Differential Equations

An equation involving derivative (derivatives) of the dependent variable with respect to independent variable is called D.E

$$\frac{d^2y}{dx^2} + 2y = 0$$

Ordinary D.E

$$\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right) + y = 0$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

(or)

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} = 0$$

Notations:

$$\frac{dy}{dx} = y' = y_1$$

$$\frac{d^2y}{dx^2} = y'' = y_2$$

$$\frac{d^3y}{dx^3} = y''' = y_3$$

$$y''' + y_2 + y = 0$$

$$\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + y = 0$$

Order of a D.E is defined as the 'Order of the Highest order derivative'.

$$\frac{dy}{dx} = e^x \Rightarrow \text{Order} = 1$$

$$\frac{d^2y}{dx^2} + y = 0 \Rightarrow \text{Order} = 2$$

$$\left(\frac{d^3y}{dx^3}\right) + x^2\left(\frac{d^2y}{dx^2}\right)^3 = 0 \Rightarrow \text{Order} = 3$$

$$\frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right)^2 - \frac{dy}{dx} + y = 0 \Rightarrow \text{Order} = 3$$

$$\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right) - \sin^2 y = 0 \quad \Rightarrow \text{Order} = 1$$

Degree of a differential Equation :

Key point : The D.E must be a polynomial equation in derivatives.

$$\frac{d^3y}{dx^3} + 2\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx} + y = 0 \quad \Rightarrow \text{Order} = 3, \text{ degree} = 1$$

$$\sin\left(\frac{dy}{dx}\right) + x = 0$$

$$\sin\left(\frac{dy}{dx}\right) = -x$$

$$\frac{dy}{dx} = \sin^{-1}(-x)$$

$$\sin\left(\frac{dy}{dx}\right) + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \sin^{-1}\left(-\frac{dy}{dx}\right)$$

$$\sqrt{\frac{dy}{dx}} - x = 0$$

$$\left(\frac{dy}{dx}\right)^{1/2} - x = 0$$

$$\left(\frac{dy}{dx}\right)^{1/2} = x$$

$$\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} - \sin^2 y = 0 \quad \Rightarrow \text{order} = 1, \text{ degree} = 2$$

$$\frac{dy}{dx} + \sin\left(\frac{dy}{dx}\right) = 0 \Rightarrow \text{Order} = 1, \text{ degree} = \text{No}$$

$$\frac{dy}{dx} = e^{x} \Rightarrow \text{Order} = 1, \text{ degree} = 1$$

$$\frac{d^3y}{dx^3} + x^2\left(\frac{d^2y}{dx^2}\right)^3 = 0 \Rightarrow \text{Order} = 3, \text{ degree} = 1$$

$$\sin\left(\frac{dy}{dx}\right) - x = 0 \Rightarrow \text{Order} = 1, \text{ degree} = 1$$

$$\frac{dy}{dx} = e^{dy/dx}$$

$$\log_e\left(\frac{dy}{dx}\right) = \log_e e^{dy/dx} \Rightarrow \text{Order} = 1, \text{ degree} = \text{No}$$

$$\log\left(\frac{dy}{dx}\right) = \frac{dy}{dx}$$

Ex. 1 : Find the Order and degree.

(i) $\left(\frac{dy}{dx}\right)^1 - \cos x = 0$

Highest order derivative $= \frac{dy}{dx}$

Order $= 1$, degree $= 1$

(ii) $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\left(\frac{dy}{dx}\right) = 0$

Highest Order derivative $= \frac{d^2y}{dx^2}$

Order $= 2$, degree $= 1$

(iii) $y''' + y^2 + e^{y'} = 0$

$$\frac{d^3y}{dx^3} + y^2 + e^{dy/dx} = 0$$

Order = 3

$\therefore$ Given D.E is not polynomial equation in derivatives, So Degree = No.

Determine Order and degree (if defined) of D.E given in Exercises 1 to 10.

(i) $\frac{d^4y}{dx^4} + \sin(y''') = 0$

$$\frac{d^4y}{dx^4} + \sin\left(\frac{d^3y}{dx^3}\right) = 0$$

Order = 4, Degree is not Defined

(ii) $y' + 5y = 0$

$$\left(\frac{dy}{dx}\right)^1 + 5y = 0$$

order = 1, degree = 1

(iii) $\left(\frac{ds}{dt}\right)^4 + 3s\left(\frac{d^2s}{dt^2}\right)^1 = 0$

order = 2, degree = 1

(iv) $\left(\frac{d^2y}{dx^2}\right)^2 + \cos\left(\frac{dy}{dx}\right) = 0$

Order = 2, Degree not defined.

