Assignment 5

Ran Liu rliu361@gatech.edu
903515184

Problem 1

question 1

Indices: $i = 1, 2, ..., N$ and $j = 1, 2, ..., N$ for different products. $s = 1, 2, ..., M$ and $t = 1, 2, ..., M$ for different inputs. $k = 1, 2, ..., q$ for different linear constraints.

Variables: The definition of variables is put before the definition of data in order to clarify the definition of data. x_i is the amount of product i that is sold. y_s is the amount of inputs s that is purchased.

Data:

- $A_{i,j}$ and b_i together form the price of the products $P_x = b - Ax$, where the ith entry of the vector $(P_x)_i = b_i - \sum_{j=1}^{N} A_{i,j} x_j$.
- $C_{s,t}$ and d_s together form the price of the inputs $P_y = d + Cy$, where the sth entry of the vector $(P_y)_s = d_s + \sum_{t=1}^{M} C_{s,t} y_t$.
- $E_{k,j}$, $F_{k,t}$, and g_k together form q linear constraints $Ex + Fy \leq g$, where for the kth constraint, $\sum_{j=1}^{N} E_{k,j} x_j + \sum_{t=1}^{M} F_{k,t} y_t \leq g_k$.

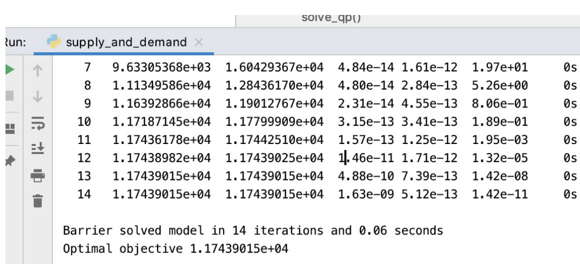
Objective: maximize $xP_x - yP_y$, which equals maximizing $\sum_{i=1}^{N}(x_i b_i - \sum_{j=1}^{N} A_{i,j} x_j x_i) - \sum_{s=1}^{M}(y_s d_s + \sum_{t=1}^{M} C_{s,t} y_t y_s)$

Constraints: $\sum_{j=1}^{N} E_{k,j} x_j + \sum_{t=1}^{M} F_{k,t} y_t \leq g_k$ for every k.

Bounds: x_i and y_s have lower bound 0.

question 2

Result as shown in Figure 1.



```
solve_qp()
Run: supply_and_demand
   7   9.63305368e+03  1.60429367e+04  4.84e-14 1.61e-12  1.97e+01     0s
   8   1.11349586e+04  1.28436170e+04  4.80e-14 2.84e-13  5.26e+00     0s
   9   1.16392866e+04  1.19012767e+04  2.31e-14 4.55e-13  8.06e-01     0s
  10   1.17187145e+04  1.17799909e+04  3.15e-13 3.41e-13  1.89e-01     0s
  11   1.17436178e+04  1.17442510e+04  1.57e-13 1.25e-12  1.95e-03     0s
  12   1.17438982e+04  1.17439025e+04  1.46e-11 1.71e-12  1.32e-05     0s
  13   1.17439015e+04  1.17439015e+04  4.88e-10 7.39e-13  1.42e-08     0s
  14   1.17439015e+04  1.17439015e+04  1.63e-09 5.12e-13  1.42e-11     0s

Barrier solved model in 14 iterations and 0.06 seconds
Optimal objective 1.17439015e+04
```

Figure 1

Problem 2

question 1

Indices: $i = 1, 2, ..., m$ for different dimensions in space R^m. $j = 1, 2, ..., n$ for different dimensions in space R^n.

Data: $A \in R^{m \times n}$ is a matrix with $A_{i,j}$ as its entry at ith row and jth column. a_i^T is the i^{th} row of the matrix A. $b \in R^m$ is a vector with b_i as its ith entry.

Variables: x_j is the decision variable. y_i is the auxiliary variable that represent each reciprocals.

Objective: minimize $\sum_{i=1}^{m} y_i$

Constraints:

- $\|(2, a_i^T x - b_i - y_i)\|_2 \leq a_i^T x - b_i + y_i$ for every i. This is because $y_i \geq \frac{1}{a_i^T x - b_i}$. Since $y_i \geq 0$ and $a_i^T x - b_i \geq 0$, $y_i \geq \frac{1}{a_i^T x - b_i}$ equals $\|(2, a_i^T x - b_i - y_i)\|_2 \leq a_i^T x - b_i + y_i$.

