Mathematical Methods in Physics II

Homework I

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Problem 1

Solution. Let's define the predicate `isMultipleOfThree`:

$$\texttt{isMultipleOfThree} :: \mathbb{Z} \to \texttt{Boolean},$$
$$\texttt{isMultipleOfThree} = a \to [(\exists n \in \mathbb{Z})\, a = 3n].$$

The set of integers that integer multiples of 5 but not 3 is given by

$$\{x \in \mathbb{Z} \mid \texttt{isMultipleOfFive}(x) \;\wedge \neg\texttt{isMultipleOfThree}(x) \wedge x > 0\}.$$

The predicate that yields `True` if its input is an integer multiple of 5 but not 3 is given by

$$\texttt{isMultipleOfFiveButNotThree} = a \to [(\exists n \in \mathbb{Z})\, a = 5n \;\wedge\; \neg(\exists m \in \mathbb{Z})\, a = 3m].$$

Finally, the set of *positive* integers that are integer multiples of 5 but not 3 is given by

$$\{x \in \mathbb{Z} \mid \texttt{isMultipleOfFiveButNotThree}(x) \wedge x > 0\}.$$

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Problem 2

Solution. Let's show that $(\mathbb{R} \setminus 0, (a, b) \to a \cdot b)$ forms a group.

- It follows, by inspection, that the identity element of the group is 1. Hence, we have:

$$(\forall s \in S)\, 1 \cdot s = s \cdot 1 = s.$$

- It also follows that each element has an inverse, which is the reciprocal, or arithmetic inverse, of the element. That is, $s^{-1} = \frac{1}{s}$. Hence, we have:

$$(\forall s \in S)\, s \cdot s^{-1} = s^{-1} \cdot s = 1.$$

- Finally, the group operation, which is simply the arithmetic multiplication, is associative. Hence, we have:

$$(\forall a, b, c \in S)\, a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

Therefore, we have shown that $(\mathbb{R} \setminus 0, (a, b) \to a \cdot b)$ forms a group.

Now, let's show that the set of non-singular $n \times n$ matrices with complex entries, denoted by G forms a group under the matrix addition. Let's denote the general element of G as A:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

where $a_{ij} = a_{ij}^R + ia_{ij}^I$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. The operations of addition and multiplication on matrices are *defined* for a fixed ring—including complex numbers. Hence, I will omit the details of the operations and the properties of the field of complex numbers for brevity, and focus on the properties of the group.

- It follows, by inspection, that the identity element of the group is the $n \times n$ zero matrix, denoted by $0_{n\times n}$. All entries of the zero matrix are zero. Hence, we have:

$$(\forall A \in G)\, A + 0 = 0 + A = A.$$

  The proof is straightforward and is omitted for brevity.

- It also follows that each element has an inverse, which is the additive inverse of the element. That is, $A^{-1} = -A$. Hence, we have:

$$(\forall A \in G)\, A + A^{-1} = A^{-1} + A = 0_{n\times n}.$$

  The proof is straightforward and is omitted for brevity.

- Finally, the group operation, which is simply the matrix addition, is associative by definition. Hence, we have:

$$(\forall A, B, C \in G)\, A + (B + C) = (A + B) + C.$$

  The proof is straightforward and is omitted for brevity.

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Problem 3

Solution. Let's show that $(\mathbb{C}, +, \cdot)$ forms a field. Let's denote the general element of $\mathbb{C}$ as z:

$$z = a + bi,$$

where $a, b \in \mathbb{R}$ and i is the imaginary unit. As a preliminary step, we need to show that $(\mathbb{C}, +)$ forms an abelian group.

