

EN1060 Signals and Systems: z Transforms

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Section 1

z -Transform

Introduction

- We developed the Laplace transform as a generalization of the continuous-time Fourier transform.
- In this lecture, we introduce the corresponding generalization of the discrete-time Fourier transform.
- The resulting transform is referred to as the z -transform.

z -Transform Motivation

- The discrete-time Fourier transform developed out of choosing complex exponentials as basic building blocks for signals because they are eigenfunctions of discrete-time LTI systems.
- A more general class of eigenfunctions consists of signals of the form z^n , where z is a general complex number. In particular, $z = re^{j\omega}$
- A representation of discrete-time signals with these more general exponentials leads to the z -transform.

Relationship between the z -Transform and the Discrete-Time Fourier Transform

- We saw that the Laplace transform is a generalization of the continuous-time Fourier transform.
- A close relationship exists between the z -transform and the discrete-time Fourier transform.
- For $z = e^{j\omega}$ or, equivalently, for the magnitude of z equal to unity, the z -transform reduces to the Fourier transform.
- More generally, the z -transform can be viewed as the Fourier transform of an exponentially weighted sequence.
- Because of this, the z -transform may converge for a given sequence even if the Fourier transform does not: the z -transform offers the possibility of transform analysis for a broader class of signals and systems.

The Region of Convergence (ROC)

- The z -transform of a signal too has associated with it both a range of values of z , referred to as the region of convergence (ROC), for which this expression is valid.
- Two different sequences can have z -transforms with identical algebraic expressions such that their z -transforms differ only in the ROC.
- Consequently, the ROC is an important part of the specification of the z -transform.

z -Plane

- z -transforms of the form of a ratio of polynomials in z^{-1} are described by poles and zeros in the complex plane, referred to as the z -plane.
- The circle of radius 1, concentric with the origin in the z -plane, is referred to as the **unit circle**.
- Since this circle corresponds to the magnitude of z equal to unity, it is the contour in the z -plane on which the z -transform reduces to the Fourier transform.
- In contrast, for continuous time it is the imaginary axis in the s -plane on which the Laplace transform reduces to the Fourier transform.
- If the sequence is known to be right-sided, for example, then the ROC must be the portion of the z -plane outside the circle bounded by the outermost pole.

Outline

z -Transform

The z -Transform

z -Transform properties

Recall: Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$
$$X(e^{j\omega}) = \sum_{-\infty}^{+\infty} x[n] e^{-j\omega n}$$

LTI systems: impulse response $h(t)$:

$$e^{j\omega n} \rightarrow \begin{array}{c} H(e^{j\omega}) e^{j\omega n} \\ \updownarrow \mathcal{F} \\ h[n] \end{array}$$

z -Transform: Eigenfunction Property

$$z^n \rightarrow \sum_{k=-\infty}^{+\infty} h[k] z^{n-k}$$

$$z^n \rightarrow z^n \sum_{k=-\infty}^{+\infty} h[k] z^{-k}$$

$$z = r e^{j\omega}$$

$$z^n \rightarrow H(z) z^n$$

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n] z^{-n}$$

z -Transform

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z)$$

z -Transform and Fourier Transform Relationship

$$X(\omega) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$z = re^{j\omega}$$

$$X(z)|_{z=e^{j\omega}} = \mathcal{F}\{x[n]\}$$

New notation:

$$\mathcal{F}\{x[n]\} = X(e^{j\omega})$$

z -Transform: Convergence Comparison

$$X(z)|_{z=e^{j\omega}} = X(e^{j\omega})$$

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$\begin{aligned} X(re^{j\omega}) &= \sum_{n=-\infty}^{+\infty} x[n] (re^{j\omega})^{-n} \\ &= \sum_{n=-\infty}^{+\infty} x[n] r^{-n} r e^{-j\omega n} \end{aligned}$$

$$X(z) = \mathcal{F} \{x[n]r^{-n}\}$$

z -Transform: Convergence Comparison

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$$X(z) = \mathcal{F} \{x[n]r^{-n}\}$$

ZT may converge when FT does not.

Example

Find the ZT of $x[n] = a^n u[n]$.

