Phys210: Mathematical Methods in Physics II

Homework 3

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Policies

- Please adhere to the *academic integrity* rules: see my explanations here for further details!
- For the overall grading scheme or any other course-related details, see the syllabus.
- Non-graded question(s) (if any) are for your own practice!
- Unless stated otherwise, you are expected to show your derivation of the results.
- The homework is due April 5th 2024, 23:59 TSI.

(1) Problem One *(2 points)*

Define $\big((\forall i \in \{1,\ldots,7\})\ a_i \in \{0,\ldots,9\}\big)$ such that the string $a_1a_2a_3a_4a_5a_6a_7$ is your student ID number. Let e_i denote some vectors of a d–dimensional vector space $\mathcal{V}$ over real numbers, and let the infix use of $\wedge$ symbol denote the wedge product if between vectors and the logical *and* operation if between Boolean types.

(1.1) (a)

Consider the multivector ω defined as

$$\omega = \left(\frac{1}{2} + a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5 \cdot a_6 \cdot a_7\right) e_1 \wedge e_2 \wedge \cdots \wedge e_7 \tag{1.1}$$

where $\cdot$ denotes the arithmetic multiplication. What is the necessary condition on d such that $\omega \neq 0$?

(1.2) (b)

Assuming e_i are linearly independent, rewrite $\left(\sum_{i=1}^{4} a_i e_i\right) \wedge \left(\sum_{i=5}^{7} a_i e_i\right)$ in terms of the independent vectors of the vector space $\Lambda^2(\mathcal{V})$

(1.3) (c)

Let f_i be the basis vectors of $\mathcal{V}$ for $d = 4$. Write down *the most generic element* of the algebra $\Lambda(\mathcal{V})$ in terms of undetermined coefficients c_i.
Hint: the most general element of the algebra would be something like $c_0 + c_1 f_1 + c_2 f_2 + \ldots$.

(2) Problem Two *(3 points)*

Consider the tensor $T :: V \otimes V \otimes V \otimes V^*$, which can be expanded in a basis as

$$T = T^{ijk}{}_m e_i \otimes e_j \otimes e_k \otimes e^m \tag{2.1}$$

where we are using Einstein's summation conventions.

(2.1) (a)

Assume that V is a 2–dimensional vector space over real numbers. In fact, the nonzero components are known to be

$$T^{111}{}_{1} = 7 \,, \quad T^{121}{}_{1} = 3 \,, \quad T^{112}{}_{2} = -5 \,, \quad T^{211}{}_{2} = -1 \,, \tag{2.2}$$

How many different tensors of the type $V \otimes V$ can we obtain by *contracting* indices of T? Compute the components of all such tensors!

(2.2) (b)

Assume that V is now a 3–dimensional vector space over real numbers, and the only nonzero components are those in (2.2). Compute the value of the scalar a defined as

$$a = T^{ijk}{}_{m} \eta_{ij} \delta^{m}_{k} \tag{2.3}$$

for the object η_{ij} for which $\eta_{11} = -1$, $\eta_{22} = \eta_{33} = 1$, and $\eta_{ij} = 0$ for $i \neq j$. Here, δ is the Kronecker symbol.

(2.3) (c)

Let us stick to a 3–dimensional vector space over real numbers, but increase the nonzero components of T as adding the following to the list in (2.2):

$$T^{133}{}_{1} = 17 \,, \quad T^{321}{}_{1} = -13 \,, \quad T^{132}{}_{2} = 1 \,, \quad T^{322}{}_{2} = -2 \,, \tag{2.4}$$

Write the explicit expression for the covector $\omega = \epsilon_{ijk} T^{ijk}{}_{m} e^{m}$ where we will take basis vectors as $e^1 = dx$, $e^2 = dy$, and $e^3 = dz$. Here, ϵ is the Levi-Civita symbol.

(3) Problem Three *(3 points)*

Consider the vector field $E(\mathbf{x}) \in \mathbb{R}^3$ as follows:

$$E(\mathbf{x}) = x \frac{\partial}{\partial x} + x y^2 \frac{\partial}{\partial y} + x y^2 z^3 \frac{\partial}{\partial z} \tag{3.1}$$

(3.1) (a)

Compute the divergence of the vector field.

(3.2) (b)

Compute the curl of the vector field.

(3.3) (c)

Can this vector field be written as *gradient* of a scalar field $\phi(\mathbf{x})$? If yes, find out $\phi(x)$. If no, argue why this is the case.