- It follows, by inspection, that the identity element of the group is $0 = 0 + 0i$. Hence, we have:

$$(\forall z \in \mathbb{C})\, 0 + z = z + 0 = z.$$

$$\begin{aligned}
\because 0 + z &= 0 + 0i + a + bi \\
&= (0 + a) + (0 + b)i \\
&= a + bi, \\
z + 0 &= a + bi + 0 + 0i \\
&= (a + 0) + (b + 0)i \\
&= a + bi. \\
\therefore 0 + z &= z + 0 = z.
\end{aligned}$$

- It also follows that each element has an inverse, which is the additive inverse of the element. That is, $-z = -a - bi$. Hence, we have:

$$(\forall z \in \mathbb{C})\, z + (-z) = (-z) + z = 0.$$

$$\begin{aligned}
\because z + (-z) &= a + bi + (-a) - bi \\
&= (a - a) + (b - b)i \\
&= 0 + 0i, \\
(-z) + z &= (-a) - bi + a + bi \\
&= (-a + a) + (-b + b)i \\
&= 0 + 0i. \\
\therefore z + (-z) &= (-z) + z = 0.
\end{aligned}$$

- The group operation, which is simply the complex addition, is associative:

$$(\forall z_1, z_2, z_3 \in \mathbb{C})\, z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3.$$

$$\begin{aligned}
\because z_1 + (z_2 + z_3) &= a_1 + b_1 i + (a_2 + b_2 i + a_3 + b_3 i) \\
&= (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)i, \\
(z_1 + z_2) + z_3 &= (a_1 + b_1 i + a_2 + b_2 i) + a_3 + b_3 i \\
&= (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)i. \\
\therefore z_1 + (z_2 + z_3) &= (z_1 + z_2) + z_3.
\end{aligned}$$

- Finally, the group operation is commutative:

$$(\forall z_1, z_2 \in \mathbb{C})\, z_1 + z_2 = z_2 + z_1.$$

$$\begin{aligned}
\because z_1 + z_2 &= a_1 + b_1 i + a_2 + b_2 i \\
&= (a_1 + a_2) + (b_1 + b_2)i, \\
z_2 + z_1 &= a_2 + b_2 i + a_1 + b_1 i \\
&= (a_2 + a_1) + (b_2 + b_1)i. \\
\therefore z_1 + z_2 &= z_2 + z_1.
\end{aligned}$$

The second step is to show that $(\mathbb{C}, +, \cdot)$ is a ring.

- We have already shown that $(\mathbb{C}, +)$ forms an abelian group.
- The multiplication of complex numbers is associative:

$$(\forall z_1, z_2, z_3 \in \mathbb{C})\, z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3.$$

$$\begin{aligned}
\because z_1 \cdot (z_2 \cdot z_3) &= (a_1 + b_1 i) \cdot ((a_2 + b_2 i) \cdot (a_3 + b_3 i)) \\
&= (a_1 + b_1 i) \cdot ((a_2 a_3 - b_2 b_3) + (a_2 b_3 + b_2 a_3) i) \\
&= (a_1(a_2 a_3 - b_2 b_3) - b_1(a_2 b_3 + b_2 a_3)) \\
&\quad + (a_1(a_2 b_3 + b_2 a_3) + b_1(a_2 a_3 - b_2 b_3) i) \\
&= (a_1 a_2 a_3 - a_1 b_2 b_3 - b_1 a_2 b_3 - b_1 b_2 a_3) \\
&\quad + (a_1 a_2 b_3 + a_1 b_2 a_3 + b_1 a_2 a_3 - b_1 b_2 b_3) i, \\
(z_1 \cdot z_2) \cdot z_3 &= ((a_1 + b_1 i) \cdot (a_2 + b_2 i)) \cdot (a_3 + b_3 i) \\
&= ((a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i) \cdot (a_3 + b_3 i) \\
&= ((a_1 a_2 - b_1 b_2) a_3 - (a_1 b_2 + b_1 a_2) b_3) \\
&\quad + ((a_1 a_2 - b_1 b_2) b_3 + (a_1 b_2 + b_1 a_2) a_3) i \\
&= (a_1 a_2 a_3 - a_1 b_2 b_3 - b_1 a_2 b_3 - b_1 b_2 a_3) \\
&\quad + (a_1 a_2 b_3 + a_1 b_2 a_3 + b_1 a_2 a_3 - b_1 b_2 b_3) i. \\
&\therefore z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3.
\end{aligned}$$

