EN1060 Signals and Systems: Introduction

Ranga Rodrigo
ranga@uom.lk

The University of Moratuwa, Sri Lanka

June 6, 2022

Section 1

Continuous-Time Fourier Series

Introduction

Introduction

- Using the Fourier techniques we can obtain the frequency-domain representation of signals.
- We use Fourier series for periodic signals, and Fourier transform for aperiodic signals.
- Each of these have continuous-time and discrete-time versions:
    1. Continuous-time Fourier series
    2. Continuous-time Fourier transform
    3. Discrete-time Fourier series
    4. Discrete-time Fourier transform
- In this part of the course, we will concentrate on how to actually compute continuous-time Fourier series and transform. Later, after we study liner, time-invariant (LTI) systems, we will study the conceptual aspects of Fourier techniques.

Figure: Jean-Baptiste Joseph Fourier, 1768–1830, French mathematician who discovered Fourier series and transform

- Every signal has a frequency distribution or a spectrum.
- Periodic signals have a line spectra, called the Fourier series.
- The French mathematician, Jean-Baptiste Joseph Fourier, discovered this representation.
- Fourier series provides a way to represent a periodic signal as a sum of complex exponentials.
- These sinusoids will be at frequencies that are integer multiples of the fundamental frequency ω_0.
- $\omega_0 = \dfrac{2\pi}{T}$, where T: fundamental period of the waveform.

Subsection 2

Fourier Series

Continuous-Time Fourier Series

Example

Let

$$x(t) = \sin \omega_0 t,$$

which has the fundamental frequency ω_0.

Euler's formula

$e^{j\theta} = \cos\theta + j\sin\theta$

$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

Example

Let

$$x(t) = 1 + \sin \omega_0 t + 2 \cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

which has the fundamental frequency ω_0.

1. Use Euler's formula to express $x(t)$ as a liner combination of complex exponentials.
2. Find the Fourier series coefficients, a_k.
3. Plot the magnitude and phase of a_k.

Example

The periodic square wave, sketched below, is defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < T/2, \end{cases}$$

This signal is periodic with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$.

1. Find the Fourier series coefficients, a_k.
2. Plot the magnitude and phase of a_k for the case $T = 4T_1$.

Subsection 3

Properties of the Continuous-Time Fourier Series

Suppose that $x(t)$ is a periodic signal with period T and fundamental frequency $\omega_0 = 2\pi/T$. Then if the Fourier series coefficients are denoted by a_k, then

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \tag{1}$$

Linearity

Let $x(t)$ and $y(t)$ denote two periodic signals with period T.

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k,$$
$$y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k.$$

Any linear combination of the two signals will also be periodic with period T. Fourier series coefficients c_k of the linear combination of $x(t)$ and $y(t)$, $z(t) = Ax(t) + By(t)$, are given by the same linear combination:

Time Shifting

$$x(t - t_0) \xleftrightarrow{\mathcal{FS}} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k \tag{2}$$

Time Reversal

If

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

then

$$x(-t) \xleftrightarrow{\mathcal{FS}} a_{-k}.$$

Time Scaling

Time scaling, in general, changes the period.
If $x(t)$ is a periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$, then $x(\alpha t)$, where α is a positive real number, is periodic with period T/α and fundamental frequency $\alpha\omega_0$.

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t} \tag{3}$$

Multiplication

Conjugation and Conjugate Symmetry

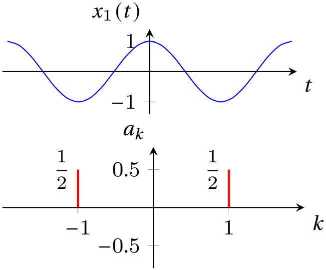
- Taking the complex conjugate of a periodic signal $x(t)$ has the effect of complex conjugation and time reversal on the corresponding Fourier series coefficients.
- If $x(t)$ is real, i.e., $x(t) = x^*(t)$: Fourier series coefficients conjugate symmetric, i.e., $a_{-k} = a_k^*$.
- If $x(t)$ is real, then a_0 is real and $|a_k| = |a_{-k}|$.
- If $x(t)$ is real and even, we know that $a_k = a_{-k}$. From above, $a_k^* = a_{-k}$, so that $a_k = a_k^*$. That is if $x(t)$ is real and even, so are its Fourier series coefficients.
- If $x(t)$ is real and odd, its Fourier series coefficients are purely imaginary and odd. Thus, e.g., $a_0 = 0$.

Example

Consider

$$x_1(t) = \cos(\omega_0 t)$$

This is real and even.

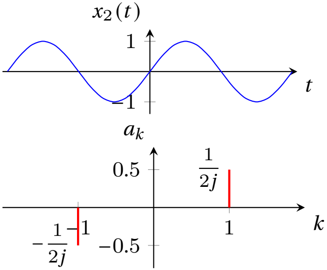


FS coefficients are real and even. (They are conjugate symmetric too.)

Consider

$$x_2(t) = \sin(\omega_0 t)$$

This is real and odd.



FS coefficients are imaginary and odd. (They are conjugate symmetric too.)

