EN1060 Signals and Systems: Fourier Transform

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Section 1

Continuous-Time Fourier Transform

Outline

Continuous-Time Fourier Transform
Introduction
Development of the Fourier Transform Representation
The Fourier Transform for Periodic Signals

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- Using the Fourier techniques we can obtain the frequency-domain representation of signals.

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  3. Discrete-time Fourier series
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- In this part of the course, we will concentrate on how to actually compute continuous-time Fourier series and transform. Later, after we study liner, time-invariant (LTI) systems, we will study the conceptual aspects of Fourier techniques.

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- The synthesis integral itself, which uses the Fourier transform to represent the signal as a linear combination (integral) of complex exponentials, is called the inverse Fourier transform.

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Fourier Series Representation for Square Wave

The continuous-time periodic square wave, sketched below, is defined over one period as

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < T/2, \end{cases}$$

This signal is periodically repeats with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$.

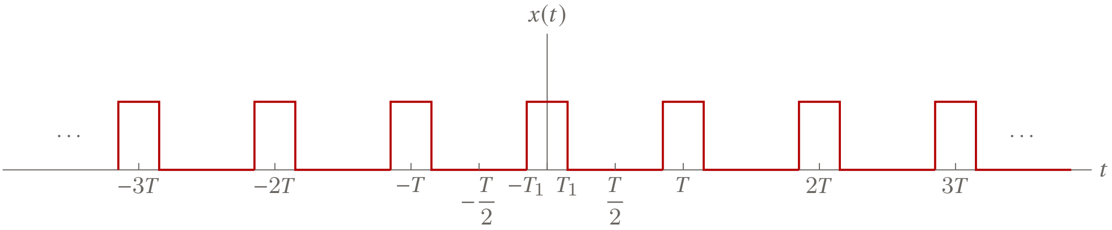

Figure: Periodic square wave

The Fourier series coefficients a_k of this wave are

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \tag{1}$$

We plotted this for a fixed value of T_1 and several values of T (shown in the next slide). An alternative wave of interpreting eq. 1 is as samples of an envelope function:

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With ω thought of as a continuous variable, the function $\dfrac{2\sin(\omega T_1)}{\omega}$ represents the envelope of $T a_k$, and the coefficients a_k are simple equally spaced samples of this envelope. For fixed T_1, the envelope of $T a_k$ is independent of T.

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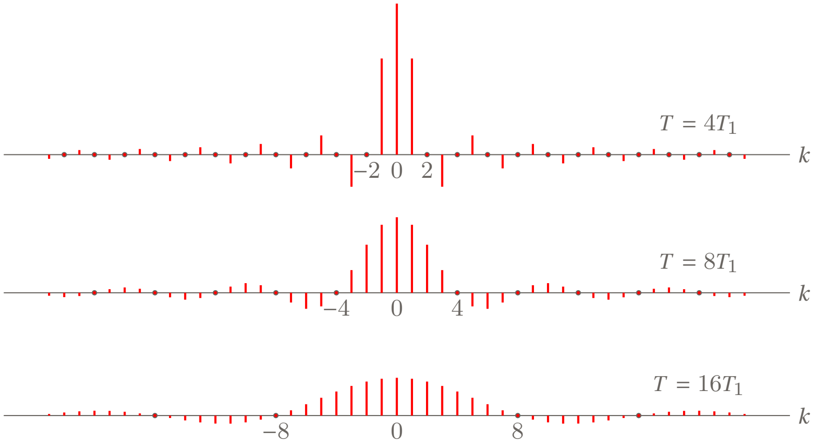
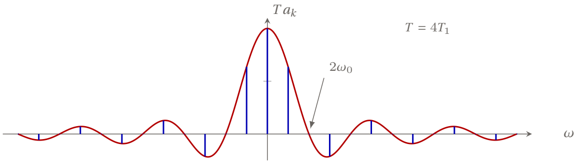


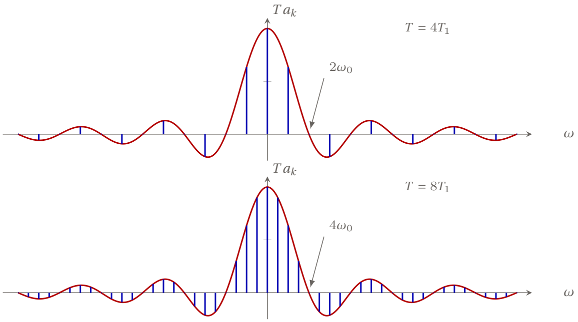
Figure: Plots of scaled Fourier series coefficients a_k

Plots of scaled Fourier series coefficients a_k for the periodic square wave with T_1 fixed and for several values of T: $T = 4T_1$, $T = 8T_1$, $T = 16T_1$.



