EN1020 Signals and Systems: Signals

Ranga Rodrigo
ranga@uom.lk

The University of Moratuwa, Sri Lanka

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Section 1

Real Signals

Outline

Continuous-Time Sinusoidal Signal

$$x(t) = A\cos(\omega_0 t + \phi). \tag{1}$$

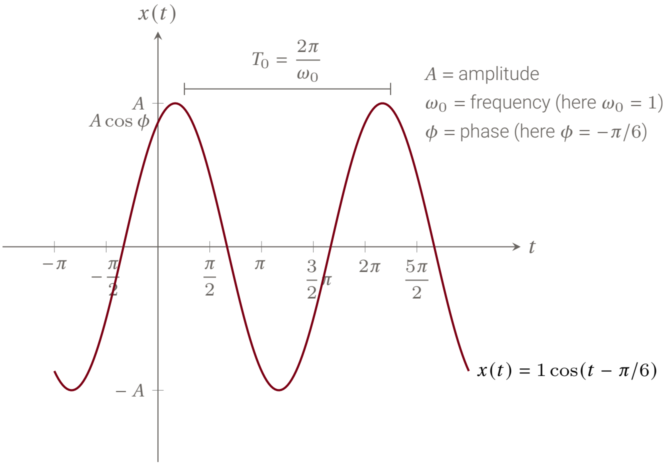


Figure: Continuous-time sinusoidal signal.

Periodicity of a Sinusoidal

Sinusoidal signal is periodic.
A periodic continuous-time signal $x(t)$ has the property that there is a positive value T for which

$$x(t) = x(t + T) \tag{2}$$

for all values of t. Under an appropriate time-shift the signal repeats itself. In this case we say that $x(t)$ is periodic with period T.
Fundamental period T_0 = smallest positive value of T for which 2 holds.
A signal that is not periodic is referred to as aperiodic.
E.g.: Consider $A\cos(\omega_0 t + \phi)$

$$\begin{aligned} A\cos(\omega_0 t + \phi) &= A\cos(\omega_0(t + T) + \phi) \quad \text{here } \omega_0 T = 2\pi m \quad \text{an integer multiple of } 2\pi \\ &= A\cos(\omega_0 t + \phi) \end{aligned}$$

$T = \dfrac{2\pi m}{\omega_0} \Rightarrow$ fundamental period $T_0 = \dfrac{2\pi}{\omega_0}$.

Phase of a Sinusoidal

A time-shift in a CT sinusoid is equivalent to a phase shift.
E.g.: Show that a time-shift of a sinusoid is equal to a phase shift.

$$A\cos[\omega_0(t+t_0)] = A\cos(\omega_0 t + \omega_0 t_0) = A\cos(\omega_0 t + \Delta\phi), \quad \Delta\phi \text{ is a change in phase.}$$

$$A\cos[\omega_0(t+t_0)+\phi] = A\cos(\omega_0 t + \omega_0 t_0 + \phi) = A\cos(\omega_0(t+t_1)), \quad t_1 = t_0 + \phi/\omega_0.$$

Even and Odd Signals

A signal $x(t)$ or $x[n]$ is referred to as an even signal if it is identical to its time-reversed counterpart, i.e., with its reflection about the origin:

$$x(-t) = x(t)$$

$$x[-n] = x[n]$$

A is referred to as an odd if

$$x(-t) = -x(t)$$

$$x[-n] = -x[n]$$

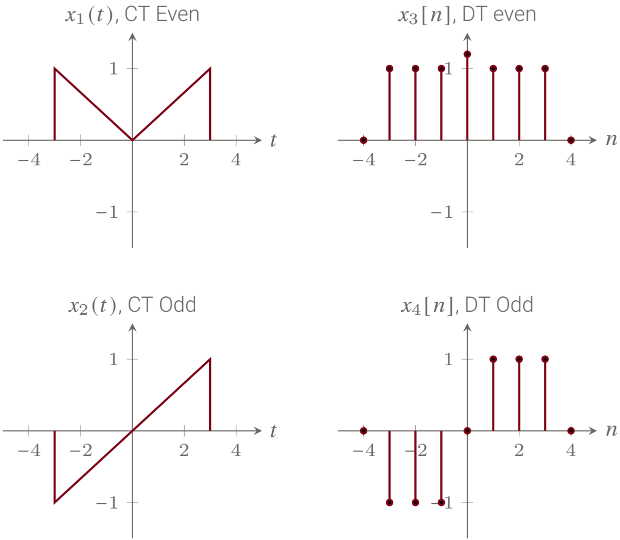
An odd signal must be) at $t = 0$ or $n = 0$.
A signal can be broken into a sum of two signals, one of which is even and one for which is odd. Even part of $x(t)$ is

$$\mathfrak{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$$

Odd part of $x(t)$ is

$$\mathfrak{Od}\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$$

Examples of Even and Odd Functions



Even and Odd Signals Contd.

Example

Show that $\mathfrak{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$.

Notation: $x_e(t)$ is even part of $x(t)$, $x_o(t)$ is odd part of $x(t)$.

$$x(t) = x_e(t) + x_o(t).$$

Even and Odd Signals Contd.

Example

Show that $\mathfrak{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$.

Notation: $x_e(t)$ is even part of $x(t)$, $x_o(t)$ is odd part of $x(t)$.

$$x(t) = x_e(t) + x_o(t).$$

$$x(-t) = x_e(-t) + x_o(-t).$$

Even and Odd Signals Contd.

Example

Show that $\mathfrak{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$.

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$$x(-t) = x_e(t) - x_o(t).$$

Even and Odd Signals Contd.

Example

Show that $\mathfrak{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$.

Notation: $x_e(t)$ is even part of $x(t)$, $x_o(t)$ is odd part of $x(t)$.

$$x(t) = x_e(t) + x_o(t).$$

$$x(-t) = x_e(-t) + x_o(-t).$$

$$x(-t) = x_e(t) - x_o(t).$$

Adding,

$$x(t) + x(-t) = x_e(t) + x_o(t) + x_e(t) - x_o(t).$$

Even and Odd Signals Contd.

Example

Show that $\mathfrak{Ev}\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$.

Notation: $x_e(t)$ is even part of $x(t)$, $x_o(t)$ is odd part of $x(t)$.

