Pricing Options with Mathematical Models

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1.1 Stocks, Bonds, and Forwards

1.1.1 Stocks and Bonds

Definition 1.1.1 (Stock)**.** A **stock** is a security that gives the owner partial ownership in a company.

Stocks are often issued by firms to finance operations. Their future price is unknown. Stocks may or may not pay **dividends**, which are profits distributed to shareholders.

Definition 1.1.2 (Bond)**.** A **bond** is a security that gives its owner the right to a fixed, predetermined payment at a future, predetermined date

The amount that a bond will pay is called its **nominal value**, **face value**, or **principal**, and the date by which it will be paid is called its **maturity**. Governments and companies issue these to raise money. They can be thought of as loans to a company or government.

Many bonds also pay intermediate payoffs called **coupons**. The price of these coupons and when they will be paid are predetermined. The price at which a bond can be sold before maturity is random and there is a risk that the issuer of the bond may default. The **coupon rate** of a bond is the percentage of the face value that will be paid as coupons each year. A bond with a coupon rate $c\%$ is called a "$c\%$ coupon bond".

1.1.2 Forwards

Definition 1.1.3 (Derivative)**.** A **derivative** is a financial contract whose value is derived from the value of an underlying asset

Derivatives can be traded at standardized exchanges or at over-the-counter (OTC) institutions. OTC institutions have much more **credit risk** (risk of default). Some types of derivatives include options, forwards, and swaps.

Derivatives are primarily used to hedge against risk. They can also be used to speculate or circumvent regulations.

Definition 1.1.4 (Forward contract)**.** A **forward contract** is an agreement to buy or sell an underlying asset at a predetermined date T, called the **maturity**, and predetermined price F, called the **forward price**.

Forwards can be used to mitigate the risk of price fluctuations by setting a predetermined future price.

Example 1.1.1. A baker who does not want to be exposed to the risk of increased sugar prices may buy a forward contract on sugar.

The price F is chosen such that the contract has zero value today. If the price of the underlying asset, known as the **spot price**, is given by S, then the payoff g at maturity for a long position is

$$g(T) = S(T) - F$$

This is because you buy the underlying asset at $S(T)$ but have to pay the forward price F. For a short position, the payoff is

$$g(T) = F - S(T)$$

This is because you are paid the forward price but have to sell the underlying asset at price $S(T)$.

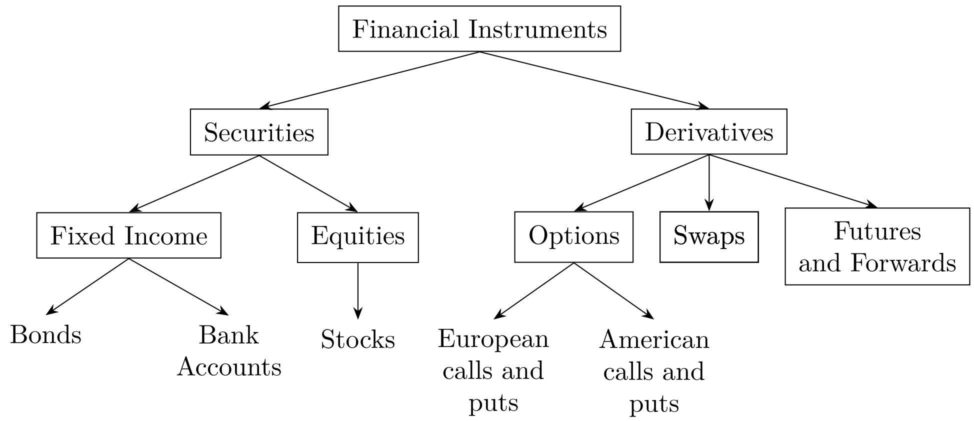

Figure 1.1: A classification of some (but not all) types of financial instruments

1.2 Swaps

Definition 1.2.1. (Swap) A **swap** is an agreement between two parties to exchange two series of payments.

Swaps can be thought of as a series of forward contracts. They are often useful when there are inefficiencies in markets, such as when two parties are under different regulations.

Example 1.2.1. In a classic **interest rate swap**, one party pays fixed interest payments on a notional amount (the notional amount is just for calculations, it is never paid) and the counterparty pays a **floating** (random) interest rate on the same notional amount. This floating rate is usually a function of the **LIBOR** (London Interbank Offer Rate).

Example 1.2.2. A CFO receives 50 000 000 USD from his company but is not allowed to sell the stock. He wants to diversify, and he may be able to do this by entering into an **equity swap**. He pays the counterparty the returns on the stock of his company in return for the returns on an investment in the S&P500.

Two parties can take advantage of **swap comparative advantage** by using swaps to borrow at better interest rates for both of them. This occurs when there are two borrowing options (for example, fixed and floating interest rates). To do this, each party borrows at the interest rate at which they have the comparative advantage and then exchanges it with the other party in a swap.

The **advantage** a party has with borrowing is how much better their rate is compared to another party. That is, the advantage is $-(r_1 - r_2)$, where r_1 is the party's interest rate and r_2 is the other party's interest rate. A party has the comparative advantage in the interest rate where it has the better advantage. The difference between the advantages can be split between both parties using swaps to reduce costs.

Example 1.2.3. Consider two firms:

1. Firm X can borrow at a 3.5% fixed rate or a LIBOR $-$ 0.2% floating rate.
2. Firm Y can borrow at a 4.5% fixed rate or a LIBOR $+$ 0.4% floating rate.

The fixed rate that X can borrow at is 1% better than the fixed rate Y can borrow at, and the floating rate that X can borrow at is 0.6% better than the floating rate that Y can borrow at. That is, X has a 1% advantage for fixed rates and a 0.6% advantage for floating rates.

X has the better advantage in fixed interest rates, so they have the comparative advantage in fixed interest rates. The 0.4% difference in advantages can be shared between the firms if X borrows at its fixed rate, Y borrows at its floating rate, and the two firms swap interest rates.

Assume they want to split the 0.4% evenly. Then, after the swap, X will borrow at (LIBOR $-$ 0.2%) $-$ 0.2% $=$ LIBOR $-$ 0.4% and Y will borrow at 4.5% $-$ 0.2% $=$ 4.3%. In the swap, X can pay the LIBOR to Y and receive 3.9% fixed interest from them to achieve this result. This is summarized in the following diagram, where an arrow pointing out of a company means they are paying that interest rate, and an arrow pointing in means they are receiving that interest rate.

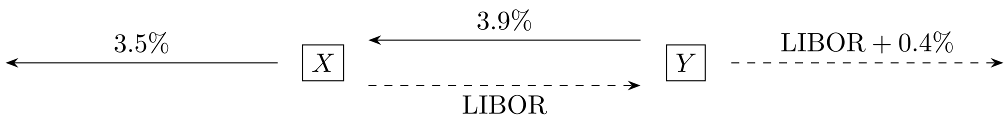

Figure 1.2: A diagram showing the interest payments

Swaptions are like swaps but they have the option to not exchange, while parties using swaps are obliged to exchange.

Swaps can be used to profit from comparative advantages, such as when one firm can borrow at a lower rate due to credit rating.

