Linear Programming

Eg : 1 Solve the linear programming problem graphically:

Maximize $Z = 4x + y$ — ①

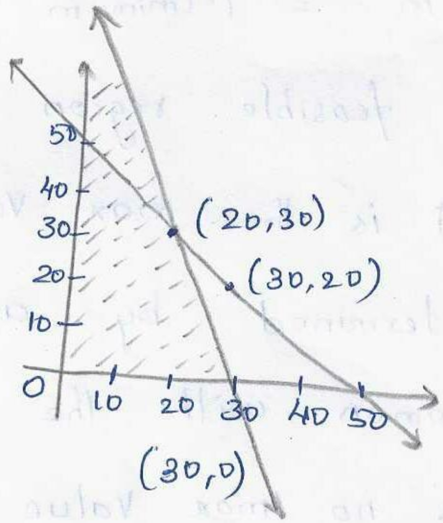
Subject to the constraints

$x + y \leq 50$ — ②

$3x + y \leq 90$ — ③

$x \geq 0 \; ; \; y \geq 0$ — ④

Let us graph the feasible region of the system of inequalities ②, ③ and ④



from ② $x + y \leq 50$

$x + y = 50$

$x = 20 \qquad x = 30$

$y = 30 \qquad y = 20$

from ③ $3x + y = 90$

$x = 20 \qquad x = 30$

$y = 30 \qquad y = 0$

$3(0) + 0 \leq 90$

$0 \leq 90$

Now we observe that OABC is feasible region.

Now we use Corner points method,

Corner point	Corresponding value of Z
A (0,0)	0
A (30,0)	120
B (20,30)	110
C (0,50)	50

Hence maximum Value of Z is 120 at point (30,0)

$Z = ax + by$

M = Maximum Value

m = Minimum Value

If feasible region is unbounded, then

a) M is the max value of Z, if open half plane determined by $ax + by > M$ has no point in Common with the feasible region. Otherwise Z has no max value.

b) m is the min value of Z, if open half plane determined by $ax + by < m$ has no point in Common with the feasible region, otherwise Z has no min value.

$$ax + by > M$$

$$ax + by = M$$

Eg: 2 $Z = -50x + 20y$ —— ①

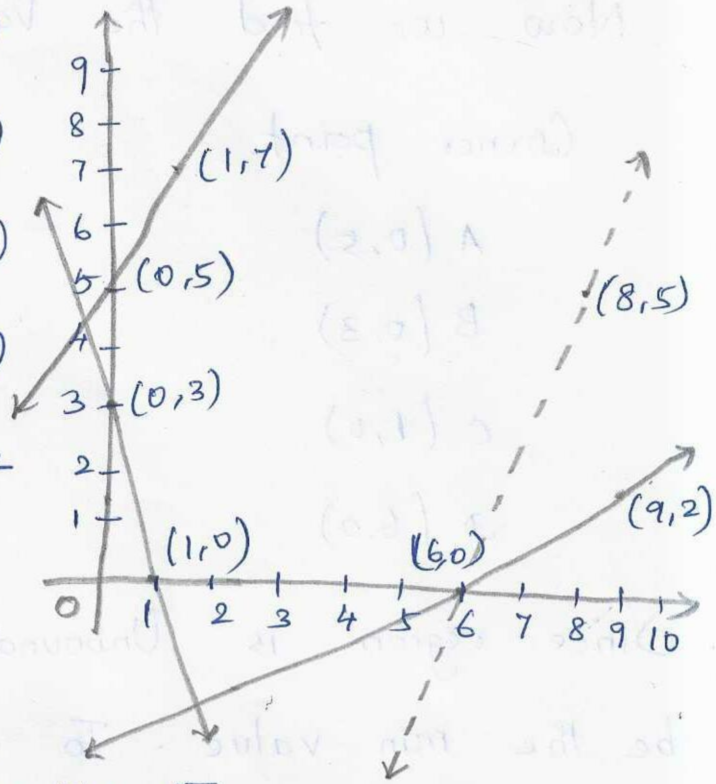
Subject to the Constraints:

