EN1020 Circuits, Signals, and Systems: Introduction to Fourier Transform

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Section 1

Continuous-Time Fourier Transform

Subsection 1

Introduction

Introduction

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.
- We use Fourier series for periodic signals, and Fourier transform for aperiodic signals.
- Each of these have continuous-time and discrete-time versions:
    1. Continuous-time Fourier series
    2. Continuous-time Fourier transform
    3. Discrete-time Fourier series
    4. Discrete-time Fourier transform
- In this part of the course, we will concentrate on how to compute continuous-time Fourier series and transform. Later, after we study liner, time-invariant (LTI) systems, we will study the conceptual aspects of Fourier techniques.
- In EN2063, we will do a more rigorous study of Fourier techniques.

Fourier Transform

- In the last lecture, we represented a periodic signal as a linear combination of complex exponentials.
- We use Fourier transform to represent aperiodic signals. A larger class of signals, including all signals with finite energy, can be represented through a linear combination of complex exponentials.
- Whereas for periodic signals the complex exponential building blocks are harmonically related, for aperiodic signals they are infinitesimally close in frequency, and the representation in terms of a linear combination takes the form of an integral rather than a sum.
- The resulting spectrum of coefficients in this representation is called the Fourier transform.
- The synthesis integral itself, which uses the Fourier transform to represent the signal as a linear combination (integral) of complex exponentials, is called the inverse Fourier transform.

Subsection 2

Development of the Fourier Transform Representation

Fourier Series Representation for Square Wave

The continuous-time periodic square wave, sketched below, is defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < T/2, \end{cases}$$

This signal periodically repeats with the fundamental period T and the fundamental frequency $\omega_0 = 2\pi/T$.

The Fourier series coefficients a_k of this wave are

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}. \tag{1}$$

We plotted this for a fixed value of T_1 and several values of T (shown in the next slide). An alternative way of interpreting Eq. 1 is as samples of an envelope function:

$$T a_k = \left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\omega=k\omega_0}.$$

With ω thought of as a continuous variable, the function $\frac{2\sin(\omega T_1)}{\omega}$ represents the envelope of $T a_k$, and the coefficients a_k are simple equally spaced samples of this envelope. For fixed T_1, the envelope of $T a_k$ is independent of T.

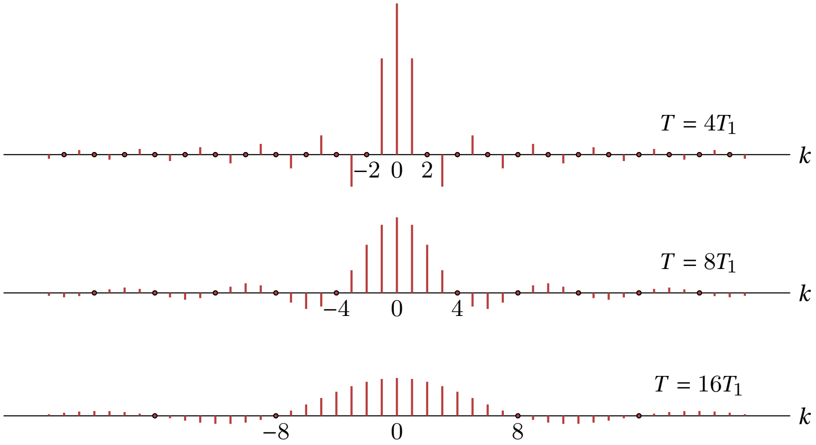


Figure: Plots of scaled Fourier series coefficients a_k

Plots of scaled Fourier series coefficients a_k for the periodic square wave with T_1 fixed and for several values of T: $T = 4T_1$, $T = 8T_1$, $T = 16T_1$.

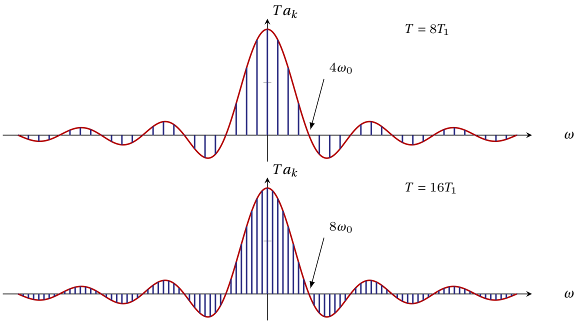
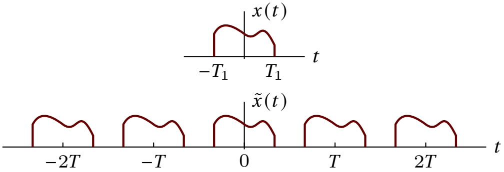


Figure: Fourier series coefficients and their envelope for periodic square wave.

