Equations

Fourier

	CT	DT
	Series (CT)	Series (DT)
Periodic	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$	$x[n] = \sum_{k=<N>} a_k e^{jk(\frac{2\pi}{N})n}$ $a_k = \frac{1}{N}\sum_{n=<N>} x[n] e^{-jk(\frac{2\pi}{N})n}$
	Transform (CT)	Transform (DT)
Aperiodic	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$ $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$ $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$

Trigonometric Fourier Series

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t).$$
$$a_0 = \frac{1}{T}\int_T x(t) dt.$$
$$a_k = \frac{2}{T}\int_T x(t) \cos(k\omega_0 t) dt.$$
$$b_k = \frac{2}{T}\int_T x(t) \sin(k\omega_0 t) dt.$$

Table 1: Properties of Continuous Time Fourier Series

Property	Periodic signal	Fourier series coefficients
	$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time shifting	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency shifting	$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	$x^*(t)$	a^*_{-k}
Time reversal	$x(-t)$	a_{-k}
Time scaling	$x(\alpha t)$, $\alpha > 0$ (periodic with period T/α)	a_k
Periodic convolution	$\int_T x(\tau) y(t-\tau) d\tau$	$T a_k b_k$
Multiplication	$x(t) y(t)$	$\sum_{l=-\infty}^{\infty} a_l b_{k-l}$
Differentiation	$\frac{d}{dt} x(t)$	$jk\omega_0 a_k$
Integration	$\int_{-\infty}^{t} x(\tau) d\tau$ finite valued and periodic only if $a_0 = 0$	$\frac{1}{jk\omega_0} a_k$
Conjugate symmetry for real signals	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathfrak{Re}\{a_k\} = \mathfrak{Re}\{a_{-k}\} \\ \mathfrak{Im}\{a_k\} = -\mathfrak{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and even signals	$x(t)$ real and even	a_k real and even
Real and odd signals	$x(t)$ real and odd	a_k purely imaginary and odd
Even-odd decomposition for real signals	$x_e(t) = \mathfrak{Ev}\{x(t)\}$, $[x(t)$ real$]$ $x_o(t) = \mathfrak{Od}\{x(t)\}$, $[x(t)$ real$]$	$\mathfrak{Re}\{a_k\}$ $j\mathfrak{Im}\{a_k\}$

Parseval's relation for periodic signals $\quad \frac{1}{T}\int_T \lvert x(t) \rvert^2 dt = \sum_{k=-\infty}^{\infty} \lvert a_k \rvert^2$

Table 2: Basic Fourier Transform Pairs

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, a_k = 0, k \neq 0$ (This is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi}\mathrm{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{\infty} \delta(t-nT)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2\sin\omega T_1}{\omega}$	–
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	
$\delta(t)$	1	–
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	–
$\delta(t-t_0)$	$e^{-j\omega t_0}$	–
$e^{-at}u(t)$, $\Re\mathfrak{e}\{a\} > 0$	$\frac{1}{a+j\omega}$	–
$te^{-at}u(t)$, $\Re\mathfrak{e}\{a\} > 0$	$\frac{1}{(a+j\omega)^2}$	–
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$, $\Re\mathfrak{e}\{a\} > 0$	$\frac{1}{(a+j\omega)^n}$	–

Table 3: Properties of the Fourier Transform

Property	Aperiodic signal	Fourier transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
Time shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time reversal	$x(-t)$	$X(-j\omega)$
Time and frequency scaling	$x(at)$	$\frac{1}{\|a\|}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}X(j\omega)*Y(j\omega)$
Differentiation in time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
Differentiation in frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate symmetry for real signals	$x(t)$ real	$\begin{cases} X(j\omega)=X^*(-j\omega) \\ \mathfrak{Re}\{X(j\omega)\}=\mathfrak{Re}\{X(-j\omega)\} \\ \mathfrak{Im}\{X(j\omega)\}=-\mathfrak{Im}\{X(-j\omega)\} \\ \|X(j\omega)\|=\|X(-j\omega)\| \\ \sphericalangle X(j\omega)=-\sphericalangle X(-j\omega) \end{cases}$
Symmetry for real and even signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for real and odd signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-odd decomposition for real signals	$x_e(t)=\mathfrak{Ev}\{x(t)\}, \quad [x(t)\text{ real}]$ $x_o(t)=\mathfrak{Od}\{x(t)\}, \quad [x(t)\text{ real}]$	$\mathfrak{Re}\{X(j\omega)\}$ $j\mathfrak{Im}\{X(j\omega)\}$

Parseval's relation for aperiodic signals $\int_{-\infty}^{\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|X(j\omega)|^2d\omega$

