Analysis of Algorithm

Assuming rounding of error & inherent error propagates through all steps.

A

$$y = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

σ. (∵ $D\phi(x)$ $D(j)$ . . .)

$$\therefore \frac{\partial\phi}{\partial a} = \cancel{-b \pm \sqrt{b^2-4ac}} \quad \frac{b}{2a^2} \cancel{\pm \frac{(b^2-4ac)}{2a^2} \pm b^2-4ac}$$

$$\frac{\partial\phi}{\partial a} = \frac{b}{2a^2} \pm \left(\frac{-\sqrt{b^2-4ac}}{2a^2} + \frac{1}{2} \cdot \frac{1}{2a} \cdot \frac{-4c}{\sqrt{b^2-4ac}} \right)$$

$$= \frac{b}{2a^2} \pm \left(\frac{-\sqrt{b^2-4ac}}{2a^2} \right) \pm \left(\frac{-c}{a\sqrt{b^2-4ac}} \right)$$

$$\frac{\partial\phi}{\partial b} = \frac{-1 \pm \frac{1}{\sqrt{b^2-4ac}}}{2a} \qquad \frac{\partial\phi}{\partial c} = \mp \frac{1}{\sqrt{b^2-4ac}}$$

Now defining step of Algorithm

$\phi(0) \Rightarrow s = b^2$

$\phi(1) \Rightarrow t = s - 4ac$

$\phi(2) \Rightarrow u = \sqrt{t}$

$\phi(3) \Rightarrow v = -b \pm u$

$\phi(4) \Rightarrow w = \frac{v}{2 \cdot a}$

inherent error in w

$$\mathcal{E}_w^{(0)} = \frac{\Delta^0 w}{w} = \frac{\partial \phi}{\partial a} \mathcal{E}_a + \frac{\partial \phi}{\partial b} \mathcal{E}_b + \frac{\partial \phi}{\partial c} \mathcal{E}_c$$

$$= -\frac{1}{a} \mathcal{E}_a + \frac{-1 \pm \sqrt{b^2-4ac}}{-b \pm \sqrt{b^2-4ac}} \mathcal{E}_b \mp \frac{2a}{(-b \pm \sqrt{b^2-4ac})(\sqrt{b^2-4ac})} \mathcal{E}_c$$

$$\pm \left(\frac{-c}{a\sqrt{b^2-4ac}} \right) \mathcal{E}_a$$

Now Round off error in ϕ_2

$$\Delta u = \epsilon \sqrt{t} = \sqrt{t}\,(1+\epsilon) \qquad \epsilon \to 2\mathcal{E}_b - \mathcal{E}_a - \mathcal{E}_c$$

$\therefore$ relative error in w as $\frac{\Delta u}{w} = \frac{2a\sqrt{b^2-4ac}}{-b \pm \sqrt{b^2-4ac}} \epsilon$

$$= \frac{-b\sqrt{b^2-4ac} \mp (b^2-4ac)}{2c} (2\mathcal{E}_b - \mathcal{E}_a - \mathcal{E}_c)$$

$\therefore$ total r. o. error

$$= -\frac{1}{a} \mathcal{E}_a - \frac{c}{a\sqrt{b^2-4ac}} \mathcal{E}_a \mp \frac{-1 \pm \sqrt{b^2-4ac}}{-b \pm \sqrt{b^2-4ac}} \mathcal{E}_b - \frac{2a}{(-b \pm \sqrt{b^2-4ac})(\sqrt{b^2-4ac})} \mathcal{E}_c$$

$$- \frac{b\sqrt{b^2-4ac} + (b^2-4ac)}{2c} (2\mathcal{E}_b - \mathcal{E}_a - \mathcal{E}_c)$$

$\therefore$ Not Numerically stable error caused by $\sqrt{b^2-4ac}$ is very large $\quad |K| >> 1$

Algorithm B

$$y = \frac{2c}{-b \pm \sqrt{b^2-4ac}}$$

$$\frac{\partial\phi}{\partial a} = \frac{\pm 2c^2}{\sqrt{b^2-4ac}\,(-b \pm \sqrt{b^2-4ac})}$$

$$\frac{\partial\phi}{\partial b} = \frac{2c}{\sqrt{b^2-4ac}\,(-b \pm \sqrt{b^2-4ac})}$$