(v) $\left(\frac{d^2y}{dx^2}\right)^1 = \cos 3x + \sin 3x$

order = 2, degree = 1

(vi) $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$

$$\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^4 + y^5 = 0$$

Order = 3, degree = 2

(vii) $y''' + 2y'' + y' = 0$

$$\frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right) + \frac{dy}{dx} = 0$$

Order = 3, degree = 1

(viii) $y' + y = e^x$

order = 1, degree = 1

(ix) $y'' + (y')^2 + 2y = 0$

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 2y = 0$$

Order = 2, degree = 1

(x) $y'' + 2y' + \sin y = 0$

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right) + \sin y = 0$$

Order = 2, degree = 1

(xi) $\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin\left(\frac{dy}{dx}\right) + 1 = 0$

Degree = Not defined.

(xii) $2x^2 \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$

Order $= 2$, degree $= 1$

Ex: 9.2

General and Particular Solutions of a Differential Equation.

$$\frac{d^2y}{dx^2} + y = 0 \quad \text{——— (1)}$$

Let $y = f(x) = a\sin(x+b)$ be the solution of equation (1)

$$\frac{dy}{dx} = a\cos(x+b)$$

$$\frac{d^2y}{dx^2} = -a\sin(x+b)$$

from equation (1)

$$-a\sin(x+b) + a\sin(x+b) = 0$$

$$0 = 0$$

LHS = RHS

General Solution $\Rightarrow$ $y = f(x) = a\sin(x+b)$

$$\left.\begin{aligned} y &= 4\sin(x+5) \\ y &= 2\sin(x+3) \\ y &= 7\sin(x+8) \end{aligned}\right\} \text{Particular Solutions}$$

The Solution which contains arbitrary constants is called the 'General Solution' of the D.E.

The Solution free from arbitrary constants i.e the

Solution obtained from the General Solution by giving particular values to the arbitrary constants is called a 'Particular Solution' of the D.E.

$$\frac{d^2y}{dx^2} + y = 0 \Rightarrow \text{Order} = 2$$

$$y = a\sin(x+b) \Rightarrow \text{No. of Arbitrary Constants} = 2$$

$$2\sin(x+3)$$

Order of DE = No. of arbitrary Constants in Solution.

Ex: 2 Verify that the function $y = e^{-3x}$ is a Solution of the differential equation $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

$$y = e^{-3x} \quad \text{—— (1)}$$

$$\frac{dy}{dx} = -3e^{-3x} \quad \text{—— (2)}$$

$$\frac{d^2y}{dx^2} = (-3)(-3)e^{-3x}$$

$$= 9e^{-3x} \quad \text{—— (3)}$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

$$9e^{-3x} - 3e^{-3x} - 6e^{-3x} = 0$$

$$9e^{-3x} - 9e^{-3x} = 0$$

Hence $y = e^{-3x}$ is a Solution of given D.E.

Ex. 3: Verify that the function $y = a\cos x + b\sin x$, where $a, b \in R$ is a solution of the D.E. $\frac{d^2y}{dx^2} + y = 0$

$$\frac{d^2y}{dx^2} + y = 0$$

$$y = a\cos x + b\sin x \quad \text{--- (1)}$$

$$\frac{dy}{dx} = -a\sin x + b\cos x$$

$$\frac{d^2y}{dx^2} = -a\cos x - b\sin x \quad \text{--- (2)}$$

$$\therefore \quad \frac{d^2y}{dx^2} + y = 0$$

$$-a\cancel{\cos x} - b\cancel{\sin x} + a\cancel{\cos x} + b\cancel{\sin x} = 0$$

Hence $y = a\cos x + b\sin x$ is the Solution of given Diff. Equation.

1. $y = e^x + 1$; $y'' - y' = 0$

Given differential Equation,

$$y'' - y' = 0$$

given function $y = e^x + 1$

$$y' = e^x \quad \text{--- (1)}$$

$$y'' = e^x \quad \text{--- (2)}$$

Taking L.H.S of D.E

$= y'' - y'$

$= e^x - e^x = 0 = RHS$

Hence given function $y = e^x + 1$ is the Solution of given differential equation.

