Mathematical Methods in Physics I
Homework 5

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1 Problem One

1.1

Solution. Let's solve the differential equation:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\frac{\mathrm{d}}{\mathrm{d}x} + 1\right) \cdot f(x) = 0 \tag{1.1}$$

The characteristic equation is:

$$r^2 + 2r + 1 = 0 \tag{1.2}$$
$$D = 4 - 4(1)(1) \tag{1.3}$$
$$D = 0 \tag{1.4}$$
$$r_{1,2} = \frac{-2 \pm 0}{2} = -1 \tag{1.5}$$

And the complete solution is:

$$f(x) = c_1 \mathrm{e}^{-x} + c_2 x \mathrm{e}^{-x} \tag{1.6}$$

1.2

Solution. Let's determine if we can rewrite the given differential equation in the form of $\mathcal{D}_1 \cdot \mathcal{D}_2 \cdot \mathcal{D}_3 \ldots \mathcal{D}_n \cdot f(x) = 0$, where $\mathcal{D}_i = \left(x^{a_i}\frac{\mathrm{d}}{\mathrm{d}x} + b_i\right)$. Since the differential operator $\mathcal{D}_i$ is of the first order, and the differential equation is of the second order, $n = 2$:

$$\left(x^{a_1}\frac{\mathrm{d}}{\mathrm{d}x} + b_1\right) \cdot \left(x^{a_2}\frac{\mathrm{d}}{\mathrm{d}x} + b_2\right) \cdot f(x) \tag{1.7}$$
$$= \left(x^{a_1}\frac{\mathrm{d}}{\mathrm{d}x} + b_1\right)\left(x^{a_2}f'(x) + b_2 f(x)\right) \tag{1.8}$$
$$= x^{a_1}\left(x^{a_2}f'(x) + b_2 f(x)\right)' + b_1\left(x^{a_2}f'(x) + b_2 f(x)\right) \tag{1.9}$$
$$= x^{a_1}(a_2 x^{a_2-1}f'(x) + x^{a_2}f''(x) + b_2 f'(x)) + b_1 x^{a_2}f'(x) + b_1 b_2 f(x) \tag{1.10}$$
$$= a_2 x^{a_1+a_2-1}f'(x) + x^{a_1+a_2}f''(x) + b_2 x^{a_1}f'(x) + b_1 x^{a_2}f'(x) + b_1 b_2 f(x) \tag{1.11}$$
$$= (x^{a_1+a_2})f''(x) + (a_2 x^{a_1+a_2-1} + b_2 x^{a_1} + b_1 x^{a_2})f'(x) + (b_1 b_2)f(x) \tag{1.12}$$

It immediately follows that $a_1 + a_2 = 6$ and $b_1 b_2 = -1$. Then $a_2 x^5 + b_1 x^{a_2} - \frac{1}{b_1} x^{a_1} = 3x^5$. Hence, $a_2 = a_1 = 3$. Then $3x^5 + x^3 \left(b_1 - \frac{1}{b_1}\right) = 3x^5$. Therefore, $b_1 - \frac{1}{b_1}$ should be equal to zero. Then,4 $b_1 = 1$ and $b_2 = -1$. The final expression is:

$$\left(x^6 \frac{\mathrm{d}^2}{\mathrm{d}x^2} + 3x^5 \frac{\mathrm{d}}{\mathrm{d}x} - 1\right) \cdot f(x) = 0 \tag{1.13}$$

Which is the original differential equation. In conclusion, we can rewrite the differential equation in the form of:

$$\left(x^3 \frac{\mathrm{d}}{\mathrm{d}x} + 1\right) \cdot \left(x^3 \frac{\mathrm{d}}{\mathrm{d}x} - 1\right) \cdot f(x) = 0 \tag{1.14}$$

