EN1020 Circuits, Signals, and Systems: Signals

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Section 1

Real Signals

Subsection 1

Sinusoids

Continuous-Time Sinusoidal Signal

$$x(t) = A\cos(\omega_0 t + \phi). \tag{1}$$

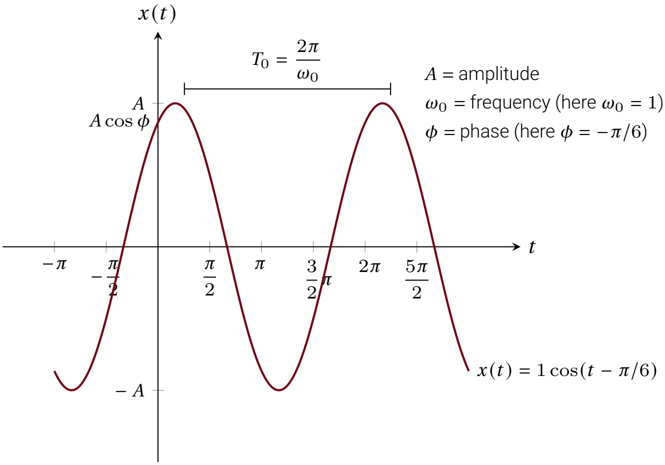


Figure: Continuous-time sinusoidal signal.

Periodicity of a Sinusoidal

Sinusoidal signal is periodic.
A periodic continuous-time signal $x(t)$ has the property that there is a positive value T for which

$$x(t) = x(t + T) \tag{2}$$

for all values of t. Under an appropriate time-shift the signal repeats itself. In this case we say that $x(t)$ is periodic with period T.
Fundamental period T_0 = smallest positive value of T for which 2 holds.
A signal that is not periodic is referred to as aperiodic.
E.g.: Consider $A\cos(\omega_0 t + \phi)$

$$\begin{aligned} A\cos(\omega_0 t + \phi) &= A\cos(\omega_0(t + T) + \phi) \quad \text{here } \omega_0 T = 2\pi m \quad \text{an integer multiple of } 2\pi \\ &= A\cos(\omega_0 t + \phi) \end{aligned}$$

Phase of a Sinusoidal

A time-shift in a CT sinusoid is equivalent to a phase shift.
E.g.: Show that a time-shift of a sinusoid is equal to a phase shift.

Even and Odd Signals

A signal $x(t)$ or $x[n]$ is referred to as an even signal if it is identical to its time-reversed counterpart, i.e., with its reflection about the origin:

$$x(-t) = x(t)$$
$$x[-n] = x[n]$$

A is referred to as an odd if

$$x(-t) = -x(t)$$
$$x[-n] = -x[n]$$

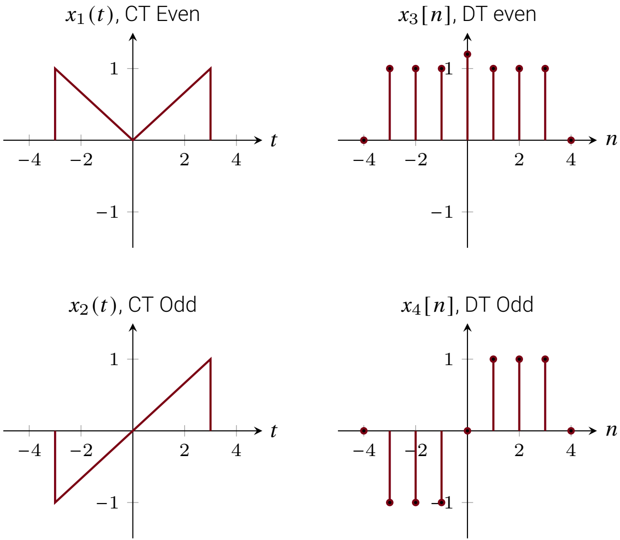
An odd signal must be) at $t = 0$ or $n = 0$.
A signal can be broken into a sum of two signals, one of which is even and one for which is odd. Even part of $x(t)$ is

$$\mathfrak{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

Odd part of $x(t)$ is

$$\mathfrak{Od}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

Examples of Even and Odd Functions



Even and Odd Signals Contd.

