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Midpoint Ellipse Algorithm

Basic Concept in ellipse:

The general equation for an ellipse with major axis 2a and minor axis 2b, centered at the origin as:

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

2a = length of major axis
2b = length of minor axis

$b^2x^2 + a^2y^2 - a^2b^2 = 0$

for any given value of x, it will given two values of y such as:

$$y = \pm b\sqrt{1 - x^2/a^2}$$

and for any given value of y, it will give two values of x such as:

$$x = \pm a\sqrt{1 - y^2/b^2}$$

Parametric eqn

$$x = a \cdot \cos\theta \qquad y/x = \frac{b \cdot \sin\theta}{a \cdot \cos\theta}$$
$$y = b \cdot \sin\theta$$

$\tan\alpha = \frac{b}{a}\tan\theta$. [$\theta$ ranges from 0 to 2π]

From coordinate geometry

$$f(x,y) = b^2x^2 + a^2y^2 - a^2b^2 \begin{cases} <0 \Rightarrow (x,y) \text{ inside the ellipse} \\ =0 \Rightarrow \text{" on the " "} \\ >0 \Rightarrow \text{" outside " "} \end{cases}$$

An ellipse in standard position is symmetric between quadrants but unlike a circle, it is not symmetric between the two octants of a quadrant. Thus we have to calculate pixel positions along the elliptical arc throughout one quadrant, then we obtain positions in the remaining 3 quadrants by using symmetry property. (see fig b)

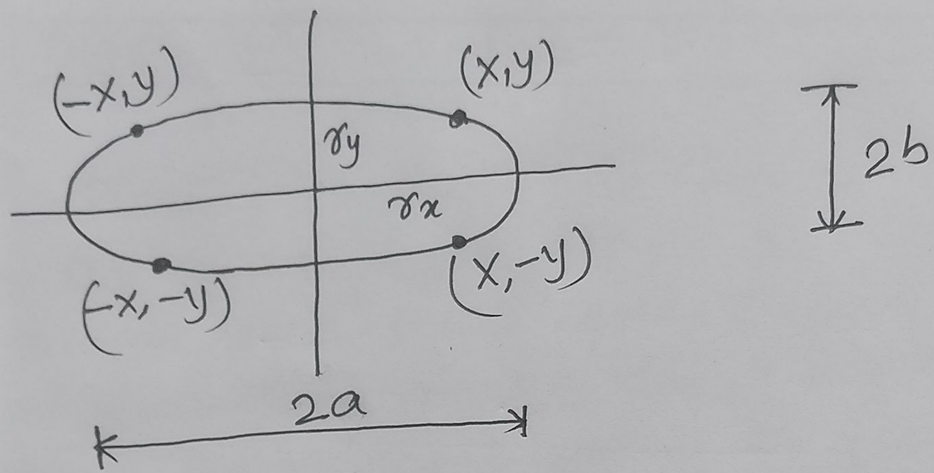
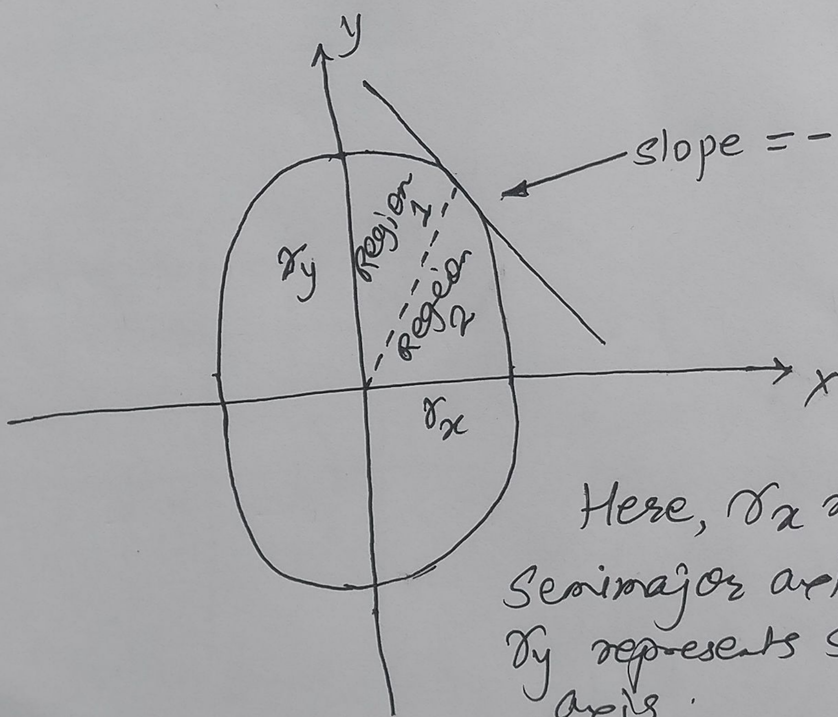


Fig: a



Here, r_x represents semimajor axis
r_y represents semiminor axis.

