# Mathematical Methods in Physics II

## Homework II

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# Problem I

*Solution.* The essence of the problem is to consider the given functions as norms of the vector space $V$ over the field $\mathbb{R}$ and determine whether they satisfy the necessary conditions for $V$ to be a normed vector space. A normed vector space is vector space $V$ over a field $F$ equipped with a norm function $\|\cdot\| : V \to \mathbb{R}$ that satisfies the following conditions for all $u, v \in V$ and $s \in F$:

- $(\forall v \in V)\ \|v\| \geq 0$ and $\|v\| = 0 \iff v = 0$.
- $(\forall v \in V)(\forall s \in F)\ \|s \odot v\| = |s| \cdot \|v\|$.
- $(\forall u, v \in V)\ \|u \oplus v\| \leq \|u\| + \|v\|$ (triangle inequality).

Let's denote two arbitrary vectors in $V$ as $u = a^i e_i$ and $v = b^i e_i$, where $a^i, b^i \in \mathbb{R}$ and $e_i$ are the basis vectors of $V$.

1) The first function is given by

$$\alpha = a^i e_i \to \left( \sum_{j=1}^{n} a_j^2 \right)^{1/2}$$

where $n$ is the dimension of the vector space $V$. Let's check the conditions for this function to be a norm.

- The first condition is satisfied since the sum of squares of real numbers is always non-negative and the square root of a non-negative number is non-negative. The only way for the norm to be zero is if all the components of the vector are zero, which means the vector itself is zero.
- The second condition is also satisfied since the norm of a scalar multiple of a vector is the absolute value of the scalar multiple times the norm of the vector.

  *Proof.*

$$\begin{aligned}
\|s \odot v\| &= \left( \sum_{j=1}^{n} (s a_j)^2 \right)^{1/2} \\
&= \left( s^2 \sum_{j=1}^{n} a_j^2 \right)^{1/2} \\
&= |s| \left( \sum_{j=1}^{n} a_j^2 \right)^{1/2} \\
&= |s| \cdot \|v\|
\end{aligned}$$

■

- The third condition is also satisfied since the norm of the sum of two vectors is less than or equal to the sum of the norms of the vectors.

*Proof.* Consider the following:

$$\|u \oplus v\| = \left(\sum_{j=1}^{n}(a_j + b_j)^2\right)^{1/2}$$

$$\|u\| + \|v\| = \left(\sum_{j=1}^{n} a_j^2\right)^{1/2} + \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}$$

Since the function always return positive values, we can square both of the expressions and compare them:

$$\|u \oplus v\|^2 = \sum_{j=1}^{n}(a_j + b_j)^2$$

$$(\|u\| + \|v\|)^2 = \sum_{j=1}^{n} a_j^2 + 2\left[\left(\sum_{j=1}^{n} a_j\right) \cdot \left(\sum_{j=1}^{n} b_j\right)\right]^{1/2} + \sum_{j=1}^{n} b_j^2$$

To show that $\|u \oplus v\| \leq \|u\| + \|v\|$, we have to prove the following inequality:

$$\sum_{j=1}^{n} a_j b_j \leq \left[\left(\sum_{j=1}^{n} a_j\right) \cdot \left(\sum_{j=1}^{n} b_j\right)\right]^{1/2}$$

Let's consider the following quadratic polynomial:

$$(a_1 x + b_1)^2 + (a_2 x + b_2)^2 + \ldots + (a_n x + b_n)^2 = x^2 \sum_{j=1}^{n} a_j^2 + x \sum_{j=1}^{n} 2a_j b_j + \sum_{j=1}^{n} b_j^2$$

Since the quadratic polynomial is non-negative for all $x$, its discriminant is non-positive:

$$\left(\sum_{j=1}^{n} 2a_j b_j\right)^2 - 4\left(\sum_{j=1}^{n} a_j^2\right)\left(\sum_{j=1}^{n} b_j^2\right) \leq 0$$

which implies

$$\sum_{j=1}^{n} a_j b_j \leq \left(\sum_{j=1}^{n} a_j^2\right)^{1/2}\left(\sum_{j=1}^{n} b_j^2\right)^{1/2}$$

Therefore, $\|u \oplus v\| \leq \|u\| + \|v\|$. ∎

Hence, the first function is a norm on the vector space $V$.

