# Assignment 6

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## Problem 1

### question 1

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**Indices**: $i = 1, 2, ..., n$ and $j = 1, 2, ..., n$ for different assets.
**Data**:

- $B$ is the budget, its unit: dollars.
- $r$ is the minimum requirement of the expected dollar return of the portfolio, its unit: dollars.
- $\mu_i \in \mathbb{R}^n$ is the $i$th entry of the mean vector $\vec{\mu}$ of $\vec{p}$.
- $\Sigma_{i,j} \in \mathbb{R}^{n\times n}$ is the $i$th row and $j$th column of the covariance matrix $\Sigma$ of $\vec{p}$.

**Variables**: $x_i$ is the total dollar amount of asset i held for this period. $y_i$ is a binary variable indicating if $x_i$ is in short position or in long position.
**Objective**: minimize $\vec{x}^T\Sigma\vec{x}$, which is minimize $\sum_{i=1}^n \sum_{j=1}^n x_i\Sigma_{i,j}x_j$
**Constraints**:

- $M(1 - y_i) \geq x_i$ for every i.
- $-My_i \leq x_i$ for every i.
- $M$ is a significantly big constant. Here $M = 2B$.
- The above 3 constraints make $y_i = 0$ an indicator of $x_i \geq 0$ and $y_i = 1$ an indicator of $x_i \leq 0$.
- $\sum_{i=1}^n \mu_i x_i \geq r$, which makes the expected dollar return be at least r dollars.
- $\sum_{i=1}^n x_i(1 - y_i) + \sum_{i=1}^n -x_i y_i \leq B$, the sum of the absolute value of portfolio is below the budget (as told by Professor Tovey).
- $\sum_{i=1}^n -y_i x_i \leq 0.2 \sum_{i=1}^n (1 - y_i)x_i$

**Bounds**: $y_i = 0$ or $y_i = 1$.

### question 2

Result as shown in Figure 1.

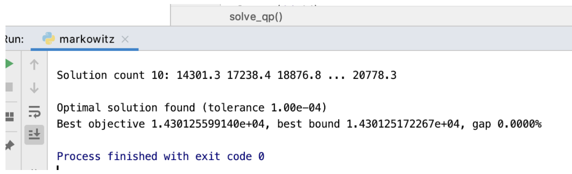


Figure 1

## Problem 2

### question 1

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**Indices**: $i = 1, 2, ..., n$ and $j = 1, 2, ..., n$ for different dimensions in space $\mathbb{R}^n$. $k = 1, 2, ..., m$ for different dimensions in space $\mathbb{R}^m$.

**Data**:

Note: Let $S^n$ denote the set of symmetric $n$−by−$n$ matrices and $S^n_+$ denote the set of symmetric positive semidefinite $n$−by−$n$ matrices. Moreover, if $X \in \mathbb{R}^{n\times n}$, let $X \succeq 0$ indicate that $X \in S^n_+$

- $Q_0$ and $\gamma$ together form the feasible region of $Q \in \mathcal{E}(\gamma)$, where $Q$ is the quadratic term of the problem. The region is defined as $\mathcal{E}(\gamma) := \{Q \in S^n : -\gamma I \preceq Q - Q_0 \preceq \gamma I\}$. $(Q_0)_{i,j}$ is the entry at the $i$th row and $j$th column of the matrix $Q_0$ and $\gamma$ is a scalar.
- $A$ and $b$ together form the constraint $Ax \le b$ ($x$ is defined later in variables section), where $A_{k,j}$ is the entry at the $k$th row and the $j$th column of the matrix $A$ and $b_k$ is the $k$th entry of the vector $b$.
- $c$ is the linear term of the problem. $c_j$ is the $j$th entry of the vector $c$.
- $d$ is the constant term of the problem.

