

# EN1060 Signals and Systems: Signals

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# Section 1

## Real Signals

# Outline

## Real Signals

- Sinusoids

- Discrete-Time Sinusoidal Signal

- Exponentials

## Complex Numbers

## Complex Signals

- CT Complex Exponentials

- DT Complex Exponentials

## Step and Impulse Functions

## Signal Energy and Power

# Continuous-Time Sinusoidal Signal

$$x(t) = A \cos(\omega_0 t + \phi). \quad (1)$$

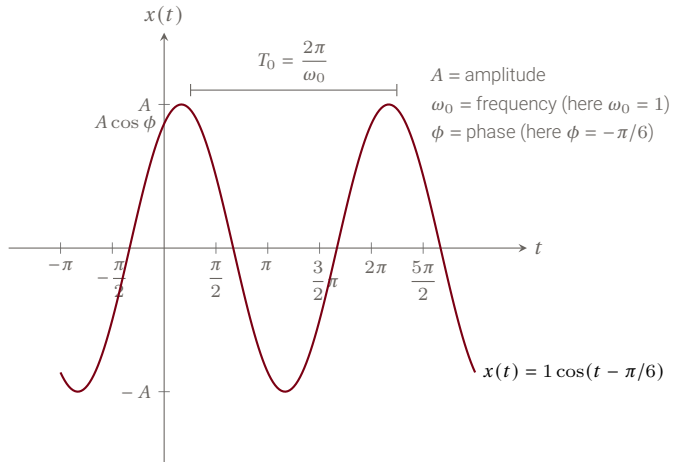


Figure: Continuous-time sinusoidal signal.

# Periodicity of a Sinusoidal

Sinusoidal signal is **periodic**.

A periodic continuous-time signal  $x(t)$  has the property that there is a positive value  $T$  for which

$$x(t) = x(t + T) \quad (2)$$

for all values of  $t$ . Under an appropriate time-shift the signal repeats itself. In this case we say that  $x(t)$  is periodic with period  $T$ .

**Fundamental period  $T_0$**  = smallest positive value of  $T$  for which 2 holds.

A signal that is not periodic is referred to as **aperiodic**.

E.g.: Consider  $A \cos(\omega_0 t + \phi)$

$$\begin{aligned} A \cos(\omega_0 t + \phi) &= A \cos(\omega_0(t + T) + \phi) \quad \text{here } \omega_0 T = 2\pi m \quad \text{an integer multiple of } 2\pi \\ &= A \cos(\omega_0 t + \phi) \end{aligned}$$

$$T = \frac{2\pi m}{\omega_0} \Rightarrow \text{fundamental period } T_0 = \frac{2\pi}{\omega_0}.$$

## Phase of a Sinusoidal

A time-shift in a CT sinusoid is equivalent to a phase shift.

E.g.: Show that a time-shift of a sinusoid is equal to a phase shift.

$$A \cos[\omega_0(t + t_0)] = A \cos(\omega_0 t + \omega_0 t_0) = A \cos(\omega_0 t + \Delta\phi), \quad \Delta\phi \text{ is a change in phase.}$$

$$A \cos[\omega_0(t + t_0) + \phi] = A \cos(\omega_0 t + \omega_0 t_0 + \phi) = A \cos(\omega_0(t + t_1)), \quad t_1 = t_0 + \phi/\omega_0.$$

# Even and Odd Signals

A signal  $x(t)$  or  $x[n]$  is referred to as an **even** signal if it is identical to its time-reversed counterpart, i.e., with its reflection about the origin:

$$x(-t) = x(t)$$

$$x[-n] = x[n]$$

A is referred to as an **odd** if

$$x(-t) = -x(t)$$

$$x[-n] = -x[n]$$

An odd signal must be 0 at  $t = 0$  or  $n = 0$ .

A signal can be broken into a sum of two signals, one of which is even and one for which is odd. Even part of  $x(t)$  is

$$\text{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

Odd part of  $x(t)$  is

$$\text{Od}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

## Even and Odd Signals Contd.

### Example

Show that  $\mathcal{E}\mathbf{v}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$ .

Notation:  $x_e(t)$  is even part of  $x(t)$ ,  $x_o(t)$  is odd part of  $x(t)$ .

$$x(t) = x_e(t) + x_o(t).$$



## Even and Odd Signals Contd.

### Example

Show that  $\mathfrak{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$ .

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$$x(-t) = x_e(t) - x_o(t).$$

## Even and Odd Signals Contd.

### Example

Show that  $\mathfrak{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$ .

Notation:  $x_e(t)$  is even part of  $x(t)$ ,  $x_o(t)$  is odd part of  $x(t)$ .

$$x(t) = x_e(t) + x_o(t).$$

$$x(-t) = x_e(-t) + x_o(-t).$$

$$x(-t) = x_e(t) - x_o(t).$$

Adding,

$$x(t) + x(-t) = x_e(t) + x_o(t) + x_e(t) - x_o(t).$$

## Even and Odd Signals Contd.

### Example

Show that  $\mathfrak{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$ .

Notation:  $x_e(t)$  is even part of  $x(t)$ ,  $x_o(t)$  is odd part of  $x(t)$ .

$$x(t) = x_e(t) + x_o(t).$$

$$x(-t) = x_e(-t) + x_o(-t).$$

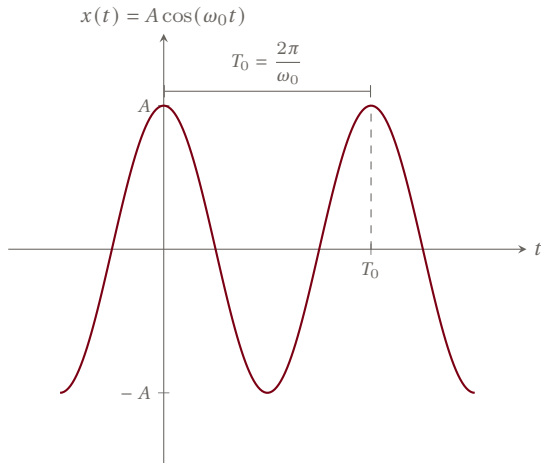
$$x(-t) = x_e(t) - x_o(t).$$

Adding,

$$x(t) + x(-t) = x_e(t) + x_o(t) + x_e(t) - x_o(t).$$

$$\mathfrak{Ev}\{x(t)\} = x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

## Phase of a Sinusoidal: $\phi = 0$

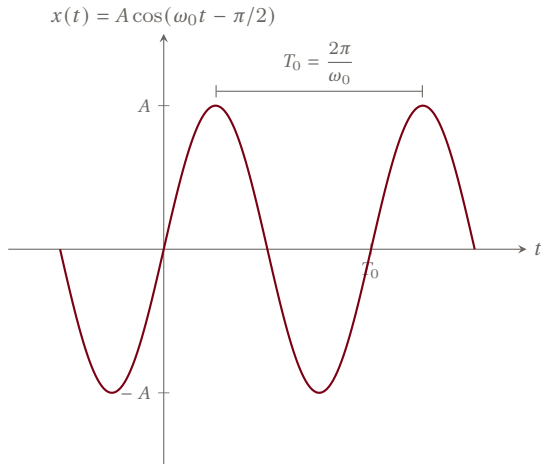


This signal is **even**. If we mirror an even signal about the time origin, it would look exactly the same.