2) The second function is given by

$$\beta = a^i e_i \to \left(\sum_{j=1}^{n} a_j^2\right)^{1/4}$$

Let's check the conditions for this function to be a norm.

- The first condition is satisfied since the sum of squares of real numbers is always non-negative and the fourth root of a non-negative number is non-negative. The only way for the norm to be zero is if all the components of the vector are zero, which means the vector itself is zero.
- The second condition is not satisfied since the norm of a scalar multiple of a vector is not equal to the absolute value of the scalar multiple times the norm of the vector.

  *Proof.*
$$
\begin{aligned}
\|s \odot v\| &= \left(\sum_{j=1}^{n} (sa_j)^2\right)^{1/4} \\
&= \left(s^2 \sum_{j=1}^{n} a_j^2\right)^{1/4} \\
&= |s|^{1/2} \left(\sum_{j=1}^{n} a_j^2\right)^{1/4} \\
&= |s|^{1/2} \cdot \|v\| \\
&\neq |s| \cdot \|v\|
\end{aligned}
$$
■

Since the second condition is not satisfied, there is no need to check the third condition. Hence, the second function is not a norm on the vector space $V$.

3) The third function is given by

$$
\gamma = a^i e_i \to \left(\sum_{j=1}^{n} a_j^2\right)
$$

Let's check the conditions for this function to be a norm:

- The first condition is satisfied since the sum of squares of real numbers is always non-negative. The only way for the norm to be zero is if all the components of the vector are zero, which means the vector itself is zero.
- The second condition is not satisfied since the norm of a scalar multiple of a vector is not equal to the absolute value of the scalar multiple times the norm of the vector.

*Proof.*

$$\begin{aligned}
\|s \odot v\| &= \left(\sum_{j=1}^{n} (sa_j)^2\right) \\
&= s^2 \sum_{j=1}^{n} a_j^2 \\
&= s^2 \cdot \|v\| \\
&\neq |s| \cdot \|v\|
\end{aligned}$$

■

Since the second condition is not satisfied, there is no need to check the third condition. Hence, the third function is not a norm on the vector space $V$.

4) The fourth function is given by

$$\lambda = a^i e_i \to \left(\sum_{j=1}^{n} a_j\right)$$

Let's check the conditions for this function to be a norm.

- The first condition is not satisfied since the sum of real numbers is not always non-negative, and the norm of a vector can be zero even if the vector itself is not zero.

  *Proof.* As a counterexample of the first statement, consider the vector $v = (-1, -1)$. Then, $\|v\| = -2 < 0$. As a counterexample of the second statement, consider the vector $v = (1, -1)$. Then, $\|v\| = 0$ even though $v \neq 0$. ■

- The second condition is not satisfied since the norm of a scalar multiple of a vector is not equal to the absolute value of the scalar multiple times the norm of the vector.

  *Proof.*

$$\begin{aligned}
\|s \odot v\| &= \left(\sum_{j=1}^{n} sa^j\right) \\
&= s \sum_{j=1}^{n} a^j \\
&= s \cdot \|v\| \\
&\neq |s| \cdot \|v\|
\end{aligned}$$

■

- The third condition is satisfied since the norm of the sum of two vectors is equal to the sum of the norms of the vectors.

*Proof.*

$$\begin{aligned}\|u \oplus v\| &= \left(\sum_{j=1}^{n}(a^j + b^j)\right) \\ &= \left(\sum_{j=1}^{n} a^j\right) + \left(\sum_{j=1}^{n} b^j\right) \\ &= \|u\| + \|v\|\end{aligned}$$

∎

Hence, the fourth function is not a norm on the vector space $V$.

e) The fifth function is given by

$$\rho = a^i e_i \to \left(\sum_{j=1}^{\lfloor n/2 \rfloor} a_j - \sum_{j=1+\lfloor n/2 \rfloor}^{n} a_j\right)$$

Let's check the conditions for this function to be a norm. Let's take $n = 3$ without loss of generality. The function takes the form:

$$\rho = a^i e_i \to a_1 - a_2 - a_3$$

- The first condition is not satisfied since the sum of real numbers is not always non-negative, and the norm of a vector can be zero even if the vector itself is not zero.

