Lecture Notes

This page is intentionally left blank.

Preface

These lecture notes were prepared for the course *PHYS210 Mathematical Methods in Physics II* at Middle East Technical University, Ankara, Turkey. The course is intended for second-year undergraduate students in the Department of Physics. The notes are based on the lectures given by Prof. Soner Albayrak in the Spring semester of the 2023–2024 academic year.

The course is a continuation of *Mathematical Methods in Physics I*, which is a prerequisite for this course. The main topics covered in this course include vector analysis, complex analysis, and partial differential equations. The notes are intended to be a useful resource for students taking the course, as well as for anyone interested in learning about the mathematical methods used in physics. If you have noticed any errors or have any suggestions for improvement, please feel free to contact me.

Last updated: May 9, 2024
E-mail: `rannyyev@gmail.com`
GitHub: `@rannyyev`
Website: `www.rahmanyazannyyev.com`

Contents

List of Figures

List of Tables

Differential Equations

Vector Analysis

1

Vector Spaces

1.1 Groups

1.1.1 Basic Definitions

A **group operation**, or **binary operation** or **law of composition** on a set S is a function $S \times S \to S$ that assigns to each pair $(a, b) \in S \times S$ a unique element $a \circ b \in S$, called the composition of a and b. Let's denote the function as o, and express the definition symbolically as follows:

$$
\begin{aligned}
&o :: S \times S \to S, \\
&o(a, b) = a \circ b.
\end{aligned}
$$

Remark 1.1 The operation o is called **binary** because it takes two arguments. The symbols "$*$" and "$\bullet$" are often used to denote the group operation.

An **inverse operation** on a set S is a function $S \to S$ that assigns to each element $s \in S$ a unique element $s^{-1} \in S$, called the inverse of s. Let's denote the function as i, and express the definition symbolically as follows:

$$
\begin{aligned}
&i :: S \to S, \\
&i(s) = s^{-1}.
\end{aligned}
$$

A **group** is a set S endowed with a group operation o and an inverse operation i that satisfies certain rules, called the **group axioms**.

Theorem 1.1 — Group Axioms. Let S be a non-empty set, endowed with a group (or *binary*) operation o. Then (S, o) is a group if and only if the following three axioms are satisfied:

1. The group operation is *associative*, that is,

$$(\forall a, b, c \in S)\; a \circ (b \circ c) = (a \circ b) \circ c.$$

2. There exists an *identity element* e (or e_S), that is,

$$(\exists e \in S)(\forall s \in S)\; e \circ s = s \circ e = s.$$

3. Every element has an *inverse*, that is,

$$(\forall s \in S)\; s \circ s^{-1} = s^{-1} \circ s = e.$$

1.1.2 Commutative Groups

A **commutative group** or **abelian group** is a group in which the group operation is commutative, that is, for all $a, b \in S$, we have $a \circ b = b \circ a$. In such a case, the "product" is often called the **addition** and is denoted by the symbol "$+$". The identity element is denoted by "0", and the inverse of s is denoted by "$-s$". The *commutative* group axioms have the following form:

$$\begin{aligned}
&(\forall a, b, c \in S)\; a + (b + c) = (a + b) + c,\\
&(\forall s \in S)\; 0 + s = s + 0 = s,\\
&(\forall s \in S)\; s + (-s) = (-s) + s = 0,\\
&(\forall a, b \in S)\; a + b = b + a.
\end{aligned}$$

Example 1.1 Let's consider the set of classrooms:

$$\begin{aligned}
&S :: \texttt{Set},\\
&S = \{\texttt{P1}, \texttt{P2}, \texttt{P3}\},\\
&<> :: (S, S) \to S.
\end{aligned}$$

Assume that $<>$ is the operation of combining two classrooms:

$$\begin{aligned}
&\texttt{P1} <> \texttt{P1} = \texttt{P3},\\
&\texttt{P2} <> \texttt{P2} = \texttt{P2},\\
&\texttt{P3} <> \texttt{P3} = \texttt{P1},\\
&\texttt{P1} <> \texttt{P2} = \texttt{P2} <> \texttt{P1} = \texttt{P1},\\
&\texttt{P1} <> \texttt{P3} = \texttt{P3} <> \texttt{P1} = \texttt{P2},\\
&\texttt{P2} <> \texttt{P3} = \texttt{P3} <> \texttt{P2} = \texttt{P3}.
\end{aligned}$$

Determine whether $(S, <>)$ is a group or not.