Periodic:  $x(t) = x(t + T)$ .

Even:  $x(t) = x(-t)$ .

## Phase of a Sinusoidal: $\phi = -\pi/2$



This signal is **odd**. If we flip an odd signal about the time origin, we also multiply it by a  $(-)$  sign to get the original signal.

Periodic:  $x(t) = x(t + T)$ .

Odd:  $x(t) = -x(-t)$ .

# Outline

## Real Signals

Sinusoids

Discrete-Time Sinusoidal Signal

Exponentials

## Complex Numbers

## Complex Signals

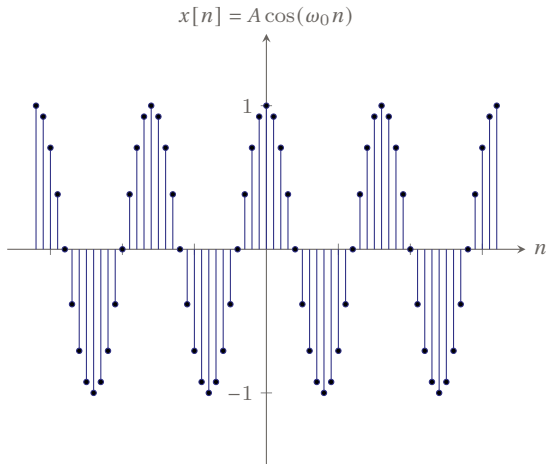
CT Complex Exponentials

DT Complex Exponentials

## Step and Impulse Functions

## Signal Energy and Power

$$x[n] = A \cos(\omega_0 n + \phi) \text{ with } \phi = 0$$



The independent variable is an integer.

The sequence takes values only at integer values of the argument.

This signal is **even**.

Even:  $x[n] = x[-n]$ .

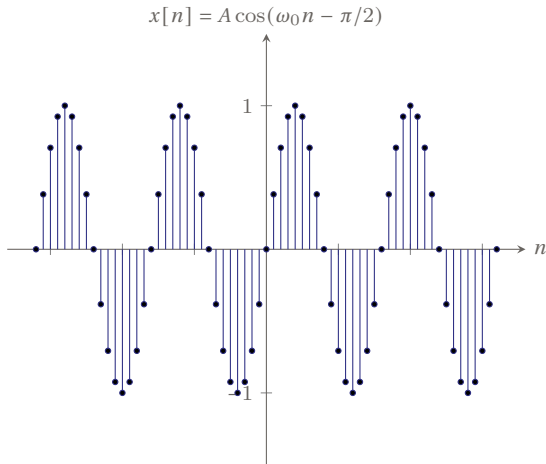
Periodic:  $x[n] = x[n + N]$ . Here,

$$N = 16$$

$$\omega_0 = \frac{2\pi}{N} = \frac{\pi}{8}.$$



$$x[n] = A \cos(\omega_0 n + \phi) \text{ with } \phi = -\pi/2$$



The independent variable is an integer.

The sequence takes values only at integer values of the argument.

This signal is **odd**.

Odd:  $x[n] = -x[-n]$ .

Periodic:  $x[n] = x[n + N]$ . Here,

$$N = 16$$

$$\omega_0 = \frac{2\pi}{N} = \frac{\pi}{8}. \quad \phi = -\pi/2, \quad x[n] =$$

$$A \cos(\omega_0 n + \phi) = A \cos(\omega_0 (n + n_0)).$$

$n_0$  must be an integer.

$$n_0 = \frac{\phi}{\omega_0} = \frac{-\pi/2}{\pi/8} = -4.$$

# Phase Change and Time Shift in DT

## Question

Does a phase change always correspond to a time shift in discrete-time signals?

# Phase Change and Time Shift in DT

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Does a phase change always correspond to a time shift in discrete-time signals?

Answer: No.

$$\begin{aligned} A \cos[\omega_0 n + \phi] &\stackrel{?}{=} A \cos[\omega_0(n + n_0)] \\ \omega_0 n + \omega_0 n_0 &= \omega_0 n + \phi \\ \omega_0 n_0 &= \phi, \quad n_0 \text{ is an integer.} \end{aligned}$$

- Depending on  $\phi$  and  $\omega_0$ ,  $n_0$  may not come out to be an integer.
- In discrete time, the amount of time shift must be an integer.

## Periodicity of a DT Signal

All continuous-time sinusoids are periodic. However, discrete-time sinusoids are not necessarily so.

$$x[n] = x[n + N], \quad \text{smallest integer } N \text{ is the fundamental period.} \quad (3)$$

$$A \cos[\omega_0(n + N) + \phi] = A \cos[\omega_0 n + \omega_0 N + \phi]$$

$\omega_0 N$  must be an integer multiple of  $2\pi$ .

Periodic  $\Rightarrow \omega_0 N = 2\pi m$

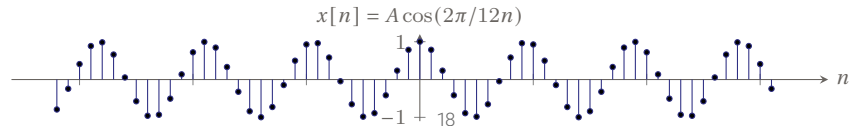
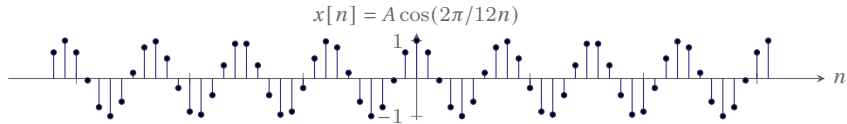
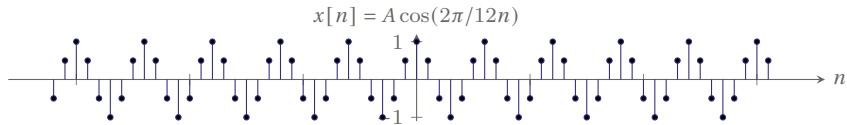
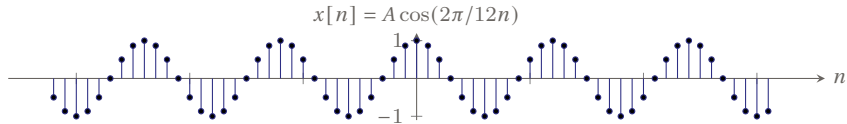
$$N = \frac{2\pi m}{\omega_0} \quad (4)$$

$N$  and  $m$  must be integers.