1.3 Call and Put Options

Definition 1.3.1 (Option)**.** An **option** is a contract that gives the purchaser the right, *but not the obligation*, to buy or sell a specific amount of an underlying asset at a fixed price, called the **strike price**, by a set date, called the **maturity**

If the option gives the right to buy the underlying, it is called a **call option**. If it gives the right to sell, it is a **put option**. A **European option** can be exercised only at maturity, while **American options** can be exercised any time at or before maturity. American and European options are called **vanilla options**. An **exotic option** is an option that is not a simple call or put European or American option.

Writing an option is another term for selling an option. The current market price of an option is its **premium**. Buying an option is called going **long** and selling one is called going **short**.

The difference between options and futures is that with futures, there is an obligation to buy/sell the underlying while with options, the purchaser has the right to buy/sell. That is, the purchaser can choose to not exercise the option.

Definition 1.3.2 (Payoff)**.** The **payoff** (g) is how much better ("better" means "higher" for puts, "lower" for calls) the strike price is than the underlying price. The payoff is 0 if the strike price is worse than the call price.

An options contract can be either

1. **In the money** (ITM) if the strike price is better than the underlying price,
2. **At the money** (ATM) if the strike price is equal to the underlying price, or
3. **Out of the money** (OTM) if the strike price is worse than the underlying price.

1.3.1 Call Options

Let the strike price be denoted K and the price of the underlying at time t be denoted $S(t)$. At maturity, if $S(T) > K$, the call will be exercised for a payoff of $S(T) - K$. However, if $S(T) \leq K$, the call will not be exercised since you can buy the underlying at the cheaper $S(T)$ instead of at K. This leaves a payoff of 0. Therefore, the payoff at maturity of a long call position is

$$g(T) = \max\{S(T) - K, 0\} = [S(T) - K]^{\oplus}$$

Let the premium of a call option with strike price K and maturity T, bought at time t, be denoted $C_K(t, T)$. The **profit** (π) at maturity for going long on a call is its payoff at maturity minus the premium that the call was bought at.

$$\pi(T) = g(T) - C_K(t, T) = [S(T) - K]^{\oplus} - C_K(t, T)$$

A **payoff diagram** is a graph of g vs. S. A **profit diagram** is a graph of π vs. S. The following graphs are the payoff and profit diagrams for a long call position at maturity.

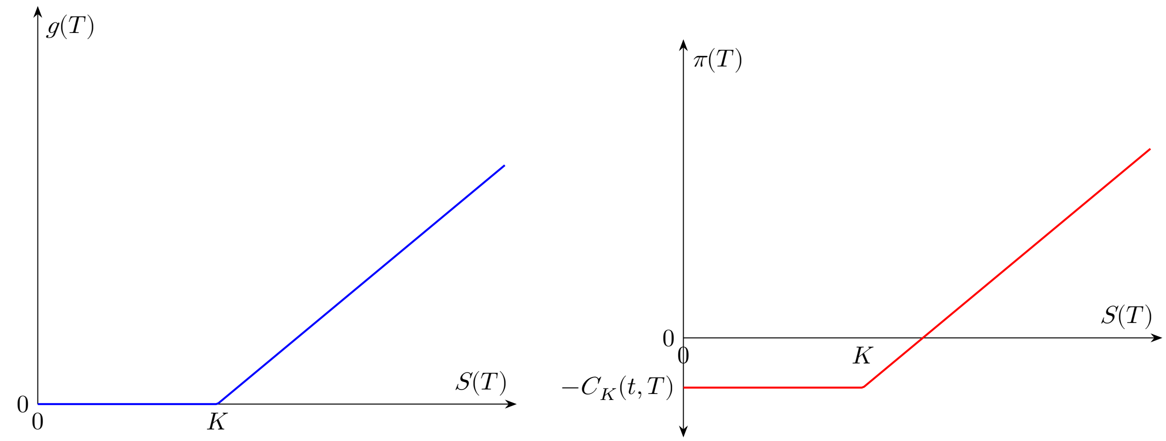

Figure 1.3: Payoff and profit diagrams for a long call position at maturity

The payoff at maturity for a short call position is given by

$$g(T) = -\max\{S(T) - K, 0\} = -[S(T) - K]^{\oplus}$$

This is because the seller of the contract will have to sell at K if $S(T) > K$, incurring a payoff of $-[S(T)-K]$. If $S(T) \leq K$, the option will not be exercised so the payoff is 0. The seller is paid the premium, so the profit for a short call position is

$$\pi(T) = g(T) + C_K(t,T) = C_K(t,T) - [S(T) - K]^{\oplus}$$

The payoff and profit diagrams for a short call position at maturity are:

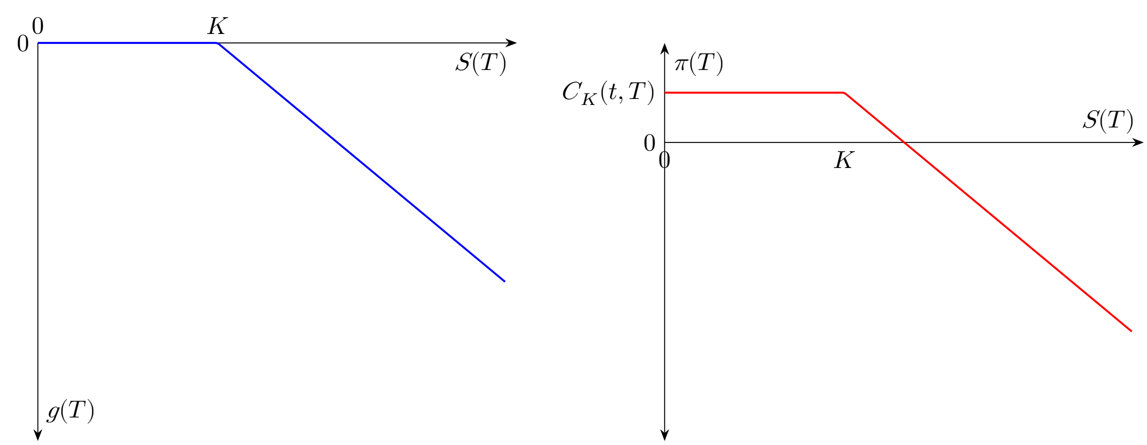

Figure 1.4: Payoff and profit diagrams for a short call position at maturity

As seen in the graphs, an important problem in short call positions is that there is potential for unlimited loss. For long call positions, the loss is limited with a maximum loss of $C_K(t,T)$, while there is no maximum loss for short call positions.