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$$x(-t) = x_e(-t) + x_o(-t).$$

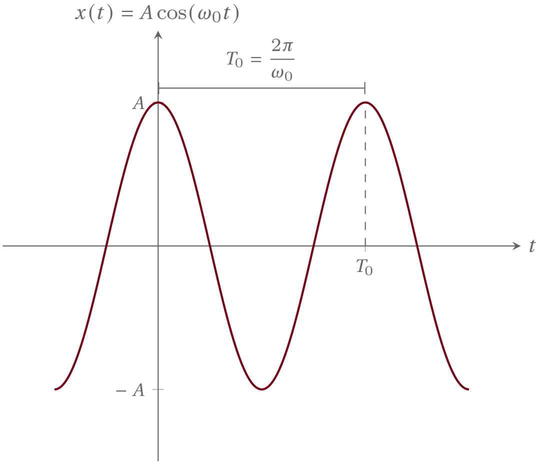
$$x(-t) = x_e(t) - x_o(t).$$

Adding,

$$x(t) + x(-t) = x_e(t) + x_o(t) + x_e(t) - x_o(t).$$

$$\mathfrak{Ev}\{x(t)\} = x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

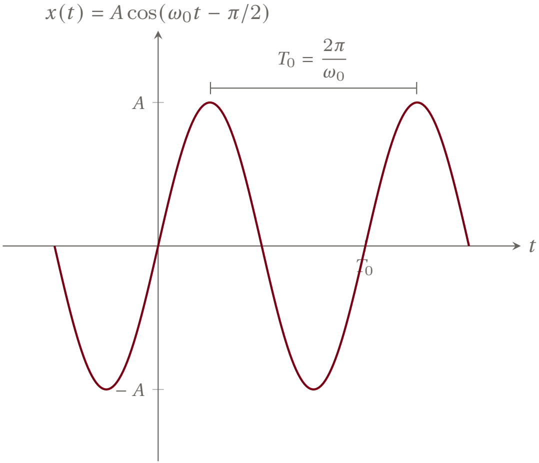
Phase of a Sinusoidal: $\phi = 0$



This signal is even. If we mirror an even signal about the time origin, it would look exactly the same.

Periodic: $x(t) = x(t + T)$.
Even: $x(t) = x(-t)$.

Phase of a Sinusoidal: $\phi = -\pi/2$



This signal is odd. If we flip an odd signal about the time origin, we also multiply it by a (–) sign to get the original signal.

Periodic: $x(t) = x(t+T)$.
Odd: $x(t) = -x(-t)$.

Outline

Real Signals
- Sinusoids
- Discrete-Time Sinusoidal Signal
- Exponentials

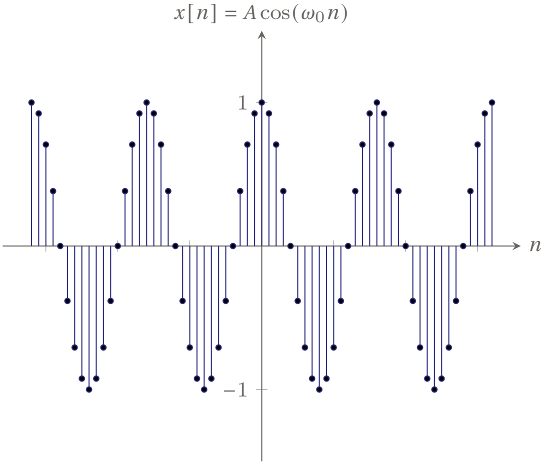
Complex Numbers

Complex Signals
- CT Complex Exponentials
- DT Complex Exponentials

Step and Impulse Functions

Signal Energy and Power

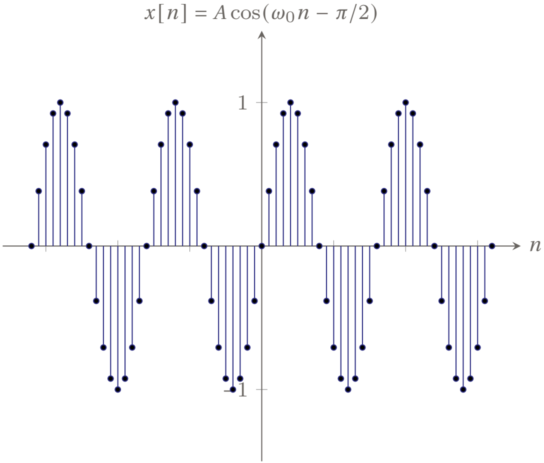
$x[n] = A\cos(\omega_0 n + \phi)$ with $\phi = 0$



The independent variable is an integer.
The sequence takes values only at integer values of the argument.
This signal is even.

Even: $x[n] = x[-n]$.
Periodic: $x[n] = x[n+N]$. Here, $N = 16$
$\omega_0 = \dfrac{2\pi}{N} = \dfrac{\pi}{8}$.

$x[n] = A\cos(\omega_0 n + \phi)$ with $\phi = -\pi/2$



The independent variable is an integer.
The sequence takes values only at inter values of he argument.
This signal is odd.

Odd: $x[n] = -x[-n]$.
Periodic: $x[n] = x[n+N]$. Here, $N = 16$
$\omega_0 = \dfrac{2\pi}{N} = \dfrac{\pi}{8}$. $\phi = -\pi/2$, $x[n] = A\cos(\omega_0 n + \phi) = A\cos(\omega_0(n + n_0))$.
n_0 must be an integer.
$n_0 = \dfrac{\phi}{\omega_0} = \dfrac{\pi/2}{\pi/8} = 4$.

Phase Change and Time Shift in DT

Question

Does a phase change always correspond to a time shift in discrete-time signals?

Phase Change and Time Shift in DT

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Does a phase change always correspond to a time shift in discrete-time signals?

Answer: No.

$$
\begin{aligned}
A\cos[\omega_0 n + \phi)] &\stackrel{?}{=} A\cos[\omega_0(n + n_0)] \\
\omega_0 n + \omega_0 n_0 &= \omega_0 n + \phi \\
\omega_0 n_0 &= \phi, \quad n_0 \text{ is an integer.}
\end{aligned}
$$

- Depending on ϕ and ω_0, n_0 many not come out to be an integer.
- In discrete time, the amount of time shift must be an integer.

