EN1020 Signals and Systems: Fourier Series

Ranga Rodrigo
ranga@uom.lk

The University of Moratuwa, Sri Lanka

January 24, 2026

Section 1

Continuous-Time Fourier Series

Outline

Introduction

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.

Introduction

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.
- We use the Fourier series for periodic signals and the Fourier transform for aperiodic signals.

Introduction

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.
- We use the Fourier series for periodic signals and the Fourier transform for aperiodic signals.
- Each of these has continuous-time and discrete-time versions:

Introduction

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.
- We use the Fourier series for periodic signals and the Fourier transform for aperiodic signals.
- Each of these has continuous-time and discrete-time versions:
    1. Continuous-time Fourier series

Introduction

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.
- We use the Fourier series for periodic signals and the Fourier transform for aperiodic signals.
- Each of these has continuous-time and discrete-time versions:
    1. Continuous-time Fourier series
    2. Continuous-time Fourier transform

Introduction

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.
- We use the Fourier series for periodic signals and the Fourier transform for aperiodic signals.
- Each of these has continuous-time and discrete-time versions:
    1. Continuous-time Fourier series
    2. Continuous-time Fourier transform
    3. Discrete-time Fourier series

Introduction

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.
- We use the Fourier series for periodic signals and the Fourier transform for aperiodic signals.
- Each of these has continuous-time and discrete-time versions:
    1. Continuous-time Fourier series
    2. Continuous-time Fourier transform
    3. Discrete-time Fourier series
    4. Discrete-time Fourier transform

Introduction

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.
- We use the Fourier series for periodic signals and the Fourier transform for aperiodic signals.
- Each of these has continuous-time and discrete-time versions:
    1. Continuous-time Fourier series
    2. Continuous-time Fourier transform
    3. Discrete-time Fourier series
    4. Discrete-time Fourier transform
- In this part of the course, we will focus on computing the continuous-time Fourier series and Fourier transform. Later, after covering linear time-invariant (LTI) systems, we will explore the conceptual aspects of Fourier techniques.

Figure: Jean-Baptiste Joseph Fourier, 1768–1830, French mathematician who discovered Fourier series and transform

- Every signal has a frequency distribution or a spectrum.

Figure: Jean-Baptiste Joseph Fourier, 1768–1830, French mathematician who discovered Fourier series and transform

- Every signal has a frequency distribution or a spectrum.
- Periodic signals have a line spectrum, called the Fourier series.

Figure: Jean-Baptiste Joseph Fourier, 1768–1830, French mathematician who discovered Fourier series and transform

- Every signal has a frequency distribution or a spectrum.
- Periodic signals have a line spectrum, called the Fourier series.
- The French mathematician, Jean-Baptiste Joseph Fourier, discovered this representation.

Figure: Jean-Baptiste Joseph Fourier, 1768–1830, French mathematician who discovered Fourier series and transform

- Every signal has a frequency distribution or a spectrum.
- Periodic signals have a line spectrum, called the Fourier series.
- The French mathematician, Jean-Baptiste Joseph Fourier, discovered this representation.
- Fourier series provides a way to represent a periodic signal as a sum of complex exponentials.

Figure: Jean-Baptiste Joseph Fourier, 1768–1830, French mathematician who discovered Fourier series and transform

- Every signal has a frequency distribution or a spectrum.
- Periodic signals have a line spectrum, called the Fourier series.
- The French mathematician, Jean-Baptiste Joseph Fourier, discovered this representation.
- Fourier series provides a way to represent a periodic signal as a sum of complex exponentials.
- These sinusoids will be at frequencies that are integer multiples of the fundamental frequency ω_0.

Figure: Jean-Baptiste Joseph Fourier, 1768–1830, French mathematician who discovered Fourier series and transform

- Every signal has a frequency distribution or a spectrum.
- Periodic signals have a line spectrum, called the Fourier series.
- The French mathematician, Jean-Baptiste Joseph Fourier, discovered this representation.
- Fourier series provides a way to represent a periodic signal as a sum of complex exponentials.
- These sinusoids will be at frequencies that are integer multiples of the fundamental frequency ω_0.
- $\omega_0 = \dfrac{2\pi}{T}$, where T: fundamental period of the waveform.

Outline

Continuous-Time Fourier Series

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\
a_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\
\omega_0 &= \frac{2\pi}{T}
\end{aligned}
\tag{1}
$$

Continuous-Time Fourier Series

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$
$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \qquad (1)$$
$$\omega_0 = \frac{2\pi}{T}$$

$$\boxed{\begin{aligned} a_k &= \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt \end{aligned}}$$

Continuous-Time Fourier Series

$$
\begin{aligned}
a_k &= \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jk\omega_0 t}dt \\
&= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
\omega_0 &= \frac{2\pi}{T}
\end{aligned}
\tag{1}
$$

The set of coefficients $\{a_k\}$ is called the Fourier series coefficients or the spectral coefficients of $x(t)$. The coefficient a_0 is the dc or constant component of $x(t)$, given by Equation 1 with $k = 0$:

$$
a_0 = \frac{1}{T}\int_T x(t)dt, \tag{2}
$$

which is simply the average of $x(t)$ over one period.

Example

Let

$$x(t) = \sin \omega_0 t,$$

which has the fundamental frequency ω_0.

Example

Let

$$x(t) = \sin \omega_0 t,$$

which has the fundamental frequency ω_0.

Euler's formula

$e^{j\theta} = \cos\theta + j\sin\theta$

$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

Example

Let

$$x(t) = \sin \omega_0 t,$$

which has the fundamental frequency ω_0.

> Euler's formula
> $e^{j\theta} = \cos\theta + j\sin\theta$
> $\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

$$\sin \omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

Comparing the right-hand side of this equation and Equation 1, we obtain

$$a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j}$$

$$a_k = 0, \qquad k \neq \pm 1.$$

Example

Let

$$x(t) = 1 + \sin \omega_0 t + 2 \cos \omega_0 t + \cos \left(2\omega_0 t + \frac{\pi}{4} \right),$$

which has the fundamental frequency ω_0.