$2x - y \geq -5$ —— ②

$3x + y \geq 3$ —— ③

$2x - 3y \leq 12$ —— ④

$x \geq 0,\ y \geq 0$ —— ⑤



First, let us find feasible region of the system of inequalities ②, ③, ④, ⑤

from ② $2x - y \geq -5$ $\quad 2x - y = -5$

$x = 0 \quad x = 1 \quad (0,0)$

$y = 5 \quad y = 7 \quad 0 \geq -5$

$(0,5)\ (1,7)$

from ③ $3x + y \geq 3$ $\quad 3x + y \geq 3$

$3x + y = 3 \quad 0 + 0 \geq 3$

$x = 0 \quad x = 1 \quad 0 \geq 3$

$y = 3 \quad y = 0$

$(0,3) \quad (1,0)$

from ④ $2x - 3y \leq 12$ $\quad 2x - 3y = 12$

$x = 6 \quad x = 9 \quad (0,0)$

$y = 0 \quad y = 2 \quad 0 \leq 12$

(6,0) (9,2)

We observe that feasible region is Unbounded.

Now we find the Value of Z at Corner points.

Corner point	$Z = -50x + 20y$
A (0,5)	100
B (0,3)	60
C (1,0)	-50
D (6,0)	-300 ← Minimum

Since region is Unbounded, -300 may or may not be the min value. To decide this issue, we graph,

$$-50x + 20y < -300$$

$$-5x + 2y < -30$$

$x = 6$ $\quad$ $x = 8$

$y = 0$ $\quad$ $y = 5$

$$-5x + 2y < -30$$

$$0 < -30$$

Now, we can see that the resulting open half and feasible region has point in common.

So, -300 will not be the min. value

Therefore there is no Min value of Z.

Ex : 12.1

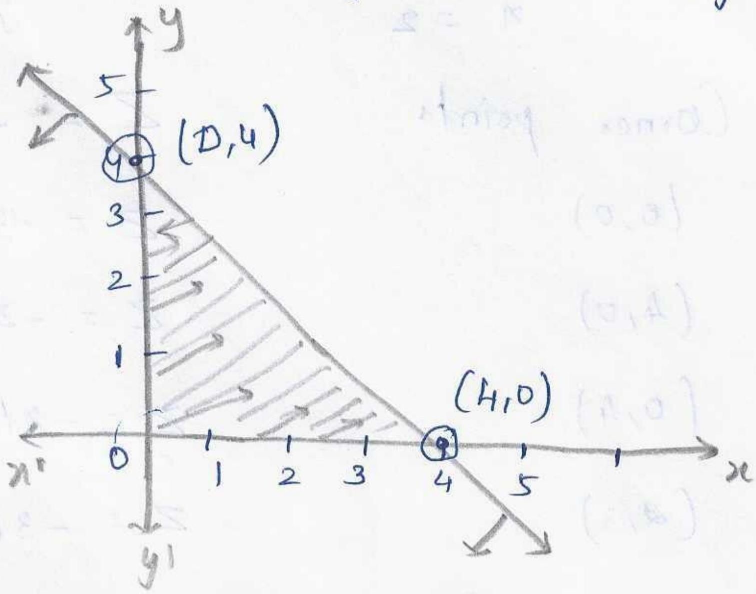
Solve the following Linear programming problems graphically:

1. Maximise $Z = 3x + 4y$

Subject to the constraints : $x + y \le 4$, $x \ge 0$, $y \ge 0$

$x + y \le 4$

x	0	4
y	4	0



Corner points	$Z = 3x + 4y$
$(0,0)$	$Z = 0 + 0 = 0$
$(4,0)$	$Z = 12 + 0 = 12$
$(0,4)$	$Z = 0 + 16 = 16$