Smallest  $N$ , if any, is the fundamental period.

$N$  may not be an integer. In this case, the signal is not periodic.

# Periodicity of a DT Signal Cntd.



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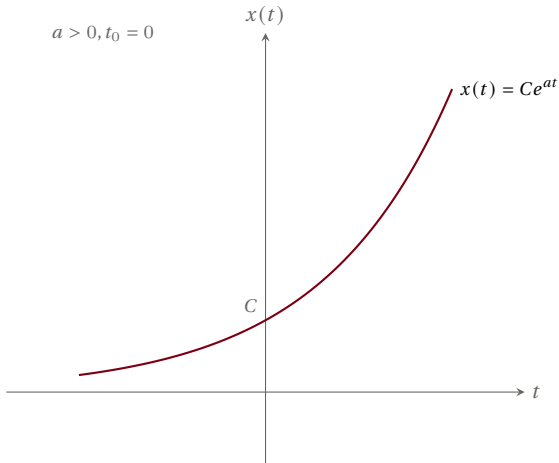
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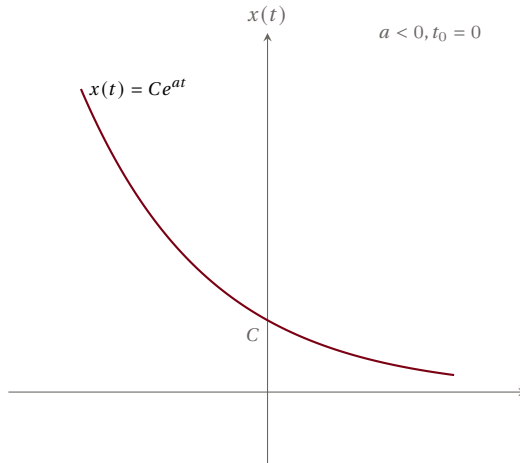
# CT Real Exponentials

$$x(t) = Ce^{a(t+t_0)}, \quad C \text{ and } a \text{ are real numbers} \\ = Ce^{at_0} e^{at}.$$

$a > 0, t_0 = 0$



$a < 0, t_0 = 0$

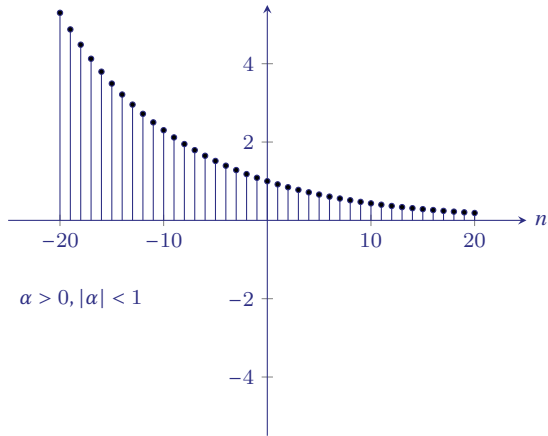


## DT Real Exponentials

$$x[n] = Ce^{\beta n} = C\alpha^n, \quad C \text{ and } \alpha \text{ are real numbers}$$

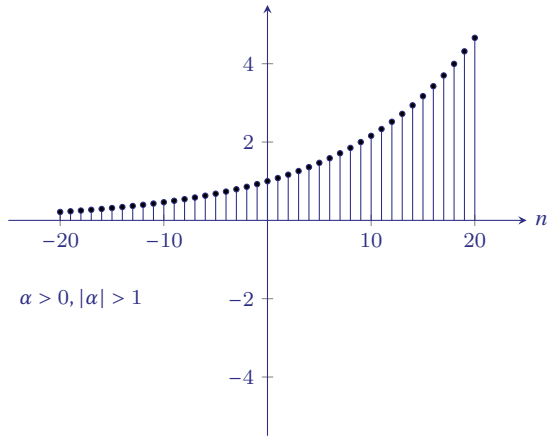


$$x[n] = C\alpha^n, \quad \alpha = 0.92$$

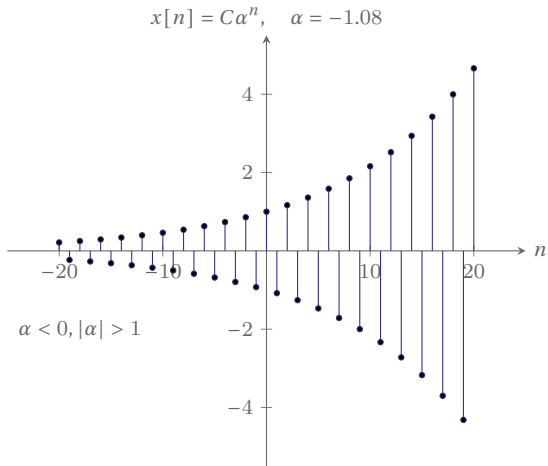
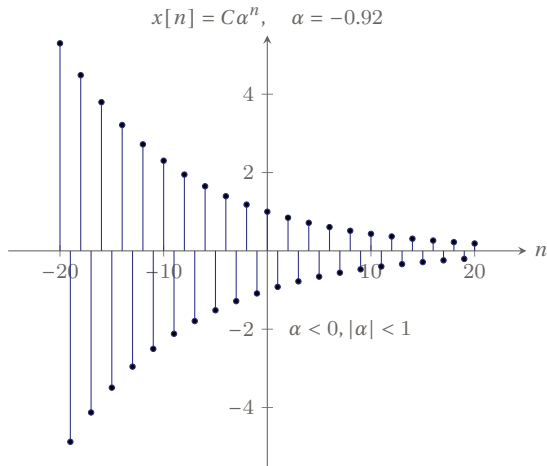


$$\alpha > 0, |\alpha| < 1$$

$$x[n] = C\alpha^n, \quad \alpha = 1.08$$



$$\alpha > 0, |\alpha| > 1$$



## Section 2

### Complex Numbers

# Representing Complex Numbers

The **Cartesian** or **rectangular** form:

$$z = x + jy,$$

where  $j = \sqrt{-1}$  and  $x$  and  $y$  are real numbers referred to respectively as the real part and the imaginary part. I.e.,

$$x = \Re\{z\}, y = \Im\{z\}$$

The **polar** form:

$$z = re^{j\theta},$$

where  $r > 0$  is the **magnitude** of  $z$  and  $\theta$  is the **angle** or **phase** of  $z$ .

