by the equations (1.7).

To obtain formula (1.13) compute the convolution of the fundamental solution $\mathscr{E}(t) = \theta(t) Z(t)$ of the operator $P\left(\frac{d}{dt}\right)$ (see Sec. 14.4(d)) with the right-hand side of (1.6):

$$\begin{aligned}\tilde{u} &= \tilde{f} * \mathscr{E} + \sum_{0\leqslant k\leqslant m-1} v_k \delta^{(k)} * \mathscr{E}\\ &= \int \tilde{f}(\tau)\,\mathscr{E}(t-\tau)\,d\tau + \sum_{0\leqslant k\leqslant m-1} v_k \mathscr{E}^{(k)}(t)\\ &= \int_0^t f(\tau)\,Z(t-\tau)\,d\tau + \theta(t) \sum_{0\leqslant k\leqslant m-1} v_k Z^{(k)}(t).\end{aligned}$$

Here we took into account the equations (see Sec. 14.4(d))

$$\mathscr{E}^{(k)}(t) = [\theta(t) Z(t)]^{(k)} = \theta(t) Z^{(k)}(t), \qquad k = 0, 1, \ldots, m-1.$$

**15.2 Retarded potential** The (generalized) functions defined by equations (4.9), (4.13) and (4.15) of Sec. 14 constitute the fundamental solution of the wave operator:

$$\mathscr{E}_n(x) = \begin{cases} \dfrac{1}{2^{2\nu-1}\pi^{\nu}\Gamma(\nu)} \square^{\nu-1}\theta(x_0)\,\delta(x^2), & n = 2\nu+1, \ \nu\geqslant 1;\\ \dfrac{1}{2^{2\nu-1}\pi^{\nu}\Gamma(\nu)} \square^{\nu-1}[\theta(x_0)\,x_+^2]^{-1/2}, & n = 2\nu;\\ \dfrac{1}{2}\theta(x_0)\,\theta(x^2), & n = 1.\end{cases} \qquad (2.1)$$

The supports of $\mathscr{E}_1$ and $\mathscr{E}_{2\nu}$ coincide with $\overline{V}^+$ and the support of $\mathscr{E}_{2\nu+1}$, $\nu \geqslant 1$, with $\partial\overline{V}^+$. These peculiarities of structure of the support of the fundamental solution are what determine the differences in the nature of wave propagation in odd-dimensional ($n \geqslant 3$) and even-dimensional spaces and on the straight line. Figures 33 to 35 depict schematically the graphs of the fundamental solutions $\mathscr{E}_n(x)$ with respect to $|\mathbf{x}|$ for a fixed $x_0 > 0$.

Lemma *The fundamental solution $\mathscr{E}_n(x)$ belongs to the class $C^\infty([0, \infty))$ with respect to $x_0$ and its restriction $\mathscr{E}_{nx_0}(\mathbf{x})$ for $x_0 > 0$ possesses the support $S_{x_0}$ (for odd $n \geqslant 3$), $\overline{U}_{x_0}$ (for even $n$ or $n = 1$), and satisfies the limiting relations, as $x_0 \to +0$,*

$$\mathscr{E}_{nx_0}(\mathbf{x}) \to 0, \quad \frac{\partial \mathscr{E}_{nx_0}(\mathbf{x})}{\partial x_0} \to \delta(\mathbf{x}), \quad \frac{\partial^2 \mathscr{E}_{nx_0}(\mathbf{x})}{\partial x_0^2} \to 0 \text{ in } \mathscr{D}'(\mathbb{R}^n). \quad (2.2)$$

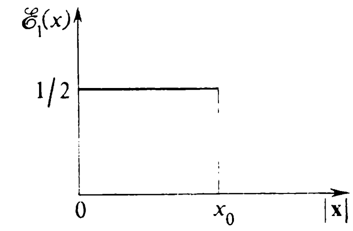



Figure 33

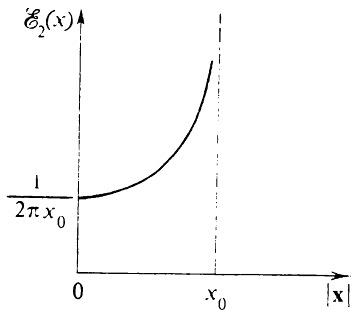



Figure 34

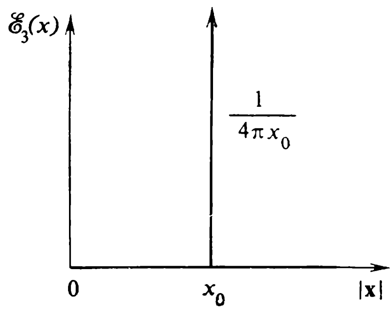



Figure 35

*Proof*. Let $n = 2\nu + 1 \geqslant 3$ be odd. We will prove that, for $x_0 > 0$,

$$\mathscr{E}_{n\varphi}(x_0) = \frac{1}{2^{2\nu}\pi^{\nu}\Gamma(\nu)} \sum_{0 \leqslant \alpha \leqslant \nu - 1} (-1)^{\nu-1-\alpha} \binom{\alpha}{\nu - 1}$$

$$\times \frac{d^{2\alpha}}{dx_0^{2\alpha}} \left[ x_0^{2\nu-1} \int_{|s|=1} ((\nabla^2)^{\nu-1-\alpha}\varphi)(x_0 s)\, ds \right], \qquad \varphi \in \mathscr{D}(\mathbb{R}^n). \qquad (2.3)$$

Indeed, for all $\varphi \in \mathscr{D}(\mathbb{R}^n)$ and $\psi \in \mathscr{D}(x_0 > 0)$ we have, by virtue of (2.1) and (4.11) of Sec. 14,

$$
\begin{aligned}
&(\mathscr{E}_{n\varphi}, \psi) \\
&= (\mathscr{E}_n, \varphi(\mathbf{x})\psi(x_0)) \\
&= \frac{1}{2^{2\nu-1}\pi^\nu\Gamma(\nu)} (\square^{\nu-1}\theta(x_0)\delta(x^2), \varphi(\mathbf{x})\psi(x_0)) \\
&= \frac{1}{2^{2\nu-1}\pi^\nu\Gamma(\nu)} \Big(\theta(x_0)\delta(x^2), \\
&\qquad \sum_{0\leqslant\alpha\leqslant\nu-1} (-1)^{\nu-1-\alpha}\binom{\alpha}{\nu-1}\psi^{(2\alpha)}(x_0)(\nabla^2)^{\nu-1-\alpha}\varphi(\mathbf{x})\Big) \\
&= \frac{1}{2^{2\nu}\pi^\nu\Gamma(\nu)} \sum_{0\leqslant\alpha\leqslant\nu-1} (-1)^{\nu-1-\alpha}\binom{\alpha}{\nu-1} \\
&\qquad \times \int_0^\infty \frac{\psi^{(2\alpha)}(x_0)}{x_0} \int_{|\mathbf{x}|=x_0} (\nabla^2)^{\nu-1-\alpha}\varphi(\mathbf{x})\, dS_{\mathbf{x}}\, dx_0 \\
&= \frac{1}{2^{2\nu}\pi^\nu\Gamma(\nu)} \sum_{0\leqslant\alpha\leqslant\nu-1} (-1)^{\nu-1-\alpha}\binom{\alpha}{\nu-1} \\
&\qquad \times \int_0^\infty \psi^{(2\alpha)}(x_0)\, x_0^{2\nu-1} \int_{|s|=1} ((\nabla^2)^{\nu-1-\alpha}\varphi)(x_0 s)\, ds\, dx_0,
\end{aligned}
$$

whence follows formula (2.3) (see the notations and techniques developed in Sec. 3.4). From (2.3) it follows that $\mathscr{E}_n \in C^\infty([0, \infty))$ with respect to $x_0$, $\operatorname{supp} \mathscr{E}_{nx_0} = S_{x_0}$ and, as $x_0 \to +0$,

$$(\mathscr{E}_{nx_0}, \varphi) = \mathscr{E}_{n\varphi}(x_0) \to 0,$$

$$
\begin{aligned}
&\left(\frac{\partial \mathscr{E}_{nx_0}}{\partial x_0}, \varphi\right) = \mathscr{E}'_{n\varphi}(x_0) \\
&\qquad = \frac{1}{2^{2\nu}\pi^\nu\Gamma(\nu)} \sum_{0\leqslant\alpha\leqslant\nu-1} (-1)^{\nu-1-\alpha}\binom{\alpha}{\nu-1} \\
&\qquad\qquad \times \frac{d^{2\alpha+1}}{dx_0^{2\alpha+1}}\left[x_0^{2\nu-1}\int_{|s|=1} ((\nabla^2)^{\nu-1-\alpha}\varphi)(x_0 s)\, ds\right] \\
&\to \frac{(2\nu-1)!}{2^{2\nu}\pi^\nu\Gamma(\nu)} \int_{|s|=1} \varphi(0)\, ds
\end{aligned}
$$

$$= \frac{\Gamma(2\nu)}{2^{2\nu}\pi^{\nu}\Gamma(\nu)} \sigma_{2\nu+1}\varphi(0)$$

$$= \frac{2\Gamma(2\nu)\pi^{\nu+1/2}}{2^{2\nu}\pi^{\nu}\Gamma(\nu)\Gamma(\nu+1/2)} \varphi(0) = \varphi(0) = (\delta, \varphi),$$

$$\left(\frac{\partial^2 \mathscr{E}_{nx_0}}{\partial x_0^2}, \varphi\right) = \mathscr{E}''_{n\varphi}(x_0)$$

$$= \frac{1}{2^{2\nu}\pi^{\nu}\Gamma(\nu)} \sum_{0 \leqslant \alpha \leqslant \nu-1} (-1)^{\nu-1-\alpha} \binom{\alpha}{\nu-1}$$

$$\times \frac{d^{2\alpha+2}}{dx_0^{2\alpha+2}} \left[ x_0^{2\nu-1} \int_{|s|=1} ((\nabla^2)^{\nu-1-\alpha}\varphi)(x_0 s)\, ds \right] \to 0.$$

In this last relation we took advantage of the fact that the function

$$\int_{|s|=1} \varphi(x_0 s)\, ds = \int_{|s|=1} \varphi(-x_0 s)\, ds$$

is even, infinitely differentiable, and therefore its first derivative with respect to $x_0$ at zero is equal to zero.

Thus the limiting relations (2.3) have been proved for odd $n \geqslant 3$. For even $n = 2\nu$ the proof is analogous; the simplest thing is to take advantage of the descent method with respect to the variable $x_{2\nu+1}$ (see Sec. 14.4(f)). For $n = 1$ the proof is trivial. The proof of the lemma is complete.

*Example.*

$$\mathscr{E}_{2\nu+1 x_0}(\mathbf{x}) = \frac{1}{2^{2\nu}\pi^{\nu}\Gamma(\nu)} \square^{\nu-1} \frac{1}{x_0} \delta_{S_{x_0}}(\mathbf{x}), \qquad x_0 > 0, \quad (2.4)$$

where $\delta_{S_{x_0}}(\mathbf{x})$ is a simple layer on the sphere $|\mathbf{x}| = x_0$ (see Sec. 1.7).

Suppose $F \in \mathscr{D}'(\overline{\mathbb{R}}^1_+ \times \mathbb{R}^n)$. The convolution $V_n = F * \mathscr{E}_n$ that exists in $\mathscr{D}'(\overline{\mathbb{R}}^1_+ \times \mathbb{R}^n)$ (see Sec. 4.5) is termed a *retarded potential* with density $F$. The retarded potential $V_n$ depends continuously on $F$ in the sense of convergence in $\mathscr{D}'(\overline{\mathbb{R}}^1_+ \times \mathbb{R}^n)$. Finally, that potential satisfies the wave equation $\square V_n = F$ (see Sec. 14.1).

The other properties of the potential $V_n$ are substantially dependent on the properties of the density $F$.

If $F = f \in \mathscr{L}^1_{\text{loc}}(\overline{\mathbb{R}}^1_+ \times \mathbb{R}^n)$, then the potential $V_n$ is given by the formulas

$$V_{2\nu+1}(x) = \frac{1}{2^{2\nu}\pi^\nu\Gamma(\nu)} \square^{\nu-1} \int\limits_{|\mathbf{x}-\boldsymbol{\xi}|<x_0} \frac{f(x_0-|\mathbf{x}-\boldsymbol{\xi}|, \boldsymbol{\xi})}{|\mathbf{x}-\boldsymbol{\xi}|}\, d\boldsymbol{\xi}, \qquad \nu \geqslant 1,$$

$$V_{2\nu}(x) = \frac{1}{2^{2\nu-1}\pi^\nu\Gamma(\nu)} \square^{\nu-1} \int\limits_0^{x_0} \int\limits_{|\mathbf{x}-\boldsymbol{\xi}|<x_0-\xi_0} \frac{f(\xi)\, d\boldsymbol{\xi}\, d\xi_0}{\sqrt{(x-\xi)^2}},$$

$$V_1(x) = \frac{1}{2} \int\limits_0^{x_0} \int\limits_{x_1-x_0+\xi_0}^{x_1+x_0-\xi_0} f(\xi)\, d\xi_1\, d\xi_0. \tag{2.5}$$

Suppose $n = 2\nu + 1 \geqslant 3$. Using the representation (5.1) of Sec. 4 for the convolution $f * \mathscr{E}_n$, we have, from (2.1) and (4.11) of Sec. 14, for all $\varphi \in \mathscr{D}(\mathbb{R}^{n+1})$,

$$\begin{aligned}
(V_n, \varphi) &= (f * \mathscr{E}_n, \varphi) = c_\nu (f * \square^{\nu-1}\theta(x_0)\,\delta(x^2), \varphi) \\
&= c_\nu (\square^{\nu-1}(f * \theta(x_0)\,\delta(x^2)), \varphi) = c_\nu (f * \theta(x_0)\,\delta(x^2), \square^{\nu-1}\varphi) \\
&= c_\nu (f(\xi) \times \theta(y_0)\,\delta(y^2), \ \eta(\xi)\,\eta_1(y)\,\square^{\nu-1}\varphi(\xi+y)) \\
&= c_\nu \left(\theta(y_0)\,\delta(y^2), \ \eta_1(y) \int f(\xi)\,\square^{\nu-1}\varphi(\xi+y)\, d\xi\right) \\
&= c_\nu \left(\theta(y_0)\,\delta(y^2), \eta_1(y) \int f(x-y)\,\square^{\nu-1}\varphi(x)\, dx \right) \\
&= \frac{c_\nu}{2} \int \left[\int f(x_0-|\mathbf{y}|, \ \mathbf{x}-\mathbf{y})\,\square^{\nu-1}\varphi(x)\, dx\right] \frac{d\mathbf{y}}{|\mathbf{y}|} \\
&= \frac{c_\nu}{2} \int \square^{\nu-1}\varphi(x) \int \frac{f(x_0-|\mathbf{x}-\boldsymbol{\xi}|, \boldsymbol{\xi})}{|\mathbf{x}-\boldsymbol{\xi}|}\, d\boldsymbol{\xi}\, dx,
\end{aligned}$$

whence follows the first of the formulas of (2.5); here, $c_\nu 2^{2\nu-}\pi^\nu \times \times \Gamma(\nu) = 1$. Similarly, and more simply, we can prove the other formulas of (2.5).

*Remark*. From (2.5), for $n = 2\nu + 1 \geqslant 3$, it follows that the potential $V_{2\nu+1}(x)$ at the point $\mathbf{x}$ at time $t = x_0 > 0$ is completely specified by the values of the source $f(\xi)$ on the lateral surface of the cone (Fig. 36)

$$\Gamma(x) = [\xi : \xi_0 - x_0 \leqslant -|\mathbf{x} - \boldsymbol{\xi}|, \ \xi_0 \geqslant 0].$$

That is, by the values of the source $f(x_0 - |\mathbf{x} - \xi|, \xi)$ in the sphere $\overline{U}(\mathbf{x}; x_0)$, which values are taken at early times $\xi_0 = x_0 - |\mathbf{x} - \xi|$; and the delay time $|\mathbf{x} - \xi|$ is the time required for the perturbation to move from point $\xi$ to point $\mathbf{x}$.

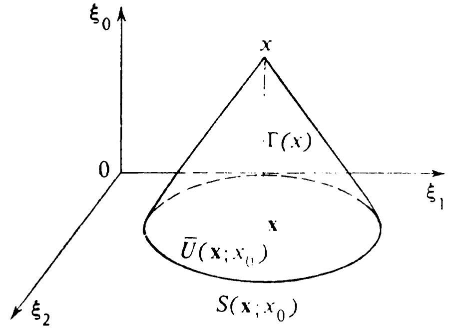



Figure 36

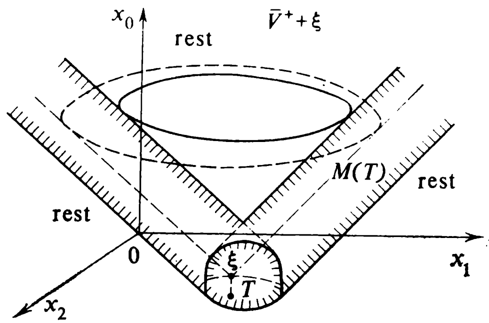



Figure 37

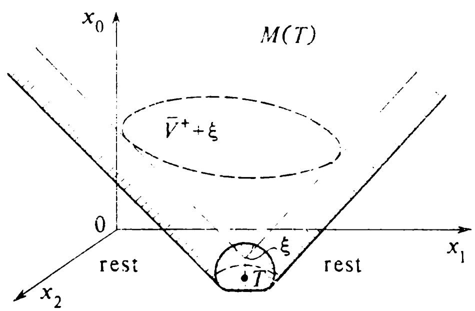



**Figure 38**

On the other hand, from (2.5), for $n = 2\nu$, it follows that the value of the potential $V_{2\nu}(x)$ is completely determined by the values of the source $f(\xi)$ on the cone $\Gamma(x)$ itself.

Let the source $f$ be concentrated on a closed set $T \subset \mathbb{R}^{n+1}$ By the foregoing, a perturbation, for odd $n \geqslant 3$, is propagated from $T$ onto the set $M(T)$, which is the union of boundaries of the light cones of the future $V^+ + \xi$ when their vertices $\xi$ run through $T$ (Fig. 37). For even $n$, the perturbation is propagated onto a union of the closed cones themselves $\overline{V^+} + \xi$, $\xi \in T$ (Fig. 38). The set $M(T)$ obtained in this fashion is called the influence region of the set $T$ It is clear that outside $M(T)$ we have a region of rest.

**15.3 Surface retarded potentials** Suppose the density $F = u_1(\mathrm{x}) \times \delta(x_0)$ or $F = u_0(\mathrm{x}) \times \delta'(x_0)$, where $u_0$ and $u_1$ are arbitrary generalized functions in $\mathscr{D}'(\mathbb{R}^n)$. The retarded potentials

$$V_n^{(0)} = [u_1(\mathrm{x}) \times \delta(x_0)] * \mathscr{E}_n, \qquad V_n^{(1)} = [u_0(\mathrm{x}) \times \delta'(x_0)] * \mathscr{E}_n$$

are called *surface retarded potentials* (of the type of a simple and double layer with densities $u_1$ and $u_0$ respectively). The retarded potential $V_n^{(1)}$ is the derivative, with respect to $x_0$, of $V_n^{(0)}$ with the same density:

$$V_n^{(1)}(x) = \frac{\partial}{\partial x_0}\{[u_0(\mathrm{x}) \times \delta(x_0)] * \mathscr{E}_n\}. \tag{3.1}$$

The surface retarded potentials $V_n^{(0)}$ and $V_n^{(1)}$ belong to the class $C^\infty([0, \infty))$ with respect to $x_0$; for $x_0 > 0$ and $k = 0, 1, \ldots$,

$$\frac{\partial^k}{\partial x_0^k} V_{nx_0}^{(0)}(\mathrm{x}) = u_1 * \frac{\partial^k \mathscr{E}_{nx_0}}{\partial x_0^k}, \tag{3.2}$$

$$\frac{\partial^k}{\partial x_0^k} V_{nx_0}^{(1)}(\mathrm{x}) = u_0 * \frac{\partial^{k+1} \mathscr{E}_{nx_0}}{\partial x_0^{k+1}}. \tag{3.3}$$

Indeed, using the lemma of Sec. 15.2, we have, for all $\varphi \in \mathscr{D}(\mathbb{R}^n)$ and $\psi \in \mathscr{D}(x_0 > 0)$ (see also Sec. 3.4),

$$\begin{aligned}
&\left(\frac{d^k V_{n\varphi}^{(0)}}{dx_0^k}, \psi\right) \\
&\quad = (-1)^k (V_{n\varphi}^{(0)}, \psi^{(k)}) \\
&\quad = (-1)^k (V_n^{(0)}, \varphi\psi^{(k)}) = (-1)^k ([u_1 \times \delta] * \mathscr{E}_n, \varphi\psi^{(k)}) \\
&\quad = \left(\frac{\partial^k \mathscr{E}_n(x)}{\partial x_0^k} \times u_1(\xi) \times \delta(\xi_0), \eta(x)\, \eta_1(\xi)\, \varphi(\mathrm{x}+\xi)\, \psi(x_0+\xi_0)\right)
\end{aligned}$$

$$= \left( \frac{\partial^k \mathscr{E}_n(x)}{\partial x_0^k}, \eta(x) \psi(x_0) (u_1(\xi), \varphi(\mathbf{x}+\xi)) \right)$$

$$= \int \left( \frac{\partial^k \mathscr{E}_{nx_0}(x)}{\partial x_0^k}, \eta(x) (u_1(\xi), \varphi(\mathbf{x}+\xi)) \right) \psi(x_0)\, dx_0$$

$$= \int \left( u_1 * \frac{\partial^k \mathscr{E}_{nx_0}}{\partial x_0^k}, \varphi \right) \psi(x_0)\, dx_0,$$

so that

$$\frac{d^k V_{n\varphi}^{(0)}(x_0)}{dx_0^k} = \left( \frac{\partial^k \mathscr{E}_{nx_0}}{\partial x_0^k} * u_1, \varphi \right). \tag{3.4}$$

From this, using the theorem on the continuity of a convolution (see Sec. 4.3) and the lemma of Sec. 15.2, we conclude that $V_{n\varphi}^{(0)} \in C^\infty([0, \infty))$ and therefore $V_n^{(0)} \in C^\infty([0, \infty))$ with respect to $x_0$. The formula (3.2) follows from (3.4) since, by (4.1) and (4.3) of Sec. 3,

$$\frac{d^k V_{n\varphi}^{(0)}(x_0)}{dx_0^k} = \frac{d^k}{dx_0^k}(V_{nx_0}^{(0)}, \varphi) = \left( \frac{\partial^k V_{nx_0}^{(0)}}{\partial x_0^k}, \varphi \right).$$

From what has been proved for the potential $V_n^{(0)}$ follow all the statements concerning the potential $V_n^{(1)}$; here, use must be made of formula (3.1).

The following limiting relations occur as $x_0 \to +0$ for the potentials $V_n^{(0)}$ and $V_n^{(1)}$:

$$V_{nx_0}^{(0)}(\mathbf{x}) \to 0, \qquad \frac{\partial V_{nx_0}^{(0)}(\mathbf{x})}{\partial x_0} \to u_1(\mathbf{x}) \quad \text{in} \quad \mathscr{D}'(\mathbb{R}^n),$$

$$V_{nx_0}^{(1)}(\mathbf{x}) \to u_0(\mathbf{x}), \qquad \frac{\partial V_{nx_0}^{(1)}(\mathbf{x})}{\partial x_0} \to 0, \quad \text{in} \quad \mathscr{D}'(\mathbb{R}^n). \tag{3.5}$$

These follow from (2.2), (3.2) and (3.3) and from the continuity of the convolution (see Sec. 4.3).

If $u_1 \in \mathscr{L}_{\text{loc}}^1$, then for $x_0 > 0$,

$$V_{2\nu+1}^{(0)}(x) = \frac{1}{2^{2\nu}\pi^\nu \Gamma(\nu)} \square^{\nu-1} \frac{1}{x_0} \int\limits_{|\mathbf{x}-\xi|=x_0} u_1(\xi)\, dS_\xi, \qquad \nu \geqslant 1,$$

$$V_{2\nu}^{(0)}(x) = \frac{1}{2^{2\nu-1}\pi^\nu \Gamma(\nu)} \square^{\nu-1} \int\limits_{|\mathbf{x}-\xi|<x_0} \frac{u_1(\xi)\, d\xi}{\sqrt{x_0^2 - |\mathbf{x}-\xi|^2}},$$

$$V_1^{(0)}(x) = \frac{1}{2} \int\limits_{x_1-x_0}^{x_1+x_0} u_1(\xi_1)\, d\xi_1. \tag{3.6}$$

To prove the formulas (3.6) for $n = 2\nu + 1 \geqslant 3$, let us make use of equalities (3.2) (for $k = 0$) and (2.4):

$$V_n^{(0)}(x) = V_{nx_0}^{(0)}(\mathbf{x}) = u_1 * \mathscr{E}_{nx_0} = \frac{1}{2^{2\nu}\pi^\nu\Gamma(\nu)} u_1 * \square^{\nu-1} \frac{1}{x_0} \delta_{S_{x_0}}$$

$$= \frac{1}{2^{2\nu}\pi^\nu\Gamma(\nu)} \square^{\nu-1} \frac{1}{x_0} (u_1 * \delta_{S_{x_0}}). \tag{3.7}$$

We will show that

$$u_1 * \delta_{S_{x_0}} = \int\limits_{|\mathbf{x}-\xi|=x_0} u_1(\xi)\, dS_\xi. \tag{3.8}$$

(By the Fubini theorem, the integral in (3.8), for each $x_0 > 0$, exists for almost all $\mathbf{x} \in \mathbb{R}^n$.)

Indeed, using (3.3) of Sec. 4, we obtain for all $\varphi$ in $\mathscr{D}(\mathbb{R}^n)$

$$(u_1 * \delta_{S_{x_0}}, \varphi) = (u_1(\xi) \times \delta_{S_{x_0}}(\mathbf{y}), \eta(\mathbf{y})\varphi(\xi + \mathbf{y}))$$

$$= \int\limits_{|\mathbf{y}|=x_0} \int u_1(\xi)\varphi(\xi + \mathbf{y})\, d\xi\, dS_{\mathbf{y}}$$

$$= \int\limits_{|\mathbf{y}|=x_0} \int u_1(\mathbf{x} - \mathbf{y})\varphi(\mathbf{x})\, d\mathbf{x}\, dS_{\mathbf{y}}$$

$$= \int \varphi(\mathbf{x}) \int\limits_{|\mathbf{y}|=x_0} u(\mathbf{x} - \mathbf{y})\, dS_{\mathbf{y}}\, d\mathbf{x}$$

$$= \int \varphi(\mathbf{x}) \int\limits_{|\mathbf{x}-\xi|=x_0} u(\xi)\, dS_\xi\, dx,$$

whence follows (3.8).

From (3.8) and (3.7) follows (3.6) for $n = 2\nu + 1 \geqslant 3$. In the other cases, the proof is analogous and simpler.

**15.4 The Cauchy problem for the wave equation** In accordance with the general theory (see Sec. 15.1), the solution of the generalized Cauchy problem for the wave equation

$$\square u = F(x), \qquad F \in \mathscr{D}'(\overline{\mathbb{R}}_+^1 \times \mathbb{R}^n) \tag{4.1}$$

exists and is unique in $\mathscr{D}'(\overline{\mathbb{R}}_+^1 \times \mathbb{R}^n)$ and is given in the form of a retarded potential

$$u = V_n = F * \mathscr{E}_n$$

with density $F$. In particular, if

$$F(x) = u_1(\mathbf{x}) \times \delta(x_0) + u_0(\mathbf{x}) \times \delta'(x_0),$$

then the appropriate solution $u \in C^\infty([0, \infty))$ with respect to $x_0$; for $x_0 > 0$ it is given as the sum of two surface retarded potentials:

$$u(x) = u_{x_0}(\mathbf{x}) = u_1 * \mathscr{E}_{nx_0} + \frac{\partial}{\partial x_0}(u_0 * \mathscr{E}_{nx_0}) \tag{4.2}$$

and, by (3.5), it satisfies the initial conditions as $x_0 \to +0$,

$$u_{x_0}(\mathbf{x}) \to u_0(\mathbf{x}), \qquad \frac{\partial u_{x_0}(\mathbf{x})}{\partial x_0} \to u_1(\mathbf{x}) \quad \text{in} \quad \mathscr{D}'(\mathbb{R}^n). \tag{4.3}$$

A question arises when the solution of the generalized Cauchy problem is classical.

Theorem *If* $f \in C^{p_n}(\overline{\mathbb{R}}^1_+ \times \mathbb{R}^n)$, $u_0 \in C^{p_n+1}(\mathbb{R}^n)$ *and* $u_1 \in C^{p_n}(\mathbb{R}^n)$, *where* $p_n = 2\left[\frac{n}{2}\right]$, $n \geqslant 2$ *and* $p_1 = 1$, *then the solution of the classical Cauchy problem (1.8)-(1.9) exists and is representable in the form of a sum of three retarded potentials (the Kirchhoff-Poisson-d'Alembert formula)*:

$$u(x) = \frac{1}{2^{2\nu}\pi^\nu\Gamma(\nu)} \Box^{\nu-1}\Bigg[\int\limits_{|\mathbf{x}-\boldsymbol{\xi}|<x_0} \frac{f(x_0 - |\mathbf{x}-\boldsymbol{\xi}|, \boldsymbol{\xi})}{|\mathbf{x}-\boldsymbol{\xi}|}\, d\boldsymbol{\xi}$$
$$+ \frac{1}{x_0}\int\limits_{|\mathbf{x}-\boldsymbol{\xi}|=x_0} u_1(\boldsymbol{\xi})\, dS_{\boldsymbol{\xi}} + \frac{\partial}{\partial x_0}\frac{1}{x_0}\int\limits_{|\mathbf{x}-\boldsymbol{\xi}|=x_0} u_0(\boldsymbol{\xi})\, dS_{\boldsymbol{\xi}}\Bigg],$$
$$n = 2\nu + 1 \geqslant 3;$$

$$u(x) = \frac{1}{2^{2\nu-1}\pi^\nu\Gamma(\nu)} \Box^{\nu-1}\Bigg[\int\limits_0^{x_0}\int\limits_{|\mathbf{x}-\boldsymbol{\xi}|<x_0-\xi_0} \frac{f(\xi)\, d\boldsymbol{\xi}\, d\xi_0}{\sqrt{(x-\xi)^2}}$$
$$+ \int\limits_{|\mathbf{x}-\boldsymbol{\xi}|<x_0} \frac{u_1(\boldsymbol{\xi})\, d\boldsymbol{\xi}}{\sqrt{x_0^2 - |\mathbf{x}-\boldsymbol{\xi}|^2}} + \frac{\partial}{\partial x_0}\int\limits_{|\mathbf{x}-\boldsymbol{\xi}|<x_0} \frac{u_0(\boldsymbol{\xi})\, d\boldsymbol{\xi}}{\sqrt{x_0^2 - |\mathbf{x}-\boldsymbol{\xi}|^2}}\Bigg],$$
$$n = 2\nu;$$

$$u(x) = \frac{1}{2}\int\limits_0^{x_0}\int\limits_{x_1-x_0+\xi_0}^{x_1+x_0-\xi_0} f(\xi)\, d\xi_1\, d\xi_0$$
$$+ \frac{1}{2}\int\limits_{x_1-x_0}^{x_1+x_0} u_1(\xi_1)\, d\xi_1 + \frac{u_0(x_1+x_0) + u_0(x_1-x_0)}{2},$$
$$n = 1. \tag{4.4}$$

*Proof*. To prove the theorem, it remains to establish, by (4.3), that all retarded potentials belong to the class $C^2(\overline{\mathbb{R}}_+^1 \times \mathbb{R}^n)$. Let us do that for $n = 2\nu + 1 \geqslant 3$. We have

$$\int\limits_{|\mathrm{x}-\xi|<x_0} \frac{f(x_0-|\mathrm{x}-\xi|,\ \xi)}{|\mathrm{x}-\xi|}\,d\xi$$

$$= x_0^{n-1} \int\limits_{|\mathrm{y}|<1} f[x_0(1-|\mathrm{y}|),\ \mathrm{x}+x_0\mathrm{y}]\,\frac{dy}{|\mathrm{y}|} \in C^{2\nu}(\overline{\mathbb{R}}_+^1 \times \mathbb{R}^n),$$

the substitution $\xi = \mathrm{x} + x_0\mathrm{y}$, $d\xi = x_0^n\,d\mathrm{y}$;

$$\frac{1}{x_0} \int\limits_{|\mathrm{x}-\xi|=x_0} u_1(\xi)\,dS_\xi = x_0^{n-2} \int\limits_{|s|=1} u_1(\mathrm{x}+x_0 s)\,ds \in C^{2\nu}(\mathbb{R}^{n+1}),$$

the substitution $\xi = \mathrm{x} + x_0 s$, $dS_\xi = x_0^{n-1}\,ds$;

$$\frac{\partial}{\partial x_0}\frac{1}{x_0} \int\limits_{|\mathrm{x}-\xi|=x_0} u_0(\xi)\,dS_\xi = \frac{\partial}{\partial x_0} x_0^{n-2} \int\limits_{|s|=1} u_0(\mathrm{x}+x_0 s)\,ds \in C^{2\nu}(\mathbb{R}^{n+1}),$$

whence follow the required properties of smoothness of the retarded potentials for $n = 2\nu + 1 \geqslant 3$. The remaining cases are considered in similar fashion. The theorem is proved.

**15.5 A statement of the generalized Cauchy problem for the heat equation** The Cauchy problem for the heat equation is studied by a method similar to that presented in Secs. 15.1 to 15.4 for the wave equation.

Let us consider the Cauchy problem

$$\frac{\partial u}{\partial t} = \nabla^2 u + f(x, t), \qquad u|_{t=0} = u_0(x) \tag{5.1}$$

where $f \in C(\overline{\mathbb{R}}_+^1 \times \mathbb{R}^n)$, $u_0 \in C(\mathbb{R}^n)$. Assuming that the classical solution $u(x, t)$ of the Cauchy problem (5.1) exists, then continuing it and the function $f$ via zero onto $t < 0$, as in Sec. 15.1, we are convinced that the extended functions $\tilde{u}$ and $\tilde{f}$ satisfy, in $\mathbb{R}^{n+1}$, the equation

$$\frac{\partial \tilde{u}}{\partial t} = \nabla^2\tilde{u} + \tilde{f}(x, t) + u_0(x) \times \delta(t). \tag{5.2}$$

This remark makes it possible to generalize the statement of the Cauchy problem for the heat equation in the following direction. Suppose $F \in \mathscr{D}'(\overline{\mathbb{R}}_+^1 \times \mathbb{R}^n)$. The *generalized Cauchy problem* for the heat equation with source $F$ is the name we will

give to the problem of finding, in $\mathbb{R}^{n+1}$, a generalized solution $u$, in $\mathcal{D}'(\overline{\mathbb{R}}^1_+ \times \mathbb{R}^n)$, of the equation

$$\frac{\partial u}{\partial t} = \nabla^2 u + F(x, t). \tag{5.3}$$

**15.6 Thermal potential** First of all, let us make a study of the properties of the fundamental solution $\mathscr{E}(x, t)$ of the heat

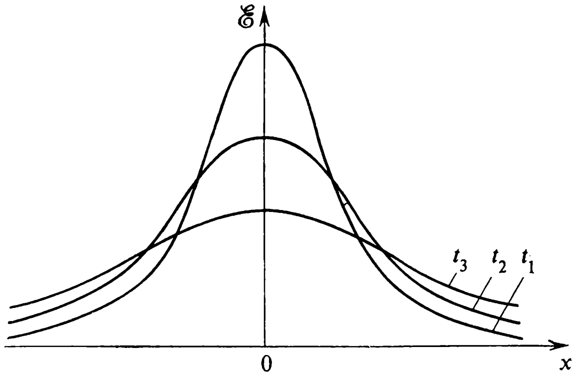



Figure 39

operator. In Sec. 14.4(e), it was shown that

$$\mathscr{E}(x, t) = \frac{\theta(t)}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}.$$

This function is nonnegative, vanishes for $t < 0$, is infinitely differentiable for $(x, t) \neq 0$ and is locally integrable in $\mathbb{R}^{n+1}$. What is more,

$$\int \mathscr{E}(x, t)\, dx = 1, \qquad t > 0, \tag{6.1}$$

by virtue of

$$\int \mathscr{E}(x, t)\, dx = \frac{1}{(4\pi t)^{n/2}} \int e^{-\frac{|x|^2}{4t}} dx = \frac{1}{\pi^{n/2}} \int e^{-|u|^2} du = 1.$$

The graph of the function $\mathscr{E}(x, t)$ for various $t$ ($t_1 < t_2 < t_3$) is depicted in Fig. 39.

Let $F \in \mathcal{D}'(\overline{\mathbb{R}}^1_+ \times \mathbb{R}^n)$. The convolution $V = F * \mathscr{E}$ is termed the *thermal potential* with density $F$. If the thermal potential $V$ exists in $\mathcal{D}'$, then it vanishes for $t < 0$, by virtue of (see Sec. 4.2(g))

$$\operatorname{supp} V \subset \overline{\operatorname{supp} F + \operatorname{supp} \mathscr{E}} \subset [(x, t) : t \geqslant 0],$$

so that $V \in \mathcal{D}'(\bar{\mathbb{R}}^1_+ \times \mathbb{R}^n)$, and it satisfies the heat equation (5.3).

Let us isolate the classes of densities $F$ for which the thermal potential definitely exists.

We denote by $\mathcal{M}$ the class of functions $\{f(x, t)\}$, measurable in $\mathbb{R}^{n+1}$, that vanish for $t < 0$ and that satisfy the following estimate in each strip $0 \leqslant t \leqslant T$, $x \in \mathbb{R}^n$:

$$|f(x, t)| \leqslant C_{T, \varepsilon}(f) e^{\varepsilon |x|^2} \tag{6.2}$$

for an arbitrary $\varepsilon > 0$. Similarly, by $\mathcal{M}_0$ we denote the class of functions $\{f(x)\}$ that are measurable in $\mathbb{R}^n$ and that satisfy the following estimate for arbitrary $\varepsilon > 0$:

$$|f(x)| \leqslant C_\varepsilon e^{\varepsilon | \ |^2}, \qquad x \in \mathbb{R}^n. \tag{6.3}$$

In (6.2) it may be assumed that the quantity $C_{T,\varepsilon}$ does not decrease with respect to $T$.

If $f \in \mathcal{M}$, then the thermal potential $V$ exists in $\mathcal{M}$, is expressed by the integral

$$V(x, t) = \int_0^t \int \frac{f(\xi, \tau)}{[4\pi(t-\tau)]^{n/2}} e^{-\frac{|x-\xi|^2}{4(t-\tau)}} d\xi \, d\tau \tag{6.4}$$

satisfies the estimate: for arbitrary $\varepsilon > 0$,

$$|V(x, t)| \leqslant \frac{tC_{t, \varepsilon}(f)}{(1-8t\varepsilon)^{n/2}} e^{2\varepsilon |x|^2}, \qquad 0 < t < \frac{1}{8\varepsilon}, \tag{6.5}$$

and satisfies the initial condition: for arbitrary $R > 0$,

$$V(x, t) \overset{|x|<R}{\Rightarrow} 0, \qquad t \to +0. \tag{6.6}$$

Indeed, since the functions $\mathcal{E}$ and $f$ are locally summable in $\mathbb{R}^{n+1}$, it follows that their convolution $V = f * \mathcal{E}$ exists, is expressed by the formula (6.4), and is a locally summable function in $\mathbb{R}^{n+1}$ if the function

$$h(x, t) = \int_0^t \int |f(\xi, \tau)| \, \mathcal{E}(x-\xi, t-\tau) \, d\xi \, d\tau$$

is locally summable in $\mathbb{R}^{n+1}$ (see Sec. 4.1). We will prove that the function $h$ satisfies the estimate (6.5). This estimate follows

from the estimate (6.2); by virtue of the Fubini theorem,

$$h(x,t) \leqslant C_{t,\varepsilon} \int_0^t \int e^{-\frac{|x-\xi|^2}{4(t-\tau)}+\varepsilon|\xi|^2} \frac{d\xi\, d\tau}{[4\pi(t-\tau)]^{n/2}}$$

$$= C_{t,\varepsilon} \int_0^t \int e^{-\frac{|y|^2}{4s}+\varepsilon|y-x|^2} \frac{dy\, ds}{(4\pi s)^{n/2}}$$

$$\leqslant C_{t,\varepsilon} e^{2\varepsilon|x|^2} \int_0^t \int e^{-|y|^2\left(\frac{1}{4s}-2\varepsilon\right)} \frac{dy\, ds}{(4\pi s)^{n/2}}.$$

Here we made use of the inequality $|y-x|^2 \leqslant 2|y|^2+2|x|^2$. Making the following substitution in the inner integral,

$$u = y\sqrt{\frac{1}{4s}-2\varepsilon}, \qquad du = \left(\frac{1}{4s}-2\varepsilon\right)^{n/2} dy,$$

we continue our estimates:

$$h(x,t) \leqslant C_{t,\varepsilon} e^{2\varepsilon|x|^2} \int_0^t \int \frac{e^{-u^2}\, du}{\pi^{n/2}} \frac{ds}{(1-8\varepsilon s)^{n/2}}$$

$$\leqslant \frac{tC_{t,\varepsilon}}{(1-8\varepsilon t)^{n/2}} e^{2\varepsilon|x|^2}, \qquad 0 < t < \frac{1}{8\varepsilon},$$

which is what was required. Since $|V| \leqslant h$, the potential $V$ also satisfies the estimate (6.5). Furthermore, $V = h = 0$ for $t < 0$, so that $V \in \mathcal{M}$. From the estimate (6.5) it follows that $V$ satisfies the initial conditions (6.6).

