EN1020 Signals and Systems

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Section 1

Introduction to Signals and Systems

Introduction to the Course

- Signals and systems find many applications in communications, and automatic control, and form the basis for signal processing, machine vision, and pattern recognition.
- Electrical signals (voltages and currents in circuits, electromagnetic communication signals), acoustic signals, image and video signals, and biological signals are all examples of signals that we encounter.
- They are functions of independent variables and carry information.

Introduction to Course Contd.

- We define a system as a mathematical relationship between an input signal and an output signal.
- We can use systems to analyze and modify signals.
- Signals and systems have enabled major advances in communication, control, and information processing technologies.
- In this course we will study the fundamentals of signals and systems.
- Types of signals in continuous time and discrete time, linear time-invariant (LTI) systems, Fourier series, and an introduction to Fourier transforms are the core components of the signals and systems part of this course.
- We will study sampling, discrete-time Fourier series and transform, Laplace transform, z-transform, and stability of systems in EN2063.

Learning Outcomes

After completing this course you will be able to do the following:

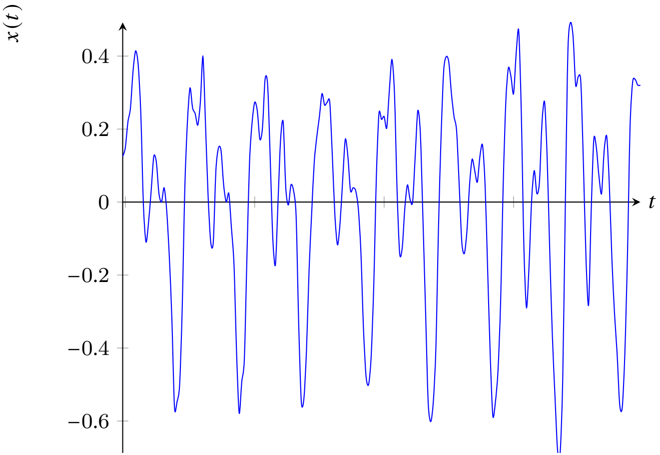
- Differentiate between continuous-time, discrete-time, and digital signals, and techniques applicable to the analysis of each type.
- Apply appropriate theoretical principles to characterize the behavior of linear time-invariant (LTI) systems.
- Use Fourier techniques to understand frequency-domain characteristics of signals.

Categories of Signals

- In this course, we study signals and systems that process these signals.
- Categories of signals:
  - Continuous-time signals: independent variable is continuous,
  - Discrete-time signals:independent variable is an integer,
- There are some very strong similarities and also some very important differences between discrete-time signals and systems and continuous-time signals and systems.

Continuous-Time Signals $x(t)$

- The independent variable is continuous.
- E.g., sound pressure at a microphone as a function of time (one-dimensional signal).
- E.g., image brightness as a function of two spatial variables (two-dimensional signal).
- For convenience, we refer to the independent variable as time.



A function of a continuous variable

A speech signal: a continuous-time, one-dimensional signal

An image on a film: a continuous-time, two-dimensional signal

Discrete-Time Signals $x[n]$

- A function of an integer variable.
- Takes on values at integer values of the argument of $x[n]$.

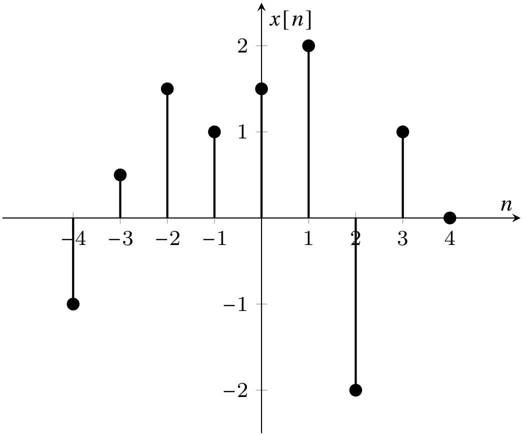


Figure 1: DT Signal

Digital Signals

- What is a digital signal?
  - A quantized discrete-time signal: A digital signal is a discrete-time signal that takes on values from a finite set of distinct, quantized levels.
- What is a digital image?
  - A two-dimensional, quantized, discrete-time signal.
  - A 600×800 image: $n \in [0, 599]$, $m \in [0, 799]$, $x[n, m] \in [0, 255]$. 8-bit image.

Systems

- A system processes signals.
- Examples of systems:
  - Dynamics of an aircraft.
  - An algorithm for analyzing financial and economic factors to predict bond prices.
  - An algorithm used in high-frequency trading, where traders use algorithms to analyze financial data and make trades in fractions of a second.
  - An algorithm for post-flight analysis of a space launch.
  - An edge detection algorithm for medical images.
  - A filter used in Electrocardiogram (ECG) signal acquisition, such as a notch filter used to remove $50\,\mathrm{Hz}$ power line interference.

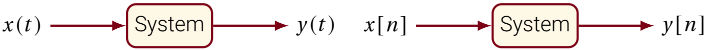


Figure 2: CT and DT Systems.

Types of Systems

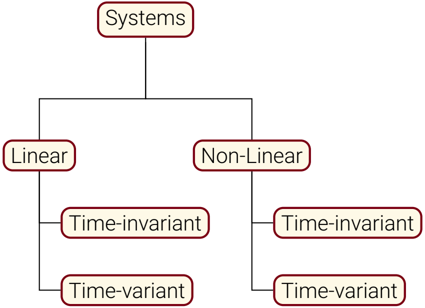


Figure 3: System types.

This classification applies to both continuous-time and discrete-time systems. This course is focused on the class of linear, time-invariant (LTI) systems.

Systems Interconnections

- To build more complex systems by interconnecting simpler subsystems.
- To modify the response of a system.
- E.g.: amplifier design, stabilizing unstable systems.

Signal-Flow (Block) Diagrams

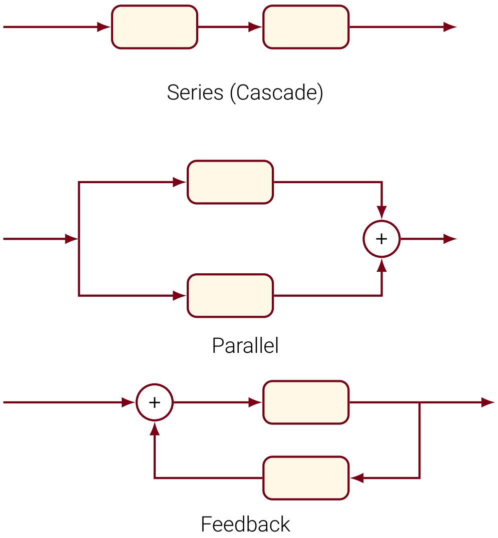
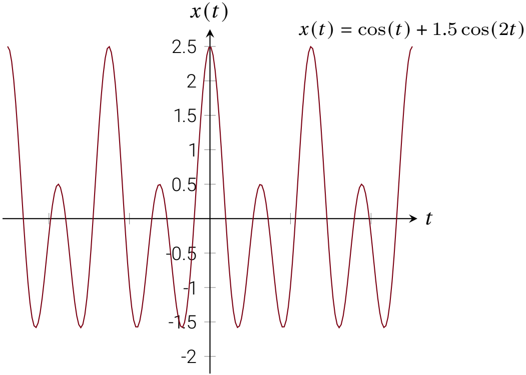
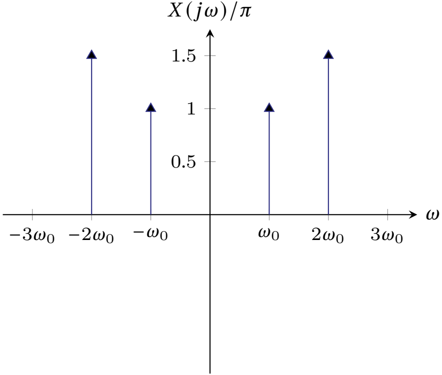


Figure 4: System interconnections.

Domains



Time domain representation.



Frequency domain representation, $\omega_0 = 1$.

Figure 5: Domains. The spectrum is shown normalized by π for visualization.

Domains

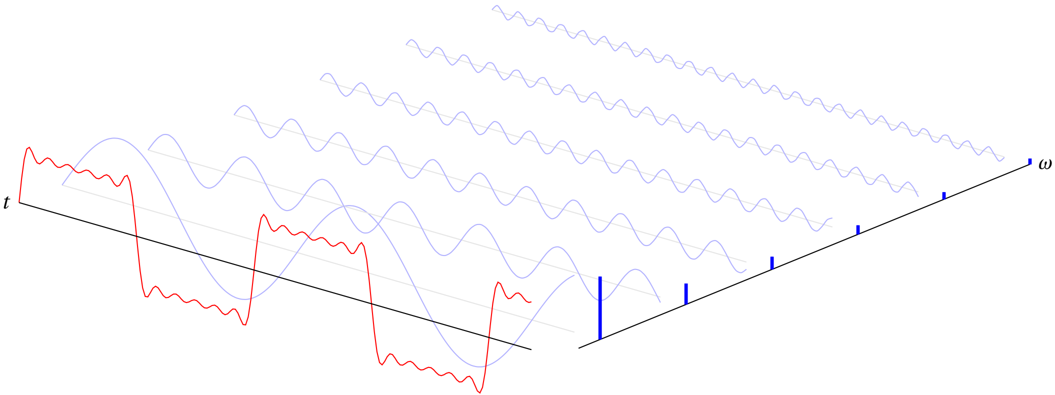


Figure 6: Square wave: time and frequency domains.

Summary

- Signals represent information as functions of time, space, or other independent variables.
- We classify signals as continuous-time, discrete-time, or digital based on their domain and amplitude.
- A system defines a mathematical relationship that transforms an input signal into an output signal.
- We interconnect systems to build complex signal-processing and control applications.
- This course focuses primarily on linear, time-invariant (LTI) systems.
- We analyze signals in both the time domain and the frequency domain.

Section 2

Signals

Continuous-Time Sinusoidal Signal

$$x(t) = A\cos(\omega_0 t + \phi). \tag{1}$$

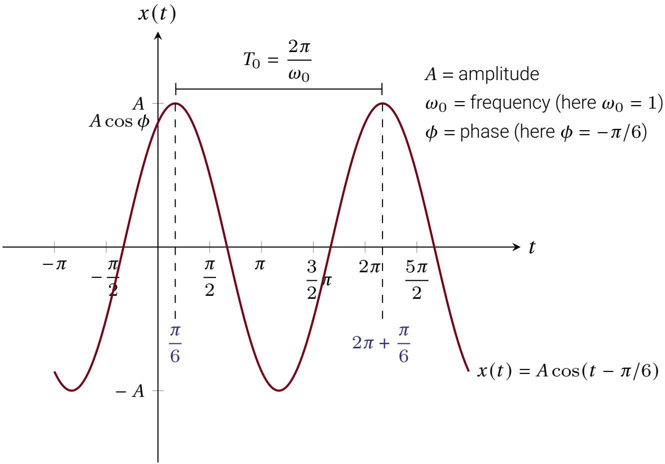


Figure 7: Continuous-time sinusoidal signal.

Periodicity of a Sinusoidal

A sinusoidal signal is periodic.
A periodic continuous-time signal $x(t)$ has the property that there is a positive value T for which

$$x(t) = x(t + T) \tag{2}$$

for all values of t.
Under an appropriate time-shift, the signal repeats itself. In this case we say that $x(t)$ is periodic with period T.
Fundamental period T_0 = smallest positive value of T for which 2 holds.
A signal that is not periodic is referred to as aperiodic.
E.g.: Consider $A\cos(\omega_0 t + \phi)$

$$\begin{aligned} A\cos(\omega_0 t + \phi) &= A\cos(\omega_0 (t + T) + \phi), \quad \text{here } \omega_0 T = 2\pi m \quad \text{an integer multiple of } 2\pi \\ &= A\cos(\omega_0 t + \phi) \end{aligned}$$

Phase of a Sinusoidal

A time-shift in a CT sinusoid is equivalent to a phase shift.
E.g.: Show that a time-shift of a sinusoid is equal to a phase shift.