1. Use Euler's formula to express $x(t)$ as a linear combination of complex exponentials.
2. Find the Fourier series coefficients, a_k.
3. Plot the magnitude and phase of a_k.

$$x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Using Euler's formula

$$x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}\right]$$

$$x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Using Euler's formula

$$x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}\right]$$

Collecting terms,

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j\pi/4}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j\pi/4}\right)e^{-j2\omega_0 t}$$

$$x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

Euler's formula
$e^{j\theta} = \cos\theta + j\sin\theta$
$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Using Euler's formula

$$x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}\right]$$

Collecting terms,

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j\pi/4}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j\pi/4}\right)e^{-j2\omega_0 t}$$

The Fourier coefficients are

$$a_0 = 1,$$
$$a_1 = \left(1 + \frac{1}{2j}\right) = \left(1 - \frac{j}{2}\right),$$
$$a_{-1} = \left(1 - \frac{1}{2j}\right) = \left(1 + \frac{j}{2}\right),$$

$$a_2 = \frac{1}{2}e^{j\pi/4} = \frac{\sqrt{2}}{4}(1 + j),$$
$$a_{-2} = \frac{1}{2}e^{-j\pi/4} = \frac{\sqrt{2}}{4}(1 - j),$$
$$a_k = 0, |k| > 2.$$

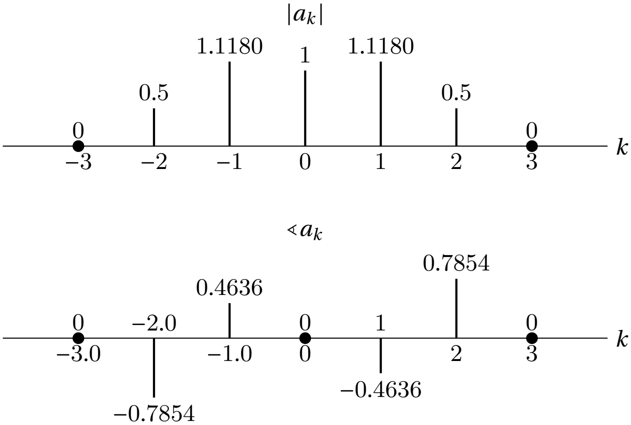


Figure: $|a_k|$, $\sphericalangle a_k$

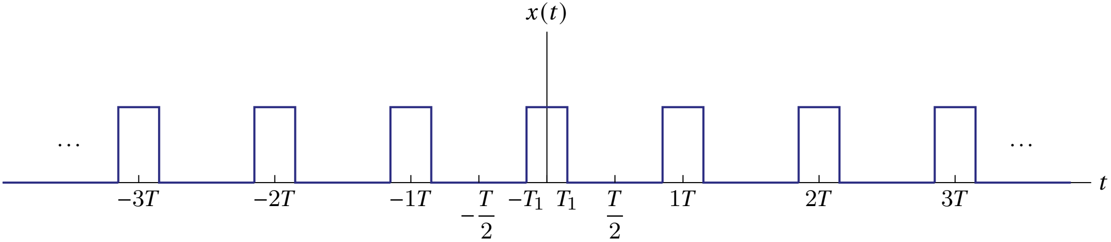
Example

The periodic square wave, sketched below, is defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < T/2, \end{cases}$$

This signal is periodic with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$.

1. Find the Fourier series coefficients, a_k.
2. Plot the magnitude and phase of a_k for the case $T = 4T_1$.



$$
\begin{aligned}
a_0 &= \frac{1}{T} \int_T x(t) dt, \\
&= \frac{1}{T} \int_{-T_1}^{T_1} 1 dt, \\
&= \frac{2T_1}{T}.
\end{aligned}
$$

$$\begin{aligned}
a_0 &= \frac{1}{T}\int_T x(t)dt, \\
&= \frac{1}{T}\int_{-T_1}^{T_1} 1dt, \\
&= \frac{2T_1}{T}.
\end{aligned}$$

$$\begin{aligned}
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt, \\
&= \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt, \\
&= \frac{1}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-T_1}^{T_1}
\end{aligned}$$

$$
\begin{aligned}
a_0 &= \frac{1}{T}\int_T x(t)dt,\\
&= \frac{1}{T}\int_{-T_1}^{T_1} 1dt,\\
&= \frac{2T_1}{T}.
\end{aligned}
$$

$$
\begin{aligned}
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt,\\
&= \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt,\\
&= \frac{1}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-T_1}^{T_1}
\end{aligned}
$$

Euler's formula

$e^{j\theta} = \cos\theta + j\sin\theta$

$\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$

$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

$$
\begin{aligned}
a_0 &= \frac{1}{T}\int_T x(t)dt, \\
&= \frac{1}{T}\int_{-T_1}^{T_1} 1dt, \\
&= \frac{2T_1}{T}.
\end{aligned}
$$

$$
\begin{aligned}
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt, \\
&= \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt, \\
&= \frac{1}{T}\left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_{-T_1}^{T_1}
\end{aligned}
$$