Suppose the density $F(x, t) = u_0(x) \times \delta(t)$, where $u_0 \in \mathcal{D}'(\mathbb{R}^n)$. The thermal potential

$$V^{(0)} = [u_0(x) \times \delta(t)] * \mathscr{E}$$

is called a *surface thermal potential* (of the type of simple layer with density $u_0$).

If $u_0 \in \mathcal{M}_0$, then the surface thermal potential $V^{(0)}$ exists in $\mathcal{M} \cap C^\infty(\mathbb{R}^1_+ \times \mathbb{R}^n)$, is expressed by the integral

$$V^{(0)}(x,t) = \frac{\theta(t)}{(4\pi t)^{n/2}} \int u_0(\xi)\, e^{-\frac{|x-\xi|^2}{4t}}\, d\xi \tag{6.7}$$

and satisfies the estimate: for arbitrary $\varepsilon > 0$,

$$|V^{(0)}(x, t)| \leqslant \frac{C_\varepsilon(u_0)}{(1-8\varepsilon t)^{n/2}} e^{2\varepsilon|x|^2}, \qquad 0 < t < \frac{1}{8\varepsilon}. \tag{6.8}$$

If, besides, $u_0 \in C$, then $V^{(0)} \in C(\overline{\mathbb{R}}^1_+ \times \mathbb{R}^n)$ and satisfies the initial condition

$$V^{(0)}|_{t=+0} = u_0(x). \tag{6.9}$$

Corollary *From (6.9) it follows that*

$$\mathscr{E}(x, t) \to \delta(x), \qquad t \to +0. \tag{6.10}$$

The representation (6.7) and the estimate (6.8) are proved in the same way as for the potential $V$. Here, use must be made of estimate (6.3). From the representation (6.7) it follows that $V^{(0)} \in \mathscr{M}$ and, besides, $V^{(0)} \in C^\infty$ for $t > 0$ and $x \in \mathbb{R}^n$. The latter again by virtue of (6.3).

It remains to prove that $V^{(0)}$ is a continuous function for $t \geqslant 0$, $x \in \mathbb{R}^n$ and satisfies the initial condition (6.9) if $u_0 \in \mathscr{M}_0 \cap C$. Suppose $(x, t) \to (x_0, 0)$, $t > 0$ and $\eta > 0$ is an arbitrary number. By continuity of the function $u_0(x)$, there is a number $\delta > 0$ such that

$$|u_0(\xi) - u_0(x_0)| < \eta \quad \text{for} \quad |\xi - x_0| < 2\delta.$$

Therefore if $|x - x_0| < \delta$ (then also $|x - y - x_0| < 2\delta$ for $|y| < \delta$), we will have, for $\varepsilon = 1$, by (6.1) and (6.3),

$$\begin{aligned}
&|V^{(0)}(x, t) - u_0(x_0)| \\
&\leqslant \int |u_0(\xi) - u_0(x_0)| \mathscr{E}(x-\xi, t)\, d\xi \\
&\qquad = \int_{|y|<\delta} |u_0(x-y) - u_0(x_0)| \mathscr{E}(y, t)\, dy \\
&\qquad\qquad + \int_{|y|>\delta} |u_0(x-y) - u_0(x_0)| \mathscr{E}(y, t)\, dy \\
&\leqslant \eta \int \mathscr{E}(y, t)\, dy + |u_0(x_0)| \int_{|y|>\delta} \mathscr{E}(y, t)\, dy \\
&\qquad\qquad + C_1 \int_{|y|>\delta} \mathscr{E}(y, t) e^{|x-y|^2}\, dy
\end{aligned}$$

$$\leqslant \eta + \frac{|u_0(x_0)|}{(4\pi t)^{n/2}} \int\limits_{|y|>\delta} e^{-\frac{|y|^2}{4t}}\, dy + C_1 \frac{e^{2|x|^2}}{(4\pi t)^{n/2}} \int\limits_{|y|>\delta} e^{-|y|^2\left(\frac{1}{4t}-2\right)}\, dy$$

$$\leqslant \eta + \frac{|u_0(x_0)|}{\pi^{n/2}} \int\limits_{|u|>\frac{\delta}{2\sqrt{t}}} e^{-|u|^2}\, du$$

$$+ \frac{C_1 e^{|x|^2}}{[\pi(1-8t)]^{n/2}} \int\limits_{|u|>\frac{\delta}{2\sqrt{t}}\sqrt{1-8t}} e^{-|u|^2}\, du. \qquad (6.11)$$

The second and third summands in (6.11) may also be made $< \eta$ for all sufficiently small $t > 0$, $t < \delta_1$. Thus,

$$|V^{(0)}(x, t) - u_0(x_0)| < 3\eta, \qquad |x - x_0| < \delta, \qquad 0 < t < \delta_1,$$

which is what we require.

## 15.7 Solution of the Cauchy problem for the heat equation

Theorem *If* $F(x, t) = f(x, t) + u_0(x) \times \delta(t)$, *where* $f \in \mathcal{M}$ *and* $u_0 \in \mathcal{M}_0$, *then the solution of the generalized Cauchy problem (5.3) exists and is unique in the class* $\mathcal{M}$ *and is given as a sum of two thermal potentials (Poisson's formula)*:

$$u(x, t) = V(x, t) + V^{(0)}(x, t)$$

$$= \int_0^t \int \frac{f(\xi, \tau)}{[4\pi(t-\tau)]^{n/2}} e^{-\frac{|x-\xi|^2}{4(t-\tau)}}\, d\xi\, d\tau$$

$$+ \frac{\theta(t)}{(4\pi t)^{n/2}} \int u_0(\xi)\, e^{-\frac{|x-\xi|^2}{4t}}\, d\xi. \qquad (7.1)$$

*If, moreover,* $f \in C^2(\overline{\mathbb{R}}^1_+ \times \mathbb{R}^n)$, $D^\alpha f \in \mathcal{M}$, $|\alpha| \leqslant 2$, $u_0 \in$ $\in \mathcal{M}_0 \cap C$, *then the formula (7.1) yields the classical solution to the Cauchy problem (5.1).*

*Proof*. By what has been proved and in accordance with the general theory developed in Sec. 4.8(c), the solution of the equation

$$\frac{\partial u}{\partial t} = \nabla^2 u + f(x, t) + u_0(x) \times \delta(t), \qquad (x, t) \in \mathbb{R}^{n+1}$$

exists and is unique in the class $\mathcal{M}$ and is given in the form of a sum of two thermal potentials:

$$u = (f + u_0 \times \delta) * \mathcal{E} = f * \mathcal{E} + (u_0 \times \delta) * \mathcal{E} = V + V^{(0)}$$

whence and also from (6.4) and (6.7) follows formula (7.1).

Making a change of variables of integration in (6.4),

$$\xi = x - 2\sqrt{s}\,y, \qquad \tau = t - s,$$

we express the potential $V$ in the form

$$V(x, t) = \frac{1}{\pi^{n/2}} \int_0^t \int f(x - 2\sqrt{s}\,y,\ t - s)\, e^{-|y|^2}\, dy\, ds. \tag{7.2}$$

Let $f \in C^2(\overline{\mathbb{R}^1_+} \times \mathbb{R}^n)$ and $D^\alpha f \in \mathcal{M}$, $|\alpha| \leqslant 2$. Using the theorems on the continuity and differentiability of integrals dependent on a parameter, we conclude, from formula (7.2) and from the equality

$$\frac{\partial V(x, t)}{\partial t} = \frac{1}{\pi^{n/2}} \int_0^t \int \frac{\partial f(x - 2\sqrt{s}\,y,\ t - s)}{\partial t}\, e^{-|y|^2}\, dy\, ds$$

$$+ \frac{1}{\pi^{n/2}} \int f(x - 2\sqrt{t}\,y,\ +0)\, e^{-|y|^2}\, dy,$$

that all functions $D^\alpha V$, $|\alpha| \leqslant 2$, with the exception of $\frac{\partial^2 V}{\partial t^2}$, are continuous for $t \geqslant 0$, $x \in \mathbb{R}^n$, and the function $\frac{\partial^2 V}{\partial t^2}$ is continuous for $t > 0$, $x \in \mathbb{R}^n$. Consequently, $V \in C^1(\overline{\mathbb{R}^1_+} \times \mathbb{R}^n) \cap C^2(\mathbb{R}^1_+ \times \mathbb{R}^n)$.

Finally, if $u_0 \in \mathcal{M}_0 \cap C$, then, by what has been proved, the potential $V^{(0)} \in C(\overline{\mathbb{R}^1_+} \times \mathbb{R}^n) \cap C^\infty(\mathbb{R}^1_+ \times \mathbb{R}^n)$.

Thus, the generalized solution $u(x, t)$ defined by (7.1) belongs to the class $C(\overline{\mathbb{R}^1_+} \times \mathbb{R}^n) \cap C^2(\mathbb{R}^1_+ \times \mathbb{R}^n)$ and therefore is the classical solution of the heat equation (5.1) for $t > 0$. Moreover, by (6.6) and (6.9), that solution satisfies the initial condition of (5.1) as well. Now this means that formula (7.1) will yield the solution to the classical Cauchy problem. The proof of the theorem is complete.

*Remark.* The uniqueness of the solution of the Cauchy problem for the heat equation may be established in a broader class, name-

ly in the class of functions that satisfy in each strip $0 \leqslant t \leqslant T$, $x \in \mathbb{R}^n$, the estimate

$$|u(x, t)| \leqslant C_T e^{a_T |x|^2}.$$

This result was obtained by Tikhonov [1].

## 16 Holomorphic Functions With Nonnegative Imaginary Part in $T^C$

**16.1 Preliminary remarks** We denote by $H_+(G)$ the class of functions that are holomorphic and have nonnegative imaginary part in the region $G$.

A function $u(x, y)$ of $2n$ variables $(x, y)$ is said to be *plurisubharmonic* in the region $G \subset \mathbb{C}^n$ if it is semicontinuous above in $G$ and its trace on every component of every open set $[\lambda : z^0 + \lambda a \subset \subset G]$, $z^0 \in G$, $a \in \mathbb{C}^n$, $a \neq 0$, is a subharmonic function with respect to $\lambda$. The function $u(x, y)$ is said to be *pluriharmonic* in the region $G$ if it is a real (or imaginary) part of some function that is holomorphic in $G$.

Concerning plurisubharmonic and convex functions, see, for example, Vladimirov [1, Chapter II].

The following statements are equivalent:

(1) A function $u(x, y)$ is pluriharmonic in $G$.

(2) A real generalized function $u(x, y)$ in $\mathscr{D}'(G)$ satisfies in $G$ the system of equations

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} = 0, \qquad 1 \leqslant j, k \leqslant n, \qquad z_j = x_j + iy_j.$$

(3) The functions $u(x, y)$ and $-u(x, y)$ are plurisubharmonic in $G$.

Here,

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right).$$

From this it follows that every pluriharmonic function in $G$ is harmonic with respect to every pair of variables $(x_j, y_j)$, $j = 1, \ldots, n$, separately and, hence, is a harmonic function in $G$,

$$\nabla^2 u = \sum_{1 \leqslant j \leqslant n} \left(\frac{\partial^2 u}{\partial x_j^2} + \frac{\partial^2 u}{\partial y_j^2}\right) = 4 \sum_{1 \leqslant j \leqslant n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j} = 0.$$

Therefore $u \in C^\infty(G)$ (see Sec. 14.6).

We denote by $\mathscr{P}_+(G)$ the class of nonnegative pluriharmonic functions in the region $G$.

Let the function $f(z)$ belong to the class $H_+(T^C)$ so that $\operatorname{Im} f \in \mathscr{P}_+(T^C)$. Without restricting generality, we may assume that the cone $C$ is convex. Indeed, by the Bochner theorem the function $f(z)$ is holomorphic (and single-valued) in the hull of holomorphicity $T^{\operatorname{ch} C}$ of the region $T^C$ and assumes the same values in $T^{\operatorname{ch} C}$ as in $T^C$ (see, for example, Vladimirov [1, Sec. 17 and Sec. 20]).

Furthermore, the cone $C$ may be assumed to be different from the entire space $\mathbb{R}^n$. Otherwise, $f(z)$ is an entire function and the condition $\operatorname{Im} f(z) \geqslant 0$ in $\mathbb{C}^n$ leads via the Liouville theorem for harmonic functions to the equation $\operatorname{Im} f(z) = \text{constant}$ in $\mathbb{C}^n$ and, hence, $f(z) = \text{constant}$ in $\mathbb{C}^n$. Finally, we may assume that $\operatorname{Im} f(z) > 0$ in $T^C$. Indeed, if $\operatorname{Im} f(z^0) = 0$ in some point $z^0 \in T^C$, then, by the maximum principle for harmonic functions, $\operatorname{Im} f(z) \equiv 0$ in $T^C$, and then $f(z) = \text{constant}$ in $T^C$.

The function $f(z)$ of the class $H_+(T^C)$ satisfies the following estimate (see Sec. 13.3): for any cone $C' \Subset C$ there is a number $M(C')$ such that

$$|f(z)| \leqslant M(C') \frac{1+|z|^2}{|y|}, \qquad z \in T^{C'}. \tag{1.1}$$

Consequently, $f \in H(C)$ (see Sec. 12.1).

Now let $C$ be a (convex) acute cone (see Sec. 4.4) and let $f \in H_+(T^C)$. By virtue of the estimate (1.1), $f(z)$ possesses a spectral function $g(\xi)$ taken from $\mathscr{S}'(C^*)$ (see Sec. 12.2), $f(z) = L[g]$. From this, using the definition of the Laplace transform (see Sec. 9.1), we have, for all $z \in T^C$,

$$\operatorname{Im} f(x+iy) = \frac{f(z)-\bar{f}(z)}{2i} = F\left[\frac{g(\xi)\, e^{-(y,\,\xi)} - g^*(\xi)\, e^{(y,\,\xi)}}{2i}\right](x) \tag{1.2}$$

where $g(\xi) \to g^*(\xi) = \overline{g(-\xi)}$. From (1.2) we derive the equation

$$\frac{1}{2i}\left[g(\xi)\, e^{-(y,\,\xi)} - g^*(\xi)\, e^{(y,\,\xi)}\right] = F_x^{-1}[\operatorname{Im} f(x+iy)](\xi), \quad y \in C \tag{1.3}$$

Let $f_+(x)$ be a boundary value of $f(z)$ in $\mathscr{S}'$, that is,

$$\int f(x+iy)\, \varphi(x)\, dx \to (f_+, \varphi), \qquad y \to 0, \quad y \in C, \quad \varphi \in \mathscr{S}. \tag{1.4}$$

Then $g = F^{-1}[f_+]$ and $\operatorname{Im} f_+$ is a nonnegative measure of slow growth (see Sec. 5.3). We denote it by $\mu = \operatorname{Im} f_+$.

Passing to the limit in (1.3) as $y \to 0$, $y \in C$ in $\mathscr{S}'$ (see Sec. 12.2), and using (1.4), we obtain

$$\frac{g-g^*}{2i} = F^{-1}[\operatorname{Im} f_+] = F^{-1}[\mu], \tag{1.5}$$

so that $-ig(\xi) + ig^*(\xi)$ is a positive definite generalized function by virtue of the Bochner-Schwartz theorem (see Sec. 8.2).

Let us now prove the following uniqueness theorem for functions of the class $H_+(T^C)$ [and the class $\mathscr{P}_+(T^C)$].

Theorem *If* $f \in H_+(T^C)$ *and* $\mu = \operatorname{Im} f_+ = 0$, *then* $f(z) = (a, z) + b$, *where* $a \in C^*$ *and* $\operatorname{Im} b = 0$.

Corollary *If* $u \in \mathscr{P}_+(T^C)$ *and its boundary value* $\mu = 0$, *then* $u(x, y) = (a, y)$, *where* $a \in C^*$.

*Proof*. Since $\mu = 0$, it follows that, by (1.5), the spectral function $g$ [in $\mathscr{S}'(C^*)$] of the function $f$ satisfies the condition $g = g^*$ and, hence, since $-C^* \cap C^* = \{0\}$ (cone $C^*$ is acute!), the supp $g = \{0\}$. By the theorem of Sec. 2.6,

$$g(\xi) = \sum_{|\alpha| \leqslant N} c_\alpha D^\alpha \delta(\xi),$$

so that $f(z)$ is a polynomial. But $f \in H_+(T^C)$ and the estimate (1.1) shows that the degree of that polynomial cannot exceed the first degree, so that $f(z) = (a, z) + b$, $z \in T^C$.

But

$$\operatorname{Im} f(z) = (\operatorname{Re} a, y) + (\operatorname{Im} a, x) + \operatorname{Im} b \geqslant 0, \qquad z \in T^C$$

and therefore $\operatorname{Re} a \in C^*$ and $\operatorname{Im} a = 0$. Furthermore, from $\operatorname{Im} f_+(x) = 0$ it follows that $\operatorname{Im} b = 0$. The proof of the theorem is complete.

*Remark*. This theorem is an elementary variant of Bogoliubov's "edge-of-the-wedge" theorem (see, for example, Vladimirov [1, Sec. 27]).

*Examples of functions of the class* $H_+(G)$. (1) If $f \in H_+(G)$, then $-\frac{1}{f} \in H_+(G)$ (see Sec. 13.3).

(2) If $C$ is an acute cone, $\mu$ a nonnegative measure on the unit sphere, $\operatorname{supp}\mu \subset \operatorname{pr} C^*$, then

$$\left[\int\limits_{\operatorname{pr} C^*} \frac{\mu(d\sigma)}{(z, \sigma)}\right]^{-1} \in H_+(T^{\operatorname{ch} C}).$$

(3) $\sqrt{z^2} \in H_+(T^{V^+})$ (see example 2 of Sec. 10.2).

**16.2 Estimates of the growth of functions of the class $\mathscr{P}_+(T^C)$** Every function $u(x, y)$ of the class $\mathscr{P}_+(T^C)$ is an imaginary part of some function $f(z)$ of the class $H_+(T^C)$. Therefore it satisfies the estimate (1.1), and its boundary value in $\mathscr{S}'$ is a nonnegative measure $\mu = \operatorname{Im} f_+ = u(x, +0)$ of slow growth, so that, by (1.4),

$$\int u(x, y)\varphi(x)\,dx \to \int \varphi(x)\mu(dx), \quad y \to 0, \quad y \in C, \quad \varphi \in \mathscr{S}. \quad (2.1)$$

However, for functions of the class $\mathscr{P}_+(T^C)$ more precise estimates of growth and boundary behaviour may be indicated in terms of the appropriate Poisson integral; namely, the following theorem holds.

Theorem *If $u \in \mathscr{P}_+(T^C)$, where $C$ is an acute (convex) cone, then we have the estimate*

$$\int \mathscr{P}_C(x - x', y)\,u(x', y')\,dx' \leqslant u(x, y + y'), \quad (2.2)$$

$$(x, y) \in T^C, \quad y' \in C,$$

*where $\mathscr{P}_C$ is the Poisson kernel of the region $T^C$. In particular, for a boundary value of the function $u(x, y)$, of the measure $\mu = u(x, +0)$, the estimate (2.2) takes the form*

$$\int \mathscr{P}_C(x - x', y)\,\mu(dx') \leqslant u(x, y), \qquad (x, y) \in T^C. \quad (2.2')$$

*The function $u(x, y)$ takes a boundary value $\mu$ in the following sense: for any $\varphi \in C \cap \mathscr{L}^\infty$,*

$$\int u(x, y')\,\mathscr{P}_C(x, y)\,\varphi(x)\,dx \to \int \varphi(x)\,\mathscr{P}_C(x, y)\,\mu(dx), \quad (2.3)$$

$$y' \to 0, \quad y' \in C', \quad \forall C' \Subset C, \quad y \in C.$$

Corollaries *The following statements hold true under the hypotheses of the theorem:*

(1) *For arbitrary* $\varepsilon > 0$ *and for the compact* $K \Subset T^C$, *there is a number* $R > 0$ *such that*

$$\int_{|x'|>R} \mathscr{P}_C(x-x', y)\, \mu(dx') < \varepsilon, \qquad (x, y) \in K. \tag{2.4}$$

(2) *If* $f \in C \cap \mathscr{L}^\infty$, *then the integral*

$$\int f(x-x')\, \mathscr{P}_C(x-x', y)\, \mu(dx') \tag{2.5}$$

*is a continuous function in* $T^C$.

(3) *For the Poisson integral*

$$\int \mathscr{P}_C(x-x', y)\, \mu(dx') = \mu * \mathscr{P}_C$$

*the Fourier transform formula*

$$F[\mu * \mathscr{P}_C] = F[\mu]\, F[\mathscr{P}_C] \tag{2.6}$$

*holds true.*

(4) *The following limiting relations hold:*

$$\int \mathscr{P}_C(x-x', y)\, \mu(dx') \to \mu, \qquad y \to 0, \quad y \in C \text{ in } \mathscr{S}', \tag{2.7}$$

$$\int \mathscr{P}_C(x-x', y')\, u(x', y)\, dx' \to u(x, y), \tag{2.8}$$

$$y' \to 0, \quad y' \in C, \quad \forall (x, y) \in T^C.$$

(5) *For any (open) convex cone* $C' \Subset C$ *there is a function* $v_{C'}(y)$ *with the following properties:*

(a) $v_{C'}(y)$ *is nonnegative and continuous in* $C'$,

(b) $v_{C'}(y) \to 0$, $y \to 0$, $y \in C'$,

(c) $$u(x, y) = \int \mathscr{P}_C(x-x', y)\, \mu(dx') + v_{C'}(y), \quad (x, y) \in T^{C'}. \tag{2.9}$$

(6) *If $C$ is a regular cone, then*

$$\int u(x', y')\mathscr{S}_C(z-x'; z^0-x')\,dx' \to \int \mathscr{S}_C(z-x'; z^0-x')\,\mu(dx'), \tag{2.10}$$

$$y' \to 0, \quad y' \in C', \quad \forall C' \Subset C, \quad z \in T^C, \quad z_0 \in T^C,$$

*where $\mathscr{S}_C$ is the Schwartz kernel of the region $T^C$* (see Sec. 12.5).

*Remark 1.* Since $u \in \mathscr{P}_+(T^C)$ implies that $u \in \mathscr{P}_+(T^{C_1})$, $C_1 \subset C$, it follows that all the above-enumerated statements hold true also for an arbitrary (open) convex cone $C_1 \subset C$.

*Remark 2.* The limiting relation (2.7) also holds on functions of the form

$$\varphi(x) = \psi(x)\mathscr{P}_{C_1}(x, y'), \quad \forall \psi \in C \cap \mathscr{L}^\infty, \; y' \in C_1, \; C_1 \subset C, \tag{2.11}$$

provided that $y \to 0$, $y \in C'$, $\forall C' \Subset C$.

*Remark 3.* The estimates (2.2) and (2.2′), for $n = 1$, $C = (0, \infty)$ (upper half-plane), follow from the Herglotz-Nevanlinna representation (see Sec. 17.2 below). In the general case, they have been proved by Vladimirov (in [7] for $C = \mathbb{R}^n_+$; in [10(II)] for $C = V^+$, $n = 4$; in [12] for the general case).

*Remark 4.* The following question arises: Does the representation (2.9) hold for $C' = C$, that is, is passage to the limit possible under the integral sign in the Poisson integral (2.9) as $C' \to C$, $C' \Subset C$? A positive answer to that question is given in two special cases by Vladimirov in [7] ($C = \mathbb{R}^n_+$) and in [10(IV)] ($C = V^+$, $n = 4$).

To prove the theorem, fix $\varepsilon > 0$ and set

$$f_\varepsilon(z) = \frac{f(z)}{1 - i\varepsilon f(z)}, \tag{2.12}$$

where $f \in H_+(T^C)$ is such that $\operatorname{Im} f = u$. Put $\operatorname{Re} f = v$, so that $f = v + iu$. The function $f_\varepsilon(z)$ has the following properties:

(a) It belongs to the class $H_+(T^C)$ since

$$\operatorname{Re}(1 - i\varepsilon f) = 1 + \varepsilon u \geqslant 1, \qquad \operatorname{Im} f_\varepsilon = \frac{u + \varepsilon(v^2 + u^2)}{(1+\varepsilon u)^2 + \varepsilon^2 v^2} \geqslant 0;$$

(b) It is bounded in $T^C$,

$$|f_\varepsilon(z)| \leqslant \min\left[\frac{1}{\varepsilon}, |f(z)|\right],$$

since

$$|f_\varepsilon|^2 = \frac{v^2+u^2}{1+2\varepsilon u+\varepsilon^2(v^2+u^2)};$$

(c) $$f_\varepsilon(z) \overset{z\in K}{\Longrightarrow} f(z), \qquad \varepsilon\to 0, \quad \forall K \Subset T^C.$$

Let $y'$ be an arbitrary fixed point in $C$. The function $f_\varepsilon(z+iy')$ is bounded and continuous in $z$ on $T^{\overline{C}}$. Therefore, for all $z^0 \in T^C$,

$$f_\varepsilon(x+iy')\,\mathcal{K}_C(x-\overline{z}^0) \in \mathcal{L}^2 = \mathcal{H}_0,$$

so that the condition (5.5) of Sec. 12 is fulfilled. By the theorem of Sec. 12.5, the function $f_\varepsilon(z+iy')$ can be represented by the Poisson integral so that

$$\operatorname{Im} f_\varepsilon(z+iy') = \int \operatorname{Im} f_\varepsilon(x'+iy')\,\mathcal{P}_C(x-x', y)\,dx', \tag{2.13}$$

$$z\in T^C, \qquad y'\in C.$$

Passing to the limit in (2.13) as $\varepsilon\to 0$, taking into account property (c), and using the Fatou lemma, we obtain inequality (2.2).

Now let us prove the estimate (2.2′). Let the sequence $\{\eta_k\}$ of functions taken from $\mathcal{D}(\mathbb{R}^n)$ converge to 1 in $\mathbb{R}^n$ (see Sec. 4.1), also $0\leqslant \eta_k(x)\leqslant 1$, $\eta_k(x)\leqslant\eta_{k+1}(x)$, $k=1, 2, \ldots$. Then from the inequality (2.2) follows the inequality

$$\int \mathcal{P}_C(x-x', y)\,\eta_x(x')\,u(x', y')\,dx' \leqslant u(x, y+y'), \qquad k=1, 2, \ldots,$$

whence and also from the limiting relation (2.1), as $y'\to 0$, $y'\in C$, we derive the inequality

$$\int \mathcal{P}_C(x-x', y)\,\eta_k(x')\,\mu(dx') \leqslant u(x, y), \qquad k=1, 2, \ldots.$$

Passing to the limit here as $k\to\infty$ and using the theorem of B. Levi, we obtain the inequality (2.2′).

Now let us prove the limiting relation (2.3) on the functions $\varphi\in C$, $\varphi(\infty)=0$, that is $\varphi\in\overline{C}_0$ (see Sec. 0.5). Since $\mathcal{D}$ is dense in $\overline{C}_0$ in norm in $C$ (see Sec. 1.2), it follows that for $\forall\varepsilon>0$, $\exists\psi\in\mathcal{D}$ such that $|\varphi(x)-\psi(x)|<\varepsilon$, $x\in\mathbb{R}^n$. From this, and also from the inequalities (2.2) and (2.2′), we derive the

inequality

$$\left|\int u(x, y')\,\mathcal{P}_C(x, y)\,\varphi(x)\,dx - \int \varphi(x)\,\mathcal{P}_C(x, y)\,\mu(dx)\right|$$

$$\leqslant \left|\int u(x, y')\,\mathcal{P}_C(x, y)\,\psi(x)\,dx - \int \psi(x)\,\mathcal{P}_C(x, y)\,\mu(dx)\right|$$

$$+\varepsilon\,[u(0, y+y') + u(0, y)],$$

from which and from (2.1) we conclude that (2.3) holds on the functions $\varphi \in \overline{C}_0$ (if $y' \to 0$, $y' \in C$).

We now prove Corollary (1). It suffices to prove it for any sufficiently small sphere $K$. Let $U(z_0; 2r_0) \Subset T^C$. By (2.2′) and (1.1) of Sec. 11 we have the inequality

$$\int |\mathcal{K}_C(x - x' + iy)|^2\,\mu(dx') \leqslant (2\pi)^n\,\mathcal{K}_C(2iy)\,u(x, y). \quad (2.14)$$

Since the function $|\mathcal{K}_C(z)|^2$ is plurisubharmonic in $T^C$, it follows, by the theorem on the spherical mean, that for all $x' \in \mathbb{R}^n$ and $z \in U(z_0; r_0)$ the following inequality holds (see, for example, Vladimirov [1, Sec. 10]):

$$|\mathcal{K}_C(z - x')|^2 \leqslant C_0 \int_{U(z;\, r_0)} |\mathcal{K}_C(x'' - x' + iy'')|^2\,dx''\,dy''$$

$$\leqslant C_0 \int_{U(z_0;\, 2r_0)} |\mathcal{K}_C(x'' - x' + iy'')|^2\,dx''\,dy'', \quad (2.15)$$

where $1/C_0 = \operatorname{mes} U(0; r_0)$. From the inequality (2.14) there follows, by the Fubini theorem, the existence of the integral

$$\int \int_{U(z_0;\, 2r_0)} |\mathcal{K}_C(x'' - x' + iy'')|^2\,dx''\,dy''\,\mu(dx')$$

$$= \int_{U(z_0;\, 2r_0)} \int |\mathcal{K}_C(x'' - x' + iy'')|^2\,\mu(dx')\,dx''\,dy''$$

$$\leqslant (2\pi)^n \int_{U(z_0;\, 2r_0)} \mathcal{K}_C(2iy'')\,u(x'', y'')\,dx''\,dy'' < \infty.$$

From this, by B. Levi's theorem,

$$\lim_{R\to\infty} \int_{|x'|>R} \int_{U(z_0;\, 2r_0)} |\mathcal{K}_C(x''-x'+iy''|^2\, dx''\, dy''\mu(dx')=0. \tag{2.16}$$

Integrating the inequality (2.15) over the region $|x'|>R$ in measure $\mu$, we derive, from (2.16),

$$\int_{x'|>R} |\mathcal{K}_C(z-x')|^2\mu(dx') \xRightarrow{z\in U(z_0;\, r_0)} 0, \qquad R\to\infty$$

and from this follows (2.4).

Let us now prove Corollary (2). By Corollary (1), the integral (2.5) can be represented as a sum of two integrals in the neighbourhood of each point of the region $T^C$: of a continuous function (for $|x'|\leqslant R$) and of an arbitrarily small function (for $|x'|>R$).

We now prove Corollary (3). Formula (2.6) follows from (2.2′) and from the Fubini theorem by virtue of the following operations:

$$\begin{aligned}(\mu*\mathcal{P}_C,\varphi)&=\int\int \mathcal{P}_C(x-x',y)\,\mu(dx')\,\varphi(x)\,dx\\ &=\int\int \mathcal{P}_C(x'-x,y)\,\varphi(x)\,dx\,\mu(dx')=(\mu,\mathcal{P}_C*\varphi)\\ &=(F[\mu],F^{-1}[\mathcal{P}_C*\varphi])=(F[\mu],F[\mathcal{P}_C]\,F^{-1}[\varphi]),\\ &=(F[\mu],F[\mathcal{P}_C],F^{-1}[\varphi])=(F^{-1}[F[\mu]\,F[\mathcal{P}_C]],\varphi),\\ &\qquad \varphi\in\mathcal{S}.\end{aligned}$$

We now prove the limiting relation (2.7) on the functions $\varphi(x)$ of the form (2.11), in which it is also assumed that $\psi(\infty)=0$. Without restricting generality, we can assume that $\varphi\geqslant 0$. We set

$$\chi(x)=\int\varphi(x+x')\,\mu(dx')=\int\psi(x+x')\,\mathcal{P}_{C_1}(x+x',y)\,\mu(dx') \tag{2.17}$$

Taking into account Remark 1, we conclude that the function $\chi\geqslant 0$, which is continuous in $\mathbb{R}^n$ [Corollary (2)], satisfies, by

virtue of (2.2′), the estimates

$$\chi(x) \leqslant \|\psi\|_{\mathcal{L}^\infty} \int \mathcal{P}_{C_1}(x+x', y')\,\mu(dx') \leqslant \|\psi\|_{\mathcal{L}^\infty} u(-x, y'), \tag{2.18}$$

$$\int \chi(x)\,\mathcal{P}_C(x, y)\,dx \leqslant \|\psi\|_{\mathcal{L}^\infty} \int u(-x, y')\,\mathcal{P}_C(x, y)\,dx$$
$$\leqslant \|\psi\|_{\mathcal{L}^\infty} u(0, y+y'). \tag{2.19}$$

The estimate (2.19) makes it possible to apply the Fubini theorem in the following chain of equalities:

$$\int \varphi(x) \int \mathcal{P}_C(x-x', y)\,\mu(dx')\,dx$$
$$= \int \mathcal{P}_C(\xi, y) \int \varphi(x'+\xi)\,\mu(dx')\,d\xi$$
$$= \int \mathcal{P}_C(\xi, y)\,\chi(\xi)\,d\xi.$$

From this, passing to the limit as $y \to 0$, $y \in C$, and using property (a), Sec. 11.3, of the Poisson kernel $\mathcal{P}_C$, we obtain the inequality

$$\underline{\lim} \int \varphi(x) \int \mathcal{P}_C(x-x', y)\,\mu(dx')\,dx$$
$$\geqslant \lim \int_{|\xi|<1} \mathcal{P}_C(\xi, y)\,\chi(\xi)\,d\xi = \chi(0) = \int \varphi(x')\,\mu(dx'). \tag{2.20}$$

On the other hand, by (2.2), for the cone $C'$ and for the limiting relation (2.3), as $y \to 0$, $y \in C$, we have

$$\overline{\lim} \int \varphi(x) \int \mathcal{P}_{C^*}(x-x', y)\,\mu(dx')\,dx$$
$$\leqslant \lim \int \varphi(x)\,u(x, y)\,dx = \lim \int \psi(x)\,\mathcal{P}_{C^*}(x, y')\,u(x, y)\,dx$$
$$= \int \psi(x)\,\mathcal{P}_{C^*}(x, y')\,\mu(dx)$$
$$= \int \varphi(x)\,\mu(dx). \tag{2.21}$$

The inequality (2.21), together with the opposite inequality (2.20), yields the limiting relation (2.7) on the functions $\varphi$ of the kind under consideration.

The case $\varphi \in \mathscr{S}$ is considered analogously and more simply.

The same method is used to prove the limiting relation (2.8) [compare (2.20) and (2.21)]: if $y' \to 0$, $y' \in C$, then

$$\overline{\lim} \int \mathscr{P}_C(x-x', y') u(x', y)\, dx'$$
$$\leqslant \underline{\lim}\, u(x, y+y') = u(x, y)$$
$$= \lim \int\limits_{|x-x'|<1} \mathscr{P}_C(x-x', y') u(x', y)\, dx'$$
$$\leqslant \underline{\lim} \int \mathscr{P}_C(x-x', y') u(x', y)\, dx.$$

Here we again made use of (2.2) and Sec. 11.3(a).

We now prove Corollary (5). Let $C' \Subset C$. By what has been proved (see Remark 1), the function

$$v_{C'}(x, y) = u(x, y) \quad \int \mathscr{P}_{C'}(x-x', y)\, \mu(dx'), \qquad (x, y) \in T^{C'} \tag{2.22}$$

is nonnegative [see (2.2′)], continuous in $T^{C'}$ (see Corollary 2), and satisfies the estimate [see (2.2)]

$$\int v_{C'}(x, y)\, \mathscr{P}_{C''}(x, y')\, dx \leqslant \int u(x, y)\, \mathscr{P}_{C''}(x, y')\, dx \leqslant u(0, y+y'),$$
$$\forall C'' \subset C', \qquad y \in C', \qquad y' \in C''. \tag{2.23}$$

Applying the Fourier transform $F_x^{-1}$ to (2.22) and using the formulas (1.3), (1.5) and (2.6), we obtain, for all $y \in C'$,

$$2iF_x^{-1}[v_{C'}](\xi)$$
$$= g(\xi)\, e^{-(y,\xi)} - g^*(\xi)\, e^{(y,\xi)} - [g(\xi) - g^*(\xi)] F_x[\mathscr{P}_{C'}](\xi),$$
$$\xi \in \mathbb{R}^n \tag{2.24}$$

where $g(\xi)$ is the spectral function of the function $f(z)$,

$$\operatorname{Im} f(z) = u(x, y), \qquad g \in \mathscr{S}'(C^*) \quad \text{(see Sec. 16.1)}.$$

Taking into consideration the equality (1.10) of Sec. 11,

$$F_x[\mathscr{P}_{C'}](\xi) = e^{-|(y,\xi)|}, \qquad \xi \in -C'^* \cup C'^*,$$

and noting that supp $g \subset C^*$, supp $g^* \subset -C^*$, and $C^* \Subset \operatorname{int} C'^*$. we derive, from (2.24), supp $F_x^{-1}[v_{C'}] = \{0\}$.

From this, via the theorem of Sec. 2.6, it follows that

$$F_x^{-1}[v_{C'}](\xi) = \sum_{|\alpha| \leqslant N(y)} C_\alpha(y) D^\alpha \delta(\xi),$$

so that $v_{C'}(x, y)$ is a polynomial in $x$. If in (2.23) we regard $C''$ as an $n$-hedral cone, we conclude that $v_{C'}(x, y) = v_{C'}(y)$ does not depend on $x$ [see Sec. 11.3(e)]. Thus the properties (a) and (c) are proved. It remains to prove property (b). Suppose $\omega \in \mathscr{D}$, $\int \omega(x)\,dx = 1$. Taking into account the limiting relations (2.1) and (2.7), we obtain, from (2.9),

$$\begin{aligned} v_{C'}(y) &= \int v_{C^*}(y)\,\omega(x)\,dx \\ &= \int u(x, y)\,\omega(x)\,dx - \int \omega(x) \int \mathscr{P}_{C^*}(x - x', y)\,\mu(dx')\,dx \\ &\to \int \omega(x)\,\mu(dx) - \int \omega(x)\,\mu(dx) = 0, \qquad y \to 0, \quad y \in C', \end{aligned}$$

which is what we set out to prove.

