

# EN1020 Signals and Systems: Fourier Series

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## Section 1

# Continuous-Time Fourier Series

# Outline

## Continuous-Time Fourier Series

Introduction

Fourier Series

Properties of the Continuous-Time Fourier Series

Convergence of Fourier Series

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  1. Continuous-time Fourier series
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  3. Discrete-time Fourier series
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- In this part of the course, we will concentrate on how to actually compute continuous-time Fourier series and transform. Later, after we study linear, time-invariant (LTI) systems, we will study the conceptual aspects of Fourier techniques.



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- $\omega_0 = \frac{2\pi}{T}$ , where  $T$ : fundamental period of the waveform.

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## Continuous-Time Fourier Series

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# Continuous-Time Fourier Series

$$\begin{aligned}x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\a_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \\ \omega_0 &= \frac{2\pi}{T}\end{aligned}\tag{1}$$

The set of coefficients  $\{a_k\}$  is called the **Fourier series coefficients** or the **spectral coefficients** of  $x(t)$ . The coefficient  $a_0$  is the dc or constant component of  $x(t)$ , given by Equation 1 with  $k = 0$ :

$$a_0 = \frac{1}{T} \int_T x(t) dt,\tag{2}$$

which is simply the average of  $x(t)$  over one period.

## Example

Let

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$$e^{j\theta} = \cos \theta + j \sin \theta$$

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$$\sin \omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

Comparing the right-hand side of this equation and Equation 1, we obtain

$$\begin{aligned} a_1 &= \frac{1}{2j} & a_{-1} &= -\frac{1}{2j} \\ a_k &= 0, & k &\neq \pm 1. \end{aligned}$$

Euler's formula

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## Example

Let

$$x(t) = 1 + \sin \omega_0 t + 2 \cos \omega_0 t + \cos \left( 2\omega_0 t + \frac{\pi}{4} \right),$$

which has the fundamental frequency  $\omega_0$ .

1. Use Euler's formula to express  $x(t)$  as a linear combination of complex exponentials.
2. Find the Fourier series coefficients,  $a_k$ .
3. Plot the magnitude and phase of  $a_k$ .

$$x(t) = 1 + \sin \omega_0 t + 2 \cos \omega_0 t + \cos \left( 2\omega_0 t + \frac{\pi}{4} \right),$$

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$$x(t) = 1 + \frac{1}{2j} [e^{j\omega_0 t} - e^{-j\omega_0 t}] + [e^{j\omega_0 t} + e^{-j\omega_0 t}] + \frac{1}{2} [e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}]$$

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Collecting terms,

$$x(t) = 1 + \left( 1 + \frac{1}{2j} \right) e^{j\omega_0 t} + \left( 1 - \frac{1}{2j} \right) e^{-j\omega_0 t} + \left( \frac{1}{2} e^{j\pi/4} \right) e^{j2\omega_0 t} + \left( \frac{1}{2} e^{-j\pi/4} \right) e^{-j2\omega_0 t}$$

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The Fourier coefficients are

$$a_0 = 1,$$

$$a_1 = \left( 1 + \frac{1}{2j} \right) = \left( 1 - \frac{j}{2} \right),$$

$$a_{-1} = \left( 1 - \frac{1}{2j} \right) = \left( 1 + \frac{j}{2} \right),$$

$$a_2 = \frac{1}{2} e^{j\pi/4} = \frac{\sqrt{2}}{4} (1 + j),$$

$$a_{-2} = \frac{1}{2} e^{-j\pi/4} = \frac{\sqrt{2}}{4} (1 - j),$$

$$a_k = 0, |k| > 2.$$

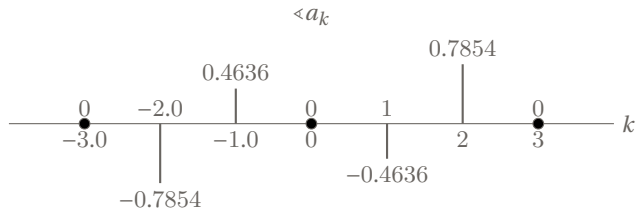
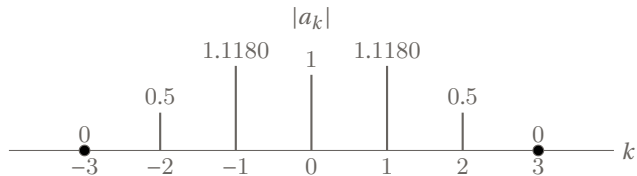