Periodicity of a DT Signal

All continuous-time sinusoids are periodic. However, discrete-time sinusoids are not necessarily so.

$$x[n] = x[n+N], \quad \text{smallest integer } N \text{ is the fundamental period.} \tag{3}$$

$$A\cos[\omega_0(n+N)+\phi] = A\cos[\omega_0 n + \omega_0 N + \phi]$$

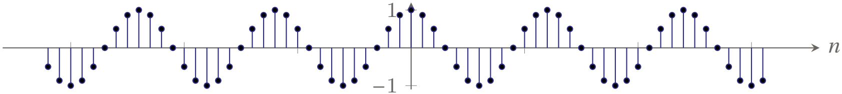
$\omega_0 N$ must be an integer multiple of 2π.
Periodic $\Rightarrow \omega_0 N = 2\pi m$

$$N = \frac{2\pi m}{\omega_0} \tag{4}$$

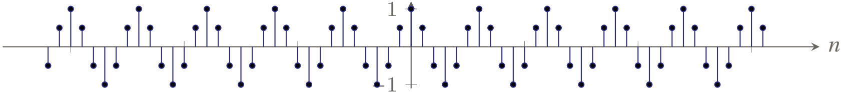
N and m must be integers.
Smallest N, if any, is the fundamental period.
N may not be an integer. In this case, the signal is not periodic.

Periodicity of a DT Signal Cntd.

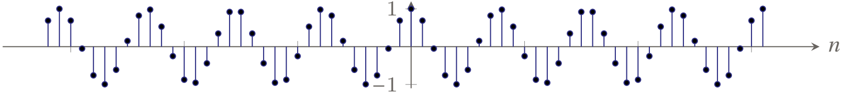
$$x[n] = A\cos(2\pi/12n)$$



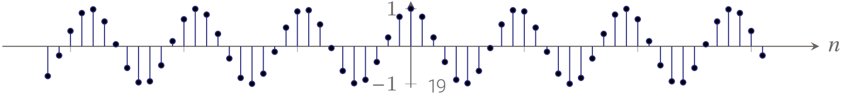
$$x[n] = A\cos(2\pi/6n)$$



$$x[n] = A\cos(2\pi/31n)$$



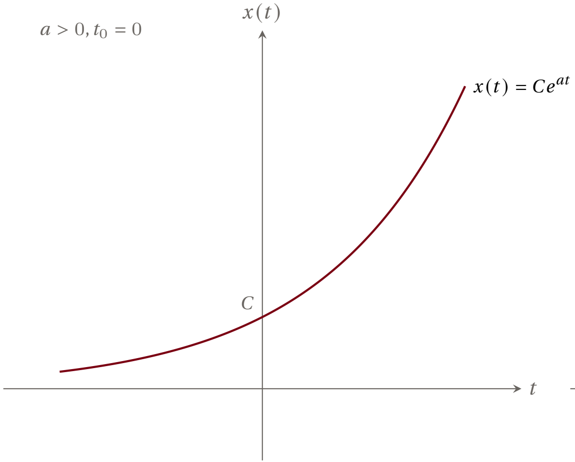
$$x[n] = A\cos(2/3n)$$

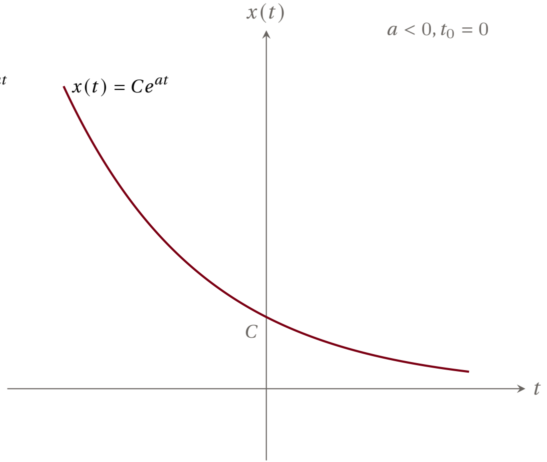


Outline

CT Real Exponentials

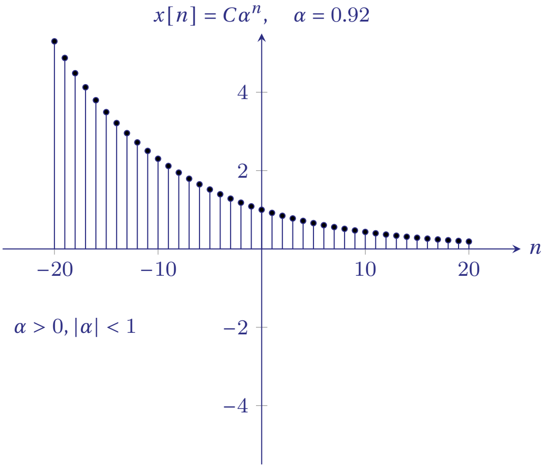
$$x(t) = Ce^{a(t+t_0)}, \quad C \text{ and } a \text{ are real numbers}$$
$$= Ce^{at_0}e^{at}.$$

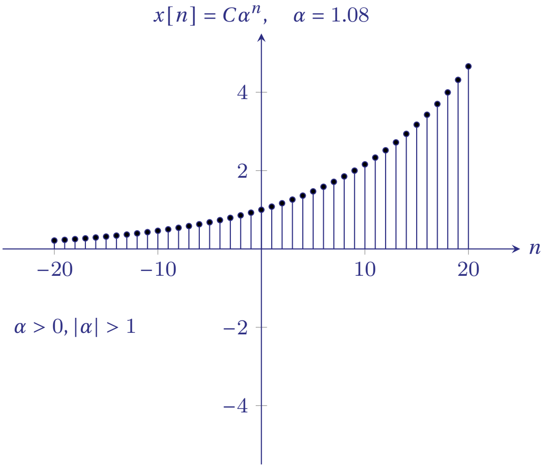


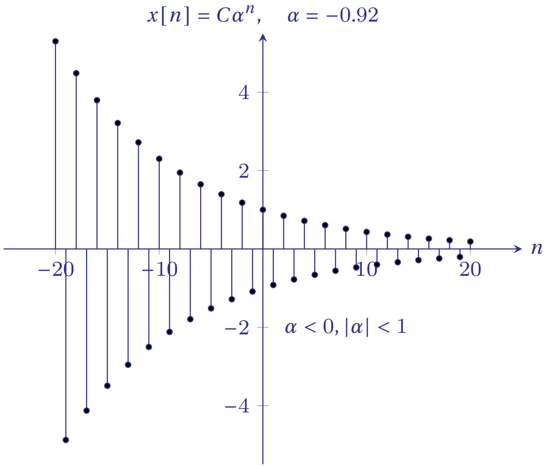


DT Real Exponentials

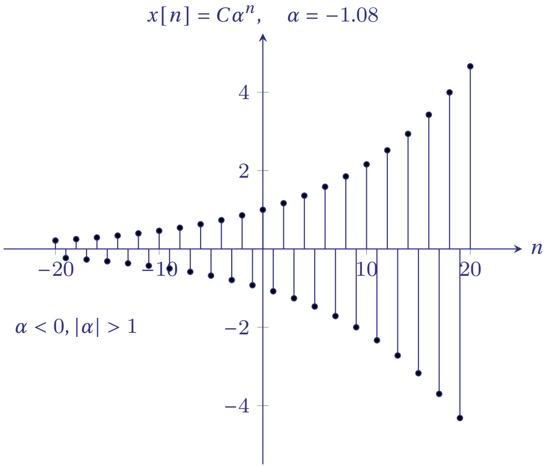
$x[n] = Ce^{\beta n} = C\alpha^n,$ C and α are real numbers