Let us now extend the limiting relation (2.3) to the functions $\varphi \in C \cap \mathscr{L}^\infty$. Suppose $C' \Subset C$. Since every cone $C' \Subset C$ may be covered with a finite number of $n$-hedral cones that are compact in $C$, it suffices to establish (2.3) for the $n$-hedral cone $C'$. Using the representation (2.9),

$$u(x, y') = \int \mathscr{P}_{C^*}(x - x', y')\,\mu(dx') + v_{C^*}(y'), \qquad (x, y') \in T^{C'},$$

where $v_{C^*}(y') \to 0$, $y' \to 0$, $y' \in C'$, we have the equation

$$\int u(x, y')\,\mathscr{P}_C(x, y)\,\varphi(x)\,dx = \int \mathscr{P}_{C'}(x, y')\,\psi(x, y)\,dx$$
$$+ v_{C'}(y') \int \mathscr{P}_C(x, y)\,\varphi(x)\,dx, \quad (2.25)$$
$$y \in C, \quad y' \in C',$$

where

$$\psi(x, y) = \int \varphi(x + x')\,\mathscr{P}_C(x + x', y)\,\mu(dx').$$

Interchanging[1] the order of integration in the integral on the right of (2.25) is possible by the Fubini theorem and the estimates

(2.2) and (2.2′), which ensure the existence of the iterated integral

$$\int \mathcal{P}_{C'}(x, y')|\psi(x, y)|\,dx$$
$$\leqslant \int \mathcal{P}_{C'}(x, y')\int |\varphi(x+x')|\mathcal{P}_C(x+x', y)\,\mu(dx')\,dx$$
$$\leqslant \|\varphi\|_{\mathcal{L}^\infty}\int \mathcal{P}_{C'}(x, y')\,u(-x, y)\,dx \leqslant \|\varphi\|_{\mathcal{L}^\infty}\,u(0, y+y'),$$
$$y\in C,\ y'\in C.$$

Furthermore, the last estimate, together with the continuity of the function $\psi(x, y)$ in $T^C$ [see Corollary (2)] permits applying the result of Sec. 11.3(d) to the integral on the right of (2.25). As a result, when $y' \to 0$, $y' \in C'$, we obtain (2.3):

$$\int u(x, y')\,\mathcal{P}_C(x, y)\,\varphi(x)\,dx \to \psi(0, y) = \int \varphi(x')\,\mathcal{P}_C(x', y)\,\mu(dx').$$

The truth of the limiting relation (2.7) on functions of the form (2.11) follows from what has been proved, provided that $y \to 0$, $y \in C'$, $\forall C' \Subset C$ (Remark 2).

Let us now prove Corollary (6). By virtue of the estimates (5.4) of Sec. 12 and (2.2), the integral (2.10) exists.

Setting

$$\varphi(x') = \frac{\mathcal{S}_C(z-x';\ z^0-x')\,|\mathcal{K}_C(z-\bar{z}^0)|}{\mathcal{K}_C(2iy)\,\mathcal{P}_C(x-x'\, y)+[\mathcal{K}_C(2iy^0)+|\mathcal{K}_C(z-\bar{z}^0)|]\,\mathcal{P}_C(x^0-x', y^0)},$$

we have $|\varphi(x')| \leqslant 1$ [see (5.4) of Sec. 12], $\varphi \in C^\infty$ and

$$\int \mathcal{S}_C(z-x'; z^0-x')\,u(x', y')\,dx'$$
$$= \int \Big\{\frac{\mathcal{K}_C(2iy)}{|\mathcal{K}_C(z-z^0)|}\,\mathcal{P}_C(x-x', y)$$
$$+\Big[\frac{\mathcal{K}_C(2iy^0]}{|\mathcal{K}_C(z-\bar{z}^0)|}+1\Big]\,\mathcal{P}_C(x^0-x', y^0)\Big\}\,\varphi(x')\,u(x', y')\,dx'.$$

From this and from (2.3) follows the limiting relation (2.10).
This completes the proof of the theorem and its corollaries.

**16.3 Estimates of the growth of functions of the class $H_+(T^C)$**
Here we will establish that, together with the estimate (1.1), any function of the class $H_+(T^C)$ is estimated in terms of its imaginary part, the estimates of the growth and boundary behaviour of which are given in Sec. 16.2.

Theorem *If $f \in H_+(T^C)$, then*

$$|\mathscr{K}_C(z-\bar{z}^0)[f(z)-\bar{f}(z^0)]|^2$$
$$\leqslant 4\mathscr{K}_C(2iy^0)\,\mathscr{K}_C(2iy)\,\mathrm{Im}\,f(z^0)\,\mathrm{Im}\,f(z), \qquad z \in T^C, \quad z^0 \in T^C. \tag{3.1}$$

Corollary *If $f \in H_+(T^C)$, then*

$$\int |\mathscr{K}_C(z-\bar{z}^0)|^4 |f(z)-\bar{f}(z^0)|^2\,dx$$
$$\leqslant 4(2\pi)^n \mathscr{K}_C(2iy^0)\,\mathscr{K}_C(2iy)\,\mathscr{K}_C(2iy_0+2iy)\,\mathrm{Im}\,f(z^0)\,\mathrm{Im}\,f(z^0+2iy),$$
$$y \in C, \qquad z^0 \in T^C. \tag{3.2}$$

To prove this, we construct, via (2.12), a function $f_\varepsilon(z)$, $\varepsilon > 0$ and apply to the function $f_\varepsilon(z+iy')$, $y' \in C$, the representation (5.9) of Sec.12:

$$\mathscr{K}_C(z-\bar{z}^0)[f_\varepsilon(z+iy')-\bar{f}_\varepsilon(z^0+iy')]$$
$$=\frac{2i}{(2\pi)^n}\int \mathrm{Im}\,f_\varepsilon(x'+iy')\,\mathscr{K}_C(z-x')\,\mathscr{K}_C(x'-\bar{z}^0)\,dx',$$
$$z^0 \in T^C, \qquad z \in T^C, \qquad y' \in C.$$

From this, using the Cauchy-Bunyakovsky inequality, we derive the inequality

$$|\mathscr{K}_C(z-\bar{z}^0)[f_\varepsilon(z+iy')-\bar{f}_\varepsilon(z^0+iy')]|^2$$
$$\leqslant \frac{4}{(2\pi)^{2n}}\int \mathrm{Im}\,f_\varepsilon(x'+iy')\,|\mathscr{K}_C(z-x')|^2\,dx'$$
$$\times \int \mathrm{Im}\,f_\varepsilon(x''+iy')\,|\mathscr{K}_C(x''-\bar{z}^0)|^2\,dx''$$
$$=4\,\mathscr{K}_C(2iy)\,\mathscr{K}_C(2iy^0)\int \mathrm{Im}\,f_\varepsilon(x'+iy')\,\mathscr{P}_C(x-x',\,y)\,dx'$$
$$\times \int \mathrm{Im}\,f_\varepsilon(x''+iy')\,\mathscr{P}_C(x^0-x'',\,y^0)\,dx''.$$

Applying the estimate (2.2) twice, we get

$$|\mathscr{K}_C(z-\bar{z}^0)[f_\varepsilon(z+iy')-\bar{f}_\varepsilon(z^0+iy')]|^2$$
$$\leqslant 4\,\mathscr{K}_C(2iy^0)\,\mathscr{K}_C(2iy)\,\mathrm{Im}\,f_\varepsilon(z+iy')\,\mathrm{Im}\,f_\varepsilon(z^0+iy'),$$
$$z^0 \in T^C, \qquad z \in T^C, \qquad y' \in C.$$

Allowing $y' \to 0$, $y' \in C$, and then $\varepsilon \to 0$, we obtain the required estimate (3.1).

To prove the inequality (3.2) we multiply the inequality (3.1) by $|\mathscr{K}_C(z - \bar{z}^0)|^2$, integrate with respect to $x$, and take advantage of the inequality (2.2):

$$\int |\mathscr{K}_C(z-\bar{z}^0)|^4 |f(z)-\overline{f(z^0)}|^2\,dx$$
$$\leqslant 4(2\pi)^n \mathscr{K}_C(2iy^0)\mathscr{K}_C(2iy)\mathscr{K}_C(2iy_0+2iy)\operatorname{Im} f(z^0)\times$$
$$\times \int \mathscr{P}_C(x-x^0,\ y+y^0)\operatorname{Im} f(x+iy)\,dx$$
$$\leqslant 4(2\pi)^n \mathscr{K}_C(2iy^0)\mathscr{K}_C(2iy)\mathscr{K}_C(2iy_0+2iy)\operatorname{Im} f(z^0)$$
$$\times \operatorname{Im} f(z^0+2iy).$$

**16.4 Smoothness of the spectral function** The estimates (3.1) and (3.2) imply a definite smoothness of the spectral functions of functions of the class $H_+(T^C)$, namely:

Theorem *If $f \in H_+(T^C)$, then its spectral function $g(\xi)$ has the property*

$$\theta^2_{C^*} * g \in \mathscr{L}^2_s(C^*), \qquad s < -\frac{3}{2}n - 1. \tag{4.1}$$

Corollary *If $f \in H_+(T^C)$, where $C$ is a regular cone, then its spectral function $g$ is uniquely representable as*

$$g = \theta^{-2}_{C^*} * g_1, \qquad g_1 \in \mathscr{L}^2_s(C^*), \quad s < -\frac{3}{2}n - 1. \tag{4.2}$$

*Remark.* The operators $\theta^{\alpha}_{C^*}*$ are introduced in Sec. 13.5.

*Example 1* [see (2.16) of Sec. 10]:

$$\theta^{-2}_{\mathbf{R}^n_+} = \frac{\partial^{2n}}{\partial\xi_1^2 \ldots \partial\xi_n^2}.$$

*Example 2* (see Sec. 13.5):

$$\theta^{-2}_{\overline{V}^+} * = 4^{-n}\pi^{-n+1}\Gamma^{-2}\left(\frac{n+1}{2}\right)\square^{n+1}.$$

To prove the theorem, in the inequality (3.2) substitute $f(z + iy^0)$ for $f(z)$ and then set $x^0 = 0$, $y^0 = y$.

After a few simple manipulations we obtain

$$\int |\mathcal{K}_C(x+2iy)|^4 |f(x+2iy)|^2 dx$$
$$\leqslant 2|f(2iy)|^2 \int |\mathcal{K}_C(x+2iy)|^4 dx$$
$$+8\pi^n \mathcal{K}_C^3(2iy) \operatorname{Im} f(2iy) \operatorname{Im} f(4iy), \qquad y \in C. \tag{4.3}$$

But by (1.7) of Sec. 11 we have, for $p=2$, the estimate

$$\int |\mathcal{K}_C(x+2iy)|^4 dx = (2\pi)^{2n} \mathcal{K}_C^2(4iy) \int \mathcal{P}_C^2(x, 2y)\, dx$$
$$\leqslant (2\pi)^{2n} \mathcal{K}_C^2(4iy) \frac{\mathcal{K}_C^2(2iy)}{(2\pi)^n \mathcal{K}_C(4iy)} = \pi^n \mathcal{K}_C^3(2iy).$$

Therefore, inequality (4.3) takes the form

$$\| \mathcal{K}_C^2(x+2iy) f(x+2iy) \|^2$$
$$\leqslant 8\pi^n \mathcal{K}_C^3(2iy) |f(2iy)| (|f(2iy)| + |f(4iy)|), \quad y \in C. \tag{4.4}$$

Now suppose an arbitrary cone $C' \Subset C$. Taking into account the estimate (1.1),

$$|f(iy)| \leqslant M(C') \frac{1+|y|^2}{|y|}, \qquad y \in C',$$

and the estimate (2.4) of Sec. 10,

$$0 < \mathcal{K}_C(iy) \leqslant M_0 \Delta^{-n}(y), \qquad y \in C,$$

from (4.4) we derive the following estimate:

$$\| \mathcal{K}_C^2(x+2iy) f(x+2iy) \|^2$$
$$\leqslant M_1(C') \Delta^{-3n}(y) \frac{1+4|y|^2}{2|y|} \left( \frac{1+4|y|^2}{2|y|} + \frac{1+16|y|^2}{4|y|} \right), \quad y \in C'. \tag{4.5}$$

But $|y| \geqslant \Delta(y) \geqslant \sigma |y|$, $y \in C'$, for some $\sigma > 0$ (see Lemma 1 of Sec. 4.4). Therefore the estimate (4.5) may be rewritten as

$$\| \mathcal{K}_C^2(x+2iy) f(x+2iy) \|^2 \leqslant M_2(C') [1 + \Delta^{-3n-2}(y)], \qquad y \in C'. \tag{4.6}$$

The estimate (4.6) holds true if distance $\Delta(y)$ is replaced by the lesser distance $\Delta'(y)$ (from $y$ to $\partial C'$). Applying the lemma of Sec. 10.5 $\left(\text{for } a = \varepsilon = 0,\ s = 0,\ \gamma = \frac{3}{2}n + 1 \text{ and } C = C'\right)$, we conclude that the function $\mathcal{K}_C^2(z) f(z)$ is the Laplace transform of the function

$$g_1(\xi) = \theta_{C^*}^2 * g \equiv \theta_{C^*} * \theta_{C^*} * g$$

[see Sec. 9.2(g)] taken from $\mathcal{L}_{s'}^2(C^*)$ for all $s' < -\frac{3}{2}n - 1$ and $C' \Subset C$, where $g$ is the spectral function of the function $f$: $f(z) = L[g]$. Hence $g_1 \in \mathcal{L}_s^2(C^*)$ for all $s < -\frac{3}{2}n - 1$. The theorem is proved.

To prove the corollary, set $g_1 = \theta_{C^*}^2 * g \in \mathcal{L}_s^2(C^*)$, $s < -\frac{3}{2}n - 1$. Then in the convolution algebra $\mathcal{S}'(C^*)$ we have, in the case of a regular cone $C$ [see Sec. 4.8(d) and Sec. 13.1],

$$\theta_{C^*}^{-2} * g_1 = \theta_{C^*}^{-2} * (\theta_{C^*}^2 * g) = (\theta_{C^*}^{-2} * \theta_{C^*}^2) * g = \delta * g = g.$$

The function $g_1$ with the indicated properties is unique.

**16.5 Indicator of growth of functions of the class $\mathcal{P}_+(T^C)$**
In Sec. 16.2 we studied the growth of functions of the class $\mathcal{P}_+(T^C)$ as $y \to 0$, $y \in C$, and as $|x| \to \infty$. Here we will investigate the growth of such functions as $|y| \to \infty$, $y \in C$. First we will prove the following lemmas.

Lemma 1 *If $u \in \mathcal{P}_+(T^C)$, where $C$ is a convex cone, then for every bounded region $D \subset \mathbb{R}^n$ and for every point $y \in C$ there is a number $t_0 \geqslant 0$ such that for all $(x^0, y^0) \in T^D$ the function $u(x^0, y^0 + ty)(t - t_0)^{-1}$ does not increase with respect to $t$ on $(t_0, \infty)$.*

*Proof.* Fix $z^0 = x^0 + iy^0 \in \mathbb{C}^n$ and $y \in C$. Since the cone $C$ is open and convex, there is a number $t_0 = t_0(y^0, y)$ such that $y^0 + ty \in C$ for all $t > t_0$. Therefore the function $u(x^0 + \sigma y, y^0 + (\tau + t_0) y)$ belongs to the class $\mathcal{P}_+(T^1)$ [with respect to the variables $(\sigma, \tau)$], and so it can be represented by the formula (see Sec. 17.1)

$$u(x^0+\sigma y,\ y^0+(\tau+t_0)\,y = \frac{\tau}{\pi}\int_{-\infty}^{\infty}\frac{\mu(x^0,\ y^0,\ y;\ d\sigma')}{(\sigma-\sigma')^2+\tau^2}$$
$$+\,a(x^0,\ y_0,\ y)\,\tau, \qquad (5.1)$$

where $a \geqslant 0$ and the measure $\mu \geqslant 0$ satisfies the condition of growth [see (2.2′)]

$$\int_{-\infty}^{\infty} \frac{\mu(x^0, y^0, y; d\sigma')}{1+\sigma'^2} < \infty.$$

Putting $\sigma = 0$ in (5.1), dividing by $\tau$, and setting $\tau = t - t_0 > 0$, we get

$$\frac{u(x^0, y^0+ty)}{t-t_0} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mu(x^0, y^0, y; d\sigma')}{\sigma'^2+(t-t_0)^2} + a(x^0, y^0, y),$$

whence, by the Levi theorem, we conclude that Lemma 1 holds true.

Lemma 2 *Suppose the function $f(x)$ is convex on the set $A$. Then for all $x^0 \in A$ and $x \in \mathbb{R}^n$ the function*

$$\frac{1}{t}[f(x^0+tx)-f(x^0)]$$

*does not decrease with respect to $t$ on the interval $[0, t_0]$ provided that all the points $x^0 + tx$, $0 \leqslant t \leqslant t_0$, are contained in $A$.*

*Proof*. By the definition of a convex function (see Sec. 0.2), the function $f(x^0 + tx)$ is convex with respect to $t$ on $[0, t_0]$ and, hence, for arbitrary $0 \leqslant t < t' \leqslant t_0$,

$$f(x^0+tx) = f\left[\frac{t}{t'}(t'x+x^0) + \left(1-\frac{t}{t'}\right)x^0\right]$$
$$\leqslant \frac{t}{t'} f(x^0+t'x) + \left(1-\frac{t}{t'}\right) f(x^0),$$

that is,

$$\frac{1}{t}[f(x^0+tx)-f(x^0)] \leqslant \frac{1}{t'}[f(x^0+t'x)-f(x^0)],$$

which completes the proof of Lemma 2.

Lemma 3 *If the function $u(x, y)$ is plurisubharmonic in the tubular region $T^D = \mathbb{R}^n + iD$ and is bounded from above on every subregion $T^{D'}$, $D' \Subset D$, then the function*

$$M(y) = \sup_x u(x, y) \tag{5.2}$$

*is convex and, hence, continuous in $D$.*

*Proof*. Suppose the points $y'$ and $y''$ in $D$ are such that $\tau y' + (1-\tau) y'' \Subset D$ for all $0 \leqslant \tau \leqslant 1$. Then the function

$$v(\sigma, \tau) = u(x + \sigma(y' - y''), \quad y'' + \tau(y' - y'')) \tag{5.3}$$

is subharmonic in the neighbourhood of the strip $0 \leqslant \tau \leqslant 1$, $\sigma \in \mathbb{R}^1$, is bounded from above and, by virtue of (5.3),

$$v(\sigma, 0) = u(x + \sigma(y' - y''), y'') \leqslant M(y''),$$
$$v(\sigma, 1) = u(x + \sigma(y' - y''), y') \leqslant M(y').$$

But then the function

$$\chi(\sigma, \tau) = v(\sigma, \tau) - \tau M(y') - (1-\tau) M(y'') \tag{5.4}$$

is subharmonic in the neighbourhood of the strip $0 \leqslant \tau \leqslant 1$, $\sigma \in \mathbb{R}^1$, is bounded above and is nonpositive on the boundary of the strip. By the Phragmén-Lindelöf theorem for subharmonic functions, $\chi(\sigma, \tau) \leqslant 0$, $0 \leqslant \tau \leqslant 1$, $\sigma \in \mathbb{R}^1$, so that by (5.4) and (5.3) (for $\sigma = 0$),

$$u(x, \tau y' + (1-\tau) y'') \leqslant \tau M(y') + (1-\tau) M(y'').$$

From this, by (5.2), we derive the inequality

$$M(\tau y' + (1-\tau) y'') \leqslant \tau M(y') + (1-\tau) M(y''),$$

which completes the proof of Lemma 3.

Let $u \in \mathscr{P}_+(T^C)$, where $C$ is a convex cone. We introduce the growth indicator $h(u; y)$ of the function $u$ via the formula

$$h(u; y) = \lim_{t \to +\infty} \frac{u(0, ty)}{t}, \qquad y \in C. \tag{5.5}$$

By Lemma 1, the limit in (5.5) exists and is nonnegative.

We introduce the function

$$\lambda(u; y) = \lim_{t \to +\infty} \frac{m(ty)}{t}, \tag{5.6}$$

where the quantity $m(y)$ is given by

$$m(y) = \inf_x u(x, y), \qquad y \in C.$$

By Lemma 3, the function $m(y)$ is nonnegative and concave (see Sec. 0.2) in $C$ and, hence, such is the function $\frac{m(ty)}{t}$ for all $t > 0$. By Lemma 2, for all $\varepsilon > 0$ and $y \in C$, the function

$$\frac{1}{t}[m(\varepsilon y + ty) - m(\varepsilon y)]$$

does not increase with respect to $t$. Therefore the limit in (5.6) exists and defines a (nonnegative) concave function $\lambda(u; y)$ in $C$ that satisfies the estimate

$$\lambda(u;\ y) \leqslant \frac{1}{t}[m(\varepsilon y + ty) - m(\varepsilon y)] \leqslant \frac{m(\varepsilon y + ty)}{t}, \qquad y \in C, \quad t > 0$$

Setting $t = 1$ here and allowing $\varepsilon \to 0$, we obtain the estimate

$$\lambda(u;\ y) \leqslant m(y) \leqslant u(x,\ y), \qquad (x,\ y) \in T^C. \tag{5.7}$$

Finally, note that the functions $h$ and $\lambda$ are homogeneous of degree of homogeneity 1; for example:

$$\begin{aligned}\lambda(u;\ ry) &= \lim_{t\to\infty} \frac{m(try)}{t} = r \lim_{t\to\infty} \frac{m(try)}{tr} \\ &= r \lim_{t'\to\infty} \frac{m(t'y)}{t'} = r\lambda(u;\ y), \qquad r > 0.\end{aligned}$$

From this, and also from (5.7) and (5.5) follows the inequality

$$\lambda(u;\ y) \leqslant h(u;\ y) \qquad y \in C. \tag{5.8}$$

We will now prove the following theorem.

Theorem *If $u \in \mathcal{P}_+(T^C)$, where $C$ is a convex cone, then the growth indicator $h(u; y)$ is nonnegative, concave, homogeneous of degree of homogeneity 1 in $C$, and*

$$\lambda(u;\ y) = h(u;\ y) = \lim_{t\to\infty} \frac{u(x^0,\ y^0 + ty)}{t}, \qquad (x^0,\ y^0) \in \mathbb{C}^n, \quad y \in C. \tag{5.9}$$

*For $(x^0, y^0) \in T^C$ the function $\frac{1}{t}u(x^0, y^0 + ty)$ does not increase with respect to $t \in (0, \infty)$ and the following inequality holds true:*

$$h(u;\ y) \leqslant u(x^0,\ y^0 + y), \qquad (x^0,\ y^0) \in T^C, \quad y \in C. \tag{5.10}$$

*Proof*. We will prove that for every $y \in C$ the function

$$\lim_{t\to\infty} \frac{u(x^0, y^0+ty)}{t} = \lim_{t\to\infty} \frac{u(x^0, y^0+ty)}{t-t_0} \tag{5.11}$$

does not depend on $(x^0, y^0)$. For this it suffices to prove, by virtue of the Liouville theorem, that for every $y \in C$ the nonnegative function (5.11) is pluriharmonic with respect to $(x^0, y^0)$ in $\mathbb{C}^n$. That is, it is pluriharmonic in every tubular region $T^D = \mathbb{R}^n + iD$, where $D \Subset \mathbb{R}^n$. By Lemma 1, the function (5.11) in the region $T^D$ is the limit of a nonincreasing sequence of functions $u(x^0, y^0 + ty)(t - t_0)^{-1}$, $t \to \infty$, $t > t_0$, of the class $\mathcal{P}_+(T^D)$ and therefore is itself pluriharmonic in $T^D$ (see Sec. 16.1). Thus, by (5.5), the second of the equalities (5.9) holds, and, by Lemma 1, the function $\frac{1}{t} u(x^0, y^0 + ty)$ does not increase with respect to $t$ for $t > 0$ if $y^\circ \in C$. Therefore

$$h(u; y) \leqslant \frac{1}{t} u(x^0, y^0+ty), \qquad (x^0, y^0) \in T^C, \quad y \in C, \quad t > 0.$$

Putting $t = 1$ here, we obtain the estimate (5.10). From this estimate we derive

$$h(u; y) \leqslant m(y), \qquad y \in C,$$

so that, by (5.6),

$$h(u; y) \leqslant \lambda(u; y), \qquad y \in C.$$

This inequality together with the inverse inequality (5.8) is what yields the first of the equalities (5.9), from which fact it follows that the indicator $h(u; y)$ is a convex function in $C$. This completes the proof of all assertions of the theorem.

*Remark*. A more general theory of growth of plurisubharmonic functions in tubular regions over convex cones is developed in Vladimirov [13].

### 16.6 An integral representation of functions of the class $H_+(T^C)$

We establish here that a function of the class $H_+(T^C)$, where $C$ is an acute cone, is representable in the form of a sum of the Schwartz integral and a linear term if and only if the corresponding Poisson integral is a pluriharmonic function in $T^C$. We first prove a lemma that generalizes the Lebesgue theorem on the limiting passage under the sign of the Lebesgue integral (see Vladimirov [10 (IV)]).

Lemma *Suppose the sequences $u_k(x)$ and $v_k(x)$, $k = 1, 2, \ldots$, of functions in $\mathcal{L}^1$ have the following properties:*

(1) $|u_k(x)| \leqslant v_k(x)$, $k = 1, 2, \ldots$, *almost everywhere in* $\mathbb{R}^n$,

(2) $u_k(x) \to u(x)$, $v_k(x) \to v(x) \in \mathcal{L}^1$, $k \to \infty$, *almost everywhere in* $\mathbb{R}^n$,

$$(3) \qquad \int v_k(x)\,dx \to \int v(x)\,dx, \qquad k \to \infty.$$

*Then* $u \in \mathcal{L}^1$ *and*

$$\int u_k(x)\,dx \to \int u(x)\,dx, \qquad k \to \infty. \tag{6.1}$$

*Proof*. From (1) and (2) it follows that $u \in \mathcal{L}^1$ and $v_k(x) \pm \pm u_k(x) \geqslant 0$, $k = 1, 2, \ldots$, almost everywhere in $\mathbb{R}^n$. Applying the Fatou lemma to the sequences of functions $v_k \pm u_k$, $k \to \infty$, and making use of (3), we derive the following chain of inequalities:

$$\begin{aligned}
&\int [v(x) \pm u(x)]\,dx \\
&\qquad \leqslant \varliminf_{k\to\infty} \int [v_k(x) \pm u_k(x)]\,dx \\
&\qquad = \varliminf_{k\to\infty} \int v_k(x)\,dx + \varliminf_{k\to\infty} \int \pm u_k(x)\,dx \\
&\qquad = \int v(x)\,dx + \varliminf_{k\to\infty} \int \pm u_k(x)\,dx,
\end{aligned}$$

whence we derive

$$\varlimsup_{k\to\infty} \int u_k(x)\,dx \leqslant \int u(x)\,dx \leqslant \varliminf_{k\to\infty} \int u_k(x)\,dx,$$

which is equivalent to the limiting relation (6.1).

Theorem *Let $f \in H_+(T^C)$, where $C$ is an acute (convex) cone. Then the following statements are equivalent:*

(1) *The Poisson integral*

$$\int \mathcal{P}_C(x-x', y)\,\mu(dx'), \qquad \mu = \operatorname{Im} f_+, \tag{6.2}$$

*is a pluriharmonic function in* $T^C$.

(2) *The function* $\operatorname{Im} f(z)$ *is representable in the form*

$$\operatorname{Im} f(z) = \int \mathcal{P}_C(x - x', y)\,\mu(dx') + (a, y), \qquad z \in T^C, \tag{6.3}$$

*for a certain* $a \in C^*$.

(3) *For all* $y' \in C$ *the following representation holds:*

$$\operatorname{Im} f(z + iy') = \int \mathcal{P}_C(x - x', y) \operatorname{Im} f(x' + iy')\,dx' + (a, y), \; z \in T^C. \tag{6.4}$$

(4) *If* $C$ *is a regular cone, then for an arbitrary* $z^0 \in T^C$, *the function* $f(z)$ *can be represented as*

$$f(z) = i \int \mathcal{S}_C(z - x'; z^0 - x')\,\mu(dx') + (a, z) + b(z^0), \qquad z \in T^C, \tag{6.5}$$

*where* $b(z^0)$ *is a real number.*

*Here,* $b(z^0) = \operatorname{Re} f(z_0) - (a, x^0)$ *and* $(a, y)$ *is the best linear minorant of the growth indicator* $h(\operatorname{Im} f; y)$ *in the cone* $C$.

*Remark.* Under the hypothesis of the theorem, the best linear minorant of the nonnegative convex function $h(\operatorname{Im} f; y)$ of degree of homogeneity 1 exists in the cone $C$ (see Sec. 16.5). For examp e, $h(\operatorname{Im} \sqrt{z^2}; y) = \sqrt{y^2}$ and $(a, y) = 0$ in $V^+$.

*Proof.* Let $f \in H_+(T^C)$. (1) $\to$ (2). The function

$$v(x, y) = \operatorname{Im} f(z) - \int \mathcal{P}_C(x - x', y)\,\mu(dx')$$

belongs to the class $\mathcal{P}_+(T^C)$ and its boundary value, as $y \to 0$, $y \in C$, is equal to 0 (see Sec. 16.2). By a corollary to the theorem of Sec. 16.1, $v(x, y) = (a, y)$ for some $a \in C^*$. The representation (6.3) is proved.

We now prove that $(a, y)$ is the best linear minorant of the function $h(\operatorname{Im} f; y)$ in the cone $C$. From (6.3) and (5.5) it follows that $(a, y)$ is a linear minorant of $h$ in $C$. Suppose $(a', y)$ is another linear minorant of $h$ in $C$, that is

$$(a', y) \leqslant h(\operatorname{Im} f; y), \qquad y \in C. \tag{6.6}$$

The function

$$f_1(z) = f(z) - (a', z), \qquad \operatorname{Im} f_1(z) = \operatorname{Im} f(z) - (a, y')$$

belongs to the class $H_+(T^C)$ since

$$\operatorname{Im} f(z) \leqslant h(\operatorname{Im} f;\ y) \geqslant (a',\ y), \qquad z \in T^C,$$

by the theorem of Sec. 16.5 and by virtue of (6.6). Furthermore, since $\operatorname{Im} f_{1+} = \operatorname{Im} f_+ = \mu$, it follows that condition (1) is fulfilled for $f_1(z)$. Applying the representation (6.3) to $\operatorname{Im} f_1(z)$, we obtain

$$\begin{aligned}\operatorname{Im} f_1(z) &= \operatorname{Im} f(z) - (a',\ y)\\ &= \int \mathcal{P}_C(x-x',\ y)\,\mu(dx') + (a'',\ y), \qquad z \in T^C,\end{aligned}$$

for some $a'' \in C^*$. Comparing that with (6.3), we derive

$$(a,\ y) = (a',\ y) + (a'',\ y) \geqslant (a',\ y), \qquad y \in C,$$

which is what we set out to prove.

(2) $\to$ (3). The function

$$\varphi(z) = f(z) - (a,\ z), \qquad \operatorname{Im} \varphi(z) = \operatorname{Im} f(z) - (a,\ y) \tag{6.7}$$

belongs to the class $H_+(T^C)$. Therefore the function

$$\begin{aligned}v(x, y, y') &= \operatorname{Im} \varphi(x+iy+iy') - \int \mathcal{P}_C(x-x', y) \operatorname{Im} \varphi(x'+iy')\,dx'\\ &= \operatorname{Im} f(x+iy+iy')\\ &\quad - \int \mathcal{P}_C(x-x',\ y) \operatorname{Im} f(x'+iy')\,dx' - (a,\ y), \qquad (6.8)\\ &\qquad (x,\ y) \in T^C, \qquad y' \in C,\end{aligned}$$

is a nonnegative function [see (2.2)] that is pluriharmonic with respect to $(x,\ y')$ in $T^C$ and, by (2.3) (for $\varphi = 1$) and (6.3),

$$v(x,\ y,\ y') \to \operatorname{Im} f(x+iy) - \int \mathcal{P}_C(x-x',\ y)\,\mu(dx') - (a,\ y) = 0,$$
$$y \to 0, \qquad y \in C' \Subset C.$$

By the corollary to the theorem of Sec. 16.1, $v(x,\ y,\ y') = (A_y,\ y')$, $(x,\ y') \in T^C$, $A_y \in C^*$ for every $y \in C$. Therefore (6.8)

takes the form

$$\operatorname{Im} f(x+iy+iy')$$

$$-\int \mathcal{P}_C(x-x'\ y)\operatorname{Im} f(x'+iy')\,dx' - (a,\ y) - (A_y,\ y') = 0, \quad (6.9)$$

$$(x,\ y) \in T^C, \qquad y \in C.$$

For $y'$ substitute $ty'$, $t > 0$, divide by $t$ and allow $t$ to go to $\infty$. As a result, using the theorem of Sec. 16.5 and the B. Levi theorem, we obtain

$$0 = \lim_{t\to\infty} \frac{1}{t} \operatorname{Im} f(x+iy+ity')$$

$$-\lim_{t\to\infty} \int \mathcal{P}_C(x-x',\ y) \frac{\operatorname{Im} f(x'+ity')}{t}\,dx'$$

$$-\lim_{t\to\infty} \left[\frac{(a,\ y)}{t} + (A_y,\ y')\right]$$

$$= h(\operatorname{Im} f;\ y') - \int \mathcal{P}_C(x-x',\ y) \lim_{t\to\infty} \frac{\operatorname{Im} f(x'+ity')}{t}\,dx' + (A_y,\ y')$$

$$= h(\operatorname{Im} f;\ y') - h(\operatorname{Im} f;\ y') \int \mathcal{P}_C(x-x',\ y)\,dx' + (A_y,\ y')$$

$$= (A_y,\ y'),$$

from which, and also from (6.9), follows the representation (6.4).

(3) $\to$ (4). Using (6.7) we introduce the function $\varphi(z)$ and then, by (2.12), we construct the function $\varphi_\varepsilon(z)$, $\varepsilon > 0$, with the properties (*a*)-(*c*). Let $y' \in C$. To the functions $\operatorname{Im}\varphi_\varepsilon(z+iy')$ and $\varphi_\varepsilon(z+iy')$ we apply the representations of Poisson and Schwartz, respectively (see Sec. 12.5):

$$\operatorname{Im}\varphi_\varepsilon(z+iy') = \int \mathcal{P}_C(x-x',\ y)\operatorname{Im}\varphi_\varepsilon(x'+iy')\,dx', \qquad z \in T^C, \quad (6.10)$$

$$\varphi_\varepsilon(z+iy') = i\int \mathcal{S}_C(z-x';\ z^0-x')\operatorname{Im}\varphi_\varepsilon(x'+iy')\,dx'$$

$$+\operatorname{Re}\varphi_\varepsilon(z^0+iy'), \qquad z \in T^C, \quad z^0 \in T^C. \quad (6.11)$$

The representation (6.4) shows that passage to the limit under the integral sign is possible in (6.10) as $\varepsilon \to 0$, since

$$\lim_{\varepsilon\to 0} \operatorname{Im} \varphi_\varepsilon (z+iy') = \operatorname{Im} \varphi (z+iy') = \operatorname{Im} f(z+iy') - (a,\ y+y')$$
$$= \int \mathscr{P}_C (x-x',\ y) [\operatorname{Im} f(x'+iy') - (a,\ y')]\, dx'$$
$$= \int \mathscr{P}_C (x-x',\ y) \lim_{\varepsilon\to 0} \varphi_\varepsilon (x'+iy')\, dx'.$$

But then, by virtue of the inequality (5.4) of Sec. 12, the lemma on the possibility of passing to the limit as $\varepsilon \to 0$ under the integral sign in (6.11) is applicable. As a result, using (5.3) of Sec. 12, we obtain the equalities

$$\varphi (z+iy') = f(z+iy') - (a,\ z+iy')$$
$$= i \int \mathscr{S}_C (z-x';\ z^0-x') \operatorname{Im} \varphi (x'+iy')\, dx' + \operatorname{Re} \varphi (z^0+iy')$$
$$= i \int \mathscr{S}_C (z-x';\ z^0-x') [\operatorname{Im} f(x'+iy') - (a,\ y')]\, dx'$$
$$+ \operatorname{Re} f(z^0+iy') - (a,\ x^0)$$
$$= i \int \mathscr{S}_C (z-x';\ z^0-x') \operatorname{Im} f(x'+iy')\, dx' - i(a,\ y')$$
$$+ \operatorname{Re} f(z^0+iy') - (a,\ x^0).$$

That is,

$$f(z+iy') = i \int \mathscr{S}_C (z-x';\ z^0-x') \operatorname{Im} f(x'+iy')\, dx'$$
$$+ (a,\ z) + \operatorname{Re} f(z^0+iy') - (a,\ x^0),$$
$$z \in T^C, \qquad z^0 \in T^C, \qquad y' \in C.$$

Passing to the limit here as $y' \to 0$, $y' \in C' \Subset C$, and making use of the limiting relation (2.10), we obtain the representation (6.5).

(4) $\to$ (1). Putting $z^0 = z$ in the representation (6.5) and making use of (5.2) of Sec. 12, and also separating the imaginary part, we obtain the representation (6.3), from which follows the pluriharmonicity in $T^C$ of the Poisson integral (6.2).

The theorem is proved.

# 17 Holomorphic Functions With Nonnegative Imaginary Part in $T^n$

In the case of the cone $C = \mathbb{R}^n_+$, the results of Sec. 16 admit of being strengthened. Here we will obtain integral representations for all functions of the classes $H_+(T^n)$ and $\mathscr{P}_+(T^n)$. We will first prove some lemmas.

**17.1 Lemmas** Set

$$\mathscr{E}_n(\xi) = \theta_n(\xi)\,\xi_1 \ldots \xi_n.$$

The fundamental solution of the operator $D_1^2 \ldots D_n^2$ [here, $\theta_n(\xi)$ is the characteristic function of the cone $\overline{\mathbb{R}^n_+}$, see Sec. 0.2].

Suppose $f \in C^{2n}$. Then the following equation holds:

$$e^{-(y,\,\xi^+)}D_1^2 \ldots D_n^2 f(\xi) = T_1 \ldots T_n\,[e^{-(y,\,\xi^+)} f(\xi)], \tag{1.1}$$

where $\xi \to \xi^+ = (|\xi_1|, \ldots, |\xi_n|)$ and

$$T_j = D_j^2 + y_j^2 + 2y_j D_j \operatorname{sign}\xi_j - 2y_j\delta(\xi_j), \qquad j = 1, \ldots, n.$$

The right-hand side of (1.1) is meaningful in $\mathscr{D}'$ and for $f \in C$, and we will take it for a definition of the generalized function in the left-hand member of (1.1).

Note that if $f \in C$ and $\operatorname{supp} f \subset \overline{\mathbb{R}^n_+}$, then

$$e^{-(y,\,\xi)}D_1^2 \ldots D_n^2 f(\xi) = e^{-(y,\,\xi^+)}D_1^2 \ldots D_n^2 f(\xi). \tag{1.2}$$

Indeed, for any $\varphi \in \mathscr{D}$, we have, by (1.1),

$$\begin{aligned}
(e^{-(y,\,\xi^+)}D_1^2 \ldots D_n^2 f, \varphi) &= (T_1 \ldots T_n\,[e^{-(y,\,\xi^+)} f], \varphi)\\
&= \int e^{-(y,\,\xi^+)} f(\xi) \prod_{1 \leqslant j \leqslant n} (D_j^2 + y_j^2 - 2y_j \operatorname{sign}\xi_j D_j)\,\varphi(\xi)\,d\xi\\
&= \int e^{-(y,\,\xi)} f(\xi) \prod_{1 \leqslant j \leqslant n} (D_j^2 + y_j^2 - 2y_j D_j)\,\varphi(\xi)\,d\xi\\
&= \int f(\xi)\,D_1^2 \ldots D_n^2\,[e^{-(y,\,\xi)}\varphi(\xi)]\,d\xi\\
&= (e^{-(y,\,\xi)}D_1^2 \ldots D_n^2 f, \varphi),
\end{aligned}$$

which is equivalent to (1.2).

**Lemma 1** *Suppose $f(\xi)$ is a continuous positive definite function in $\mathbb{R}^n$. Then for all $y \in \mathbb{R}^n_+$, the generalized function*

$$e^{-(y,\xi^+)}(1-D_1^2)\dots(1-D_n^2)f(\xi)$$

*is positive definite and the following equation holds:*

$$F[e^{-(y,\xi^+)}(1-D_1^2)\dots(1-D_n^2)f]$$
$$=\int \mathcal{P}_n(x-x',\ y)(1+x_1'^2)\dots(1+x_n'^2)\,\sigma(dx'), \qquad (1.3)$$

where the measure $\sigma = F[f]$, and $\mathcal{P}_n(x,\ y)$ is the Poisson kernel of the region $T^n$ [see (1.2) of Sec. 11].