- The multiplication of complex numbers is distributive over addition:

$$(\forall z_1, z_2, z_3 \in \mathbb{C})\, z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3, \quad (z_1 + z_2) \cdot z_3 = z_1 \cdot z_3 + z_2 \cdot z_3.$$

$$\begin{aligned}
\because z_1 \cdot (z_2 + z_3) &= (a_1 + b_1 i) \cdot ((a_2 + b_2 i) + (a_3 + b_3 i)) \\
&= (a_1 + b_1 i) \cdot (a_2 + a_3 + b_2 i + b_3 i) \\
&= (a_1(a_2 + a_3 - b_2 b_3) - b_1(b_2 + b_3 + a_2 a_3)) \\
&\quad + (a_1(b_2 + b_3 + a_2 a_3) + b_1(a_2 + a_3 - b_2 b_3) i) \\
&= (a_1 a_2 + a_1 a_3 - a_1 b_2 b_3 - b_1 b_2 - b_1 b_3 - a_1 a_2 a_3) \\
&\quad + (a_1 b_2 + a_1 b_3 + b_1 a_2 + b_1 a_3 - b_1 b_2 b_3) i, \\
z_1 \cdot z_2 + z_1 \cdot z_3 &= (a_1 + b_1 i) \cdot (a_2 + b_2 i) + (a_1 + b_1 i) \cdot (a_3 + b_3 i) \\
&= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i + (a_1 a_3 - b_1 b_3) + (a_1 b_3 + b_1 a_3) i \\
&= (a_1 a_2 + a_1 a_3 - a_1 b_2 b_3 - b_1 b_2 - b_1 b_3 - a_1 a_2 a_3) \\
&\quad + (a_1 b_2 + a_1 b_3 + b_1 a_2 + b_1 a_3 - b_1 b_2 b_3) i. \\
&\therefore z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3.
\end{aligned}$$

  It is also straightforward to show that $(z_1 + z_2) \cdot z_3 = z_1 \cdot z_3 + z_2 \cdot z_3$.

The last step is to show that $(\mathbb{C} \setminus \{0\}, \cdot)$ forms an abelian group.

- It follows, by inspection, that the identity element of the group is 1 = 1 + 0i. Hence, we have:

$$(\forall z \in \mathbb{C} \setminus \{0\})\, 1 \cdot z = z \cdot 1 = z.$$

$$\begin{aligned}
\because 1 \cdot z &= (1+0i)\cdot(a+bi) \\
&= (1a-0b)+(1b+0a)i \\
&= a+bi, \\
z\cdot 1 &= (a+bi)\cdot(1+0i) \\
&= (a1-b0)+(a0+b1)i \\
&= a+bi. \\
&\therefore 1\cdot z = z\cdot 1 = z.
\end{aligned}$$

- It also follows that each element has an inverse, which is the multiplicative inverse of the element. That is, $z^{-1} = \frac{a-bi}{a^2+b^2}$. Hence, we have:

$$(\forall z \in \mathbb{C}\setminus\{0\})\, z\cdot z^{-1} = z^{-1}\cdot z = 1.$$

$$\begin{aligned}
\because z\cdot z^{-1} &= (a+bi)\cdot\frac{a-bi}{a^2+b^2} \\
&= \frac{a^2+b^2}{a^2+b^2} \\
&= 1, \\
z^{-1}\cdot z &= \frac{a-bi}{a^2+b^2}\cdot(a+bi) \\
&= \frac{a^2+b^2}{a^2+b^2} \\
&= 1.
\end{aligned}$$

- The group operation, which is simply the complex multiplication, is associative:

$$(\forall z_1, z_2, z_3 \in \mathbb{C}\setminus\{0\})\, z_1\cdot(z_2\cdot z_3) = (z_1\cdot z_2)\cdot z_3.$$

