Mathematical Methods in Physics I
Homework 8

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1 Question One

1.1

Solution. Let's combine the following equations:

$$c''(t) = \frac{1}{a} - \frac{1}{a}p(t) - \frac{b^2}{a}c'(t) \tag{1.1}$$
$$p(t) = \left(\frac{\mathrm{d}^3}{\mathrm{d}t^3} + ab^2\right)c(t) \tag{1.2}$$

If we substitute the second equation into the first one, we get:

$$c''(t) = \frac{1}{a} - \frac{1}{a}(c'''(t) + ab^2c(t)) - \frac{b^2}{a}c'(t) \tag{1.3}$$
$$ac''(t) = 1 - c'''(t) - ab^2c(t) - b^2c'(t) \tag{1.4}$$
$$c'''(t) + ac''(t) + b^2c'(t) + ab^2c(t) = 1 \tag{1.5}$$

1.2

Solution. By inspection, it follows that one of the solutions is $c_1(t) = \dfrac{1}{ab^2}$:

$$\left(\frac{1}{ab^2}\right)''' + a\left(\frac{1}{ab^2}\right)'' + b^2\left(\frac{1}{ab^2}\right)' + ab^2\frac{1}{ab^2} = 1 \tag{1.6}$$
$$0 + 0 + 0 + 1 = 1 \tag{1.7}$$
$$1 = 1 \tag{1.8}$$

1.3

Solution. Assume that $c_2(t) = \mathrm{e}^{-\alpha t}$ is a solution. Hence, let's denote $c(t) = g(t)c_2(t) = g(t)\mathrm{e}^{-\alpha t}$. For the sake of brevity, let's denote $g(t)$ as g. Then, we have:

$$(g\mathrm{e}^{-\alpha t})''' + a(g\mathrm{e}^{-\alpha t})'' + b^2(g\mathrm{e}^{-\alpha t})' + ab^2(g\mathrm{e}^{-\alpha t}) = 1 \tag{1.9}$$

Let's expand the derivatives:

$$(g\mathrm{e}^{-\alpha t})' = g'\mathrm{e}^{-\alpha t} - \alpha g\mathrm{e}^{-\alpha t} \tag{1.10}$$
$$(g\mathrm{e}^{-\alpha t})'' = g''\mathrm{e}^{-\alpha t} - 2\alpha g'\mathrm{e}^{-\alpha t} + \alpha^2 g\mathrm{e}^{-\alpha t} \tag{1.11}$$
$$(g\mathrm{e}^{-\alpha t})''' = g'''\mathrm{e}^{-\alpha t} - 3\alpha g''\mathrm{e}^{-\alpha t} + 3\alpha^2 g'\mathrm{e}^{-\alpha t} - \alpha^3 g\mathrm{e}^{-\alpha t} \tag{1.12}$$

For this differential equation to be reduced, the terms with g must cancel out. Hence, we have:

$$\alpha^3 - a\alpha^2 + b^2\alpha - ab^2 = 0 \tag{1.13}$$
$$\alpha^2(\alpha - a) + b^2(\alpha - a) = 0 \tag{1.14}$$
$$(\alpha^2 + b^2)(\alpha - a) = 0 \tag{1.15}$$

Hence, we have three solutions:

$$\alpha_1 = a \tag{1.16}$$
$$\alpha_{2,3} = \pm \mathrm{i}b \tag{1.17}$$

Note that $\alpha_{2,3}$ are not suitable because the solutions would not be decreasing exponentially. Therefore, $\alpha = a$. If we substitute α, the reduced differential equation is:

$$g'''\mathrm{e}^{-at} - 3ag''\mathrm{e}^{-at} + 3a^2g'\mathrm{e}^{-at} + a(g''\mathrm{e}^{-at} - 2ag'\mathrm{e}^{-at}) + b^2(g'\mathrm{e}^{-at}) = 1 \tag{1.18}$$
$$g'''\mathrm{e}^{-at} - 3ag''\mathrm{e}^{-at} + 3a^2g'\mathrm{e}^{-at} + ag''\mathrm{e}^{-at} - 2a^2g'\mathrm{e}^{-at} + b^2g'\mathrm{e}^{-at} = 1 \tag{1.19}$$
$$g'''\mathrm{e}^{-at} - 2ag''\mathrm{e}^{-at} + a^2g'\mathrm{e}^{-at} + b^2g'\mathrm{e}^{-at} = 1 \tag{1.20}$$
$$g''' - 2ag'' + (a^2 + b^2)g' = \mathrm{e}^{at} \tag{1.21}$$

Let's denote $g' = h$. Then, we have:

$$h'' - 2ah' + (a^2 + b^2)h = \mathrm{e}^{at} \tag{1.22}$$

1.4

Solution. Let's find the homogeneous solution:

$$h'' - 2ah' + (a^2 + b^2)h = 0 \tag{1.23}$$
$$\lambda^2 - 2a\lambda + (a^2 + b^2) = 0 \tag{1.24}$$
$$\lambda_{1,2} = \frac{2a \pm \sqrt{4a^2 - 4(a^2 + b^2)}}{2} \tag{1.25}$$
$$\lambda_{1,2} = a \pm \mathrm{i}b \tag{1.26}$$

Hence, the homogeneous solution is:

$$h_h = c_1\mathrm{e}^{(a+\mathrm{i}b)t} + c_2\mathrm{e}^{(a-\mathrm{i}b)t} \tag{1.27}$$

1.5

Solution. The general solution is:

$$c(t) = \frac{1}{ab^2} + c_1\mathrm{e}^{-at} + c_2\mathrm{e}^{(a+\mathrm{i}b)t} + c_3\mathrm{e}^{(a-\mathrm{i}b)t} \tag{1.28}$$