*Proof*. Note that by the Bochner theorem (see Sec. 8.2), $\sigma = F[f]$ is a measure that is nonnegative and finite on $\mathbb{R}^n$, and equalities of the following type hold:

$$f(\xi_1,\ \dots,\ \xi_k,\ 0,\ \dots,\ 0)=\frac{1}{(2\pi)^n}\int e^{-ix_1\xi_1-\dots-ix_k\xi_k}\sigma(dx). \qquad (1.4)$$

Let us prove (1.3) for $n = 1$. From (1.1), for all $y > 0$, we have

$$F[e^{-y|\xi|}(1-D^2)f]=F[(1-T)(e^{-y|\xi|}f)]$$
$$=F[(1-D^2-y^2-2yD\operatorname{sign}\xi+2y\delta(\xi))(e^{-y|\xi|}f(\xi))]$$
$$=(1+x^2-y^2)F[e^{-y|\xi|}f(\xi)]+2ixyF[\operatorname{sign}\xi e^{-y|\xi|}f(\xi)]+2yf(0). \qquad (1.5)$$

Taking into account the equalities

$$F[e^{-y|\xi|}](x)=2\int_0^\infty e^{-y\xi}\cos x\xi\,d\xi=\frac{2y}{x^2+y^2},$$

$$F[\operatorname{sign}\xi e^{-y|\xi|}](x)=2i\int_0^\infty e^{-y\xi}\sin x\xi\,d\xi=\frac{2xi}{x^2+y^2},$$

we obtain

$$F[f(\xi)e^{-y|\xi|}]=\frac{1}{2\pi}F[f]*F[e^{-y|\xi|}]$$
$$=\frac{y}{\pi}\int\frac{\sigma(dx')}{(x-x')^2+y^2},$$

$$F[f(\xi)\operatorname{sign}\xi e^{-y|\xi|}]=\frac{1}{2\pi}F[f]*F[\operatorname{sign}\xi e^{-y|\xi|}]$$
$$=\frac{i}{\pi}\int\frac{(x-x')\,\sigma(dx')}{(x-x')^2+y^2}$$

Substituting the resulting expressions into (1.5) and taking into account (1.4) for $k = 0$, we obtain (1.3) for $n = 1$.

The case of $n > 1$ is considered in similar fashion if one notices that every operator $T_j$ operates only on its own variable $\xi_j$, if one applies the Fourier transform technique with respect to a part of the variables (see Sec. 6.2 and Sec. 6.3), and if one takes advantage of equations of the type (1.4). The proof of Lemma 1 is complete.

Lemma 2 *Let the function* $v(\xi)$ *be continuous and bounded in* $\mathbb{R}^n$ *and let* $\operatorname{supp} v \subset \overline{\mathbb{R}}^n_+$. *Then the solution of the equation*

$$D_1^2 \ldots D_n^2 u(\xi) = (1 - D_1^2) \ldots (1 - D_n^2) v(\xi) \tag{1.6}$$

*exists and is unique in the class of continuous functions in* $\mathbb{R}^n$ *with support in* $\overline{\mathbb{R}}^n_+$, *which functions satisfy the estimate*

$$|u(\xi)| \leqslant C(1 + \xi_1^2) \ldots (1 + \xi_n^2). \tag{1.7}$$

*Proof*. The solution of equation (1.6) is unique even in the algebra $\mathscr{D}'(\overline{\mathbb{R}}^n_+)$ and is representable in the form (see Sec. 4.8(d))

$$u = \mathscr{E}_n * (1 - D_1^2) \ldots (1 - D_n^2) v. \tag{1.8}$$

Let us represent the right-hand side of (1.8) as

$$\begin{aligned}
u(\xi) &= (1 - D_1^2) \ldots (1 - D_n^2) \mathscr{E}_n * v \\
&= \{[\theta(\xi_1)\xi_1 - \delta(\xi_1)] \times \ldots \times [\theta(\xi_n)\xi_n - \delta(\xi_n)]\} * v \\
&= (-1)^n v(\xi) \\
&\quad + \sum_{\substack{1 \leqslant k \leqslant n \\ j_1 < \ldots < j_k}} (-1)^{n-k} \int_0^{\xi_{j_1}} \ldots \int_0^{\xi_{j_k}} v(\ldots, \xi'_{j_1}, \ldots, \xi'_{j_k}, \ldots) \\
&\qquad \times (\xi_{j_1} - \xi'_{j_1}) \ldots (\xi_{j_k} - \xi'_{j_k}) \, d\xi'_{j_1} \ldots d\xi'_{j_k}.
\end{aligned}$$

It remains to note that each summand in the last sum is a continuous function that satisfies the estimate (1.7). This completes the proof of Lemma 2.

Lemma 3 *Suppose* $u(\xi)$ *is a continuous function of slow growth in* $\mathbb{R}^n$. *Then the solution of the equation*

$$(1 - D_1^2) \ldots (1 - D_n^2) v(\xi) = D_1^2 \ldots D_n^2 u(\xi) \tag{1.9}$$

*exists and is unique in the class of continuous functions of slow growth.*

*Proof*. The solution of equation (1.9) is unique even in the class $\mathscr{S}'$ since the Fourier transform of the generalized function $(1 - D_1^2) \ldots (1 - D_n^2)\,\delta(\xi)$, equal to $(1 + x_1^2) \ldots (1 + x_n^2)$, does not vanish anywhere in $\mathbb{R}^n$. We will prove its existence:

$$\mathscr{E}(\xi) = \frac{1}{2} e^{-|\xi_1| - \ldots - |\xi_n|}$$

is the fundamental solution of the operator $(1 - D_1^2) \ldots (1 - D_n^2)$. Since $u(\xi)$ is of slow growth, the convolution $\mathscr{E} * u$ exists (see Sec. 4.1). Therefore, the solution $v$ of equation (1.9) can be expressed in the form of a convolution:

$$\begin{aligned}
v &= \mathscr{E} * D_1^2 \ldots D_n^2 u = D_1^2 \ldots D_n^2 \mathscr{E} * u \\
&= \left\{\left[-\delta(\xi_1) + \frac{1}{2} e^{-|\xi_1|}\right] \ldots \left[-\delta(\xi_n) + \frac{1}{2} e^{-|\xi_n|}\right]\right\} * u \\
&= (-1)^n u(\xi) \\
&\quad + \sum_{\substack{1 \leqslant k \leqslant n \\ j_1 < \ldots < j_k}} \frac{(-1)^{n-k}}{2^k} \int_{\mathbb{R}^k} u(\ldots, \xi'_{j_1}, \ldots, \xi'_{j_k}, \ldots) \\
&\qquad\qquad \times e^{-|\xi_{j_1} - \xi'_{j_1}| - \ldots - |\xi_{j_k} - \xi'_{j_k}|}\, d\xi'_{j_1} \ldots d\xi'_{j_k}.
\end{aligned}$$

It remains to note that each term in the last sum is a continuous function of slow growth. Lemma 3 is proved.

Lemma 4 *If the function $v(\xi)$ is continuous, bounded in $\overline{\mathbb{R}}_+^2 \times \mathbb{R}^{n-2}$, $n \geqslant 2$, and satisfies the equation*

$$(1 - D_1^2) \ldots (1 - D_n^2)\, v(\xi) = 0, \qquad \xi \in \mathbb{R}_+^2 \times \mathbb{R}^{n-2}, \qquad (1\ 10)$$

*then it can be expressed as*

$$v(\xi) = e^{-\xi_1} v(0, \xi_2, \tilde{\xi}) + e^{-\xi_2} v(\xi_1, 0, \tilde{\xi}) - e^{-\xi_1 - \xi_2} v(0, 0, \tilde{\xi}) \qquad (1.11)$$

*where* $\tilde{\xi} = (\xi_3, \ldots, \xi_n)$.

*Proof*. We continue the function $v(\xi)$ by zero onto the whole space $\mathbb{R}^n$ and we construct for the function the mean function

$$v_\varepsilon(\xi) = \int v(\xi')\, \omega_\varepsilon(\xi - \xi')\, d\xi' = v * \omega_\varepsilon$$

with the following properties (see Sec. 1.2 and Sec. 4.6):

$$D^\alpha v_\varepsilon \in C^\infty \cap \mathcal{L}^\infty, \quad \forall\alpha; \qquad v_\varepsilon(\xi) \to v(\xi), \quad \varepsilon \to 0, \quad \xi \in \mathbb{R}_+^2 \times \mathbb{R}^{n-2},$$

$$(1-D_1^2)\ldots(1-D_n^2)\, v_\varepsilon(\xi) = 0, \qquad \xi_1 > 2\varepsilon, \quad \xi_2 > 2\varepsilon, \quad \tilde{\xi} \in \mathbb{R}^{n-2}.$$

Put

$$\chi_\varepsilon(\xi) = (1-D_3^2)\ldots(1-D_n^2)\, v_\varepsilon(\xi). \tag{1.12}$$

Then $D^\alpha \chi_\varepsilon \in C^\infty \cap \mathcal{L}^\infty$ for $\forall\alpha$, and $\chi_\varepsilon$ satisfies the equation

$$(1-D_1^2)(1-D_2^2)\,\chi_\varepsilon(\xi) = 0, \qquad \xi_1 > 2\varepsilon, \quad \xi_2 > 2\varepsilon, \quad \tilde{\xi} \in \mathbb{R}^{n-2}. \tag{1.13}$$

Fix $\delta > 0$ and let $2\varepsilon < \delta$. From the equation (1.13) and from the boundedness of the function $(1 - D_2^2)\, \chi_\varepsilon(\xi)$ we derive the relation

$$(1-D_2^2)\,\chi_\varepsilon(\xi_1, \xi_2, \tilde{\xi}) = (1-D_2^2)\,\chi_\varepsilon(\delta, \xi_2, \tilde{\xi})\, e^{-(\xi_1-\delta)},$$

that is,

$$(1-D_2^2)\,[\chi_\varepsilon(\xi_1, \xi_2, \tilde{\xi}) - \chi_\varepsilon(\delta, \xi_2, \tilde{\xi})\, e^{-(\xi_1-\delta)}] = 0, \quad \xi_1 \geqslant \delta, \quad \xi_2 \geqslant \delta. \tag{1.14}$$

Similarly, from the equation (1.14) we derive the relation

$$\chi_\varepsilon(\xi_1, \xi_2, \tilde{\xi}) - \chi_\varepsilon(\delta, \xi_2, \tilde{\xi})\, e^{-(\xi_1-\delta)} = [\chi_\varepsilon(\xi_1, \delta, \tilde{\xi}) - \chi_\varepsilon(\delta, \delta, \tilde{\xi})\, e^{-(\xi_1-\delta)}]\, e^{-(\xi_2-\delta)}.$$

That is, by (1.12),

$$(1-D_3^2)\ldots(1-D_n^2)\,[v_\varepsilon(\xi_1, \xi_2, \tilde{\xi}) - v_\varepsilon(\delta, \xi_2, \tilde{\xi})\, e^{-(\xi_1-\delta)} - v_\varepsilon(\xi_1, \delta, \tilde{\xi})\, e^{-(\xi_2-\delta)} + v_\varepsilon(\delta, \delta, \tilde{\xi})\, e^{-(\xi_1+\xi_2-2\delta)}] = 0,$$

$$\xi_1 \geqslant \delta, \qquad \xi_2 \geqslant \delta, \qquad \tilde{\xi} \in \mathbb{R}^{n-2}.$$

From this, by uniqueness of the solution of the last equation (via Lemma 3), follows the equality

$$v_\varepsilon(\xi) = v_\varepsilon(\delta, \xi_2, \tilde{\xi})\, e^{-(\xi_1-\delta)} + v_\varepsilon(\xi_1, \delta, \tilde{\xi})\, e^{-(\xi_2-\delta)} - v_\varepsilon(\delta, \delta, \tilde{\xi})\, e^{-(\xi_1+\xi_2-2\delta)},$$

$$\xi_1 \geqslant \delta, \qquad \xi_2 \geqslant \delta, \qquad \tilde{\xi} \in \mathbb{R}^{n-2}.$$

Passing to the limit here as $\varepsilon \to 0$ and, furthermore, as $\delta \to 0$, we obtain the representation (1.10). This completes the proof of Lemma 4.

Lemma 5 *The equation*

$$D_1^2 \dots D_n^2 u(\xi) + \sum_{1 \leqslant |\alpha| \leqslant N} a_\alpha D^\alpha \delta(\xi) = 0, \tag{1.15}$$

*provided that* $u \in C$, supp $u \subset \overline{\mathbb{R}}^n_+$, *is possible only when* $u(\xi) = 0$ *and* $a_\alpha = 0$, $1 \leqslant |\alpha| \leqslant N$.

*Proof*. In the algebra $\mathscr{D}'(\overline{\mathbb{R}}^n_+)$, the equation (1.15) is equivalent (see Sec. 4.8(d)) to

$$\begin{aligned} u(\xi) &= -\mathscr{E}_n * \sum_{1 \leqslant |\alpha| \leqslant N} a_\alpha D^\alpha \delta = -\sum_{1 \leqslant |\alpha| \leqslant N} a_\alpha D^\alpha \mathscr{E}_n(\xi) \\ &= -\sum_{1 \leqslant k \leqslant n-1} \sum_{1 \leqslant j_1 < \dots < j_k \leqslant n} a_{j_1 \dots j_k} \xi_{j_1} \theta(\xi_{j_1}) \dots \xi_{j_k} \theta(\xi_{j_k}) \\ &\qquad \times \theta(\xi_{j_{k+1}}) \dots \theta(\xi_{j_n}) \\ &\qquad - \sum_{j=1}^{n} \sum_{\substack{\alpha_j \geqslant 0 \\ 2 \leqslant |\alpha| \leqslant N}} a_\alpha D^\alpha \mathscr{E}_n(\xi). \end{aligned} \tag{1.16}$$

Each term in the second sum of (1.16) contains at least one $\delta$ function with respect to any one of the variables $\xi_j$, $1 \leqslant j \leqslant n$, and the combinations of $\delta$ functions in all terms are distinct. The other summands in (1.16) are locally integrable functions, whence we conclude that $a_\alpha = 0$ if there is a $j$ such that $\alpha_j \geqslant 2$, and (1.16) takes the form

$$\begin{aligned} u(\xi) = -\sum_{1 \leqslant k \leqslant n-1} \sum_{1 \leqslant j_1 < \dots < j_k \leqslant n} a_{j_1 \dots j_k} \xi_{j_1} \theta(\xi_{j_1}) \dots \xi_{j_k} \theta(\xi_{j_k}) \\ \times \theta(\xi_{j_{k+1}}) \dots \theta(\xi_{j_n}). \end{aligned}$$

From this, taking into account the properties of the function $u$, it is easy to derive, by induction on $n$, that all $a_{j_1 \dots j_k} = 0$ and $u(\xi) = 0$. Lemma 5 is proved.

Lemma 6 *The general solution of the equation*

$$D_1^2 \dots D_n^2 u(\xi) = 0 \tag{1.17}$$

*in the class of continuous functions with support in* $-\overline{\mathbb{R}}^n_+ \cup \overline{\mathbb{R}}^n_+$ *is expressed by the formula*

$$u(x) = C[\mathscr{E}_n(x) - \mathscr{E}_n(-x)], \tag{1.18}$$

*where* $C$ *is an arbitrary constant.*

*Proof*. Function (1.18) satisfies (1.17) since $D_1^2 \ldots D_n^2 \mathscr{E}_n(\pm\xi) = \delta(\xi)$. Let $u(\xi)$ be an arbitrary solution to (1.17) taken from the class under consideration. Then the function $u^+(\xi) = \theta_n(\xi)\, u(\xi)$ satisfies (1.17) in $\mathbb{R}^n \setminus \{0\}$ and hence (see Sec. 2.6)

$$\begin{aligned} D_1^2 \ldots D_n^2 u^+(\xi) &= \sum_{0 \leqslant |\alpha| \leqslant N} c_\alpha D^\alpha \delta(\xi) \\ &= c_0 D_1^2 \ldots D_n^2 \mathscr{E}_n(\xi) + \sum_{1 \leqslant |\alpha| \leqslant N} c_\alpha D^\alpha \delta(\xi) \end{aligned} \tag{1.19}$$

for certain $N$ and $c_\alpha$. By Lemma 5, the equation (1.19) is possible only for $c_\alpha = 0$, $|\alpha| \geqslant 1$ and $u^+(\xi) = c_0 \mathscr{E}_n(\xi)$. Similarly, we derive that $u^-(\xi) = \theta_n(-\xi)\, u(\xi) = c_0' \mathscr{E}_n(-\xi)$ so that $u(\xi) = u^+(\xi) + u^-(\xi) = c_0 \mathscr{E}_n(\xi) + c_0' \mathscr{E}_n(-\xi)$. But by virtue of (1.17)

$$\begin{aligned} D_1^2 \ldots D_n^2 u(\xi) &= c_0 D_1^2 \ldots D_n^2 \mathscr{E}_n(\xi) + c_0' D_1^2 \ldots D_n^2 \mathscr{E}_n(-\xi) \\ &= (c_0 + c_0')\, \delta(\xi) = 0, \end{aligned}$$

so that $c_0' = -c_0$ and the representation (1.18) is proved. The proof of Lemma 6 is complete.

**17.2 Functions of the classes $H_+(T^1)$ and $\mathscr{P}_+(T^1)$** We first consider the case $n = 1$. Suppose the function $f \in H_+(T^1)$, that is, $f(z)$ is holomorphic and $\operatorname{Im} f(z) = u(x, y) \geqslant 0$ in the upper half-plane $T^1$ so that $\operatorname{Im} f \in \mathscr{P}_+(T^1)$.

Recall that $f(z)$ satisfies the estimate (see Sec. 13.3)

$$|f(z)| \leqslant M \frac{1 + |z|^2}{y}, \qquad y > 0, \tag{2.1}$$

and the measure $\mu = \operatorname{Im} f_+ = u(x, +0)$ satisfies the condition (see Sec. 16.2)

$$\int \frac{\mu(dx)}{1 + x^2} < \infty. \tag{2.2}$$

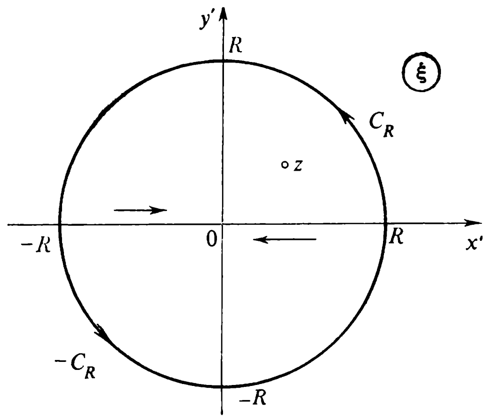
Let $\varepsilon > 0$ and $R > 1$ and denote by $C_R$ and $-C_R$ semicircles of radius $R$ centred at 0, as depicted in Fig. 40. By the residue

theorem we have

$$\frac{f(z+i\varepsilon)}{1+z^2}=\frac{1}{2\pi i}\left(\int_{-R}^{R}+\int_{C_R}\right)\frac{f(\xi+i\varepsilon)\,d\xi}{(1+\xi^2)(\xi-z)}+\frac{f(i+i\varepsilon)}{2i(z-i)},\quad y>0,\quad |z|<R. \tag{2.3}$$

Analogously, for the function $\frac{\bar{f}(\bar{z}+i\varepsilon)}{1+z^2}$, which is meromorphic

Figure 40




in the lower half-plane $y<0$ with the sole simple pole $-i$, we have

$$0=\frac{1}{2\pi i}\left(\int_{-R}^{R}+\int_{-C_R}\right)\frac{\bar{f}(\xi+i\varepsilon)\,d\xi}{(1+\xi^2)(\xi-z)}-\frac{\bar{f}(i+i\varepsilon)}{2i(z+i)},\quad y>0,\quad |z|<R. \tag{2.4}$$

Sending $R$ to $\infty$ in (2.3) and (2.4), and using the estimate (2.1), according to which

$$\left|\int_{C_R}\frac{f(\xi+i\varepsilon)\,d\xi}{(1+\xi)^2(\xi-z)}\right|=\left|\int_0^{\pi}\frac{f(Re^{i\varphi}+i\varepsilon)\,iRe^{i\varphi}\,d\varphi}{(1+R^2e^{2i\varphi})(Re^{i\varphi}-z)}\right|$$

$$\leqslant M\int_0^{\pi}\frac{(1+|Re^{i\varphi}+i\varepsilon|^2)\,R\,d\varphi}{(R\sin\varphi+\varepsilon)\,|1+R^2e^{2i\varphi}|\,|Re^{i\varphi}-z|}$$

$$\leqslant\frac{MR\,[1+(R+\varepsilon)^2]}{(R^2-1)(R-|z|)}\int_0^{\pi}\frac{d\varphi}{R\sin\varphi+\varepsilon}\to 0,\quad R\to\infty$$

(and similarly for the contour $-C_R$), we obtain

$$f(z+i\varepsilon)=\frac{1+z^2}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{f(x'+i\varepsilon)\,dx'}{(1+x'^2)(x'-z)}+\frac{f(i+i\varepsilon)}{2i}(z+i),\qquad y>0,$$

$$0=-\frac{1+z^2}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{\bar f(x'+i\varepsilon)\,dx'}{(1+x'^2)(x'-z)}-\frac{\bar f(i+i\varepsilon)}{2i}(z-i),\qquad y>0.$$

Adding together the resulting equalities, we derive an integral representation for the function $f(z+i\varepsilon)$:

$$f(z+i\varepsilon)=\frac{1+z^2}{\pi}\int\limits_{-\infty}^{\infty}\frac{u(x',\varepsilon)\,dx'}{(1+x'^2)(x'-z)}+zu(0,1+\varepsilon)+\operatorname{Re}f(i+i\varepsilon),$$
$$y>0. \tag{2.5}$$

Separating the imaginary part in (2.5), we obtain an integral representation for the function $u(x, y+\varepsilon)$:

$$u(x,y+\varepsilon)=\frac{y}{\pi}\int\limits_{-\infty}^{\infty}u(x',\varepsilon)\left[\frac{1}{(x-x')^2+y^2}-\frac{1}{1+x'^2}\right]dx'$$
$$+yu(0,1+\varepsilon),\quad y>0. \tag{2.6}$$

Passing to the limit in (2.5) and (2.6) as $\varepsilon\to 0$, and making use of the limiting relation (2.3) of Sec. 16, we obtain the necessity of the conditions in the Herglotz-Nevanlinna theorem (see Nevanlinna [1]).

Theorem I *For the function $f(z)$ to belong to the class $H_+(T^1)$, it is necessary and sufficient that it be representable in the form*

$$f(z)=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{(1+x'z)\,\mu(dx')}{(1+x'^2)(x'-z)}+az+b$$
$$=i\int\limits_{-\infty}^{\infty}\mathscr{S}_1(z-x';i-x')\,\mu(dx')+az+b,\qquad y>0, \tag{2.7}$$

*where the measure $\mu$ is nonnegative and satisfies the condition (2.2), $a\geqslant 0$, and $b$ is a real number. The representation (2.6) is unique,*

*and* $\mu = \operatorname{Im} f_+$, $b = \operatorname{Re} f(i)$,

$$a = \operatorname{Im} f(i) - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mu(dx)}{1+x^2} = \lim_{y\to\infty} \frac{\operatorname{Im} f(iy)}{y}, \qquad (2.8)$$

$$\operatorname{Im} f(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\mu(dx')}{(x-x')^2+y^2} + ay, \qquad y>0. \qquad (2.9)$$

The sufficiency of the conditions of Theorem I is straightforward.

Corollary *For the function* $u(x, y)$ *to belong to the class* $\mathscr{P}_+(T^1)$, *it is necessary and sufficient that it be representable in the form*

$$u(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\mu(dx')}{(x-x')^2+y^2} + ay, \qquad y>0, \qquad (2.9)$$

*where* $a \geqslant 0$ *and the measure* $\mu$ *is nonnegative and satisfies the condition (2.2); here,*

$$\mu = u(x, +0) \quad \text{and} \quad a = \lim_{y\to\infty} \frac{u(0, y)}{y}.$$

*Remark.* From the representation (2.9) it follows that the Poisson integral is a harmonic function in $T^1$. By the theorem of Sec. 16.6, the representation with the Schwartz kernel holds with respect to any point $z^0 \in T^1$ [formula (2.7) for $z^0 = i$].

In terms of the spectral function $g(\xi)$ of the function $f(z)$ (see Sec. 16.1), the class $H_+(T^1)$ is characterized by the following theorem (König and Zemanian [1]).

Theorem II *For a function* $f(z)$ *to belong to the class* $H_+(T^1)$, *it is necessary and sufficient that its spectral function* $g(\xi)$ *have the following properties:*

(a) $-ig(\xi) + ig^*(\xi) \gg 0$,

(b) $g(\xi) = iu''(\xi) + ia\delta'(\xi)$,

*where* $a \geqslant 0$ *and* $u(\xi)$ *is a continuous function with support in* $[0, \infty)$, *which function satisfies the growth condition*

$$|u(\xi)| \leqslant C(1+\xi^2). \qquad (2.10)$$

*Here, the expansion* (b) *is unique, the number* $a$ *is defined by (2.8), and* $\operatorname{Im} f(z)$ *is defined by (2.9).*

Corollary *For the measure* $\mu$ *to be a boundary value of the function* $u(x, y)$ *of the class* $\mathcal{P}_+(T^1)$, $\mu = u(x, +0)$, *it is necessary and sufficient that* $\mu = F[v'']$, *where* $v'' \gg 0$, $v$ *is a continuous* $*$*-Hermitian function satisfying the growth condition (2.10), and* $v(0) = 0$. *In this case the function* $v$ *with the indicated properties is unique to within the summand* $ic\xi$, *where* $c$ *is an arbitrary real number.*

This follows from Theorem II for $v = u + u^*$ (necessity) and for $u = \theta v$ (sufficiency) if we take advantage of (1.5) of Sec. 16, $\mu = \frac{1}{2} F[-ig + ig^*]$.

*Proof of Theorem II. Necessity.* Let $f \in H_+(T^1)$. Condition (a) was proved in Sec. 16.1. To prove condition (b), rewrite representation (2.7) as [compare with (2.5)]

$$f(z) = \frac{1+z^2}{\pi} \int_{-\infty}^{\infty} \frac{\mu(dx')}{(1+x'^2)(x'-z)} + \operatorname{Im} f(i)\, z + b$$

$$= i(1+z^2)(\sigma * \mathcal{K}_1(x'+iy)) + \operatorname{Im} f(i)\, z + b, \qquad \sigma = \frac{\mu}{\pi(1+x^2)}. \tag{2.11}$$

Since $\mathcal{K}_1(x+iy) \in \mathcal{H}_s$ (for all $s$ and $y > 0$) [see (2.5) of Sec. 11] and $\int \sigma(dx') < \infty$ [see (2.2)], the Fourier transform formula of the convolution $\sigma * \mathcal{K}_1$ holds:

$$F^{-1}[\sigma * \mathcal{K}_1] = F[\sigma](-\xi)\, F^{-1}[\mathcal{K}_1](\xi) = e^{-y\xi}\theta(\xi)\, v(\xi), \tag{2.12}$$

where $v(\xi) = F[\sigma](-\xi)$ a continuous positive definite (and, hence, bounded) function (see Sec. 8). Now, using (2.11) and (2.12), we compute the spectral function $g(\xi)$ (see Sec. 9):

$$g(\xi) = i(1 - D^2)[\theta(\xi)\, v(\xi)] + i \operatorname{Im} f(i)\, \delta'(\xi) + b\delta(\xi). \tag{2.13}$$

By Lemma 2 of Sec. 17.1 there exists a continuous function $u_1(\xi)$ with support in $[0, \infty)$ that satisfies the estimate (2.10) and is such that

$$(1 - D^2)\{\theta(\xi)[v(\xi) - v(0)\} = D^2 u_1(\xi).$$

Therefore, (2.13) takes the form

$$g(\xi) = iD^2\left[u_1(\xi) + \frac{v(0)}{2}\xi^2\theta(\xi) - ib\xi\theta(\xi)\right] - i[\operatorname{Im} f(i) - v(0)]\delta'(\xi). \tag{2.14}$$

Setting

$$u(\xi)=u_1(\xi)+\frac{v(0)}{2}\xi^2\theta(\xi)-ib\xi\theta(\xi)$$

and noting that, by (2.8),

$$\operatorname{Im} f(i)-v(0)=\operatorname{Im} f(i)-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mu(dx)}{1+x^2}=a,$$

we obtain, from (2.14), the representation (b). The uniqueness of expansion (b) follows from the uniqueness of the spectral function $g$ and from Lemma 5 of Sec. 17.1.

*Sufficiency*. Suppose the generalized function $g(\xi)$ satisfies conditions (a) and (b). Then $g \in \mathcal{S}'(\mathbb{R}^1_+)$ and its Laplace transform $f(z) = L[g]$ is a function of the class $H(\mathbb{R}^1_+)$ (see Sec. 12.2). It remains to prove that $\operatorname{Im} f(z) > 0$, $y > 0$.

Using the formulas (1.2) and (1.5) of Sec. 16.1, we have

$$\begin{aligned}\operatorname{Im} f(z)&=\frac{1}{2i}F[g(\xi)e^{-y\xi}-g^*(\xi)e^{y\xi}]\\&=\frac{1}{2}F(u''(\xi)e^{-y\xi}+u^{*''}(\xi)e^{y\xi}]\\&\quad+\frac{a}{2}F[\delta'(\xi)e^{-y\xi}+\delta'(-\xi)e^{y\xi}],\quad y>0,\end{aligned}\tag{2.15}$$

$$\operatorname{Im} f_+=\frac{1}{2}F[-ig+ig^*]=\frac{1}{2}F[u''-u^{*''}].\tag{2.16}$$

The equation (2.16) shows, by virtue of the Bochner-Schwartz theorem (see Sec. 8.2), that $\mu = \operatorname{Im} f_+$ is a nonnegative measure and $(u + u^*)'' \gg 0$. By Lemma 3 of Sec. 17.1 there exists a continuous function $v$ of slow growth such that

$$(1-D^2)v(\xi)=\frac{1}{2}[u(\xi)+u^*(\xi)]''.\tag{2.17}$$

From (2.17) it follows that $v(\xi)$ is a continuous positive definite function, and, by (2.16),

$$(1+x^2)F[v]=\operatorname{Im} f_+=\mu.\tag{2.18}$$

Using the formulas (1.2), (1.3) and (2.17), we now obtain the following chain of equalities:

$$\begin{aligned}\frac{1}{2}F[u''(\xi)e^{-y\xi}+u^{*''}(\xi)e^{y\xi}]&=\frac{1}{2}F[e^{-y|\xi|}(u+u^*)'']\\&=F[e^{-y|\xi|}(1-D^2)v]=\frac{y}{\pi}\int\frac{\mu(dx')}{(x-x')^2+y^2}.\end{aligned}\tag{2.19}$$

Finally, taking into account

$$a(\xi)\,\delta'(\xi) = -a'(0)\,\delta(\xi) + a(0)\,\delta'(\xi), \qquad \delta'(-\xi) = -\delta'(\xi),$$

we obtain

$$\delta'(\xi)\,e^{-y\xi} + \delta'(-\xi)\,e^{y\xi} = \delta'(\xi)\,(e^{-y\xi} - e^{y\xi}) = 2y\delta(\xi). \qquad (2.20)$$

Substituting the expressions (2.19) and (2.20) into (2.15), we obtain the representation (2.9) for $\operatorname{Im} f(z)$, from which it follows that $\operatorname{Im} f(z) \geqslant 0$, $y > 0$. Theorem II is proved.

**17.3 Functions of the class $\mathscr{P}_+ T(^n)$** The case $n > 1$ may be considered in similar fashion to Sec. 17.2 with use made of the residue theorem (see Vladimirov [7]). However, here we will apply a different method, one which makes use of Lemma 4 of Sec. 17.1 on the general form of a bounded continuous solution of the differential equation (1.10).

Suppose the function $u(x, y)$ belongs to the class $\mathscr{P}_n(T^n)$ and the measure $\mu = u(x, +0) \geqslant 0$ is its boundary value (see Sec. 16.2).

By the theorem of Sec. 16.2, the measure $\mu$ has the following properties

$$\frac{1}{\pi^n}\int \frac{\mu(dx)}{(1+x_1^2)\dots(1+x_n^2)} \leqslant u(0, \mathbf{1}), \qquad (3.1)$$

where $\mathbf{1} = (1, \dots, 1)$; for any $\varphi \in C \cap \mathscr{L}^\infty$,

$$\int \frac{u(x, y)\,\varphi(x)\,dx}{(1+x_1^2)\dots(1+x_n^2)} \to \int \frac{\varphi(x)\,\mu(dx)}{(1+x_1^2)\dots(1+x_n^2)}, \qquad (3.2)$$
$$y \to 0, \qquad y \in \forall C' \Subset \mathbb{R}_+^n.$$

For the measure $\mu$ we construct $2^n - 2$ measures $\mu_{j_1\dots j_k}$, $1 \leqslant k \leqslant n-1$, $1 \leqslant j_1 < \dots < j_k \leqslant n$, of the variables $x_{j_1\dots j_k} = (x_{j_1}, \dots, x_{j_k}) \in \mathbb{R}^k$ via the following formula: for arbitrary $\varphi \in C \cap \mathscr{L}^\infty$,

$$\int \frac{\varphi(x_{j_1\dots j_k})\,\mu_{j_1\dots j_k}(dx_{j_1\dots j_k})}{(1+x_{j_1}^2)\dots(1+x_{j_k}^2)} = \int \frac{\varphi(x_{j_1\dots j_k})\,\mu(dx)}{(1+x_1^2)\dots(1+x_n^2)}. \qquad (3.3)$$

From this definition (for $\varphi \equiv 1$) and from (3.1) it follows, by the Fubini theorem, that the measures $\mu_{j_1\dots j_k}$ are nonnega-

tive and satisfy the condition

$$\frac{1}{\pi^n}\int\frac{\mu_{j_1\ldots j_k}(dx_{j_1\ldots j_k})}{(1+x_{j_1}^2)\ldots(1+x_{j_k}^2)}\leqslant u(0,1). \tag{3.4}$$

Set

$$\sigma_{j_1\ldots j_k}=\frac{\mu_{j_1\ldots j_k}}{(1+x_{j1}^2)\ldots(1+x_{j_k}^2)}, \tag{3.5}$$

$$\chi_{j_1\ldots j_k}(\xi_{j_1\ldots j_k})=(2\pi)^{n-k}F^{-1}[\sigma_{j_1\ldots j_k}], \tag{3.6}$$

$$\mu_{1\ldots n}=\mu, \qquad \sigma_{1\ldots n}=\sigma, \qquad \chi_{1\ldots n}=\chi. \tag{3.7}$$

By virtue of (3.4) to (3.7), the functions $\chi_{j_1\ldots j_k}$ are continuous positive definite in $\mathbb{R}^k$ and, hence, are bounded in $\mathbb{R}^k$ (see Sec. 8). Furthermore, the following equalities hold true:

$$\chi_{j_1\ldots j_k}(\xi_{j_1\ldots j_k})=\chi(\xi)\,|_{\xi_{j_{k+1}}=\ldots=\xi_{j_n}=0}. \tag{3.8}$$

Indeed, using (3.3) to (3.7), we obtain (3.8):

$$\chi(\xi)\,|_{\xi_{j_{k+1}}=\ldots=\xi_{j_n}=0}=\frac{1}{(2\pi)^n}\int\frac{\exp(-i\xi_{j_1}\chi_{j_1}-\ldots-i\xi_{j_k}\chi_{j_k})\,\mu(dx)}{(1+x_1^2)\ldots(1+x_n^2)}$$

$$=\frac{1}{(2\pi)^n}\int\frac{\exp(-i\xi_{j_1}x_{j_1}-\ldots-i\xi_{j_k}x_{j_k})\,\mu_{j_1\ldots j_h}(dx_{j_1\ldots j_k})}{(1+x_{j_1}^2)\ldots(1+x_{j_k}^2)}$$

$$=\frac{1}{(2\pi)^n}\int\exp(-i\xi_{j_1}x_{j_1}-\ldots-i\xi_{j_k}x_{j_k})\,\sigma_{j_1\ldots j_k}(dx_{j_1\ldots j_k})$$

$$=\chi_{j_1\ldots j_k}(\xi_{j_1\ldots j_k}).$$

We now prove that

$$F_k^{-1}[\mu_{j_1\ldots j_k}](\xi_{j_1\ldots j_k})=0, \qquad \xi_{j_1\ldots j_k}\overline{\in}-\overline{\mathbb{R}}_+^k\cup\overline{\mathbb{R}}_+^k, \tag{3.9}$$

where $F_k$ is the Fourier transform operation with respect to $k$ variables $(\xi_{j_1},\ldots,\xi_{j_k})=\xi_{j_1\ldots j_k}$.

For the measure $\mu=\mu_{1\ldots n}$, (3.9) follows from (1.5) of Sec. 16, where $g\in\mathscr{S}'(\overline{\mathbb{R}}_+^n)$ [$g$ is the spectral function of the function $f$ taken from $H_+(T^n)$, for which function $\operatorname{Im} f=u$].

Now suppose

$$\varphi(x_{j_1\ldots j_k})=(1+x_{j_1}^2)\ldots(1+x_{j_k}^2)\,F_k^{-1}[\alpha], \tag{3.10}$$

where $\alpha(\xi_{j_1\ldots j_k})$ is an arbitrary function in $\mathscr{D}(\mathbb{R}^k)$ with support outside $-\overline{\mathbb{R}^k_+} \cup \overline{\mathbb{R}^k_+}$. Substituting the expression (3.10) into (3.3) and rewriting it in terms of Fourier transforms, we obtain

$$(\mu_{j_1\ldots j_k}, F_k^{-1}[\alpha]) = (F_k^{-1}[\mu_{j_1\ldots j_k}], \alpha)$$
$$= \left(F^{-1}[\mu], F\left[F_k^{-1}[\alpha]\frac{1}{(1+x_{j_{k+1}}^2)\ldots(1+x_{j_n}^2)}\right]\right)$$
$$= \left(F^{-1}[\mu], \alpha(\xi_{j_1}\ldots{}_{j_k}) F_{n-k}\left[\frac{1}{(1+x_{k+1}^2)\ldots(1+x_{j_n}^2)}\right]\right).$$

Using the formula

$$F\left[\frac{1}{1+x^2}\right] = \pi e^{-|\xi|},$$

we can rewrite the last equalities in the form

$$(F_k^{-1}[\mu_{j_1}\ldots{}_{j_k}], \alpha)$$
$$= \pi^{n-k}(F^{-1}[\mu], \alpha(\xi_{j_1}\ldots{}_{j_k})\exp(-|\xi_{j_{k+1}}| - \ldots - |\xi_{j_n}|).$$

The right-hand side of this equation vanishes because the support of the function

$$\alpha(\xi_{j_1\ldots j_k})\exp(-|\xi_{j_{k+1}} - \ldots - |\xi_{j_n}|)$$

lies outside $(-\overline{\mathbb{R}^k_+} \cup \overline{\mathbb{R}^k_+}) \times \mathbb{R}^{n-k}$, and $F^{-1}[\mu]$, by what has been proved, vanishes outside $-\overline{\mathbb{R}^n_+} \cup \overline{\mathbb{R}^n_+}$. That precisely is what proves (3.9).

From the equations (3.9), (3.5) to (3.7) there follow differential equations for the function $\chi_{j_1\ldots j_k}$:

$$(1-D_{j_1}^2)\ldots(1-D_{j_k}^2)\chi_{j_1\ldots j_k}(\xi_{j_1\ldots j_k}) = 0,$$
$$\xi_{j_1\ldots j_k} \bar{\in} -\overline{\mathbb{R}^k_+} \cup \overline{\mathbb{R}^k_+}. \tag{3.11}$$

**Theorem I** *If $u \in \mathscr{P}_+(T^n)$, $n \geqslant 2$, then the function*

$$\chi(\xi) = F^{-1}\left[\frac{\mu}{(1+x_1^2)\ldots(1+x_n^2)}\right], \qquad \mu = u(x, +0), \tag{3.12}$$

*may be uniquely represented in the form*

$$\chi\ (\xi) = \sum_{2\leqslant k\leqslant n}\ \sum_{1\leqslant j_1<\ldots<j_k\leqslant n} \exp(-|\xi_{j_{k+1}}|-\ldots-|\xi_{j_n}|)$$
$$\times\,\Phi_{j_1\ldots j_k}(\xi_{j_1\ldots j_k})$$
$$+\exp(-|\xi_2|-\ldots-|\xi_n|)\,\chi(\xi_1, 0, \ldots, 0)$$
$$+\ldots+\exp(-|\xi_1|-\ldots-|\xi_{n-1}|)\,\chi(0, \ldots, 0, \xi_n)$$
$$=(n-1)\exp(-|\xi_1|-\ldots-|\xi_n|)\,\chi(0), \tag{31.3}$$
$$\xi\in\mathbb{R}^n,$$

*where* $\Phi_{j_1\ldots j_k}$ *are continuous bounded functions in* $\mathbb{R}^k$ *with support in* $-\overline{\mathbb{R}}^k_+ \cup \overline{\mathbb{R}}^k_+$.