$$
a_k = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right]
$$

$$
a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, k \neq 0.
$$

Euler's formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

For $T = 4T_1$

$$a_k = 0, \quad k \text{ even.}$$

$$a_0 = \frac{1}{2}$$

$$a_1 = a_{-1} = \frac{1}{\pi}$$

$$a_3 = a_{-3} = -\frac{1}{3\pi}$$

$$a_5 = a_{-5} = \frac{1}{5\pi}$$

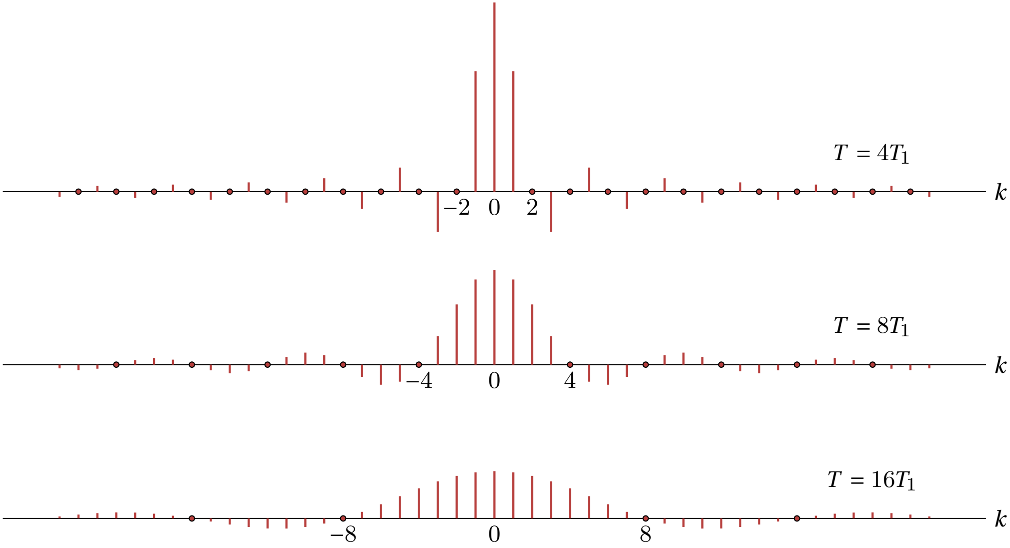


Figure: Plots of scaled Fourier series coefficients Ta_k

Outline

Properties of the Continuous-Time Fourier Series

Suppose that $x(t)$ is periodic signal with period T and fundamental frequency $\omega_0 = 2\pi/T$. If the Fourier series coefficients are denoted by a_k, then

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \tag{3}$$

Linearity

Let $x(t)$ and $y(t)$ denote two periodic signals with period T.

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k,$$
$$y(t) \xleftrightarrow{\mathcal{FS}} b_k.$$

Any linear combination of the two signals will also be periodic with period T. Fourier series coefficients c_k of the linear combination of $x(t)$ and $y(t)$, $z(t) = Ax(t) + By(t)$, are given by the same linear combination:

Linearity

Let $x(t)$ and $y(t)$ denote two periodic signals with period T.

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k,$$
$$y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k.$$

Any linear combination of the two signals will also be periodic with period T. Fourier series coefficients c_k of the linear combination of $x(t)$ and $y(t)$, $z(t) = Ax(t) + By(t)$, are given by the same linear combination:

$$z(t) = Ax(t) + By(t) \overset{\mathcal{FS}}{\longleftrightarrow} c_k = Aa_k + Bb_k. \tag{4}$$

Time Shifting

$$x(t-t_0) \xleftrightarrow{\mathcal{FS}} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k \tag{5}$$

Proof:

Fourier series:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

Time Shifting

$$x(t-t_0) \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k \tag{5}$$

Fourier series:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

Proof:

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k, \quad x(t-t_0) \overset{\mathcal{FS}}{\longleftrightarrow} b_k,$$

$$\begin{aligned} b_k &= \frac{1}{T}\int_T x(t-t_0) e^{-jk\omega_0 t} dt, \\ &= \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0(\tau+t_0)} d\tau, \\ &= e^{-jk\omega_0 t_0} \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0 \tau} d\tau, \\ &= e^{-jk\omega_0 t_0} a_k. \end{aligned}$$

$$x(t-t_0) \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k.$$

Note: $|a_k| = |b_k|$

Time Reversal

If

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

then

$$x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}.$$

Fourier series:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}.$$

Substitution: $k = -m$

$$x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\pi t/T}.$$

Right-hand side of this equation has the form of the Fourier series synthesis equation for $x(-t)$, where the Fourier seriescoefficients b_k are

$$b_k = a_{-k}.$$

- Time reversal applied to a continuous-time signal results in a reversal of the indices of the corresponding Fourier series coefficients.

- Time reversal applied to a continuous-time signal results in a reversal of the indices of the corresponding Fourier series coefficients.
- If $x(t)$ is even, i.e., $x(-t) = x(t)$, then its Fourier series coefficients are also even, i.e., $a_{-k} = a_k$.

- Time reversal applied to a continuous-time signal results in a reversal of the indices of the corresponding Fourier series coefficients.
- If $x(t)$ is even, i.e., $x(-t) = x(t)$, then its Fourier series coefficients are also even, i.e., $a_{-k} = a_k$.
- If $x(t)$ is odd, i.e., $x(-t) = -x(t)$, then its Fourier series coefficients are also odd, i.e., $a_{-k} = -a_k$.

Time Scaling

Time scaling, in general, changes the period.
If $x(t)$ is a periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$, then $x(\alpha t)$, where α is a positive real number, is periodic with period T/α and fundamental frequency $\alpha\omega_0$.

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t} \tag{6}$$

While Fourier coefficients have not changed, the Fourier series representation has changed because of the change in the fundamental frequency.

Multiplication

Let $x(t)$ and $y(t)$ denote two periodic signals with period T.

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k,$$
$$y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k.$$

Since the product $x(t)y(t)$ is also periodic with period T, its Fourier series coefficients h_k are

$$x(t)y(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}. \tag{7}$$

Conjugation and Conjugate Symmetry

- Taking the complex conjugate of a periodic signal $x(t)$ has the effect of complex conjugation and time reversal on the corresponding Fourier series coefficients.

