# EN1060 Signals and Systems: Sampling

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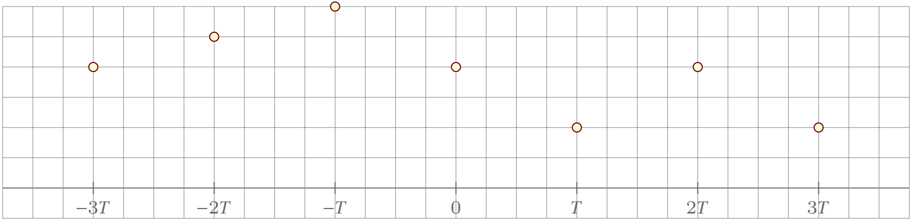
# Section 1

## Sampling and Reconstruction

# Introduction

- Under ceratin conditions, a continuous-time (CT) signal can be completely represented by and recoverable from knowledge of its values at points equally spaced in time.
- These values are called samples.
- This somewhat surprising property follows from a basic result that is referred to as the alertsampling theorem.
- Sampling theorem is extremely important, particulary as it forms the bridge between CT signals and discrete-time (DT) signals.
- Under ceratin conditions, a CT signal can be completely recovered from a sequence of its samples. This provides a mechanism for representing CT signals by a DT signal.
- We exploit sampling to convert a CT signal to a DT signal , process the DT signal using a DT system, and then convert back to CT.

In general, in the absence of any additional information, we would not expect that a signal could be uniquely specified by a sequence of equally spaced samples.




However, if the signal is band-limited—i.e., if its Fourier transform is zero outside a finite band of frequencies—and if the samples are taken sufficiently close together in relation to the highest frequency present in the signal, then the samples uniquely specify the signal, and we can reconstruct it perfectly.
This result is known as the sampling theorem.

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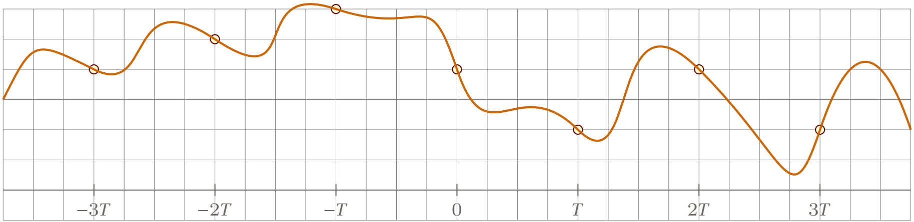


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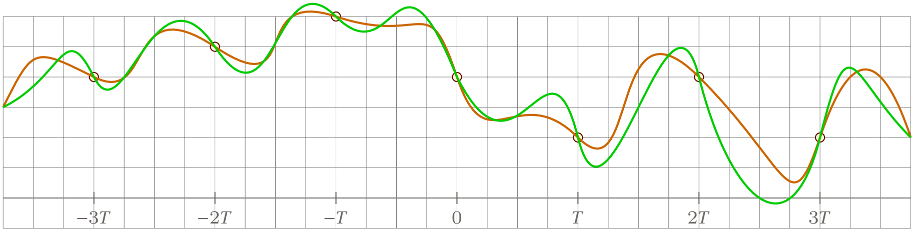


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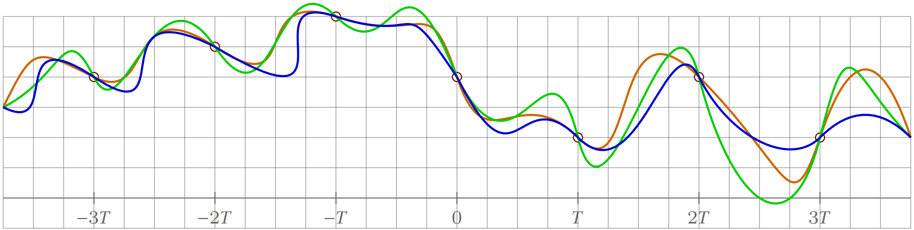


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# Impulse-Train Sampling

A convenient way to represent sampling of a CT signal at regular intervals is to use an impulse train multiplied by the CT signal $x(t)$ that we wish to sample.