*Proof*. The uniqueness of the representation (3.13) in each octant follows from the properties of continuity and of the support of the functions $\Phi_{j_1\ldots j_k}$. We carry out the existence proof of the representation (3.13) by induction on $n$. The function $\chi(\xi) = \chi_{1\ldots n}(\xi)$ and all the functions $\chi_{j_1\ldots j_k}(\xi_{j_1\ldots j_k})$ are, by (3.8), continuous and bounded in $\mathbb{R}^k$ and satisfy equation (3.11) outside $-\overline{\mathbb{R}}^k_+ \cup \overline{\mathbb{R}}^k_+$. Therefore when $n = 2$ Theorem I holds, by virtue of Lemma 4 of Sec. 17.1, if in the representation (3.13) we put

$$\Phi_{12}(\xi)=\chi(\xi)-e^{|\xi_1|}\chi(0, \xi_2)-e^{-|\xi_2|}\chi(\xi_1, 0)+e^{-|\xi_1|-|\xi_2|}\chi(0).$$

Suppose the representation (3.13) holds for all dimensions $k < n$, so that the functions $\chi_{j_1\ldots j_k}$ in $\mathbb{R}^k$ are representable in the form of the corresponding formulas (3.13). We now prove the representation (3.13) in the region

$$G_{+-}=[\xi:\xi_1>0,\ \xi_2<0,\ \tilde{\xi}\in\mathbb{R}^{n-2}].$$

By Lemma 4 of Sec. 17.1, the function $\chi(\xi)$ can be represented as

$$\chi(\xi)=e^{-|\xi_1|}\chi(0, \xi_2, \tilde{\xi})+e^{-|\xi_2|}\chi(\xi_1, 0, \tilde{\xi})-e^{-|\xi_1|-|\xi_2|}\chi(0, 0, \tilde{\xi}). \tag{3.14}$$
$$\xi\in G_{+-}.$$

In accordance with the induction hypothesis, for the functions that follow

$$\chi(0, \xi_2, \tilde{\xi}) = \chi_{2\ldots n}(\xi_2, \tilde{\xi}),$$
$$\chi(\xi_1, 0, \tilde{\xi}) = \chi_{13\ldots n}(\xi_1, \tilde{\xi}),$$
$$\chi(0, 0, \tilde{\xi}) = \chi_{3\ldots n}(\tilde{\xi}),$$

the corresponding representations (3.13) hold true. Substituting them into (3.14), we obtain (3.13) in the region $G_{+-}$. The representation (3.13) occurs also in other regions of the type $G_{+-}$ that do not contain $-\bar{\mathbb{R}}^n_+ \cup \bar{\mathbb{R}}^n_+$. From the uniqueness of the representation (3.13) in the indicated regions of the type $G_{+-}$ if follows that the appropriate representations (3.13) coincide in the intersections of those regions. Hence, the representation (3.13) holds true everywhere outside $-\bar{\mathbb{R}}^n_+ \cup \bar{\mathbb{R}}^n_+$. By introducing the function

$$\Phi_{1\ldots n}(\xi) = \chi(\xi) - \sum_{2\leqslant k\leqslant n-1}\ \sum_{1\leqslant j_1<\ldots<j_k\leqslant n} \exp(-|\xi_{j_{k+1}}| - \ldots - |\xi_{j_n}|)$$
$$\times \Phi_{j_1\ldots j_k}(\xi_{j_1\ldots j_k})$$
$$-\exp(-|\xi_2| - \ldots - |\xi_n|)\,\chi(\xi_1, 0, \ldots, 0)$$
$$-\ldots-\exp(-|\xi_1| - \ldots - |\xi_{n-1}|)\,\chi(0, \ldots, 0, \xi_n)$$
$$+(n-1)\exp(-|\xi_1| - \ldots - |\xi_n|)\,\chi(0),$$

which is continuous and bounded in $\mathbb{R}^n$ with support in $-\bar{\mathbb{R}}^n_+ \cup \bar{\mathbb{R}}^n_+$, we are convinced of the truth of (3.13) throughout the space $\mathbb{R}^n$. This completes the proof of Theorem I.

Theorem II. *For the measure* $\mu$ *to be a boundary value of the function* $u(x, y)$ *of the class* $\mathscr{P}_+(T^n)$, $\mu = u(x, +0)$, *it is necessary and sufficient that*

$$\mu = F[D_1^2 \ldots D_n^2 v], \tag{3.15}$$

*where* $D_1^2 \ldots D_n^2 v$ *is much greater than zero,* $v$ *is a continuous* *-*Hermitian function that satisfies the growth conditions (1.7), and* $\operatorname{supp} v \subset \bar{\mathbb{R}}^n_+ \cup \bar{\mathbb{R}}^n_+$.

*Here, the function* $v$ *having the indicated properties is unique up to the additive term* $iC[\mathscr{E}_n(\xi) - \mathscr{E}_n(-\xi)]$, *where* $C$ *is an arbitrary real number.*

*Proof*. For $n=1$, Theorem II has already been proved in Sec. 17.2. Sufficiency for $n \geqslant 2$ follows from the theorem of Sec. 17.4.

*Necessity for* $n \geqslant 2$. Suppose $u \in \mathcal{F}_+(T^n)$ and $\mu = u(x, +0)$. From (3.5) and (3.6) we obtain

$$F^{-1}[\mu] = F^{-1}[(1+x_1^2)\ldots(1+x_n^2)\sigma] = (1-D_1^2)\ldots(1-D_n^2)\chi(\xi). \tag{3.16}$$

Noting that

$$(1-D^2)e^{-|\xi|} = 2\delta(\xi)$$

and using (3.13), we continue the equalities (3.16):

$$\begin{aligned} F^{-1}[\mu] = \sum_{2\leqslant k\leqslant n} 2^{n-k} \sum_{1\leqslant j_1<\ldots<j_k\leqslant n} & \delta(\xi_{j_{k+1}})\times\ldots\times\delta(\xi_{j_n}) \\ & \times(1-D_{j_1}^2)\ldots(1-D_{j_k}^2)\Phi_{j_1\ldots j_k}(\xi_{j_1},\ldots,\xi_{j_k}) \\ & +2^{n-1}\delta(\xi_2)\times\ldots\times\delta(\xi_n) \\ & \qquad\times(1-D_1^2)[\chi(\xi_1,0,\ldots,0)-\chi(0)e^{-|\xi_1|}] \\ & +\ldots+2^{n-1}\delta(\xi_1)\times\ldots\times\delta(\xi_{n-1}) \\ & \qquad\times(1-D_n^2)[\chi(0,\ldots,0,\xi_n)-\chi(0)e^{-|\xi_n|}) \\ & +2^n\chi(0)\delta(\xi), \end{aligned} \tag{3.17}$$

where $\Phi_{j_1\ldots j_k}$ are continuous bounded unctions in $\mathbb{R}^k$ with support in $-\overline{\mathbb{R}}_+^k \cup \overline{\mathbb{R}}_+^k$. Every term under the summation sign on the right in (3.17) can, by Lemma 2 of Sec. 17.1, be represented as

$$\begin{aligned} & 2^{n-k}D_{j_{k+1}}^2\ldots D_{j_n}^2[\theta(\xi_{j_{k+1}})\xi_{j_{k+1}}\ldots\theta(\xi_{j_n})\xi_{j_n}] \\ & \quad\times(1-D_{j_1}^2)\ldots(1-D_{j_k}^2)[\theta(\xi_{j_1})\ldots\theta(\xi_{j_k})\Phi_{j_1\ldots j_k}(\xi_{j_1}\ldots\xi_{j_k})] \\ & +(-2)^{n-k}D_{j_{k+1}}^2\ldots D_{j_n}^2[\theta(-\xi_{j_{k+1}})\xi_{j_{k+1}}\ldots\theta(-\xi_{j_n})\xi_{j_n}] \\ & \quad\times(1-D_{j_1}^2)\ldots(1-D_{j_k}^2)[\theta(-\xi_{j_1})\ldots\theta(-\xi_{j_k})\Phi(\xi_{j_1},\ldots,\xi_{j_k})] \\ & =D_1^2\ldots D_n^2 v_{j_1\ldots j_k}(\xi), \end{aligned} \tag{3.18}$$

where $v_{j_1\ldots j_k}$ is a continuous function in $\mathbb{R}^n$ with support in $-\overline{\mathbb{R}}_+^n \cup \overline{\mathbb{R}}_+^n$, the function satisfying (1.7). By the same reasoning, the other terms on the right of (3.17) can also be represented

in the form of (3.18):

$$2^{n-1}D_{j_2}^2 \dots D_{j_n}^2 [\theta(\xi_{j_2})\xi_{j_2} \dots \theta(\xi_{j_n})\xi_{j_n}]$$
$$\times (1-D_{j_1}^2)\{\theta(\xi_{j_1})[\chi(0, \dots, \xi_{j_1}, \dots, 0)-\chi(0)e^{-|\xi_{j_1}|}]\}$$
$$+(-2)^{n-1}D_{j_2}^2 \dots D_{j_n}^2 [\theta(-\xi_{j_2})\xi_{j_2} \dots \theta(-\xi_{j_n})\xi_{j_n}]$$
$$\times (1-D_{j_1}^2)\{\theta(-\xi_{j_1})[\chi(0, \dots, \xi_{j_1}, \dots, 0)-\chi(0)e^{-|\xi_{j_1}|}]\}$$
$$= D_1^2 \dots D_n^2 v_{j_1}(\xi); \tag{3.19}$$

$$2^n\chi(0)\delta(\xi) = 2^n\chi(0) D_1^2 \dots D_n^2 [\theta_n(\xi)\xi_1 \dots \xi_n]$$
$$= D_1^2 \dots D_n^2 v_0(\xi). \tag{3.20}$$

Putting

$$v(\xi) = \sum_{0\leqslant k\leqslant n} \sum_{1\leqslant j_1<\dots<j_k\leqslant n} v_{j_1\dots j_k}(\xi),$$

we obtain, by (3.17) to (3.20), the representation (3.15), where the function $v$ is continuous with support in $-\overline{\mathbb{R}}_+^n \cup \overline{\mathbb{R}}_+^n$ and satisfies (1.7). If $v$ is not $*$-Hermitian, then it may be replaced by $\frac{1}{2}(v + v^*)$, since the measure $\mu$ is real.

The conclusion that the function $v$ is unique follows from Lemma 6 of Sec. 17.1. The proof of Theorem II is complete.

**17.4 Functions of the class $H_+(T^n)$** We recall that the Poisson kernel $\mathcal{P}_n(x, y)$ and the Schwartz kernel $\mathcal{S}_n(z; z^0)$ for the region $T^n$ have been written out in Sec. 11.1 and Sec. 12.5 respectively.

Theorem *The following conditions are equivalent*:

(1) *The function $f(z)$ belongs to the class $H_+(T^n)$.*

(2) *Its spectral function $g(\xi)$ has the following properties*:

(a) $-ig(\xi) + ig^*(\xi) \gg 0$,

(b) $g(\xi) = iD_1^2 \dots D_n^2 u(\xi) + i(a, D)\delta(\xi)$,

*where $a \in \overline{\mathbb{R}}_+^n$ and $u(\xi)$ is a continuous function in $\mathbb{R}^n$ with support in $\overline{\mathbb{R}}_+^n$, which function satisfies the growth condition (1.7). The expansion* (b) *is unique.*

(3) *The following representation holds*:

$$\operatorname{Im} f(z) = \int \mathcal{P}_n(x-x', y)\,\mu(dx') + (a, y), \qquad z \in T^n. \tag{4.1}$$

(4) *For all* $z^0 \in T^n$ *the following representation holds*:

$$f(z) = i \int \mathscr{S}_n(z - x'; xz^0 - x')\,\mu(dx') + (a, z) + b(z^0), \qquad z \in T^n. \tag{4.2}$$

*Here*, $\mu = \operatorname{Im} f_+$, $b(z^0) = \operatorname{Re} f(z^0) - (a, x^0)$,

$$a_j = \lim_{y_j \to \infty} \frac{\operatorname{Im} f(iy)}{y_j}, \qquad j = 1, \ldots, n, \quad y \in \overline{\mathbb{R}}^n_+, \tag{4.3}$$

$(a, y)$ *is the best linear minorant of the indicator* $h(\operatorname{Im} f; y)$ *in the cone* $\mathbb{R}^n_+$.

*Proof*. For $n = 1$, the theorem has already been proved in Sec. 17.2. Suppose $n \geqslant 2$.

(1) → (2). Suppose $f \in H_+(T^n)$. Then $f(z) = L[g]$, $g \in \mathscr{S}'(\overline{\mathbb{R}}^n_+)$, $f_+ = F[g]$, $\mu = \operatorname{Im} f_+$ and

$$F^{-1}[\mu] = \frac{g - g^*}{2i} \tag{4.4}$$

(see Sec. 16.2). From (4.4) follows the condition (a) (see Sec. 8).

To prove condition (b), let us make use of Theorem II of Sec. 17.3 (the necessity of its hypotheses has already been proved). By (3.15) equation (4.4) takes the form

$$\frac{1}{2i}[g(\xi) - g^*(\xi)] = D_1^2 \ldots D_n^2 v(\xi), \tag{4.5}$$

where $v = v^* \in C^{(\mathbb{R}^n)}$, supp $v c - \mathbb{R}^n_+ \cup \overline{\mathbb{R}}^n_+$ and $v$ satisfies the growth condition (1.7). The generalized function

$$g_0(\xi) = 2iD_1^2 \ldots D_n^2 [\theta_n(\xi)\, v(\xi)] \tag{4.6}$$

satisfies (4.5) in $\mathbb{R}^n$. The general solution of the homogeneous equation (4.5), $g - g^* = 0$, has, in the class $\mathscr{S}'(-\overline{\mathbb{R}}^n_+ \cup \overline{\mathbb{R}}^n_+)$, support 0 and, hence, can be represented in the form (see Sec. 2.6)

$$a_0 \delta(\xi) + \sum_{1 \leqslant |\alpha| \leqslant N} i^{|\alpha|} a_\alpha D^\alpha \delta(\xi),$$

where $a_\alpha$ are arbitrary real constants. From this and from (4.6) it follows that the spectral function $g$ is representable as

$$g(\xi) = iD_1^2 \ldots D_n^2 [2\theta_n(\xi)\, v(\xi) - ia_0 \mathscr{E}_n(\xi)] + \sum_{1 \leqslant |\alpha| \leqslant N} i^{|\alpha|} a_\alpha D^\alpha \delta(\xi). \tag{4.7}$$

Set

$$u(\xi)=2\theta_n(\xi)\,v(\xi)-ia_0\mathscr{E}_n(\xi).$$

The function $u(\xi)$ satisfies conditions (b) of the theorem. Here, (4.7) takes the form

$$g(\xi)=iD_1^2\ldots D_n^2u(\xi)+\sum_{1\leqslant|\alpha|\leqslant N}i^{|\alpha|}a_\alpha D^\alpha\delta(\xi) \qquad (4.8)$$

Thus (see Sec. 9.2)

$$f(z)=L[g]=i(-1)^n z_1^2\ldots z_n^2\int_{\mathbb{R}_+^n}u(\xi)\,e^{i(z,\,\xi)}\,d\xi+\sum_{1\leqslant|\alpha|\leqslant N}a_\alpha z^\alpha,\ z\in T^n. \qquad (4.9)$$

Setting $z=iy$, $y\in\mathbb{R}_+^n$ in (4.9), we obtain

$$f(iy)=iy_1^2\ldots y_n^2\int_{\mathbb{R}_+^n}n(\xi)\,e^{-(y,\,\xi)}d\xi+\sum_{1\leqslant|\alpha|\leqslant N}i^{\|\alpha|}a_\alpha y^\alpha. \qquad (4.10)$$

We now prove that for every $j=1,\ldots,n$,

$$\lim_{y_j\to\infty}y_1\ldots y_n\int_{\mathbb{R}_+^n}u(\xi)\,e^{-(y,\,\xi)}d\xi=0. \qquad (4.11)$$

Indeed, from the properties of the function $u(\xi)$ it follows (by the Lebesgue theorem) that passage to the limit under the integral sign is valid:

$$\lim_{y_j\to\infty}y_1\ldots y_n\int_0^\infty\ldots\int_0^\infty|u(\xi)|\,e^{-(y,\,\xi)}d\xi$$

$$=\lim_{y_j\to\infty}\int_0^\infty\ldots\int_0^\infty\left|u\left(\frac{x_1}{y_1},\ldots,\frac{x_n}{y_n}\right)\right|e^{-x_1-\ldots-x_n}dx=0.$$

Taking into account the estimate (3.1) of Sec. 13, we obtain, from (4.10), the inequality

$$\left|iy_1^2\ldots y_n^2\int_{\mathbb{R}_+^n}u(\xi)\,e^{-(y,\,\xi)}d\xi+i(a,y)+\sum_{2\leqslant|\alpha|\leqslant N}i^{|\alpha|}a_\alpha y^\alpha\right|$$

$$\leqslant M(C')\frac{1+|y|^2}{|y|},\quad y\in C'\Subset\mathbb{R}_+^n,$$

from which, and also from the limiting relations (4.11), we conclude that $a_\alpha = 0$, $|\alpha| \geqslant 2$, and for the numbers $a_j$ the formula (4.3) holds, so that $a_j \geqslant 0$, $j = 1, \ldots, n$, that is, $a \in \overline{\mathbb{R}}^n_+$. This, by (4.8), proves the representation (b). Its uniqueness follows from Lemma 5 of Sec. 17.1

(2) $\to$ (3). The proof is literally the same as for the one-dimensional case in the sufficiency proof in Theorem II of Sec. 17.2. There, use is made of Lemma 1 and Lemma 3 of Sec. 17.1.

(3) $\to$ (4) $\to$ (1). From the representation (4.1) it follows that the corresponding Poisson integral is a pluriharmonic function in $T^n$. All other assertions of the theorem follow from this and from the theorem of Sec. 16.6. This completes the proof of the theorem.

Corollary *If $f \in H_+(T^n)$, then the best linear minorant $(a, y)$ of the indicator $h(\operatorname{Im} f;\, y)$ in the cone $\mathbb{R}^n_+$ is given by*

$$a_j = \lim_{y \to e_j,\, y \in \mathbb{R}^n_+} h(\operatorname{Im} f;\, y), \qquad j = 1, \ldots, n, \tag{4.12}$$

where $e_j$ are unit vectors in $\mathbb{R}^n$.

Indeed, from the inequality

$$(a, y) \leqslant h(\operatorname{Im} f;\, y)$$

it follows that

$$a_j \leqslant \lim_{y \to e_j,\, y \in \mathbb{R}^n_+} h(\operatorname{Im} f;\, y). \tag{4.13}$$

The function

$$\frac{1}{t} \operatorname{Im} f(\mathrm{i} + ity), \qquad y \in \mathbb{R}^n_+ \cup \{e_1, \ldots, e_n\}, \quad t > 0,$$

is continuous in $y$, does not increase with respect to $t > 0$, and tends (as $t \to \infty$) to the (semicontinuous above) function

$$\tilde{h}(y) = \begin{cases} h(\operatorname{Im} f;\, y), & y \in \mathbb{R}^n_+, \\ a_j & y = e_j, \quad j = 1, \ldots, n. \end{cases} \tag{4.14}$$

For $y \in \mathbb{R}^n_+$ this assertion has been proved (see the theorem in Sec. 16.5). For $y = e_j$ it follows from the representation (4.1),

$$\frac{1}{t} \operatorname{Im} f(\mathrm{i} + ite_j) = \frac{1}{\pi^n}\left(1 + \frac{1}{t}\right) \int \frac{1}{x_j^2 + (1+t)^2} \prod_{\substack{1 \leqslant k \leqslant n \\ k \neq j}} \frac{1}{1 + x_k^2}\, \mu(dx)$$
$$+ \frac{1}{t} \sum_{1 \leqslant k \leqslant n} a_k + a_j, \tag{4.15}$$

because, by virtue of B. Levi's theorem, no summand in the right member of (4.15) increases with respect to $t > 0$. From the fact that the function $\hbar(y)$ is semicontinuous above on the set $\mathbb{R}^n_+ \cup \{e_1, \ldots, e_n\}$ and from (4.14) follows the inequality

$$\overline{\lim_{y \to e_j,\, y \in \mathbb{R}^+_n}} h(\operatorname{Im} f; y) \leqslant \overline{\lim_{y \to e_j,\, y \in \mathbb{R}^n_+ \cup \{e_j\}}} \hbar(y) = \hbar(e_j) = a_j,$$

which, together with the inequality (4.13), implies equation (4.12).

*Remark 1.* The representation (b) strengthens the results of Sec. 16.4 concerning the smoothness of the spectral function and worsens the estimate of its growth in the case of the cone $\mathbb{R}^n_+$:

$$g(\xi) = D_1^2 \ldots D_n^2 g_1(\xi),$$

$$g_1(\xi) = iu(\xi) + i\theta_n(\xi) \sum_{1 \leqslant j \leqslant n} a_j \xi_1 \ldots \xi_{j-1} \xi_{j+1} \ldots \xi_n,$$

$$g_1 \in \mathcal{L}^2_s(\overline{\mathbb{R}^n_+}), \qquad s < -\frac{5}{2} n \quad \left(\text{in Sec. } 16.4\ s < -\frac{3}{2} n - 1\right).$$

*Remark 2.* A description of the functions of the class $H_+(G)$ in the polycircle has been given by Korányi and Pukánszky [1] and Vladimirov and Drozhzhinov [1]; in the "generalized unit circle" (in the set of $2 \times 2$ matrices $w$ that satisfy the condition $ww^* < I$) by Vladimirov [10]; in bounded strictly star regions, in particular, in the classical symmetric regions, by Aizenberg and Dautov [1]; in the "tube of the future" $\tau^+ = T^{V^+}$ ($n = 3$, see Sec. 4.4) by Vladimirov [10]. In the last case it was established that the Poisson integral for $f \in H_+(\tau^+)$ is a pluriharmonic function (and, hence, the theorem of Sec. 16.6 holds) if and only if the indicator $h(\operatorname{Im} f; y)$ possesses the properties

$$h(\operatorname{Im} f; y) = h_0(y) + (a, y), \qquad h_0(y) \geqslant 0, \quad a \in \overline{V^+}; \quad y \in V^+;$$

$$\lim_{|y| \to 1-0} \int_{|s|-1} h_0(1, s\,|\mathbf{y}|)\, ds = 0.$$

# 18 Positive Real Matrix Functions in $T^C$

Suppose $A(x) = (A_{kj}(x))$ is a square matrix with elements $A_{kj}$ taken from $\mathcal{D}'$. We use the following terminology: $A^*(x) = \overline{A}^T(-x)$ is the *$*$-Hermitian conjugate of* $A$; $A^+(x) = \overline{A}^T(x)$

is the $+$-*Hermitian conjugate of* $A$;

$$\operatorname{Re} A = \frac{1}{2}(A + A^+), \qquad \operatorname{Im} A = \frac{1}{2i}(A - A^+)$$

are the real part and imaginary part of $A$ (compare Sec. 1.3).

If $A = A^*$ or $A = A^+$, then $A$ will be called $*$-*Hermitian* (compare Sec. 8.1) or $+$-*Hermitian* respectively. For constant matrices, both concepts of Hermiticity coincide, and in that case we will simply call them *Hermitian matrices* or *Hermitian conjugate matrices.*

Clearly, if $A(x)$ is a matrix of slow growth (that is, $A_{kj} \in \mathscr{S}'$), then

$$F[A^+] = F[A]^*, \qquad F[A^*] = F[A]^+,$$

where the Fourier transform $F[A]$ of matrix $A$ signifies a matrix with components $F[A_{kj}]$.

The matrix function $A(z)$, holomorphic in the tubular region $T^C$ is said to be *positive real in* $T^C$ if it satisfies the conditions: (a) $\operatorname{Re} A(z) \geqslant 0$, $z \in T^C$, (b) $A(iy)$ is real for all $y \in C$ [and then $A(z) = \bar{A}(-\bar{z})$, $z \in T^C$, by virtue of the Schwartz symmetry principle]. It is clear that if $A(z)$ is positive real in $T^C$, then $A(z)$ is positive real in any $T^{C'}$, $C' \subset C$, as well.

We term the matrix $Z(\xi)$, $Z_{kj} \in \mathscr{D}'$, for which $A(z) = L[Z]$, the *spectral matrix function* of the matrix $A(z)$.

Our problem is to give a description of positive real matrix functions in $T^C$, where $C$ is an acute convex cone. Let us first consider the scalar case, that is, positive real functions in $T^C$.

**18.1 Positive real functions in $T^C$** A function $f(z)$ is positive real in $T^C$ if and only if $if \in H_+(T^C)$ and its spectral function $g$ is real. The last assertion is due to the equalities

$$f(z) = L[g] = F[g(\xi)e^{-(y,\xi)}] = \bar{f}(-\bar{z}) = F[\bar{g}(\xi)e^{i(y,\xi)}].$$

Suppose $C' = [y\colon (e_1, y) > 0, \ldots, (e_n, y) > 0]$ is an $n$-hedral acute cone. Then (see Sec. 4.4)

$$C'^* = \left[\xi : \xi = \sum_{1 \leqslant j \leqslant n} \lambda_j e_j,\ \lambda_j \geqslant 0\right].$$

We denote by $A$ the (nonsingular) linear transformation

$$z \to \zeta = (\zeta_1 = (e_1, z), \ldots, \zeta_n = (e_n, z)) = Az. \tag{1.1}$$

The transformation $\zeta = Az$ maps biholomorphly the region $T^{C'}$ onto the region $T^n$, and the transformation $\xi' = A^{-1T}\xi$ maps the cone $C'^*$ onto the cone $\overline{\mathbb{R}}^n_+$. In the process, the derivatives $D = (D_1, \ldots, D_n)$ pass into the derivatives $D' = (D'_1, \ldots, D'_n)$, $D'_j = \dfrac{\partial}{\partial \xi'_j}$, via the formulas

$$D'_j = \sum_{1 \leqslant k \leqslant n} \frac{\partial \xi_k}{\partial \xi'_j} D_k = (e_j, D) = (AD)_j, \qquad j = 1, \ldots, n. \quad (1.2)$$

That is, $D' = AD$.

Furthermore (see Sec. 1.9),

$$\delta(\xi) = \delta(A^T\xi') = \frac{\delta(\xi')}{|\det A|}. \quad (1.3)$$

Lemma *If the vectors $e_1, \ldots, e_n$ are linearly independent, then*

$$(e_1, D)^2 \ldots (e_n, D)^2 u(\xi) + (a, D)\,\delta(\xi) = 0, \quad (1.4)$$

*where* $u \in C(\mathbb{R}^n)$, supp $u \subset C'^*$, *is possible only for* $u(\xi) = 0$ and $a = 0$.

*Proof.* In the variables $\xi' = A^{-1T}\xi$, the equation (1.4) becomes, by virtue of (1.2) and (1.3),

$$D_1'^2 \ldots D_n'^2 \tilde{u}(\xi') + (\tilde{a}, D')\,\delta(\xi') = 0, \quad (1.5)$$

where

$$\tilde{u}(\xi') = |\det A|\, u(A^T\xi'), \qquad \tilde{a} = A^{-1T}a, \quad (1.6)$$

and $\tilde{u} \in C(\mathbb{R}^n)$, supp $\tilde{u} \subset \overline{\mathbb{R}}^n_+$. By Lemma 5 of Sec. 17.1, $\tilde{u}(\xi') = 0$ and $\tilde{a} = 0$, whence, by (1.6), we obtain $u(\xi) = 0$ and $a = 0$, which proves the lemma.

Theorem *For a function $f(z)$ to be positive real in $T^C$, where $C$ is an acute (convex) cone in $\mathbb{R}^n$, it is necessary and sufficient that its spectral function $g(\xi)$ have the following properties:*

(a) $g(\xi) + g^*(\xi) \gg 0$,

(b) *for any n-hedral cone* $C' = [y\colon (e_1, y) > 0, \ldots, (e_n, y) > 0]$ *contained in the cone $C$, that it be (uniquely) representable in the form*

$$g(\xi) = (e_1, D)^2 \ldots (e_n, D)^2 u_{C'}(\xi) + (a_{C'}, D)\,\delta(\xi), \quad (1.7)$$

*where* $a_{C'} \in C'^*$ *and* $u_{C'}(\xi)$ *is a real continuous function of slow growth in* $\mathbb{R}^n$ *with support in the cone* $C'^*$.

*Proof. Necessity.* Let $f(z)$ be positive real in $T^C$ so that —if $\in$ $\in H_+(T^C)$ and $f(z) = L[g]$, and the spectral function $g(\xi)$ is real in $\mathcal{S}'(C^*)$. From this fact and from (1.5) of Sec. 16 there follows the condition (a). To prove the representation (1.7) for the $n$-hedral cone $C'$, let us perform a biholomorphic mapping $\zeta = Az$ [see (1.1)] of the region $T^{C'}$ onto $T^n$; in the process, the function $f(z)$ passes into the positive real function $f(A^{-1}\zeta)$ in $T^n$. By the theorem of Sec. 16.4 we conclude that there exist a vector $a_1 \in \overline{\mathbb{R}}^n_+$ and a continuous function $u_1(\xi')$ of slow growth with support in $\overline{\mathbb{R}}^n_+$ such that the spectral function $g_1(\xi')$ of the function $f(A^{-1}\zeta)$ is representable as

$$g_1(\xi') = D_1'^2 \ldots D_n'^2 u_1(\xi') + (a_1, D')\,\delta(\xi'). \qquad (1.8)$$

Let us now pass to the old variables $z = A^{-1}\zeta$ and $\xi = A^T\xi'$. The spectral functions $g(\xi)$ and $g_1(\xi')$ are connected by the relation (see Sec. 9.2(e))

$$|\det A\,|\,g(\xi) = g_1(\xi') = g_1(A^{-1T}\xi). \qquad (1.9)$$

Using the formulas (1.2) and (1.3), we derive from (1.8) and (1.9) the representation (1.7) for $g(\xi)$ in which

$$u_{C'}(\xi) = \frac{1}{|\det A|}\, u_1(A^{-1T}\xi), \qquad a_{C'} = A^T a_1. \qquad (1.10)$$

Taking into account that the representation $A^T$ carries the cone $\overline{\mathbb{R}}^n_+$ onto the cone $C'^*$, we conclude from (1.10) that $a_{C'} = C'^*$ and $u_{C'} \in C(\mathbb{R}^n)$ are of slow growth, $\operatorname{supp} u_{C'} \subset C'^*$.

The uniqueness of the expansion (1.7) and the real nature of the function $u_{C'}$ follow from the reality of the spectral function $g$ and the vector $a$ by virtue of the lemma of Sec. 17.1.

*Sufficiency.* Suppose the generalized function $g(\xi)$ has properties (a) and (b). Then from the representation (1.7) it follows that $g$ is real and $g \in \mathcal{S}'(C'^*)$ for all $n$-hedral cones $C' \subset C$, so that $g \in \mathcal{S}'(C^*)$. Therefore the function $f(z) = L[g]$ is holomorphic in $T^C$ and $f(iy)$ is real in $C$. It remains to prove that $\operatorname{Re} f(z) \geqslant 0$, $z \in T^C$. Let us take an arbitrary $n$-hedral cone $C' \subset C$ and pass to the new variables $\zeta = Az$ and $\xi' = A^{-1T}\xi$. Then, as in the proof of necessity, we conclude that the spectral functions $g(\xi)$ and $g_1(\xi')$ of the functions $f(z)$ and $f(A^{-1}\zeta)$ are connected by the relation (1.9) and therefore $g_1(\xi')$ can be represented as (1.8),

where $u_1(\xi')$ and $a_1$ are expressed in terms of $u_{C'}(\xi)$ and $a_{C'}$ via the formulas (1.10), so that $a_1 \in \overline{\mathbb{R}}^n_+$ and $u_1 \in C(\mathbb{R}^n)$ are of slow growth, supp $u_1 \subset \overline{\mathbb{R}}^n_+$. Besides, by (1.9)

$$g_1(\xi') + g_1^*(\xi') = |\det A| [g(\xi) + g^*(\xi)] \gg 0.$$

From this, by the theorem of Sec. 16.4, we conclude that *if* $(A^{-1}\zeta) \in H_+(T^n)$, that is, Re $f(z) \geqslant 0$ in $T^C$, whence, by the arbitrariness of $C' \subset C$, it follows that Re $f(z) \geqslant 0$ in $T^C$, which is what we set out to prove. The theorem is proved.

### 18.2 Positive real matrix functions in $T^C$

Theorem *For an $N \times N$ matrix function $A(z)$ to be positive real in $T^C$, where $C$ is an acute (convex) cone in $\mathbb{R}^n$, it is necessary and sufficient that its spectral matrix function $Z(\xi)$ have the following properties*:

(a) $\langle Z(\xi) a + Z^*(\xi) a, a \rangle \gg 0, \qquad a \in \mathbb{C}^N,$ (2.1)

(b) *for any n-hedral cone* $C' = [y : (e_1, y) > 0, \ldots, (e_n, y) > 0]$ *contained in the cone $C$, it is (uniquely) representable in the form*

$$Z(\xi) = (e_1, D)^2 \ldots (e_n, D)^2 Z_{C'}(\xi) + \sum_{1 \leqslant j \leqslant n} Z_{C'}^{(j)} D_j \delta(\xi), \quad (2.2)$$

*where the matrix function $Z_{C'}(\xi)$ is a continuous real function of slow growth in $\mathbb{R}^n$ with support in $\overline{\mathbb{R}}^n_+$, and the matrices $Z_{C'}^{(j)}$, $j = 1, \ldots, n$, are real symmetric and such that*

$$\sum_{1 \leqslant j \leqslant n} y_j Z_{C'}^{(j)} \geqslant 0, \qquad y \in \overline{C}'. \quad (2.3)$$

*Here, the following equation holds*:

$$\text{(a}'\text{)} \quad \mathrm{Re} \int \langle Z * \varphi, \varphi \rangle \, d\xi \geqslant 0, \qquad \varphi = (\varphi_1, \ldots, \varphi_N) \in \mathscr{S}^{\times N}. \quad (2.4)$$

*Proof. Sufficiency*. From (b) of the theorem it follows that the spectral function $Z(\xi)$ is real and its elements $Z_{kj} \in \mathscr{S}'(C^*)$, so that the matrix function $A(z) = L[Z]$ is holomorphic in the region $T^C$, where $C = \operatorname{int} C^{**}$ (see Sec. 12.2), and satisfies the condition of reality $A(z) = \overline{A}(-\overline{z})$.

Let us now verify that the generalized function $g_a(\xi) = \langle Z(\xi) \times a, a \rangle$ satisfies, for all $a \in \mathbb{C}^N$, the conditions (a) and (b) of the theorem of Sec. 17.1.

Condition (a) is fulfilled by virtue of (2.1):

$$g_a(\xi) + g_a^*(\xi) = \langle Z(\xi)\, a + Z^*(\xi)\, a,\ a\rangle \gg 0.$$

Conditions (b) are fulfilled by virtue of (2.2) and (2.3):

$$g_a(\xi) = (e_1, D)^2 \dots (e_n, D)^2 \langle Z_{C'}(\xi)\, a, a\rangle + \sum_{1\leqslant j\leqslant n} \langle Z_{C'}^{(j)} a, a\rangle D_j \delta(\xi),$$

where $\langle Z_{C'}(\xi)\, a,\ a\rangle$ is a continuous function of slow growth in $\mathbb{R}^n$ with support in $C'^*$, and

$$\sum_{1\leqslant j\leqslant n} y_j \langle Z_{C'}^{(j)} a, a\rangle \geqslant 0, \qquad y \in \bar{C}'.$$

That is, the vector

$$(\langle Z_{C'}^{(1)} a, a\rangle, \dots, \langle Z_{C'}^{(n)} a, a\rangle) \in C'^*.$$

Noting that $g_a(\xi)$ is the spectral function of the function $\langle A(z)\, a,\ a\rangle$, we derive from the theorem of Sec. 17.1 that $\operatorname{Re}\langle A(z)\, a,\ a\rangle \geqslant 0$, $z \in T^C$, $a \in \mathbb{C}^N$. That is, $\operatorname{Re} A(z) \geqslant 0$, $z \in T^C$. Thus, the matrix function $A(z)$ is positive real in $T^C$.

*Necessity*. Let $A(z)$ be a positive real matrix function in $T^C$. Then for every vector $a \in \mathbb{C}^N$ the function $\langle A(z)\, a,\ a\rangle$ is positive real in $T^C$. By the theorem of Sec. 17.1, its spectral function $g_a(\xi)$ taken from $\mathscr{S}'(C^*)$ has the following properties:

(a′) $g_a(\xi) + g_a^*(\xi) \gg 0$,

(b′) for any $n$-hedral cone $C' \subset C$ it can be represented as

$$g_a(\xi) = (e_1, D)^2 \dots (e_n, D)^2 U_{C'}(\xi; a) + \sum_{1\leqslant j\leqslant n} A_{C'}^{(j)}(a) D_j \delta(\xi), \tag{2.5}$$

where the function $U_{C'}(\xi;\ a)$ is a continuous function of slow growth in $\mathbb{R}^n$ with support in the cone $C'^*$, and the vector

$$(A_{C'}^{(1)}(a), \dots, A_{C'}^{(n)}(a)) \in C'^*. \tag{2.6}$$

Furthermore, since $g_a(\xi)$ is the spectral function of the quadratic form $\langle A(z)\, a, a\rangle$, $a \in \mathbb{C}^N$, it follows that $g_a(\xi)$ is a quadratic form with respect to the vector $a$, so that there exists an $N \times N$ matrix $Z(\xi)$ [the spectral matrix function of the matrix $A(z)$]

such that

$$\langle Z(\xi)\,a, a\rangle = g_a(\xi), \qquad Z_{k_j} \in \mathscr{S}'(C^*). \tag{2.7}$$

From this and from the condition (a′) it follows that the matrix $Z(\xi)$ satisfies the condition (a). Furthermore, from the equality $A(z) = \overline{A}(-\overline{z})$ follows the real nature of the matrix $Z(\xi)$.

Now, using the equation

$$\begin{aligned}\langle Z(\xi)\,a, b\rangle &= \frac{1}{4}\langle Z(\xi)(a+b), a+b\rangle \\ &\quad - \frac{1}{4}\langle Z(\xi)(a-b), a-b\rangle \\ &\quad + \frac{i}{4}\langle Z(\xi)(a+ib), a+ib\rangle \\ &\quad - \frac{i}{4}\langle Z(\xi)(a-ib), a-ib\rangle, \qquad a, b \in \mathbb{C}^N,\end{aligned}$$

we derive, from (2.5) and (2.6),

$$\begin{aligned}\langle Z(\xi)\,a, b\rangle &= \frac{1}{4}(e_1, D)^2 \ldots (e_n, D)^2 \,[U_{C'}(\xi; a+b) \\ &\quad + U_{C'}(\xi; a-b) + iU_{C'}(\xi; a+ib) - iU_{C'}(\xi; a-ib)] \\ &\quad + \frac{1}{4}\sum_{1\le j\le n} [A_{C'}^{(j)}(a+b) + A_{C'}^{(j)}(a-b) + iA_{C'}^{(j)}(a+ib) \\ &\qquad - iA_{C'}^{(j)}(a-ib)]\, D_j\delta(\xi).\end{aligned}$$

This implies the existence of the $N \times N$ matrix function $Z_{C'}(\xi)$, which is a continuous function of slow growth in $\mathbb{R}^n$ with support in the cone $C'^*$, and the existence of $N \times N$ matrices $Z_{C'}^{(j)}$, $j = 1, \ldots, n$, such that the representation (2.2) holds. From the lemma of Sec. 17.1 follow the uniqueness of the representation (2.2), the reality of the matrices $Z_{C'}(\xi)$ and $Z_{C'}^{(j)}$, $j = 1, \ldots, n$ [by virtue of the reality of the matrix $Z(\xi)$], and the equalities [by virtue of (2.5) and (2.7)]

$$U_{C'}(\xi; a) = \langle Z_{C'}(\xi)\,a, a\rangle, \qquad A_{C'}^{(j)}(a) = \langle Z_{C'}^{(j)} a, a\rangle,$$
$$j = 1, \ldots, n, \quad a \in \mathbb{C}^N.$$

From this and from (2.6) it follows that the matrices $Z_{C'}^{(j)}$ are symmetric and satisfy the condition (2.3):

$$\left\langle \sum_{1\leqslant j\leqslant n} y_j Z_{C'}^{(j)} a, a \right\rangle = \sum_{1\leqslant j\leqslant n} y_j \langle Z_{C'}^{(j)} a, a\rangle$$
$$= \sum_{1\leqslant j\leqslant n} y_i A_{C'}^{(j)}(a) \geqslant 0.$$

Thus, the spectral function $Z(\xi)$ satisfies conditions (b) as well.