1.3.2 Put Options

At maturity, if $S(T) < K$, the put will be exercised for a payoff of $K - S(T)$. If $S(T) \geq K$, the buyer of the put will just sell the underlying on the market at $S(T)$ instead of exercising the put and selling at the lower price K, so the payoff would be 0. Therefore, the payoff of a long put position at maturity is

$$g(T) = \max\{K - S(T), 0\} = [K - S(T)]^{\oplus}$$

Let the premium of a put option with strike price K and maturity T, bought at time t, be denoted $P_K(t,T)$. The profit at maturity is again the payoff at maturity minus the premium.

$$\pi(T) = g(T) - P_K(t,T) = [K - S(T)]^{\oplus} - P_K(t,T)$$

The payoff and profit diagrams for a long put position at maturity are:

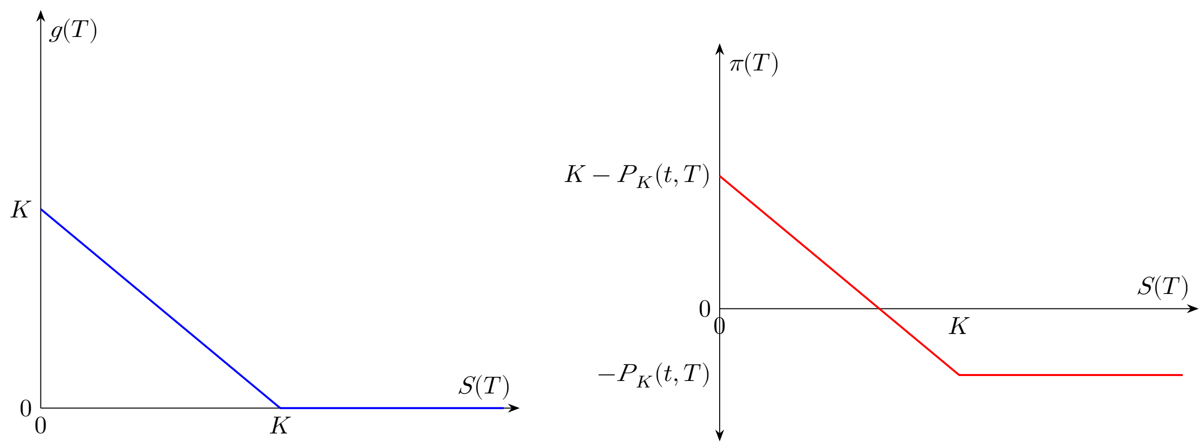

Figure 1.5: Payoff and profit diagrams for a long put position at maturity

The option will be exercised at maturity if $S(T) < K$, so the seller of the put will need to sell for a payoff of $-[K - S(T)]$. If the option is not exercised, the seller will have a payoff of 0. Therefore, the payoff of a short put position is

$$g(T) = -\max\{K - S(T), 0\} = -[K - S(T)]^{\oplus}$$

Since the seller is being paid the premium, the profit at maturity is the premium added to the payment at maturity:

$$\pi(T) = g(T) + P_K(t,T) = P_K(t,T) - [K - S(T)]^{\oplus}$$

The payoff and profit diagrams for a short put position are:

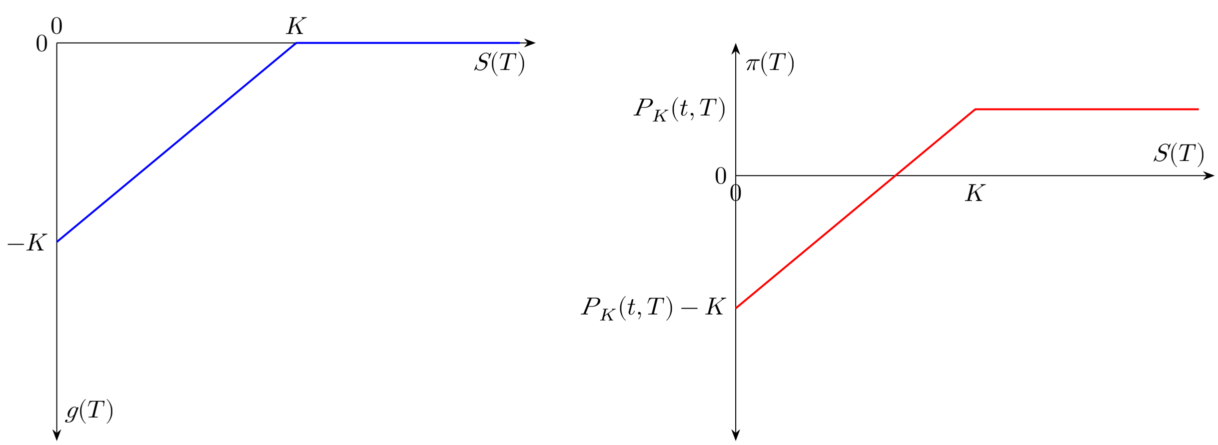

Figure 1.6: Payoff and profit diagrams for a short put position at maturity

1.3.3 Implicit Leverage and Hedging

Definition 1.3.3 (Implicit leverage)**. Implicit leverage** is the ability to make relatively large profits and losses using a relatively small initial investment using options.

Trading options has the potential for very high profits and losses. This is because the cost of an option is the premium, which is relatively small compared to the underlying price, so many more options can be bought at the same price.

Example 1.3.1. Consider a stock with $S(0) = 100$ and an long call option on that stock with $K = 100$ and $C_{100}(0, T) = 2.5$. If we invest \$100, we can buy one stock and 40 options.

If $S(T) = 105$, then investing in the stock gives a profit of 5% the initial investment. Investing in options however gives a profit of

$$\pi(T) = 40 \cdot \{[S(T) - K]^{\oplus} - C_{100}(0, T)\} = 40 \cdot [(105 - 100)^{\oplus} - 2.5] = 100$$

or 100% the initial investment, clearly much more than if we just invested in the underlying.

However, if the underlying instead increased to $S(T) = \$101$, investing in the underlying has a profit of 1% while investing in the calls gives a negative profit (a loss) of -60%. Even if the underlying increases, if it increases too little to cover the premium there is a loss.

Finally, if the underlying drops to $S(T) = 98\%$, investing in the underlying returns -2% while investing in the calls returns -100%.

As shown in the example, investing in options has the potential for horrible losses after just slight movements in the underlying, which makes them relatively risky.

Anytime an option expires OTM, the loss on a long position is 100%. For long calls, if the position is OTM at maturity,

$$\pi(T) = [S(T) - K]^{\oplus} - C_K(t, T) = -C_K(t, T)$$

which is the price the options were bought at. The same phenomenon occurs with long puts OTM at maturity.

Definition 1.3.4 (Hedging)**.** **Hedging** is a technique of limiting losses on an investment by buying options against a position on the underlying that you hold. If you lose money on the underlying, you can exercise the option to recover some of the losses.

If the price of the underlying goes down, the options used to hedge limit the losses. This is why options are helpful with risk management. If the price of the underlying goes up, you pay the premium.

Example 1.3.2. Suppose that you have a position of 100 shares of a company at worth $S(0) = 150$. To hedge against this, you can buy 60 puts against the underlying with strike price $K = 150$. If the value of the stock falls to \$100, you lose \$5000 in stock but then exercise your puts gaining \$3000, limiting your losses to just \$2000 minus the premium.

1.4 Options Combinations

Options on the same underlying are often combined to achieve a specific payoff at maturity. To calculate the payoff/profit of an options combination, add the payoffs/profits of the options that make up the combination.

1.4.1 Bull Spreads

Definition 1.4.1 (Bull spread)**.** A **bull spread** is a type of options combination where you make money when the price of the underlying increases, but your profits and losses are limited.