Conjugation and Conjugate Symmetry

- Taking the complex conjugate of a periodic signal $x(t)$ has the effect of complex conjugation and time reversal on the corresponding Fourier series coefficients.
- If $x(t)$ is real, i.e., $x(t) = x^*(t)$: Fourier series coefficients conjugate symmetric, i.e., $a_{-k} = a_k^*$.

Conjugation and Conjugate Symmetry

- Taking the complex conjugate of a periodic signal $x(t)$ has the effect of complex conjugation and time reversal on the corresponding Fourier series coefficients.
- If $x(t)$ is real, i.e., $x(t) = x^*(t)$: Fourier series coefficients conjugate symmetric, i.e., $a_{-k} = a_k^*$.
- If $x(t)$ is real, then a_0 is real and $|a_k| = |a_{-k}|$.

Conjugation and Conjugate Symmetry

- Taking the complex conjugate of a periodic signal $x(t)$ has the effect of complex conjugation and time reversal on the corresponding Fourier series coefficients.
- If $x(t)$ is real, i.e., $x(t) = x^*(t)$: Fourier series coefficients conjugate symmetric, i.e., $a_{-k} = a_k^*$.
- If $x(t)$ is real, then a_0 is real and $|a_k| = |a_{-k}|$.
- If $x(t)$ is real and even, we know that $a_k = a_{-k}$. From above, $a_k^* = a_{-k}$, so that $a_k = a_k^*$. That is if $x(t)$ is real and even, so are its Fourier series coefficients.

Conjugation and Conjugate Symmetry

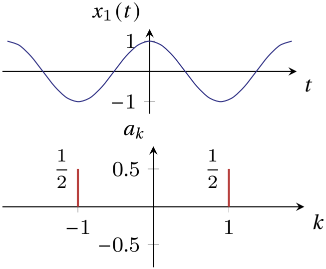
- Taking the complex conjugate of a periodic signal $x(t)$ has the effect of complex conjugation and time reversal on the corresponding Fourier series coefficients.
- If $x(t)$ is real, i.e., $x(t) = x^*(t)$: Fourier series coefficients conjugate symmetric, i.e., $a_{-k} = a_k^*$.
- If $x(t)$ is real, then a_0 is real and $|a_k| = |a_{-k}|$.
- If $x(t)$ is real and even, we know that $a_k = a_{-k}$. From above, $a_k^* = a_{-k}$, so that $a_k = a_k^*$. That is if $x(t)$ is real and even, so are its Fourier series coefficients.
- If $x(t)$ is real and odd, its Fourier series coefficients are purely imaginary and odd. Thus, e.g., $a_0 = 0$.

Example

Consider

$$x_1(t) = \cos(\omega_0 t)$$

This is real and even.

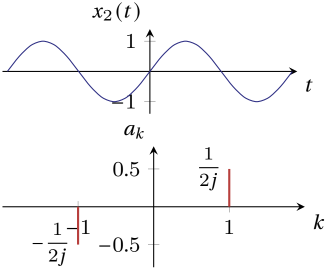


FS coefficients are real and even. (They are conjugate symmetric too.)

Consider

$$x_2(t) = \sin(\omega_0 t)$$

This is real and odd.



FS coefficients are imaginary and odd. (They are conjugate symmetric too.)

Parseval's Relation for Continuous-Time Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2. \tag{8}$$

Parseval's Relation for Continuous-Time Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2. \tag{8}$$

Note: Left-hand side of equation 8 is the average power (i.e., energy per unit time) in one period of the periodic signal $x(t)$.

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2\, dt = |a_k|^2. \tag{9}$$

So, $|a_k|^2$ is the average power in the kth harmonic component of $x(k)$.
Thus, what Parseval's relation states is that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

Example

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 5.

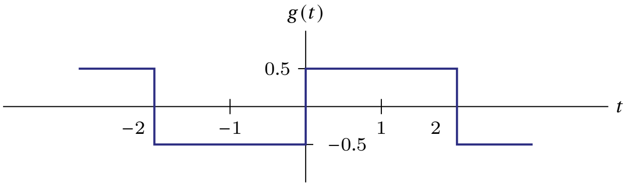


Figure: Figure for example

Determine the Fourier series representation of $g(t)$

1. directly from the analysis equation.
2. by assuming that the Fourier series coefficients of the symmetric periodic square wave are known.

Solution: Direct

$$a_0 = 0, \quad a_k = \frac{1}{2\pi j k}(1 - \cos k\pi)$$

$$a_1 = -j/\pi, a_2 = 0, a_3 = -j/(3\pi), a_4 = 0, a_5 = -j/(5\pi), \ldots$$

We notice that

$$g(t) = x(t-1) - 1/2,$$

with $T = 4$ and $T_1 = 1$.

We notice that

$$g(t) = x(t-1) - 1/2,$$

with $T = 4$ and $T_1 = 1$. If FS coefficients of $x(t)$ are denoted by a_k, the FS coefficients of $x(t-1)$ may be expressed as

$$b_k = a_k e^{-jk\pi/2},$$

The FS coefficients of the dc offset $-1/2$

$$c_k = \begin{cases} 0, & \text{for } k \neq 0, \\ -\dfrac{1}{2} & \text{for } k = 0. \end{cases}$$

Applying the linearity property, the FS coefficients of $g(t)$ may be expressed as

We notice that

$$g(t) = x(t-1) - 1/2,$$

with $T = 4$ and $T_1 = 1$. If FS coefficients of $x(t)$ are denoted by a_k, the FS coefficients of $x(t-1)$ may be expressed as

$$b_k = a_k e^{-jk\pi/2},$$

The FS coefficients of the dc offset $-1/2$

$$c_k = \begin{cases} 0, & \text{for } k \neq 0, \\ -\dfrac{1}{2} & \text{for } k = 0. \end{cases}$$

Applying the linearity property, the FS coefficients of $g(t)$ may be expressed as

$$d_k = \begin{cases} a_k e^{-jk\pi/2}, & \text{for } k \neq 0, \\ a_0 - \dfrac{1}{2} & \text{for } k = 0. \end{cases}$$

yielding

$$d_k = \begin{cases} \dfrac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0, \\ 0 & \text{for } k = 0. \end{cases}$$

Example

Consider the triangular wave signal $x(t)$ with period $T = 4$ and fundamental frequency $\omega_0 = \pi/2$, shown in Figure 6. The derivative signal is the signal $g(t)$ in Figure 5. Using this information, find the Fourier series coefficients of $x(t)$.