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{+\infty} x[n]z^{-n} \\ &= \sum_{n=-\infty}^{+\infty} a^n z^{-n} u[n] \end{aligned}$$

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |az^{-1}| < 1$$

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$$a^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

Example

Find the ZT of

$$x[n] = -a^n u[-n - 1].$$

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Example

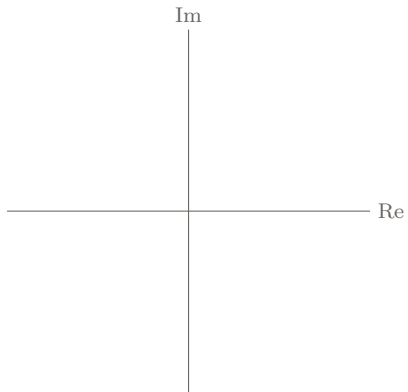
Find the ZT of

$$x[n] = -a^n u[-n - 1].$$

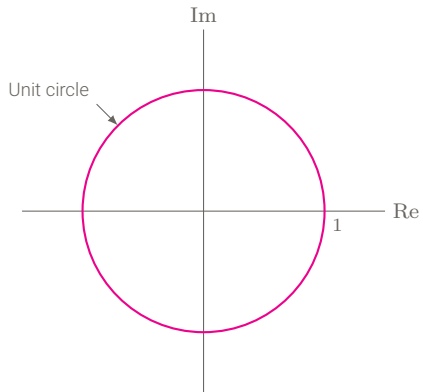
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$$-a^n u[-n - 1] \xleftrightarrow{\mathcal{Z}} \frac{1}{1 - az^{-1}}, \quad |z| < |a|$$

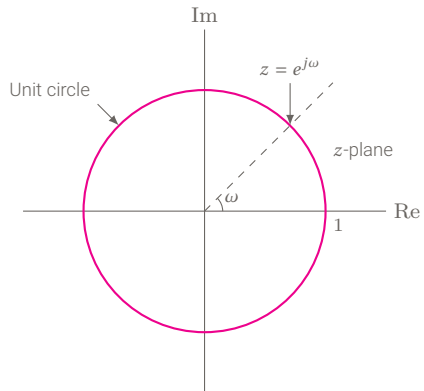
z -Plane and the Unit Circle



z -Plane and the Unit Circle

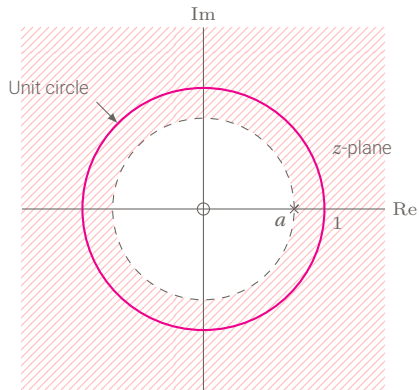


z -Plane and the Unit Circle



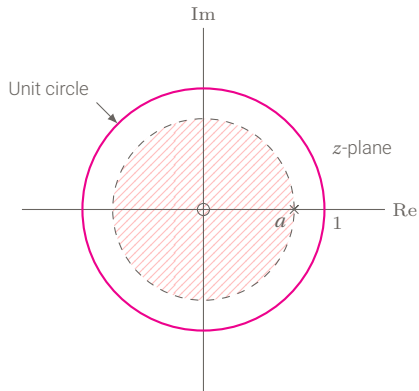
Pole-Zero Plot for a Right-Handed Sequence

$$a^n u[n] \xleftrightarrow{\mathcal{Z}} \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a|$$



Pole-Zero Plot for a Left-Handed Sequence

$$-a^n u[-n-1] \xleftrightarrow{\mathcal{Z}} \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| < |a|$$



First-Order Difference Equation

$$y[n] - ay[n - 1] = x[n]$$

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$$H(z) = \frac{1}{1 - az^{-1}}$$

Causality:

First-Order Difference Equation

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Causality: $|z| > |a|$

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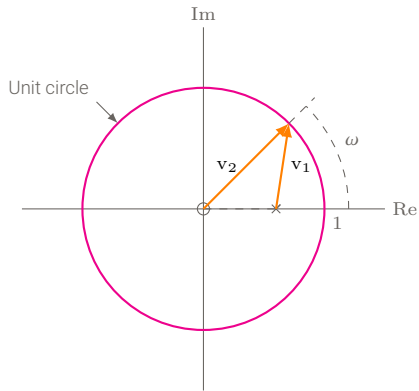
Causality: $|z| > |a|$