1.3

Solution. From the given facts, it follows that we can derive a general solution for the differential equation:

$$\left(x^3 \frac{\mathrm{d}}{\mathrm{d}x} + 1\right) \cdot f(x) = 0 \tag{1.15}$$

$$x^3 f'(x) + f(x) = 0 \tag{1.16}$$

$$x^3 f'(x) = -f(x) \tag{1.17}$$

$$\frac{f'(x)}{f(x)} = -\frac{1}{x^3} \tag{1.18}$$

If we integrate both sides, we obtain:

$$\ln f(x) = \frac{1}{2x^2} + c_0 \tag{1.19}$$

Where c_0 is the integration constant. If we exponentiate both sides, we obtain:

$$f(x) = \mathrm{e}^{c_0} \mathrm{e}^{1/(2x^2)} \tag{1.20}$$

$$= c_1 \mathrm{e}^{1/(2x^2)} \tag{1.21}$$

Hence, the general solution is:

$$f(x) = c_1 \mathrm{e}^{1/(2x^2)} + c_2 \mathrm{e}^{-1/(2x^2)} \tag{1.22}$$

1.4

Solution. Let's parametrize the given differential equation in terms of $u(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \tag{1.23}$$

$$= u' \frac{\mathrm{d}}{\mathrm{d}u} \tag{1.24}$$

It also follows that:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) \tag{1.25}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(u'\frac{\mathrm{d}}{\mathrm{d}u}\right) \tag{1.26}$$

$$= u''\frac{\mathrm{d}}{\mathrm{d}u} + u'\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}u} \tag{1.27}$$

$$= u''\frac{\mathrm{d}}{\mathrm{d}u} + (u')^2\frac{\mathrm{d}^2}{\mathrm{d}u^2} \tag{1.28}$$

Therefore, the equation obtains the form:

$$\left(u''\frac{\mathrm{d}}{\mathrm{d}u} + (u')^2\frac{\mathrm{d}^2}{\mathrm{d}u^2} + u'\frac{\mathrm{d}}{\mathrm{d}u} + \mathrm{e}^{-2x}\right)\cdot f(u) = \mathrm{e}^{-2x} \tag{1.29}$$

If we divide both sides by $(u')^2$, we have:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}u^2} + \frac{u''+u'}{(u')^2}\frac{\mathrm{d}}{\mathrm{d}u} + \frac{\mathrm{e}^{-2x}}{(u')^2}\right)\cdot f(u) = \frac{\mathrm{e}^{-2x}}{(u')^2} \tag{1.30}$$

For this differential equation to be of the form of constant coefficient, $\frac{u''+u'}{(u')^2}\frac{\mathrm{d}}{\mathrm{d}u}$ and $\frac{\mathrm{e}^{-2x}}{u''}$ must be constants. Assume that the latter is equal to 1. Therefore, we have:

$$u = \int \sqrt{\mathrm{e}^{-2x}}\,\mathrm{d}x \tag{1.31}$$

$$= -\mathrm{e}^{-x} + C \tag{1.32}$$

Where C is an arbitrary constant. Also, we have:

$$u' = \mathrm{e}^{-x},\ u'' = -\mathrm{e}^{-x},\ (u')^2 = \mathrm{e}^{-2x} \tag{1.33}$$

And it follows that:

$$\frac{u''+u'}{(u')^2} = 0 \tag{1.34}$$

The preliminary conditions are satisfied. Hence, we can rewrite the differential equation as follows:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}u^2} + 1\right)\cdot f(u) = 1 \tag{1.35}$$

This differential equation is non-homogeneous. The homogeneous solution is obtained as follows:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}u^2} + 1\right)\cdot f(u) = 0 \tag{1.36}$$

The characteristic equation is:

$$r^2 + 1 = 0 \tag{1.37}$$

$$r^2 = -1 \tag{1.38}$$

$$r_{1,2} = \pm i \tag{1.39}$$

Hence, the homogeneous solution is:

$$f_h(u) = a_1\mathrm{e}^{-iu} + a_2\mathrm{e}^{iu} \tag{1.40}$$

The particular solution is obtained as follows:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}u^2} + 1\right) \cdot \mathbb{i}(u) = \delta(u) \tag{1.41}$$

Applying Laplace transform to both sides, we obtain:

$$\left(s^2\mathbb{I}(s) - s\mathbb{i}(0) - \mathbb{i}'(0)\right) + \mathbb{I}(s) = 1 \tag{1.42}$$

