Mathematical Methods in Physics I
Homework 11

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1 Question One

1.1

Solution. Let's define $f(x) = \cos(x)$ and $g(x) = \sin(x)$. We will find their Fourier transforms using Euler's identity, $e^{ix} = \cos(x) + i\sin(x)$. The Fourier transform of $f(x)$ is:

$$\hat{f}(k) = \int_{\mathbb{R}} f(x)e^{-ikx}\mathrm{d}x \tag{1.1}$$
$$= \int_{\mathbb{R}} \cos(x)e^{-ikx}\mathrm{d}x \tag{1.2}$$
$$= \int_{\mathbb{R}} \frac{e^{ix}+e^{-ix}}{2}e^{-ikx}\mathrm{d}x \tag{1.3}$$
$$= \frac{1}{2}\int_{\mathbb{R}} e^{ix(1-k)}\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} e^{-ix(1+k)}\mathrm{d}x \tag{1.4}$$

Let's use the following identity:

$$\delta(k) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{ixk}\mathrm{d}x \tag{1.5}$$

Note that the Dirac delta function is even. Therefore, we can rewrite the above equation as:

$$\hat{f}(k) = \frac{1}{2}(2\pi\delta(k-1)) + \frac{1}{2}(2\pi\delta(k+1)) \tag{1.6}$$
$$= \pi\delta(k-1) + \pi\delta(k+1) \tag{1.7}$$
$$= \pi\left(\delta(k-1) + \delta(k+1)\right) \tag{1.8}$$

Following the same reasoning, we can find the Fourier transform of $g(x)$:

$$\hat{g}(k) = \int_{\mathbb{R}} g(x)e^{-ikx}\mathrm{d}x \tag{1.9}$$
$$= \int_{\mathbb{R}} \sin(x)e^{-ikx}\mathrm{d}x \tag{1.10}$$
$$= \int_{\mathbb{R}} \frac{e^{ix}-e^{-ix}}{2i}e^{-ikx}\mathrm{d}x \tag{1.11}$$
$$= \frac{1}{2i}\int_{\mathbb{R}} e^{ix(1-k)}\mathrm{d}x - \frac{1}{2i}\int_{\mathbb{R}} e^{-ix(1+k)}\mathrm{d}x \tag{1.12}$$
$$= \frac{1}{2i}(2\pi\delta(k-1)) - \frac{1}{2i}(2\pi\delta(k+1)) \tag{1.13}$$
$$= \frac{\pi}{i}\delta(k-1) - \frac{\pi}{i}\delta(k+1) \tag{1.14}$$
$$= \pi\left(i\delta(k+1) - i\delta(k-1)\right) \tag{1.15}$$

1.2

Solution. Suppose that $f,\ \hat{f} \in L^1(\mathbb{R}) \cap C^0(\mathbb{R})$. Note that k in the inverse Fourier transform is a dummy variable, and we can replace it with a without changing the meaning of the equation. The same is true for x in the Fourier transform, so let's change it to b. The motivation is to avoid confusion when we substitute $\hat{f}$ into f. Therefore, we have:

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(a)e^{iax}\mathrm{d}a && (1.16)\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(b)e^{-iab}\,\mathrm{d}b\right) e^{iax}\mathrm{d}a && (1.17)\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(b)e^{ia(x-b)}\,\mathrm{d}b\right)\,\mathrm{d}a && (1.18)\end{aligned}$$

It is not directly possible to change the order of integration, since the hypotheses of the Fubini-Tonelli theorem are not satisfied. In particular, the function $F(b,a) = f(b)e^{ia(x-b)} \notin L^1(\mathbb{R}^2)$. But we can consider the following integral:

$$I_\epsilon(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(b)e^{ia(x-b)}e^{-\epsilon^2 a^2/4}\,\mathrm{d}b\right)\,\mathrm{d}a \qquad (1.19)$$

It follows that the integrand is now integrable, and we can change the order of integration. The steps will be omitted due to the advanced nature of the techniques used. Hence, we have:

$$\begin{aligned} f(x) &= \int_{\mathbb{R}} f(b)\left(\frac{1}{2\pi}\int_{\mathbb{R}} e^{ia(x-b)}\,\mathrm{d}a\right)\,\mathrm{d}b && (1.20)\\ &= \int_{\mathbb{R}} f(b)\delta(x-b)\,\mathrm{d}b && (1.21)\\ &= f(x) && (1.22)\end{aligned}$$

Therefore, we have shown that the Fourier transform and its inverse are consistent with each other.