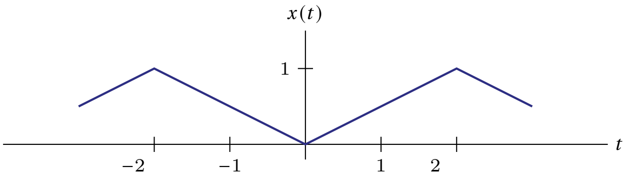


Figure: Figure for example

The derivative of this signal is the signal $g(t)$ in the previous example. Denoting the Fourier coefficients of $g(t)$ by d_k and those of $x(t)$ by e_k we see that the differentiation property indicates that

The derivative of this signal is the signal $g(t)$ in the previous example. Denoting the Fourier coefficients of $g(t)$ by d_k and those of $x(t)$ by e_k we see that the differentiation property indicates that

$$d_k = jk(\pi/2)e_k.$$

This equation can be used to express e_k in terms of d_k except when $k = 0$. Specifically,

The derivative of this signal is the signal $g(t)$ in the previous example. Denoting the Fourier coefficients of $g(t)$ by d_k and those of $x(t)$ by e_k we see that the differentiation property indicates that

$$d_k = jk(\pi/2)e_k.$$

This equation can be used to express e_k in terms of d_k except when $k = 0$. Specifically,

$$e_k = \frac{2d_k}{jk\pi} = \frac{2\sin(\pi k/2)}{j(k\pi)^2}e^{-jk\pi/2}, \quad k \neq 0.$$

For $k = 0$, e_0 can be determined by finding the area under one period of $x(t)$ and dividing by the length of the period:

The derivative of this signal is the signal $g(t)$ in the previous example. Denoting the Fourier coefficients of $g(t)$ by d_k and those of $x(t)$ by e_k we see that the differentiation property indicates that

$$d_k = jk(\pi/2)e_k.$$

This equation can be used to express e_k in terms of d_k except when $k = 0$. Specifically,

$$e_k = \frac{2d_k}{jk\pi} = \frac{2\sin(\pi k/2)}{j(k\pi)^2}e^{-jk\pi/2}, \quad k \neq 0.$$

For $k = 0$, e_0 can be determined by finding the area under one period of $x(t)$ and dividing by the length of the period:

$$e_0 = \frac{1}{2}.$$

Example

Obtain the Fourier series coefficients of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{10}$$

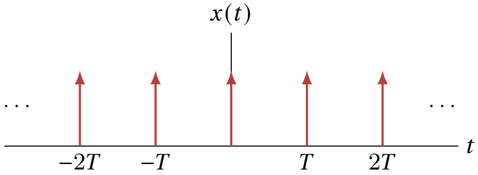


Figure: Impulse train

To determine the Fourier series coefficients a_k, we select the interval of integration to be $-T/2 \leq t \leq T/2$. Within this interval, $x(t)$ is the same as $\delta(t)$.

To determine the Fourier series coefficients a_k, we select the interval of integration to be $-T/2 \leq t \leq T/2$. Within this interval, $x(t)$ is the same as $\delta(t)$.

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk2\pi/T} dt = \frac{1}{T}.$$

To determine the Fourier series coefficients a_k, we select the interval of integration to be $-T/2 \le t \le T/2$. Within this interval, $x(t)$ is the same as $\delta(t)$.

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk2\pi/T}dt = \frac{1}{T}.$$

In other words, all the Fourier series coefficients of the impulse train are identical. These coefficients are also real valued and even (with respect to the index k).

Example

By expressing the derivative of a square wave signal in terms of impulses, obtain the Fourier series coefficients of the square wave signal.

