Mathematical Methods in Physics I
Homework 12

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1 Question One

1.1

Solution. The action of $\mathcal{D}$ acting on f at x is given by:

$$\left(\frac{\mathrm{d}^3}{\mathrm{d}x^3} + \cos(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^3\frac{\mathrm{d}}{\mathrm{d}x} + 1\right) f(x) = 0 \tag{1.1}$$

Hence, if we rewrite it as a matrix equation $\mathcal{A}$, we get:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} f(x) \\ f'(x) \\ f''(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -x^3 & -\cos(x) \end{pmatrix}\begin{pmatrix} f(x) \\ f'(x) \\ f''(x) \end{pmatrix} \tag{1.2}$$

2 Question Two

2.1

Solution. For the sake of simplicity, let's denote $f(x) = y$. Now, let's introduce new functions $y_1, y_2, \ldots, y_n$ such that $y_1 = y, y_2 = y', \ldots, y_n = y^{(n-1)}$. Hence, if we differentiate $y_1, y_2, \ldots, y_n$ with respect to x, we get:

$$y_1' = y' = y_2 \tag{2.1}$$

$$y_2' = y'' = y_3 \tag{2.2}$$

$$\vdots \tag{2.3}$$

$$y_{n-1}' = y^{(n-1)} = y_n \tag{2.4}$$

$$y_n' = y^{(n)} \tag{2.5}$$

Now, let's rewrite the differential equation as a matrix equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} \tag{2.6}$$

Or, if we write it in terms of $y = f(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{pmatrix} f(x) \\ f'(x) \\ \vdots \\ f^{(n-2)}(x) \\ f^{(n-1)}(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{pmatrix}\begin{pmatrix} f(x) \\ f'(x) \\ \vdots \\ f^{(n-2)}(x) \\ f^{(n-1)}(x) \end{pmatrix} \tag{2.7}$$

2.2

Solution. The given expression is:

$$\sum_{i_2=1}^{n} \cdots \sum_{i_{n-1}=1}^{n} \sum_{i_n=1}^{n} \epsilon_{2i_2\dots i_n} \mathcal{M}_{2i_2} \dots \mathcal{M}_{n-1 i_{n-1}} \mathcal{M}_{ni_n} \tag{2.8}$$

Let's rewrite it by expanding the summation $\sum_{i_n=1}^{n}$ and ignoring the terms that are zero. The range of values for i_n is 1 to n. Let's consider the case where $i_n = 1$:

$$\sum_{i_2=1}^{n} \cdots \sum_{i_{n-1}=1}^{n} \epsilon_{2i_2\dots i_{n-1}1} \mathcal{M}_{2i_2} \dots \mathcal{M}_{n-1 i_{n-1}} \mathcal{M}_{n1} \tag{2.9}$$

$\mathcal{M}_{n1} = -a_n$, so the expression becomes:

$$-a_n \sum_{i_2=1}^{n} \cdots \sum_{i_{n-1}=1}^{n} \epsilon_{2i_2\dots i_{n-1}1} \mathcal{M}_{2i_2} \dots \mathcal{M}_{n-1 i_{n-1}} \tag{2.10}$$

Note that $\epsilon_{2i_2\dots i_{n-1}1}$ is non-zero when the values that $i_2, i_3, \dots, i_{n-1}$ take are all different and are in the range 3 to n. The product $\mathcal{M}_{2i_2} \dots \mathcal{M}_{n-1 i_{n-1}}$ is not always zero. Now, let's consider the case where $i_n = 2$. It immediately follows that the expression is zero, since $\epsilon_{2i_2\dots i_{n-1}2}$ is always zero. Let's consider the case where $i_n = 3$. The expression becomes:

$$\sum_{i_2=1}^{n} \cdots \sum_{i_{n-1}=1}^{n} \epsilon_{2i_2\dots i_{n-1}3} \mathcal{M}_{2i_2} \dots \mathcal{M}_{n-1 i_{n-1}} \mathcal{M}_{n3} \tag{2.11}$$