Bull spreads can be used when you believe there will only be a moderate increase in the price of the underlying because buying a bull spread is cheaper than just buying the stock. To create a bull spread with call options, known as a **bull call spread**, enter

1. 1 long call position with strike price K_1 and maturity T
2. 1 short call position with strike price $K_2 > K_1$ and maturity T

The payoff and profit at maturity for a bull call spread are

$$g(T) = [S(T) - K_1]^{\oplus} - [S(T) - K_2]^{\oplus} = \begin{cases} 0 & \Longleftarrow \ S(T) \leq K_1 \\ S(T) - K_1 & \Longleftarrow \ K_1 < S(T) \leq K_2 \\ K_2 - K_1 & \Longleftarrow \ S(T) > K_2 \end{cases}$$

$$\begin{aligned} \pi(T) &= [S(T) - K_1]^{\oplus} - [S(T) - K_2]^{\oplus} + \left(C_{K_2} - C_{K_1}\right)(t,T) \\ &= \begin{cases} \left(C_{K_2} - C_{K_1}\right)(t,T) & \Longleftarrow \ S(T) \leq K_1 \\ S(T) - K_1 + \left(C_{K_2} - C_{K_1}\right)(t,T) & \Longleftarrow \ K_1 < S(T) \leq K_2 \\ K_2 - K_1 + \left(C_{K_2} - C_{K_1}\right)(t,T) & \Longleftarrow \ S(T) > K_2 \end{cases} \end{aligned}$$

The profit diagram for a bull call spread at maturity

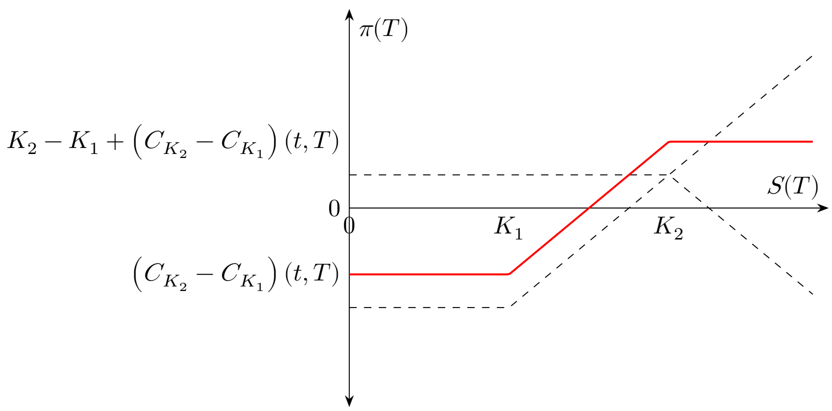

Figure 1.7: Profit diagram for a bull call spread at maturity

Bull spreads can also be created with put options. These are called **bull put spreads**. To make one, enter

1. 1 long put position with strike price K_1 and maturity T
2. 1 short put position with strike price $K_2 > K_1$ and maturity T

The payoff and profit at maturity for a bull put spread are

$$g(T) = [K_1 - S(T)]^{\oplus} - [K_2 - S(T)]^{\oplus} = \begin{cases} K_1 - K_2 & \Longleftarrow \ S(T) \leq K_1 \\ S(T) - K_2 & \Longleftarrow \ K_1 < S(T) \leq K_2 \\ 0 & \Longleftarrow \ S > K_2 \end{cases}$$

$$\begin{aligned} \pi(T) &= [K_1 - S(T)]^{\oplus} - [K_2 - S(T)]^{\oplus} + \left(P_{K_2} - P_{K_1}\right)(t,T) \\ &= \begin{cases} K_1 - K_2 + \left(P_{K_2} - P_{K_1}\right)(t,T) & \Longleftarrow \ S(T) \leq K_1 \\ S(T) - K_2 + \left(P_{K_2} - P_{K_1}\right)(t,T) & \Longleftarrow \ K_1 < S(T) \leq K_2 \\ \left(P_{K_2} - P_{K_1}\right)(t,T) & \Longleftarrow \ S > K_2 \end{cases} \end{aligned}$$

The profit diagram for a bull put spread at maturity is

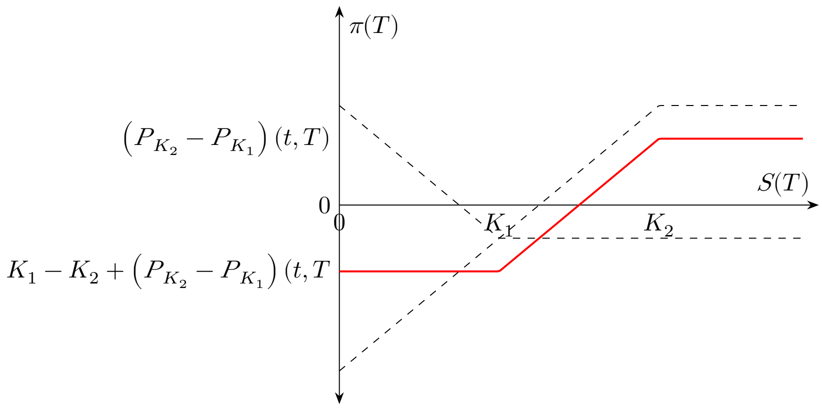

Figure 1.8: Profit diagram for a bull put spread at maturity

1.4.2 Bear Spreads

Definition 1.4.2 (Bear spread)**.** A **bear spread** is a type of options combination where you make money when the price of the underlying decreases, but your profits and losses are limited.

Bear spreads are useful when you believe there will be a moderate decrease in the price of the underlying because buying a bear spread is cheaper than just buying the stock. To create a **bear call spread**, enter

1. 1 short call position with strike price K_1 and maturity T
2. 1 long call position with strike price $K_2 > K_1$ and maturity T

The payoff and profit at maturity for a bear call spread are

$$g(T) = [S(T) - K_2]^{\oplus} - [S(T) - K_1]^{\oplus} = \begin{cases} 0 & \Longleftarrow \; S(T) \leq K_1 \\ K_1 - S(T) & \Longleftarrow \; K_1 < S(T) \leq K_2 \\ K_1 - K_2 & \Longleftarrow \; S(T) < K_2 \end{cases}$$

$$\begin{aligned} \pi(T) &= [S(T) - K_2]^{\oplus} - [S(T) - K_1]^{\oplus} + \left(C_{K_1} - C_{K_2}\right)(t, T) \\ &= \begin{cases} \left(C_{K_1} - C_{K_2}\right)(t, T) & \Longleftarrow \; S(T) \leq K_1 \\ K_1 - S(T) + \left(C_{K_1} - C_{K_2}\right)(t, T) & \Longleftarrow \; K_1 < S(T) \leq K_2 \\ K_1 - K_2 + \left(C_{K_1} - C_{K_2}\right)(t, T) & \Longleftarrow \; S(T) > K_2 \end{cases} \end{aligned}$$

