EN1060 Signals and Systems: Laplace Transform

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Section 1

Laplace Transform

Introduction

- Using the Fourier transform, we represented a signal as a linear combination of basic signals using the eigenfunctions $e^{j\omega t}$.
- Then we could represent a given LTI system as a spectrum of eigenvalues as a function of ω, which is the change in amplitude that the system applies to each of the basic inputs $e^{j\omega t}$.
- Now we study a generalization of the Fourier transform, referred to as the Laplace transform.
- The Laplace transform converges for a broader class of signals than does the Fourier transform.

The Laplace Transform

- The general class of eigenfunctions for LTI systems consists of the complex exponential e^{st}, where s is a complex number. In particular, $s = \sigma + j\omega$.
- When s is purely imaginary, $s = j\omega$, the Laplace transform reduces to the Fourier transform.
- The Laplace transform is the Fourier transform of an exponentially weighted signal. Therefore, the Laplace transform can converge for signals for which the Fourier transform does not converge.
- The range of values of s for which the Laplace transform converges is the region of convergence (ROC).
- Two different signals can have Laplace transforms with identical algebraic expressions and differing only in the ROC.

Recall: Continuous-Time Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$
$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

LTI systems: impulse response $h(t)$:

$$\begin{array}{ccc} e^{j\omega t} & \longrightarrow & H(\omega) e^{j\omega t} \\ & & \updownarrow \mathcal{F} \\ & & h(t) \end{array}$$

Laplace Transform: Eigenfunction Property

$$e^{st} \longrightarrow \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau$$

$$e^{st} \longrightarrow e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

$$s = \sigma + j\omega$$

$$e^{st} \longrightarrow H(s)e^{st}$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau$$

Laplace Transform

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt$$

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

Laplace Transform and Fourier Transform Relationship

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}dt$$

$$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

$$s = \sigma + j\omega$$

$$X(s)|_{s=j\omega} = \mathcal{F}\{x(t)\}$$

New notation:

$$\mathcal{F}\{x(t)\} = X(j\omega)$$

Laplace Transform: Convergence Comparison

$$X(s)|_{s=j\omega} = X(j\omega)$$

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt$$

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-(\sigma+j\omega)t}\,dt$$

$$= \int_{-\infty}^{+\infty} x(t)e^{-\sigma t}e^{-j\omega t}\,dt$$

$$X(s) = \mathcal{F}\left\{x(t)e^{-\sigma t}\right\}$$

Laplace Transform: Convergence Comparison

$$
\begin{aligned}
X(s)\big|_{s=j\omega} &= X(j\omega) \\
X(s) &= \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt \\
X(\sigma + j\omega) &= \int_{-\infty}^{+\infty} x(t)e^{-(\sigma+j\omega)t}\,dt \\
&= \int_{-\infty}^{+\infty} x(t)e^{-\sigma t}e^{-j\omega t}\,dt \\
X(s) &= \mathcal{F}\left\{x(t)e^{-\sigma t}\right\}
\end{aligned}
$$

LT may converge when FT does not.

Example

Find the LT of

$$x(t) = e^{-at}u(t).$$

$$\begin{aligned} X(s) &= \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt \\ &= \int_{0}^{+\infty} e^{-at}e^{-st}\,dt \\ X(s) &= \frac{1}{s+a}, \quad \mathrm{Re}\{s\} > -a \end{aligned}$$

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathrm{Re}\{s\} > -a$$

Example

Find the LT of

$$x(t) = -e^{-at}u(-t).$$

$$-e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathrm{Re}\{s\} < -a$$

Note: Two time functions generate the same algebraic expression for the LT. The difference is only in the ROC.

Example

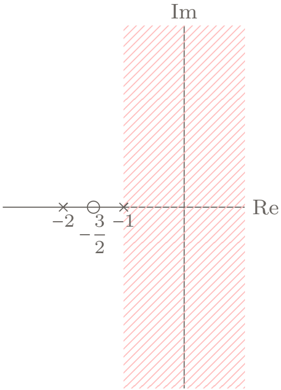
Find the LT of

$$x(t) = e^{-t}u(t) + e^{-2t}u(t).$$

$$e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+1}, \quad \mathrm{Re}\{s\} > -1$$

$$e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+2}, \quad \mathrm{Re}\{s\} > -2$$

$$e^{-t}u(t) + e^{-2t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{2s+3}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1$$



$$X(s) = \frac{N(s)}{D(s)}$$

$N(s) = 0$ zeros of $X(s)$

$D(s) = 0$ poles of $X(s)$

Properties of the Region of Convergence

- The ROC contains no poles
- The ROC of $X(S)$ consists of s trip parallel to the $j\omega$ axis in the s-plane.
- $\mathcal{F}\{x(t)\}$ converges $\Leftrightarrow$ ROC includes the $j\omega$-axis in the s-plane.