The Fourier series coefficients and their envelope for periodic square wave for several values of T (with T_1 fixed): $T = 4T_1$, $T = 8T_1$, $T = 16T_1$. The coefficients are regularly-spaced samples of the envelope $(2 \sin \omega T_1)/\omega$, where the spacing between samples, $2\pi/T$, decreases as T increases.

Figure: Fourier series coefficients and their envelope for periodic square wave.



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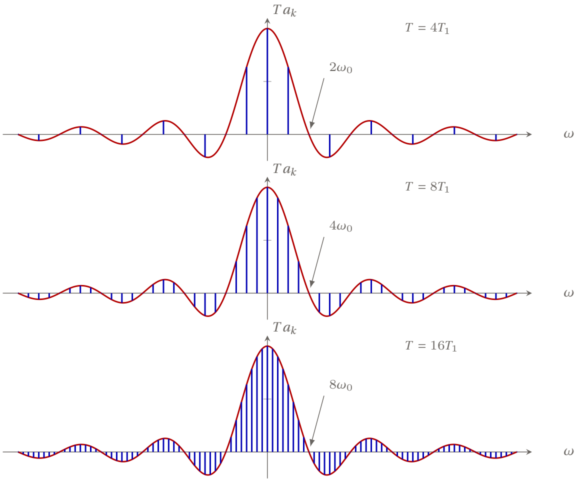

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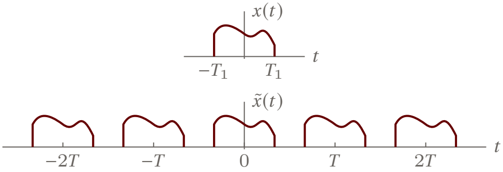
As T increases, or equivalently, as the fundamental frequency $\omega_0 = 2\pi/T$ decreases, the envelope is sampled with a close and closer spacing. As T becomes arbitrarily large, the original periodic square waveform approaches the rectangular pulse. Also, the Fourier series coefficients, multiplied by T, become more and more closely spaced sampled of the envelope. So, in some sense, the set of Fourier series coefficients approaches the envelope function as $T \to \infty$.



$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \qquad \omega_0 = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t}\, dt.$$

As $\tilde{x}(t) = x(t)$ for $|t| < T/2$, and also, as $x(t) = 0$ outside this interval,

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t}\, dt.$$

Defining the envelope $X(j\omega)$ of Ta_k as

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt,$$

we have, for the coefficients a_k,

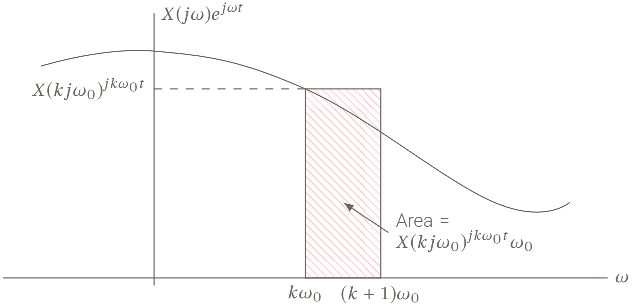
$$a_k = \frac{1}{T} X(jk\omega_0).$$

Combining and expressing $\tilde{x}(t)$ in terms of $X(j\omega)$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t},$$

or, as $\omega_0 = 2\pi/T$

$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t}\omega_0. \qquad (2)$$



$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0.$$

Inverse Fourier transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega.$$

Fourier transform or Fourier integral:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt.$$

As $T \to \infty$, $\tilde{x}(t)$ approaches $x(t)$, and consequently, Eq. 2 becomes a representation of $x(t)$. Furthermore, as $\omega_0 \to 0$ as $T \to \infty$, and the right-hand side of Eq. 2 passes to an integral. As $\omega_0 \to 0$, the summation converges to the integral of $X(j\omega)e^{j\omega t}$.

