EN1060 Signals and Systems: Fourier Transform Properties

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Section 1

Fourier Transform Properties

Fourier Transform: Recall

Synthesis equation:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega. \tag{1}$$

Analysis equation:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt. \tag{2}$$

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega). \tag{3}$$

Linearity

If

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega).$$

and

$$y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega).$$

then

$$ax(t) + by(t) \overset{\mathcal{F}}{\longleftrightarrow}$$

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Time Shifting

If

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Time Shifting

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$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega).$$

then

$$x(t - t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0} X(j\omega).$$

Proof

$$
\begin{aligned}
x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \, d\omega. \\
x(t - t_0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega(t - t_0)} \, d\omega. \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(e^{-j\omega t_0} X(j\omega)\right) e^{j\omega t} \, d\omega.
\end{aligned}
$$

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x(t-t_0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t-t_0)}\,d\omega. \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(e^{-j\omega t_0}X(j\omega)\right)e^{j\omega t}\,d\omega.
\end{aligned}
$$

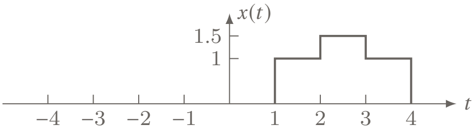
This is the synthesis equation for $x(t-t_0)$. Therefore,

$$\mathcal{F}\{x(t-t_0)\} = e^{-j\omega t_0}X(j\omega).$$

Magnitude of the Fourier transform not altered. Time shift introduces a phase shift $-\omega t_0$, which is a linear function of ω.

Example

Evaluate the Fourier transform of $x(t)$.



$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5)$$

$$X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega}$$

$$X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}$$

$$X(j\omega) = e^{-j5\omega/2}\left[\frac{\sin(\omega/2) + 2\sin(3\omega/2)}{\omega}\right].$$

For the rectangular pulse

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & |t| > T_1 \end{cases},$$

$$X(j\omega) = 2\frac{\sin \omega T_1}{\omega}$$

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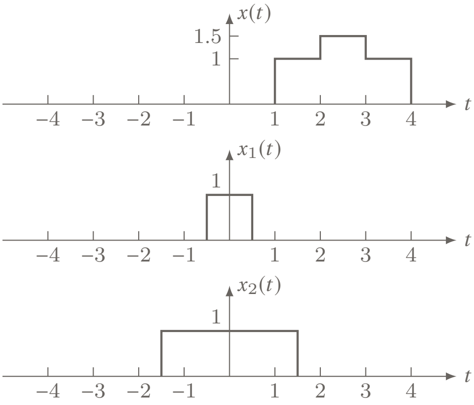

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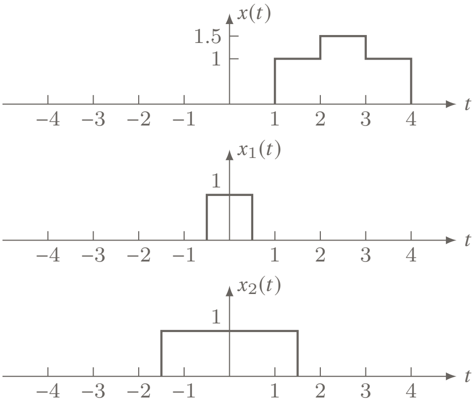

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Conjugation and Conjugate Symmetry

If

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega).$$

then

$$x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-j\omega).$$

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$$\begin{aligned} X^*(j\omega) &= \left[\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt\right]^* \\ &= \int_{-\infty}^{\infty} x^*(t)e^{j\omega t}\,dt \end{aligned}$$

Replacing ω by $-\omega$

$$X^*(-j\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}\,dt$$

which is the analysis equation for $x^*(t)$.

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which is the analysis equation for $x^*(t)$.

If $x(t)$ is real, i.e., $x(t) = x^*(t)$, $X(j\omega)$ has conjugate symmetry.

$$X(-j\omega) = X^*(j\omega) \quad [x(t) \text{ real}]$$

Using Conjugate Symmetry

Use the conjugate property to comment about the symmetry of Fourier transform of a signal $x(t)$ if

1. $x(t)$ is real,
2. $x(t)$ is real and even, and
3. $x(t)$ is real and odd.