Even and Odd Signals

A signal $x(t)$ or $x[n]$ is referred to as an even signal if it is identical to its time-reversed counterpart, i.e., with its reflection about the origin:

$$x(-t) = x(t),$$
$$x[-n] = x[n].$$

A signal is referred to as an signal odd if

$$x(-t) = -x(t),$$
$$x[-n] = -x[n].$$

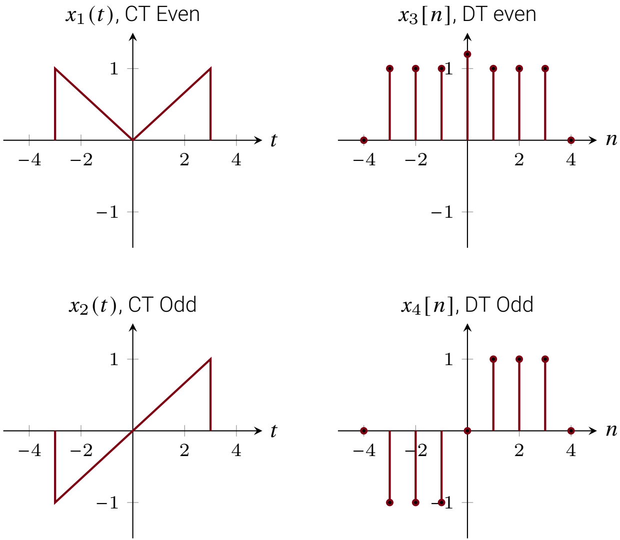
An odd signal must be 0 at $t = 0$ or $n = 0$.
A signal can be broken into a sum of two signals, one of which is even and one for which is odd. Even part of $x(t)$ is

$$\mathfrak{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)].$$

Odd part of $x(t)$ is

$$\mathfrak{Od}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)].$$

Examples of Even and Odd Functions



Even and Odd Signals Contd.

Example

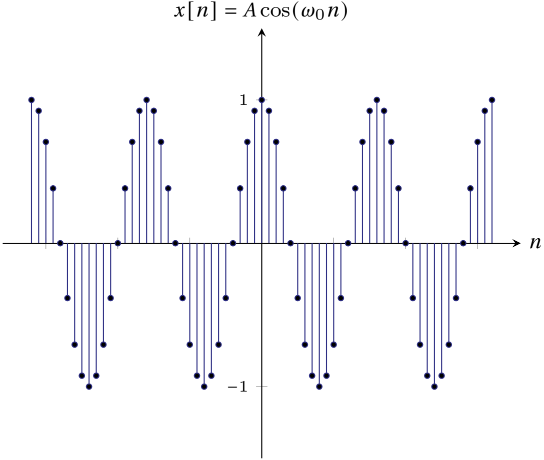
Show that $\mathfrak{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$.

Notation: $x_e(t)$ is even part of $x(t)$, $x_o(t)$ is odd part of $x(t)$.

Phase of a Sinusoidal: $\phi = 0$

Phase of a Sinusoidal: $\phi = -\pi/2$

$$x[n] = A\cos(\omega_0 n + \phi) \text{ with } \phi = 0$$



The independent variable is an integer.
The sequence takes values only at integer values of the argument.
This signal is even.

Even: $x[n] = x[-n]$.
Periodic: $x[n] = x[n+N]$. Here, $N = 16$

$$\omega_0 = \frac{2\pi}{N} = \frac{\pi}{8}.$$

$$x[n] = A\cos(\omega_0 n + \phi) \text{ with } \phi = -\pi/2$$

Phase Change and Time Shift in DT

Question

Does a phase change always correspond to a time shift in discrete-time signals?

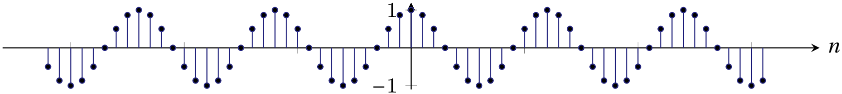
Periodicity of a DT Signal

All continuous-time sinusoids are periodic. However, discrete-time sinusoids are not necessarily so.

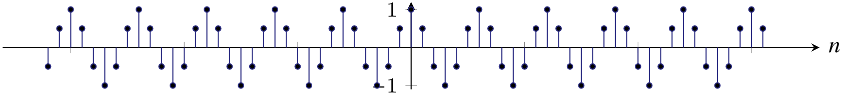
$$x[n] = x[n+N], \quad \text{smallest integer } N \text{ is the fundamental period.} \tag{3}$$

Periodicity of a DT Signal Contd.

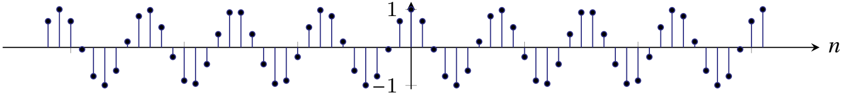
$x[n] = A\cos(2\pi/12n)$



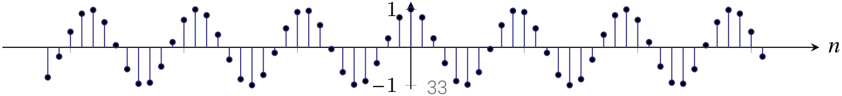
$x[n] = A\cos(2\pi/6n)$



$x[n] = A\cos(8\pi/31n)$



$x[n] = A\cos(2/3n)$

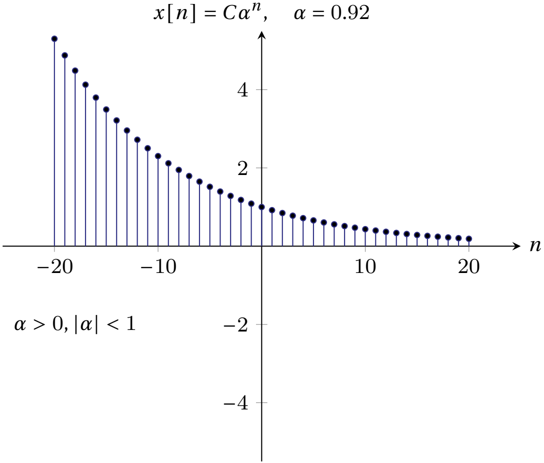
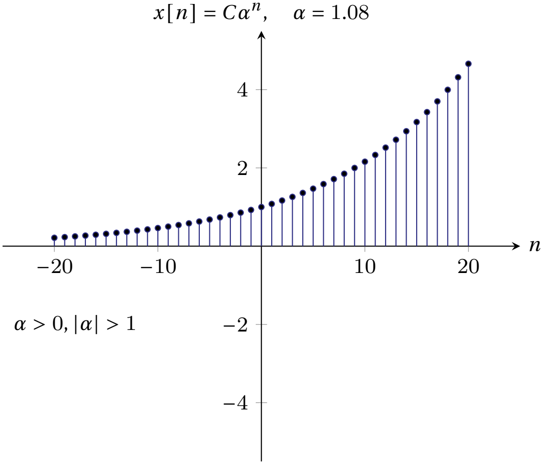


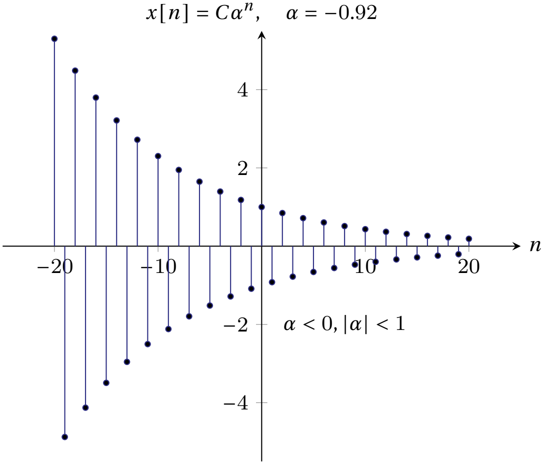
CT Real Exponentials

$$
\begin{aligned}
x(t) &= Ce^{a(t+t_0)}, \quad C \text{ and } a \text{ are real numbers} \\
&= Ce^{at_0}e^{at}.
\end{aligned}
$$

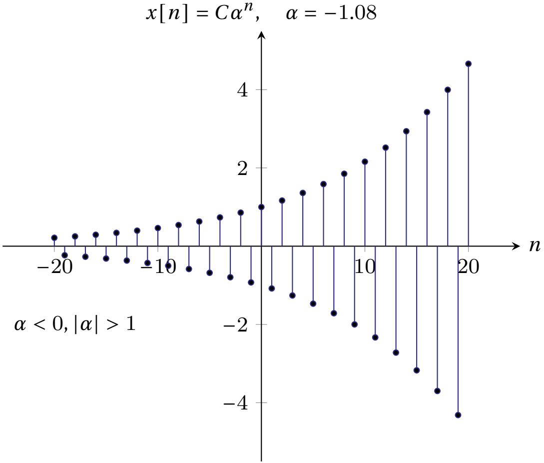
DT Real Exponentials

$$x[n] = Ce^{\beta n} = C\alpha^n, \quad C \text{ and } \alpha \text{ are real numbers}$$


$x[n] = C\alpha^n, \quad \alpha = 0.92$
n
4
2
−2
−4
−20
−10
10
20
$\alpha > 0, |\alpha| < 1$

$x[n] = C\alpha^n, \quad \alpha = 1.08$
n
4
2
−2
−4
−20
−10
10
20
$\alpha > 0, |\alpha| > 1$



$x[n] = C\alpha^n, \quad \alpha = -0.92$
4
2
−2
−4
−20
−10
10
20
n
$\alpha < 0, |\alpha| < 1$



$x[n] = C\alpha^n, \quad \alpha = -1.08$
4
2
−2
−4
−20
−10
10
20
n
$\alpha < 0, |\alpha| > 1$

Representing Complex Numbers

The Cartesian or rectangular form:

$$z = x + jy,$$

where $j = \sqrt{-1}$ and x and y are real numbers referred to respectively as the real part and the imaginary part. I.e.,

$$x = \Re\{z\}, y = \Im\{z\}$$

The polar form:

$$z = re^{j\theta},$$

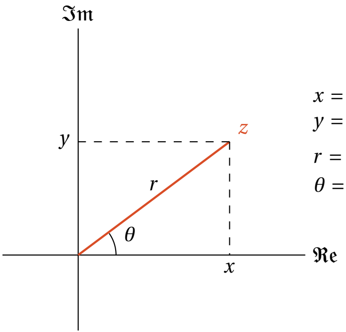
where $r > 0$ is the magnitude of z and θ is the angle or phase of z.

$$r = |z|, \theta = \sphericalangle z.$$

The relationship between these two representations can be determined from Euler's formula:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

or by plotting z in the complex plane.



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \operatorname{atan2}(y, x)$$

Example Let z_0 be a complex number with polar coordinates (r_0, θ_0) and Cartesian coordinates (x_0, y_0). Determine expressions for the Cartesian coordinates of the following complex numbers in terms of x_0 and y_0. Plot the points z_0, z_1, z_2, z_3, z_4, and z_5 in the complex plane when $r_0 = 2$ and $\theta_0 = \pi/4$ and when $r_0 = 2$ and $\theta_0 = \pi/2$. Indicate on the plot the real and imaginary parts of each point.

