5D Motion Subspaces for Planar Motions: Supplemental Material

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1 Extracting Rank-Degenerate Solutions

The linear system in Eq. 11 is concisely formulated as

$$\left[\hat{\mathbf{M}}^T \odot \hat{\mathbf{M}}^T\right]^T \mathbf{K}_5 \operatorname{vecs}(\mathbf{Q}) = \mathbf{1}, \tag{1}$$

where $\odot$ denotes the Khatri-Rao product with column-wise block-partitioning (i.e. column-wise Kronecker product), $\operatorname{vecs}()$ vectorizes the upper triangular part of a matrix, and $\mathbf{K}_5$ is the duplicity matrix s.t. $\operatorname{vec}(\mathbf{Q}) = \mathbf{K}_5 \operatorname{vecs}(\mathbf{Q})$ (we refer to reference [1] for more details about these operators). Eq. (1) can be solved in the least squares sense. The solution will in general have rank 3. Let $\mathbf{Q}_0 \in \mathbb{R}^{5\times 5}$ denote a particular solution and $\mathbf{N} \in \mathbb{R}^{5\times 5}$ denote the nullspace of the linear system in Eq. (1). The particular solution and the solution of the nullspace will be of rank 3 and will have the following parameterization $b(\mathbf{q}_1\mathbf{q}_1^T + \mathbf{q}_3\mathbf{q}_3^T) + (1-b)(\mathbf{q}_1+\mathbf{q}_2)(\mathbf{q}_1+\mathbf{q}_2)^T$ with $b \in \mathbb{R}$ in the unknown $b_{\mathbf{Q}_0}$ resp. $b_{\mathbf{N}}$. In order to find the rank deficient solutions, a 3^{rd}-order polynomial constraint in x could be imposed on all the 3×3 subdeterminants of $\mathbf{Q}_0 + x\mathbf{N}$. However, it is difficult to robustly combine the constraints of all the 3-by-3 subdeterminants in one polynomial constraint. Another approach is based on the fact, that we can readily solve $\hat{\mathbf{M}}(\mathbf{q}_1+\mathbf{q}_2) = \mathbf{1}_{F\times 1}$ for the vector $\mathbf{q}_1+\mathbf{q}_2$. Then we have

$$\begin{aligned}
\mathbb{P}^{\perp}_{\mathbf{q}_1+\mathbf{q}_2}\left[\mathbf{Q}_0 + x\mathbf{N}\right] = \mathbb{P}^{\perp}_{\mathbf{q}_1+\mathbf{q}_2}\Big[&\left(b_{\mathbf{Q}_0} + x b_{\mathbf{N}}\right)\left[\mathbf{q}_1\mathbf{q}_1^T + \mathbf{q}_3\mathbf{q}_3^T\right] \\
&+ \left(1 - b_{\mathbf{Q}_0} + x\left(1 - b_{\mathbf{N}}\right)\right)\left(\mathbf{q}_1+\mathbf{q}_2\right)\left(\mathbf{q}_1+\mathbf{q}_2\right)^T\Big] \\
= \mathbb{P}^{\perp}_{\mathbf{q}_1+\mathbf{q}_2}&\left[\left(b_{\mathbf{Q}_0} + x b_{\mathbf{N}}\right)\left[\mathbf{q}_1\mathbf{q}_1^T + \mathbf{q}_3\mathbf{q}_3^T\right]\right].
\end{aligned} \tag{2}$$

The row space of the resulting matrix reveals the span of the rank-2 matrix $\mathbf{q}_1\mathbf{q}_1^T + \mathbf{q}_3\mathbf{q}_3^T$. This allows us to compute

$$\begin{aligned}
\mathbb{P}^{\perp}_{\mathbf{q}_1\mathbf{q}_1^T+\mathbf{q}_3\mathbf{q}_3^T}\mathbf{Q}_0\mathbb{P}^{\perp}_{\mathbf{q}_1\mathbf{q}_1^T+\mathbf{q}_3\mathbf{q}_3^T} &= \left(1 - b_{\mathbf{Q}_0}\right)\mathbb{P}^{\perp}_{\mathbf{q}_1\mathbf{q}_1^T+\mathbf{q}_3\mathbf{q}_3^T}\left(\mathbf{q}_1+\mathbf{q}_2\right)\left(\mathbf{q}_1+\mathbf{q}_2\right)^T\mathbb{P}^{\perp}_{\mathbf{q}_1\mathbf{q}_1^T+\mathbf{q}_3\mathbf{q}_3^T} \\
\mathbb{P}^{\perp}_{\mathbf{q}_1\mathbf{q}_1^T+\mathbf{q}_3\mathbf{q}_3^T}\mathbf{N}\mathbb{P}^{\perp}_{\mathbf{q}_1\mathbf{q}_1^T+\mathbf{q}_3\mathbf{q}_3^T} &= \left(1 - b_{\mathbf{N}}\right)\mathbb{P}^{\perp}_{\mathbf{q}_1\mathbf{q}_1^T+\mathbf{q}_3\mathbf{q}_3^T}\left(\mathbf{q}_1+\mathbf{q}_2\right)\left(\mathbf{q}_1+\mathbf{q}_2\right)^T\mathbb{P}^{\perp}_{\mathbf{q}_1\mathbf{q}_1^T+\mathbf{q}_3\mathbf{q}_3^T},
\end{aligned}$$