It remains to prove the inequality (2.4). Let $\varphi \in \mathscr{S}^{\times N}$; set $\psi = F[\varphi] \in \mathscr{S}^{\times N}$. Taking into account the equalities

$$A_+(x) = F[Z], \qquad \operatorname{Re} A_+(x) = \lim_{y\to 0,\, y\in C} \operatorname{Re} A(x+iy) \text{ in } \mathscr{S}'$$

and using the properties of the Fourier transform (see Sec. 6.3 and Sec. 6.5), we have the following chain of equalities:

$$\operatorname{Re}\int \langle Z * \varphi, \varphi\rangle\, d\xi = \operatorname{Re} \sum_{1\leqslant k,\, j\leqslant N} (Z_{kj} * \varphi_j, \bar{\varphi}_k)\, d\xi$$
$$= \operatorname{Re} \sum_{1\leqslant k,\, j\leqslant N} (Z_{kj}(-\xi), \varphi_j * \varphi_k^*)$$
$$= \operatorname{Re} \sum_{1\leqslant k,\, j\leqslant N} (F[Z_{kj}], F^{-1}[(\varphi_j * \varphi_k^*)(-\xi)])$$
$$= \frac{1}{(2\pi)^n} \operatorname{Re} \sum_{1\leqslant k,\, j\leqslant N} (A_{+kj}, \psi_j \bar{\psi}_k)$$
$$= \frac{1}{2(2\pi)^n} \sum_{1\leqslant k,\, j\leqslant N} (A_{+kj} + \bar{A}_{+jk}, \psi_j \bar{\psi}_k)$$
$$= \frac{1}{2(2\pi)^n} \lim_{y\to 0,\, y\in C} \sum_{1\leqslant k,\, j\leqslant N} \int [A_{kj}(x+iy) + \bar{A}_{jk}(x+iy)]\, \psi_j(x)\, \bar{\psi}_k(x)\, dx$$
$$= \frac{1}{(2\pi)^n} \lim_{y\to 0,\, y\in C} \int \langle \operatorname{Re} A(x+iy)\, \psi(x), \psi(x)\rangle\, dx$$

which is greater than zero. The proof of the theorem is complete.

*Remark 1.* For $n = 1$ the theorem has been proved by König and Zemanian [1]; for $n \geqslant 2$ by Vladimirov [9].

*Remark 2.* In $\mathbb{R}^2$, any convex open cone $C$ is dihedral, that is, $C = [y\colon (e_1, y) > 0,\ (e_2, y) > 0]$, and for that reason we can take the cone $C$ itself for the cone $C'$ in the representation (2.2).

# 19 Linear Passive Systems

**19.1 Introduction** We consider a physical system obeying the following scheme. Suppose the original in-perturbation $u(x) = (u_1(x), \ldots, u_N(x))$ is acting on the system, as a result of which there arises an out-perturbation (response of the system) $f(x) = (f_1(x), \ldots, f_N(x))$. Here, by $x = (x_1, \ldots, x_n)$ are to be understood the temporal, spatial and other variables. Suppose the following conditions have been fulfilled:

(a) *Linearity*: if to the original perturbations $u_1$ and $u_2$ there correspond perturbations $f_1$ and $f_2$ then their linear combination $au_1 + \beta u_2$ is associated with the perturbation $\alpha f_1 + \beta f_2$.

(b) *Reality*: if the original perturbation $u$ is real, then the response perturbation $f$ is real.

(c) *Continuity*: if all components of the original perturbation $u(x)$ tend to 0 in $\mathscr{E}'$, then so do all components of the response perturbation $f(x)$ tend to 0 in $\mathscr{D}'$.

(d) *Translational invariance*: if a response perturbation $f(x)$ is associated with the original perturbation $u(x)$, then, for any translation $h \in \mathbb{R}^n$, to the original perturbation $u(x+h)$ there corresponds a response perturbation $f(x+h)$.

The conditions (a)-(d) are equivalent to the existence of a unique $N \times N$ matrix $Z(x) = (Z_{kj}(x))$, $Z_{kj} \in \mathscr{D}'(\mathbb{R}^n)$, which connects the original $u(x)$ perturbation and the response perturbation $f(x)$ via the formula (see Sec. 4.7)

$$Z * u = f. \tag{1.1}$$

Let us impose on the system (1.1) yet another requirement, the so-called condition of *passivity relative to the cone* $\Gamma$. Suppose $\Gamma$ is a closed, convex, solid cone in $\mathbb{R}^n$ (with vertex at 0).

(e) *Passivity relative to the cone* $\Gamma$: for any vector function $\varphi(x)$ in $\mathscr{D}^{\times N}$ the following inequality holds:

$$\operatorname{Re} \int_{-\Gamma} \langle Z * \varphi, \varphi \rangle \, dx \geqslant 0. \tag{1.2}$$

Note that the function $\langle Z * \varphi, \varphi \rangle \in \mathscr{D}$ (see Sec. 4.6), so that the integral in (1.2) always exists. Furthermore, because of the reality of the matrix $Z(x)$ the condition of passivity (1.2) is equivalent to the condition

$$\int_{-\Gamma} \langle Z * \varphi, \varphi \rangle \, dx \geqslant 0, \qquad \varphi \in \mathscr{D}_r^{\times N} \tag{1.2'}$$

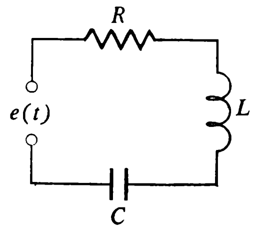
where $\mathscr{D}_r^{\times N}$ consists of real $N$ vectors with components in $\mathscr{D}$.

The inequality (1.2′) is of the energy type: it reflects the ability of a physical system to absorb energy, but not generate it. Here, causality relative to the cone $\Gamma$ is taken into account (see below, Sec. 19.2).

The convolution operator $Z*$ is termed a *passive operator relative to the cone* $\Gamma$, and the corresponding matrix function $\tilde{Z}(\zeta)$ — the Laplace transform of the matrix $Z(x)$—is called the *impedance* of the physical system.

To illustrate the proposed scheme, let us consider a one-dimensional passive system ($n = N = 1$): an elementary electric circuit consisting of a resistance $R$, a self-inductance $L$, a capacitance

Figure 41




$C$, and a source of electromotive force $e(t)$ that is switched on at time $t = 0$ (Fig. 41). Then, by the Kirchhoff law, the current $i(t)$ in the circuit satisfies the integro-differential equation

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(\tau)\,d\tau = e(t),$$

that is,

$$Z * i = e$$

where

$$Z(t) = L\delta'(t) + R\delta(t) + \frac{1}{C}\theta(t)$$

is the generalized "resistance" of the circuit. We now verify that the operator $Z*$ satisfies the condition of passivity (1.2′) relative to the cone $\Gamma = [0, \infty)$:

$$\begin{aligned}
&\int_{-\infty}^{0} (Z*\varphi)\,\varphi\,dt \\
&\quad = \int_{-\infty}^{0} \left[L\varphi'(t) + R\varphi(t) + \frac{1}{C}\int_0^t \varphi(\tau)\,d\tau\right]\varphi(t)\,dt \\
&\quad = \frac{L}{2}\varphi^2(0) + R\int_{-\infty}^{0}\varphi^2(t)\,dt + \frac{1}{2C}\left[\int_{-\infty}^{0}\varphi(t)\,dt\right]^2 \geqslant 0, \quad \varphi \in \mathscr{D}_r.
\end{aligned}$$

One-dimensional ($n = 1$) linear passive systems describe the relationship between currents and voltage in complex electric circuits. They also describe linear thermodynamic systems, the scattering of electromagnetic waves and elementary particles (see König and Meixner [1], Youla, Castriota and Carlin [1], Wu [1], Zemanian [1], Beltrami and Wohlers [1], Güttinger [1]). One-dimensional passive operators have been studied by many authors and the results of their investigations have been summarized in two monographs (1965-1966): Zemanian [1] and Beltrami and Wohlers [1]. This theory has been extended by Hackenbroch [1] and Zemanian [2] from the matrix case to the case of operators in Hilbert space.

Multidimensional ($n \geqslant 2$) linear passive systems are frequently encountered in mathematical physics: they describe physical systems with account taken of their spatio-temporal dynamics (some instances of such systems are given below in Sec. 19.7). The theory of multidimensional linear passive systems has been elaborated by Vladimirov [9, 11] on the basis of the theory of positive real matrix functions (see Sec. 18).

### 19.2 Corollaries to the condition of passivity

(a) The condition of passivity (1.2) is fulfilled in the strong form:

$$\operatorname{Re} \int\limits_{-\Gamma+x_0} \langle Z * \varphi, \varphi \rangle \, dx \geqslant 0, \qquad \varphi \in \mathscr{D}^{\times N}, \quad x_0 \in \mathbb{R}^n. \tag{2.1}$$

Indeed, if $\varphi \in \mathscr{D}^{\times N}$, for every $x_0 \in \mathbb{R}^n$ the vector function $\varphi_{x_0}(x) = \varphi(x + x_0) \in \mathscr{D}^{\times N}$ and therefore, by virtue of the property of translational invariance of a convolution (see Sec. 4.2 (c)), from the inequality (1.2) there follows the inequality (2.1):

$$0 \leqslant \operatorname{Re} \int\limits_{-\Gamma} \langle Z * \varphi_{x_0}, \varphi_{x_0} \rangle \, dx = \operatorname{Re} \int\limits_{-\Gamma} \langle (Z * \varphi)(x + x_0), \varphi(x + x_0) \rangle \, dx$$

$$= \operatorname{Re} \int\limits_{-\Gamma+x_0} \langle Z * \varphi, \varphi \rangle \, dx', \qquad x' = x + x_0.$$

(b) *Dissipation*:

$$\operatorname{Re} \int \langle Z * \varphi, \varphi \rangle \, dx \geqslant 0, \qquad \varphi \in \mathscr{D}^{\times N}. \tag{2.2}$$

Indeed, putting $x_0 = \lambda e$, $e \in \operatorname{int} \Gamma$, in (2.1) and passing to the limit as $\lambda \to +\infty$ (so that $-\Gamma + \lambda e \to \mathbb{R}^n$, Fig. 42), we obtain from (2.1) the inequality (2.2).

(c) *Causality with respect to the cone* $\Gamma$:

$$\operatorname{supp} Z(x) \subset \Gamma. \tag{2.3}$$

True enough, let $\varphi$ and $\psi \in \mathscr{D}_r^{\times N}$ and let $\lambda$ be a real number.

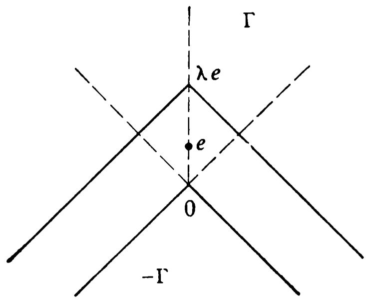



Figure 42

Substituting $\varphi + \lambda\psi$ for $\varphi$ in the inequality (1.2′), we get the inequality

$$\int_{-\Gamma} \langle Z * \varphi, \varphi\rangle \, dx + \lambda \int_{-\Gamma} [\langle Z * \psi, \varphi\rangle + \langle Z * \varphi, \psi\rangle] \, dx + \lambda^2 \int_{-\Gamma} \langle Z * \psi, \psi\rangle \, dx \geqslant 0,$$

which holds true for all real $\lambda$. We therefore have the inequality

$$\left[\int_{-\Gamma} \langle Z * \varphi, \psi\rangle \, dx + \int_{-\Gamma} \langle Z * \psi, \varphi\rangle \, dx\right]^2 \leqslant 4 \int_{-\Gamma} \langle Z * \varphi, \varphi\rangle \, dx \int_{-\Gamma} \langle Z * \psi, \psi\rangle \, dx. \tag{2.4}$$

Suppose $\operatorname{supp} \varphi \subset \mathbb{R}^n \setminus (-\Gamma)$. Then it follows from the inequality (2.4) that

$$\int_{-\Gamma} \langle Z * \varphi, \psi\rangle \, dx = 0$$

for all $\psi \in \mathscr{D}_r^{\times N}$. By the Du Bois Reymond lemma (see Sec. 1.6), we conclude from this that $Z * \varphi = 0$, $x \in -\Gamma$, and therefore (see Sec. 4.6)

$$(Z_{kj} * \varphi_0)(x) = (Z_{kj}(x'), \varphi_0(x - x')) = 0, \qquad x \in -\Gamma$$

for all $\varphi_0 \in \mathscr{D}_r(\mathbb{R}^n \setminus (-\Gamma))$. Putting $x = 0$ in the last equality, we obtain $(Z_{kj}(-x'), \varphi_0(x')) = 0$ so that $Z_{kj}(-x) = 0$, $x \in \mathbb{R}^n \setminus (-\Gamma)$, and therefore $\operatorname{supp} Z_{kj} \subset \Gamma$ (see Sec. 1.5). The inclusion (2.3) is proved.

(d) *Positive definiteness*:

$$\langle Za + Z^*a, a \rangle \geqslant 0, \qquad a \in \mathbb{C}^N \tag{2.5}$$

or, in an equivalent form,

$$\operatorname{Re}(\langle Za, a \rangle, \varphi * \varphi^*) \geqslant 0, \qquad a \in \mathbb{C}^N, \quad \varphi \in \mathscr{D}. \tag{2.5'}$$

It follows from the inequality (2.2) for $\varphi = a\varphi_0(-x)$, where $a \in \mathbb{C}^N$ and $\varphi_0 \in \mathscr{D}$, that

$$\begin{aligned} 0 &\leqslant \operatorname{Re} \int \langle Z * a\varphi_0(-x), a\varphi_0(-x) \rangle \, dx \\ &= \operatorname{Re} \int [\langle Za, a \rangle * \varphi_0(-x)] \, \varphi_0^*(x) \, dx = \operatorname{Re}(\langle Za, a \rangle, \varphi_0 * \varphi_0^*) \\ &= \frac{1}{2}(\langle Z(x)a, a \rangle, \varphi_0 * \varphi_0^*) + \frac{1}{2}(\overline{\langle Z(x)a, a \rangle}, \overline{\varphi_0 * \varphi_0^*}) \\ &= \frac{1}{2}(\langle Z(x)a + Z^T(-x)a, a \rangle, \varphi_0 * \varphi_0^*), \end{aligned}$$

which proves the inequalities (2.5) and (2.5′) (see Sec. 8.1). Here we made use of the property of the convolution (6.4) of Sec. 4.

In what follows we assume that the cone $\Gamma$ is acute.

(e) *Restriction to growth*: $Z \in (\mathscr{S}')^{\times N^2}$.

Indeed, by the Bochner-Schwartz theorem (see Sec. 8.2), the generalized function $\langle Z(x)a + Z^*(x)a, a \rangle$ belongs to $\mathscr{S}'$ for all $a \in \mathbb{C}^N$, and from this it follows that the generalized function $\langle Z(x)a + Z^*(x)a, b \rangle \in \mathscr{S}'$ for all $a$ and $b$ in $\mathbb{C}^N$, so that

$$f_{kj}(x) = Z_{kj}(x) + Z_{jk}(-x) \in \mathscr{S}', \qquad 1 \leqslant k, j \leqslant N.$$

From the causality condition (2.3), $\operatorname{supp} Z_{kj} \subset \Gamma$, it follows that $\operatorname{supp} f_{kj} \subset -\Gamma \cup \Gamma$. Suppose $\eta \in C^\infty$, $\eta(t) = 1$, $t > 1$, $\eta(t) = 0$, $t < 0$ and $e \in \operatorname{int} \Gamma^*$. Then the function $\eta((e, x)) \in \theta_M$ and for that reason $\eta((e, x)) f_{kj} \in \mathscr{S}'$ (see Sec. 5.3). Furthermore, the support of the generalized function

$$s_{kj}(x) = Z_{kj} - \eta((e, x)) f_{kj}(x)$$

is a compact, by virtue of Lemma 1 of Sec. 4.4 (see Fig. 22; the cone $\Gamma$ is assumed to be acute!), so that $s_{kj} \in \mathscr{S}'$ (see Sec. 5.3). Conclusion: $Z_{kj} \in \mathscr{S}'$.

(f) The condition of passivity (1.2) holds in the strong form:

$$\operatorname{Re} \int_{-\Gamma} \langle Z * \varphi, \varphi \rangle \, dx \geqslant 0, \qquad \varphi \in \mathscr{S}^{\times N}. \tag{2.6}$$

Indeed, fix $\varphi$ taken from $\mathscr{S}^{\times N}$ and let $\varphi_\nu \in \mathscr{D}^{\times N}$, $\varphi_\nu \to \varphi$, $\nu \to \infty$ in $\mathscr{S}^{\times N}$ (see Sec. 5.1). Then by (1.2)

$$\operatorname{Re} \int_{-\Gamma} \langle Z * \varphi_\nu, \varphi_\nu \rangle \, dx \geqslant 0, \qquad \nu = 1, 2, \ldots \tag{2.7}$$

By (e), $Z_{kj} \in \mathscr{S}'$, $1 \leqslant k$, $j \leqslant N$, and therefore $Z_{kj}$ is of finite order (see Sec. 5.2). Denoting by $m$ the largest of the orders and using the estimate (6.4) of Sec. 5, we obtain, for $k = 1, \ldots, N$,

$$(\alpha) \quad | (Z * \varphi_\nu)_k (x) | \leqslant C (1 + | x |^2)^{m/2} \max_{1 \leqslant j \leqslant N} \| \varphi_{\nu j} \|_m, \quad \nu = 1, 2, \ldots$$

$$(\beta) \quad (Z * \varphi_\nu)_k (x) \overset{|x| \leqslant R}{\Longrightarrow} (Z * \varphi)_k (x), \quad \nu \to \infty$$

for arbitrary $R > 0$, from which we conclude that passage to the limit is possible as $\nu \to \infty$ under the integral sign in the inequality (2.7); we thus obtain the inequality (2.6).

(g) *The existence of impedance*:

$$\tilde{Z} (\zeta) = L [Z] = F [Z (x) e^{-(q,x)}] (p), \qquad \zeta = p + iq$$

is a holomorphic matrix function in the region $T^C = \mathbb{R}^n + iC$, where $C = \operatorname{int} \Gamma^*$.

This follows from the properties (c) and (e): $Z_{kj} \in \mathscr{S}' (\Gamma)$ (see Sec. 9.1).

(h) *The condition of reality of impedance*:

$$\tilde{Z} (\zeta) = \overline{\tilde{Z}} (- \bar{\zeta}), \qquad \zeta \in T^C. \tag{2.8}$$

This follows from the reality of the matrix $Z (x)$.

(i) *The property of positivity of impedance*:

$$\operatorname{Re} \tilde{Z} (\zeta) \geqslant 0, \qquad \zeta \in T^C. \tag{2.9}$$

Indeed, let the function $\eta_\varepsilon(x)$ be such that $\eta_\varepsilon \in C^\infty$; $\eta_\varepsilon(x) = 1$, $x \in \Gamma^{\varepsilon/2}$; $\eta_\varepsilon(x) = 0$, $x \bar{\in} \Gamma^\varepsilon$; $|D^\alpha \eta_\varepsilon(x)| \leqslant C_{\alpha\varepsilon}$. Then, for all $\zeta \in T^C$ (see Sec. 9.1),

$$\varphi(\zeta; x) = \eta_\varepsilon(-x)\, e^{-i(\zeta, x)} \in \mathscr{S},$$

and for all $a \in \mathbb{C}^N$ the vector function $a\varphi \in \mathscr{S}^{\times N}$; therefore, using the formula (6.4) of Sec. 4, we have

$$\begin{aligned}\langle Z * a\varphi, a\varphi\rangle &= [\langle Za, a\rangle * \varphi](x)\, \overline{\varphi}(\zeta; x)\\ &= (\langle Z(x')\, a, a\rangle, \varphi(\zeta; x - x'))\, \overline{\varphi}(\zeta; x)\\ &= e^{2(q, x)}\, \eta_\varepsilon(-x)\, (\langle Z(x')\, a, a\rangle, \eta_\varepsilon(x' - x)\, e^{i(\zeta, x')}).\end{aligned} \quad (2.10)$$

But $\eta_\varepsilon(x' - x) = 1$ for $x' \in \Gamma^{\varepsilon/2}$ and $x \in -\Gamma$ because by Lemma 2 of Sec. 4.4 $x' - x \in \Gamma + U_{\varepsilon/2} + \Gamma = \Gamma + U_{\varepsilon/2} = \Gamma^{\varepsilon/2}$. From this, taking into account that $\operatorname{supp} Z(x') \subset \Gamma$ and using (10.2) of Sec. 1, we continue the chain of equalities (2.10) for $x \in -\Gamma$:

$$\langle Z * (a\varphi), a\varphi\rangle = e^{2(q, x)}\, (\langle Z(x')\, a, a\rangle, \eta_\varepsilon(x')\, e^{i(\zeta, x')}). \quad (2.11)$$

Now let us take advantage of (1.4) of Sec. 9 for the Laplace transform and let us integrate (2.11) with respect to the cone $\Gamma$. As a result, using property (f), we obtain the inequality

$$0 \leqslant \operatorname{Re} \int_{-\Gamma} \langle Z * (a\varphi), a\varphi\rangle\, dx = \operatorname{Re} \langle \widetilde{Z}(\zeta)\, a, a\rangle \int_{-\Gamma} e^{2(q, x)} dx. \quad (2.12)$$

Noting that for all $q \in C$ the last integral exists and is positive (see Sec. 10.2), we conclude from (2.12) that

$$\operatorname{Re} \langle \widetilde{Z}(\zeta)\, a, a\rangle = \frac{1}{2} \langle \widetilde{Z}(\zeta)\, a + \widetilde{Z}^+(\zeta)\, a, a\rangle \geqslant 0,$$

which is equivalent to (2.9).

From the properties (g), (h), and (i) it follows that the impedance $\widetilde{Z}(\zeta)$ belongs to the class of positive real matrix functions in $T^C$, the description of which was given in Sec. 18.2 (the cone $\Gamma$ is assumed to be acute).

*Remark.* It is readily seen, if we slightly modify the foregoing reasoning, that the corollaries (c), (d), (e), (g), (h), (i) remain true

even when the weak condition of passivity is carried out relative to the cone $\Gamma$:

$$\operatorname{Re} \int\limits_{-\Gamma} [\langle Z(x)\, a, a\rangle * \varphi]\, \overline{\varphi}\, dx \geqslant 0, \qquad \varphi \in \mathscr{D}, \quad a \in \mathbb{C}^N. \quad (2.13)$$

## 19.3 The necessary and sufficient conditions for passivity

Theorem I *For a matrix $Z(x)$ to define a passive operator with respect to an acute cone $\Gamma$, it is necessary and sufficient that its impedance $\widetilde{Z}(\zeta)$ be a positive real matrix function in the domain $T^C$, where $C = \operatorname{int} \Gamma^*$.*

Corollary *If a system is passive relative to an acute cone $\Gamma$, then it is also passive relative to any acute cone containing $\Gamma$.*

*Remark.* For $n = 1$, Theorem I was proved by Zemanian [3], and for $n \geqslant 2$ it was proved by Vladimirov [9].

First let us prove the following lemma.

Lemma *Suppose an $N \times N$ matrix $Z(x)$ has the following properties:*

(a) *it defines a passive operator relative to a certain cone $\Gamma_0$ containing the cone $C^*$; the boundary of $\Gamma_0$ is assumed to be piecewise smooth;*

(b) *for any n-hedral cone $C' = [q : (e_1, q) > 0, \ldots, (e_n, q) > 0]$ contained in a convex acute cone $C$, it is given in the form*

$$Z(x) = (e_1, D)^2 \ldots (e_n, D)^2 Z_{C'}(x) + \sum_{1 \leqslant j \leqslant n} Z_{C'}^{(j)} D_j \delta(x), \quad (3.1)$$

*where the matrix function $Z_{C'}(x)$ is continuous, real, and of slow growth in $\mathbb{R}^n$ with support in the cone $C'^*$; the matrices $Z_{C'}^{(j)}$, $j = 1, \ldots, n$, are real, symmetric and such that*

$$\sum_{1 \leqslant j \leqslant n} q_j Z_{C'}^{(j)} \geqslant 0, \qquad q \in \overline{C}'. \quad (3.2)$$

*Then the matrix $Z(x)$ defines a passive operator relative to the cones $\Gamma_e = [x : (e, x) \geqslant 0,\ x \in \Gamma_0]$, where $e$ is any unit vector taken from the cone $\overline{C}$.*

*Proof.* From conditions (b) it follows that the matrix $Z(x)$ is real and of slow growth with support in the cone $C^*$.

The lemma is nontrivial if the cone

$$\Gamma_e' = \Gamma_0 \setminus \Gamma_e = [x : (e, x) < 0,\ x \in \Gamma_0]$$

is a solid cone. Clearly, $\Gamma_0 = \Gamma_e \cup \Gamma_e'$ (Fig. 43).

Let $\eta \in C^{\infty}$; $0 \leqslant \eta(t) \leqslant 1$; $\eta(t) = 1$, $t < \frac{1}{2}$; $\eta(t) = 0$, $t > 1$. We set

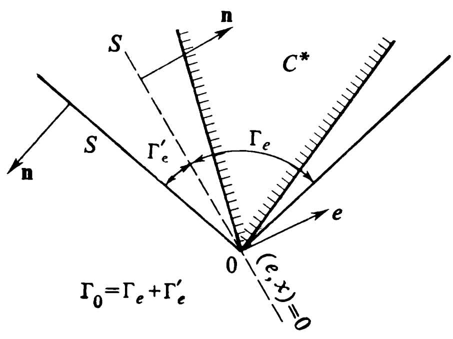
$$\eta_\varepsilon(x) = \eta\left[\frac{(e, x)}{\varepsilon}\right].$$

Then for all $\varphi \in \mathscr{D}_r^{\times N}$ we have

$$\begin{aligned}\int_{-\Gamma_0} \langle Z * \varphi, \varphi\rangle\, dx = & \int_{-\Gamma_0} \langle Z * (\varphi\eta_\varepsilon), \varphi\eta_\varepsilon\rangle\, dx \\ & + \int_{-\Gamma_e} [\langle Z * \varphi, \varphi\rangle - \langle Z * (\varphi\eta_\varepsilon), \varphi\eta_\varepsilon\rangle]\, dx \\ & - \int_{-\Gamma_0'} \langle Z * (\varphi\eta_\varepsilon), \varphi\eta_\varepsilon\rangle\, dx. \end{aligned} \tag{3.3}$$

The first summand in the right-hand member of (3.3) is non-negative for all $\varepsilon > 0$ by virtue of the condition (a). Furthermore,

Figure 43




since supp $Z(x') \subset C^*$ and $\eta_\varepsilon(x - x') = 1$, $x \in -\Gamma_e$, $x' \in (C^*)^{\varepsilon/2}$ [because $(e, x - x') \leqslant -(e, x') = -(e, x_1) - (e, x_2) \leqslant |(e, x_2)| \leqslant \varepsilon/2$, where $x' = x_1 + x_2$, $x_1 \in C^*$, $|x_2| < \varepsilon/2$], it follows that

$$\begin{aligned}(Z * (\varphi\eta_\varepsilon))_k(x) &= \sum_{1 \leqslant j \leqslant N} (Z_{kj}(x'), \varphi_j(x - x')\eta_\varepsilon(x - x')) \\ &= \sum_{1 \leqslant j \leqslant N} (Z_{kj}(x'), \varphi_j(x - x')) \\ &= (Z * \varphi)_k(x), \qquad x \in -\Gamma_e,\end{aligned}$$

and, hence, the second summand on the right of (3.3) is zero. Thus, for all $\varepsilon > 0$ we have the inequality

$$\int_{-\Gamma_e} \langle Z * \varphi, \varphi \rangle \, dx \geqslant - \int_{-\Gamma'_e} \langle Z * (\varphi\eta_\varepsilon), \varphi\eta_\varepsilon \rangle \, dx, \qquad \varphi \in \mathscr{D}_r^{\times N}. \tag{3.4}$$

We choose linearly independent vectors $e'_1 = e, e'_2, \ldots, e_n$ from the cone $\overline{C}$ and let $\{e_k\}$ be a system of vectors that is biorthogonal to the system $\{e'_j\}$, $(e_k, e'_j) = \delta_{kj}$. We set $C' = [q : (e_1, q) > > 0, \ldots, (e_n, q) > 0]$. Then $C' \subset C$, $k \in \overline{C}'$ and $C'^* = = [x : (e, x) \geqslant 0, \ldots, (e'_n, x) \geqslant 0]$. For the cone $C'$, the representation (3.1) holds true.

Taking into account that representation, we transform the right-hand side of the inequality (3.4) to the following form:

$$\begin{aligned}
&- \int_{-\Gamma'_e} \langle Z * (\varphi\eta_\varepsilon), \varphi\eta_\varepsilon \rangle \, dx \\
&\qquad = - \int_{-\Gamma'_e} (e_1, D)^2 \ldots (e_n, D)^2 \langle Z_{C'} * (\varphi\eta_\varepsilon), \varphi\eta_\varepsilon \rangle \, dx \\
&\qquad - \sum_{1 \leqslant j \leqslant n} \int_{-\Gamma'_e} \left\langle Z_{C'}^{(j)} \frac{\partial}{\partial x_j} (\varphi\eta_\varepsilon), \varphi\eta_\varepsilon \right\rangle dx \\
&\qquad = I_1(\varepsilon) + I_2(\varepsilon).
\end{aligned} \tag{3.5}$$

For the quantity $I_1(\varepsilon)$ we have the estimate

$$\begin{aligned}
|I_1(\varepsilon)| \leqslant c_1 \sum_{1 \leqslant k,\, j \leqslant N} \int_{\substack{|x| < R \\ 0 < (e,\, x) < \varepsilon}} \Bigg| &(e_1, D)^2 \ldots (e_n, D)^2 \\
&\times \int_{C'^*} Z_{C',\, kj}(x') \, \varphi_j(x - x') \, \eta\left[\frac{(e, x - x')}{\varepsilon}\right] dx' \Bigg| \, dx,
\end{aligned} \tag{3.6}$$

where $c_1 = \max\limits_{1 \leqslant k \leqslant N} |\varphi_k(x)|$ and the number $R > 0$ is such that $\operatorname{supp} \varphi \subset U_R$. In the inner and outer integrals in (3.6) we make a change of the variables of integration via the following formulas, respectively:

$$x \to Bx = y = [y_1 = (e, x), \ldots, y_n = (e'_n, x)], \qquad x' \to Bx' = y'.$$

Then the cone $C'^*$ goes into the cone $\overline{\mathbb{R}}_+^n = [y' : y'_1 \geqslant 0, \ldots \ldots, y'_n \geqslant 0]$, the ball $U_R$ goes into a bounded region contained

in some ball $U_{R_1}$, the strip $0 < (x, e) < \varepsilon$ goes into the strip $0 < y_1 < \varepsilon$, and the derivative $(e_k, D)$ into the derivative $D_k$ (see Sec. 18.2).

Setting

$$Z_{C', kj}(B^{-1}y') = v_{kj}(y'), \qquad \varphi_1(B^{-1}y) = \psi_1(y),$$
$$D_2^2 \ldots D_n^2 \psi_j(y) = u_j(y),$$

we obtain from (3.6) the estimate

$$|I_1(\varepsilon)| \leqslant \frac{c_1}{(\det B)^2} \sum_{1 \leqslant k,\, j \leqslant N} \int_{\substack{|y|<R_1 \\ 0<y_1<\varepsilon}} \Bigg| D_1^2 \int_{\substack{\mathbb{R}_+^n \\ |y-y'|<R_1}} v_{kj}(y')\, u_j(y-y') \times \eta\left(\frac{y_1-y_1'}{\varepsilon}\right) dy' \Bigg| dy$$

$$= \frac{c_1}{(\det B)^2} \sum_{1 \leqslant k,\, j \leqslant N} \int_{\substack{|y|<R_1 \\ 0<y_1<\varepsilon}} \int_{\substack{|y'|<2R_1 \\ y' \in \mathbb{R}_+^n}} |v_{kj}(y')| \times \Bigg| \eta\left(\frac{y_1-y_1'}{\varepsilon}\right) \frac{\partial^2 u_j(y-y')}{\partial y_1^2} + \frac{2}{\varepsilon}\eta'\left(\frac{y_1-y_1'}{\varepsilon}\right) \frac{\partial u_j(y-y')}{\partial y_1} + \frac{1}{\varepsilon^2}\eta''\left(\frac{y_1-y_1'}{\varepsilon}\right) u_j(y-y') \Bigg| dy'\, dy.$$

We set

$$\chi(y_1') = \sum_{1 \leqslant k,\, j \leqslant N} \int_{\substack{y_2'>0, \ldots, y_n'>0 \\ y_2'^2+\ldots+y_n'^2<4R_1^2}} |v_{kj}(y_1', y_2', \ldots, y_n')|\, dy_2' \ldots dy_n'.$$

Since the functions $v_{kj}$ are continuous in $\mathbb{R}^n$ with supports in $\overline{\mathbb{R}}_+^n$, it follows that the function $\chi(y_1')$ is continuous in $\mathbb{R}^1$ and is zero for $y_1' < 0$. Using this notation, we continue our estimate:

$$|I_1(\varepsilon)| \leqslant c_2 \int_0^{\varepsilon} \left[ \int_0^{2R_1} \chi(y_1')\, dy_1' \right] dy_1 + \left(\frac{c_3}{\varepsilon} + \frac{c_4}{\varepsilon^2}\right) \int_0^{\varepsilon} \left[ \int_0^{y_1+\varepsilon/2} \chi(y_1')\, dy_1' \right] dy_1$$

$$\leqslant c_5\varepsilon + \frac{c_4}{\varepsilon} \int_0^{3\varepsilon/2} \chi(y_1')\, dy_1',$$

whence it follows that

$$\lim_{\varepsilon\to+0} I_1(\varepsilon)=0. \tag{3.7}$$

We now consider the quantity $I_2(\varepsilon)$. Taking into account that the matrices $Z_{C'}^{(j)}$, $j=1, \ldots, n$ are real and symmetric, we have

$$\begin{aligned} I_2(\varepsilon) &= -\sum_{1\leqslant j\leqslant n}\int_{-\Gamma'_e}\left\langle Z_{C'}^{(j)}\frac{\partial}{\partial x_j}(\varphi\eta_\varepsilon), \varphi\eta_\varepsilon\right\rangle dx \\ &= \sum_{1\leqslant j\leqslant n}\sum_{1\leqslant k,\, s\leqslant N} Z_{C',\,ks}^{(j)}\int_{-\Gamma'_e}\frac{\partial}{\partial x_j}(\varphi_s\eta_\varepsilon)\,\varphi_k\eta_\varepsilon\,dx \\ &= -\frac{1}{2}\sum_{1\leqslant j\leqslant n}\sum_{1\leqslant k,\, s\leqslant N} Z_{C'\,ks}^{(j)}\int_{-\Gamma'_e}\frac{\partial}{\partial x_j}(\varphi_s\varphi_k\eta_\varepsilon^2)\,dx \\ &= -\frac{1}{2}\int_{-\Gamma'_e}\sum_{1\leqslant j\leqslant n}\frac{\partial}{\partial x_j}[\langle Z_{C'}^{(j)}\varphi, \varphi\rangle\,\eta_\varepsilon^2]\,dx. \end{aligned} \tag{3.8}$$

The cone $\Gamma'_e$ has a piecewise smooth boundary, which we denote by $S$; let n be an outer normal to $S$ (see Fig. 43). Applying the Gauss-Ostrogradsky formula to the integral in (3.8), we obtain

$$I_2(\varepsilon) = -\frac{1}{2}\int_S\sum_{1\leqslant j\leqslant n}\langle Z_{C'}^{(j)}\varphi, \varphi\rangle\,\eta_\varepsilon^2\cos(\widehat{\mathbf{n}x_j})\,dS.$$

Passing to the limit here as $\varepsilon\to+0$ and noting that

$$\eta_\varepsilon(x)=\eta\left[\frac{(e, x)}{\varepsilon}\right]\to\theta[-(e, x)], \qquad 0\leqslant\eta_\varepsilon(x)\leqslant 1,$$

we obtain

$$\lim_{\varepsilon\to+0} I_2(\varepsilon) = -\frac{1}{2}\int_{S\cap[(e,\,x)\leqslant 0]}\sum_{1\leqslant j\leqslant n}\langle Z_{C'}^{(j)}\varphi, \varphi\rangle\cos(\widehat{\mathbf{n}x_j})\,dS. \tag{3.9}$$

But $(e, x)<0$ in interior points of the cone $\Gamma'_e$ (see Fig. 43), and so $(e, x)\leqslant 0$ on $\partial\Gamma'_e$ and then $(e, x)\geqslant 0$ on $S$. For this reason, actually only that part of the boundary $S$ is left in (3.9), where $(e, x)=0$ [and there, $e=-\mathbf{n}=(-\cos(\widehat{\mathbf{n}x_1}), \ldots, -\cos(\widehat{\mathbf{n}x_n}))$]

(see Fig. 43), so that (3.9) becomes

$$\lim_{\varepsilon\to+0} I_2(\varepsilon) = \frac{1}{2} \int\limits_{S\cap[(e,\, x)=0]} \left\langle \sum_{1\leqslant j\leqslant n} e_j Z_{C'}^{(j)} \varphi, \varphi \right\rangle dS.$$

Now the last quantity is nonnegative by virtue of the condition (3.2). From this and also from (3.7), (3.5) and (3.4) follows the condition for passivity relative to the cone $\Gamma_e$. The proof of the lemma is complete.

*Proof of Theorem I.* Necessity was proved in Sec. 19.2. We will prove sufficiency. Suppose the matrix function $\widetilde{Z}\,(\zeta)$ is positive real in $T^C$. Then by the theorem of Sec. 18.2 it is the Laplace transform of the matrix $Z\,(x)$ that satisfies the conditions of the lemma for $\Gamma_0 = \mathbb{R}^n$. Therefore the matrix $Z\,(x)$ defines a passive operator relative to the half-plane $\Gamma_1 = [x : (e_1,\, x) \geqslant 0]$, where $e_1$ is any unit vector in $\overline{C}$. Again applying the lemma to the cone $\Gamma_1$ and to any vector $e_2 \in \overline{C}$, $|\, e_2 \,| = 1$, we obtain the passivity of $Z\,(x)$ relative to the cone $\Gamma_2 = [x : (e_1,\, x) \geqslant 0,\ \ (e_2,\, x) \geqslant 0]$ and so forth. By means of an $m$-fold repetition of that process we obtain that the matrix $Z\,(x)$ defines a passive operator relative to the cone $\Gamma_m = [x : (e_1,\, x) \geqslant 0, \ldots, (e_m,\, x) \geqslant 0]$,

$$\int\limits_{-\Gamma_m} \langle Z * \varphi, \varphi\rangle \, dx \geqslant 0, \qquad \varphi \in \mathcal{D}_r^{\times N}. \tag{3.10}$$

But the convex cone $C^* = [x : (x,\, q) \geqslant 0,\ q \in \overline{C}]$ may be approximated from above by arbitrarily close $m$-hedral cones $\Gamma_m$ as $m \to \infty$. Therefore, passing to the limit as $\Gamma_m \to C^*$ under the condition of passivity (3.10), we obtain the condition for passivity for the cone $C^* = (\operatorname{int} \Gamma^*)^* = \Gamma$, which is what we set out to prove.

