Chapter Fourteen

TWO BEAM INTERFERENCE BY DIVISION OF WAVE FRONT

'The wave nature of light was demonstrated convincingly for the first time in 1801 by Thomas Young by a wonderfully simple experiment . . . He let a ray of sunlight into a dark room, placed a dark screen in front of it, pierced with two small pinholes, and beyond this, at some distance a white screen. He then saw two darkish lines at both sides of a bright line, which gave him sufficient encouragement to repeat the experiment, this time with spirit flame as light source, with a little salt in it, to produce the bright yellow sodium light. This time he saw a number of dark lines, regularly spaced; the first clear proof that light added to light can produce darkness. This phenomenon is called interference. Thomas Young had expected it because he believed in the wave theory of light.

—Dennis Gabor in his Nobel Lecture, December 11, 1971

Thomas Young had amazing broad interests and talents . . . From his discoveries in medicine and science, Helmholtz concluded: 'His was one of the most profound minds that the world has ever seen.'

—From the Internet

14.1 INTRODUCTION

In Chap.13, we had considered the superposition of one-dimensional waves propagating on a string and showed that there is a variation of energy density along the length of the string due to the interference of two waves (see Fig. 13.5). In general, whenever two waves superpose, one obtains an intensity distribution which is known as the interference pattern. In this chapter, we will consider the interference pattern produced by waves emanating from two point sources. We may note that with sound waves the interference pattern can be observed without much difficulty because the two interfering waves maintain a constant phase relationship; this is also the case for microwaves. However, for light waves, due to the very process of emission, one cannot observe interference between the waves from two independent sources,[1] although the interference does take place (see Sec. 14.4). Thus, one tries to derive interfering waves from a single wave so that the phase relationship is maintained. The methods to achieve this can be classified under two broad categories. Under the first category, in a typical arrangement, a beam is allowed to fall on two closely spaced holes, and the two beams emanating from the holes, interfere. This method is known as division of wave front and will be discussed in detail in this chapter. In the other method, known as division of amplitude, a beam is divided at two or more reflecting surfaces and the reflected beams interfere. This will be discussed in Chap.15. We must, however, emphasize that the present and the following chapters are based on one underlying principle, namely, the superposition principle.

It is also possible to observe interference by using multiple-beams; this is known as multiple-beam interferometry and will be discussed in Chap. 16. It will be shown that multiple beam interferometry offers some unique advantages over two-beam interferometry.

[1] It is difficult to observe the interference pattern even with two laser beams unless they are phase-locked.

14.2 INTERFERENCE PATTERN PRODUCED ON THE SURFACE OF WATER

We consider surface waves emanating from two point sources in a water tank. We may have, for example, two sharp needles vibrating up and down at points S_1 and S_2 (see Fig. 14.1). Although water waves are not really transverse, we will, for the sake of simplicity, assume water waves to produce displacements which are transverse to the direction of propagation.

If there were only one needle (say, at S_1) vibrating with a certain frequency ν, then circular ripples would have spread out from point S_1. The wavelength would have been v/ν, and the crests and troughs would have moved outward. Similarly for the vibrating needle at S_2. However, if both needles are vibrating, then waves emanating from S_1 will interfere with the waves emanating from S_2. We assume that the needle at S_2 vibrates in phase with the needle at S_1; i.e., S_1 and S_2 go up simultaneously, and they also reach the lowest position at the same time. Thus, if at a certain instant, the disturbance emanating from the source S_1 produced a crest at a distance ρ from S_1 then the disturbance from S_2 would also produce a crest at a distance ρ from S_2, etc. This is explicitly shown in Fig. 14.1, where the solid curves represent (at a particular instant) the positions of the crests due to disturbances emanating from S_1 and S_2. Similarly, the dashed curves represent (at the same instant) the positions of the troughs. Notice that at all points on the perpendicular bisector OY, the disturbances reaching from S_1 and from S_2 will always be in phase. Consequently, at an arbitrary point A (on the perpendicular bisector) we may write the resultant disturbance as

$$\begin{aligned} y &= y_1 + y_2 \\ &= 2a\cos\omega t \end{aligned} \tag{1}$$

where y_1 (= a cos ωt) and y_2 (= a cos ωt) represent the displacements at point A due to S_1 and S_2, respectively. We see that the amplitude at A is twice the amplitude produced by each one of the sources. At $t = T/4$ $(= 1/4\nu = \pi/2\omega)$ the displacements produced at point A by each of the sources will be zero, and the resultant will also be zero. This is also obvious from Eq. (1).

Next, let us consider a point B such that

$$S_2B - S_1B = \lambda/2 \tag{2}$$

At such a point the disturbance reaching from source S_1 will always be out of phase with the disturbance reaching from S_2. This follows from the fact that the disturbance reaching point B from source S_2 must have started one-half of a period (= $T/2$) earlier than the disturbance reaching B from S_1. Consequently, if the displacement at B due to S_1 is given by

$$y_1 = a\cos\omega t$$

then the displacement at B due to S_2 is given by

$$y_2 = a\cos(\omega t - \pi) = -a\cos\omega t$$

and the resultant $y = y_1 + y_2$ is zero at all times. Such a point corresponds to destructive interference and is known as a node and corresponds to minimum intensity. The amplitudes of the two vibrations reaching the point B will not really be equal, as it is at different distances from S_1 and S_2. However, if the distances involved are large (in comparison to the wavelength), the two amplitudes will be very nearly equal and the resultant intensity will be very nearly zero.

In a similar manner we may consider a point C such that

$$S_2C - S_1C = \lambda$$

where the phases of the vibration (reaching from S_1 and S_2) are exactly the same as at point A. Consequently we will again have constructive interference. In general, if a point P is such that

$$S_2P \sim S_1P = n\lambda \qquad \text{(maxima)} \tag{3}$$

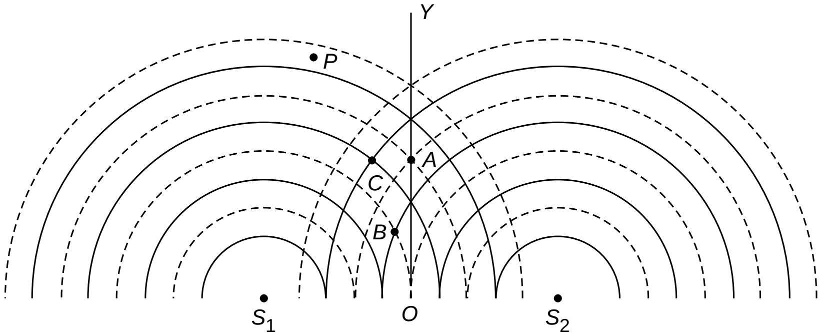

Fig. 14.1 Waves emanating from two point sources S_1 and S_2 vibrating in phase. The solid and the dashed curves represent the positions of the crests and troughs, respectively.

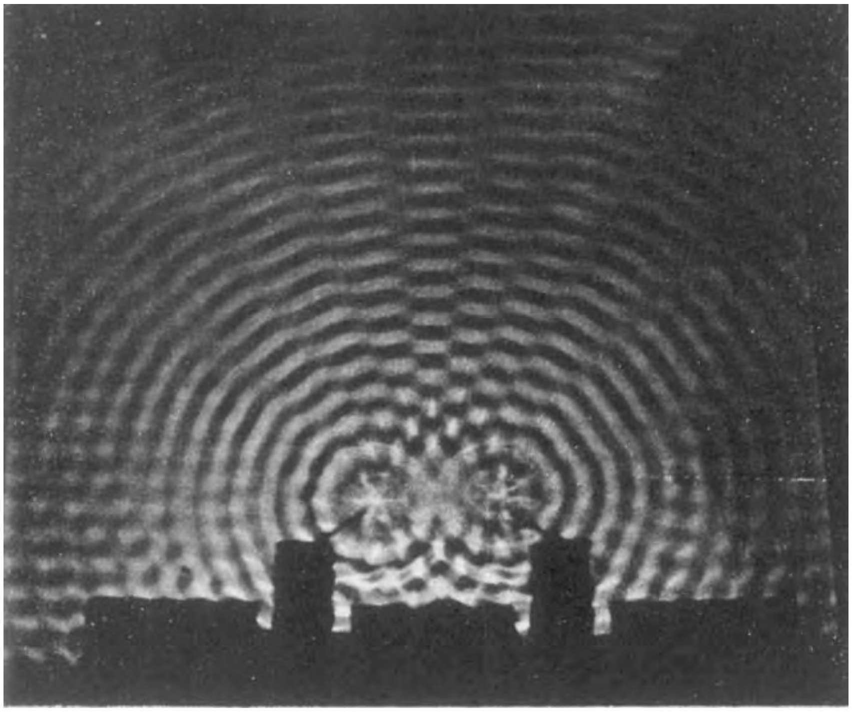

Fig. 14.2 The actual interference pattern produced from two point sources vibrating in phase in a ripple tank (After Ref. 9, used with permission).

$n = 0, 1, 2, \ldots$, then the disturbances reaching point P from the two sources will be in phase, the interference will be constructive, and the intensity will be maximum. On the other hand, if point P is such that

$$S_2P \sim S_1P = \left(n + \frac{1}{2}\right)\lambda \qquad \text{(minima)} \qquad \textbf{(4)}$$

then the disturbances reaching point P from the two sources will be out of phase, the interference will be destructive, and the intensity will be minimum. The actual interference pattern produced from two point sources vibrating in phase in a ripple tank is shown in Fig. 14.2.

Example 14.1 The intensity at the point which statisfies neither Eq. (3) nor Eq. (4) will not be a maximum or zero. Consider a point P such that $S_2P - S_1P = \lambda/3$. Find the ratio of the intensity at point P to that at a maximum.

Solution: If the disturbance reaching point P from S_1 is given by

$$y_1 = a \cos \omega t$$

then the disturbance from S_2 is given by

$$y_2 = a \cos \left(\omega t - \frac{2\pi}{3}\right)$$

because a path difference of $\lambda/3$ corresponds to a phase difference of $2\pi/3$.

Thus the resultant displacement is

$$y = y_1 + y_2$$

$$= a \left[\cos \omega t + \cos\left(\omega t - \frac{2\pi}{3}\right)\right]$$

$$= 2a \cos \left(\omega t - \frac{\pi}{3}\right) \cos \frac{\pi}{3}$$

$$= a \cos \left(\omega t - \frac{\pi}{3}\right)$$

The intensity is therefore one-fourth of the intensity at the maxima. In a similar manner one can calculate the intensity at any other point.