Table 4: Properties of the Discrete-Time Fourier Series

Property	Periodic signal	Fourier series coefficients
$\left.\begin{matrix} x[t] \\ y[t] \end{matrix}\right\}$	Periodic with period N fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time reversal	$x[-n]$	a_{-k}
Time scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$	$\frac{1}{m} a_k$ $\left(\begin{matrix}\text{viewed as periodic} \\ \text{with period } mN\end{matrix}\right)$
Periodic convolution	$\sum_{r=<N>} x[r] y[n-r]$	$N a_k b_k$
Multiplication	$x[n] y[n]$	$\sum_{l=<N>} a_l b_{k-l}$
First difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)}) a_k$
Running sum	$\sum_{k=-\infty}^{\infty} x[k]$ $\left(\begin{matrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{matrix}\right)$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate symmetry for real signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathfrak{Re}\{a_k\} = \mathfrak{Re}\{a_{-k}\} \\ \mathfrak{Im}\{a_k\} = -\mathfrak{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and even signals	$x[n]$ real and even	a_k real and even
Real and odd signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-odd decomposition of real signals	$x_e[n] = \mathfrak{Ev}\{x[n]\}, \quad [x[n] \text{ real}]$ $x_o[n] = \mathfrak{Od}\{x[n]\}, \quad [x[n] \text{ real}]$	$\mathfrak{Re}\{a_k\}$ $j\mathfrak{Im}\{a_k\}$

Parseval's relation for aperiodic signals

$$\frac{1}{N} \sum_{n=<N>} |x[n]|^2 = \sum_{k=<N>} |a_k|^2$$

Table 5: Basic Laplace Transform Pairs

$x(t)$	$X(s)$	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\text{Re}(s) > 0$
$-u(-t)$	$\frac{1}{s}$	$\text{Re}(s) < 0$
$tu(t)$	$\frac{1}{s^2}$	$\text{Re}(s) > 0$
$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\text{Re}(s) > 0$
$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\text{Re}(s) < 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\text{Re}(s) > -a$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\text{Re}(s) < -a$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\text{Re}(s) > -a$
$-te^{-at}u(-t)$	$\frac{1}{(s+a)^2}$	$\text{Re}(s) < -a$
$(\cos\omega_0 t)u(t)$	$\frac{s}{s^2+\omega^2}$	$\text{Re}(s) > 0$
$(\sin\omega_0 t)u(t)$	$\frac{\omega_0}{s^2+\omega^2}$	$\text{Re}(s) > 0$
$(e^{-at}\cos\omega_0 t)u(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\text{Re}(s) > -a$
$(e^{-at}\sin\omega_0 t)u(t)$	$\frac{\omega_0}{(s+a)^2+\omega^2}$	$\text{Re}(s) > -a$

Table 6: Properties of the Laplace Transform

Property	Signal	Laplace transform	ROC
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0} X(s)$	R
Shifting in s domain	$e^{s_0 t} x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in ROC if $s - s_0$ is in R)
Time scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in ROC if s/a is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the time domain	$\frac{dx(t)}{dt}$	$sX(s)$	At least R
Differentiation in the s-domain	$-tx(t)$	$\frac{dX(s)}{ds}$	R
Integration in the time domain	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s} X(s)$	At least $R \cap \{\mathfrak{Re}\{s\} > 0\}$

Initial- and final-value theorems: If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$x(0^+) = \lim_{s\to\infty} sX(s)$

$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

Table 7: z-Transform Pairs

Signal	Transfrom	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z, except 0 (if $m>0$) or ∞ (if $m<0$)
$\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
$n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
$(\cos\omega_0 n)u[n]$	$\frac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$(\sin\omega_0 n)u[n]$	$\frac{1-(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$(r^n\cos\omega_0 t)u[n]$	$\frac{1-(r\cos\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$
$(r^n\sin\omega_0 t)u[n]$	$\frac{1-(r\sin\omega_0)z^{-1}}{1-(2r\cos\omega_0)z^{-1}+r^2z^{-2}}$	$\lvert z\rvert > r$

Table 8: Properties of the z-Transform

Property	Aperiodic signal	z-transform	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	at least $R_1 \cap R_2$
Time shifting	$x[n-n_0]$	$z^{-n_0}X(z)$	R
Scaling in z domain	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$\lvert z_0\rvert R$
	$e^{j\omega n}x[n]$	$X(e^{-j\omega n}z)$	R
	$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\lvert a\rvert R$, the set of points $\{\lvert a\rvert z\}$ for z in R)
Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R).
Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r.	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	at least $R_1 \cap R_2$
Conjugation	$x^*[n]$	$X^*(z^*)$	R
First difference	$x[n] - x[n-1]$	$(1-z^{-1})X(z)$	At least the intersection of R and $\lvert z\rvert > 0$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1-z^{-1}}X(z)$	At least the intersection of R and $\lvert z\rvert > 1$
Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R
Initial value theorem:	If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z\to\infty} X(z)$		