Inverse Trigonometric functions

	0°	30° $(\pi/6)$	45° $(\pi/4)$	60° $(\pi/3)$	90° $(\pi/2)$
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞
$\mathrm{cosec}\,\theta$	∞	2	$\sqrt{2}$	$2/\sqrt{3}$	1
$\sec\theta$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	∞
$\cot\theta$	∞	$\sqrt{3}$	1	$1/\sqrt{3}$	0

$$\frac{1}{0} = \infty$$

$$\frac{1}{\infty} = 0$$

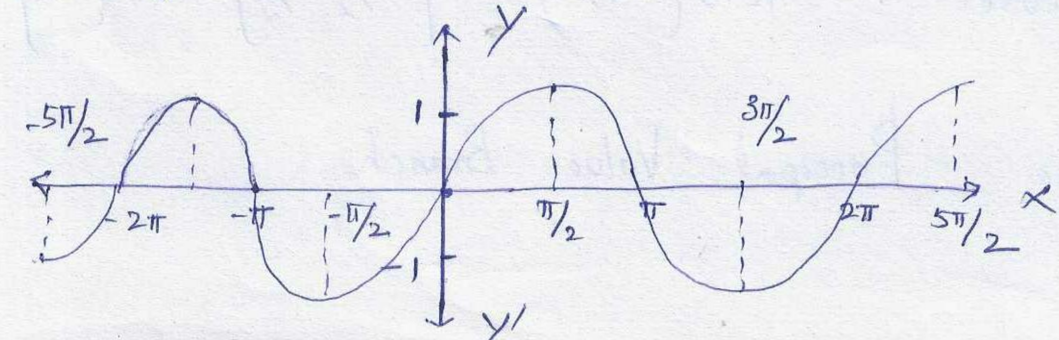
Trigonometric fns - $\sin x$, $\cos x$, $\tan x$, $\mathrm{cosec}\,x$, $\sec x$, $\cot x$.

Inverse Trig. fn - $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\mathrm{cosec}^{-1}x$, $\sec^{-1}x$, $\cot^{-1}x$.

f^{-1} exist only when Bijective (One-One, onto)

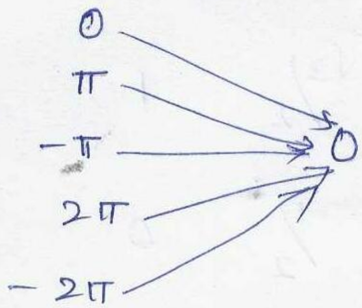
$f: A \to B$

$f^{-1}: B \to A$



$y = \sin x$

$\sin x$: consider $R \rightarrow$ Real Numbers.



$\sin x \rightarrow$ Range $\rightarrow [-1, 1]$

Domain $- \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Trigonometric functions are neither one-one nor onto over their natural domains and Ranges.

$\sin^{-1} x : [-1, 1] \longrightarrow \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\cos^{-1} x : [-1, 1] \longrightarrow [0, \pi]$ [ ] closed interval

$\tan^{-1} : R \longrightarrow \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ ( ) open interval

$\cot^{-1} : R \longrightarrow (0, \pi)$

$\sec^{-1} : R \rightarrow (-1, 1) \quad [0, \pi] - \left\{\frac{\pi}{2}\right\}$

$\operatorname{cosec}^{-1} : R \rightarrow (-1, 1) \quad \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Range : Principal Value Branch.

$\sin^{-1}(-x) = -\sin^{-1}x$

$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$

$\tan^{-1}(-x) = -\tan^{-1}x$

$\cos^{-1}(-x) = \pi - \cos^{-1}x$

$\sec^{-1}(-x) = \pi - \sec^{-1}x$

$\cot^{-1}(-x) = \pi - \cot^{-1}x$

Ex : 2.1

Find the principal Values of following :