Expressing $X(j\omega)$ in rectangular form as

$$X(j\omega) = \mathfrak{Re}\left\{X(j\omega)\right\} + j\mathfrak{Im}\left\{X(j\omega)\right\},$$

then if $x(t)$ is real $[x(t) = x^*(t)]$

$$\mathfrak{Re}\left\{X(j\omega)\right\} = \mathfrak{Re}\left\{X(-j\omega)\right\} \quad \text{and}$$
$$\mathfrak{Im}\left\{X(j\omega)\right\} = -\mathfrak{Im}\left\{X(-j\omega)\right\}$$

That is, the real part of the Fourier transform is an even function of frequency, and the imaginary part is an odd function of frequency.

Expressing $X(j\omega)$ in rectangular form as

$$X(j\omega) = \Re e\left\{X(j\omega)\right\} + j\Im m\left\{X(j\omega)\right\},$$

then if $x(t)$ is real $[x(t) = x^*(t)]$

$$\begin{aligned} \Re e\left\{X(j\omega)\right\} &= \Re e\left\{X(-j\omega)\right\} \quad \text{and} \\ \Im m\left\{X(j\omega)\right\} &= -\Im m\left\{X(-j\omega)\right\} \end{aligned}$$

That is, the real part of the Fourier transform is an even function of frequency, and the imaginary part is an odd function of frequency. Considering

$$X(j\omega) = |X(j\omega)|e^{\angle X(j\omega)},$$

we see that $|X(j\omega)|$ is an even function of frequency, and $\angle X(j\omega)$ is an odd function of frequency.

If $x(t)$ is both real and even, then $X(j\omega)$ will also be real and even.
Proof:

$$X(-j\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}\,dt$$

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With the substitution $\tau = -t$

$$X(-j\omega) = \int_{-\infty}^{\infty} x(-\tau)e^{-j\omega\tau}\,d\tau$$

Since $x(-\tau) = x(\tau)$ we have

$$X(-j\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\,d\tau$$

$$X(-j\omega) = X(j\omega)$$

In a similar manner, it can be shown that if $x(t)$ is a real and odd function of time, so that $x(t) = -x(-t)$, then $X(j\omega)$ is purely imaginary and odd.

Fourier Transforms of Odd and Even Parts

A real function $x(t)$ can be expressed as

$$x(t) = x_e(t) + x_o(t),$$

where $x_e(t) = \mathfrak{Ev}\{x(t)\}$ is the even part of $x(t)$ and $x_o(t) = \mathfrak{Od}\{x(t)\}$ is the odd part of $x(t)$. Express Fourier transforms of

1. $x_e(t) = \mathfrak{Ev}\{x(t)\}$, and
2. $x_o(t) = \mathfrak{Od}\{x(t)\}$.

in terms of $X(j\omega)$.

From the linearity of the Fourier transform,

$$\mathfrak{F}\left\{x(t)\right\} = \mathfrak{F}\left\{x_e(t)\right\} + \mathfrak{F}\left\{x_o(t)\right\},$$

and from the preceding discussion, $\mathfrak{F}\left\{x_e(t)\right\}$ is a real function and $\mathfrak{F}\left\{x_o(t)\right\}$ is purely imaginary. Thus, we can conclude that, with $x(t)$ real, From the linearity of the Fourier transform,

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and from the preceding discussion, $\mathfrak{F}\left\{x_e(t)\right\}$ is a real function and $\mathfrak{F}\left\{x_o(t)\right\}$ is purely imaginary. Thus, we can conclude that, with $x(t)$ real,

$$\begin{aligned} x(t) &\overset{\mathcal{F}}{\longleftrightarrow} X(j\omega), \\ \mathfrak{Ev}\left\{x(t)\right\} &\overset{\mathcal{F}}{\longleftrightarrow} \mathfrak{Re}\left\{X(j\omega)\right\}, \\ \mathfrak{Od}\left\{x(t)\right\} &\overset{\mathcal{F}}{\longleftrightarrow} j\mathfrak{Im}\left\{X(j\omega)\right\}. \end{aligned}$$

Example

Use the symmetry properties of the Fourier transform to evaluate the Fourier transform of

$$x(t) = e^{-a|t|}, \quad a > 0.$$

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We have already found that

$$e^{-at} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a + j\omega}.$$

$$\begin{aligned} x(t) = e^{-a|t|} &= e^{-at}u(t) + e^{at}u(-t) \\ &= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right] \\ &= 2\mathfrak{Ev}\left\{e^{-at}u(t)\right\}. \end{aligned}$$