Figure:  $|a_k|$ ,  $\angle a_k$

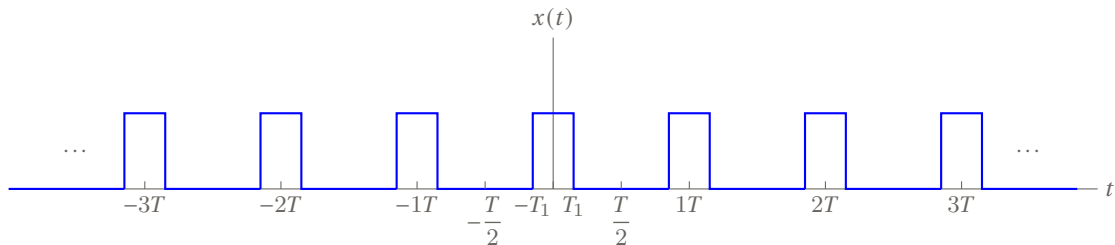
## Example

The periodic square wave, sketched below, is defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < T/2, \end{cases}$$

This signal is periodic with fundamental period  $T$  and fundamental frequency  $\omega_0 = 2\pi/T$ .

1. Find the Fourier series coefficients,  $a_k$ .
2. Plot the magnitude and phase of  $a_k$  for the case  $T = 4T_1$ .



$$\begin{aligned}a_0 &= \frac{1}{T} \int_T x(t) dt, \\&= \frac{1}{T} \int_{-T_1}^{T_1} 1 dt, \\&= \frac{2T_1}{T}.\end{aligned}$$

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 a_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt, \\
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$$e^{j\theta} = \cos \theta + j \sin \theta$$

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 \end{aligned}$$

$$\begin{aligned}
 a_k &= \frac{2}{k\omega_0 T} \left[ \frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j} \right] \\
 a_k &= \frac{2 \sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, k \neq 0.
 \end{aligned}$$

