Mathematical Methods in Physics I

Homework III

Rahmanyaz Annyyev
October 23, 2023

Department of Physics
Middle East Technical University
[Revised May 10, 2024]

Problem 1

Solution. (a) The differential equation that describes the system is

$$Li'(t) + Ri(t) + v_C(t) = v_i(t)$$

If we differentiate both sides, we obtain

$$Li''(t) + Ri'(t) + v_C'(t) = v_i'(t)$$

or equivalently

$$\left(L\frac{d^2}{dt^2} + R\frac{d}{dt} + \frac{1}{C}\right) i(t) = v_i'(t)$$

(b) The characteristic equation for the differential equation is

$$Lr^2 + Rr + \frac{1}{C} = 0$$

(c) The roots of the characteristic equation are

$$r_{1,2} = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$$

(d) If we rewrite the characteristic equation in terms of β and ω, we obtain

$$\begin{aligned} r_{1,2} &= -\frac{R}{2L} \pm \frac{\sqrt{R^2 - \frac{4L}{C}}}{2L} \\ &= -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \\ &= -\beta \pm \sqrt{\beta^2 - \omega^2} \end{aligned}$$

(e) If $\beta \neq \omega$, the homogeneous solution to the system is

$$f_h(t) = c_1 e^{\left(-\beta + \sqrt{\beta^2 - \omega^2}\right)t} + c_2 e^{\left(-\beta - \sqrt{\beta^2 - \omega^2}\right)t}$$

(f) If $\beta = \omega$, the homogeneous solution to the system is

$$f_h(t) = e^{-\beta t}\left(c_1 + c_2 t\right)$$

(g) If we apply the Laplace transform to the differential equation $L\mathbb{i}''(t) + R\mathbb{i}'(t) + \frac{1}{C}\mathbb{i} = \delta(t)$, we obtain

$$L\left(s^2\mathbb{I}(s) - s\mathbb{i}(0) - \mathbb{i}'(0)\right) + R\left(s\mathbb{I}(s) - \mathbb{i}(0)\right) + \frac{1}{C}\mathbb{I}(s) = 1$$

from which we can solve for $\mathbb{I}(s)$:

$$\mathbb{I}(s) = \frac{1}{Ls^2 + Rs + \frac{1}{C}}$$

(h) If we rewrite the previous equation in terms of β, ω, and L, we obtain

$$\mathbb{I}(s) = \frac{1}{Ls^2 + 2L\beta s + L\omega^2}$$

For $\beta \neq \omega$, let's rewrite the previous equation the following way:

$$\mathbb{I}(s) = \frac{A}{s - r_1} + \frac{B}{s - r_2}$$

Therefore, $As - Ar_2 + Bs - Br_1 = (A + B)s - (Ar_2 + Br_1) = 1$. It follows that $A = -B$ and $Ar_2 + Br_1 = -1$. The latter can be rewritten as $Ar_2 - Ar_1 = A(r_2 - r_1) = -1$, or $A(r_1 - r_2) = 1$. Consequently, $A = \frac{1}{r_1 - r_2}$ and $B = -\frac{1}{r_1 - r_2}$. From (c), it follows that $r_1 - r_2 = 2\sqrt{\beta^2 - \omega^2}$. The equation can be rewritten as

$$\begin{aligned}\mathbb{I}(s) &= \frac{1}{2}\frac{1}{\sqrt{\beta^2 - \omega^2}}\frac{1}{s - r_1} - \frac{1}{2}\frac{1}{\sqrt{\beta^2 - \omega^2}}\frac{1}{s - r_2} \\ &= \frac{1}{2}\frac{1}{\sqrt{\beta^2 - \omega^2}}\left(\frac{1}{s - r_1} - \frac{1}{s - r_2}\right)\end{aligned}$$

For $\beta = \omega$, we have:

$$\mathbb{I}(s) = \frac{1}{(s - r)^2}$$

(i) If $\beta \neq \omega$, the impulse response $\mathbb{i}(s)$ is

$$\mathbb{i}(t) = \frac{1}{2}\frac{1}{\sqrt{\beta^2 - \omega^2}}\left(e^{\left(-\beta + \sqrt{\beta^2 - \omega^2}\right)t} - e^{\left(-\beta - \sqrt{\beta^2 - \omega^2}\right)t}\right)$$