Example

Show that $\mathfrak{Ev}\{x(t)\} = \dfrac{1}{2}[x(t) + x(-t)]$.

Notation: $x_e(t)$ is even part of $x(t)$, $x_o(t)$ is odd part of $x(t)$.

Phase of a Sinusoidal: $\phi = 0$

Phase of a Sinusoidal: $\phi = -\pi/2$

Subsection 2

Discrete-Time Sinusoidal Signal

$$x[n] = A\cos(\omega_0 n + \phi) \text{ with } \phi = 0$$

$x[n] = A\cos(\omega_0 n + \phi)$ with $\phi = -\pi/2$

Phase Change and Time Shift in DT

Question

Does a phase change always correspond to a time shift in discrete-time signals?

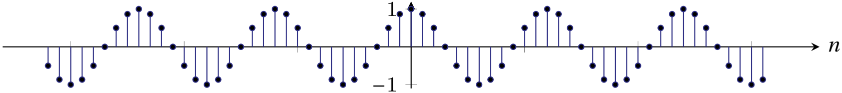
Periodicity of a DT Signal

All continuous-time sinusoids are periodic. However, discrete-time sinusoids are not necessarily so.

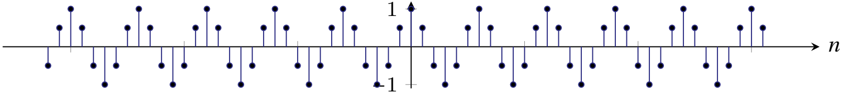
$$x[n] = x[n+N], \quad \text{smallest integer } N \text{ is the fundamental period.} \tag{3}$$

Periodicity of a DT Signal Cntd.

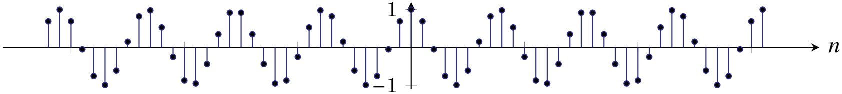
$x[n] = A\cos(2\pi/12n)$



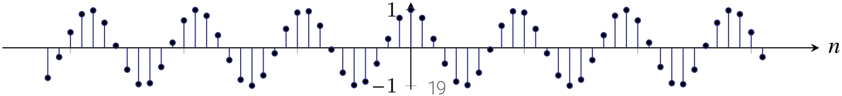
$x[n] = A\cos(2\pi/6n)$



$x[n] = A\cos(2\pi/31n)$



$x[n] = A\cos(2/3n)$



Subsection 3

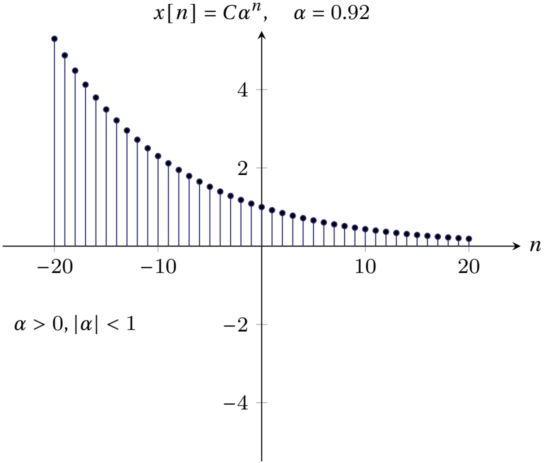
Exponentials

CT Real Exponentials

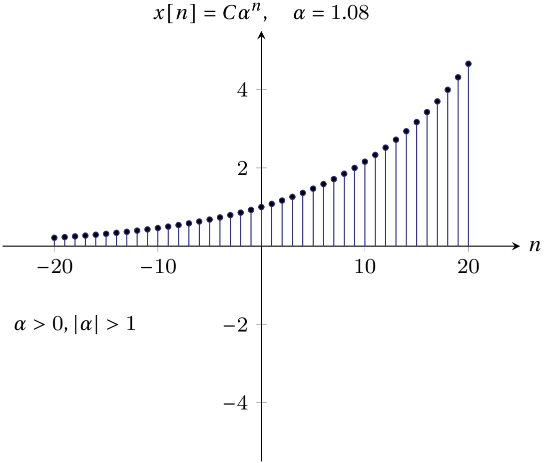
$$\begin{aligned} x(t) &= Ce^{a(t+t_0)}, \quad C \text{ and } a \text{ are real numbers} \\ &= Ce^{at_0}e^{at}. \end{aligned}$$