**Variables**: $x$ with $x_j$ as its $j$th entry.
**Objective**: minimize $\frac{1}{2}x^T(Q_0 + \gamma I)x - c^T x + d$.
This equals $\sup_{Q\in\mathcal{E}(\gamma)} \left\{\frac{1}{2}x^T Qx - c^T x + d\right\}$ because $x^T \gamma I x \ge 0$ always holds.
This is equivalent to minimize$\frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n x_i(Q_0)_{i,j}x_j + \frac{1}{2}\sum_{i=1}^n \gamma x_i^2 - \sum_{j=1}^n c_j x_j + d$
**Constraints**: $Ax \le b$. This is equivalent to $\sum_{j=1}^n A_{k,j}x_j \le b_k$ for every k.
**Bounds**: NA.

### question 2

Result as shown in Figure 2.

## Problem 3

### question 1

**Indices**: $i = 1, 2, ..., m$ for different dimensions in space $\mathbb{R}^m$. $j = 1, 2, ..., n$ and $k = 1, 2, ..., n$ for different dimensions in space $\mathbb{R}^n$.

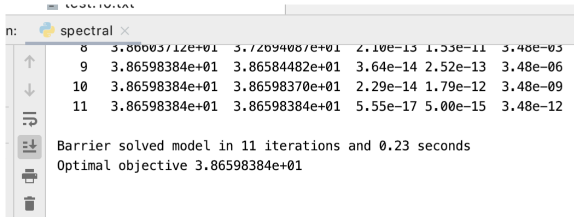


Figure 2

**Data**:

- $A$ and $b$ together form the constraint $Ax = b$ ($x$ is defined later in variables section), where $A_{i,j}$ is the entry at the $i$th row and the $j$th column of the matrix $A$ and $b_i$ is the $i$th entry of the vector $b$.
- $\mu$ and $\Sigma$, respectively, are the mean and covariance matrix of a normally distributed $c \in \mathbb{R}^n$.
- $\alpha$ is a constant.

**Variables**:

- $x_j$ are the decision variables.
- $y_j = \frac{x_j}{\mu^T x - \alpha} = \frac{x_j}{\sum_{j=1}^n (\mu_j x_j) - \alpha}$ are auxiliary variables.
- $s = \frac{1}{\mu^T x - \alpha} = \frac{1}{\sum_{j=1}^n (\mu_j x_j) - \alpha}$ are auxiliary variables.

**Objective**: minimize $y^T \Sigma y$, which equals minimize $\sum_{j=1}^n \sum_{k=1}^n y_j \Sigma_{j,k} y_k$
**Constraints and Bounds**:

- $Ay = bs$, which equals $\sum_{j=1}^n A_{i,j} y_j = b_i s$ for every i.
- $\mu^T y - \alpha s = 1$, which equals $\sum_{j=1}^n \mu_j y_j - \alpha s = 1$.
- $s \geq 0$
- $y_j \geq 0$ for every j.

**Reasons for this formulation**

Assume a random variable $v = c^T x$ is normally distributed with a mean of $\mu^T x$ and a variance of $x^T \Sigma x$, then $(\text{prob}(v \geq \alpha))^2 = (\text{prob}\left(\frac{v - \mu^T x}{\sqrt{x^T \Sigma x}} \geq \frac{\alpha - \mu^T x}{\sqrt{x^T \Sigma x}}\right))^2 = (1 - \Phi\left(\frac{\alpha - \mu^T x}{\sqrt{x^T \Sigma x}}\right))^2$, where $\Phi(\cdot)$ is the distribution function of a normal random variable with zero mean and unit covariance. Thus, in order to maximize prob $(v \geq \alpha)$, we can minimize $\left(\alpha - \mu^T x\right)^2 / x^T \Sigma x$. Hence, by setting $y = \frac{x}{\mu^T x - \alpha}$ and $s = \frac{1}{\mu^T x - \alpha}$, this problem becomes the problem as formulated above.

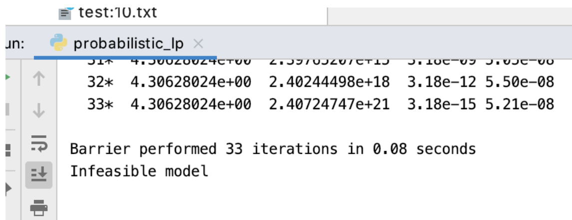


Figure 3

**question 2**

Result as shown in Figure 3. The model is infeasible when m is big, but there should be some results when setting $m = 3$ (as told by piazza).