1. $z_1 = r_0 e^{-j\theta_0}$
2. $|z_2| = r_0$
3. $z_3 = r_0 e^{j(\theta_0+\pi)}$
4. $z_4 = r_0 e^{j(-\theta_0+\pi)}$
5. $z_5 = r_0 e^{j(\theta_0+2\pi)}$

$$r = 2, \theta = \pi/4$$

$z_1 = r_0 e^{-j\theta_0}$, $|z_2| = r_0$, $z_3 = r_0 e^{j(\theta_0+\pi)}$, $z_4 = r_0 e^{j(-\theta_0+\pi)}$, $z_5 = r_0 e^{j(\theta_0+2\pi)}$

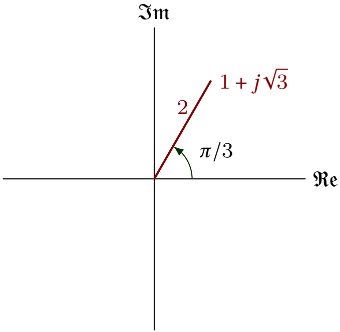
$$r = 2, \theta = \pi/2$$

$z_1 = r_0 e^{-j\theta_0}$, $|z_2| = r_0$, $z_3 = r_0 e^{j(\theta_0+\pi)}$, $z_4 = r_0 e^{j(-\theta_0+\pi)}$, $z_5 = r_0 e^{j(\theta_0+2\pi)}$

Example Express each of the following complex numbers in polar form, and plot them in the complex plane, indicating the magnitude and angle of each number.

1. $1 + j\sqrt{3}$
2. -5
3. $-5 - 5j$
4. $3 + 4j$
5. $(1 - j\sqrt{3})^3$
6. $\dfrac{e^{j\pi/3} - 1}{1 + j\sqrt{3}}$

$$
\begin{aligned}
1 + j\sqrt{3} &= \sqrt{1^2 + (\sqrt{3})^2}\left(\frac{1}{\sqrt{1^2 + (\sqrt{3})^2}} + j\frac{\sqrt{3}}{\sqrt{1^2 + (\sqrt{3})^2}}\right) \\
&= 2e^{j\,\mathrm{atan2}(\sqrt{3},1)} \\
&= 2e^{j\pi/3}
\end{aligned}
$$



$\cos\theta$ and $\sin\theta$

Using Euler's fromula, derive the following relationships:

1. $\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$
2. $\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$

Complex Conjugate

Let z denote a complex variable; i.e.,

$$z = x + jy = re^{j\theta}.$$

The complex conjugate of z is

$$z^* = x - jy = re^{-j\theta}.$$

Show that

1. $zz^* = r^2$
2. $z + z^* = 2\mathfrak{Re}\{z\}$
3. $z - z^* = 2j\mathfrak{Im}\{z\}$

1. $zz^* = re^{j\theta}re^{-j\theta} = r^2e^0 = r^2$
2. $z + z^* = x + jy + x - jy = 2x = 2\mathfrak{Re}\{z\}$
3. $z - z^* = x + jy - (x - jy) = 2jy = 2j\mathfrak{Im}\{z\}$

List the values of

CT Complex Exponentials

$$x(t) = Ce^{at}, \quad C \text{ and } a \text{ are complex numbers.}$$

$$C = |C|e^{j\theta}$$

$$a = r + j\omega_0$$

$$x(t) = |C|e^{j\theta}e^{(r+j\omega_0)t}$$

$$= |C|e^{rt}\, e^{j(\omega_0 t+\theta)}$$

$$= |C|e^{rt}\, [\cos(\omega_0 t + \theta) + j\, \sin(\omega_0 t + \theta)]$$

CT Complex Exponentials Plot

DT Complex Exponentials

$$
\begin{aligned}
x[n] &= C\alpha^n, \quad C \text{ and } \alpha \text{ are complex numbers.} \\
C &= |C|e^{j\theta} \\
\alpha &= |\alpha|e^{j\omega_0} \\
x[n] &= |C|e^{j\theta}\left(|\alpha|e^{j\omega_0}\right)^n \\
&= |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta)
\end{aligned}
$$

Comments:

- When $|\alpha| = 1$: sinusoidal real and imaginary parts.
- $e^{j\omega_0 n}$ may or may not be periodic depending on the value of ω_0.
- Sinusoidal, exponential, step, and impulse signal form the cornerstones for signals and systems analysis.

DT Complex Exponentials Plot

Periodicity Properties of Discrete-Time Complex Exponentials $e^{j\omega_0 n}$

- For the CT counterpart $e^{j\omega_0 t}$,
  1. The larger the magnitude of ω_0, the higher is the rate of oscillation in the signal.
  2. $e^{j\omega_0 t}$ is periodic for any value of ω_0.
- In DT, as

$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}$$

  the exponential at frequency $\omega_0 + 2\pi$ is the same as that at frequency ω_0 .
- Although in CT $e^{j\omega_0 t}$ are all distinct for distinct values of ω_0, In DT, these signals are not distinct, as the signal with frequency ω_0 is identical to the signals with frequencies $\omega_0 + 2\pi$, $\omega_0 + 4\pi$, and so on. Therefore, in considering DT complex exponentials, we need only consider a frequency interval of length 2π in which to choose ω_0.
- In DT, as we increase ω_0 from 0, we obtain signals that oscillate more and more rapidly until we reach $\omega_0 = \pi$. As we continue to increase ω_0, we decrease the rate of oscillation until we reach $\omega_0 = 2\pi$. Note: $e^{j\pi n} = \left(e^{j\pi}\right)^n = (-1)^n$.

Comparison of the Signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of $e^{j\omega_0 t}$	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $2\pi/\omega_0$	Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m(2\pi/\omega_0)$

Discrete-Time Unit Step $u[n]$

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \tag{4}$$

Discrete-Time Unit Impulse (Unit Sample) $\delta[n]$

$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \tag{5}$$

DT Step and Impulse

Unit impulse is the first backward difference of the unit step sequence.

$$\delta[n] = u[n] - u[n-1]. \tag{6}$$

DT Step and Impulse

The unit step sequence is the running sum of the unit impulse.

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]. \tag{7}$$

DT Step and Impulse

The unit step sequence is a superposition of delayed unit impulses.

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]. \tag{8}$$

Continuous-Time Unit Step Function $u(t)$

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{9}$$

Continuous-Time Unit Impulse Function $\delta(t)$

$$\delta(t) = \frac{du(t)}{dt}. \tag{10}$$

CT Unit Step Function and Unit Impulse Function

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau. \tag{11}$$

Energy I

The total energy over a time interval $t_1 \leq t \leq t_2$ in a continuous-time signal $x(t)$ is

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

The total energy over a time interval $n_1 \leq n \leq n_2$ in a discrete-time signal $x[n]$ is

$$\sum_{n=n_1}^{n_2} |x[n]|^2 dt$$

Total energy over an infinite interval in a CT signal:

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt. \tag{12}$$

Energy II

Total energy over an infinite interval in a DT signal:

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2. \tag{13}$$

Note that this integral and may not converge for some signals. Such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.

Power

Time-averaged power over an infinite interval in a CT signal:

$$P_\infty \triangleq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt. \tag{14}$$

In a DT signal:

$$P_\infty \triangleq \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2. \tag{15}$$

With these definitions, we can identify three important classes of signals:

Examples

Determine whether the following signals are energy signals, power signals, or neither.

1. $x(t) = e^{-at}u(t), \quad a > 0$
2. $x(t) = A\cos(\omega_0 t + \theta)$
3. $x(t) = tu(t)$

$$x(t) = e^{-at}u(t), \quad a > 0$$

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$$

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^\infty e^{-2at} dt$$

$$= \frac{-1}{2a}\left[e^{-2at}\right]_0^\infty = \frac{-1}{2a}[0-1] = \frac{1}{2a}$$

This is an energy signal.

Section 3

Continuous-Time Fourier Series

Introduction

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.
- We use the Fourier series for periodic signals and the Fourier transform for aperiodic signals.
- Each of these has continuous-time and discrete-time versions:
    1. Continuous-time Fourier series
    2. Continuous-time Fourier transform
    3. Discrete-time Fourier series
    4. Discrete-time Fourier transform
- In this part of the course, we will focus on computing the continuous-time Fourier series and Fourier transform. Later, after covering linear time-invariant (LTI) systems, we will explore the conceptual aspects of Fourier techniques.

Figure 8: Jean-Baptiste Joseph Fourier, 1768–1830, French mathematician who discovered Fourier series and transform

- Every signal has a frequency distribution or a spectrum.
- Periodic signals have a line spectrum, called the Fourier series.
- The French mathematician, Jean-Baptiste Joseph Fourier, discovered this representation.
- Fourier series provides a way to represent a periodic signal as a sum of complex exponentials.
- These sinusoids will be at frequencies that are integer multiples of the fundamental frequency ω_0.
- $\omega_0 = \dfrac{2\pi}{T}$, where T: fundamental period of the waveform.

Continuous-Time Fourier Series

Example

Let

$$x(t) = \sin \omega_0 t,$$

which has the fundamental frequency ω_0.

Euler's formula

$e^{j\theta} = \cos\theta + j\sin\theta$

$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

Example

Let

$$x(t) = 1 + \sin \omega_0 t + 2 \cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

which has the fundamental frequency ω_0.

1. Use Euler's formula to express $x(t)$ as a linear combination of complex exponentials.
2. Find the Fourier series coefficients, a_k.
3. Plot the magnitude and phase of a_k.

Example

The periodic square wave, sketched below, is defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < T/2, \end{cases}$$

This signal is periodic with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$.

1. Find the Fourier series coefficients, a_k.
2. Plot the magnitude and phase of a_k for the case $T = 4T_1$.

Properties of the Continuous-Time Fourier Series

Suppose that $x(t)$ is periodic signal with period T and fundamental frequency $\omega_0 = 2\pi/T$. If the Fourier series coefficients are denoted by a_k, then

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \tag{16}$$

Linearity

Let $x(t)$ and $y(t)$ denote two periodic signals with period T.

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k,$$
$$y(t) \xleftrightarrow{\mathcal{FS}} b_k.$$

Any linear combination of the two signals will also be periodic with period T. Fourier series coefficients c_k of the linear combination of $x(t)$ and $y(t)$, $z(t) = Ax(t) + By(t)$, are given by the same linear combination:

Time Shifting

$$x(t - t_0) \overset{\mathcal{FS}}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k \tag{17}$$

Time Reversal

If

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

then

$$x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} a_{-k}.$$

Time Scaling

Time scaling, in general, changes the period.
If $x(t)$ is a periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$, then $x(\alpha t)$, where α is a positive real number, is periodic with period T/α and fundamental frequency $\alpha\omega_0$.

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t} \tag{18}$$

Multiplication

Let $x(t)$ and $y(t)$ denote two periodic signals with period T.

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k,$$

$$y(t) \xleftrightarrow{\mathcal{FS}} b_k.$$

Conjugation and Conjugate Symmetry

- Taking the complex conjugate of a periodic signal $x(t)$ has the effect of complex conjugation and time reversal on the corresponding Fourier series coefficients.
- If $x(t)$ is real, i.e., $x(t) = x^*(t)$: Fourier series coefficients conjugate symmetric, i.e., $a_{-k} = a_k^*$.
- If $x(t)$ is real, then a_0 is real and $|a_k| = |a_{-k}|$.
- If $x(t)$ is real and even, we know that $a_k = a_{-k}$. From above, $a_k^* = a_{-k}$, so that $a_k = a_k^*$. That is if $x(t)$ is real and even, so are its Fourier series coefficients.
- If $x(t)$ is real and odd, its Fourier series coefficients are purely imaginary and odd. Thus, e.g., $a_0 = 0$.

Example

Consider

$$x_1(t) = \cos(\omega_0 t)$$

Consider

$$x_2(t) = \sin(\omega_0 t)$$

Parseval's Relation for Continuous-Time Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2. \tag{19}$$

Example

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 11.