which in turn enables the computation of the fraction $\frac{1-b_{\mathbf{Q}_0}}{1-b_{\mathbf{N}}}$. Finally, this leads to a valid rank-2 solution

$$\mathbf{Q}_0 - \frac{1-b_{\mathbf{Q}_0}}{1-b_{\mathbf{N}}}\mathbf{N} = \left(b_{\mathbf{Q}_0} - \frac{1-b_{\mathbf{Q}_0}}{1-b_{\mathbf{N}}}b_{\mathbf{N}}\right)\left[\mathbf{q}_1\mathbf{q}_1^T + \mathbf{q}_3\mathbf{q}_3^T\right] \tag{3}$$

$$+ \underbrace{\left(1 - b_{\mathbf{Q}_0} - \frac{1-b_{\mathbf{Q}_0}}{1-b_{\mathbf{N}}}\left(1-b_{\mathbf{N}}\right)\right)}_{=0}\left(\mathbf{q}_1+\mathbf{q}_2\right)\left(\mathbf{q}_1+\mathbf{q}_2\right). \tag{4}$$

The last step consists in decomposing the solution $\mathbf{Q}_2 = \mathbf{q}_1\mathbf{q}_1^T + \mathbf{q}_3^T\mathbf{q}_3^T$ into the vectors $\mathbf{q}_1$ and $\mathbf{q}_2$. This can be done with an eigenvalue decomposition of $\mathbf{Q}_2$ and assigning $\mathbf{q}_1$ and $\mathbf{q}_3$ the eigenvectors scaled by the square root of its corresponding eigenvalue.

A small detail needs to be mentioned. Because $\cos^2\alpha_f + (-1-\cos\alpha_f)^2 + 2\cos\alpha_f(1-\cos\alpha_f) = 1$ (compare with Eq. 10) the second column of $\hat{\mathbf{M}}\mathbf{Q}_{trig}$ might correspond to $-1-\cos\alpha_f$ rather than $1-\cos\alpha_f$. However, if this happens (which is easy to check since $-1-\cos\alpha_f \leq 0 \leq 1-\cos\alpha_f$), $\mathbf{q}_2$ is replaced with $-\mathbf{q}_2 - 2\mathbf{q}_1$ (because $-(-1-\cos\alpha_f) - 2\cos\alpha_f = 1-\cos\alpha_f$).

2 Projection onto Plane of Rotation

This section shows how feature trajectories of planar motions can be projected onto the plane of rotation knowing neither the camera matrices nor the 3D coordinates of the points. The derivation starts by subtracting the first row (the mean of the rows could be subtracted instead as well) from the data matrix

$$\left[\mathbf{I}_F - \mathbf{1}_{F\times 1}\left[1, \mathbf{0}_{1\times F-1}\right]\right]\mathbf{W} = \left[\mathbf{I}_F - \mathbf{1}_{F\times 1}\left[1, \mathbf{0}_{1\times F-1}\right]\right]\mathbf{M}\mathcal{C}_{(f)}\mathbf{S}\otimes\mathbf{P}^T \tag{5}$$

$$= \left[\Downarrow_f \cos\alpha_f - \cos\alpha_1, \sin\alpha_f - \sin\alpha_1, \mathbf{t}_f^T - \mathbf{t}_1^T\right]\begin{bmatrix} 1 & -1 & 0 & \mathbf{0}_{1\times 2} \\ 0 & 0 & 1 & \mathbf{0}_{1\times 2} \\ \mathbf{0}_{2\times 1} & \mathbf{0}_{2\times 1} & \mathbf{0}_{2\times 1} & \mathbf{I}_2 \end{bmatrix}\mathcal{C}_{(f)}\mathbf{S}\otimes\mathbf{P}^T.$$

