

# EN1020 Circuits, Signals, and Systems: Introduction to Fourier Transform

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February 24, 2023



## Section 1

# Continuous-Time Fourier Transform

## Subsection 1

### Introduction

# Introduction

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.
- We use Fourier series for periodic signals, and Fourier transform for aperiodic signals.
- Each of these have continuous-time and discrete-time versions:
  1. Continuous-time Fourier series
  2. Continuous-time Fourier transform
  3. Discrete-time Fourier series
  4. Discrete-time Fourier transform
- In this part of the course, we will concentrate on how to actually compute continuous-time Fourier series and transform. Later, after we study linear, time-invariant (LTI) systems, we will study the conceptual aspects of Fourier techniques.
- In EN2063, we will do a more rigorous study of Fourier techniques.

# Fourier Transform

- In the last lecture, we represented a periodic signal as a linear combination of complex exponentials.
- We use Fourier transform to represent aperiodic signals. A larger class of signals, including all signals with finite energy, can be represented through a linear combination of complex exponentials.
- Whereas for periodic signals the complex exponential building blocks are harmonically related, for aperiodic signals they are **infinitesimally close in frequency**, and the representation in terms of a linear combination takes the form of an integral rather than a sum.
- The resulting spectrum of coefficients in this representation is called the **Fourier transform**.
- The synthesis integral itself, which uses the Fourier transform to represent the signal as a linear combination (integral) of complex exponentials, is called the **inverse Fourier transform**.

## Subsection 2

### Development of the Fourier Transform Representation

# Fourier Series Representation for Square Wave

The continuous-time periodic square wave, sketched below, is defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < T/2, \end{cases}$$

This signal is periodically repeats with fundamental period  $T$  and fundamental frequency  $\omega_0 = 2\pi/T$ .

The Fourier series coefficients  $a_k$  of this wave are

$$a_k = \frac{2 \sin(k \omega_0 T_1)}{k \omega_0 T} \quad (1)$$

We plotted this for a fixed value of  $T_1$  and several values of  $T$  (shown in the next slide). An alternative way of interpreting Eq. ?? is as samples of an envelope function:

$$T a_k = \left. \frac{2 \sin(\omega T_1)}{\omega} \right|_{\omega = k \omega_0}$$

With  $\omega$  thought of as a continuous variable, the function  $\frac{2 \sin(\omega T_1)}{\omega}$  represents the envelope of  $T a_k$ , and the coefficients  $a_k$  are simple equally spaced samples of this envelope. For fixed  $T_1$ , the envelope of  $T a_k$  is independent of  $T$ .



Plots of scaled  
Fourier series  
coefficients  $a_k$  for  
the periodic square  
wave with  $T_1$  fixed  
and for several  
values of  $T$ :  $T = 4T_1$ ,  
 $T = 8T_1$ ,  $T = 16T_1$ .

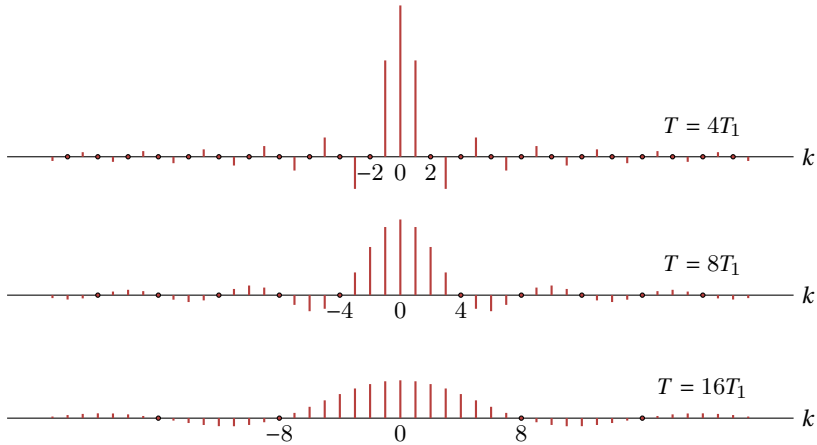


Figure: Plots of scaled Fourier series coefficients  $a_k$

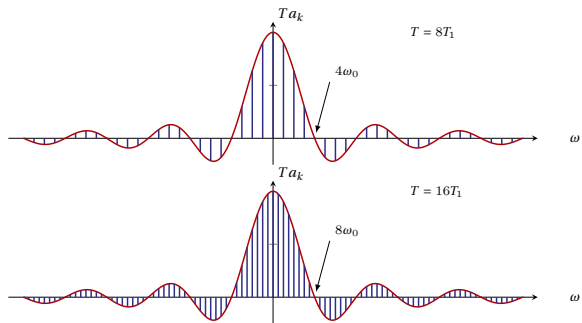
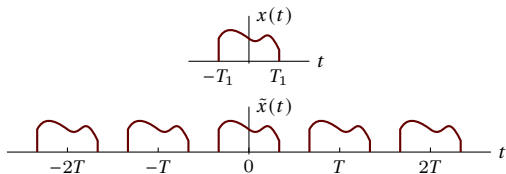


Figure: Fourier series coefficients and their envelope for periodic square wave.