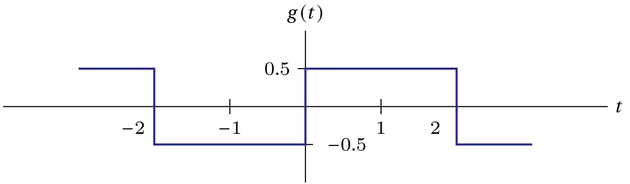


Figure 11: Figure for example

Determine the Fourier series representation of $g(t)$

1. directly from the analysis equation.
2. by assuming that the Fourier series coefficients of the symmetric periodic square wave are known.

Solution: Direct

Example

Consider the triangular wave signal $x(t)$ with period $T = 4$ and fundamental frequency $\omega_0 = \pi/2$, shown in Figure 12. The derivative signal is the signal $g(t)$ in Figure 11. Using this information, find the Fourier series coefficients of $x(t)$.

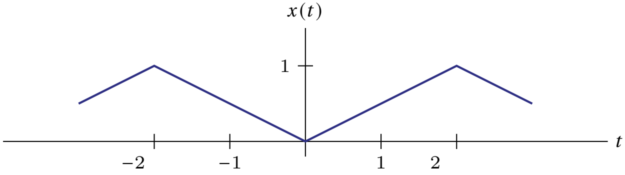


Figure 12: Figure for example

Example

Obtain the Fourier series coefficients of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{20}$$

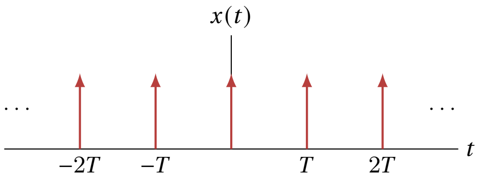


Figure 13: Impulse train

Example

By expressing the derivative of a square wave signal in terms of impulses, obtain the Fourier series coefficients of the square wave signal.

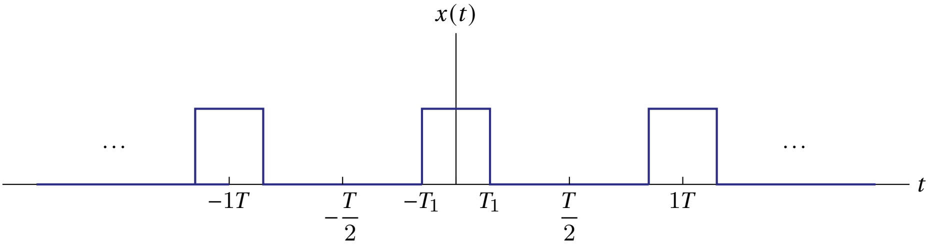


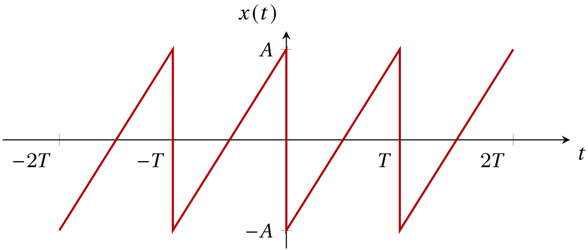
Figure 14: Figure for example

Example

For the waveform $x(t)$,

1. Obtain expression for the exponential Fourier series coefficients a_k.
2. Compute the average power
$$\frac{1}{T}\int_T |x(t)|^2 dt.$$
3. Verify Parseval's relation.

Given: Sum of the reciprocals of the positive square integers is $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.



$$x(t) = A\left(\frac{2t}{T} - 1\right), \quad 0 < t <\leq T.$$

Example: Computing a_k

Example: Computing the Average Power

Example: Verifying Parseval's relation

Other Forms of Fourier Series

Complex Exponential Fourier Series

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$
$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \tag{21}$$

Trigonometric Fourier Series

$$x(t) = A_0 + 2\sum_{k=1}^{+\infty} A_k \cos k\omega_0 t + B_k \sin k\omega_0 t$$
$$A_k = \frac{1}{T}\int_T x(t)A_k \cos k\omega_0 t dt$$
$$B_k = \frac{1}{T}\int_T x(t)A_k \sin k\omega_0 t dt \tag{22}$$

Harmonic Form Fourier Series (for Real $x(t)$)

$$x(t) = C_0 + 2\sum_{k=1}^{+\infty} C_k \cos(k\omega_0 t - \theta_k)$$
$$C_0 = A_0 \tag{23}$$
$$C_k = \sqrt{A_k^2 + B_k^2} \quad \theta_k = \tan^{-1}\left(\frac{B_k}{A_k}\right)$$

Relationship

$$A_0 = a_0$$
$$A_k = \frac{a_k + a_{-k}}{2}$$
$$B_k = j\frac{a_k - a_{-k}}{2} \tag{24}$$
$$\omega_0 = \frac{2\pi}{T}$$

Convergence of Fourier Series

FS synthesis and analysis equations:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Consider the finite series of the form

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

Let $e_N(t)$ denote the approximation error, that is,

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

A quantitative measure of approximation error is

$$E_N = \int_T |e_N(t)|^2 \, dt$$

Convergence of Fourier Series

- If $x(t)$ has a Fourier series representation, then the limit of E_N as $N \to \infty$ is zero.
- If $x(t)$ does not have a Fourier series representation, then the integral that computes a_k may diverge. Moreover, even if all of the coefficients a_k obtained are finite, when these coefficients are substituted into the synthesis equation, the resulting infinite series may not converge to the original signal $x(t)$.
- Fortunately, there are no convergence difficulties for large classes of periodic signals, continuous and discontinuous.

Finite-Energy Convergence Criterion

One class of periodic signals that are representable through the Fourier series is those signals which have finite energy over a single period:

$$\int_T |x(t)|^2 \, dt < \infty \tag{25}$$

- In this case coefficients a_k are finite.
- As $N \to \infty$, $E_N \to 0$.
- This does not imply that the signal $x(t)$ and its Fourier series representation are equal at every value of t. What it does say is that there is no energy in their difference.
- However, since physical systems respond to signal energy, from this perspective $x(t)$ and its Fourier series representation are indistinguishable.

Alternative Conditions (Dirichlet Conditions)

Dirichlet conditions guarantee that $x(t)$ equals its Fourier series representation, except at isolated values of t for which $x(t)$ is discontinuous. At these values, the infinite series converges to the average of the values on either side of the discontinuity.

Condition 1
Over any period, $x(t)$ must be absolutely integrable

$$\int_T |x(t)|\, dt < \infty. \tag{26}$$

This guarantees that a_ks are finite.

Condition 2
In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

Condition 3
In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

Examples of Functions that Violate Dirichlet Conditions

Cond. 1 The periodic signal with period 1 with one period defined as

$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1.$$

Cond. 2 The periodic signal with period 1 with one period defined as

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1.$$

For this

$$\int_0^1 |x(t)|\, dt < 1$$

The function has, however, an infinite number of maxima and minima in the interval.

Cond. 3 The signal, of period T = 8, is composed of an infinite number of sections, each of which is half the height and half the width of the previous section. Thus, the area under one period of the function is clearly less than 8. However, there are an infinite number of discontinuities in each period, thereby violating Condition 3.

Gibbs Phenomenon

- When a function with jump discontinuities is approximated by a finite number of Fourier series terms, oscillations appear near the discontinuities.
- These oscillations:
  - do not disappear as the number of harmonics increases,
  - become more localized around the discontinuity,
  - have a maximum overshoot of approximately 9% of the jump magnitude.
- This effect is known as the Gibbs phenomenon.
- Away from the discontinuities, the Fourier series converges to the original signal.

Gibbs Phenomenon

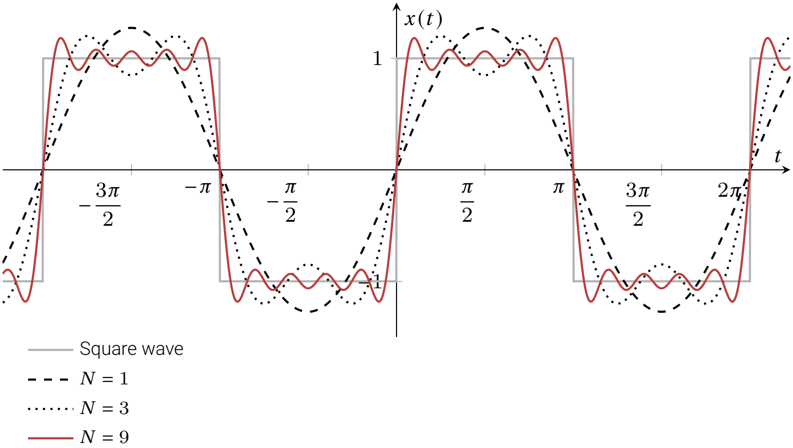


Figure 15: Gibbs phenomenon

Section 4

Continuous-Time Fourier Transform

Introduction

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.
- We use Fourier series for periodic signals, and Fourier transform for aperiodic signals.
- Each of these have continuous-time and discrete-time versions:
    1. Continuous-time Fourier series
    2. Continuous-time Fourier transform
    3. Discrete-time Fourier series
    4. Discrete-time Fourier transform
- In this part of the course, we will concentrate on how to compute continuous-time Fourier series and transform. Later, after we study liner, time-invariant (LTI) systems, we will study the conceptual aspects of Fourier techniques.
- In EN2063, we will do a more rigorous study of Fourier techniques.

Fourier Transform

- In the last lecture, we represented a periodic signal as a linear combination of complex exponentials.
- We use Fourier transform to represent aperiodic signals. A larger class of signals, including all signals with finite energy, can be represented through a linear combination of complex exponentials.
- Whereas for periodic signals the complex exponential building blocks are harmonically related, for aperiodic signals they are infinitesimally close in frequency, and the representation in terms of a linear combination takes the form of an integral rather than a sum.
- The resulting spectrum of coefficients in this representation is called the Fourier transform.
- The synthesis integral itself, which uses the Fourier transform to represent the signal as a linear combination (integral) of complex exponentials, is called the inverse Fourier transform.

Fourier Series Representation for Square Wave

The continuous-time periodic square wave, sketched below, is defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < T/2, \end{cases}$$

This signal periodically repeats with the fundamental period T and the fundamental frequency $\omega_0 = 2\pi/T$.

The Fourier series coefficients a_k of this wave are

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}. \tag{27}$$

We plotted this for a fixed value of T_1 and several values of T (shown in the next slide). An alternative way of interpreting Eq. 27 is as samples of an envelope function:

$$T a_k = \left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\omega=k\omega_0}.$$

With ω thought of as a continuous variable, the function $\dfrac{2\sin(\omega T_1)}{\omega}$ represents the envelope of Ta_k, and the coefficients a_k are simple equally spaced samples of this envelope. For fixed T_1, the envelope of Ta_k is independent of T.

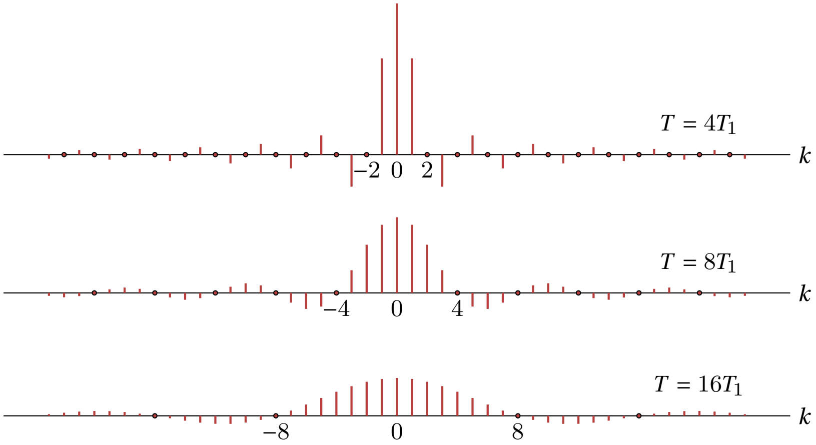


Figure 17: Plots of scaled Fourier series coefficients a_k

Plots of scaled Fourier series coefficients a_k for the periodic square wave with T_1 fixed and for several values of T: $T = 4T_1$, $T = 8T_1$, $T = 16T_1$.