Maximum Value is 16 at $(0,4)$

2. Minimise $Z = -3x + 4y$

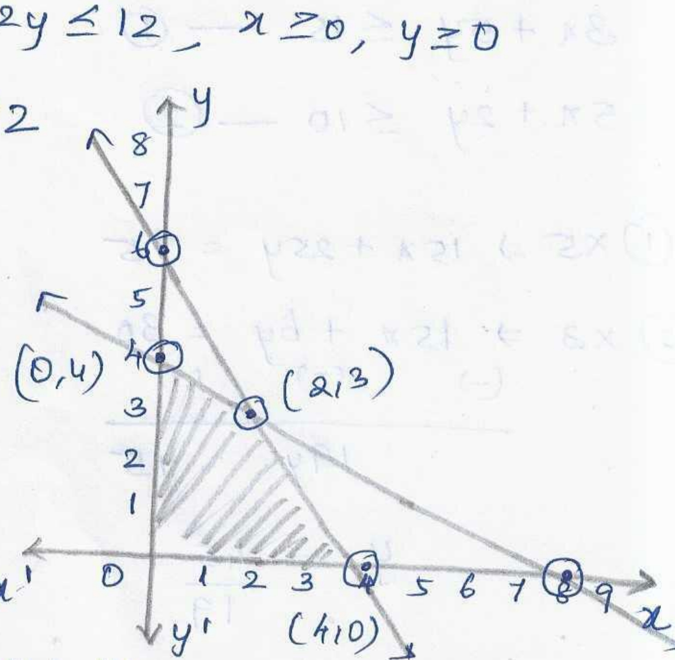
Subject to $x + 2y \le 8$, $3x + 2y \le 12$, $x \ge 0$, $y \ge 0$

$x + 2y \le 8$

x	0	8
y	4	0

$3x + 2y \le 12$

x	0	4
y	6	0



$$3x + 2y = 12$$
$$x + 2y = 8$$
$$(-) \quad (-) \quad (-)$$
$$2x = 4$$
$$x = 2$$

$$2 + 2y = 8$$
$$2y = 8 - 2$$
$$2y = 6$$
$$y = 3$$

Corner points

Corner points	$Z = -3x + 4y$
$(0,0)$	$Z = -3(0) + 4(0) = 0$
$(4,0)$	$Z = -3(4) + 4(0) = -12$
$(0,4)$	$Z = -3(0) + 4(4) = 16$
$(2,3)$	$Z = -3(2) + 4(3) = 6$

Minimum Value is -12 at $(4,0)$

3. Maximise $Z = 5x + 3y$

Subject to $3x + 5y \le 15$, $5x + 2y \le 10$, $x \ge 0, y \ge 0$

$3x + 5y \le 15$

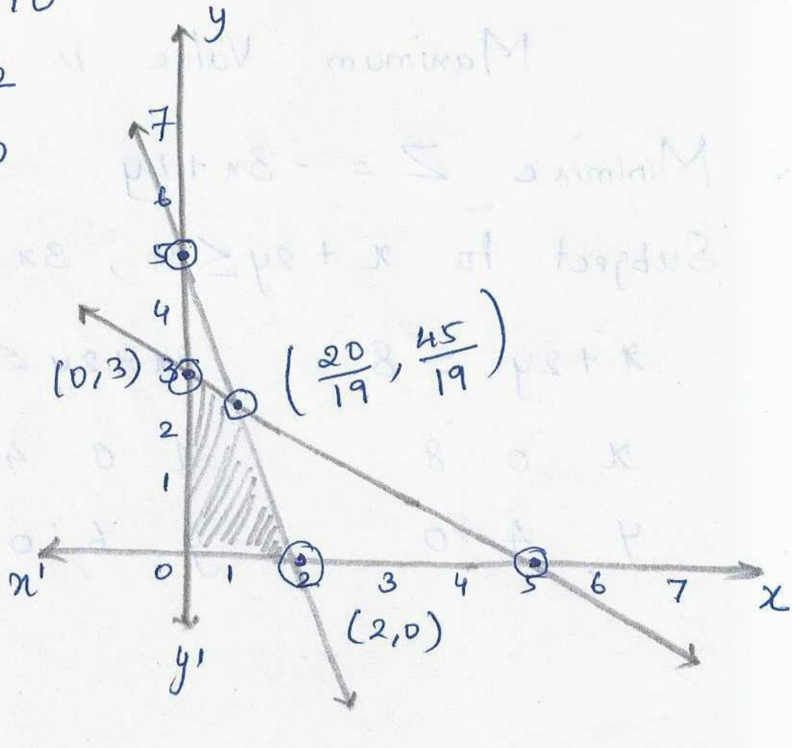
x	0	5
y	3	0

$5x + 2y \le 10$

x	0	2
y	5	0

$3x + 5y \le 15$ — (1)

$5x + 2y \le 10$ — (2)