Note that $\epsilon_{2i_2\dots i_{n-1}3}$ is non-zero when the values that $i_2, i_3, \dots, i_{n-1}$ take are all different and are in $\{1, 4, 5, \dots, n\}$. Nonetheless, it follows that the product $\mathcal{M}_{2i_2} \dots \mathcal{M}_{n-1 i_{n-1}}$ is always zero since $\mathcal{M}_{2i_2}$ is zero for all values in $\{1, 4, 5, \dots, n\}$. Hence, we can simply ignore the expression when $i_n = 3$. We can repeat this process for all values of $i_n = j$ for $3 < j \leq n$ and conclude that the expression is always zero. Hence, the given expression can be rewritten as:

$$-a_n \sum_{i_2=1}^{n} \cdots \sum_{i_{n-1}=1}^{n} \epsilon_{2i_2\dots i_{n-1}1} \mathcal{M}_{2i_2} \dots \mathcal{M}_{n-1 i_{n-1}} \tag{2.12}$$

2.3

Solution. Let's further simplify the expression we found in the previous part by considering the values that i_2 can take. The range of values for i_2 is 1 to n. Let's consider the case where $i_2 = 1$. It immediately follows that the expression is zero, since $\epsilon_{21\dots i_{n-1}1}$ is always zero. The same is true when $i_2 = 2$. Let's consider the case where $i_2 = 3$. The expression becomes:

$$-a_n \sum_{i_2=1}^{n} \cdots \sum_{i_{n-1}=1}^{n} \epsilon_{23i_3\dots i_{n-1}1} \mathcal{M}_{23} \dots \mathcal{M}_{n-1 i_{n-1}} \tag{2.13}$$

Note that $\epsilon_{23i_3\ldots i_{n-1}1}$ is non-zero when the values that $i_3, i_4, \ldots, i_{n-1}$ take are all different and are in the range 4 to n. The product $\mathcal{M}_{2i_2} \ldots \mathcal{M}_{n-1i_{n-1}}$ is not always zero and $\mathcal{M}_{23} = 1$. Let's consider the case where $i_2 = 4$. The expression becomes:

$$-a_n \sum_{i_2=1}^{n} \cdots \sum_{i_{n-1}=1}^{n} \epsilon_{24i_3\ldots i_{n-1}1} \mathcal{M}_{24} \ldots \mathcal{M}_{n-1i_{n-1}} \tag{2.14}$$

Note that $\epsilon_{23i_3\ldots i_{n-1}1}$ is non-zero when the values that $i_3, i_4, \ldots, i_{n-1}$ take are all different and are in $\{3, 5, 6, \ldots, n\}$. Nonetheless, it follows that the product $\mathcal{M}_{24} \ldots \mathcal{M}_{n-1i_{n-1}}$ is always zero since $\mathcal{M}_{24}$ is zero. We can repeat this process for all values of $i_2 = j$ for $4 < j \leq n$ and conclude that the expression is always zero. Hence, the given expression can be rewritten as:

$$-a_n \sum_{i_3=1}^{n} \cdots \sum_{i_{n-1}=1}^{n} \epsilon_{23i_3\ldots i_{n-1}1} \mathcal{M}_{3i_3} \ldots \mathcal{M}_{n-1i_{n-1}} \tag{2.15}$$

Note that we can repeat this whole process for every i_n for $3 \leq i_n \leq n$. This means that the given expression can be rewritten as:

$$-a_n \epsilon_{2,3,4\ldots n-1,n,1} \tag{2.16}$$

2.4

Solution. Without loss of generality, let's assume that $n = 3$—an odd number. Therefore, the expression we found in the previous part is:

$$\begin{gathered} -a_4 \epsilon_{2,3,1} \\ -a_4 \epsilon_{2,1,3} \\ -a_4 \epsilon_{1,2,3} \end{gathered} \tag{2.17}$$

Hence, the parity of the permutation is 2. The sign of the permutation is $(-1)^2$. Let's consider the case where $n = 4$—an even number. Hence, the expression takes the form:

$$\begin{gathered} -a_4 \epsilon_{2,3,4,1} \\ -a_4 \epsilon_{2,4,1,3} \\ -a_4 \epsilon_{2,1,3,4} \\ -a_4 \epsilon_{1,2,3,4} \end{gathered} \tag{2.18}$$