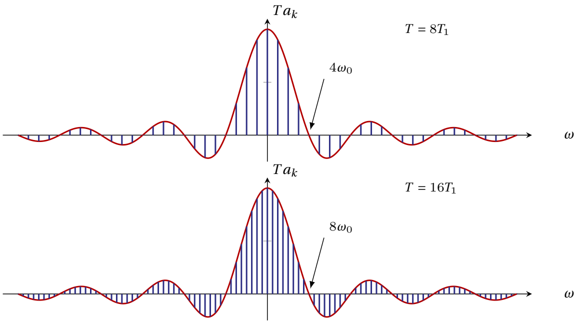
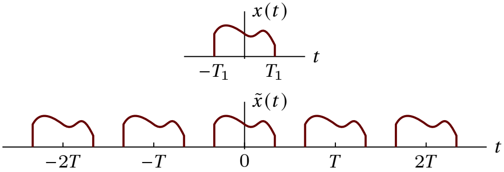


Figure 18: Fourier series coefficients and their envelope for periodic square wave.

The Fourier series coefficients and their envelope for periodic square wave for several values of T (with T_1 fixed): $T = 4T_1$, $T = 8T_1$, $T = 16T_1$. The coefficients are regularly-spaced samples of the envelope $(2 \sin \omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

As T increases, or equivalently, as the fundamental frequency $\omega_0 = 2\pi/T$ decreases, the envelope is sampled with a close and closer spacing. As T becomes arbitrarily large, the original periodic square waveform approaches the rectangular pulse. Also, the Fourier series coefficients, multiplied by T, become more and more closely spaced samples of the envelope. So, in some sense, the set of Fourier series coefficients approaches the envelope function as $T \to \infty$.



$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt.$$

As $\tilde{x}(t) = x(t)$ for $|t| < T/2$, and also, as $x(t) = 0$ outside this interval,

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt.$$

Defining the envelope $X(j\omega)$ of Ta_k as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt,$$

we have, for the coefficients a_k,

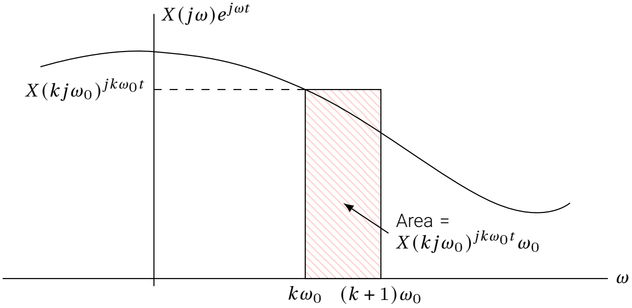
$$a_k = \frac{1}{T} X(jk\omega_0).$$

Combining and expressing $\tilde{x}(t)$ in terms of $X(j\omega)$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t},$$

or, as $\omega_0 = 2\pi/T$

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0. \tag{28}$$



$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0.$$

Inverse Fourier transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

Fourier transform or Fourier integral:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

As $T \to \infty$, $\tilde{x}(t)$ approaches $x(t)$, and consequently, Eq. 28 becomes a representation of $x(t)$. Furthermore, as $\omega_0 \to 0$ as $T \to \infty$, and the right-hand side of Eq. 28 passes to an integral. As $\omega_0 \to 0$, the summation converges to the integral of $X(j\omega)e^{j\omega t}$.

Fourier Transform: Synthesis and Analysis Equations

FS synthesis and analysis equations:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t}\,dt$$

Relation with a_k

Assume that the Fourier transform of $x(t)$ is $X(j\omega)$.
If we construct a periodic signal $\tilde{x}(t)$ by repeating the aperiodic signals $x(t)$ with period T, its Fourier series coefficients are

FS synthesis and analysis equations:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

FT synthesis and analysis equations:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Convergence of Fourier Transform

Assume that we evaluated $X(j\omega)$ according to eq. 118, and left $\hat{x}(t)$ denote the signal obtained by using $X(j\omega)$ in 118:

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

When is $\hat{x}(t)$ a valid representation of the original signal $x(t)$? We define the error between $\hat{x}(t)$ and $x(t)$ as

$$e(t) = \hat{x}(t) - x(t).$$

If $x(t)$ has finite energy (square integrable), i.e.,

(29)

$X(j\omega)$ is finite, and

(30)

If $x(t)$ has finite energy, then, although $x(t)$ and its Fourier representation $\hat{x}(t)$ may differ significantly at individual values of t, there is no energy in their difference.

Convergence of Fourier Transform: Dirichlet Conditions

There are alternative conditions sufficient to ensure that $\hat{x}(t)$ is qual to $x(t)$ for any t except at a discontinuity, where it is equal to the average of the values on either side of the discontinuity.

1. $x(t)$ is absolutely integrable, i.e.,

(31)

2. $x(t)$ has a finite number of maxima and minima within any finite interval.
3. $x(t)$ has a finite number of discontinuities within any finite interval. Furthermore, each of these discontinuities must be finite.

Therefore, absolutely integrable signals that are continuous or that have finite number of discontinuities have a Fourier transform.

Example

Example

Find the Fourier transform of the signal

$$x(t) = e^{-at}u(t), \quad a > 0.$$

Example Cntd. FT of $e^{-at}u(t), \quad a > 0$

Example

Find the Fourier transform of the signal

$$x(t) = e^{-a|t|}, \quad a > 0.$$

Example

Determine the Fourier transform of the unit impulse

$$x(t) = \delta(t).$$

Rectangular Pulse

Example

Determine the Fourier transform of the signal

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & |t| > T_1. \end{cases}$$

Example

Consider the signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \begin{cases} 1, & |\omega| < W, \\ 0, & |\omega| > W. \end{cases}$$

Determine $x(t)$.

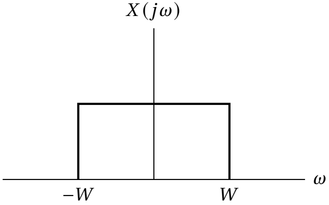


Figure 19: Fourier transform for $x(t)$.

The sinc Function

$$\mathrm{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}. \tag{32}$$

Express

$$\frac{2\sin\omega T_1}{\omega}$$

and

$$\frac{\sin Wt}{\pi t}$$

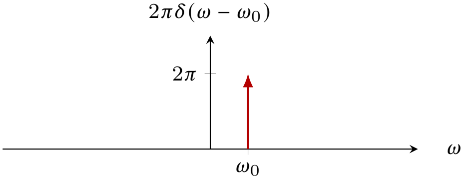
as sinc functions.

What Happens when W Increases?

The Fourier Transform for Periodic Signals: Introduction

In the previous section, we studied the Fourier transform representation, paying attention to aperiodic signals. We can also develop Fourier transform representations for periodic signals. This allows us to consider periodic and aperiodic signals in a unified context. We can construct the Fourier transform of a periodic signal directly from its Fourier series representation.

Consider a signal $x(t)$ with the Fourier transform $X(j\omega)$ that is a single impulse of area 2π at $\omega = \omega_0$, i.e.,



Let’s determine the signal $x(t)$:

Example

Find the Fourier transformof the square wave signal whose Fourier series coefficients are

$$a_k = \frac{\sin k\omega_0 T_1}{\pi k}.$$

Method: Multiply the Fourier seriescoefficients a_k by 2π, place them using the impulse function $\delta(\omega - k\omega_0)$, and sum.

$$\boxed{X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)}$$

Example

Find the Fourier transformof

$$x(t) = \sin \omega_0 t,$$

and

$$x(t) = \cos \omega_0 t.$$

Example

Find the Fourier transform of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

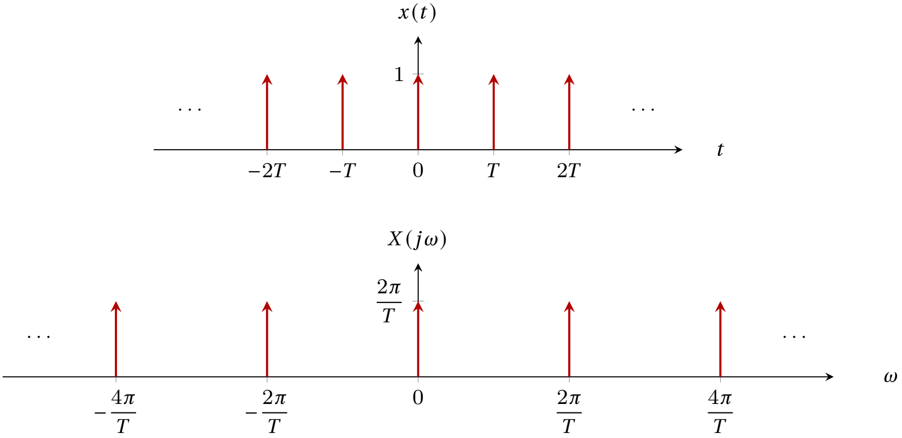


Figure 20: Periodic impulse train and its Fourier transform.

Section 5

Linear, Time-Invariant Systems

Systems

- A system processes signals.
- Examples of systems:
  - Dynamics of an aircraft.
  - An algorithm for analyzing financial and economic factors to predict bond prices.
  - An algorithm for post-flight analysis of a space launch.
  - An edge detection algorithm for medical images.

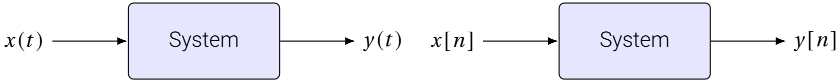


Figure 21: CT and DT Systems.

Types of Systems

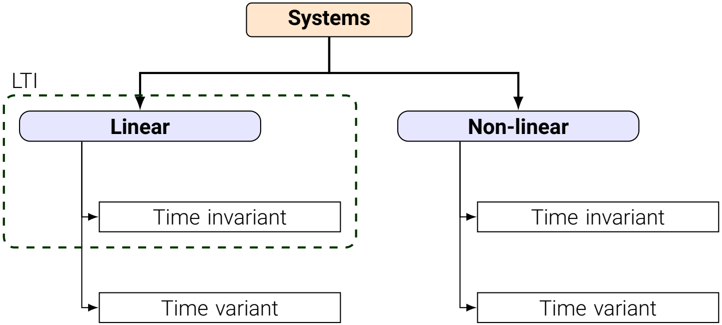


Figure 22: System types.

This course is focused on the class of linear, time-invariant (LTI) systems.

Systems Interconnections

- To build more complex systems by interconnecting simpler subsystems.
- To modify the response of a system.
- E.g.: amplifier design, stabilizing unstable systems.

Signal-Flow (Block) Diagrams

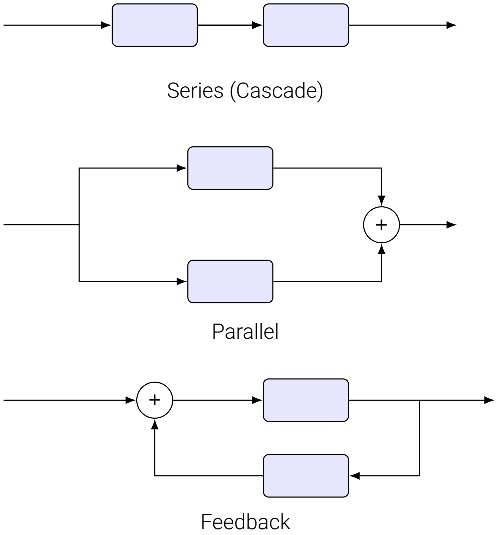


Figure 23: System interconnections.

Systems with and without Memory

A system is said to be memoryless if its output for each value of the independent variable at a given time is dependent only on the input at the same time instant.