The Fourier series coefficients and their envelope for periodic square wave for several values of  $T$  (with  $T_1$  fixed):  $T = 4T_1$ ,  $T = 8T_1$ ,  $T = 16T_1$ . The coefficients are regularly-spaced samples of the envelope  $(2 \sin \omega T_1) / \omega$ , where the spacing between samples,  $2\pi/T$ , decreases as  $T$  increases.

As  $T$  increases, or equivalently, as the fundamental frequency  $\omega_0 = 2\pi/T$  decreases, the envelope is sampled with a close and closer spacing. As  $T$  becomes arbitrarily large, the original periodic square waveform approaches the rectangular pulse. Also, the Fourier series coefficients, multiplied by  $T$ , become more and more closely spaced sampled of the envelope. So, in some sense, the set of Fourier series coefficients approaches the envelope function as  $T \rightarrow \infty$ .



$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \quad \omega_0 = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt.$$

As  $\tilde{x}(t) = x(t)$  for  $|t| < T/2$ , and also, as  $x(t) = 0$  outside this interval,

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t} dt = \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt.$$

Defining the envelope  $X(j\omega)$  of  $Ta_k$  as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt,$$

we have, for the coefficients  $a_k$ ,

$$a_k = \frac{1}{T} X(jk\omega_0).$$

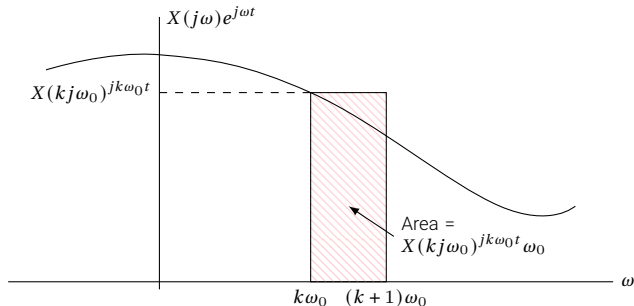
Combining and expressing  $\tilde{x}(t)$  in terms of  $X(j\omega)$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t},$$

or, as  $\omega_0 = 2\pi/T$

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0. \quad (2)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt.$$



As  $T \rightarrow \infty$ ,  $\tilde{x}(t)$  approaches  $x(t)$ , and consequently, Eq. ?? becomes a representation of  $x(t)$ . Furthermore, as  $\omega_0 \rightarrow 0$  as  $T \rightarrow \infty$ , and the right-hand side of Eq. ?? passes to an integral. As  $\omega_0 \rightarrow 0$ , the summation converges to the integral of  $X(j\omega)e^{j\omega t}$ .

# Fourier Transform: Synthesis and Analy

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$
$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

## Relation with $a_k$

Assume that the Fourier transform of  $x(t)$  is  $X(j\omega)$ .

If we construct a periodic signal  $\tilde{x}(t)$  by repeating the aperiodic signal  $x(t)$ , its Fourier series coefficients are

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

FT synthesis and analysis equations:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

# Convergence of Fourier Transform

Assume that we evaluated  $X(j\omega)$  according to eq. ??, and let  $\hat{x}(t)$  denote the signal obtained by using  $X(j\omega)$  in ??:

$$\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega.$$

When is  $\hat{x}(t)$  a valid representation of the original signal  $x(t)$ ? We define the error between  $\hat{x}(t)$  and  $x(t)$  as

$$e(t) = \hat{x}(t) - x(t).$$

If  $x(t)$  has finite energy (square integrable), i.e.,

(3)

$X(j\omega)$  is finite, and

(4)

If  $x(t)$  has finite energy, then, although  $x(t)$  and its Fourier representation  $\hat{x}(t)$  may differ significantly at individual values of  $t$ , there is no energy in their difference.



# Convergence of Fourier Transform: Dirichlet Conditions

There are alternative conditions sufficient to ensure that  $\hat{x}(t)$  is equal to  $x(t)$  for any  $t$  except at a discontinuity, where it is equal to the average of the values on either side of the discontinuity.