(1) $\times 5 \Rightarrow 15x + 25y = 75$

(2) $\times 3 \Rightarrow 15x + 6y = 30$

$$(-) \quad (-) \quad (-)$$

$$19y = 45$$

$$y = \frac{45}{19}$$

$① \times 2 \Rightarrow 6x + 10y = 30$

$② \times 5 \Rightarrow 25x + 10y = 50$

$\quad (-) \quad (-) \quad (-)$

$\cancel{+}19x = \cancel{+}20$

$x = 20/19$

Corner points $\qquad Z = 5x + 3y$

$(0,0) \qquad Z = 5(0) + 3(0) = 0$

$(2,0) \qquad Z = 5(2) + 3(0) = 10$

$(0,3) \qquad Z = 5(0) + 3(3) = 9$

$\left(\frac{20}{19}, \frac{45}{19}\right) \qquad Z = 5\left(\frac{20}{19}\right) + 3\left(\frac{45}{19}\right)$

$= \frac{100}{19} + \frac{135}{19}$

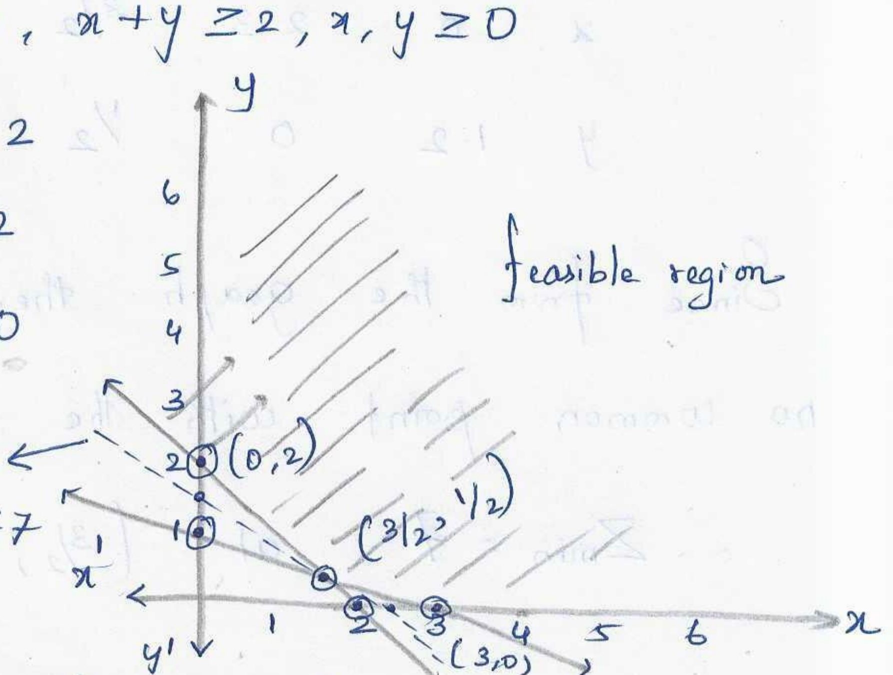
$= \frac{235}{19}$

$Z_{max} = \frac{235}{19}$ at $\left(\frac{20}{19}, \frac{45}{19}\right)$ //

4. Minimise $Z = 3x + 5y$

Such that $x + 3y \geq 3$, $x + y \geq 2$, $x, y \geq 0$

$x + 3y \geq 3$			$x + y \geq 2$		
x	0	3	x	0	2
y	1	0	y	2	0



$$x + 3y = 3$$
$$x + y = 2$$
$$\underline{(-) \quad (-) \quad (-)}$$
$$2y = 1$$
$$y = 1/2$$

$$x + 1/2 = 2$$
$$x = 2 - 1/2$$
$$x = \frac{4-1}{2}$$
$$x = 3/2$$

Corner points $\quad Z = 3x + 5y$

$(0, 2)$ $\quad Z = 3(0) + 5(2) = 10$

$(3, 0)$ $\quad Z = 3(3) + 5(0) = 9$

$(3/2, 1/2)$ $\quad Z = 3(3/2) + 5(1/2)$

$$= 9/2 + 5/2$$
$$= 14/2 = 7 \text{ (Minimum)}$$

As the region is unbounded.
So, 7 may or may not be the minimum

$$3x + 5y < 7$$

x	0	2.3	3/2
y	1.2	0	1/2

$$5y = 7$$
$$y = 7/5 = 1.2$$
$$3x = 7$$
$$x = 7/3 = 2.3$$

Since from the graph there is no common point with the feasible region.