Examples of memoryless systems

$$y[n] = (2x[n] - x^2[n])^2,$$

$$y(t) = Rx(t),$$

where $x(t)$ the current through the resistor R and $y(t)$ taken as the voltage across the resistor.

$$y(t) = x(t),$$

which is called the identity system. In DT

$$y[n] = x[n].$$

Squarer: $y(t) = x^2(t).$

Examples of systems with memory

Accumulator or summer:

$$y[n] = \sum_{k=-\infty}^{n} x[k].$$

Unit delay:

$$y[n] = x[n-1].$$

Capacitor with current as the input and the output taken as the voltage:

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau.$$

Invertibility and Inverse Systems

A system is invertible if distinct inputs lead to distinct outputs (one-to-one). If a system is invertible, the inverse system exists, and when cascaded with the original system, yields an output equal to the input to the first system.

Examples of invertible systems:

If $y(t) = 2x(t)$, the inverse system is

$$w(t) = \frac{1}{2}y(t),$$

If (accumulator) $y[n] = \sum_{k=-\infty}^{n} x[k]$.

the inverse system is

$$w[n] = y[n] - y[n-1].$$

Examples of non-invertible systems:

The differentiator is not invertible in general because the constant value (the DC offset) is lost during differentiation. Therefore, the integrator is not a true inverse system unless additional information (e.g., initial conditions) is available.

Inte.: $y_1(t) = \int_{-\infty}^{t} x_1(\tau)d\tau$ Diff.: $y_2(t) = \dfrac{dx_2(t)}{dt}$

$$y[n] = 0.$$

$$y(t) = x^2(t)$$

Causality

A system is said to be causal if it only responds when you "kick it." Its response at any time depends only on that input prior or equal to that time. The system cannot anticipate future inputs.

$$x_1(t) \to y_1(t)$$
$$x_2(t) \to y_2(t)$$

If

$$x_1(t) = x_2(t), \quad t < t_0,$$

then

$$y_1(t) = y_2(t), \quad t < t_0.$$

If inputs are identical until t_0, the outputs are identical until t_0. Same for DT.

Example

$$y[n] = \frac{1}{3}\left[x[n-1] + x[n] + x[n+1]\right] : \text{not causal.}$$

$$y[n] = \frac{1}{3}\left[x[n-2] + x[n-1] + x[n]\right] : \text{causal.}$$

Stability

Many forms. We choose Bounded Input Bounded Output (BIBO) stability.
If a system is stable in BIBO sense, for every bounded input the output is bounded.
We can (carefully) use feedback to stabilize systems.

Time Invariance

The system does not really care what we call the origin. If the input is shifted by any amount of time t_0, the output is also shifted by the same amount of time.

Linearity

If

$$x_1(t) \to y_1(t)$$
$$x_2(t) \to y_2(t)$$

then

$$ax_1(t) + bx_2(t) \to ay_1(t) + by_2(t)$$

If the system is linear, if we give a linear combination of inputs, the output will also be a similar linear combination of the original outputs.

Linear Time-Invariant Systems

1. Systems that are both linear and time invariant are called Linear Time-Invariant (LTI) systems.
2. With systems that are linear and time invariant, using the impulse function in CT and DT, produces an important and useful mechanism for characterizing those systems.
3. In this lecture we develop in detail the representation of both continuous-time and discrete-time signals as a linear combination of delayed impulses and the consequences for representing linear, time-invariant systems.
4. The resulting representation is referred to as convolution.

Introduction (from Oppenheim)

- A linear system: the response to a linear combination of inputs is the same linear combination of the individual responses.
- Time invariance: the system is not sensitive to the time origin. If the input is shifted in time by some amount, then the output is simply shifted by the same amount.
- For a linear system, if the system inputs can be decomposed as a linear combination of some basic inputs and the system response is known for each of the basic inputs, then the response can be constructed as the same linear combination of the responses to each of the basic inputs.
- Signals can be decomposed as a linear combination of basic signals in a variety of ways (e.g., Taylor series expansion that expresses a function in polynomial form.) However, in the context of signals and systems, it is important to choose the basic signals in the expansion so that, in some sense, the response is easy to compute.
- For systems that are both linear and time-invariant, there are two particularly useful choices for these basic signals: delayed impulses and complex exponentials.

Introduction (from Oppenheim)

- In this lecture we develop in detail the representation of both continuous-time and discrete-time signals as a linear combination of delayed impulses and the consequences for representing linear, time-invariant systems. The resulting representation is referred to as convolution.
- Earlier, we developed in detail the decomposition of signals as linear combinations of complex exponentials (referred to as Fourier analysis) and the consequence of that representation for linear, time-invariant systems.

Introduction

- Using the convolution we can express the response of an LTI system to an arbitrary input in terms of the system's response to the unit impulse.
- An LTI system is completely characterized by its response to a single signal, namely, its response to the unit impulse.
- In discrete time, we have the convolution sum. In continuous time, we have the convolution integral.

Strategy for Exploiting Linearity and Time Invariance

A DT Signal as Superposition of Weighted Delayed Impulses

- We can express a DT signal as a linear combination of weighted delayed impulses.
- If we have a linear system, and a signal expressed as above as a linear combination of basic signals, the response would be the same linear combination the responses for individual basic signals.

Convolution Sum

Convolution Sum: Summary

The convolution of the sequence $x[n]$ and $h[n]$ is given by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \tag{33}$$

which we represent symbolically as

$$y[n] = x[n] * h[n]. \tag{34}$$

Example

Compute $y[n] = x[n] * h[n]$ for $x[n]$ and $h[n]$ as shown in Figure 24.

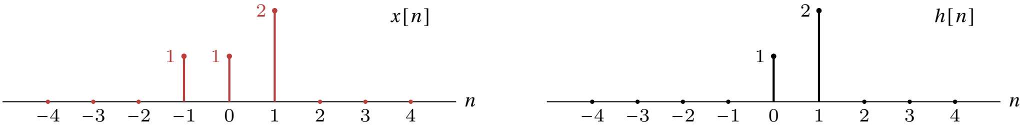


Figure 24: Computing convolution

Example

Consider and input $x[n]$ and a unit impulse response $h[n]$ given by

$$\begin{aligned} x[n] &= \alpha^n u[n] \\ h[n] &= u[n], \end{aligned} \tag{35}$$

which $0 < \alpha < 1$. Find $y[n]$ and sketch.

$$x[k]h[n-k] = \begin{cases} \alpha^k, & 0 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

Thus for $n \ge 0$,

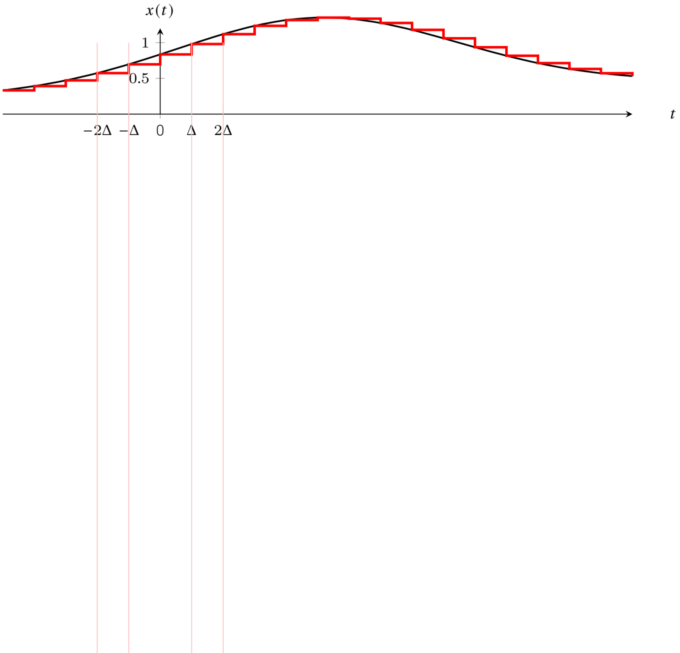
$$y[n] = \sum_{k=0}^{n} \alpha^k$$

$$y[n] = \frac{1-\alpha^{n+1}}{1-\alpha} \quad \text{for } n \ge 0.$$

$$y[n] = \frac{1-\alpha^{n+1}}{1-\alpha} u[n].$$

Continuous-Time Systems: The Convolution Integral

1. Similar to what we did in DT, in this section we obtain a complete characterization of a continuous-time LTI system in terms of its unit impulse response.
2. In discrete time, the key to developing the convolution sum was the sifting property of the DT unit impulse—i.e., mathematical representation of a signal as a superposition of scaled and shifted unit impulse functions.
3. We begin by considering the staircase approximation $\hat{x}(t)$ of a CT signal $x(t)$.


$x(t)$
1
0.5
t
-2Δ
$-\Delta$
0
Δ
2Δ

The approximation that we saw can be expressed as a linear combination of delayed impulses. Define

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise.} \end{cases}$$

Since $\Delta\delta_\Delta(t)$ has unit amplitude, we have

As $\Delta \to 0$, the summation approaches an integral. Consequently,

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

This is known as the sifting property of the continuous time impulse.

Example:

Use the sifting property to express $u(t)$ in terms of $\delta(t)$.

The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI Systems

Let's define $\hat{h}_{k\Delta}(t)$ as the response of an LTI system to the input $\delta_\Delta(t - k\Delta)$.

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta.$$

Since the pulse $\delta_\Delta(t - k\Delta)$ corresponds to a shifted unit impulse as $\Delta \to 0$, the response $\hat{h}_{k\Delta}(t)$ to this input pulse becomes the response to an impulse in the limit. If we let $h_\tau(t)$ denote the response at time t to a unit impulse $\delta(t - \tau)$ located at time τ, then

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)h_{k\Delta}(t)\Delta.$$

As a $\Delta \to 0$, the summation on the right-hand side becomes an integral.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_\tau(t)d\tau$$

In addition to being linear, the system is time-invariant, the response of the LTI system to the unit impulse $\delta(t-\tau)$

$$h_\tau(t) = h_0(t-\tau).$$

Defining unit impulse response $h(t)$ as

$$h(t) = h_0(t),$$

we have

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

which is referred to as the convolution integral or the superposition integral. This corresponds to the representation of a continuous-time LTI system in terms of its response to a unit impulse.

$$y(t) = x(t) * h(t).$$

As in discrete time, a continuous-time LTI system is completely characterized by its impulse response—i.e., by its response to a.single elementary signal, the unit impulse $\delta(t)$.

Example: Let $x(t)$ be the input to an LTI system with unit impulse response $h(t)$, where

$$x(t) = e^{-at}u(t), a > 0$$

and

$$h(t) = u(t).$$

For $t < 0$, the product $x(\tau)$ and $h(t-\tau)$ is zero, consequently $y(t)$ is zero.
For $t > 0$,

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t, \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} y(t) &= \int_0^t e^{-a\tau} d\tau = -\frac{1}{a} e^{-a\tau} \Big|_0^t \\ &= \frac{1}{a}(1 - e^{-at}) \end{aligned}$$

Thus for all t,

$$y(t) = \frac{1}{a}(1 - e^{-at})u(t)$$

Example: Consider the convolution of the following two signals:

$$x(t) = \begin{cases} 1, & 0 < t < T, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$h(t) = \begin{cases} t, & 0 < t < 2T, \\ 0, & \text{otherwise.} \end{cases}$$

Example: Find $y(t)$, the convolution of the following two signals:

$$x(t) = e^{2t}u(-t),$$

and

$$x(t) = u(t-3).$$

When $t - 3 \leq 0$, the product of $x(\tau)$ and $h(t - \tau)$ is nonzero for $-\infty < \tau < t - 3$, and the convolving integral becomes

$$y(t) = \int_{-\infty}^{t-3} e^{2\tau} d\tau = \frac{1}{2} e^{2(t-3)}.$$

F0r $t - 3 \geq 0$, the product of $x(\tau)h(t - \tau)$ is nonzero for $-\infty < \tau < 0$, and the convolving integral becomes

$$y(t) = \int_{-\infty}^{0} e^{2\tau} d\tau = \frac{1}{2}.$$

Recapitulation

1. In discrete time the representation takes the form of the convolution sum, while its continuous-time counterpart is the convolution integral:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

2. Characteristics of an LTI system are completely determined by its impulse response ($h(t)$ in CT, $h[n]$ in DT.).

The Commutative Property

DT

$$x[n] * h[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k].$$

CT

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau.$$

Verify the commutative property for DT.