Since $e^{-at}u(t)$is real valued, the symmetry properties of the Fourier transform lead us to conclude that

$$\mathfrak{Ev}\left\{e^{-at}u(t)\right\} \overset{\mathcal{F}}{\longleftrightarrow} \mathfrak{Re}\left\{\frac{1}{a + j\omega}\right\}.$$

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$$\mathfrak{Ev}\left\{e^{-at}u(t)\right\} \overset{\mathcal{F}}{\longleftrightarrow} \mathfrak{Re}\left\{\frac{1}{a + j\omega}\right\}.$$

$$X(j\omega) = 2\mathfrak{Re}\left\{\frac{1}{a + j\omega}\right\} = \frac{2a}{a^2 + \omega^2}.$$

Differentiation and Integration

Synthesis equation:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega.$$

Differentiating both sides of the equation

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega X(j\omega) e^{j\omega t} d\omega.$$

Therefore,

$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega).$$

Integration:

$$\int_{-\infty}^{t} x(\tau) d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega} X(j\omega) + \pi X(0) \delta(\omega).$$

Example

Determine the Fourier transform of the unit step $x(t) = u(t)$ making use of the knowledge that

$$g(t) = \delta(t) \xleftrightarrow{\mathcal{F}} G(j\omega) = 1.$$

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$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau$$

we obtain that

$$X(j\omega) = \frac{1}{j\omega}G(j\omega) + \pi G(0)\delta(\omega).$$

Since $G(j\omega) = 1$

$$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

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Since $G(j\omega) = 1$

$$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

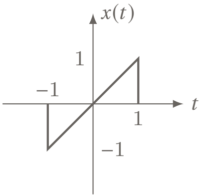
Observe that, we can apply the differentiation property to recover the transform of the impulse:

$$\delta(t) = \frac{du(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega \left[\frac{1}{j\omega} + \pi\delta(\omega) \right] = 1.$$

Note: $\omega\delta(\omega) = 0$

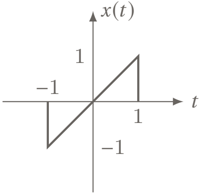
Example

Determine the Fourier transform of the signal $x(t)$ shown below:

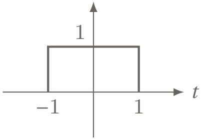


Example

Determine the Fourier transform of the signal $x(t)$ shown below:



$g(t) = \dfrac{dx(t)}{dt} =$



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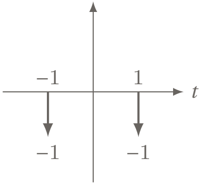
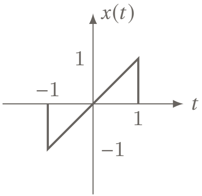


Example

Determine the Fourier transform of the signal $x(t)$ shown below:

$$G(j\omega) = \left(\frac{2\sin\omega}{\omega}\right) - e^{j\omega} - e^{-j\omega}$$



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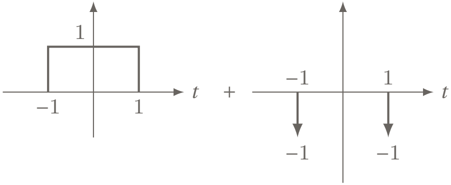

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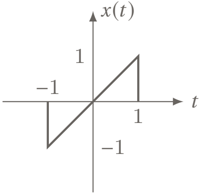

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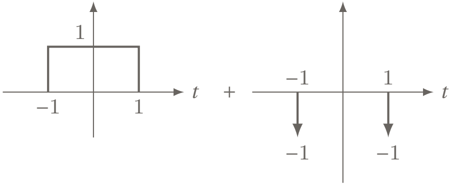

$$G(j\omega) = \left(\frac{2\sin\omega}{\omega}\right) - e^{j\omega} - e^{-j\omega}$$

$$X(j\omega) = \frac{1}{j\omega}G(j\omega) + \pi G(0)\delta(\omega).$$

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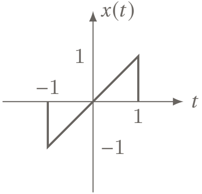

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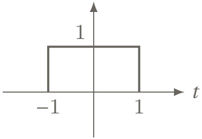

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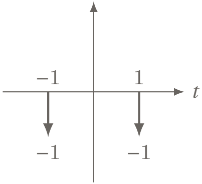

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$$X(j\omega) = \frac{1}{j\omega}G(j\omega) + \pi G(0)\delta(\omega).$$

As $G(0) = 0$

$$X(j\omega) = \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}$$

Note: $X(j\omega)$ is purely imaginary and odd.