1. $\sin^{-1}(-1/2)$

$y = \sin^{-1}(-1/2)$ $\qquad \sin^{-1}(-x) = -\sin^{-1}x$

$y = -\sin^{-1}(1/2)$

$y = -\cancel{\sin^{-1}}(\cancel{\sin}\,\pi/6)$

$y = -\pi/6$ $\qquad \left[-\pi/2, \pi/2\right]$

Principal Value of $\sin^{-1}(-1/2)$ is $-\pi/6$

2. $\cos^{-1}\left(\sqrt{3}/2\right)$

$y = \cos^{-1}\left(\sqrt{3}/2\right)$ $\qquad 30^\circ = \pi/6$

$\cos y = \sqrt{3}/2$ $\qquad 60^\circ = \pi/3$

$\cos y = \cos \pi/6$

$y = \pi/6$ //

Range $[0, \pi]$

3. $\operatorname{cosec}^{-1}(2)$

$y = \operatorname{cosec}^{-1} 2$

$\operatorname{cosec} y = 2$

$\operatorname{cosec} y = \operatorname{cosec}(\pi/6)$

$y = \pi/6$

Range of $\operatorname{cosec}^{-1} [-\pi/2, \pi/2] - \{0\}$

principal value is $\pi/6$

4. $\tan^{-1}(-\sqrt{3})$

$y = \tan^{-1}(-\sqrt{3})$

$\tan^{-1}(-x) = -\tan^{-1} x$

$y = -\tan^{-1}(\sqrt{3})$

$\tan y = -\sqrt{3}$

$\tan y = -\tan \pi/3$

$y = -\pi/3$

$(-\pi/2, \pi/2)$

principal Value is $-\pi/3$

5. $\cos^{-1}(-1/2)$

$y = \cos^{-1}(-1/2)$

$$\cos^{-1}(-x) = \pi - \cos^{-1} x$$

$$y = \pi - \cos^{-1}(1/2)$$

$$= \pi - \cancel{\cos}^{-1}(\cancel{\cos}\, \pi/3)$$

$$= \pi - \pi/3$$

$$y = 3\pi - \pi/3$$

$$y = 2\pi/3 \in [0, \pi]$$

6. $\tan^{-1}(-1)$

$$y = \tan^{-1}(-1)$$

$$\tan^{-1}(-x) = -\tan^{-1} x$$

$$y = -\tan^{-1}(1)$$

$$\tan y = -1$$

$$\tan y = -\tan \pi/4$$

$$y = -\pi/4 \in (-\pi/2, \pi/2)$$

principal value is $-\pi/4$

7. $\sec^{-1}(2/\sqrt{3})$

$$y = \sec^{-1}(2/\sqrt{3})$$

$$\sec y = 2/\sqrt{3}$$

$$\sec y = \sec \pi/6$$

$y = \pi/6 \in [0, \pi], \{\pi/2\}$

principal value is $\pi/6$.

8. $\cot^{-1}(\sqrt{3})$

$y = \cot^{-1}(\sqrt{3})$

$\cot y = \sqrt{3}$

$\cot y = \cot \pi/6$

$y = \pi/6 \in (0, \pi)$

p. value is $\pi/6$

9. $\cos^{-1}(-1/\sqrt{2})$

$y = \cos^{-1}(-1/\sqrt{2})$

$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

$y = \pi - \cos^{-1}(1/\sqrt{2})$

$= \pi - \cancel{\cos}^{-1}(\cancel{\cos}\,\pi/4)$

$y = \pi - \pi/4$

$= 4\pi - \pi/4$

$y = 3\pi/4 \in [0, \pi)$

p. value is $3\pi/4$

10. $\operatorname{cosec}^{-1}(-\sqrt{2})$

$$y = \operatorname{cosec}^{-1}(-\sqrt{2})$$

$$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}(x)$$

$$y = -\operatorname{cosec}^{-1}(\sqrt{2})$$

$$\operatorname{cosec} y = -\sqrt{2}$$

$$\operatorname{cosec} y = -\operatorname{cosec} \pi/4$$

$$y = -\pi/4 \in [-\pi/2, \pi/2] - \{0\}$$

principal value is $-\pi/4$

11. Find the value of following.

$\tan^{-1}(1) + \cos^{-1}(-1/2) + \sin^{-1}(-1/2)$

$$y = \tan^{-1}(1)$$

$$y = \pi/4$$

$$y = \cos^{-1}(-1/2)$$

$$y = \pi - \cos^{-1}(1/2)$$

$$y = \pi - \cos^{-1}(\cos \pi/3)$$

$$= \pi - \pi/3$$

$$= 3\pi - \pi/3$$

$$y = 2\pi/3$$

$$y = \sin^{-1}(-1/2)$$

$$= -\sin^{-1}(1/2)$$

$$= -\pi/6$$

$$\frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6}$$

$$\frac{3\pi + 8\pi - 2\pi}{12} = \frac{9\pi}{12} = 3\pi/4$$

12. $\cos^{-1}(1/2) + 2\sin^{-1}(1/2)$

$y = \cos^{-1}(1/2)$ $\qquad$ $y = \sin^{-1}(1/2)$

$\cos y = 1/2$ $\qquad$ $\sin y = 1/2$

$\cos y = \cos \pi/3$ $\qquad$ $\sin y = \sin \pi/6$

$y = \pi/3$ $\qquad$ $y = \pi/6$

$$= \frac{\pi}{3} + \cancel{2}\left(\frac{\pi}{\cancel{6}_{3}}\right)$$

$$= \frac{\pi}{3} + \frac{\pi}{3} = 2\pi/3$$

13. If $\sin^{-1}x = y$, then Range is

$y = \sin^{-1}x$

Range is $[-\pi/2, \pi/2]$

14. Find the value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ is equal to.

$y = \tan^{-1}\sqrt{3}$

$\tan y = \sqrt{3}$

$\tan y = \tan \pi/3$ $\qquad$ $y = \sec^{-1}(-2)$

$y = \pi/3$ $\qquad$ $y = \pi - \sec^{-1}(2)$

$= \pi - \pi/3$

$= 3\pi - \pi/3$

$y = 2\pi/3$

$$= \frac{\pi}{3} - \frac{2\pi}{3} = -\pi/3$$

Ex : 2.2

1. prove the following,

$$3 \sin^{-1} x = \sin^{-1}(3x - 4x^3), \quad x \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right]$$

Let $x = \sin\theta$, $\quad \theta = \sin^{-1} x$

R.H.S, $\sin^{-1}(3x - 4x^3)$

$\sin^{-1}(3\sin\theta - 4\sin^3\theta)$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$

$\sin^{-1}(\sin 3\theta)$

$= 3\theta$

$= 3\sin^{-1} x = L.H.S$

Hence proved.