Example

Sketch the choices of the ROC associated with

$$X(s) = \frac{1}{(s+1)(s+2)}.$$

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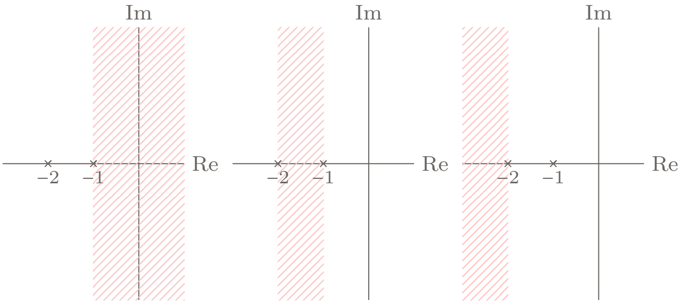
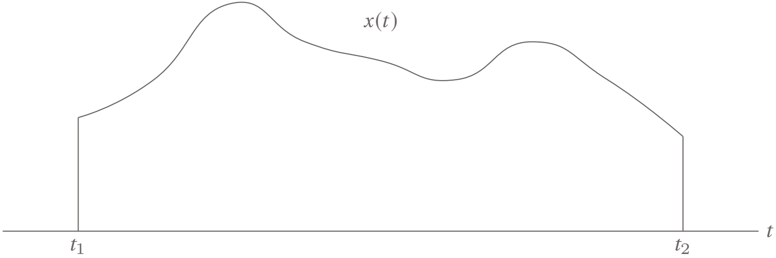


ROC of a Finite-Duration Signal

If $x(t)$ is a finite-duration signal, then the ROC is the entire s-plane.



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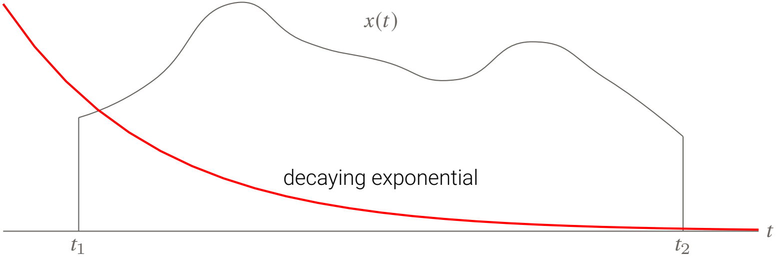

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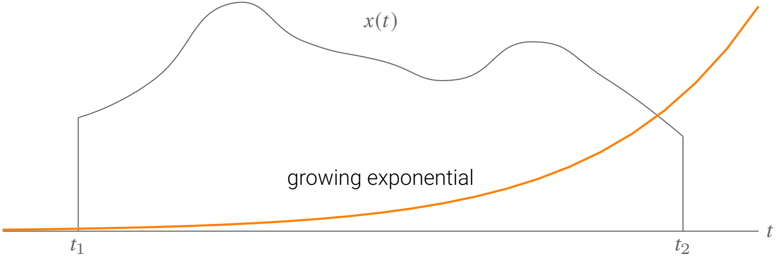
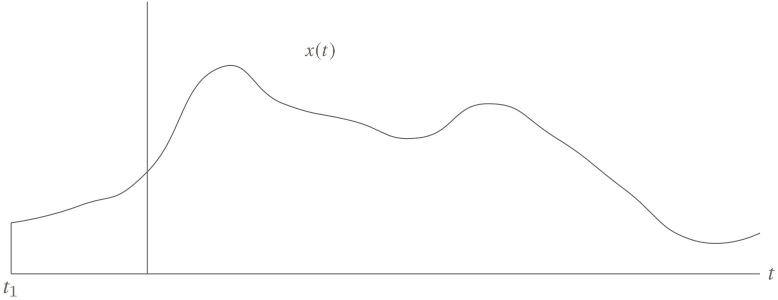


ROC of a Right-Sided Signal

If $x(t)$ is right-sided and $\mathrm{Re}\{s\} = \sigma_0$ is in ROC, then all values for which $\mathrm{Re}\{s\} > \sigma_0$ are in ROC.