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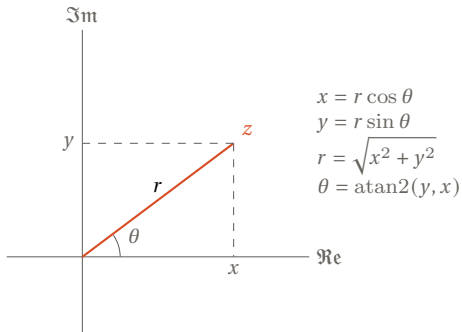
where  $r > 0$  is the **magnitude** of  $z$  and  $\theta$  is the **angle** or **phase** of  $z$ .

$$r = |z|, \theta = \angle z.$$

The relationship between these two representations can be determined from **Euler's relation**:

$$e^{j\theta} = \cos \theta + j \sin \theta$$

or by plotting  $z$  in the complex plane.



**Example** Let  $z_0$  be a complex number with polar coordinates  $(r_0, \theta_0)$  and Cartesian coordinates  $(x_0, y_0)$ . Determine expressions for the Cartesian coordinates of the following complex numbers in terms of  $x_0$  and  $y_0$ . Plot the points  $z_0, z_1, z_2, z_3, z_4$ , and  $z_5$  in the complex plane when  $r_0 = 2$  and  $\theta_0 = \pi/4$  and when  $r_0 = 2$  and  $\theta_0 = \pi/2$ . Indicate on the plot the real and imaginary parts of each point.

1.  $z_1 = r_0 e^{-j\theta_0}$

2.  $z_2 = r_0$

3.  $z_3 = r_0 e^{j(\theta_0 + \pi)}$

4.  $z_4 = r_0 e^{j(-\theta_0 + \pi)}$

5.  $z_5 = r_0 e^{j(\theta_0 + 2\pi)}$

$$\begin{aligned} z_0 &= r_0 e^{j\theta_0} = r_0 (\cos \theta_0 + j \sin \theta_0) \\ &= r_0 \cos \theta_0 + j r_0 \sin \theta_0 = x_0 + j y_0. \end{aligned}$$

$$z_1 = r_0 e^{-j\theta} = r_0 (\cos(-\theta_0) + j \sin(-\theta_0)) = x_0 - j y_0.$$

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$$z_1 = r_0 e^{-j\theta} = r_0 (\cos(-\theta_0) + j \sin(-\theta_0)) = x_0 - j y_0.$$

$$z_2 = r_0 = \sqrt{x_0^2 + y_0^2}$$

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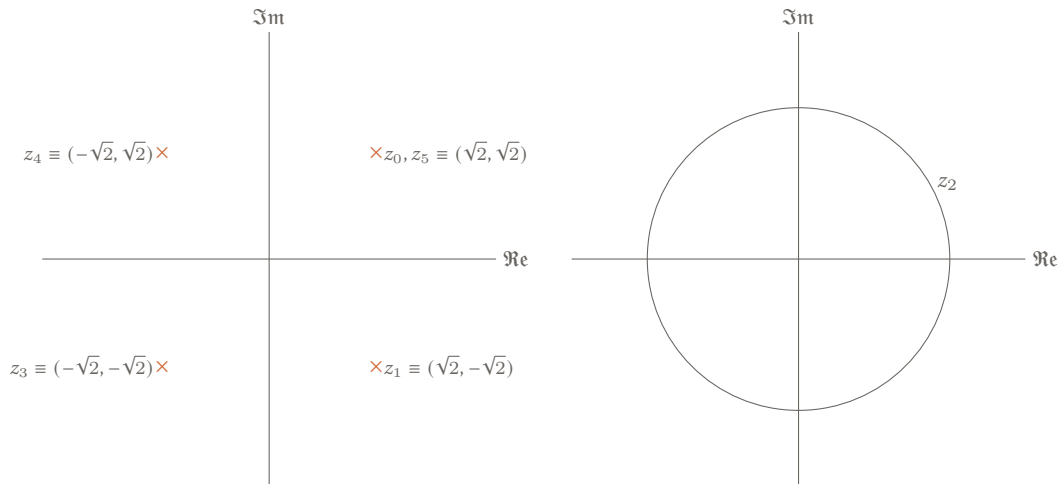
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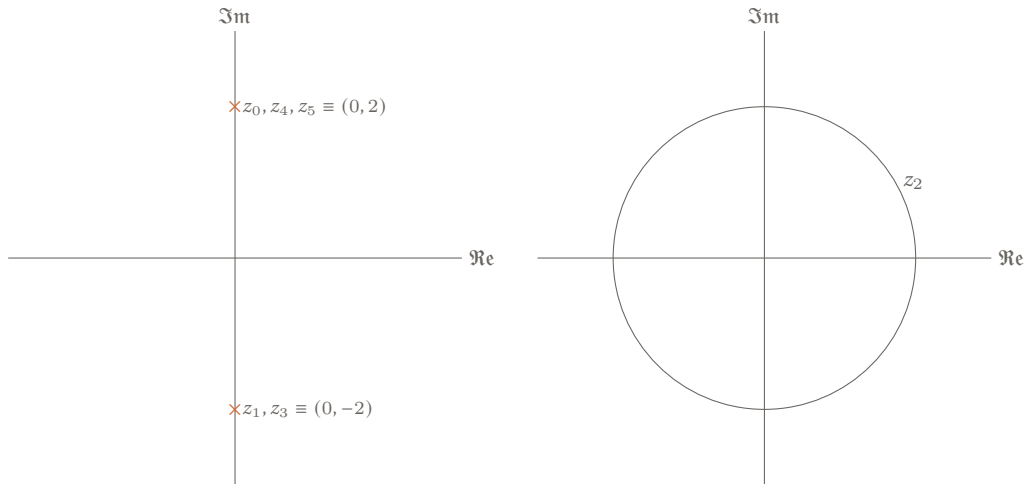
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$$r = 2, \theta = \pi/2$$

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**Example** Express each of the following complex numbers in polar form, and plot them in the complex plane, indicating the magnitude and angle of each number.