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For  $T = 4T_1$

$$a_k = 0, \quad k \text{ even.}$$

$$a_0 = \frac{1}{2}$$

$$a_1 = a_{-1} = \frac{1}{\pi}$$

$$a_3 = a_{-3} = -\frac{1}{3\pi}$$

$$a_5 = a_{-5} = \frac{1}{5\pi}$$

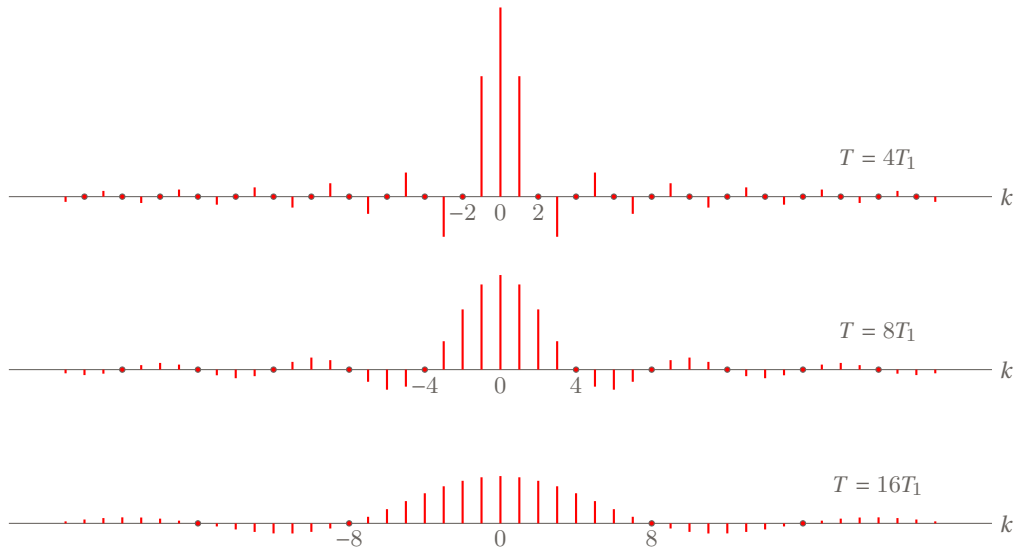


Figure: Plots of scaled Fourier series coefficients  $T a_k$

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Suppose that  $x(t)$  is a periodic signal with period  $T$  and fundamental frequency  $\omega_0 = 2\pi/T$ . Then if the Fourier series coefficients are denoted by  $a_k$ , then

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \quad (3)$$

# Linearity

Let  $x(t)$  and  $y(t)$  denote two periodic signals with period  $T$ .

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k,$$

$$y(t) \xleftrightarrow{\mathcal{FS}} b_k.$$

Any linear combination of the two signals will also be periodic with period  $T$ . Fourier series coefficients  $c_k$  of the linear combination of  $x(t)$  and  $y(t)$ ,  $z(t) = Ax(t) + By(t)$ , are given by the same linear combination:

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$$z(t) = Ax(t) + By(t) \xleftrightarrow{\mathcal{FS}} c_k = Aa_k + Bb_k. \quad (4)$$

# Time Shifting

Proof:

$$x(t - t_0) \xleftrightarrow{\mathcal{FS}} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k \quad (5)$$

Fourier series:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

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Proof:

$$\begin{aligned} x(t) &\xleftrightarrow{\mathcal{FS}} a_k, \quad x(t - t_0) \xleftrightarrow{\mathcal{FS}} b_k, \\ b_k &= \frac{1}{T} \int_T x(t - t_0) e^{-jk\omega_0 t} dt, \\ &= \frac{1}{T} \int_T x(\tau) e^{-jk\omega_0(\tau+t_0)} d\tau, \\ &= e^{-jk\omega_0 t_0} \frac{1}{T} \int_T x(\tau) e^{-jk\omega_0 \tau} d\tau, \\ &= e^{-jk\omega_0 t_0} a_k. \\ x(t - t_0) &\xleftrightarrow{\mathcal{FS}} e^{-jk\omega_0 t_0} a_k. \end{aligned}$$

Fourier series:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\ a_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt \end{aligned}$$

Note:  $|a_k| = |b_k|$



# Time Reversal

If

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

then

$$x(-t) \xleftrightarrow{\mathcal{FS}} a_{-k}.$$

Fourier series:

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$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}.$$

Substitution:  $k = -m$

$$x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\pi t/T}.$$

Right-hand side of this equation has the form of the Fourier series synthesis equation for  $x(-t)$ , where the Fourier series coefficients  $b_k$  are

$$b_k = a_{-k}.$$

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- If  $x(t)$  is even, i.e.,  $x(-t) = x(t)$ , then its Fourier series coefficients are also even, i.e.,  $a_{-k} = a_k$ .
- If  $x(t)$  is odd, i.e.,  $x(-t) = -x(t)$ , then its Fourier series coefficients are also odd, i.e.,  $a_{-k} = -a_k$ .

# Time Scaling

Time scaling, in general, changes the period.

If  $x(t)$  is a periodic with period  $T$  and fundamental frequency  $\omega_0 = 2\pi/T$ , then  $x(\alpha t)$ , where  $\alpha$  is a positive real number, is periodic with period  $T/\alpha$  and fundamental frequency  $\alpha\omega_0$ .

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t} \quad (6)$$

While Fourier coefficients have not changes, the Fourier series representation **has** changed because of the change in the fundamental frequency.

# Multiplication

Let  $x(t)$  and  $y(t)$  denote two periodic signals with period  $T$ .

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k,$$

$$y(t) \xleftrightarrow{\mathcal{FS}} b_k.$$

Since the product  $x(t)y(t)$  is also periodic with period  $T$ , its Fourier series coefficients  $h_k$  are

$$x(t)y(t) \xleftrightarrow{\mathcal{FS}} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}. \quad (7)$$

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- If  $x(t)$  is real and even, we know that  $a_k = a_{-k}$ . From above,  $a_k^* = a_{-k}$ , so that  $a_k = a_k^*$ . That is if  $x(t)$  is real and even, so are its Fourier series coefficients.