Fig: b

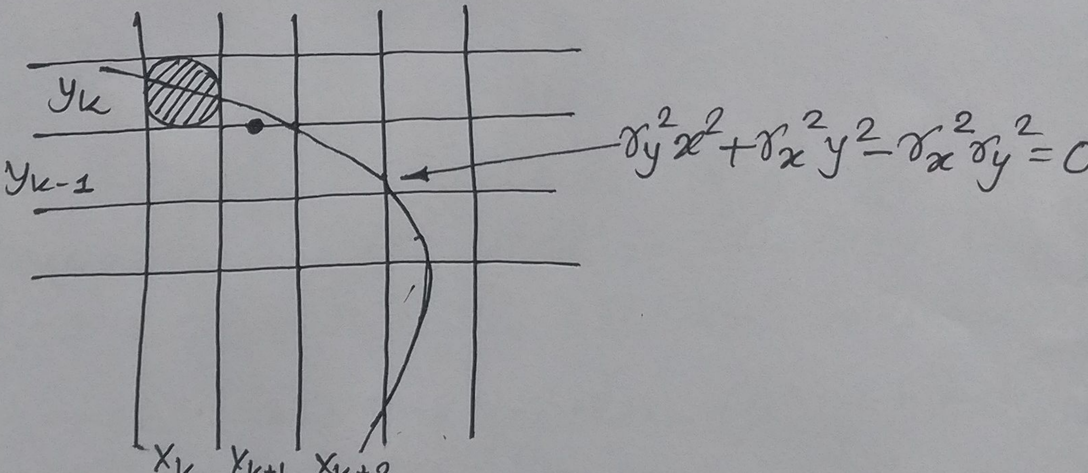
Midpoint Ellipse Algorithm

The midpoint ellipse method is applied throughout the first quadrant in two parts i.e region 1 and region 2. We are ~~[illegible]~~ forming these regions by considering the slope of the curve.

When: slope < 1 — we are in region 1
slope > 1 — " " " " 2.

Starting at $(0, r_y)$ we have to take unit steps in the x-direction until we reach the boundary between region 1 and region 2. Then we have to switch to unit steps in the y-direction over the remainder of the curve in the first quadrant. At each step, we need to test the value of the slope of curve. The slope of the ellipse is given as:

$$\frac{dy}{dx} = -\frac{2r_y^2 x}{2r_x^2 y} \quad \left[\because f(x,y) = r_y^2 x^2 + r_x^2 y^2 - r_x^2 r_y^2\right]$$



At the boundary between region 1 and region 2

$$\frac{dy}{dx} = -1 \quad \text{and} \quad 2r_y^2 x = 2r_x^2 y$$

Therefore, we move out the region 1 whenever

$$2r_y^2 x \geq 2r_x^2 y$$

and switch to unit steps in the y direction over the remainder of the curve in the first quadrant.

Figure shows the mid point between the two candidate pixels at sampling position x_k+1 in the first region.

The next position along the ellipse path can be evaluated by decision parameter at this midpoint

$$P1_k = f_{ellipse}\left(x_k+1, y_k-\frac{1}{2}\right)$$

$$= r_y^2(x_k+1)^2 + r_x^2\left(y_k-\frac{1}{2}\right)^2 - r_x^2 r_y^2$$

If $(P1_k < 0)$

– midpoint is inside the ellipse and the pixel on scan line y_k is closer to the ellipse boundary

- Otherwise :

– the midpoint is outside or on the ellipse boundary and the pixel on the scan line $y_k - 1$ is closer to the ellipse boundary.

For the next sampling position $(x_{k+1}+1 = x_k+2)$ the decision parameter for region 1 is evaluated as:

$$P1_{k+1} = f_{ellipse}\left(x_{k+1}+1, y_{k+1}-\frac{1}{2}\right)$$

$$= r_y^2\left[(x_k+1)+1\right]^2 + r_x^2\left(y_{k+1}-\frac{1}{2}\right)^2 - r_x^2 r_y^2$$

$$P1_{k+1} = P1_k + 2r_y^2(x_k+1) + r_y^2 + r_x^2\left[\left(y_{k+1}-\frac{1}{2}\right)^2 - \left(y_k-\frac{1}{2}\right)^2\right]$$

where,

y_{k+1} is either y_k or $y_k - 1$ depending on the sign of $P1_k$

if $P1_k < 0$, $y_{k+1} = y_k$

$$\therefore \boxed{P1_{k+1} = P1_k + 2r_y^2 x_{k+1} + r_y^2}$$

if $P1_k \geq 0$, $y_{k+1} = y_k - 1$

$$P1_{k+1} = P1_k + 2r_y^2 x_{k+1} r_y^2 - 2r_x^2 y_{k+1}$$

Thus, decision parameters are incremental by the following amounts

$$\text{increment} = \begin{cases} 2r_y^2 x_{k+1} + r_y^2 & \text{if } P1_k < 0 \\ 2r_y^2 x_{k+1} + r_y^2 - 2r_x^2 y_{k+1} & \text{if } P1_k \geq 0 \end{cases}$$