DT Real Exponentials

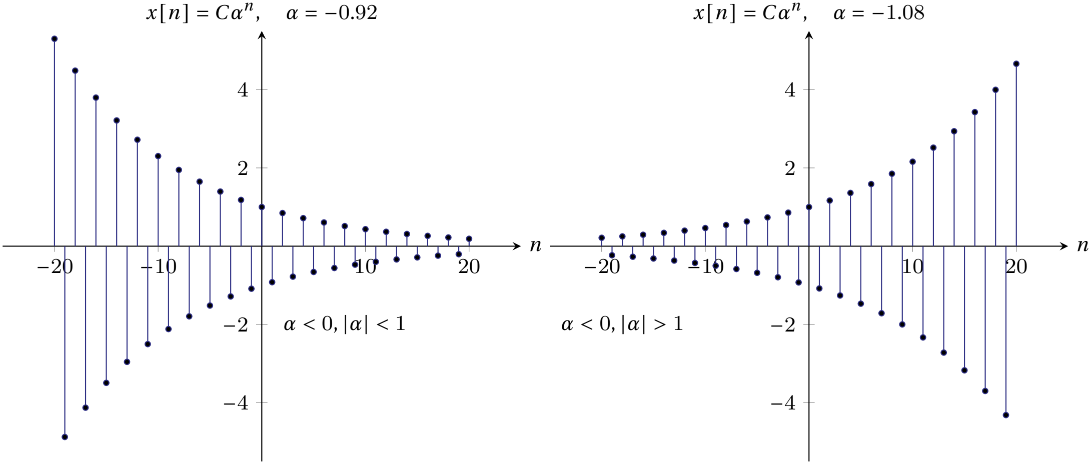
$$x[n] = Ce^{\beta n} = C\alpha^n, \quad C \text{ and } \alpha \text{ are real numbers}$$



$x[n] = C\alpha^n, \quad \alpha = 0.92$
4
2
−2
−4
−20
−10
10
20
n
$\alpha > 0, |\alpha| < 1$



$x[n] = C\alpha^n, \quad \alpha = 1.08$
4
2
−2
−4
−20
−10
10
20
n
$\alpha > 0, |\alpha| > 1$



$x[n] = C\alpha^n, \quad \alpha = -0.92$
$\alpha < 0, |\alpha| < 1$
$x[n] = C\alpha^n, \quad \alpha = -1.08$
$\alpha < 0, |\alpha| > 1$
n

Section 2

Complex Numbers

Representing Complex Numbers

The Cartesian or rectangular form:

$$z = x + jy,$$

where $j = \sqrt{-1}$ and x and y are real numbers referred to respectively as the real part and the imaginary part. I.e.,

$$x = \Re\mathfrak{e}\{z\}, y = \Im\mathfrak{m}\{z\}$$

The polar form:

$$z = re^{j\theta},$$

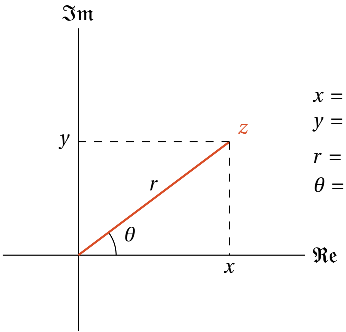
where $r > 0$ is the magnitude of z and θ is the angle or phase of z.

$$r = |z|, \theta = \sphericalangle z.$$

The relationship between these two representations can be determined from Euler's relation:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

or by plotting z in the complex plane.



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \mathrm{atan2}(y, x)$$

Example Let z_0 be a complex number with polar coordinates (r_0, θ_0) and Cartesian coordinates (x_0, y_0). Determine expressions for the Cartesian coordinates of the following complex numbers in terms of x_0 and y_0. Plot the points z_0, z_1, z_2, z_3, z_4, and z_5 in the complex plane when $r_0 = 2$ and $\theta_0 = \pi/4$ and when $r_0 = 2$ and $\theta_0 = \pi/2$. Indicate on the plot the real and imaginary parts of each point.