# Conjugation and Conjugate Symmetry

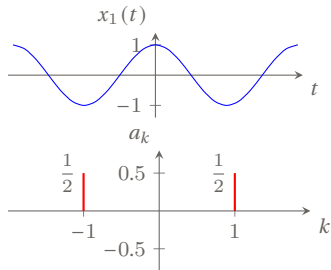
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- If  $x(t)$  is real and odd, its Fourier series coefficients are purely imaginary and odd. Thus, e.g.,  $a_0 = 0$ .

## Example

Consider

$$x_1(t) = \cos(\omega_0 t)$$

This is real and even.

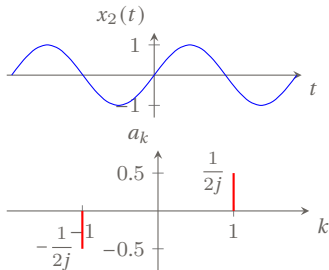


FS coefficients are real and even. (They are conjugate symmetric too.)

Consider

$$x_2(t) = \sin(\omega_0 t)$$

This is real and odd.



FS coefficients are imaginary and odd. (They are conjugate symmetric too.)

## Parseval's Relation for Continuous-Time Periodic Signals

$$\frac{1}{T} \int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2. \quad (8)$$

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$$\frac{1}{T} \int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2. \quad (8)$$

Note: Left-hand side of equation 8 is the average power (i.e., energy per unit time) in one period of the periodic signal  $x(t)$ .

$$\frac{1}{T} \int_T |a_k e^{jk\omega_0 t}|^2 dt = \frac{1}{T} \int_T |a_k|^2 dt = |a_k|^2. \quad (9)$$

So,  $|a_k|^2$  is the average power in the  $k$ th harmonic component of  $x(k)$ .

Thus, what Parseval's relation states is that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

## Example

Consider the signal  $g(t)$  with a fundamental period of 4, shown in Figure 5.

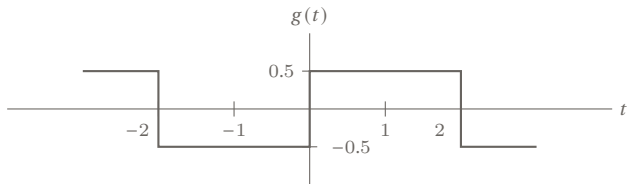


Figure: Figure for example

Determine the Fourier series representation of  $g(t)$

1. directly from the analysis equation.
2. by assuming that the Fourier series coefficients of the symmetric periodic square wave are known.

## Solution: Direct

$$a_0 = 0, \quad a_k = \frac{1}{2\pi j k} (1 - \cos k\pi)$$

$$a_1 = -j/\pi, a_2 = 0, a_3 = -j/(3\pi), a_4 = 0, a_5 = -j/(5\pi), \dots$$



We notice that

$$g(t) = x(t - 1) - 1/2,$$

with  $T = 4$  and  $T_1 = 1$ .

Applying the linearity property, the FS coefficients of  $g(t)$  may be expressed as

We notice that

$$g(t) = x(t - 1) - 1/2,$$

with  $T = 4$  and  $T_1 = 1$ . If FS coefficients of  $x(t)$  are denoted by  $a_k$ , the FS coefficients of  $x(t - 1)$  may be expressed as

$$b_k = a_k e^{-jk\pi/2},$$

The FS coefficients of the dc offset  $-1/2$

$$c_k = \begin{cases} 0, & \text{for } k \neq 0, \\ -\frac{1}{2} & \text{for } k = 0. \end{cases}$$

We notice that

$$g(t) = x(t - 1) - 1/2,$$

with  $T = 4$  and  $T_1 = 1$ . If FS coefficients of  $x(t)$  are denoted by  $a_k$ , the FS coefficients of  $x(t - 1)$  may be expressed as