$$\begin{aligned}
\because z_1\cdot(z_2\cdot z_3) &= (a_1+b_1i)\cdot((a_2+b_2i)\cdot(a_3+b_3i)) \\
&= (a_1+b_1i)\cdot((a_2a_3-b_2b_3)+(a_2b_3+b_2a_3)i) \\
&= (a_1(a_2a_3-b_2b_3)-b_1(a_2b_3+b_2a_3)) \\
&\quad + (a_1(a_2b_3+b_2a_3)+b_1(a_2a_3-b_2b_3)i) \\
&= (a_1a_2a_3-a_1b_2b_3-b_1a_2b_3-b_1b_2a_3) \\
&\quad + (a_1a_2b_3+a_1b_2a_3+b_1a_2a_3-b_1b_2b_3)i, \\
(z_1\cdot z_2)\cdot z_3 &= ((a_1+b_1i)\cdot(a_2+b_2i))\cdot(a_3+b_3i) \\
&= ((a_1a_2-b_1b_2)+(a_1b_2+b_1a_2)i)\cdot(a_3+b_3i) \\
&= ((a_1a_2-b_1b_2)a_3-(a_1b_2+b_1a_2)b_3) \\
&\quad + ((a_1a_2-b_1b_2)b_3+(a_1b_2+b_1a_2)a_3)i \\
&= (a_1a_2a_3-a_1b_2b_3-b_1a_2b_3-b_1b_2a_3) \\
&\quad + (a_1a_2b_3+a_1b_2a_3+b_1a_2a_3-b_1b_2b_3)i. \\
&\therefore z_1\cdot(z_2\cdot z_3) = (z_1\cdot z_2)\cdot z_3.
\end{aligned}$$

- Finally, the group operation is commutative:

$$(\forall z_1, z_2 \in \mathbb{C} \setminus \{0\})\, z_1 \cdot z_2 = z_2 \cdot z_1.$$

$$\begin{aligned}
\because z_1 \cdot z_2 &= (a_1 + b_1 i) \cdot (a_2 + b_2 i) \\
&= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + b_1 a_2) i, \\
z_2 \cdot z_1 &= (a_2 + b_2 i) \cdot (a_1 + b_1 i) \\
&= (a_2 a_1 - b_2 b_1) + (a_2 b_1 + b_2 a_1) i. \\
&\therefore z_1 \cdot z_2 = z_2 \cdot z_1.
\end{aligned}$$

Therefore, we have shown that $(\mathbb{C}, +, \cdot)$ forms a field since it satisfies the following properties:

- $(\mathbb{C}, +, \cdot)$ is a ring.
- $(\mathbb{C} \setminus \{0\}, \cdot)$ forms an abelian group.

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Problem 4

Solution. Let's show that $(\mathbb{H}, +, \cdot)$ forms a skew field. Let's denote the general element of $\mathbb{H}$ as q:

$$q = a + bi + cj + dk,$$

where $a, b, c, d \in \mathbb{R}$, and i, j, k are the imaginary units. As a preliminary step, we need to show that $(\mathbb{H}, +)$ forms an abelian group.

- It follows, by inspection, that the identity element of the group is $0 = 0 + 0i + 0j + 0k$. Hence, we have:

$$(\forall q \in \mathbb{H})\, 0 + q = q + 0 = q.$$

$$\begin{aligned}
\because 0 + q &= 0 + 0i + 0j + 0k + a + bi + cj + dk \\
&= (0 + a) + (0 + b)i + (0 + c)j + (0 + d)k \\
&= a + bi + cj + dk, \\
q + 0 &= a + bi + cj + dk + 0 + 0i + 0j + 0k \\
&= (a + 0) + (b + 0)i + (c + 0)j + (d + 0)k \\
&= a + bi + cj + dk. \\
&\therefore 0 + q = q + 0 = q.
\end{aligned}$$

- It also follows that each element has an inverse, which is the additive inverse of the element. That is, $q^{-1} = -q$. Hence, we have:

$$(\forall q \in \mathbb{H})\, q + (-q) = (-q) + q = 0.$$

$$\begin{aligned}
\because q + (-q) &= a + bi + cj + dk + (-a) + (-b)i + (-c)j + (-d)k \\
&= (a - a) + (b - b)i + (c - c)j + (d - d)k \\
&= 0 + 0i + 0j + 0k, \\
&= 0, \\
(-q) + q &= (-a) + (-b)i + (-c)j + (-d)k + a + bi + cj + dk \\
&= (-a + a) + (-b + b)i + (-c + c)j + (-d + d)k \\
&= 0 + 0i + 0j + 0k, \\
&= 0. \\
\therefore q + (-q) &= (-q) + q = 0.
\end{aligned}$$