2. $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x) \quad x \in [1/2, 1]$

$x = \cos\theta \quad , \quad \theta = \cos^{-1}x$

R.H.S , $\cos^{-1}(4x^3 - 3x)$

$\cos^{-1}(4\cos^3\theta - 3\cos\theta)$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$= \cancel{\cos}^{-1}(\cancel{\cos} 3\theta)$

$= 3\theta$

$= 3\cos^{-1}x = L.H.S$

3. $\tan^{-1}\dfrac{\sqrt{1+x^2} - 1}{x} \quad , \quad x \neq 0$

Whenever $\sqrt{1+x^2}$

$x = \tan\theta$.

$\tan^{-1}\left(\dfrac{\sqrt{1+\tan^2\theta} - 1}{\tan\theta}\right)$

$\tan^{-1}\left(\dfrac{\sqrt{\sec^2\theta} - 1}{\tan\theta}\right)$

$\tan^{-1}\left(\dfrac{\sec\theta - 1}{\tan\theta}\right)$

$\tan^{-1}\left(\dfrac{\frac{1}{\cos\theta} - 1}{\frac{\sin\theta}{\cos\theta}}\right)$

$$\tan^{-1}\left(\frac{1-\frac{\cos\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$1-\cos 2\theta = 2\sin^2\theta$

$1-\cos\theta = 2\sin^2\theta/_2$

$\sin 2\theta = 2\sin\theta . \cos\theta$

$\sin\theta = 2\sin\theta/_2 . \cos\theta/_2$

$$= \tan^{-1}\left(\frac{\cancel{2}\sin^2\theta/_2}{\cancel{2}\cancel{\sin\theta/_2} . \cos\theta/_2}\right)$$

$$= \tan^{-1}\left(\frac{\sin\theta/_2}{\cos\theta/_2}\right)$$

$$= \cancel{\tan}^{-1}\left(\frac{\cancel{\tan}\,\theta/_2}{\cancel{\tan}}\right)$$

$$= \cancel{\tan}^{-1}(\cancel{\tan}\,\theta/_2)$$

$$= \frac{\theta}{2} = \frac{\tan^{-1}x}{2} = \frac{1}{2}\tan^{-1}x$$

4. Write in Simplest form,

$$\tan^{-1}\left(\frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}\right), \quad x < \pi$$

$1-\cos x = 2\sin^2 x/_2$

$1+\cos x = 2\cos^2 x/_2$

$$\tan^{-1}\left(\sqrt{\frac{\cancel{2}\sin^2 x/_2}{\cancel{2}\cos^2 x/_2}}\right)$$

$$\tan^{-1}\left(\frac{\sin x/2}{\cos x/2}\right)$$

$$\cancel{\tan}^{-1}\left(\cancel{\tan} x/2\right)$$

$$= x/2 \;//$$

5. $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right), \quad -\pi/4 < x < 3\pi/4$

$\div$ by $\cos x$

$$\tan^{-1}\left(\frac{\frac{\cancel{\cos x}}{\cancel{\cos x}} - \frac{\sin x}{\cos x}}{\frac{\cancel{\cos x}}{\cancel{\cos x}} + \frac{\sin x}{\cos x}}\right)$$

$$\tan^{-1}\left(\frac{1 - \tan x}{1 + 1\tan x}\right)$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \cdot \tan B}$$

$$\tan^{-1}\left(\frac{\tan \pi/4 - \tan x}{1 + \tan \pi/4 \cdot \tan x}\right) \qquad \tan \pi/4 = 1.$$

$$A = \pi/4, \; B = x$$

$$\cancel{\tan}^{-1}\left(\cancel{\tan}\left(\pi/4 - x\right)\right)$$

$$= \pi/4 - x$$

$\sin^2\theta + \cos^2\theta = 1$

$1 - \cos^2\theta = \sin^2\theta$

$1 - \sin^2\theta = \cos^2\theta$

$\sec^2\theta - \tan^2\theta = 1$

$1 + \tan^2\theta = \sec^2\theta$

$1 - \sec^2\theta = -\tan^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$

$\operatorname{cosec}^2\theta - \cot^2\theta = 1$

$\operatorname{cosec}^2\theta = 1 + \cot^2\theta$

$\cot^2\theta = \operatorname{cosec}^2\theta - 1$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$\sin 2\theta = 2\sin\theta \cdot \cos\theta$

$\sin\theta = 2\sin\theta/_2 \cdot \cos\theta/_2$

$1 - \cos 2\theta = 2\sin^2\theta \Rightarrow 1 - \cos\theta = 2\sin^2\theta/_2$

$1 + \cos 2\theta = 2\cos^2\theta \qquad 1 + \cos\theta = 2\cos^2\theta/_2$

6. $\tan^{-1} \dfrac{x}{\sqrt{a^2-x^2}}$, $|x| < a$

$\sqrt{1-x^2}$

$x = \cos\theta$ (or) $\sin\theta$

$x = a\sin\theta$

$\dfrac{x}{a} = \sin\theta$.