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Applying the linearity property, the FS coefficients of  $g(t)$  may be expressed as

$$d_k = \begin{cases} a_k e^{-jk\pi/2}, & \text{for } k \neq 0, \\ a_0 - \frac{1}{2} & \text{for } k = 0. \end{cases}$$

yielding

$$d_k = \begin{cases} \frac{\sin(\pi k/2)}{k\pi} e^{-jk\pi/2}, & \text{for } k \neq 0, \\ 0 & \text{for } k = 0. \end{cases}$$

## Example

Consider the triangular wave signal  $x(t)$  with period  $T = 4$  and fundamental frequency  $\omega_0 = \pi/2$ , shown in Figure 6. The derivative signal is the signal  $g(t)$  in Figure 5. Using this information, find the Fourier series coefficients of  $x(t)$ .

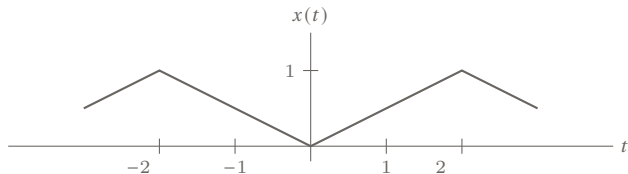


Figure: Figure for example

The derivative of this signal is the signal  $g(t)$  in the previous example. Denoting the Fourier coefficients of  $g(t)$  by  $d_k$  and those of  $x(t)$  by  $e_k$  we see that the **differentiation property** indicates that

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$$e_0 = \frac{1}{2}.$$



## Example

Obtain the Fourier series coefficients of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT). \quad (10)$$

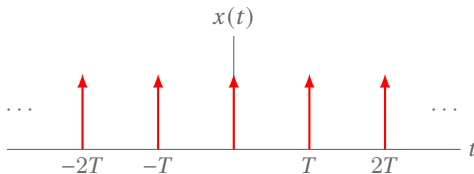


Figure: Impulse train

To determine the Fourier series coefficients  $a_k$ , we select the interval of integration to be  $-T/2 \leq t \leq T/2$ . Within this interval,  $x(t)$  is the same as  $\delta(t)$ .

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In other words, all the Fourier series coefficients of the impulse train are identical. These coefficients are also real valued and even (with respect to the index  $k$ ).

## Example

By expressing the derivative of a square wave signal in terms of impulses, obtain the Fourier series coefficients of the square wave signal.