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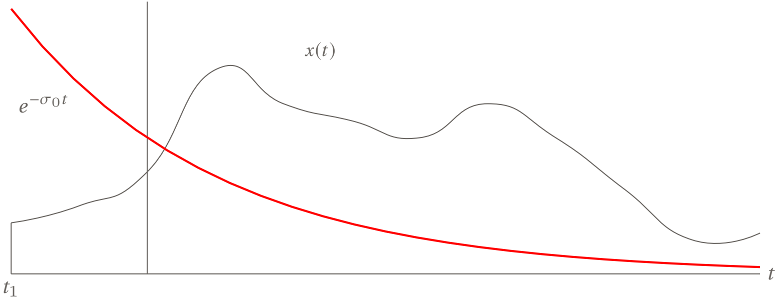

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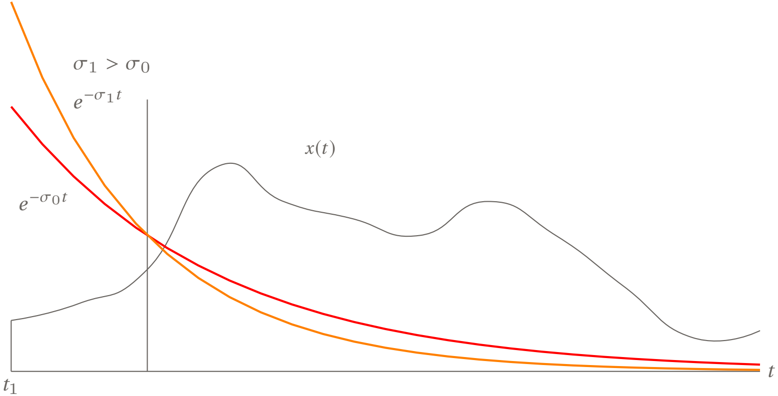

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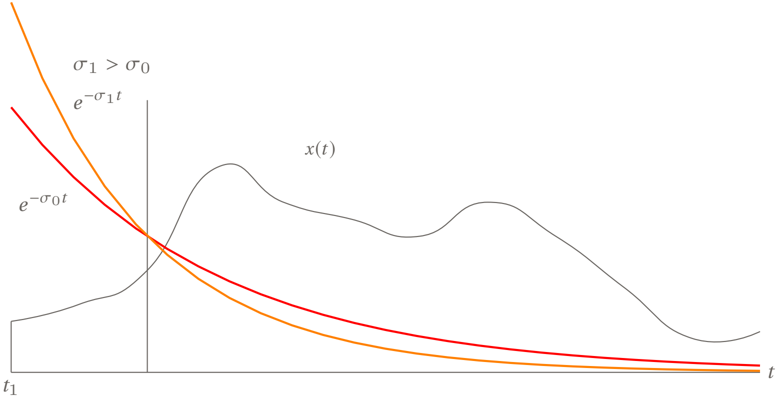


If $x(t)$ is right-sided and $X(s)$ is rational, then ROC is the right of the rightmost pole.

ROC of a Left-Sided Signal

If $x(t)$ is left-sided and $\mathrm{Re}\{s\} = \sigma_0$ is in ROC, then all values for which $\mathrm{Re}\{s\} < \sigma_0$ are in ROC.

ROC of a Left-Sided Signal

If $x(t)$ is left-sided and $\mathrm{Re}\{s\} = \sigma_0$ is in ROC, then all values for which $\mathrm{Re}\{s\} < \sigma_0$ are in ROC.
If $x(t)$ is left-sided and $X(s)$ is rational, then ROC is the left of the leftmost pole.

ROC of a Two-Sided Signal

If $x(t)$ is two-sided and $\mathrm{Re}\{s\} = \sigma_0$ is in ROC, then ROC is the strip in the s-plane.

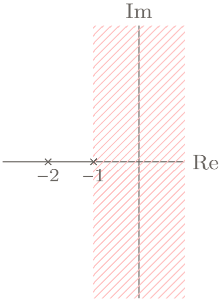
If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right sided, the ROC is the resign in the s-plane to the right of the rightmost pole. If $x(t)$ is left sided, the ROC is the region in the s-plane to the left of the leftmost pole.

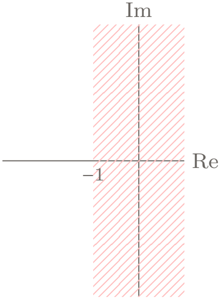
Example

A Laplace transform is specified by

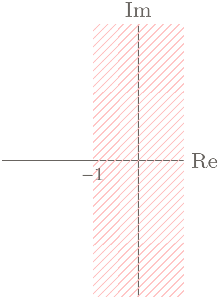
$$X(s) = \frac{1}{(s+1)(s+2)}, \qquad \mathrm{Re}\{s\} > -1.$$

Find the inverse laplace transform.