$\theta = \sin^{-1}(x/a)$

$\tan^{-1}\left(\dfrac{a\sin\theta}{\sqrt{a^2 - a^2\sin^2\theta}}\right)$

$\tan^{-1}\left(\dfrac{a\sin\theta}{\sqrt{a^2(1-\sin^2\theta)}}\right)$

$\tan^{-1}\left(\dfrac{a\sin\theta}{a\sqrt{1-\sin^2\theta}}\right)$ $1-\sin^2\theta = \cos^2\theta$

$\tan^{-1}\left(\dfrac{\cancel{a}\sin\theta}{\cancel{a}\sqrt{\cos^2\theta}}\right)$

$\tan^{-1}(\tan\theta)$

$= \theta = \sin^{-1}(x/a)$

7. $\tan^{-1}\left(\dfrac{3a^2x - x^3}{a^3 - 3ax^2}\right)$ $a > 0$; $-a/\sqrt{3} < x < \dfrac{a}{\sqrt{3}}$.

$\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

$x = a\tan\theta$

$$\frac{x}{a} = \tan\theta$$

$$\theta = \tan^{-1}(x/a)$$

$$\tan^{-1}\left(\frac{3a^2(a\tan\theta) - (a\tan\theta)^3}{a^3 - 3a(a\tan\theta)^2}\right)$$

$$= \tan^{-1}\left(\frac{3a^3\tan\theta - a^3\tan^3\theta}{a^3 - 3a^3\tan^2\theta}\right)$$

$$= \tan^{-1}\left(\frac{\cancel{a^3}(3\tan\theta - \tan^3\theta)}{\cancel{a^3}(1 - 3\tan^2\theta)}\right)$$

$$= \cancel{\tan^{-1}}(\cancel{\tan}\, 3\theta)$$

$$= 3\theta = 3\tan^{-1}(x/a)$$

Q. Find the value of $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

$$\tan^{-1}\left[2\cos\left(2\,\cancel{\sin^{-1}}(\cancel{\sin}\,\pi/6)\right)\right]$$

$$\tan^{-1}\left[2\cos\left(\cancel{2}\times \pi/\cancel{6}_{3}\right)\right]$$

$$\tan^{-1}\left[2\cos\pi/3\right]$$

$$\tan^{-1}\left[\cancel{2}\times 1/\cancel{2}\right]$$

$$\tan^{-1}[1] = \tan^{-1}(1) = \pi/4$$

9. $\tan \frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right]$

① $\sin^{-1}\frac{2x}{1+x^2}$ $\quad x = \tan\theta$

$\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$\sin^{-1}\left(\frac{2\tan\theta}{\sec^2\theta}\right)$

$\sin^{-1}\left(\frac{2\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}}\right)$

$\sin^{-1}\left(2\frac{\sin\theta}{\cancel{\cos\theta}}\times\frac{\cos^{\cancel{2}}\theta}{1}\right)$

$\sin^{-1}\left(2\sin\theta\cos\theta\right)$

$\sin^{-1}\left(\cancel{2}\sin 2\theta\right)$

$= 2\theta$

② $\cos^{-1}\frac{1-y^2}{1+y^2}$ $\quad y = \tan\phi$

$\cos^{-1}\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right)$

$\cos^{-1}\left(\frac{1-\tan^2\phi}{\sec^2\phi}\right)$

$$= \cos^{-1}\left(\frac{1 - \frac{\sin^2\phi}{\cos^2\phi}}{\frac{1}{\cos^2\phi}}\right)$$

$$= \cos^{-1}\left(\frac{\frac{\cos^2\phi - \sin^2\phi}{\cos^2\phi}}{1/\cos^2\phi}\right)$$

$$= \cos^{-1}\left(\frac{\cos^2\phi - \sin^2\phi}{\cancel{\cos^2\phi}} \times \frac{\cancel{\cos^2\phi}}{1}\right)$$

$$= \cos^{-1}\left(\cos^2\phi - \sin^2\phi\right) \qquad \cos 2x = \cos^2 x - \sin^2 x$$

$$= \cancel{\cos}^{-1}\left(\cancel{\cos}\, 2\phi\right)$$

$$= 2\phi$$

Solving

$$\tan \frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right]$$

$$\tan \frac{1}{2}\left[2\theta + 2\phi\right] \qquad \tan(A+B)$$

$$\tan \frac{1}{\cancel{2}}\left(\cancel{2}(\theta+\phi)\right)$$

$$\tan \theta + \phi = \frac{\tan\theta + \tan\phi}{1 - \tan\theta \cdot \tan\phi}$$

$x = \tan\theta, \quad y = \tan\phi$

$$\therefore \frac{x+y}{1-xy} \quad ''$$

10. Find the Values of $\sin^{-1}\left(\sin 2\pi/3\right)$

$y = \cancel{\sin^{-1}}\left(\cancel{\sin}\ 2\pi/3\right)$

$-90 \quad +90$

$y = 2\pi/3$ Range is $\left[-\pi/2, \pi/2\right]$

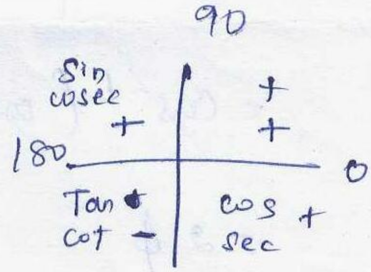
$\frac{2 \times 180}{3} = 120^\circ$

$y = 2\pi/3$ does not belong

to range $\notin \left[-\pi/2, \pi/2\right]$

$\Downarrow$

To adjust to range.



