The Generalized Trace-Norm and its Application to Structure-from-Motion Problems: Supplemental Material

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Abstract

This document provides a short proof that the generalized trace norm with a positive-definite block-diagonal weight matrix $\mathbf{D}$ *is indeed a proper norm.*

1. Proof: $\|\mathbf{X}\|_{*\mathbf{D}}$ is a Norm

For the case of positive-definite block-diagonal

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_c \end{bmatrix}$$

the proof starts with the Eigenvalue-decomposition of $\mathbf{D}_r = \mathbf{V}_r \mathbf{\Lambda}_r \mathbf{V}_r^T$ and $\mathbf{D}_c = \mathbf{V}_c \mathbf{\Lambda}_c \mathbf{V}_c^T$. These eigenvalue decompositions can be assembled into an eigenvalue decomposition for $\mathbf{D} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T$ with

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_c \end{bmatrix} \text{ and } \mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_c \end{bmatrix}. \tag{1}$$

For later reference we introduce

$$\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{V}^T = \begin{bmatrix} \mathbf{\Lambda}_r^{\frac{1}{2}}\mathbf{V}_r^T & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_c^{\frac{1}{2}}\mathbf{V}_c^T \end{bmatrix} = \begin{bmatrix} \mathbf{C}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_c \end{bmatrix}.$$

Then due to the cyclic property of the trace we get

$$\begin{aligned}
\|\mathbf{X}\|_{*\mathbf{D}} &= \frac{1}{2}\min_{\mathbf{X}_1,\mathbf{X}_2} \operatorname{trace}\left(\mathbf{D}^T \begin{bmatrix} \mathbf{X}_1 & \mathbf{X} \\ \mathbf{X}^T & \mathbf{X}_2 \end{bmatrix}\right) \\
&= \frac{1}{2}\min_{\mathbf{X}_1,\mathbf{X}_2} \operatorname{trace}\left(\mathbf{\Lambda}^{\frac{1}{2}}\mathbf{V}^T \begin{bmatrix} \mathbf{X}_1 & \mathbf{X} \\ \mathbf{X}^T & \mathbf{X}_2 \end{bmatrix} \mathbf{V}\mathbf{\Lambda}^{\frac{1}{2}}\right) \\
&= \frac{1}{2}\min_{\mathbf{X}_1,\mathbf{X}_2} \operatorname{trace}\left(\begin{bmatrix} \mathbf{C}_r\mathbf{X}_1\mathbf{C}_r^T & \mathbf{C}_r\mathbf{X}\mathbf{C}_c^T \\ \mathbf{C}_c\mathbf{X}^T\mathbf{C}_r^T & \mathbf{C}_c\mathbf{X}_2\mathbf{C}_c^T \end{bmatrix}\right).
\end{aligned}$$

Since $\mathbf{C}_r \in \mathbb{R}^{m\times m}$ and $\mathbf{C}_c \in \mathbb{R}^{n\times n}$ are both regular matrices, we can absorb them and change the variables of the minimization to $\mathbf{Y}_1 = \mathbf{C}_r\mathbf{X}_1\mathbf{C}_r^T$ and $\mathbf{Y}_2 = \mathbf{C}_c\mathbf{X}_2\mathbf{C}_c^T$ and get an equivalent problem which looks like

$$\begin{aligned}
\|\mathbf{X}\|_{*\mathbf{D}} &= \frac{1}{2}\min_{\mathbf{Y}_1,\mathbf{Y}_2} \operatorname{trace}\left(\begin{bmatrix} \mathbf{Y}_1 & \mathbf{C}_r\mathbf{X}\mathbf{C}_c^T \\ \mathbf{C}_c\mathbf{X}^T\mathbf{C}_r^T & \mathbf{Y}_2 \end{bmatrix}\right) \\
&= \left\|\mathbf{C}_r^T\mathbf{X}\mathbf{C}_c\right\|_* .
\end{aligned} \tag{2}$$

The generalized trace-norm $\|\mathbf{X}\|_{*\mathbf{D}}$ with a block-diagonal matrix $\mathbf{D}$ reduces to the standard trace norm $\left\|\mathbf{C}_r\mathbf{X}\mathbf{C}_c^T\right\|_*$ of the matrix $\mathbf{C}_r\mathbf{X}\mathbf{C}_c^T$. Equipped with this insight, it is straight-forward to verify that the three axioms for a norm indeed hold for our definition of the generalized trace-norm.