The Fourier series coefficients and their envelope for periodic square wave for several values of T (with T_1 fixed): $T = 4T_1$, $T = 8T_1$, $T = 16T_1$. The coefficients are regularly-spaced samples of the envelope $(2\sin\omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

As T increases, or equivalently, as the fundamental frequency $\omega_0 = 2\pi/T$ decreases, the envelope is sampled with a close and closer spacing. As T becomes arbitrarily large, the original periodic square waveform approaches the rectangular pulse. Also, the Fourier series coefficients, multiplied by T, become more and more closely spaced samples of the envelope. So, in some sense, the set of Fourier series coefficients approaches the envelope function as $T \rightarrow \infty$.



$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t}\,dt.$$

As $\tilde{x}(t) = x(t)$ for $|t| < T/2$, and also, as $x(t) = 0$ outside this interval,

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}\,dt.$$

Defining the envelope $X(j\omega)$ of Ta_k as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt,$$

we have, for the coefficients a_k,

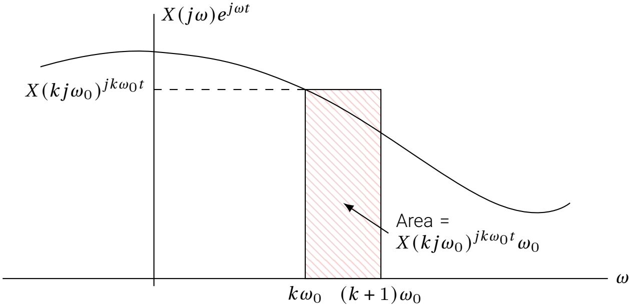
$$a_k = \frac{1}{T}X(jk\omega_0).$$

Combining and expressing $\tilde{x}(t)$ in terms of $X(j\omega)$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T}X(jk\omega_0)e^{jk\omega_0 t},$$

or, as $\omega_0 = 2\pi/T$

$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0. \qquad (2)$$



$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0.$$

Inverse Fourier transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

Fourier transform or Fourier integral:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt.$$

As $T \to \infty$, $\tilde{x}(t)$ approaches $x(t)$, and consequently, Eq. 2 becomes a representation of $x(t)$. Furthermore, as $\omega_0 \to 0$ as $T \to \infty$, and the right-hand side of Eq. 2 passes to an integral. As $\omega_0 \to 0$, the summation converges to the integral of $X(j\omega)e^{j\omega t}$.

Fourier Transform: Synthesis and Analysis Equations

FS synthesis and analysis equations:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

Relation with a_k

Assume that the Fourier transform of $x(t)$ is $X(j\omega)$.
If we construct a periodic signal $\tilde{x}(t)$ by repeating the aperiodic signals $x(t)$ with period T, its Fourier series coefficients are

FS synthesis and analysis equations:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

FT synthesis and analysis equations:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Convergence of Fourier Transform

Assume that we evaluated $X(j\omega)$ according to eq. 15, and left $\hat{x}(t)$ denote the signal obtained by using $X(j\omega)$ in 15:

$$\hat{x}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega.$$

When is $\hat{x}(t)$ a valid representation of the original signal $x(t)$? We define the error between $\hat{x}(t)$ and $x(t)$ as

$$e(t) = \hat{x}(t) - x(t).$$

If $x(t)$ has finite energy (square integrable), i.e.,

(3)

$X(j\omega)$ is finite, and

(4)

If $x(t)$ has finite energy, then, although $x(t)$ and its Fourier representation $\hat{x}(t)$ may differ significantly at individual values of t, there is no energy in their difference.

Convergence of Fourier Transform: Dirichlet Conditions

There are alternative conditions sufficient to ensure that $\hat{x}(t)$ is qual to $x(t)$ for any t except at a discontinuity, where it is equal to the average of the values on either side of the discontinuity.