1.  $1 + j\sqrt{3}$

2.  $-5$

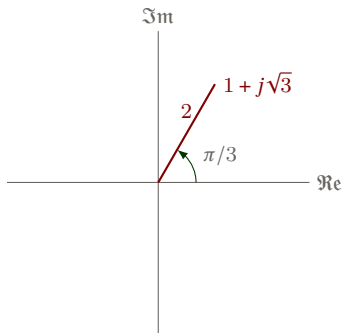
3.  $-5 - 5j$

4.  $3 + 4j$

5.  $(1 - j\sqrt{3})^3$

6.  $\frac{e^{j\pi/3} - 1}{1 + j\sqrt{3}}$

$$\begin{aligned} 1 + j\sqrt{3} &= \sqrt{1^2 + (\sqrt{3})^2} \left( \frac{1}{\sqrt{1^2 + (\sqrt{3})^2}} + j \frac{\sqrt{3}}{\sqrt{1^2 + (\sqrt{3})^2}} \right) \\ &= 2e^{j\operatorname{atan2}(\sqrt{3},1)} \\ &= 2e^{j\pi/3} \end{aligned}$$



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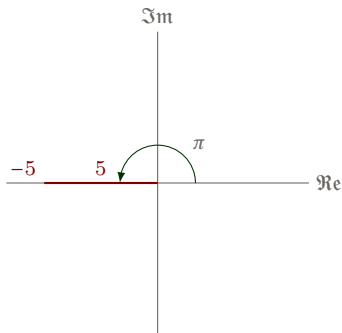
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5.  $(1 - j\sqrt{3})^3$

6.  $\frac{e^{j\pi/3} - 1}{1 + j\sqrt{3}}$

$$\begin{aligned} -5 &= 5(-1 + j0) \\ &= 5e^{j\operatorname{atan2}(0,-1)} \\ &= 5e^{j\pi} \end{aligned}$$



**Example** Express each of the following complex numbers in polar form, and plot them in the complex plane, indicating the magnitude and angle of each number.

1.  $1 + j\sqrt{3}$

2.  $-5$

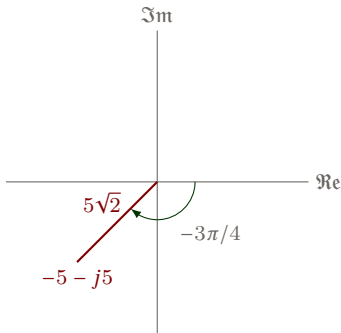
3.  $-5 - 5j$

4.  $3 + 4j$

5.  $(1 - j\sqrt{3})^3$

6.  $\frac{e^{j\pi/3} - 1}{1 + j\sqrt{3}}$

$$\begin{aligned} -5 - 5j &= 5(-1 + j(-1)) \\ &= 5e^{j\text{atan2}(-1,-1)} \\ &= 5e^{-j3\pi/4} \end{aligned}$$



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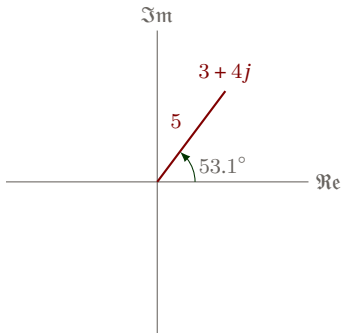
5.  $(1 - j\sqrt{3})^3$

6.  $\frac{e^{j\pi/3} - 1}{1 + j\sqrt{3}}$

$$3 + 4j = 5(3/5 + j4/5)$$

$$= 5e^{j\text{atan2}(4,3)}$$

$$= 5e^{-j3\pi/4}$$





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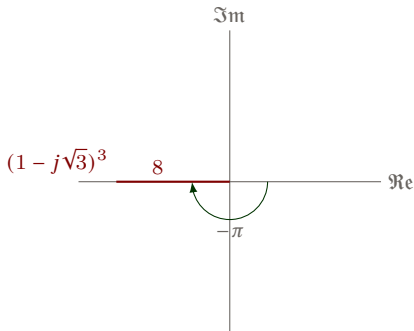
3.  $-5 - 5j$

4.  $3 + 4j$

5.  $(1 - j\sqrt{3})^3$

6.  $\frac{e^{j\pi/3} - 1}{1 + j\sqrt{3}}$

$$\begin{aligned}(1 - j\sqrt{3})^3 &= \left(2e^{-j\pi/3}\right)^3 \\ &= 8e^{-j\pi}\end{aligned}$$



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## $\cos \theta$ and $\sin \theta$

Using Euler's relations, derive the following relationships:

1.  $\cos \theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$
2.  $\sin \theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$

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Adding

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Subtracting

$$\begin{aligned}e^{j\theta} &= \cos \theta + j \sin \theta \\e^{-j\theta} &= \cos \theta - j \sin \theta\end{aligned}$$

$$\begin{aligned}e^{j\theta} - e^{-j\theta} &= 2j \sin \theta \\ \sin \theta &= \frac{1}{2j}(e^{j\theta} - e^{-j\theta})\end{aligned}$$

Adding

$$\begin{aligned}e^{j\theta} + e^{-j\theta} &= 2 \cos \theta \\ \cos \theta &= \frac{1}{2}(e^{j\theta} + e^{-j\theta})\end{aligned}$$

# Complex Conjugate

Let  $z$  denote a complex variable; i.e.,

$$z = x + jy = re^{j\theta}.$$

The **complex conjugate** of  $z$  is

$$z^* = x - jy = re^{-j\theta}.$$

Show that

1.  $zz^* = r^2$
2.  $z + z^* = 2\Re\{z\}$
3.  $z - z^* = 2j\Im\{z\}$



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1.  $zz^* = re^{j\theta}re^{-j\theta} = r^2e^0 = r^2$

2.  $z + z^* = x + jy + x - jy = 2x = 2\Re\{z\}$

3.  $z - z^* = x + jy - (x - jy) = 2jy = 2j\Im\{z\}$

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List the values of

1.  $e^{j0}$

2.  $e^{j\pi/2}$

3.  $e^{j\pi}$

4.  $e^{j3\pi/2}$

5.  $e^{j2\pi}$

Show that

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List the values of

1.  $e^{j0} = 1$

2.  $e^{j\pi/2} = j$

3.  $e^{j\pi} = -1$

4.  $e^{j3\pi/2} = -j$

5.  $e^{j2\pi} = 1$

## Section 3

### Complex Signals

# Outline

## Real Signals

- Sinusoids

- Discrete-Time Sinusoidal Signal

- Exponentials

## Complex Numbers

## Complex Signals

- CT Complex Exponentials

- DT Complex Exponentials

## Step and Impulse Functions

## Signal Energy and Power

# CT Complex Exponentials

$$x(t) = Ce^{at} \quad C \text{ and } a \text{ are complex numbers.}$$

$$C = |C|e^{j\theta}$$

$$a = r + j\omega_0$$

$$x(t) = |C|e^{j\theta}e^{(r+j\omega_0)t}$$

$$= |C|e^{rt}e^{j(\omega_0 t + \theta)}$$

$$= |C|e^{rt} [\cos(\omega_0 t + \theta) + j \sin(\omega_0 t + \theta)]$$

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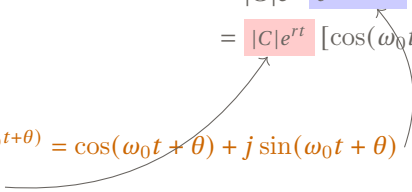
$$a = r + j\omega_0$$

$$x(t) = |C|e^{j\theta}e^{(r+j\omega_0)t}$$

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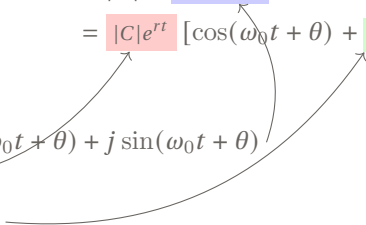
$$x(t) = |C|e^{j\theta}e^{(r+j\omega_0)t}$$

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- Real





# CT Complex Exponentials

$x(t) = Ce^{at}$   $C$  and  $a$  are complex numbers.