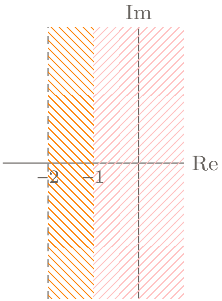
$$\begin{aligned} X(s) &= \frac{1}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1, \\ &= \frac{1}{(s+1)} - \frac{1}{(s+2)}, \quad \mathrm{Re}\{s\} > -1. \end{aligned}$$



$$X(s) = \frac{1}{(s+1)(s+2)}, \quad \mathrm{Re}\{s\} > -1,$$
$$= \frac{1}{(s+1)} - \frac{1}{(s+2)}, \quad \mathrm{Re}\{s\} > -1.$$

Consider

$$X_1(s) = \frac{1}{(s+1)}, \quad \mathrm{Re}\{s\} > -1,$$
$$x_1(t) = e^{-t}u(t).$$



Consider

$$X_2(s) = -\frac{1}{(s+2)}, \quad \mathrm{Re}\{s\} > -1,$$
$$x_2(t) = -e^{-2t}u(t).$$

So

$$x(t) = (e^{-t} - e^{-2t})u(t).$$

Example

Find the inverse laplace transform of

1. $$X(s) = \frac{2s+4}{s^2+4s+3}, \quad \mathrm{Re}\{s\} > -1,$$

2. $$X(s) = \frac{2s+4}{s^2+4s+3}, \quad \mathrm{Re}\{s\} < -3,$$

3. $$X(s) = \frac{2s+4}{s^2+4s+3}, \quad -3 < \mathrm{Re}\{s\} < -1,$$

$$X(s) = \frac{2s + 4}{s^2 + 4s + 3} = \frac{1}{s + 1} + \frac{1}{s + 3}$$

$$X(s) = \frac{2s+4}{s^2+4s+3} = \frac{1}{s+1} + \frac{1}{s+3}$$

1. The ROC of $X(s)$ is $\mathrm{Re}\{s\} > -1$. Thus, $x(t)$ is a right-sides signal. We obtain

$$x(t) = e^{-t}u(t) + e^{-3t}u(t) = (e^{-t} + e^{-3t})u(t).$$

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2. The ROC of $X(s)$ is $\mathrm{Re}\{s\} < -3$. Thus, $x(t)$ is a left-sides signal. We obtain

$$x(t) = -e^{-t}u(-t) - e^{-3t}u(-t) = -(e^{-t} + e^{-3t})u(-t).$$

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3. The ROC of $X(s)$ is $-3 < \mathrm{Re}\{s\} < -1$. Thus, $x(t)$ is a double-sided signal. We obtain

$$x(t) = -e^{-t}u(-t) + e^{-3t}u(t).$$

Some Laplace Transform Pairs I

Instead of having to reevaluate the transform of a given signal, we can simply refer to the Laplace transform table and and read out the desired transform.

$x(t)$	$X(s)$	ROC
$\delta(t)$	1	All s
$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}(s) < 0$
$tu(t)$	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathrm{Re}(s) > 0$
$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathrm{Re}(s) < 0$

Some Laplace Transform Pairs II

$e^{-at}u(t)$	$\dfrac{1}{s+a}$	$\mathrm{Re}(s) > -a$
$-e^{-at}u(-t)$	$\dfrac{1}{s+a}$	$\mathrm{Re}(s) < -a$
$te^{-at}u(t)$	$\dfrac{1}{(s+a)^2}$	$\mathrm{Re}(s) > -a$
$-te^{-at}u(-t)$	$\dfrac{1}{(s+a)^2}$	$\mathrm{Re}(s) < -a$
$(\cos\omega_0 t)u(t)$	$\dfrac{s}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
$(\sin\omega_0 t)u(t)$	$\dfrac{\omega_0}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
$(e^{-at}\cos\omega_0 t)u(t)$	$\dfrac{s+a}{(s+a)^2+\omega^2}$	$\mathrm{Re}(s) > -a$
$(e^{-at}\sin\omega_0 t)u(t)$	$\dfrac{\omega_0}{(s+a)^2+\omega^2}$	$\mathrm{Re}(s) > -a$

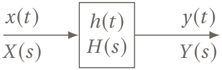
Some Laplace Transform Pairs III

Outline

Properties

$$ax_1(t) + bx_2(t) \overset{\mathcal{L}}{\longleftrightarrow} aX_1(s) + bX_2(s).$$

$$\frac{dx(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} sX(s).$$



$$Y(s) = H(s)X(s).$$

Stable, causal $\Leftrightarrow$ all poles in left-half s-plane [later].

