A Distributional Perspective on Reinforcement Learning

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Abstract

In this paper we argue for the fundamental importance of the *value distribution*: the distribution of the random return received by a reinforcement learning agent. This is in contrast to the common approach to reinforcement learning which models the expectation of this return, or *value*. Although there is an established body of literature studying the value distribution, thus far it has always been used for a specific purpose such as implementing risk-aware behaviour. We begin with theoretical results in both the policy evaluation and control settings, exposing a significant distributional instability in the latter. We then use the distributional perspective to design a new algorithm which applies Bellman's equation to the learning of approximate value distributions. We evaluate our algorithm using the suite of games from the Arcade Learning Environment. We obtain both state-of-the-art results and anecdotal evidence demonstrating the importance of the value distribution in approximate reinforcement learning. Finally, we combine theoretical and empirical evidence to highlight the ways in which the value distribution impacts learning in the approximate setting.

1. Introduction

One of the major tenets of reinforcement learning states that, when not otherwise constrained in its behaviour, an agent should aim to maximize its expected utility Q, or *value* (Sutton & Barto, 1998). Bellman's equation succinctly describes this value in terms of the expected reward and expected outcome of the random transition $(x, a) \to (X', A')$:

$$Q(x, a) = \mathbb{E}\, R(x, a) + \gamma\, \mathbb{E}\, Q(X', A').$$

In this paper, we aim to go beyond the notion of value and argue in favour of a distributional perspective on reinforcement learning. Specifically, the main object of our study is the random return Z whose expectation is the value Q. This random return is also described by a recursive equation, but one of a distributional nature:

$$Z(x, a) \overset{D}{=} R(x, a) + \gamma Z(X', A').$$

The *distributional Bellman equation* states that the distribution of Z is characterized by the interaction of three random variables: the reward R, the next state-action (X', A'), and its random return $Z(X', A')$. By analogy with the well-known case, we call this quantity the *value distribution*.

Although the distributional perspective is almost as old as Bellman's equation itself (Jaquette, 1973; Sobel, 1982; White, 1988), in reinforcement learning it has thus far been subordinated to specific purposes: to model parametric uncertainty (Dearden et al., 1998), to design risk-sensitive algorithms (Morimura et al., 2010b;a), or for theoretical analysis (Azar et al., 2012; Lattimore & Hutter, 2012). By contrast, we believe the value distribution has a central role to play in reinforcement learning.

Contraction of the policy evaluation Bellman operator. Basing ourselves on results by Rösler (1992) we show that, for a fixed policy, the Bellman operator over value distributions is a contraction in a maximal form of the Wasserstein (also called Kantorovich or Mallows) metric. Our particular choice of metric matters: the same operator is not a contraction in total variation, Kullback-Leibler divergence, or Kolmogorov distance.

Instability in the control setting. We will demonstrate an instability in the distributional version of Bellman's optimality equation, in contrast to the policy evaluation case. Specifically, although the optimality operator is a contraction in expected value (matching the usual optimality result), it is not a contraction in any metric over distributions. These results provide evidence in favour of learning algorithms that model the effects of nonstationary policies.

Better approximations. From an algorithmic standpoint, there are many benefits to learning an approximate distribution rather than its approximate expectation. The distributional Bellman operator preserves multimodality in value distributions, which we believe leads to more stable learning. Approximating the full distribution also mitigates the effects of learning from a nonstationary policy. As a whole,

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关于加固学习的分配观点

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抽象的

在本文中，我们主张*value distribution*的基本重要性：增强学习代理人收到的随机回报的分布。这与加固学习的方法相反，该方法模拟了此回报的期望或*value*。尽管有一个既定的文献都在研究价值分布，但到目前为止，它一直用于特定目的，例如实施风险感知行为。我们从政策评估和控制设置的理论结果开始，暴露了后者的显着分布不稳定性。然后，我们使用分布观点来设计一种新算法，该算法将贝尔曼的方程应用于学习近似价值分布的方程。我们使用街机学习环境中的游戏套件来评估我们的算法。我们获得了最先进的资源和轶事证据，证明了价值分布在近似强化学习中的重要性。最后，我们综合理论和经验证据，以高度阐明价值分布在大概环境中学习的方式。

1。简介

强化学习的主要原则之一指出，如果不以其他方式限制其行为，则代理应旨在最大化其预期的效用Q或*value*（Sutton＆Barto，1998）。贝尔曼的方程式用随机过渡（x, a}）的预期奖励和表达的结果来缩短描述了这一值→ (X', A')：

$$Q(x,a)=\mathbb{E}\,R(x,a)+\gamma\,\mathbb{E}\,Q(X',A').$$

在本文中，我们旨在超越价值概念，并主张对加强的分配观点 - 学习。特别是，我们研究的主要对象是随机返回Z，其期望为Q。递归方程也描述了这种随机回报，但分布性质之一：

$$Z(x,a)\overset{D}{=}R(x,a)+\gamma Z(X',A').$$

*distributional Bellman equation*指出Z的分布的特点是三个随机变量的相互作用：奖励R，下一个状态 - action（X', A'）和其随机返回$Z(X', A')$。通过类似于众所周知的情况，我们将此数量称为*value distribution*。

尽管分布观点几乎与贝尔曼方程本身一样古老（Jaquette，1973; Sobel，1982; White，1988），但在强化学习中，迄今已将其从属于特定目的：用于模拟参数性不确定性（Dearden等人。，1998年），设计对风险敏感的al-gorithm（Morimura等，2010b; a）或用于理论分析（Azar等，2012; Lattimore＆Hutter，2012）。通过Concontast，我们认为价值分布在强化学习中起着核心作用。

政策评估的收缩Bellman操作员。基于R ¨Osler（1992）的结果，我们表明，对于固定的策略，对价值分布的Bellman运营商是Waseserstein（也称为Kantorovich或Mallows）指标的最大形式的收缩。我们的指标事项的特定选择：同一操作员在总变化，kullback-leibler Divergence或Kolmogorov距离上不是收缩。

在控制设置中不稳定。与政策评估案例相比，我们将证明Bellman的Opti-timeliality方程式的发行版本不稳定。特别地，尽管最优运算符是预期值的违反（与通常的最优性相匹配），但在任何分布上的任何度量中，它都不是收缩。这些结果提供了支持学习非组织政策影响的算术算法的证据。

更好的近似值。从算法的角度来看，学习近似分布而不是其近似期望有许多好处。分布式钟楼操作员保留了价值分布的多模式，我们认为这会导致更稳定的学习。近似完整的分布还减轻了从非组织政策中学习的影响。总体而言

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we argue that this approach makes approximate reinforcement learning significantly better behaved.

We will illustrate the practical benefits of the distributional perspective in the context of the Arcade Learning Environment (Bellemare et al., 2013). By modelling the value distribution within a DQN agent (Mnih et al., 2015), we obtain considerably increased performance across the gamut of benchmark Atari 2600 games, and in fact achieve state-of-the-art performance on a number of games. Our results echo those of Veness et al. (2015), who obtained extremely fast learning by predicting Monte Carlo returns.

From a supervised learning perspective, learning the full value distribution might seem obvious: why restrict ourselves to the mean? The main distinction, of course, is that in our setting there are no given targets. Instead, we use Bellman's equation to make the learning process tractable; we must, as Sutton & Barto (1998) put it, "learn a guess from a guess". It is our belief that this guesswork ultimately carries more benefits than costs.

2. Setting

We consider an agent interacting with an environment in the standard fashion: at each step, the agent selects an action based on its current state, to which the environment responds with a reward and the next state. We model this interaction as a time-homogeneous Markov Decision Process $(\mathcal{X}, \mathcal{A}, R, P, \gamma)$. As usual, $\mathcal{X}$ and $\mathcal{A}$ are respectively the state and action spaces, P is the transition kernel $P(\cdot \,|\, x, a)$, $\gamma \in [0, 1]$ is the discount factor, and R is the reward function, which in this work we explicitly treat as a random variable. A stationary policy π maps each state $x \in \mathcal{X}$ to a probability distribution over the action space $\mathcal{A}$.

2.1. Bellman's Equations

The *return* Z^π is the sum of discounted rewards along the agent's trajectory of interactions with the environment. The value function Q^π of a policy π describes the expected return from taking action $a \in \mathcal{A}$ from state $x \in \mathcal{X}$, then acting according to π:

$$Q^\pi(x, a) := \mathbb{E}\, Z^\pi(x, a) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R(x_t, a_t)\right], \qquad (1)$$

$$x_t \sim P(\cdot \,|\, x_{t-1}, a_{t-1}), a_t \sim \pi(\cdot \,|\, x_t), x_0 = x, a_0 = a.$$

Fundamental to reinforcement learning is the use of Bellman's equation (Bellman, 1957) to describe the value function:

$$Q^\pi(x, a) = \mathbb{E}\, R(x, a) + \gamma \mathop{\mathbb{E}}_{P,\pi} Q^\pi(x', a').$$

In reinforcement learning we are typically interested in acting so as to maximize the return. The most common approach for doing so involves the optimality equation

$$Q^*(x, a) = \mathbb{E}\, R(x, a) + \gamma\, \mathbb{E}_P \max_{a' \in \mathcal{A}} Q^*(x', a').$$

This equation has a unique fixed point Q^*, the optimal value function, corresponding to the set of optimal policies Π^* (π^* is optimal if $\mathbb{E}_{a\sim\pi^*} Q^*(x, a) = \max_a Q^*(x, a)$).

We view value functions as vectors in $\mathbb{R}^{\mathcal{X}\times\mathcal{A}}$, and the expected reward function as one such vector. In this context, the *Bellman operator* $\mathcal{T}^\pi$ and *optimality operator* $\mathcal{T}$ are

$$\mathcal{T}^\pi Q(x, a) := \mathbb{E}\, R(x, a) + \gamma \mathop{\mathbb{E}}_{P,\pi} Q(x', a') \qquad (2)$$

$$\mathcal{T} Q(x, a) := \mathbb{E}\, R(x, a) + \gamma\, \mathbb{E}_P \max_{a' \in \mathcal{A}} Q(x', a'). \qquad (3)$$

These operators are useful as they describe the expected behaviour of popular learning algorithms such as SARSA and Q-Learning. In particular they are both contraction mappings, and their repeated application to some initial Q_0 converges exponentially to Q^π or Q^*, respectively (Bertsekas & Tsitsiklis, 1996).

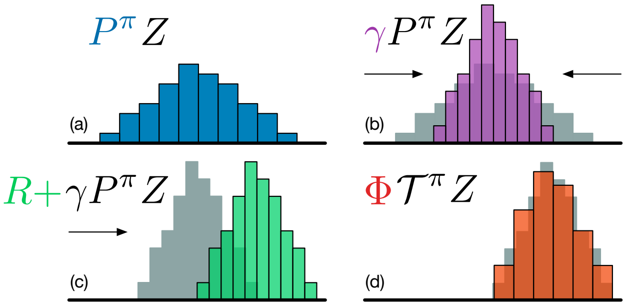


Figure 1. A distributional Bellman operator with a deterministic reward function: (a) Next state distribution under policy π, (b) Discounting shrinks the distribution towards 0, (c) The reward shifts it, and (d) Projection step (Section 4).

3. The Distributional Bellman Operators

In this paper we take away the expectations inside Bellman's equations and consider instead the full distribution of the random variable Z^π. From here on, we will view Z^π as a mapping from state-action pairs to distributions over returns, and call it the *value distribution*.

Our first aim is to gain an understanding of the theoretical behaviour of the distributional analogues of the Bellman operators, in particular in the less well-understood control setting. The reader strictly interested in the algorithmic contribution may choose to skip this section.

3.1. Distributional Equations

It will sometimes be convenient to make use of the probability space $(\Omega, \mathcal{F}, \Pr)$. The reader unfamiliar with mea-

我们认为，这种方法使近似的强化学习表现得更好。

我们将在街机学习环境的背景下说明分布观点的实际收益（Bellemare等，2013）。通过对DQN代理中的价值划分进行建模（Mnih等，2015），我们在基准Atari 2600游戏的范围内大大提高了性能游戏数。我们的结果与Veness等人的结果相呼应。（2015年），他通过预测蒙特卡洛回报获得了非常快的学习。

从监督的学习角度来看，学习全价分布似乎很明显：为什么将我们的自我限制为卑鄙？当然，主要的区别是在我们的环境中没有给定的目标。取而代之的是，我们使用贝尔曼方程来使学习过程可进行。正如Sutton＆Barto（1998）所说，我们必须“从猜测中学习猜测”。我们相信，这种猜测最终比成本更具好处。

2。设置

我们考虑一个以标准方式与环境相互作用的代理：在每个步骤中，代理商根据其当前状态选择一个符合条件，环境以奖励和下一个状态对其进行了。我们将此反应建模为时间均匀的马尔可夫决策过程（$\mathcal{X}, \mathcal{A}, R, P, \gamma$）。像往常一样，$\mathcal{X}$和$\mathcal{A}$分别是状态和动作空间，$P$是过渡内核$P(\cdot \,|\, x, a)$，$\gamma \in$，{0，1]是折现因子，$R$是奖励功能，在这项工作中，我们明确将其视为一个随机变量。固定策略π将每个状态$x \in \mathcal{X}$映射到动作空间$\mathcal{A}$上的概率分布。

2.1。贝尔曼方程

return Z^π是沿代理与环境交互轨迹的折扣奖励之和。策略π的值函数Q^π描述了从状态$x \in \mathcal{X}$采取action $a \in \mathcal{A}$的预期重新转向，然后根据π进行行动：

$$Q^\pi(x, a) := \mathbb{E}\, Z^\pi(x, a) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t R(x_t, a_t)\right], \qquad (1)$$

$$x_t \sim P(\cdot \,|\, x_{t-1}, a_{t-1}), a_t \sim \pi(\cdot \,|\, x_t), x_0 = x, a_0 = a.$$

强化学习的基础是使用贝尔曼方程（Bellman，1957）来描述价值的功能：

$$Q^\pi(x, a) = \mathbb{E}\, R(x, a) + \gamma \mathop{\mathbb{E}}_{P,\pi} Q^\pi(x', a').$$

在加强学习中，我们通常对采取行动感兴趣，以最大程度地提高回报。最常见的approach

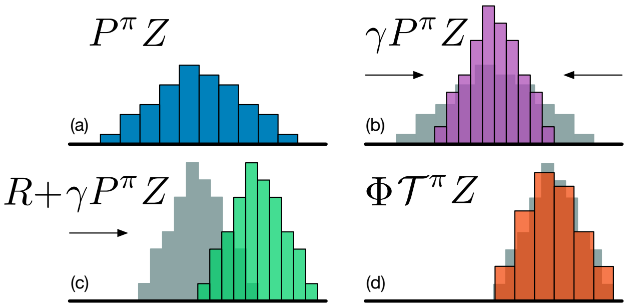


图1。具有确定性奖励函数的分配钟手操作员：（a）策略下的下一个状态分布π，（b）打折将分布缩小到0，（c）奖励会移动它，并且（d）投影步骤（d）投影步骤（第4节）。

这样做的过程涉及最佳方程

$$Q^*(x, a) = \mathbb{E}\, R(x, a) + \gamma\, \mathbb{E}_P \max_{a' \in \mathcal{A}} Q^*(x', a').$$

该方程具有唯一的固定点Q^*，即对应于最佳策略Π^* (π^*集合的最佳值函数，如果$\mathbb{E}_{a \sim \pi^*} Q^*(x, a) =$ Max Max $_a\, Q^*(x, a)$)是最佳的。

我们将值函数视为$\mathbb{R}^{\mathcal{X} \times \mathcal{A}}$中的向量，而被指出的奖励函数则是这样的向量。在这种情况下，*Bellman operator* $\mathcal{T}^\pi$和*optimality operator* $\mathcal{T}$是

$$\mathcal{T}^\pi Q(x, a) := \mathbb{E}\, R(x, a) + \gamma \mathop{\mathbb{E}}_{P,\pi} Q(x', a') \qquad (2)$$

$$\mathcal{T} Q(x, a) := \mathbb{E}\, R(x, a) + \gamma\, \mathbb{E}_P \max_{a' \in \mathcal{A}} Q(x', a'). \qquad (3)$$

这些操作员很有用，因为它们描述了流行学习算法（例如SARSA和Q-LEARNING）的预期行为。特别是它们既是收缩映射，并且对某些初始Q_0的重复应用分别将指数收敛到Q^π或Q^*（bert- sekas＆tsitsiklis，1996）。

3。发行的贝尔曼操作员

在本文中，我们消除了贝尔曼方程内的期望，而是考虑随机变量Z^π的完整分布。从这里开始，我们将Z^π视为从状态行动对到分布的映射，并将其称为*value distribution*。

我们的第一个目的是了解Bellman操作员的分布类似物的理论行为，尤其是在不太理解的控制环境中。严格对算法贡献感兴趣的读者可能会选择跳过本节。

3.1。分布方程

有时使用概率空间（ω, $\mathcal{F}$, pr）会很方便。读者不熟悉measure theory may think of Ω as the space of all possible outcomes of an experiment (Billingsley, 1995). We will write $\|\mathbf{u}\|_p$ to denote the L_p norm of a vector $\mathbf{u} \in \mathbb{R}^{\mathcal{X}}$ for $1 \leq p \leq \infty$; the same applies to vectors in $\mathbb{R}^{\mathcal{X} \times \mathcal{A}}$. The L_p norm of a random vector $U : \Omega \to \mathbb{R}^{\mathcal{X}}$ (or $\mathbb{R}^{\mathcal{X} \times \mathcal{A}}$) is then $\|U\|_p := \left[\mathbb{E}\left[\|U(\omega)\|_p^p\right]\right]^{1/p}$, and for $p = \infty$ we have $\|U\|_\infty = \operatorname{ess\,sup} \|U(\omega)\|_\infty$ (we will omit the dependency on $\omega \in \Omega$ whenever unambiguous). We will denote the c.d.f. of a random variable U by $F_U(y) := \Pr\{U \leq y\}$, and its inverse c.d.f. by $F_U^{-1}(q) := \inf\{y : F_U(y) \geq q\}$.

A distributional equation $U \overset{D}{:=} V$ indicates that the random variable U is distributed according to the same law as V. Without loss of generality, the reader can understand the two sides of a distributional equation as relating the distributions of two independent random variables. Distributional equations have been used in reinforcement learning by Engel et al. (2005); Morimura et al. (2010a) among others, and in operations research by White (1988).

3.2. The Wasserstein Metric

The main tool for our analysis is the Wasserstein metric d_p between cumulative distribution functions (see e.g. Bickel & Freedman, 1981, where it is called the Mallows metric). For F, G two c.d.fs over the reals, it is defined as

$$d_p(F, G) := \inf_{U,V} \|U - V\|_p,$$

where the infimum is taken over all pairs of random variables (U, V) with respective cumulative distributions F and G. The infimum is attained by the inverse c.d.f. transform of a random variable $\mathcal{U}$ uniformly distributed on $[0, 1]$:

$$d_p(F, G) = \|F^{-1}(\mathcal{U}) - G^{-1}(\mathcal{U})\|_p.$$

For $p < \infty$ this is more explicitly written as

$$d_p(F, G) = \left(\int_0^1 \left|F^{-1}(u) - G^{-1}(u)\right|^p du\right)^{1/p}.$$

Given two random variables U, V with c.d.fs F_U, F_V, we will write $d_p(U, V) := d_p(F_U, F_V)$. We will find it convenient to conflate the random variables under consideration with their versions under the inf, writing

$$d_p(U, V) = \inf_{U,V} \|U - V\|_p.$$

whenever unambiguous; we believe the greater legibility justifies the technical inaccuracy. Finally, we extend this metric to vectors of random variables, such as value distributions, using the corresponding L_p norm.

Consider a scalar a and a random variable A independent of U, V. The metric d_p has the following properties:

$$d_p(aU, aV) \leq |a| d_p(U, V) \tag{P1}$$
$$d_p(A + U, A + V) \leq d_p(U, V) \tag{P2}$$
$$d_p(AU, AV) \leq \|A\|_p d_p(U, V). \tag{P3}$$

We will need the following additional property, which makes no independence assumptions on its variables. Its proof, and that of later results, is given in the appendix.

Lemma 1 (Partition lemma)**.** *Let $A_1, A_2, \dots$ be a set of random variables describing a partition of Ω, i.e. $A_i(\omega) \in \{0, 1\}$ and for any ω there is exactly one A_i with $A_i(\omega) = 1$. Let U, V be two random variables. Then*

$$d_p(U, V) \leq \sum\nolimits_i d_p(A_i U, A_i V).$$

Let $\mathcal{Z}$ denote the space of value distributions with bounded moments. For two value distributions $Z_1, Z_2 \in \mathcal{Z}$ we will make use of a maximal form of the Wasserstein metric:

$$\bar{d}_p(Z_1, Z_2) := \sup_{x,a} d_p(Z_1(x, a), Z_2(x, a)).$$

We will use $\bar{d}_p$ to establish the convergence of the distributional Bellman operators.

Lemma 2. *$\bar{d}_p$ is a metric over value distributions.*

3.3. Policy Evaluation

In the *policy evaluation* setting (Sutton & Barto, 1998) we are interested in the value function V^π associated with a given policy π. The analogue here is the value distribution Z^π. In this section we characterize Z^π and study the behaviour of the policy evaluation operator $\mathcal{T}^\pi$. We emphasize that Z^π describes the intrinsic randomness of the agent's interactions with its environment, rather than some measure of uncertainty about the environment itself.