The algebraic structure of $\mathbf{M} = \left[\Downarrow_f \cos\alpha_f, 1-\cos\alpha_f, \sin\alpha_f, \mathbf{t}_f^T\right]$ together with $(1-\cos\alpha_f) - (1-\cos\alpha_1) = -\cos\alpha_f + \cos\alpha_1$ has been used to replace the motion matrix $\mathbf{M}$ of rank 5 by a rank 4 matrix which is right multiplied with a suitable matrix in order to get the motion matrix with subtracted first row. It is interesting to see what happens if this matrix is left multiplied with the second factor $\mathbf{A} = \mathcal{C}_{(f)}\mathbf{S}\otimes\mathbf{P}^T$ of the rank-5 decomposition

$$\begin{bmatrix} 1 & -1 & 0 & \mathbf{0}_{1\times 2} \\ 0 & 0 & 1 & \mathbf{0}_{1\times 2} \\ \mathbf{0}_{2\times 1} & \mathbf{0}_{2\times 1} & \mathbf{0}_{2\times 1} & \mathbf{I}_2 \end{bmatrix}\mathcal{C}_{(f)}\mathbf{S}\otimes\mathbf{P}^T = \begin{bmatrix} \operatorname{vec}\left(\mathbf{I}_3 - \mathbf{a}\mathbf{a}^T\right) & \mathbf{0}_{1\times 3} & 0 \\ \operatorname{vec}\left([\mathbf{a}]_\times\right) & \mathbf{0}_{1\times 3} & 0 \\ \mathbf{0}_{2\times 9} & \mathbf{V}^T & \mathbf{0}_{2\times 1} \end{bmatrix}\mathbf{S}\otimes\mathbf{P}^T \tag{6}$$

$$= \begin{bmatrix} \operatorname{vec}\left(\mathbb{P}_{\mathbf{a}}^\perp\right) & \mathbf{0}_{1\times 3} & 0 \\ \operatorname{vec}\left([\mathbf{a}]_\times\right) & \mathbf{0}_{1\times 3} & 0 \\ \mathbf{0}_{2\times 9} & \mathbf{V}^T\mathbb{P}_{\mathbf{V}} & \mathbf{0}_{2\times 1} \end{bmatrix}\mathbf{S}\otimes\mathbf{P}^T \tag{7}$$

$$= \begin{bmatrix} \operatorname{vec}\left(\mathbb{P}_{\mathbf{V}}\right) & \mathbf{0}_{1\times 3} \\ \operatorname{vec}\left(\mathbb{P}_{\mathbf{V}}[\mathbf{a}]_\times\mathbb{P}_{\mathbf{V}}\right) & \mathbf{0}_{1\times 3} \\ \mathbf{0}_{2\times 9} & \mathbf{V}^T\mathbb{P}_{\mathbf{V}} \end{bmatrix}\left[\begin{bmatrix} \mathbb{P}_{\mathbf{V}} & \mathbf{0}_{3\times 1} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}\mathbf{S}\right]\otimes\left[\mathbb{P}_{\mathbf{V}}\mathbf{P}_{[:,1:3]}^T\right]. \tag{8}$$

The properties $\mathbf{I}_3 - \mathbf{a}\mathbf{a}^T = \mathbb{P}_{\mathbf{a}}^{\perp} = \mathbb{P}_{\mathbf{V}}$, $[\mathbf{a}]_{\times} = \mathbb{P}_{\mathbf{V}} [\mathbf{a}]_{\times} \mathbb{P}_{\mathbf{V}}$, $\mathbf{V}^T = \mathbf{V}^T \mathbb{P}_{\mathbf{V}}$, and the symmetry and idempotence of orthogonal projection matrices have been used. This final formula actually has a very intuitive explanation. By subtracting the first row (or the mean of all the rows) the non-dynamic aspect in the data is removed. The coordinates of the points along the rotation axis remain constant, so does the camera translation. Both the point coordinates along the rotation axis and the camera translation are thus removed by subtracting the first row.

3 Polynomial Solution to Orthogonality and Equality of Norm Constraints

For notational reasons, the symmetric 2-by-2 matrix $\mathbf{P}^k_{[:,1:3]} \mathbb{P}_{\mathbf{V}} {\mathbf{P}^k_{[:,1:3]}}^T$ in

$$\lambda_k^2 \mathbf{I}_2 = \mathbf{P}^k_{[:,1:3]} {\mathbf{P}^k}^T_{[:,1:3]} = \mathbf{P}^k_{[:,1:3]} \mathbb{P}_{\mathbf{V}} {\mathbf{P}^k_{[:,1:3]}}^T + \mathbf{P}^k_{[:,1:3]} \mathbb{P}_{\mathbf{a}} {\mathbf{P}^k_{[:,1:3]}}^T$$

is denoted as $\mathbf{G}^k$. Thus, it follows

$$\mathbf{G}^k + \mathbf{w}_k \mathbf{w}_k^T = \begin{bmatrix} \mathbf{G}^k_{[1,1]} & \mathbf{G}^k_{[1,2]} \\ \mathbf{G}^k_{[1,2]} & \mathbf{G}^k_{[2,2]} \end{bmatrix} + \begin{bmatrix} \mathbf{w}^2_{k,[1]} & \mathbf{w}_{k,[1]} \mathbf{w}_{k,[2]} \\ \mathbf{w}_{k,[1]} \mathbf{w}_{k,[2]} & \mathbf{w}^2_{k,[2]} \end{bmatrix} = \lambda_k^2 \mathbf{I}_2. \tag{9}$$