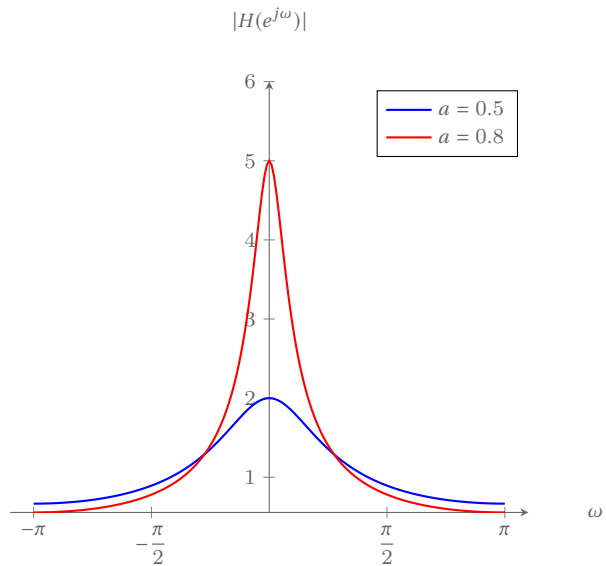
$$h[n] = a^n u[n]$$

Pole-Zero Plot for a DT First-Order System

This illustrates the determination of the Fourier transform from the pole-zero plot.

$$H(z) = \frac{z}{z - a}, \quad |z| > |a|.$$





Second-Order Difference Equation

$$y[n] + 2r \cos \theta y[n - 1] + r^2 y[n - 2] = x[n]$$

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$\cos \theta < 1 \Rightarrow$ complex poles

Poles are at

Second-Order Difference Equation

$$y[n] + 2r \cos \theta y[n-1] + r^2 y[n-2] = x[n]$$

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$\cos \theta < 1 \Rightarrow$ complex poles

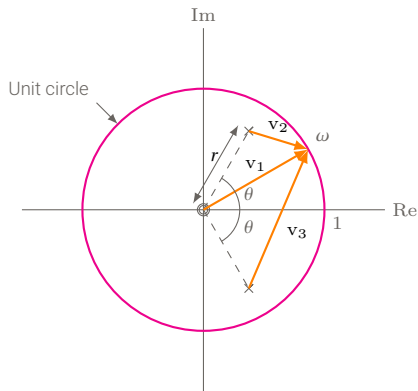
Poles are at

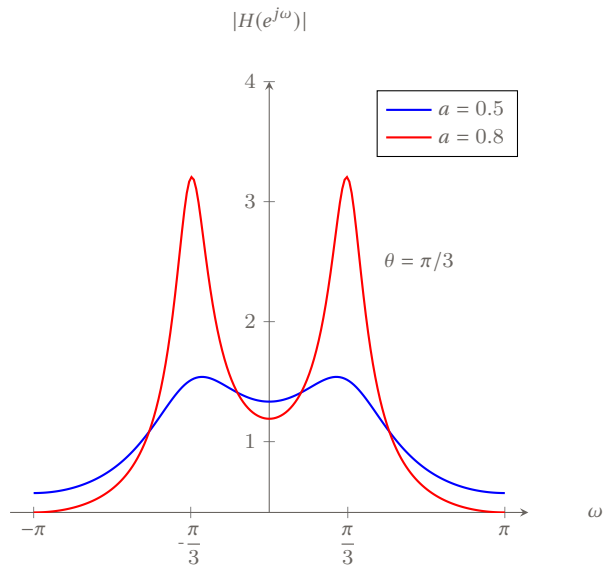
$$re^{\pm j\theta}$$

Pole-Zero Plot for a DT Under-Damped Second-Order System

This illustrates the determination of the Fourier transform from the pole-zero plot.

$$H(z) = \frac{1}{1 - (2r \cos \theta)z^{-1} + r^2 z^{-2}}, \quad |z| > |a|.$$





Properties of the ROC of the z -Transform

- The ROC does not contain poles
- The ROC of $X(z)$ consists of a ring in the z -plane centered about the origin
- $\mathcal{F}\{x[n]\}$ converges \Leftrightarrow ROC includes the unit circle in the z -plane
- $x[n]$ finite duration \Rightarrow ROC is entire z -plane with the possible exception of $z = 0$ or $z = \infty$

Properties of the ROC for a Right-Sided Sequence

- $x[n]$ right-sided and $|z| = r_0$ is in ROC \Rightarrow all finite values of z for which $|z| > r_0$ are in ROC.
- $x[n]$ right-sided and $X(z)$ rational \Rightarrow ROC is outside the outermost pole.