We view the reward function as a random vector $R \in \mathcal{Z}$, and define the transition operator $P^\pi : \mathcal{Z} \to \mathcal{Z}$

$$P^\pi Z(x, a) \overset{D}{:=} Z(X', A') \tag{4}$$
$$X' \sim P(\cdot \,|\, x, a),\ A' \sim \pi(\cdot \,|\, X'),$$

where we use capital letters to emphasize the random nature of the next state-action pair (X', A'). We define the distributional Bellman operator $\mathcal{T}^\pi : \mathcal{Z} \to \mathcal{Z}$ as

$$\mathcal{T}^\pi Z(x, a) \overset{D}{:=} R(x, a) + \gamma P^\pi Z(x, a). \tag{5}$$

While $\mathcal{T}^\pi$ bears a surface resemblance to the usual Bellman operator (2), it is fundamentally different. In particular, three sources of randomness define the compound distribution $\mathcal{T}^\pi Z$:

当然，理论可以将ω视为实验所有可能结果的空间（Billingsley，1995）。我们将编写$\|\mathbf{u}\|_p$来表示1 $\leq p \leq \infty$的向量$\mathbf{u} \in \mathbb{R}^{\mathcal{X}}$的$L_p$ norm; $\mathbb{R}^{\mathcal{X} \times \mathcal{A}}$中的向量也是如此。然后，随机向量$U$的$L_p$标准：ω→ $\mathbb{R}^{\mathcal{X}}$ (或$\mathbb{R}^{\mathcal{X} \times \mathcal{A}}$)是$\|U\|_p$：$= \left[\mathbb{E}\left[\|U(\omega)\|_p^p\right]\right]^{1/p}$：$= \left[\mathbb{E}\left[\|U(\omega)\|_p^p\right]\right]^{1/p}$，对于$p = \infty$，我们有$\|U\|_\infty =$ ess sup $\|U(\omega)\|_\infty$ (，我们将省略每当明确)时，对ω ∈ω的依赖性。我们将表示C.D.F. $F_U(y)$的随机变量U：$=$ pr $\{U \leq y\}$及其倒数c.d.f.由$F_U^{-1}(q)$：$=$ inf $\{y$：$F_U(y) \geq q\}$。

分布方程$U \overset{D}{}$：$=$ V表示random variable U是根据与V相同的定律分配的。在不失去一般性的情况下，读者可以理解分布方程的两个侧面，将两个独立的随机变量的分离关联。 Engel等人已将分布方程用于加强学习。（2005）; Morimura等。（2010a）以及怀特（White）（1988）的运营研究中。

3.2。 Wasserstein指标

我们分析的主要工具是累积分布函数之间的Wasserstein Metric d_p（参见例如Bickel＆Freedman，1981，其中称为Mallows Metric）。对于F，G两个c.d.fs在真实方面，它被定义为

$$d_p(F, G) := \inf_{U,V} \|U - V\|_p,$$

如果将所有随机变量（U, V）与各个累积分布F和G相应的所有对摄影。 Informum由C.D.F.随机变量$\mathcal{U}$的trans形式均匀分布在[0 , 1]:

$$d_p(F, G) = \|F^{-1}(\mathcal{U}) - G^{-1}(\mathcal{U})\|_p.$$

对于$p < \infty$这更明确地写成

$$d_p(F, G) = \left(\int_0^1 \left|F^{-1}(u) - G^{-1}(u)\right|^p du\right)^{1/p}.$$

给定两个随机变量U，V，带有c.d.fs F_U，F_V，我们将编写$d_p(U, V)$：$= d_p(F_U, F_V)$。我们将发现它可以提出相结合的随机变量，并在INF下使用其版本，写作

$$d_p(U, V) = \inf_{U,V} \|U - V\|_p.$$

每当明确时；我们认为，更大的知名度是有理由的。最后，我们使用相应的L_p narm将此度量扩展到随机变量的向量，例如值分布。

考虑标量a和一个随机变量A独立U, V的of。公制d_p具有以下属性:

$$d_p(aU, aV) \leq |a| d_p(U, V) \tag{P1}$$

$$d_p(A + U, A + V) \leq d_p(U, V) \tag{P2}$$

$$d_p(AU, AV) \leq \|A\|_p d_p(U, V). \tag{P3}$$

我们将需要以下额外的属性，这在其变量上没有独立假设。它的证明和后来的结果是在附录中给出的。

引理1（分区引理）。 *Let $A_1, A_2, \ldots$ be a set of random variables describing a partition of*ω*, i.e. $A_i(\omega) \in \{0, 1\}$ and for any* ω *there is exactly one A_i with $A_i(\omega) = 1$. Let U, V be two random variables. Then*

$$d_p(U, V) \leq \sum\nolimits_i d_p(A_i U, A_i V).$$

令$\mathcal{Z}$用有限的矩表示价值分布的空间。对于两个值分布$Z_1, Z_2 \in \mathcal{Z}$，我们将使用Wasserstein Metric的最大形式：

$$\bar{d}_p(Z_1, Z_2) := \sup_{x,a} d_p(Z_1(x, a), Z_2(x, a)).$$

我们将使用$\bar{d}_p$来建立分销钟手运营商的收敛性。

引理2。*$\bar{d}_p$ is a metric over value distributions.*

3.3。政策评估

在*policy evaluation*设置（Sutton＆Barto，1998）中，我们对与给定策略π关联的值函数V^π感兴趣。这里的类似物是值分布Z^π。在本节中，我们表征Z^π并研究策略评估操作员$\mathcal{T}^\pi$的行为。我们认为Z^π描述了代理与环境相互作用的固有随机性，而不是对环境本身的不确定性的某种量度。

我们将奖励函数视为随机向量$R \in \mathcal{Z}$，并定义过渡操作员P^π：$\mathcal{Z} \rightarrow \mathcal{Z}$

$$\begin{aligned} P^\pi Z(x, a) &\overset{D}{:=} Z(X', A') \\ X' &\sim P(\cdot \,|\, x, a),\, A' \sim \pi(\cdot \,|\, X'), \end{aligned} \tag{4}$$

我们使用大写字母来强调下一个状态行动对的随机性（X', A'）。我们定义分销的贝尔曼操作员$\mathcal{T}^\pi$：$\mathcal{Z} \rightarrow \mathcal{Z}$为

$$\mathcal{T}^\pi Z(x, a) \overset{D}{:=} R(x, a) + \gamma P^\pi Z(x, a). \tag{5}$$

$\mathcal{T}^\pi$与通常的贝尔曼操作员（2）具有表面相似之处，但这根本不同。在特定的三种随机性来源中定义化合物分配$\mathcal{T}^\pi Z$:

a) The randomness in the reward R,
b) The randomness in the transition P^π, and
c) The next-state value distribution $Z(X', A')$.

In particular, we make the usual assumption that these three quantities are independent. In this section we will show that (5) is a contraction mapping whose unique fixed point is the random return Z^π.

3.3.1. Contraction in $\bar{d}_p$

Consider the process $Z_{k+1} := \mathcal{T}^\pi Z_k$, starting with some $Z_0 \in \mathcal{Z}$. We may expect the limiting expectation of $\{Z_k\}$ to converge exponentially quickly, as usual, to Q^π. As we now show, the process converges in a stronger sense: $\mathcal{T}^\pi$ is a contraction in $\bar{d}_p$, which implies that all moments also converge exponentially quickly.

Lemma 3. *$\mathcal{T}^\pi : \mathcal{Z} \to \mathcal{Z}$ is a γ-contraction in $\bar{d}_p$.*

Using Lemma 3, we conclude using Banach's fixed point theorem that $\mathcal{T}^\pi$ has a unique fixed point. By inspection, this fixed point must be Z^π as defined in (1). As we assume all moments are bounded, this is sufficient to conclude that the sequence $\{Z_k\}$ converges to Z^π in $\bar{d}_p$ for $1 \leq p \leq \infty$.

To conclude, we remark that not all distributional metrics are equal; for example, Chung & Sobel (1987) have shown that $\mathcal{T}^\pi$ is not a contraction in total variation distance. Similar results can be derived for the Kullback-Leibler divergence and the Kolmogorov distance.

3.3.2. Contraction in Centered Moments

Observe that $d_2(U, V)$ (and more generally, d_p) relates to a coupling $C(\omega) := U(\omega) - V(\omega)$, in the sense that

$$d_2^2(U, V) \leq \mathbb{E}[(U - V)^2] = \mathbb{V}(C) + \big(\mathbb{E}\, C\big)^2.$$

As a result, we cannot directly use d_2 to bound the variance difference $|\mathbb{V}(\mathcal{T}^\pi Z(x, a)) - \mathbb{V}(Z^\pi(x, a))|$. However, $\mathcal{T}^\pi$ is in fact a contraction in variance (Sobel, 1982, see also appendix). In general, $\mathcal{T}^\pi$ is not a contraction in the p^{th} centered moment, $p > 2$, but the centered moments of the iterates $\{Z_k\}$ still converge exponentially quickly to those of Z^π; the proof extends the result of Rösler (1992).

3.4. Control

Thus far we have considered a fixed policy π, and studied the behaviour of its associated operator $\mathcal{T}^\pi$. We now set out to understand the distributional operators of the *control* setting – where we seek a policy π that maximizes value – and the corresponding notion of an optimal value distribution. As with the optimal value function, this notion is intimately tied to that of an optimal policy. However, while all optimal policies attain the same value Q^*, in our case a difficulty arises: in general there are many optimal value distributions.

In this section we show that the distributional analogue of the Bellman optimality operator converges, in a weak sense, to the set of optimal value distributions. However, this operator is not a contraction in any metric between distributions, and is in general much more temperamental than the policy evaluation operators. We believe the convergence issues we outline here are a symptom of the inherent instability of greedy updates, as highlighted by e.g. Tsitsiklis (2002) and most recently Harutyunyan et al. (2016).

Let Π^* be the set of optimal policies. We begin by characterizing what we mean by an *optimal value distribution*.

Definition 1 (Optimal value distribution)**.** *An optimal value distribution is the v.d. of an optimal policy. The set of optimal value distributions is $\mathcal{Z}^* := \{Z^{\pi^*} : \pi^* \in \Pi^*\}$.*

We emphasize that not all value distributions with expectation Q^* are optimal: they must match the full distribution of the return under some optimal policy.

Definition 2. *A greedy policy π for $Z \in \mathcal{Z}$ maximizes the expectation of Z. The set of greedy policies for Z is*

$$\mathcal{G}_Z := \{\pi : \textstyle\sum_a \pi(a \,|\, x)\, \mathbb{E}\, Z(x, a) = \max_{a' \in \mathcal{A}} \mathbb{E}\, Z(x, a')\}.$$

Recall that the expected Bellman optimality operator $\mathcal{T}$ is

$$\mathcal{T} Q(x, a) = \mathbb{E}\, R(x, a) + \gamma\, \mathbb{E}_P \max_{a' \in \mathcal{A}} Q(x', a'). \quad (6)$$

The maximization at x' corresponds to some greedy policy. Although this policy is implicit in (6), we cannot ignore it in the distributional setting. We will call a *distributional Bellman optimality operator* any operator $\mathcal{T}$ which implements a greedy selection rule, i.e.

$$\mathcal{T} Z = \mathcal{T}^\pi Z \text{ for some } \pi \in \mathcal{G}_Z.$$

As in the policy evaluation setting, we are interested in the behaviour of the iterates $Z_{k+1} := \mathcal{T} Z_k$, $Z_0 \in \mathcal{Z}$. Our first result is to assert that $\mathbb{E}\, Z_k$ behaves as expected.

Lemma 4. *Let $Z_1, Z_2 \in \mathcal{Z}$. Then*

$$\left\|\mathbb{E}\, \mathcal{T} Z_1 - \mathbb{E}\, \mathcal{T} Z_2\right\|_\infty \leq \gamma \left\|\mathbb{E}\, Z_1 - \mathbb{E}\, Z_2\right\|_\infty,$$

and in particular $\mathbb{E}\, Z_k \to Q^*$ exponentially quickly.

By inspecting Lemma 4, we might expect that Z_k converges quickly in $\bar{d}_p$ to some fixed point in $\mathcal{Z}^*$. Unfortunately, convergence is neither quick nor assured to reach a fixed point. In fact, the best we can hope for is pointwise convergence, not even to the set $\mathcal{Z}^*$ but to the larger set of *nonstationary optimal value distributions*.

a）奖励R，b）过渡P^π和c）的随机性，c）
下一态值分布$Z(X', A')$。

特别是，我们通常假设这三个数量是独立的。在本节中，我们将证明（5）是一个收缩映射，其唯一固定点是随机返回Z^π。

3.3.1。$\bar{d}_p$中的收缩

考虑过程Z_{k+1}：$= \mathcal{T}^\pi Z_k$，从一些$Z_0 \in \mathcal{Z}$开始。我们可能希望$\{Z_k\}$的限制期望能像往常一样快速地汇总为Q^π。正如我们现在显示的那样，该过程从更强的意义上收敛：$\mathcal{T}^\pi$是$\bar{d}_p$的收缩，这意味着所有矩也很快就会成倍收敛。

引理3。$\mathcal{T}^\pi$：$\mathcal{Z} \to \mathcal{Z}$ *is a* γ*-contraction in* $\bar{d}_p$*.*

使用引理3，我们使用Banach的固定点定理得出结论，$\mathcal{T}^\pi$具有独特的固定点。通过检查，该固定点必须按（1）中定义为Z^π。正如我们假设所有矩都是有界的，这是有足够的结论，可以得出结论，$\{Z_k\}$的序列$\bar{d}_p$在$\bar{d}_p$中收敛到$1 \le p \le \infty$。

总而言之，我们指出，并非所有的分布指标都是平等的。例如，Chung＆Sobel（1987）表明$\mathcal{T}^\pi$在总变化距离上不是收缩。可以为Kullback-Leibler Divergence和Kolmogorov距离得出模拟结果。

3.3.2。以中心的时刻收缩

观察$d_2(U, V)$ (，更普遍地，d_p)与一个耦合$C(\omega)$：$= U(\omega) - V(\omega)$，从某种意义上说

$$d_2^2(U,V) \le \mathbb{E}[(U-V)^2] = \mathbb{V}(C) + \big(\mathbb{E}\, C\big)^2.$$

结果，我们不能直接使用d_2来绑定方差差$|\mathbb{V}(\mathcal{T}^\pi Z(x,a)) - \mathbb{V}(Z^\pi(x,a))|$。但是，$\mathcal{T}^\pi$实际上是方差的收缩（Sobel，1982，另请参见附录）。通常，$\mathcal{T}^\pi$并不是p^{th}居中的时刻，$p > 2$的收缩，但是迭代的焦点$\{Z_k\}$仍然会迅速地与Z^π的矩阵呈指数融合；证明扩展了R ̈Osler（1992）的结果。

3.4。控制

到目前为止，我们已经考虑了固定的策略π，并研究了其关联的操作员$\mathcal{T}^\pi$的行为。现在，我们着手了解*control*设置的分布运算符 - 我们寻求最大化价值的策略π以及最佳价值分布的相应概念。与最佳价值函数一样，此概念与最佳策略的概念密切相关。但是，尽管所有最佳策略都达到相同的值Q^*，但在我们的情况下很难出现：总的来说，有许多最佳价值分布。

在本节中，我们表明，贝尔曼最佳算法的分布类似物在薄弱的意义上会收敛于一组最佳价值分布。但是，该操作员在分配之间的任何指标中都不是收缩，并且通常比政策评估运营商更气质。我们认为，我们在这里概述的趋势问题是贪婪更新固有不稳定的症状，例如Tsitsiklis（2002）和最近的Harutyunyan等人。（2016）。

令Π^*为最佳策略集。我们首先要描述
*optimal value distribution*的含义。

定义1（最佳值分布）。*An optimal value distribution is the v.d. of an optimal policy. The set of optimal value distributions is* $\mathcal{Z}^*$：$= \{Z^{\pi^*}$：$\pi^* \in \Pi^*\}$.

我们强调的是，并非所有具有预期Q^*的价值分布是最佳的：它们必须在某些最佳策略下与退货的完整分布匹配。

定义2。*A greedy policy* π *for* $Z \in \mathcal{Z}$ *maximizes the expectation of* Z*. The set of greedy policies for* Z *is*

$$\mathcal{G}_Z := \{\pi : \sum_a \pi(a\,|\,x)\,\mathbb{E}\,Z(x,a) = \max_{a'\in\mathcal{A}} \mathbb{E}\,Z(x,a')\}.$$

回想一下，预期的贝尔曼最佳操作员$\mathcal{T}$是

$$\mathcal{T}Q(x,a) = \mathbb{E}\,R(x,a) + \gamma\,\mathbb{E}_P \max_{a'\in\mathcal{A}} Q(x',a'). \qquad (6)$$

x'的最大化对应于某些贪婪的策略。尽管此策略在（6）中是隐含的，但我们不能在分配环境中忽略它。我们将称为*distributional Bellman optimality operator*任何操作员$\mathcal{T}$，该操作员将其赋予贪婪的选择规则，即

$$\mathcal{T}Z = \mathcal{T}^\pi Z \text{ for some } \pi \in \mathcal{G}_Z.$$

与策略评估设置一样，我们对迭代Z_{k+1}的行为感兴趣：$= \mathcal{T}Z_k$，$Z_0 \in \mathcal{Z}$。我们的第一个结果是断言$\mathbb{E}\,Z_k$的行为如预期。

引理4。*Let* $Z_1, Z_2 \in \mathcal{Z}$*. Then*

$$\|\mathbb{E}\,\mathcal{T}Z_1 - \mathbb{E}\,\mathcal{T}Z_2\|_\infty \le \gamma\,\|\mathbb{E}\,Z_1 - \mathbb{E}\,Z_2\|_\infty\,,$$

and in particular $\mathbb{E}\,Z_k \to Q^*$ *exponentially quickly.*

通过检查引理4，我们可能希望Z_k在$\bar{d}_p$中快速到$\mathcal{Z}^*$中的某个固定点。不胜非想地，收敛既不是快速或确保达到固定点的。实际上，我们所希望的最好的是侧重融合，甚至不是集合$\mathcal{Z}^*$，而是与较大的
*nonstationary optimal value distributions*集合。

Definition 3. *A nonstationary optimal value distribution Z^{**} is the value distribution corresponding to a sequence of optimal policies. The set of n.o.v.d. is $\mathcal{Z}^{**}$.*

Theorem 1 (Convergence in the control setting)**.** *Let $\mathcal{X}$ be measurable and suppose that $\mathcal{A}$ is finite. Then*

$$\lim_{k\to\infty} \inf_{Z^{**}\in\mathcal{Z}^{**}} d_p(Z_k(x,a), Z^{**}(x,a)) = 0 \quad \forall x, a.$$

If $\mathcal{X}$ is finite, then Z_k converges to $\mathcal{Z}^{**}$ uniformly. Furthermore, if there is a total ordering $\prec$ on Π^*, such that for any $Z^* \in \mathcal{Z}^*$,

$$\mathcal{T} Z^* = \mathcal{T}^\pi Z^* \text{ with } \pi \in \mathcal{G}_{Z^*},\ \pi \prec \pi' \ \ \forall \pi' \in \mathcal{G}_{Z^*} \setminus \{\pi\}.$$

Then $\mathcal{T}$ has a unique fixed point $Z^* \in \mathcal{Z}^*$.

Comparing Theorem 1 to Lemma 4 reveals a significant difference between the distributional framework and the usual setting of expected return. While the mean of Z_k converges exponentially quickly to Q^*, its distribution need not be as well-behaved! To emphasize this difference, we now provide a number of negative results concerning $\mathcal{T}$.

Proposition 1. *The operator $\mathcal{T}$ is not a contraction.*

Consider the following example (Figure 2, left). There are two states, x_1 and x_2; a unique transition from x_1 to x_2; from x_2, action a_1 yields no reward, while the optimal action a_2 yields $1 + \epsilon$ or $-1 + \epsilon$ with equal probability. Both actions are terminal. There is a unique optimal policy and therefore a unique fixed point Z^*. Now consider Z as given in Figure 2 (right), and its distance to Z^*:

$$\bar{d}_1(Z, Z^*) = d_1(Z(x_2, a_2), Z^*(x_2, a_2)) = 2\epsilon,$$

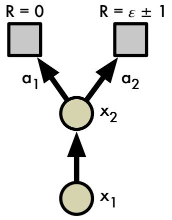
where we made use of the fact that $Z = Z^*$ everywhere except at (x_2, a_2). When we apply $\mathcal{T}$ to Z, however, the greedy action a_1 is selected and $\mathcal{T}Z(x_1) = Z(x_2, a_1)$. But

$$\begin{aligned} d_1(\mathcal{T}Z, \mathcal{T}Z^*) &= d_1(\mathcal{T}Z(x_1), Z^*(x_1)) \\ &= \tfrac{1}{2}|1-\epsilon| + \tfrac{1}{2}|1+\epsilon| > 2\epsilon \end{aligned}$$

for a sufficiently small ϵ. This shows that the undiscounted update is not a nonexpansion: $\bar{d}_1(\mathcal{T}Z, \mathcal{T}Z^*) > \bar{d}_1(Z, Z^*)$. With $\gamma < 1$, the same proof shows it is not a contraction. Using a more technically involved argument, we can extend this result to any metric which separates Z and $\mathcal{T}Z$.

Proposition 2. *Not all optimality operators have a fixed point $Z^* = \mathcal{T}Z^*$.*

To see this, consider the same example, now with $\epsilon = 0$, and a greedy operator $\mathcal{T}$ which breaks ties by picking a_2 if $Z(x_1) = 0$, and a_1 otherwise. Then the sequence $\mathcal{T}Z^*(x_1), (\mathcal{T})^2 Z^*(x_1), \ldots$ alternates between $Z^*(x_2, a_1)$ and $Z^*(x_2, a_2)$.



	x_1	x_2, a_1	x_2, a_2
Z^*	$\epsilon \pm 1$	0	$\epsilon \pm 1$
Z	$\epsilon \pm 1$	0	$-\epsilon \pm 1$
$\mathcal{T}Z$	0	0	$\epsilon \pm 1$

Figure 2. Undiscounted two-state MDP for which the optimality operator $\mathcal{T}$ is not a contraction, with example. The entries that contribute to $\bar{d}_1(Z, Z^*)$ and $\bar{d}_1(\mathcal{T}Z, Z^*)$ are highlighted.

Proposition 3. *That $\mathcal{T}$ has a fixed point $Z^* = \mathcal{T}Z^*$ is insufficient to guarantee the convergence of $\{Z_k\}$ to $\mathcal{Z}^*$.*

Theorem 1 paints a rather bleak picture of the control setting. It remains to be seen whether the dynamical eccentricies highlighted here actually arise in practice. One open question is whether theoretically more stable behaviour can be derived using stochastic policies, for example from conservative policy iteration (Kakade & Langford, 2002).

4. Approximate Distributional Learning

In this section we propose an algorithm based on the distributional Bellman optimality operator. In particular, this will require choosing an approximating distribution. Although the Gaussian case has previously been considered (Morimura et al., 2010a; Tamar et al., 2016), to the best of our knowledge we are the first to use a rich class of parametric distributions.

4.1. Parametric Distribution

We will model the value distribution using a discrete distribution parametrized by $N \in \mathbb{N}$ and $V_{\text{MIN}}, V_{\text{MAX}} \in \mathbb{R}$, and whose support is the set of atoms $\{z_i = V_{\text{MIN}} + i\triangle z : 0 \leq i < N\}$, $\triangle z := \frac{V_{\text{MAX}} - V_{\text{MIN}}}{N-1}$. In a sense, these atoms are the "canonical returns" of our distribution. The atom probabilities are given by a parametric model $\theta : \mathcal{X} \times \mathcal{A} \to \mathbb{R}^N$

$$Z_\theta(x, a) = z_i \quad \text{w.p. } \ p_i(x, a) := \frac{e^{\theta_i(x,a)}}{\sum_j e^{\theta_j(x,a)}}.$$

The discrete distribution has the advantages of being highly expressive and computationally friendly (see e.g. Van den Oord et al., 2016).

4.2. Projected Bellman Update

Using a discrete distribution poses a problem: the Bellman update $\mathcal{T}Z_\theta$ and our parametrization Z_θ almost always have disjoint supports. From the analysis of Section 3 it would seem natural to minimize the Wasserstein metric (viewed as a loss) between $\mathcal{T}Z_\theta$ and Z_θ, which is also

定义3。*A nonstationary optimal value distribution Z^{**} is the value distribution corresponding to a sequence of optimal policies. The set of n.o.v.d. is $\mathcal{Z}^{**}$.*

定理1（控制设置中的收敛）。*Let $\mathcal{X}$ be measurable and suppose that $\mathcal{A}$ is finite. Then*

$$\lim_{k\to\infty} \inf_{Z^{**}\in\mathcal{Z}^{**}} d_p(Z_k(x,a), Z^{**}(x,a)) = 0 \quad \forall x, a.$$

If $\mathcal{X}$ is finite, then Z_k converges to $\mathcal{Z}^{**}$ uniformly. Furthermore, if there is a total ordering $\prec$ on Π^*, such that for any $Z^* \in \mathcal{Z}^*$,

$$\mathcal{T}Z^* = \mathcal{T}^{\pi}Z^* \textit{ with } \pi \in \mathcal{G}_{Z^*},\ \pi \prec \pi' \quad \forall \pi' \in \mathcal{G}_{Z^*} \setminus \{\pi\}.$$

Then $\mathcal{T}$ has a unique fixed point $Z^* \in \mathcal{Z}^*$.