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$$a = r + j\omega_0$$

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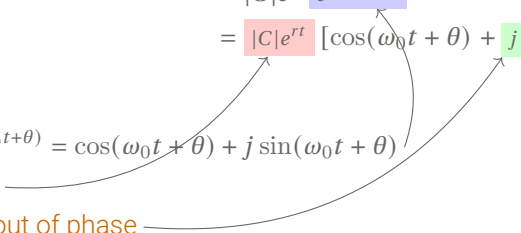
$$= |C|e^{rt}e^{j(\omega_0 t + \theta)}$$

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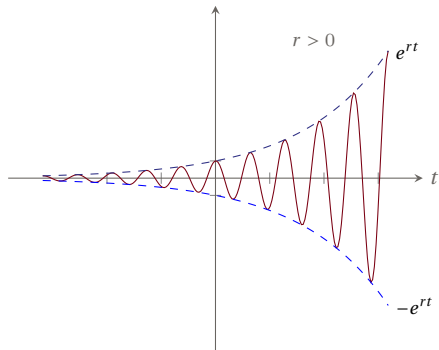
- $e^{j(\omega_0 t + \theta)} = \cos(\omega_0 t + \theta) + j \sin(\omega_0 t + \theta)$

- Real

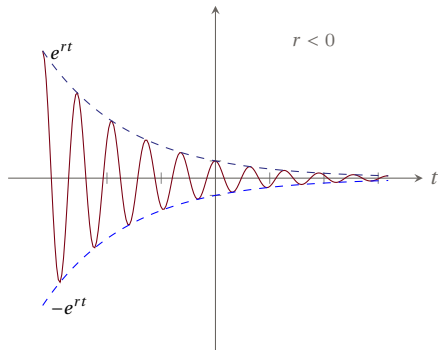
- 90° out of phase



$$x(t) = |C|e^{rt} \cos(\omega_0 t + \phi)$$



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# Outline

## Real Signals

- Sinusoids

- Discrete-Time Sinusoidal Signal

- Exponentials

## Complex Numbers

## Complex Signals

- CT Complex Exponentials

- DT Complex Exponentials

## Step and Impulse Functions

## Signal Energy and Power

# DT Complex Exponentials

$$x[n] = C\alpha^n, \quad C \text{ and } \alpha \text{ are complex numbers.}$$

$$C = |C|e^{j\theta}$$

$$\alpha = |\alpha|e^{j\omega_0}$$

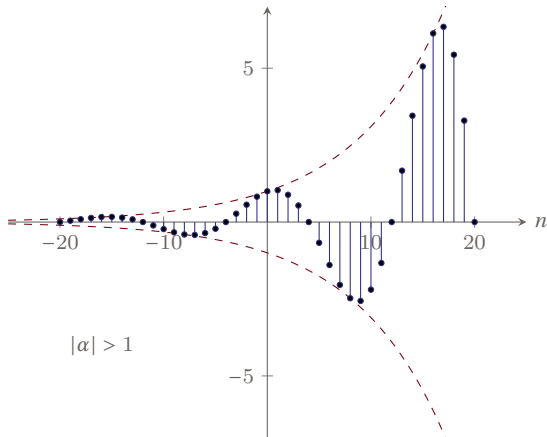
$$\begin{aligned} x[n] &= |C|e^{j\theta} \left( |\alpha|e^{j\omega_0} \right)^n \\ &= |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta) \end{aligned}$$

Comments:

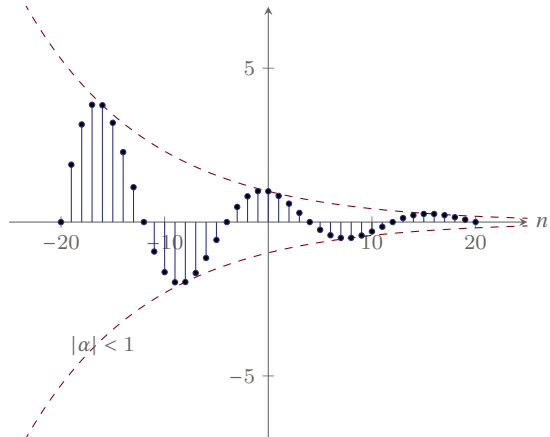
- When  $|\alpha| = 1$ : sinusoidal real and imaginary parts.
- $e^{j\omega_0 n}$  may or may not be periodic depending on the value of  $\omega_0$ .
- Sinusoidal, exponential, step, and impulse signal form the cornerstones for signals and systems analysis.

# DT Complex Exponentials Plot

$$x[n] = |C||\alpha|^n \cos(\omega_0 n + \theta), \quad |\alpha| = 1.12, \theta = 0$$



$$x[n] = |C||\alpha|^n \cos(\omega_0 n + \theta), \quad |\alpha| = 0.92, \theta = 0$$



# Periodicity Properties of Discrete-Time Complex Exponentials

$e^{j\omega_0 n}$

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- In DT, as

$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n} e^{j\omega_0 n} = e^{j\omega_0 n}$$

the exponential at frequency  $\omega_0 + 2\pi$  is the same as that at frequency  $\omega_0$  .

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- Although in CT  $e^{j\omega_0 t}$  are all distinct for distinct values of  $\omega_0$ , In DT, these signals are not distinct, as the signal with frequency  $\omega_0$  is identical to the signals with frequencies  $\omega_0 + 2\pi$ ,  $\omega_0 + 4\pi$ , and so on. Therefore, in considering DT complex exponentials, we need only consider a frequency interval of length  $2\pi$  in which to choose  $\omega_0$ .

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- In DT, as we increase  $\omega_0$  from 0, we obtain signals that oscillate more and more rapidly until we reach  $\omega_0 = \pi$ . As we continue to increase  $\omega_0$ , we decrease the rate of oscillation until we reach  $\omega_0 = 2\pi$ . Note:  $e^{j\pi n} = \left(e^{j\pi}\right)^n = (-1)^n$ .

