EN1060 Signals and Systems: Discrete-Time Fourier Series

Ranga Rodrigo
ranga@uom.lk

The University of Moratuwa, Sri Lanka

March 15, 2021

Section 1

Discrete-Time Fourier Series

Introduction

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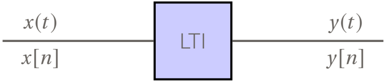
- In this lesson we will develop a similar tool for discrete time.
- Specifically, we consider the representation of discrete-time signals through a decomposition as a linear combination of complex exponentials.
  - DT periodic signals $\longrightarrow$ DT Fourier series
  - DT aperiodic signals $\longrightarrow$ DT Fourier transform
- Fourier series representation of a DT periodic signal is a finite series, as opposed to the infinite series representation required for CT periodic signals.

Philosophy



Decompose the input as

$$x = a_1\phi_1 + a_2\phi_2 + \cdots \quad \text{linear combination of basic inputs}$$

Then

$$y = a_1\psi_1 + a_2\psi_2 + \cdots \quad \text{linear combination of corresponding outputs}$$

Choose $\phi_k(t)$ or $\phi_k[n]$ such that

- Broad class of signals can be constructed, and
- Response to ϕ_ks easy to compute.

Eigenfunction Property

Continuous-Time:
$\phi_k(t) = e^{j\omega_k t}$:

$$e^{j\omega_k t} \longrightarrow H(j\omega_k)e^{j\omega_k t} \quad \text{(a scaled-version of the input)}$$

“Discrete-Time”:

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"Discrete-Time": $\phi_k[n] = e^{j\omega_k n}$

$$e^{j\omega_k n} \longrightarrow \underset{\substack{\uparrow \\ \text{eigenfunction}}}{e^{j\omega_k n}} \underbrace{\sum_{r=-\infty}^{\infty} h[r]e^{-j\omega_k r}}_{\text{eigenvalue}}$$

Lin. Comb. of Harmonically Related Comp. Exponentials I

Consider $x[n]$ to be periodic with period N, i.e.,

$$x[n] = x[n+N]$$

Fundamental frequency: $\omega_0 = \dfrac{2\pi}{N}$

E.g., the complex exponential $e^{j(2\pi/N)n}$ is periodic with period N. Furthermore, the set of all DT complex exponential signals that are periodic with period N given by

$$\phi_k[n] = e^{jk\omega_0 n} = e^{j(k+N)\omega_0 n},\ k = 0, \pm 1, \pm 2, \ldots. \tag{1}$$

All these signals have fund. frequencies that are multiples of $2\pi/N$ and thus are harmonically related. There are only N distinct signals in the set given by eq. 1. This is a consequence of of DT complex exponentials which differ in frequency by 2π are identical, e.g., $\phi_0[n] = \phi_N[n]$, $\phi_1[n] = \phi_{N+1}[n]$, and, in general,

$$\phi_k[n] = \phi_{k+rN}[n].$$

Lin. Comb. of Harmonically Related Comp. Exponentials II

I.e., when k is changed by any integer multiple of N, the identical sequence is generated. This is different in CT, in which signals $\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t}$, $k = 0, \pm 1, \pm 2, \ldots$ are all different from one another.

Discrete-Time Fourier Series

Consider the representation of more general preiodic sequences in terms of linear combination fo sequences $\phi_k[n]$

$$x[n] = \sum_k \phi_k[n] = \sum_k a_k e^{jk\omega_0 n} = \sum_k a_k e^{jk(2\pi/N)n},$$

As the sequence $\phi_k[n]$ are distinct only over a ragen of N successive values of k, the above summation need only include terms over this range.

$$x[n] = \sum_{k=<N>} \phi_k[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n} \tag{2}$$

E.g., k could take values $k = 0, 1, 2, \ldots, N-1$, or $k = 3, 4, 5, \ldots, N+2$. Eq. 2 is referred to as the discrete-time Fourier series and the coefficients a_k as the Fourier series coefficients.

Discrete-Time Fourier Series

Continuous-Time

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

Discrete-Time Fourier Series

Synthesis

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}.$$

Analysis

$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=<N>} x[n]e^{-jk(2\pi/N)n}.$$

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Note the duality.