The profit diagram for a bear call spread at maturity is

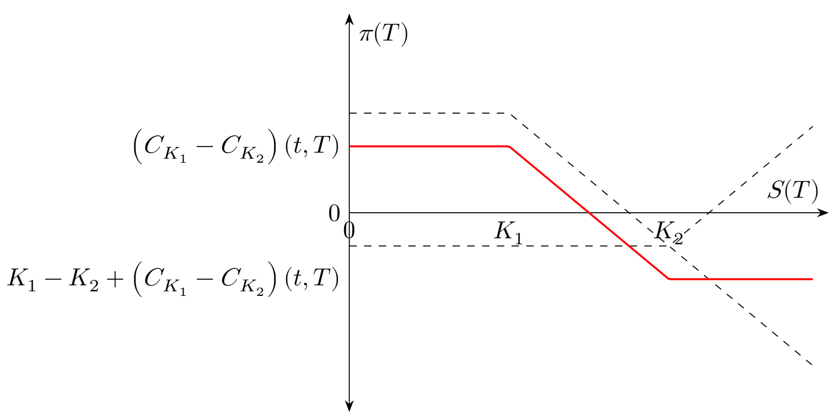

Figure 1.9: Profit diagram for a bear call spread at maturity

To make a **bear put spread**, enter

1. 1 short put position with strike price K_1 and maturity T
2. 1 long put position with strike price $K_2 > K_1$ and maturity T

The payoff and profit at maturity for a bear put spread are

$$g(T) = [K_2 - S(T)]^{\oplus} - [K_1 - S(T)]^{\oplus} = \begin{cases} K_2 - K_1 & \Longleftarrow \ S(T) \leq K_1 \\ K_2 - S(T) & \Longleftarrow \ K_1 < S(T) \leq K_2 \\ 0 & \Longleftarrow \ S(T) > K_2 \end{cases}$$

$$\begin{aligned} \pi(T) &= [K_2 - S(T)]^{\oplus} - [K_1 - S(T)]^{\oplus} + \left(P_{K_1} - P_{K_2}\right)(t,T) \\ &= \begin{cases} K_2 - K_1 + \left(P_{K_1} - P_{K_2}\right)(t,T) & \Longleftarrow \ S(T) \leq K_1 \\ K_2 - S(T) + \left(P_{K_1} - P_{K_2}\right)(t,T) & \Longleftarrow \ K_1 < S(T) \leq K_2 \\ \left(P_{K_1} - P_{K_2}\right)(t,T) & \Longleftarrow \ S(T) > K_2 \end{cases} \end{aligned}$$

The profit diagram for a bear put spread at maturity is

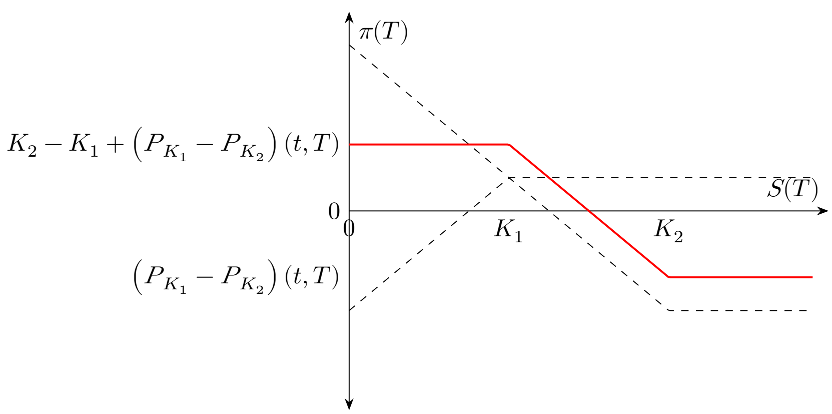

Figure 1.10: Profit diagram for a bear put spread at maturity

1.4.3 Butterfly Spreads

Definition 1.4.3. A **butterfly spread** is a type of options combination with a payoff that is 0 when $S(T)$ is more than δ away from K. The payoff within $K - \delta, K + \delta$ is further from 0 as $S(T)$ is closer to K.

Long butterfly spreads are used to profit off of low volatility; they are used to bet that $S(T)$ will stay near K. They are created by combining a bear and a bull spread. A **long butterfly call spread** can be made by entering

1. 1 long call position with strike price $K - \delta$ and maturity T
2. 2 short call positions with strike price K and maturity T
3. 1 long call position with strike price $K + \delta$ and maturity T

With $C(t,T) = \left(2C_K - C_{K+\delta} - C_{K-\delta}\right)(t,T) < 0$, the payoff and profit at maturity for a long butterfly call spread are

$$\begin{aligned} g(T) &= [S(T) - (K - \delta)]^{\oplus} - 2[S(T) - K]^{\oplus} + [S(T) - (K + \delta)]^{\oplus} = [\delta - |S(T) - K|]^{\oplus} \\ &= \begin{cases} 0 & \Longleftarrow \ S(T) \notin (K - \delta, K + \delta) \\ S(T) - K + \delta & \Longleftarrow \ K - \delta < S(T) \leq K \\ K - S(T) + \delta & \Longleftarrow \ K < S(T) < K + \delta \end{cases} \end{aligned}$$

$$\pi(T) = g(T) + C(t,T) = [\delta - |S(T) - K|]^{\oplus} + C(t,T) = \begin{cases} C(t,T) & \Longleftarrow \; S(T) \notin (K-\delta, K+\delta) \\ [S(T) - K + \delta] + C(t,T) & \Longleftarrow \; K - \delta < S(T) \leq K \\ [K - S(T) + \delta] + C(t,T) & \Longleftarrow \; K < S(T) < K + \delta \end{cases}$$

The profit diagram for a long butterfly call spread at maturity is

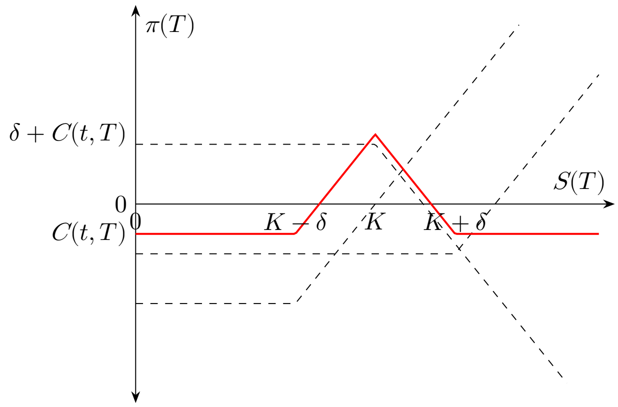

Figure 1.11: Profit diagram for a long butterfly call spread at maturity

A **long butterfly put spread** can be made by entering

1. 1 long put position with strike price $K - \delta$ and maturity T
2. 2 short put positions with strike price K and maturity T
3. 1 long put position with strike price $K + \delta$ and maturity T

The payoff at maturity for a long butterfly put spread is the same as that for a long butterfly call spread. With $P(t,T) = (2P_K - P_{K+\delta} - P_{K-\delta})\,(t,T) < 0$, the profit is given by

$$\pi(T) = g(T) + P(t,T)$$

The profit diagram for a long butterfly put spread at maturity is

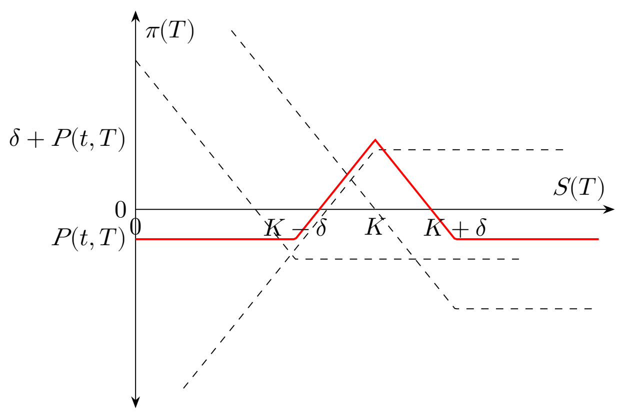

Figure 1.12: Profit diagram for a long butterfly put spread at maturity

Short butterfly spreads can also be made. They are used to bet that $S(T)$ will move far from K. That is, they are used to profit off of high volatility. To create a **short butterfly call spread**, enter