# Comparison of the Signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$

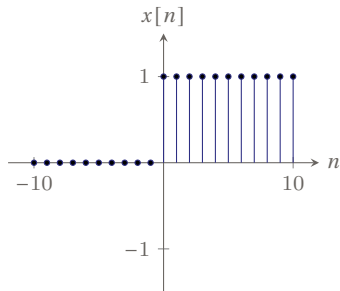
$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of $\omega_0$	Identical signals for values of $\omega_0$ separated by multiples of $2\pi$
Periodic for any choice of $e^{j\omega_0 t}$	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and $m$ .
Fundamental frequency $\omega_0$	Fundamental frequency $\omega_0/m$
Fundamental period $\omega_0 = 0$ : undefined $\omega_0 \neq 0$ : $2\pi/\omega_0$	Fundamental period $\omega_0 = 0$ : undefined $\omega_0 \neq 0$ : $m(2\pi/\omega_0)$

## Section 4

### Step and Impulse Functions

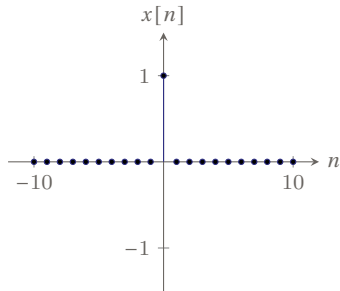
## Discrete-Time Unit Step $u[n]$

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \quad (5)$$



## Discrete-Time Unit Impulse (Unit Sample) $\delta[n]$

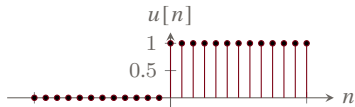
$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \quad (6)$$



## DT Step and Impulse

Unit impulse is the first backward difference of the unit step sequence.

$$\delta[n] = u[n] - u[n - 1]. \quad (7)$$

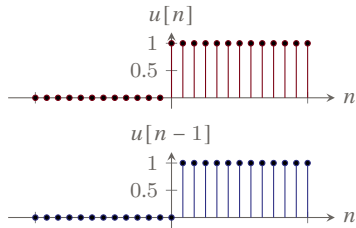




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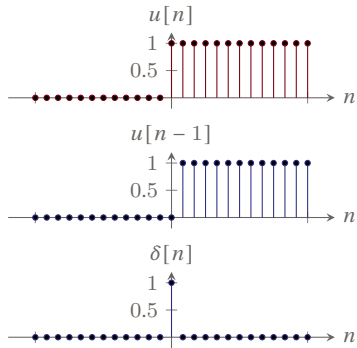
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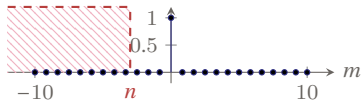
The unit step sequence is the running sum of the unit impulse.

$$u[n] = \sum_{m=-\infty}^n \delta[m]. \quad (8)$$

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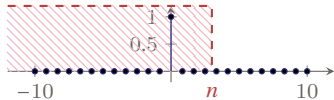
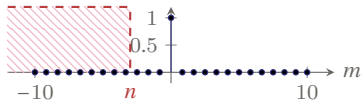
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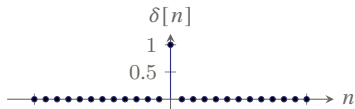
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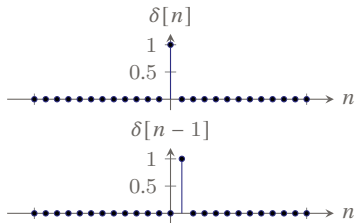
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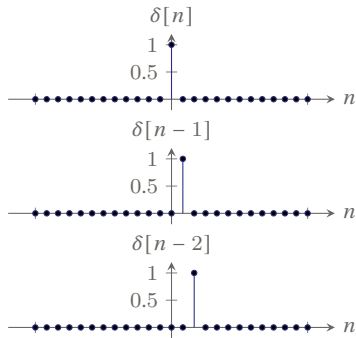




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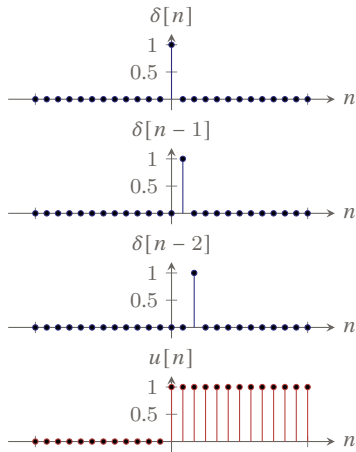
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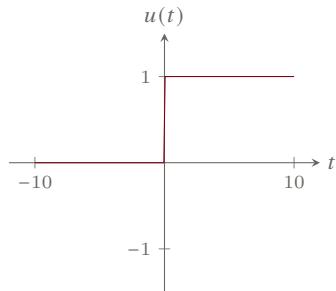
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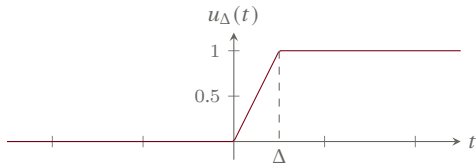


## Continuous-Time Unit Step Function $u(t)$

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \quad (10)$$

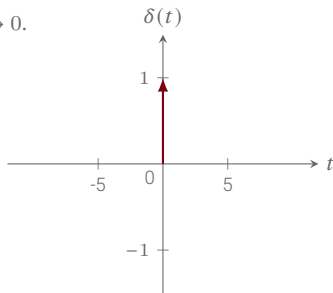


# Continuous-Time Unit Impulse Function $\delta(t)$



$u_\Delta(t) \rightarrow u(t)$  as  $\Delta \rightarrow 0$ .

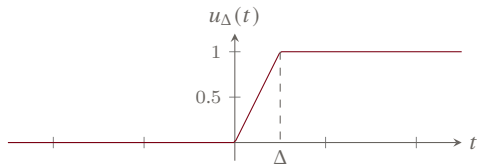
$$\delta(t) = \frac{du(t)}{dt}. \quad (11)$$



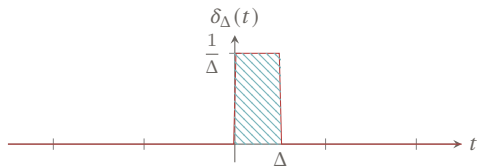
$$x(t)\delta(t) = x(0)\delta(t)$$

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

# Continuous-Time Unit Impulse Function $\delta(t)$

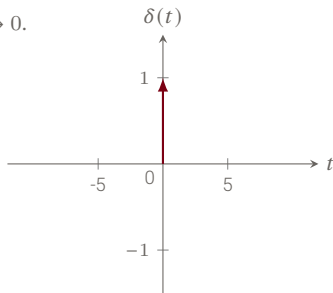


$u_\Delta(t) \rightarrow u(t)$  as  $\Delta \rightarrow 0$ .