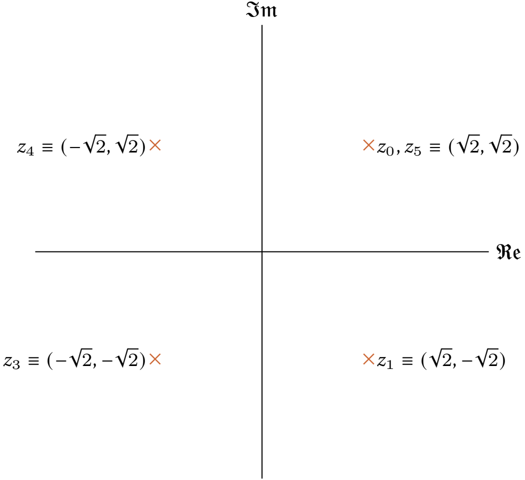
1. $z_1 = r_0 e^{-j\theta_0}$
2. $z_2 = r_0$
3. $z_3 = r_0 e^{j(\theta_0+\pi)}$
4. $z_4 = r_0 e^{j(-\theta_0+\pi)}$
5. $z_5 = r_0 e^{j(\theta_0+2\pi)}$

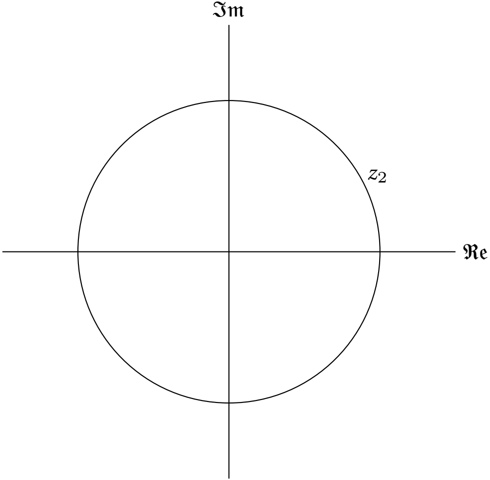
$$
\begin{aligned}
z_0 &= r_0 e^{j\theta_0} = r_0(\cos\theta_0 + j\sin\theta_0) \\
&= r_0\cos\theta_0 + j r_0 \sin\theta_0 = x_0 + j y_0.
\end{aligned}
$$

$$
z_1 = r_0 e^{-j\theta} = r_0(\cos(-\theta_0) + j\sin(-\theta_0)) = x_0 - j y_0.
$$

$$r = 2, \theta = \pi/4$$

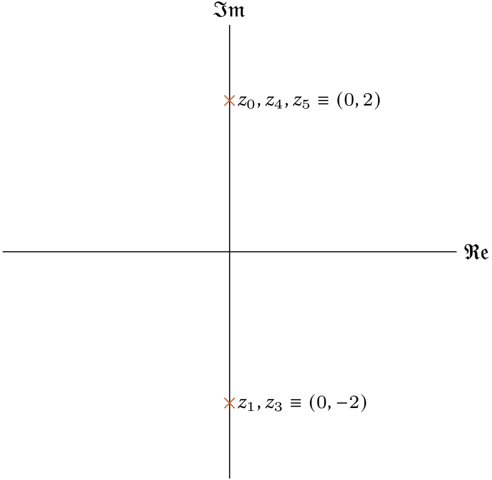
$$z_1 = r_0 e^{-j\theta_0},\ z_2 = r_0,\ z_3 = r_0 e^{j(\theta_0+\pi)},\ z_4 = r_0 e^{j(-\theta_0+\pi)},\ z_5 = r_0 e^{j(\theta_0+2\pi)}$$