Periodicity

$x[n]$	periodic in n,	true for CT
$e^{jk\omega_0 n}$	periodic in n,	true for CT
$e^{jk\omega_0 n}$	periodic in k,	not true for CT
a_k	periodic in k,	not true for CT

Convergence

Continuous-time:

- $x(t)$ square-integrable OR
- Dirichlet condition

Discrete-time

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}.$$

$$\hat{x}[n] = \sum_{p \text{ terms}} a_k e^{jk\omega_0 n}.$$

$p = N$

$$\hat{x}[n] \equiv x[n].$$

There is no issue of convergence in DT.

Example

Consider the signal $x[n] = \sin \omega_0 n$.

1. When is this signal periodic?
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$x[n]$ is periodic with fundamental period N.
Expanding the signal as a sum of two complex exponentials,

$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}. \quad (3)$$

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$$a_1 = \frac{1}{2j}, \quad a_{-1} = -\frac{1}{2j}.$$

Fourier Coefficients for $x[n] = \sin(2\pi/N)n$ for $N = 5$

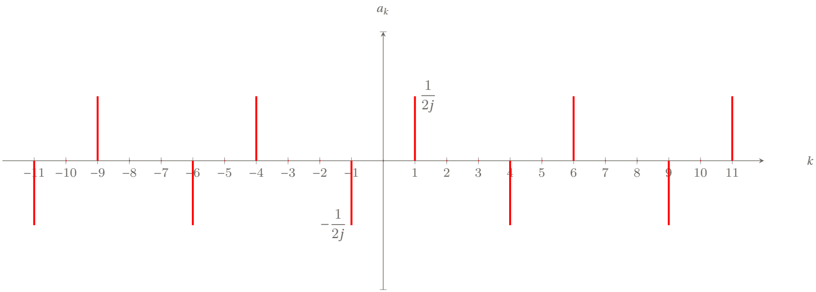

Figure: Fourier coefficients for $x[n] = \sin(2\pi/5)n$.

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Determine and sketch the DTFS of

$$x[n] = 1 + \sin \omega_0 n + 3 \cos \omega_0 n + \cos\left(2\omega_0 n + \frac{\pi}{2}\right).$$

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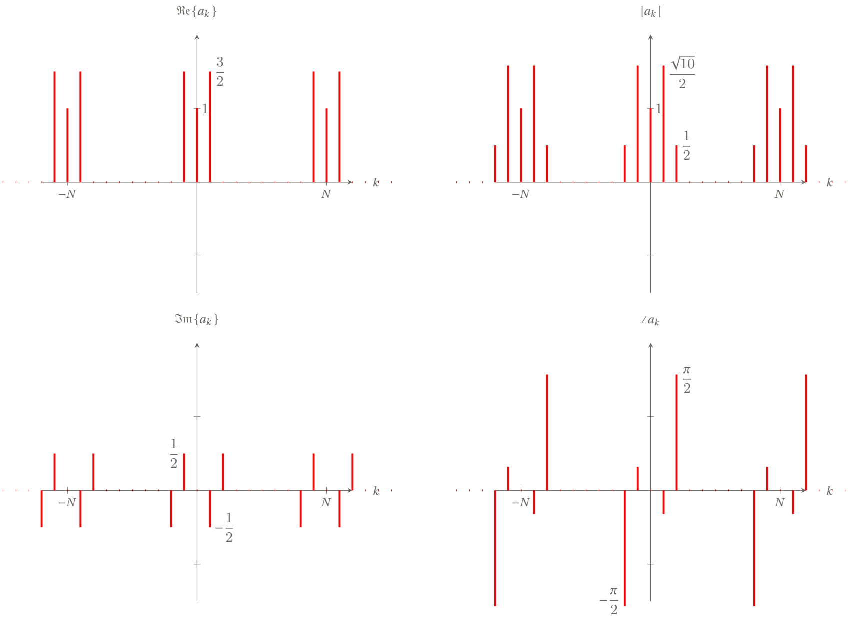
$\Re\mathfrak{e}\{a_k\}$
$\frac{3}{2}$
1
k
$-N$
N
$|a_k|$
$\frac{\sqrt{10}}{2}$
1
$\frac{1}{2}$
k
$-N$
N
$\Im\mathfrak{m}\{a_k\}$
$\frac{1}{2}$
$-\frac{1}{2}$
k
$-N$
N
$\angle a_k$
$\frac{\pi}{2}$
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k
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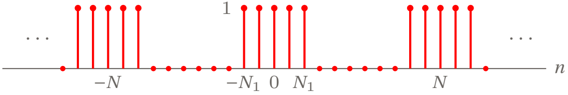
N