Properties of the ROC for a Left-Sided and for a Two-Sided Sequence

- $x[n]$ left-sided and $|z| = r_0$ is in ROC \Rightarrow all values of z for which $0 < |z| < r_0$ will also be in ROC.
- $x[n]$ left-sided and $X(z)$ rational \Rightarrow ROC is inside the innermost pole.
- $x[n]$ two-sided and $|z| = r_0$ is in ROC \Rightarrow ROC is a ring in the z -plane which includes the circle $|z| = r_0$.

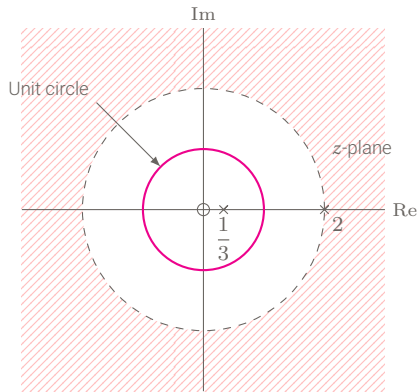
Example

Show the choices of the ROC for

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z - 2)}.$$

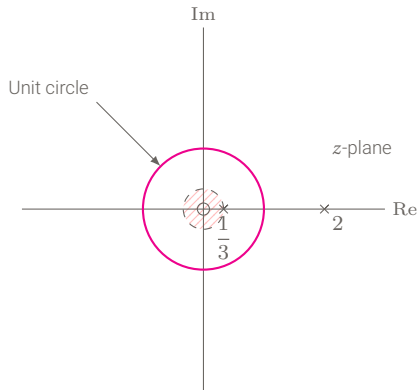
ROC If the Sequence Is Right-Sided.

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z - 2)}.$$



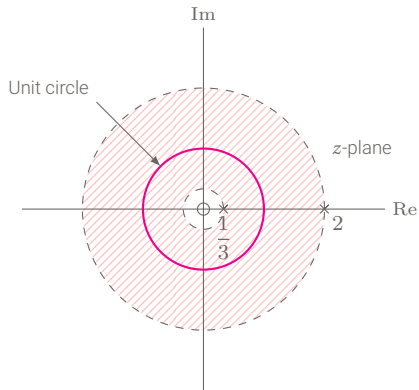
ROC If the Sequence Is Left-Sided.

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ROC If the Sequence Is Two-Sided.

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z - 2)}.$$



Inverse z -Transform

$$X(z) = \mathcal{F} \{x[n]r^{-n}\}$$

$$x[n]r^{-n} = \mathcal{F}^{-1} \{X(z)\}$$

$$= \frac{1}{2\pi} \int_{2\pi} X(re^{j\omega}) e^{-j\omega n} d\omega$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(re^{j\omega}) (re^{j\omega})^n d\omega$$

$$z = re^{j\omega}, \quad dz = jre^{j\omega} d\omega$$

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^{n-1} dz$$

Example

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z - 2)}, \quad |z| > 2.$$

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$$\begin{aligned} X(z) &= \frac{z}{\left(z - \frac{1}{3}\right)(z - 2)}, \quad |z| > 2, \\ &= \frac{z^{-1}}{\left(1 - \frac{1}{3}z^{-1}\right)(1 - 2z^{-1})}, \quad |z| > 2, \\ &= \frac{-\frac{3}{5}}{\left(1 - \frac{1}{3}z^{-1}\right)} + \frac{-\frac{3}{5}}{(1 - 2z^{-1})}, \quad |z| > 2. \end{aligned}$$

$$x[n] = -\frac{3}{5} \left(\frac{1}{3}\right)^n u[n] + \frac{3}{5} (2)^n u[n].$$

Example

$$X(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z - 2)}, \quad |z| > 2.$$