Time and Frequency Scaling

If

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega).$$

then

$$x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right).$$

where a is a real constant.

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Letting $a = -1$

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where a is a real constant.
Letting $a = -1$

$$x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X(-j\omega).$$

The scaling property is another example of the inverse relationship between time and frequency.

Duality

Because of the similarity between the synthesis equation

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \, d\omega. \tag{4}$$

and the analysis equation,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt. \tag{5}$$

for any transform pair, there is a dual pair with the time and frequency variables interchanged.

We determined the Fourier transform of the square pulse as

$$x_1(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & |t| > T_1, \end{cases} \xleftrightarrow{\mathcal{F}} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}$$

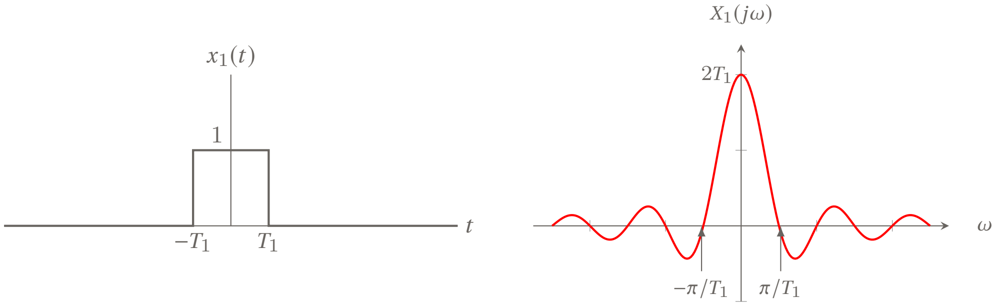


Figure: Rectangular pulse and the Fourier transform.

We also determined that for a time-domain signal that is similar in shape to the $X_1(j\omega)$ as

$$x_2(t) = \frac{\sin Wt}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} X_2(j\omega) = \begin{cases} 1, & |\omega| < W, \\ 0, & |\omega| > W. \end{cases}$$

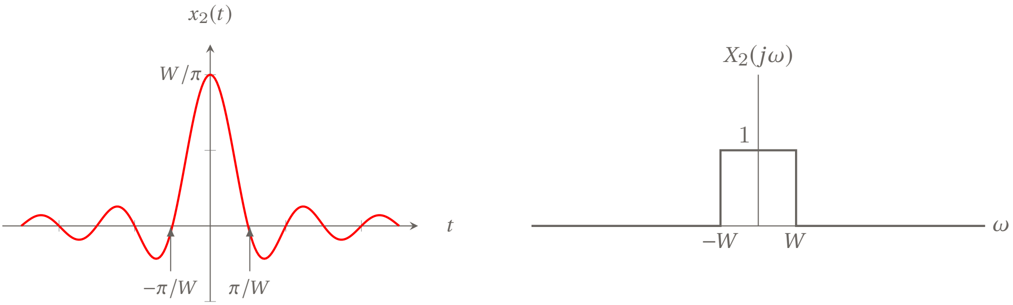
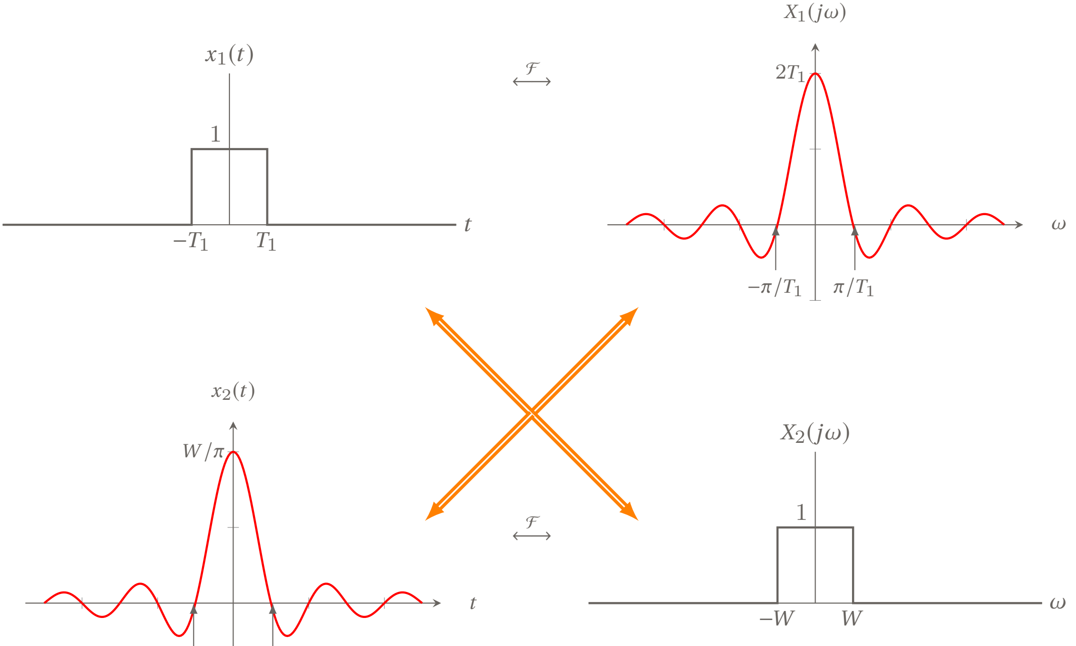