$$\frac{\partial\phi}{\partial c} = \frac{2}{-b + \sqrt{b^2-4ac}}$$

$\phi(0) \Rightarrow s = b^2$

$\phi(1) \Rightarrow t = s - 4ac$

$\phi(2) \Rightarrow u = \sqrt{t}$

$\phi(3) \Rightarrow v = -b \pm u$

$\phi(4) \Rightarrow w = \frac{2.c}{v}$

inherent error in w $\quad \epsilon_w^{(0)} = \frac{\Delta^0 w}{w} = \frac{\partial\phi}{\partial a}\epsilon_a + \frac{\partial\phi}{\partial b}\epsilon_b + \frac{\partial\phi}{\partial c}\epsilon_c$

$$= \frac{\pm c}{\sqrt{b^2-4ac}}\epsilon_a + \frac{1}{\sqrt{b^2-4ac}}\epsilon_b \mp \frac{a/c}{\sqrt{b^2-4ac}}\epsilon_c$$

in $\phi(2)$ $-\Delta u = \epsilon\sqrt{t} = \epsilon\sqrt{b^2-4ac}$ (floating point error

$= \sqrt{t}\,(1+\epsilon)$ $\qquad \epsilon \to 2\epsilon_b - \epsilon_a - \epsilon_c$

define $\psi(u) = \frac{2c}{-b \pm u}$

$\therefore$ relative error in W $\quad \dfrac{\frac{\partial \psi}{\partial u} \Delta u}{W}$

$$= \frac{-V\sqrt{b^2-4ac}}{(-b \pm u)^2} = -\frac{\sqrt{b^2-4ac}}{-b \pm \sqrt{b^2-4ac}} (2\epsilon_b - \epsilon_a - \epsilon_c)$$

$\therefore$ total r. o. error

$$= \frac{c}{\sqrt{b^2-4ac}} \epsilon_a + \frac{1}{\sqrt{b^2-4ac}} \epsilon_b - \frac{a/c \; \epsilon_c}{\sqrt{b^2-4ac}} - \frac{\sqrt{b^2-4ac}\,(2\epsilon_b - \epsilon_a - \epsilon_c)}{-b+\sqrt{b^2-4ac}}$$

Numerically stable $|K| < 1$

Algoritm C

$$q = -\frac{1}{2}\left[b + \mathrm{Sgn}(b)\sqrt{b^2-4ac}\right]$$

$$\frac{\partial \phi}{\partial a} = \frac{bc}{\sqrt{b^2-4ac}} \qquad \frac{\partial \phi}{\partial b} = -\frac{1}{2}\left(\frac{b+\sqrt{b^2-4ac}}{\sqrt{b^2-4ac}}\right)$$

$$\frac{\partial \phi}{\partial c} = \frac{a}{\sqrt{b^2-4ac}}$$

$\phi(0) \Rightarrow S = b^2$

$\phi(1) \Rightarrow u = S - 4ac$

$\phi(2) = V = \sqrt{u}$

$\phi(3) = W = -\frac{1}{2}(b+V)$

$$\frac{\partial \phi}{\partial a}\epsilon_a + \frac{\partial \phi}{\partial b}\epsilon_b + \frac{\partial \phi}{\partial c}\epsilon_c = \frac{bc}{\sqrt{b^2-4ac}}\epsilon_a - \frac{1}{2}\left(\frac{b+\sqrt{b^2-4ac}}{\sqrt{b^2-4ac}}\right)\epsilon_b + \frac{a\,\epsilon_c}{\sqrt{b^2-4ac}}$$

r. o. error in $\phi(2)$

$$\Delta u = \epsilon\sqrt{t} = \sqrt{t}(1+\epsilon)$$

$$\epsilon \rightarrow 2\epsilon_b - \epsilon_a - \epsilon_c$$

$\therefore$ relative error in w

$$\frac{\Delta u}{v} = \frac{-2\sqrt{b^2-4ac}}{b+\sqrt{b^2-4ac}}(2\epsilon_b - \epsilon_a - \epsilon_c)$$

$\therefore$ Total r. o. error $\rightarrow \frac{bc}{\sqrt{b^2-4ac}}\epsilon_a - \frac{1}{2}\left(\frac{b+\sqrt{b^2-4ac}}{\sqrt{b^2-4ac}}\right)\epsilon_b$

$$+\frac{a}{\sqrt{b^2-4ac}}\epsilon_c - \frac{2\sqrt{b^2-4ac}}{b+\sqrt{b^2-4ac}}(2\epsilon_b - \epsilon_a - \epsilon_c)$$

Not Numerically stable bcz $|k| >> 1$ due to large round off error in compare to inharent error.

lowest round off error $\rightarrow$ B Algorith

$\therefore$ β is more numerically stable & trustworthy.