- $a_i^T x - b_i \geq 0$ for every i, so that $Ax > b$.

- $a_i^T x - b_i + y_i \geq 0$ for every i, this constraint may be redundant but is a must for Gurobi program.

Bounds: $y_i \geq 0$.

Note: Assume x_0 is the optimal solution, $\frac{1}{x_0}$ is the optimal solution for the original question.

question 2

Result as shown in Figure 2.

Note: Assume x_0 is the optimal solution, $\frac{1}{x_0}$ is the optimal solution for the original question.

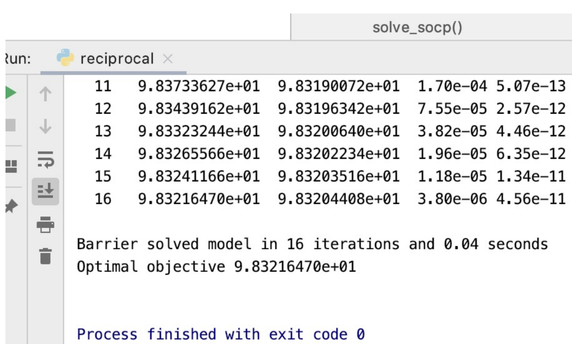


Figure 2

Problem 3

question 1

Indices: $i = 1, 2, ..., m$ for different dimensions in space R^m, and $j = 1, 2, ..., n$ for different dimensions in space R^n.

Data: $M \in R^{m \times n}$ is a matrix with $M_{i,j}$ as its entry at ith row and jth column. $b \in R^m$ is a vector with b_i as its ith entry.

Variables: z_j is the decision variable. x_i is the auxiliary variable that represent the square of the ith entry of $Mz - b$.

Objective: minimize $\sum_{i=1}^{m} x_i x_i$

Constraints: $x_i \geq (\sum_{j=1}^{n} M_{i,j} z_j - b_i)(\sum_{j=1}^{n} M_{i,j} z_j - b_i)$ for every i.

Bounds: $x_i \geq 0$.

Note: Assume x_0 is the optimal solution, $x_0^{1/4}$ is the optimal solution for the original question.

question 2

Result as shown in Figure 3.

Note: Assume x_0 is the optimal solution, $x_0^{1/4}$ is the optimal solution for the original question.

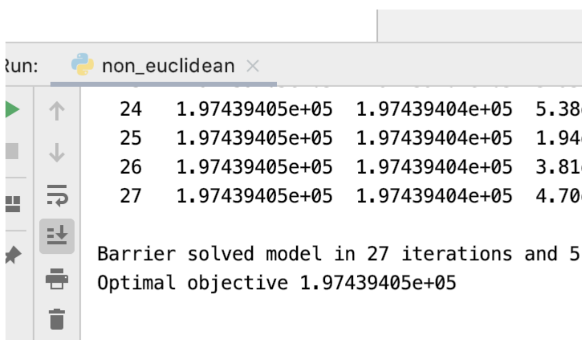


Figure 3

Problem 4

question 1

-

The KKT conditions for Q

- $\vec{c} - E^T\vec{\mu} + A^T\vec{\lambda} = 0$
- $\vec{d} - E\vec{x} \leq 0$
- $A\vec{x} - \vec{b} = 0$

- $\vec{\mu} \geq \vec{0}$
- $\vec{\mu}^T(\vec{d} - E\vec{x}) = 0$

The KKT conditions for Q_ρ

- $\vec{c} + \rho A^T(A\vec{x} - \vec{b}) - E^T\vec{\mu} = 0$
- $\vec{d} - E\vec{x} \leq 0$
- $\vec{\mu} \geq \vec{0}$
- $\vec{\mu}^T(\vec{d} - E\vec{x}) = 0$

The conditions that a Q_ρ KKT point can be used to generate a Q KKT point

Combine the KKT conditions of Q, we can get the KKT point would satisfy the equation $\vec{x^T}\vec{c} - \vec{d^T}\vec{\mu} + \vec{b^T}\vec{\lambda} = 0$. Similarly, the KKT conditions of Q_ρ can be combined, and we can get the KKT point of Q_ρ would satisfy the equation $\vec{x^T}\vec{c} + \rho\vec{x^T}A^T(A\vec{x} - b) - \vec{d^T}\vec{\mu} = 0$.