Fourier Transform: Synthesis and Analysis Equations

Inverse Fourier transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega. \tag{3}$$

Fourier transform or Fourier integral:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt. \tag{4}$$

The transform $X(j\omega)$ of an aperiodic signal $x(t)$ is referred to as the spectrum of $x(t)$.

FS synthesis and analysis equations:

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$$

Relation with a_k

Assume that the Fourier transform of $x(t)$ is $X(j\omega)$.
If we construct a periodic signal $\tilde{x}(t)$ by repeating the aperiodic signals $x(t)$ with period T, its Fourier series coefficients are

$$a_k = \frac{1}{T} X(j\omega)\Big|_{\omega = k\omega_0} \tag{5}$$

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FT synthesis and analysis equations:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Convergence of Fourier Transform

Assume that we evaluated $X(j\omega)$ according to eq. 4, and left $\hat{x}(t)$ denote the signal obtained by using $X(j\omega)$ in 3:

$$\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega.$$

When is $\hat{x}(t)$ a valid representation of the original signal $x(t)$? We define the error between $\hat{x}(t)$ and $x(t)$ as

$$e(t) = \hat{x}(t) - x(t).$$

If $x(t)$ has finite energy (square integrable), i.e.,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty, \tag{6}$$

$X(j\omega)$ is finite, and

$$\int_{-\infty}^{\infty} |e(t)|^2 dt = 0, \tag{7}$$

If $x(t)$ has finite energy, then, although $x(t)$ and its Fourier representation $\hat{x}(t)$ may differ significantly at individual values of t, there is no energy in their difference.

Convergence of Fourier Transform: Dirichlet Conditions

There are alternative conditions sufficient to ensure that $\hat{x}(t)$ is qual to $x(t)$ for any t except at a discontinuity, where it is equal to the average of the values on either side of the discontinuity.

1. $x(t)$ is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty, \tag{8}$$

2. $x(t)$ has a finite number of maxima and minima within any finite interval.
3. $x(t)$ has a finite number of discontinuities within any finite interval. Furthermore, each of these discontinuities must be finite.

Therefore, absolutely integrable signals that are continuous or that have finite number of discontinuities have a Fourier transform.

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Find the Fourier transform of the signal

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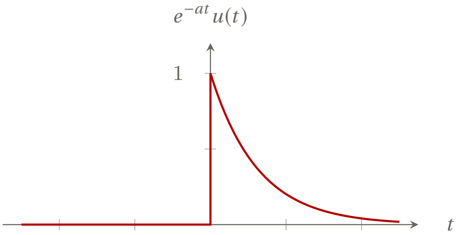

Figure: $e^{-at}u(t), \quad a > 0.$

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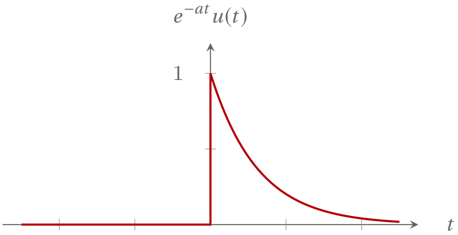

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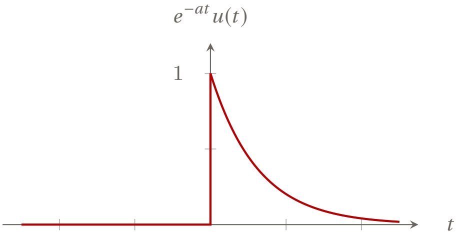

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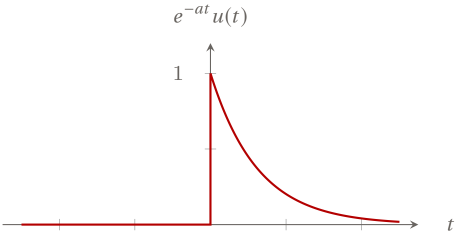

Figure: $e^{-at}u(t), \qquad a > 0.$

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Example Cntd. FT of $e^{-at}u(t), \quad a > 0$

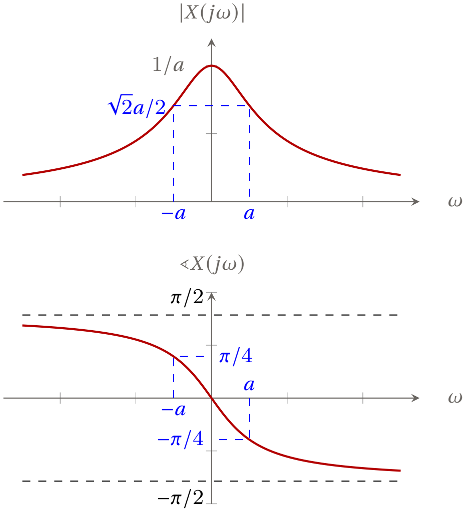

Figure: Fourier transform of the signal $e^{-at}u(t), \quad a > 0$.