Combining Theorem I, the theorem of Sec. 18.2, and the remark of Sec. 19.2, we obtain

Theorem II *The following conditions are equivalent:*

(a) *The matrix $Z\,(x)$ defines a passive operator relative to an acute cone $\Gamma$.*

(b) *The matrix $Z\,(x)$ satisfies the weak condition of passivity (2.13) relative to the cone $\Gamma$.*

(c) *The matrix $Z\,(x)$ satisfies the condition (2.5) and the conditions* (b) *of the lemma.*

(d) *The matrix $Z\,(x)$ satisfies the condition of dissipation (2.2) and the conditions (b) of the lemma.*

**19.4 Multidimensional dispersion relations** The results obtained in Sec. 19.3 permit deriving (multidimensional) dispersion relations (see Sec. 10.6) that connect the real and imaginary parts of the matrix $\widetilde{Z}(p)$—the boundary value of the impedance $\widetilde{Z}(\zeta)$. For the sake of simplicity of exposition, we confine ourselves to the case of the cone $C = \mathbb{R}^n_+$.

Let us first prove the following lemma.

Lemma *The general solution of the matrix equation*

$$D_1^2 \ldots D_n^2 Z(x) = 0 \tag{4.1}$$

*in the class of real continuous $*$-Hermitian matrix functions in $\mathbb{R}^n$ with support in $-\overline{\mathbb{R}}^n_+ \cup \overline{\mathbb{R}}^n_+$ is given by the formula*

$$Z(x) = [\mathscr{E}_n(x) - \mathscr{E}_n(-x)] Z_0, \tag{4.2}$$

*where $Z_0$ is an arbitrary real skew-symmetric matrix.*

*Proof.* By Lemma 6 of Sec. 17.1 we have

$$Z_{kj}(x) = Z_{0,\,kj} [\mathscr{E}_n(x) - \mathscr{E}_n(-x)], \qquad 1 \leqslant k,\ j \leqslant N,$$

where $Z_{0,\,kj}$ are arbitrary real numbers. From this and from the conditions $Z_{jk}(x) = Z_{kj}(-x)$ it follows that $Z_{0,\,kj} = -Z_{0,\,jk}$, that is, $Z_0 = -Z_0^T$. The representation (4.2) is proved. The lemma is proved.

We denote by $N(-\overline{\mathbb{R}}^n_+ \cup \overline{\mathbb{R}}^n_+)$ the class of $+$-Hermitian matrices that are Fourier transforms of real continuous matrix functions of slow growth in $\mathbb{R}^n$ with support in $-\overline{\mathbb{R}}^n_+ \cup \overline{\mathbb{R}}^n_+$.

For a matrix of the class $N(-\overline{\mathbb{R}}^n_+ \cup \overline{\mathbb{R}}^n_+)$, all matrix elements belong to the space of generalized fuctions $\mathscr{D}'_{\mathscr{L}^2}$ (see Sec. 10.1).

From the lemma just proved it follows that the general solution of the matrix equation

$$p_1^2 \ldots p_n^2 M(p) = 0$$

in the class $N(-\overline{\mathbb{R}}^n_+ \cup \overline{\mathbb{R}}^n_+)$ is given by the formula

$$M(p) = -iZ^{(0)} D_1 \ldots D_n \operatorname{Im}[i^n \mathscr{K}_n(p)], \tag{4.3}$$

where $Z^{(0)}$ is an arbitrary real skew-symmetric matrix.

Indeed, passing to Fourier transforms in (4.2) and using the definition of the kernel $\mathscr{K}_n(p)$ (see Sec. 10.2), we have

$$\begin{aligned} M(p) &= Z_0\{F[\mathscr{E}_n] - \bar{F}[\mathscr{E}_n]\} = 2iZ_0 \operatorname{Im} F[\mathscr{E}_n](p) \\ &= 2iZ_0 \operatorname{Im} F[\theta_n(x) x_1 \ldots x_n] \\ &= iZ^{(0)} D_1 \ldots D_n \operatorname{Im}[i^n \mathscr{K}_n(p)], \end{aligned} \tag{4.4}$$

where $Z^{(0)} = 2(-1)^n Z_0$.

**Theorem** *In order that the matrix $Z(x)$ should define a passive operator relative to the cone $\overline{\mathbb{R}}^n_+$, it is necessary and sufficient that its Fourier transform $\tilde{Z}(p)$ satisfy the dissipative relation*

$$\operatorname{Im} \tilde{Z}(p) = \frac{2}{(2\pi)^n} p_1^2 \ldots p_n^2 (M * \operatorname{Im} \mathscr{K}_n) + iZ^{(0)} - \sum_{1 \leqslant j \leqslant n} Z^{(j)} p_j, \tag{4.5}$$

*where the matrix $M(p)$ is a solution in the class $N(-\overline{\mathbb{R}}^n_+ \cup \overline{\mathbb{R}}^n_+)$ of the equation*

$$p_1^2 \ldots p_n^2 M(p) = \operatorname{Re} \tilde{Z}(p). \tag{4.6}$$

*Here, the matrix $\operatorname{Re} \tilde{Z}(p)$ is such that for all $a \in \mathbb{C}^N$ the generalized function $\langle \operatorname{Re} \tilde{Z}(p) a, a \rangle$ is a nonnegative measure of slow growth in $\mathbb{R}^n$; the matrix $Z^{(0)}$ is real, skew-symmetric, and the matrices $Z^{(j)}$, $j = 1, \ldots, n$, are real and positive.*

*In the dispersion relation (4.5), matrices $[M(p), Z^{(0)}, Z^{(1)}, \ldots, Z^{(n)}]$ are unique up to additive terms of the form*

$$[iAD_1 \ldots D_n \operatorname{Im}[i_n \mathscr{K}_n(p)], A, 0, \ldots, 0], \tag{4.7}$$

*where $A$ is an arbitrary constant real skew-symmetric matrix.*

*Remark 1.* For $n = 1$ the theorem was proved by Beltrami and Wohlers [1]; for $n \geqslant 2$, it was proved by Vladimirov [9].

*Remark 2.* The actual growth of the measure $\langle \operatorname{Re} \tilde{Z}(p) a, a \rangle$ is such that the measure

$$\frac{\langle \operatorname{Re} Z(p) a, a \rangle}{(1 + p_1^2) \ldots (p + p_n^2)}$$

is finite on $\mathbb{R}^n$ (see the theorem of Sec. 17.4).

*Proof of the theorem. Necessity.* Suppose the matrix $Z(x)$ defines a passive operator relative to the cone $\overline{\mathbb{R}}^n_+$. By Theorem II of

Sec. 19.3, the matrix $Z(x)$ has the following properties:

$$\text{(a)} \qquad \langle Z(x)a + Z^*(x)a, a\rangle \gg 0, \qquad a \in \mathbb{C}^N; \tag{4.8}$$

$$\text{(b)} \qquad Z(x) = D_1^2 \dots D_n^2 Z_0(x) + \sum_{1 \leqslant j \leqslant n} Z^{(j)} D_j \delta(x), \tag{4.9}$$

where the matrix function $Z_0(x)$ is continuous, real, and of slow growth in $\mathbb{R}^n$ with support in the cone $\overline{\mathbb{R}}_+^n$; the matrices $Z^{(j)}$, $j = 1, \dots, n$, are real and positive. Passing to the Fourier transform in (4.8) and (4.9), we conclude that for all $a \in \mathbb{C}^n$ the generalized function

$$\langle \operatorname{Re} \widetilde{Z}(p) a, a\rangle = \frac{1}{2} F[\langle Z(x)a + Z^*(x)a, a\rangle] \tag{4.10}$$

is a nonnegative measure of slow growth in $\mathbb{R}^n$ (by the Bochner-Schwartz theorem; see Sec. 8.2) and

$$\widetilde{Z}(p) = (-1)^n p_1^2 \dots p_n^2 F[Z_0](p) - i \sum_{1 \leqslant j \leqslant n} Z^{(j)} p_j, \tag{4.11}$$

$$\operatorname{Re} \widetilde{Z}(p) = \frac{(-1)^n}{2} p_1^2 \dots p_n^2 F[Z_0(x) + Z_0^*(x)](p). \tag{4.12}$$

We set

$$M(p) = \frac{(-1)^n}{2} F[Z_0(x) + Z_0^*(x)](p). \tag{4.13}$$

The matrix $M(p)$ belongs to the class $N(-\overline{\mathbb{R}}_+^n \cup \overline{\mathbb{R}}_+^n)$ and by (4.12) it satisfies the equation (4.6). Furthermore, taking into account the equalities (see Sec. 10.1 and Sec. 10.2)

$$\begin{aligned} F[Z_0](p) &= F[(Z_0(x) + Z_0^*(x))\,\theta_n(x)] \\ &= \frac{1}{(2\pi)^n} F[Z_0(x) + Z_0^*(x)] * F[\theta_n] = 2(-1)^n M * \mathscr{K}_n, \end{aligned}$$

we rewrite relations (4.11) as

$$\widetilde{Z}(p) = \frac{2}{(2\pi)^n} p_1^2 \dots p_n^2 (M * \mathscr{K}_n) - i \sum_{1 \leqslant j \leqslant n} Z^{(j)} p_j. \tag{4.14}$$

Separating the real and imaginary parts in (4.14), we obtain the dispersion relation (4.5) (for $Z^{(0)} = 0$) and the relation

$$\operatorname{Re} \widetilde{Z}(p) = \frac{2}{(2\pi)^n} p_1^2 \dots p_n^2 (M * \operatorname{Re} \mathscr{K}_n), \tag{4.15}$$

which is equivalent to the relation (4.6) by virtue of (4.6) of Sec. 10:

$$M = \frac{2}{(2\pi)^n} M * \operatorname{Re} \mathscr{K}_n. \tag{4.16}$$

*Sufficiency*. Suppose the matrix $Z(x)$ is such that its Fourier transform $\tilde{Z}(p)$ satisfies the dispersion relation (4.5), where the matrix $M(p)$ is a solution in the class $N(-\overline{\mathbb{R}}^n_+ \cup \overline{\mathbb{R}}^n_+)$ of the equation (4.6), and the matrix is such that for any $a \in \mathbb{C}^n$ the generalized function $\langle \operatorname{Re} \tilde{Z}(p) a, a\rangle$ is a nonnegative measure of slow growth; the matrix $Z^{(0)}$ is real and skew-symmetric, and the matrices $Z^{(j)}$, $j = 1, \ldots, n$, are real and positive.

By (4.16), the equation (4.6) is equivalent to equation (4.15), which, together with the dispersion relation (4.5), yields

$$\tilde{Z}(p) = \frac{2}{(2\pi)^n} p_1^2 \ldots p_n^2 (M * \mathscr{K}_n) - Z^{(0)} - i \sum_{1 \leqslant j \leqslant n} Z^{(j)} p_j, \tag{4.17}$$

whence, using the inverse Fourier transform, we obtain

$$Z(x) = D_1^2 \ldots D_n^2 Z_1(x) - Z^{(0)} \delta(x) + \sum_{1 \leqslant j \leqslant n} Z^{(j)} D_j \delta(x), \tag{4.18}$$

where $Z_1(x) = 2(-1)^n F^{-1}[M](x)\, \theta_n(x)$ is a real continuous function of slow growth in $\mathbb{R}^n$ with support in the cone $\overline{\mathbb{R}}^n_+$. Noticing that

$$D_1^2 \ldots D_n^2 Z_1(x) - Z^{(0)} \delta(x) = D_1^2 \ldots D_n^2 [Z_1(x) - Z^{(0)} \mathscr{E}_n(x)],$$

we are convinced that the matrix $Z(x)$ satisfies the condition (4.9). The condition (4.8) is also fulfilled, by virtue of (4.10) and the Bochner-Schwartz theorem (see Sec. 8.2). By Theorem II of Sec. 19.3, the matrix $Z(x)$ defines a passive operator relative to the cone $\overline{\mathbb{R}}^n_+$.

We now prove the uniqueness of the dispersion relation (4.5) up to additive terms of the form (4.7). Suppose the representation (4.5) occurs with other matrices $[M_1, Z_1^{(0)}, Z_1^{(1)}, \ldots, Z_1^{(n)}]$. Then, by what has been proved,

$$M(p) - M_1(p) = iAD_1 \ldots D_n \operatorname{Im} [i^n \mathscr{K}_n(p)],$$

where $A$ is some constant real skew-symmetric matrix. From this, by subtracting the distinct representations (4.5) for $\operatorname{Im} \tilde{Z}(p)$,

we obtain

$$\frac{2i}{(2\pi)^n} Ap_1^2\ldots p_n^2\,[D_1\ldots D_n \operatorname{Im}(i^n\mathscr{K}_n) * \operatorname{Im}\mathscr{K}_n]$$
$$+i\,[Z^{(0)}-Z_1^{(0)}]-\sum_{1\leqslant j\leqslant n}[Z^{(j)}-Z_1^{(j)}]\,p_j=0. \qquad (4.19)$$

Passing to the inverse Fourier transform in (4.19) and using the formulas (4.4) and (2.8) of Sec. 10, we obtain the equality

$$-\frac{i}{2} AD_1^2\ldots D_n^2\{[\mathscr{E}_n(x)-\mathscr{E}_n(-x)]\,[\theta_n(x)-\theta_n(-x)]\}$$
$$+i\,[Z^{(0)}-Z_1^{(0)}]\,\delta(x)+\sum_{1\leqslant j\leqslant n}[Z^{(j)}-Z_1^{(j)}]\,D_j\delta(x)$$
$$=-\frac{i}{2}AD_1^2\ldots D_n^2\,[\mathscr{E}_n(x)+\mathscr{E}_n(-x)]$$
$$+i\,[Z^{(0)}-Z_1^{(0)}]\,\delta(x)+\sum_{1\leqslant j\leqslant n}[Z^{(j)}-Z_1^{(j)}]\,D_j\delta(x)$$
$$=i\,[Z^{(0)}-Z_1^{(0)}-A]\,\delta(x)+\sum_{1\leqslant j\leqslant n}[Z^{(j)}-Z_1^{(j)}]\,D_j\delta(x)=0,$$

which is only possible for

$$Z^{(0)}=Z_1^{(0)}+A,\qquad Z^{(j)}=Z_1^{(j)},\quad j=1,\ldots,n.$$

This completes the proof of the theorem.

**19.5 The fundamental solution and the Cauchy problem** The fundamental solution of the passive operator $Z*$ relative to the cone $\Gamma$ is any matrix $A\,(x)$, $A_{kj}\in\mathscr{D}'$, that satisfies the convolution matrix equation

$$Z*A=I\delta(x). \qquad (5.1)$$

The operator $A*$ is also said to be the *inverse* of $Z*$ (compare Sec. 4.8(d)), and the matrix function $\widetilde{A}(\zeta)$—the Laplace transform of the matrix $A\,(x)$—is called the *admittance* of the physical system.

The passive operator $Z*$ relative to the cone $\Gamma$ is said to be *nonsingular* (respectively, *completely nonsingular*) if $\Gamma$ is an acute cone and $\det\widetilde{Z}(\zeta)\neq 0$, $\zeta\in T^C$, where $C=\operatorname{int}\Gamma^*$ (and, respectively, if for any $a\in\mathbb{C}^N$, $a\neq 0$, there exists a point $\zeta_0\in T^C$

such that

$$\operatorname{Re} \langle \widetilde{Z}(\zeta_0)\, a,\ a\rangle > 0.) \tag{5.2}$$

If the operator $Z*$ that is passive relative to the cone $\Gamma$ is completely nonsingular, then

$$\operatorname{Re} \widetilde{Z}(\zeta) > 0, \qquad \zeta \in T^C. \tag{5.3}$$

Indeed, by Theorem I of Sec. 19.3, the function $\langle \widetilde{Z}(\zeta)\, a,\ a\rangle$ is holomorphic and $\operatorname{Re} \langle \widetilde{Z}(\zeta)\, a,\ a\rangle \geqslant 0$ in $T^C$. But then, by (5.2), the inequality $\operatorname{Re} \langle \widetilde{Z}(\zeta)\, a,\ a\rangle > 0$ holds if $a \neq 0$ (see the reasoning in Sec. 16.1), which is equivalent to (5.3).

From this it follows that any completely nonsingular passive operator is also a nonsingular passive operator relative to the same cone.

Furthermore, for an operator $Z*$ that is passive with respect to an acute cone $\Gamma$ to be completely nonsingular, it is necessary and sufficient that the equality

$$\langle Z(x)\, a,\ a\rangle = ig\delta(x) \tag{5.4}$$

be impossible for any $a \in \mathbb{C}^N$, $a \neq 0$, and for any real $g$.

Indeed, if the operator $Z*$ that is passive relative to the cone $\Gamma$ is completely nonsingular, then (5.4), which is equivalent to the equality

$$\langle \widetilde{Z}(\zeta)\, a, a\rangle = ig, \qquad \zeta \in T^C,$$

is impossible by (5.2) for any $a \neq 0$ and for any real $g$. Conversely, suppose the operator $Z*$ that is passive relative to the acute cone $\Gamma$ is not completely nonsingular. Then, for some $a \neq 0$, we would have $\operatorname{Re} \langle \widetilde{Z}(\zeta)\, a,\ a\rangle \leqslant 0$, $\zeta \in T^C$. On the other hand, by Theorem I of Sec. 19.3, the function $\langle Z(\zeta)\, a,\ a\rangle$ is holomorphic and $\operatorname{Re} \langle \widetilde{Z}(\zeta)\, a,\ a\rangle \geqslant 0$ in $T^C$ and therefore $\operatorname{Re} \langle \widetilde{Z}(\zeta)\, a,\ a\rangle = 0$ in $T^C$. Hence, $\langle \widetilde{Z}(\zeta)\, a,\ a\rangle = ig$, where $g$ is a real number so that (5.4) has turned out possible for certain $a \neq 0$ and for certain real $g$.

Theorem I *Every nonsingular passive operator relative to a cone $\Gamma$ has a unique fundamental solution that determines a nonsingular passive operator relative to that same cone $\Gamma$.*

*Proof.* Let $Z*$ be a nonsingular passive operator relative to a cone $\Gamma$ so that $\widetilde{Z}(\zeta)$ is a positive real matrix in $T^C$ (by Theorem I of Sec. 19.3) and $\det \widetilde{Z}(\zeta) \neq 0$, $\zeta \in T^C$. We will prove the existence

and uniqueness of the solution of equation (5.1) in the class of matrices $A(x)$ that define nonsingular passive operators relative to $\Gamma$. Applying the Laplace transform to equation (5.1), we obtain an equivalent matrix equation

$$\tilde{Z}(\zeta)\tilde{A}(\zeta)=I, \qquad \zeta\in T^C. \tag{5.5}$$

Equation (5.5) is unambiguously solvable for all $\zeta \in T^C$ and its solution—the matrix function $\tilde{A}(\zeta) = \tilde{Z}^{-1}(\zeta)$—is holomorphic and $\det \tilde{A}(\zeta) \neq 0$ in $T^C$. Furthermore, from the equality $\tilde{Z}(\zeta) = \bar{\tilde{\bar{Z}}}(-\bar{\zeta})$, $\zeta \in T^C$, and from (5.5) it follows that $\tilde{Z}(\zeta)\bar{\tilde{\bar{A}}}(-\bar{\zeta}) = I$, that is,

$$\tilde{Z}^{-1}(\zeta)=\tilde{A}(\zeta)=\bar{\tilde{\bar{A}}}(-\bar{\zeta}), \qquad \zeta\in T^C.$$

Finally, from the condition $\operatorname{Re}\tilde{Z}(\zeta) \geqslant 0$, $\zeta \in T^C$, and from (5.5) we derive

$$\operatorname{Re}\tilde{A}(\zeta)=\tilde{A}^{+}(\zeta)\,[\operatorname{Re}\tilde{Z}(\zeta)]\,\tilde{A}(\zeta)\geqslant 0, \qquad \zeta\in T^C. \tag{5.6}$$

Consequently, the matrix $\tilde{A}(\zeta)$ is positive real in $T^C$. By Theorem I of Sec. 19.3 the matrix $A(x)$ defines a nonsingular passive operator relative to the cone $\Gamma$. The matrix $A(x)$ is unique. The proof of Theorem I is complete.

Corollary *If the passive operator $Z*$ is completely nonsingular, then its inverse operator $A*$ is completely nonsingular.*

Indeed, since $\operatorname{Re}\tilde{Z}(\zeta) > 0$ and $\det \tilde{A}(\zeta) \neq 0$, it follows, by (5.6), that $\operatorname{Re}\tilde{A}(\zeta) > 0$, $\zeta \in T^C$.

Let $\Gamma$ be a closed convex acute cone, $C = \operatorname{int}\Gamma^*$, let $S$ be a $C$-like surface, and let $S_+$ be a region lying above $S$ (see Sec. 4.4).

By analogy with Sec. 15.1 we introduce the following definition. We use the term generalized problem of Cauchy for an operator $Z*$ that is passive relative to the cone $\Gamma$ with source $f \in \mathscr{D}'(\bar{S}_+)^{\times N}$ to designate the problem of finding, in $\mathbb{R}^n$, a solution $u(x)$ taken from $\mathscr{D}'(\bar{S}_+)^{\times N}$ of the system (1.1).

As in Sec. 15.1, the following theorem is readily proved.

Theorem II *If a passive operator $Z*$ is nonsingular relative to a (solid) cone $\Gamma$, then the solution of the generalized Cauchy problem for it exists for any $f$ in $\mathscr{D}'(\bar{S}_+)^{\times N}$, is unique, and is given by the formula*

$$u = A * f. \tag{5.7}$$

Corollary *If $S$ is a strictly $C$-like surface and $f \in \mathcal{S}'(\overline{S}_+)^{\times N}$, then the solution of the generalized Cauchy problem for the operator $Z*$ exists and is unique in the class $\mathcal{S}'(\overline{S}_+)^{\times N}$* [and is given by the formula (5.7)].

This follows from Theorem II and from the results of Sec. 5.6(b).

Thus, passive systems behave in similar fashion to hyperbolic systems (see Sec. 15.1, Hörmander [1, Chapter 5], Friedrichs [1], Desin [1]).

**19.6 What differential and difference operators are passive operators?** A system of $N$ linear differential equations of order $\leqslant m$ (with constant coefficients) is determined by the matrix (compare Sec. 14.1)

$$Z(x) = \sum_{0 \leqslant |\alpha| \leqslant m} Z_\alpha D^\alpha \delta(x), \tag{6.1}$$

where $Z_\alpha$ are (constant) $N \times N$ matrices.

Theorem I *For a system of $N$ linear differential equations with constant coefficients to be passive relative to an acute cone $\Gamma$, it is necessary and sufficient that*

$$Z(x) = \sum_{1 \leqslant j \leqslant n} Z_j D_j \delta(x) + Z_0 \delta(x), \tag{6.2}$$

*where $Z_1, \ldots, Z_n$ are real symmetric $N \times N$ matrices such that $\sum_{1 \leqslant j \leqslant n} q_j Z_j \geqslant 0$ for all $q \in C = \operatorname{int} \Gamma^*$; the matrix $Z_0$ is real and* $\operatorname{Re} Z_0 \geqslant 0$.

*Proof. Necessity.* Suppose the differential operator $Z*$ defined by formula (6.1) is passive relative to the cone $\Gamma$. Then by Theorem I of Sec. 19.3 the matrix function

$$\tilde{Z}(\zeta) = \sum_{0 \leqslant |\alpha| \leqslant m} (-i\zeta)^\alpha Z_\alpha \tag{6.3}$$

is positive real in $T^C$. Therefore, for every $a \in \mathbb{C}^N$ the function $\langle \tilde{Z}(\zeta) a, a\rangle$ is holomorphic and $\operatorname{Re} \langle \tilde{Z}(\zeta) a, a\rangle \geqslant 0$ in $T^C$. Therefore, that function satisfies the estimate (1.1) of Sec. 16 and, hence, all matrix elements $\tilde{Z}_{kj}(\zeta)$ satisfy that estimate:

$$\left| \sum_{0 \leqslant |\alpha| \leqslant m} (-i\zeta)^\alpha Z_{\alpha, kj} \right| \leqslant M(C') \frac{1+|\zeta|^2}{|q|}, \qquad \zeta \in T^{C'},$$

which is only possible for $Z_{\alpha,kj} = 0$, $1 \leqslant k, j \leqslant N$, $|\alpha| \geqslant 2$. Therefore, the matrix (6.3) takes the form

$$\widetilde{Z}(\zeta) = -i \sum_{1 \leqslant j \leqslant n} Z_j \zeta_j + Z_0 \tag{6.4}$$

and the representation (6.2) is proved. Writing out the conditions for positive reality of the matrix $\widetilde{Z}(\zeta)$,

$$\sum_{1 \leqslant j \leqslant n} q_j Z_j + Z_0 = \sum_{1 \leqslant j \leqslant n} q_j \overline{Z}_j + \overline{Z}_0, \qquad q \in C,$$

$$-i \sum_{1 \leqslant j \leqslant n} \zeta_j Z_j + Z_0 + i \sum_{1 \leqslant j \leqslant n} \overline{\zeta}_j Z_j^+ + Z_0^+ \geqslant 0, \qquad \zeta \in T^C,$$

we conclude that the matrices $Z_j$, $j = 0, 1, \ldots, n$, are real. and the matrices $Z_j$, $j = 1, \ldots, n$, are symmetric, $\sum_{1 \leqslant j \leqslant n} q_j Z_j \geqslant$ $\geqslant 0$ and $\operatorname{Re} Z_0 \geqslant 0$.

*Sufficiency*. Let the matrix $Z(x)$ satisfy the conditions of Theorem I. Then its Laplace transform $\widetilde{Z}(\zeta)$ is of the form (6.4) and

$$\operatorname{Re} \widetilde{Z}(\zeta) = \sum_{1 \leqslant j \leqslant n} q_j Z_j + \operatorname{Re} Z_0 \geqslant 0, \qquad \zeta \in T^C. \tag{6.5}$$

Therefore the matrix function $\widetilde{Z}(\zeta)$ is positive real in $T^C$ and by Theorem I of Sec. 19.3 the operator $Z*$ is passive relative to the cone $\Gamma$. The proof of the theorem is complete.

Suppose we are given real symmetric $N \times N$ matrices $Z_1, \ldots$ $\ldots, Z_n$ having the property that for a certain vector $l \in \mathbb{R}^n$, $\sum_{1 \leqslant j \leqslant n} l_j Z_j > 0$. We set

$$\Gamma_c = [x : x_1 = \langle Z_1 a, a \rangle, \ldots, x_n = \langle Z_n a, a \rangle, a \in \mathbb{C}^N],$$

$$\Gamma_r = [x : x_1 = \langle Z_1 a, a \rangle, \ldots, x_n = \langle Z_n a, a \rangle, a \in \mathbb{R}^N].$$

Under the mapping

$$a \to x = (\langle Z_1 a, a \rangle, \ldots, \langle Z_n a, a \rangle) \tag{6.6}$$

the preimage of 0 is 0 by virtue of the inequality

$$(l, x) = \langle \sum_{1 \leqslant j \leqslant 0} l_j Z_j a, a \rangle \geqslant \varkappa |a|^2, \qquad \varkappa > 0. \tag{6.7}$$

Clearly, $\Gamma_c$ and $\Gamma_r$ are cones with vertex at 0, and $\Gamma_r \subset \Gamma_c$.

Lemma *The cones $\Gamma_c$ and $\Gamma_r$ are closed and acute*; $\Gamma_c = \Gamma_r + \Gamma_r$; $l \in \operatorname{int} \Gamma_c^*$.

*Proof*. The mapping (6.6) is continuous from $\mathbb{C}^N$ (in $\mathbb{R}^N$) into $\mathbb{R}^n$ and, by virtue of the inequality (6.7), is of a compact nature, that is, the preimage of any compact is a compact. Therefore the cones $\Gamma_c$ and $\Gamma_r$ are closed. Furthermore, from the equalities

$$\langle Z_j a, a\rangle = \langle Z_j b, b\rangle + \langle Z_j c, c\rangle, \qquad a = b + ic, \qquad j = 1, \ldots, n,$$

we conclude that $\Gamma_c = \Gamma_r + \Gamma_r$. Finally, by the inequality (6.6) the plane $(l, x) = 0$ has only one point in common with the cone $\Gamma_c$—its vertex. Therefore $\Gamma_c$ and $\Gamma_r$ are acute cones and $l \in \operatorname{int} \Gamma_c^*$ (see lemma 1 of Sec. 4.4). The lemma is proved.

Notice that the cones $\Gamma_c$ and $\Gamma_r$ may not be solid, as the following instance shows: $Z_1 > 0$, $Z_2 = 0$ and $\Gamma_c$ and $\Gamma_r$ lie in the plane $x_2 = 0$.

Theorem II *In order that the matrix* (*6.2*) *define a passive completely nonsingular operator, it is necessary and sufficient that the matrices $Z_1, \ldots, Z_n$ be real symmetric, that the matrix $Z_0$ be real and* $\operatorname{Re} Z_0 \geqslant 0$, *and that there exists a vector $l \in \mathbb{R}^n$ such that*

$$\sum_{1\leqslant j\leqslant n} l_j Z_j + \operatorname{Re} Z_0 > 0. \tag{6.8}$$

*Here, the passivity and the complete nonsingularity of the operator $Z*$ occur in the case of any acute cone $\Gamma$ that contains the cone $\Gamma_c$, and* $l \in \operatorname{int} \Gamma^*$.

*Proof*. *Necessity*. Suppose the matrix (6.2) defines a passive and completely nonsingular operator with respect to a certain (acute) cone $\Gamma$. Then the conditions of Theorem I are fulfilled and, by (5.3) and (6.5),

$$\operatorname{Re} \tilde{Z}(\zeta) = \sum_{1\leqslant j\leqslant n} q_j Z_j + \operatorname{Re} Z_0 > 0, \qquad \zeta \in T^C, \quad C = \operatorname{int} \Gamma^*,$$

so that the condition (6.8) holds for all $q \in C$.

*Sufficiency*. Let the matrices $Z_0, \ldots, Z_n$ in (6.2) satisfy the conditions of Theorem II. Suppose $\Gamma$ is an acute cone containing the cone $\Gamma_c$ and such that $l \in C = \operatorname{int} \Gamma^*$. From this it follows that $(q, x) \geqslant 0$ for all $q \in C$, $x \in \Gamma_c \subset \Gamma$, that is,

$$(q, x) = \sum_{1\leqslant j\leqslant n} q_j \langle Z_j a, a\rangle \geqslant 0, \qquad q \in C, \quad a \in \mathbb{C}^N.$$

This means that $\sum\limits_{1\leqslant j\leqslant n} q_j Z_j \geqslant 0$, $q \in C$. By Theorem I, the matrix $Z(x)$ defines a passive operator relative to the cone $\Gamma$. Furthermore, it is given that $l \in C$ and so, by (6.5) and (6.8),

$$\operatorname{Re} Z(il) = \sum_{1\leqslant j\leqslant n} l_j Z_j + \operatorname{Re} Z_0 > 0,$$

so that the operator $Z*$ is completely nonsingular relative to the cone $\Gamma$ (see Sec. 19.5). The proof of Theorem II is complete.

*Remark.* Theorem II states that the matrices $\sum\limits_{1\leqslant j\leqslant n} Z_j D_j \delta(x)$, which define passive completely nonsingular differential operators $\sum\limits_{1\leqslant j\leqslant n} Z_j \frac{\partial}{\partial x_j}$, coincide with the principal parts of Friedrichs-symmetric (see Friedrichs [1]) differential operators with constant coefficients.

A system of $N$ linear difference equations with the number of steps $\leqslant m$ is given by the matrix

$$Z(x) = \sum_{1\leqslant \nu\leqslant m} Z_\nu \delta(x - h_\nu). \tag{6.9}$$

Theorem III *For a system of $N$ linear difference equations (when $h_\nu \neq h_k$, $\nu \neq k$) to be passive relative to an acute cone $\Gamma$, it is necessary and sufficient that the $N \times N$ matrices $Z_1, \ldots, Z_m$ be real and for all $\zeta \in T^C$, $C = \operatorname{int} \Gamma^*$, the matrix*

$$\sum_{1\leqslant \nu\leqslant m} e^{-(q,\, h_\nu)} [\cos(p, h_\nu) \operatorname{Re} Z_\nu - \sin(p, h_\nu) \operatorname{Im} Z_\nu] \geqslant 0. \tag{6.10}$$

*Proof. Necessity.* Suppose the difference operator $Z*$ that is defined by the matrix (6.9) is passive relative to the acute cone $\Gamma$. From the fact that the matrix $Z(x)$ is real and from the condition $h_\nu \neq h_k$ for $\nu \neq k$ it follows that the matrices $Z_1, \ldots, Z_m$ are real. By Theorem I of Sec. 19.3,

$$\operatorname{Re} \widetilde{Z}(\zeta) = \operatorname{Re} \sum_{1\leqslant \nu\leqslant m} e^{i(\zeta,\, h_\nu)} Z_\nu \geqslant 0, \qquad \zeta \in T^C, \tag{6.11}$$

and the condition (6.10) is fulfilled.

*Sufficiency.* Suppose the matrices $Z_1, \ldots, Z_m$ in (6.9) satisfy the conditions of Theorem III. Then, by (6.11), the matrix $\widetilde{Z}(\zeta)$

is positive real in $T^C$ and by Theorem I of Sec. 19.3 the matrix $Z(x)$ defines a passive operator relative to the cone $C^* = \Gamma$. Theorem III is proved.

*Remark.* We make a special note of the necessity condition of passivity of the matrix (6.9): the smallest convex cone containing the points $\{0, h_1, \ldots, h_m\}$ must be acute.

**19.7 Examples** Let us denote by $V^+(a) = [(x, t) : at > |x|]$ the future cone in $\mathbb{R}^4$, which corresponds to the rate of propagation $a$: $V^+ = V^+(1)$ (compare Sec. 4.4).

(1) *Maxwell's equations.* The principal part of the appropriate differential operator is of the form[§]

$$\frac{\partial \mathbf{D}}{\partial x_0} - \operatorname{rot} \mathbf{H}, \qquad \frac{\partial \mathbf{B}}{\partial x_0} + \operatorname{rot} \mathbf{E}, \tag{7.1}$$

where $x_0 = ct$, $c$ is the speed of light in vacuum, $x = (x_0, \mathbf{x})$ and

$$\mathbf{D} = \varepsilon \times \mathbf{E}, \qquad \mathbf{B} = \mu * \mathbf{H}, \tag{7.2}$$

where $\varepsilon$ and $\mu$ are $3 \times 3$ matrices called tensors of permittivity and magnetic permeability respectively.

If $\varepsilon$ and $\mu$ are constant matrices that are multiples of the unit matrix, $\varepsilon = \varepsilon I \delta(x)$, $\mu = \mu I \delta(x)$, then the system (7.1)-(7.2) becomes

$$\varepsilon \frac{\partial \mathbf{E}}{\partial x_0} - \operatorname{rot} \mathbf{H}, \qquad \mu \frac{\partial \mathbf{H}}{\partial x_0} + \operatorname{rot} \mathbf{E}. \tag{7.3}$$

The system (7.3) is passive with respect to the cone $\overline{V}^+(1/\sqrt{\varepsilon\mu})$ by virtue of the inequality

$$\int\limits_{-V^+(1/\sqrt{\varepsilon\mu})} \left[\varepsilon\left(\frac{\partial \mathbf{E}}{\partial x_0}, \mathbf{E}\right) - (\mathbf{E}, \operatorname{rot} \mathbf{H}) + \mu\left(\frac{\partial \mathbf{H}}{\partial x_0}, \mathbf{H}\right) + (\mathbf{H}, \operatorname{rot} \mathbf{E})\right] dx \geqslant 0, \tag{7.4}$$

which holds for all $\mathbf{E} \subset \mathcal{D}_r(\mathbb{R}^4)^{\times 3}$ and $\mathbf{H} \in \mathcal{D}_r(\mathbb{R}^4)^{\times 3}$. Here, $N = 6$, $n = 4$.

To prove the inequality (7.4) we make use of the identity

$$(\mathbf{H}, \operatorname{rot} \mathbf{E}) - (\mathbf{E}, \operatorname{rot} \mathbf{H}) = \operatorname{div}(\mathbf{E} \times \mathbf{H}),$$

§ Specification of div **D** and div **B** in the system of Maxwell's equations is not essential for our purposes; actually, these are matching conditions.

by virtue of which the left-hand member of (7.4) is equal to

$$\int\limits_{\mathbb{R}^3}\int\limits_{-\infty}^{-\sqrt{\varepsilon\mu}|\mathbf{x}|}\frac{\partial}{\partial x_0}(\varepsilon|\mathbf{E}|^2+\mu|\mathbf{H}|^2)\,dx_0d\mathbf{x}+\int\limits_{-\infty}^{0}\int\limits_{|\mathbf{x}|<-x_0/\sqrt{\varepsilon\mu}}\operatorname{div}(\mathbf{E}\times\mathbf{H})\,dx$$

$$=\frac{1}{2}\int\limits_{\mathbb{R}^3}[\varepsilon|\mathbf{E}|^2+\mu|\mathbf{H}|^2+2\sqrt{\varepsilon\mu}\,((\mathbf{E}\times\mathbf{H}),\mathbf{n})]\Big|_{x_0=-\sqrt{\varepsilon\mu}|\mathbf{x}|}d\mathbf{x}$$

$$\geqslant\frac{1}{2}\int\limits_{\mathbb{R}^3}(\varepsilon|\mathbf{E}|^2+\mu|\mathbf{H}|^2-2\sqrt{\varepsilon\mu}|\mathbf{E}||\mathbf{H}|)\Big|_{x_0=-\sqrt{\varepsilon\mu}\,|\mathbf{x}|}d\mathbf{x}$$

$$=\frac{1}{2}\int\limits_{\mathbb{R}^3}(\sqrt{\varepsilon}\,|\mathbf{E}|-\sqrt{\mu}\,|\mathbf{H}|)^2\Big|_{x_0=-\sqrt{\varepsilon\mu}|\mathbf{x}|}d\mathbf{x}\geqslant 0.$$

Let us verify that when $\varepsilon=\mu=1$ the cone $\Gamma_c=\Gamma_r=\bar{V}^+$. Indeed, if $a\in\mathbb{R}^6$, then the mapping (6.6) takes the form

$$x_0=a_1^2+\ldots+a_6^2,\qquad x_1=-2a_2a_6+2a_3a_5,$$
$$x_2=2a_1a_6-2a_3a_4,\qquad x_3=-2a_1a_5+2a_2a_4,$$

so that $x_0\geqslant 0$ and

$$x^2=x_0^2-|\mathbf{x}|^2$$
$$=(a_1^2+a_2^2+a_3^2-a_4^2-a_5^2-a_6^2)^2+4\,(a_1a_4+a_2a_5+a_3a_6)^2\geqslant 0.$$

But if the tensors $\varepsilon$ and $\mu$ are nontrivial, then, depending on the properties of the medium, it is natural to postulate passivity with respect to an acute cone of certain of the operators

$$\varepsilon *,\quad \mu *,\quad \frac{\partial\varepsilon}{\partial x_0}*,\quad \frac{\partial\mu}{\partial x_0}*\qquad (N=3,n=4).$$

Relative to appropriate impedance and admittance matrices, all propositions of the theory developed in Secs. 19.3 to 19.5 hold true; in particular, the four-dimensional dispersion relations (see Sec. 19.4, compare Silin and Rukhadze [1]).

(2) *Dirac's equation.* The appropriate operator is

$$i\sum_{0\leqslant\mu\leqslant 3}\gamma^\mu D_\mu-m,\tag{7.5}$$

where $\gamma^\mu$ is a $4\times 4$ Dirac matrix; in the Majorana basis (see, for example, Bogoliubov, Logunov, and Todorov [1, Chapter 2]),

they are of the form

$$\gamma^0 = \begin{pmatrix} 0 & i\sigma_1 \\ -i\sigma_1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix},$$
$$\gamma^3 = \begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix},$$

where $\sigma_k$ are $2 \times 2$ Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We now prove that the operator (7.5), after multiplication by the matrix $-i\gamma^0$,

$$\sum_{0 \leqslant \mu \leqslant 3} \gamma^0\gamma^\mu D_\mu + im\gamma^0, \tag{7.6}$$

is passive and completely nonsingular relative to the cone $\bar{V}^+$.

