THE UNIVERSITY *of* EDINBURGH
School of Engineering

Partial Differential Equations 3

Finite differences

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Taylor series & 1D Forward difference

- Starting from Taylor series (Forward)

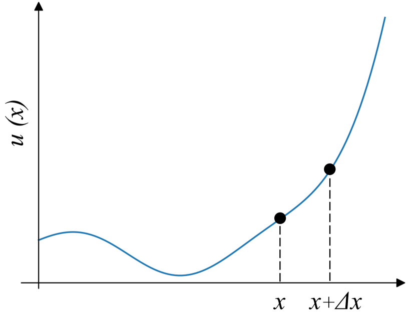
$$u(x+\Delta x) = u(x) + \frac{\Delta x}{1!}\frac{\partial u}{\partial x} + \frac{\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^3}{3!}\frac{\partial^3 u}{\partial x^3} + \dots$$

- Truncate the series after the first derivative

$$u(x+\Delta x) = u(x) + \frac{\Delta x}{1!}\frac{\partial u}{\partial x} + O(\Delta x)$$

- Solve for first derivative and neglect higher order terms

$$\frac{\partial u}{\partial x} = \frac{u(x+\Delta x) - u(x)}{\Delta x}$$



1D Backward difference

- Starting from Taylor series (Backward)

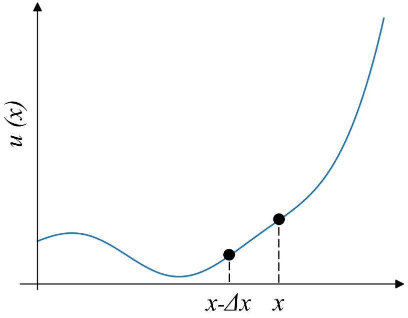
$$u(x-\Delta x) = u(x) - \frac{\Delta x}{1!}\frac{\partial u}{\partial x} + \frac{\Delta x^2}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{\Delta x^3}{3!}\frac{\partial^3 u}{\partial x^3} + \dots$$

- Truncate the series after the first derivative

$$u(x-\Delta x) = u(x) - \frac{\Delta x}{1!}\frac{\partial u}{\partial x} + O(\Delta x)$$

- Solve for first derivative and neglect higher order terms

$$\frac{\partial u}{\partial x} = \frac{u(x) - u(x-\Delta x)}{\Delta x}$$



1D Central difference

- By taking the difference between $u(x + \Delta x)$ and $u(x - \Delta x)$, we get the central difference

$$\frac{\partial u}{\partial x} = \frac{u(x + \Delta x) - u(x - \Delta x)}{2\Delta x}$$

- Similarly, if we keep up to the $\partial^2 u / \partial x^2$ term, we get a central difference approximation to the second derivative

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x + \Delta x) - 2u(x) + u(x - \Delta x)}{\Delta x^2}$$

Partial derivatives in 2 dimensions

- Now, we have $u(x, y)$, where u is a function of both x and y
- Similarly, we can derive the partial derivatives with respect to x and y

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x+\Delta x, y) - 2u(x, y) + u(x-\Delta x, y)}{\Delta x^2}$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{u(x, y+\Delta y) - 2u(x, y) + u(x, y-\Delta y)}{\Delta y^2}$$

- Substituting these into Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\frac{u(x+\Delta x, y) - 2u(x, y) + u(x-\Delta x, y)}{\Delta x^2} + \frac{u(x, y+\Delta y) - 2u(x, y) + u(x, y-\Delta y)}{\Delta y^2} = 0$$

Partial derivatives in 2 dimensions

- Rearrange to solve for $u(x,y)$ and for $\beta = \frac{\Delta x}{\Delta y}$

$$u(x,y) = \frac{1}{2(1+\beta^2)}\left(u(x-\Delta x, y) + u(x+\Delta x, y) + \beta^2(u(x, y-\Delta y) + u(x, y+\Delta y))\right)$$

- For $\Delta x = \Delta y$

$$u(x,y) = \frac{1}{4}\left(u(x-\Delta x, y) + u(x+\Delta x, y) + u(x, y-\Delta y) + u(x, y+\Delta y)\right)$$

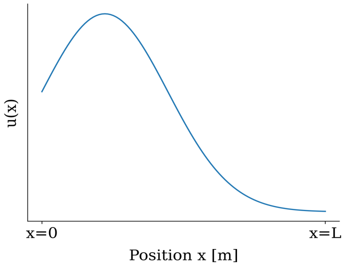
- Because we are implementing a numerical method and we use matrices, we change to indices

$$u_{i,j} = \frac{1}{4}\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}\right)$$

which is the approximate solution $u_{i,j}$ to point i,j

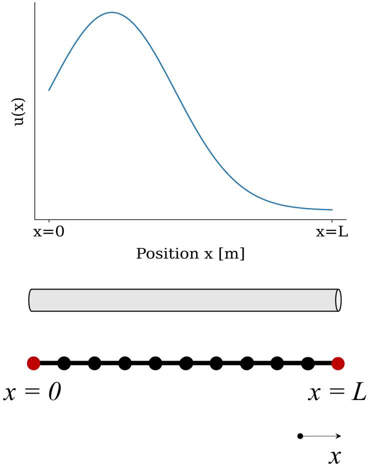
1D Finite Differences

- We need to find solution $u(x)$ to a partial differential equation



1D Finite Differences

- We need to find solution $u(x)$ to a partial differential equation
- With a numerical method, we find approximate solutions at discrete points