Solution. It follows by inspection that $(S, <>)$ satisfies the first group axiom since

$$\texttt{P1} <> (\texttt{P2} <> \texttt{P3}) = \texttt{P1} <> \texttt{P3} = \texttt{P2}$$

is equal to

$$(\texttt{P1} <> \texttt{P2}) <> \texttt{P3} = \texttt{P1} <> \texttt{P3} = \texttt{P2}.$$

It also follows by inspection that $(S, <>)$ satisfies the second group axiom, where $\texttt{P2}$ is the identity element:

$$\begin{aligned} &\texttt{P1} <> \texttt{P2} = \texttt{P2} <> \texttt{P1} = \texttt{P1}, \\ &\texttt{P3} <> \texttt{P2} = \texttt{P2} <> \texttt{P3} = \texttt{P3}, \\ &\texttt{P2} <> \texttt{P2} = \texttt{P2}. \end{aligned}$$

Finally, it follows by inspection that every element has an inverse:

$$\begin{aligned} &\texttt{P1} <> \texttt{P3} = \texttt{P3} <> \texttt{P1} = \texttt{P2}, \\ &\texttt{P2} <> \texttt{P2} = \texttt{P2} \end{aligned}$$

Hence, $(S, <>)$ satisfies the third group axiom. Therefore, $(S, <>)$ is a group. Moreover, $(S, <>)$ is a commutative group since the group operation $<>$ is commutative. ■

Some of the most important examples of groups are $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, and $(\mathbb{C}, +)$. All of them are commutative groups. $(\mathbb{Z}, \times)$ is not a group because it does not satisfy the third group axiom. $(\mathbb{R}, \times)$ is not a group because it does not satisfy the second group axiom. However, $(\mathbb{R}^+, \times)$ and $(\mathbb{R} \setminus \{0\}, \times)$ are groups.

1.2 Rings

Let's introduce another algebraic structure called a **ring**. First, we define the **multiplication** operation on a set S, denoted by the symbol "$\cdot$", as a function $S \times S \to S$ that assigns to each pair $(a, b) \in S \times S$ a unique element $a \cdot b \in S$.

Definition 1.1 A **ring** is a set S endowed with two operations, called **addition** and **multiplication** that satisfy **ring axioms**.

Theorem 1.2 — Ring Axioms. Let S be a non-empty set, endowed with two operations, *addition* and *multiplication*. Then $(S, +, \cdot)$ is a ring if and only if the following three axioms are satisfied:

1. The set $(S, +)$ is a *commutative group*.
2. The multiplication operation is *associative*, that is,
$$(\forall a, b, c \in S)\ a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$
3. The multiplication operation is *distributive* with respect to the addition operation, that is,
$$(\forall a, b, c \in S)\ a \cdot (b + c) = a \cdot b + a \cdot c, \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

Example 1.2 We have already seen that $(\mathbb{Z}, +)$ is a commutative group. Check if $(\mathbb{Z}, +, \cdot)$ is a ring.

Solution. The first axiom is satisfied. The second axiom is satisfied because multiplication is associative:
$$(\forall a, b, c \in \mathbb{Z})\ a \cdot (b \cdot c) = a \cdot b \cdot c.$$

The third axiom is satisfied because multiplication is distributive with respect to addition:
$$(\forall a, b, c \in \mathbb{Z})\ a \cdot (b + c) = a \cdot b + a \cdot c, \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

Therefore, $(\mathbb{Z}, +, \cdot)$ is a ring. ■

1.3 Division Rings

A **division ring** is a ring in which every non-zero element has a multiplicative inverse. A **skew field** is a division ring in which the multiplication operation is not necessarily commutative.

Definition 1.2 A **skew field** is a set S endowed with two operations, **addition** and **multiplication**, that satisfy **skew field axioms**.

Theorem 1.3 — Skew Field Axioms. Let S be a non-empty set, endowed with two operations, *addition* and *multiplication*. Then $(S, +, \cdot)$ is a skew field if and only if the following two axioms are satisfied:

1. The set $(S, +)$ is a *commutative group*.
2. Every non-zero element has a *multiplicative inverse*, that is,
$$(\forall s \in S \setminus \{0\})(\exists s^{-1} \in S \setminus \{0\})\ s \cdot s^{-1} = s^{-1} \cdot s = 1.$$

Consider the ring $(\mathbb{Z}, +, \cdot)$. The set $(\mathbb{Z}, +)$ is a commutative group. However, the set $(\mathbb{Z} \setminus \{0\}, \cdot)$ is not a group because not every non-zero element has a multiplicative inverse. For example, there is no integer x such that $2 \cdot x = 1$. Therefore, $(\mathbb{Z}, +, \cdot)$ is not a skew field. The ring $(\mathbb{R}, +, \cdot)$ is a skew field. The ring $(\mathbb{C}, +, \cdot)$ is also a skew field.