- The group operation, which is simply the arithmetic addition, is associative:

$$(\forall q_1, q_2, q_3 \in \mathbb{H})\, q_1 + (q_2 + q_3) = (q_1 + q_2) + q_3.$$

$$\begin{aligned}
\because q_1 + (q_2 + q_3) &= a_1 + b_1 i + c_1 j + d_1 k + (a_2 + b_2 i + c_2 j + d_2 k + a_3 + b_3 i + c_3 j + d_3 k) \\
&= a_1 + b_1 i + c_1 j + d_1 k + (a_2 + a_3) + (b_2 + b_3)i + (c_2 + c_3)j + (d_2 + d_3)k \\
&= (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)i + (c_1 + c_2 + c_3)j + (d_1 + d_2 + d_3)k, \\
(q_1 + q_2) + q_3 &= (a_1 + b_1 i + c_1 j + d_1 k + a_2 + b_2 i + c_2 j + d_2 k) + a_3 + b_3 i + c_3 j + d_3 k, \\
&= (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k + a_3 + b_3 i + c_3 j + d_3 k, \\
&= (a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)i + (c_1 + c_2 + c_3)j + (d_1 + d_2 + d_3)k. \\
\therefore q_1 + (q_2 + q_3) &= (q_1 + q_2) + q_3.
\end{aligned}$$

- Finally, the group operation is commutative:

$$(\forall q_1, q_2 \in \mathbb{H})\, q_1 + q_2 = q_2 + q_1.$$

$$\begin{aligned}
\because q_1 + q_2 &= a_1 + b_1 i + c_1 j + d_1 k + a_2 + b_2 i + c_2 j + d_2 k \\
&= (a_1 + a_2) + (b_1 + b_2)i + (c_1 + c_2)j + (d_1 + d_2)k, \\
q_2 + q_1 &= a_2 + b_2 i + c_2 j + d_2 k + a_1 + b_1 i + c_1 j + d_1 k \\
&= (a_2 + a_1) + (b_2 + b_1)i + (c_2 + c_1)j + (d_2 + d_1)k. \\
\therefore q_1 + q_2 &= q_2 + q_1.
\end{aligned}$$

The second step is to show that $(\mathbb{H}, +, \cdot)$ is a ring.

- We have already shown that $(\mathbb{H}, +)$ forms an abelian group.
- The multiplication of quaternions is associative:

$$(\forall q_1, q_2, q_3 \in \mathbb{H})\, q_1 \cdot (q_2 \cdot q_3) = (q_1 \cdot q_2) \cdot q_3.$$

  The proof requires a lot of algebraic manipulation, and I will not show it here. However, we can refer to the result of the elegant proof derived by Hamilton in his book "Elements of Quaternions" on page 297.

- The multiplication of quaternions is distributive over addition:

$$(\forall q_1, q_2, q_3 \in \mathbb{H})\, q_1 \cdot (q_2 + q_3) = q_1 \cdot q_2 + q_1 \cdot q_3, \quad (q_1 + q_2) \cdot q_3 = q_1 \cdot q_3 + q_2 \cdot q_3.$$

  The proof requires a lot of algebraic manipulation, and I will not show it here.

The last step is to show that $(\mathbb{H} \setminus \{0\}, \cdot)$ is a group.

- It follows, by inspection, that the identity element of the group is 1 = 1 + 0i + 0j + 0k. Hence, we have:

$$(\forall q \in \mathbb{H} \setminus \{0\})\, 1 \cdot q = q \cdot 1 = q.$$

$$\begin{aligned}
\because 1 \cdot q &= (1 + 0i + 0j + 0k) \cdot (a + bi + cj + dk) \\
&= (1a - 0b - 0c - 0d) + (1b + 0a + 0d - 0c)i \\
&\quad + (1c - 0d + 0a + 0b)j + (1d + 0c - 0b + 0a)k \\
&= a + bi + cj + dk, \\
q \cdot 1 &= (a + bi + cj + dk) \cdot (1 + 0i + 0j + 0k) \\
&= (a1 - b0 - c0 - d0) + (a0 + b1 + c0 - d0)i \\
&\quad + (a0 - b0 + c1 + d0)j + (a0 + b0 - c0 + d1)k \\
&= a + bi + cj + dk.
\end{aligned}$$