Example

Determine and sketch the DTFS of $x[n]$ of which is shown in the figure.



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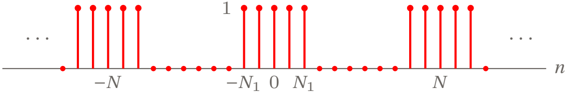

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}$$

Letting $m = n + N_1$

$$= \frac{1}{N} \sum_{m=0}^{2N_1} e^{-jk(2\pi/N)/(m-N_1)}$$

$2N_1 + 1$ terms in a geometric series

$$= \frac{1}{N} e^{jk(2\pi/N)N_1} \left[\frac{1 - e^{-jk(2\pi/N)(2N_1+1)}}{1 - e^{-jk(2\pi/N)}} \right]$$

$$= \frac{1}{N} e^{jk(2\pi/N)N_1} \cdot \frac{e^{-j\frac{k(2\pi/N)(2N_1+1)}{2}}}{e^{-j\frac{k(2\pi/N)}{2}}} \left[\frac{e^{j\frac{k(2\pi/N)(2N_1+1)}{2}} - e^{-j\frac{k(2\pi/N)(2N_1+1)}{2}}}{e^{j\frac{k(2\pi/N)}{2}} - e^{-j\frac{k(2\pi/N)}{2}}} \right]$$

$$= \frac{1}{N} \frac{\sin\left[\frac{k(2\pi/N)(2N_1+1)}{2}\right]}{\sin\left[\frac{k(2\pi/N)}{2}\right]}$$

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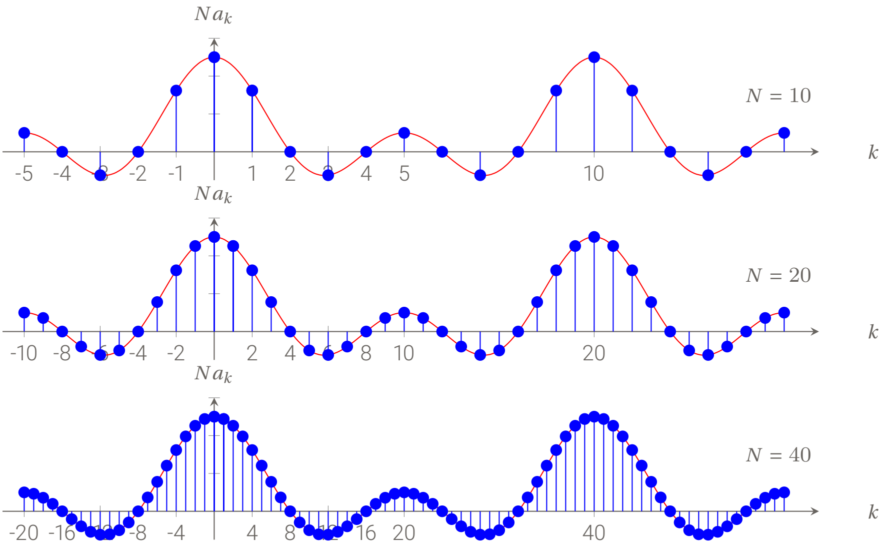
$$a_k = \frac{1}{N} \frac{\sin\left[2\pi k(N_1 + 1/2)/N\right]}{\sin(\pi k/N)} \quad k \neq 0, \pm N, \pm 2N, \ldots$$