$x[n] = C\alpha^n, \quad \alpha = 0.92$
4
2
−2
−4
−20
−10
10
20
n
$\alpha > 0, |\alpha| < 1$


$x[n] = C\alpha^n, \quad \alpha = 1.08$
4
2
−2
−4
−20
−10
10
20
n
$\alpha > 0, |\alpha| > 1$



$x[n] = C\alpha^n, \quad \alpha = -0.92$
4
2
−2
−4
−20
−10
10
20
n
$\alpha < 0, |\alpha| < 1$



$x[n] = C\alpha^n, \quad \alpha = -1.08$
4
2
−2
−4
−20
−10
10
20
n
$\alpha < 0, |\alpha| > 1$

Section 2

Complex Numbers

Representing Complex Numbers

The Cartesian or rectangular form:

$$z = x + jy,$$

where $j = \sqrt{-1}$ and x and y are real numbers referred to respectively as the real part and the imaginary part. I.e.,

$$x = \mathfrak{Re}\{z\}, y = \mathfrak{Im}\{z\}$$

The polar form:

$$z = re^{j\theta},$$

where $r > 0$ is the magnitude of z and θ is the angle or phase of z.

$$r = |z|, \theta = \sphericalangle z.$$

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The polar form:

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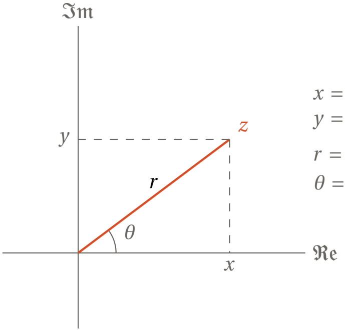
where $r > 0$ is the magnitude of z and θ is the angle or phase of z.

$$r = |z|, \theta = \sphericalangle z.$$

The relationship between these two representations can be determined from Euler's relation:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

or by plotting z in the complex plane.



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \operatorname{atan2}(y, x)$$

Example Let z_0 be a complex number with polar coordinates (r_0, θ_0) and Cartesian coordinates (x_0, y_0). Determine expressions for the Cartesian coordinates of the following complex numbers in terms of x_0 and y_0. Plot the points z_0, z_1, z_2, z_3, z_4, and z_5 in the complex plane when $r_0 = 2$ and $\theta_0 = \pi/4$ and when $r_0 = 2$ and $\theta_0 = \pi/2$. Indicate on the plot the real and imaginary parts of each point.

1. $z_1 = r_0 e^{-j\theta_0}$
2. $z_2 = r_0$
3. $z_3 = r_0 e^{j(\theta_0+\pi)}$
4. $z_4 = r_0 e^{j(-\theta_0+\pi)}$
5. $z_5 = r_0 e^{j(\theta_0+2\pi)}$

$$
\begin{aligned}
z_0 &= r_0 e^{j\theta_0} = r_0(\cos\theta_0 + j\sin\theta_0) \\
&= r_0\cos\theta_0 + j r_0 \sin\theta_0 = x_0 + j y_0.
\end{aligned}
$$

$$
z_1 = r_0 e^{-j\theta} = r_0(\cos(-\theta_0) + j\sin(-\theta_0)) = x_0 - j y_0.
$$

Example Let z_0 be a complex number with polar coordinates (r_0, θ_0) and Cartesian coordinates (x_0, y_0). Determine expressions for the Cartesian coordinates of the following complex numbers in terms of x_0 and y_0. Plot the points z_0, z_1, z_2, z_3, z_4, and z_5 in the complex plane when $r_0 = 2$ and $\theta_0 = \pi/4$ and when $r_0 = 2$ and $\theta_0 = \pi/2$. Indicate on the plot the real and imaginary parts of each point.

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z_0 &= r_0 e^{j\theta_0} = r_0(\cos\theta_0 + j\sin\theta_0) \\
&= r_0\cos\theta_0 + jr_0\sin\theta_0 = x_0 + jy_0.
\end{aligned}
$$

$$z_1 = r_0 e^{-j\theta} = r_0(\cos(-\theta_0)+j\sin(-\theta_0)) = x_0 - jy_0.$$

$$z_2 = r_0 = \sqrt{x_0^2 + y_0^2}$$

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z_3 &= r_0 e^{j(\theta_0+\pi)} \\
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$$\begin{aligned} z_0 &= r_0 e^{j\theta_0} = r_0(\cos\theta_0 + j\sin\theta_0) \\ &= r_0\cos\theta_0 + jr_0\sin\theta_0 = x_0 + jy_0. \end{aligned}$$

$$z_1 = r_0 e^{-j\theta} = r_0(\cos(-\theta_0)+j\sin(-\theta_0)) = x_0 - jy_0.$$

$$z_2 = r_0 = \sqrt{x_0^2 + y_0^2}$$

$$\begin{aligned} z_3 &= r_0 e^{j(\theta_0+\pi)} \\ &= r_0(\cos(\theta_0+\pi) + \sin(\theta_0+\pi)) = -x_0 - jy_0 = -z_0. \end{aligned}$$

$$z_4 = -x_0 + jy_0.$$

Example Let z_0 be a complex number with polar coordinates (r_0, θ_0) and Cartesian coordinates (x_0, y_0). Determine expressions for the Cartesian coordinates of the following complex numbers in terms of x_0 and y_0. Plot the points z_0, z_1, z_2, z_3, z_4, and z_5 in the complex plane when $r_0 = 2$ and $\theta_0 = \pi/4$ and when $r_0 = 2$ and $\theta_0 = \pi/2$. Indicate on the plot the real and imaginary parts of each point.