$\sin^{-1}\left(\sin\left(\pi - \pi/3\right)\right)$

$\cancel{\sin^{-1}}\left(\cancel{\sin}\ \pi/3\right)$

$= \pi/3$

$\sin 120 = \sin(180-60)$
$= \sin\theta$
$= \sin 60^\circ$
$= \sin \pi/3.$

11. $\tan^{-1}\left(\tan 3\pi/4\right)$

$y = \cancel{\tan^{-1}}\left(\cancel{\tan}\ 3\pi/2\right)$

$y = 3\pi/2$

$\frac{3\pi}{4} \notin \left(-\pi/2, \pi/2\right)$ NOO—Total

$\tan^{-1}\left(\tan \pi - \pi/4\right)$ $\qquad \tan^{-1}\left(\tan(180-45)\right)$

$\tan^{-1}\left(-\tan \pi/4\right)$ $\qquad \tan^{-1}(-\tan 45)$

$= -45^\circ = -\pi/4$

$= -\pi/4$ //

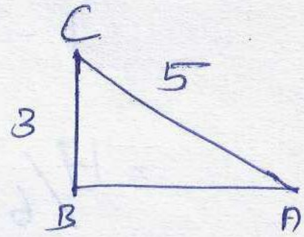
12. $\tan\left(\sin^{-1} 3/5 + \cot^{-1} 3/2\right)$

$$\tan A+B = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B.}$$

Finding $\tan A$

$A = \sin^{-1} 3/5$

$\sin A = 3/5$ $\quad$ opp/hy



$\tan A = \frac{\sin A}{\cos A}$

$AB^2 + BC^2 = AC^2$

$AB^2 + 3^2 = 5^2$

$AB = \pm 4 = 4$

$\cos A = \frac{adj}{hyp} = 4/5$

$\tan A = \frac{3/5}{4/5} = \frac{3}{5} \times \frac{5}{4} = 3/4$

$\tan A = 3/4$

Finding $\tan B$

$B = \cot^{-1} 3/2$

$$\cot B = 3/2 = \frac{adj}{opp}$$

$$\tan B = 2/3$$

$$\tan\left(\sin^{-1} 3/5 + \cot^{-1} 3/2\right) = \tan(A+B)$$

$$= \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

$$= \frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{\cancel{3}}{\cancel{4}_2} \times \frac{\cancel{2}}{\cancel{3}}}$$

$$= \frac{\frac{9+8}{12}}{1 - 1/2} = \frac{17/12}{2 - 1/2} = \frac{17}{\cancel{12}_6} \times \frac{\cancel{2}}{1}$$

$$= 17/6$$ //

13. Find Values of $\cos^{-1}\left(\cos \frac{7\pi}{6}\right)$ is,

$$y = \cancel{\cos}^{-1}\left(\cancel{\cos}\ 7\pi/6\right)$$

$$\frac{7 \times \cancel{180}^{30}}{\cancel{6}} = 210$$

$$y = \frac{7\pi}{6} \quad \text{Range} \rightarrow (0, \pi)$$

0 180

$y = 210$ is not possible. So not in range.

$$y = \cos^{-1}\left(\cos 7\pi/6\right)$$

$$\cos y = \cos(360 - 150)$$

$\cos(360-\theta) = \cos\theta$

$= \cos 150^\circ$

$\cos y = \cos 5\pi/6$

$y = 5\pi/6$ //

14. Find Value of $\sin\left(\pi/3 - \sin^{-1}(-1/2)\right)$

$\sin^{-1}(-x) = -\sin^{-1}x$

$\sin^{-1}(-1/2) = -\sin^{-1}(1/2)$

$= -\pi/6$

$\sin\left(\pi/3 - (-\pi/6)\right)$

$\sin\left(\pi/3 + \pi/6\right) = \sin(60^\circ + 30^\circ)$

$= \sin 90^\circ = 1$ //

15. Find the Value of $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$

$y = \tan^{-1}\sqrt{3}$

$\tan y = \tan \pi/3$

$y = \pi/3$ Range $(-\pi/2, \pi/2)$

$\cot^{-1}(-\sqrt{3})$

$$x = \cot^{-1}(-\sqrt{3})$$

$$\cot x = (-\sqrt{3}) \qquad \cot^{-1}(-x) = \pi - \cot^{-1}x$$

$$x = \pi - \cot^{-1}(\sqrt{3})$$

$$x = \pi - \pi/6$$

$$= \frac{6\pi - \pi}{6} = 5\pi/6 \qquad \text{Range } (0, \pi)$$

$$\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$$

$$= \pi/3 - 5\pi/6$$

$$= \frac{2\pi - 5\pi}{6} = \frac{-3\pi}{6} = -\pi/2$$

Miscellaneous.