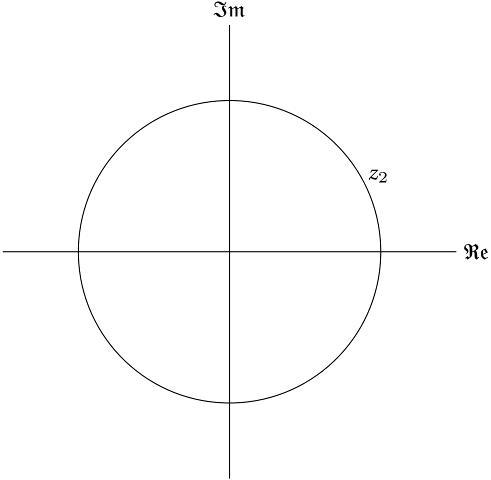




$r = 2, \theta = \pi/2$

$z_1 = r_0 e^{-j\theta_0}$, $z_2 = r_0$, $z_3 = r_0 e^{j(\theta_0+\pi)}$, $z_4 = r_0 e^{j(-\theta_0+\pi)}$, $z_5 = r_0 e^{j(\theta_0+2\pi)}$

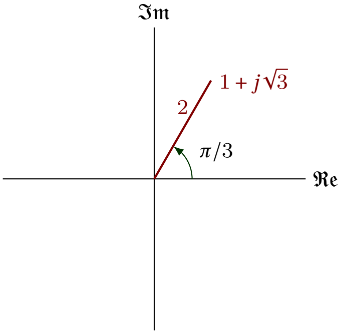




Example Express each of the following complex numbers in polar form, and plot them in the complex plane, indicating the magnitude and angle of each number.

1. $1 + j\sqrt{3}$
2. -5
3. $-5 - 5j$
4. $3 + 4j$
5. $(1 - j\sqrt{3})^3$
6. $\dfrac{e^{j\pi/3} - 1}{1 + j\sqrt{3}}$

$$\begin{aligned}
1 + j\sqrt{3} &= \sqrt{1^2 + (\sqrt{3})^2}\left(\frac{1}{\sqrt{1^2 + (\sqrt{3})^2}} + j\frac{\sqrt{3}}{\sqrt{1^2 + (\sqrt{3})^2}}\right) \\
&= 2e^{j\,\mathrm{atan2}(\sqrt{3},1)} \\
&= 2e^{j\pi/3}
\end{aligned}$$



$\cos\theta$ and $\sin\theta$

Using Euler's relations, derive the following relationships:

1. $\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$
2. $\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$

$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$

Adding

$$e^{j\theta} + e^{-j\theta} = 2\cos\theta$$
$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

Subtracting

$$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$$
$$\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

Complex Conjugate

Let z denote a complex variable; i.e.,

$$z = x + jy = re^{j\theta}.$$

The complex conjugate of z is

$$z^* = x - jy = re^{-j\theta}.$$

Show that

1. $zz^* = r^2$
2. $z + z^* = 2\mathfrak{Re}\{z\}$
3. $z - z^* = 2j\mathfrak{Im}\{z\}$

1. $zz^* = re^{j\theta}re^{-j\theta} = r^2e^0 = r^2$
2. $z + z^* = x + jy + x - jy = 2x = 2\mathfrak{Re}\{z\}$
3. $z - z^* = x + jy - (x - jy) = 2jy = 2j\mathfrak{Im}\{z\}$

List the values of

Section 3

Complex Signals

Subsection 1

CT Complex Exponentials

CT Complex Exponentials

$x(t) = Ce^{at}$ $\quad C$ and a are complex numbers.

$$C = |C|e^{j\theta}$$

$$a = r + j\omega_0$$

$$x(t) = |C|e^{j\theta}e^{(r+j\omega_0)t}$$

$$= |C|e^{rt}\, e^{j(\omega_0 t+\theta)}$$

$$= |C|e^{rt}\, [\cos(\omega_0 t + \theta) + j\, \sin(\omega_0 t + \theta)]$$

Subsection 2

DT Complex Exponentials

DT Complex Exponentials

$$
\begin{aligned}
x[n] &= C\alpha^n, \quad C \text{ and } \alpha \text{ are complex numbers.} \\
C &= |C|e^{j\theta} \\
\alpha &= |\alpha|e^{j\omega_0} \\
x[n] &= |C|e^{j\theta}\left(|\alpha|e^{j\omega_0}\right)^n \\
&= |C||\alpha|^n\cos(\omega_0 n + \theta) + j|C||\alpha|^n\sin(\omega_0 n + \theta)
\end{aligned}
$$

Comments:

- When $|\alpha| = 1$: sinusoidal real and imaginary parts.
- $e^{j\omega_0 n}$ may or may not be periodic depending on the value of ω_0.
- Sinusoidal, exponential, step, and impulse signal form the cornerstones for signals and systems analysis.