## Problem 4

### question 1: formulate a one-variable NLP to find the maximum profit

Formulation:

**Data**: $c$ is the fixed cost, $L$ and $U$, respectively, are the lower bound and upper bound of price $p$ (defined later), $\alpha$, $\beta$, $a$, and $b$ determine the distribution of the demand $D$ together.

**Variables**: $p$ is the price of scallops.

**Objective**: maximize $\mathbb{E}(f(p))$, where $\mathbb{E}(f(p))$ is the expectation of the profit function $f(p)$. The profit function $f(p)$ is a function with $p$ as its single variable.

**Constraints and Bounds**: $L \leq p \leq U$

The profit function $f(p)$ and its expectation $\mathbb{E}(f(p))$ can be written as below:

$$\begin{aligned}
\mathbb{E}(f(p)) &= \mathbb{E}(pD^* - cq^*) \\
&= \mathbb{E}(pD) - cq^* \\
&= \mathbb{P}(D \leq q^*)pD + \mathbb{P}(D > q^*)pq^* - cq^* \\
&= \frac{q^* - x}{y - x} p \frac{q^* + x}{2} + \frac{y - q^*}{y - x} pq^* - cq^* \\
&= \frac{p}{2}(y + x + (y - x)\frac{c^2}{p^2} - 2y\frac{c}{p}) \\
\mathbb{E}(f(p)) &= \frac{p}{2}(a + b) - \frac{p^2}{2}(\alpha + \beta) + \frac{c^2}{2p}(b - a - \beta p + \alpha p) - c(b - \beta p)
\end{aligned} \tag{1}$$

The first-order derivative and the second-order derivative can be written as below, where we use the $\mathbb{E}(f(p))' = 0$ to compute the Newton's method:

$$\mathbb{E}(f(p))' = \frac{1}{2}(a + b) - p(\alpha + \beta) + c\beta - \frac{c^2}{2p^2}(b - a) \tag{2}$$

$$\mathbb{E}(f(p))'' = -(\alpha + \beta) + \frac{c^2}{p^3}(b - a) \tag{3}$$

The Newton's method solved $p = 416.2$ with a profit of 230943.6. However, this is out of the boundary, after adjusting the number inside boundary, the profit is 48029.2, as shown in figure 4.

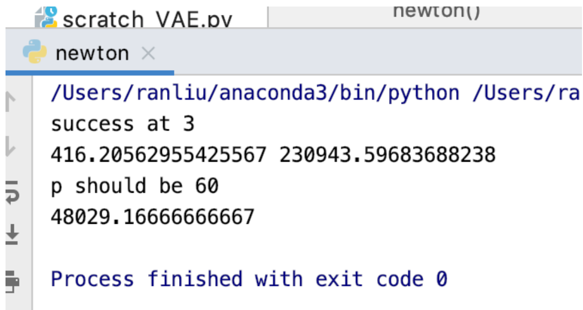


Figure 4

**Compare the convergence**: the secant quasi-Newton method takes more convergence steps and has a less precise convergence path comparing with Newton's method, since it does not need a second derivative. The secant quasi-Newton method gives a result with a larger margin of errors.

**Why it would be absurd to solve the model to such precision**: 1) The method is overall slow since it takes more convergence steps. 2) The final solution solved has a large margin of error, which could cause a lot of money loss. 3) In the real-life situation there are more factors to consider other than the profit. 4) In real-life situation, people would typically select a normal number such as an integer to be the price. It would be absurd to have 1.23456 to be a price.

**question 2**

Formulation:

**Data**: $c$ is the fixed cost, $L$ and $U$, respectively, are the lower bound and upper bound of price $p$ (defined later), $\alpha$, $\beta$, $a$, and $b$ determine the distribution of the demand $D$ together.