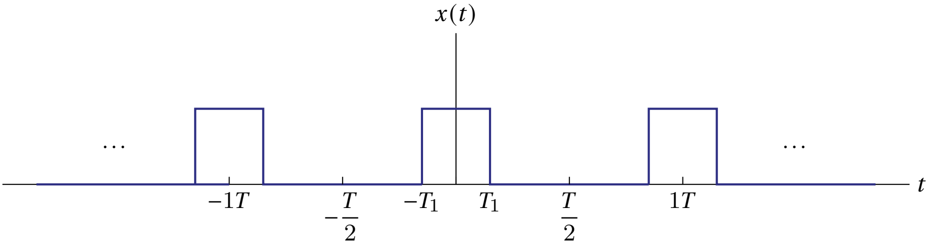


Figure: Figure for example

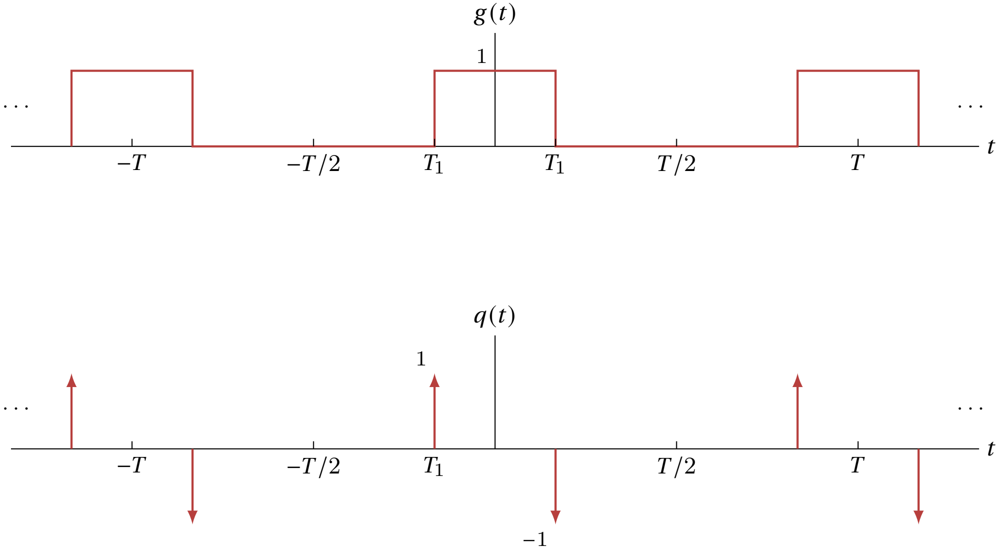


Figure: Periodic square wave and its derivative.

We may interpret $q(t)$ as the difference of two shifted versions of the impulse train $x(t)$.
That is,

$$q(t) = x(t + T_1) - x(t - T_1).$$

We may interpret $q(t)$ as the difference of two shifted versions of the impulse train $x(t)$.
That is,

$$q(t) = x(t + T_1) - x(t - T_1).$$

Fourier series coefficients b_k of $q(t)$ may be expressed in terms of the Fourier series coefficients a_k of $x(t)$; that is,

$$\begin{aligned} b_k &= e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k, \quad \omega_0 = 2\pi/T, \\ &= \frac{1}{T}\left[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right] = \frac{2j\sin(k\omega_0 T_1}{T}). \end{aligned}$$

We may interpret $q(t)$ as the difference of two shifted versions of the impulse train $x(t)$. That is,

$$q(t) = x(t + T_1) - x(t - T_1).$$

Fourier series coefficients b_k of $q(t)$ may be expressed in terms of the Fourier series coefficients a_k of $x(t)$; that is,

$$\begin{aligned} b_k &= e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k, \quad \omega_0 = 2\pi/T, \\ &= \frac{1}{T}\left[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right] = \frac{2j\sin(k\omega_0 T_1}{T}). \end{aligned}$$

Since $q(t)$ is the derivative of $g(t)$, we can use the differentiation property:

$$b_k = jk\omega_0 c_k,$$

where the c_k are the Fourier series coefficients of $g(t)$. Thus,

$$c_k = \frac{b_k}{jk\omega_0} = \frac{2j\sin(k\omega_0 T_1)}{jk\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0$$

Since c_0 is just the average value of $g(t)$ over one period,

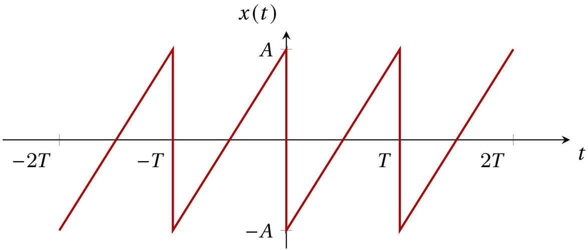
$$c_0 = \frac{2T_1}{T}.$$

Example

For the waveform $x(t)$,

1. Obtain expression for the exponential Fourier series coefficients a_k.
2. Compute the average power
$$\frac{1}{T}\int_T |x(t)|^2 dt.$$
3. Verify Parseval's relation.

Given: Sum of the reciprocals of the positive square integers is $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$.

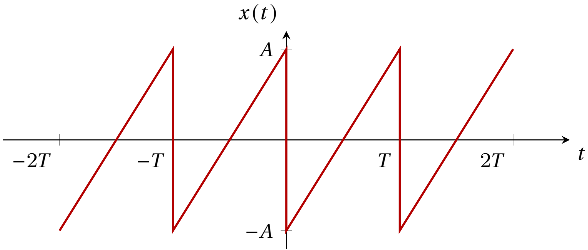


Example

For the waveform $x(t)$,

1. Obtain expression for the exponential Fourier series coefficients a_k.
2. Compute the average power
$$\frac{1}{T}\int_T |x(t)|^2 dt.$$
3. Verify Parseval's relation.

Given: Sum of the reciprocals of the positive square integers is $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.



$$x(t) = A\left(\frac{2t}{T} - 1\right), \quad 0 < t <\leq T.$$

Example: Computing a_k

$$
\begin{aligned}
a_0 &= \frac{1}{T}\int_T x(t)dt \\
&= \frac{A}{T}\int_0^T \left(\frac{2t}{T} - 1\right) dt \\
&= \left[\frac{2t^2}{2T} - t\right]_0^T \\
&= 0
\end{aligned}
$$

Example: Computing a_k

$$
\begin{aligned}
a_0 &= \frac{1}{T}\int_T x(t)dt \\
&= \frac{A}{T}\int_0^T \left(\frac{2t}{T}-1\right)dt \\
&= \left[\frac{2t^2}{2T}-t\right]_0^T \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \frac{A}{T}\int_0^T \overbrace{\left(\frac{2t}{T}-1\right)}^{u} \overbrace{e^{-jk\omega_0 t}dt}^{dv} \\
&= \frac{A}{T}\left\{\left[\left(\frac{2t}{T}-1\right)\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_0^T - \left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\frac{2}{T}\right]_0^T\right\}
\end{aligned}
$$