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$$= \int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t} dt.$$
$$X(j\omega) = \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt$$

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X(j\omega) &= \int_{-\infty}^{0} e^{at}e^{-j\omega t}\,dt + \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt \\
X(j\omega) &= \frac{1}{a - j\omega} + \frac{1}{a + j\omega}, \\
&= \frac{2a}{a^2 + \omega^2}.
\end{aligned}$$

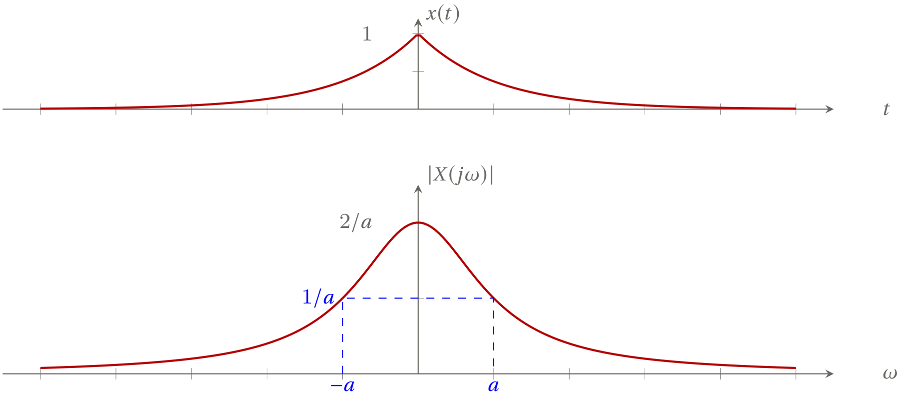

Figure: Fourier transform of the signal $e^{-a|t|}$, $\quad a > 0$.

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Determine the Fourier transform of the unit impulse

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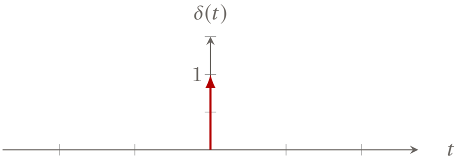

Figure: $\delta(t)$

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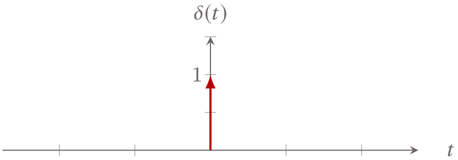

Figure: $\delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1.$$

The unit impulse has a Fourier transform consisting of qual contributions at all frequencies.

Rectangular Pulse

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Determine the Fourier transform of the signal

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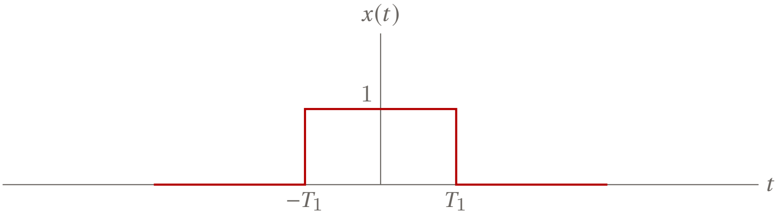

Figure: Rectangular pulse and the Fourier transform.

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt.$$

$$
\begin{aligned}
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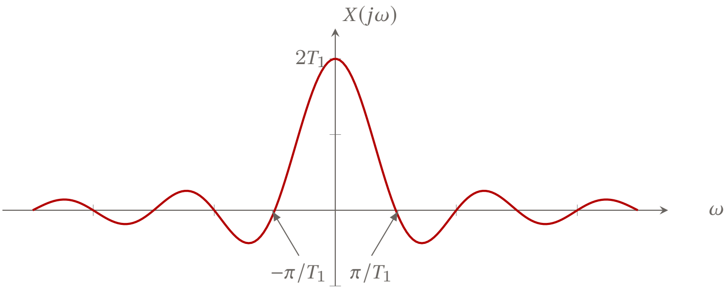


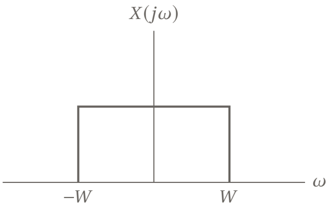
Figure: Fourier transform of the rectangular pulse.