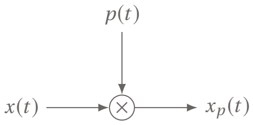
$$x_p(t) = x(t)p(t)$$

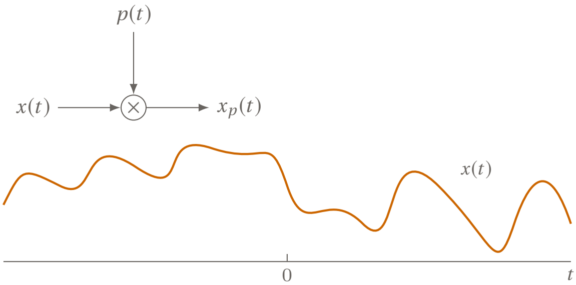
where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$p(t)$: sampling function, $T$: sampling period, $\omega_s = 2\pi/T$: sampling frequency.

$$\begin{aligned} x_p(t) &= x(t) \sum_{n=-\infty}^{+\infty} \delta(t - nT) \\ &= \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT) \end{aligned}$$


p(t)
x(t)
x_p(t)

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$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

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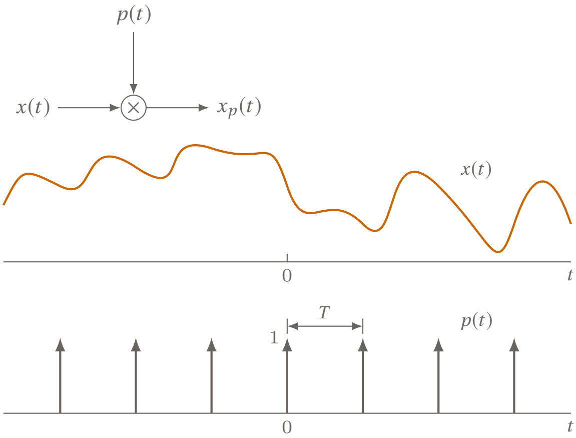
$$= \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$

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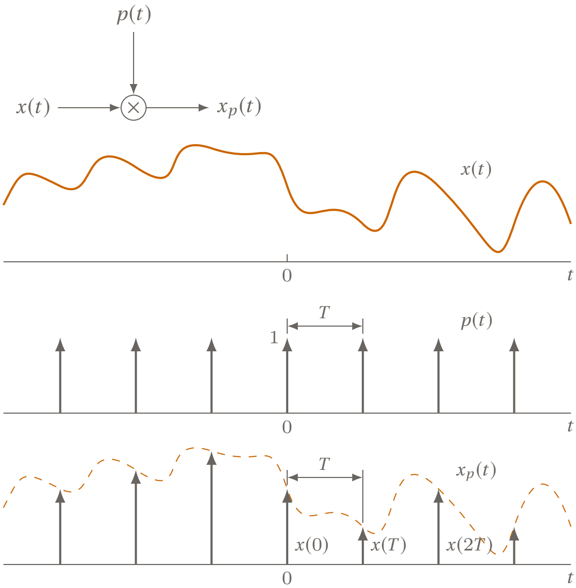


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# Impulse-Train Sampling Cntd.

$$X_p(j\omega) = \frac{1}{2\pi}\left[X(\omega) * P(\omega)\right]$$

$$P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{+\infty}\delta\left(\omega - k\omega_s\right)$$