The Distributive Property

Convolution distributes over addition.

DT

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n].$$

CT

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t).$$

The Associative Property

DT

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n].$$

CT

$$x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t).$$

As a consequence,

$$y[n] = x[n] * h_1[n] * h_2[n]$$

and

$$y(t) = x(t) * h_1(t) * h_2(t).$$

are unambiguous.
Using the commutative property together with the associative property we can see that the order in which they are cascaded does not matter as far as the overall system impulse response is concerned.

LTI Systems with and without Memory

1. A system is memoryless if its output at any time depends only on the value of the input at that same time.
2. The only way that this can be true for a discrete-time LTI system is if $h[n] = 0$ for $n \neq 0$.
3. In this case the impulse response has the form

$$h[n] = K\delta[n],$$

   where $K = h[0]$ is a constant.
4. The convolution sum reduces to the relation

$$y[n] = Kx[n].$$

5. If a discrete-time LTI system has an impulse response $h[n]$ that is not identically zero for $n \neq 0$, then the system has memory.
6. For CT:

$$h(t) = K\delta(t).$$
$$y(t) = Kx(t).$$

Invertibility of LTI Systems

An LTI system is invertible only if an inverse system exists that, when connected in series with the original system, produces an output equal to the input to the first system.

Example
Consider the following relationship of a pure time shift:

$$y(t) = x(t - t_0)$$

Is the corresponding system memoryless? What is the inverse system of the system?

Example

Determine $y[n]$, and find the inverse system of the following LTI system with impulse response

$$h[n] = u[n].$$

Example ctd.

Verify that $h_1[n] = \delta[n] - \delta[n-1]$ indeed is the inverse of $h[n] = u[n]$.

$$\begin{aligned} h[n] * h_1[n] &= u[n] * \{\delta[n] - \delta[n-1]\} \\ &= u[n] * \delta[n] - u[n] * \delta[n-1] \\ &= u[n] - u[n-1] \\ &= \delta[n] \end{aligned}$$

Causality for LTI Systems

1. The output of a causal system depends only on the present and past values of the input to the system.
2. For a DT LTI system, $y[n]$ must not depend on $x[k]$ for $k > n$.
3. For this to be true, all the coefficients $h[n-k]$ that multiply values of $x[k]$ for $k > n$ must be zero.
4. This then requires that the impulse response of a causal discrete-time LTI system satisfy the condition
$$h[n] = 0 \quad \text{for } n < 0.$$
5. The impulse response of a causal LTI system must be zero before the impulse occurs, which is consistent with the intuitive concept of causality.
6. More generally, causality for a linear system is equivalent to the condition of initial rest; i.e., if the input to a causal system is 0 up to some point in time, then the output must also be 0 up to that time.
7. The equivalence of causality and the condition of initial rest applies only to linear systems.

Causality for LTI Systems

1. A continuous-time LTI system is causal if

$$h(t) = 0 \quad \text{for } t < 0.$$

2. Causality of an LTI system is equivalent to its impulse response being a causal signal.

For a causal DT LTI system, the condition $h[n] = 0 \quad \text{for } n < 0$ implies that the convolution sum becomes

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k].$$

and as

$$y[n] = h[n] * x[n] = \sum_{k=0}^{\infty} h[k]x[n-k].$$

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k].$$

For a causal CT system, $h(t) = 0 \quad \text{for } t < 0$, convolution integral is

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau.$$

Stability for LTI Systems

A system is stable if every bounded input produces a bounded output. Consider an input $x[n]$ that is bounded in magnitude:

$$|x[n]| < B \quad \text{for all } n.$$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

$$|y[n]| \leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

$$|y[n]| \leq B \sum_{k=-\infty}^{\infty} |h[k]| \quad \text{for all } n$$

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty.$$

If the impulse response is absolutely summable, then $y[n]$ is bounded in magnitude, and hence, the system is stable.

Stability for LTI Systems

In CT a system is stable if the impulse response is absolutely integrable.

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty.$$

Examples Determine whether the following systems are stable: 1. Pure time shift in DT. 2. Pure time shift in CT. 3. Accumulator in DT. 4. CT counterpart of the accumulator.

The Unit Step Response of an LTI System

There is another signal that is also used in describing the behavior of LTI systems: the unit step response, $s[n]$ or $s(t)$, corresponding to the output when $x[n] = u[n]$ or $x(t) = u(t)$.

$$s[n] = u[n] * h[n]$$

Commutative property:

$$s[n] = h[n] * u[n]$$

$s[n]$ can be viewed as the response to the input $h[n]$ of a discrete-time LTI system with unit impulse response $u[n]$.

$u[n]$ is the unit impulse response of the accumulator. Therefore,

$$s[n] = \sum_{k=-\infty}^{\infty} h[k]$$

$h[n]$ can be recovered from $s[n]$ using the relation

$$h[n] = s[n] - s[n-1].$$

That is, the step response of a discrete-time LTI system is the running sum of its impulse response. Conversely, the impulse response of a discrete-time LTI system is the first difference of its step response.

Similarly, in CT, the step response of an LTI system with impulse response $h(t)$ is given by $s(t) = u(t) * h(t)$, which also equals the response of an integrator [with impulse response $u(t)$] to the input $h(t)$. That is, the unit step response of a continuous-time LTI system is the running integral of its impulse response, or

Zero-Input Response

For a linear system (time-invariant or not), if we put nothing into it, we get nothing out of it.

$$x(t) = 0 \quad \text{for all } t, \quad \text{then}$$
$$y(t) = 0 \quad \text{for all } t,$$
$$x[n] = 0 \quad \text{for all } n, \quad \text{then}$$
$$y[n] = 0 \quad \text{for all } n,$$

“Proof”: If the system is linear and

$$x(t) \rightarrow y(t), \quad \text{then if we scale}$$

$$ax(t) \rightarrow ay(t).$$

Select the sale factor $a = 0$.
Not all systems are like this, e.g., even if a battery is not connected to anything, the output is 1.5 V.

Implications for Causality

The system cannot anticipate the input.
I.e., If

$$x_1(t) = x_2(t), \quad \text{for } t < t_0,$$

then

$$y_1(t) = y_2(t), \quad \text{for } t < t_0,$$

Same for DT.

Implications for Causality for a Linear System

For linear systems, if

Initial rest: The system does not respond until an input is given.
For a linear system to be causal it must have the property of initial rest.
Why? For linear systems zero in $\rightarrow$ zero out.

Causality for Linear Time Invariant Systems

For LTI systems,

Causality $\Leftrightarrow$

$$\begin{aligned} h(t) &= 0, \quad t < 0 \\ h[n] &= 0, \quad n < 0 \end{aligned}$$

"Proof": $\Rightarrow$: Why does causality imply the above? Ans:

$\Leftarrow$:Why does $h(t) = 0,\ t < 0$ $(h[n] = 0,\ n < 0)$, imply the system is causal? Ans:

Example: Accumulator

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

The accumulator is an LTI system. Also, we saw that its impulse response is

$$h[n] = u[n].$$

1. Does the accumulator have memory?
2. Is the accumulator causal?
3. Is accumulator stable in the BIBO sense?
4. If invertible, what is the inverse?

Example

$$y[n] - ay[n-1] = x[n]$$

under the assumption of initial rest $\Rightarrow$ LTI. Memory? Causal? Stable?

Example

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

under the assumption of initial rest $\Rightarrow$ LTI. Memory? Causal? Stable?

Impulses

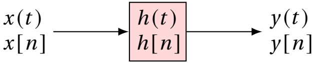


Figure 26

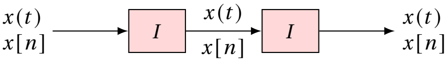


Figure 28

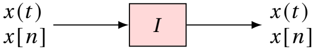
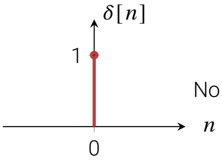


Figure 27



No problem in DT.

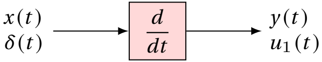
Figure 29

Operational Definition

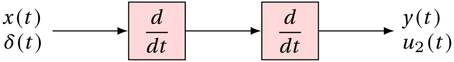
We use operational definitions through convolution to handle derivatives and integrals of impulse, which are badly behaved functions. This leads to a set of singularity functions. Impulse and step are examples of these.

$$x(t) * \delta(t) = x(t)$$

$$\frac{d}{dt}\left[\delta(t)\right]$$



$$x(t) * u_1(t) = \frac{dx(t)}{dt}$$



$$u_2(t) = u_1(t) * u_1(t)$$

$$x(t) * u_2(t) = \frac{d^2x(t)}{dt^2}$$

$$u_k(t) = u_1(t) * u_1(t) * \cdots k \text{ times}$$

$$x(t) * u_k(t) = \frac{d^kx(t)}{dt^k}$$

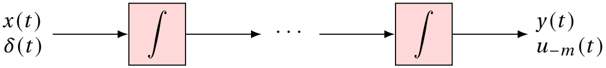
$$u_0(t) = \delta(t)$$

$u_{-1}(t)$ = unit step

$u_{-2}(t)$ = unit ramp

$$u_0(t) = \delta(t)$$

$$u_{-1}(t) = u(t)$$



$$u_k(t) * u_l(t) = u_{k+l}(t).$$

$x(t) * u_{-m}(t) = m^{\mathrm{th}}$ running integral

$u_k(t)$ defined by

$$x(t) * u_k(t) = \frac{d^k x(t)}{dt^k}$$

Linear, Constant-Coefficient Differential and Difference Equations

- An important class of CT systems is that for which the input and output are related through a linear constant-coefficient differential equation.
- These arise in the description of a wide variety of systems and physical phenomena. E.g., the response of the RC circuit, the motion of a vehicle subject to acceleration inputs and frictional forces.
- Correspondingly, an important class of DT systems is that for which the input and output are related through a linear constant-coefficient difference equation.
- These are used to describe the sequential behavior of many different processes. E.g., accumulation of savings in a bank account, a digital simulation of a continuous-time system, DT designed to perform particular operations on the input signal such as a system that calculates the difference between successive input values, of computes the average value of the input over an interval.

Linear, Constant-Coefficient Differential Equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{dx(t)}{dt^k} \tag{36}$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0. \quad \text{Homogeneous equation.} \tag{37}$$

Given $x(t)$, if $y_p(t)$ satisfies 36, so does $y_p(t) + y_h(t)$ where $y_h(t)$ satisfies 37.

$$y_p(t) \triangleq \text{ particular solution}$$
$$y_h(t) \triangleq \text{ homogeneous solution}$$

Homogeneous Solution

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0.$$

Guess a solution of the form

$$y_h(t) = Ae^{st}, \quad \text{a complex exponential}$$

$$y_h(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \cdots + A_N e^{s_N t}$$

Coefficients $A_1, A_2, \ldots, A_N$ are undetermined. We need N auxiliary conditions to determined them.

$$y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{N-1}y(t)}{dt^{N-1}} \quad \text{at } t = t_0.$$

Linear system $\Longleftrightarrow$ auxiliary conditions = 0
Linear system $\Rightarrow$ zero in, zero out.
Causal and LTI $\Longleftrightarrow$ initial rest
If $x(t) = 0$, $t < t_0$ then
$y(t) = 0$, $t < t_0$ then

Example: First-Order Differential Equation

$$\frac{dy(t)}{dt} + ay(t) = x(t).$$

Homogeneous equation:

$$\frac{dy_h(t)}{dt} + ay_h(t) = 0.$$

Now guess a solution

$$y_h(t) = Ae^{st}.$$

Obtain the impulse response of the above system.