Example

Consider the signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \begin{cases} 1, & |\omega| < W, \\ 0, & |\omega| > W. \end{cases}$$

Determine $x(t)$.



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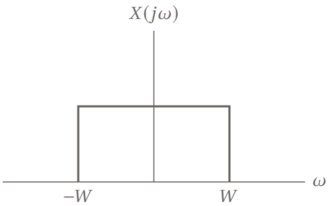

Figure: Fourier transform for $x(t)$.

Using the synthesis equation:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \, d\omega.$$

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$$\begin{aligned} x(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \, d\omega. \\ &= \frac{1}{2\pi} \int_{-W}^{W} e^{j\omega t} \, d\omega. \end{aligned}$$

Using the synthesis equation:

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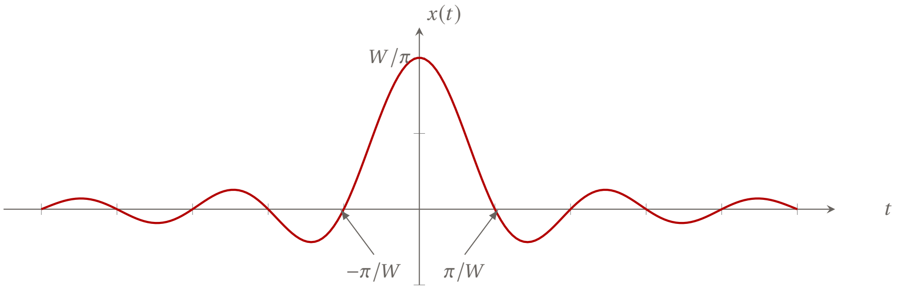

Figure: Time function.

The sinc Function

$$\mathrm{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}. \tag{9}$$

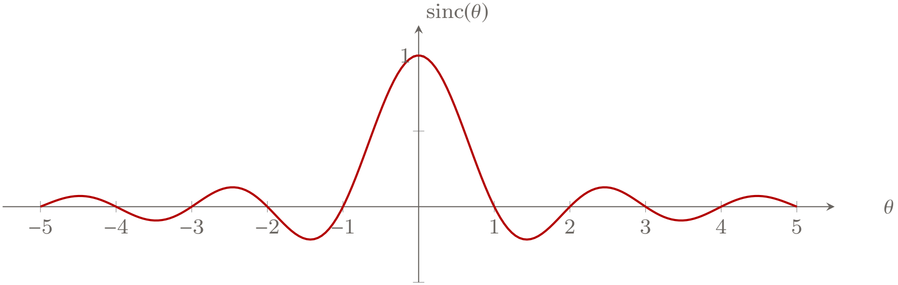

Figure: Fourier transform for $x(t)$.

Express

$$\frac{2\sin\omega T_1}{\omega}$$

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What Happens when W Increases?

- As W increases, $X(j\omega)$ becomes broader, while the main peak of $x(t)$ at $t = 0$ becomes higher and the width of the first lobe of this signal (i.e., the part of the signal for $|t| < \pi/W$) becomes narrower.