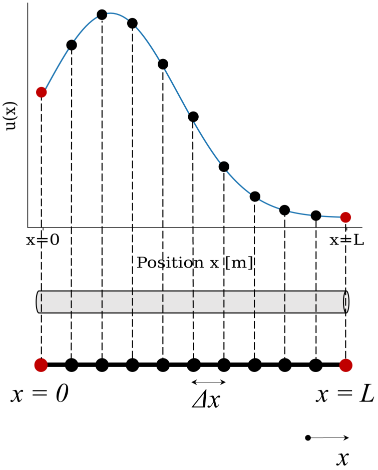
1D Discretization

- We need to find solution $u(x)$ to a partial differential equation
- With a numerical method, we find approximate solutions at discrete points
- We select the uniform mesh spacing Δx (distance between points) and we split the domain in points

$$x = \begin{bmatrix} x_0 & x_1 & x_2 & \dots & x_{\mathrm{n}} \end{bmatrix}$$

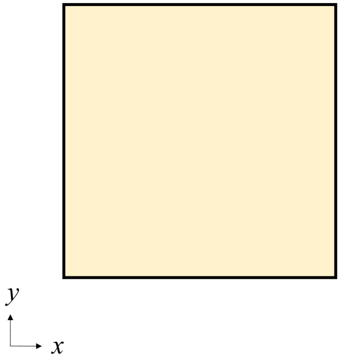
where $x_i = x_0 + i\Delta x$

- We find the approximate solution u_i at each point x_i
- Boundary conditions at the edge points



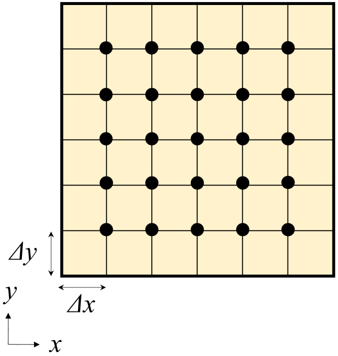
2D Discretization

- Assume a 2D domain, where $u(x, y)$ is a function of both x and y



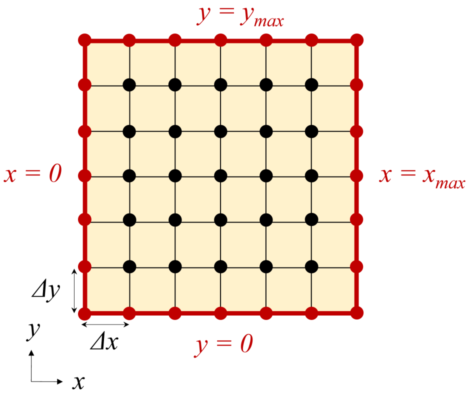
2D Discretization

- Assume a 2D domain, where $u(x, y)$ is a function of both x and y
- In this case, we discretize at both directions into a number of cells.
- We use mesh spacing Δx for the x direction and Δy for the y direction



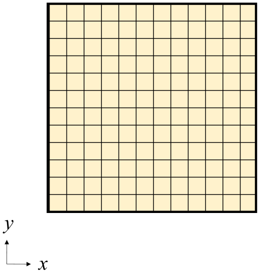
2D Discretization

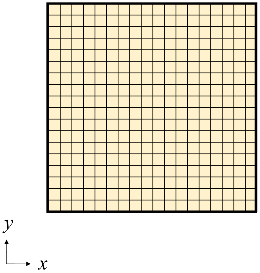
- Assume a 2D domain, where $u(x, y)$ is a function of both x and y
- In this case, we discretize at both directions into a number of cells.
- We use mesh spacing Δx for the x direction and Δy for the y direction
- Boundary conditions at the four edges of the domain
  $\Rightarrow$ Need to apply boundary condition at whole edge (multiple points)
- We find the solution $u_{i,j}$ at the interior at each point (x_i, y_j), where $x_i = x_0 + i\Delta x$ and $y_j = y_0 + j\Delta y$

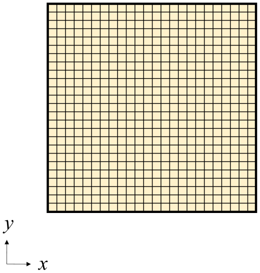


Grid and convergence

- In general, increasing the number of cells leads to better accuracy







- Accuracy is increased up to a limit $\Rightarrow$ Convergence analysis