将定理1与引理4进行比较，揭示了分布框架与通常的预期收益设置之间的显着差异。 Z_k的平均值会迅速收敛到Q^*，但它的分布不一定是良好的！为了强调这种差异，我们现在提供了许多有关$\mathcal{T}$的负面结果。

命题1。*The operator $\mathcal{T}$ is not a contraction.*

考虑以下示例（图2，左）。有两个状态，x_1和x_2;从x_1到x_2的唯一过渡;从x_2，Action a_1不会产生奖励，而最佳影响a_2产生$1+\epsilon$或$-1+\epsilon$，具有均等的可能性。这两个动作都是终端。有一个独特的最佳策略，因此有一个唯一的固定点Z^*。现在考虑图2（右）中给出的Z，其距离Z^*：

$$\bar{d}_1(Z, Z^*) = d_1(Z(x_2,a_2), Z^*(x_2,a_2)) = 2\epsilon,$$

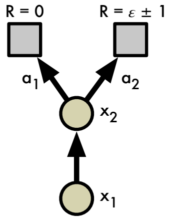
除了（x_2, a_2）以外，我们利用$Z = Z^*$到处的事实。但是，当我们将$\mathcal{T}$应用于Z时，选择了贪婪的动作a_1并$\mathcal{T}Z(x_1) = Z(x_2,a_1)$。但

$$\begin{aligned} d_1(\mathcal{T}Z, \mathcal{T}Z^*) &= d_1(\mathcal{T}Z(x_1), Z^*(x_1)) \\ &= \tfrac{1}{2}|1-\epsilon| + \tfrac{1}{2}|1+\epsilon| > 2\epsilon \end{aligned}$$

对于有足够的小ϵ。这表明未估计的更新不是一个非跨性别的：$\bar{d}_1(\mathcal{T}Z, \mathcal{T}Z^*) > \bar{d}_1(Z, Z^*)$。使用$\gamma < 1$，相同的证明表明它不是收缩。使用更技术上的参数，我们可以将此结果扩展到将Z和$\mathcal{T}Z$分开的任何度量。

命题2。*Not all optimality operators have a fixed point $Z^* = \mathcal{T}Z^*$.*

要查看这一点，请考虑使用$\epsilon = 0$的同一示例，以及一个贪婪的操作员$\mathcal{T}$，该$\mathcal{T}$如果$Z(x_1)$ $= 0$，则通过选择a_2的打破领带，否则$Z(x_1)$ $= 0$，a_1}否则。然后序列$\mathcal{T}Z^*(x_1), (\mathcal{T})^2Z^*(x_1), \ldots$在$Z^*(x_2,a_1)$和$Z^*(x_2,a_2)$之间交替。



	x_1	x_2, a_1	x_2, a_2
Z^*	$\epsilon \pm 1$	0	$\epsilon \pm 1$
Z	$\epsilon \pm 1$	0	$-\epsilon \pm 1$
$\mathcal{T}Z$	0	0	$\epsilon \pm 1$

图2。未交流的两态MDP，最佳操作员$\mathcal{T}$并不是一个缩写。突出显示了有助于$\bar{d}_1(Z, Z^*)$和$\bar{d}_1(\mathcal{T}Z, Z^*)$的条目。

命题3。*That $\mathcal{T}$ has a fixed point $Z^* = \mathcal{T}Z^*$ is insufficient to guarantee the convergence of $\{Z_k\}$ to $\mathcal{Z}^*$.*

定理1描绘了控制设置的相当黯淡的图片。在实践中实际上出现了动态化的影响，这还有待观察。一个开放的问题是，从理论上讲，是否可以使用随机策略来得出更稳定的行为，例如从保密政策迭代（Kakade＆Langford，2002）。

4。近似分布学习

在本节中，我们提出了一种基于分歧式贝尔曼最佳操作员的算法。特别是，这将需要选择一个近似分布。尽管以前已经考虑过高斯案例（Morimura等，2010a; Tamar等，2016），据我们所知，我们首先是使用丰富的守则分布。

4.1。参数分布

我们将使用由$N \in \mathbb{N}$和$V_{\text{MIN}}, V_{\text{MAX}} \in \mathbb{R}$参数参数的离散分布来对价值分布进行建模，其支持是原子$\{z_i = V_{\text{MIN}} + i\triangle z\text{：}\ 0 \le i < N\}$，$\triangle z$，$\triangle z$：$= \frac{V_{\text{MAX}} - V_{\text{MIN}}}{N-1}$。从某种意义上说，这些原子是我们分布的“规范回报”。原子概率由参数模型θ给出：$\mathcal{X} \times \mathcal{A} \to \mathbb{R}^N$

$$Z_\theta(x,a) = z_i \quad \text{w.p.} \;\; p_i(x,a) := \frac{e^{\theta_i(x,a)}}{\sum_j e^{\theta_j(x,a)}}.$$

离散分布具有高度表达和计算友好的优势（例如，参见van den oord等，2016）。

4.2。投影的贝尔曼更新

使用离散分发提出了一个问题：贝尔曼更新$\mathcal{T}Z_\theta$，我们的参数化Z_θ几乎具有不相交的支持。根据第3节的分析，将$\mathcal{T}Z_\theta$和Z_θ之间的Wasserstein Metic（被视为损失）最小化似乎很自然，这也是

conveniently robust to discrepancies in support. However, a second issue prevents this: in practice we are typically restricted to learning from sample transitions, which is not possible under the Wasserstein loss (see Prop. 5 and toy results in the appendix).

Instead, we project the sample Bellman update $\hat{\mathcal{T}} Z_\theta$ onto the support of Z_θ (Figure 1, Algorithm 1), effectively reducing the Bellman update to multiclass classification. Let π be the greedy policy w.r.t. $\mathbb{E}\, Z_\theta$. Given a sample transition (x, a, r, x'), we compute the Bellman update $\hat{\mathcal{T}} z_j := r + \gamma z_j$ for each atom z_j, then distribute its probability $p_j(x', \pi(x'))$ to the immediate neighbours of $\hat{\mathcal{T}} z_j$. The i^{th} component of the projected update $\Phi \hat{\mathcal{T}} Z_\theta(x, a)$ is

$$(\Phi\hat{\mathcal{T}} Z_\theta(x,a))_i = \sum_{j=0}^{N-1} \left[1 - \frac{|[\hat{\mathcal{T}} z_j]_{V_{\text{MIN}}}^{V_{\text{MAX}}} - z_i|}{\triangle z}\right]_0^1 p_j(x', \pi(x')), \tag{7}$$

where $[\cdot]_a^b$ bounds its argument in the range $[a, b]$.[1] As is usual, we view the next-state distribution as parametrized by a fixed parameter $\tilde{\theta}$. The sample loss $\mathcal{L}_{x,a}(\theta)$ is the cross-entropy term of the KL divergence

$$D_{\text{KL}}(\Phi\hat{\mathcal{T}} Z_{\tilde{\theta}}(x,a) \,\|\, Z_\theta(x,a)),$$

which is readily minimized e.g. using gradient descent. We call this choice of distribution and loss the *categorical algorithm*. When $N = 2$, a simple one-parameter alternative is $\Phi\hat{\mathcal{T}} Z_\theta(x,a) := [\mathbb{E}[\hat{\mathcal{T}} Z_\theta(x,a)] - V_{\text{MIN}})/\triangle z]_0^1$; we call this the *Bernoulli algorithm*. We note that, while these algorithms appear unrelated to the Wasserstein metric, recent work (Bellemare et al., 2017) hints at a deeper connection.

Algorithm 1 Categorical Algorithm

```
input  A transition $x_t, a_t, r_t, x_{t+1}, \gamma_t \in [0, 1]$
  $Q(x_{t+1}, a) := \sum_i z_i p_i(x_{t+1}, a)$
  $a^* \leftarrow \arg\max_a Q(x_{t+1}, a)$
  $m_i = 0, \quad i \in 0, \dots, N-1$
  for $j \in 0, \dots, N-1$ do
    # Compute the projection of $\hat{\mathcal{T}} z_j$ onto the support $\{z_i\}$
    $\hat{\mathcal{T}} z_j \leftarrow [r_t + \gamma_t z_j]_{V_{\text{MIN}}}^{V_{\text{MAX}}}$
    $b_j \leftarrow (\hat{\mathcal{T}} z_j - V_{\text{MIN}})/\Delta z$   # $b_j \in [0, N-1]$
    $l \leftarrow \lfloor b_j \rfloor, u \leftarrow \lceil b_j \rceil$
    # Distribute probability of $\hat{\mathcal{T}} z_j$
    $m_l \leftarrow m_l + p_j(x_{t+1}, a^*)(u - b_j)$
    $m_u \leftarrow m_u + p_j(x_{t+1}, a^*)(b_j - l)$
  end for
output  $-\sum_i m_i \log p_i(x_t, a_t)$   # Cross-entropy loss
```

5. Evaluation on Atari 2600 Games

To understand the approach in a complex setting, we applied the categorical algorithm to games from the Arcade Learning Environment (ALE; Bellemare et al., 2013). While the ALE is deterministic, stochasticity does occur in a number of guises: 1) from state aliasing, 2) learning from a nonstationary policy, and 3) from approximation errors. We used five training games (Fig 3) and 52 testing games.

For our study, we use the DQN architecture (Mnih et al., 2015), but output the atom probabilities $p_i(x, a)$ instead of action-values, and chose $V_{\text{MAX}} = -V_{\text{MIN}} = 10$ from preliminary experiments over the training games. We call the resulting architecture *Categorical DQN*. We replace the squared loss $(r + \gamma Q(x', \pi(x')) - Q(x, a))^2$ by $\mathcal{L}_{x,a}(\theta)$ and train the network to minimize this loss.[2] As in DQN, we use a simple ϵ-greedy policy over the expected action-values; we leave as future work the many ways in which an agent could select actions on the basis of the full distribution. The rest of our training regime matches Mnih et al.'s, including the use of a target network for $\tilde{\theta}$.

Figure 4 illustrates the typical value distributions we observed in our experiments. In this example, three actions (those including the button press) lead to the agent releasing its laser too early and eventually losing the game. The corresponding distributions reflect this: they assign a significant probability to 0 (the terminal value). The safe actions have similar distributions (LEFT, which tracks the invaders' movement, is slightly favoured). This example helps explain why our approach is so successful: the distributional update keeps separated the low-value, "losing" event from the high-value, "survival" event, rather than average them into one (unrealizable) expectation.[3]

One surprising fact is that the distributions are not concentrated on one or two values, in spite of the ALE's determinism, but are often close to Gaussians. We believe this is due to our discretizing the diffusion process induced by γ.

5.1. Varying the Number of Atoms

We began by studying our algorithm's performance on the training games in relation to the number of atoms (Figure 3). For this experiment, we set $\epsilon = 0.05$. From the data, it is clear that using too few atoms can lead to poor behaviour, and that more always increases performance; this is not immediately obvious as we may have expected to saturate the network capacity. The difference in performance between the 51-atom version and DQN is particularly striking: the latter is outperformed in all five games, and in SEAQUEST we attain state-of-the-art performance. As an additional point of the comparison, the single-parameter Bernoulli algorithm performs better than DQN in 3 games out of 5, and is most notably more robust in ASTERIX.

[1] Algorithm 1 computes this projection in time linear in N.

[2] For $N = 51$, our TensorFlow implementation trains at roughly 75% of DQN's speed.

[3] Video: http://youtu.be/yFBwyPuO2Vg.

方便地对支持的差异非常强大。但是，第二个问题阻止了这一点：实际上，我们通常仅限于从样本过渡中学习，这是在瓦斯坦损失下不可能的（请参阅附录5和玩具结果）。

取而代之的是，我们将示例Bellman Update$\hat{\mathcal{T}} Z_\theta$投影到$Z_\theta$ (图1，算法1)的支持上，有效地将Bellman更新重新定义为多类分类。令π为贪婪的策略W.R.T. $\mathbb{E}\, Z_\theta$。给定样品转移（x, a, r, x'），我们为每个原子z_j计算Bellman Update$\hat{\mathcal{T}} z_j \hat{\mathcal{T}} z_j\, r + \gamma z_j$，然后将其概率$p_j(x', \pi(x'))$分配给$\hat{\mathcal{T}} z_j$的直接邻居。投影更新的$i^{th}$组件$\Phi \hat{\mathcal{T}} Z_\theta(x, a)$是

$$(\Phi \hat{\mathcal{T}} Z_\theta(x, a))_i = \sum_{j=0}^{N-1} \left[1 - \frac{|[\hat{\mathcal{T}} z_j]_{V_{\text{MIN}}}^{V_{\text{MAX}}} - z_i|}{\triangle z} \right]_0^1 p_j(x', \pi(x')), \tag{7}$$

其中$[\cdot]\,_a^b$在$[a, b]$范围内界定其参数。[1]与通常一样，我们查看了由xuded参数$\tilde{\theta}$参数化的下一个状态分布。样本损失$\mathcal{L}_{x,a}(\theta)$是kl发散的跨渗透项

$$D_{\text{KL}}(\Phi \hat{\mathcal{T}} Z_{\tilde{\theta}}(x, a) \,\|\, Z_\theta(x, a)),$$

这很容易最小化，例如使用梯度下降。我们称这种分配和损失的选择为*categorical al- gorithm*。当$N = 2$时，一个简单的单参数替代方案是$\Phi \hat{\mathcal{T}} Z_\theta(x, a)$：$= [\mathbb{E}\,[\hat{\mathcal{T}} Z_\theta(x, a)] - V_{\text{MIN}}$）$- V_{\text{MIN}}$）$/\triangle z]\,_0^1$;我们将其称为*Bernoulli algorithm*。我们注意到，尽管这些属性似乎与Wasserstein指标无关，但最近的工作（Bellemare等，2017）暗示了更深的联系。

Algorithm 1 Categorical Algorithm

```
input  A transition $x_t, a_t, r_t, x_{t+1}, \gamma_t \in [0, 1]$
  $Q(x_{t+1}, a) := \sum_i z_i p_i(x_{t+1}, a)$
  $a^* \leftarrow \arg\max_a Q(x_{t+1}, a)$
  $m_i = 0, \quad i \in 0, \dots, N-1$
  for $j \in 0, \dots, N-1$ do
    # Compute the projection of $\hat{\mathcal{T}} z_j$ onto the support $\{z_i\}$
    $\hat{\mathcal{T}} z_j \leftarrow [r_t + \gamma_t z_j]_{V_{\text{MIN}}}^{V_{\text{MAX}}}$
    $b_j \leftarrow (\hat{\mathcal{T}} z_j - V_{\text{MIN}})/\Delta z$   # $b_j \in [0, N-1]$
    $l \leftarrow \lfloor b_j \rfloor, u \leftarrow \lceil b_j \rceil$
    # Distribute probability of $\hat{\mathcal{T}} z_j$
    $m_l \leftarrow m_l + p_j(x_{t+1}, a^*)(u - b_j)$
    $m_u \leftarrow m_u + p_j(x_{t+1}, a^*)(b_j - l)$
  end for
output  $-\sum_i m_i \log p_i(x_t, a_t)$   # Cross-entropy loss
```

5。对Atari 2600游戏的评估

为了理解复杂的环境中的方法，我们将分类算法与AR的游戏相关联Cade学习环境（ALE; Bellemare等，2013）。虽然啤酒是确定性的，但随机性确实发生在许多伪造中：1）从状态混轴承，2）从非组织政策中学习和3）从近似错误中学习。我们使用了五个训练游戏（图3）和52场测试游戏。

在我们的研究中，我们使用DQN体系结构（Mnih等，2015），但输出原子概率$p_i(x, a)$而不是动作值，然后从训练游戏中的初步实验中选择$V_{\text{MAX}} = -V_{\text{MIN}} = 10$。我们称之为生成的体系结构*Categorical DQN*。我们用$\mathcal{L}_{x,a}(\theta)$替换平方损失（$r + \gamma Q(x', \pi(x')) - Q(x, a)$）[2]，然后训练网络以最小化此损失。[2]如DQN中，我们使用的是一个简单的ϵ - 固定策略，对预期的动作值；我们将作为未来的工作离开，代理商可以根据完整的分布选择行动的许多方式。我们其余的培训制度与Mnih等人的匹配，包括将目标网络用于$\tilde{\theta}$。

图4说明了我们在实验中所遵循的典型价值分布。在此示例中，三个动作（包括按钮按下的动作）导致代理会过早释放其激光，并最终失去游戏。相应的分布反映了这一点：他们为0（终端值）分配了一个可能的概率。安全的动作具有相似的分布（左侧跟踪入侵者运动的左侧是有些青睐的）。这个示例有助于解释为什么我们的方法如此成功：分歧更新使低价值，“失去”事件与高价值，“生存”事件分开。[3]

一个令人惊讶的事实是，尽管淡啤酒的确定性iSM，但通常与高斯人相近，但并未对一个或两个值进行汇总。我们认为这是由于我们离散的γ引起的扩散过程。

5.1。改变原子数

我们首先研究了与原子数量有关的算法在训练游戏中的表现（图3）。对于此实验，我们设置$\epsilon = 0\,.\,05$。从数据来看，很明显，使用过多的原子会导致行为不良，并且更多地会提高性能。这并不明显，因为我们可能期望将网络容量饱和。 51个原子版本和DQN之间的性能差异特别引人注目：后者在所有五场比赛中都表现出色，而在Seaquest中，我们获得了最先进的性能。作为比较的另一个点，单参数Bernoulli al-Gorithm在5场中的3场比赛中的表现优于DQN，并且最著名的是Asterix更健壮。

[1] Algorithm 1 computes this projection in time linear in N.

[2] For $N = 51$, our TensorFlow implementation trains at roughly 75% of DQN's speed.

[3] Video: `http://youtu.be/yFBwyPuO2Vg`.

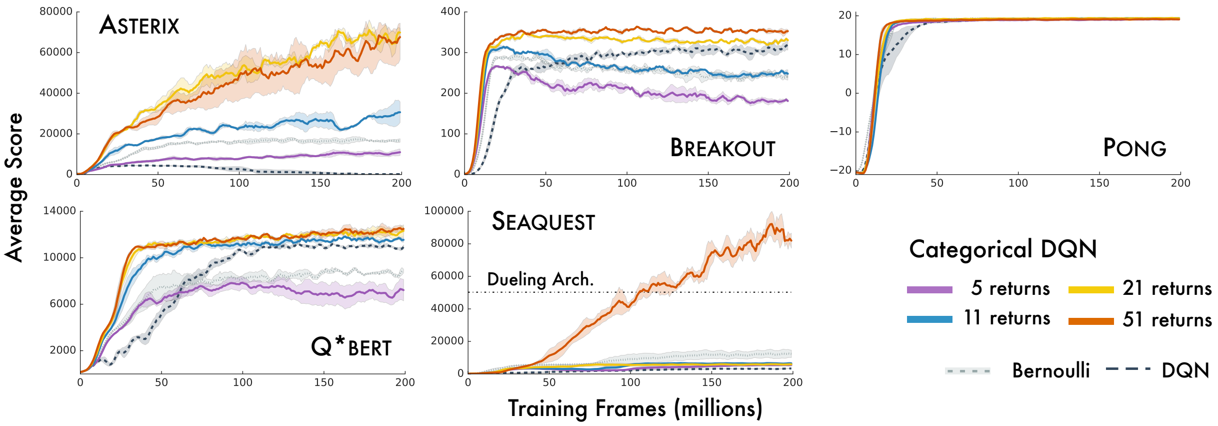


Figure 3. Categorical DQN: Varying number of atoms in the discrete distribution. Scores are moving averages over 5 million frames.

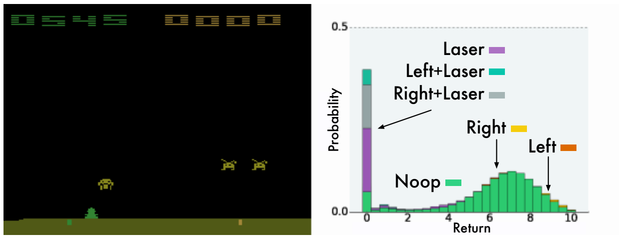


Figure 4. Learned value distribution during an episode of SPACE INVADERS. Different actions are shaded different colours. Returns below 0 (which do not occur in SPACE INVADERS) are not shown here as the agent assigns virtually no probability to them.

One interesting outcome of this experiment was to find out that our method does pick up on stochasticity. PONG exhibits intrinsic randomness: the exact timing of the reward depends on internal registers and is truly unobservable. We see this clearly reflected in the agent's prediction (Figure 5): over five consecutive frames, the value distribution shows two modes indicating the agent's belief that it has yet to receive a reward. Interestingly, since the agent's state does not include past rewards, it cannot even extinguish the prediction after receiving the reward, explaining the relative proportions of the modes.

5.2. State-of-the-Art Results

The performance of the 51-atom agent (from here onwards, C51) on the training games, presented in the last section, is particularly remarkable given that it involved none of the other algorithmic ideas present in state-of-the-art agents. We next asked whether incorporating the most common hyperparameter choice, namely a smaller training ϵ, could lead to even better results. Specifically, we set $\epsilon = 0.01$ (instead of 0.05); furthermore, every 1 million frames, we evaluate our agent's performance with $\epsilon = 0.001$.

We compare our algorithm to DQN ($\epsilon = 0.01$), Double DQN (van Hasselt et al., 2016), the Dueling architecture (Wang et al., 2016), and Prioritized Replay (Schaul et al., 2016), comparing the best evaluation score achieved during training. We see that C51 significantly outperforms these other algorithms (Figures 6 and 7). In fact, C51 surpasses the current state-of-the-art by a large margin in a number of games, most notably SEAQUEST. One particularly striking fact is the algorithm's good performance on sparse reward games, for example VENTURE and PRIVATE EYE. This suggests that value distributions are better able to propagate rarely occurring events. Full results are provided in the appendix.

We also include in the appendix (Figure 12) a comparison, averaged over 3 seeds, showing the number of games in which C51's training performance outperforms fully-trained DQN and human players. These results continue to show dramatic improvements, and are more representative of an agent's average performance. Within 50 million frames, C51 has outperformed a fully trained DQN agent on 45 out of 57 games. This suggests that the full 200 million training frames, and its ensuing computational cost, are unnecessary for evaluating reinforcement learning algorithms within the ALE.

The most recent version of the ALE contains a stochastic execution mechanism designed to ward against trajectory overfitting.Specifically, on each frame the environment rejects the agent's selected action with probability $p = 0.25$. Although DQN is mostly robust to stochastic execution, there are a few games in which its performance is reduced. On a score scale normalized with respect to the random and DQN agents, C51 obtains mean and median score improvements of 126% and 21.5% respectively, confirming the benefits of C51 beyond the deterministic setting.