1. 1 short call position with strike price $K - \delta$ and maturity T
2. 2 long call positions with strike price K and maturity T

3. 1 short call position with strike price $K+\delta$ and maturity T

With $C(t,T) = \left(C_{K+\delta} + C_{K-\delta} - 2C_K\right)(t,T) > 0$, the payoff and profit at maturity for a short butterfly call spread are

$$g(T) = -[S(T)-(K-\delta)]^{\oplus} + 2[S(T)-K]^{\oplus} - [S(T)-(K+\delta)]^{\oplus} = -[\delta - |K-S(T)|]^{\oplus}$$
$$= \begin{cases} 0 & \Longleftarrow\ S(T) \notin (K-\delta, K+\delta) \\ K - S(T) - \delta & \Longleftarrow\ K-\delta < S(T) \le K \\ S(T) - K - \delta & \Longleftarrow\ K < S(T) < K+\delta \end{cases}$$

$$\pi(T) = g(T) + C(t,T) = -[\delta - |K-S(T)|]^{\oplus} + C(t,T)$$
$$= \begin{cases} C(t,T) & \Longleftarrow\ S(T) \notin (K-\delta, K+\delta) \\ K - S(T) - \delta + C(t,T) & \Longleftarrow\ K-\delta < S(T) \le K \\ S(T) - K - \delta + C(t,T) & \Longleftarrow\ K < S(T) < K+\delta \end{cases}$$

The profit diagram for a short butterfly call spread at maturity is

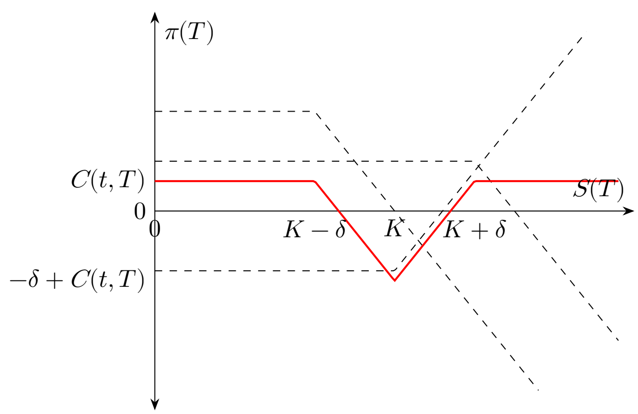

Figure 1.13: Profit diagram for a short butterfly call spread at maturity

A **short butterfly put spread** can be made by entering

1. 1 short put position with strike price $K-\delta$ and maturity T
2. 2 long put positions with strike price K and maturity T
3. 1 short put position with strike price $K+\delta$ and maturity T

The payoff at maturity for a short butterfly put spread is the same as that for a short butterfly call spread. With $P(t,T) = \left(P_{K+\delta} + P_{K-\delta} - 2P_K\right)(t,T) > 0$, the profit at maturity for a short butterfly put spread is

$$g(T) = \pi(T) + P(t,T)$$

The profit diagram for a short butterfly put spread at maturity is The profit diagram for a short butterfly call spread at maturity is

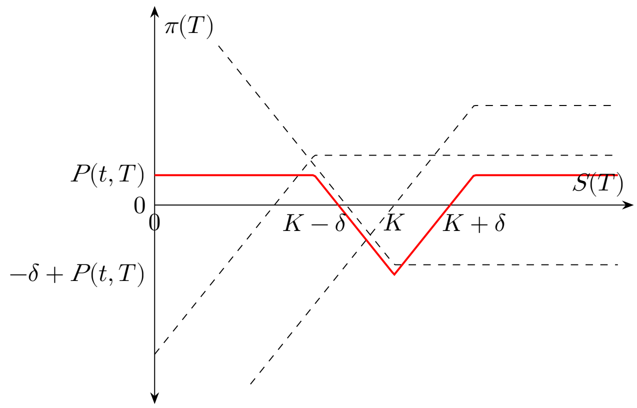

Figure 1.14: Profit diagram for a short butterfly call spread at maturity

1.4.4 Straddles and Strangles

Definition 1.4.4 (Straddle)**.** A **straddle** is a type of options combination with a payoff that is 0 when $S(T)$ is between two strike prices K_1 and K_2 and linear outside that range.

Straddles are made with a put position at K_1 and a call position at K_2. A **strangle** is a type of straddle where $K_1 = K_2$. **Long straddles** are used to profit off of high volatility. A long straddle can be made by entering

1. 1 long put position with strike price K_1 and maturity T
2. 1 long call position with strike price K_2 and maturity T

The payoff and profit at maturity for a long straddle are

$$g(T) = [K_1 - S(T)]^{\oplus} + [S(T) - K_2]^{\oplus} = \begin{cases} K_1 - S(T) & \Longleftarrow \; S(T) \leq K_1 \\ 0 & \Longleftarrow \; K_1 < S(T) \leq K_2 \\ S(T) - K_2 & \Longleftarrow \; S(T) > K_2 \end{cases}$$

$$\begin{aligned} \pi(T) &= [K_1 - S(T)]^{\oplus} + [S(T) - K_2]^{\oplus} + \left(-P_{K_1} - C_{K_2}\right)(t,T) \\ &= \begin{cases} K_1 - S(T) + \left(-P_{K_1} - C_{K_2}\right)(t,T) & \Longleftarrow \; S(T) \leq K_1 \\ \left(-P_{K_1} - C_{K_2}\right)(t,T) & \Longleftarrow \; K_1 < S(T) \leq K_2 \\ S(T) - K_2 + \left(-P_{K_1} - C_{K_2}\right)(t,T) & \Longleftarrow \; S(T) > K_2 \end{cases} \end{aligned}$$

The profit diagram for a long straddle at maturity is

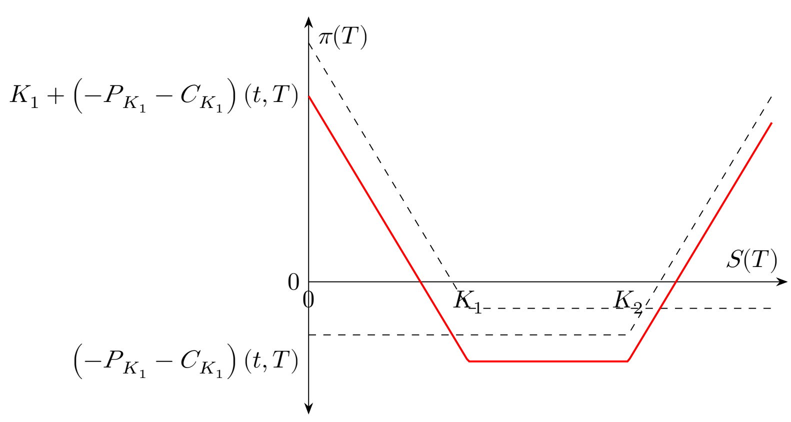

Figure 1.15: Profit diagram for a long straddle at maturity

A **short straddle**, used to profit off of low volatility, can be made by entering