Parseval's Relation for Continuous-Time Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2. \tag{4}$$

Example

Consider the signal $g(t)$ with a fundamental period of 4, shown in Figure 4.

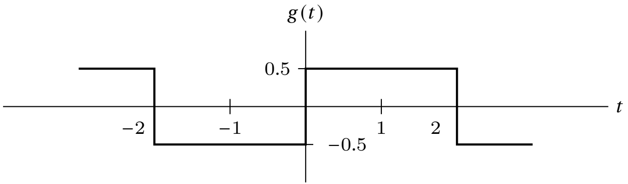


Figure: Figure for example

Determine the Fourier series representation of $g(t)$

1. directly from the analysis equation.
2. by assuming that the Fourier series coefficients of the symmetric periodic square wave are known.

Solution: Direct

Example

Consider the triangular wave signal $x(t)$ with period $T = 4$ and fundamental frequency $\omega_0 = \pi/2$, shown in Figure 5. The derivative signal is the signal $g(t)$ in Figure 4. Using this information, find the Fourier series coefficients of $x(t)$.

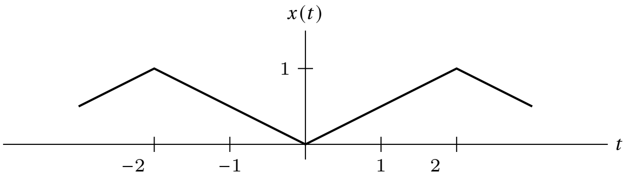


Figure: Figure for example

Example

Obtain the Fourier series coefficients of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{5}$$

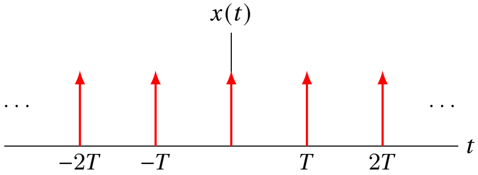


Figure: Impulse train

Example

By expressing the derivative of a square wave signal in terms of impulses, obtain the Fourier series coefficients of the square wave signal.

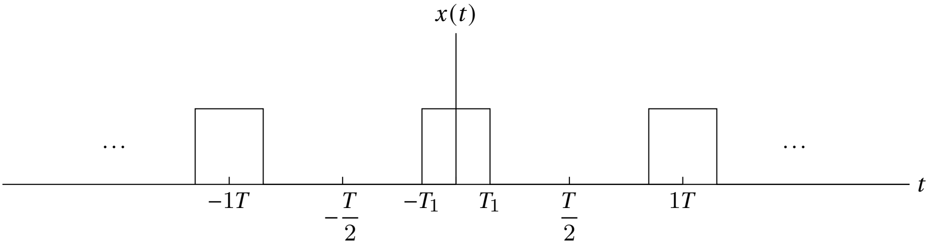


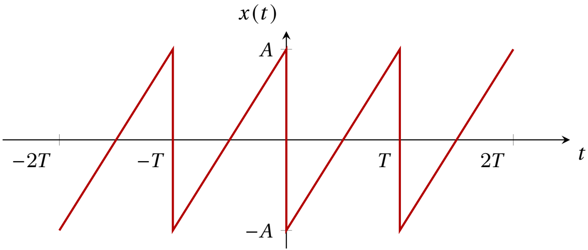
Figure: Figure for example

Example

For the waveform $x(t)$,

1. Obtain expression for the exponential Fourier series coefficients a_k.
2. Compute the average power
$$\frac{1}{T}\int_T |x(t)|^2 dt.$$
3. Verify Parseval's relation.

Given: Sum of the reciprocals of the positive square integers is $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.



$$x(t) = A\left(\frac{2t}{T} - 1\right), \quad 0 < t <\leq T.$$

Example: Computing a_k

$$\boxed{\omega_0 T = 2\pi}$$

$$
\begin{aligned}
a_0 &= \frac{1}{T}\int_T x(t)dt \\
&= \frac{A}{T}\int_0^T \left(\frac{2t}{T} - 1\right) dt \\
&= \left[\frac{2t^2}{2T} - t\right]_0^T \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \\
&= \frac{A}{T}\int_0^T \overbrace{\left(\frac{2t}{T} - 1\right)}^{u} \overbrace{e^{-jk\omega_0 t}dt}^{dv} \\
&= \frac{A}{T}\left\{\left[\left(\frac{2t}{T} - 1\right)\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\right]_0^T - \left[\frac{e^{-jk\omega_0 t}}{-jk\omega_0}\frac{2}{T}\right]_0^T\right\}
\end{aligned}
$$

$$
\begin{aligned}
a_k &= \frac{A}{T}\left\{\left[(2-1)\frac{e^{-jk\omega_0 T}}{-jk\omega_0} - \frac{(-1)}{-jk\omega_0}\right]\right. \\
&\quad \left. - \frac{2}{-jk\omega_0 T}\left[e^{-jk\omega_0 T} - 1\right]\right\} \\
&= \frac{A}{T}\left\{\frac{-2}{jk\omega_0}\right\} \\
&= \frac{Aj}{\pi k}.
\end{aligned}
$$

$$
a_k = \begin{cases} 0, & k = 0, \\ \dfrac{Aj}{\pi k}, & k \neq 0. \end{cases}
$$