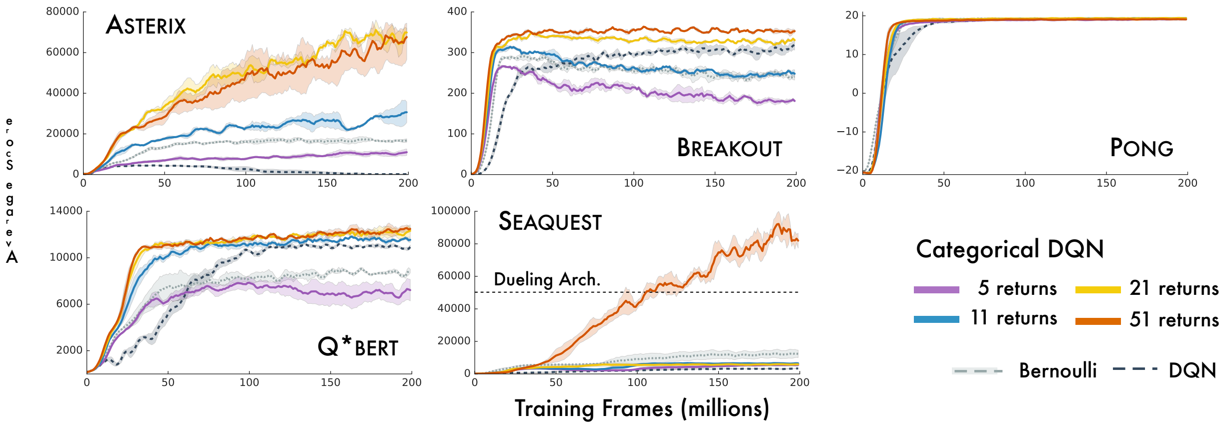


图3。分类DQN：离散分布中的原子数量不同。分数正在移动超过500万帧的平均值。

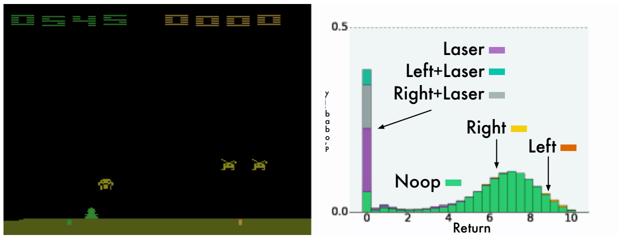


图4。在太空入侵者的一集中学习的价值分布。不同的动作被阴影不同。重新转弯低于0（在空间入侵者中不发生），因为代理几乎没有向其分配概率。

该实验的一个有趣的结果是找出我们的方法确实在随机性方面提取。乒乓球表现出内在的随机性：重新定位的确切时机取决于内部寄存器，并且确实无法观察到。我们看到这在代理的预测中明显反映了这一点（图5）：连续五个框架，价值分布显示了两种模式，表明代理人相信它尚未获得奖励。有趣的是，由于代理商的状态不包括过去的奖励，因此在收到奖励后甚至无法占用预测，从而解释了模式的相对比例。

5.2。最先进的结果

上一节中介绍的51个原子代理（从此处开始，C51）在培训游戏中的性能尤其引人注目，鉴于它没有涉及到最先进的代理商中其他算法的想法。接下来，我们询问合并最常见的超级参数选择，即较小的培训ϵ是否会带来更好的结果。具体而言，我们设置$\epsilon = 0.01$（而不是0.05）;此外，每100万帧，我们用$\epsilon = 0.001$评估我们的代理商的性能。

我们将算法与DQN（$\epsilon = 0.01$），Double DQN（Van Hasselt等，2016），决斗结构（Wang等，2016）和优先级重播（Schaul等，，2016年），比较培训期间达到的最佳评估评分。我们看到C51显着胜过这些其他算法（图6和7）。实际上，在许多游戏中，C51超过了当前的最新水平，最著名的是Seaquest。一个特别惊人的事实是该算法在稀疏奖励游戏中的良好表现，例如冒险和私人眼睛。这表明价值分布能够更好地提出很少发生的事件。附录中提供了完整的结果。

我们还包括一个比较的附录（图12），平均有3种种子，显示了C51的训练性能优于训练有素的DQN和人类玩家的游戏数量。这些结果继续显示出巨大的改进，并且更代表了代理商的平均表现。在5000万帧之内，C51在57场比赛中的45场比赛中表现出了全面训练的DQN代理。这表明，整个200万个训练框架及其随之而来的计算成本对于评估啤酒中的强化学习算法是不必要的。

ALE的最新版本包含一种随机执行机制，旨在防止轨迹上的拟合轨迹。确切地，在每个帧中，环境都以概率$p = 0.25$的概率重新启用了代理所选的动作。尽管DQN大多对随机执行非常强大，但在一些游戏中，其性能会降低。在相对于随机和DQN代理的得分量表上，C51分别获得了平均值和中位数得分，分别为126%和21.5%，确认了超出确定性环境的C51的好处。

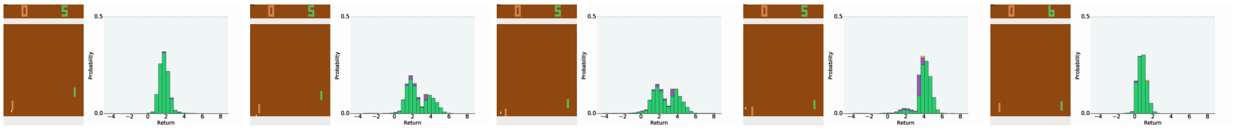


Figure 5. Intrinsic stochasticity in PONG.

	Mean	Median	> H.B.	> DQN
DQN	228%	79%	24	0
DDQN	307%	118%	33	43
DUEL.	373%	151%	37	50
PRIOR.	434%	124%	39	48
PR. DUEL.	592%	172%	39	44
C51	**701%**	**178%**	**40**	**50**
UNREAL[†]	880%	250%	-	-

Figure 6. Mean and median scores across 57 Atari games, measured as percentages of human baseline (H.B., Nair et al., 2015).

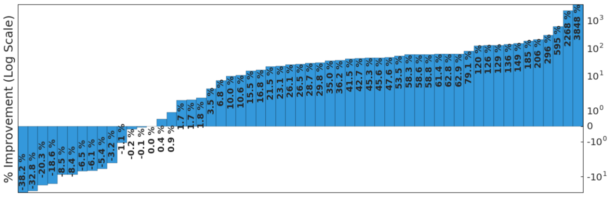


Figure 7. Percentage improvement, per-game, of C51 over Double DQN, computed using van Hasselt et al.'s method.

6. Discussion

In this work we sought a more complete picture of reinforcement learning, one that involves value distributions. We found that learning value distributions is a powerful notion that allows us to surpass most gains previously made on Atari 2600, without further algorithmic adjustments.

6.1. Why does learning a distribution matter?

It is surprising that, when we use a policy which aims to maximize expected return, we should see any difference in performance. The distinction we wish to make is that *learning distributions matters in the presence of approximation*. We now outline some possible reasons.

Reduced chattering. Our results from Section 3.4 highlighted a significant instability in the Bellman optimality operator. When combined with function approximation, this instability may prevent the policy from converging, what Gordon (1995) called *chattering*. We believe the gradient-based categorical algorithm is able to mitigate these effects by effectively averaging the different distributions, similar to conservative policy iteration (Kakade & Langford, 2002). While the chattering persists, it is integrated to the approximate solution.

State aliasing. Even in a deterministic environment, state aliasing may result in effective stochasticity. McCallum (1995), for example, showed the importance of coupling representation learning with policy learning in partially observable domains. We saw an example of state aliasing in PONG, where the agent could not exactly predict the reward timing. Again, by explicitly modelling the resulting distribution we provide a more stable learning target.

A richer set of predictions. A recurring theme in artificial intelligence is the idea of an agent learning from a multitude of predictions (Caruana 1997; Utgoff & Stracuzzi 2002; Sutton et al. 2011; Jaderberg et al. 2017). The distributional approach naturally provides us with a rich set of auxiliary predictions, namely: the probability that the return will take on a particular value. Unlike previously proposed approaches, however, the accuracy of these predictions is tightly coupled with the agent's performance.

Framework for inductive bias. The distributional perspective on reinforcement learning allows a more natural framework within which we can impose assumptions about the domain or the learning problem itself. In this work we used distributions with support bounded in $[V_{\text{MIN}}, V_{\text{MAX}}]$. Treating this support as a hyperparameter allows us to change the optimization problem by treating all extremal returns (e.g. greater than V_{MAX}) as equivalent. Surprisingly, a similar value clipping in DQN significantly degrades performance in most games. To take another example: interpreting the discount factor γ as a proper probability, as some authors have argued, leads to a different algorithm.

Well-behaved optimization. It is well-accepted that the KL divergence between categorical distributions is a reasonably easy loss to minimize. This may explain some of our empirical performance. Yet early experiments with alternative losses, such as KL divergence between continuous densities, were not fruitful, in part because the KL divergence is insensitive to the values of its outcomes. A closer minimization of the Wasserstein metric should yield even better results than what we presented here.

In closing, we believe our results highlight the need to account for distribution in the design, theoretical or otherwise, of algorithms.

[†] The UNREAL results are not altogether comparable, as they were generated in the asynchronous setting with per-game hyperparameter tuning (Jaderberg et al., 2017).

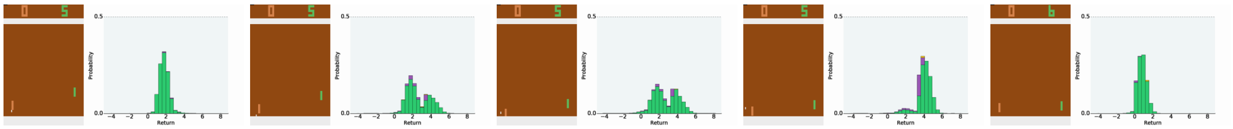


图5。乒乓球中的内在随机性。

	Mean	Median	> H.B.	> DQN
DQN	228%	79%	24	0
DDQN	307%	118%	33	43
DUEL.	373%	151%	37	50
PRIOR.	434%	124%	39	48
PR. DUEL.	592%	172%	39	44
C51	**701%**	**178%**	**40**	**50**
UNREAL[†]	880%	250%	-	-

图6。在57场Atari游戏中的平均值和中位数得分为人类基线的百分比（H.B.，Nair等，2015）。

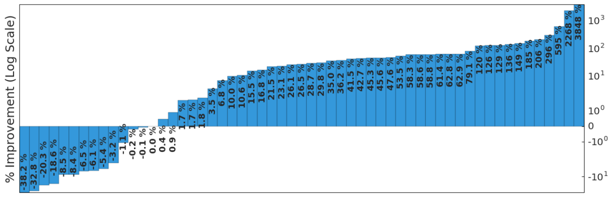


图7。使用van Hasselt等人的方法计算的C51比DQN的C51的百分比提高百分比。

6。讨论

在这项工作中，我们寻求更完整的re绳学习图片，其中涉及价值分布。我们发现，学习价值分布是一个强大的疾病，它使我们能够超过以前在Atari 2600上获得的大多数收益，而无需进一步的算法调整。

6.1。为什么学习分布很重要?

令人惊讶的是，当我们使用旨在最大程度地提高预期回报的策略时，我们应该看到绩效的任何差异。我们希望做出的区别是 *learning distributions matters in the presence of approximation*。我们现在概述了一些可能的原因。

减少chat不休。我们的结果是第3.4节高点，在Bellman最佳操作员中显着不稳定性。当与函数大约结合使用时，这种不稳定可能会阻止策略收敛，Gordon（1995）称为*chattering*。我们认为，基于梯度的分类算法能够通过有效平均不同的分辨率来减轻这些影响

Butions，类似于保守的政策迭代（Kakade＆Langford，2002）。尽管聊天持续存在，但它与近似解决方案有关。

国家的混叠。即使在确定性的环境中，状态混叠也可能导致有效的随机性。例如，麦卡勒姆（McCallum，1995）表明，在部分可用的领域中，耦合表示与政策学习的重要性。我们看到了一个在乒乓球的国家别名的例子，在那里代理人无法准确预测重新定时。同样，通过明确建模产生的分布，我们提供了一个更稳定的学习目标。

一组更丰富的预测。人工智能中的一个反复出现的主题是代理从预测的程度学习的想法（Caruana 1997; Utgoff＆Stracuzzi 2002; Sutton etal。2011; Jaderberg etal。2017）。分歧的方法自然为我们提供了一系列辅助预测，即回报会带有特定价值的概率。但是，与以前提出的方法不同，这些预写的准确性与代理商的性能紧密相结合。

感应偏见的框架。强化学习的分布表现出了一个更自然的框架，在该框架中，我们可以对域或学习问题本身施加假设。在这项工作中，我们使用了$[V_{\text{MIN}}, V_{\text{MAX}}]$中的支持的分布。将此支持视为超参数，使我们能够通过将所有极端回报（例如大于V_{MAX}）视为等效来改变优化问题。令人惊讶的是，在大多数游戏中，DQN中类似的价值剪辑显着降低。举一个例子：正如一些作者所说的那样，将折现因子γ作为适当的概率导致了不同的算法。

行为良好的优化。众所周知，分类分布之间的KL差异是最小化的损失。这可以解释我们的一些经验表现。然而，早期的实验损失的早期实验，例如持续密度之间的KL差异并不富有成果，部分原因是Kl di-di-Gergence对其结果的值不敏感。 Wasserstein度量的近距离最小化应该比我们在这里提出的更好的结果更好。

在结束时，我们相信我们的结果凸显了需要在设计，理论或其他方面的算法中分布的必要性。

[†] The UNREAL results are not altogether comparable, as they were generated in the asynchronous setting with per-game hyperparameter tuning (Jaderberg et al., 2017).

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Erratum

The camera-ready copy of this paper incorrectly reported a mean score of 1010% for C51. The corrected figure stands at 701%, which remains higher than the other comparable baselines. The median score remains unchanged at 178%.

The error was due to evaluation episodes in one game (Atlantis) lasting over 30 minutes; in comparison, the other results presented here cap episodes at 30 minutes, as is standard. The previously reported score on Atlantis was 3.7 million; our 30-minute score is 841,075, which we believe is close to the achievable maximum in this time frame. Capping at 30 minutes brings our human-normalized score on Atlantis from 22824% to a mere (!) 5199%, unfortunately enough to noticeably affect the mean score, whose sensitivity to outliers is well-documented.

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勘误

本文的相机就绪副本不正确地报告了C51的平均得分为1010%。校正后的数字为701%，比其他可比基线高。中位数得分保持不变，为178%。

错误是由于一场比赛（ATLANTIS）持续30分钟的评估发作。相比之下，此处呈现的其他结果在30分钟内（标准）呈现。先前报道的亚特兰蒂斯分数为370万；我们的30分钟分数是841,075，我们的得分接近了这个时间范围的最大值。 30分钟时的盖帽使我们对亚特兰蒂斯的人体范围分数从22824%提高到仅（！）5199%，不幸的是，足以明显影响平均得分，其对离群值的敏感性已得到充分纪录。

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A. Related Work

To the best of our knowledge, the work closest to ours are two papers (Morimura et al., 2010b;a) studying the distributional Bellman equation from the perspective of its cumulative distribution functions. The authors propose both parametric and nonparametric solutions to learn distributions for risk-sensitive reinforcement learning. They also provide some theoretical analysis for the policy evaluation setting, including a consistency result in the nonparametric case. By contrast, we also analyze the control setting, and emphasize the use of the distributional equations to improve approximate reinforcement learning.

The variance of the return has been extensively studied in the risk-sensitive setting. Of note, Tamar et al. (2016) analyze the use of linear function approximation to learn this variance for policy evaluation, and Prashanth & Ghavamzadeh (2013) estimate the return variance in the design of a risk-sensitive actor-critic algorithm. Mannor & Tsitsiklis (2011) provides negative results regarding the computation of a variance-constrained solution to the optimal control problem.

The distributional formulation also arises when modelling uncertainty. Dearden et al. (1998) considered a Gaussian approximation to the value distribution, and modelled the uncertainty over the parameters of this approximation using a Normal-Gamma prior. Engel et al. (2005) leveraged the distributional Bellman equation to define a Gaussian process over the unknown value function. More recently, Geist & Pietquin (2010) proposed an alternative solution to the same problem based on unscented Kalman filters. We believe much of the analysis we provide here, which deals with the intrinsic randomness of the environment, can also be applied to modelling uncertainty.

Our work here is based on a number of foundational results, in particular concerning alternative optimality criteria. Early on, Jaquette (1973) showed that a *moment optimality* criterion, which imposes a total ordering on distributions, is achievable and defines a stationary optimal policy, echoing the second part of Theorem 1. Sobel (1982) is usually cited as the first reference to Bellman equations for the higher moments (but not the distribution) of the return. Chung & Sobel (1987) provides results concerning the convergence of the distributional Bellman operator in total variation distance. White (1988) studies "nonstandard MDP criteria" from the perspective of optimizing the state-action pair occupancy.

A number of probabilistic frameworks for reinforcement learning have been proposed in recent years. The *planning as inference* approach (Toussaint & Storkey, 2006; Hoffman et al., 2009) embeds the return into a graphical model, and applies probabilistic inference to determine the sequence of actions leading to maximal expected reward. Wang et al. (2008) considered the dual formulation of reinforcement learning, where one optimizes the stationary distribution subject to constraints given by the transition function (Puterman, 1994), in particular its relationship to linear approximation. Related to this dual is the Compress and Control algorithm Veness et al. (2015), which describes a value function by learning a return distribution using density models. One of the aims of this work was to address the question left open by their work of whether one could be design a practical distributional algorithm based on the Bellman equation, rather than Monte Carlo estimation.

B. Proofs

Lemma 1 (Partition lemma). *Let $A_1, A_2, \dots$ be a set of random variables describing a partition of Ω, i.e. $A_i(\omega) \in \{0, 1\}$ and for any ω there is exactly one A_i with $A_i(\omega) = 1$. Let U, V be two random variables. Then*

$$d_p(U, V) \leq \sum\nolimits_i d_p(A_i U, A_i V).$$

Proof. We will give the proof for $p < \infty$, noting that the same applies to $p = \infty$. Let $Y_i \overset{D}{:=} A_i U$ and $Z_i \overset{D}{:=} A_i V$, respectively. First note that

$$\begin{aligned} d_p^p(A_i U, A_i V) &= \inf_{Y_i, Z_i} \mathbb{E}\left[|Y_i - Z_i|^p\right] \\ &= \inf_{Y_i, Z_i} \mathbb{E}\left[\mathbb{E}\left[|Y_i - Z_i|^p \,|\, A_i\right]\right]. \end{aligned}$$

Now, $|A_i U - A_i V|^p = 0$ whenever $A_i = 0$. It follows that we can choose Y_i, Z_i so that also $|Y_i - Z_i|^p = 0$ whenever $A_i = 0$, without increasing the expected norm. Hence

$$\begin{aligned} &d_p^p(A_i U, A_i V) = \\ &\qquad \inf_{Y_i, Z_i} \Pr\{A_i = 1\} \, \mathbb{E}\left[|Y_i - Z_i|^p \,|\, A_i = 1\right]. \end{aligned} \quad (8)$$

Next, we claim that

$$\begin{aligned} &\inf_{U,V} \sum\nolimits_i \Pr\{A_i = 1\} \, \mathbb{E}\left[\left|A_i U - A_i V\right|^p \,|\, A_i = 1\right] \quad (9) \\ &\quad \leq \inf_{\substack{Y_1, Y_2, \dots \\ Z_1, Z_2, \dots}} \sum\nolimits_i \Pr\{A_i = 1\} \, \mathbb{E}\left[\left|Y_i - Z_i\right|^p \,|\, A_i = 1\right]. \end{aligned}$$

Specifically, the left-hand side of the equation is an infimum over all r.v.'s whose cumulative distributions are F_U and F_V, respectively, while the right-hand side is an infimum over sequences of r.v's $Y_1, Y_2, \dots$ and $Z_1, Z_2, \dots$ whose cumulative distributions are $F_{A_i U}, F_{A_i V}$, respectively. To prove this upper bound, consider the c.d.f. of U:

$$\begin{aligned} F_U(y) &= \Pr\{U \leq y\} \\ &= \sum\nolimits_i \Pr\{A_i = 1\} \Pr\{U \leq y \,|\, A_i = 1\} \\ &= \sum\nolimits_i \Pr\{A_i = 1\} \Pr\{A_i U \leq y \,|\, A_i = 1\}. \end{aligned}$$

A.相关工作

据我们所知，最接近我们的工作是两篇论文（Morimura等，2010b； a）从其作用分布函数的角度研究了分散的Bellman方程。作者提出了参数和非参数解决方案，以学习对风险敏感的增强学习的分布。他们还为政策评估设置提供了一些理论分析，包括在非参数案例中的一致性结果。相比之下，我们还分析了控制设置，并强调使用分布方程来证明近似强化学习。

回报的差异已在风险敏感的环境中广泛介绍。值得注意的是，Tamar等人。（2016年）分析了线性函数近似来学习这种差异进行策略评估，Prashanth＆Ghavamzadeh（2013）估计了对风险敏感的参与者 - 批判算法的设计回报差异。 Mannor＆tsitsiklis（2011）就计算差异解决方案的计算对光控制问题的解决方案提供了负面结果。

在建模不确定性时，也会出现分布式配方。 Dearden等。（1998年）考虑了对值分布的高斯近似，并建模了与正常γ先验的近似参数的不确定性。 Engel等。（2005年）利用分布的钟形方程将高斯过程定义为未知的值函数。最近，Geist＆Pietquin（2010）提出了基于无味的卡尔曼过滤器的同一问题的替代解决方案。我们认为，我们在这里提供的大部分分析，这些分析涉及环境的固有随机性，也可以应用于建模不确定性。

我们在这里的工作是基于许多基础知识，尤其是关于替代性最佳症状的基础。一开始，Jaquette（1973）表明*moment opti- mality*标准是可以实现的，该标准是可以实现的，它是可以实现的，并定义了一个固定的最佳策略，回应了定理的第二部分。）通常将其作为对较高时刻（但不是分布）转弯的钟声方程的第一个引用。 Chung＆Sobel（1987）提供了有关在总变化距离中分配钟形操作员收敛的结果。 White（1988）从优化国家行动对占用的角度研究了“非标准MDP标准”。

近年来，已经提出了许多用于加强学习的概率框架。*plan- ning as inference*方法（Toussaint＆Storkey，2006; Hoffman等，2009）将返回嵌入到图形模型中，并应用概率推理来确定一系列行动，导致最大的预期奖励。 Wang等。（2008年）考虑了对潜在学习的双重表述，其中一种优化的固定分布受到过渡函数给出的约束（Puterman，1994），特别是其与线性近似的关系。与此双重相关的是压缩和控制算法Veness等。（2015年），它通过使用DENSITY模型来学习返回分布来描述价值函数。这项工作的目的之一是解决他们是否可以根据钟声方程设计的实用分配算法而不是蒙特卡洛估计的实用分配算法所留下的问题。

B.证明

引理1（分区引理）。 *Let* $A_1, A_2, \dots$ *be a set of random variables describing a partition of*ω*, i.e.* $A_i(\omega) \in$ { 0 , 1
} *and for any* ω *there is exactly one* A_i *with* $A_i(\omega) = 1$
. *Let* U, V *be two random variables. Then*

$$d_p\big(U, V\big) \leq \sum\nolimits_i d_p(A_i U, A_i V).$$

Proof. 我们将提供$p < \infty$的证明，并指出相同的$p = \infty$。令$Y_i \overset{D}{:} = A_i U$和$Z_i \overset{D}{}$分别分别：$= A_i V$。第一注意

$$\begin{aligned} d_p^p(A_i U, A_i V) &= \inf_{Y_i, Z_i} \mathbb{E}\left[|Y_i - Z_i|^p\right] \\ &= \inf_{Y_i, Z_i} \mathbb{E}\Big[\mathbb{E}\left[|Y_i - Z_i|^p \mid A_i\right]\Big]. \end{aligned}$$

现在，$|A_i U - A_i V|^p = 0$时，每当$A_i = 0$时。随之而来的是，我们可以选择Y_i, Z_i，以便每当$|Y_i - Z_i|^p = 0$时$|Y_i - Z_i|^p = 0$ $A_i = 0$，而无需增加预期的规范。因此

$$\begin{aligned} d_p^p(A_i U, A_i V) = \\ \inf_{Y_i, Z_i} \Pr\{A_i = 1\} \mathbb{E}\left[|Y_i - Z_i|^p \mid A_i = 1\right]. \end{aligned} \tag{8}$$

