Finite difference schemes for the solution of Laplace's equation

We have seen so far three FD schemes for solving Laplace's equation: 1) Jacobi, 2) Gauss-Siedel and 3) SOR scheme, all of which approximate the second derivatives using a 2nd order central difference.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{u_E - 2u_O + u_W}{\Delta x^2} + \frac{u_N - 2u_O + u_S}{\Delta y^2} = 0$$

Jacobi Iteration scheme

At each iteration need to find (update) the solution $u_O(u_{i,j})$ at each interior point i, j based on information from its neighbouring points. The Jacobi scheme uses information only from the previous iteration n to update the solution at the current iteration $n+1$.

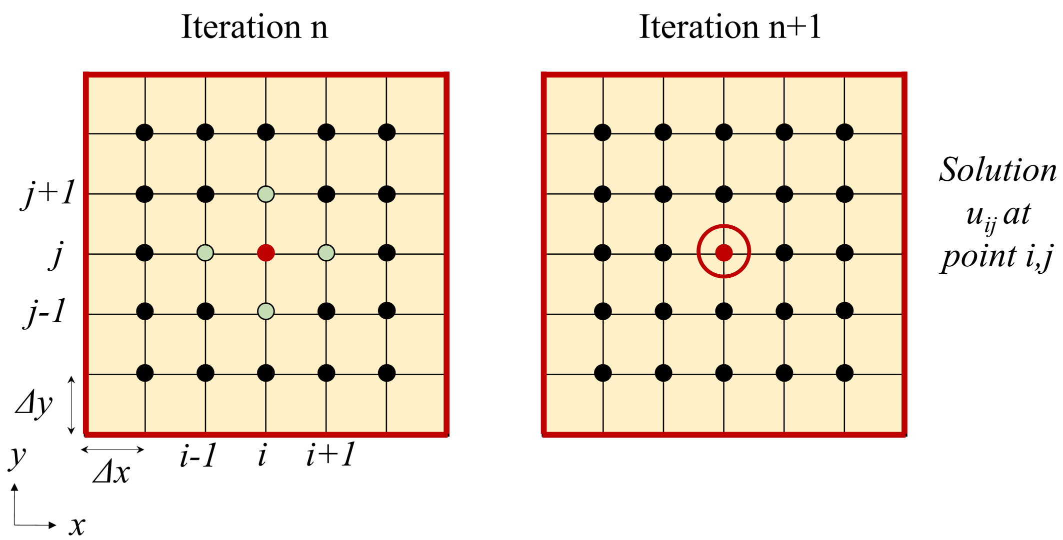
$$u_O^{n+1} = \frac{1}{2(1+\beta^2)}\left(u_E^n + u_W^n + \beta^2(u_N^n + u_S^n)\right)$$

or

$$u_{i,j}^{n+1} = \frac{1}{2(1+\beta^2)}\Big(u_{i+1,j}^n + u_{i-1,j}^n + \beta^2(u_{i,j+1}^n + u_{i,j-1}^n)\Big)$$

where

$$\beta = \frac{\Delta x}{\Delta y}$$



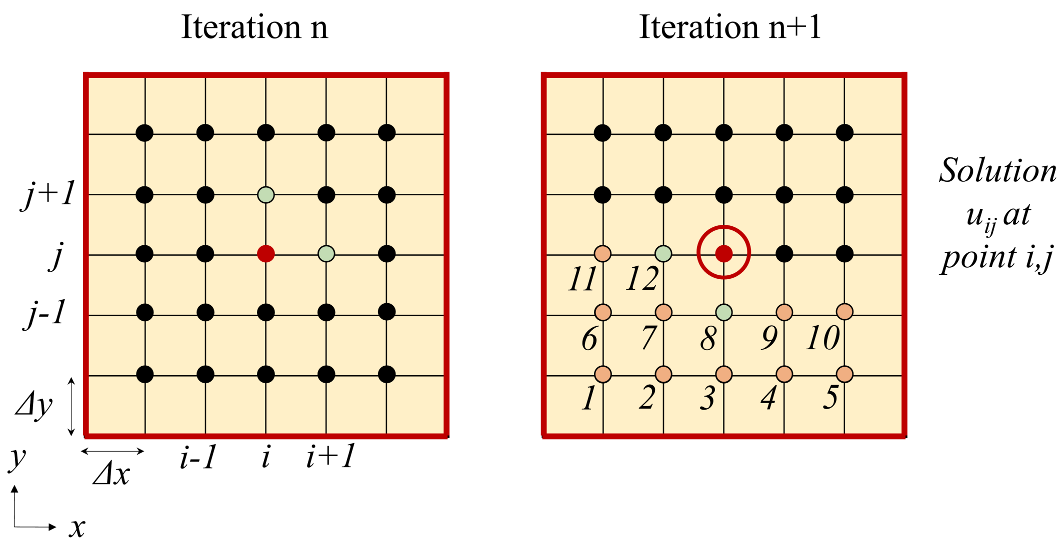
Gauss-Seidel scheme

The Gauss-Seidel scheme uses information of the neighbouring points from the previous and the current iteration to find the solution. Solution at each point is updated in an order and at the time we update solution u_O, the west and south values of u are already know.

$$u_O^{n+1} = \frac{1}{2(1+\beta^2)} \left(u_E^n + u_W^{n+1} + \beta^2 (u_N^n + u_S^{n+1}) \right) .$$

or

$$u_{i,j}^{n+1} = \frac{1}{2(1+\beta^2)} \left(u_{i+1,j}^n + u_{i-1,j}^{n+1} + \beta^2 (u_{i,j+1}^n + u_{i,j-1}^{n+1}) \right) .$$



Successive over-relaxation (SOR) scheme

The SOR scheme takes a weighted average of the old solution u_O and the update step using information from the neighbouring points.

$$u_O^{n+1} = (1-\omega) u_O^n + \frac{\omega}{2(1+\beta^2)} \left(u_E^n + u_W^{n+1} + \beta^2 (u_N^n + u_S^{n+1}) \right)$$

or

$$u_{i,j}^{n+1} = (1-\omega) u_{i,j}^n + \frac{\omega}{2(1+\beta^2)} \left(u_{i+1,j}^n + u_{i-1,j}^{n+1} + \beta^2 (u_{i,j+1}^n + u_{i,j-1}^{n+1}) \right)$$

where $1 \leqslant \omega < 2$. The optimal value of ω is:

$$\omega = \frac{2}{1 + \sqrt{1-\lambda}}$$

where λ is the dominant Eigenvalue of the itteration matrix. On a regular Cartesian grid this is

$$\lambda = \frac{1}{4}\left(\cos\frac{\pi}{N_i} + \cos\frac{\pi}{N_J}\right)^2$$