  *Proof.* As a counterexample of the first statement, consider the vector $v = (1, 1, 1)$. Then, $\|v\| = -1 < 0$. As a counterexample of the second statement, consider the vector $v = (2, 1, 1)$. Then, $\|v\| = 0$ even though $v \neq 0$. ∎

  Since the first condition is not satisfied, there is no need to check the second and third conditions. Hence, the fifth function is not a norm on the vector space $V$.

In conclusion, the subset of $\{(V, \alpha), (V, \beta), (V, \gamma), (V, \lambda), (V, \rho)\}$ that includes normed vector spaces is $\{(V, \alpha)\}$.

∎

# Problem II

*Solution.* An inner product space is a vector space $V$ over a field $F$ equipped with an inner product function $\langle \cdot, \cdot \rangle : V \times V \to F$ that satisfies the following conditions for all $u, v, w \in V$ and $\alpha, \beta \in F$:

- $(\forall u, v \in V)\, \langle u, v \rangle = \langle v, u \rangle^*$ (conjugate symmetry).
- $(\forall u, v, w \in V)(\forall \alpha, \beta \in F)\, \langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$.
- $(\forall v \in V \setminus \{0\})\, \langle v, v \rangle > 0$.
- $(\forall v \in V)\, \langle v, v \rangle = 0 \iff v = 0$.

Let's denote three arbitrary vectors in two-dimensional vector space $\mathcal{V}$ as $u = a^i e_i$, $v = b^i e_i$, and $w = c^i e_i$, where $a_i, b_i, c_i \in \mathbb{C}$ and $e_i$ are the basis vectors of $\mathcal{V}$. The inner product in $\mathcal{V}$ is defined as

$$\langle a^i e_i, b^j e_j \rangle = a^i (b^j)^* \delta_{ij},$$

where $\delta_{ij}$ is the Kronecker delta. Let's also denote the general form of a complex number as $a_i = a_{i1} + i a_{i2}$, where $a_{i1}, a_{i2} \in \mathbb{R}$. Since $\dim \mathcal{V} = 2$, the definition of the inner product can be written as

$$\begin{aligned}\langle a^i e_i, b^j e_j \rangle &= \sum_{i=1}^{2} \sum_{j=1}^{2} a_i (b_j)^* \delta_{ij} \\ &= (a_{11} + i a_{12})(b_{11} - i b_{12}) + (a_{21} + i a_{22})(b_{21} - i b_{22}) \\ &= a_{11} b_{11} + a_{12} b_{12} + a_{21} b_{21} + a_{22} b_{22}.\end{aligned}$$

1) Let's check whether this definition of the inner product satisfies the necessary conditions for $V$ to be an inner product space.

- The first condition is satisfied since the inner product is defined as the complex conjugate of the inner product of the vectors in the opposite order.

  *Proof.* The inner product of $u$ and $v$ is

  $$\langle u, v \rangle = a^i (b^j)^* \delta_{ij}.$$

  The complex conjugate of the inner product of $u$ and $v$ is

  $$\begin{aligned}\langle v, u \rangle^* &= (b^i (a^j)^* \delta_{ij})^* \\ &= (b^i)^* (a^j) \delta_{ij} \\ &= a^i (b^j)^* \delta_{ij}.\end{aligned}$$

  $\therefore \langle u, v \rangle = \langle v, u \rangle^*$. ∎

- The second condition is satisfied since the inner product of a linear combination of two vectors with a third vector is equal to the linear combination of the inner products of the vectors with the third vector.

  *Proof.* The inner product of $au + bv$ and $w$ is

  $$\begin{aligned}\langle \alpha u + \beta v, w \rangle &= (\alpha a^i + \beta b^i)(c^j)^* \delta_{ij} \\ &= \alpha a^i (c^j)^* \delta_{ij} + \beta b^i (c^j)^* \delta_{ij} \\ &= \alpha \langle u, w \rangle + \beta \langle v, w \rangle.\end{aligned}$$

  The linear combination of the inner products of the vectors with the third vector is

  $$\alpha \langle u, w \rangle + \beta \langle v, w \rangle.$$

  $\therefore \langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$. ∎

- The third condition is satisfied since the inner product of a vector with itself is always non-negative and the only way for the inner product to be zero is if the vector itself is zero.