1. $\cos^{-1}\left(\cos \frac{13\pi}{6}\right)$

180, 360 → No change
90°, 270° → change

$\cos^{-1}(\cos x) = x; \quad x \in [0, \pi]$

$\frac{13 \times 180}{6} = 30°$
$= 390°$

$$\cos^{-1}\left[\cos\left(\frac{12\pi + \pi}{6}\right)\right]$$

$$\cos^{-1}\left[\cos\left(2\pi + \pi/6\right)\right] \qquad \cos 2\pi + \theta = \cos\theta$$

$$\cos \pi/6 \qquad \pi/6 \in [0, \pi]$$

2. $\tan^{-1}\left(\tan\frac{7\pi}{6}\right)$ $\tan^{-1}$ range $\left(-\pi/2, \pi/2\right)$

$\tan^{-1}(\tan x) = x$ if $x \in \left(-\pi/2, \pi/2\right)$

$\frac{7 \times 180}{6} = 210$

$210 = 180 + 30$

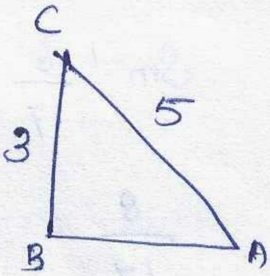
$\pi + \pi/6$

$\tan^{-1}\left(\tan \pi + \pi/6\right) = x$

$\tan^{-1}\left(\tan \pi/6\right) = x$

$x = \pi/6 \in \left(-\pi/2, \pi/2\right)$

3. $2\sin^{-1} 3/5 = \tan^{-1}\left(\frac{24}{7}\right)$



$\sin^{-1} 3/5 = A$

$3/5 = \sin A$

$AC^2 = BC^2 + AB^2$

$5^2 = 3^2 + AB^2$

$16 = AB^2$, $AB = 4$

$\tan A = \frac{3}{4}$

$A = \tan^{-1}\left(3/4\right)$

$2A = \tan^{-1}\left(3/4\right)$

$2\sin^{-1} 3/5 = 2\tan^{-1}\left(3/4\right)$

$2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2}$

$2\tan^{-1}\left(\frac{3}{4}\right) = \frac{\tan^{-1} \cancel{2}\left(3/\cancel{4}_2\right)}{1-\left(3/4\right)^2} = \frac{\tan^{-1} 3/2}{1 - 9/16}$

$= \tan^{-1} \dfrac{3/2}{\dfrac{16-9}{16}}$

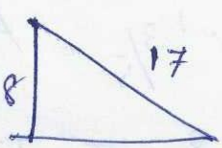
$= \tan^{-1} \dfrac{3}{\cancel{2}} \times \dfrac{\cancel{16}^{8}}{7}$

$= \tan^{-1} \dfrac{3}{\cancel{4}} \times \dfrac{8}{7}$

$= \tan^{-1} \dfrac{24}{7} = R.H.S$

4. $\sin^{-1} \dfrac{8}{17} + \sin^{-1} \dfrac{3}{5} = \tan^{-1} \dfrac{77}{36}$

$\sin^{-1} \dfrac{8}{17} = A$ —— ①



$\dfrac{8}{17} = \sin A$

AB = 15

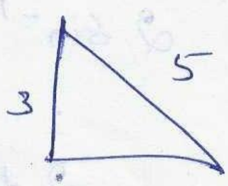
$\tan A = \dfrac{8}{15}$

$A = \tan^{-1} 8/15$ —— ②

from ① & ②

$\sin^{-1} \dfrac{8}{17} = \tan^{-1} \dfrac{8}{15}$

$\sin^{-1} \dfrac{3}{5} = B$ —— ③



AB = 4

$\dfrac{3}{5} = \sin B$

$\tan B = \dfrac{3}{4}$ —— ④

$B = \tan^{-1} (3/4)$ —— ④

$\therefore \sin^{-1} 3/5 = \tan^{-1} 3/4$

$\tan^{-1} \frac{8}{15} + \tan^{-1} 3/4$

$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\tan^{-1}\left(\frac{\frac{8}{15} + \frac{3}{4}}{1 - \frac{8}{15} \times \frac{3}{4}}\right)$

$\tan^{-1}\left(\frac{\frac{32+45}{60}}{1 - 2/5}\right)$

$\tan^{-1}\left(\frac{77/60}{3/5}\right)$

$= \tan^{-1}\left(\frac{77}{60} \times 5/3\right)$

$= \tan^{-1}\left(\frac{77}{36}\right) = R.H.S.$

5. $\cos^{-1} 4/5 + \cos^{-1} 12/13 = \cos^{-1} \frac{33}{65}$

$\cos^{-1} 4/5 = A \qquad \cos^{-1} \frac{12}{13} = B$

$\frac{4}{5} = \cos A \qquad \sin B = 5/13$

$\sin A = 3/5$

$A + B = \cos^{-1} \frac{33}{65}$

$$\cos(A+B) = \frac{33}{65}$$

$$\cos A \cos B - \sin A \sin B = \cos(A+B)$$

$$\frac{4}{5} \times \frac{12}{13} - \frac{3}{\cancel{5}} \times \frac{\cancel{5}}{13}$$

$$= \frac{48}{65} - \frac{3}{13} = \frac{33}{65}$$

$$\cos(A+B) = \frac{33}{65}$$

$$A+B = \cos^{-1} \frac{33}{65} = R.H.S.$$

6. $$\cos^{-1} \frac{12}{13} + \sin^{-1} \frac{3}{5} = \sin^{-1} \frac{56}{65}$$

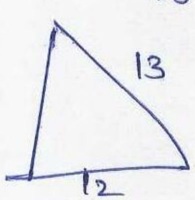
$$\cos^{-1} \frac{12}{13} = A, \qquad \sin^{-1} \frac{3}{5} = B$$

