Phys209: Mathematical Methods in Physics I

Homework 4

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Policies

- Please adhere to the *academic integrity* rules: see my explanations here for further details!
- For the overall grading scheme or any other course-related details, see the syllabus.
- Non-graded question(s) are for your own practice!
- Unless stated otherwise, you are expected to show your derivation of the results.
- The homework is due November 3rd 2023, 23:59 TSI.

(1) Problem One

(3 points)

We have seen in class that the solution for *a linear ordinary differential equation with constant coefficients* for the unknown function f is

$$\left(a_n \frac{\mathrm{d}^n}{\mathrm{d}x^n} + \cdots + a_1 \frac{\mathrm{d}}{\mathrm{d}x} + a_0\right) f(x) = h(x)$$

$$\Downarrow$$

$$f(x) = \sum_i \left[\left(\sum_{k=0}^{\alpha_i} c_{ik} x^k\right) e^{r_i x}\right] + \int_0^\infty h(x-y) i(y) dy \tag{1.1}$$

for the impulse response $i(x)$ and the roots of the characteristic equation r_i.

A similar general solution can not be found for differential equations with *functional coefficients*; nevertheless, there are some exceptions: we will discuss a few such cases in this homework.

(1.1) (0.6pt)

Consider the differential operator

$$\text{🦇}_a :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{1.2a}$$

$$\text{🦇}_a = (x \to f(x)) \to (x \to [x f'(x) - a f(x)]) \tag{1.2b}$$

and remember the zero function (denoted with pumpkin in this homework) defined as follows

$$\text{🎃} :: \mathbb{C} \to \mathbb{C} \tag{1.3a}$$

$$\text{🎃} = x \to 0 \tag{1.3b}$$

We claim that the "ghost" function 👻_Δ defined as

$$\text{👻}_\Delta :: \mathbb{C} \to \mathbb{C} \tag{1.4a}$$

$$\text{👻}_\Delta = x \to x^\Delta \tag{1.4b}$$

solves the following differential equation

$$\text{🦇}_a \cdot \text{👻}_\Delta = \text{🎃} \tag{1.5}$$

for *a particular value of* Δ. What is that value?
Hint: This question is equivalent to asking "What is the parameter Δ *for which* $\left(x \frac{\mathrm{d}}{\mathrm{d}x} - a\right) x^\Delta = 0$*?"*

Answer: As $\frac{\mathrm{d}}{\mathrm{d}x} x^{\Delta} = \Delta x^{\Delta - 1}$, we see that the solution is $\Delta = a$.

(1.2) (0.6pt)

Two operators $\mathcal{D}_1$ and $\mathcal{D}_2$ are said to commute if their action on a function can be interchanged, i.e. $\mathcal{D}_1 \cdot \mathcal{D}_2 \cdot f = \mathcal{D}_2 \cdot \mathcal{D}_1 \cdot f$ for any function f if $\mathcal{D}_1$ and $\mathcal{D}_2$ commute.

The commutation of the differential operators (or its lack of) is vitally important in many physical applications; in the case of differential equations, it actually helps us solve them generically to any order.

Remember the linear ordinary differential equations *with constant coefficients*, which can be rewritten in the form

$$💀_{a_1} \cdot 💀_{a_2} \cdots 💀_{a_n} \cdot f = 🎃 \tag{1.6}$$

if we define the following differential operator

$$💀_a :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{1.7a}$$

$$💀_a = (x \to f(x)) \to (x \to [f'(x) - af(x)]) \tag{1.7b}$$

The main reason we can solve such equations generically is because **(a)** differential equations with constant coefficients can be rewritten in this product form, and **(b)** these operators commute with themselves: if we define *the commutator of two operators A and B as*

$$[A, B] \cdot f := A \cdot B \cdot f - B \cdot A \cdot f \tag{1.8}$$

for any function f, then we can actually show that

$$\left[💀_a \, , \, 💀_b\right] \cdot f = 🎃 \tag{1.9}$$

Prove this relation!

Answer: Observe that

$$💀_a \cdot f = x \to (f'(x) - af(x)) \tag{1.10}$$

which means

$$💀_b \cdot 💀_a \cdot f = x \to (f''(x) - (a+b)f'(x) + abf(x)) \tag{1.11}$$

Note that the right hand side is *symmetric* under the exchange $a \leftrightarrow b$: this means it is same independent of the order of application, i.e.

$$💀_b \cdot 💀_a \cdot f - 💀_a \cdot 💀_b \cdot f = x \to 0 \tag{1.12}$$

which leads to the required relation!

(1.3) (0.6pt)

The relation in (1.9) is the reason why we can solve differential equations with constant coefficients to any generic order! To see that, observe the following: if f satisfies the relation

$$💀_{a_n} \cdot f = 🎃 \tag{1.13}$$

then it immediately satisfies the equation in (1.6) as the following relation is trivially satisfied:

$$💀_{a_1} \cdot 💀_{a_2} \cdots 💀_{a_{n-1}} \cdot 🎃 = 🎃 \tag{1.14}$$

In values, this equation simply means ***"multiplication with any constant" and "taking any order of derivatives" would always take*** 0 ***to*** 0.