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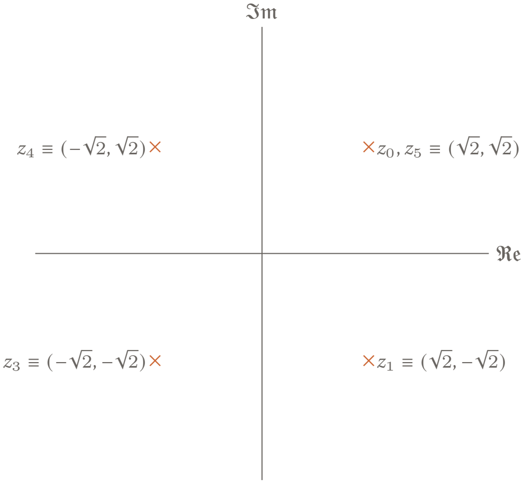
\end{aligned}
$$

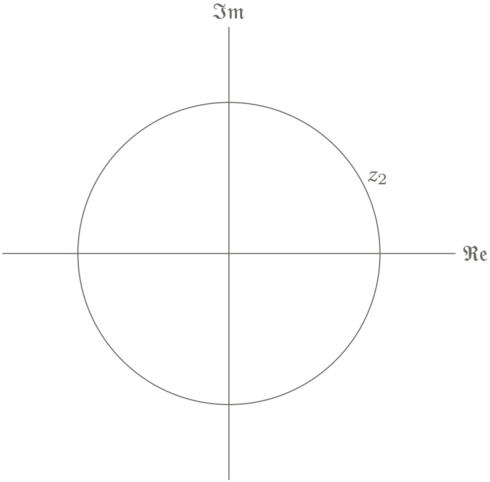
$$z_4 = -x_0 + jy_0.$$

$$z_5 = z_0.$$

$r = 2, \theta = \pi/4$

$z_1 = r_0 e^{-j\theta_0}$, $z_2 = r_0$, $z_3 = r_0 e^{j(\theta_0+\pi)}$, $z_4 = r_0 e^{j(-\theta_0+\pi)}$, $z_5 = r_0 e^{j(\theta_0+2\pi)}$





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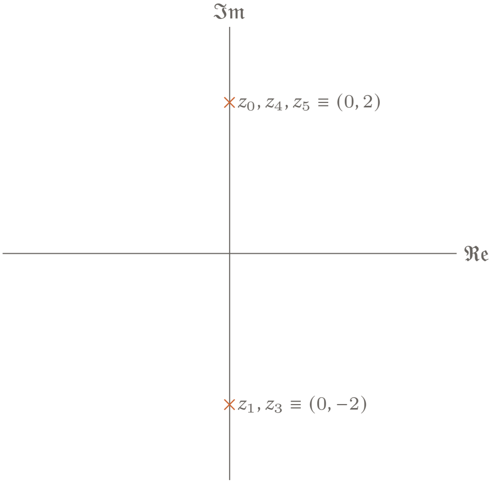
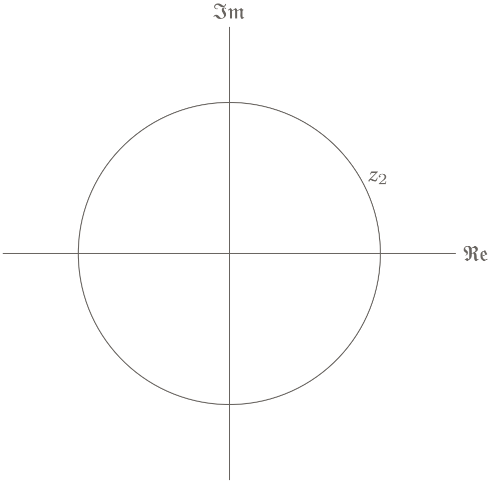
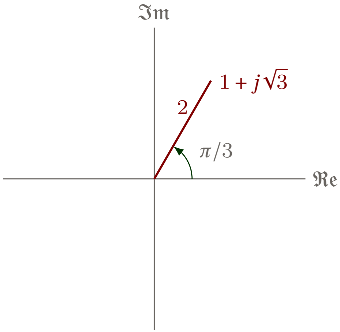




Example Express each of the following complex numbers in polar form, and plot them in the complex plane, indicating the magnitude and angle of each number.

1. $1 + j\sqrt{3}$
2. -5
3. $-5 - 5j$
4. $3 + 4j$
5. $(1 - j\sqrt{3})^3$
6. $\dfrac{e^{j\pi/3} - 1}{1 + j\sqrt{3}}$

$$
\begin{aligned}
1 + j\sqrt{3} &= \sqrt{1^2 + (\sqrt{3})^2}\left(\frac{1}{\sqrt{1^2 + (\sqrt{3})^2}} + j\frac{\sqrt{3}}{\sqrt{1^2 + (\sqrt{3})^2}}\right) \\
&= 2e^{j\,\mathrm{atan2}(\sqrt{3},1)} \\
&= 2e^{j\pi/3}
\end{aligned}
$$



Example Express each of the following complex numbers in polar form, and plot them in the complex plane, indicating the magnitude and angle of each number.

1. $1 + j\sqrt{3}$
2. -5
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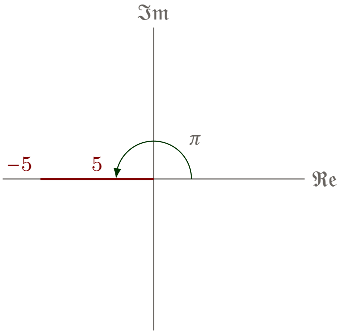
4. $3 + 4j$
5. $(1 - j\sqrt{3})^3$
6. $\dfrac{e^{j\pi/3} - 1}{1 + j\sqrt{3}}$

$$\begin{aligned} -5 &= 5(-1 + j0) \\ &= 5e^{j\,\mathrm{atan2}(0,-1)} \\ &= 5e^{j\pi} \end{aligned}$$



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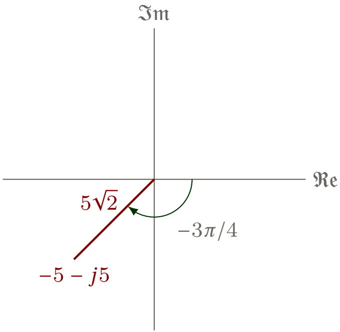
$$\begin{aligned} -5 - 5j &= 5(-1 + j(-1)) \\ &= 5e^{j\,\mathrm{atan2}(-1,-1)} \\ &= 5e^{-j3\pi/4} \end{aligned}$$



