Mathematical Methods in Physics II

Homework III

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Problem 1

Solution. a) My student ID number is 2604619. Hence, the multivector ω is given by

$$\omega = \frac{1}{2}\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 \wedge \mathbf{e}_5 \wedge \mathbf{e}_6 \wedge \mathbf{e}_7$$

Since a multivector in an n-dimensional space is non-zero if and only if it is a product of n or fewer vectors, the multivector ω is non-zero if and only if $\dim(\mathcal{V}) \geq 7$. Therefore, the multivector ω is non-zero if and only if $d \geq 7$.

b) Let's take the wedge product of the expression

$$\left(\sum_{i=1}^{5} a_i e_i\right) \wedge \left(\sum_{i=5}^{7} a_i e_i\right)$$

We have

$$\left(\sum_{i=1}^{5} a_i e_i\right) \wedge \left(\sum_{i=5}^{7} a_i e_i\right) = \sum_{i=1}^{5}\sum_{j=5}^{7} a_i a_j e_i \wedge e_j$$

If we write the expression in terms of the basis vectors, we have

$$\begin{aligned}\left(\sum_{i=1}^{5} a_i e_i\right) \wedge \left(\sum_{i=5}^{7} a_i e_i\right) &= a_1a_5e_1 \wedge e_5 + a_1a_6e_1 \wedge e_6 + a_1a_7e_1 \wedge e_7 \\ &+ a_2a_5e_2 \wedge e_5 + a_2a_6e_2 \wedge e_6 + a_2a_7e_2 \wedge e_7 \\ &+ a_3a_5e_3 \wedge e_5 + a_3a_6e_3 \wedge e_6 + a_3a_7e_3 \wedge e_7 \\ &+ a_4a_5e_4 \wedge e_5 + a_4a_6e_4 \wedge e_6 + a_4a_7e_4 \wedge e_7\end{aligned}$$

c) The most generic element of the algebra $\Lambda(\mathcal{V})$, where $d = \dim(\mathcal{V}) = 4$ and $\mathbf{f}_i$ is a basis vector, is given by

$$\begin{aligned}v = c_0 &+ c_1\mathbf{f}_1 + c_2\mathbf{f}_2 + c_3\mathbf{f}_3 + c_4\mathbf{f}_4 \\ &+ c_5\mathbf{f}_1 \wedge \mathbf{f}_2 + c_6\mathbf{f}_1 \wedge \mathbf{f}_3 + c_7\mathbf{f}_1 \wedge \mathbf{f}_4 + c_8\mathbf{f}_2 \wedge \mathbf{f}_3 + c_9\mathbf{f}_2 \wedge \mathbf{f}_4 + c_{10}\mathbf{f}_3 \wedge \mathbf{f}_4 \\ &+ c_{11}\mathbf{f}_1 \wedge \mathbf{f}_2 \wedge \mathbf{f}_3 + c_{12}\mathbf{f}_1 \wedge \mathbf{f}_2 \wedge \mathbf{f}_4 + c_{13}\mathbf{f}_1 \wedge \mathbf{f}_3 \wedge \mathbf{f}_4 + c_{14}\mathbf{f}_2 \wedge \mathbf{f}_3 \wedge \mathbf{f}_4 \\ &+ c_{15}\mathbf{f}_1 \wedge \mathbf{f}_2 \wedge \mathbf{f}_3 \wedge \mathbf{f}_4\end{aligned}$$

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Problem 2

Solution. Let's denote the tensor as

$$\mathbf{T} = T^{ijk}_m \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}^m$$

a) First, let's act on $\mathbf{e}_i$ with $\mathbf{e}^m$. We have

$$\mathbf{e}^m \cdot \mathbf{e}_i = \delta^m_i$$

The tensor becomes

$$\mathbf{T} = T^{ijk}_m \mathbf{e}_j \otimes \mathbf{e}_k \delta^m_i$$

Since $T^{ijk}_m \delta^m_i = T^{ijk}_i$, we have

$$\mathbf{T} = T^{ijk}_i \mathbf{e}_j \otimes \mathbf{e}_k$$

The component can be written as

$$T^{ijk}_i = T^{1jk}_1 + T^{2jk}_2$$

Considering all of the non-zero components, we have

$$\begin{aligned} T^{ijk}_i &= T^{111}_1 + T^{121}_1 + T^{211}_2 \\ &= 7 + 3 + (-1) \\ &= 9 \end{aligned}$$