Figure: Fourier transform for $x(t)$.



$x_1(t)$
1
$-T_1$
T_1
t
$\overset{\mathcal{F}}{\longleftrightarrow}$
$X_1(j\omega)$
$2T_1$
ω
$-\pi/T_1$
π/T_1
$x_2(t)$
W/π
t
$\overset{\mathcal{F}}{\longleftrightarrow}$
$X_2(j\omega)$
1
$-W$
W
ω

Example

Use the duality property to find the Fourier transform $G(j\omega)$ of the signal

$$g(t) = \frac{2}{1+t^2}.$$

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Consider the signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \frac{2}{1+\omega^2}.$$

$$x(t) = e^{-|t|} \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega) = \frac{2}{1+\omega^2}.$$

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Now interchanging the names of variables t and ω

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$$2\pi e^{-|t|} = \int_{-\infty}^{\infty} \left(\frac{2}{1+\omega^2} \right) e^{-j\omega t} d\omega.$$

Now interchanging the names of variables t and ω

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty} \left(\frac{2}{1+t^2} \right) e^{-j\omega t} dt.$$

The synthesis equation for this Fourier transform pair is

$$e^{-|t|} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{2}{1+\omega^2} \right) e^{j\omega t} d\omega.$$

Multiplying this equation by 2π and replacing t by $-t$

$$2\pi e^{-|t|} = \int_{-\infty}^{\infty} \left(\frac{2}{1+\omega^2} \right) e^{-j\omega t} d\omega.$$

Now interchanging the names of variables t and ω

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty} \left(\frac{2}{1+t^2} \right) e^{-j\omega t} dt.$$

The right-hand side of this expression is the Fourier transform analysis equation for $2/(1+t^2)$. Thus

$$\mathcal{F} \left\{ \frac{2}{1+t^2} \right\} = 2\pi e^{-|\omega|}.$$

More Properties Using Duality

$$-jtx(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{dX(j\omega)}{d\omega}.$$

$$e^{j\omega_0 t}x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j(\omega - \omega_0)).$$

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\omega} x(\eta)d\eta.$$

Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega.$$

Outline

Fourier Transform Properties
    The Convolution Property

Convolution Property

$$y(t) = h(t) * x(t) \xleftrightarrow{\mathcal{F}} Y(j\omega) = H(j\omega)X(j\omega)$$

This equation is of major importance in signal and system analysis. This says that the Fourier transform maps the convolution of two signals into the product of their Fourier transforms.

Figure: Convolution property.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = h(t) * x(t)$$

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

Figure: Convolution property.

Example

An LTI system has the impulse response

$$h(t) = \delta(t - t_0).$$

If the Fourier transform of the input signal $x(t)$ is $X(j\omega)$, what is the Fourier transform of the output?

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Example

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$$h(t) = \delta(t - t_0).$$

If the Fourier transform of the input signal $x(t)$ is $X(j\omega)$, what is the Fourier transform of the output? $x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

$$\begin{aligned} h(t) &= \delta(t - t_0) \\ H(j\omega) &= e^{-j\omega t_0} \\ Y(j\omega) &= H(j\omega)X(j\omega) \\ &= e^{-j\omega t_0}X(j\omega) \end{aligned}$$

Example

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The input output relationship of the differentiator is