The unknown scale factor λ_k^2 can be eliminated by subtracting the two equations on the diagonal from each other which leads to the system

$$\mathbf{G}^k_{[1,1]} - \mathbf{G}^k_{[2,2]} + \mathbf{w}^2_{k,[1]} - \mathbf{w}^2_{k,[2]} = 0 \tag{10}$$

$$\mathbf{G}^k_{[1,2]} + \mathbf{w}_{k,[1]} \mathbf{w}_{k,[2]} = 0. \tag{11}$$

The second equation Eq. (11) can be solved for $\mathbf{w}_{k,[1]} = -\frac{\mathbf{G}^k_{[1,2]}}{\mathbf{w}_{k,[2]}}$ (if either $\mathbf{w}_{k,[2]} = 0$ or $\mathbf{w}_{k,[1]} = 0$ the above system becomes a second-order polynom in one unknown which is trivial to solve). Substituting $\mathbf{w}_{k,[1]}$ in Eq. (10) leads to a polynom in the monomials $\mathbf{w}^2_{k,[2]}$ and $\mathbf{w}^4_{k,[2]}$. This polynom can be solved for $\mathbf{w}^2_{k,[2]}$ which implicitly gives $\mathbf{w}_{k,[2]}$ and $\mathbf{w}_{k,[1]}$. This approach provides four solutions, two of them are conjugate complex and the remaining two are equal up to the sign. Hence, the solution is unique up to the sign.

4 General Remark on Solving Linear Matrix Equations in the Least-Squares Sense

The closed-form solution presented in the paper requires solving linear matrix equations in the least squares sense. A typical instance of a linear matrix equation in $\mathbf{X}$ is $\mathbf{AXB} = \mathbf{C}$. This section provides some hints on how to reformulate linear matrix equations into standard form $\mathbf{Ax} = \mathbf{b}$ with vectorized unknown $\mathbf{x} = \operatorname{vec}(\mathbf{X})$, system matrix $\mathbf{A}$, and right-hand side $\mathbf{b}$. Systems in standard form can be readily solved with any least-squares method of choice (e.g. with

QR-decomposition or singular-value decomposition). Knowing how to rewrite the two following instances of matrix equations allows to express all the matrix equations mentioned in the paper in standard form. For more details, see [1].

i) **The Matrix Equation AXB:** The Jacobian of $\mathbf{AXB}$ w.r.t. $\mathbf{x} = \mathrm{vec}(\mathbf{X})$ is $\mathbf{J_x} = \mathbf{B}^T \otimes \mathbf{A}$ which leads to the linear system in standard form $\mathbf{J_x x} = \mathrm{vec}(\mathbf{C})$.

ii) **The Matrix Equation $\mathbf{X} \otimes \mathbf{Y}$:** Let $\mathbf{X} \in \mathbb{R}^{m\times n}$ and $\mathbf{Y} \in \mathbb{R}^{p\times q}$. Then the Jacobian w.r.t. the vectorized unknowns $\mathbf{x} = \mathrm{vec}(\mathbf{X})$ and $\mathbf{y} = \mathrm{vec}(\mathbf{Y})$ is

$$\mathbf{J_{x,y}} = [\mathbf{I}_n \otimes \mathbf{K}_{q,m} \otimes \mathbf{I}_p] [\mathbf{I}_{mn} \otimes \mathrm{vec}(\mathbf{Y}), \mathrm{vec}(\mathbf{X}) \otimes \mathbf{I}_{pq}], \tag{12}$$

where $\mathbf{K}_{q,m}$ denotes the commutator matrix which enjoys the identity

$$\mathbf{K}_{q,m}\, \mathrm{vec}(\mathbf{A}) = \mathrm{vec}\left(\mathbf{A}^T\right) \tag{13}$$

for any $\mathbf{A} \in \mathbb{R}^{q\times m}$. The bilinear matrix equation $\mathbf{X} \otimes \mathbf{Y} = \mathbf{C}$ is thus equivalent to

$$\mathbf{J_{x,y}} \begin{pmatrix} \mathrm{vec}(\mathbf{X}) \\ \mathrm{vec}(\mathbf{Y}) \end{pmatrix} = \mathrm{vec}(\mathbf{C}). \tag{14}$$

References

1. Magnus, J.R., Neudecker, H.: Matrix differential calculus with applications in statistics and econometrics. 2nd edn. John Wiley & Sons (1999)