Since $\mathbb{i}(0) = \mathbb{i}'(0) = 0$, we have:

$$\mathbb{I}(s) = \frac{1}{s^2+1} \tag{1.43}$$

Which is the Laplace transform of $\sin(u)$:

$$\mathbb{i}(u) = \sin(u) \tag{1.44}$$

For $u \geq 0$. Therefore, the complete solution of the differential equation is:

$$f(u) = a_1\mathrm{e}^{-iu} + a_2\mathrm{e}^{iu} + \int_0^u \sin(u-y)h(y)\,\mathrm{d}y \tag{1.45}$$

Note that $h(y) = 1$ no matter what y is. Consequently, we have:

$$\begin{aligned} f(u) &= a_1\mathrm{e}^{-iu} + a_2\mathrm{e}^{iu} + \int_0^u \sin(u-y)\,\mathrm{d}y && (1.46)\\ &= a_1\mathrm{e}^{-iu} + a_2\mathrm{e}^{iu} + \left[\cos(u-y)\Big|_0^u\right] && (1.47)\\ &= a_1\mathrm{e}^{-iu} + a_2\mathrm{e}^{iu} + 1 - \cos(u) && (1.48)\\ &= a_1\cos(u) + a_2\sin(u) + 1 - \cos(u) && (1.49)\\ &= c_1\cos(u) + c_2\sin(u) + 1 && (1.50) \end{aligned}$$

Where $c_1 = a_1 - 1$ and $c_2 = a_2$. In conclusion, the solution of the given differential equation is:

$$f(x) = c_1\cos(-\mathrm{e}^{-x}) + c_2\sin(-\mathrm{e}^{-x}) + 1 \tag{1.51}$$

1.5

Solution. The given differential equation can be rewritten as:

$$\begin{aligned} h''(x) + 2h'(x) + h(x) = 0 && (1.52)\\ \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\frac{\mathrm{d}}{\mathrm{d}x} + 1\right) \cdot h(x) = 0 && (1.53) \end{aligned}$$

Where $h(x) = g''(x)$. The characteristic equation is:

$$r^2 + 2r + 1 = 0 \tag{1.54}$$
$$D = 4 - 4(1)(1) \tag{1.55}$$
$$D = 0 \tag{1.56}$$
$$r_{1,2} = \frac{-2 \pm 0}{2} = -1 \tag{1.57}$$

Hence, the solution in terms of $h(x)$ is:

$$h(x) = c_1 \mathrm{e}^{-x} + c_2 x \mathrm{e}^{-x} \tag{1.58}$$

It follows that:

$$g(x) = \int \left[\int h(x) \,\mathrm{d}x \right] \mathrm{d}x \tag{1.59}$$

Therefore, we have:

$$g(x) = \int \left[\int (c_1 \mathrm{e}^{-x} + c_2 x \mathrm{e}^{-x}) \,\mathrm{d}x \right] \mathrm{d}x \tag{1.60}$$

Let's denote the inner integral as A, and the outer integral as B:

$$A = \int (c_1 \mathrm{e}^{-x} + c_2 x \mathrm{e}^{-x}) \,\mathrm{d}x \tag{1.61}$$
$$= (-c_1 - c_2 x - c_2)\mathrm{e}^{-x} + a \tag{1.62}$$

Where a is the integration constant. Then, we have:

$$B = \int A \,\mathrm{d}x \tag{1.63}$$
$$= \int \left[(-c_1 - c_2 x - c_2)\mathrm{e}^{-x} + a \right] \mathrm{d}x \tag{1.64}$$
$$= ax + (c_1 + c_2 x + 2c_2)\mathrm{e}^{-x} + b \tag{1.65}$$

Where b is the integration constant.

1.6

Solution. To show that the given differential equation is exact, let's find the condition that it must satisfy. The general form of a third-order homogeneous linear differential equation is:

$$\left(p(x)\frac{\mathrm{d}^3}{\mathrm{d}x^3} + q(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + r(x)\frac{\mathrm{d}}{\mathrm{d}x} + s(x) \right) \cdot f(x) = 0 \tag{1.66}$$
$$pf''' + qf'' + rf' + sf = 0 \tag{1.67}$$

Then, we have:

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\left[\alpha(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta(x)\frac{\mathrm{d}}{\mathrm{d}x} + \gamma(x) \right] \cdot f(x) \right) = 0 \tag{1.68}$$