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From the differentiation property

$$Y(j\omega) = j\omega X(j\omega).$$

Consequently, the frequency response of the differentiator is

$$H(j\omega) = j\omega.$$

Example

Consider the response of an LTI system with impulse response

$$h(t) = e^{-at}u(t), \qquad a > 0,$$

to the input signal

$$x(t) = e^{-bt}u(t), \qquad b > 0.$$

Rather than computing $y(t) = x(t) * h(t)$ directly, find $y(t)$ by transfroming the problem into the frequency domain.

$$X(j\omega) = \frac{1}{b + j\omega}$$

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Therefore,

$$Y(j\omega) = \frac{1}{(a + j\omega)(b + j\omega)}$$

To determine the output $y(t)$, we wish to obtain the inverse transform of $Y(j\omega)$. This is most simply done by expanding $Y(j\omega)$ in a partial-fraction expansion.

$$Y(j\omega) = \frac{A}{a + j\omega} + \frac{B}{b + j\omega}$$

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$b \neq a$

$$A = \frac{1}{b - a} = -B,$$

$$Y(j\omega) = \frac{1}{b - a}\left[\frac{1}{a + j\omega} - \frac{1}{b + j\omega}\right]$$

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By inspection

$$y(t) = \frac{1}{b - a}\left[e^{-at}u(t) - e^{-bt}u(t)\right].$$

For the case $a = b$,

$$Y(j\omega) = \frac{1}{(a + j\omega)^2}.$$

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$$Y(j\omega) = \frac{1}{(a + j\omega)^2}.$$

Recognizing this as

$$\frac{1}{(a + j\omega)^2} = j\frac{d}{d\omega}\left[\frac{1}{a + j\omega}\right],$$

we can use the dual of the differentiation property,

$$e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \frac{1}{a + j\omega}$$

$$te^{-at}u(t) \xleftrightarrow{\mathcal{F}} j\frac{d}{d\omega}\left[\frac{1}{a + j\omega}\right] = \frac{1}{(a + j\omega)^2},$$

and consequently,

$$y(t) = te^{-at}u(t).$$

Multiplication Property

The convolution property states that convolution in time domain corresponds to multiplication in frequency domain. Because of the duality between time and frequency domains, we would expect a dual property also to hold (i.e., that multiplication in the time domain corresponds to convolution in the frequency domain). Specifically,

$$r(t) = s(t)p(t) \stackrel{\mathcal{F}}{\longleftrightarrow} R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)].$$

Multiplication of one signal by another can be thought of as using one signal to scale or modulate the amplitude of the other. Consequently, the multiplication of two signals is often referred to as amplitude modulation. For this reason, this equation is sometime referred to as the modulation property.

Example

Let $s(t)$ be a signal whose spectrum is depicted in the figure below. Also consider the signal

$$p(t) = \cos \omega_0 t\,.$$

Show the spectrum of $r(t) = s(t)p(t)$.

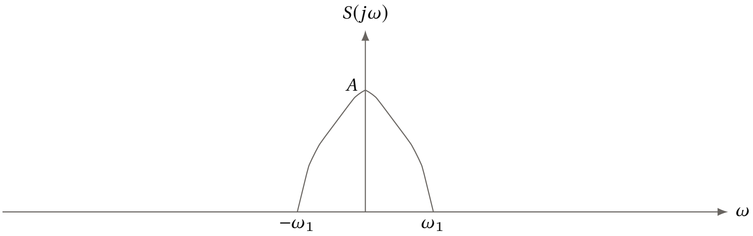

Figure: Spectrum of signal $s(t)$.

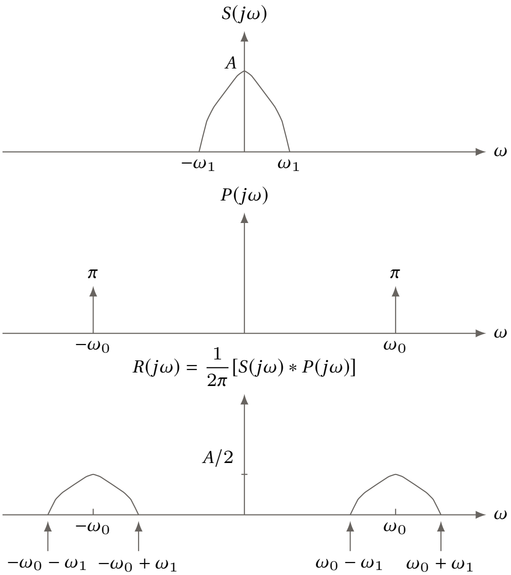

Figure: Fourier transform of $r(t) = s(t)p(t)$.