*Proof.* The inner product of $v$ with itself is

$$\begin{aligned}\langle v, v\rangle &= b^i (b^j)^* \delta_{ij} \\ &= b_{11}^2 + b_{12}^2 + b_{21}^2 + b_{22}^2.\end{aligned}$$

The only way for the inner product to be zero is if all the components of the vector are zero, which means the vector itself is zero. Therefore, the third condition is satisfied. ∎

- The fourth condition is satisfied following the same logic as the third condition.

2) A normed vector space is vector space $V$ over a field $F$ equipped with a norm function $\|\cdot\| : V \to \mathbb{R}$ that satisfies the following conditions for all $u, v \in V$ and $\alpha \in \mathbb{R}$:

- $(\forall v \in V)\, \|v\| \geq 0$ and $\|v\| = 0 \iff v = 0$.
- $(\forall v \in V)(\forall \alpha \in \mathbb{R})\, \|\alpha \odot v\| = |\alpha| \cdot \|v\|$.
- $(\forall u, v \in V)\, \|u \oplus v\| \leq \|u\| + \|v\|$ (triangle inequality).

The norm of a vector $v$ is defined as

$$\|v\| = \langle v, v\rangle^k,$$

where $k$ is a real number that is to be determined. The definition of the norm has to satisfy the aforementioned conditions for $\mathcal{V}$ to be a normed vector space.

- The first condition is satisfied for any $k$ since the inner product of a vector with itself is always non-negative and the $k$th power of a non-negative number is non-negative. The only way for the norm to be zero is if the vector itself is zero.

  *Proof.* The norm of $v$ is

  $$\|v\| = \left(a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2\right)^k,$$

  which is non-negative for any $k$. The only way for the norm to be zero is if all the components of the vector are zero, which means the vector itself is zero. Therefore, any $k$ satisfies the first condition. ∎

- The second condition is satisfied for $k = \frac{1}{2}$.

  *Proof.* It follows that

  $$\alpha \odot v = (\alpha a_{11} + i\alpha a_{12}, \alpha a_{21} + i\alpha a_{22}),$$

  and its norm is

  $$\begin{aligned}\|\alpha \odot v\| &= \left((\alpha a_{11} + i\alpha a_{12})^2 + (\alpha a_{21} + i\alpha a_{22})^2\right)^k \\ &= \left(\alpha^2 a_{11}^2 + \alpha^2 a_{12}^2 + \alpha^2 a_{21}^2 + \alpha^2 a_{22}^2\right)^k \\ &= \left(\alpha^2\right)^k \left(a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2\right)^k \\ &= |\alpha| \|v\|.\end{aligned}$$