Example: Computing a_k

$$\boxed{\omega_0 T = 2\pi}$$

$$\begin{aligned}
a_0 &= \frac{1}{T}\int_T x(t)dt \\
&= \frac{A}{T}\int_0^T \left(\frac{2t}{T} - 1\right) dt \\
&= \left[\frac{2t^2}{2T} - t\right]_0^T \\
&= 0
\end{aligned}$$

$$\begin{aligned}
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \frac{A}{T}\int_0^T \underbrace{\left(\frac{2t}{T} - 1\right)}_{u} \underbrace{e^{-jk\omega_0 t}dt}_{dv} \\
&= \frac{A}{T}\left\{\left[\left(\frac{2t}{T} - 1\right)\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_0^T - \left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\frac{2}{T}\right]_0^T\right\}
\end{aligned}$$

$$\begin{aligned}
a_k &= \frac{A}{T}\left\{\left[(2-1)\frac{e^{-jk\omega_0 T}}{-jk\omega_0} - \frac{(-1)}{-jk\omega_0}\right]\right. \\
&\quad \left. - \frac{2}{-jk\omega_0 T}\left[e^{-jk\omega_0 T} - 1\right]\right\} \\
&= \frac{A}{T}\left\{\frac{-2}{jk\omega_0}\right\} \\
&= \frac{Aj}{\pi k}.
\end{aligned}$$

Example: Computing a_k

$$\boxed{\omega_0 T = 2\pi}$$

$$
\begin{aligned}
a_0 &= \frac{1}{T}\int_T x(t)dt \\
&= \frac{A}{T}\int_0^T \left(\frac{2t}{T} - 1\right) dt \\
&= \left[\frac{2t^2}{2T} - t\right]_0^T \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \frac{A}{T}\int_0^T \overbrace{\left(\frac{2t}{T} - 1\right)}^{u} \overbrace{e^{-jk\omega_0 t}dt}^{dv} \\
&= \frac{A}{T}\left\{\left[\left(\frac{2t}{T} - 1\right)\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_0^T - \left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\frac{2}{T}\right]_0^T\right\}
\end{aligned}
$$

$$
\begin{aligned}
a_k &= \frac{A}{T}\left\{\left[(2-1)\frac{e^{-jk\omega_0 T}}{-jk\omega_0} - \frac{(-1)}{-jk\omega_0}\right]\right. \\
&\quad \left. - \frac{2}{-jk\omega_0 T}\left[e^{-jk\omega_0 T} - 1\right]\right\} \\
&= \frac{A}{T}\left\{\frac{-2}{jk\omega_0}\right\} \\
&= \frac{Aj}{\pi k}.
\end{aligned}
$$

$$
a_k = \begin{cases} 0, & k = 0, \\ \dfrac{Aj}{\pi k}, & k \neq 0. \end{cases}
$$

Example: Computing the Average Power

$$
\begin{aligned}
\frac{1}{T}\int_T |x(t)|^2 dt &= \frac{A^2}{T}\int_0^T \left(\frac{2t}{T} - 1\right)^2 dt \\
&= \frac{A^2}{T}\int_0^T \left[\frac{4t^2}{T^2} - 4\frac{t}{T} + 1\right] \\
&= \frac{A^2}{T}\int_0^T \left[\frac{4t^3}{3T^2} - 4\frac{t^2}{2T} + t\right] \\
&= \frac{A^2}{T}\left[\frac{4}{3T} - 2T + T\right]_0^T \\
&= \frac{A^2}{3}
\end{aligned}
$$

Example: Verifying Parseval's relation

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = \sum_{k\neq 0} \left| \frac{Aj}{\pi k} \right|^2$$

$$= 2\frac{A^2}{\pi^2} \sum_{k\neq 1}^{\infty} \frac{1}{k^2}$$

$$= 2\frac{A^2}{\pi^2} \frac{\pi^2}{6}$$

$$= \frac{A^2}{3}$$

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}}$$

Other Forms of Fourier Series

Complex Exponential Fourier Series

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$
$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \tag{11}$$

Trigonometric Fourier Series

$$x(t) = A_0 + 2\sum_{k=1}^{+\infty} A_k \cos k\omega_0 t + B_k \sin k\omega_0 t$$
$$A_k = \frac{1}{T}\int_T x(t)A_k \cos k\omega_0 t dt$$
$$B_k = \frac{1}{T}\int_T x(t)A_k \sin k\omega_0 t dt \tag{12}$$

Harmonic Form Fourier Series (for Real $x(t)$)

$$x(t) = C_0 + 2\sum_{k=1}^{+\infty} C_k \cos(k\omega_0 t - \theta_k)$$
$$C_0 = A_0$$
$$C_k = \sqrt{A_k^2 + B_k^2} \quad \theta_k = \tan^{-1}\left(\frac{B_k}{A_k}\right) \tag{13}$$

Relationship

$$A_0 = a_0$$
$$A_k = \frac{a_k + a_{-k}}{2}$$
$$B_k = j\frac{a_k - a_{-k}}{2}$$
$$\omega_0 = \frac{2\pi}{T} \tag{14}$$

Outline

Convergence of Fourier Series

FS synthesis and analysis equations:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Consider the finite series of the form

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

Let $e_N(t)$ denote the approximation error, that is,

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

A quantitative measure of approximation error is

$$E_N = \int_T |e_N(t)|^2 \, dt$$

Convergence of Fourier Series

- If $x(t)$ has a Fourier series representation, then the limit of E_N as $N \to \infty$ is zero.

Convergence of Fourier Series

- If $x(t)$ has a Fourier series representation, then the limit of E_N as $N \to \infty$ is zero.
- If $x(t)$ does not have a Fourier series representation, then the integral that computes a_k may diverge. Moreover, even if all of the coefficients a_k obtained are finite, when these coefficients are substituted into the synthesis equation, the resulting infinite series may not converge to the original signal $x(t)$.

Convergence of Fourier Series

- If $x(t)$ has a Fourier series representation, then the limit of E_N as $N \to \infty$ is zero.
- If $x(t)$ does not have a Fourier series representation, then the integral that computes a_k may diverge. Moreover, even if all of the coefficients a_k obtained are finite, when these coefficients are substituted into the synthesis equation, the resulting infinite series may not converge to the original signal $x(t)$.
- Fortunately, there are no convergence difficulties for large classes of periodic signals, continuous and discontinuous.