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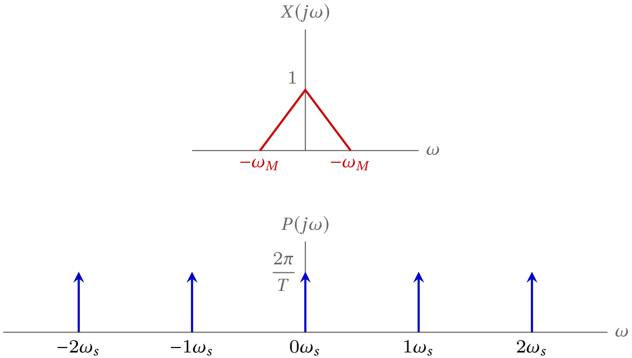
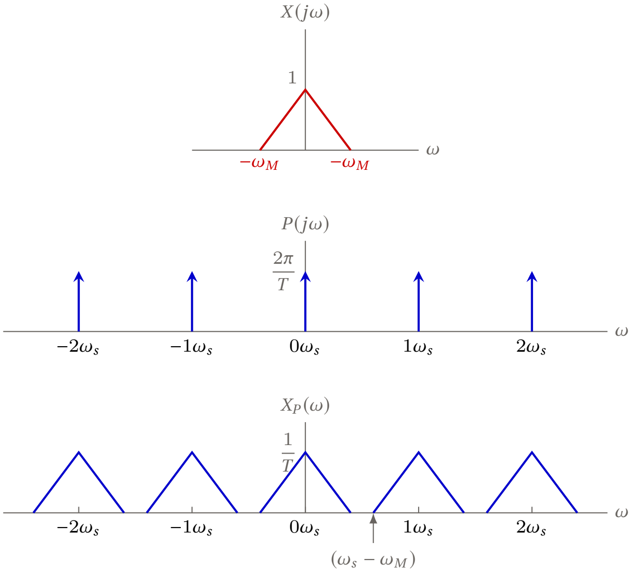
$$X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{+\infty}X\left(j(\omega - k\omega_s)\right)$$

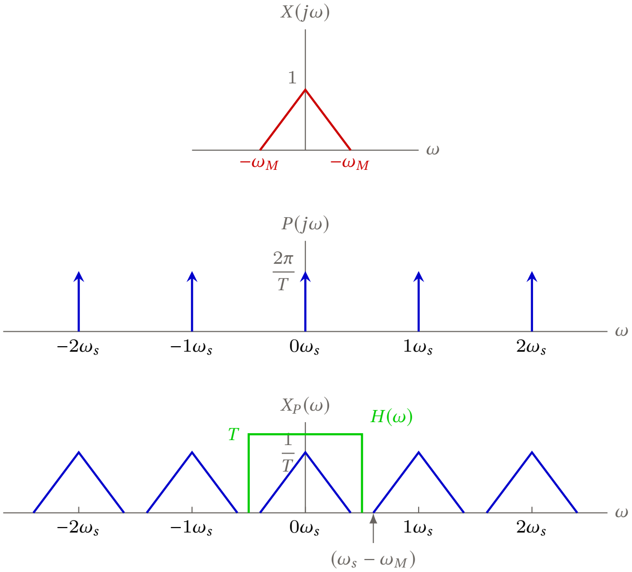
That is, $X_p(j\omega)$ is a periodic function of $\omega$ consiting of superposition of shifted replicas of $X(j\omega)$, scaled by $1/T$.



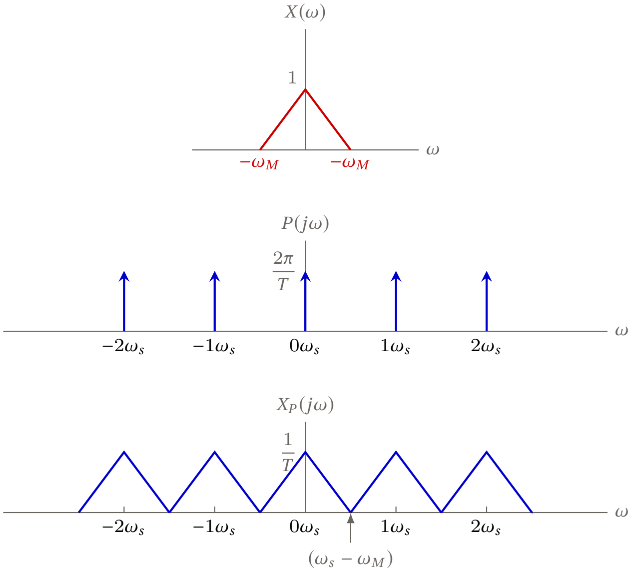
X(jω)
1
ω
−ω_M
−ω_M
P(jω)
2π/T
ω
−2ω_s
−1ω_s
0ω_s
1ω_s
2ω_s



