Mathematical Methods in Physics II
Homework V

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Problem 1

Solution. Since the region is over the xy-plane, and the integrand is of the form

$$\int_{\partial\Omega} L(x,y)\,dx + M(x,y)\,dy$$

where L and M are differentiable functions, we can use Green's theorem to convert the line integral to a double integral over the region Ω enclosed by the curve $\partial\Omega$. Green's theorem states that

$$\int_{\partial\Omega} L(x,y)\,dx + M(x,y)\,dy = \iint_{\Omega} \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dx\,dy$$

Let's calculate the partial derivatives of L and M:

$$\begin{aligned}
\frac{\partial}{\partial x} M(x,y) &= -y\sin(xy)\left(1 + x^b y^a e^{xy}\right) + \cos(xy)\left(bx^{b-1}y^a e^{xy} + x^b y^{a+1} e^{xy}\right) \\
\frac{\partial}{\partial y} L(x,y) &= -x\sin(xy)\left(1 + x^a y^b e^{xy}\right) + \cos(xy)\left(bx^a y^{b-1} e^{xy} + x^{a+1} y^b e^{xy}\right)
\end{aligned}$$

For the value of $\mathcal{I}(a)(b)$ to be independent of b, the terms involving b in the partial derivatives must cancel out. to achieve this, we must have

$$\begin{aligned}
y(x^b y^a) - x(x^a y^b) = 0 \\
x^b y^{a+1} - x^{a+1} y^b = 0
\end{aligned}$$

and

$$bx^{b-1}y^a + x^b y^{a+1} - bx^a y^{b-1} - x^{a+1}y^b = 0$$

From the first two equations, it follows that $b = a + 1$. In terms of b, the third equation becomes

$$\begin{aligned}
bx^{b-1}y^{b-1} + x^b y^b - bx^{b-1}y^{b-1} - x^b y^b = 0 \\
0 = 0
\end{aligned}$$

which is satisfied for all b. Therefore, the value of $\mathcal{I}(a)(b)$ is independent of b for $a = b - 1$. ■

Problem 2

Solution. A general three-dimensional vector field is given by

$$\mathbf{F} = P(x,y,z)\hat{\imath} + Q(x,y,z)\hat{\jmath} + R(x,y,z)\hat{k}$$

For it to be *path independent*, its curl must be zero:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \mathbf{0}$$

In our case, the coefficients of the vector field are given by

$$\begin{aligned}
P(x,y,z) &= -\left(\left((a-2)\,yz + 1\right)\sin\left(x + y^2\right)\right) \\
Q(x,y,z) &= 5z\cos\left(x + y^2\right) - 2y(b + 5yz)\sin\left(x + y^2\right) \\
R(x,y,z) &= cy\cos\left(x + y^2\right)
\end{aligned}$$

If we expand the determinant, we get

$$\nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} - \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

For the curl to be zero, each component must be zero. Therefore, we have the following equations

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 0$$
$$\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = 0$$
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

Let's calculate the partial derivatives:

$$\frac{\partial R}{\partial y} = c\text{cos}\left(x+y^2\right) - 2cy^2\text{sin}\left(x+y^2\right)$$
$$\frac{\partial Q}{\partial z} = 5\text{cos}\left(x+y^2\right) - 10y^2\text{sin}\left(x+y^2\right)$$
$$\frac{\partial P}{\partial z} = 2y\text{sin}\left(x+y^2\right) - ay\text{sin}\left(x+y^2\right)$$
$$\frac{\partial R}{\partial x} = -cy\text{sin}\left(x+y^2\right)$$
$$\frac{\partial Q}{\partial x} = -2by\text{cos}\left(x+y^2\right) - 5z\text{sin}\left(x+y^2\right) - 10y^2z\text{cos}\left(x+y^2\right)$$
$$\frac{\partial P}{\partial y} = -(a-2)z\text{sin}\left(x+y^2\right) - 2y\left((a-2)\,yz+1\right)\cos\left(x+y^2\right)$$

Let's substitute the first two equations into the first equation:

$$c\cos\left(x+y^2\right) - 2cy^2\sin\left(x+y^2\right) - 5\cos\left(x+y^2\right) + 10y^2\sin\left(x+y^2\right) = 0$$

It immediately follows that $c = 5$. Therefore, we have

$$2y\sin\left(x+y^2\right) - ay\sin\left(x+y^2\right) - 5y\sin\left(x+y^2\right) = 0$$

From this equation, we find that $a = 7$. Lastly, let's find the value of b:

$$\begin{aligned} -2by\cos\left(x+y^2\right) - 5z\sin\left(x+y^2\right) - 10y^2z\cos\left(x+y^2\right) \\ + 5z\sin\left(x+y^2\right) + 2y(5yz+1)\cos\left(x+y^2\right) = 0 \end{aligned}$$

Simplifying, we get

$$-2by\cos\left(x+y^2\right) + 2y\cos\left(x+y^2\right) = 0$$

which implies that $b = 1$. Therefore, the vector field is path independent for $a = 7$, $b = 1$, and $c = 5$.

(b) Since the vector field is path independent, it must have a potential function $U(x,y,z)$ such that

$$\frac{\partial U}{\partial x} = P(x,y,z), \quad \frac{\partial U}{\partial y} = Q(x,y,z), \quad \frac{\partial U}{\partial z} = R(x,y,z)$$

Let's integrate all three equations to find the potential function:

$$\begin{aligned} U(x,y,z) &= -\int (5yz+1)\sin\bigl(x+y^2\bigr)\,dx \\ &= (5yz+1)\cos\bigl(x+y^2\bigr) + h(y,z) \end{aligned}$$

where $h(y,z)$ is an arbitrary function of y and z. Let's integrate the second equation:

$$\begin{aligned} U(x,y,z) &= \int 5z\cos\bigl(x+y^2\bigr) - 2y(1+5yz)\sin\bigl(x+y^2\bigr)\,dy \\ &= (5yz+1)\cos\bigl(x+y^2\bigr) + g(x,z) \end{aligned}$$

where $g(x,z)$ is an arbitrary function of x and z. Let's integrate the third equation:

$$\begin{aligned} U(x,y,z) &= \int 5y\cos\bigl(x+y^2\bigr)\,dz \\ &= 5yz\cos\bigl(x+y^2\bigr) + k(x,y) \end{aligned}$$

where $k(x,y)$ is an arbitrary function of x and y. Equating the three expressions for $U(x,y,z)$, we find that

$$h(y,z) = g(x,z) = k(x,y) = 0$$

since the potential function must be unique. Therefore, the potential function is

$$U(x,y,z) = (5yz+1)\cos\bigl(x+y^2\bigr)$$

Finally, we can calculate the line integral of the vector field along the path C from $(0,0,0)$ to $(\pi, 1/10, -2)$:

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{r} &= U(\pi, 1/10, -2) - U(0,0,0) \\ &= \left(5\cdot\frac{1}{10}\cdot(-2)+1\right)\cos\left(\pi+\frac{1}{100}\right) - \cos(0) \\ &= -1 \end{aligned}$$

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