Mathematical Methods in Physics I
Homework 10

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1 Question One

1.1

Solution. The complex conjugate of $z = e^{i\pi\theta}$ is $z^* = e^{-i\pi\theta}$. At $\theta = 0$, $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$, the complex conjugates are 1, $e^{-i\pi/4}$, $e^{-i\pi/2}$, and $e^{-3i\pi/4}$, respectively. Let's compute the values of $f(z^*)$:

$$\sin(1) + \cos(1) = 1.38177329068 \tag{1.1}$$

$$\sin(e^{-i\pi/4}) + \cos(e^{-i\pi/4}) = 0.76536686473 \tag{1.2}$$

$$\sin(e^{-i\pi/2}) + \cos(e^{-i\pi/2}) = 0.38177329068 \tag{1.3}$$

$$\sin(e^{-3i\pi/4}) + \cos(e^{-3i\pi/4}) = -0.38177329068 \tag{1.4}$$

1.2

Solution. It follows that $f(z^*) = \sin(z^*) + \cos(z^*)$. To find $(f(z))^*$, let's first rewrite $f(z)$ as follows employing the Euler's formula:

$$f(z) = \sin(z) + \cos(z) \tag{1.5}$$

$$= \frac{e^{iz} - e^{-iz}}{2i} + \frac{e^{iz} + e^{-iz}}{2} \tag{1.6}$$

Now, we can find $(f(z))^*$:

$$(f(z))^* = \left(\frac{e^{iz} - e^{-iz}}{2i} + \frac{e^{iz} + e^{-iz}}{2}\right)^* \tag{1.7}$$

$$= \frac{(e^{iz} - e^{-iz})^*}{(2i)^*} + \frac{(e^{iz} + e^{-iz})^*}{2^*} \tag{1.8}$$

$$= \frac{e^{-iz^*} - e^{iz^*}}{-2i} + \frac{e^{-iz^*} + e^{iz^*}}{2} \tag{1.9}$$

$$= \frac{e^{iz^*} - e^{-iz^*}}{2i} + \frac{e^{iz^*} + e^{-iz^*}}{2} \tag{1.10}$$

$$= \sin(z^*) + \cos(z^*) \tag{1.11}$$

$$= f(z^*) \tag{1.12}$$

1.3

Solution. It follows that $g(z^*) = \cos(iz^*)$. To find $(g(z))^*$, let's first rewrite $g(z)$ as follows employing the Euler's formula:

$$\begin{aligned} g(z) &= \cos(iz) && (1.13) \\ &= \frac{e^{i(iz)} + e^{-i(iz)}}{2} && (1.14) \\ &= \frac{e^{-z} + e^{z}}{2} && (1.15) \end{aligned}$$

Hence, $(g(z))^*$ is:

$$\begin{aligned} (g(z))^* &= \left(\frac{e^{-z} + e^{z}}{2}\right)^* && (1.16) \\ &= \frac{(e^{-z} + e^{z})^*}{2^*} && (1.17) \\ &= \frac{e^{-z^*} + e^{z^*}}{2} && (1.18) \\ &= \cos(iz^*) && (1.19) \\ &= g(z^*) && (1.20) \end{aligned}$$

It also follows that $h(z^*) = \sin(iz^*)$. To find $(h(z))^*$, let's first rewrite $h(z)$ as follows once again employing the Euler's formula:

$$\begin{aligned} h(z) &= \sin(iz) && (1.21) \\ &= \frac{e^{i(iz)} - e^{-i(iz)}}{2i} && (1.22) \\ &= \frac{e^{-z} - e^{z}}{2i} && (1.23) \end{aligned}$$

Hence, $(h(z))^*$ is:

$$\begin{aligned} (h(z))^* &= \left(\frac{e^{-z} - e^{z}}{2i}\right)^* && (1.24) \\ &= \frac{(e^{-z} - e^{z})^*}{(2i)^*} && (1.25) \\ &= \frac{e^{-z^*} - e^{z^*}}{-2i} && (1.26) \\ &= -\sin(iz^*) && (1.27) \\ &= -h(z^*) && (1.28) \\ &\neq h(z^*) && (1.29) \end{aligned}$$

2 Question Two

2.1

Solution. Let's denote the 3×3 matrix as B. Then, we can write B as follows:

$$B = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix} \tag{2.1}$$

Note that $z_{ij} = a_{ij} + ib_{ij}$. Hence, $B^\dagger$ is:

$$B^\dagger = \begin{pmatrix} z_{11}^* & z_{21}^* & z_{31}^* \\ z_{12}^* & z_{22}^* & z_{32}^* \\ z_{13}^* & z_{23}^* & z_{33}^* \end{pmatrix} \tag{2.2}$$

For the matrix B to be Hermitian, $B = B^\dagger$. Therefore, we can write the following equations:

$$z_{11} = z_{11}^* \tag{2.3}$$
$$z_{12} = z_{21}^* \tag{2.4}$$
$$z_{13} = z_{31}^* \tag{2.5}$$
$$z_{21} = z_{12}^* \tag{2.6}$$
$$z_{22} = z_{22}^* \tag{2.7}$$
$$z_{23} = z_{32}^* \tag{2.8}$$
$$z_{31} = z_{13}^* \tag{2.9}$$
$$z_{32} = z_{23}^* \tag{2.10}$$
$$z_{33} = z_{33}^* \tag{2.11}$$

It follows that $a_{ij} = a_{ji}$ and $b_{ij} = -b_{ji}$ for $i \neq j$. For $i = j$, $b_{ij} = b_{ji} = 0$. Hence, B has the form:

$$B = \begin{pmatrix} a_{11} & a_{12} + ib_{12} & a_{13} + ib_{13} \\ a_{12} - ib_{12} & a_{22} & a_{23} + ib_{23} \\ a_{13} - ib_{13} & a_{23} - ib_{23} & a_{33} \end{pmatrix} \tag{2.12}$$