When a Q_ρ KKT point can be used to generate a Q KKT point, the KKT point $\vec{x_\rho}$ that satisfy the equation $\vec{x_\rho^T}\vec{c} + \rho\vec{x_\rho^T}A^T(A\vec{x_\rho} - \vec{b}) = \vec{d^T}\vec{\mu}$ would also satisfy $\vec{x_\rho^T}\vec{c} + \vec{b^T}\vec{\lambda} = \vec{d^T}\vec{\mu}$. From the KKT conditions, we can see that $A\vec{x_\rho} - b = 0$ must be satisfied to be a Q KKT point candidate. Thus, $\vec{b^T}\vec{\lambda} = 0$ and thus $\vec{\lambda} = 0$.

Conclusion: when $A\vec{x_\rho} - b = 0$, a Q_ρ KKT point can be used to generate a Q KKT point and at that time, $\vec{\lambda} = 0$ should be satisfied for that Q KKT point.

question 2

Result as shown in Figure 4.

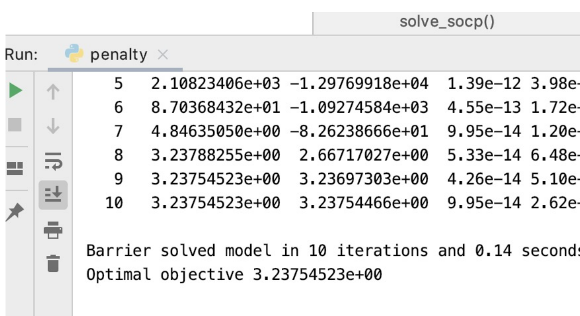


Figure 4

question 3

Result as shown in Figure 5 and 6.

Observation: when the value of ρ increases, the distance/difference between z and z_ρ decreases to zero as well as the value of the norm of $Az_\rho - b$ decreases to zero. This result supports the conclusion in question (a), where the conclusion says that when $Az_\rho - b = 0$,

z_ρ is a KKT point of Q as well. It also indicates that in Gurobi (or any other computer programs), the optimal solution is not exactly the theoretical solution but a reasonable approximation. When ρ is comparably small, the product of ρ and $Az_\rho - b$ would be comparably small and close to zero so that there still remains a noticeable $Az_\rho - b$ value. However, when ρ is comparably big, $Az_\rho - b$ would need to be closer to zero so that the product of ρ and $Az_\rho - b$ could be comparably small. Hence, the raw value of $Az_\rho - b$ would be close to zero.

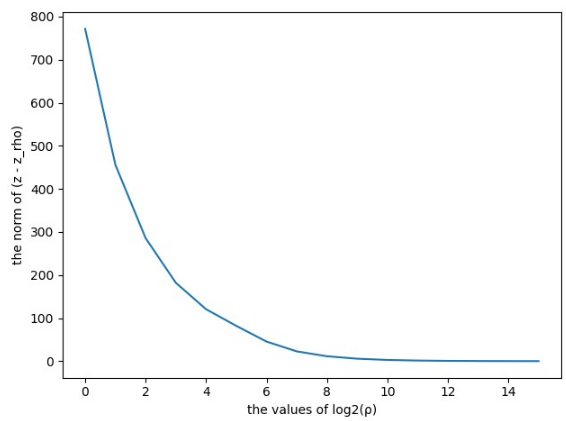


Figure 5

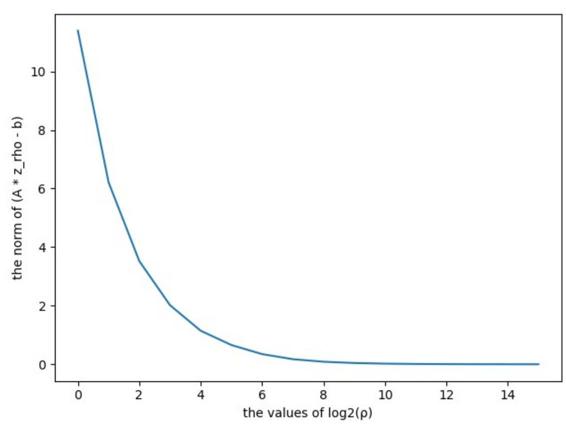


Figure 6