1.3

Solution. One of the properties of the Fourier transform is that the Fourier transform of a product of two functions is the convolution of their Fourier transforms. Hence, $m(t)$ is a product of two functions, namely, $\cos(ft)$ and $s(t)$. Then, the Fourier transform of $m(t)$ is the convolution of the Fourier transforms of $\cos(ft)$ and $s(t)$. Let's find the Fourier transform of $\cos(ft)$:

$$\begin{aligned} \mathcal{F}\{\cos(ft)\}(k) &= \int_{\mathbb{R}} \cos(ft)e^{-ikt}\,\mathrm{d}t && (1.23)\\ &= \int_{\mathbb{R}} \frac{e^{ift}+e^{-ift}}{2}e^{-ikt}\,\mathrm{d}t && (1.24)\\ &= \frac{1}{2}\int_{\mathbb{R}} e^{it(f-k)}\,\mathrm{d}t + \frac{1}{2}\int_{\mathbb{R}} e^{-it(f+k)}\,\mathrm{d}t && (1.25)\\ &= \frac{1}{2}\left[2\pi\delta(f-k) + 2\pi\delta(f+k)\right] && (1.26)\\ &= \pi\left[\delta(f-k)+\delta(f+k)\right] && (1.27)\end{aligned}$$

Let's denote the Fourier transform of $s(t)$ as $S(k)$. Hence, the Fourier transform of $m(t)$, $M(k)$, is:

$$\begin{aligned} M(k) &= \frac{1}{2\pi}(\mathcal{F}\{\cos(ft)\} * \mathcal{F}\{s(t)\})(k) && (1.28)\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \pi\left[\delta(f-\tau)+\delta(f+\tau)\right]S(k-\tau)\,\mathrm{d}\tau && (1.29)\\ &= \frac{1}{2}\left[S(k-f)+S(k+f)\right] && (1.30)\end{aligned}$$

2 Question Two

2.1

Solution. The fundamental period of $\cos(x)$ is 2π, and the fundamental period of $\sin(2x)$ is π. Since their ratio is rational, the fundamental period of $\cos(x) + \sin(2x)$ is 2π since it is the least common multiple of fundamental periods of the terms. Hence, for every integral multiple of 2π, the function $f_P(x) = \cos(x) + \sin(2x)$ will have the same value, and the same must be true for the function $f(x)$. The smallest value of T, therefore, is 2π.

2.2

Solution. Let $T = 4\pi$. Hence, f takes the form $f\left(x - 4\pi\left\lfloor \frac{x}{4\pi} \right\rfloor\right)$. For $a = \{10, 11, 21, 22\}$, we have:

$$f_P(10) = f\left(10 - 4\pi\left\lfloor \frac{10}{4\pi} \right\rfloor\right) = f(10) \tag{2.1}$$

$$f_P(11) = f\left(11 - 4\pi\left\lfloor \frac{11}{4\pi} \right\rfloor\right) = f(11) \tag{2.2}$$

$$f_P(21) = f\left(21 - 4\pi\left\lfloor \frac{21}{4\pi} \right\rfloor\right) = f(21 - 4\pi) \tag{2.3}$$

$$f_P(22) = f\left(22 - 4\pi\left\lfloor \frac{22}{4\pi} \right\rfloor\right) = f(22 - 4\pi) \tag{2.4}$$

2.3

Solution. Let's define $g(x) = x + x^2 + x^3$ with the fundamental period of 2π. Hence, $L = \pi$, and the Fourier series of $g(x)$ is:

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left(a_m \cos(mx) + b_m \sin(mx)\right) \tag{2.5}$$

Let's employ the Euler-Fourier formulas to evaluate the coefficients. The coefficient a_0 is:

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos(0)\, \mathrm{d}x && (2.6)\\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2 + x^3)\, \mathrm{d}x && (2.7)\\
&= \frac{1}{\pi} \left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}\right]_{-\pi}^{\pi} && (2.8)\\
&= \frac{1}{\pi} \left[\cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3} + \cancel{\frac{\pi^4}{4}} - \cancel{\frac{\pi^2}{2}} + \frac{\pi^3}{3} - \cancel{\frac{\pi^4}{4}}\right] && (2.9)\\
&= \frac{2\pi^2}{3} && (2.10)
\end{aligned}$$