Example 14.2 The locus of points which correspond to minima is known as nodal lines. Show that the equation of a nodal line is a hyperbola. Also obtain the locus of points which correspond to maxima.

Solution: For the sake of generality we find the locus of point P which satisfies the following equation:

$$S_1P - S_2P = \Delta \qquad \textbf{(5)}$$

Thus, if $\Delta = n\lambda$, we have a maximum; and if $\Delta = \left(n+\frac{1}{2}\right)\lambda$, we have a minimum. We choose the midpoint of S_1S_2 as the origin,

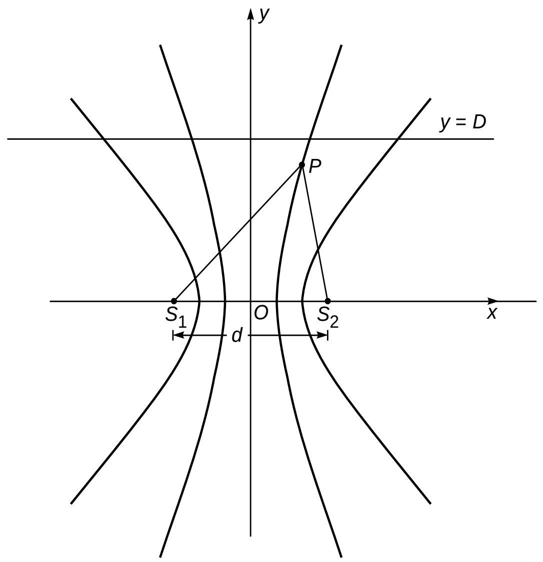

Fig. 14.3 The nodal curves.

with the x axis along S_1S_2 and the y axis perpendicular to it (see Fig. 14.3). If the distance between S_1 and S_2 is d, then the coordinates of points S_1 and S_2 are $(-d/2, 0)$ and $(+d/2, 0)$ respectively. Let the coordinates of the point P be (x, y). Then

$$S_1P = \left[\left(x+\frac{d}{2}\right)^2 + y^2\right]^{1/2}$$

and
$$S_2P = \left[\left(x-\frac{d}{2}\right)^2 + y^2\right]^{1/2}$$

Therefore,

$$S_1P - S_2P = \left[\left(x+\frac{d}{2}\right)^2 + y^2\right]^{1/2} - \left[\left(x-\frac{d}{2}\right)^2 + y^2\right]^{1/2} = \Delta$$

or

$$\left(x+\frac{d}{2}\right)^2 + y^2 = \Delta^2 + \left(x-\frac{d}{2}\right)^2 + y^2 + 2\Delta\left[\left(x-\frac{d}{2}\right)^2 + y^2\right]^{1/2}$$

or
$$2xd - \Delta^2 = 2\Delta\left[\left(x-\frac{d}{2}\right)^2 + y^2\right]^{1/2}$$

On squaring, we obtain

$$4x^2d^2 - 4xd\Delta^2 + \Delta^4 = 4\Delta^2\left(x^2 - xd + \frac{d^2}{4} + y^2\right)$$

Thus we obtain

$$\frac{x^2}{\frac{1}{4}\Delta^2} - \frac{y^2}{\frac{1}{4}(d^2-\Delta^2)} = 1 \qquad \textbf{(6)}$$

which is the equation of a hyperbola. When $\Delta = \left(n+\frac{1}{2}\right)\lambda$, the curves correspond to minima; and when $\Delta = n\lambda$, the curves correspond to maxima. For large values of x and y, the curves asymptotically tend to the straight lines

$$y = \pm\left(\frac{d^2-\Delta^2}{\Delta^2}\right)^{1/2} x \qquad \textbf{(7)}$$

There is no point P for which $S_1P \sim S_2P > d$ ($S_1P \sim S_2P$ equals d on the x axis only). Now, it appears from Eq. (6) that when $\Delta > d$, the resulting equation is an ellipse, which we know is impossible. The fallacy is a result of the fact that because of a few squaring operations, Eq. (6) also represents the locus of all those points for which $S_1P + S_2P = \Delta$, and obviously in this case Δ can exceed d.

Example 14.3 Consider a line parallel to the x axis at a distance D from the origin (see Fig. 14.3). Assume $D >>> \lambda$. Find the points on this line where minimum intensity will occur.

Solution: The equation of this line is

$$y = D \qquad \textbf{(8)}$$

Further, at large distances from the origin the equation of the nodal lines is

$$y = \pm\left(\frac{d^2-\Delta_n^2}{\Delta_n^2}\right)^{1/2} x \qquad \textbf{(9)}$$

where $\Delta_n = \left(n+\frac{1}{2}\right)\lambda$; $n = 0, 1, 2, \ldots$ Clearly the points at which minima will occur (on the line $y = D$) are given by

$$x_n = \pm\left(\frac{\Delta_n^2}{d^2-\Delta_n^2}\right)^{1/2} D$$

$$= \pm\frac{\Delta_n}{d}\left(1-\frac{\Delta_n^2}{d^2}\right)^{-1/2} D$$

$$\approx \pm\left(n+\frac{1}{2}\right)\frac{\lambda D}{d} \qquad \textbf{(10)}$$

where we have assumed $\Delta_n << d$. Thus the points corresponding to minima will be equally spaced with a spacing of $\lambda D/d$.

Example 14.4 Until now we have assumed the needles at S_1 and S_2 (see Fig. 14.1) to vibrate in phase. Assume now that the needles vibrate with a phase difference of π, and obtain the nodal lines. Generalize the result for an arbitrary phase difference between the vibrations of the two needles.

Solution: The two needles S_1 and S_2 vibrate out of phase. Thus if, at any instant, the needle at S_1 produces a crest at a distance R

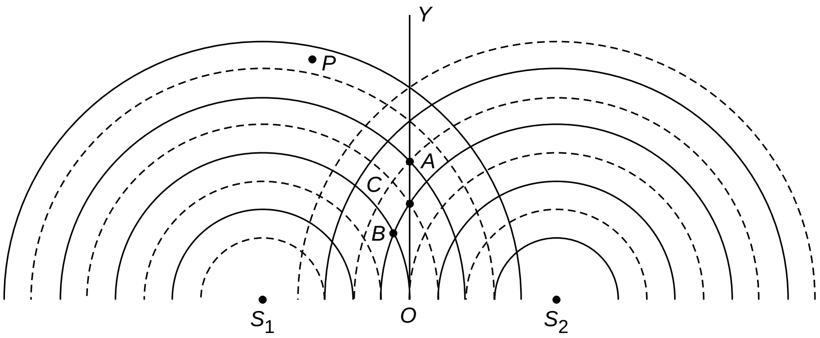

Fig. 14.4 Waves emanating from two point sources S_1 and S_2 vibrating out of phase.

from it, then the needle at S_2 produces a trough at a distance R from S_2. Therefore, at all points on the perpendicular bisector OY (see Fig. 14.4), the two vibrations will always be out of phase and we will have a minimum. On the other hand, at point B which satisfies the equation

$$S_2B - S_1B = \lambda/2$$

the two vibrations will be in phase, and we will have a maximum. Thus, because of the initial phase difference of π, the conditions for maxima and minima are reversed; i.e., when

$$S_2P \sim S_1P = \left(n + \frac{1}{2}\right)\lambda \qquad \text{(maxima)}$$

the interference will be constructive and we will have maxima, and when

$$S_2P \sim S_1P = n\lambda \qquad \text{(minima)}$$

the interference will be destructive and we will have minima. Notice that one again obtains a stationary interference pattern with nodal lines as hyperbolas.

The above analysis can be easily generalized for an arbitrary phase difference between the two needles. Assume, for example, that there is a phase difference of $\pi/3$; i.e., if there is a crest at a distance R from S_1, then there is a crest at a distance $R - \lambda/6$ from S_2. Consequently, the condition

$$S_1P - S_2P = n\lambda + \frac{\lambda}{6} \qquad n = 0, \pm 1, \pm 2, \ldots$$

will correspond to maxima.

14.3 COHERENCE

From the above examples we find that whenever the two needles vibrate with a constant phase difference, a stationary interference pattern is produced. The positions of the maxima and minima will, however, depend on the phase difference in the vibration of the two needles. Two sources which vibrate with a fixed phase difference between them are said to be *coherent.*

We next assume that the two needles are sometimes vibrating in phase, sometimes vibrating out of phase, sometimes vibrating with a phase difference of $\pi/3$, etc.; then the interference pattern will keep on changing. If the phase difference changes with such great rapidity that a stationary interference cannot be observed, then the sources are said to be *incoherent.*

Let the displacement produced by the sources at S_1 and S_2 be given by

$$\begin{aligned} y_1 &= a\cos\omega t \\ y_2 &= a\cos(\omega t + \phi) \end{aligned} \tag{11}$$

Then the resultant displacement is

$$y = y_1 + y_2 = 2a\cos\frac{\phi}{2}\cos\left(\omega t + \frac{\phi}{2}\right) \tag{12}$$

The intensity I which is proportional to the square of the amplitude can be written in the form

$$I = 4I_0\cos^2\frac{\phi}{2} \tag{13}$$

where I_0 is the intensity produced by each one of the sources individually. Clearly if $\phi = \pm\pi, \pm 3\pi, \ldots$, the resultant intensity will be zero and we will have minima. On the other hand, when $\phi = 0, \pm 2\pi, \pm 4\pi, \ldots$, the intensity will be maximum ($= 4I_0$). However, if the phase difference between sources S_1 and S_2 (i.e., ϕ) is changing with time, the observed intensity is given by

$$I = 4I_0\left\langle\cos^2\frac{\phi}{2}\right\rangle \tag{14}$$

where $\langle\ldots\rangle$ denotes the time average of the quantity inside the angular brackets; the time average of a time-dependent function is defined by the relation

$$\langle f(t)\rangle = \frac{1}{\tau}\int_{-\tau/2}^{+\tau/2} f(t)\,dt \tag{15}$$

where τ represents the time over which the averaging is carried out. For example, if the interference pattern is viewed by a normal eye, this averaging will be over about 0.1; for a camera with exposure time 0.001 s, τ = 0.001 s, etc. Clearly, if ϕ varies in a random manner in times which are small compared to τ, then $\cos^2(\phi/2)$ will randomly vary between 0 and 1 and $\left\langle \cos^2(\phi/2) \right\rangle$ will be $\frac{1}{2}$ (see also Sec. 14.6). For such a case

$$I = 2I_0 \tag{16}$$

which implies that if the sources are incoherent, then the resultant intensity is the sum of the two intensities and there is no variation of intensity! Thus, if one (or both) of the two vibrating sources is such that the phase difference between the vibrations of the two sources varies rapidly, then the interference phenomenon will not be observed. We will discuss this point again in Sec. 14.6 and in Chap. 17.