The terms $2r_y^2 x$ and $2r_x^2 y$ can be incrementally calculated as:

$$2r_y^2 x_{k+1} = 2r_y^2 x_k + 2r_y^2 \text{ and}$$
$$2r_x^2 y_{k+1} = 2r_x^2 y_k - 2r_x^2$$

In region 1, the initial value of the decision parameter can be obtained by evaluating the ellipse function at the start position $(x_0, y_0) = (0, r_y)$

$$P1_0 = f_{ellipse}\left(1, r_y - \frac{1}{2}\right)$$
$$= r_y^2 + r_x^2\left(r_y - \frac{1}{2}\right)^2 - r_x^2 r_y^2$$

$$= r_y^2 + r_x^2 r_y^2 + \frac{1}{4} r_x^2 - r_x^2 r_y - r_x^2 r_y^2$$

$$P1_0 = r_y^2 - r_x^2 r_y + \frac{1}{4} r_x^2$$

For region 2, we sample at unit steps in the negative y direction and the midpoint is now taken between horizontal pixels at each step. For this region, the decision parameter is evaluated as:

$$P2_k = f_{ellipse}\left(x_k + \frac{1}{2}, y_k - 1\right)$$

$$= r_y^2 \left(x_k + \frac{1}{2}\right)^2 + r_x^2 (y_k - 1)^2 - r_x^2 r_y^2$$

If $P2_k > 0$, the mid point is outside the ellipse boundary and we select the pixel at x_k.

If $P2_k \leq 0$, the mid point is inside or on the ellipse boundary and we select the pixel position x_{k+1}

The incremental decision parameters for region 2 can be given as:

$$P2_{k+1} = f_{ellipse}\left(x_{k+1} + \frac{1}{2}, y_{k+1} - 1\right)$$

$$= r_y^2 \left(x_{k+1} + \frac{1}{2}\right)^2 + r_x^2 \left[(y_k - 1) - 1\right]^2 - r_x^2 r_y^2$$

Substituting value of $P2_k$ in the above expression we get,

$$P2_{k+1} = P2_k - 2r_x^2 (y_k - 1) + r_x^2 + r_y^2 \left[\left(x_{k+1} + \frac{1}{2}\right)^2 - \left(x_k + \frac{1}{2}\right)^2 \cdot 2\right]$$

where x_{k+1} set either to x_k or to x_{k+1} depending on the sign of $P2_k$.

In region 2, the initial value of the decision parameter can be obtained by evaluating the ellipse function at the last position in the region 1.

$$P2_0 = f_{ellipse}\left(x_0+\frac{1}{2}, y_0-1\right)$$

$$= r_y^2\left(x_0+\frac{1}{2}\right)^2 + r_x^2\left(y_0-1\right)^2 - r_x^2 r_y^2$$

Midpoint ellipse algorithm

1. Read radii r_x and r_y
2. Initialize starting points as:

$$x=0$$
$$y=r_y$$

plot (x,y).

3. Calculate the initial value of decision parameter in region 1 as:

$$P_1 = r_y^2 - r_x^2 r_y + \frac{1}{4} r_x^2$$

4. Initialize dx and dy as:

$$dx = 2r_y^2 x$$
$$dy = 2r_x^2 y$$

5. do
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{   plot (x,y)
    if (P1 < 0)
       {  x = x + 1
          y = y
```

$dx = dx + 2r_y^2$
$P1 = P1 + dx + r_y^2$
}
else
{ $x = x+1$
$y = y-1$
$dx = dx + 2r_y^2$
$dy = dy - 2r_x^2$
$P1 = P1 + dx - dy + r_y^2$
}
} While ($dx < dy$)

6. Calculate the initial value of decision parameter in region 2 as:

$$P_2 = r_y^2\left(x+\frac{1}{2}\right)^2 + r_x^2(y-1)^2 - r_x^2 r_y^2$$

7. do
{ plot (x, y)
if ($P2 > 0$)
{ $x = x$
$y = y-1$
$dy = dy - 2r_x^2$
$P2 = P2 - dy + r_x^2$
}
else
{ $x = x+1$
$y = y-1$
$dy = dy - 2r_x^2$
$dx = dx + 2r_y^2$
$P2 = P2 + dx - dy + r_x^2$
}
} While ($y > 0$)

8. Determine symmetrical points in other three quadrants.