  For this to be true, $k$ has to be equal to $\frac{1}{2}$. ∎

- Since $k = \frac{1}{2}$ from the second condition, let's check whether the third condition is satisfied for this value of $k$. It follows that
$$\begin{aligned}\|u \oplus v\| &= \langle u \oplus v, u \oplus v\rangle^{1/2},\\ \|u\| + \|v\| &= \langle u, u\rangle^{1/2} + \langle v, v\rangle^{1/2}.\end{aligned}$$
Since both expressions always return positive values, we can square both of the expressions and compare them:
$$\begin{aligned}\|u \oplus v\|^2 &= \langle u \oplus v, u \oplus v\rangle,\\ (\|u\| + \|v\|)^2 &= \langle u, u\rangle + 2\langle u, u\rangle^{1/2}\langle v, v\rangle^{1/2} + \langle v, v\rangle.\end{aligned}$$
If we expand the first expression, we obtain
$$\begin{aligned}\|u \oplus v\|^2 &= \langle u \oplus v, u \oplus v\rangle\\ &= \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle.\end{aligned}$$
Next, if we compare the expressions, we obtain
$$\begin{aligned}\langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle &\le \langle u, u\rangle + 2\langle u, u\rangle^{1/2}\langle v, v\rangle^{1/2} + \langle v, v\rangle\\ \langle u, v\rangle + \langle v, u\rangle &\le 2\langle u, u\rangle^{1/2}\langle v, v\rangle^{1/2}.\end{aligned}$$
To show the inequality, we have to prove the following two inequalities:
$$\begin{aligned}\langle u, v\rangle &\le \langle u, u\rangle^{1/2}\langle v, v\rangle^{1/2},\\ \langle v, u\rangle &\le \langle u, u\rangle^{1/2}\langle v, v\rangle^{1/2}.\end{aligned}$$
If we square both sides of the inequalities, we obtain
$$\begin{aligned}\langle u, v\rangle^2 &\le \langle u, u\rangle\langle v, v\rangle,\\ \langle v, u\rangle^2 &\le \langle u, u\rangle\langle v, v\rangle.\end{aligned}$$
These inequalities are true from the Cauchy-Schwarz inequality. Therefore, $\|u \oplus v\| \le \|u\| + \|v\|$ for $k = \frac{1}{2}$.

3) Let's redefine the inner product as
$$\langle a^i e_i, b^j e_j\rangle = b^i (a^j)^* \delta_{ij}.$$

Let's check whether this definition of the inner product satisfies the necessary conditions for $V$ to be an inner product space.

- The first condition is satisfied since the inner product is defined as the complex conjugate of the inner product of the vectors in the opposite order.

  *Proof.* The inner product of $u$ and $v$ is
$$\langle u, v\rangle = b^i (a^j)^* \delta_{ij}.$$

  The complex conjugate of the inner product of $u$ and $v$ is
$$\begin{aligned}\langle v, u\rangle^* &= (a^i (b^j)^* \delta_{ij})^*\\ &= (a^i)^* b^j \delta_{ij}\\ &= b^i (a^j)^* \delta_{ij}.\end{aligned}$$
  $\therefore \langle u, v\rangle = \langle v, u\rangle^*$. ∎

- The second condition is not satisfied since the inner product of a linear combination of two vectors with a third vector is not equal to the linear combination of the inner products of the vectors with the third vector.

  *Proof.* The inner product of $au + bv$ and $w$ is

  $$\begin{aligned}\langle \alpha u + \beta v, w\rangle &= (\alpha a^i + \beta b^i)^*(c^j)\delta_{ij} \\ &= (\alpha a^i)^*(c^j)\delta_{ij} + (\beta b^i)^*(c^j)\delta_{ij} \\ &= \alpha^*\langle u, w\rangle + \beta^*\langle v, w\rangle.\end{aligned}$$

  The linear combination of the inner products of the vectors with the third vector is

  $$\alpha\langle u, w\rangle + \beta\langle v, w\rangle.$$

  $\therefore \langle \alpha u + \beta v, w\rangle \neq \alpha\langle u, w\rangle + \beta\langle v, w\rangle.$ ■

  Hence, the second definition of the inner product does not satisfy the necessary conditions for $V$ to be an inner product space.

4) The function `angle` : $\mathcal{V} \times \mathcal{V} \to \mathbb{R}$ between two vectors $u$ and $v$ is defined as

$$\texttt{angle} = (u, v) \to \arccos\left(\frac{\operatorname{Re}(\langle u, v\rangle)}{\sqrt{\langle u, u\rangle\langle v, v\rangle}}\right).$$

Let's calculate $\texttt{angle}(e_1 + ie_2, e_1 - ie_2)$. The inner product of the two vectors is

$$\begin{aligned}\langle e_1 + ie_2, e_1 - ie_2\rangle &= (1)(1) + (i)(-i) \\ &= 1 + 1 \\ &= 2.\end{aligned}$$

The inner product of each vector with itself is

$$\begin{aligned}\langle e_1 + ie_2, e_1 + ie_2\rangle &= (1)(1) + (i)(-i) \\ &= 1 + 1 \\ &= 2, \\ \langle e_1 - ie_2, e_1 - ie_2\rangle &= (1)(1) + (-i)(i) \\ &= 1 + 1 \\ &= 2.\end{aligned}$$

