Exercises – Markov Chain

3.2.1 A Markov chain $\{X_n\}$ on the states 0, 1, 2 has the transition probability matrix

$$\mathbf{P} = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ 0 & 0.1 & 0.2 & 0.7 \\ 1 & 0.2 & 0.2 & 0.6 \\ 2 & 0.6 & 0.1 & 0.3 \end{array}.$$

(a) Compute the two-step transition matrix P^2.
(b) What is $\Pr\{X_3 = 1 | X_1 = 0\}$?
(c) What is $\Pr\{X_3 = 1 | X_0 = 0\}$?

3.2.2 A particle moves among the states 0, 1, 2 according to a Markov process whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 2 & \frac{1}{2} & \frac{1}{2} & 0 \end{array}.$$

Let X_n denote the position of the particle at the nth move. Calculate $\Pr\{X_n = 0 | X_0 = 0\}$ for $n = 0, 1, 2, 3, 4$.

3.2.3 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ 0 & 0.7 & 0.2 & 0.1 \\ 1 & 0 & 0.6 & 0.4 \\ 2 & 0.5 & 0 & 0.5 \end{array}.$$

Determine the conditional probabilities

$$\Pr\{X_3 = 1 | X_0 = 0\} \quad \text{and} \quad \Pr\{X_4 = 1 | X_0 = 0\}.$$

3.2.4 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ 0 & 0.6 & 0.3 & 0.1 \\ 1 & 0.3 & 0.3 & 0.4 \\ 2 & 0.4 & 0.1 & 0.5 \end{array}.$$

If it is known that the process starts in state $X_0 = 1$, determine the probability $\Pr\{X_2 = 2\}$.

3.2.5 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ 0 & 0.1 & 0.1 & 0.8 \\ 1 & 0.2 & 0.2 & 0.6 \\ 2 & 0.3 & 0.3 & 0.4 \end{array}.$$

Determine the conditional probabilities

$$\Pr\{X_3 = 1|X_1 = 0\} \quad \text{and} \quad \Pr\{X_2 = 1|X_0 = 0\}.$$

3.2.6 A Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\|\begin{array}{ccc} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.1 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{array}\right\| \end{array}$$

and initial distribution $p_0 = 0.5$ and $p_1 = 0.5$. Determine the probabilities $\Pr\{X_2 = 0\}$ and $\Pr\{X_3 = 0\}$.

Problems

3.2.1 Consider the Markov chain whose transition probability matrix is given by

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \left\|\begin{array}{cccc} 0.4 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.4 & 0.3 & 0.2 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.2 & 0.1 & 0.4 & 0.3 \end{array}\right\| \end{array}.$$

Suppose that the initial distribution is $p_i = \frac{1}{4}$ for $i = 0, 1, 2, 3$. Show that $\Pr\{X_n = k\} = \frac{1}{4}, k = 0, 1, 2, 3$, for all n. Can you deduce a general result from this example?

3.2.2 Consider the problem of sending a binary message, 0 or 1, through a signal channel consisting of several stages, where transmission through each stage is subject to a fixed probability of error α. Let X_0 be the signal that is sent, and let X_n be the signal that is received at the nth stage. Suppose X_n is a Markov chain with transition probabilities $P_{00} = P_{11} = 1 - \alpha$ and $P_{01} = P_{10} = \alpha$, $(0 < \alpha < 1)$. Determine $\Pr\{X_5 = 0|X_0 = 0\}$, the probability of correct transmission through five stages.

3.2.3 Let X_n denote the quality of the nth item produced by a production system with $X_n = 0$ meaning "good" and $X_n = 1$ meaning "defective." Suppose that X_n evolves as a Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \left\|\begin{array}{cc} 0.99 & 0.01 \\ 0.12 & 0.88 \end{array}\right\| \end{array}.$$

What is the probability that the fourth item is defective given that the first item is defective?

3.2.4 Suppose X_n is a two-state Markov chain whose transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & \quad 1 \end{array} \\ \left\| \begin{array}{cc} \alpha & 1-\alpha \\ 1-\beta & \beta \end{array} \right\| \end{array}.$$

Then, $Z_n = (X_{n-1}, X_n)$ is a Markov chain having the four states (0, 0), (0, 1), (1, 0), and (1, 1). Determine the transition probability matrix.

3.2.5 A Markov chain has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & \;\;1 & \;\;2 \end{array} \\ \left\| \begin{array}{ccc} 0.7 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \\ 0 & 0 & 1 \end{array} \right\| \end{array}.$$

The Markov chain starts at time zero in state $X_0 = 0$. Let

$$T = \min\{n \geq 0; X_n = 2\}$$

be the first time that the process reaches state 2. Eventually, the process will reach and be absorbed into state 2. If in some experiment we observed such a process and noted that absorption had not yet taken place, we might be interested in the conditional probability that the process is in state 0 (or 1), given that absorption had not yet taken place. Determine $\Pr\{X_3 = 0 | X_0, T > 3\}$.

Hint: The event $\{T > 3\}$ is exactly the same as the event $\{X_3 \neq 2\} = \{X_3 = 0\} \cup \{X_3 = 1\}$.