$$A + B = \sin^{-1} \frac{56}{65}$$

$$\sin(A+B) = \frac{56}{65}$$

$$\sin A \cos B + \cos A \cdot \sin B$$

$$\frac{12}{13} = \cos A$$



$$\sin A = 5/13$$

$$\frac{3}{5} = \sin B$$

$$\cos A = 4/5$$

$$\sin A \cos B + \cos A \cdot \sin B = \sin^{-1} \frac{56}{65}$$

$$\frac{5}{13} \times \frac{4}{5} + \frac{12}{13} \times \frac{3}{5}$$

$$\frac{20}{65} + \frac{36}{65} = \frac{56}{65} = R.H.S.$$

$$\sin(A+B) = \frac{56}{65}$$

$$A+B = \sin^{-1}\left(\frac{56}{65}\right)$$

7: $\tan^{-1}\frac{63}{16} = \sin^{-1}5/13 + \cos^{-1}3/5$

$A = \sin^{-1}5/13$ $\qquad$ $\cos^{-1}\frac{3}{5} = B$

$\sin A = 5/13$ $\qquad$ $\cos B = 3/5$

$\tan A = 5/12$ $\qquad$ $\tan B = 4/3$

$A = \tan^{-1}(5/12)$ $\qquad$ $B = \tan^{-1}(4/3)$

$$\tan^{-1}\frac{63}{16} = A+B$$

$$= \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{4}{3}$$

$$= \tan^{-1}x + \tan^{-1}y$$

$$= \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$= \tan^{-1}\left(\frac{5/12 + 4/3}{1 - 5/12 \times 4/3}\right)$$

$$= \tan^{-1}\left(\frac{\frac{15+48}{36}}{1 - 20/36}\right)$$

$$= \tan^{-1}\left(\frac{\frac{63}{36}}{\frac{36-20}{36}}\right)$$

$$= \tan^{-1}\left(\frac{63}{\cancel{36}} \times \frac{\cancel{36}}{16}\right)$$

$$= \tan^{-1}\left(\frac{63}{16}\right) = R.H.S.$$

8. $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\frac{1-x}{1+x}$

$x = \tan^2\theta$

$\sqrt{x} = \tan\theta$

$\tan^{-1}\sqrt{x} = \theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

R.H.S. $\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

$$= \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$= \frac{1}{\cancel{2}}\cancel{\cos^{-1}}(\cancel{\cos}\,\cancel{2}\theta)$$

$$= \theta = \tan^{-1}\sqrt{x} = L.H.S.$$

9. $\cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right) = \frac{x}{2}, \ x \in \left(0, \pi/2\right)$

$\sin x = 2\sin x/2 . \cos x/2$

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x/2 + \cos^2 x/2 = 1$

$$1+\sin x = \sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin x/2 \cdot \cos x/2$$

$$1+\sin x = \left(\sin x/2 + \cos x/2\right)^2$$

$$\sqrt{1+\sin x} = \sin\frac{x}{2} + \cos x/2$$

$$1-\sin x = \cos^2 x/2 + \sin^2 x/2 - 2\cos x/2\ \sin x/2$$

$$1-\sin x = \left(\cos x/2 - \sin x/2\right)^2$$

$$\sqrt{1-\sin x} = \cos x/2 - \sin x/2$$

$$\cot^{-1}\left(\frac{\cancel{\sin x/2} + \cos x/2 + \cos x/2 - \cancel{\sin x/2}}{\sin x/2 + \cancel{\cos x/2} - \cancel{\cos x/2} + \sin x/2}\right)$$

$$\cot^{-1}\left(\frac{\cancel{2}\cos x/2}{\cancel{2}\sin x/2}\right)$$

$$\cancel{\cot^{-1}}\left(\cancel{\cot} x/2\right)$$

$$= \frac{x}{2} = R.H.S.$$

10. $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \pi/4 - \frac{1}{2}\cos^{-1}x,$

$-1/\sqrt{2} \le x \le 1$ [put $x = \cos 2\theta$]

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$1+\cos 2\theta = 2\cos^2\theta$$

$$\tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$$

$\cos 2\theta = 1-2\sin^2\theta$

$2\sin^2\theta = 1-\cos 2\theta$

$$\tan^{-1}\left(\frac{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}\right)$$

$$\tan^{-1}\left(\frac{\sqrt{2}\cos\theta-\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta+\sqrt{2}\sin\theta}\right)$$

$$\tan^{-1}\left(\frac{\cancel{\sqrt{2}}(\cos\theta-\sin\theta)}{\cancel{\sqrt{2}}(\cos\theta+\sin\theta)}\right)$$