If $\beta = \omega$, it is

$$\mathbb{i}(t) = te^{-\beta t}$$

(j) If $\beta \neq \omega$, then the particular solution is

$$f_p(t) = \int_0^\infty v_i'(t - t')\left(\frac{1}{2}\frac{1}{\sqrt{\beta^2 - \omega^2}}\left(e^{\left(-\beta + \sqrt{\beta^2 - \omega^2}\right)t'} - e^{\left(-\beta - \sqrt{\beta^2 - \omega^2}\right)t'}\right)\right) dt'$$

The complete solution is

$$\begin{aligned}i(t) =& c_1 e^{\left(-\beta + \sqrt{\beta^2 - \omega^2}\right)t} + c_2 e^{\left(-\beta - \sqrt{\beta^2 - \omega^2}\right)t} \\ &+ \int_0^\infty v_i'(t - t')\left(\frac{1}{2\sqrt{\beta^2 - \omega^2}}\left(e^{\left(-\beta + \sqrt{\beta^2 - \omega^2}\right)t'} - e^{\left(-\beta - \sqrt{\beta^2 - \omega^2}\right)t'}\right)\right) dt'\end{aligned}$$

If $\beta = \omega$, then the particular solution is

$$f_p(t) = \int_0^\infty v_i'(t - t')t'\mathrm{e}^{-\beta t'}\, dt'$$

The complete solution is

$$i(t) = e^{-\beta t}\left(c_1 + c_2 t\right) + \int_0^{\infty} v_i'(t-t')t'e^{-\beta t'}\,dt'$$

(k) The expression of $i(0)$ is

$$i(0) = c_1 + c_2 + \int_0^{\infty} v_i'(-t')\left(\frac{1}{2\sqrt{\beta^2-\omega^2}}\left(e^{r_1 t'} - e^{r_2 t'}\right)\right)dt'$$

Since $-t'$ is always less than zero, by definition, $v_i(-t') = v_i'(-t') = 0$. Therefore, the definite integral is equal to zero:

$$c_1 + c_2 = 0$$

The derivative of $i(t)$ is

$$\begin{aligned} i'(t) &= c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t} \\ &\quad + \frac{d}{dt}\int_0^{\infty} v_i'(t-t')\left(\frac{1}{2\sqrt{\beta^2-\omega^2}}\left(e^{r_1 t'} - e^{r_2 t'}\right)\right)dt' = 0 \end{aligned}$$

And $i'(0)$ is

$$i'(0) = c_1 r_1 + c_2 r_2$$

It follows that $c_2 = -c_1$, so $c_1(r_1 - r_2) = 0$. Hence, c_1 and c_2 are equal to zero. From the definitions,

$$v_o(t) = \frac{1}{C}\int_0^t i(t)\,dt$$

If we substitute $i(t)$ into the equation, we obtain

$$v_o(t) = \int_0^{\rho}\frac{1}{C}\left[\int_0^{\infty} v_i'(t-\tau)\left(\frac{1}{2\sqrt{\beta^2-\omega^2}}\left(e^{r_1\tau} - e^{r_2\tau}\right)\right)d\tau\right]d\rho$$

The derivative of $v_i(t-\tau)$ is

$$\begin{aligned} v_i'(t-\tau) &= \frac{d}{dt}\left(-\frac{\cos(f(t-\tau))}{f}\right) \\ &= \sin(f(t-\tau)) \end{aligned}$$

Hence, the final expression of the output voltage $v_o(t)$ is

$$v_o(t) = \frac{1}{C}\int_0^{\rho}\left[\int_0^t \sin(f(t-\tau))\left(\frac{1}{2}\frac{1}{\sqrt{\beta^2-\omega^2}}\left(e^{r_1\tau} - e^{r_2\tau}\right)\right)d\tau\right]d\rho$$

Note that $t-\tau$ cannot be less than zero; otherwise, $v_i'(t)$ and $\sin(f(t-\tau))$ would both be equal to zero, resulting in the inner definite integral being zero as well, which contradicts the given conditions. Thus, the upper limit of the inner definite integral is t. ∎