$$\therefore Z_{min} = 7 \text{ at } (3/2, 1/2)$$

//

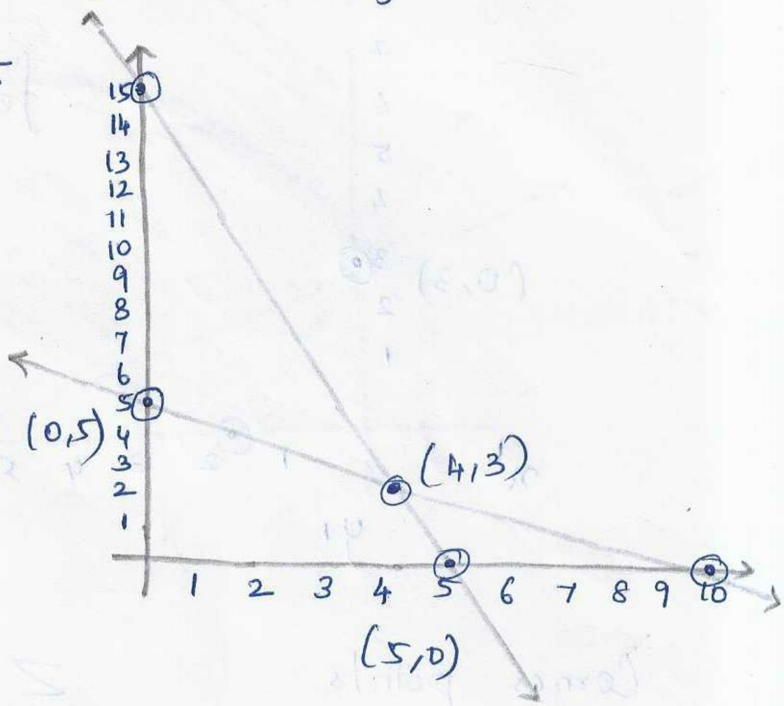
5. Maximise $Z = 3x + 2y$

Subject to $x + 2y \le 10$, $3x + y \le 15$, $x, y \ge 0$

$x + 2y \le 10$ $\quad$ $3x + y \le 15$

x	0	10
y	5	0

x	0	5
y	15	0



① $\times 3 \Rightarrow 3x + 6y = 30$

② $\Rightarrow 3x + y = 15$

$\quad (-) \quad (-) \quad (-)$

$5y = 15$

$y = 3$

$x + 2(3) = 10$

$x + 6 = 10$

$x = 4$

Corner points	$Z = 3x + 2y$
$(0,0)$	$Z = 3(0) + 2(0) = 0$
$(5,0)$	$Z = 3(5) + 2(0) = 15$
$(0,5)$	$Z = 3(0) + 2(5) = 10$
$(4,3)$	$Z = 3(4) + 2(3) = 18$

