Mathematical Methods in Physics I

Homework II

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Problem 1

Solution. (a) The type and definition of d^n/dx^n are as follows:

$$\frac{d^n}{dx^n} :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C})$$
$$\frac{d^n}{dx^n} = (x \mapsto f(x)) \mapsto \left(x \mapsto f^{(n)}(x)\right)$$

(b) The definition of $\mathcal{D}$ acting on cos is as follows:

$$\mathcal{D} \cdot \cos = x \mapsto (a_0 \cos(x) - a_1 \sin(x) - a_2 \cos(x) + a_3 \sin(x) + a_4 \cos(x) - \dots)$$

Note that the nth derivative of $\cos(x)$ is $\cos\left(x + \frac{n\pi}{2}\right)$, where $n \in \mathbb{Z}$. Therefore, we can rewrite the previous equation as

$$\mathcal{D} \cdot \cos = x \mapsto \sum_{i=0}^{n} a_i \cos\left(x + \frac{i\pi}{2}\right)$$

The same logic applies to $\sin(x)$, where the nth derivative of $\sin(x)$ is $\sin\left(x + \frac{n\pi}{2}\right)$, $n \in \mathbb{Z}$:

$$\begin{aligned}\mathcal{D} \cdot \sin &= x \mapsto (a_0 \sin(x) + a_1 \cos(x) - a_2 \sin(x) - a_3 \cos(x) + a_4 \sin(x) + \dots) \\ &= x \mapsto \sum_{i=0}^{n} a_i \sin\left(x + \frac{i\pi}{2}\right)\end{aligned}$$

Finally, the definition of $\mathcal{D}$ acting on exp is as following:

$$\begin{aligned}\mathcal{D} \cdot \exp &= x \mapsto (a_0 e^x + a_1 e^x + a_2 e^x + a_3 e^x + a_4 e^x + \dots) \\ &= e^x \sum_{i=0}^{n} a_i\end{aligned}$$

(c) The kernel of transformation of the cosine function is

$$\left\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots\right\}$$

Since it is an infinite sequence, it can be rewritten as

$$\left\{\frac{\pi}{2} + k\pi\right\}$$

where $k \in \mathbb{Z}$. Therefore, the cosine function has a map

$$\left\{\frac{\pi}{2} + k\pi\right\} \xrightarrow{\cos} \{0\}$$

As for the kernel of the differential operator $\left(\frac{d}{dx} - a\right)$, let's denote the operator as L:

$$L = \frac{d}{dx} - a$$

The kernel of this operator, denoted as $\ker(L)$, is the set of functions $y = f(x)$ such that

$$L[y] = \frac{dy}{dx} - ay = 0$$

To find the functions in the kernel, we have to solve the differential equation. The solution is $y = ce^{ax}$, where c is an arbitrary constant. Therefore, $\ker(L)$ is the infinite set of solutions $y = ce^{ax}$. The type and definition of the kernel are

$$\ker\left(\frac{d}{dx} - a\right) :: \{\mathbb{C} \to \mathbb{C}\}$$
$$\ker\left(\frac{d}{dx} - a\right) = \{x \to ce^{ax}\}$$

(d) The characteristic equation of the given differential equation is

$$r^5 - 3r^4 - 23r^3 + 51r^2 + 94r - 120 = 0$$

Three roots of the characteristic were given: 5, -4, and 3. Hence, we can use Vieta's formulas to obtain the remaining two roots, r_4 and r_5:

$$4 + r_4 + r_5 = 3$$
$$-60r_4r_5 = 120$$

By performing simple algebraic operations, it is obvious that $r_4 = -2$ and $r_5 = 1$. The general solution of the given differential equation is

$$y = c_1e^{5x} + c_2e^{-4x} + c_3e^{3x} + c_4e^{-2x} + c_5e^x$$

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Problem 2

Solution. The Laplace transform of $f_{n,a}$ is

$$\begin{aligned}\mathcal{L}\{f_{n,a}\} &= \int_0^\infty e^{-sx}\left(x^ne^{ax}\right)dx \\ &= \int_0^\infty e^{(a-s)x}x^n\,dx\end{aligned}$$

To evaluate this integral, we should manipulate it to bring it to the form of the gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$$

where $\Re(z) > 0$. Let's substitute $-(a-s)x$ with u. It follows that $du = -(a-s)\,dx = (s-a)\,dx$. Therefore, we have

$$\begin{aligned}\mathcal{L}\{f_{n,a}\} &= \int_0^\infty e^{-u}\left(\frac{u}{s-a}\right)^n\frac{du}{(s-a)} \\ &= (s-a)^{-(n+1)}\int_0^\infty e^{-u}u^n\,du\end{aligned}$$

From the definition of the gamma function, it follows that

$$\mathcal{L}\left\{f_{n,a}\right\} = (s-a)^{-(n+1)}\,\Gamma\left(n+1\right)$$

Therefore, the type and definition of the Laplace transform of $f_{n,a}$ are

$$\begin{aligned}
\mathcal{L}\cdot f_{n,a} &:: \mathbb{C} \to \mathbb{C} \\
\mathcal{L}\cdot f_{n,a} &= s \mapsto (s-a)^{-(n+1)}\,\Gamma\left(n+1\right)
\end{aligned}$$

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