Let's find a_1:

$$\begin{aligned}
a_1 &= \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos(x)\, \mathrm{d}x && (2.11)\\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} (x + x^2 + x^3) \cos(x)\, \mathrm{d}x && (2.12)\\
&= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x \cos(x)\, \mathrm{d}x + \int_{-\pi}^{\pi} x^2 \cos(x)\, \mathrm{d}x + \int_{-\pi}^{\pi} x^3 \cos(x)\, \mathrm{d}x\right] && (2.13)
\end{aligned}$$

Let's denote the integrals as I_1, I_2, and I_3, respectively. We can evaluate these integrals using integration by parts. Let's find I_1, by denoting $u = x$ and $\mathrm{d}v = \cos(x)\,\mathrm{d}x$:

$$\begin{aligned} I_1 &= x\sin(x)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \sin(x)\,\mathrm{d}x && (2.14)\\ &= x\sin(x) + \cos(x)\Big|_{-\pi}^{\pi} && (2.15)\\ &= 0 && (2.16) \end{aligned}$$

Now, let's find I_2, by denoting $u = x^2$ and $\mathrm{d}v = \cos(x)\,\mathrm{d}x$. It follows that $\mathrm{d}u = 2x\,\mathrm{d}x$ and $v = \sin(x)$:

$$I_2 = x^2\sin(x)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x\sin(x)\,\mathrm{d}x \tag{2.17}$$

Let $u = x$ and $\mathrm{d}v = \sin(x)\,\mathrm{d}x$. It follows that $\mathrm{d}u = \mathrm{d}x$ and $v = -\cos(x)$:

$$\begin{aligned} I_2 &= -2\left[-x\cos(x)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos(x)\,\mathrm{d}x\right] && (2.18)\\ &= -2\left[-x\cos(x) + \sin(x)\right]\Big|_{-\pi}^{\pi} && (2.19)\\ &= -4\pi && (2.20) \end{aligned}$$

Finally, let's find I_3, by denoting $u = x^3$ and $\mathrm{d}v = \cos(x)\,\mathrm{d}x$. It follows that $\mathrm{d}u = 3x^2\,\mathrm{d}x$ and $v = \sin(x)$:

$$\begin{aligned} I_3 &= x^3\sin(x)\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 3x^2\sin(x)\,\mathrm{d}x && (2.21)\\ &= -3\int_{-\pi}^{\pi} x^2\sin(x)\,\mathrm{d}x && (2.22) \end{aligned}$$

Let's denote $u = x^2$ and $\mathrm{d}v = \sin(x)\,\mathrm{d}x$. It follows that $\mathrm{d}u = 2x\,\mathrm{d}x$ and $v = -\cos(x)$:

$$\begin{aligned} I_3 &= -3\left[-x^2\cos(x)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} 2x\cos(x)\,\mathrm{d}x\right] && (2.23)\\ &= -3\left[-x^2\cos(x) + 2x\sin(x) + 2\cos(x)\right]\Big|_{-\pi}^{\pi} && (2.24)\\ &= 0 && (2.25) \end{aligned}$$

Hence, a_1 is:

$$\begin{aligned} a_1 &= \frac{1}{\pi}\left[I_1 + I_2 + I_3\right] && (2.26)\\ &= \frac{1}{\pi}\left[0 - 4\pi + 0\right] && (2.27)\\ &= -4 && (2.28) \end{aligned}$$

Now, let's find b_1:

$$\begin{aligned} b_1 &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin(x)\,\mathrm{d}x && (2.29)\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2 + x^3)\sin(x)\,\mathrm{d}x && (2.30)\\ &= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin(x)\,\mathrm{d}x + \int_{-\pi}^{\pi} x^2\sin(x)\,\mathrm{d}x + \int_{-\pi}^{\pi} x^3\sin(x)\,\mathrm{d}x\right] && (2.31) \end{aligned}$$

Let's denote the integrals as I_1, I_2, and I_3 respectively. We can evaluate these integrals using integration by parts once again. For the sake of brevity, the steps will be omitted. It follows that $I_1 = 2\pi$, $I_2 = 0$, and $I_3 = 2\pi^3 - 12\pi$. Hence, b_1 is:

$$\begin{aligned} b_1 &= \frac{1}{\pi}\left[I_1 + I_2 + I_3\right] && (2.32) \\ &= \frac{1}{\pi}\left[2\pi + 0 + 2\pi^3 - 12\pi\right] && (2.33) \\ &= 2\pi^2 - 10 && (2.34) \end{aligned}$$

Therefore, the Fourier series of $g(x)$ is:

$$f(x) = \frac{\pi^2}{3} - 4\cos(x) + (2\pi^2 - 10)\sin(x) + \cdots \tag{2.35}$$