接下来，我们声称

$$\begin{aligned} &\inf_{U,V} \sum\nolimits_i \Pr\{A_i = 1\} \mathbb{E}\Big[\big|A_i U - A_i V\big|^p \mid A_i = 1\Big] \\ &\quad \leq \inf_{\substack{Y_1, Y_2, \dots \\ Z_1, Z_2, \dots}} \sum\nolimits_i \Pr\{A_i = 1\} \mathbb{E}\Big[\big|Y_i - Z_i\big|^p \mid A_i = 1\Big]. \end{aligned} \tag{9}$$

特定地，方程的左侧是所有r.v.的含量，其累积分布分别为F_U和F_V，而右侧则是r序列的内部。 V's $Y_1, Y_2, \dots$和$Z_1, Z_2, \dots$的累积分布是$F_{A_i U}, F_{A_i V}$，分别为$F_{A_i U}, F_{A_i V}$。要证明这种上限，请考虑C.D.F. U:

$$\begin{aligned} F_U(y) &= \Pr\{U \leq y\} \\ &= \sum\nolimits_i \Pr\{A_i = 1\} \Pr\{U \leq y \mid A_i = 1\} \\ &= \sum\nolimits_i \Pr\{A_i = 1\} \Pr\{A_i U \leq y \mid A_i = 1\}. \end{aligned}$$

Hence the distribution F_U is equivalent, in an almost sure sense, to one that first picks an element A_i of the partition, then picks a value for U conditional on the choice A_i. On the other hand, the c.d.f. of $Y_i \overset{D}{=} A_i U$ is

$$\begin{aligned} F_{A_iU}(y) &= \Pr\{A_i = 1\} \Pr\{A_i U \leq y \mid A_i = 1\} \\ &\quad + \Pr\{A_i = 0\} \Pr\{A_i U \leq y \mid A_i = 0\} \\ &= \Pr\{A_i = 1\} \Pr\{A_i U \leq y \mid A_i = 1\} \\ &\quad + \Pr\{A_i = 0\} \mathbb{I}\left[y \geq 0\right]. \end{aligned}$$

Thus the right-hand side infimum in (9) has the additional constraint that it must preserve the conditional c.d.fs, in particular when $y \geq 0$. Put another way, instead of having the freedom to completely reorder the mapping $U : \Omega \to \mathbb{R}$, we can only reorder it within each element of the partition. We now write

$$\begin{aligned} d_p^p(U, V) &= \inf_{U,V} \|U - V\|_p \\ &= \inf_{U,V} \mathbb{E}\left[|U - V|^p\right] \\ &\overset{(a)}{=} \inf_{U,V} \sum\nolimits_i \Pr\{A_i = 1\} \mathbb{E}\left[|U - V|^p \mid A_i = 1\right] \\ &= \inf_{U,V} \sum\nolimits_i \Pr\{A_i = 1\} \mathbb{E}\left[|A_i U - A_i V|^p \mid A_i = 1\right], \end{aligned}$$

where (a) follows because $A_1, A_2, \dots$ is a partition. Using (9), this implies

$$\begin{aligned} & d_p^p(U, V) \\ &= \inf_{U,V} \sum\nolimits_i \Pr\{A_i = 1\} \mathbb{E}\left[\left|A_i U - A_i V\right|^p \mid A_i = 1\right] \\ &\leq \inf_{\substack{Y_1, Y_2, \dots \\ Z_1, Z_2, \dots}} \sum\nolimits_i \Pr\{A_i = 1\} \mathbb{E}\left[\left|Y_i - Z_i\right|^p \mid A_i = 1\right] \\ &\overset{(b)}{=} \sum\nolimits_i \inf_{Y_i, Z_i} \Pr\{A_i = 1\} \mathbb{E}\left[\left|Y_i - Z_i\right|^p \mid A_i = 1\right] \\ &\overset{(c)}{=} \sum\nolimits_i d_p(A_i U, A_i V), \end{aligned}$$

because in (b) the individual components of the sum are independently minimized; and (c) from (8). □

Lemma 2. *$\bar{d}_p$ is a metric over value distributions.*

Proof. The only nontrivial property is the triangle inequality. For any value distribution $Y \in \mathcal{Z}$, write

$$\begin{aligned} & \bar{d}_p(Z_1, Z_2) = \sup_{x,a} d_p(Z_1(x,a), Z_2(x,a)) \\ &\overset{(a)}{\leq} \sup_{x,a} \left[d_p(Z_1(x,a), Y(x,a)) + d_p(Y(x,a), Z_2(x,a))\right] \\ &\leq \sup_{x,a} d_p(Z_1(x,a), Y(x,a)) + \sup_{x,a} d_p(Y(x,a), Z_2(x,a)) \\ &= \bar{d}_p(Z_1, Y) + \bar{d}_p(Y, Z_2), \end{aligned}$$

where in (a) we used the triangle inequality for d_p. □

Lemma 3. *$\mathcal{T}^\pi : \mathcal{Z} \to \mathcal{Z}$ is a γ-contraction in $\bar{d}_p$.*

Proof. Consider $Z_1, Z_2 \in \mathcal{Z}$. By definition,

$$\bar{d}_p(\mathcal{T}^\pi Z_1, \mathcal{T}^\pi Z_2) = \sup_{x,a} d_p(\mathcal{T}^\pi Z_1(x,a), \mathcal{T}^\pi Z_2(x,a)). \tag{10}$$

By the properties of d_p, we have

$$\begin{aligned} & d_p(\mathcal{T}^\pi Z_1(x,a), \mathcal{T}^\pi Z_2(x,a)) \\ &\quad = d_p(R(x,a) + \gamma P^\pi Z_1(x,a), R(x,a) + \gamma P^\pi Z_2(x,a)) \\ &\quad \leq \gamma d_p(P^\pi Z_1(x,a), P^\pi Z_2(x,a)) \\ &\quad \leq \gamma \sup_{x',a'} d_p(Z_1(x',a'), Z_2(x',a')), \end{aligned}$$

where the last line follows from the definition of P^π (see (4)). Combining with (10) we obtain

$$\begin{aligned} \bar{d}_p(\mathcal{T}^\pi Z_1, \mathcal{T}^\pi Z_2) &= \sup_{x,a} d_p(\mathcal{T}^\pi Z_1(x,a), \mathcal{T}^\pi Z_2(x,a)) \\ &\leq \gamma \sup_{x',a'} d_p(Z_1(x',a'), Z_2(x',a')) \\ &= \gamma \bar{d}_p(Z_1, Z_2). \end{aligned}$$

□

Proposition 1 (Sobel, 1982)**.** *Consider two value distributions $Z_1, Z_2 \in \mathcal{Z}$, and write $\mathbb{V}(Z_i)$ to be the vector of variances of Z_i. Then*

$$\begin{aligned} \left\|\mathbb{E}\,\mathcal{T}^\pi Z_1 - \mathbb{E}\,\mathcal{T}^\pi Z_2\right\|_\infty &\leq \gamma \left\|\mathbb{E}\,Z_1 - \mathbb{E}\,Z_2\right\|_\infty, \textit{ and} \\ \left\|\mathbb{V}(\mathcal{T}^\pi Z_1) - \mathbb{V}(\mathcal{T}^\pi Z_2)\right\|_\infty &\leq \gamma^2 \left\|\mathbb{V} Z_1 - \mathbb{V} Z_2\right\|_\infty. \end{aligned}$$

Proof. The first statement is standard, and its proof follows from $\mathbb{E}\,\mathcal{T}^\pi Z = \mathcal{T}^\pi\,\mathbb{E}\,Z$, where the second $\mathcal{T}^\pi$ denotes the usual operator over value functions. Now, by independence of R and $P^\pi Z_i$:

$$\begin{aligned} \mathbb{V}(\mathcal{T}^\pi Z_i(x,a)) &= \mathbb{V}\Big(R(x,a) + \gamma P^\pi Z_i(x,a)\Big) \\ &= \mathbb{V}(R(x,a)) + \gamma^2 \mathbb{V}(P^\pi Z_i(x,a)). \end{aligned}$$

And now

$$\begin{aligned} & \left\|\mathbb{V}(\mathcal{T}^\pi Z_1) - \mathbb{V}(\mathcal{T}^\pi Z_2)\right\|_\infty \\ &\quad = \sup_{x,a} \left|\mathbb{V}(\mathcal{T}^\pi Z_1(x,a)) - \mathbb{V}(\mathcal{T}^\pi Z_2(x,a))\right| \\ &\quad = \sup_{x,a} \gamma^2 \left| \left[\mathbb{V}(P^\pi Z_1(x,a)) - \mathbb{V}(P^\pi Z_2(x,a))\right] \right| \\ &\quad = \sup_{x,a} \gamma^2 \left| \mathbb{E}\left[\mathbb{V}(Z_1(X',A')) - \mathbb{V}(Z_2(X',A'))\right] \right| \\ &\quad \leq \sup_{x',a'} \gamma^2 \left|\mathbb{V}(Z_1(x',a')) - \mathbb{V}(Z_2(x',a'))\right| \\ &\quad \leq \gamma^2 \left\|\mathbb{V} Z_1 - \mathbb{V} Z_2\right\|_\infty. \end{aligned}$$

□

Lemma 4. *Let $Z_1, Z_2 \in \mathcal{Z}$. Then*

$$\left\|\mathbb{E}\,\mathcal{T} Z_1 - \mathbb{E}\,\mathcal{T} Z_2\right\|_\infty \leq \gamma \left\|\mathbb{E}\,Z_1 - \mathbb{E}\,Z_2\right\|_\infty,$$

and in particular $\mathbb{E}\,Z_k \to Q^*$ exponentially quickly.

因此，在几乎可以肯定的是，分布F_U是等于第一个选择分区的元素A_i的分布，然后选择U有条件的A_i条件的值。另一方面，C.D.F. $Y_i \overset{D}{=} A_iU$是

$$\begin{aligned} F_{A_iU}(y) &= \Pr\{A_i = 1\}\Pr\{A_iU \le y \mid A_i = 1\} \\ &\quad + \Pr\{A_i = 0\}\Pr\{A_iU \le y \mid A_i = 0\} \\ &= \Pr\{A_i = 1\}\Pr\{A_iU \le y \mid A_i = 1\} \\ &\quad + \Pr\{A_i = 0\}\mathbb{I}\left[y \ge 0\right]. \end{aligned}$$

因此，（9）中的右侧右侧具有附加的约束，即必须保留条件件的c.d.fs，特别是当$y \ge 0$时。换句话说，而不是自由地完全重新安排映射U：ω→ ℝ，我们只能在分区的每个元素中重新排序。我们现在写

$$\begin{aligned} d_p^p(U, V) &= \inf_{U,V} \|U - V\|_p \\ &= \inf_{U,V} \mathbb{E}\left[|U - V|^p\right] \\ &\overset{(a)}{=} \inf_{U,V} \sum\nolimits_i \Pr\{A_i = 1\}\,\mathbb{E}\left[|U - V|^p \mid A_i = 1\right] \\ &= \inf_{U,V} \sum\nolimits_i \Pr\{A_i = 1\}\,\mathbb{E}\left[|A_iU - A_iV|^p \mid A_i = 1\right], \end{aligned}$$

其中（a）之所以跟随，是因为$A_1, A_2, \ldots$是一个分区。使用（9），这意味着

$$\begin{aligned} &d_p^p(U, V) \\ &= \inf_{U,V} \sum\nolimits_i \Pr\{A_i = 1\}\,\mathbb{E}\left[\left|A_iU - A_iV\right|^p \mid A_i = 1\right] \\ &\le \inf_{\substack{Y_1, Y_2, \ldots \\ Z_1, Z_2, \ldots}} \sum\nolimits_i \Pr\{A_i = 1\}\,\mathbb{E}\left[\left|Y_i - Z_i\right|^p \mid A_i = 1\right] \\ &\overset{(b)}{=} \sum\nolimits_i \inf_{Y_i, Z_i} \Pr\{A_i = 1\}\,\mathbb{E}\left[\left|Y_i - Z_i\right|^p \mid A_i = 1\right] \\ &\overset{(c)}{=} \sum\nolimits_i d_p(A_iU, A_iV), \end{aligned}$$

因为在（b）中，总和的各个组成部分被独立最小化；（c）来自（8）。 □

引理2。$\bar{d}_p$ *is a metric over value distributions.*

Proof. 唯一的非平凡特性是三角形不平等。对于任何值分发$Y \in \mathcal{Z}$，写

$$\begin{aligned} &\bar{d}_p(Z_1, Z_2) = \sup_{x,a} d_p(Z_1(x, a), Z_2(x, a)) \\ &\overset{(a)}{\le} \sup_{x,a}\left[d_p(Z_1(x, a), Y(x, a)) + d_p(Y(x, a), Z_2(x, a))\right] \\ &\le \sup_{x,a} d_p(Z_1(x, a), Y(x, a)) + \sup_{x,a} d_p(Y(x, a), Z_2(x, a)) \\ &= \bar{d}_p(Z_1, Y) + \bar{d}_p(Y, Z_2), \end{aligned}$$

在（a）中，我们将三角形不等式用于d_p。 □

引理3。$\mathcal{T}^\pi$： $\mathcal{Z} \to \mathcal{Z}$ *is a γ-contraction in $\bar{d}_p$*。

Proof. 考虑$Z_1, Z_2 \in \mathcal{Z}$。根据定义，

$$\bar{d}_p(\mathcal{T}^\pi Z_1, \mathcal{T}^\pi Z_2) = \sup_{x,a} d_p(\mathcal{T}^\pi Z_1(x, a), \mathcal{T}^\pi Z_2(x, a)). \tag{10}$$

根据d_p的属性，我们有

$$\begin{aligned} &d_p(\mathcal{T}^\pi Z_1(x, a), \mathcal{T}^\pi Z_2(x, a)) \\ &\quad = d_p(R(x, a) + \gamma P^\pi Z_1(x, a), R(x, a) + \gamma P^\pi Z_2(x, a)) \\ &\quad \le \gamma d_p(P^\pi Z_1(x, a), P^\pi Z_2(x, a)) \\ &\quad \le \gamma \sup_{x', a'} d_p(Z_1(x', a'), Z_2(x', a')), \end{aligned}$$

最后一行是根据P^π (的定义，请参见（4)）。结合（10）我们获得

$$\begin{aligned} \bar{d}_p(\mathcal{T}^\pi Z_1, \mathcal{T}^\pi Z_2) &= \sup_{x,a} d_p(\mathcal{T}^\pi Z_1(x, a), \mathcal{T}^\pi Z_2(x, a)) \\ &\le \gamma \sup_{x', a'} d_p(Z_1(x', a'), Z_2(x', a')) \\ &= \gamma \bar{d}_p(Z_1, Z_2). \end{aligned}$$ □

命题1（Sobel，1982）。*Consider two value distributions $Z_1, Z_2 \in \mathcal{Z}$, and write $\mathbb{V}(Z_i)$ to be the vector of variances of Z_i. Then*

$$\begin{aligned} \left\|\mathbb{E}\,\mathcal{T}^\pi Z_1 - \mathbb{E}\,\mathcal{T}^\pi Z_2\right\|_\infty &\le \gamma \left\|\mathbb{E}\,Z_1 - \mathbb{E}\,Z_2\right\|_\infty, \textit{ and} \\ \left\|\mathbb{V}(\mathcal{T}^\pi Z_1) - \mathbb{V}(\mathcal{T}^\pi Z_2)\right\|_\infty &\le \gamma^2 \left\|\mathbb{V}Z_1 - \mathbb{V}Z_2\right\|_\infty. \end{aligned}$$

Proof. 第一个语句是标准的，其证明来自$\mathbb{E}\,\mathcal{T}^\pi Z = \mathcal{T}^\pi\,\mathbb{E}\,Z$，其中第二个$\mathcal{T}^\pi$表示通常的operator在值函数上。现在，通过$R$和$P^\pi Z_i$的独立性：

$$\begin{aligned} \mathbb{V}(\mathcal{T}^\pi Z_i(x, a)) &= \mathbb{V}\Big(R(x, a) + \gamma P^\pi Z_i(x, a)\Big) \\ &= \mathbb{V}(R(x, a)) + \gamma^2 \mathbb{V}(P^\pi Z_i(x, a)). \end{aligned}$$

现在

$$\begin{aligned} &\left\|\mathbb{V}(\mathcal{T}^\pi Z_1) - \mathbb{V}(\mathcal{T}^\pi Z_2)\right\|_\infty \\ &\quad = \sup_{x,a} \left|\mathbb{V}(\mathcal{T}^\pi Z_1(x, a)) - \mathbb{V}(\mathcal{T}^\pi Z_2(x, a))\right| \\ &\quad = \sup_{x,a} \gamma^2 \left|\left[\mathbb{V}(P^\pi Z_1(x, a)) - \mathbb{V}(P^\pi Z_2(x, a))\right]\right| \\ &\quad = \sup_{x,a} \gamma^2 \left|\mathbb{E}\left[\mathbb{V}(Z_1(X', A')) - \mathbb{V}(Z_2(X', A'))\right]\right| \\ &\quad \le \sup_{x', a'} \gamma^2 \left|\mathbb{V}(Z_1(x', a')) - \mathbb{V}(Z_2(x', a'))\right| \\ &\quad \le \gamma^2 \left\|\mathbb{V}Z_1 - \mathbb{V}Z_2\right\|_\infty. \end{aligned}$$ □

引理4。*Let $Z_1, Z_2 \in \mathcal{Z}$. Then*

$$\left\|\mathbb{E}\,\mathcal{T}Z_1 - \mathbb{E}\,\mathcal{T}Z_2\right\|_\infty \le \gamma \left\|\mathbb{E}\,Z_1 - \mathbb{E}\,Z_2\right\|_\infty,$$

and in particular $\mathbb{E}\,Z_k \to Q^*$ exponentially quickly.

Proof. The proof follows by linearity of expectation. Write $\mathcal{T}_D$ for the distributional operator and $\mathcal{T}_E$ for the usual operator. Then

$$\begin{aligned}\|\mathbb{E}\,\mathcal{T}_D Z_1 - \mathbb{E}\,\mathcal{T}_D Z_2\|_\infty &= \|\mathcal{T}_E\,\mathbb{E}\,Z_1 - \mathcal{T}_E\,\mathbb{E}\,Z_2\|_\infty \\ &\le \gamma\,\|Z_1 - Z_2\|_\infty\,. \qquad \square\end{aligned}$$

Theorem 1 (Convergence in the control setting)**.** *Let $Z_k := \mathcal{T} Z_{k-1}$ with $Z_0 \in \mathcal{Z}$. Let $\mathcal{X}$ be measurable and suppose that $\mathcal{A}$ is finite. Then*

$$\lim_{k\to\infty} \inf_{Z^{**}\in\mathcal{Z}^{**}} d_p(Z_k(x,a), Z^{**}(x,a)) = 0 \quad \forall x, a.$$

If $\mathcal{X}$ is finite, then Z_k converges to $\mathcal{Z}^{**}$ uniformly. Furthermore, if there is a total ordering $\prec$ on Π^*, such that for any $Z^* \in \mathcal{Z}^*$,

$$\mathcal{T} Z^* = \mathcal{T}^\pi Z^* \text{ with } \pi \in \mathcal{G}_{Z^*},\ \pi \prec \pi' \ \ \forall \pi' \in \mathcal{G}_{Z^*} \setminus \{\pi\},$$

then $\mathcal{T}$ has a unique fixed point $Z^* \in \mathcal{Z}^*$.

The gist of the proof of Theorem 1 consists in showing that for every state x, there is a time k after which the greedy policy w.r.t. Q_k is mostly optimal. To clearly expose the steps involved, we will first assume a unique (and therefore deterministic) optimal policy π^*, and later return to the general case; we will denote the optimal action at x by $\pi^*(x)$. For notational convenience, we will write $Q_k := \mathbb{E}\,Z_k$ and $\mathcal{G}_k := \mathcal{G}_{Z_k}$. Let $B := 2\sup_{Z\in\mathcal{Z}} \|Z\|_\infty < \infty$ and let $\epsilon_k := \gamma^k B$. We first define the set of states $\mathcal{X}_k \subseteq \mathcal{X}$ whose values must be sufficiently close to Q^* at time k:

$$\mathcal{X}_k := \Big\{ x : Q^*(x, \pi^*(x)) - \max_{a \ne \pi^*(x)} Q^*(x,a) > 2\epsilon_k \Big\}. \tag{11}$$

Indeed, by Lemma 4, we know that after k iterations

$$|Q_k(x,a) - Q^*(x,a)| \le \gamma^k |Q_0(x,a) - Q^*(x,a)| \le \epsilon_k.$$

For $x \in \mathcal{X}$, write $a^* := \pi^*(x)$. For any $a \in \mathcal{A}$, we deduce that

$$Q_k(x,a^*) - Q_k(x,a) \ge Q^*(x,a^*) - Q^*(x,a) - 2\epsilon_k.$$

It follows that if $x \in \mathcal{X}_k$, then also $Q_k(x,a^*) > Q_k(x,a')$ for all $a' \ne \pi^*(x)$: for these states, the greedy policy $\pi_k(x) := \arg\max_a Q_k(x,a)$ corresponds to the optimal policy π^*.

Lemma 5. *For each $x \in \mathcal{X}$ there exists a k such that, for all $k' \ge k$, $x \in \mathcal{X}_{k'}$, and in particular $\arg\max_a Q_k(x,a) = \pi^*(x)$.*

Proof. Because $\mathcal{A}$ is finite, the gap

$$\Delta(x) := Q^*(x, \pi^*(x)) - \max_{a \ne \pi^*(x)} Q^*(x,a)$$

is attained for some strictly positive $\Delta(x) > 0$. By definition, there exists a k such that

$$\epsilon_k = \gamma^k B < \frac{\Delta(x)}{2},$$

and hence every $x \in \mathcal{X}$ must eventually be in $\mathcal{X}_k$. □

This lemma allows us to guarantee the existence of an iteration k after which sufficiently many states are well-behaved, in the sense that the greedy policy at those states chooses the optimal action. We will call these states "solved". We in fact require not only these states to be solved, but also most of their successors, and most of the successors of those, and so on. We formalize this notion as follows: fix some $\delta > 0$, let $\mathcal{X}_{k,0} := \mathcal{X}_k$, and define for $i > 0$ the set

$$\mathcal{X}_{k,i} := \big\{ x : x \in \mathcal{X}_k, P(\mathcal{X}_{k-1,i-1} \,|\, x, \pi^*(x)) \ge 1 - \delta \big\},$$

As the following lemma shows, any x is eventually contained in the recursively-defined sets $\mathcal{X}_{k,i}$, for any i.

Lemma 6. *For any $i \in \mathbb{N}$ and any $x \in \mathcal{X}$, there exists a k such that for all $k' \ge k$, $x \in \mathcal{X}_{k',i}$.*

Proof. Fix i and let us suppose that $\mathcal{X}_{k,i} \uparrow \mathcal{X}$. By Lemma 5, this is true for $i = 0$. We infer that for any probability measure P on $\mathcal{X}$, $P(\mathcal{X}_{k,i}) \to P(\mathcal{X}) = 1$. In particular, for a given $x \in \mathcal{X}_k$, this implies that

$$P(\mathcal{X}_{k,i} \,|\, x, \pi^*(x)) \to P(\mathcal{X} \,|\, x, \pi^*(x)) = 1.$$

Therefore, for any x, there exists a time after which it is and remains a member of $\mathcal{X}_{k,i+1}$, the set of states for which $P(\mathcal{X}_{k-1,i} \,|\, x, \pi^*(x)) \ge 1 - \delta$. We conclude that $\mathcal{X}_{k,i+1} \uparrow \mathcal{X}$ also. The statement follows by induction. □

Proof of Theorem 1. The proof is similar to policy iteration-type results, but requires more care in dealing with the metric and the possibly infinite state space. We will write $W_k(x) := Z_k(x, \pi_k(x))$, define W^* similarly and with some overload of notation write $\mathcal{T} W_k(x) := W_{k+1}(x) = \mathcal{T} Z_k(x, \pi_{k+1}(x))$. Finally, let $S_i^k(x) := \mathbb{I}\,[x \in \mathcal{X}_{k,i}]$ and $\bar{S}_i^k(x) = 1 - S_i^k(x)$.