DT Complex Exponentials Plot

Periodicity Properties of Discrete-Time Complex Exponentials $e^{j\omega_0 n}$

- For the CT counterpart $e^{j\omega_0 t}$,
  1. The larger the magnitude of ω_0, the higher is the rate of oscillation in the signal.
  2. $e^{j\omega_0 t}$ is periodic for any value of ω_0.
- In DT, as

$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}$$

  the exponential at frequency $\omega_0 + 2\pi$ is the same as that at frequency ω_0 .
- Although in CT $e^{j\omega_0 t}$ are all distinct for distinct values of ω_0, In DT, these signals are not distinct, as the signal with frequency ω_0 is identical to the signals with frequencies $\omega_0 + 2\pi$, $\omega_0 + 4\pi$, and so on. Therefore, in considering DT complex exponentials, we need only consider a frequency interval of length 2π in which to choose ω_0.
- In DT, as we increase ω_0 from 0, we obtain signals that oscillate more and more rapidly until we reach $\omega_0 = \pi$. As we continue to increase ω_0, we decrease the rate of oscillation until we reach $\omega_0 = 2\pi$. Note: $e^{j\pi n} = \left(e^{j\pi}\right)^n = (-1)^n$.

Comparison of the Signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of $e^{j\omega_0 t}$	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $2\pi/\omega_0$	Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m(2\pi/\omega_0)$

Section 4

Step and Impulse Functions

Discrete-Time Unit Step $u[n]$

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \tag{4}$$

Discrete-Time Unit Impulse (Unit Sample) $\delta[n]$

$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \tag{5}$$

DT Step and Impulse

Unit impulse is the first backward difference of the unit step sequence.

$$\delta[n] = u[n] - u[n-1]. \tag{6}$$

DT Step and Impulse

The unit step sequence is the running sum of the unit impulse.

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]. \tag{7}$$

DT Step and Impulse

The unit step sequence is a superposition of delayed unit impulses.

$$u[n] = \sum_{k=0}^{\infty} \delta[n-k]. \tag{8}$$

Continuous-Time Unit Step Function $u(t)$

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{9}$$

Continuous-Time Unit Impulse Function $\delta(t)$

$$\delta(t) = \frac{du(t)}{dt}. \tag{10}$$

CT Unit Step Function and Unit Impulse Function

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau. \tag{11}$$

Section 5

Signal Energy and Power

Energy I

The total energy over a time interval $t_1 \leq t \leq t_2$ in a continuous-time signal $x(t)$ is

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

The total energy over a time interval $n_1 \leq n \leq n_2$ in a discrete-time signal $x[n]$ is

$$\sum_{n=n_1}^{n_2} |x[n]|^2 dt$$

Total energy over an infinite interval in a CT signal:

$$E_\infty \triangleq \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt. \tag{12}$$

Energy II

Total energy over an infinite interval in a DT signal:

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2. \tag{13}$$

Note that this integral and may not converge for some signals. Such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.

Power

Time-averaged power over an infinite interval in a CT signal:

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt. \tag{14}$$

Total energy in a DT signal:

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2. \tag{15}$$

With these definitions, we can identify three important classes of signals:

Examples

Determine whether the following signals are energy signals, power signals, or neither.

1. $x(t) = e^{-at}u(t), \quad a > 0$
2. $x(t) = A\cos(\omega_0 t + \theta)$
3. $x(t) = tu(t)$

$$x(t) = e^{-at}u(t), \quad a > 0$$

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$$

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^\infty e^{-2at} dt$$

$$= \frac{-1}{2a}\left[e^{-at}\right]_0^\infty = \frac{-1}{2a}[0-1] = \frac{1}{2a}$$

This is an energy signal.