14.4 INTERFERENCE OF LIGHT WAVES

Until now we have considered interference of waves produced on the surface of water. We will now discuss the interference pattern produced by light waves; however, for light waves it is difficult to observe a stationary interference pattern. For example, if we use two conventional light sources (such as two sodium lamps) illuminating two pinholes (see Fig. 14.5), we will not observe any interference pattern on the screen. This can be understood from the following reasoning: In a conventional light source, light comes from a large number of independent atoms, each atom emitting light for about 10^{-10} s, i.e., light emitted by an atom is essentially a pulse lasting for only 10^{-10} s. However, since the optical frequencies are of the order of 10^{15} s^{-1}, such a short pulse consists of about 1 million oscillations; thus it is almost monochromatic (see Chap. 17). Even if the atoms were emitting under similar conditions, waves from different atoms would differ in their initial phases.

Consequently, light coming out from holes S_1 and S_2 will have a fixed phase relationship for about 10^{-10} s, hence the interference pattern will keep on changing every billionth of a second. The eye can notice intensity changes which last at least for 0.1s, and hence we will observe a uniform intensity over the screen. However, if we have a camera whose time of shutter opening can be made less than 10^{-10} s, then the film will record an interference pattern.[2] We summarize the above results by noting that light beams from two independent sources do not have any fixed relationship, as such, they do not produce any stationary interference pattern.

Thomas Young in 1801 devised an ingenious but simple method to lock the phase relationship between the two sources. The trick lies in the division of a single wave front into two; these two split wave fronts act as if they emanated from two sources having a fixed phase relationship, and therefore when these two waves were allowed to interfere, a stationary interference pattern was obtained. In the actual experiment a light source illuminates pinhole S (see Fig. 14.6). Light diverging from this pinhole fell on a barrier which contained two pinholes S_1 and S_2 that were very close to each other and were located equidistant from S. Spherical

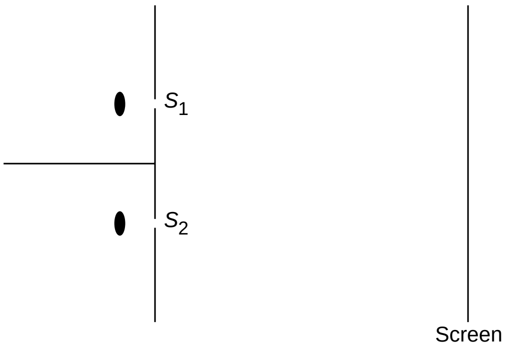

Fig. 14.5 If two sodium lamps illuminate two pinholes S_1 and S_2, no interference pattern will be observed on the screen.

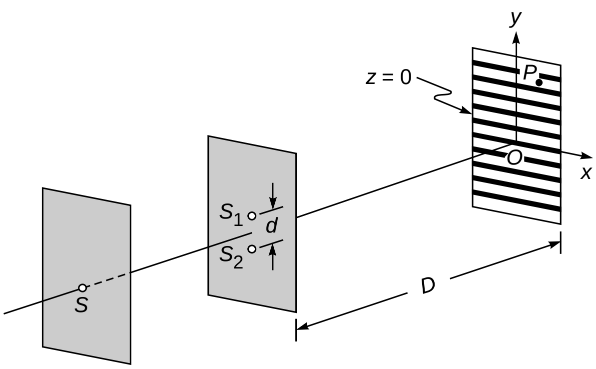

Fig. 14.6 Young's arrangement to produce interference pattern.

[2] This interference pattern will be a set of dark and bright bands only if the light waves have the same state of polarization. This can, however, be easily done by putting two Polaroids in front of S_1 and S_2 (see Fig. 14.5).

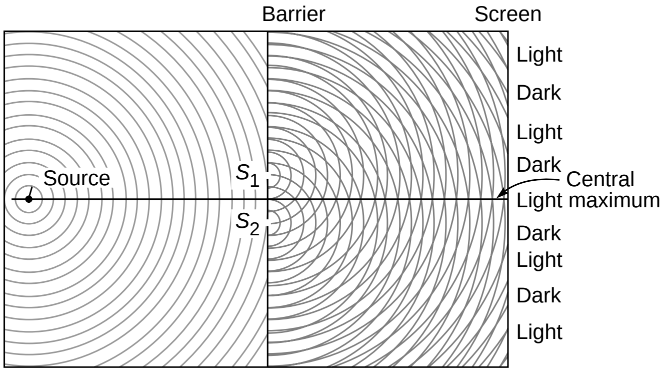

Fig. 14.7 Sections of the spherical wave fronts emanating from S, S_1, and S_2 (Adapted from Ref. 7; used with permission).

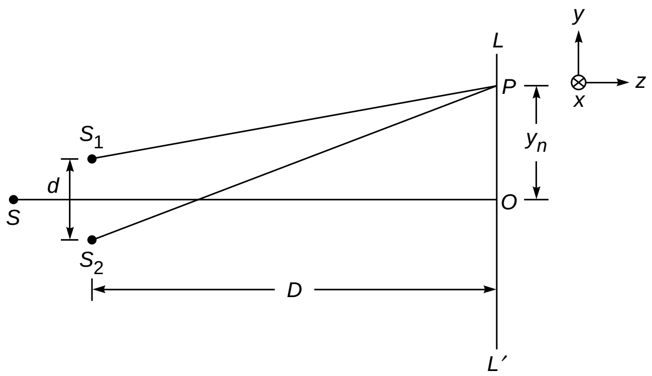

Fig. 14.8 Arrangement for producing Young's interference pattern.

waves emanating from S_1 and S_2 (see Fig. 14.7) were coherent, and on the screen beautiful interference fringes were obtained. To show that this was indeed an interference effect, Young showed that the fringes on the screen disappear when S_1 (or S_2) is covered up. Young explained the interference pattern by considering the principle of superposition, and by measuring the distance between the fringes he calculated the wavelength. Figure 14.7 shows the section of the wave front on the plane containing S, S_1, and S_2.

14.5 THE INTERFERENCE PATTERN

Let S_1 and S_2 represent the two pinholes of Young's interference experiment. We want to determine the positions of maxima and of minima on line LL' which is parallel to the y axis and lies in the plane containing points S, S_1, and S_2 (see Fig. 14.8). We will show that the interference pattern (around point O) consists of a series of dark and bright lines perpendicular to the plane of Fig. 14.8; point O is the foot of the perpendicular from point S on the screen.

For an arbitrary point P (on line LL') to correspond to a maximum, we must have

$$S_2P - S_1P = n\lambda \qquad n = 0, 1, 2, \ldots \tag{17}$$

Now,

$$(S_2P)^2 - (S_1P)^2 = \left[D^2 + \left(y_n + \frac{d}{2}\right)^2\right] - \left[D^2 + \left(y_n - \frac{d}{2}\right)^2\right] = 2y_n d$$

where

$$S_1S_2 = d \qquad \text{and} \qquad OP = y_n$$

Thus

$$S_2P - S_1P = \frac{2y_n d}{S_2P + S_1P} \tag{18}$$

If y_n, $d << D$, then negligible error will be introduced if $S_2P + S_1P$ is replaced by $2D$. For example, for $d = 0.02$ cm, D=50 cm, and $OP = 0.5$ cm (which corresponds to typical values for a light interference experiment)

$$S_2P + S_1P = [(50)^2 + (0.51)^2]^{1/2} + [(50)^2 + (0.49)^2]^{1/2} \approx 100.005 \text{ cm}$$

Thus if we replace $S_2P + S_1P$ by $2D$, the error involved is about 0.005%. In this approximation, Eq. (18) becomes

$$S_2P - S_1P \approx \frac{y_n d}{D} \tag{19}$$

Using Eq. (17), we obtain

$$y_n = \frac{n\lambda D}{d} \tag{20}$$

Thus the dark and bright fringes are equally spaced, and the distance between two consecutive dark (or bright) fringes is given by

$$\beta = y_{n+1} - y_n = \frac{(n+1)\lambda D}{d} - \frac{n\lambda D}{d}$$

or

$$\beta = \frac{\lambda D}{d} \tag{21}$$

which is the expression for the fringe width.

To determine the shape of the interference pattern, we first note that the locus of point P such that

$$S_2P - S_1P = \Delta \tag{22}$$

is a hyperbola in any plane containing points S_1 and S_2 (see Example 14.2). Consequently, the locus is a hyperbola of revolution obtained by rotating the hyperbola about the axis S_1S_2. To find the shape of the fringe on the screen, we assume the origin to be at point O and the z axis to be perpendicular to the plane of the screen as shown in Fig. 14.6. The y axis is assumed to be parallel to S_2S_1. We consider an arbitrary point P on the plane of the screen (i.e., z = 0) (see Fig. 14.6). Let its coordinates be (x, y, 0). The coordinates of points S_1 and S_2 are (0, d/2, D) and (0, –d/2, D) respectively. Thus

$$S_2P - S_1P = \left[x^2 + \left(y + \frac{d}{2}\right)^2 + D^2\right]^{1/2} - \left[x^2 + \left(y - \frac{d}{2}\right)^2 + D^2\right]^{1/2} = \Delta \quad \text{(say)}$$

or

$$\left[x^2 + \left(y + \frac{d}{2}\right)^2 + D^2\right] = \left\{\Delta + \left[x^2 + \left(y - \frac{d}{2}\right)^2 + D^2\right]^{1/2}\right\}^2$$

or $$(2yd - \Delta^2)^2 = (2\Delta)^2\left[x^2 + \left(y - \frac{d}{2}\right)^2 + D^2\right]$$

Hence,

$$(d^2 - \Delta^2)y^2 - \Delta^2 x^2 = \Delta^2\left[D^2 + \frac{1}{4}(d^2 - \Delta^2)\right]$$

which is the equation of a hyperbola. Thus the shape of the fringes is hyperbolic. On rearranging, we get

$$y = \pm\left(\frac{\Delta^2}{d^2 - \Delta^2}\right)^{1/2}\left[x^2 + D^2 + \frac{1}{4}(d^2 - \Delta^2)\right]^{1/2} \qquad \textbf{(23)}$$

For values of x such that

$$x^2 << D^2 \qquad \textbf{(24)}$$

the loci are straight lines parallel to the x axis. Thus we obtain approximately straight-line fringes on the screen. It should be emphasized that the fringes are straight lines although sources S_1 and S_2 are point sources. It is easy to see that if we had slits instead of the point sources, we would have obtained again straight-line fringes with increased intensities.