1. $x(t)$ is absolutely integrable, i.e.,

(5)

2. $x(t)$ has a finite number of maxima and minima within any finite interval.
3. $x(t)$ has a finite number of discontinuities within any finite interval. Furthermore, each of these discontinuities must be finite.

Therefore, absolutely integrable signals that are continuous or that have finite number of discontinuities have a Fourier transform.

Example

Example

Find the Fourier transform of the signal

$$x(t) = e^{-at}u(t), \quad a > 0.$$

Example Cntd. FT of $e^{-at}u(t), \quad a > 0$

Example

Find the Fourier transform of the signal

$$x(t) = e^{-a|t|}, \quad a > 0.$$

Example

Determine the Fourier transform of the unit impulse

$$x(t) = \delta(t).$$

Rectangular Pulse

Example

Determine the Fourier transform of the signal

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & |t| > T_1. \end{cases}$$

Example

Consider the signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \begin{cases} 1, & |\omega| < W, \\ 0, & |\omega| > W. \end{cases}$$

Determine $x(t)$.

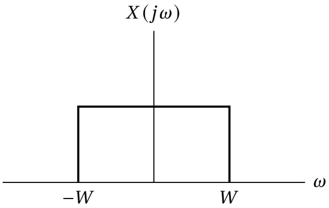


Figure: Fourier transform for $x(t)$.

The sinc Function

$$\mathrm{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}. \tag{6}$$

Express

$$\frac{2\sin\omega T_1}{\omega}$$

and

$$\frac{\sin Wt}{\pi t}$$

as sinc functions.

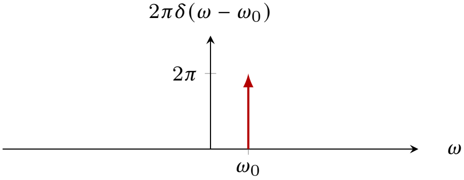
What Happens when W Increases?

Subsection 3

The Fourier Transform for Periodic Signals

The Fourier Transform for Periodic Signals: Introduction

In the previous section, we studied the Fourier transform representation, paying attention to aperiodic signals. We can also develop Fourier transform representations for periodic signals. This allows us to consider periodic and aperiodic signals in a unified context. We can construct the Fourier transform of a periodic signal directly from its Fourier series representation.
Consider a signal $x(t)$ with the Fourier transform $X(j\omega)$ that is a single impulse of area 2π at $\omega = \omega_0$, i.e.,



Let's determine the signal $x(t)$:

Example

Find the Fourier transformof the square wave signal whose Fourier series coefficients are

$$a_k = \frac{\sin k\omega_0 T_1}{\pi k}.$$

Method: Multiply the Fourier seriescoefficients a_k by 2π, place them using the impulse function $\delta(\omega - k\omega_0)$, and sum.

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Example

Find the Fourier transformof

$$x(t) = \sin \omega_0 t,$$

and

$$x(t) = \cos \omega_0 t.$$

Example

Find the Fourier transform of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

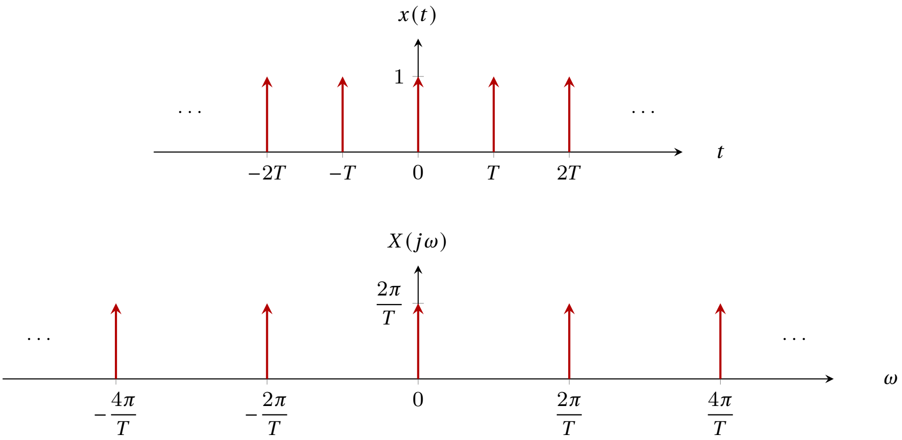


Figure: Periodic impulse train and its Fourier transform.