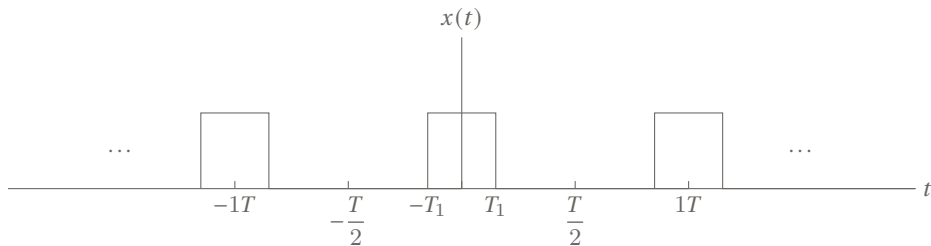


Figure: Figure for example

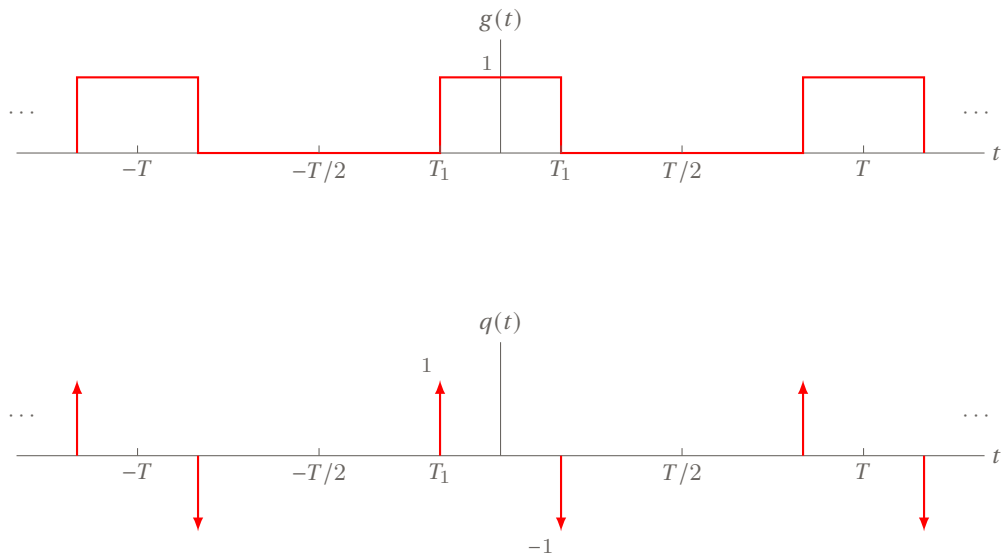


Figure: Periodic square wave and its derivative.

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$$q(t) = x(t + T_1) - x(t - T_1).$$

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$$\begin{aligned} b_k &= e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k, \quad \omega_0 = 2\pi/T, \\ &= \frac{1}{T} [e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}] = \frac{2j \sin(k\omega_0 T_1)}{T}. \end{aligned}$$



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Since  $q(t)$  is the derivative of  $g(t)$ , we can use the differentiation property:

$$b_k = jk\omega_0 c_k,$$

where the  $c_k$  are the Fourier series coefficients of  $g(t)$ . Thus,

$$c_k = \frac{b_k}{jk\omega_0} = \frac{2j \sin(k\omega_0 T_1)}{jk\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \quad k \neq 0$$

Since  $c_0$  is just the average value of  $g(t)$  over one period,

$$c_0 = \frac{2T_1}{T}.$$

## Example

For the waveform  $x(t)$ ,

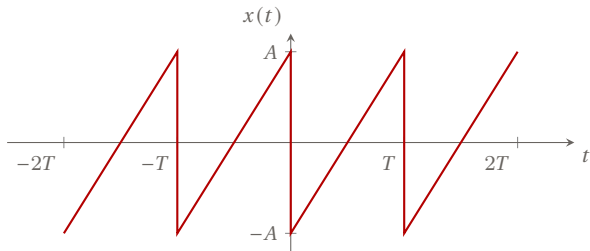
1. Obtain expression for the exponential Fourier series coefficients  $a_k$ .

2. Compute the average power

$$\frac{1}{T} \int_T |x(t)|^2 dt.$$

3. Verify Parseval's relation.

Given: Sum of the reciprocals of the positive square integers is  $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ .



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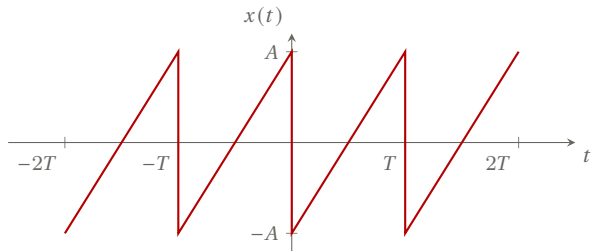
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$$x(t) = A \left( \frac{2t}{T} - 1 \right), \quad 0 < t \leq T.$$

Example: Computing  $a_k$

$$\begin{aligned} a_0 &= \frac{1}{T} \int_T x(t) dt \\ &= \frac{A}{T} \int_0^T \left( \frac{2t}{T} - 1 \right) dt \\ &= \left[ \frac{2t^2}{2T} - t \right]_0^T \\ &= 0 \end{aligned}$$

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$$a_k = \begin{cases} 0, & k = 0, \\ \frac{Aj}{\pi k}, & k \neq 0. \end{cases}$$

## Example: Computing the Average Power

$$\begin{aligned}\frac{1}{T} \int_T |x(t)|^2 dt &= \frac{A^2}{T} \int_0^T \left( \frac{2t}{T} - 1 \right)^2 dt \\&= \frac{A^2}{T} \int_0^T \left[ \frac{4t^2}{T^2} - 4\frac{t}{T} + 1 \right] \\&= \frac{A^2}{T} \int_0^T \left[ \frac{4t^3}{3T^2} - 4\frac{t^2}{2T} + t \right] \\&= \frac{A^2}{T} \left[ \frac{4}{3T} - 2T + T \right]_0^T \\&= \frac{A^2}{3}\end{aligned}$$