The fringes so produced are said to be nonlocalized; they can be photographed by just placing a film on the screen; they can also be seen through an eyepiece.

14.6 THE INTENSITY DISTRIBUTION

Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be the electric fields produced at point P by S_1 and S_2, respectively (see Fig. 14.8). The electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ will, in general, have different directions and different magnitudes. However, if the distances S_1P and S_2P are very large in comparison to the distance S_1S_2, the two fields will almost be in the same direction. Thus, we may write

$$\mathbf{E_1} = \hat{\mathbf{i}} E_{01} \cos\left(\frac{2\pi}{\lambda}S_1P - \omega t\right)$$

and

$$\mathbf{E_2} = \hat{\mathbf{i}} E_{02} \cos\left(\frac{2\pi}{\lambda}S_2P - \omega t\right) \qquad \textbf{(25)}$$

where $\hat{\mathbf{i}}$ represents the unit vector along the direction of either of the electric fields. The resultant field is given by

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \hat{\mathbf{i}}\left[E_{01}\cos\left(\frac{2\pi}{\lambda}S_1P - \omega t\right) + E_{02}\cos\left(\frac{2\pi}{\lambda}S_2P - \omega t\right)\right] \qquad \textbf{(26)}$$

The intensity I is proportional to the square of the electric field and is given by

$$I = KE^2 \qquad \textbf{(27)}$$

or

$$I = K\left[E_{01}^2\cos^2\left(\frac{2\pi}{\lambda}S_1P - \omega t\right) + E_{02}^2\cos^2\left(\frac{2\pi}{\lambda}S_2P - \omega t\right) + E_{01}E_{02}\left\{\cos\left[\frac{2\pi}{\lambda}(S_2P - S_1P)\right] + \cos\left[2\omega t - \frac{2\pi}{\lambda}(S_2P + S_1P)\right]\right\}\right] \qquad \textbf{(28)}$$

where K is a proportionality constant.[3] For an optical beam the frequency is very large ($\omega \approx 10^{15}$ s^{-1}), and all the terms

[3] Equation (27) will be derived in Sec. 23.5. In free space the constant K will be shown to be equal to $\varepsilon_0 c^2$, where ε_0 (=8.854 × 10^{-12} C^2 N^{-1} m^{-2}) represents the permittivity of free space and c is the speed of light in free space.

depending on ωt will vary with extreme rapidity (10^{15} times per second); consequently, any detector would record an average value of various quantities. Now

$$\left\langle \cos^2 (\omega t - \theta) \right\rangle = \frac{1}{2\tau} \int_{-\tau}^{+\tau} \frac{1 + \cos [2(\omega t - \theta)]}{2} dt$$

$$= \frac{1}{2} + \frac{1}{16\pi} \frac{T}{\tau} \left\{ \left[\sin 2(\omega t - \theta) \right]_{-\tau}^{+\tau} \right\}$$

where $T = 2\pi/\omega$ ($\approx 2\pi \times 10^{-15}$ s for an optical beam). For any practical detector T/τ <<< 1, and since the quantity within the curly braces will always be between –2 and +2, we may write

$$\langle \cos^2(\omega t - \theta) \rangle \approx \tfrac{1}{2} \qquad \textbf{(29)}$$

For the normal eye, $\tau \approx 0.1$ s; thus $T/\tau \approx 6 \times 10^{-14}$; even for a detector having 1 ns as the resolution time, $T/\tau \approx 6 \times 10^{-5}$.

The factor $\cos (2\omega t - \phi)$ will oscillate between +1 and –1, and its average will be zero as can indeed be shown mathematically. Thus the intensity, that a detector will record, will be given by

$$I = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos \delta \qquad \textbf{(30)}$$

where

$$\delta = \frac{2\pi}{\lambda} (S_2P - S_1P) \qquad \textbf{(31)}$$

represents the phase difference between the displacements reaching point P from S_1 and S_2. Further

$$I_1 = \tfrac{1}{2} K E_{01}^2$$

represents the intensity produced by source S_1 if no light from S_2 is allowed to fall on the screen; similarly, $I_2 = \frac{1}{2} K E_{02}^2$ represents the intensity produced by source S_2 if no light from S_1 is allowed to fall on the screen. From Eq. (30) we may deduce the following:

1. The maximum and minimum values of $\cos \delta$ are +1 and –1, respectively; as such, the maximum and minimum values of I are given by

$$I_{\max} = (\sqrt{I_1} + \sqrt{I_2})^2$$

and

$$I_{\min} = (\sqrt{I_1} - \sqrt{I_2})^2 \qquad \textbf{(32)}$$

The maximum intensity occurs when

$$\delta = 2n\pi \qquad n = 0, 1, 2, \ldots$$

or

$$S_2P \sim S_1P = n\lambda$$

and the minimum intensity occurs when

$$\delta = (2n + 1)\pi \qquad n = 0, 1, 2, \ldots$$

or

$$S_2P \sim S_1P = \left(n + \frac{1}{2}\right)\lambda$$

When $I_1 = I_2$, the intensity minimum is zero. In general, $I_1 \neq I_2$ and the minimum intensity is not zero.

2. If holes S_1 and S_2 are illuminated by different light sources (see Fig. 14.4), then the phase difference δ will remain constant for about 10^{-10} s (see discussion in Sec. 14.3) and thus δ would also vary with time[4] in a random way. If we now carry out the averaging over time scales which are of the order of 10^{-8} s, then

$$\langle \cos \delta \rangle = 0$$

and we obtain

$$I = I_1 + I_2$$

Thus, for two incoherent sources, the resultant intensity is the sum of the intensities produced by each one of the sources independently, and no interference pattern is observed.

3. In the arrangement shown in Fig. 14.6, if the distances S_1P and S_2P are extremely large in comparison to d, then

$$I_1 \approx I_2 = I_0 \qquad \text{(say)}$$

and

$$I = 2I_0 + 2I_0 \cos \delta = 4I_0 \cos^2 \frac{\delta}{2} \qquad \textbf{(33)}$$

The intensity distribution (which is often termed the $\cos^2$ pattern) is shown in Fig. 14.9. The actual fringe pattern (as it will appear on the screen) is shown in Fig. 14.10. Figure 14.10(a) and (b) corresponds to $d = 0.005$ mm ($\beta \approx 5$ mm) and $d = 0.025$ mm ($\beta \approx 1$ mm), respectively. Both figures correspond to $D = 5$ cm and $\lambda = 5 \times 10^{-5}$ cm. The values of the parameters are such that one can see the hyperbolic nature of the fringe pattern in Fig. 14.10(a).

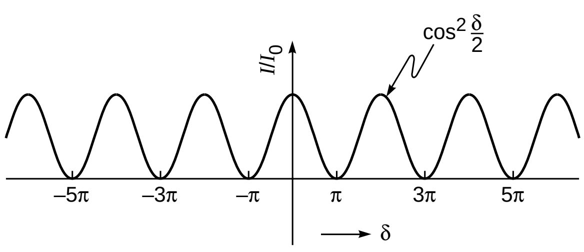

Fig. 14.9 The variation of intensity with δ.

[4] Notice that this variation occurs in times of the order of 10^{-10} s which is about 1 million times longer than the times for variation of the intensity due to the terms depending on ωt. Thus we are justified in first carrying out the averaging which leads to Eq. (30).

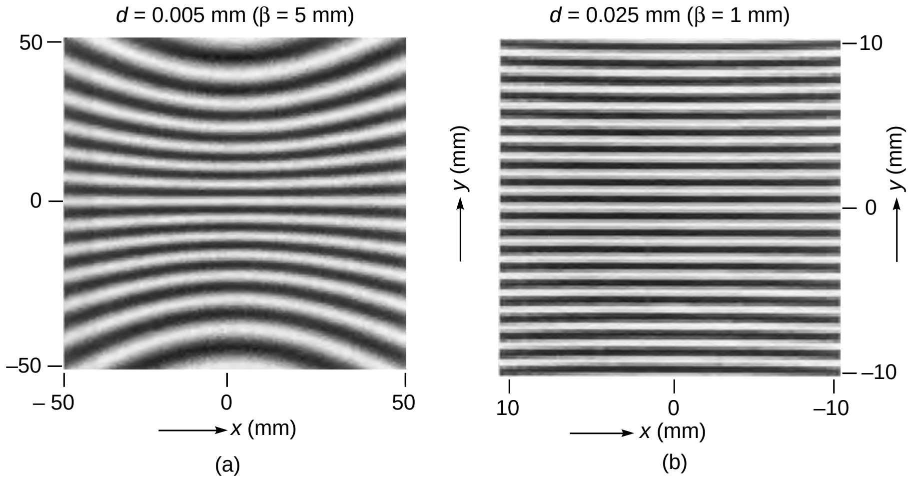

Fig. 14.10 Computer-generated fringe pattern produced by two point sources S_1 and S_2 on the screen LL' (see Fig. 14.8); (a) and (b) correspond to d = 0.005 and 0.025 mm, respectively (both figures correspond to D = 5 cm and $\lambda = 5 \times 10^{-5}$ cm).