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Outline

z -Transform

The z -Transform

z -Transform properties

Recall: z -Transform

$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

$$x[n] = \frac{1}{2\pi j} \oint X(z)z^{-n}dz$$

$$X(z)|_{z=e^{j\omega}} = \mathcal{F}\{x[n]\}$$

$$z = re^{j\omega}$$

$$X(z) = \mathcal{F}\{x[n]r^{-n}\}$$

z-Transform Properties

Property	Signal	Transform	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	at least $R_1 \cap R_2$
Time shifting	$x[n - n_0]$	$z^{-n_0} X(z)$	R
Scaling in z domain	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	$ z_0 R$
	$e^{j\omega n} x[n]$	$X(e^{-j\omega n} z)$	R
	$a^n x[n]$	$X(a^{-1} z)$	Scaled version of R (i.e., $ a R$, the set of points $\{a z \}$ for z in R)
Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1} , where z is in R).
Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r .	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	at least $R_1 \cap R_2$

z-Transform Properties II

Property	Signal	Transform	ROC
Conjugation	$x^*[n]$	$X^*(z^*)$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $ z > 0$
Accumulation	$\sum_{k=-\infty}^n x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $ z > 1$
Differentiation in the z -domain	$nx[n]$	$-z \frac{dX(z)}{dz}$	R

Initial value theorem:

If $x[n] = 0$ for $n < 0$, then $x[0] = \lim_{z \rightarrow \infty} X(z)$

Example

Consider an LTI system for which

$$y[n] = h[n] * x[n],$$

where

$$h[n] = \delta[n] - \delta[n - 1].$$

1. Find $H(z)$.
2. Find $y[n]$ in terms of $x[n]$.

Note that

$$\delta[n] - \delta[n - 1] \xleftrightarrow{\mathcal{Z}} 1 - z^{-1},$$

with ROC equal to the entire z -plane, except the origin. Also, this z -transform has a zero at $z = 1$. If

$$x[n] \xleftrightarrow{\mathcal{Z}} X(z), \quad \text{with ROC} = R. \quad (1)$$

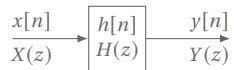
then

$$y[n] \xleftrightarrow{\mathcal{Z}} (1 - z^{-1})X(z), \quad (2)$$

with ROC equal to R with the possible deletion of $z = 0$ and or addition of $z = 1$.
Note for this system

$$y[n] = [\delta[n] - \delta[n - 1]] * x[n] = x[n] - x[n - 1]. \quad (3)$$

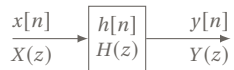
System Stability



$$Y[n] = h[n] * x[n]$$

$$Y(z) = H(z)X(z)$$

System Stability



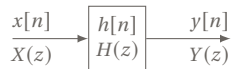
$$Y[n] = h[n] * x[n]$$

$$Y(z) = H(z)X(z)$$

$$\text{stable} \Leftrightarrow \sum_{-\infty}^{\infty} |h[n]| < \infty$$

$$\mathcal{F}\{h[n]\} \Leftrightarrow \sum_{-\infty}^{\infty} |h[n]| < \infty$$

System Stability



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The condition for stability and the existence of the Fourier transform are the same.

Stability, Causality, and ROC

stable \Leftrightarrow ROC of $H(z)$ includes unit circle in z -plane

causal $\Rightarrow h[n]$ is right-sided

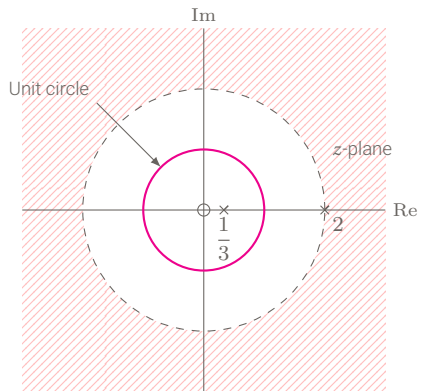
\Rightarrow ROC of $H(z)$ outside the outermost pole

causal and stable \Leftrightarrow All poles inside unit circle

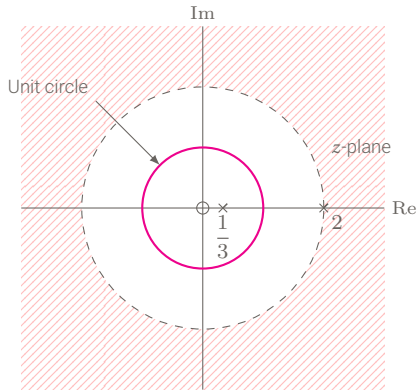
Example

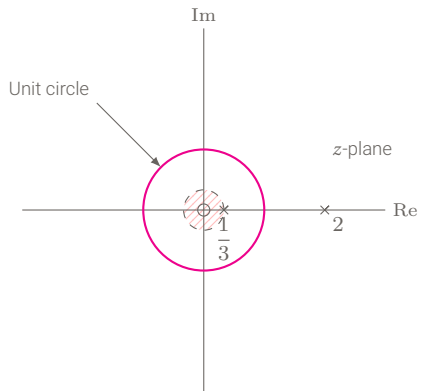
Discuss the stability and causality of the system represented by the following system function with respect to different regions of convergence.