Example: Computing the Average Power

$$
\begin{aligned}
\frac{1}{T}\int_{T}|x(t)|^2 dt &= \frac{A^2}{T}\int_0^T \left(\frac{2t}{T}-1\right)^2 dt \\
&= \frac{A^2}{T}\int_0^T \left[\frac{4t^2}{T^2}-4\frac{t}{T}+1\right] \\
&= \frac{A^2}{T}\int_0^T \left[\frac{4t^3}{3T^2}-4\frac{t^2}{2T}+t\right] \\
&= \frac{A^2}{T}\left[\frac{4}{3T}-2T+T\right]_0^T \\
&= \frac{A^2}{3}
\end{aligned}
$$

Example: Verifying Parseval's relation

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = \sum_{k\neq 0} \left| \frac{Aj}{\pi k} \right|^2$$

$$= 2\frac{A^2}{\pi^2} \sum_{k\neq 1}^{\infty} \frac{1}{k^2}$$

$$= 2\frac{A^2}{\pi^2} \frac{\pi^2}{6}$$

$$= \frac{A^2}{3}$$

$$\boxed{\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}}$$

Other Forms of Fourier Series

Complex Exponential Fourier Series

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \\
a_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt
\end{aligned}
\tag{6}
$$

Trigonometric Fourier Series

$$
\begin{aligned}
x(t) &= A_0 + 2\sum_{k=1}^{+\infty} A_k \cos k\omega_0 t + B_k \sin k\omega_0 t \\
A_k &= \frac{1}{T}\int_T x(t)A_k \cos k\omega_0 t dt \\
B_k &= \frac{1}{T}\int_T x(t)A_k \sin k\omega_0 t dt
\end{aligned}
\tag{7}
$$

Harmonic Form Fourier Series (for Real $x(t)$)

$$
\begin{aligned}
x(t) &= C_0 + 2\sum_{k=1}^{+\infty} C_k \cos(k\omega_0 t - \theta_k) \\
C_0 &= A_0 \\
C_k &= \sqrt{A_k^2 + B_k^2} \quad \theta_k = \tan^{-1}\left(\frac{B_k}{A_k}\right)
\end{aligned}
\tag{8}
$$

Relationship

$$
\begin{aligned}
A_0 &= a_0 \\
A_k &= \frac{a_k + a_{-k}}{2} \\
B_k &= j\frac{a_k - a_{-k}}{2} \\
\omega_0 &= \frac{2\pi}{T}
\end{aligned}
\tag{9}
$$

Subsection 4

Convergence of Fourier Series

Convergence of Fourier Series

FS synthesis and analysis equations:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

Fourier series representation:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Consider the finite series of the form

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

Let $e_N(t)$ denote the approximation error, that is,

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$$

A quantitative measure of approximation error is

$$E_N = \int_T |e_N(t)|^2\, dt$$

Convergence of Fourier Series

- If $x(t)$ has a Fourier series representation, then the limit of E_N as $N \to \infty$ is zero.
- If $x(t)$ does not have a Fourier series representation, then the integral that computes a_k may diverge. Moreover, even if all of the coefficients a_k obtained are finite, when these coefficients are substituted into the synthesis equation, the resulting infinite series may not converge to the original signal $x(t)$.
- Fortunately, there are no convergence difficulties for large classes of periodic signals, continuous and discontinuous.

Finite-Energy Convergence Criterion

One class of periodic signals that are representable through the Fourier series is those signals which have finite energy over a single period:

$$\int_T |x(t)|^2 \, dt < \infty \tag{10}$$

- In this case coefficients a_k are finite.
- As $N \to \infty$, $E_N \to 0$.
- This does not imply that the signal $x(t)$ and its Fourier series representation are equal at every value of t. What it does say is that there is no energy in their difference.
- However, since physical systems respond to signal energy, from this perspective $x(t)$ and its Fourier series representation are indistinguishable.

Alternative Conditions (Dirichlet Conditions)

Dirichlet conditions guarantee that $x(t)$ equals its Fourier series representation, except at isolated values of t for which $x(t)$ is discontinuous. At these values, the infinite series converges to the average of the values on either side of the discontinuity.

Condition 1
Over any period, $x(t)$ must be absolutely integrable

$$\int_T |x(t)|\, dt < \infty. \tag{11}$$

This guarantees that a_ks are finite.

Condition 2
In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

Condition 3
In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

Examples of Functions that Violate Dirichlet Conditions

Cond. 1 The periodic signal with period 1 with one period defined as

$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1.$$

Cond. 2 The periodic signal with period 1 with one period defined as

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \quad 0 < t \leq 1.$$

For this

$$\int_0^1 |x(t)|\, dt < 1$$

The function has, however, an infinite number of maxima and minima in the interval.

Cond. 3 The signal, of period T = 8, is composed of an infinite number of sections, each of which is half the height and half the width of the previous section. Thus, the area under one period of the function is clearly less than 8. However, there are an infinite number of discontinuities in each period, thereby violating Condition 3.