1.  $x(t)$  is absolutely integrable, i.e.,

(5)

2.  $x(t)$  has a finite number of maxima and minima within any finite interval.
3.  $x(t)$  has a finite number of discontinuities within any finite interval. Furthermore, each of these discontinuities must be finite.

Therefore, absolutely integrable signals that are continuous or that have finite number of discontinuities have a Fourier transform.

# Example

## Example

Find the Fourier transform of the signal

$$x(t) = e^{-at}u(t), \quad a > 0.$$

Example Cntd. FT of  $e^{-at}u(t)$ ,  $a > 0$

## Example

Find the Fourier transform of the signal

$$x(t) = e^{-a|t|}, \quad a > 0.$$



## Example

Determine the Fourier transform of the unit impulse

$$x(t) = \delta(t).$$

# Rectangular Pulse

## Example

Determine the Fourier transform of the signal

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & |t| > T_1. \end{cases}$$





## Example

Consider the signal  $x(t)$  whose Fourier transform is

$$X(j\omega) = \begin{cases} 1, & |\omega| < W, \\ 0, & |\omega| > W. \end{cases}$$

Determine  $x(t)$ .

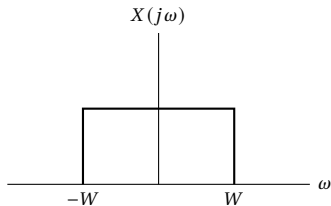


Figure: Fourier transform for  $x(t)$ .



# The sinc Function

$$\text{sinc}(\theta) = \frac{\sin \pi \theta}{\pi \theta}. \quad (6)$$

Express

$$\frac{2 \sin \omega T_1}{\omega}$$

and

$$\frac{\sin W t}{\pi t}$$

as sinc functions.

What Happens when  $W$  Increases?



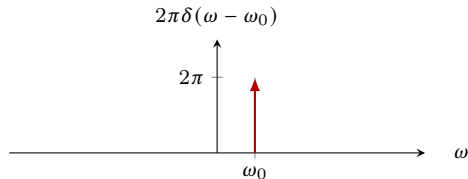
## Subsection 3

### The Fourier Transform for Periodic Signals

# The Fourier Transform for Periodic Signals: Introduction

In the previous section, we studied the Fourier transform representation, paying attention to aperiodic signals. We can also develop Fourier transform representations for periodic signals. This allows us to consider periodic and aperiodic signals in a unified context. We can construct the Fourier transform of a periodic signal directly from its Fourier series representation.

Consider a signal  $x(t)$  with the Fourier transform  $X(j\omega)$  that is a single impulse of area  $2\pi$  at  $\omega = \omega_0$ , i.e.,





Let's determine the signal  $x(t)$ :

## Example

Find the Fourier transform of the square wave signal whose Fourier series coefficients are

$$a_k = \frac{\sin k\omega_0 T_1}{\pi k}.$$

Method: Multiply the Fourier series coefficients  $a_k$  by  $2\pi$ , place them using the impulse function  $\delta(\omega - k\omega_0)$ , and sum.

## Example

Find the Fourier transform of

$$x(t) = \sin \omega_0 t,$$

and

$$x(t) = \cos \omega_0 t.$$



## Example

Find the Fourier transform of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$



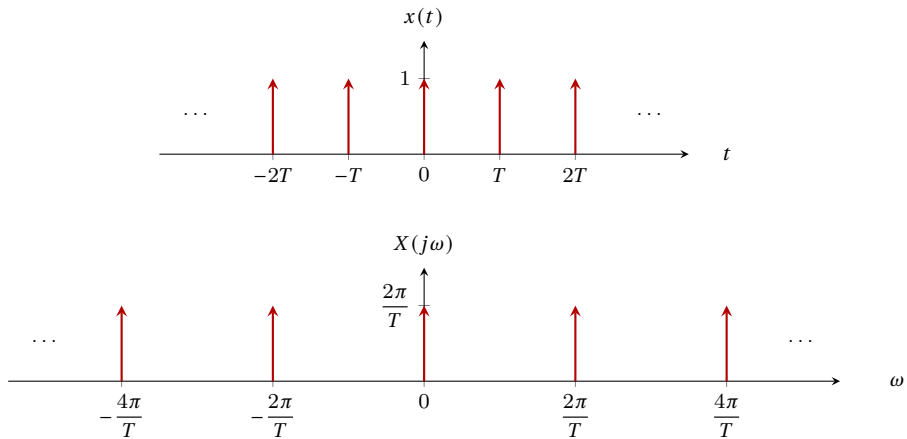


Figure: Periodic impulse train and its Fourier transform.