2. $y = x^2 + 2x + c$; $y' - 2x - 2 = 0$

$y = x^2 + 2x + c$

$y' = 2x + 2$

D.E, $y' - 2x - 2 = 0$

$(2x + 2) - 2x - 2 = 0$

$2x + 2 - 2x - 2 = 0$

$0 = 0$

$LHS = R.H.S$

hence $y = x^2 + 2x + c$ is the Solution of given D.E

3. $y = \cos x + c$; $y' + \sin x = 0$

$y = \cos x + c$

$y' = -\sin x$

taking LHS of DE

$= y' + \sin x$

$= -\sin x + \cancel{\sin x}$

$= 0 = RHS$

Hence given function $y = \cos x + c$ is the Solution of given D.E.

4. $y = \sqrt{1+x^2}$

$y' = \frac{xy}{1+x^2}$

given function,

$f(x) = y = \sqrt{1+x^2}$

$y = (1+x^2)^{1/2}$ ——— (A)

and differential Equation,

$y' = \frac{xy}{1+x^2}$

$\frac{dy}{dx} = \frac{xy}{1+x^2}$ ——— (1)

We have $y = (1+x^2)^{1/2}$

$\frac{dy}{dx} = \frac{1}{\cancel{2}}(1+x^2)^{1/2 - 1} \times \cancel{2}x$

$= x(1+x^2)^{-1/2}$

$\frac{dy}{dx} = \frac{x}{\sqrt{1+x^2}}$ ——— (B)

from equation (1)

$\frac{dy}{dx} = \frac{xy}{1+x^2}$

$$\frac{x}{\sqrt{1+x^2}} = \frac{x(1+x^2)^{1/2}}{(1+x^2)}$$

$$\frac{x}{(1+x^2)^{1/2}} = \frac{x}{(1+x^2)^{1/2}}$$

from eq (A) and (B), LHS = RHS

Hence given $y = (1+x^2)^{1/2}$ is the Solution of given D.E.

5. $y = Ax$; $xy' = y$ $(x \neq 0)$

Given: $y = Ax$

We have to Verify the y is the Solution of D.E

$$y = Ax$$

$$y' = \frac{dy}{dx} = A$$

So, $xy' = y$

$$xA = Ax$$

$$Ax = Ax$$

LHS = RHS

Hence y is the Solution of given D.E

6. $y = x\sin x$; $xy' = y + x\sqrt{x^2 - y^2}$ $(x \neq 0$ and $x > y$ or $x < -y)$