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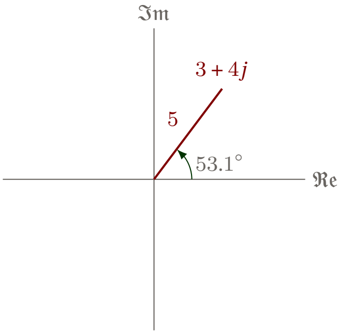
$$\begin{aligned} 3 + 4j &= 5(3/5 + j4/5) \\ &= 5e^{j\,\mathrm{atan2}(4,3)} \\ &= 5e^{-j3\pi/4} \end{aligned}$$



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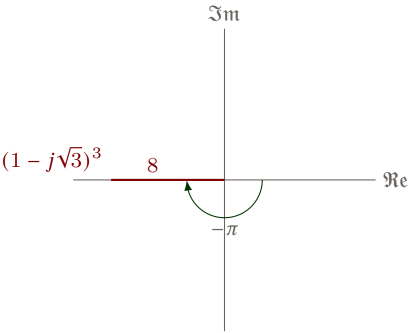
$$\begin{aligned}(1 - j\sqrt{3})^3 &= \left(2e^{-j\pi/3}\right)^3 \\ &= 8e^{-j\pi}\end{aligned}$$



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$\cos\theta$ and $\sin\theta$

Using Euler's relations, derive the following relationships:

1. $\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$
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Subtracting

$$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$$
$$\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

Complex Conjugate

Let z denote a complex variable; i.e.,

$$z = x + jy = re^{j\theta}.$$

The complex conjugate of z is

$$z^* = x - jy = re^{-j\theta}.$$

Show that

1. $zz^* = r^2$
2. $z + z^* = 2\mathfrak{Re}\{z\}$
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List the values of

1. e^{j0}
2. $e^{j\pi/2}$
3. $e^{j\pi}$
4. $e^{j3\pi/2}$
5. $e^{j2\pi}$

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List the values of

1. $e^{j0} = 1$
2. $e^{j\pi/2} = j$
3. $e^{j\pi} = -1$
4. $e^{j3\pi/2} = -j$
5. $e^{j2\pi} = 1$

Section 3

Complex Signals

Outline

CT Complex Exponentials

$$
\begin{aligned}
x(t) &= Ce^{at} \quad C \text{ and } a \text{ are complex numbers.} \\
C &= |C|e^{j\theta} \\
a &= r + j\omega_0 \\
x(t) &= |C|e^{j\theta}e^{(r+j\omega_0)t} \\
&= |C|e^{rt}\, e^{j(\omega_0 t+\theta)} \\
&= |C|e^{rt}\, [\cos(\omega_0 t + \theta) + j \sin(\omega_0 t + \theta)]
\end{aligned}
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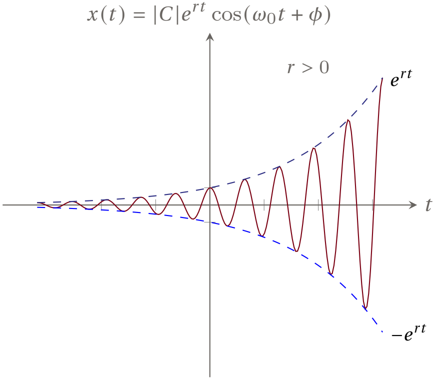
$$

- $e^{j(\omega_0 t+\theta)} = \cos(\omega_0 t + \theta) + j\sin(\omega_0 t + \theta)$
- Real
- 90° out of phase



$x(t) = |C|e^{rt}\cos(\omega_0 t + \phi)$
$r > 0$
e^{rt}
$-e^{rt}$
t
$x(t) = |C|e^{rt}\cos(\omega_0 t + \phi)$
$r < 0$
e^{rt}
$-e^{rt}$
t

Outline

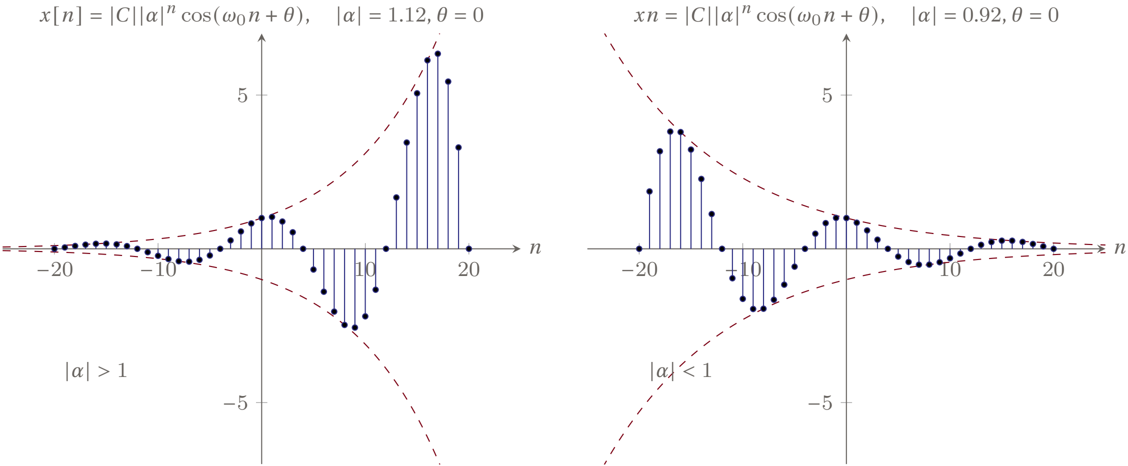
DT Complex Exponentials

$$
\begin{aligned}
x[n] &= C\alpha^n, \quad C \text{ and } \alpha \text{ are complex numbers.} \\
C &= |C|e^{j\theta} \\
\alpha &= |\alpha|e^{j\omega_0} \\
x[n] &= |C|e^{j\theta}\left(|\alpha|e^{j\omega_0}\right)^n \\
&= |C||\alpha|^n\cos(\omega_0 n + \theta) + j|C||\alpha|^n\sin(\omega_0 n + \theta)
\end{aligned}
$$

Comments:

- When $|\alpha| = 1$: sinusoidal real and imaginary parts.
- $e^{j\omega_0 n}$ may or may not be periodic depending on the value of ω_0.
- Sinusoidal, exponential, step, and impulse signal form the cornerstones for signals and systems analysis.

