Multivariable Calculus

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Contents

1 Parametric Equations and Polar Coordinates

1.1 Curved Defined by Parametric Equations

Definition 1.1.1 (Parametric equations)**. Parametric equations** are equations that define **dependent variables** as functions of one or more independent variables, called **parameters**.

Restrictions on the domains of the parameters are often imposed. Curves whose coordinates are specified by parametric equations are called **parametric curves**. A set of parametric equations that specify a curve are a **parameterization** of the curve. Parameterizing a curve has several advantages over specifying it with relations between the coordinates, such as

1. Many curves that can be described with parametric equations cannot be described simply as a relationship between coordinates.

2. Parametric curves can describe **orientation** as the direction in which the parameter increases.

Example 1.1.1. The graph of the parametric equations

$$\begin{cases} x = t^2 - 2t \\ y = t + 1 \end{cases}$$

traces out the following parabola:

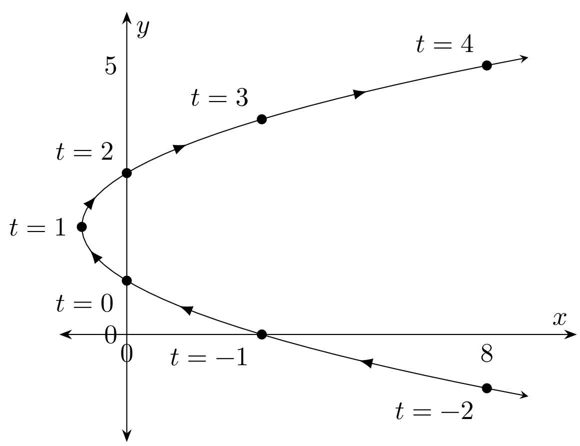

Figure 1.1

The orientation of the parabola is shown with arrows on the curve.

Eliminating the parameter of parametric equations refers to finding equations relating the dependent variables without the parameters. It is not always possible or practical to eliminate the parameter, as there are many parametric equations that do not have equivalent representations as standard equations of the dependent variables.

Example 1.1.2. The parameter from the parametric equations in the previous example can be eliminated by rearranging the second equation into $t = y - 1$ then substituting into the first equation:

$$x = t^2 - 2t = (y-1)^2 - 2(y-1) = y^2 - 4y + 3$$

An n-dimensional parametric curve $\left(x_k\right)_{k=1}^n = \left(f_k(t)\right)_{k=1}^n$ dependent on one parameter $t \in [a, b]$, has **initial point** $\left(x_k\right)_{k=1}^n = \left(f_k(a)\right)_{k=1}^n$ and **terminal point** $\left(x_k\right)_{k=1}^n = \left(f_k(b)\right)_{k=1}^n$.

There is a subtle distinction between a curve and a parametric curve. A curve is a shape made by a set of coordinate points, while a parametric curve is a curve that is traced in a specific way, which is specified by the parameter. The parameterization of a curve is not unique. Two different parametric curves can coincide with the same curve if they have the same coordinate points but the points were traced out in a different way, such as with a different orientation.

Example 1.1.3. The parametric equations

$$\begin{cases} x = t^3 \\ y = t \end{cases} \qquad \text{and} \qquad \begin{cases} x = -t^3 \\ y = -t \end{cases}$$

describe the same curve but different parametric curves they have different orientations, as shown by their graphs:

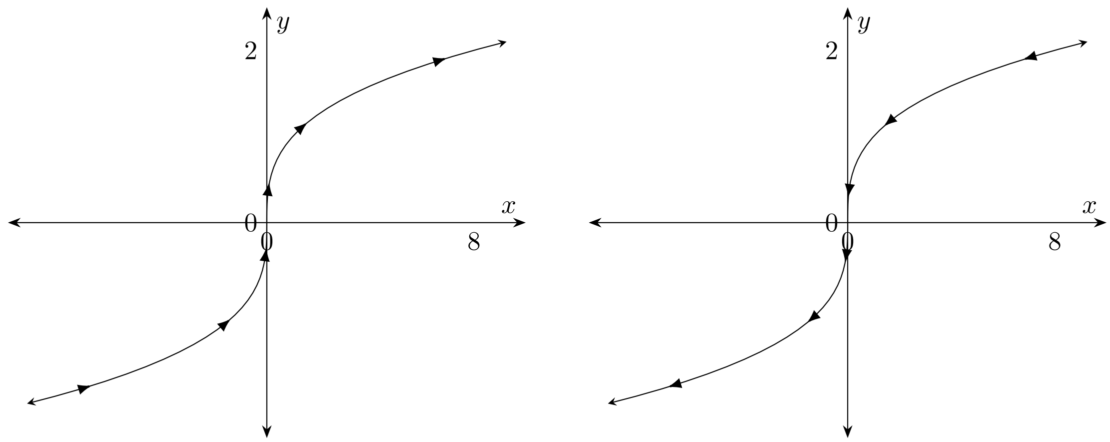

Figure 1.2: Different parametric curves corresponding to the same curve

1.2 Calculus with 2-Dimensional Parametric Curves

A dot above the variable is often used to represent the derivative w.r.t. a parameter. In this case, for curve with coordinates (x, y),

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} \qquad \dot{y} = \frac{\mathrm{d}y}{\mathrm{d}t} \tag{1}$$

The derivative of y w.r.t. x is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}}$$

where $\dot{x} \neq 0$. This is derived using the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \quad \therefore \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\dot{y}}{\dot{x}}$$