Example 14.5 Instead of considering two point sources, we consider the superposition of two plane waves as shown in Fig. 14.11(a). The wave vectors for the two waves are given by

$$\mathbf{k}_1 = -\hat{\mathbf{y}} k \sin\theta_1 + \hat{\mathbf{z}} k \cos\theta_1$$

and

$$\mathbf{k}_2 = +\hat{\mathbf{y}} k \sin\theta_2 + \hat{\mathbf{z}} k \cos\theta_2$$

where $k = 2\pi/\lambda$ and θ_1 and θ_2 are defined in Fig. 14.11(a). Thus the electric fields of the two waves are described by the equations

$$E_1 = E_{01} \cos(\mathbf{k_1} \cdot \mathbf{r} - \omega t)$$
$$= E_{01} \cos(-ky \sin\theta_1 + kz \cos\theta_1 - \omega t)$$
$$E_2 = E_{02} \cos(\mathbf{k_2} \cdot \mathbf{r} - \omega t)$$
$$= E_{02} \cos(ky \sin\theta_2 + kz \cos\theta_2 - \omega t)$$

where we have assumed both electric fields along the same direction (say, along the x axis); if we further assume $E_{01} = E_{02} = E_0$ and $\theta_1 = \theta_2 = \theta$, then the resultant field is given by

$$E = 2E_0 \cos(ky \sin\theta) \cos(kz \cos\theta - \omega t)$$

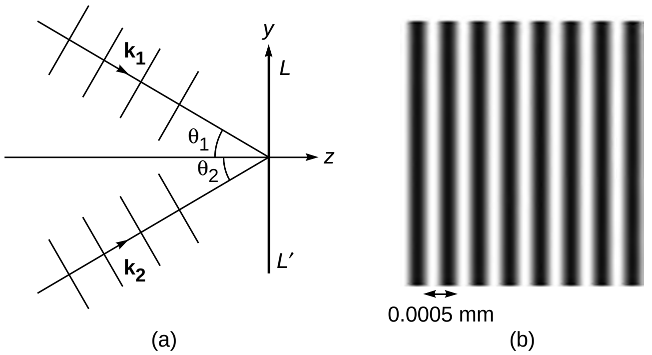

Fig. 14.11 (a) The superposition of two plane waves on LL'. (b) Computer-generated interference pattern on the screen LL' for $\theta_1 = \theta_2 = \pi/6$ and λ = 5000 Å. The fringes are parallel to the x axis.

Thus the intensity distribution on the photograph plate LL' is given by

$$I = 4I_0 \cos^2(ky \sin\theta)$$

and the fringe pattern will be strictly straight lines (parallel to the x-axis) with fringe width given by

$$\beta = \frac{\lambda}{2\sin\theta}$$

Figure 14.11(b) shows the computer-generated interference pattern on the screen LL' for $\theta = \pi/6$ and λ = 5000 Å. Thus $\beta = \lambda$ = 0.0005 mm.

Example 14.6 In this example, we consider the interference pattern produced by two point sources S_1 and S_2 on a plane PP' which is perpendicular to the line joining S_1 and S_2 [see Fig. 14.12(a)]. Obviously, on plane PP', the locus of point P for which

$$S_1P - S_2P = \text{constant}$$

will be a circle. Figure 14.12(b) and (c) shows the fringe patterns for D = 20 and 10 cm; for both figures $S_1S_2 = d$ = 0.05 mm and λ = 5000 Å. Obviously, if O represents the center of the fringe pattern, then

$$S_1O - S_2O = d = 100\lambda$$

Thus (for this value of d) the central spot will be bright for all values of D and will correspond to n = 100. The first and second bright circles will correspond to a path difference of 99λ and 98λ, respectively. Similarly, the first and second dark rings in the interference pattern will correspond to a path difference of 99.5λ and 98.5λ, respectively. The radii of the fringes can be calculated by using the formula given in Prob. 14.10.

Example 14.7 We finally consider the interference pattern produced on PP' by the superposition of a plane wave incident

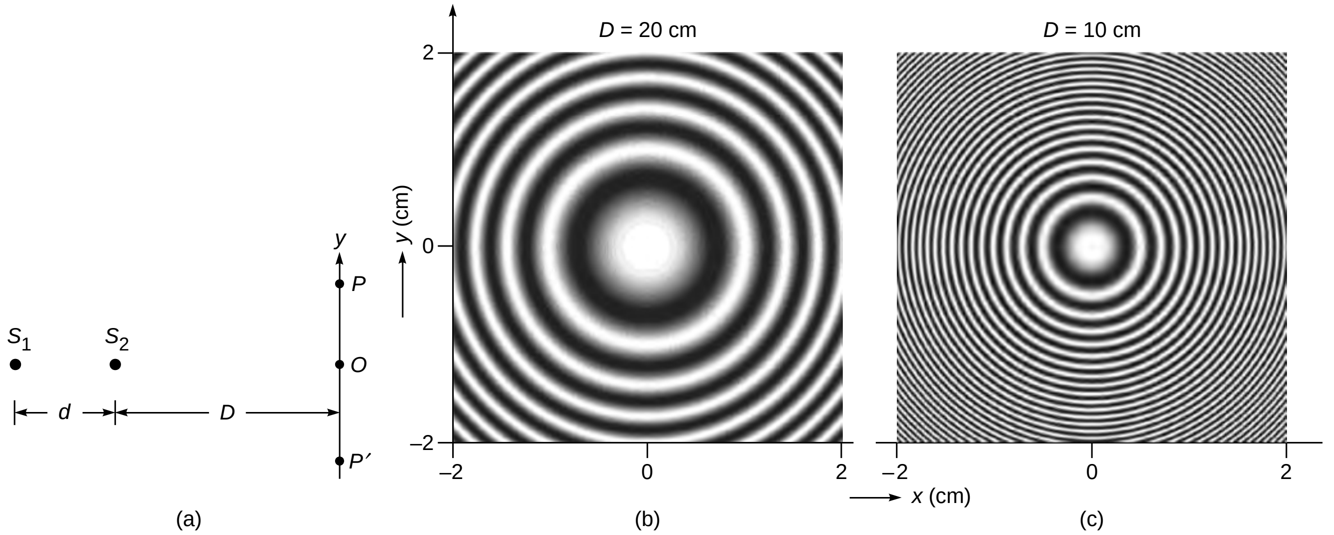

Fig. 14.12 (a) S_1 and S_2 represent two coherent sources. (b) and (c) Interference fringes observed on the screen PP' when D = 20 cm and D = 10 cm, respectively.

normally and a spherical wave emanating from point O (see Fig.14.13). The plane wave is given by

$$E_1 = E_0 \cos (kz - \omega t + \phi)$$

and the spherical wave is given by

$$E_2 = \frac{A_0}{r} \cos (kr - \omega t)$$

where r is the distance measured from point O which is assumed to be the origin. Now, on the plane PP' ($z = D$)

$$r = (x^2 + y^2 + D^2)^{1/2} \approx D\left(1 + \frac{x^2 + y^2}{2D^2}\right)$$

$$\approx D + \frac{x^2 + y^2}{2D}$$

where we have assumed $x, y << D$. On the plane $z = D$, the resultant field is given by

$$E = E_1 + E_2$$

$$\approx E_0 \cos (kD - \omega t + \phi)$$

$$+ \frac{A_0}{D} \cos\left[kD + \frac{k}{2D}(x^2 + y^2) - \omega t\right]$$

Thus

$$\langle E^2 \rangle = \frac{1}{2} E_0^2 + \frac{1}{2}\left(\frac{A_0}{D}\right)^2 + E_0 \frac{A_0}{D} \cos\left[\frac{k}{2D}(x^2 + y^2) - \phi\right]$$

If we assume that

$$\frac{A_0}{D} \approx E_0$$

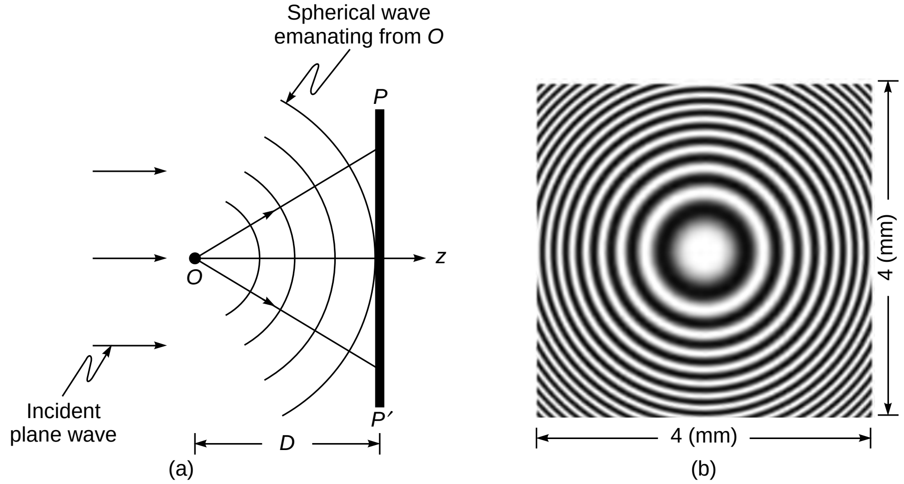

Fig. 14.13 (a) Superposition of a plane wave and a spherical wave emanating from point O; (b) interference fringes observed on the screen PP'.

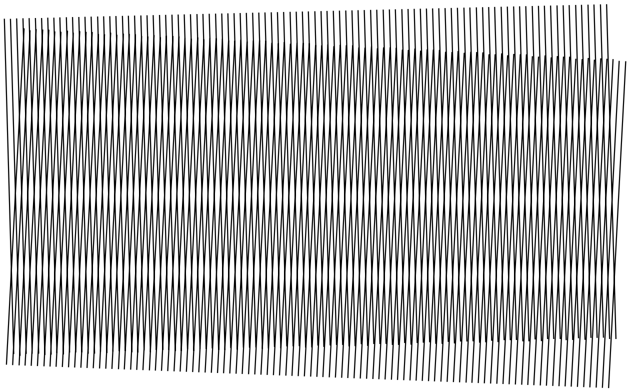

Fig. 14.14 The moiré pattern produced by two overlapping straight-line patterns.

i.e., the amplitude of the spherical wave (on plane PP') is the same as the amplitude of the plane wave, then

$$\langle E^2 \rangle \approx 2E_0^2 \cos^2\left[\frac{k}{4D}(x^2+y^2) - \frac{1}{2}\phi\right]$$

and we obtain circular interference fringes as shown in Fig. 14.13(b). If r_m and r_{m+p} denote the radii of the mth and $(m + p)$th bright rings, then

$$r_{m+p}^2 - r_m^2 = 2p\lambda D$$

14.6.1 Moiré Fringes

Moiré fringes can be very effectively used to study the formation of fringe patterns. In Fig. 14.14 we have shown the overlapping of two simple patterns from which one can understand the formation of bright and dark fringes when two plane waves propagate in slightly different directions. In a classroom, it can be easily demonstrated by having a periodic pattern on a transparency and overlapping it with its own photocopy at different angles. Similarly, if one overlaps a circular pattern (on a transparency) with its own copy, one obtains the hyperbolic fringes as shown in Fig. 14.15. (To get a clearer fringe pattern, you may have to view the patterns from a greater distance.) In Sec. 17.5 we have shown how the beat phenomenon can be understood by observing the Moiré fringes obtained by the overlapping of two patterns of slightly different periods (see Fig. 17.13).