Properties of the Laplace Transform I

Property	Signal	Laplace transform	ROC
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
Time shifting	$x(t - t_0)$	$e^{-st_0}X(s)$	R
Shifting in s domain	$e^{s_0 t}x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in ROC if $s - s_0$ is in R)
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in ROC if s/a is in R)
Conjugation	$x^*(t)$	$X^*(s^*)$	R
Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the time domain	$\frac{dx(t)}{dt}$	$sX(s)$	At least R
Differentiation in the s-domain	$-tx(t)$	$\frac{dX(s)}{ds}$	R
Integration in the time domain	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \{\Re\{s\} > 0\}$

Initial- and final-value theorems: If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s\to\infty} sX(s)$$

Properties of the Laplace Transform II

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

Example

Verify the time-shifting property

$$x(t - t_0) \longleftrightarrow e^{-st_0} X(s), \quad R' = R.$$

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$$\mathcal{L}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)e^{-st}\,dt$$

By the change of variables $\tau = t - t_0$, we obtain

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By the change of variables $\tau = t - t_0$, we obtain

$$\begin{aligned}
\mathcal{L}\{x(t - t_0)\} &= \int_{-\infty}^{\infty} x(\tau) e^{-s(\tau + t_0)} dt \\
&= e^{-st_0} \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} dt \\
&= e^{-st_0} X(s)
\end{aligned}$$

with the same ROC as for $X(s)$ itself.

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Example

Using the various Laplace transform properties, derive the Laplace transforms of the following signals from the Laplace transform of $u(t)$.

1. $\delta(t)$
2. $\delta'(t)$
3. $tu(t)$
4. $e^{-at}u(t)$
5. $te^{-at}u(t)$
6. $\cos\omega_0 t u(t)$
7. $e^{-at}\cos\omega_0 t u(t)$

1.

$$
\begin{aligned}
\mathcal{L}\{u(t)\} &= \int_{-\infty}^{\infty} u(t)e^{-st}\,dt \\
&= \int_{0}^{\infty} e^{-st}\,dt \\
&= -\left.\frac{1}{s}e^{-st}\right|_{0}^{\infty} \\
&= \frac{1}{s}, \quad \operatorname{Re}(s) > 0.
\end{aligned}
$$

$$
\delta(t) = \frac{du(t)}{dt}
$$

Thus, using time differentiation property, we obtain

$$
\delta(t) \longleftrightarrow s\frac{1}{s} = 1, \quad \text{all } s
$$

2. Again applying the time-differentiation property to the result above), we obtain

$$\delta'(t) \longleftrightarrow s, \quad \text{all } s$$

3. Using the differentiation in s property, we have

$$tu(t) \longleftrightarrow -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2}, \quad \mathrm{Re}(s) > 0$$

4. Using the shifting in the s-domain property, we have

$$e^{-at}u(t) \longleftrightarrow \frac{1}{s+a}, \quad \mathrm{Re}(s) > -a$$

Outline

Laplace Transform

- Properties of the Laplace Transform
- Analysis of LTI Systems Using the Laplace Transform
- The Unilateral Laplace Transform

Introduction

- The properties of the Laplace transform make it useful in analyzing LTI systems that are represented by linear constant-coefficient differential equations.
- Applying the Laplace transform to a differential equation converts it to an algebraic equation relating the Laplace transform of the system output to the product of the Laplace transform of the system input and the Laplace transform of the system impulse response, referred to as the system function.
- The system function is readily obtained by inspection of the differential equation, and the system impulse response can be obtained by evaluating the inverse Laplace transform of the system function.
- Alternatively, the response for any other input can be evaluated by first multiplying the Laplace transform of the input by the system function and then applying the inverse Laplace transform.

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- An LTI system is stable if and only if the ROC of its system function $H(s)$ includes the entire $j\omega$-axis [i.e., $\Re e(s) = 0$]
- A causal system with rational system function $H(s)$ is stable if and only if all of the poles of $H(s)$ lie in the left-half of the s-plane—i.e., all he pose have negative real parts.

First- and Second-Order Systems

- Two particularly important classes of systems described by linear constant-coefficient differential equations are first-order and second-order systems.
- In implementing higher-order systems, it is very common to use first and second-order systems as building blocks.
- First-order systems are represented by a single pole in the s-plane, and second-order systems by a pair of poles. There may or may not also be zeros in the transfer function, depending on whether there are derivative terms on the right-hand side of the differential equation.
- From the differential equation, the system function can be written directly.
- If we assume that the systems are causal, so that the impulse response is right-sided, then the ROC of the system function is implicitly specified to be to the right of the rightmost pole in the s-plane.