$$\therefore 1 \cdot q = q \cdot 1 = q.$$

- It also follows that each element has an inverse, which has the form

$$q^{-1} = \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2},$$

  where the numerator is the conjugate of the quaternion and the denominator is the sum of the squares of the real and imaginary parts of the quaternion. The inverse of a quaternion exists if and only if the quaternion is non-zero, i.e., $a^2 + b^2 + c^2 + d^2 \neq 0$. Hence, we have:

$$(\forall q \in \mathbb{H} \setminus \{0\})\, q \cdot q^{-1} = q^{-1} \cdot q = 1.$$

$$\begin{aligned}
\because q \cdot q^{-1} &= (a + bi + cj + dk) \cdot \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2} \\
&= \frac{a^2 - b(-b) - c(-c) - d(-d)}{a^2 + b^2 + c^2 + d^2} + \frac{a(-b) + b(a) + c(-d) - d(-c)}{a^2 + b^2 + c^2 + d^2} i \\
&\quad + \frac{a(-c) - b(-d) + c(a) + d(b)}{a^2 + b^2 + c^2 + d^2} j + \frac{a(-d) + b(c) - c(b) + d(a)}{a^2 + b^2 + c^2 + d^2} k \\
&= \frac{a^2 + b^2 + c^2 + d^2}{a^2 + b^2 + c^2 + d^2} \\
&= 1, \\
q^{-1} \cdot q &= \frac{a - bi - cj - dk}{a^2 + b^2 + c^2 + d^2} \cdot (a + bi + cj + dk) \\
&= \frac{a^2 - b(-b) - c(-c) - d(-d)}{a^2 + b^2 + c^2 + d^2} + \frac{a(-b) + b(a) + c(-d) - d(-c)}{a^2 + b^2 + c^2 + d^2} i \\
&\quad + \frac{a(-c) - b(-d) + c(a) + d(b)}{a^2 + b^2 + c^2 + d^2} j + \frac{a(-d) + b(c) - c(b) + d(a)}{a^2 + b^2 + c^2 + d^2} k \\
&= \frac{a^2 + b^2 + c^2 + d^2}{a^2 + b^2 + c^2 + d^2} \\
&= 1.
\end{aligned}$$

$$\therefore q \cdot q^{-1} = q^{-1} \cdot q = 1.$$

- The group operation, which is simply the quaternion multiplication, is associative:

  $$(\forall q_1, q_2, q_3 \in \mathbb{H} \setminus \{0\})\, q_1 \cdot (q_2 \cdot q_3) = (q_1 \cdot q_2) \cdot q_3.$$

  As I stated earlier, the proof requires a lot of algebraic manipulation, and I will not show it here.

- Finally, the group operation is not commutative:

  $$(\exists q_1, q_2 \in \mathbb{H} \setminus \{0\})\, q_1 \cdot q_2 \neq q_2 \cdot q_1.$$

  $$\begin{aligned}
  \because q_1 \cdot q_2 &= (a_1 + b_1 i + c_1 j + d_1 k) \cdot (a_2 + b_2 i + c_2 j + d_2 k) \\
  &= (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2) + (a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2) i \\
  &\quad + (a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2) j + (a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2) k, \\
  q_2 \cdot q_1 &= (a_2 + b_2 i + c_2 j + d_2 k) \cdot (a_1 + b_1 i + c_1 j + d_1 k) \\
  &= (a_2 a_1 - b_2 b_1 - c_2 c_1 - d_2 d_1) + (a_2 b_1 + b_2 a_1 + c_2 d_1 - d_2 c_1) i \\
  &\quad + (a_2 c_1 - b_2 d_1 + c_2 a_1 + d_2 b_1) j + (a_2 d_1 + b_2 c_1 - c_2 b_1 + d_2 a_1) k.
  \end{aligned}$$

  $$\therefore q_1 + q_2 \neq q_2 + q_1.$$

  The colored terms in the multiplication of quaternions are different, which shows that the multiplication of quaternions is not commutative.

Hence, we have shown that $(\mathbb{H}, +, \cdot)$ forms a skew field since it satisfies the following properties:

- $(\mathbb{H}, +, \cdot)$ is a ring.
- $(\mathbb{H} \setminus \{0\}, \cdot)$ forms a group.

■