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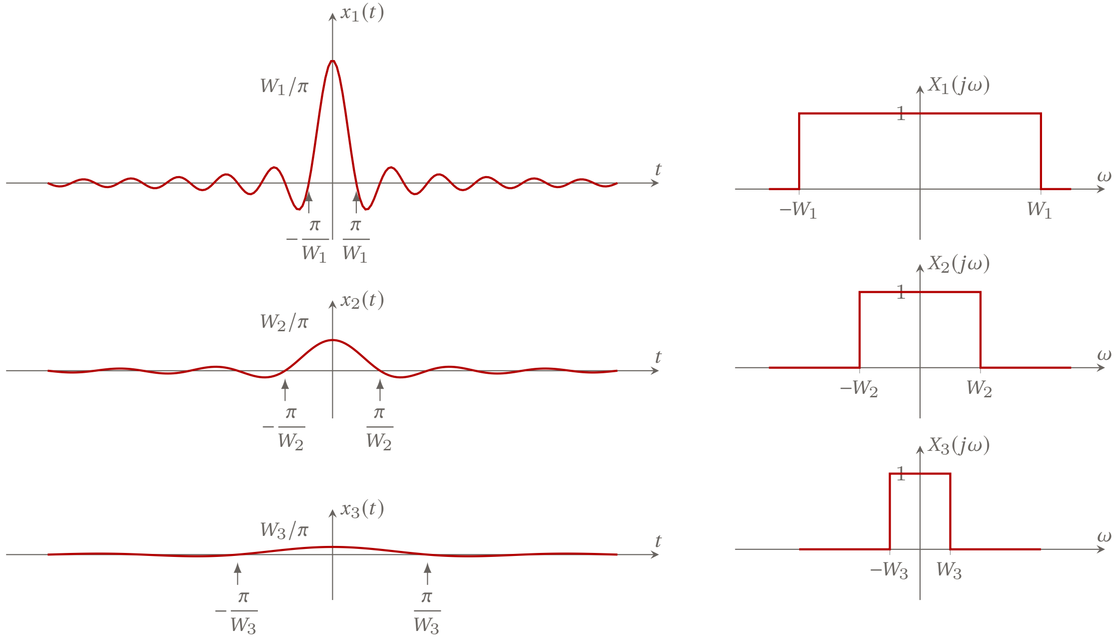
- In fact, in the limit as $W \to \infty$, $X(j\omega) = 1$ for all ω, and consequently, we see that $x(t)$ converges to an impulse as $W \to \infty$.
- The behavior is an example of the inverse relationship that exists between the time and frequency domains.



$x_1(t)$
W_1/π
t
$-\frac{\pi}{W_1}$
$\frac{\pi}{W_1}$
$x_2(t)$
W_2/π
t
$-\frac{\pi}{W_2}$
$\frac{\pi}{W_2}$
$x_3(t)$
W_3/π
t
$-\frac{\pi}{W_3}$
$\frac{\pi}{W_3}$
$X_1(j\omega)$
1
ω
$-W_1$
W_1
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1
ω
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ω
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W_3

Outline

Continuous-Time Fourier Transform

- Introduction
- Development of the Fourier Transform Representation
- The Fourier Transform for Periodic Signals

The Fourier Transform for Periodic Signals: Introduction

In the previous section, we studied the Fourier transform representation, paying attention to aperiodic signals. We can also develop Fourier transform representations for periodic signals. This allows us to consider periodic and aperiodic signals in a unified context. We can construct the Fourier transformof a periodic signal directly from its Fourier series representation.

Consider a signal $x(t)$ with Fourier transform $X(j\omega)$ that is a single impulse of area 2π at $\omega = \omega_0$, i.e.,

$$X(j\omega) = 2\pi\delta(\omega - \omega_0) \tag{10}$$

Let's determine the signal $x(t)$:

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$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{j\omega t}\,d\omega, \\ &= e^{j\omega_0 t}. \end{aligned}$$

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More generally, if $X(j\omega)$ is of the form of a linear combination of impulses equally spaced in frequency, i.e.,

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then

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}. \tag{12}$$

which is exactly the Fourier series representation of a periodic signal.
Thus, the Fourier transformof a periodic signal with Fourier series coefficients $\{a_k\}$ can be interpreted as a train of impulses occurring at the harmonically related frequencies and for which the area of the impulse at the kth harmonic frequency $k\omega_0$ is 2π times the kth Fourier series coefficient a_k.

Example

Find the Fourier transformof the square wave signal whose Fourier series coefficients are

$$a_k = \frac{\sin k\omega_0 T_1}{\pi k}.$$

Method: Multiply the Fourier seriescoefficients a_k by 2π, place them using the impulse function $\delta(\omega - k\omega_0)$, and sum.

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

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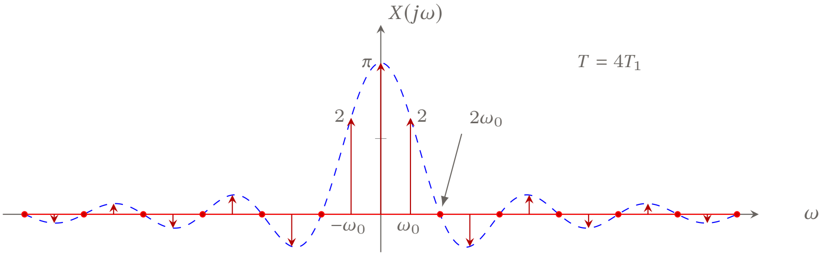

Figure: Fourier transform of a symmetric periodic square wave.