The angle between the two vectors is

$$\begin{aligned}\texttt{angle}(e_1 + ie_2, e_1 - ie_2) &= \arccos\left(\frac{\operatorname{Re}(\langle e_1 + ie_2, e_1 - ie_2\rangle)}{\sqrt{\langle e_1 + ie_2, e_1 + ie_2\rangle\langle e_1 - ie_2, e_1 - ie_2\rangle}}\right) \\ &= \arccos\left(\frac{2}{\sqrt{2 \cdot 2}}\right) \\ &= \arccos\left(\frac{2}{2}\right) \\ &= \arccos 1 \\ &= 0.\end{aligned}$$

5) Let's denote the vectors $u$ and $v$ in $\mathcal{V}$ as $u = r_1 e^{i\alpha_1}\cos(\theta_1)e_1 + r_1 e^{i\beta_1}\sin(\theta_1)e_2$ and $v = r_2 e^{i\alpha_2}\cos(\theta_2)e_1 + r_2 e^{i\beta_2}\sin(\theta_2)e_2$, where $r_1, r_2, \in \mathbb{R}^+$, $\alpha_1, \alpha_2, \beta_1, \beta_2, \theta_1, \theta_2 \in [0, 2\pi)$, and $e^{i\alpha}$ and $e^{i\beta}$ are complex numbers. The inner product of the two vectors is

$$\begin{aligned}\langle u, v\rangle &= \left(r_1 e^{i\alpha_1}\cos(\theta_1)\right)\cdot\left(r_2 e^{i\alpha_2}\cos(\theta_2)\right)^* + \left(r_1 e^{i\beta_1}\sin(\theta_1)\right)\cdot\left(r_2 e^{i\beta_2}\sin(\theta_2)\right)^*\\ &= r_1 r_2\cos(\theta_1)\cos(\theta_2)e^{i(\alpha_1-\alpha_2)} + r_1 r_2 \sin(\theta_1)\sin(\theta_2)e^{i(\beta_1-\beta_2)}.\end{aligned}$$

Let's employ the identity $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ to express the inner product in terms of the angle between the two vectors:

$$\begin{aligned}\langle u, v\rangle = r_1 r_2\cos(\theta_1)\cos(\theta_2)(\cos(\alpha_1-\alpha_2) + \sin(\alpha_1-\alpha_2)i)\\ + r_1 r_2\sin(\theta_1)\sin(\theta_2)(\cos(\beta_1-\beta_2) + \sin(\beta_1-\beta_2)i).\end{aligned}$$

If we take the real part of this expression, we obtain

$$\operatorname{Re}(\langle u, v\rangle) = r_1 r_2\left[\cos(\theta_1)\cos(\theta_2)\cos(\alpha_1-\alpha_2) + \sin(\theta_1)\sin(\theta_2)\sin(\beta_1-\beta_2)\right].$$

The inner product of each vector with itself is

$$\begin{aligned}\langle u, u\rangle &= r_1^2\cos^2(\theta_1) + r_1^2\sin^2(\theta_1)\\ &= r_1^2,\\ \langle v, v\rangle &= r_2^2\cos^2(\theta_2) + r_2^2\sin^2(\theta_2)\\ &= r_2^2.\end{aligned}$$

Therefore, the angle between the two vectors is

$$\begin{aligned}\texttt{angle}(u, v) &= \arccos\left(\frac{\operatorname{Re}(\langle u, v\rangle)}{\sqrt{\langle u, u\rangle\langle v, v\rangle}}\right)\\ &= \arccos\left(\frac{r_1 r_2\left(\cos(\theta_1)\cos(\theta_2)\cos(\alpha_1-\alpha_2) + \sin(\theta_1)\sin(\theta_2)\sin(\beta_1-\beta_2)\right)}{\sqrt{r_1^2\cdot r_2^2}}\right)\\ &= \arccos\left(\cos(\theta_1)\cos(\theta_2)\cos(\alpha_1-\alpha_2) + \sin(\theta_1)\sin(\theta_2)\sin(\beta_1-\beta_2)\right).\end{aligned}$$

■

# Problem IV

*Solution.* Let $V$ be a vector space over a field $F$ with a basis $\{e_1, e_2, \ldots, e_n\}$. Let's denote three arbitrary vectors in $V$ as $u = a^i e_i$, $v = b^i e_i$, and $w = c^i e_i$, where $a_i, b_i, c_i \in F$.