$$y = x\sin x$$

$$y' = \frac{dy}{dx} = x\cos x + \sin x$$

Taking L.H.S of D.E

$$= x y'$$

$$= x(x\cos x + \sin x)$$

$$= x^2 \cos x + x \sin x$$

Taking RHS of DE

$$= y + x\sqrt{x^2 - y^2}$$

$$= x\sin x + x\sqrt{x^2 - x^2 \sin^2 x}$$

$$= x\sin x + x\sqrt{x^2(1-\sin^2 x)}$$

$$= x\sin x + x\sqrt{x^2 \cdot \cos^2 x}$$

$$= x\sin x + \sqrt{x^2\cos^2 x} \cdot x$$

$$= x\sin x + x \cdot x \cdot \cos x$$

$$= x\sin x + x^2 \cos x$$

$$= x^2\cos x + x\sin x = RHS$$

Hence y is the Solution of D.E

7. $xy = \log y + c \;;\; y' = \dfrac{y^2}{1-xy} \quad (xy \neq 1)$

Given : $xy = \log y + c$

$$y + x \cdot y' = \frac{1}{y} \cdot y'$$

$$y = \frac{y'}{y} - xy'$$

$$y = \frac{y' - xyy'}{y}$$

$$y^2 = y' - xyy'$$

$$y^2 = y'(1 - xy)$$

$$\frac{y^2}{1-xy} = y'$$

Taking LHS of D.E

$$= y' = \frac{y^2}{1-xy} = RHS$$

8. $y - \cos y = x \quad ; \left(y \sin y + \cos y + x\right)y' = y$

$$y - \cos y = x$$

$$y = x + \cos y$$

$$y' + \sin y \cdot y' = 1$$

$$y'(1 + \sin y) = 1$$

$$y' = \frac{1}{1+\sin y}$$

Taking LHS of D.E

$$= \left(y \sin y + \cos y + x\right) y'$$

$$= \left(y\sin y + \cos y + x\right) \cdot \frac{1}{(1+\sin y)}$$

$$= \frac{y\sin y + \cos y + x}{1 + \sin y}$$

$$= \frac{y\sin y + \cos y + (y - \cos y)}{1 + \sin y}$$

$$= \frac{y\sin y + \cancel{\cos y} + y - \cancel{\cos y}}{1 + \sin y}$$

$$= \frac{y\sin y + y}{1 + \sin y}$$

$$= \frac{y\,(\cancel{\sin y} + 1)}{1 + \cancel{\sin y}}$$

$$= y = RHS$$

9. $x + y = \tan^{-1} y \quad ; \quad y^2 y' + y^2 + 1 = 0$

$$x + y = \tan^{-1} y$$

$$1 + y' = \frac{1}{1+y^2} \times y'$$

$$1 + y' = \frac{y'}{1+y^2}$$

$$(1 + y')(1 + y^2) = y'$$

$$1 + y' + y^2 + y^2 y' = y'$$

$$1 + \cancel{y'} + y^2 + y^2 y' - \cancel{y'} = 0$$

$$y^2 y' + y^2 + 1 = 0$$

10. $y = \sqrt{a^2 - x^2}$, $x \in (-a, a)$; $x + y\dfrac{dy}{dx} = 0$ $(y \neq 0)$

$$y = (a^2 - x^2)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{\cancel{2}}(a^2 - x^2)^{-1/2}(-\cancel{2}x)$$

$$= \frac{-x}{(a^2 - x^2)^{1/2}}$$

$$= \frac{-x}{\sqrt{a^2 - x^2}}$$

Taking LHS of D.E

$$= x + y\frac{dy}{dx}$$

$$= x + \cancel{\sqrt{a^2 - x^2}}\left(\frac{-x}{\cancel{\sqrt{a^2 - x^2}}}\right)$$

$$= x - x$$

$$= 0 = RHS$$

11. No. of arbitrary Constants in the general Solution of a differential equation of fourth order are '4'

12. No. of arbitrary constants in the particular solution of a differential equation of third order are '0'

Formation of a differential Equation whose general Solution is given:

1. No. of arbitrary Constant = Order of D.E
2. Eliminate constants.

form the D.E representing the family and curves $y = mx$ where m is arbitrary Constant.

Given $y = mx$ —— (1)

$$\frac{dy}{dx} = m$$

from eq (1)

$$\frac{y}{x} = m$$

$$\frac{dy}{dx} = \frac{y}{x}$$

$$x\frac{dy}{dx} = y$$

$$x\frac{dy}{dx} - y = 0$$

Form the D.E representing the family of curves $y = a\sin(x+b)$ where a and b are arbitrary constants.

$$y = a\sin(x+b) \quad \text{---} \ ①$$

$$\frac{dy}{dx} = a\cos(x+b) \quad \text{---} \ ②$$

$$\frac{d^2y}{dx^2} = -a\sin(x+b)$$

$$\frac{d^2y}{dx^2} = -y$$

$$\frac{d^2y}{dx^2} + y = 0$$

form the D.E Representing the family of Ellipse having foci on x-axis and Centre at origin.

W.k.t ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{x}{a^2} + \frac{y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{y}{b^2}\frac{dy}{dx} = -\frac{x}{a^2}$$

$$y\frac{dy}{dx} = \frac{-x}{a^2} \times b^2$$

$$\frac{y}{x}\frac{dy}{dx} = \frac{-b^2}{a^2}$$

$$\frac{y}{x}\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(\frac{x\frac{dy}{dx}-y\times 1}{x^2}\right)=0$$

$$\frac{y}{x}\frac{d^2y}{dx^2}+\frac{x}{x^2}\left(\frac{dy}{dx}\right)^2-\frac{y}{x^2}\left(\frac{dy}{dx}\right)=0$$

$$\frac{y}{x}\frac{d^2y}{dx^2}+\frac{1}{x}\left(\frac{dy}{dx}\right)^2-\frac{y}{x^2}\left(\frac{dy}{dx}\right)=0$$

Form the D.E Representing the family of parabolas having vertex at origin and axis along the direction of X-axis.

$$y^2=4ax$$

$$4a=\frac{y^2}{x}$$

$$\therefore \frac{y^2}{x}=4a$$

$$2y.\frac{dy}{dx}=4a$$

$$2y\cdot\frac{dy}{dx}=\frac{y^2}{x}$$

$$2y\frac{dy}{dx}-\frac{y^2}{x}=0$$

$$\frac{2xy\frac{dy}{dx}-y^2}{x}=0$$