$$H(z) = \frac{z}{\left(z - \frac{1}{3}\right)(z - 2)}.$$

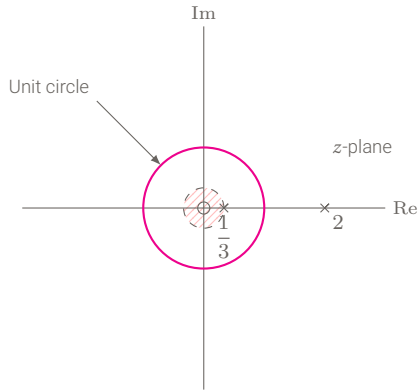


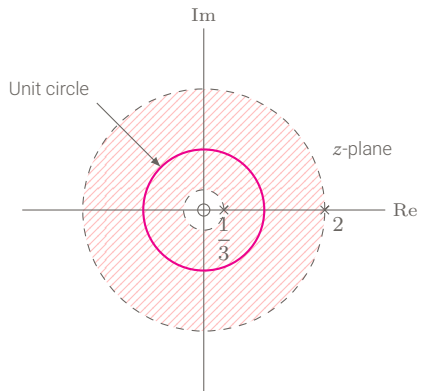
The system is causal and unstable.



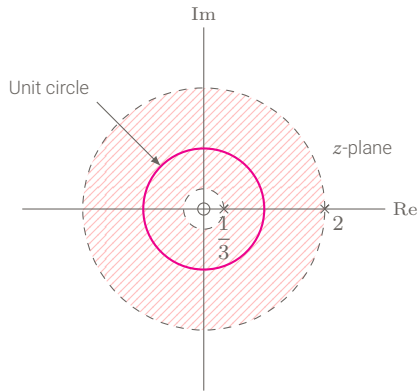


The system is unstable and not causal.





The system is stable and not causal.



Example

Consider the LTI system for which the input $x[n]$ and the output $y[n]$ satisfy the linear constant-coefficient difference equation

$$y[n] - \frac{1}{2}y[n-1] = x[n] + \frac{1}{3}x[n-1].$$

1. Obtain an expression for the system function $H(z)$.
2. What are the two choices for the region of convergence?
3. Obtain $h[n]$ for each of these cases and comment on the stability and causality.

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z)$$

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z)$$

$$Y(z) = X(z) \left[\frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}} \right]$$

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = X(z) + \frac{1}{3}z^{-1}X(z)$$

$$Y(z) = X(z) \left[\frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}} \right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3}z^{-1}}{1 - \frac{1}{2}z^{-1}}$$

There are two choices for the RoC:

$|z| > 1/2$: with the assumption that $h[n]$ is right sided.

$|z| < 1/2$: with the assumption that $h[n]$ is left sided.

If $|z| > 1/2$: with the assumption that $h[n]$ is right sided.

$$H(z) = \left(1 + \frac{1}{3}z^{-1}\right) \frac{1}{1 - \frac{1}{2}z^{-1}}$$

If $|z| > 1/2$: with the assumption that $h[n]$ is right sided.

$$H(z) = \left(1 + \frac{1}{3}z^{-1}\right) \frac{1}{1 - \frac{1}{2}z^{-1}}$$

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3} \left(\frac{1}{2}\right)^{n-1} u[n-1]$$

If $|z| > 1/2$: with the assumption that $h[n]$ is right sided.

$$H(z) = \left(1 + \frac{1}{3}z^{-1}\right) \frac{1}{1 - \frac{1}{2}z^{-1}}$$

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{3} \left(\frac{1}{2}\right)^{n-1} u[n-1]$$

$|z| < 1/2$: with the assumption that $h[n]$ is left sided.

$$h[n] = -\left(\frac{1}{2}\right)^n u[-n-1] - \frac{1}{3} \left(\frac{1}{2}\right)^{n-1} u[-n]$$

In this case the system is anticausal ($h[n] = 0$ for $n > 0$) and unstable.