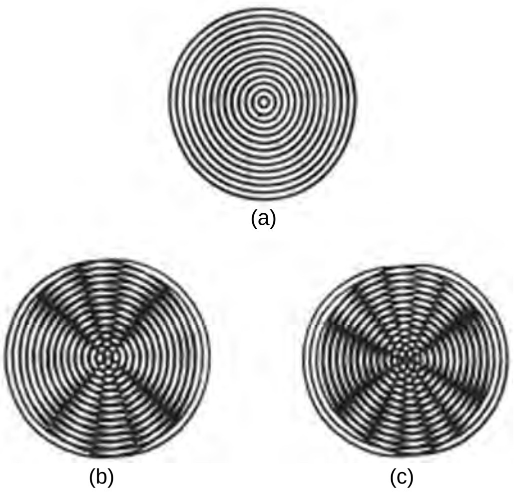

Fig. 14.15 The moiré pattern produced by two overlapping circular patterns. You will see clear hyperbolic fringes if you put the pattern at a greater distance from the eye. The circular pattern was provided by Dr. R. E. Bailey.

Example 14.8 Consider a parallel beam of light (from a distant source S' such as a star) incident (at an angle θ) on two slits S_1 and S_2 as shown in Fig. 14.16. Obviously the path difference between the waves emanating from slits S_1 and S_2 is given by

$$XS_2 = d \sin\theta$$

Therefore the intensity distribution on the screen due to S' is given by

$$I = I_0 \cos^2\frac{\delta}{2}$$

where

$$\begin{aligned}\delta &= \frac{2\pi}{\lambda}\left(XS_2 + S_2P - S_1P\right)\\ &= \frac{2\pi}{\lambda}\left[(S_2P - S_1P) + d\sin\theta\right]\\ &= \frac{2\pi}{\lambda}\left(\frac{xd}{D} + d\sin\theta\right)\end{aligned}$$

Thus the intensity distribution (due to light coming from the distant source S') is given by

$$I' = I_0 \cos^2\left[\frac{\pi}{\lambda}\left(\frac{xd}{D} + d\sin\theta\right)\right]$$

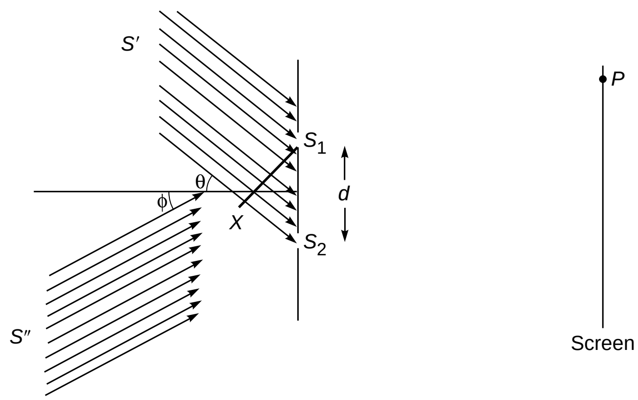

Fig. 14.16 Two distant sources illuminating the slits S_1 and S_2.

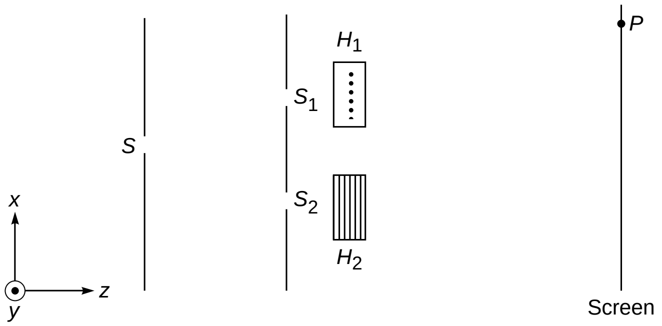

Fig. 14.17 H_1 and H_2 are half wave plates placed in front of S_1 and S_2. The optic axis of H_1 and H_2 are along y and x directions respectively.

Similarly, if there is light incident from another distant source S'' (at an angle ϕ), then the corresponding intensity distribution on the screen is given by

$$I'' = I_0 \cos^2\left[\frac{\pi}{\lambda}\left(\frac{\text{xd}}{\text{D}} - d\sin\phi\right)\right]$$

The resultant intensity distribution is given by

$$I = I' + I''$$

Example 14.9 This example presupposes the knowledge of half wave plates (see Sec. 22.6), and therefore readers may skip this example until they have gone through Chap. 22.

Consider a y-polarized light beam incident on a double-hole system as shown in Fig. 14.17. Behind the hole S_1 we have put a half wave plate H_1 whose optic axis is along the y direction, and behind the hole S_2 we have put a half wave plate H_2 whose optic axis is along the x direction. Thus as discussed in Sec. 22.6, in H_1 a y-polarized beam will propagate with velocity c/n_e; and in H_2 a y-polarized beam will propagate with velocity c/n_o. In calcite. $n_e < n_o$; and in a half wave plate, a phase change of π is introduced between the o wave and the e wave. Thus the whole fringe pattern will shift by $\beta/2$, where β is the fringe width. What will happen if the incident light beam is x-polarized?

14.7 FRESNEL'S TWO-MIRROR ARRANGEMENT

After Young's double-hole interference experiment, Fresnel devised a series of arrangements to produce the interference pattern. One of the experimental arrangements, known as the Fresnel two-mirror arrangement, is shown in Fig. 14.18; it consists of two plane mirrors which are inclined to each other at a small angle θ and touching at the point M. Point S represents a narrow slit placed perpendicular to the plane of the paper.

A portion of the wave front from S gets reflected from M_1M and illuminates the region AD of the screen. Another portion of the wave front gets reflected from the mirror MM_2 and illuminates the region BC of the screen. Since these two wave fronts are derived from the same source, they are coherent. Thus in the region BC, one observes interference fringes. The formation of the fringes can also be understood as being due to the interference of the wave fronts from the virtual sources S_1 and S_2 of S formed by mirrors M_1 and M_2, respectively. From simple geometric considerations, it can be shown that points S, S_1, and S_2 lie on a circle whose center is at point M. Further, if the angle between the mirrors is θ, then the angle S_1SS_2 is also θ and the angle S_1MS_2 is 2θ. Thus S_1S_2 is $2R\theta$, where R is the radius of the circle.

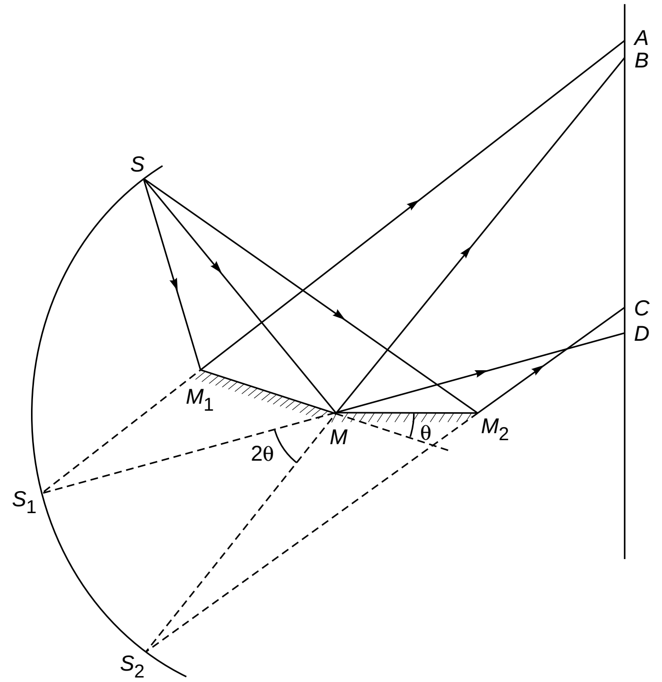

Fig. 14.18 Fresnel's two-mirror arrangement.

14.8 FRESNEL BIPRISM

Fresnel devised yet another simple arrangement for the production of interference pattern. He used a biprism, which was actually a simple prism, the base angles of which are extremely small (~20′). The base of the prism is shown in Fig. 14.19, and the prism is assumed to stand perpendicular to the plane of the paper. Point S represents the slit which is also placed perpendicular to the plane of the paper. Light from slit S gets refracted by the prism and produces two virtual images S_1 and S_2. These images act as coherent sources and produce interference fringes on the right of the biprism. The fringes can be viewed through an eyepiece. If n represents the refractive

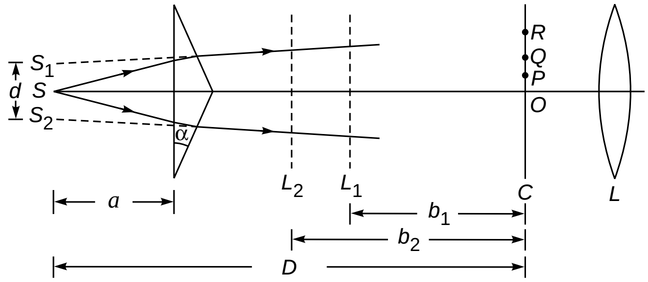

Fig. 14.19 Fresnel's biprism arrangement. Points C and L represent the positions of the crosswires and the eyepiece, respectively. To determine d, one introduces a lens between the biprism and the crosswires; L_1 and L_2 represent the two positions of the lens where the slits are clearly seen.

index of the material of the biprism and α the base angle, then $(n - 1)\alpha$ is approximately the angular deviation produced by the prism, and therefore the distance S_1S_2 is $2a(n - 1)\alpha$, where a represents the distance from S to the base of the prism. Thus, for $n = 1.5$, $\alpha \approx 20' \approx 5.8 \times 10^{-3}$ radians, $a \approx 2$ cm, one gets $d = 0.012$ cm.