We can solve $💀_{a_n} \cdot f = 🎃$ as it is a simple first order differential equation —we have derived this in class, also see the notes for explicit computation. This leads to the well known result $f = x \to e^{a_n x}$. The commutation in equation (1.9) is then sufficient to get the full solution: by symmetry, we also should have solutions such as $f = x \to e^{a_1 x}$, and by linearity the full solution should be the superposition: $\sum_{i=1}^{n} c_i e^{a_i x}$ for arbitrary coefficients c_i. As this solution contains n unknowns, our order$-n$ differential equation cannot have any other homogeneous solution, which concludes the derivation of the most general solution.

We discussed in class that there are some differential equations *with functional coefficients* that we can solve for generic order. The main reason for this is because they can be recast in the product form

$$\mathcal{D}_{a_1} \cdot \mathcal{D}_{a_2} \cdots \mathcal{D}_{a_n} \cdot f = \text{🎃} \tag{1.15}$$

for some first order differential operator $\mathcal{D}_a$ which commutes with itself. Indeed, we can also solve the differential equation

$$\text{🦇}_{a_1} \cdot \text{🦇}_{a_2} \cdots \text{🦇}_{a_n} \cdot f = \text{🎃} \tag{1.16}$$

as

$$[\text{🦇}_a \, , \, \text{🦇}_b] \cdot f = \text{🎃} \tag{1.17}$$

Prove this commutation!

Answer: Observe that

$$\text{🦇}_a \cdot f = x \to (x f'(x) - a f(x)) \tag{1.18}$$

which means

$$\text{🦇}_b \cdot \text{🦇}_a \cdot f = x \to \left(x^2 f''(x) - (a+b-1) x f'(x) + ab f(x) \right) \tag{1.19}$$

Note that the right hand side is *symmetric* under the exchange $a \leftrightarrow b$, leading to

$$\text{🦇}_b \cdot \text{🦇}_a \cdot f - \text{🦇}_a \cdot \text{🦇}_b \cdot f = x \to 0 \tag{1.20}$$

which leads to the required relation!

(1.4) (0.6pt)

Derive the following commutator:

$$\left[\text{💀}_a , \text{🦇}_b \right] = ??? \tag{1.21}$$

You will see that this commutator is not the zero function: this means, we cannot write down a generic result for an arbitrary order differential equation of the form

$$\text{💀}_{a_1} \cdot \text{💀}_{a_2} \cdot \text{🦇}_{b_1} \cdot \text{💀}_{a_3} \cdots \cdot \text{🦇}_{b_k} \cdot \text{💀}_{a_\ell} \cdot f = \text{🎃} \tag{1.22}$$

Answer: We see that

$$\mathbat_b \cdot f = x \to (xf'(x) - bf(x)) \tag{1.23a}$$
$$\mathskull_a \cdot f = x \to (f'(x) - af(x)) \tag{1.23b}$$

which then leads to

$$\mathskull_a \cdot \mathbat_b \cdot f = x \to (xf''(x) - (ax + b - 1)f'(x) - abf(x)) \tag{1.24a}$$
$$\mathbat_b \cdot \mathskull_a \cdot f = x \to (x'f'(x) - (ax + b)f'(x) - abf(x)) \tag{1.24b}$$

Therefore

$$\left[\mathskull_a, \mathbat_b\right] \cdot f = f'(x) \tag{1.25}$$

meaning

$$\left[\mathskull_a, \mathbat_b\right] :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{1.26a}$$
$$\left[\mathskull_a, \mathbat_b\right] = (x \to f(x)) \to (x \to f'(x)) \tag{1.26b}$$

or equivalently

$$\left[\mathskull_a, \mathbat_b\right] = \frac{\mathrm{d}}{\mathrm{d}x} \tag{1.27}$$

(1.5) (0.6pt)

In § 1.1, you solved the differential equation

$$\mathbat_a \cdot \mathghost_\Delta = \mathpumpkin \tag{1.28}$$

and we showed in § 1.3 that

$$[\mathbat_a\,,\ \mathbat_b] \cdot f = \mathpumpkin \tag{1.29}$$

Using these information, solve the generic differential equation

$$\mathbat_{a_1} \cdot \mathbat_{a_2} \cdots \mathbat_{a_n} \cdot f = \mathpumpkin \tag{1.30}$$

Answer: As discussed above, the symmetry due to the zero commutator ensures that the answer is the superposition of all individual answers:

$$f :: \mathbb{C} \to \mathbb{C} \tag{1.31a}$$

$$f = \sum_{i=1}^{n} \mathghost_{a_i} \tag{1.31b}$$

(2) Problem Two *(3 points)*

A second order linear ordinary differential equation can generically be brought to the form

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x)\right] f(x) = 0 \tag{2.1}$$

which can be rewritten as a linear ordinary differential equation *with constant coefficients* via the change of variables from x to $u(x)$ for

$$u(x) = \int \sqrt{q(x)}dx \tag{2.2}$$

if the following equality is satisfied as we have derived in class:

$$\frac{q'(x) + 2p(x)q(x)}{2q(x)^{3/2}} = \text{constant} \tag{2.3}$$

Since we know how to solve differential equations with constant coefficients, we can then immediately compute the result. In this problem, we are going to review this methodology.