Example

Find the Fourier transformof

$$x(t) = \sin \omega_0 t,$$

and

$$x(t) = \cos \omega_0 t.$$

$$x(t) = \sin \omega_0 t\,.$$

The Fourier series coefficients for this signal are

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j},$$

$a_k = 0,\ k \neq 1$ or 1

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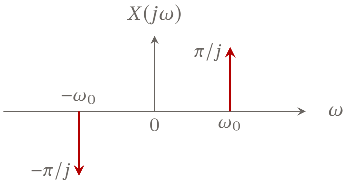

Figure: Fourier transform of the $x(t) = \sin \omega_0 t$.

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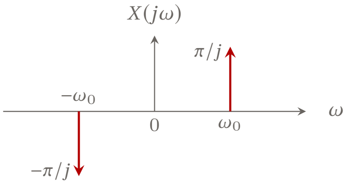

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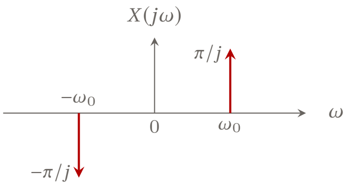

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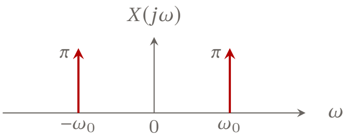

Figure: Fourier transform of the $x(t) = \cos \omega_0 t$.

Example

Find the Fourier transform of the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT).$$

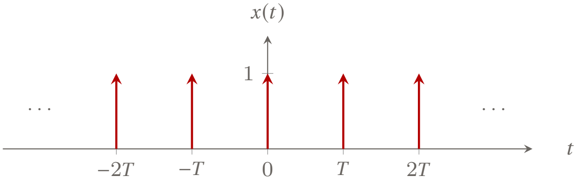

Figure: Pulse train.

The Fourier series coefficients for this signal:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta t e^{-j\omega_0 t} dt = \frac{1}{T}.$$

That is, every Fourier coefficient of the periodic impulse train has the same value, $1/T$. Substituting this value for a_k

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Thus, the Fourier transform of a periodic impulse train in the time domain with period T is a periodic impulse train in the frequency domain with period $2\pi/T$,

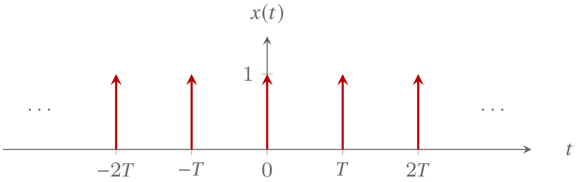
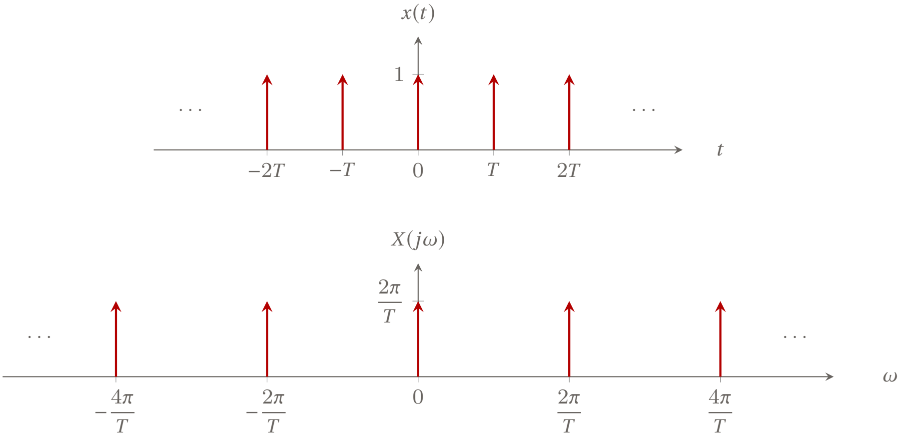


Figure: Periodic impulse train and its Fourier transform.



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