Recall: Laplace Transform

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt$$

$$x(t) \leftrightarrow X(s)$$

$$X(s)|_{s=j\omega} = X(j\omega) = \mathcal{F}\{x(t)\}$$

$$s = \sigma + j\omega$$

$$X(s) = \mathcal{F}\left\{x(t)e^{-\sigma t}\right\}$$

LT converges for some values of σ and not others: ROC.

First-Order System

$$\frac{dy(t)}{dt} + ay(t) = x(t)$$

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$$\downarrow \qquad \downarrow \qquad \downarrow$$

First-Order System

$$\begin{array}{ccccc} \dfrac{dy(t)}{dt} & + & ay(t) & = & x(t) \\ \downarrow & & \downarrow & & \downarrow \\ sY(s) & + & aY(s) & = & X(s) \end{array}$$

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$$Y(s) = \frac{1}{s+a} X(s), \quad \mathrm{Re}\{s\} > -a$$

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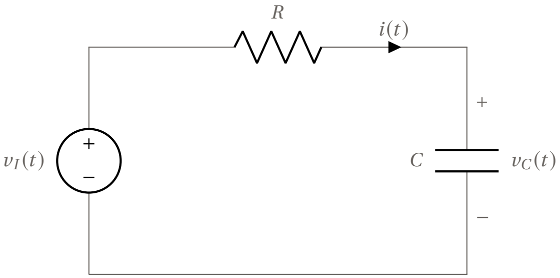
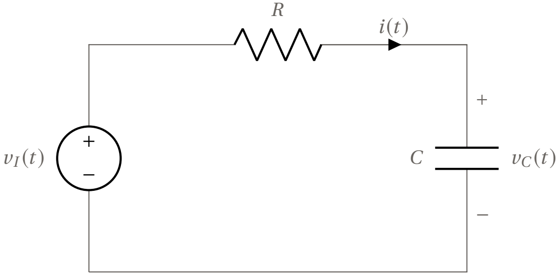
$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathrm{Re}\{s\} > -a$$

$$-e^{-at}u(-t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad \mathrm{Re}\{s\} < -a$$

Example: RC Step Response



Example: RC Step Response



$$v_I(t) - v_R(t) - v_C(t) = 0.$$

$$i(t) = C\frac{dv_C(t)}{dt}.$$

$$v_I(t) - RC\frac{dv_C(t)}{dt} - v_C(t) = 0.$$

$$\frac{dv_C(t)}{dt} + \frac{1}{RC}v_C(t) = \frac{1}{RC}v_I(t).$$

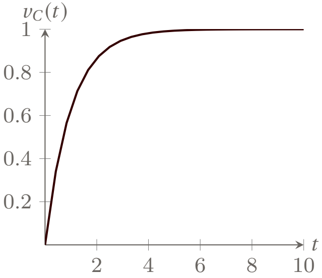
Assume initial rest.

$$\frac{dv_C(t)}{dt} + \frac{1}{RC}v_C(t) = \frac{1}{RC}v_I(t).$$
$$\frac{dv_C(t)}{dt} + \frac{1}{RC}v_C(t) = \frac{1}{RC}u(t)$$

$$sV_C(s) + \frac{1}{RC}V_C(s) = \frac{1}{RC}\frac{1}{s}$$
$$V_C(s) = \frac{\frac{1}{RC}}{s(s+\frac{1}{RC})}$$
$$= \frac{1}{s} - \frac{1}{s+\frac{1}{RC}}$$

$$v_c(t) = u(t) - e^{-\frac{1}{RC}t}u(t)$$
$$v_c(t) = \left(1 - e^{-\frac{1}{RC}t}\right)u(t)$$

Plot with $RC = 1$:



Continuous Time Second-Order System

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t).$$

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$$\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right] Y(s) = \omega_n^2 X(s).$$