$Z_{max} = 18$ at $(4,3)$

6. Minimise $Z = x + 2y$

Subject to $2x + y \ge 3$, $x + 2y \ge 6$, $x, y \ge 0$

$2x + \cancel{2}y \ge 3$ $\quad$ $x + 2y \ge 6$

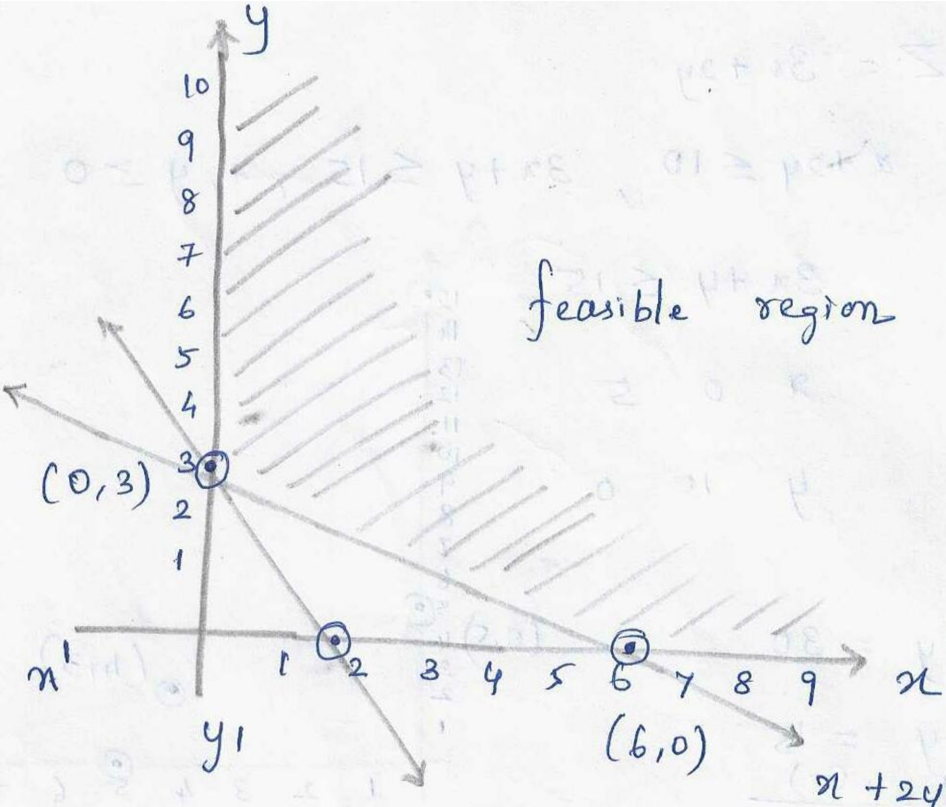
x	0	1.5
y	3	0

x	0	6
y	3	0

$2x + y \ge 3$

$2(4) + 4 \ge 3$

$12 \ge 3$



$x + 2y < 6$

Corner points | $Z = x + 2y$

$(0,3)$ | $Z = 0 + 2(3) = 6$

$(6,0)$ | $Z = 6 + 2(0) = 6$

Since the region is unbounded, So we need to Check 6 is minimum Value or not

$x + 2y < 6$

x	0	6
y	3	0

$Z_{min} = 6$

There is no Common region with the feasible graph & at all points in the line Segment joining (0,3) and (6,0).

7. Minimise and Maximise $Z = 5x + 10y$

Subject to $x + 2y \leq 120$, $x + y \geq 60$, $x - 2y \geq 0$, $x, y \geq 0$

$x + 2y \leq 120$

x	0	120
y	60	0

$x + y \geq 60$

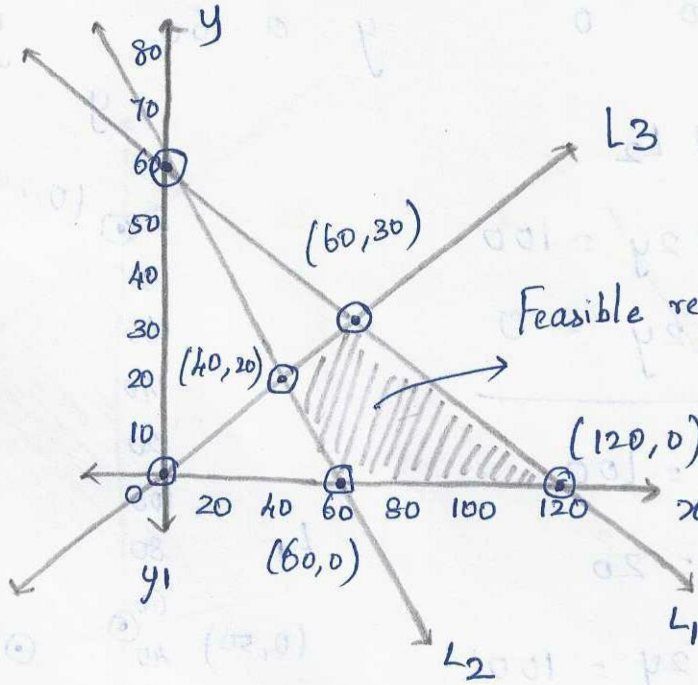
x	0	60
y	60	0

$x - 2y \geq 0$

x	0	40
y	0	20

L_3 and L_2

$$x + y = 60$$
$$x - 2y = 0$$
$$- \quad (+) \quad -$$
$$3y = 60$$
$$y = 60/3$$
$$y = 20$$
$$x + 20 = 60$$
$$x = 60 - 20 = 40$$
$$x = 40$$



