Grid convergence analysis

What is the order of convergence of our finite difference scheme?

The order of convergence of a finite difference scheme refers to how quickly the numerical solution approaches the exact solution of a partial differential equation as the grid spacing decreases. A “second-order” solution (using central difference approximations) would have p = 2.

For a constant grid refinement ratio $r = \frac{h_2}{h_1}$ (h_1 is the spacing of the finer grid and h_2 is the spacing of the coarser grid), the order of convergence is:

$$p = ln\left(\frac{f_3 - f_2}{f_2 - f_1}\right)/ln(r)$$

where f_i can be individual points of the solution, e.g.

$$f_1 = u_{2,2}$$

which is the solution at point (2,2) at grid 1, or

f_i can be integrated quantities of the solution at many points at the ith grid, e.g.

$$f_1 = \int_\Omega u^2 dxdy$$

Let's see an example. The integrated quantity of the solution has been evaluated for 5 grid with different sizes

Grid	Grid size	Spacing	f
1	641 × 321	0.003	1.000000
2	321 × 161	0.006	0.999000
3	161 × 81	0.012	0.995000
4	81 × 41	0.0024	0.982000
5	41 × 21	0.0048	0.923000

Taking the ratio between the first two spacings, r = 0.006/0.003 = 2, and similarly for the other grids, we can see that we have a constant refinement ratio. Substitute values to calculate the order of convergence.

$$p = ln\left(\frac{0.995 - 0.999}{0.999 - 1.00}\right)/ln(2) \approx 2$$

Thus, the error of the numerical solution decreases proportionally to Δx^2

Is my solution in the asymptotic range?

A solution in the asymptotic range is a mesh independent solution. To find if a solution is in the asymptotic range, we use the Grid Convergence Index (GCI). The GCI is a measure of the percentage the computed value is away from the value of the asymptotic numerical value. It can be calculated as:

$$GCI_{ij} = \frac{F_s \left| \frac{f_j - f_i}{f_i} \right|}{r^p - 1}$$

where i is the finer and j is the coarser grid. Usually, we calculate this for comparisons between 3 grids and using a safety factor of $F_s = 1.25$.

First, we find this for grids 1-2

$$GCI_{12} = \frac{1.25 \left| \frac{0.999 - 1}{1} \right|}{2^2 - 1} = 0.0004166$$

Then, we find the GCI for grids 2-3

$$GCI_{23} = \frac{1.25 \left| \frac{0.995 - 0.999}{0.999} \right|}{2^2 - 1} = 0.00166$$

If the ratio

$$\frac{r^p GCI_{12}}{GCI_{23}} \approx 1$$

then we are at the asymptotic range. Calculating for this example

$$\frac{4 * 0.0004166}{0.00166} \approx 1$$

therefore, the solution is in the asymptotic range.