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$$H(s) = \frac{\omega_n^2}{(s - c_1)(s - c_2)}.$$

$$c_1 =$$

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$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

$$c_2 =$$

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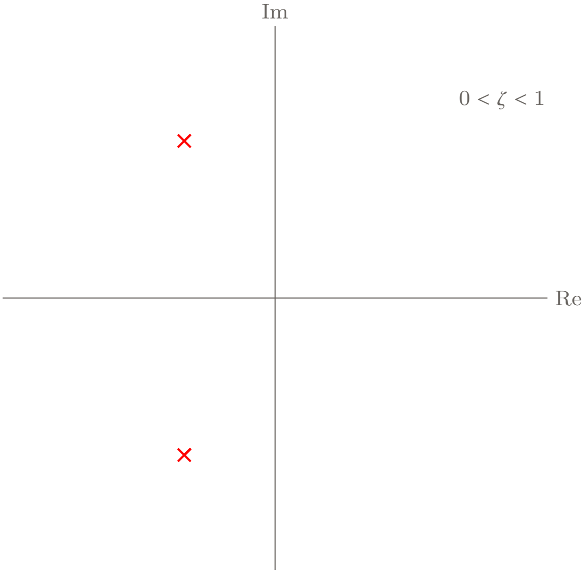
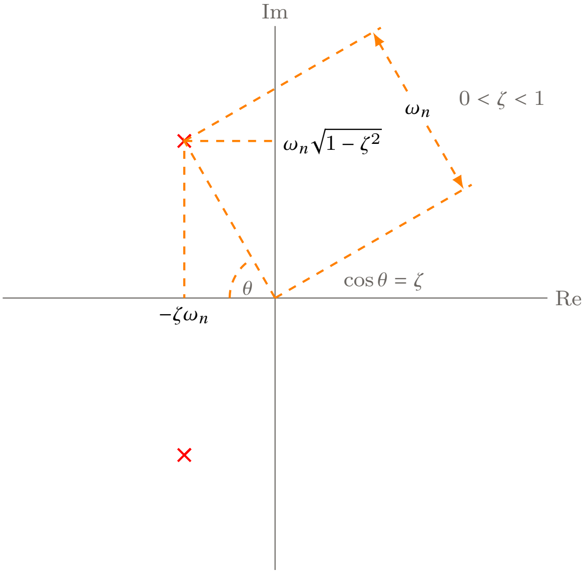
$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

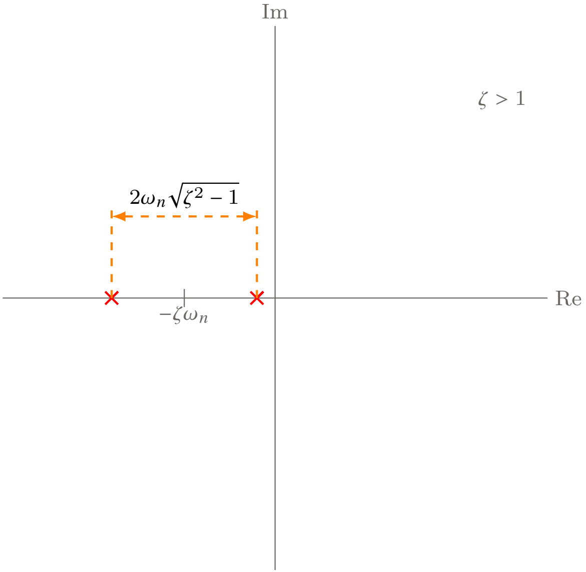
$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

For $\zeta < 1$,

$$\begin{aligned} c_1 &= c_2^* \\ &= -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \end{aligned}$$


Im
$0 < \zeta < 1$
Re


Im
$0 < \zeta < 1$
ω_n
$\omega_n\sqrt{1-\zeta^2}$
$\cos\theta = \zeta$
θ
Re
$-\zeta\omega_n$


Im
$\zeta > 1$
$2\omega_n\sqrt{\zeta^2 - 1}$
Re
$-\zeta\omega_n$

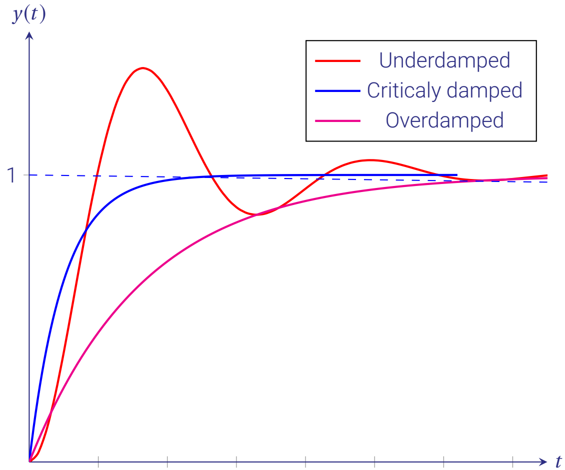


Figure: Second-order system responses.

Example

The transfer function of a network is

$$H(s) = \frac{s + 10}{s^2 + 4s + 8}$$

Determine the pole-zero plot of $H(s)$, the type of damping exhibited by the network, and the unit step response of the.