L_1 and L_3

$$x + 2y = 120$$
$$x - 2y = 0$$
$$2x = 120$$
$$x = 60$$
$$60 - 2(\cancel{60}\,y) = 0$$
$$x = 130$$

Corner points	$Z = 5x + 10y$
$(60, 0)$	$Z = 5(60) + 10(0) = 300$
$(120, 0)$	$Z = 5(120) + 10(0) = 600$
$(40, 20)$	$Z = 5(40) + 10(20) = 400$
$(60, 30)$	$Z = 5(60) + 10(30) = 600$

$Z_{min} = 300$ at $(60, 0)$

$Z_{max} = 600$ at all points joining the line segment $(120, 0)$ and $(60, 30)$

8. Minimise and Maximize $Z = x + 2y$

Subject to $x + 2y \geq 1000$, $2x - y \leq 0$; $2x + y \leq 200$; $x, y \geq 0$.

$x + 2y \geq 100$

x	0	100
y	50	0

$2x - y \leq 0$

x	0	30
y	0	60

$2x + y \leq 200$

x	0	100
y	200	0

L_1 and L_2

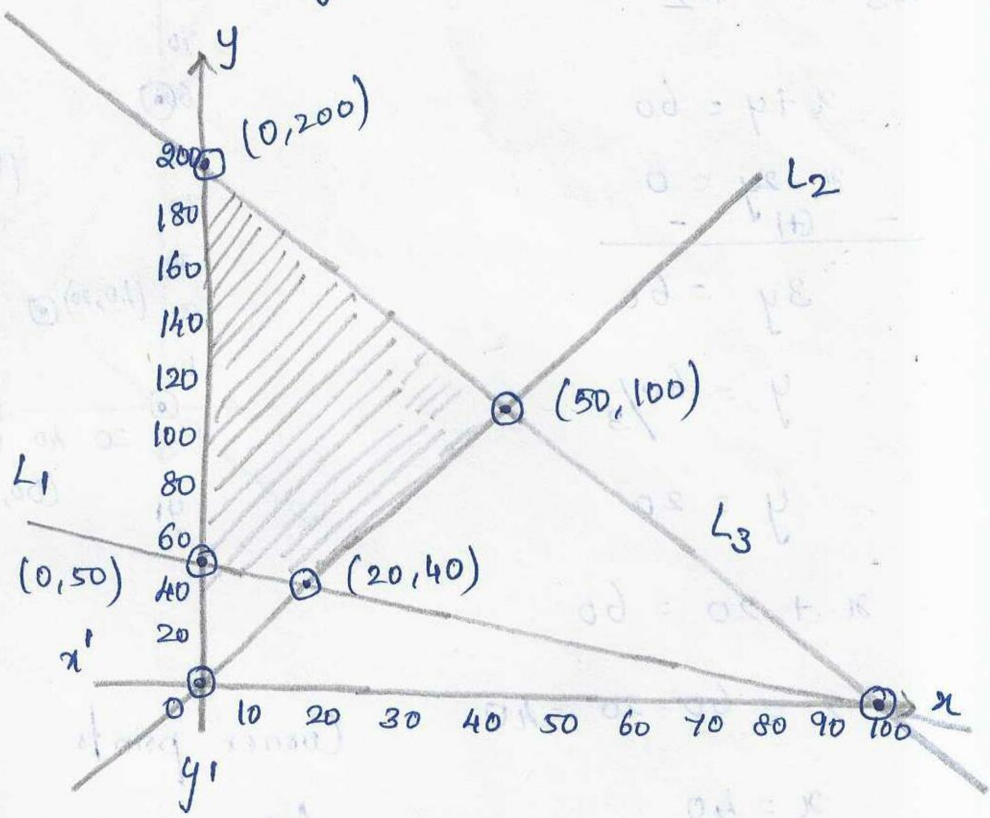
$$x + 2y = 100$$
$$4x - 2y = 0$$

$$5x = 100$$
$$x = 20$$
$$20 + 2y = 100$$
$$2y = 100 - 20$$
$$2y = 80$$
$$y = 40$$