Fix $i > 0$ and $x \in \mathcal{X}_{k+1,i+1} \subseteq \mathcal{X}_k$. We begin by using Lemma 1 to separate the transition from x into a solved term and an unsolved term:

$$P^{\pi_k} W_k(x) = S_i^k W_k(X') + \bar{S}_i^k W_k(X'),$$

where X' is the random successor from taking action $\pi_k(x) := \pi^*(x)$, and we write $S_i^k = S_i^k(X'), \bar{S}_i^k = \bar{S}_i^k(X')$ to ease the notation. Similarly,

$$P^{\pi_k} W^*(x) = S_i^k W^*(X') + \bar{S}_i^k W^*(X').$$

Proof. 证明是根据期望的线性性。为分布运算符编写$\mathcal{T}_D$，为通常的操作器编写$\mathcal{T}_E$。然后

$$\begin{aligned}\left\|\mathbb{E}\,\mathcal{T}_D Z_1 - \mathbb{E}\,\mathcal{T}_D Z_2\right\|_\infty &= \left\|\mathcal{T}_E\,\mathbb{E}\,Z_1 - \mathcal{T}_E\,\mathbb{E}\,Z_2\right\|_\infty \\ &\leq \gamma \left\|Z_1 - Z_2\right\|_\infty .\end{aligned}$$

□

定理1（控制设置中的收敛）。 *Let* Z_k: $= \mathcal{T} Z_{k-1}$ *with* $Z_0 \in \mathcal{Z}$. *Let* $\mathcal{X}$ *be measurable and suppose that* $\mathcal{A}$ *is finite. Then*

$$\lim_{k\to\infty} \inf_{Z^{**}\in\mathcal{Z}^{**}} d_p(Z_k(x,a), Z^{**}(x,a)) = 0 \quad \forall x, a.$$

If $\mathcal{X}$ *is finite, then* Z_k *converges to* $\mathcal{Z}^{**}$ *uniformly. Furthermore, if there is a total ordering* $\prec$ *on* Π^**, such that for any* $Z^* \in \mathcal{Z}^*$*,*

$$\mathcal{T} Z^* = \mathcal{T}^\pi Z^* \text{ with } \pi \in \mathcal{G}_{Z^*},\ \pi \prec \pi' \ \ \forall \pi' \in \mathcal{G}_{Z^*} \setminus \{\pi\},$$

then $\mathcal{T}$ *has a unique fixed point* $Z^* \in \mathcal{Z}^*$*.*

定理1证明的要旨包括表明，对于每个状态x，贪婪的策略W.R.T.都有一个时间k。 Q_k主要是最佳的。为了清楚地揭示所涉及的步骤，我们将首先假设一个独特的（又是确定性的）最佳策略π^*，然后返回一般情况；我们将用$\pi^*(x)$在x上表示最佳动作。为了方便起见，我们将编写Q_k：$= \mathbb{E}\,Z_k$和$\mathcal{G}_k$：$= \mathcal{G}_{Z_k}$。令B：$=$ 2 sup $_{Z\in\mathcal{Z}} \|Z\|_\infty < \infty$，让$\epsilon_k$：$= \gamma^k B$。我们首先定义一组状态$\mathcal{X}_k \subseteq \mathcal{X}$，其值必须足够接近$Q^*$时$k$：

$$\mathcal{X}_k := \Big\{x : Q^*(x, \pi^*(x)) - \max_{a\neq\pi^*(x)} Q^*(x,a) > 2\epsilon_k\Big\}. \tag{11}$$

确实，通过引理4，我们知道k迭代之后

$$|Q_k(x,a) - Q^*(x,a)| \leq \gamma^k |Q_0(x,a) - Q^*(x,a)| \leq \epsilon_k.$$

对于$x \in \mathcal{X}$，写a^*：$= \pi^*(x)$。对于任何$a \in \mathcal{A}$，我们推论

$$Q_k(x,a^*) - Q_k(x,a) \geq Q^*(x,a^*) - Q^*(x,a) - 2\epsilon_k.$$

因此，如果$x \in \mathcal{X}_k$，则所有$Q_k(x,a^*) > Q_k(x,a')$也为所有$a' \neq \pi^*(x)$：对于这些状态，贪婪的策略$\pi_k(x)$：= arg max max $_a Q_k(x,a)$对应于最佳策略π^*。

引理5。*For each* $x \in \mathcal{X}$ *there exists a* k *such that, for all* $k' \geq k$*,* $x \in \mathcal{X}_{k'}$*, and in particular* arg max $_a Q_k(x,a) = \pi^*(x)$*.*

Proof. 因为$\mathcal{A}$是有限的，所以差距

$$\Delta(x) := Q^*(x, \pi^*(x)) - \max_{a\neq\pi^*(x)} Q^*(x,a)$$

通过定义，有一些严格的阳性$\Delta(x) > 0$。存在一个k，这样

$$\epsilon_k = \gamma^k B < \frac{\Delta(x)}{2},$$

因此，每个$x \in \mathcal{X}$最终都必须在$\mathcal{X}_k$中。 □

这种引理使我们能够保证存在迭代k的存在，然后有足够的表现良好，从某种意义上说，这些州的贪婪政策选择了最佳行动。我们将这些国家称为“解决”。实际上，我们不仅要求解决这些国家，还要求其大多数继任者以及这些州的大多数继任者，等等。我们将此概念形式化如下：x- x x bir $\delta > 0$，让$\mathcal{X}_{k,0}$：$= \mathcal{X}_k$，并为$i > 0$定义。

$$\mathcal{X}_{k,i} := \big\{x : x \in \mathcal{X}_k, P(\mathcal{X}_{k-1,i-1} \,|\, x, \pi^*(x)) \geq 1 - \delta\big\},$$

如以下引理所示，对于任何i，最终都包含在递归确定的集合$\mathcal{X}_{k,i}$的集合中。

引理6。*For any* $i \in \mathbb{N}$ *and any* $x \in \mathcal{X}$*, there exists a* k *such that for all* $k' \geq k$*,* $x \in \mathcal{X}_{k',i}$*.*

Proof. 修复i，让我们假设$\mathcal{X}_{k,i} \uparrow \mathcal{X}$。通过引理5，对于$i = 0$。我们在$\mathcal{X}$上的任何概率度量$P$，$P(\mathcal{X}_{k,i}) \to P(\mathcal{X}) = 1$。尤其是给定的$x \in \mathcal{X}_k$上推断出来，这意味着这意味着它

$$P(\mathcal{X}_{k,i} \,|\, x, \pi^*(x)) \to P(\mathcal{X} \,|\, x, \pi^*(x)) = 1.$$

因此，对于任何x，存在一段时间，然后仍然是$\mathcal{X}_{k,i+1}$的成员，$P(\mathcal{X}_{k-1,i} \,|\, x, \pi^*(x)) \geq 1 - \delta$的一组状态。我们得出结论，$\mathcal{X}_{k,i+1} \uparrow \mathcal{X}$。该声明以归纳为由。 □

Proof of Theorem 1. 该证明类似于政策迭代类型的结果，但需要更多的护理来处理指标和可能的国家空间。我们将同样地编写$W_k(x)$：$= Z_k(x, \pi_k(x))$，定义W^*，并在一定的符号中写入$\mathcal{T} W_k(x)$：$= W_{k+1}(x) = \mathcal{T} Z_k(x, \pi_{k+1}(x))$。最后，令$S_i^k(x)$：$= \mathbb{I}\,[x \in \mathcal{X}_{k,i}]$和$\bar{S}_i^k(x) = 1 - S_i^k(x)$。

修复$i > 0$和$x \in \mathcal{X}_{k+1,i+1} \subseteq \mathcal{X}_k$。我们首先使用引理1将$x$的过渡分为求解的术语和未解决的术语：

$$P^{\pi_k} W_k(x) = S_i^k W_k(X') + \bar{S}_i^k W_k(X'),$$

其中X'是采取动作$\pi_k(x)$的随机后继：$= \pi^*(x)$，然后我们编写$S_i^k = S_i^k(X'), \bar{S}_i^k = \bar{S}_i^k(X')$来简化符号。相似地，

$$P^{\pi_k} W^*(x) = S_i^k W^*(X') + \bar{S}_i^k W^*(X').$$

Now

$$\begin{aligned} d_p(W_{k+1}(x), W^*(x)) &= d_p(\mathcal{T}W_k(x), \mathcal{T}W^*(x)) \\ &\overset{(a)}{\le} \gamma d_p(P^{\pi_k}W_k(x), P^{\pi^*}W^*(x)) \\ &\overset{(b)}{\le} \gamma d_p(S_i^k W_k(X'), S_i^k W^*(X')) \\ &\quad + \gamma d_p(\bar{S}_i^k W_k(X'), \bar{S}_i^k W^*(X')), \end{aligned} \tag{12}$$

where in (a) we used Properties P1 and P2 of the Wasserstein metric, and in (b) we separate states for which $\pi_k = \pi^*$ from the rest using Lemma 1 ($\{S_i^k, \bar{S}_i^k\}$ form a partition of Ω). Let $\delta_i := \Pr\{X' \notin \mathcal{X}_{k,i}\} = \mathbb{E}\{\bar{S}_i^k(X')\} = \|\bar{S}_i^k(X')\|_p$. From property P3 of the Wasserstein metric, we have

$$\begin{aligned} d_p(\bar{S}_i^k W_k(X'), \bar{S}_i^k W^*(X')) & \\ &\le \sup_{x'} d_p(\bar{S}_i^k(X')W_k(x'), \bar{S}_i^k(X')W^*(x')) \\ &\le \|\bar{S}_i^k(X')\|_p \sup_{x'} d_p(W_k(x'), W^*(x')) \\ &\le \delta_i \sup_{x'} d_p(W_k(x'), W^*(x')) \\ &\le \delta_i B. \end{aligned}$$

Recall that $B < \infty$ is the largest attainable $\|Z\|_\infty$. Since also $\delta_i < \delta$ by our choice of $x \in \mathcal{X}_{k+1,i+1}$, we can upper bound the second term in (12) by $\gamma\delta B$. This yields

$$\begin{aligned} d_p(W_{k+1}(x), W^*(x)) \le & \\ &\gamma d_p(S_i^k W_k(X'), S_i^k W^*(X')) + \gamma\delta B. \end{aligned}$$

By induction on $i > 0$, we conclude that for $x \in \mathcal{X}_{k+i,i}$ and some random state X'' i steps forward,

$$\begin{aligned} d_p(W_{k+i}(x), W^*(x)) \le & \\ &\gamma^i d_p(S_0^k W_k(X''), S_0^k W^*(X'')) + \frac{\delta B}{1-\gamma} \\ &\le \gamma^i B + \frac{\delta B}{1-\gamma}. \end{aligned}$$

Hence for any $x \in \mathcal{X}$, $\epsilon > 0$, we can take δ, i, and finally k large enough to make $d_p(W_k(x), W^*(x)) < \epsilon$. The proof then extends to $Z_k(x,a)$ by considering one additional application of $\mathcal{T}$.

We now consider the more general case where there are multiple optimal policies. We expand the definition of $\mathcal{X}_{k,i}$ as follows:

$$\mathcal{X}_{k,i} := \big\{x \in \mathcal{X}_k : \forall \pi^* \in \Pi^*, \mathop{\mathbb{E}}_{a^* \sim \pi^*(x)} P(\mathcal{X}_{k-1,i-1} \mid x, a^*) \ge 1-\delta\big\},$$

Because there are finitely many actions, Lemma 6 also holds for this new definition. As before, take $x \in \mathcal{X}_{k,i}$, but now consider the sequence of greedy policies $\pi_k, \pi_{k-1}, \ldots$ selected by successive applications of $\mathcal{T}$, and write

$$\mathcal{T}^{\bar{\pi}_k} := \mathcal{T}^{\pi_k}\mathcal{T}^{\pi_{k-1}} \cdots \mathcal{T}^{\pi_{k-i+1}},$$

such that

$$Z_{k+1} = \mathcal{T}^{\bar{\pi}_k} Z_{k-i+1}.$$

Now denote by $\mathcal{Z}^{**}$ the set of nonstationary optimal policies. If we take any $Z^* \in \mathcal{Z}^*$, we deduce that

$$\inf_{Z^{**} \in \mathcal{Z}^{**}} d_p(\mathcal{T}^{\bar{\pi}_k} Z^*(x,a), Z^{**}(x,a)) \le \frac{\delta B}{1-\gamma},$$

since Z^* corresponds to some optimal policy π^* and $\bar{\pi}_k$ is optimal along most of the trajectories from (x,a). In effect, $\mathcal{T}^{\bar{\pi}_k} Z^*$ is close to the value distribution of the nonstationary optimal policy $\bar{\pi}_k \pi^*$. Now for this Z^*,

$$\begin{aligned} \inf_{Z^{**}} d_p(Z_k(x,a), Z^{**}(x,a)) & \\ &\le d_p(Z_k(x,a), \mathcal{T}^{\bar{\pi}_k} Z^*(x,a)) \\ &\quad + \inf_{Z^{**}} d_p(\mathcal{T}^{\bar{\pi}_k} Z^*(x,a), Z^{**}(x,a)) \\ &\le d_p(\mathcal{T}^{\bar{\pi}_k} Z_{k-i+1}(x,a), \mathcal{T}^{\bar{\pi}_k} Z^*(x,a)) + \frac{\delta B}{1-\gamma} \\ &\le \gamma^i B + \frac{2\delta B}{1-\gamma}, \end{aligned}$$

using the same argument as before with the newly-defined $\mathcal{X}_{k,i}$. It follows that

$$\inf_{Z^{**} \in \mathcal{Z}^{**}} d_p(Z_k(x,a), Z^{**}(x,a)) \to 0.$$

When $\mathcal{X}$ is finite, there exists a fixed k after which $\mathcal{X}_k = \mathcal{X}$. The uniform convergence result then follows.

To prove the uniqueness of the fixed point Z^* when $\mathcal{T}$ selects its actions according to the ordering $\prec$, we note that for any optimal value distribution Z^*, its set of greedy policies is Π^*. Denote by π^* the policy coming first in the ordering over Π^*. Then $\mathcal{T} = \mathcal{T}^{\pi^*}$, which has a unique fixed point (Section 3.3). □

Proposition 4. *That $\mathcal{T}$ has a fixed point $Z^* = \mathcal{T}Z^*$ is insufficient to guarantee the convergence of $\{Z_k\}$ to $\mathcal{Z}^*$.*

We provide here a sketch of the result. Consider a single state x_1 with two actions, a_1 and a_2 (Figure 8). The first action yields a reward of $1/2$, while the other either yields 0 or 1 with equal probability, and both actions are optimal. Now take $\gamma = 1/2$ and write $R_0, R_1, \ldots$ for the received rewards. Consider a stochastic policy that takes action a_2 with probability p. For $p = 0$, the return is

$$Z_{p=0} = \frac{1}{1-\gamma}\frac{1}{2} = 1.$$

For $p = 1$, on the other hand, the return is random and is given by the following fractional number (in binary):

$$Z_{p=1} = R_0.R_1R_2R_3\cdots.$$

现在

$$\begin{aligned} d_p(W_{k+1}(x), W^*(x)) &= d_p(\mathcal{T}W_k(x), \mathcal{T}W^*(x)) \\ &\overset{(a)}{\le} \gamma d_p(P^{\pi_k}W_k(x), P^{\pi^*}W^*(x)) \\ &\overset{(b)}{\le} \gamma d_p(S_i^k W_k(X'), S_i^k W^*(X')) \\ &\quad + \gamma d_p(\bar{S}_i^k W_k(X'), \bar{S}_i^k W^*(X')), \end{aligned} \tag{12}$$

在（a）中，我们使用了Wasser- Stein Metric的属性P1和P2，在（B）中，我们使用Lemma 1（$\{S_i^k, \bar{S}_i^k\}$形成parti-partimi- ω）。令δ_i：＝pr
$\{X' \notin \mathcal{X}_{k,i}\} = \mathbb{E}\{\bar{S}_i^k(X')\} = \|\bar{S}_i^k(X')\|_p$。从Wasserstein公制的财产P3中，我们有

$$\begin{aligned} d_p(\bar{S}_i^k W_k(X'), \bar{S}_i^k W^*(X')) \\ &\le \sup_{x'} d_p(\bar{S}_i^k(X')W_k(x'), \bar{S}_i^k(X')W^*(x')) \\ &\le \|\bar{S}_i^k(X')\|_p \sup_{x'} d_p(W_k(x'), W^*(x')) \\ &\le \delta_i \sup_{x'} d_p(W_k(x'), W^*(x')) \\ &\le \delta_i B. \end{aligned}$$

回想一下$B < \infty$是最大的可达到的$\|Z\|_\infty$。由于$\delta_i < \delta$也可以选择$x \in \mathcal{X}_{k+1,i+1}$，因此我们可以在（12）中通过$\gamma\delta B$在（12）中绑定第二项。这会产生

$$\begin{aligned} d_p(W_{k+1}(x), W^*(x)) &\le \\ &\gamma d_p(S_i^k W_k(X'), S_i^k W^*(X')) + \gamma\delta B. \end{aligned}$$

通过$i > 0$的归纳，我们得出结论，对于$x \in \mathcal{X}_{k+i,i}$和一些随机状态$X''$ i向前迈进，

$$\begin{aligned} d_p(W_{k+i}(x), W^*(x)) &\le \\ &\gamma^i d_p(S_0^k W_k(X''), S_0^k W^*(X'')) + \frac{\delta B}{1-\gamma} \\ &\le \gamma^i B + \frac{\delta B}{1-\gamma}. \end{aligned}$$

因此，对于任何$x \in \mathcal{X}$，$\epsilon > 0$，我们可以采用δ，i，并且最终k足以使$d_p(W_k(x), W^*(x)) < \epsilon$大。然后，通过考虑$\mathcal{T}$的另一种副本来扩展到$Z_k(x,a)$。

现在，我们考虑有多种最佳政策的更普遍的情况。我们扩展了$\mathcal{X}_{k,i}$的定义，如下所示:

$$\mathcal{X}_{k,i} := \big\{x \in \mathcal{X}_k : \forall \pi^* \in \Pi^*, \mathop{\mathbb{E}}_{a^* \sim \pi^*(x)} P(\mathcal{X}_{k-1,i-1} \,|\, x, a^*) \ge 1 - \delta\big\},$$

因为有很多行动，因此引理6也适用于这一新的定义。和以前一样，请$x \in \mathcal{X}_{k,i}$，但现在考虑通过$\mathcal{T}$连续应用程序选择的贪婪政策的顺序，然后写入

$$\mathcal{T}^{\bar{\pi}_k} := \mathcal{T}^{\pi_k}\mathcal{T}^{\pi_{k-1}} \cdots \mathcal{T}^{\pi_{k-i+1}},$$

这样

$$Z_{k+1} = \mathcal{T}^{\bar{\pi}_k} Z_{k-i+1}.$$

现在用$\mathcal{Z}^{**}$非组织最佳策略的集合表示。如果我们采用任何$Z^* \in \mathcal{Z}^*$，我们将其推断

$$\inf_{Z^{**} \in \mathcal{Z}^{**}} d_p(\mathcal{T}^{\bar{\pi}_k} Z^*(x,a), Z^{**}(x,a)) \le \frac{\delta B}{1-\gamma},$$

因为Z^*对应于某些最佳策略π^*，并且$\bar{\pi}_k$沿（x, a）的大多数轨迹是最佳的。实际上，$\mathcal{T}^{\bar{\pi}_k} Z^*$接近非局部 - 最佳策略$\bar{\pi}_k\pi^*$的价值分布。现在为此$Z^*$，

$$\begin{aligned} \inf_{Z^{**}} d_p(Z_k(x,a), Z^{**}(x,a)) \\ &\le d_p(Z_k(x,a), \mathcal{T}^{\bar{\pi}_k} Z^*(x,a)) \\ &\quad + \inf_{Z^{**}} d_p(\mathcal{T}^{\bar{\pi}_k} Z^*(x,a), Z^{**}(x,a)) \\ &\le d_p(\mathcal{T}^{\bar{\pi}_k} Z_{k-i+1}(x,a), \mathcal{T}^{\bar{\pi}_k} Z^*(x,a)) + \frac{\delta B}{1-\gamma} \\ &\le \gamma^i B + \frac{2\delta B}{1-\gamma}, \end{aligned}$$

使用与以前相同的参数与新固定的$\mathcal{X}_{k,i}$使用相同的参数。遵循

$$\inf_{Z^{**} \in \mathcal{Z}^{**}} d_p(Z_k(x,a), Z^{**}(x,a)) \to 0.$$

当$\mathcal{X}$是有限的时，存在一个固定的k，然后$\mathcal{X}_k = \mathcal{X}$。然后随后均匀的收敛结果。

为了证明当$\mathcal{T}$根据订购$\prec$进行动作时，固定点Z^*的唯一性，我们注意到，对于任何最佳价值分布Z^*，其贪婪的派别集为Π^*。用π^*表示策略在Π^*上首次提出。然后$\mathcal{T} = \mathcal{T}^{\pi^*}$，它具有唯一的固定点（第3.3节）。

□

命题4。*That $\mathcal{T}$ has a fixed point $Z^* = \mathcal{T}Z^*$ is insufficient to guarantee the convergence of $\{Z_k\}$ to $\mathcal{Z}^*$.*

我们在这里提供结果的草图。考虑一个单个状态x_1，其中两个动作a_1和a_2 (图8)。第一个动作产生的奖励为1 / 2，而另一个动作则以同等的概率产生0或1的奖励，并且两种动作都是最佳的。现在，以γ = 1 / 2为2，并为接收到的奖励编写$R_0, R_1, \ldots$。考虑一种随机策略，该策略将动作a_2带有概率p。对于$p = 0$，返回是

$$Z_{p=0} = \frac{1}{1-\gamma}\frac{1}{2} = 1.$$

另一方面，对于$p = 1$，返回是随机的，由以下分数编号（以二进制为单位）给出:

$$Z_{p=1} = R_0.R_1R_2R_3\cdots.$$

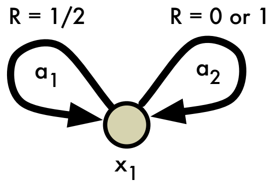

Figure 8. A simple example illustrating the effect of a nonstationary policy on the value distribution.

As a result, $Z_{p=1}$ is uniformly distributed between 0 and 2! In fact, note that

$$Z_{p=0} = 0.11111 \cdots = 1.$$

For some intermediary value of p, we obtain a different probability of the different digits, but always putting some probability mass on all returns in $[0, 2]$.

Now suppose we follow the nonstationary policy that takes a_1 on the first step, then a_2 from there on. By inspection, the return will be uniformly distributed on the interval $[1/2, 3/2]$, which does not correspond to the return under any value of p. But now we may imagine an operator $\mathcal{T}$ which alternates between a_1 and a_2 depending on the exact value distribution it is applied to, which would in turn converge to a nonstationary optimal value distribution.

