Mathematical Methods in Physics I
Homework 7

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1 Question One

2 Question Two

Let's find α in the follownig expression:

$$\det C = \alpha(\det A)(\det B) \tag{2.1}$$

Note that:

$$C = A \cdot B \tag{2.2}$$

$$C = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{2.3}$$

Therefore:

$$C = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \tag{2.4}$$

The determinant of A is:

$$\begin{aligned} \det A &= \sum_{i=1}^{2}\sum_{j=1}^{2} \epsilon_{ij} a_{1j} a_{2j} && (2.5) \\ &= \epsilon_{11}a_{11}a_{12} + \epsilon_{12}a_{11}a_{22} + \epsilon_{21}a_{12}a_{21} + \epsilon_{22}a_{12}a_{22} && (2.6) \\ &= a_{11}a_{22} - a_{12}a_{21} && (2.7) \end{aligned}$$

Since $\epsilon_{11} = \epsilon_{22} = 0$, $\epsilon_{12} = 1$, and $\epsilon_{21} = -1$. Similarly, it follows that $\det B = b_{11}b_{22} - b_{12}b_{21}$ and $\det C = c_{11}c_{22} - c_{12}c_{21}$. Therefore:

$$c_{11}c_{22} - c_{12}c_{21} = \alpha(a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21}) \tag{2.8}$$

Let's compute the left-hand side of the equation by substituting the values of c_{ij} from the matrix C:

$$\begin{aligned} c_{11}c_{22} - c_{12}c_{21} &= (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21}) && (2.9) \\ &= a_{11}a_{22}b_{11}b_{22} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21} && (2.10) \\ & && (2.11) \end{aligned}$$

And the right-hand side:

$$\alpha(a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21}) = \alpha(a_{11}a_{22}b_{11}b_{22} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21}) \tag{2.12}$$

Hence, it follows that $\alpha = 1$.

3 Question Three

To solve this question, let's compute the following:

$$f_1^{(1)}(x) = -x^{-2} \tag{3.1}$$

$$f_1^{(2)}(x) = 2x^{-3} \tag{3.2}$$

$$f_2^{(1)}(x) = -x^{-2}\mathrm{e}^{-x/2} - x^{-1}\frac{\mathrm{e}^{-x/2}}{2} \tag{3.3}$$

$$f_2^{(2)}(x) = 2x^{-3}\mathrm{e}^{-x/2} + x^{-2}\frac{\mathrm{e}^{-x/2}}{2} + x^{-2}\frac{\mathrm{e}^{-x/2}}{2} + x^{-1}\frac{\mathrm{e}^{-x/2}}{4} \tag{3.4}$$

$$f_3^{(1)}(x) = 1 \tag{3.5}$$

$$f_3^{(2)}(x) = 0 \tag{3.6}$$

For $x = 1$, the Wronskian is:

$$W = \begin{vmatrix} 1 & \frac{1}{\sqrt{\mathrm{e}}} & -3 \\ -1 & -\frac{3}{2\sqrt{\mathrm{e}}} & 1 \\ 2 & \frac{13}{4\sqrt{\mathrm{e}}} & 0 \end{vmatrix} \tag{3.7}$$

Solving this determinant, we obtain:

$$W = -3\left(-\frac{13}{4\sqrt{\mathrm{e}}} + \frac{3}{\sqrt{\mathrm{e}}}\right) + \left(\frac{13}{4\sqrt{\mathrm{e}}} - \frac{2}{\sqrt{\mathrm{e}}}\right) \tag{3.8}$$

$$= -\frac{1}{2\sqrt{\mathrm{e}}} \tag{3.9}$$