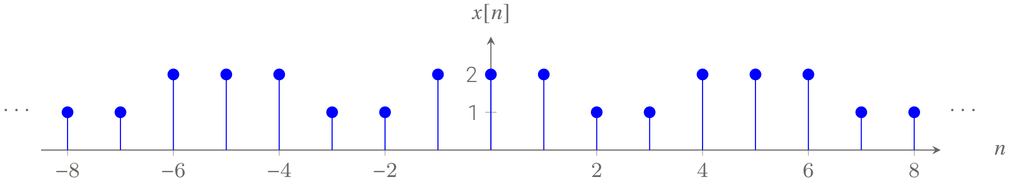
$$a_k = \frac{2N_1 + 1}{N} \quad k = 0, \pm N, \pm 2N, \ldots.$$


Na_k
$N = 10$
k
-5 -4 -2 -1 1 2 4 5 10
Na_k
$N = 20$
k
-10 -8 -4 -2 2 4 8 10 20
Na_k
$N = 40$
k
-20 -16 -8 -4 4 8 16 20 40

Outline

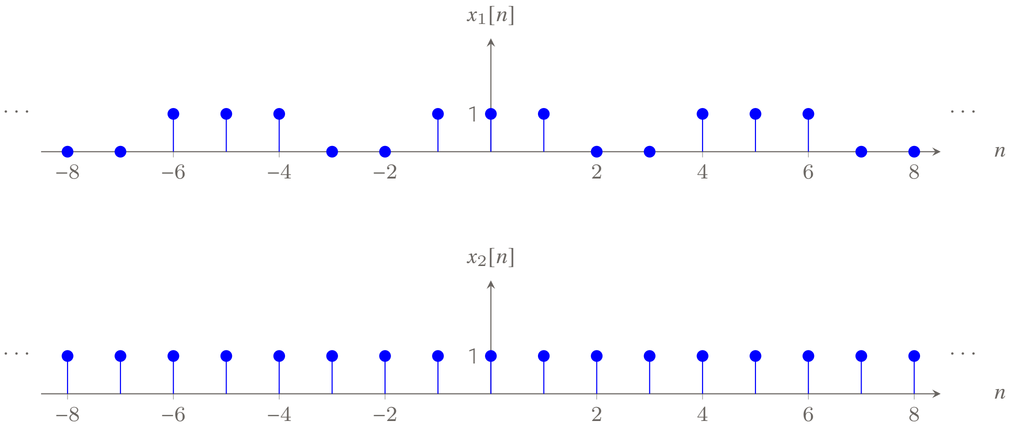
Example

Find the Fourier series coefficients a_k of $x[n]$.



Denoting the Fourier series coefficients of $x_1[n]$ by b_k and those of $x_2[n]$ by c_k. We use the linearity property of to conclude that

$$a_k = b_k + c_k.$$



From the previous work, (with $N_1 = 1$ and $N = 5$), the Fourier series coefficients b_k corresponding to $x_1[n]$ can be expressed as

$$b_k = \begin{cases} \dfrac{1}{5}\dfrac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & \text{for } k \neq 0, \pm 5, \pm 10, \dots \\ \dfrac{3}{5}, & \text{for } k = 0, \pm 5, \pm 10, \dots \end{cases}$$

Example

Suppose that we are given the following facts about a sequence $x[n]$:

1. $x[n]$ is periodic with period $n = 6$.
2. $\sum_{n=0}^{5} x[n] = 2$.
3. $\sum_{n=2}^{7} (-1)^n x[n] = 1$.
4. $x[n]$ has the minimum power per period among the set of signals satisfying the proceeding three conditions.

Determine the sequence $x[n]$.

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From Parseval's relation, the average power in $x[n]$ is

$$P = \sum_{k=0}^{5} |a_k|^2.$$

Since each nonzero coefficient contributes a positive amount to P, and since the values of a_0 and $a3$ are pre-specified, the value of P is minimized by choosing $a_1 = a_2 = a_4 = a5 = 0$. It then follows that

$$x[n] = a_0 + a_3 e^{j\pi n} = (1/3) + (1/6)(-1)^n.$$

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