(2.1) (0.5pt)

Consider the following differential equation:

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \pi\tan(\pi x)\frac{\mathrm{d}}{\mathrm{d}x} + \pi^4\cos^2(\pi x)\right] f(x) = 0 \tag{2.4}$$

What should be the new parameter $u(x)$?

Answer:

$$u(x) = \int \sqrt{\pi^4\cos(\pi x)^2}dx = \pi\sin(\pi x) \tag{2.5}$$

(2.2) (0.5pt)

Is the necessary condition in equation (2.3) satisfied for this differential equation?

Answer: As

$$q'(x) + 2p(x)q(x) = -2\pi^5 \cos(\pi x)\sin(\pi x) \\ + 2\pi \tan(\pi x)\pi^4 \cos(\pi x)^2 = 0 \quad (2.6)$$

is a constant, yes it is satisfied!

(2.3) (0.5pt)

What is the new differential equation in terms of the parameter u?

Answer: We can explicitly check that the differential equation takes the form

$$\left[(u'(x))^2 \frac{\mathrm{d}^2}{\mathrm{d}u^2} + (u''(x) + p(x)u'(x)) \frac{\mathrm{d}}{\mathrm{d}x} + q(x)\right] f(u) = 0 \quad (2.7)$$

which becomes

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}u^2} + 1\right) f(u) = 0 \quad (2.8)$$

(2.4) (0.5pt)

What is the solution to this differential equation? Write down the function f in terms of the variable u.
Hint: Any function of the form $c_1 \exp(ix) + c_2 \exp(-ix)$ *for arbitrary* c_i *can be rewritten as* $d_1 \cos(x) + d_2 \sin(x)$ *for arbitrary* d_i*. Both expressions are mathematically same, but the second one makes more intuitive sense in many physical systems of real parameters.*

Answer: The answer is simply

$$f(u) = c_1 \cos(u) + c_2 \sin(u) \quad (2.9)$$

(2.5) (0.5pt)

Write down the answer in terms of the original variable x.

Answer:

$$f(x) = c_1 \cos(\pi \sin(\pi x)) + c_2 \sin(\pi \sin(\pi x)) \tag{2.10}$$

(2.6) (0.5pt)

Assume that you are given the initial conditions

$$f(x=0) = 12000212\sqrt{2} \tag{2.11a}$$
$$f'(x=0) = 17101711\sqrt{2}\pi^2 \tag{2.11b}$$

What would be the value of this function at $x = \text{asin}(1/4)/\pi$, i.e

$$f\left(x = \frac{\text{asin}(1/4)}{\pi}\right) = ??? \tag{2.12}$$

Answer: We have

$$f(x) = c_1 \cos(\pi \sin(\pi x)) + c_2 \sin(\pi \sin(\pi x)) \tag{2.13}$$

which means

$$\begin{aligned} f(0) &= c_1 \\ f'(0) &= c_2 \pi^2 \end{aligned} \tag{2.14}$$

With the given initial conditions, then our function becomes

$$f(x) = 12000212\sqrt{2}\cos(\pi \sin(\pi x)) + 11101711\sqrt{2}\sin(\pi \sin(\pi x)) \tag{2.15}$$

At $x = \text{asin}(1/4)/\pi$, we have $\sin(\pi x) = 1/4$, which leads to

$$f(\text{asin}(1/4)/\pi) = 12000212\sqrt{2}\cos(\pi/4) + 11101711\sqrt{2}\sin(\pi/4) \tag{2.16}$$

hence

$$f(\text{asin}(1/4)/\pi) = 23101923 \tag{2.17}$$

(2.7) Bonus question *(not graded)*

The result of the previous question is a 8-digit number $\#_1\#_2\#_3\#_4\#_5\#_6\#_7\#_8$ which can be reinterpreted as a date, i.e. $\#_1\#_2$ / $\#_3\#_4$ / $\#_5\#_6\#_7\#_8$. Let's talk about it in this part.

For international students, I suggest that you google this date: it leads to a Wikipedia page with English as a language option. It would be good for you to know a little bit about the country that you are currently residing, so I suggest that you check out that Wikipedia page!

I expect that all non-international students are already aware of this date and (hopefully) are celebrating it! Enjoy being in a part of the modern world!

In these non-graded questions, we usually introduce some Mathematica code to solve some problems; let us do something similar in this homework. You may explicitly check that the following code finds out which day of the week $\#_1\#_2$ / $\#_3\#_4$ / $\#_5\#_6\#_7\#_8$ was:

```
DateObject[{year, month, day}, "Day", "Gregorian", 0]
```

Answer: Monday

Or you may wanna check out the population of Turkey on that year:

```
CountryData["Turkey", {"Population", year}]
```

```
Quantity[13877000, "People"]
```

If you are interested, you may play with similar commands to get more information about the history of Turkey. Enjoy!