Mathematical Methods in Physics I
Homework 9

Rahmanyaz Annyyev

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1 Question One

1.1

Solution. It immediately follows:

$$f'(x) = \sum_{k=0}^{\infty} a_k k(x+1)^{k-1} \tag{1.1}$$

$$f''(x) = \sum_{k=0}^{\infty} a_k k(k-1)(x+1)^{k-2} \tag{1.2}$$

Hence, the differential equation becomes:

$$\sum_{k=0}^{\infty} a_k k(k-1)(x+1)^{k-2} - (x+1)\sum_{k=0}^{\infty} a_k (x+1)^k = 0 \tag{1.3}$$

$$\sum_{k=0}^{\infty} a_k k(k-1)(x+1)^{k-2} - \sum_{k=0}^{\infty} a_k (x+1)^{k+1} = 0 \tag{1.4}$$

The first two terms of the first sum are zero. Therefore, we can rewrite the equation as:

$$\sum_{k=2}^{\infty} a_k k(k-1)(x+1)^{k-2} - \sum_{k=0}^{\infty} a_k (x+1)^{k+1} = 0 \tag{1.5}$$

We can now change the index of the first sum to $k-2$:

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1)(x+1)^{k} - \sum_{k=0}^{\infty} a_k (x+1)^{k+1} = 0 \tag{1.6}$$

For $k=0$, the first sum is $2a_2$. Therefore, we can rewrite the equation as:

$$2a_2 + \sum_{k=1}^{\infty} a_{k+2}(k+2)(k+1)(x+1)^{k} - \sum_{k=0}^{\infty} a_k (x+1)^{k+1} = 0 \tag{1.7}$$

We can now change the index of the first sum to $k-1$:

$$2a_2 + \sum_{k=0}^{\infty} a_{k+3}(k+3)(k+2)(x+1)^{k+1} - \sum_{k=0}^{\infty} a_k (x+1)^{k+1} = 0 \tag{1.8}$$

$$2a_2 + \sum_{k=0}^{\infty} [a_{k+3}(k+3)(k+2) - a_k](x+1)^{k+1} = 0 \tag{1.9}$$

Note that $a_2 = 0$. Therefore, we can rewrite the equation as:

$$\sum_{k=0}^{\infty}[a_{k+3}(k+3)(k+2) - a_k](x+1)^{k+1} = 0 \tag{1.10}$$

1.2

Solution. From orthogonality, it follows that:

$$a_{k+3}(k+3)(k+2) - a_k = 0 \tag{1.11}$$
$$a_{k+3} = \frac{a_k}{(k+3)(k+2)} \tag{1.12}$$

We can obtain the recursion relation in terms of a_0, a_1, a_2. Since $a_2 = 0$, we have:

$$a_3 = \frac{a_0}{3 \cdot 2} \tag{1.13}$$

Let's find several more terms:

$$a_4 = \frac{a_1}{4 \cdot 3} \tag{1.14}$$
$$a_5 = \frac{a_2}{5 \cdot 4} = 0 \tag{1.15}$$
$$a_6 = \frac{a_3}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2} \tag{1.16}$$
$$a_7 = \frac{a_4}{7 \cdot 6} = \frac{a_1}{7 \cdot 6 \cdot 4 \cdot 3} \tag{1.17}$$
$$a_8 = \frac{a_5}{8 \cdot 7} = 0 \tag{1.18}$$
$$\tag{1.19}$$

Hence, the general form of a_k is:

$$a_k = \begin{cases} \dfrac{a_0}{(3 \cdot 2)(6 \cdot 5)\dots(3n \cdot (3n-1))}, & \text{if } k = 3n \\ \dfrac{a_1}{(4 \cdot 3)(7 \cdot 6)\dots((3n+1) \cdot (3n))}, & \text{if } k = 3n+1 \\ 0, & \text{if } k = 3n+2 \end{cases} \tag{1.20}$$

for $n \in \mathbb{N}$. We can rewrite the general form of a_k as:

$$a_k = \begin{cases} \dfrac{a_0}{\prod_{i=1}^{n}(3i)(3i-1)}, & \text{if } k = 3n \\ \dfrac{a_1}{\prod_{i=1}^{n}(3i+1)(3i)}, & \text{if } k = 3n+1 \\ 0, & \text{if } k = 3n+2 \end{cases} \tag{1.21}$$

1.3

Solution. It follows that:

$$f_1(x) = \sum_{n=0}^{\infty} \frac{(x+1)^{3n}}{\prod_{i=1}^{n}(3i)(3i-1)} \tag{1.22}$$
$$f_2(x) = \sum_{n=0}^{\infty} \frac{(x+1)^{3n+1}}{\prod_{i=1}^{n}(3i+1)(3i)} \tag{1.23}$$

Hence, we have:

$$f(x) = a_0 \sum_{n=0}^{\infty} \frac{(x+1)^{3n}}{\prod_{i=1}^{n}(3i)(3i-1)} + a_1 \sum_{n=0}^{\infty} \frac{(x+1)^{3n+1}}{\prod_{i=1}^{n}(3i+1)(3i)} \tag{1.24}$$

1.4

Solution. Let's compute the Wronskian. It follows that:

$$f_1'(x) = \sum_{n=0}^{\infty} \frac{(3n)(x+1)^{3n-1}}{\prod_{i=1}^{n}(3i)(3i-1)} \tag{1.25}$$

$$f_2'(x) = \sum_{n=0}^{\infty} \frac{(3n+1)(x+1)^{3n}}{\prod_{i=1}^{n}(3i+1)(3i)} \tag{1.26}$$

Hence, the Wronskian is:

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix} = f_1(x)f_2'(x) - f_2(x)f_1'(x) \tag{1.27}$$

$$\tag{1.28}$$

Therefore:

$$\begin{aligned} W(x) = &\left[\sum_{n=0}^{\infty} \frac{(x+1)^{3n}}{\prod_{i=1}^{n}(3i)(3i-1)}\right]\left[\sum_{n=0}^{\infty} \frac{(3n+1)(x+1)^{3n}}{\prod_{i=1}^{n}(3i+1)(3i)}\right] - \\ &- \left[\sum_{n=0}^{\infty} \frac{(x+1)^{3n+1}}{\prod_{i=1}^{n}(3i+1)(3i)}\right]\left[\sum_{n=0}^{\infty} \frac{(3n)(x+1)^{3n-1}}{\prod_{i=1}^{n}(3i)(3i-1)}\right] \end{aligned} \tag{1.29}$$

Note that $f_1(-1) = f_2'(-1) = 1$ and $f_2(-1) = f_1'(-1) = 0$. Therefore, the Wronskian is 1, and the solutions are linearly independent.