This allows us to find the slope of the line tangent to the curve without having to eliminate the parameter. Additionally, it is now clear that the curve has a horizontal tangent when $\dot{y} = 0 \neq \dot{x}$ and a vertical tangent when $\dot{x} = 0 \neq \dot{y}$. When $\dot{y} = 0 = \dot{x}$, check for a vertical or horizontal tangent by checking if the limit of the derivative is 0 or $\pm\infty$

The second derivative can be found easily by plugging $\frac{\mathrm{d}y}{\mathrm{d}x}$ in for y in Equation 1:

$$\frac{\mathrm{d}^2 y}{(\mathrm{d}x)^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{\dot{x}}$$

Consider a curve given by $y = F(x)$, parameterized by $x = f(t)$ and $y = g(t)$. The integral of this curve from a to b is

$$\int_a^b y\,\mathrm{d}x = \int_{t_1}^{t_2} g(t)f'(t)\,\mathrm{d}t$$

where $g(t_1) = a$ and $g(t_2) = b$. This fact is found using the Substitution Rule for Definite Integrals.

Theorem 1.1. *If a curve $\mathcal{C}$ is parameterized by $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where f' and g' are continuous over $[\alpha, \beta]$ and $\mathcal{C}$ is traversed exactly once over $t \in [\alpha, \beta]$, then the length of $\mathcal{C}$ is*

$$S = \int_{\alpha}^{\beta} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$$

Proof. Divide $[\alpha, \beta]$ into n equal subintervals $T_k = [t_{k-1}, t_k]$ for $k \in [1..n]$ with width Δt. For each k, applying the MVT for derivatives to f over T_k gives $t_k^{\ast} \in T_k$ s.t.

$$f(t_k) - f(t_{k-1}) = f'(t_k^{\ast})(t_k - t_{k-1})$$

Applying the MVT for derivatives to g over T_k gives $t_k^{\star} \in T_k$ s.t.

$$g(t_k) - g(t_{k-1}) = g'(t_k^{\star})(t_k - t_{k-1})$$

With $(\Delta x)_k = x_k - x_{x-1}$ and $(\Delta y)_k = y_k - y_{k-1}$, the above equations can be written as

$$(\Delta x)_k = f'\left(t_k^{\ast}\right) \Delta t \qquad (\Delta y)_k = g'\left(t_k^{\star}\right) \Delta t$$

$(\Delta x)_k$ is the horizontal displacement from $(f(t_{k-1}), g(t_{k-1}))$ to $(f(t_k), g(t_k))$ and $(\Delta y)_k$ is the vertical displacement. Therefore, by the Pythagorean Theorem, the total distance S_k of the line segment between $(f(t_{k-1}), g(t_{k-1}))$ and $(f(t_k), g(t_k))$ is

$$S_k = \sqrt{(\Delta x)_k^2 + (\Delta y)_k^2}$$

The sum of all S_k provides an approximation for S that gets closer and closer to S as $n \to \infty$. The formula for the approximation of S with n subintervals is

$$S \approx \sum_{k=1}^{n} S_k = \sum_{k=1}^{n} \sqrt{(\Delta x)_k^2 + (\Delta y)_k^2} = \sum_{k=1}^{n} \sqrt{\left[f'\left(t_k^{\ast}\right) \Delta t\right]_k^2 + \left[g'\left(t_k^{\star}\right) \Delta t\right]_k^2}$$

The exact value of S is the value this approximation approaches as $n \to \infty$:

$$S = \lim_{n \to \infty} \sum_{k=1}^{n} \sqrt{\left[f'\left(t_k^{\ast}\right)\right]_k^2 + \left[g'\left(t_k^{\star}\right)\right]_k^2} \Delta t$$

This is not a Riemann sum because, in general, $t_k^{\ast} \neq t_k^{\star}$. However, it can be shown, since f' and g' are continuous, that the limit is the same as if they were equal. That is, even if this is not a Riemann sum it still converges to the integral as $n \to \infty$. Therefore,

$$S = \int_{\alpha}^{\beta} \sqrt{\left[f'(t)\right]^2 + \left[g'(t)\right]^2} = \int_{\alpha}^{\beta} \sqrt{\dot{x}^2 + \dot{y}^2}$$

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Theorem 1.1 can easily be extended to a higher dimensional curve with coordinates given by $(x_k)_{k=1}^{n} = (f_k(t))_{k=1}^{n}$ with $t \in [\alpha, \beta]$. The arc length of this curve is given by

$$S = \int_{\alpha}^{\beta} \sqrt{\sum_{k=1}^{n} \left(\frac{\mathrm{d}x_k}{\mathrm{d}t}\right)^2}\, \mathrm{d}t$$

The **arc length function** for a curve parameterized by t is the arc length of the section of the curve from $t = \alpha$ to t. For a 2-dimensional parametric curve, it is given by

$$s(t) = \int_{\alpha}^{t} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2}\, \mathrm{d}u$$

In general, the arc length of any curve with arc length function $s(t)$ over $t \in [\alpha, \beta]$ is

$$S = \int_{\alpha}^{\beta} \mathrm{d}s$$

where $\mathrm{d}s$ is the differential of $s(t)$. In the case of 2-dimensional parametric curves,

$$\mathrm{d}s = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}u}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}u}\right)^2}\,\mathrm{d}u$$

This fact is summarized with the following abuse of notation:

$$(\mathrm{d}s)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2$$