Example 1.3 Determine whether $(M_{2\times 2}(\mathbb{R}), +, \cdot)$ is a skew field or not.

Solution. The set $(M_{2\times 2}(\mathbb{R}), +)$ is a commutative group. Let's consider the set $(M_{2\times 2}(\mathbb{R}) \setminus \{0\}, \cdot)$. Not every non-zero element has a multiplicative inverse. For example, the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

does not have a multiplicative inverse. Therefore, $(M_{2\times 2}(\mathbb{R}), +, \cdot)$ is not a skew field. ■

A division ring that is multiplicative commutative is called a **field**. The ring $(\mathbb{R}, +, \cdot)$ is a field. The ring $(\mathbb{C}, +, \cdot)$ is also a field.

1.4 Vector Spaces

A **vector**, or **linear space** is a set whose elements, called *vectors*, can be added together and multiplied, or scaled, by numbers, called *scalars*. Scalars are often taken to be real numbers, but there are also vector spaces with scalar multiplication by complex numbers, rational numbers, or generally any field. Let's define a linear space more formally.

Definition 1.3 — Vector Space. A set V, whose elements are called *vectors*, along with a vector addition operation $\oplus : V \times V \to V$ and a scalar multiplication operation $\odot : S \times V \to V$, is a *vector space* over a field $F = (S, +, \cdot)$ if the following conditions are satisfied:

- $(V, \oplus)$ is an *abelian*, or *commutative* group.
- $(\forall v \in V)\, 1 \odot v = v$, where 1 is the identity element of the field F.
- $(\forall v \in V)(\forall s \in S)\, s \odot v \in V$.
- $(\forall v \in V)(\forall a, b \in S)\, (a \cdot b) \odot v = a \odot (b \odot v)$.
- $(\forall v \in V)(\forall a, b \in S)\, (a + b) \odot v = (a \odot v) \oplus (b \odot v)$.
- $(\forall v, w \in V)(\forall s \in S)\, s \odot (v \oplus w) = (s \odot v) \oplus (s \odot w)$.

1.5 Linear Algebra

A **linear algebra** is a vector space over a field F with an additional operation $\otimes : V \times V \to V$.

Definition 1.4 — Linear Algebra. A set V, whose elements are called *vectors*, along with a vector addition operation $\oplus : V \times V \to V$, a scalar multiplication operation $\odot : S \times V \to V$, and a vector multiplication operation $\otimes : V \times V \to V$, is a *linear algebra* over a field $F = (S, +, \cdot)$

if the following conditions are satisfied:

1. $(V, \oplus, \otimes)$ is a *ring*.
2. $(V, \oplus, \odot)$ is a *linear space*.
3. $(\forall s \in S)(\forall v, w \in V)\ s \odot (v \otimes w) = (s \odot v) \otimes w$.

1.6 Lie Algebra

1.7 Basis of Vector Spaces

Let B be a set of vectors in V. B is called a **basis** of V if every element of V can be expressed as a finite linear combination of the elements of B in a unique way. The coefficients of the linear combination are called the **coordinates** of the vector with respect to the basis B. Equivalently, B is a basis if it is a linearly independent set that spans V. Alternatively, we can define a basis more formally.

1.8 Normed Vector Space

1.9 Inner Product Vector Space

1.10 Dual Vector Spaces

Let V be a vector space over a field $\mathbb{K}$. The **dual**, or **conjugate space** of V, denoted V^*, is the set of all linear maps from V to $\mathbb{K}$. The elements of V^* are called **linear functionals** on V, that is, they are maps $\mathbf{f}^* : V \to \mathbb{K}$ having the following properties:

$$\mathbf{f}^*(\mathbf{u}+\mathbf{v}) = \mathbf{f}^*(\mathbf{u}) + \mathbf{f}^*(\mathbf{v}) \quad \text{and} \quad \mathbf{f}^*(\lambda\mathbf{v}) = \lambda\mathbf{f}^*(\mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V,\ \forall \lambda \in \mathbb{K}$$

The elements of V^* are also called **dual vectors**, **covectors**, or **linear forms**.