1) Let's consider the space

$$T^2 V = V \otimes V,$$

which is the tensor product of V with itself. Such a space must satisfy certain axioms that define its structure:

- $(\forall u, v \in V)(\forall \alpha \in F)\, \alpha \odot (u \otimes v) = (\alpha \odot u) \times v = u \otimes (\alpha \odot v)$.
- $(\forall u, v, w \in V)\, u \otimes (v \oplus w) = u \otimes v + u \otimes w$ (distributive property).

An element of $T^2V$ is a tensor of the form

$$t = a^i e_i \otimes b^j e_j = a^i b^j e_i \otimes e_j.$$

Let's check whether $T^2V$ is a vector space over $F$. As a preliminary step, let's check whether $(V \otimes V, \oplus)$ is an abeian group.

- $(\forall t_1, t_2, t_3 \in V \otimes V)\, t_1 \oplus (t_2 \oplus t_3) = (t_1 \oplus t_2) \oplus t_3$.

  *Proof.*

$$\begin{aligned} t_1 \oplus (t_2 \oplus t_3) &= (a^i b^j e_i \otimes e_j) \oplus ((c^i d^j e_i \otimes e_j) \oplus (e^i f^j e_i \otimes e_j)) \\ &= (a^i b^j e_i \otimes e_j) \oplus (c^i d^j e_i \otimes e_j) \oplus (e^i f^j e_i \otimes e_j) \\ &= ((a^i b^j e_i \otimes e_j) \oplus (c^i d^j e_i \otimes e_j)) \oplus (e^i f^j e_i \otimes e_j) \\ &= (t_1 \oplus t_2) \oplus t_3. \end{aligned}$$

  $\therefore t_1 \oplus (t_2 \oplus t_3) = (t_1 \oplus t_2) \oplus t_3$. ■

- $(\forall t \in V \otimes V)\, t \oplus 0 = t$.

  *Proof.*

$$\begin{aligned} t \oplus 0e_i &= (a^i b^j e_i \otimes e_j) \oplus (0e_i \otimes e_j) \\ &= (a^i b^j + 0) e_i \otimes e_j \\ &= a^i b^j e_i \otimes e_j \\ &= t. \end{aligned}$$

  $\therefore t \oplus 0 = t$. ■

- $(\forall t \in V \otimes V)\, t \oplus (-t) = 0$.

  *Proof.*

$$\begin{aligned} t \oplus (-t) &= (a^i b^j e_i \otimes e_j) \oplus ((-a^i b^j) e_i \otimes e_j) \\ &= (a^i b^j - a^i b^j) e_i \otimes e_j \\ &= 0. \end{aligned}$$

  $\therefore t \oplus (-t) = 0$. ■

- $(\forall t_1, t_2 \in V \otimes V)\, t_1 \oplus t_2 = t_2 \oplus t_1$.

  *Proof.*

$$\begin{aligned} t_1 \oplus t_2 &= (a^i b^j e_i \otimes e_j) \oplus (c^i d^j e_i \otimes e_j) \\ &= (a^i b^j + c^i d^j) e_i \otimes e_j \\ &= (c^i d^j + a^i b^j) e_i \otimes e_j \\ &= (c^i d^j e_i \otimes e_j) \oplus (a^i b^j e_i \otimes e_j) \\ &= t_2 \oplus t_1. \end{aligned}$$

  $\therefore t_1 \oplus t_2 = t_2 \oplus t_1$. ■

Therefore, $(V \otimes V, \oplus)$ is an abelian group. Let's check the other axioms that define the structure of $T^2V$:

- $(V \otimes V, \oplus)$ is a *commutative*, or *abelian* group.

  *Proof.* Proof is given above. ∎

- $(\forall t \in V \otimes V)\, 1 \odot v = v$, where 1 is the identity element of the field $F$.

