Probability and Stochastic Processes (2023-24)
Problem Sheet 4

You are free to use either C/C++ or Python for the assignment. However, the only library functions that you can use are:

- **Python:** random() from random library

- **C/C++:** Basic random number generators like rand() or erand48()

1. Write a program to simulate a biased coin. Using this biased coin, write a program to simulate a coin with $P(\text{Chance of Head}) = \frac{1}{3}$.

2. Write a program to draw samples from:

   - Binomial distribution with n = 100, p = 0.5
   - Normal distribution with $\mu = 0, \sigma = 1$ up to two floating points.

3. Write a program to estimate the area of the union of these 3 circles using monte carlo method. Check how simulation performs as number of samples grow from 10 to 1000 at steps of size 10:

   (a) A circle of radius 2 centered at $[2, 2]$.

   (b) A circle of radius 1 centered at $[2, 4]$

   (c) A circle of radius 1 centered at $[2, 0]$

4. Create an array containing the first 100 natural numbers. Select 35 numbers from this array uniformly at random.

5. Here is the algorithm for a sampling technique known as reservoir sampling:

Algorithm 1 Vitter's Reservoir Algorithm

Require: A sequence $\sigma = \{x_1, x_2, ..., x_m\}$, and a size k
    function INITIALIZE
        Create a reservoir R of length k and put the first k elements in stream $x_1, x_2, ..., x_k$ in it

    function UPDATE(x_i)
        Select x_i with probability $\frac{k}{i}$ to be included in R
        if x_i is selected to be included in R **then**
            Replace an element chosen uniformly at random from R with x_i

    function REPORT
        Return R

Note that during any point during its execution, the reservoir maintains a k-sized uniform sample from the sequence it has processed till that point. Write a code to simulate this algorithm. Run the algorithm with $m = 100$ and $k = 5$ and check the frequency of elements in the reservoir at $m = 20, 40, 60, 80, 100$ over 1000 runs.