Next, let's act on $\mathbf{e}_j$ with $\mathbf{e}^m$. We have

$$\mathbf{e}^m(\mathbf{e}_j) = \delta^m_j$$

The tensor becomes

$$\mathbf{T} = T^{ijk}_m \mathbf{e}_i \otimes \mathbf{e}_k \delta^m_j$$

Since $T^{ijk}_m \delta^m_j = T^{ijk}_j$, we have

$$\mathbf{T} = T^{ijk}_j \mathbf{e}_i \otimes \mathbf{e}_k$$

The component can be written as

$$T^{ijk}_j = T^{i1k}_1 + T^{i2k}_2$$

Considering all of the non-zero components, we have

$$\begin{aligned} T^{ijk}_j &= T^{111}_1 \\ &= 7 \end{aligned}$$

Finally, let's act on $\mathbf{e}_k$ with $\mathbf{e}^m$. We have

$$\mathbf{e}^m(\mathbf{e}_k) = \delta^m_k$$

The tensor becomes

$$\mathbf{T} = T^{ijk}_m \mathbf{e}_i \otimes \mathbf{e}_j \delta^m_k$$

Since $T^{ijk}_m \delta^m_k = T^{ijk}_k$, we have

$$\mathbf{T} = T^{ijk}_k \mathbf{e}_i \otimes \mathbf{e}_j$$

The component can be written as

$$T^{ijk}_k = T^{ij1}_1 + T^{ij2}_2$$

Considering all of the non-zero components, we have

$$\begin{aligned} T^{ijk}_{k} &= T^{111}_{1} + T^{121}_{1} + T^{112}_{2} \\ &= 7 + 3 + (-5) \\ &= 5 \end{aligned}$$

Therefore, three different tensors can be obtained by contracting the tensor $\mathbf{T}$.

b) The scalar a is given by

$$a = T^{ijk}_{m} \eta_{ij} \delta^{m}_{k}$$

Note that $T^{ijk}_{m} \delta^{m}_{k} = T^{ijk}_{k}$, so we have

$$a = T^{ijk}_{k} \eta_{ij}$$

If we sum over k, we have

$$a = T^{ij1}_{1} \eta_{ij} + T^{ij2}_{2} \eta_{ij} + T^{ij3}_{3} \eta_{ij}$$

Since $\eta_{ij} = 0$ for $i \neq j$, we have

$$\begin{aligned} a &= T^{111}_{1} \eta_{11} + T^{222}_{2} \eta_{22} + T^{333}_{3} \eta_{33} \\ &= (7)(-1) + (0)(1) + (0)(1) \\ &= -7 \end{aligned}$$

c) The covector ω is given by

$$\omega = \epsilon_{ijk} T^{ijk}_{m} \mathbf{e}^{m}$$

If we sum over m, we have

$$\omega = \epsilon_{ijk} T^{ijk}_{1} \mathbf{e}^{1} + \epsilon_{ijk} T^{ijk}_{2} \mathbf{e}^{2} + \epsilon_{ijk} T^{ijk}_{3} \mathbf{e}^{3}$$

or

$$\omega = \epsilon_{ijk} T^{ijk}_{1}\, dx + \epsilon_{ijk} T^{ijk}_{2}\, dy + \epsilon_{ijk} T^{ijk}_{3}\, dz$$

Considering the non-zero components, we have

$$\begin{aligned} \omega = &\left(\epsilon_{111} T^{111}_{1} + \epsilon_{121} T^{121}_{1} + \epsilon_{133} T^{133}_{1} + \epsilon_{321} T^{321}_{1}\right) dx \\ &+ \left(\epsilon_{112} T^{112}_{2} + \epsilon_{211} T^{211}_{2} + \epsilon_{132} T^{132}_{2} + \epsilon_{322} T^{322}_{2}\right) dy \end{aligned}$$

Since $\epsilon_{ijk} = 0$ if there are two equal indices, we have

$$\begin{aligned} \omega &= \epsilon_{321} T^{321}_{1}\, dx + \epsilon_{132} T^{132}_{2}\, dy \\ &= -T^{321}_{1}\, dx - T^{132}_{2}\, dy \\ &= 13\, dx - dy \end{aligned}$$

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Problem 3

Solution. We are given a vector field

$$E(\mathbf{x}) = x\frac{\partial}{\partial x} + xy^2\frac{\partial}{\partial y} + xy^2z^3\frac{\partial}{\partial z}$$

a) Let's find the divergence of the vector field:

$$\begin{aligned}\nabla \cdot E &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \\ &= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}\left(xy^2\right) + \frac{\partial}{\partial z}\left(xy^2z^3\right) \\ &= 1 + 2xy + 3xy^2z^2\end{aligned}$$

b) Let's find the curl of the vector field:

$$\begin{aligned}\nabla \times E &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & xy^2 & xy^2z^3 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}\left(xy^2z^3\right) - \frac{\partial}{\partial z}\left(xy^2\right)\right)\mathbf{e}_x \\ &\quad + \left(\frac{\partial}{\partial z}(x) - \frac{\partial}{\partial x}\left(xy^2z^3\right)\right)\mathbf{e}_y \\ &\quad + \left(\frac{\partial}{\partial x}\left(xy^2\right) - \frac{\partial}{\partial y}(x)\right)\mathbf{e}_z \\ &= \left(2xyz^3 - 0\right)\mathbf{e}_x + \left(0 - y^2z^3\right)\mathbf{e}_y + \left(y^2 - 0\right)\mathbf{e}_z \\ &= 2xyz^3\mathbf{e}_x - y^2z^3\mathbf{e}_y + y^2\mathbf{e}_z\end{aligned}$$

c) A vector field F is a gradient, or *conservative* field if there exists a scalar function f such that $\nabla f = F$. Several theorems can be used to determine if a vector field is conservative. One of them is the *Poincaré's Lemma*, which states that if the curl of a vector field is zero, then the vector field is conservative. In other words, if $\nabla \times F = 0$, then F is conservative. In this case, the curl of the vector field is **not** zero, so the vector field is **not** conservative, and it cannot be written as a gradient of some scalar field. ∎