  *Proof.*
  $$1 \odot t = (1 \cdot a^i b^j) e_i \otimes e_j = a^i b^j e_i \otimes e_j = t.$$
  $\therefore 1 \odot t = t.$ ∎

- $(\forall t \in V \otimes V)(\forall \alpha \in S)\, \alpha \odot t \in V \otimes V$.

  *Proof.*
  $$\alpha \odot t = (\alpha \cdot a^i b^j) e_i \otimes e_j = \alpha a^i b^j e_i \otimes e_j.$$
  $\therefore \alpha \odot t \in V \otimes V.$ ∎

- $(\forall t \in V \otimes V)(\forall \alpha, \beta \in S)\, (\alpha \cdot \beta) \odot t = \alpha \odot (\beta \odot t)$.

  *Proof.*
  $$\begin{aligned}(\alpha \cdot \beta) \odot t &= ((\alpha \cdot \beta) \cdot a^i b^j) e_i \otimes e_j \\ &= (\alpha \cdot \beta \cdot a^i b^j) e_i \otimes e_j \\ &= (\alpha \cdot (\beta \cdot a^i b^j)) e_i \otimes e_j \\ &= \alpha \odot (\beta \odot t).\end{aligned}$$
  $\therefore (\alpha \cdot \beta) \odot t = \alpha \odot (\beta \odot t).$ ∎

- $(\forall t \in V \otimes V)(\forall \alpha, \beta \in S)\, (\alpha + \beta) \odot t = (\alpha \odot t) \oplus (\beta \odot t)$.

  *Proof.*
  $$\begin{aligned}(\alpha + \beta) \odot t &= ((\alpha + \beta) \cdot a^i b^j) e_i \otimes e_j \\ &= ((\alpha \cdot a^i b^j) + (\beta \cdot a^i b^j)) e_i \otimes e_j \\ &= (\alpha \cdot a^i b^j) e_i \otimes e_j + (\beta \cdot a^i b^j) e_i \otimes e_j \\ &= (\alpha \odot t) \oplus (\beta \odot t).\end{aligned}$$
  $\therefore (\alpha + \beta) \odot t = (\alpha \odot v) \oplus (\beta \odot t).$ ∎

- $(\forall t_1, t_2 \in V \otimes V)(\forall \alpha \in S)\, \alpha \odot (t_1 \oplus t_2) = (\alpha \odot t_1) \oplus (\alpha \odot t_2)$.

  *Proof.*
  $$\begin{aligned}\alpha \odot (t_1 \oplus t_2) &= \alpha \odot (a^i b^j e_i \otimes e_j + c^i d^j e_i \otimes e_j) \\ &= \alpha \odot ((a^i b^j + c^i d^j) e_i \otimes e_j) \\ &= \alpha (a^i b^j + c^i d^j) e_i \otimes e_j \\ &= \alpha a^i b^j e_i \otimes e_j + \alpha c^i d^j e_i \otimes e_j \\ &= \alpha \odot t_1 \oplus \alpha \odot t_2.\end{aligned}$$
  $\therefore \alpha \odot (t_1 \oplus t_2) = \alpha \odot t_1 \oplus \alpha \odot t_2.$ ∎

Therefore, $T^2V$ is a vector space over $F$ with the standard basis $\{e_i \otimes e_j\}$ which represents all possible tensor products of the basis vectors of $V$ with themselves.

2) Let's consider the space

$$\Lambda V = V \wedge V \wedge V,$$

which is the exterior product of $V$ with itself. It is a subspace of $T^3V = V \otimes V \otimes V$ consisting of all antisymmetric tensors of the form

$$t_1 \otimes t_2 \otimes t_3 - t_2 \otimes t_1 \otimes t_3 + t_3 \otimes t_1 \otimes t_2 - t_3 \otimes t_2 \otimes t_1 + t_2 \otimes t_3 \otimes t_1 - t_1 \otimes t_3 \otimes t_2.$$

We can define this expression to be equivalent to $w_1 \wedge w_2 \wedge w_3$ for $w_i = a^i e_i$ and $a_i \in F$. Let's check whether $\Lambda V$ is a vector space over $F$. ∎