## Example: Verifying Parseval's relation

$$\begin{aligned}\sum_{k=-\infty}^{\infty} |a_k|^2 &= \sum_{k \neq 0} \left| \frac{A j}{\pi k} \right|^2 \\ &= 2 \frac{A^2}{\pi^2} \sum_{k \neq 1}^{\infty} \frac{1}{k^2} \\ &= 2 \frac{A^2}{\pi^2} \frac{\pi^2}{6} \\ &= \frac{A^2}{3}\end{aligned}$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

# Other Forms of Fourier Series

Complex Exponential Fourier Series

$$\begin{aligned}x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\a_k &= \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt\end{aligned}\quad (11)$$

Trigonometric Fourier Series

$$\begin{aligned}x(t) &= A_0 + 2 \sum_{k=1}^{+\infty} A_k \cos k\omega_0 t + B_k \sin k\omega_0 t \\A_k &= \frac{1}{T} \int_T x(t) \cos k\omega_0 t dt \\B_k &= \frac{1}{T} \int_T x(t) \sin k\omega_0 t dt\end{aligned}\quad (12)$$

Harmonic Form Fourier Series (for Real  $x(t)$ )

$$\begin{aligned}x(t) &= C_0 + 2 \sum_{k=1}^{+\infty} C_k \cos(k\omega_0 t - \theta_k) \\C_0 &= A_0 \\C_k &= \sqrt{A_k^2 + B_k^2} \quad \theta_k = \tan^{-1} \left( \frac{B_k}{A_k} \right)\end{aligned}\quad (13)$$

Relationship

$$\begin{aligned}A_0 &= a_0 \\A_k &= \frac{a_k + a_{-k}}{2} \\B_k &= j \frac{a_k - a_{-k}}{2} \\\omega_0 &= \frac{2\pi}{T}\end{aligned}\quad (14)$$

# Outline

## Continuous-Time Fourier Series

Introduction

Fourier Series

Properties of the Continuous-Time Fourier Series

Convergence of Fourier Series

# Convergence of Fourier Series

FS synthesis and analysis equations:

Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

Consider the **finite** series of the form

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

Let  $e_N(t)$  denote the approximation error, that is,

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

A quantitative measure of approximation error is

$$E_N = \int_T |e_N(t)|^2 dt$$

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- Fortunately, there are no convergence difficulties for large classes of periodic signals, continuous and discontinuous.

# Finite-Energy Convergence Criterion

One class of periodic signals that are representable through the Fourier series is those signals which have finite energy over a single period:

$$\int_T |x(t)|^2 dt < \infty \quad (15)$$

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- However, since physical systems respond to signal energy, from this perspective  $x(t)$  and its Fourier series representation are indistinguishable.

# Alternative Conditions (Dirichlet Conditions)

Dirichlet conditions guarantee that  $x(t)$  equals its Fourier series representation, except at isolated values of  $t$  for which  $x(t)$  is discontinuous. At these values, the infinite series converges to the average of the values on either side of the discontinuity.

## Condition 1

Over any period,  $x(t)$  must be absolutely integrable

$$\int_T |x(t)| dt < \infty. \quad (16)$$

This guarantees that  $a_k$ s are finite.

## Condition 2

In any finite interval of time,  $x(t)$  is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

## Condition 3

In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

# Examples of Functions that Violate Dirichlet Conditions

Cond. 1 The periodic signal with period 1 with one period defined as

$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1.$$

Cond. 2 The periodic signal with period 1 with one period defined as

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1.$$

For this

$$\int_0^1 |x(t)| dt < 1$$

The function has, however, an infinite number of maxima and minima in the interval.

Cond. 3 The signal, of period  $T = 8$ , is composed of an infinite number of sections, each of which is half the height and half the width of the previous section. Thus, the area under one period of the function is clearly less than 8. However, there are an infinite number of discontinuities in each period, thereby violating Condition 3.