L_2 and L_3

$$2x - y = 0$$
$$2x + y = 200$$

$$4x = 200$$
$$x = 50$$
$$2(50) - y = 0 \quad \therefore y = 100$$



Corner points	$Z = x + 2y$
$(0, 200)$	$Z = 0 + 2(200) = 400$
$(0, 50)$	$Z = 0 + 2(50) = 100$
$(20, 40)$	$Z = 20 + 2(40) = 100$
$(50, 100)$	$Z = 50 + 2(100) = 250$

$Z_{max} = 400$ at $(0, 200)$

$Z_{min} = 100$ at all points joining $(0, 50)$ and $(20, 40)$

9. Maximise $Z = -x + 2y$, subject to the constraints;

$x \geq 3,\ x + y \geq 5,\ x + 2y \geq 6,\ y \geq 0$

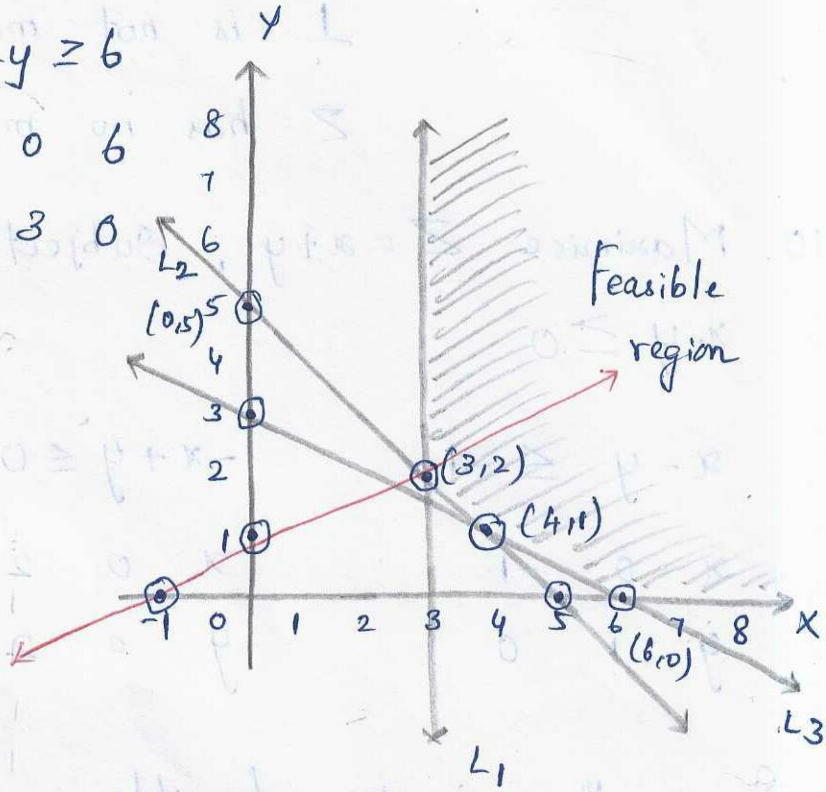
$x \geq 3$

$x + y \geq 5$

x	0	5
y	5	0

$x + 2y \geq 6$

x	0	6
y	3	0



L_2 and L_3

$$x + y = 5$$
$$x + 2y = 6$$
$$(-)\ (-)\quad (-)$$
$$-y = -1$$
$$y = 1$$
$$x + 1 = 5$$
$$x = 4$$

L_1

$$x = 3$$
$$x + y = 5$$
$$3 + y = 5$$
$$y = 5 - 3$$
$$y = 2$$

Corner points

$Z = -x + 2y$

$(3,2)$	$Z = -3 + 2(2) = -3 + 4 = 1$
$(4,1)$	$Z = -4 + 2(1) = -4 + 2 = -2$
$(6,0)$	$Z = -6 + 2(0) = -6$

Since the region is unbounded, so 1 may or may not be maximum value.

$$-x + 2y > 1$$

x	0	-1
y	0.5	0

$$2y = 1$$
$$y = 0.5$$
$$-x = 1$$
$$x = -1$$

Graph of $-x+2y \geq 1$ has common points with the feasible region.

$\therefore$ 1 is not max Value

Z has no max Value.

10. Maximise $Z = x+y$, Subject to $x-y \leq -1$, $-x+y \leq 0$, $x, y \geq 0$

$x-y \leq -1$		
x	0	-1
y	1	0

$-x+y \leq 0$		
x	0	2
y	0	2

Since there is no feasible region, No solution.