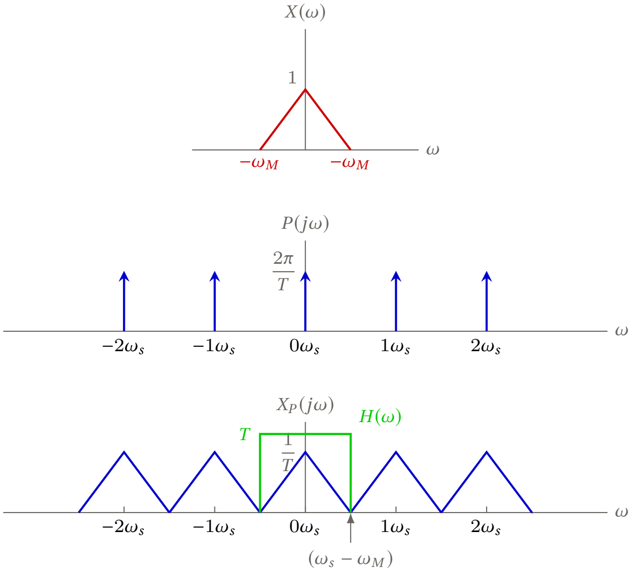
$X(j\omega)$
1
$\omega$
$-\omega_M$
$-\omega_M$
$P(j\omega)$
$\frac{2\pi}{T}$
$\omega$
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$X_P(\omega)$
$\frac{1}{T}$
$\omega$
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$(\omega_s - \omega_M)$



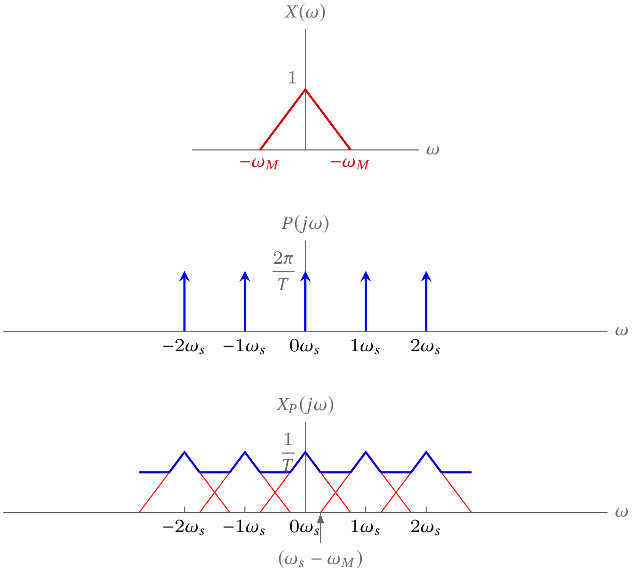
$X(j\omega)$
1
$-\omega_M$
$-\omega_M$
$\omega$
$P(j\omega)$
$\frac{2\pi}{T}$
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$\omega$
$X_P(\omega)$
$H(\omega)$
$T$
$\frac{1}{T}$
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$\omega$
$(\omega_s - \omega_M)$



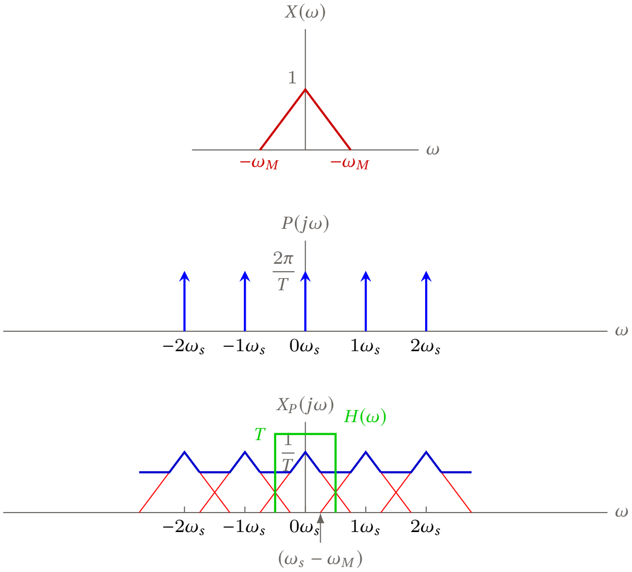
$X(\omega)$
1
$-\omega_M$
$-\omega_M$
$\omega$
$P(j\omega)$
$\frac{2\pi}{T}$
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$\omega$
$X_P(j\omega)$
$\frac{1}{T}$
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$\omega$
$(\omega_s - \omega_M)$