**Variables**: $p$ is the price of scallops. $q$ is the quantity of scallops to buy.

**Objective**: maximize $\mathbb{E}(g(p,q))$, where $\mathbb{E}(g(p,q))$ is the expectation of the profit function $g(p,q)$. The profit function $g(p,q)$ is a function with $p$ and $q$ as its two variables.

**Constraints and Bounds**: $L \leq p \leq U$.

The profit function $g(p,q)$ and its expectation $\mathbb{E}(g(p,q))$ can be written as below:

$$
\begin{aligned}
&\text{Situation 1: when } q \geq y \text{, which equals } q \geq b - \beta p \\
&\mathbb{E}(g(p,q)) = p\frac{x+y}{2} - cq = p\frac{a+b-\alpha p - \beta p}{2} - cq \\
&\text{Situation 2: when } q \leq x \text{, which equals } q \leq a - \alpha p \\
&\mathbb{E}(g(p,q)) = (p-c)q \\
&\text{Situation 3: when } x \leq q \leq y \text{, which equals } a - \alpha p \leq q \leq b - \beta p \\
&\mathbb{E}(g(p,q)) = \frac{2yq - x^2 - q^2}{2(y-x)}p - cq = \frac{2(b-\beta p)q - (a-\alpha p)^2 - q^2}{2(b - \beta p - (a - \alpha p))}p - cq
\end{aligned} \tag{4}
$$

**We change the notations to standard KKT condition notations in below KKT conditions.**

**Situation 1**

$$
\begin{aligned}
\text{maximize } f = \frac{1}{2}((a+b)p - (\alpha+\beta)p^2) - cq \\
g_1 = p - U \leq 0 \\
g_2 = L - p \leq 0 \\
g_3 = b - \beta p - q \leq 0
\end{aligned} \tag{5}
$$

**KKT conditions**

- $\frac{a+b}{2} - (\alpha+\beta)p - \mu_1 + \mu_2 + \beta\mu_3 = 0$
- $-c + \mu_3 = 0$
- $p - U \leq 0$, $L - p \leq 0$, $b - \beta p - q \leq 0$
- $\mu_1 \geq 0$, $\mu_2 \geq 0$, $\mu_3 \geq 0$
- $\mu_1(p-U) + \mu_2(L-p) + \mu_3(b - \beta p - q) = 0$

**Solution** The KKT conditions make $b - \beta p - q = 0$, which equals $q = y$.
When $p \neq U$ and $p \neq L$, $p = \frac{1}{\alpha+\beta}(\frac{a+b}{2} + \beta c)$ and $q = b - \frac{\beta}{\alpha+\beta}(\frac{a+b}{2} + \beta c)$.
When $p = U$, $q = b - \beta U$.
When $p = L$, $q = b - \beta L$.

**Situation 2**

$$
\begin{aligned}
\text{maximize } f = (p-c)q \\
g_1 = p - U \leq 0 \\
g_2 = L - p \leq 0 \\
g_3 = q - a + \alpha p \leq 0
\end{aligned} \tag{6}
$$

**KKT conditions**

- $q - \mu_1 + \mu_2 - \mu_3\alpha = 0$
- $p - c - \mu_3 = 0$
- $p - U \leq 0$, $L - p \leq 0$, $q - a + \alpha p \leq 0$

- $\mu_1 \geq 0,\ \mu_2 \geq 0,\ \mu_3 \geq 0$
- $\mu_1(p-U) + \mu_2(L-p) + \mu_3(q-a+\alpha p) = 0$

**Solution** Similarly, the KKT conditions make $(p-c)(q-a+\alpha p) = 0$, which equals $q = x$ or $p = c$.

When $p = c$, $q = 0$.

When $q = x$, $p \neq U$, $p \neq L$, the optimal solution can be found at $p = \frac{a+c\alpha}{2\alpha}$ with $q = \frac{a-c\alpha}{2}$.

When $q = x$, $p = U$, $q = a - \alpha U$.

When $q = x$, $p = L$, $q = a - \alpha L$.