Definition 1.5 A **covector** is a linear map $\mathbf{f}^* : V \to \mathbb{K}$. The set of all covectors on V is the dual space V^*. The zero covector is the linear function that maps every vector to zero. Covectors are equal if

$$\mathbf{f}^*(\mathbf{v}) = \mathbf{g}^*(\mathbf{v}), \quad \forall \mathbf{v} \in V$$

Example 1.4 Suppose $V = \mathbb{R}^2$ and $\mathbf{v} \equiv (x, y)$, and $\mathbf{f}^*(\mathbf{v}) = x + 2y$. Is $\mathbf{f}^*$ a covector on V?

Solution. Let's check whether the properties of a covector hold for $\mathbf{f}^*$. Let $\mathbf{u} = (x_1, y_1)$ and $\mathbf{v} = (x_2, y_2)$. Then

$$\begin{aligned}
\mathbf{f}^*(\mathbf{u} + \mathbf{v}) &= \mathbf{f}^*((x_1 + x_2, y_1 + y_2)) \\
&= x_1 + x_2 + 2(y_1 + y_2) \\
&= (x_1 + 2y_1) + (x_2 + 2y_2) \\
&= \mathbf{f}^*(\mathbf{u}) + \mathbf{f}^*(\mathbf{v})
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{f}^*(\lambda \mathbf{v}) &= \mathbf{f}^*(\lambda x, \lambda y) \\
&= \lambda x + 2\lambda y \\
&= \lambda(x + 2y) \\
&= \lambda \mathbf{f}^*(\mathbf{v})
\end{aligned}$$

Thus, $\mathbf{f}^*$ is a covector on V. ■

One says that a covector $\mathbf{f}^*$ *is applied* to a vector $\mathbf{v}$ and yields a number $\mathbf{f}^*(\mathbf{v})$. This is similar to writing $\sin(0) = 1$ and saying that the sine function is applied to the number 0, or "acts" on the number 0, and yields the number 1. The number $\mathbf{f}^*(\mathbf{v})$ is called the **value** of the covector $\mathbf{f}^*$ at the vector $\mathbf{v}$. Other notations for a covector acting on a vector are $\langle \mathbf{f}^*, \mathbf{v} \rangle$, $\mathbf{f}^* \cdot \mathbf{v}$, and $\iota_{\mathbf{f}^*}(\mathbf{v})$ (where the symbol ι stands for "insertion").

1.10.1 Basis

Let $\{\mathbf{e}_i\}$ be a basis for V. Then the dual basis $\{\mathbf{e}^i\}$ for V^* is defined by

$$\mathbf{e}^i(\mathbf{e}_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

where δ^i_j is the Kronecker delta. Since the number of elements in a basis for V is equal to the number of elements in a basis for V^*, the dimension of V is equal to the dimension of V^*.

Remark 1.2 Another notation for the dual basis is $\mathbf{e}^i = \mathbf{e}^*_i$.

1.11 Algebras Over Fields

1.11.1 Tensor Algebra

Tensor Product

As a preliminary step, we will define the concept of **formal product** that will serve as an intermediate step in the definition of the tensor product.

Definition 1.6 A **formal product** of two vector spaces V and W with a common field $\mathbb{K}$ is a vector space $V * W$ over $\mathbb{K}$ defined by

$$V * W = \operatorname{span}_{\mathbb{K}}\{\mathbf{v} * \mathbf{w} \mid \mathbf{v} \in V, \mathbf{w} \in W\}$$

where $*$ is a symbol that denotes the formal product.

Now, let's construct a quotient space of $V * W$. Let I be a vector subspace of $V * W$ that is spanned by the elements of the following form:

- $(\lambda \mathbf{v}) * \mathbf{w} - \lambda(\mathbf{v} * \mathbf{w})$
- $\mathbf{v} * (\lambda \mathbf{w}) - \lambda(\mathbf{v} * \mathbf{w})$
- $(\mathbf{v_1} + \mathbf{v_2}) * \mathbf{w} - (\mathbf{v_1} * \mathbf{w} + \mathbf{v_2} * \mathbf{w})$
- $\mathbf{v} * (\mathbf{w_1} + \mathbf{w_2}) - (\mathbf{v} * \mathbf{w_1} + \mathbf{v} * \mathbf{w_2})$

where $\mathbf{v}, \mathbf{v_1}, \mathbf{v_2} \in V$, $\mathbf{w}, \mathbf{w_1}, \mathbf{w_2} \in W$, and $\lambda \in \mathbb{K}$. The quotient space $V \otimes W$ is defined as

$$V \otimes W = (V * W)/I$$

which is called the **tensor product space** of V and W. The elements of $V \otimes W$ are called **tensors** defined by

$$\mathbf{v} \otimes \mathbf{w} = \mathbf{v} * \mathbf{w} + I$$

From here, the properties of the tensor product can be derived:

- $\lambda(\mathbf{v} \otimes \mathbf{w}) = (\lambda \mathbf{v}) \otimes \mathbf{w} = \mathbf{v} \otimes (\lambda \mathbf{w})$
- $(\mathbf{v_1} + \mathbf{v_2}) \otimes \mathbf{w} = \mathbf{v_1} \otimes \mathbf{w} + \mathbf{v_2} \otimes \mathbf{w}$
- $\mathbf{v} \otimes (\mathbf{w_1} + \mathbf{w_2}) = \mathbf{v} \otimes \mathbf{w_1} + \mathbf{v} \otimes \mathbf{w_2}$

A tensor product space is a space of all linear combinations of tenors of the form

$$\mathbf{v_1} \otimes \mathbf{w_1} + \mathbf{v_2} \otimes \mathbf{w_2} + \cdots + \mathbf{v_n} \otimes \mathbf{w_n}$$

This means that every tensor can be written as a linear combination of some tensors of the form $\mathbf{v} \otimes \mathbf{w}$. This tensor is called a **simple tensor**.

Remark 1.3 What we essentially did was define a general structure that allowed us to lay the foundation for the tensor product. The tensor product is a smaller space than the formal product confined by the constraints imposed by the subspace I.

Theorem 1.4 If $B_V = \{\mathbf{e}_i\}$ and $B_W = \{\mathbf{f}_j\}$ are bases for V and W, respectively, then $B_{V \otimes W} = \{\mathbf{e}_i \otimes \mathbf{f}_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ is a basis for $V \otimes W$. The dimension of $V \otimes W$ is equal to the product of the dimensions of V and W.

Higher Order Tensors

The tensor product can be extended to more than two vector spaces. For example, the tensor product of three vector spaces U, V, and W is defined as

$$U \otimes V \otimes W = (U \otimes V) \otimes W = U \otimes (V \otimes W)$$

In such a case, the tensor product space is a space of all linear combinations of the form

$$\mathbf{u_1} \otimes \mathbf{v_1} \otimes \mathbf{w_1} + \mathbf{u_2} \otimes \mathbf{v_2} \otimes \mathbf{w_2} + \cdots + \mathbf{u_n} \otimes \mathbf{v_n} \otimes \mathbf{w_n}$$

Consider a case when we have only one vector space V and its dual space V^*. A tensor of type (p, q) is an element of the tensor product space

$$\underbrace{V \otimes V \otimes \cdots \otimes V}_{p \text{ times}} \otimes \underbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}_{q \text{ times}}$$

Definition of Tensor Algebra

1.11.2 Exterior Algebra

1.12 One-forms

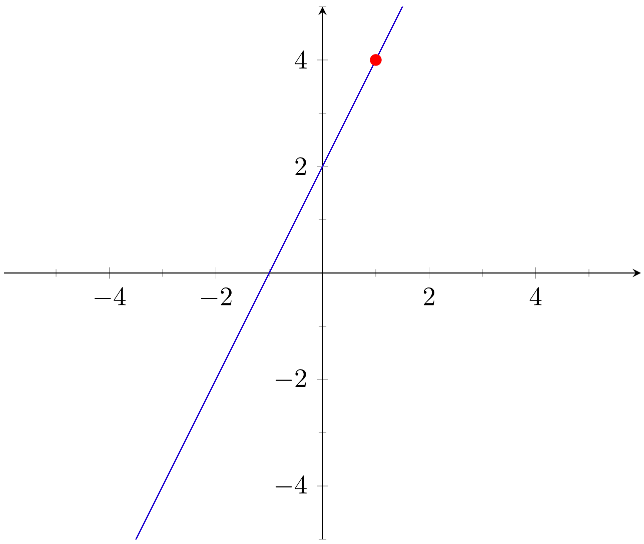

Figure 1.1: A linear function $f(x) = 2x + 2$ and a point $(1, 4)$ on the graph of the function.