Linear, Constant-Coefficient Difference Equations

Consider the N^{th}-order difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \tag{38}$$

$$\sum_{k=0}^{N} a_k y[n-k] = 0 : \quad \text{homogeneous equation.} \tag{39}$$

If $y_p[n]$ satisfies 38 so does $y_p[n] + y_h[n]$ where $y_h[n]$ satisfies 39.

$$\begin{aligned} y_p[n] &\triangleq \text{ particular solution} \\ y_h[n] &\triangleq \text{ homogeneous solution} \end{aligned}$$

Homogeneous Solution

$$\sum_{k=0}^{N} a_k y[n-k] = 0.$$

"Guess" a solution of the form

$$y_h[n] = Az^n.$$

$$\sum_{k=0}^{N} a_k A z^n z^{-k} = 0.$$

$$\sum_{k=0}^{N} a_k z^{-k} = 0. \quad N \text{ roots }, z_1, z_2, \ldots, z_N.$$

$$y_h[n] = A_1 z_1^n + A_2 z_2^n + \cdots + A_N z_N^n.$$

$$y[n] = A_1 z_1^n + A_2 z_2^n + \cdots + A_N z_N^n + y_p[n]$$

The undetermined constants A_1 to A_N are to be found using the N auxiliary conditions, $y[n_0]$, $y[n_0 - 1]$, $y[n_0 - N + 1]$.
Linear system $\Longleftrightarrow$ auxiliary conditions 0
Causal, LTI $\Longleftrightarrow$ initial rest
If $x[n] = 0$, $n < n_0$ then
$y[n] = 0$, $n < n_0$ then

Explicit Solution to Difference Equations

Assume causality.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k].$$

$$y[n] = \frac{1}{a_0}\left[\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right]$$

This is said to be a recursive solution to linear constant-coefficient difference equations. To get the computation of $y[n_0]$ started we need the initial conditions or boundary conditions $y[n_0-1]$, $y[n_0-N]$. Then we compute $y[n_0+1]$ and so on.

Example: First-Order Difference Equation

$$y[n] - ay[n-1] = x[n].$$

Causal, LTI $\Longleftrightarrow$ initial rest

$$y[n] = x[n] + ay[n-1].$$

Set $x[n] = \delta[n]$.

$$h[n] = \delta[n] + ah[n-1].$$

$$h[n] = 0, n < 0 \quad \text{initial rest.}$$

$$h[0] = 1$$
$$h[1] = a$$
$$h[2] = a^2 \vdots \qquad = \vdots$$

$$h[n] = a^n u[n].$$

So if causal and LTI

$$\delta[n] \to a^n u[n].$$

Family of solutions

$$\delta[n] \to a^n u[n] + y_h[n].$$

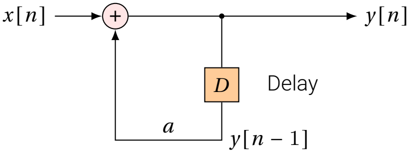
$$y_h[n] = Az^n.$$

$$Az^n - aAz^{n-1} = 0.$$

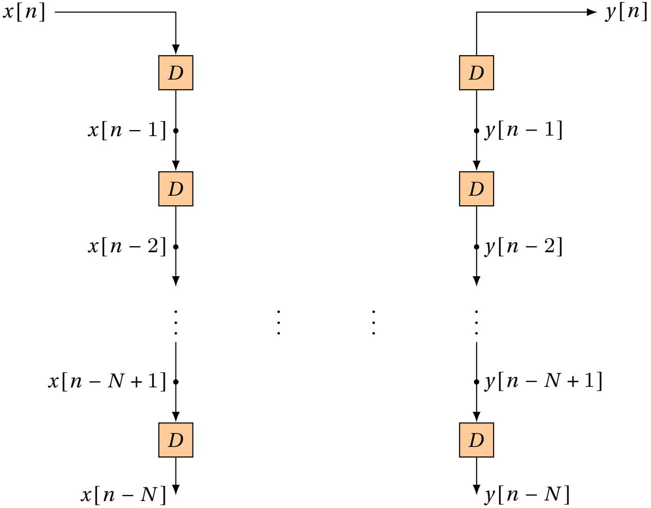
$$a - z^{-1} = 0 \Rightarrow z = a.$$

$$y_h[n] = Aa^n.$$

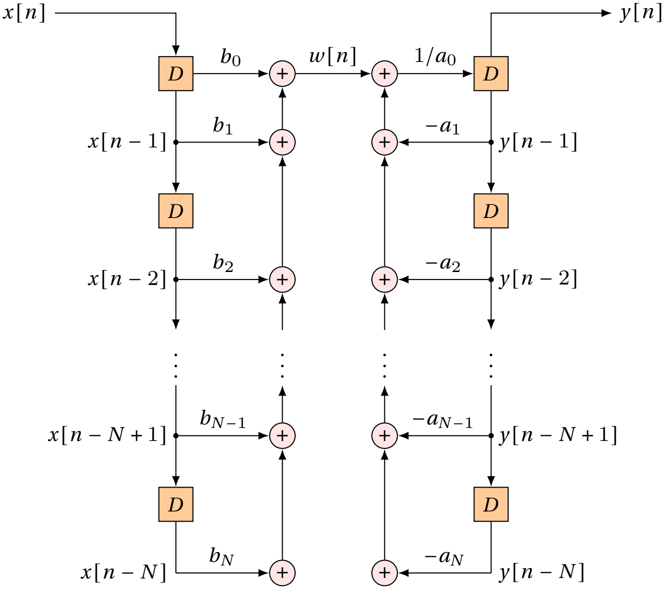
$$\delta[n] \to a^n u[n] + Aa^n.$$



$x[n]$
$+$
$y[n]$
D
Delay
a
$y[n-1]$

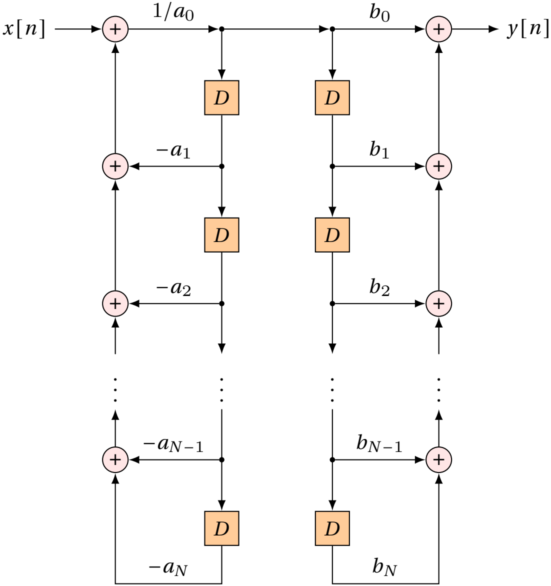


$x[n]$
$y[n]$
D
$x[n-1]$
$y[n-1]$
$x[n-2]$
$y[n-2]$
$x[n-N+1]$
$y[n-N+1]$
$x[n-N]$
$y[n-N]$



Direct form I implementation

This is a cascade of two linear systems. We can interchange the order of the two segments.


$x[n]$
$1/a_0$
b_0
$y[n]$
D
D
$-a_1$
b_1
D
D
$-a_2$
b_2
$-a_{N-1}$
b_{N-1}
D
D
$-a_N$
b_N

Convolution

1. In representing and analyzing LTI systems, our approach has been to decompose the system inputs into a liner combination of basic signals and exploit the fact that for a linear system, the response is the same linear combination of the responses to the basic inputs.
2. The convolution sum and the convolution integral resulted out of the particular choice of the basic signals, delayed unit impulses.
3. This choice has the advantage that for systems that are time invariant in addition to being linear, once the response to an impulse at one time position is known, then the response is know at all time positions.

Complex Exponentials with Unity Magnitude as Basic Signals

1. When we select complex exponential with unity magnitude as the basic signals, the decomposition of this form of a periodic signal is the Fourier series.
2. For aperiodic signals, it becomes the Fourier transform.
3. In latter lectures, we will generalize this representation to Laplace tansform for continuous-time signals and z-transform for discrete-time signals.

Consider a linear system

$$\begin{array}{c} x(t) \\ x[n] \end{array} \longrightarrow \boxed{\begin{array}{c} h(t) \\ h[n] \end{array}} \longrightarrow \begin{array}{c} y(t) \\ y[n] \end{array}$$

If

$$x(t) = a_1\phi_1(t) + a_2\phi_2(t) + \cdots$$

and

$$\phi_k(t) \longrightarrow \psi_k(t), \quad \text{(output due to } \phi_k(t)\text{)}$$

then

$$y(t) = a_1\psi_1(t) + a_2\psi_2(t) + \cdots$$

Identical for DT. So

$$\begin{aligned} \text{If} \quad & x(t) = a_1\phi_1 + a_2\phi_2 + \cdots, \\ \text{then} \quad & y(t) = a_1\psi_1 + a_2\psi_2 + \cdots. \end{aligned}$$

Choose $\phi_k(t)$ or $\phi_k[n]$ so that

1. A broad class of signals can be constructed as a linear combination of ϕ_ks
2. Response to ϕ_ks easy to compute.

Choice of signals $\delta(t - k\Delta)$ and $\delta[n - k]$ lead to the convolution integral and convolution sum.

$$\begin{aligned} \text{CT} \quad & \phi_k(t) = \delta(t - k\Delta) \\ & \psi_k(t) = h(t - k\Delta) \quad \Rightarrow \text{ convolution integral} \\ \text{DT} \quad & \phi_k[n] = \delta[n - k] \\ & \psi_k[n] = h[n - k] \quad \Rightarrow \text{ convolution sum} \end{aligned}$$

Here, we choose complex exponentials as the set of basic signals.

$$\begin{aligned} \phi_k(t) &= e^{s_k t}, \quad s_k \text{ complex} \\ \phi_k[n] &= z_k^n, \quad z_k \text{ complex} \end{aligned}$$

Fourier Analysis

CT $\quad s_k = j\omega k$ purely imaginary $\quad \phi_k(t) = e^{j\omega_k t}$

DT $\quad |z_k| = 1 \quad \phi_k[n] = e^{j\omega_k n}$

s_k complex $\Rightarrow$ Laplace transform
z_k complex $\Rightarrow$ z-transform

Eigenfunction Property

Consider $\phi_k(t) = e^{j\omega_k t}$:

$$e^{j\omega_k t} \longrightarrow H(\omega_k)e^{j\omega_k t} \quad \text{(a scaled-version of the input)}$$

“Proof”:

CT Fourier Series

$$x(t) = x(t+T), \quad T = \text{ period}, \quad \omega_0 = \frac{2\pi}{T}$$

Consider the periodic exponential signal with the same fundamental period:

$$e^{j\omega_0 t}, \quad T = \frac{2\pi}{\omega_0}$$

$$e^{jk\omega_0 t}, \quad T = \frac{2\pi}{k\omega_0}, \quad \text{harmonically related}$$

Fourier series: $\displaystyle x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

1. How do we find a_ks?
2. What are the signal that can be expressed in this form? [already answered, see convergence]

Example: Show that the complex exponential e^{st}, where s is a complex variable[1], is an eigenfunction of the LTI system with impulse response $h(t)$. Find en expression for the eigenvalue $H(s)$. Hence, show that $e^{j\omega t}$ is an eigenfunction of the LTI system with impulse response $h(t)$ and find the eigenvalue $H(j\omega)$.

[1] $s = \sigma + j\omega$

Example: Consider an LTI system for which the input and the output are related by

$$y(t) = x(t-3),$$

1. If the input is the exponential signal $x(t) = e^{j2t}$, find the output $y(t)$ and the eigenvalue $H(s)$.
2. Sate $h(t)$ and, hence, find $H(j\omega)$.