Indeed, the matrices $\gamma^0\gamma^\mu$ are real and symmetric, and the matrix $im\gamma^0$ is real and skew-symmetric. Furthermore, the cone $\Gamma_r$ coincides with the boundary of the cone $V^+$ since the mapping (6.6) is, for $a \in \mathbb{R}^4$, of the form

$$x_0 = \langle \gamma^0\gamma^0 a, a\rangle = a_1^2 + \ldots + a_4^2, \quad x_1 = \langle \gamma^0\gamma^1 a, a\rangle = 2a_1a_4 + 2a_2a_3,$$
$$x_2 = a_1^2 - a_2^2 + a_3^2 - a_4^2, \qquad x_3 = -2a_1a_2 + 2a_3a_4,$$

so that $x_0 \geqslant 0$ and

$$\begin{aligned} x_0^2 - |\mathbf{x}|^2 &= (a_1^2 + \ldots + a_4^2)^2 - 4(a_1a_4 + a_2a_3)^2 \\ &\quad - (a_1^2 - a_2^2 + a_3^2 - a_4^2)^2 - 4(a_1a_2 - a_3a_4)^2 \\ &= 0. \end{aligned}$$

By Theorem II of Sec. 19.6 the operator (7.6) is passive and completely nonsingular relative to the cone $\Gamma_c = \bar{V}^+$, the convex hull of the cone $\Gamma_r$ (compare with the lemma of Sec. 19.6).

(3) *The equations of a rotating fluid and acoustics:*§

$$\alpha\frac{\partial p}{\partial t} + \rho \operatorname{div} \mathbf{v}, \qquad \rho\frac{\partial v}{\partial t} + \operatorname{grad} p + \mathbf{v} \times \mathbf{w}. \tag{7.7}$$

Here $N = n = 4$. For all $\alpha > 0$, the system (7.7) is passive and completely nonsingular relative to the cone $\bar{V}^+(1/\sqrt{\alpha})$. The

§ See Maslennikova [1] and Drozhzhinov and Galeev [1].

mapping (6.6) is, for $a \in \mathbb{R}^4$, of the form

$$t = \alpha a_1^2 + a_2^2 + a_3^2 + a_4^2, \quad x_1 = 2a_1a_2, \quad x_2 = 2a_1a_3, \quad x_3 = 2a_1a_4,$$

so that $t \geqslant 0$ and

$$\begin{aligned} t^2 - \alpha \mid x \mid^2 &= (\alpha a_1^2 + a_2^2 + a_3^2 + a_4^2)^2 - 4\alpha a_1^2 (a_2^2 + a_3^2 + a_4^2) \\ &= (\alpha a_1^2 - a_2^2 - a_3^2 - a_4^2)^2 \geqslant 0. \end{aligned}$$

Therefore $\Gamma_c = \Gamma_r = \overline{V}^+ (1/\sqrt{\alpha})$.

(4) *Equations of magnetic hydrodynamics*:§

$$\frac{\partial p}{\partial t} + a^2 \rho \operatorname{div} \mathbf{v}, \qquad \frac{\partial \mathbf{H}}{\partial t} - \operatorname{rot} (\mathbf{v} \times \mathbf{B}),$$

$$\rho \frac{\partial v}{\partial t} + \operatorname{grad} p - \frac{1}{4\pi} (\operatorname{rot} \mathbf{H}) \times \mathbf{B}, \tag{7.8}$$

where $\mathbf{B}$ is a specified vector; here, $N = 7$, $n = 4$. The system (7.8) is passive and completely nonsingular relative to a certain cone.

(5) *Equations of the theory of elasticity*:§§

$$\frac{\partial^2 w_i}{\partial t^2} = \sum_{1 \leqslant j,\, m,\, n \leqslant 3} c_{mn}^{ij} \frac{\partial^2 w_m}{\partial x_j \, \partial x_n}, \qquad i = 1, 2, 3, \tag{7.9}$$

where $c_{mn}^{ij} = c_{mn}^{ji} = c_{nm}^{ij} = c_{ji}^{nm}$. If we introduce the velocity vector $\left( v_i = \frac{\partial w_i}{\partial t}, \ i = 1, 2, 3 \right)$ and the stress tensor

$$\left\{ \sum_{ij} = \sum_{1 \leqslant n,\, m \leqslant 3} c_{mn}^{ij} \frac{\partial w_m}{\partial x_n}, \quad 1 \leqslant i, j \leqslant 3 \right\},$$

then the operator (7.9) becomes passive and completely nonsingular relative to a certain cone. Here, $N = 9$, $n = 4$.

(6) *Transfer equation*:

$$\frac{\partial}{\partial t} + (\Omega, \operatorname{grad}) \tag{7.10}$$

where $\Omega$ is a constant vector in $\mathbb{R}^3$; here, $N = 1$, $n = 4$. The operator (7.10) is passive and completely nonsingular relative to any (acute) cone containing the vector $(1, \Omega)$ in $\mathbb{R}^4$. If we

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§ See Leonard [1] and Drozhzhinov [1].

§§ See Wilcox [1].

apply the method of spherical harmonics to the operator (7.10), then to any $\mathscr{P}_N$-approximation what is obtained is a passive completely nonsingular system relative to a certain cone (dependent on $N$). The appropriate operators are written out in Godunov and Sultangazin [1].

## 20 Abstract Scattering Operator

Let us apply the results concerning linear passive systems to the study of a finite-dimensional scattering matrix. As above, $\Gamma$ is a closed convex acute solid cone in $\mathbb{R}^n$, $C = \operatorname{int} \Gamma^*$. For $n = 1$, see Beltrami and Wohlers [1], Lax and Phillips [1], Güttinger [1], and Rañada [1].

**20.1 The definition and properties of an abstract scattering matrix** We use the term *abstract scattering matrix* relative to a cone $\Gamma$ for the real $N \times N$ matrix $S(x) = (S_{kj}(x))$, $S_{kj} \in \mathscr{D}'(\mathbb{R}^n)$ that satisfies the conditions: of causality relative to the cone $\Gamma$,

$$\operatorname{supp} S(x) \subset \Gamma; \tag{1.1}$$

of boundedness,

$$\int \langle S * \varphi, S * \varphi \rangle \, dx \leqslant \int \langle \varphi, \varphi \rangle \, dx, \qquad \varphi \in \mathscr{D}^{\times N}. \tag{1.2}$$

The corresponding operator $S*$ is called the *scattering operator* (relative to the cone $\Gamma$).

Properties of an abstract scattering matrix.

(a) The operator $S*$ admits of extension onto $(\mathscr{L}^2)^{\times N}$ with the inequality (1.2) preserved:

$$\sum_{1 \leqslant k \leqslant N} \Big\| \sum_{1 \leqslant j \leqslant N} S_{kj} * \varphi_j \Big\|^2 \leqslant \sum_{1 \leqslant k \leqslant N} \| \varphi_k \|^2, \qquad \varphi_k \in \mathscr{L}^2. \tag{1.2'}$$

This follows from (1.2) and from the density $\mathscr{D}$ in $\mathscr{L}^2$ (see Sec. 1.2).

(b) *Restrictions to growth*:

$$S \in (\mathscr{D}'_{\mathscr{L}^2})^{\times N^2}. \tag{1.3}$$

Indeed, from (1.2′) it follows that if $\varphi \in \mathscr{L}^2$, then also $S_{kj} * \varphi \in \mathscr{L}^2$. Let $\mathscr{E}_{n,m}(x)$ be a fundamental solution of the operator $(\nabla^2)^m$, which solution belongs to $\mathscr{L}^2_{\text{loc}}$. [By virtue of Sec. 14.4 (g),

such $m$ exist for every $n$, and $\mathscr{E}_{n,\,m} \in C^\infty(\mathbb{R}^n \setminus \{0\})$.] Suppose $\alpha \in \mathscr{D}$, $\alpha(x) = 1$ in the neighbourhood of the point 0. Then

$$(\nabla^2)^m (\alpha \mathscr{E}_{n,\,m}) = \delta(x) + \eta(x),$$

where $\eta \in \mathscr{D}$ and $\alpha\mathscr{E}_{n,\,m} \in \mathscr{L}^2$. Therefore

$$\begin{aligned} S_{kj} = S_{kj} * \delta &= S_{kj} * (\nabla^2)^m (\alpha\mathscr{E}_{n,\,m}) - S_{kj} * \eta \\ &= (\nabla^2)^m (S_{kj} * (\alpha\mathscr{E}_{n,\,m})) - S_{kj} * \eta. \end{aligned}$$

But $S_{kj} * (\alpha\mathscr{E}_{n,\,m}) \in \mathscr{L}^2$, $S_{kj} * \eta \in \mathscr{L}^2$, and therefore $S_{kj} \in \mathscr{H}_s$ for some $s < 0$ (see Sec. 10.1). The inclusion (1.3) is proved.
(c) The following inequality holds true:

$$\int_{-\Gamma} [\langle\varphi, \varphi\rangle - \langle S * \varphi, S * \varphi\rangle]\, dx \geqslant 0, \qquad \varphi \in (\mathscr{L}^2)^{\times N}. \tag{1.4}$$

Indeed, let $\varphi \in (\mathscr{L}^2)^{\times N}$; then $\psi = \theta_{-\Gamma}\varphi \in (\mathscr{L}^2)^{\times N}$, $\operatorname{supp}\psi \subset -\Gamma$ and, hence, $S * \varphi = S * \varphi$ almost everywhere in $-\Gamma$ because, by Sec. 4.2(g) and (1.1),

$$\operatorname{supp}(S * \varphi - S * \psi) = \operatorname{supp} S * [(1 - \theta_{-\Gamma})\varphi]$$

$$\subset \overline{\operatorname{supp} S + \operatorname{supp}(1 - \theta_{-\Gamma})} \subset \overline{\Gamma + \mathbb{R}^n \setminus (-\Gamma)} = \overline{\mathbb{R}^n \setminus (-\Gamma)}.$$

From this and from (1.2′) follows the inequality (1.4):

$$\begin{aligned} \int_{-\Gamma} [\langle\varphi, \varphi\rangle - \langle S * \varphi, S * \varphi\rangle]\, dx &= \int \langle\psi, \psi\rangle\, dx - \int_{-\Gamma} \langle S * \psi, S * \psi\rangle\, dx \\ &\geqslant \int [\langle\psi, \psi\rangle - \langle S * \psi, S * \psi\rangle]\, dx \geqslant 0. \end{aligned}$$

(d) The inequality (1.4) holds true in strong form:

$$\int_{-\Gamma + x_0} [\langle\varphi, \varphi\rangle - \langle S * \varphi, S * \varphi\rangle]\, dx \geqslant 0, \quad \varphi \in (\mathscr{L}^2)^{\times N},\ x_0 \in \mathbb{R}^n. \tag{1.5}$$

This follows from the inequality (1.4) via reasoning similar to that given in Sec. 19.2(a).

From the inequality (1.5), as in Sec. 19.2(b), follows the inequality (1.2).

*Remark.* It is also interesting to find out whether the causality condition (1.1) follows from the inequality (1.4), as is the case for passive operators (see Sec. 19.2(c)).

(e) *Positive definiteness*:

$$|a|^2\delta - \langle (S^* * S)\, a,\ a\rangle \gg 0, \qquad a \in \mathbb{C}^N. \tag{1.6}$$

Indeed, setting $\varphi = a\varphi_0$, $a \in \mathbb{C}^N$, and $\varphi_0 \in \mathscr{D}$ in (1.2), we obtain (1.6) (see Sec. 8.1):

$$\begin{aligned}
0 &\leqslant \int [|a|^2\,|\varphi_0|^2 - \langle S * a\varphi_0,\ S * a\varphi_0\rangle]\,dx \\
&= \Big(\delta,\ |a|^2\,\varphi_0 * \varphi_0^* - \sum_{1\leqslant j\leqslant N} (S * a\varphi_0)_j * (S * a\varphi_0)_j^*\Big) \\
&= \Big(\delta,\ |a|^2\,\varphi_0 * \varphi_0^* - \sum_{1\leqslant j\leqslant N} (S(x) * a\varphi_0)_j * (S(-x) * \bar{a}\varphi_0^*)_j\Big) \\
&= \Big(\delta,\ |a|^2\,\varphi_0 * \varphi_0^* - \sum_{1\leqslant j,\,k,\,l\leqslant N} S_{jk} * S_{jl}^* * \varphi_0 * \varphi_0^* a_k \bar{a}_l\Big) \\
&= (|a|^2\delta - \langle (S^* * S)\, a,\ a\rangle,\ \varphi_0 * \varphi_0^*).
\end{aligned}$$

Here we made use of the existence of convolutions for generalized functions taken from $\mathscr{D}'_{\mathscr{L}^2}$ [property (b)] and of the properties of convolutions (see Sec. 10.1, Sec. 4.2, and Sec. 4.6).

(f) The matrix $S(x)$ has the Laplace transform $\tilde{S}(\zeta)$ that is holomorphic in $T^C$, $\tilde{S}_{kj} \in H(C)$, and that satisfies the reality condition:

$$\tilde{S}(\zeta) = \bar{\tilde{S}}(-\bar{\zeta}), \qquad \zeta \in T^C. \tag{1.7}$$

This follows from (1.1) and (1.3) by virtue of the results of Sec. 12.2.

(g) The boundary value $\tilde{S}(p) = F[S]$ of the matrix $\tilde{S}(\zeta)$ satisfies the inequality (for almost all $p \in \mathbb{R}^n$)

$$I - \tilde{S}^+(p)\,\tilde{S}(p) \geqslant 0. \tag{1.8}$$

In particular, for almost all $p \in \mathbb{R}^n$,

$$\sum_{1\leqslant k\leqslant N} |\tilde{S}_{kj}(p)|^2 \leqslant 1, \qquad j = 1, \ldots, N. \tag{1.9}$$

This follows from (1.6) by the Bochner-Schwartz theorem (see Sec. 8.2):

$$\langle \tilde{S}^+(p)\,\tilde{S}(p)\, a,\ a\rangle = |\tilde{S}(p)\, a|^2 \leqslant |a|^2, \qquad a \in \mathbb{C}^N. \tag{1.8'}$$

(h) The matrix function $\tilde{S}(\zeta)$ satisfies the inequality

$$I-\tilde{S}^{+}(\zeta)\tilde{S}(\zeta)\geqslant 0, \qquad \zeta\in T^{C}. \tag{1.10}$$

In particular,

$$\sum_{1\leqslant k\leqslant N} |\tilde{S}_{kj}(\zeta)|^{2}\leqslant 1, \qquad \zeta\in T^{C}, \quad j=1, \ldots, N. \tag{1.11}$$

Indeed, from (1.8′) for all $a$ and $b$ in $\mathbb{C}^N$ we have

$$\langle\tilde{S}(p)a, b\rangle\leqslant|a||b| \quad \text{for almost all } p\in\mathbb{R}^{n}. \tag{1.12}$$

Furthermore, the function $\langle\tilde{S}(\zeta)a, b\rangle$ belongs to the class $H(C)$ [see (f)], and its boundary value $\langle\tilde{S}(p)a, b\rangle$ satisfies the estimate (1.12). By the Phragmén-Lindelöf theorem (see Sec. 12.6) the function $\langle\tilde{S}(\zeta)a, b\rangle$ satisfies the estimate

$$|\langle\tilde{S}(\zeta)a, b\rangle|\leqslant|a||b|, \qquad \zeta\in T^{C},$$

from which, for $b=\tilde{S}(\zeta)a$, follows the estimate

$$|\tilde{S}(\zeta)a|^{2}\leqslant|a|^{2}, \qquad \zeta\in T^{C}, \quad a\in\mathbb{C}^{N}, \tag{1.10′}$$

which is equivalent to the estimate (1.10).

Every matrix that is holomorphic in $T^C$ and satisfies the conditions of reality (1.7) and boundedness (1.10) is said to be *bounded-real* in $T^C$.

We have thus proved that the matrix function $\tilde{S}(\zeta)$ is bounded-real in $T^C$.

## 20.2 A description of abstract scattering matrices

Theorem *In order that the matrix $S(x)$ should define a scattering operator relative to the cone $\Gamma$, it is necessary and sufficient that its Laplace transform $\tilde{S}(\zeta)$ be a bounded-real matrix function in the region of $T^C$, where $C=$ int $\Gamma^*$.*

*Proof.* Necessity was proved in Sec. 20.1. We now prove sufficiency. Suppose $\tilde{S}(\zeta)$ is a bounded-real matrix function in $T^C$. Then it is a Laplace transform,

$$\tilde{S}(\zeta)=L[S]=F[Se^{-(q,x)}],$$

of a real matrix $S(x)$ with elements taken from $\mathscr{S}'$ that satisfies the causality condition (1.1) relative to the cone $C^* = \Gamma$; here, $\tilde{S}(p) = F[S]$, where $\tilde{S}(p)$ is the boundary value of $\tilde{S}(\zeta)$ as $q \to 0$, $q \in C$, in $\mathscr{S}'$ (see Sec. 12.2). Furthermore, since the elements of the matrix $\tilde{S}(\zeta)$ are uniformly bounded in $T^C$ and converge weakly as $q \to 0$, $q \in C$, on the set $\mathscr{S}$ that is dense in $\mathscr{L}'$ (see Sec. 1.2), it follows that the elements of the boundary matrix $\tilde{S}(p)$ may be identified with functions taken from $\mathscr{L}^\infty$.

Suppose $\varphi \in \mathscr{D}^{\times N}$ and then $F[\varphi] = \tilde{\varphi} \in \mathscr{S}^{\times N}$. From the condition (1.10′) we have, for every $q \in C$, the inequality

$$\int \langle \tilde{S}(\zeta)\tilde{\varphi}(p), \tilde{S}(\zeta)\tilde{\varphi}(p)\rangle \, dp \leqslant \int \langle \tilde{\varphi}, \tilde{\varphi}\rangle \, dp.$$

From this, using the Parseval equation (see Sec. 6.6(c)) and, also the Fourier transform theorem of a convolution (see Sec. 6.5), we obtain

$$\int \langle [Se^{-(q,x)}] * \varphi, [Se^{-(q,x)}] * \varphi\rangle \, dx \leqslant \int \langle \varphi, \varphi\rangle \, dx, \qquad q \in C. \quad (2.1)$$

We now apply formula (2.9) of Sec. 9,

$$[Se^{-(q,x)}] * \varphi = e^{-(q,x)} (S * [\varphi e^{(q,x)}]).$$

As a result, inequality (2.1) becomes

$$\int e^{-2(q,x)} \langle S * [\varphi e^{(q,x)}], S * [\varphi e^{(q,x)}]\rangle \, dx \leqslant \int \langle \varphi, \varphi\rangle \, dx, \qquad q \in C. \quad (2.2)$$

We now prove the possibility of passing to the limit as $q \to 0$, $q \in C$, under the sign of the integral on the left-hand side of (2.2). For this purpose, note that since all elements of the matrix $\tilde{S}(p)$ belong to $\mathscr{L}^\infty$ and, hence, all elements of the matrix $S = F^{-1}[\tilde{S}(p)]$ belong to $\mathscr{D}'_{\mathscr{L}^2}$ (see Sec. 10.1), and the representation (1.2) of Sec. 10 holds for the latter, it suffices to consider the case where all elements of the matrix $S$ belong to $\mathscr{L}^2$. In that case, the following properties hold true:

(a) $|S_{kj} * [\varphi_j e^{(q,x)}]| \leqslant |S_{kj}| * [|\varphi_j| e^{|x|}] \in \mathscr{L}^2$, $|q| \leqslant 1$ (see Sec. 4.1(b));

(b) $$S * [\varphi e^{(q,x)}] = \int S(x') \varphi(x - x') e^{(q, x-x')} \, dx' \to \int S(x')\varphi(x-x')\, dx' = S * \varphi, \qquad q \to 0;$$

(c) $\qquad \operatorname{supp} S * [\varphi e^{(q,x)}] \subset \overline{\operatorname{supp} S + \operatorname{supp} \varphi} \subset \Gamma + \bar{U}_R,$

if $\operatorname{supp} \varphi \subset \bar{U}_R$ (see Sec. 4.2(g));

(d) $\qquad e^{-2(q,x)} \leqslant e^{2R}, \qquad x \in \Gamma + \bar{U}_R, \quad |q| \leqslant 1, \quad q \in C.$

The properties from (a) to (d) are what ensure, by the Lebesgue theorem, the possibility of passing to the limit as $q \to 0$, $q \in C$, under the integral sign in the left-hand member of (2.2). We thus obtain the boundedness condition (1.2). The theorem is proved.

*Remark 1.* This theorem was proved for $n = 1$ by Beltrami and Wohlers [1] and for $n \geqslant 2$ by Vladimirov [9].

*Remark 2.* The proof of sufficiency in the theorem is simplified if use is made of the well-known Fatou theorem, according to which $\tilde{S}(p + iq) \to \tilde{S}(p)$, $q \to 0$, $q \in C$, for almost all $p \in \mathbb{R}^n$. It will be noticed that we did not make use of the Fatou theorem.

**20.3 The relationship between passive operators and scattering operators** In order to establish the relationship between the scattering matrix $S(x)$ and passive operators, it is convenient to introduce a new matrix $T(x)$ via the formula

$$S(x) = I\delta(x) + 2iT(x). \tag{3.1}$$

It is called the *scattering amplitude.*

The scattering operator $S *$ relative to the cone $\Gamma$ is said to be *nonsingular* if

$$\det [I - \tilde{S}(\zeta)] \neq 0, \qquad \zeta \in T^C. \tag{3.2}$$

In (3.1) let us pass to the Laplace transform

$$\tilde{S}(\zeta) = I + 2i\tilde{T}(\zeta), \qquad \zeta \in T^C. \tag{3.3}$$

By the theorem of Sec. 20.2 the matrix $\tilde{S}(\zeta)$ is bounded-real in $T^C$, that is, it is holomorphic in $T^C$ and satisfies the conditions (1.7) and (1.10). Therefore, it follows from (3.2) and (3.3) that the matrix $i\tilde{T}(\zeta)$ is holomorphic in $T^C$, satisfies the reality condition (1.7), $\det \tilde{T}(\zeta) \neq 0$, $\zeta \in T^C$, and

$$\operatorname{Re} [-i\tilde{T}(\zeta)] = -\frac{i}{2} [\tilde{T}(\zeta) - \tilde{T}^+(\zeta)] \geqslant \tilde{T}^+(\zeta)\, \tilde{T}(\zeta) > 0, \qquad \zeta \in T^C. \tag{3.4}$$

Thus the matrix $i\widetilde{T}(\zeta)$ is positive real in $T^C$ and, by Theorem I of Sec. 19.3, the matrix $iT(x)$ defines the passive operator $-iT*$ relative to the cone $\Gamma$; by virtue of (3.4), that operator is completely nonsingular (see Sec. 19.5). By Theorem I of Sec. 19.5, there exists an inverse operator $(-iT)^{-1}*$ that is a completely nonsingular passive operator relative to the cone $\Gamma$.

Let us now introduce new vector quantities—$j$ for "current" and $v$ for "voltage"—via the formulas

$$j = u - S * u, \qquad v = u + S * u. \tag{3.5}$$

By (3.1), $S * u = u + 2iT * u$, whence

$$j = -2iT * u = -iT * (v + j)$$

and therefore $j$ and $v$ are connected by the relation

$$v = Z * j \tag{3.6}$$

where

$$\begin{aligned} Z * &= (-iT)^{-1} * -I\delta * \\ &= 2(I\delta - S)^{-1} * -I\delta * = (I\delta - S)^{-1} * (I\delta + S) *. \end{aligned} \tag{3.7}$$

We now prove that the matrix $Z(x)$ defines a passive operator relative to the cone $\Gamma$.

Indeed, the matrix $Z(x)$ is real and, by (3.7) and (3.4), for all $\zeta \in T^C$,

$$\begin{aligned} \operatorname{Re}\widetilde{Z}(\zeta) &= \frac{1}{2}[\widetilde{Z}(\zeta) + \widetilde{Z}^+(\zeta)] = \frac{1}{2i}[\widetilde{T}^{+^{-1}}(\zeta) - \widetilde{T}^{-1}(\zeta)] - I \\ &= \widetilde{T}^{+^{-1}}(\zeta)\left\{-\frac{i}{2}[\widetilde{T}(\zeta) - \widetilde{T}^+(\zeta)] - \widetilde{T}^+(\zeta)\widetilde{T}(\zeta)\right\}\widetilde{T}^{-1}(\zeta) \geqslant 0. \end{aligned}$$

Thus the matrix function $\widetilde{Z}(\zeta)$ is positive real in $T^C$ and, hence, the matrix $Z(x)$ defines a passive operator relative to the cone $\Gamma$.

*Remark.* If, in addition to the condition (3.2), we require fulfillment of the condition

$$\det[I + \widetilde{S}(\zeta)] \neq 0, \qquad \zeta \in T^C,$$

then the operator $Z*$ is nonsingular by (3.3) and (3.7),

$$\det\widetilde{Z}(\zeta) = \left(\frac{i}{2}\right)^n \frac{\det[I + \widetilde{S}(\zeta)]}{\det\widetilde{T}(\zeta)} \neq 0, \qquad \zeta \in T^C.$$

Conversely, suppose we have a passive operator $Z *$ relative to the cone $\Gamma$. Then the operator $Z * + I\delta *$ is passive and completely nonsingular relative to the cone $\Gamma$. Therefore there exists the inverse operator

$$Q * = (Z + I\delta)^{-1} * \tag{3.8}$$

which is a passive, completely nonsingular operator relative to $\Gamma$ (see Sec. 19.5). We now prove the inequality

$$\operatorname{Re} \tilde{Q}(\zeta) \geqslant \tilde{Q}^{+}(\zeta)\, \tilde{Q}(\zeta), \qquad \zeta \in T^{C}. \tag{3.9}$$

Indeed, by (3.8), the matrix $Q(x)$ is real and (see Sec. 19.3)

$$\tilde{Z}(\zeta) = \tilde{Q}^{-1}(\zeta) - I, \qquad \operatorname{Re} \tilde{Z}(\zeta) \geqslant 0, \quad \zeta \in T^{C},$$

whence follows the inequality (3.9):

$$\begin{aligned}
\operatorname{Re} \tilde{Q}(\zeta) &= \frac{1}{2}[\tilde{Q}(\zeta) + \tilde{Q}^{+}(\zeta)] \\
&= \tilde{Q}^{+}(\zeta)\, \tilde{Q}(\zeta) + \frac{1}{2}\{\tilde{Q}^{+}(\zeta)\, [\tilde{Q}^{-1}(\zeta) - I]\, \tilde{Q}(\zeta) \\
&\qquad + \tilde{Q}^{+}(\zeta)\, [\tilde{Q}^{+^{-1}}(\zeta) - I]\, \tilde{Q}(\zeta)\} \\
&= \tilde{Q}^{+}(\zeta)\, \tilde{Q}(\zeta) + \operatorname{Re} \tilde{Q}^{+}(\zeta)\, \tilde{Z}(\zeta)\, \tilde{Q}(\zeta) \geqslant \tilde{Q}^{+}(\zeta)\, \tilde{Q}(\zeta).
\end{aligned}$$

Let us now prove that the operator

$$S * = I\delta * - 2Q * = (Z + I\delta)^{-1} * (Z - I\delta) * \tag{3.10}$$

is a scattering operator relative to the cone $\Gamma$.

Indeed, $S(x)$ is a real matrix and

$$\tilde{S}(\zeta) = I - 2\tilde{Q}(\zeta), \qquad \zeta \in T^{C}$$

and so, by virtue of (3.9), we have

$$\begin{aligned}
\tilde{S}^{+}(\zeta)\, \tilde{S}(\zeta) &= [I - 2\tilde{Q}^{+}(\zeta)]\, [I - 2\tilde{Q}(\zeta)] \\
&= I - 4\operatorname{Re} \tilde{Q}(\zeta) + 4\tilde{Q}^{+}(\zeta)\, \tilde{Q}(\zeta) \leqslant I.
\end{aligned}$$

Thus the matrix function $\tilde{S}(\zeta)$ is bounded-real in $T^{C}$. By the theorem of Sec. 20.2, the matrix $S(x)$ defines a scattering operator relative to the cone $\Gamma$.

We have thus proved the following theorem.

Theorem *Every nonsingular abstract scattering operator* $S\,*$ *defines, by the formula*

$$Z\,* = (I\delta - S)^{-1} * (I\delta + S)\,*,$$

*a passive operator; conversely, every passive operator* $Z\,*$ *defines, via the formula*

$$S\,* = (Z + I\delta)^{-1} * (Z - I\delta)\,*,$$

*an abstract scattering operator.*

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### by S. P. Strelkov, D. Sc.

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# Contents

21. The totality of (measurable) functions $f(x)$ for which $|f|^p$ is integrable on $G$ is denoted by $L_p(G)$. In $L_p(G)$ the norm is

$$\| f \|_{L_p(G)} = \left[ \int_G |f|^p \, dx \right]^{1/p}, \quad 1 \leqslant p < \infty$$

$$\| f \|_{L_\infty(G)} = \operatorname*{vrai\,sup}_{x \in G} |f(x)|, \quad p = \infty$$

The scalar product in $L_2(G)$ is introduced thus:

$$(f, g) = \int_G f\bar{g} \, dx, \quad f, g \in L_2(G).$$

22. Let $\rho(x)$ be a continuous positive-valued function in a region $G$. The totality of (measurable) functions $f(x)$ for which $\rho(x) \, |f(x)|^2$ is integrable on $G$ is denoted by $L_{2,\rho}(G)$ and constitutes a Hilbert space with the scalar product

$$(f, g)_{L_{2,\rho}}(G) = \int_G \rho f \bar{g} \, dx.$$

23. Cylinder functions.

(a) Bessel functions:

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\nu+1)\,\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+\nu}, \quad -\infty < x < \infty;$$

(b) Neumann's Bessel functions:

$$N_\nu(x) = \frac{1}{\sin \pi\nu} [J_\nu(x) \cos \pi\nu - J_{-\nu}(x)], \quad \nu \neq n,$$

$$N_n(x) = \frac{1}{\pi} \left[ \frac{\partial J_\nu(x)}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}(x)}{\partial \nu} \right], \quad \nu = n;$$

(c) Hankel functions:

$$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x), \quad H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x);$$

(d) modified Bessel and Hankel functions:

$$I_\nu(x) = e^{-\frac{\pi}{2}\nu i} J_\nu(ix), \quad K_\nu(x) = \frac{\pi i}{2} e^{\frac{\pi}{2}\nu i} H_\nu^{(1)}(ix).$$

of motion (vibrations) of the string we examine the portion of the string between $x$ and $x + \Delta x$ and project all the forces acting on this portion (including the forces of inertia) on the axes of coordinates. According to D'Alembert's principle, the sum of the projections of all forces on each axis must be zero. Since we are studying only transverse vibrations, we can assume that the external forces and the force of inertia act along the $u$ axis. We will consider only small

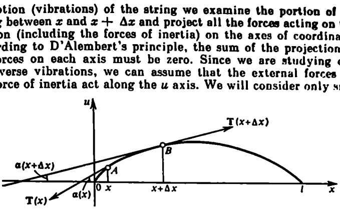


Fig. 1

vibrations, which means that in deriving our equation we will neglect squares of $u_x(x, t)$. The length $S$ of the arc $AB$ is expressed by the integral $S = \int\limits_x^{x+\Delta x} (1 + u_x^2)^{1/2}\, dx \cong \Delta x$. This means that no portion of the string changes its length during vibrations and, hence, by Hooke's law the magnitude of the tensile force $T_0 = |\mathbf{T}|$ depends neither on time nor on $x$. Let us find the projections of all the forces at time $t$ onto the $u$ axis. Up to first-order infinitesimals, the projection of the tensile force is (see Fig. 1)

$$\begin{aligned} &T_0[\sin\alpha(x+\Delta x) - \sin\alpha(x)] \\ &= T_0\left[\frac{\tan\alpha(x+\Delta x)}{\sqrt{1+\tan^2\alpha(x+\Delta x)}} - \frac{\tan\alpha(x)}{\sqrt{1+\tan^2\alpha(x)}}\right] \\ &= T_0\left[\frac{u_x(x+\Delta x, t)}{\sqrt{1+u_x^2(x+\Delta x, t)}} - \frac{u_x(x, t)}{\sqrt{1+u_x^2(x, t)}}\right] \\ &\cong T_0[u_x(x+\Delta x, t) - u_x(x, t)] \\ &\cong T_0 u_{xx}(x, t)\,\Delta x. \end{aligned}$$

Let $p(x, t)$ be the continuous linear density of the external forces acting on the string. Then the force that acts on the portion $AB$ along the $u$ axis is $p(x, t)\,\Delta x$. To find the force of inertia acting on $AB$, we recall the expression $-mu_{tt}$, with $m$ the mass of that portion of the string. If $\rho(x)$ is the continuous linear density of the material of the string, then $m = \rho\,\Delta x$. Thus the projection of the force of inertia on the $u$ axis is $-\rho u_{tt}\,\Delta x$, and the projection of all the forces on the same axis obeys the equation

$$[T_0 u_{xx} + p(x, t) - \rho(x)\, u_{tt}]\,\Delta x = 0. \qquad (1.1)$$

Hence,

$$T_0 u_{xx} - \rho(x)\, u_{tt} + p(x, t) = 0,$$

from time $t = 0$ a magnetic field $H = H_0 \sin \Omega t$ is maintained at the surface $x = 0$ (the magnetic field is directed parallel to the surface).

**1.34.** State the boundary value problem of determining the temperature of a rod $0 \leqslant x \leqslant l$ with heat-insulated lateral surface. Consider the cases where (a) the ends of the rod are kept at a given temperature, (b) at the ends of the rod a given heat flux is maintained, and (c) at the ends of the rod there is convective heat exchange by Newton's law with a medium whose temperature is given.

**1.35.** Derive the equation of diffusion in a medium at rest assuming that the planes perpendicular to the $x$ axis are at each moment of time $t$ the surfaces of constant density. Write the boundary conditions assuming that diffusion occurs in the flat layer $0 \leqslant x \leqslant l$ and consider the cases where (a) the concentration of the diffusing substance is kept equal to zero at the boundary planes, (b) the boundary planes are impenetrable, and (c) the boundary surfaces are semipermeable, with diffusion through these planes governed by a law similar to Newton's law for convective heat exchange.

**1.36.** Derive the equation of diffusion for a disintegrating gas (the number of molecules decaying every second at a given point is proportional to the density with a positive proportionality factor $\alpha$).

**1.37.** Suppose we have a thin homogeneous rod of length $l$ whose initial temperature is $f(x)$. State the boundary value problem of determining the temperature of the rod if a constant temperature $u_0$ is maintained at the end $x = 0$, while the lateral surface and the end $x = l$ are involved in convective heat exchange (governed by Newton's law) with the surrounding medium whose temperature is zero.

**1.38.** State the problem of determining the temperature distribution in a thin heat-insulated rod of infinite length along which a point heat source begins to move at time $t = 0$. The source moves in the positive direction, its velocity is $v_0$, and it generates $q$ units of heat every second.

**1.39.** State the boundary value problem of the cooling off of a thin homogeneous ring of radius $R$ whose surface is involved in convective heat exchange with the surrounding medium whose temperature is known. The nonuniformity of the temperature distribution in the ring can be ignored.

**1.40.** Derive the equation that describes the diffusion of suspended particles and allows for precipitation. Assume that the velocity of the particles caused by gravity is constant and the density of the particles depends only on the height $z$ and the time $t$. Write the boundary condition that corresponds to an impenetrable partition.

**1.41.** State the boundary value problem of the cooling off of a uniformly heated rod in the form of a frustum of a cone (the warpage

1.16. $\left.\begin{array}{l} u^1_{tt} = a_1^2 u^1_{xx}, \quad -\infty < x < 0 \\ u^2_{tt} = a_2^2 u^2_{xx}, \quad 0 < x < +\infty \end{array}\right\} t > 0$, $u^1(0, t) = u^2(0, t)$, $E_1 u^1_x(0, t) = E_2 u^2_x(0, t)$, $t > 0$, $u^1(x, 0) = f(x)$, $u^1_t(x, 0) = F(x)$, $-\infty < x < 0$, $u^2(x, 0) = f(x)$, $u^2_t(x, 0) = F(x)$, $x > 0$, where $u^1$ and $u^2$ are the deflections of the points of the left and right rods, and $a_i^2 = E_i/\rho_i$, $i = 1, 2$.

1.18. $\frac{\partial^2 u}{\partial t^2} = g\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)$, $0<x<l$, $t>0$, $|u(0, t)| < \infty$, $u(l, t) = 0$, $t > 0$, $u|_{t=0} = f(x)$, $u_t|_{t=0} = F(x)$, $0 \leqslant x \leqslant l$.

1.19. $\nabla^2\Phi^{(e)} = 0$, $r > R$, $\nabla^2\Phi^{(i)} = 0$, $0 \leqslant r < R$, $\operatorname{grad}\Phi \parallel H$. $\Phi_r^{(e)}|_{r=R} = \Phi_r^{(i)}|_{r=R}$, $\Phi_\varphi^{(e)}|_{r=R} = \left(\Phi_\varphi^{(i)} + \frac{4\pi}{c} j_{sur}\right)\Big|_{r=R}$, $|\Phi^{(i)}(0, t)| < \infty$, $j_{sur} = J/2\pi R$ is the surface current density, and $\Phi^{(i)}$ and $\Phi^{(e)}$ are the magnetic field potentials inside and outside the conductor, respectively.

1.20. $J_x = -Cv_t$, $v_x = -LJ_t$, $0<x<l$, $t > 0$, $v(0, t) = \frac{1}{C}\int_0^t J\,dt$ at the grounded end and $v_x(l, t) = 0$ at the insulated end.

1.21. $\theta_{tt} = a^2\theta_{xx}$, $0 < x < l$, $t > 0$, $\theta|_{t=0} = \alpha x/l$, $\theta_t|_{t=0} = 0$, $0 \leqslant x \leqslant l$, $\theta|_{x=0} = 0$, $\theta_x|_{x=l} = -\frac{\Phi}{JG}\theta_{tt}$, where the constants, $a^2$, $\Phi$, $J$, and $G$ have the same meaning as in Problem 1.4.

1.22. $u_{tt} = a^2 u_{xx} + g$, $0 < x < l$, $t > 0$, $u(x, 0) = u_t(x, 0) = 0$, $0 \leqslant x \leqslant l$, $u(0, t) = 0$, $u_x(l, t) = 0$, $t > 0$, $a^2 = E/\rho$.

1.23. $u_{tt} = a^2 u_{xx} + g$, $0 < x < l$, $t > 0$, $u|_{t=0} = u_t|_{t=0} = 0$, $0 \leqslant x \leqslant l$, $u|_{x=0} = 0$, $\frac{Q}{g}u_{tt}|_{x=l} = -ESu_x|_{x=l} - Q$.

1.25. $u_{tt} = a^2\left(u_{rr} + \frac{1}{r}u_r\right)$, $0<r<R$, $t>0$, $u(r, 0) = f(r)$, $u_t(r, 0) = F(r)$, $0 \leqslant r \leqslant R$, $|u(0, t)| < \infty$, $u_r|_{r=R} = 0$.

1.26. $\nabla^2\varphi = 0$, $r > R$, $t > 0$, $\left.\frac{\partial\varphi}{\partial r}\right|_{r=R} = 0$, $t>0$, $\lim\limits_{r\to\infty} v = \lim\limits_{r\to\infty}\operatorname{grad}\varphi = v_0$, where $v_0$ is the flow velocity at infinity.

1.27. $u_{tt} = a^2\left(u_{rr} + \frac{2}{r}u_r\right)$, $0 \leqslant r < R$, $t>0$, $u(r, 0) = f(r)$, $u_t|_{t=0} = F(r)$, $0 \leqslant r \leqslant R$, $|u(0, t)| < \infty$, $u_r|_{r=R} = 0$, where $a^2 = \gamma p_0/\rho_0$.

1.29. $u_{tt} + ku_t = a^2\nabla^2 u + f(r, \varphi, t)/\rho$, $0 \leqslant r < R$, $0 \leqslant \varphi < 2\pi$, $t > 0$, $u|_{t=0} = u_t|_{t=0} = 0$, $|u(0, \varphi, t)| < \infty$, $u(R, \varphi, t) = 0$, where $a^2 = T/\rho$, $k = \alpha/\rho$, and $\alpha$ is the coefficient of the elastic resistance of the medium.