Lemma 7 (Sample Wasserstein distance)**.** *Let $\{P_i\}$ be a collection of random variables, $I \in \mathbb{N}$ a random index independent from $\{P_i\}$, and consider the mixture random variable $P = P_I$. For any random variable Q independent of I,*

$$d_p(P, Q) \le \mathop{\mathbb{E}}_{i \sim I} d_p(P_i, Q),$$

and in general the inequality is strict and

$$\nabla_Q d_p(P_I, Q) \ne \mathop{\mathbb{E}}_{i \sim I} \nabla_Q d_p(P_i, Q).$$

Proof. We prove this using Lemma 1. Let $A_i := \mathbb{I}[I = i]$. We write

$$\begin{aligned} d_p(P, Q) &= d_p(P_I, Q) \\ &= d_p\Big(\sum\nolimits_i A_i P_i, \sum\nolimits_i A_i Q\Big) \\ &\le \sum\nolimits_i d_p(A_i P_i, A_i Q) \\ &\le \sum\nolimits_i \Pr\{I = i\} d_p(P_i, Q) \\ &= \mathbb{E}_I \, d_P(P_i, Q). \end{aligned}$$

where in the penultimate line we used the independence of I from P_i and Q to appeal to property P3 of the Wasserstein metric.

To show that the bound is in general strict, consider the mixture distribution depicted in Figure 9. We will simply consider the d_1 metric between this distribution P and another distribution Q. The first distribution is

$$P = \begin{cases} 0 & \text{w.p. } 1/2 \\ 1 & \text{w.p. } 1/2. \end{cases}$$

In this example, $i \in \{1, 2\}$, $P_1 = 0$, and $P_2 = 1$. Now consider the distribution with the same support but that puts probability p on 0:

$$Q = \begin{cases} 0 & \text{w.p. } p \\ 1 & \text{w.p. } 1 - p. \end{cases}$$

The distance between P and Q is

$$d_1(P, Q) = |p - \tfrac{1}{2}|.$$

This is $d_1(P, Q) = \frac{1}{2}$ for $p \in \{0, 1\}$, and strictly less than $\frac{1}{2}$ for any other values of p. On the other hand, the corresponding expected distance (after sampling an outcome x_1 or x_2 with equal probability) is

$$\mathbb{E}_I \, d_1(P_i, Q) = \tfrac{1}{2} p + \tfrac{1}{2}(1 - p) = \tfrac{1}{2}.$$

Hence $d_1(P, Q) < \mathbb{E}_I \, d_1(P_i, Q)$ for $p \in (0, 1)$. This shows that the bound is in general strict. By inspection, it is clear that the two gradients are different. □

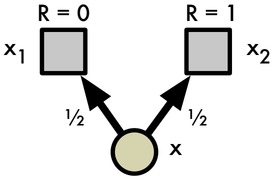

Figure 9. Example MDP in which the expected sample Wasserstein distance is greater than the Wasserstein distance.

Proposition 5. *Fix some next-state distribution Z and policy π. Consider a parametric value distribution Z_θ, and and define the Wasserstein loss*

$$\mathcal{L}_W(\theta) := d_p(Z_\theta(x, a), R(x, a) + \gamma Z(X', \pi(X'))).$$

Let $r \sim R(x, a)$ and $x' \sim P(\cdot \,|\, x, a)$ and consider the sample loss

$$L_W(\theta, r, x') := d_p(Z_\theta(x, a), r + \gamma Z(x', \pi(x'))).$$

Its expectation is an upper bound on the loss $\mathcal{L}_W$:

$$\mathcal{L}_W(\theta) \le \mathop{\mathbb{E}}_{R, P} L_W(\theta, r, x'),$$

in general with strict inequality.

The result follows directly from the previous lemma.

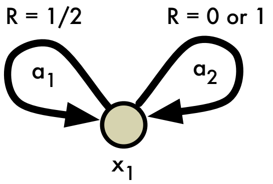

图8。一个简单的示例说明了非机构策略对价值分布的影响。

结果，$Z_{p=1}$在0到2之间均匀分布！实际上，请注意

$$Z_{p=0} = 0.11111 \cdots = 1.$$

对于p的某些中间值，我们获得了不同数字的不同概率，但始终在[0，2]中的所有返回上都放置一些概率质量。

现在假设我们遵循第一个步骤a_1的非组织策略，然后从那里开始a_2。通过检查，返回将在间隔[1 / 2，3 / 2]的间隔上均匀分布，这与p的任何值下的返回不符。但是现在我们可以想象一个操作员$\mathcal{T}$，该操作员在a_1和a_2之间交替，具体取决于它应用于的外部值分布，这又会收敛到非平稳的最佳价值分布。

引理7（样本瓦斯坦距离）。*Let $\{P_i\}$ be a collection of random variables, $I \in \mathbb{N}$ a random index independent from $\{P_i\}$, and consider the mixture random variable $P = P_I$. For any random variable Q independent of I,*

$$d_p(P, Q) \le \mathop{\mathbb{E}}_{i \sim I} d_p(P_i, Q),$$

and in general the inequality is strict and

$$\nabla_Q d_p(P_I, Q) \ne \mathop{\mathbb{E}}_{i \sim I} \nabla_Q d_p(P_i, Q).$$

Proof. 我们使用引理1。LETA_i：$= \mathbb{I}[I = i]$证明了这一点。我们写

$$\begin{aligned} d_p(P, Q) &= d_p(P_I, Q) \\ &= d_p\Big(\sum\nolimits_i A_i P_i, \sum\nolimits_i A_i Q\Big) \\ &\le \sum\nolimits_i d_p(A_i P_i, A_i Q) \\ &\le \sum\nolimits_i \Pr\{I = i\} d_p(P_i, Q) \\ &= \mathbb{E}_I \, d_P(P_i, Q). \end{aligned}$$

在倒数第二行中，我们使用I和Q的I的独立性吸引了Wasserstein Metric的属性P3。

为了表明界限通常是严格的，请考虑图9中描述的混合物分布。我们将简单地考虑此分布P和An-其他分布Q之间的d_1度量。第一个分布是

$$P = \begin{cases} 0 & \text{w.p. } 1/2 \\ 1 & \text{w.p. } 1/2. \end{cases}$$

在此示例中，$i \in \{ 1 , 2 \}$，$P_1 = 0$和$P_2 = 1$。现在考虑具有相同支持的分布，但在0：0：在0：v39}的情况下。

$$Q = \begin{cases} 0 & \text{w.p. } p \\ 1 & \text{w.p. } 1 - p. \end{cases}$$

P和Q之间的距离是

$$d_1(P, Q) = |p - \tfrac{1}{2}|.$$

对于$p \in \{ 0 , 1 \}$，这是$d_1(P, Q) = \frac{1}{2}$，严格小于$\frac{1}{2}$的任何其他值p。另一方面，相关的预期距离（在采样结果x_1或x_2以均等概率为

$$\mathbb{E}_I \, d_1(P_i, Q) = \tfrac{1}{2} p + \tfrac{1}{2}(1 - p) = \tfrac{1}{2}.$$

因此，$d_1(P, Q) < \mathbb{E}_I \, d_1(P_i, Q)$对于$p \in (0 , 1)$。这表明界限通常是严格的。通过检查，很明显，这两个梯度不同。 □

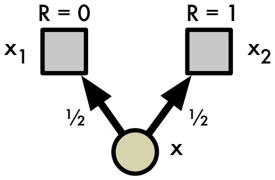

图9。示例MDP，其中预期的样品遗漏距离大于Wasserstein距离。

命题5。*Fix some next-state distribution Z and policy π. Consider a parametric value distribution Z_θ, and and define the Wasserstein loss*

$$\mathcal{L}_W(\theta) := d_p(Z_\theta(x, a), R(x, a) + \gamma Z(X', \pi(X'))).$$

Let $r \sim R(x, a)$ and $x' \sim P(\cdot \,|\, x, a)$ and consider the sample loss

$$L_W(\theta, r, x') := d_p(Z_\theta(x, a), r + \gamma Z(x', \pi(x'))).$$

Its expectation is an upper bound on the loss $\mathcal{L}_W$:

$$\mathcal{L}_W(\theta) \le \mathop{\mathbb{E}}_{R, P} L_W(\theta, r, x'),$$

in general with strict inequality.

结果直接来自先前的引理。

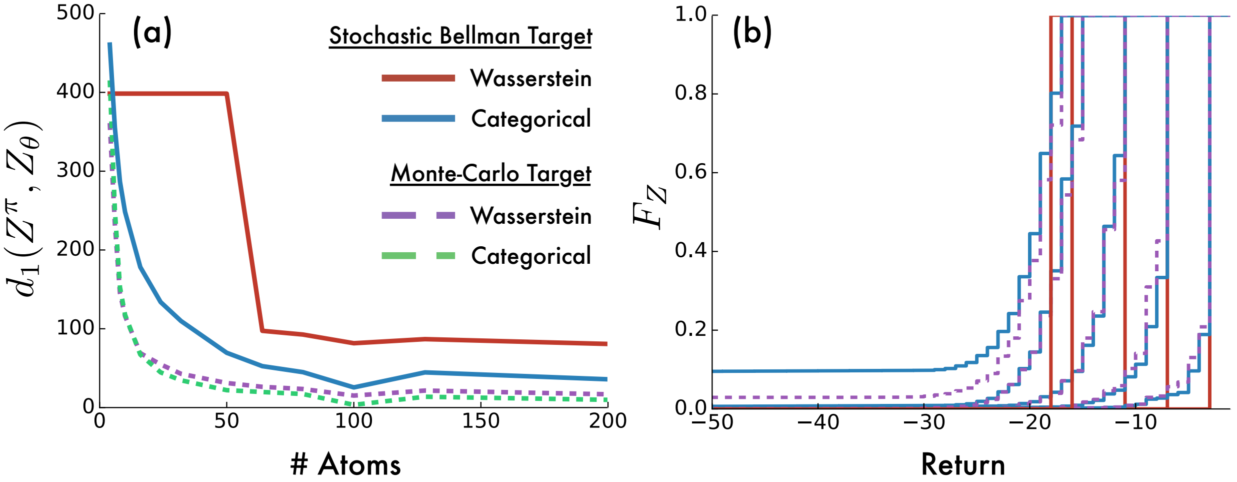

Figure 10. (a) Wasserstein distance between ground truth distribution Z^π and approximating distributions Z_θ. Varying number of atoms in approximation, training target, and loss function. (b) Approximate cumulative distributions for five representative states in CliffWalk.

C. Algorithmic Details

While our training regime closely follows that of DQN (Mnih et al., 2015), we use Adam (Kingma & Ba, 2015) instead of RMSProp (Tieleman & Hinton, 2012) for gradient rescaling. We also performed some hyperparameter tuning for our final results. Specifically, we evaluated two hyperparameters over our five training games and choose the values that performed best. The hyperparameter values we considered were $V_{\text{MAX}} \in \{3, 10, 100\}$ and $\epsilon_{adam} \in \{1/L, 0.1/L, 0.01/L, 0.001/L, 0.0001/L\}$, where $L = 32$ is the minibatch size. We found $V_{\text{MAX}} = 10$ and $\epsilon_{adam} = 0.01/L$ performed best. We used the same step-size value as DQN ($\alpha = 0.00025$).

Pseudo-code for the categorical algorithm is given in Algorithm 1. We apply the Bellman update to each atom separately, and then project it into the two nearest atoms in the original support. Transitions to a terminal state are handled with $\gamma_t = 0$.

D. Comparison of Sampled Wasserstein Loss and Categorical Projection

Lemma 3 proves that for a fixed policy π the distributional Bellman operator is a γ-contraction in $\bar{d}_p$, and therefore that $\mathcal{T}^\pi$ will converge in distribution to the true distribution of returns Z^π. In this section, we empirically validate these results on the CliffWalk domain shown in Figure 11. The dynamics of the problem match those given by Sutton & Barto (1998). We also study the convergence of the distributional Bellman operator under the sampled Wasserstein loss and the categorical projection (Equation 7) while following a policy that tries to take the safe path but has a 10% chance of taking another action uniformly at random.

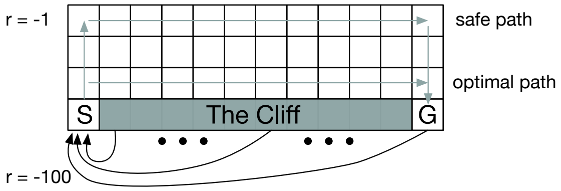

Figure 11. CliffWalk Environment (Sutton & Barto, 1998).

We compute a ground-truth distribution of returns Z^π using 10000 Monte-Carlo (MC) rollouts from each state. We then perform two experiments, approximating the value distribution at each state with our discrete distributions.

In the first experiment, we perform supervised learning using either the Wasserstein loss or categorical projection (Equation 7) with cross-entropy loss. We use Z^π as the supervised target and perform 5000 sweeps over all states to ensure both approaches have converged. In the second experiment, we use the same loss functions, but the training target comes from the one-step distributional Bellman operator with sampled transitions. We use $V_{\text{MIN}} = -100$ and $V_{\text{MAX}} = -1$.[4] For the sample updates we perform 10 times as many sweeps over the state space. Fundamentally, these experiments investigate how well the two training regimes

[4] Because there is a small probability of larger negative returns, some approximation error is unavoidable. However, this effect is relatively negligible in our experiments.

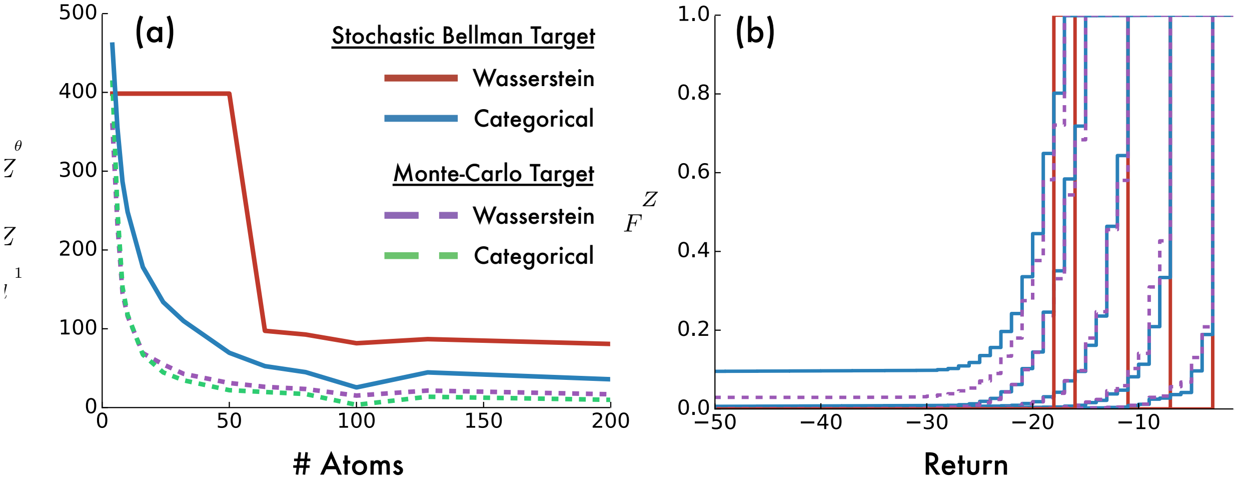

图10。（a）地面真相分布Z^{π}和近似分布Z_{θ}之间的距离。近似，训练目标和损失功能的原子数量有所不同。（b）在悬崖上五个代表状态的近似累积分布。

C.算法细节

尽管我们的培训制度紧密遵循DQN（Mnih等人，2015年），但我们使用Adam（Kingma＆Ba，2015年）而不是RMSPROP（Tieleman＆Hinton，2012）进行续订。我们还为最终结果进行了一些HyperParam-ETER调整。具体而言，我们在五个训练游戏中评估了两个超参数，并选择表现最佳的值。我们认为的HyperPa- Rameter值为$V_{\text{MAX}} \in \{3, 10, 100\}$和$\epsilon_{adam} \in \{1/L, 0.1.1/L, 0.01/L, 01/L, 0$ {V13 {V13 $.001/L, 0.0001/L\}$，其中$L = 32$是MiniBatch大小。我们发现$V_{\text{MAX}} = 10$和$\epsilon_{adam} = 0.01/L$执行最佳。我们使用了与DQN（$\alpha = 0.00025$）相同的步长值。

分类算法的伪代码在算法1中给出。我们将Bellman更新应用于每个原子sepa，然后将其投影到原始支持中的两个最近的原子中。$\gamma_t = 0$处理到终端状态的过渡。

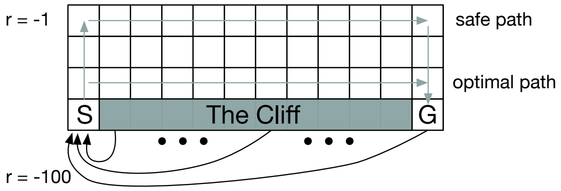

图11。悬崖环境（Sutton＆Barto，1998）。

D.采样的瓦斯林损失和分类投影的比较

引理3证明，对于固定策略π，分布式贝尔曼操作员是$\bar{d}_p$中的γ-contraction，因此$\mathcal{T}^{\pi}$将在分布中收敛到返回的真实分布Z^{π}。在本节中，我们从图11所示的悬崖域上进行了经验验证这些结果。问题的动力学匹配了Sutton＆Barto（1998）给出的动力学。我们还研究了在采样的瓦斯坦损失和分类投影下（等式7）下的分散式贝尔曼操作员的收敛性（方程7）降低试图采取安全道路但有10%的机会随机采取另一项行动的政策。

我们使用每个状态的10000 Monte-Carlo（MC）推出，计算回报Z^{π}的基础真相分布。然后，我们执行两个实验，以我们的离散分布近似于每个状态的值分布。

在第一个实验中，我们进行了监督的学习，以学习瓦斯汀损失或分类投影（方程7），并具有跨膜损失。我们使用Z^{π}作为监督目标，并对所有状态进行5000次扫描，以确保两种方法都收敛。在第二个实验中，我们使用相同的损失函数，但是训练目标来自带有采样过渡的一步分配钟形操作器。我们使用$V_{\text{MIN}} = -100$和$V_{\text{MAX}} = -1$.[4]用于示例更新，我们在状态空间上执行的扫描的10倍。从根本上讲，这些实验研究了两个培训制度

[4] Because there is a small probability of larger negative returns, some approximation error is unavoidable. However, this effect is relatively negligible in our experiments.

(minimizing the Wasserstein or categorical loss) minimize the Wasserstein metric under both ideal (supervised target) and practical (sampled one-step Bellman target) conditions.

In Figure 10a we show the final Wasserstein distance $d_1(Z^\pi, Z_\theta)$ between the learned distributions and the ground-truth distribution as we vary the number of atoms. The graph shows that the categorical algorithm does indeed minimize the Wasserstein metric in both the supervised and sample Bellman setting. It also highlights that minimizing the Wasserstein loss with stochastic gradient descent is in general flawed, confirming the intuition given by Proposition 5. In repeat experiments the process converged to different values of $d_1(Z^\pi, Z_\theta)$, suggesting the presence of local minima (more prevalent with fewer atoms).

Figure 10 provides additional insight into why the sampled Wasserstein distance may perform poorly. Here, we see the cumulative densities for the approximations learned under these two losses for five different states along the safe path in CliffWalk. The Wasserstein has converged to a fixed-point distribution, but not one that captures the true (Monte Carlo) distribution very well. By comparison, the categorical algorithm captures the variance of the true distribution much more accurately.

E. Supplemental Videos and Results

In Figure 13 we provide links to supplemental videos showing the C51 agent during training on various Atari 2600 games. Figure 12 shows the relative performance of C51 over the course of training. Figure 14 provides a table of evaluation results, comparing C51 to other state-of-the-art agents. Figures 15–18 depict particularly interesting frames.

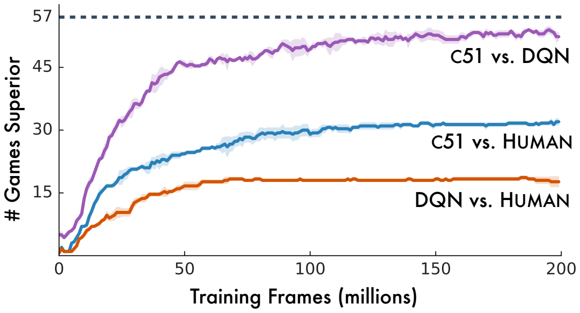

Figure 12. Number of Atari games where an agent's training performance is greater than a baseline (fully trained DQN & human). Error bands give standard deviations, and averages are over number of games.

GAMES	VIDEO URL
Freeway	`http://youtu.be/97578n9kFIk`
Pong	`http://youtu.be/vIz5P6s80qA`
Q*Bert	`http://youtu.be/v-RbNX4uETw`
Seaquest	`http://youtu.be/d1yz4PNFUjI`
Space Invaders	`http://youtu.be/yFBwyPuO2Vg`

Figure 13. Supplemental videos of C51 during training.