$\delta_\Delta(t) \rightarrow \delta(t)$  as  $\Delta \rightarrow 0$ .  
area = 1

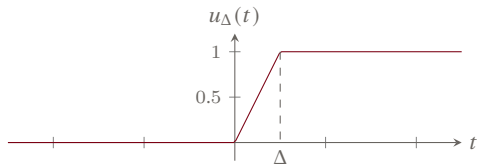
$$\delta(t) = \frac{du(t)}{dt}. \quad (11)$$



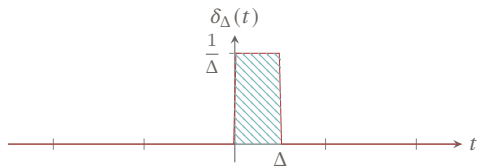
$$x(t)\delta(t) = x(0)\delta(t)$$

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

# Continuous-Time Unit Impulse Function $\delta(t)$

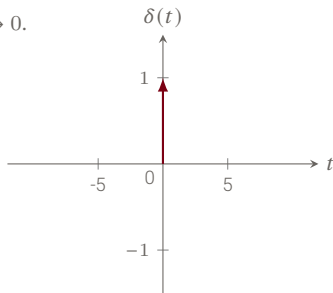


$u_\Delta(t) \rightarrow u(t)$  as  $\Delta \rightarrow 0$ .



$\delta_\Delta(t) \rightarrow \delta(t)$  as  $\Delta \rightarrow 0$ .  
area = 1

$$\delta(t) = \frac{du(t)}{dt}. \quad (11)$$

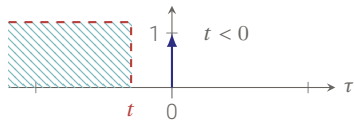


$$x(t)\delta(t) = x(0)\delta(t)$$

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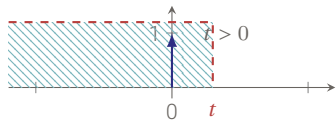
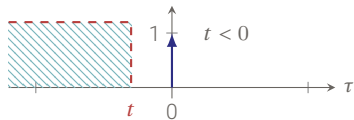
## CT Unit Step Function and Unit Impulse Function

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau. \quad (12)$$



# CT Unit Step Function and Unit Impulse Function

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau. \quad (12)$$





## Section 5

### Signal Energy and Power

# Energy I

The total energy over a time interval  $t_1 \leq t \leq t_2$  in a continuous-time signal  $x(t)$  is

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

The total energy over a time interval  $n_1 \leq n \leq n_2$  in a discrete-time signal  $x[n]$  is

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

Total energy over an infinite interval in a CT signal:

$$E_{\infty} \triangleq \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt. \quad (13)$$

## Energy II

Total energy over an infinite interval in a DT signal:

$$E_{\infty} \triangleq \lim_{N \rightarrow \infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2. \quad (14)$$

Note that this integral and may not converge for some signals. Such signals have infinite energy, while signals with  $E_{\infty} < \infty$  have finite energy.

# Power

Time-averaged power over an infinite interval in a CT signal:

$$P_{\infty} \triangleq \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt. \quad (15)$$

Total energy in a DT signal:

$$P_{\infty} \triangleq \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2. \quad (16)$$

With these definitions, we can identify three important classes of signals:

1. Energy signals: Signals with finite total energy  $E_{\infty} < \infty$ . These have zero average power.
2. Power signals: Signals with finite average power  $0 < P_{\infty} < \infty$ . As  $P_{\infty} > 0$ ,  $E_{\infty} = \infty$ .
3. Signals with neither  $E_{\infty}$  nor  $P_{\infty}$  are finite.

# Examples

Determine whether the following signals are energy signals, power signals, or neither.

1.  $x(t) = e^{-at}u(t), \quad a > 0$

2.  $x(t) = A \cos(\omega_0 t + \theta)$

3.  $x(t) = tu(t)$

$$x(t) = e^{-at}u(t), \quad a > 0$$

$$E_{\infty} = \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt$$

$$\begin{aligned} E_{\infty} &= \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{\infty} e^{-2at} dt \\ &= \frac{-1}{2a} [e^{-at}]_0^{\infty} = \frac{-1}{2a} [0 - 1] = \frac{1}{2a} \end{aligned}$$

This is an energy signal.

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$$x(t) = A \cos(\omega_0 t + \theta)$$

$$\begin{aligned} E_\infty &= \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt \\ &= \lim_{T \rightarrow \infty} \int_{-T}^T A^2 \cos^2(\omega_0 t + \theta) dt \\ &= \frac{A^2}{2} \lim_{T \rightarrow \infty} \int_{-T}^T [1 + \cos(2\omega_0 t + 2\theta)] dt \\ &= \frac{A^2}{2} \lim_{T \rightarrow \infty} \left[ t - \frac{\cos(2\omega_0 t + 2\theta)}{2\omega_0} \right]_{-T}^T \end{aligned}$$

Considering  $T$  as an integer multiple of  $2\pi/\omega_0$

$$E_\infty = A^2 \lim_{T \rightarrow \infty} T \rightarrow \infty.$$

This is not an energy signal.

$$\begin{aligned} P_\infty &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt \\ &= A^2 \lim_{T \rightarrow \infty} \frac{1}{2T} T = \frac{A^2}{2} < \infty \end{aligned}$$

This is a power signal.

# Examples

Determine whether the following signals are energy signals, power signals, or neither.

1.  $x(t) = e^{-at}u(t), \quad a > 0$

2.  $x(t) = A \cos(\omega_0 t + \theta)$

3.  $x(t) = tu(t)$

$$x(t) = tu(t)$$

$$\begin{aligned} E_\infty &= \lim_{T \rightarrow \infty} \int_{-T}^T |x(t)|^2 dt \\ &= \lim_{T \rightarrow \infty} \int_0^T t^2 dt = \lim_{T \rightarrow \infty} \left[ \frac{t^3}{3} \right]_0^T \\ &= \lim_{T \rightarrow \infty} \frac{T^3}{3} \rightarrow \infty. \end{aligned}$$

This is not an energy signal.

$$x(t) = tu(t)$$

$$\begin{aligned} P_\infty &= \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T |x(t)|^2 dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{2T} \frac{T^3}{3} \\ &= \lim_{T \rightarrow \infty} \frac{T^2}{6} \rightarrow \infty. \end{aligned}$$

This is not a power signal either.