DT Complex Exponentials Plot



Periodicity Properties of Discrete-Time Complex Exponentials $e^{j\omega_0 n}$

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- In DT, as we increase ω_0 from 0, we obtain signals that oscillate more and more rapidly until we reach $\omega_0 = \pi$. As we continue to increase ω_0, we decrease the rate of oscillation until we reach $\omega_0 = 2\pi$. Note: $e^{j\pi n} = \left(e^{j\pi}\right)^n = (-1)^n$.

Comparison of the Signals $e^{j\omega_0 t}$ and $e^{j\omega_0 n}$

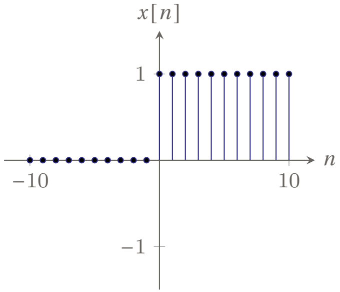
$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0	Identical signals for values of ω_0 separated by multiples of 2π
Periodic for any choice of $e^{j\omega_0 t}$	Periodic only if $\omega_0 = 2\pi m/N$ for some integers $N > 0$ and m.
Fundamental frequency ω_0	Fundamental frequency ω_0/m
Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $2\pi/\omega_0$	Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m(2\pi/\omega_0)$

Section 4

Step and Impulse Functions

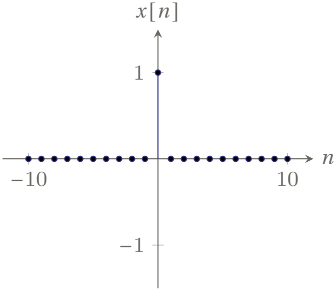
Discrete-Time Unit Step $u[n]$

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \tag{5}$$



Discrete-Time Unit Impulse (Unit Sample) $\delta[n]$

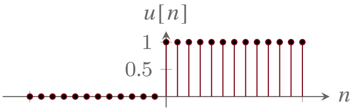
$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases} \tag{6}$$



DT Step and Impulse

Unit impulse is the first backward difference of the unit step sequence.

$$\delta[n] = u[n] - u[n-1]. \tag{7}$$



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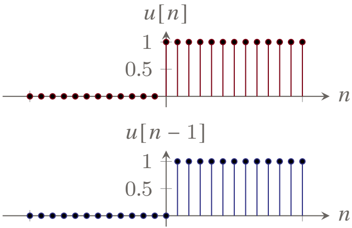

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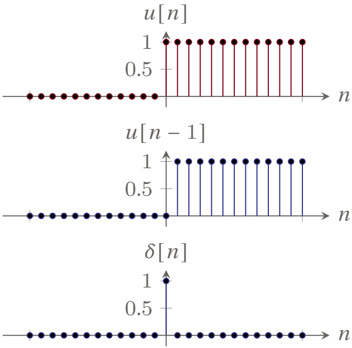

DT Step and Impulse

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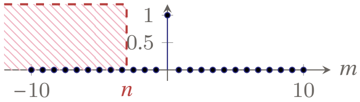

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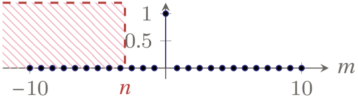

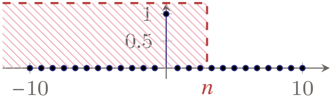

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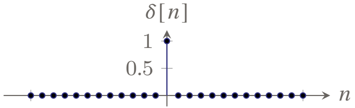

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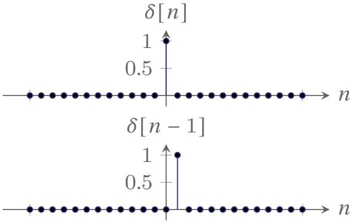

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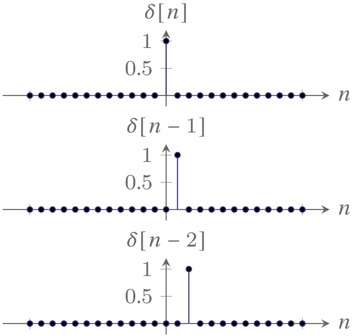

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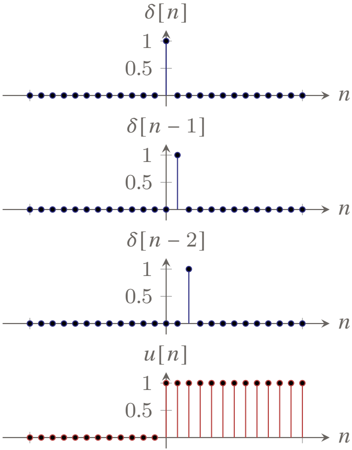
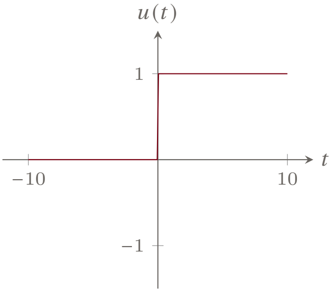
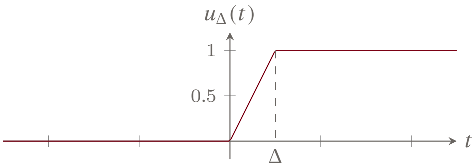


Continuous-Time Unit Step Function $u(t)$

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases} \tag{10}$$

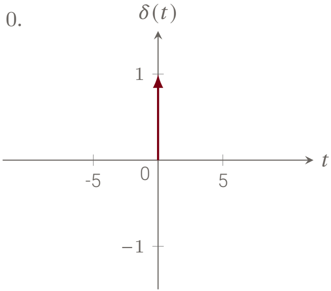


Continuous-Time Unit Impulse Function $\delta(t)$



$u_\Delta(t) \to u(t)$ as $\Delta \to 0$.

$$\delta(t) = \frac{du(t)}{dt}. \tag{11}$$



$$x(t)\delta(t) = x(0)\delta(t)$$

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

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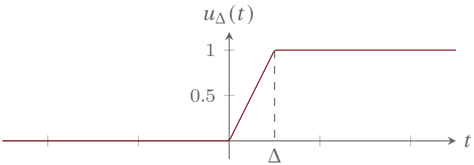

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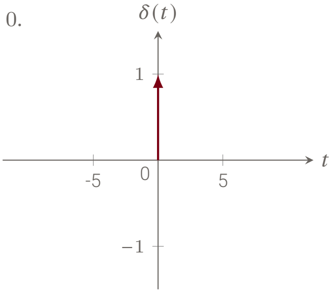
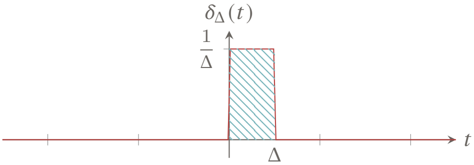