Finite-Energy Convergence Criterion

One class of periodic signals that are representable through the Fourier series is those signals which have finite energy over a single period:

$$\int_T |x(t)|^2 \, dt < \infty \tag{15}$$

- In this case coefficients a_k are finite.

Finite-Energy Convergence Criterion

One class of periodic signals that are representable through the Fourier series is those signals which have finite energy over a single period:

$$\int_T |x(t)|^2 \, dt < \infty \tag{15}$$

- In this case coefficients a_k are finite.
- As $N \to \infty$, $E_N \to 0$.

Finite-Energy Convergence Criterion

One class of periodic signals that are representable through the Fourier series is those signals which have finite energy over a single period:

$$\int_T |x(t)|^2 \, dt < \infty \tag{15}$$

- In this case coefficients a_k are finite.
- As $N \to \infty$, $E_N \to 0$.
- This does not imply that the signal $x(t)$ and its Fourier series representation are equal at every value of t. What it does say is that there is no energy in their difference.

Finite-Energy Convergence Criterion

One class of periodic signals that are representable through the Fourier series is those signals which have finite energy over a single period:

$$\int_T |x(t)|^2 \, dt < \infty \tag{15}$$

- In this case coefficients a_k are finite.
- As $N \to \infty$, $E_N \to 0$.
- This does not imply that the signal $x(t)$ and its Fourier series representation are equal at every value of t. What it does say is that there is no energy in their difference.
- However, since physical systems respond to signal energy, from this perspective $x(t)$ and its Fourier series representation are indistinguishable.

Alternative Conditions (Dirichlet Conditions)

Dirichlet conditions guarantee that $x(t)$ equals its Fourier series representation, except at isolated values of t for which $x(t)$ is discontinuous. At these values, the infinite series converges to the average of the values on either side of the discontinuity.

Condition 1

Over any period, $x(t)$ must be absolutely integrable

$$\int_T |x(t)|\,dt < \infty. \tag{16}$$

This guarantees that a_ks are finite.

Condition 2

In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

Condition 3

In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

Examples of Functions that Violate Dirichlet Conditions

Cond. 1 The periodic signal with period 1 with one period defined as

$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1.$$

Cond. 2 The periodic signal with period 1 with one period defined as

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1.$$

For this

$$\int_0^1 |x(t)|\, dt < 1$$

The function has, however, an infinite number of maxima and minima in the interval.

Cond. 3 The signal, of period T = 8, is composed of an infinite number of sections, each of which is half the height and half the width of the previous section. Thus, the area under one period of the function is clearly less than 8. However, there are an infinite number of discontinuities in each period, thereby violating Condition 3.

Gibbs Phenomenon

- When a function with jump discontinuities is approximated by a finite number of Fourier series terms, oscillations appear near the discontinuities.
- These oscillations:
  - do not disappear as the number of harmonics increases,
  - become more localized around the discontinuity,
  - have a maximum overshoot of approximately 9% of the jump magnitude.
- This effect is known as the Gibbs phenomenon.
- Away from the discontinuities, the Fourier series converges to the original signal.

Increasing the number of Fourier coefficients improves global accuracy but cannot eliminate local overshoot near discontinuities.

Gibbs Phenomenon

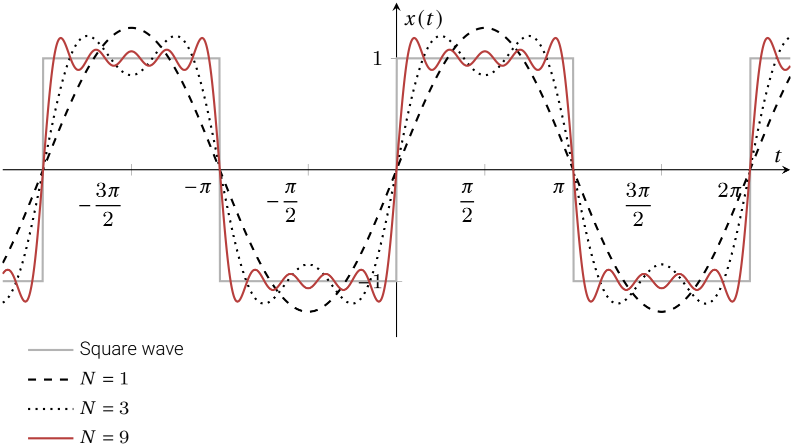


Figure: Gibbs phenomenon

Gibbs Phenomenon

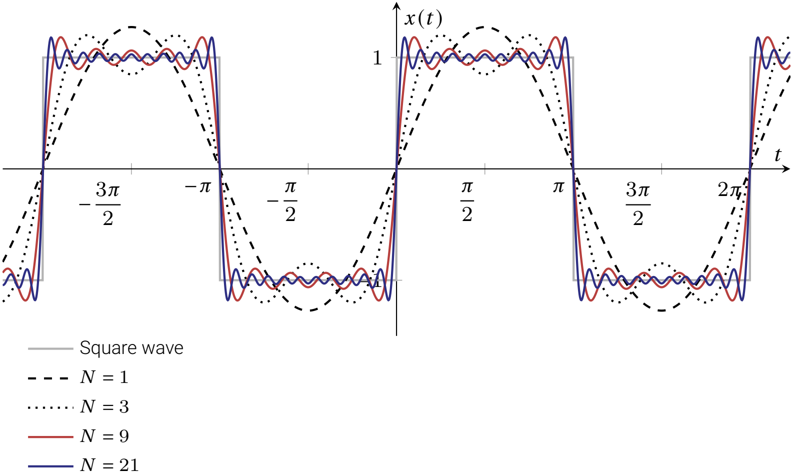


Figure: Gibbs phenomenon