**Situation 3**

$$\begin{aligned} \text{maximize } f = \frac{2(b-\beta p)q - (a-\alpha p)^2 - q^2}{2(b-\beta p-(a-\alpha p))}p - cq \\ g_1 = p - U \leq 0 \\ g_2 = L - p \leq 0 \\ g_3 = a - \alpha p - q \leq 0 \\ g_4 = q - b + \beta p \leq 0 \end{aligned} \tag{7}$$

**KKT conditions**

- $\frac{(b-\beta p)p-qp}{b-\beta p-a+\alpha p} - c + \mu_3 - \mu_4 = 0$
- $\frac{\partial f}{\partial p} - \mu_1 + \mu_2 + \mu_3\alpha - \mu_4\beta = 0$, where $\frac{\partial f}{\partial p} = q - \frac{(a-q-\alpha p)[(a-q-3\alpha p)(b-a)-2\alpha p^2(\alpha-\beta)]}{2(b-a-\beta p+\alpha p)^2}$
- $p-U \leq 0,\ L-p \leq 0,\ a-\alpha p-q \leq 0,\ q-b+\beta p \leq 0$
- $\mu_1 \geq 0,\ \mu_2 \geq 0,\ \mu_3 \geq 0,\ \mu_4 \geq 0$
- $\mu_1(p-U) + \mu_2(L-p) + \mu_3(a-\alpha p-q) + \mu_4(q-b+\beta p) = 0$

**Solution**

Situation 1, when $q = a - \alpha p$ and $\mu_1 = \mu_2 = \mu_4 = 0$, $p = \frac{a+c\alpha}{2\alpha}$ and $q = \frac{a-c\alpha}{2}$.

Situation 2, when $q = b - \beta p$ and $\mu_1 = \mu_2 = \mu_3 = 0$, $\mu_4 = -c$, no solution. As long as $q = b - \beta p$, there is no solution.

Situation 3, when $q = a - \alpha p$, $p = U$, and $\mu_2 = \mu_4 = 0$, $q = a - \alpha U$.

Situation 4, when $q = a - \alpha p$, $p = L$, and $\mu_1 = \mu_4 = 0$, $q = a - \alpha L$.

Situation 5, the interior case when $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$. Now we want to solve $\frac{\partial f}{\partial p} = 0$ and $\frac{\partial f}{\partial q} = c$. Using $x = a - \alpha p$ and $y = b - \beta p$ to represent the variables, from $\frac{\partial f}{\partial q} = c$, we get $\frac{yp-qp}{y-x} = c$ and thus $q = y - (y-x)\frac{c}{p}$, which is same as question a. Solving $\frac{\partial f}{\partial p} = 0$, we get $q = \frac{1}{2}(1-\frac{c}{p})[(1-\frac{c}{p})(b-a)+2\alpha p]$. This equals $2p^3(\beta+\alpha) - 2\beta cp^2 = a(p^2+c^2) + b(p^2-c^2)$. I used Newton's method to solve the above equation and got an optimal solution that is out of the bound of p (which is same as the situation in question a). The function is as plotted in Figure 5, where the optimal solution is the same as the optimal solution in part a.

**Conclusion: The result I get in part b is same as the result I get in part a.**

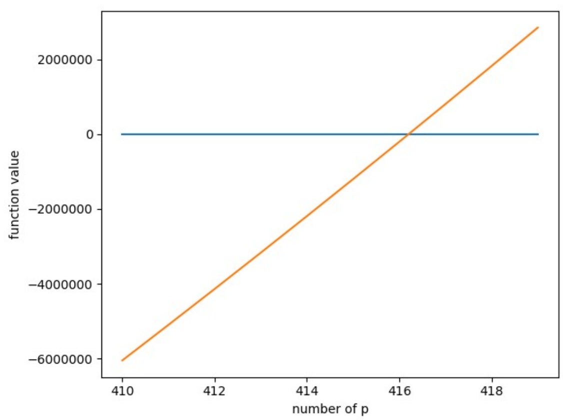


Figure 5: The x value when function equals zero is the same as in part a.