1. 1 short put position with strike price K_1 and maturity T
2. 1 short call position with strike price K_2 and maturity T

The payoff and profit at maturity for a short straddle are given by

$$g(T) = -[K_1 - S(T)]^{\oplus} - [S(T) - K_2]^{\oplus} = \begin{cases} S(T) - K_1 & \Longleftarrow \; S(T) \leq K_1 \\ 0 & \Longleftarrow \; K_1 < S(T) \leq K_2 \\ K_2 - S(T) & \Longleftarrow \; S(T) > K_2 \end{cases}$$

$$\begin{aligned} \pi(T) &= -[K_1 - S(T)]^{\oplus} - [S(T) - K_2]^{\oplus} + \left(P_{K_1} + C_{K_2}\right)(t,T) \\ &= \begin{cases} S(T) - K_1 + \left(P_{K_1} + C_{K_2}\right)(t,T) & \Longleftarrow \; S(T) \leq K_1 \\ \left(P_{K_1} + C_{K_2}\right)(t,T) & \Longleftarrow \; K_1 < S(T) \leq K_2 \\ K_2 - S(T) + \left(P_{K_1} + C_{K_2}\right)(t,T) & \Longleftarrow \; S(T) > K_2 \end{cases} \end{aligned}$$

The profit diagram for a short straddle at maturity is

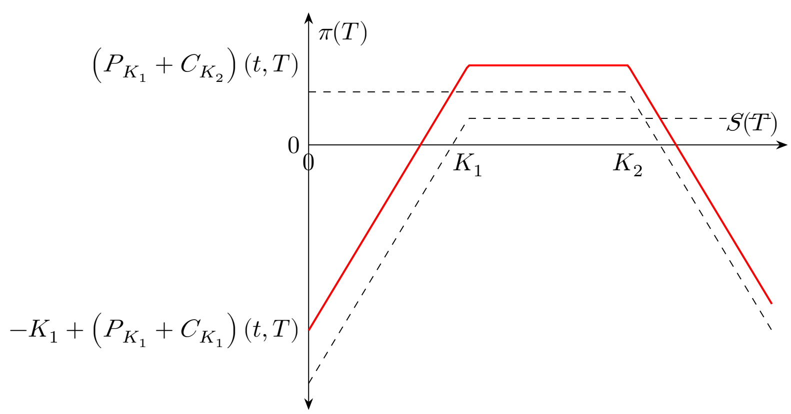

Figure 1.16: Profit diagram for a short straddle at maturity

1.4.5 Arbitrary Payoff Shapes

Under the (idealized) assumption that options with underlying $S(T)$ can be traded for all possible strike prices $K \in \mathbb{R}^+$, then any arbitrary payoff function at maturity can be made using options combinations.

Theorem 1.1. *For a smooth function* $f\colon \mathbb{R}^+ \to \mathbb{R}$ *with* $\left[\lim_{s\to\infty} f(s)\right] \cdot 0 = 0$,

$$f(s) = f(0) + f'(0)s + \int_0^{\infty} f''(K)(s-K)^{\oplus}\, dK$$

Proof. Using integration by parts to solve the integral on the RHS with $u = (s-K)^{\oplus}$ and $\mathrm{d}v = f''(K)\,\mathrm{d}K$ yields

$$\begin{aligned} \int_0^{\infty} f''(K)(s-K)^{\oplus}\,\mathrm{d}K &= \left[f'(K)(s-K)^{\oplus}\right]\Big|_{K=0}^{\infty} - \int_0^{\infty} f'(K)\,\mathrm{d}(s-K)^{\oplus} \\ &= \left[\lim_{s\to\infty} f(s)\right] \cdot 0 - f'(0)s - \int_0^{\infty} f'(K) \begin{Bmatrix} -1 & \Longleftarrow \; K \leq s \\ 0 & \Longleftarrow \; K > s \end{Bmatrix} \mathrm{d}K \\ &= -f'(0)s + \int_0^{s} f'(K)\,\mathrm{d}K = -f'(0)s + f(s) - f(0) \end{aligned}$$

Substituting this into the RHS of the theorem yields the LHS of the theorem, completing the proof. ■

In theory, an arbitrary payoff at maturity $g(T) = f[S(T)]$ can be made by holding $f(0)$ in cash, buying $f'(0)$ units of the underlying asset, and buying $f''(0)$ call options at each strike price K. This is impossible in reality because options cannot be bought at any strike price and you cannot buy an uncountably infinite number of options. In reality, arbitrary payoff functions can be approximated by using a sum of available strike prices instead of an integral of infinitely many strike prices.

2 Interest Rates, Forward Rates, and Bond Yields

2.1 Pricing Deterministic Payoffs

2.1.1 Interest

Definition 2.1.1 (Simple interest)**. Simple interest** occurs when interest payments are calculated as a percentage of the initial investment.

Suppose you can lend money at a deterministic interest rate r for a certain period (monthly, yearly, etc.). If X_0 is lent with a simple interest rate, after t periods (where t can be fractional), the investment is worth

$$V(t) = X_0(1 + rt)$$

Definition 2.1.2 (Compound interest)**. Compound interest** occurs when interest payments are calculated as a percentage of the initial investment and any interest previously earned on that.

Compounding interest is more common in the real world. If the interest is compounded n times per period during t periods, the investment at time t is

$$V(t) = X_0 \left(1 + \frac{r}{n}\right)^{nt}$$

For interest compounding n times annually, the **effective annual interest rate** r' is the equivalent simple annual interest rate over one year. It is useful for comparing interest rates compounded at different frequencies. It is given by

$$1 + r' = \left(1 + \frac{r}{n}\right)^n$$

Example 2.1.1. If an investment has a quarterly compounding annual interest rate of $r = 8\%$, the effective annual interest rate is

$$r' = \left(1 + \frac{0.08}{4}\right)^4 - 1 \approx 8.24\%$$

If an investment of X_0 is compounded continuously, the value at time t is

$$V(t) = X_0 \left[\lim_{n\to\infty} \left(1 + \frac{r}{n}\right)^{nt}\right] = X_0 \mathsf{e}^{rt}$$

2.1.2 Present Value

Definition 2.1.3 (Price)**.** The **price** of an amount $X(T)$ is the amount V_0 that can be invested now such that the investment is expected to have the value $X(T)$ at time T.

Theorem 2.1 (Law of One Price)**.** *Two cash flows that deliver the same payments in the future have the same value today.*

This result comes naturally from the No Arbitrage Assumption. For deterministic cash flows, the price is called the **present value**. The formula for present value comes from the interest rate formulae, since earning interest on the present value yields $X(T)$.

$$V_0 = \frac{X(T)}{(1 + \frac{r}{n})^{nT}} \xrightarrow{n\to\infty} X(T)\mathsf{e}^{-rT}$$

The values $\left(1+\dfrac{r}{n}\right)^{-nT}$ and e^{-rT} are called **discount factors**.