(最大程度地减少Wasserstein或分类损失)在理想（监督目标）和实用（采样一步的Bellman目标）条件下，将Wasserstein度量最小化。

在图10a中，我们显示了最终的瓦斯汀距离$d_1(Z^\pi, Z_\theta)$，因为我们在改变原子的数量时，学到的分布与地面真相分布之间的最终距离$d_1(Z^\pi, Z_\theta)$。该图表明，在监督和样本的贝尔曼设置中，分类算法确实确实使Wasserstein指标最小化。它还强调说，将瓦斯汀损失与随机梯度下降最小化是一般的浮躁，并确定了预言5所给出的直觉。在重复实验中，该过程融合到了$d_1(Z^\pi, Z_\theta)$的不同值，表明存在局部最小值（更普遍（更普遍）原子更少）。

图10提供了有关为何采样的瓦斯汀距离可能表现不佳的进一步见解。在这里，我们看到沿悬崖旁的安全路径的五个不同状态在这两个损失下学到的近似值的累积密度。 Wasserstein已融合到固定点分布，但没有很好地捕获真实（Monte Carlo）分布的分布。相比之下，分类算法更准确地捕获了真实分布的方差。

E.补充视频和结果

在图13中，我们提供了在各种Atari 2600游戏培训期间显示C51代理的补充视频的链接。图12显示了C51在训练过程中的相对性能。图14提供了评估结果表，将C51与其他最先进的代理进行了比较。图15–18描绘了特别有趣的帧。

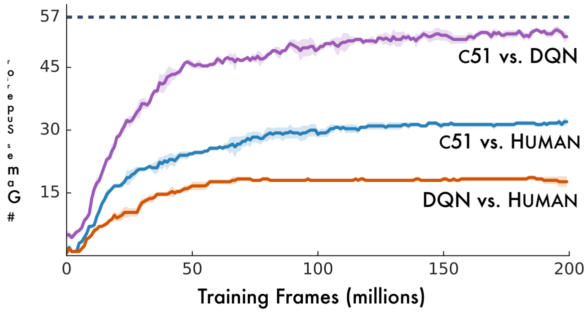

图12。代理商的训练能力大于基线（完全训练的DQN＆Human）的Atari游戏数量。错误频段会产生标准偏差，平均值超过了游戏。

GAMES	VIDEO URL
Freeway	http://youtu.be/97578n9kFIk
Pong	http://youtu.be/vIz5P6s80qA
Q*Bert	http://youtu.be/v-RbNX4uETw
Seaquest	http://youtu.be/d1yz4PNFUjI
Space Invaders	http://youtu.be/yFBwyPuO2Vg

图13。培训期间C51的补充视频。

GAMES	RANDOM	HUMAN	DQN	DDQN	DUEL	PRIOR. DUEL.	C51
Alien	227.8	**7,127.7**	1,620.0	3,747.7	4,461.4	3,941.0	3,166
Amidar	5.8	1,719.5	978.0	1,793.3	**2,354.5**	2,296.8	1,735
Assault	222.4	742.0	4,280.4	5,393.2	4,621.0	**11,477.0**	7,203
Asterix	210.0	8,503.3	4,359.0	17,356.5	28,188.0	375,080.0	**406,211**
Asteroids	719.1	**47,388.7**	1,364.5	734.7	2,837.7	1,192.7	1,516
Atlantis	12,850.0	29,028.1	279,987.0	106,056.0	382,572.0	395,762.0	**841,075**
Bank Heist	14.2	753.1	455.0	1,030.6	**1,611.9**	1,503.1	976
Battle Zone	2,360.0	**37,187.5**	29,900.0	31,700.0	37,150.0	35,520.0	28,742
Beam Rider	363.9	16,926.5	8,627.5	13,772.8	12,164.0	**30,276.5**	14,074
Berzerk	123.7	2,630.4	585.6	1,225.4	1,472.6	**3,409.0**	1,645
Bowling	23.1	**160.7**	50.4	68.1	65.5	46.7	81.8
Boxing	0.1	12.1	88.0	91.6	**99.4**	98.9	97.8
Breakout	1.7	30.5	385.5	418.5	345.3	366.0	**748**
Centipede	2,090.9	**12,017.0**	4,657.7	5,409.4	7,561.4	7,687.5	9,646
Chopper Command	811.0	7,387.8	6,126.0	5,809.0	11,215.0	13,185.0	**15,600**
Crazy Climber	10,780.5	35,829.4	110,763.0	117,282.0	143,570.0	162,224.0	**179,877**
Defender	2,874.5	18,688.9	23,633.0	35,338.5	42,214.0	41,324.5	**47,092**
Demon Attack	152.1	1,971.0	12,149.4	58,044.2	60,813.3	72,878.6	**130,955**
Double Dunk	-18.6	-16.4	-6.6	-5.5	0.1	-12.5	**2.5**
Enduro	0.0	860.5	729.0	1,211.8	2,258.2	2,306.4	**3,454**
Fishing Derby	-91.7	-38.7	-4.9	15.5	**46.4**	41.3	8.9
Freeway	0.0	29.6	30.8	33.3	0.0	33.0	**33.9**
Frostbite	65.2	4,334.7	797.4	1,683.3	4,672.8	**7,413.0**	3,965
Gopher	257.6	2,412.5	8,777.4	14,840.8	15,718.4	**104,368.2**	33,641
Gravitar	173.0	**3,351.4**	473.0	412.0	588.0	238.0	440
H.E.R.O.	1,027.0	30,826.4	20,437.8	20,130.2	20,818.2	21,036.5	**38,874**
Ice Hockey	-11.2	**0.9**	-1.9	-2.7	0.5	-0.4	-3.5
James Bond	29.0	302.8	768.5	1,358.0	1,312.5	812.0	**1,909**
Kangaroo	52.0	3,035.0	7,259.0	12,992.0	**14,854.0**	1,792.0	12,853
Krull	1,598.0	2,665.5	8,422.3	7,920.5	**11,451.9**	10,374.4	9,735
Kung-Fu Master	258.5	22,736.3	26,059.0	29,710.0	34,294.0	**48,375.0**	48,192
Montezuma's Revenge	0.0	**4,753.3**	0.0	0.0	0.0	0.0	0.0
Ms. Pac-Man	307.3	**6,951.6**	3,085.6	2,711.4	6,283.5	3,327.3	3,415
Name This Game	2,292.3	8,049.0	8,207.8	10,616.0	11,971.1	**15,572.5**	12,542
Phoenix	761.4	7,242.6	8,485.2	12,252.5	23,092.2	**70,324.3**	17,490
Pitfall!	-229.4	**6,463.7**	-286.1	-29.9	0.0	0.0	0.0
Pong	-20.7	14.6	19.5	20.9	**21.0**	20.9	20.9
Private Eye	24.9	**69,571.3**	146.7	129.7	103.0	206.0	15,095
Q*Bert	163.9	13,455.0	13,117.3	15,088.5	19,220.3	18,760.3	**23,784**
River Raid	1,338.5	17,118.0	7,377.6	14,884.5	**21,162.6**	20,607.6	17,322
Road Runner	11.5	7,845.0	39,544.0	44,127.0	**69,524.0**	62,151.0	55,839
Robotank	2.2	11.9	63.9	65.1	**65.3**	27.5	52.3
Seaquest	68.4	42,054.7	5,860.6	16,452.7	50,254.2	931.6	**266,434**
Skiing	-17,098.1	**-4,336.9**	-13,062.3	-9,021.8	-8,857.4	-19,949.9	-13,901
Solaris	1,236.3	**12,326.7**	3,482.8	3,067.8	2,250.8	133.4	8,342
Space Invaders	148.0	1,668.7	1,692.3	2,525.5	6,427.3	**15,311.5**	5,747
Star Gunner	664.0	10,250.0	54,282.0	60,142.0	89,238.0	**125,117.0**	49,095
Surround	-10.0	6.5	-5.6	-2.9	4.4	1.2	**6.8**
Tennis	-23.8	-8.3	12.2	-22.8	5.1	0.0	**23.1**
Time Pilot	3,568.0	5,229.2	4,870.0	8,339.0	**11,666.0**	7,553.0	8,329
Tutankham	11.4	167.6	68.1	218.4	211.4	245.9	**280**
Up and Down	533.4	11,693.2	9,989.9	22,972.2	**44,939.6**	33,879.1	15,612
Venture	0.0	1,187.5	163.0	98.0	497.0	48.0	**1,520**
Video Pinball	16,256.9	17,667.9	196,760.4	309,941.9	98,209.5	479,197.0	**949,604**
Wizard Of Wor	563.5	4,756.5	2,704.0	7,492.0	7,855.0	**12,352.0**	9,300
Yars' Revenge	3,092.9	54,576.9	18,098.9	11,712.6	49,622.1	**69,618.1**	35,050
Zaxxon	32.5	9,173.3	5,363.0	10,163.0	12,944.0	**13,886.0**	10,513

Figure 14. Raw scores across all games, starting with 30 no-op actions. Reference values from Wang et al. (2016).

GAMES	RANDOM	HUMAN	DQN	DDQN	DUEL	PRIOR. DUEL.	C51
Alien	227.8	**7,127.7**	1,620.0	3,747.7	4,461.4	3,941.0	3,166
Amidar	5.8	1,719.5	978.0	1,793.3	**2,354.5**	2,296.8	1,735
Assault	222.4	742.0	4,280.4	5,393.2	4,621.0	**11,477.0**	7,203
Asterix	210.0	8,503.3	4,359.0	17,356.5	28,188.0	375,080.0	**406,211**
Asteroids	719.1	**47,388.7**	1,364.5	734.7	2,837.7	1,192.7	1,516
Atlantis	12,850.0	29,028.1	279,987.0	106,056.0	382,572.0	395,762.0	**841,075**
Bank Heist	14.2	753.1	455.0	1,030.6	**1,611.9**	1,503.1	976
Battle Zone	2,360.0	**37,187.5**	29,900.0	31,700.0	37,150.0	35,520.0	28,742
Beam Rider	363.9	16,926.5	8,627.5	13,772.8	12,164.0	**30,276.5**	14,074
Berzerk	123.7	2,630.4	585.6	1,225.4	1,472.6	**3,409.0**	1,645
Bowling	23.1	**160.7**	50.4	68.1	65.5	46.7	81.8
Boxing	0.1	12.1	88.0	91.6	**99.4**	98.9	97.8
Breakout	1.7	30.5	385.5	418.5	345.3	366.0	**748**
Centipede	2,090.9	**12,017.0**	4,657.7	5,409.4	7,561.4	7,687.5	9,646
Chopper Command	811.0	7,387.8	6,126.0	5,809.0	11,215.0	13,185.0	**15,600**
Crazy Climber	10,780.5	35,829.4	110,763.0	117,282.0	143,570.0	162,224.0	**179,877**
Defender	2,874.5	18,688.9	23,633.0	35,338.5	42,214.0	41,324.5	**47,092**
Demon Attack	152.1	1,971.0	12,149.4	58,044.2	60,813.3	72,878.6	**130,955**
Double Dunk	-18.6	-16.4	-6.6	-5.5	0.1	-12.5	**2.5**
Enduro	0.0	860.5	729.0	1,211.8	2,258.2	2,306.4	**3,454**
Fishing Derby	-91.7	-38.7	-4.9	15.5	**46.4**	41.3	8.9
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Gravitar	173.0	**3,351.4**	473.0	412.0	588.0	238.0	440
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Ice Hockey	-11.2	**0.9**	-1.9	-2.7	0.5	-0.4	-3.5
James Bond	29.0	302.8	768.5	1,358.0	1,312.5	812.0	**1,909**
Kangaroo	52.0	3,035.0	7,259.0	12,992.0	**14,854.0**	1,792.0	12,853
Krull	1,598.0	2,665.5	8,422.3	7,920.5	**11,451.9**	10,374.4	9,735
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Montezuma’s Revenge	0.0	**4,753.3**	0.0	0.0	0.0	0.0	0.0
Ms. Pac-Man	307.3	**6,951.6**	3,085.6	2,711.4	6,283.5	3,327.3	3,415
Name This Game	2,292.3	8,049.0	8,207.8	10,616.0	11,971.1	**15,572.5**	12,542
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Pitfall!	-229.4	**6,463.7**	-286.1	-29.9	0.0	0.0	0.0
Pong	-20.7	14.6	19.5	20.9	**21.0**	20.9	20.9
Private Eye	24.9	**69,571.3**	146.7	129.7	103.0	206.0	15,095
Q*Bert	163.9	13,455.0	13,117.3	15,088.5	19,220.3	18,760.3	**23,784**
River Raid	1,338.5	17,118.0	7,377.6	14,884.5	**21,162.6**	20,607.6	17,322
Road Runner	11.5	7,845.0	39,544.0	44,127.0	**69,524.0**	62,151.0	55,839
Robotank	2.2	11.9	63.9	65.1	**65.3**	27.5	52.3
Seaquest	68.4	42,054.7	5,860.6	16,452.7	50,254.2	931.6	**266,434**
Skiing	-17,098.1	**-4,336.9**	-13,062.3	-9,021.8	-8,857.4	-19,949.9	-13,901
Solaris	1,236.3	**12,326.7**	3,482.8	3,067.8	2,250.8	133.4	8,342
Space Invaders	148.0	1,668.7	1,692.3	2,525.5	6,427.3	**15,311.5**	5,747
Star Gunner	664.0	10,250.0	54,282.0	60,142.0	89,238.0	**125,117.0**	49,095
Surround	-10.0	6.5	-5.6	-2.9	4.4	1.2	**6.8**
Tennis	-23.8	-8.3	12.2	-22.8	5.1	0.0	**23.1**
Time Pilot	3,568.0	5,229.2	4,870.0	8,339.0	**11,666.0**	7,553.0	8,329
Tutankham	11.4	167.6	68.1	218.4	211.4	245.9	**280**
Up and Down	533.4	11,693.2	9,989.9	22,972.2	**44,939.6**	33,879.1	15,612
Venture	0.0	1,187.5	163.0	98.0	497.0	48.0	**1,520**
Video Pinball	16,256.9	17,667.9	196,760.4	309,941.9	98,209.5	479,197.0	**949,604**
Wizard Of Wor	563.5	4,756.5	2,704.0	7,492.0	7,855.0	**12,352.0**	9,300
Yars’ Revenge	3,092.9	54,576.9	18,098.9	11,712.6	49,622.1	**69,618.1**	35,050
Zaxxon	32.5	9,173.3	5,363.0	10,163.0	12,944.0	**13,886.0**	10,513

图14。从30个无操作动作开始，所有游戏的原始分数。 Wang等人的参考值。（2016）。

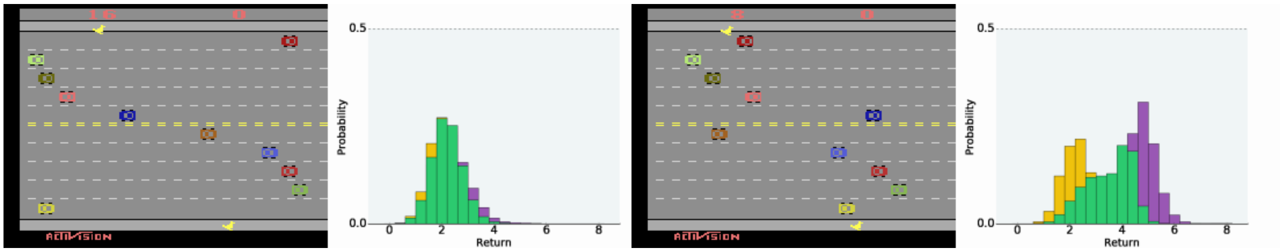

Figure 15. FREEWAY: Agent differentiates action-value distributions under pressure.

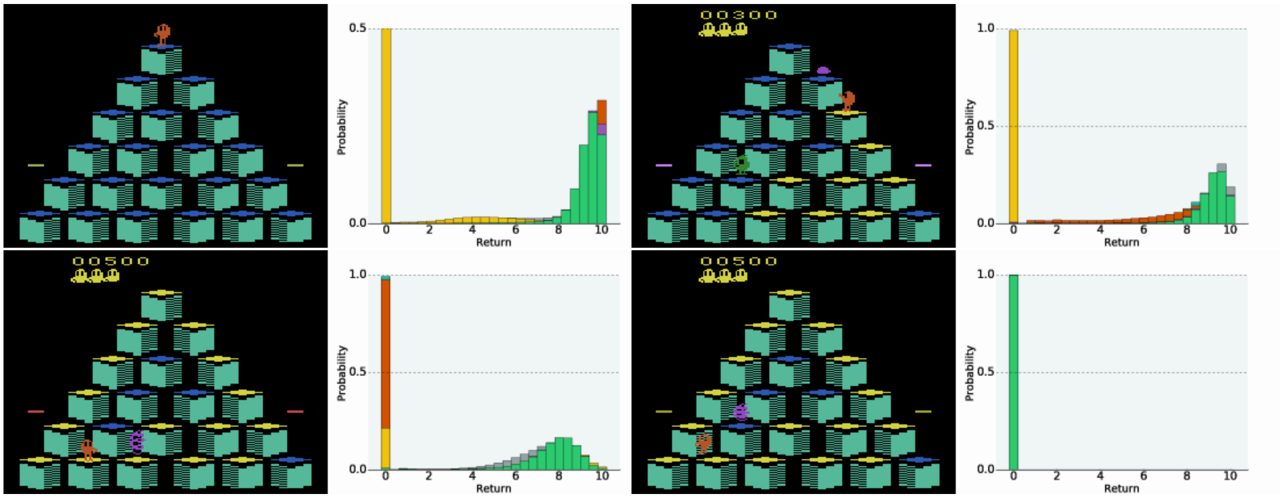

Figure 16. Q*BERT: Top, left and right: Predicting which actions are unrecoverably fatal. Bottom-Left: Value distribution shows steep consequences for wrong actions. Bottom-Right: The agent has made a huge mistake.

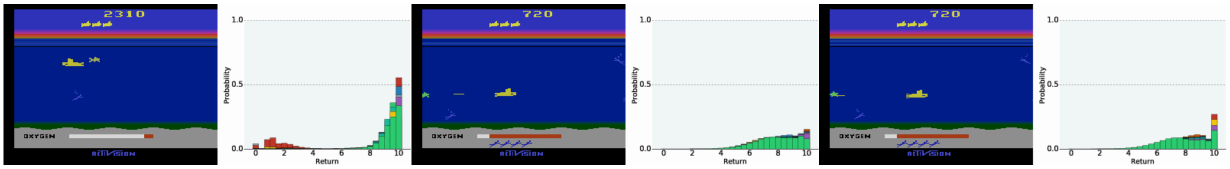

Figure 17. SEAQUEST: Left: Bimodal distribution. Middle: Might hit the fish. Right: Definitely going to hit the fish.

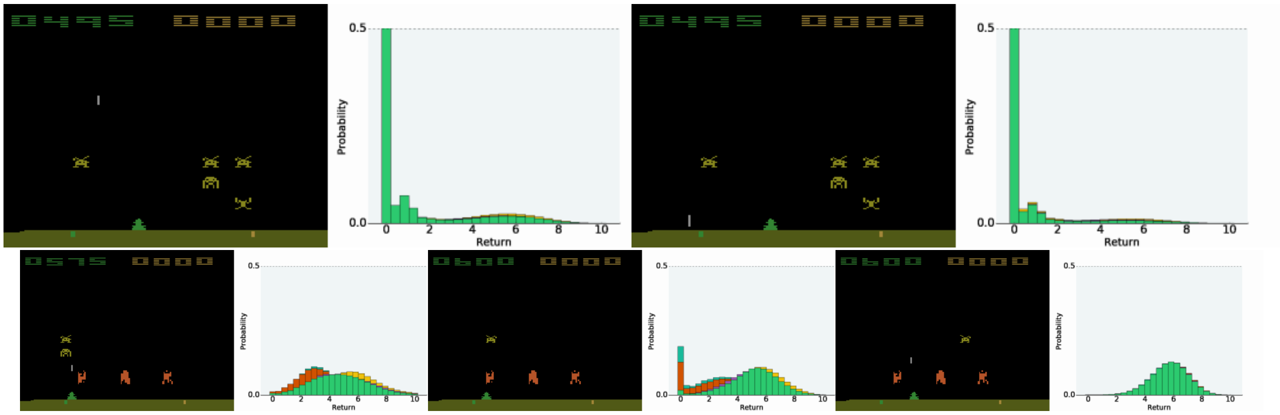

Figure 18. SPACE INVADERS: Top-Left: Multi-modal distribution with high uncertainty. Top-Right: Subsequent frame, a more certain demise. Bottom-Left: Clear difference between actions. Bottom-Middle: Uncertain survival. Bottom-Right: Certain success.

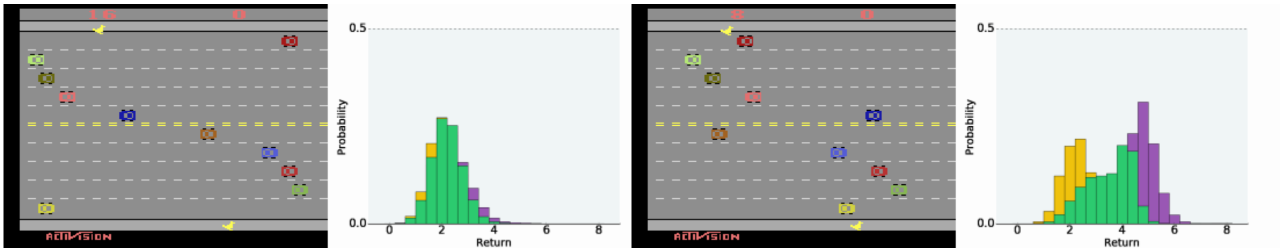

图15。高速公路：代理在压力下区分了动作值分布。

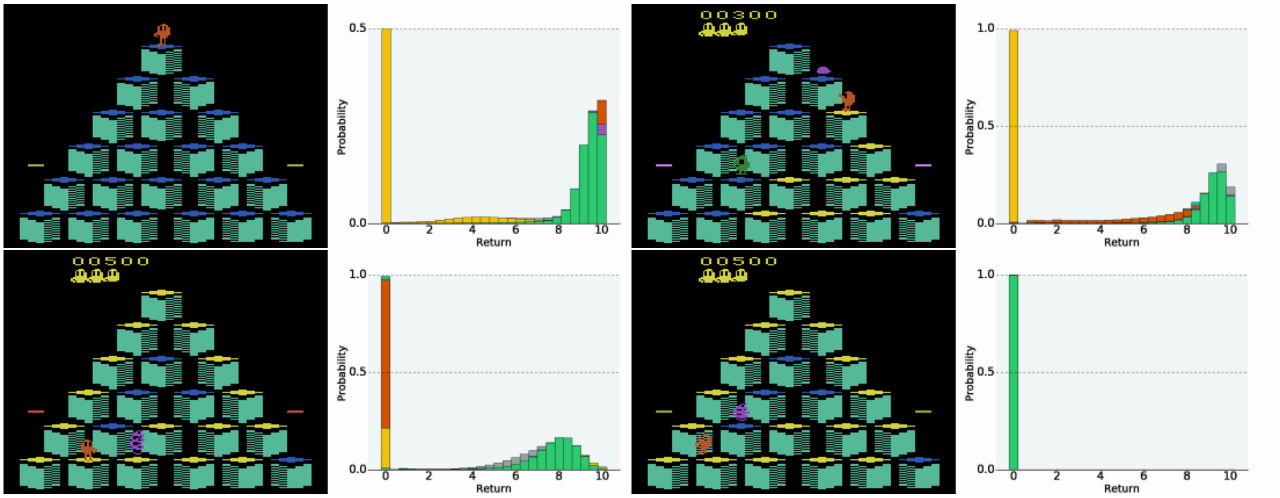

图16。Q*BERT：顶部，左右：预测哪些行动是不可恢复的致命。左下：价值分布显示出对错误动作的严重后果。右下角：代理商犯了一个巨大的错误。

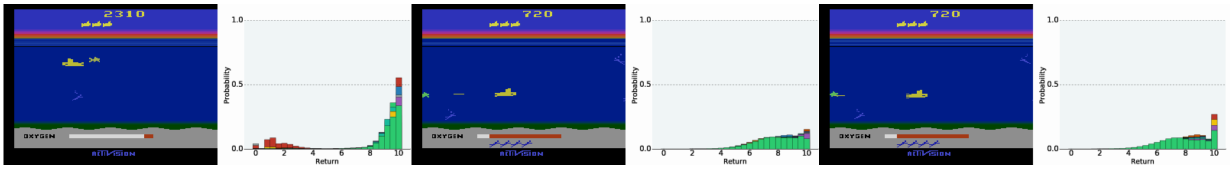

图17。Seaquest：左：双峰分布。中间：可能会击中鱼。右：绝对要击中鱼。

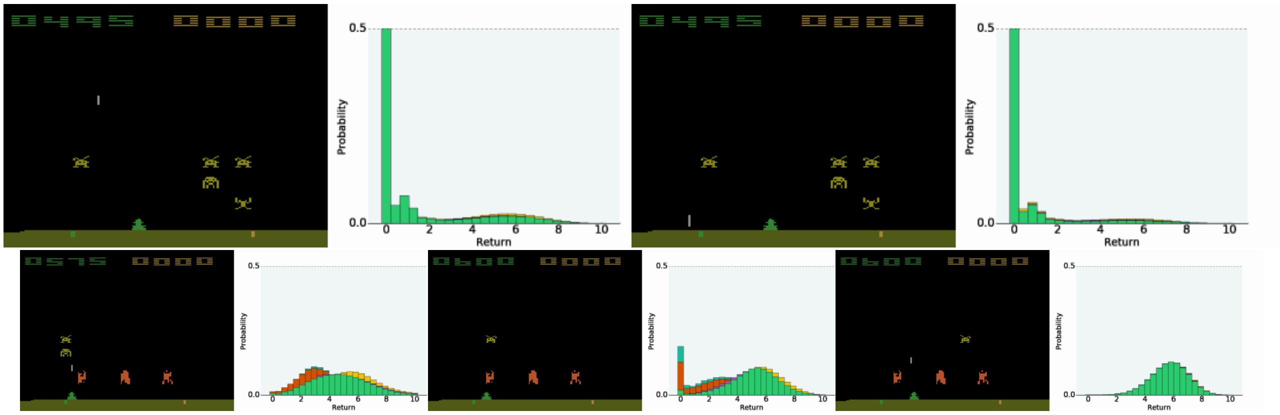

图18。空间入侵者：左上：多模式分布，不确定性高。右翼：随后的框架，更确定的灭亡。左下：动作之间的明显差异。底部中间：不确定的生存。右右：一定的成功。