Divide by $\cos\theta$

$$\tan^{-1}\left(\frac{\frac{\cos\theta-\sin\theta}{\cos\theta}}{\frac{\cos\theta+\sin\theta}{\cos\theta}}\right)$$

$$\tan^{-1}\left(\frac{\frac{\cos\theta}{\cos\theta}-\frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}\right)$$

$$=\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x . \tan y}.$$

$$= \tan^{-1}\left(\frac{1 - \tan\theta}{1 + 1.\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\tan \pi/4 - \tan\theta}{1 + \tan \pi/4 \tan\theta}\right)$$

$$= \tan^{-1}\left(\tan\left(\pi/4 - \theta\right)\right)$$

$$= \pi/4 - \theta$$

$$, \quad x = \cos 2\theta$$

$$\cos^{-1} x = 2\theta$$

$$\frac{1}{2}\cos^{-1} x = \theta$$

$$= \pi/4 - 1/2 \cos^{-1} x = \text{RHS}.$$

11. Solve $2\tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$

$$2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$2\tan^{-1}(\cos x) = \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right)$$

$$= \tan^{-1}\left(\frac{2\cos x}{\sin^2 x}\right)$$

$$\tan^{-1}\left(\frac{2\cos x}{\sin^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x)$$

$$\tan^{-1}\left(\frac{2\cos x}{\sin^2 x}\right) = \tan^{-1}\left(\frac{2}{\sin x}\right)$$

$$\frac{\cancel{2}\cos x}{\sin^2 x} = \frac{\cancel{2}}{\cancel{\sin x}}$$

$$\frac{\cos x}{\sin x} = 1$$

$$\cot x = 1$$

$$\cot x = \cot \pi/4$$

$$x = \pi/4$$

12. $$\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x, \quad x > 0$$

$$2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \tan^{-1}x$$

$$2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

Sub $x = \frac{1-x}{1+x}$

$$\tan^{-1}\left(\frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2}\right) = \tan^{-1}x$$

$$\tan^{-1}\left(\frac{2\left(\frac{1-x}{1+x}\right)}{\frac{(1+x)^2-(1-x)^2}{(1+x)2}}\right) = \tan^{-1}x$$

$$\tan^{-1}\left(\frac{2(1-x)}{(1+x)} \times \frac{(1+x)^2}{(1+x)^2-(1-x)^2}\right) = \tan^{-1}x$$

$$\tan^{-1}\left(\frac{2(1-x)(1+x)}{(1+x)^2-(1-x)^2}\right) = \tan^{-1}x$$

$$(a+b)(a-b) = a^2-b^2$$

$$\tan^{-1}\left(\frac{2(1-x^2)}{(1+x+1-x)(1+x-1+x)}\right) = \tan^{-1}x$$

$$\tan^{-1}\left(\frac{\cancel{2}(1-x^2)}{\cancel{2}\times 2x}\right) = \tan^{-1}x$$

$$\tan^{-1}\left(\frac{1-x^2}{2x}\right) = \tan^{-1}x$$

$$\frac{1-x^2}{2x} = x$$

$$1-x^2 = 2x^2$$

$$1-3x^2 = 0$$

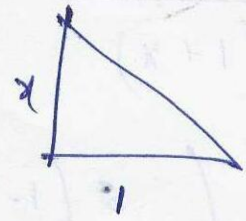
$$3x^2 = 1$$

$$x^2 = 1/3$$

$$x = \pm\sqrt{1/3} \text{ (or) } \pm 1/\sqrt{3}$$

13. $\sin\left(\tan^{-1}x\right)$, $|x|<1$ is equal to

$$\tan^{-1}x = A$$



$$\frac{x}{1} = \tan A$$

$$\sin\left(\tan^{-1}x\right)$$

$$\sin(A)$$

$$\cancel{\sin}\left(\cancel{\sin^{-1}}\left(\frac{x}{\sqrt{1+x^2}}\right)\right)$$

$$= \frac{x}{\sqrt{1+x^2}}$$

14. $\sin^{-1}(1-x) - 2\sin^{-1}x = \pi/2$, then x is equal to,

$$\sin^{-1}(1-x) = \pi/2 + 2\sin^{-1}x$$

$$1 - x = \sin\left(\frac{\pi}{2} + 2\sin^{-1}x\right)$$

$$\sin\left(\pi/2 + \theta\right) = \cos\theta \qquad \cos 2\theta = 1 - 2\sin^2\theta$$

$$1 - x = \cos\left(2\sin^{-1}x\right)$$

$$1 - x = 1 - 2\sin^2\left(\sin^{-1}x\right)$$

$$1 - x = 1 - 2\left(\cancel{\sin}\left(\cancel{\sin^{-1}}x\right)\right)^2$$

$$1 \mp x = 1 \cancel{+} - 2x^2$$

$$x = 2x^2$$

$x(1-2x) = 0$

$1 - 2x = 0$

$1 = 2x$

$\therefore x = 1/2$

$x = 0, \quad x = 1/2$

at $x = 1/2$

$\sin^{-1}(1 - 1/2) - 2\sin^{-1} 1/2$

$\sin^{-1}(1/2) - 2\sin^{-1}(1/2)$

$= \frac{\pi}{6} - \frac{2\pi}{6}$

$= -\pi/6 \neq \pi/2$

at $x = 0$

$\sin^{-1}(1 - 0) - 2\sin^{-1}(0)$

$\sin^{-1}(1) = -2\sin^{-1}(0)$

$\cancel{\sin^{-1}}(\cancel{\sin}\,\pi/2) = -2\,\cancel{\sin^{-1}}(\cancel{\sin}\,0)$

$\pi/2 = 0$ // $\boxed{x = 0}$