The biprism arrangement can be used for the determination of wavelength of an almost monochromatic light such as the one coming from a sodium lamp. Light from the sodium lamp illuminates slit S, and interference fringes can be easily viewed through the eyepiece. The fringe width β can be determined by means of a micrometer attached to the eyepiece. Once β is known, λ can be determined by using the following relation:

$$\lambda = \frac{d\beta}{D} \tag{34}$$

To determine d, one need not measure the value of α. In fact the distances d and D can be easily determined by placing a convex lens between the biprism and the eyepiece. For a fixed position of the eyepiece there will be two positions of the lens (shown as L_1 and L_2 in Fig. 14.19) where the images of S_1 and S_2 can be seen at the eyepiece. Let d_1 be the distance between the two images when the lens is at position L_1 (at a distance b_1 from the eyepiece). Let d_2 and b_2 be the corresponding distances when the lens is at L_2. Then it can be easily shown that

$$d = \sqrt{d_1 d_2}$$

and

$$D = b_1 + b_2$$

Typically for $d \approx 0.01$ cm, $\lambda \approx 6 \times 10^{-5}$ cm, $D \approx 50$ cm, and $\beta \approx 0.3$ cm.

In the above we considered here a slit instead of a point source. Since each pair of points S_1 and S_2 produces (approximately) straight-line fringes, the slit will also produce straight-line fringes of increased intensity.

14.9 INTERFERENCE WITH WHITE LIGHT

We will now discuss the interference pattern when the slit is illuminated by white light. The wavelengths corresponding to the violet and red ends of the spectrum are about 4×10^{-5} cm and 7×10^{-5} cm, respectively. Clearly, the central fringe produced at point O (Fig. 14.19) will be white because all wavelengths will constructively interfere here. Now, slightly below (or above) point O the fringes will become colored. For example, if point P is such that

$$S_2P \sim S_1P = 2 \times 10^{-5} \text{ cm } \left(= \frac{\lambda_{\text{violet}}}{2}\right)$$

then complete destructive interference will occur only for the violet color. Partial destructive interference will occur for other wavelengths. Consequently we will have a line devoid of the violet color that will appear reddish. The point Q which satisfies

$$S_2Q \sim S_1Q = 3.5 \times 10^{-5} \text{ cm } \left(= \frac{\lambda_{\text{red}}}{2}\right)$$

will be devoid of the red color. It will correspond to almost constructive interference for the violet color. No other wavelength (in the visible region) will either constructively or destructively interfere. Thus following the white central fringe we will have colored fringes; when the path difference is about 2×10^{-5} cm, the fringe will be red, then the color will gradually change to violet. The colored fringes will soon disappear because at points far away from O there will be so many wavelengths (in the visible region) which will constructively interfere that we will observe uniform white illumination. For example, at a point R, such that $S_2R \sim S_1R = 30 \times 10^{-5}$ cm, wavelengths corresponding to $30 \times 10^{-5}/n$ ($n = 1, 2, \ldots$) will constructively interfere. In the visible region these wavelengths will be 7.5×10^{-5} cm (red), 6×10^{-5} cm (yellow), 5×10^{-5} cm (greenish yellow), and 4.3×10^{-5} cm (violet). Further, wavelengths corresponding to $30 \times 10^{-5}/\left(n + \frac{1}{2}\right)$ will destructively interfere; thus, in the visible region, the wavelengths 6.67×10^{-5} cm (orange), 5.5×10^{-5} cm (yellow), 4.6×10^{-5} cm (indigo) and 4.0×10^{-5} cm (violet) will be absent. The color of such light, as seen by the unaided eye, will be white. Thus, with white light one gets a white central fringe at the point of zero path difference along with a few colored fringes on both the sides, the color soon fading off to white. While using a white light source, if we put a red (or green) filter in front of our eye, we will see the interference pattern corresponding to the red (or green) light.

As discussed above, when we observe an interference pattern using a white light source, we will see only a few colored fringes. However, if we put a red filter in front of our eye, the fringe pattern (corresponding to the red color) will suddenly appear. If we replace the red filter by a green filter in front of our eye, the fringe pattern corresponding to the green color will appear.

In the usual interference pattern with a nearly monochromatic source (such as a sodium lamp), a large number of interference fringes are obtained, and it is extremely difficult to determine the position of the central fringe. In many interference experiments it is necessary to determine the position of the central fringe, and as has been discussed above, this can be easily done by using white light as a source.

14.10 DISPLACEMENT OF FRINGES

We will now discuss the change in the interference pattern produced by introducing a thin transparent plate in the path of one of the two interference beams as shown in Fig. 14.20. Let t be the thickness of the plate, and let n be its refractive index. It is easily seen from the figure that light reaching

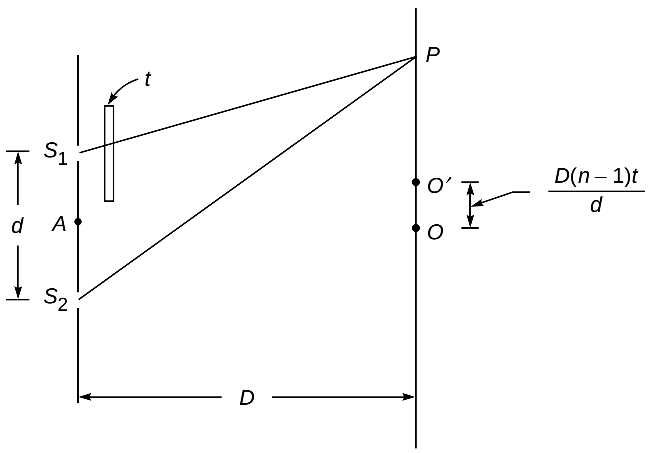

Fig. 14.20 If a thin transparent sheet (of thickness t) is introduced in one of the beams, the fringe pattern gets shifted by a distance $(n - 1)tD/d$.

point P from S_1 has to traverse a distance t in the plate and a distance $S_1P - t$ in air. Thus the time required for the light to reach from S_1 to point P is given by

$$\frac{S_1P - t}{c} + \frac{t}{v} = \frac{1}{c}\left(S_1P - t + nt\right)$$

$$= \frac{1}{c}[S_1P + (n-1)t] \quad \textbf{(35)}$$

where $v\ (= c/n)$ represents the speed of light in the plate. Equation (35) shows that by introducing the thin plate the effective optical path increases by $(n - 1)t$. Thus, when the thin plate is introduced, the central fringe (which corresponds to equal optical path from S_1 and S_2) is formed at point O' where

$$S_1O' + (n-1)t = S_2O'$$

Since [see Eq. (19)]

$$S_2O' - S_1O' \approx \frac{d}{D}OO'$$

therefore

$$(n-1)t = \frac{d}{D}OO' \quad \textbf{(36)}$$

Thus the fringe pattern gets shifted by a distance Δ which is given by

$$\Delta = \frac{D(n-1)\,t}{d} \quad \textbf{(37)}$$

The above principle enables us to determine the thickness of extremely thin transparent sheets (such as that of mica) by measuring the displacement of the central fringe. Further, if white light is used as a source, the displacement of the central fringe is easy to measure.

Example 14.10 In a double-slit interference arrangement one of the slits is covered by a thin mica sheet whose refractive index is 1.58. The distances S_1S_2 and AO (see Fig. 14.20) are 0.1 and 50 cm, respectively. Due to the introduction of the mica sheet the central fringe gets shifted by 0.2 cm. Determine the thickness of the mica sheet.

Solution: $\Delta = 0.2$ cm $\quad d = 0.1$ cm $\quad D = 50$ cm

Hence

$$t = \frac{d\,\Delta}{D(n-1)} = \frac{0.1 \times 0.2}{50 \times 0.58}$$

$$\approx 6.9 \times 10^{-4} \text{ cm}$$

Example 14.11 In an experimental arrangement similar to that discussed in Example 14.10, one finds that by introducing the mica sheet the central fringe occupies the position that was originally occupied by the eleventh bright fringe. If the source of light is a sodium lamp (λ = 5893 Å), determine the thickness of the mica sheet.

Solution: The point O' (see Fig. 14.20) corresponds to the eleventh bright fringe, thus

$$S_2O' - S_1O' = 11\lambda = (n - 1)t = 0.58t$$

14.11 LLOYD'S MIRROR ARRANGEMENT

In this arrangement, light from a slit S_1 is allowed to fall on a plane mirror at grazing incidence (see Fig. 14.21). The light directly coming from slit S_1 interferes with the light reflected from the mirror, forming an interference pattern in the region BC of the screen. One may thus consider slit S_1 and its virtual image S_2 to form two coherent sources which produce the interference pattern. Note that at grazing incidence one really need not have a mirror; even a dielectric surface has very high reflectivity (see Chap. 23).

As can be seen from Fig. 14.21, the central fringe cannot be observed on the screen unless the latter is moved to the position $L'_1L'_2$, where it touches the end of the reflector. Alternatively, one may introduce a thin mica sheet in the path of the direct beam so that the central fringe appears in the region BC. (This is discussed in detail in Prob. 14.2.) Indeed, if the central fringe is observed with white light, it is found to be dark. This implies that the reflected beam undergoes a sudden phase change of π on reflection. Consequently, when point P on the screen is such that

$$S_2P - S_1P = n\lambda \qquad n = 0, 1, 2, 3, \ldots$$

we will get minima (i.e., destructive interference). On the other hand, if

$$S_2P - S_1P = \left(n + \frac{1}{2}\right)\lambda$$

we will get maxima.

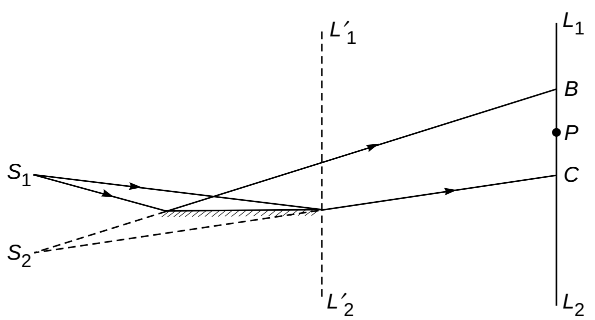

Fig. 14.21 The Lloyd's mirror arrangement.