The **net present value** of a cash flow is the sum of the present values of the payments in the cash flow. That is, for cash flow $(X(t_k))_{k=0}^{N}$,

$$V = \sum_{k=0}^{N} \frac{X(t_k)}{(1+\frac{r}{n})^{nt_k}}$$

When the cash flow has $X(0) = 0$ and pays N evenly spaced payments of value X, the net present value can be found using a geometric series:

$$V_0 = X \sum_{k=1}^{N} \left(1+\frac{r}{n}\right)^{-k} = X \cdot \frac{n}{r}\left(1 - \frac{1}{(1+\frac{r}{n})^N}\right)$$

A **perpetual annuity** is a cash flow that pays X at the end of each period forever. This is just the limit of the above formula with $n = 1$ as $N \to \infty$. That is, for a perpetual annuity,

$$V_0 = \sum_{k=1}^{\infty} \frac{X}{(1+r)^k} = \frac{X}{r}$$

Definition 2.1.4 (Internal rate of return)**.** The **internal rate of return** of a cash flow is the interest rate which would make the net present value of the cash flow equal to 0.

For a cash flow $(X_k)_{k=0}^{N}$, where the payments are evenly spaced, the **internal rate of return** is the number r^* that satisfies

$$0 = \sum_{k=0}^{\infty} \frac{X_k}{(1+r^*)^k}$$

Equivalently, it is the number r^* satisfying

$$0 = \sum_{k=0}^{\infty} X_k c^k$$

where $c_k = \dfrac{1}{1+r^*}$. This equation definition is more practical for calculating r^*. However, there is not always one unique r^*, or even a valid (non-negative, real) value for r^*. The internal rate of return is a useful way to compare cash flow streams.

2.2 Bonds

2.2.1 Yield to Maturity

Definition 2.2.1 (Yield to maturity)**.** The **yield to maturity** (YTM) of a bond is the internal rate of return of the bond payments. This rate makes the price of the bond equal to the present value of the payments. It is also the rate of return on your investment after all coupons and the face value are collected.

Suppose we have a bond that pays a face value V at maturity T (measured in coupon payment periods). This bond pays n coupons per year with a total yearly value of C (so each coupon payment is $\frac{C}{n}$). The bond's cash flow stream is $\left(-P, \frac{C}{n}, \ldots, \frac{C}{n}, V\right)$, where T total coupon payments of $\frac{C}{n}$ are made. The annualized YTM λ of this stream satisfies

$$P = \frac{V}{\left(1+\frac{\lambda}{n}\right)^T} + \sum_{k=1}^{T} \frac{\frac{C}{n}}{\left(1+\frac{\lambda}{n}\right)^k} = \frac{V}{\left(1+\frac{\lambda}{n}\right)^T} + \frac{C}{\lambda}\left[1 - \frac{1}{\left(1+\frac{\lambda}{n}\right)^T}\right]$$

Clearly, a higher yield implies a lower price.

A **zero-coupon bond** (z.c. bond) is a bond that does not pay coupons (only pays the face value). Its yield is called it's **spot rate**.

Since there is assumed to be no arbitrage, one way of pricing an investment (such as a bond) is to replicate its payoff using a portfolio of other investments and calculate values from that. This method is called **replication**.

Example 2.2.1. Consider the following bonds:

1. A 6-month zero coupon bond with face value \$100 that costs \$98 (bond 1)
2. A 12-month coupon bond with one coupon payment of \$3 at 6 months with face value \$103 that costs \$101.505 (bond 2)

The yield and price of a 12-month zero coupon bond with a face value of \$103 can be calculated from this. The cash flows of bond 1 and bond 2 are (-98, 100, 0) and (-101.505, 3, 103).

To replicate the cash flow, we need to cancel the coupon payment of bond 2. To do this, go long one unit of bond 2 and short 0.03 units of bond 1. This gives the cash flow of the 12-month zero coupon bond:

$$(-101.505 - 0.03 \cdot (-98), 3 - 0.03 \cdot 100, 103) = (-98.565, 0, 103)$$

From this cash flow, we can see that the price of the 12-month zero coupon bond is \$98.565. We can now find the yield:

$$98.565 = \frac{103}{1+\lambda} \quad \therefore \quad \lambda \approx 4.49956\%$$

2.2.2 Forward Rates

Definition 2.2.2 (Forward rate)**.** The theoretical interest rate that will be observed on a loan or investment from period i to period j is the **forward rate** from i to j.

Z.c. bonds pay a fixed amount at a fixed date in the future, so the ratio of this payment to the current price is the interest rate charged for money held until the maturity of the bond. Therefore, the forward rate from i to j can be derived from the i and j period z.c. bond yields.

Theorem 2.2. *If λ_k is the annualized spot rate for k periods from now, compounding n times per year, then the forward rate $f_{i,j}$ between periods i and j is given by*

$$\left(1+\frac{\lambda_j}{n}\right)^j = \left(1+\frac{\lambda_i}{n}\right)^i \left(1+\frac{f_{i,j}}{n}\right)^{j-i}$$

Proof. Suppose you buy a j period z.c. bond. The investment increases by a factor of $\left(1+\frac{\lambda_j}{n}\right)^j$ from 0 to j.

Instead, you can also buy an i period z.c. bond then invest the coupons and notional amount from that at $f_{i,j}$ for the remaining $j-i$ periods. In this second case, the investment increases by a factor of $\left(1+\frac{\lambda_i}{n}\right)^i$ from 0 to i and then increases by a factor of $\left(1+\frac{f_{i,j}}{n}\right)^{j-i}$ from i to j. Therefore, the investment increases by a total of $\left(1+\frac{\lambda_i}{n}\right)^i \left(1+\frac{f_{i,j}}{n}\right)^{j-i}$.

The factors by which the investments increase are summarized in the following diagram:

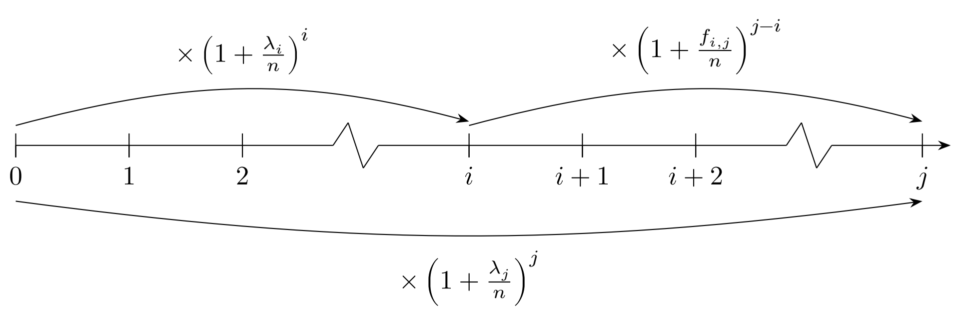

Figure 2.1: The factors by which the investments increase under each strategy

By the No Arbitrage Assumption, the increases in investments from the two investments must be equal, or else there would be an arbitrage opportunity. Therefore, the theorem is true. ■