Hence, the parity of the permutation is 3. The sign of the permutation is $(-1)^3$. Hence, the parity of the permutation is $n-1$. The sign of the permutation is $(-1)^{n-1}$. The expression we found in the previous part can be rewritten as:

$$-a_n \epsilon_{2,3,4\ldots n-1,n,1} = (-a_n)(-1)^{n-1} \tag{2.19}$$

The determinant takes the form:

$$\det{}_n \mathcal{M} = -\lambda \sum_{i_2=1}^{n} \cdots \sum_{i_n=1}^{n} \epsilon_{1i_2\ldots i_n} \mathcal{M}_{2i_2} \ldots \mathcal{M}_{ni_n} + a_n(-1)^n \tag{2.20}$$

2.5

Solution. To prove $\epsilon_{1,i_2,i_3,\ldots,i_n} = \epsilon_{k_1,k_2,\ldots,k_{n-1}}$, we need to consider the properties of permutations. The indices $i_2, i_3, \ldots, i_n$ and $k_1, k_2, \ldots, k_{n-1}$ represent permutations of the set $\{2, 3, \ldots, n\}$. The left side of the equation represents the permutation of the set $\{1, 2, 3, \ldots, n\}$, where 1 is fixed. The right side of the equation represents the permutation of the set $\{1, 2, 3, \ldots, n\}$ minus one element. The crucial observation is that adding 1 at the beginning of any permutation $2, 3, \ldots, n$ does not change the parity of the permutation (whether it is an even or odd permutation). The parity of a permutation is defined by the number of swaps required to return it to the original ordering. Inserting 1 at the beginning does not require any additional swaps, as it is already in its correct position. Therefore, if $(i_2, i_3, \ldots, i_n)$ is an even permutation of $\{2, 3, \ldots, n\}$, then $(1, i_2, i_3, \ldots, i_n)$ is an even perumtation of $\{1, 2, 3, \ldots, n\}$, and $\epsilon_{1,i_2,i_3,\ldots,i_n} = 1$. Similarly, if it is an odd permutation, then $\epsilon_{1,i_2,i_3,\ldots,i_n} = -1$. The same is true for the right side of the equation. Hence, the equation holds. Note that, following the reasoning of previous parts, $\epsilon_{k_1,k_2,\ldots,k_{n-1}}$ denotes a determinant of a matrix with $n-1$ rows and columns.

2.6

Solution. Let's start with the case where $n = 1$. The determinant takes the form:

$$\det{}_1\mathcal{M} = -\lambda - a_1 \tag{2.21}$$

If $n = 2$, the determinant takes the form:

$$\begin{aligned} \det{}_2\mathcal{M} &= -\lambda \det{}_1\mathcal{M} + a_2 && (2.22)\\ &= -\lambda(-\lambda - a_1) + a_2 && (2.23)\\ &= \lambda^2 + \lambda a_1 + a_2 && (2.24) \end{aligned}$$

If $n = 3$, the determinant takes the form:

$$\begin{aligned} \det{}_3\mathcal{M} &= -\lambda \det{}_2\mathcal{M} - a_3 && (2.25)\\ &= -\lambda(\lambda^2 + \lambda a_1 + a_2) - a_3 && (2.26)\\ &= -\lambda^3 - \lambda^2 a_1 - \lambda a_2 - a_3 && (2.27) \end{aligned}$$

If $n = 4$, the determinant takes the form:

$$\begin{aligned} \det{}_4\mathcal{M} &= -\lambda \det{}_3\mathcal{M} + a_4 && (2.28)\\ &= -\lambda(-\lambda^3 - \lambda^2 a_1 - \lambda a_2 - a_3) + a_4 && (2.29)\\ &= \lambda^4 + \lambda^3 a_1 + \lambda^2 a_2 + \lambda a_3 + a_4 && (2.30) \end{aligned}$$

Hence, we can conclude that the determinant is:

$$\det{}_n\mathcal{M} = \begin{cases} -\lambda^n - \lambda^{n-1}a_1 - \ldots - \lambda a_{n-1} - a_n, & \text{if } n \text{ is odd} \\ \lambda^n + \lambda^{n-1}a_1 + \ldots + \lambda a_{n-1} + a_n, & \text{if } n \text{ is even} \end{cases} \tag{2.31}$$

Or more compactly:

$$\det{}_n\mathcal{M} = (-1)^n \left(\lambda^n + \lambda^{n-1}a_1 + \ldots + \lambda a_{n-1} + a_n\right) \tag{2.32}$$