In the next section, using the principle of optical reversibility, we will show that if there is an abrupt phase change of π when light gets reflected by a denser medium, then no such abrupt phase change occurs when reflection takes place at a rarer medium.

14.12 PHASE CHANGE ON REFLECTION

We will now investigate the reflection of light at an interface between two media, using the principle of optical reversibility. According to this principle, in the absence of any absorption, a light ray that is reflected or refracted will retrace its original path if its direction is reversed.[5]

Consider a light ray incident on an interface of two media of refractive indices n_1 and n_2 as shown in Fig. 14.22(a). Let the amplitude reflection and transmission coefficients be r_1 and t_1, respectively. Thus, if the amplitude of the incident ray is a, then the amplitudes of the reflected and refracted rays are ar_1 and at_1, respectively.

We now reverse the rays, and we consider a ray of amplitude at_1 incident on medium 1 and a ray of amplitude ar_1 incident on medium 2 as shown in Fig. 14.22(b). The ray of amplitude at_1 will give rise to a reflected ray of amplitude at_1r_2 and a transmitted ray of amplitude at_1t_2, where r_2 and t_2 are the amplitude reflection and transmission coefficients, respectively, when a ray is incident from medium 2 on medium 1. Similarly, the ray of amplitude ar_1 will give rise to a ray of amplitude ar_1^2 and a refracted ray of amplitude ar_1t_1. According to the principle of optical reversibility, the two rays of amplitudes ar_1^2 and at_1t_2 must combine to give the incident ray of Fig. 14.22(a); thus

$$ar_1^2 + at_1t_2 = a$$

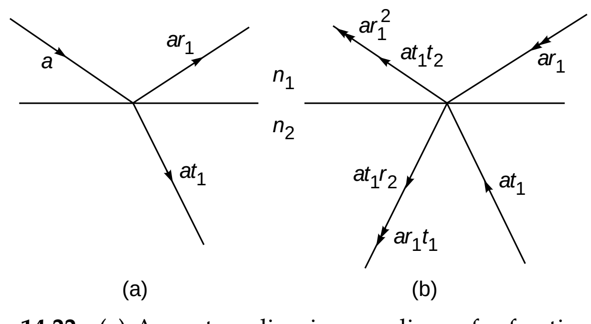

Fig. 14.22 (a) A ray traveling in a medium of refractive index n_1 incident on a medium of refractive index n_2. (b) Rays of amplitude ar_1 and at_1 incident on a medium of refractive index n_1.

[5] This principle is a consequence of time reversal invariance according to which processes can run either way in time; for more details see Refs. 3 and 8.

or

$$t_1 t_2 = 1 - r_1^2 \quad \textbf{(38)}$$

Further, the two rays of amplitudes at_1r_2 and ar_1t_1 must cancel each other, i.e.,

$$at_1r_2 + ar_1t_1 = 0$$

or

$$r_2 = -r_1 \quad \textbf{(39)}$$

Since we know from Lloyd's mirror experiment that an abrupt phase change of π occurs when light gets reflected by a denser medium, we may infer from Eq. (39) that no such abrupt phase change occurs when light gets reflected by a rarer medium. This is indeed borne out by experiments. Equations (38) and (39) are known as Stokes' relations.

In Chap. 24, we will calculate the amplitude reflection and transmission coefficients for plane waves incident on a dielectric and also on a conductor. It will be shown that the coefficients satisfy Stokes' relations; the phase change on reflection will also be discussed there.

Summary

- In 1801, Thomas Young devised an ingenious but simple method to lock the phase relationship between two sources of light. The trick lies in the division of a single wave front into two; these two split wave fronts act as if they emanated from two sources having a fixed phase relationship, and therefore when these two waves are allowed to interfere, a stationary interference pattern is obtained.
- For two coherent point sources, almost straight-line interference fringes are formed on some planes, and by measuring the fringe width (which represents the distance between two consecutive fringes) one can calculate the wavelength.
- On a plane which is normal to the line joining the two coherent point sources, the fringe pattern is circular.
- In Young's double-slit interference pattern, if we use a white light source, we get a white central fringe at the point of zero path difference along with a few colored fringes on both the sides, the color soon fading off to white. If we now introduce a very thin slice of transparent material (such as mica) in the path of one of the interfering beams, the fringes get displaced; and by measuring the displacement of fringes, we can calculate the thickness of the mica sheet.

Problems

14.1 In Young's double-hole experiment (see Fig. 14.6), the distance between the two holes is 0.5 mm, $\lambda = 5 \times 10^{-5}$ cm, and $D = 50$ cm. What will be the fringe width?

14.2 Figure 14.23 represents the layout of Lloyd's mirror experiment. Point S is a point source emitting waves of frequency 6×10^{14} s^{-1}. Points A and B represent the two ends of a mirror placed horizontally, and LOM represents the screen. The distances SP, PA, AB, and BO are 1 mm, 5 cm, 5 cm, and 190 cm, respectively. (a) Determine the position of the region where the fringes will be visible, and calculate the number of fringes. (b) Calculate the thickness of a mica sheet ($n = 1.5$) which should be introduced in the path of the direct ray so that the lowest fringe becomes the central fringe. The velocity of light is 3×10^{10} cm s^{-1}.

[**Ans:** (a) 2 cm, 40 fringes, (b) 38 μm]

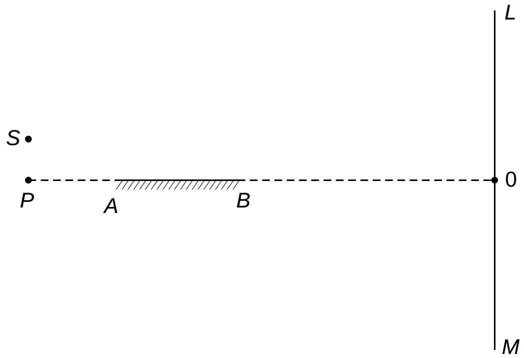

Fig. 14.23 For Prob. 14.2.

14.3 **(a)** In Fresnel's biprism arrangement, show that $d = 2(n-1)a\alpha$, where a represents the distance from the source to the base of the prism (see Fig. 14.19), α is the angle of the biprism, and n is the refractive index of the material of the biprism.

(b) In a typical biprism arrangement $b/a = 20$, and for sodium light ($\lambda \simeq 5893$ Å) one obtains a fringe width of 0.1 cm; here b is the distance between the biprism and the screen. Assuming $n = 1.5$, calculate the angle α.

[**Ans:** $\simeq$ 0.71°]

14.4 In Young's double-hole experiment, a thin mica sheet ($n = 1.5$) is introduced in the path of one of the beams. If the central fringe gets shifted by 0.2 cm, calculate the thickness of the mica sheet. Assume $d = 0.1$ cm and $D = 50$ cm.

14.5 To determine the distance between the slits in the Fresnel biprism experiment, one puts a convex lens in between the biprism and the eyepiece. Show that if $D > 4f$, one will obtain two positions of the lens where the image of the slits will be formed at the eyepiece; here f is the focal length of the convex lens, and D is the distance between the slit and the eyepiece. If d_1 and d_2 are the distances between the images (of the slits) as measured by the eyepiece, then show that $d = \sqrt{d_1 d_2}$. What would happen if $D < 4f$?

14.6 In Young's double-hole experiment, interference fringes are formed using sodium light which predominantly comprises two wavelengths (5890 and 5896 Å). Obtain the regions on the screen where the fringe pattern will disappear. You may assume $d = 0.5$ mm and $D = 100$ cm.

14.7 If one carries out Young's double-hole interference experiment using microwaves of wavelength 3 cm, discuss the nature of the fringe pattern if $d = 0.1$, 1, and 4 cm. You may assume $D = 100$ cm. Can you use Eq. (21) for the fringe width?

14.8 In Fresnel's two-mirror arrangement (see Fig. 14.18) show that points S, S_1, and S_2 lie on a circle and $S_1S_2 = 2b\theta$, where $b = MS$ and θ is the angle between the mirrors.

14.9 In the double-hole experiment using white light, consider two points on the screen, one corresponding to a path difference of 5000 Å and the other corresponding to a path difference of 40,000 Å. Find the wavelengths (in the visible region) which correspond to constructive and destructive interference. What will be the color of these points?

14.10 **(a)** Consider a plane which is normal to the line joining two point coherent sources S_1 and S_2 as shown in Fig. 14.14. If $S_1P - S_2P = \Delta$, then show that

$$y = \frac{1}{2\Delta}(d^2 - \Delta^2)^{1/2}[4D^2 + 4Dd + (d^2 - \Delta^2)]^{1/2}$$

$$\approx \frac{D}{\Delta}\sqrt{(d-\Delta)(d+\Delta)}$$

where the last expression is valid for $D >> d$.

(b) For $\lambda = 0.5$ µm, $d = 0.4$ mm and $D = 20$ cm; $S_1O - S_2O = 800\ \lambda$. Calculate the value of $S_1P - S_2P$ for point P to be the first dark ring and first bright ring.

[**Ans:** 0.39975 mm, 0.3995 mm]

14.11 In continuation of Prob. 14.10, calculate the radii of the first two dark rings for **(a)** $D = 20$ cm and **(b)** $D = 10$ cm.

[**Ans:** **(a)** ≈ 0.71 cm, **(b)** 1.22 cm]

14.12 In continuation of Prob. 14.10, assume that $d = 0.5$ mm, $\lambda = 5 \times 10^{-5}$ cm, and $D = 100$ cm. Thus the central (bright) spot will correspond to $n = 1000$. Calculate the radii of the first, second, and third bright rings which will correspond to $n = 999$, 998, and 997, respectively.

14.13 Using the expressions for the amplitude reflection and transmission coefficients [see Eqs. (67) to (72) of Chap. 24], show that they satisfy Stokes' relations.

14.14 Assume a plane wave incident normally on a plane containing two holes separated by a distance d. If we place a convex lens behind the slits, show that the fringe width, as observed on the focal plane of the lens, will be $f\lambda/d$, where f is the focal length of the lens.

14.15 In Prob. 14.14, show that if the plane (containing the holes) lies in the front focal plane of the lens, then the interference pattern will consist of exactly parallel straight lines. However, if the plane does not lie on the front focal plane, the fringe pattern will be hyperbolas.

14.16 In Young's double-hole experiment, calculate I/I_{max} where I represents the intensity at a point where the path difference is $\lambda/5$.

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