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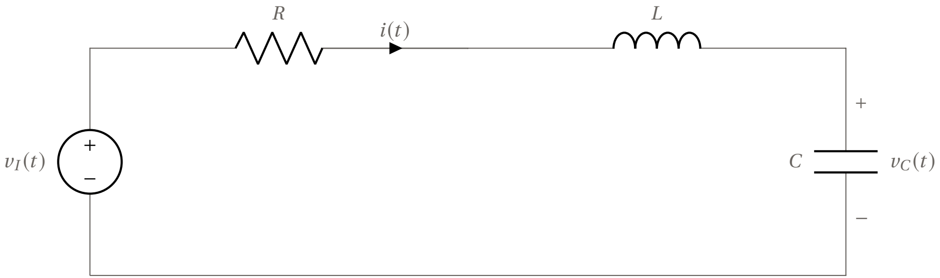
Determine the pole-zero plot of $H(s)$, the type of damping exhibited by the network, and the unit step response of the.

Solution: The network is underdamped.

$$y(t) = \left[\frac{10}{8} + 1.4e^{-2t}\cos(2t - 210.96^\circ)\right]u(t)$$

Example

Consider an RLC series network in which the capacitor voltage and inductor current are initially zero.



1. Obtain the voltage transfer function.
2. If $\omega_n = 2000\ \mathrm{rad/s}$ and $\zeta = 0.25$, 0.50, 0.75, and 1.0, sketch the pole-zero plots.
3. Sketch the step response for each case.

Take $x(t) = v_i(t)$, and $y(t) = v_C(t) = v_o(t)$.

$$v_R(t) + v_L(t) + v_C(t) = x(t).$$

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$$RC\frac{dy(t)}{dt} + LC\frac{d^2y(t)}{dt^2} + y(t) = x(t).$$

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Taking the LT of both the sides

$$RCsY(s) + LCs^2LCY(s) + Y(s) = X(s).$$

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Taking the LT of both the sides

$$RCsY(s) + LCs^2LCY(s) + Y(s) = X(s).$$

$$H(s) = \frac{1/LC}{s^2 + (R/L)s + (1/LC)}.$$

Outline

Introduction to The Unilateral Laplace Transform

- In the preceding sections, we have dealt with what is commonly called the bilateral Laplace transform.
- In this section, we briefly study the unilateral Laplace transform.
- It is of considerable value in analyzing causal systems and, particularly, systems specified by linear constant-coefficient differential equations with nonzero initial conditions (i.e., systems that are not initially at rest).

The Unilateral Laplace Transform

$$\mathcal{X}(s) \triangleq \int_{0^-}^{\infty} x(t)e^{-st}\,dt$$

where the lower limit of integration, 0^-, signifies that we include in the interval of integration any impulses or higher order singularity functions concentrated at $t = 0$.

$$x(t) \xleftrightarrow{\mathcal{UL}} \mathcal{X}(s)$$

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$$x(t) \overset{\mathcal{UL}}{\longleftrightarrow} \mathcal{X}(s)$$

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt$$

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$$

The system analysis tools and system function algebra developed and used in this lecture apply without change to unilateral transforms, as long as we deal with causal LTI systems (for which the system function is both the bilateral and the unilateral transform of the impulse response) with inputs that are identically zero fort $t < 0$.

Example

A causal LTI system is described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t).$$

Suppose that the system is at initial rest.

1. Find the system function $\mathcal{H}(s)$.
2. Find the Laplace transform of the output $\mathcal{Y}(s)$ if the input is $x(t) = \alpha u(t)$.
3. Find the output $y(t)$.

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t).$$

System function is

$$\mathcal{H}(s) = \frac{1}{s^2 + 3s + 1}$$

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t).$$

System function is

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$$x(t) = \alpha u(t)$$
$$X(s) = \alpha \frac{1}{s}$$

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System function is

$$\mathcal{H}(s) = \frac{1}{s^2 + 3s + 1}$$

$$x(t) = \alpha u(t)$$

$$X(s) = \alpha\frac{1}{s}$$

$$\mathcal{Y}(s) = \mathcal{H}(s)\mathcal{X}(s) = \frac{\alpha}{s(s^2 + 3s + 1)}$$

$$= \frac{\alpha/2}{s} - \frac{\alpha}{s+1} + \frac{\alpha/2}{s+2}$$

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t).$$

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$$= \frac{\alpha/2}{s} - \frac{\alpha}{s+1} + \frac{\alpha/2}{s+2}$$

$$y(t) = \alpha\left[\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\right]u(t)$$