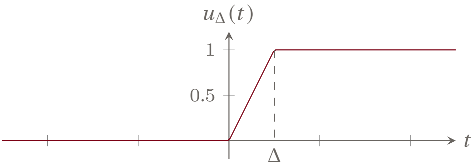


$\delta_\Delta(t) \to \delta(t)$ as $\Delta \to 0$.
area = 1

$$x(t)\delta(t) = x(0)\delta(t)$$

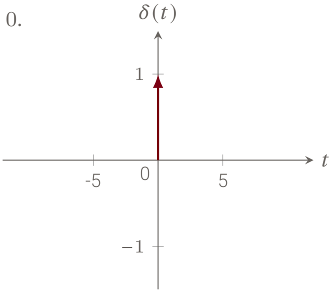
$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

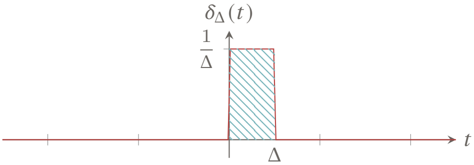
Continuous-Time Unit Impulse Function $\delta(t)$



$$\delta(t) = \frac{du(t)}{dt}. \tag{11}$$

$u_\Delta(t) \rightarrow u(t)$ as $\Delta \rightarrow 0$.





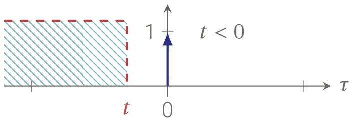
$\delta_\Delta(t) \rightarrow \delta(t)$ as $\Delta \rightarrow 0$.
area = 1

$$x(t)\delta(t) = x(0)\delta(t)$$

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

CT Unit Step Function and Unit Impulse Function

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau. \tag{12}$$



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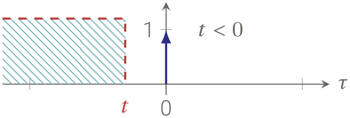

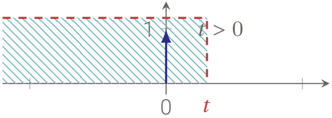

Section 5

Signal Energy and Power

Energy I

The total energy over a time interval $t_1 \leq t \leq t_2$ in a continuous-time signal $x(t)$ is

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

The total energy over a time interval $n_1 \leq n \leq n_2$ in a discrete-time signal $x[n]$ is

$$\sum_{n=n_1}^{n_2} |x[n]|^2 dt$$

Total energy over an infinite interval in a CT signal:

$$E_\infty \triangleq \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |x(t)|^2 dt. \tag{13}$$

Energy II

Total energy over an infinite interval in a DT signal:

$$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2. \tag{14}$$

Note that this integral and may not converge for some signals. Such signals have infinite energy, while signals with $E_\infty < \infty$ have finite energy.

Power

Time-averaged power over an infinite interval in a CT signal:

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt. \tag{15}$$

Total energy in a DT signal:

$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2. \tag{16}$$

With these definitions, we can identify three important classes of signals:

1. Energy signals: Signals with finite total energy $E_\infty < \infty$. These have zero average power.
2. Power signals: Signals with finite average power $0 < P_\infty < \infty$. As $P_\infty > 0$, $E_\infty = \infty$.
3. Signals with neither E_∞ nor P_∞ are finite.

Examples

Determine whether the following signals are energy signals, power signals, or neither.

1. $x(t) = e^{-at}u(t), \quad a > 0$
2. $x(t) = A\cos(\omega_0 t + \theta)$
3. $x(t) = tu(t)$

$$x(t) = e^{-at}u(t), \quad a > 0$$

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt$$

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^\infty e^{-2at} dt$$

$$= \frac{-1}{2a}\left[e^{-at}\right]_0^\infty = \frac{-1}{2a}[0-1] = \frac{1}{2a}$$

This is an energy signal.

Examples

Determine whether the following signals are energy signals, power signals, or neither.

1. $x(t) = e^{-at}u(t), \quad a > 0$
2. $x(t) = A\cos(\omega_0 t + \theta)$
3. $x(t) = tu(t)$

$$x(t) = A\cos(\omega_0 t + \theta)$$

$$\begin{aligned}
E_\infty &= \lim_{T\to\infty}\int_{-T}^{T}|x(t)|^2dt \\
&= \lim_{T\to\infty}\int_{-T}^{T}A^2\cos^2(\omega_0 t + \theta)dt \\
&= \frac{A^2}{2}\lim_{T\to\infty}\int_{-T}^{T}[1 + \cos(2\omega_0 t + 2\theta)]dt \\
&= \frac{A^2}{2}\lim_{T\to\infty}\left[t - \frac{\cos(2\omega_0 t + 2\theta)}{2\omega_0}\right]_{-T}^{T}
\end{aligned}$$

Considering T as an integer multiple of $2\pi/\omega_0$

$$E_\infty = A^2\lim_{T\to\infty}T \to \infty.$$

This is not an energy signal.

$$\begin{aligned}
P_\infty &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2dt \\
&= A^2\lim_{T\to\infty}\frac{1}{2T}T = \frac{A^2}{2} < \infty
\end{aligned}$$

This is a power signal.

Examples

Determine whether the following signals are energy signals, power signals, or neither.

1. $x(t) = e^{-at}u(t), \quad a > 0$
2. $x(t) = A\cos(\omega_0 t + \theta)$
3. $x(t) = tu(t)$

$$x(t) = tu(t)$$

$$\begin{aligned} E_\infty &= \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt \\ &= \lim_{T\to\infty} \int_{0}^{T} t\,dt = \lim_{T\to\infty} \left[\frac{t^2}{2}\right]_0^T \\ &= \lim_{T\to\infty} \frac{T^2}{2} \to \infty. \end{aligned}$$

This is not an energy signal.

$$x(t) = tu(t)$$

$$\begin{aligned} P_\infty &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt \\ &= \lim_{T\to\infty} \frac{1}{2T} \frac{T^2}{2} \\ &= \lim_{T\to\infty} \frac{T}{4} \to \infty. \end{aligned}$$

This is not a power signal either.

Real Signals
- Sinusoids
- Discrete-Time Sinusoidal Signal
- Exponentials

Complex Numbers

Complex Signals
- CT Complex Exponentials
- DT Complex Exponentials

Step and Impulse Functions

Signal Energy and Power