Hence, it follows that:

$$\alpha' f'' + \alpha f''' + \beta' f' + \beta f'' + \gamma' f + \gamma f' = 0 \tag{1.69}$$
$$\alpha f''' + (\alpha' + \beta) f'' + (\beta' + \gamma) f' + \gamma' f = 0 \tag{1.70}$$

Therefore, $\alpha = p$, $\alpha' + \beta = q$, $\beta' + \gamma = r$, and $\gamma' = s$. We have:

$$\alpha = p \tag{1.71}$$
$$\beta = q - p' \tag{1.72}$$
$$\gamma = \int s \, \mathrm{d}x \tag{1.73}$$

The condition is:

$$r = q' - p'' + \int s \, \mathrm{d}x \tag{1.74}$$

Consequently, for an arbitrary third-order homogeneous linear differential equation to be exact, it must satisfy the derived condition. If we substitute α, β, and γ, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(\left[p(x) \frac{\mathrm{d}^2}{\mathrm{d}x^2} + (q(x) - p'(x)) \frac{\mathrm{d}}{\mathrm{d}x} + \left(\int s(x) \, \mathrm{d}x \right) \right] \cdot f(x) \right) = 0 \tag{1.75}$$

For this equation to be true, it follows that:

$$\left[p(x) \frac{\mathrm{d}^2}{\mathrm{d}x^2} + (q(x) - p'(x)) \frac{\mathrm{d}}{\mathrm{d}x} + \left(\int s(x) \, \mathrm{d}x \right) \right] \cdot f(x) = \text{constant} \tag{1.76}$$

Hence, if the aforementioned condition is satisfied, we can rewrite an arbitrary third-order homogeneous linear differential equation in this form. Let's check whether the given differential equation is exact or not. It follows that:

$$p(x) = x \tag{1.77}$$
$$q(x) = 1 \tag{1.78}$$
$$r(x) = \frac{1}{x} \tag{1.79}$$
$$s(x) = -\frac{1}{x^2} \tag{1.80}$$

Therefore, we obtain:

$$\frac{1}{x} = 0 - 0 + \int -\frac{1}{x^2} \, \mathrm{d}x \tag{1.81}$$
$$\frac{1}{x} = \frac{1}{x} \tag{1.82}$$

The condition is satisfied. In conclusion, the given differential equation is exact, and it can be rewritten as:

$$\left(x \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{x} \right) \cdot f(x) = \text{constant} \tag{1.83}$$

1.7

Solution. To rewrite the given differential equation, let's define $f(x) = g(x)f_1(x) = g(x)\mathrm{e}^x$. Hence, we obtain:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(g(x)\mathrm{e}^x\right) && (1.84)\\
&= g'(x)\mathrm{e}^x + g(x)\mathrm{e}^x && (1.85)\\
\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x} f(x)\right) && (1.86)\\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left(g'(x)\mathrm{e}^x + g(x)\mathrm{e}^x\right) && (1.87)\\
&= g''(x)\mathrm{e}^x + 2g'(x)\mathrm{e}^x + g(x)\mathrm{e}^x && (1.88)
\end{aligned}
$$

If we substitute the derived expression into the differential equation, we obtain:

$$
\begin{aligned}
&(x-1)(g''(x)\mathrm{e}^x + 2g'(x)\mathrm{e}^x + g(x)\mathrm{e}^x) - x(g'(x)\mathrm{e}^x + g(x)\mathrm{e}^x) + g(x)\mathrm{e}^x = 0 && (1.89)\\
&(x-1)(g''(x) + 2g'(x) + g(x)) - x(g'(x) + g(x)) + g(x) = 0 && (1.90)\\
&xg''(x) + 2xg'(x) + xg(x) - g''(x) - 2g'(x) - \cancel{g(x)} - g'(x) - xg(x) + \cancel{g(x)} = 0 && (1.91)\\
&(x-1)g''(x) + (2x-3)g'(x) = 0 && (1.92)\\
& && (1.93)
\end{aligned}
$$

If $g'(x) = h(x)$, then:

$$
(x-1)h'(x) + (2x-3)h(x) = 0 \tag{1.94}
$$

Therefore, we can rewrite the given second-order differential equation as the first-order one, given the solution $f_1(x) = \mathrm{e}^x$.