$X(\omega)$
1
$\omega$
$-\omega_M$
$-\omega_M$
$P(j\omega)$
$\frac{2\pi}{T}$
$\omega$
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$X_P(j\omega)$
$H(\omega)$
$T$
$\frac{1}{T}$
$\omega$
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$(\omega_s - \omega_M)$



$X(\omega)$
1
$-\omega_M$
$-\omega_M$
$\omega$
$P(j\omega)$
$\frac{2\pi}{T}$
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$\omega$
$X_P(j\omega)$
$\frac{1}{T}$
$-2\omega_s$
$-1\omega_s$
$0\omega_s$
$1\omega_s$
$2\omega_s$
$\omega$
$(\omega_s - \omega_M)$



$X(\omega)$
1
$-\omega_M$
$-\omega_M$
$\omega$
$P(j\omega)$
$\frac{2\pi}{T}$
$-2\omega_s$ $-1\omega_s$ $0\omega_s$ $1\omega_s$ $2\omega_s$
$\omega$
$X_P(j\omega)$
$H(\omega)$
$T$
$\frac{1}{T}$
$-2\omega_s$ $-1\omega_s$ $0\omega_s$ $1\omega_s$ $2\omega_s$
$\omega$
$(\omega_s - \omega_M)$

# Sampling Theorem

If $\omega_M < (\omega_s - \omega_M)$, or equivalently $\omega_s > 2\omega_M$, there is no overlap between shifted replicas of $X(j\omega)$. If $\omega_s > 2\omega_M$, $x(t)$ can be recovered exactly from $x_p(t)$ by means of a lowpass filter with gain $T$ and a cutoff frequency greater than $\omega_M$ and less than $\omega_s - \omega_M$.

## Theorem

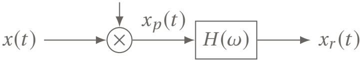
Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = 0, \pm 1, \pm 2, \ldots$, if

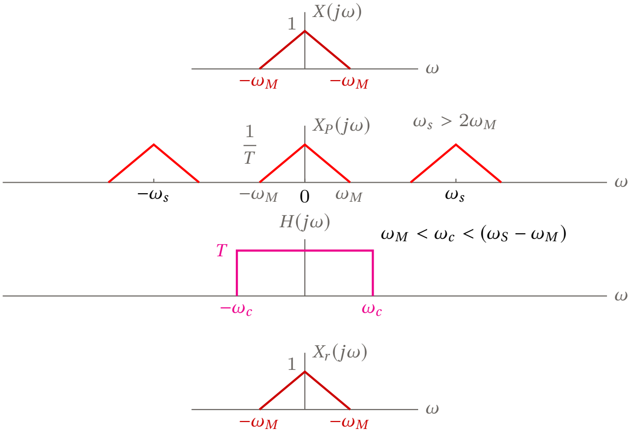
$$\omega_2 > 2\omega_M,$$

where $\omega_2 = 2\pi/T$

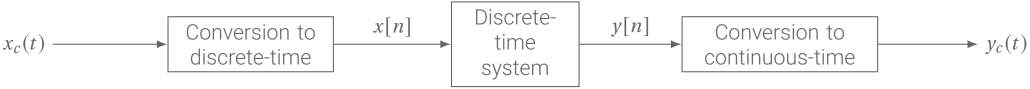
Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values, The impulse train is then passed through an ideal lowpass filter with gain $T$ and cutoff frequency greater than $\omega_M$ and less than $\omega_s - \omega_M$. the resulting output signal will exactly equal $x(t)$.

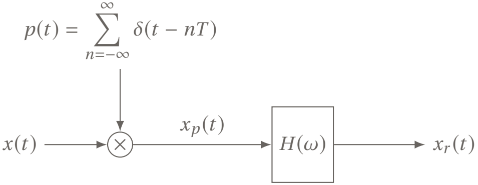
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



# Discrete-Time Processing of Continuous-Time Signals






$$
\begin{aligned}
x_p(t) &= x(t)p(t) \\
&= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\
&= \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)
\end{aligned}
$$

$$
\begin{aligned}
x_r(t) &= x_p(t) * h(t) \\
&= \sum_{n=-\infty}^{+\infty} x(nT)h(t - nT)
\end{aligned}
$$