Phys209: Mathematical Methods in Physics I

Homework 2

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Policies

- Please adhere to the *academic integrity* rules: see my explanations here for further details!
- For the overall grading scheme or any other course-related details, see the syllabus.
- Non-graded question(s) are for your own practice!
- The homework is due October 20th 2023, 23:59 TSI.

(1) Problem One *(4 points)*

We have seen in class that exponential functions form the homogeneous solutions to the *linear ordinary differential equations with constant coefficients*. Let us expand more on that in this question.

(1.1) (0.8pt)

We can define a *higher order differential operator* $\mathcal{D}$ as

$$\mathcal{D} ::(\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{1.1a}$$

$$\mathcal{D} = a_0\mathcal{I} + a_1\frac{\mathrm{d}}{\mathrm{d}x} + \cdots + a_n\frac{\mathrm{d}^n}{\mathrm{d}x^n} \tag{1.1b}$$

where $\mathcal{I}$ is the identity higher order function

$$\mathcal{I} :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{1.2a}$$

$$\mathcal{I} = \Big(x \to f(x)\Big) \to \Big(x \to f(x)\Big) \tag{1.2b}$$

We will drop $\mathcal{I}$ in the rest of the homework as it does not create any ambiguity (see also homework 1 for more on this operator).

The operator $\mathcal{D}$ acts on a function and creates another function; for instance, if we define *the square function* sqr as

$$\mathrm{sqr} :: \mathbb{C} \to \mathbb{C} \tag{1.3a}$$

$$\mathrm{sqr} = x \to x^2 \tag{1.3b}$$

we immediately see that

$$\mathcal{D} \cdot \mathrm{sqr} :: \mathbb{C} \to \mathbb{C} \tag{1.4a}$$

$$\mathcal{D} \cdot \mathrm{sqr} = x \to \Big(a_0x^2 + 2a_1x + 2a_2\Big) \tag{1.4b}$$

To show this, we need to know how $\frac{\mathrm{d}^n}{\mathrm{d}x^n}$ acts: what should be the ??? below?

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} :: (\mathbb{C} \to \mathbb{C}) \to ??? \tag{1.5a}$$

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} = ??? \tag{1.5b}$$

Hint: see eqn. 1.2 of homework 1

Answer:

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{1.6a}$$
$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} = (x \to f(x)) \to (x \to f^{(n)}(x)) \tag{1.6b}$$

(1.2) (0.8pt)

Please find out ??? below:

$$\mathcal{D} \cdot \cos = ??? \tag{1.7a}$$
$$\mathcal{D} \cdot \sin = ??? \tag{1.7b}$$
$$\mathcal{D} \cdot \exp = ??? \tag{1.7c}$$

Answer:

$$\begin{aligned}\mathcal{D} \cdot \cos = x \to\ & (a_0 - a_2 + a_4 + \dots) \cos(x) \\ & - (a_1 - a_3 + a_5 + \dots) \sin(x)\end{aligned} \tag{1.8a}$$
$$\begin{aligned}\mathcal{D} \cdot \sin = x \to\ & (a_0 - a_2 + a_4 + \dots) \sin(x) \\ & + (a_1 - a_3 + a_5 + \dots) \cos(x)\end{aligned} \tag{1.8b}$$
$$\mathcal{D} \cdot \exp = x \to (a_0 + a_1 + \dots + a_n)\, e^x \tag{1.8c}$$

(1.3) (1.6pt)

Consider the simple function

$$\mathrm{egFunc1} :: \mathbb{C} \to \mathbb{C} \tag{1.9a}$$
$$\mathrm{egFunc1} = x \to x(x-1) \tag{1.9b}$$

which has the abbreviated name for *exempli gratia function one*. We observe that this function maps the set $\{0, 1\}$ to the set $\{0\}$:

$$\{0, 1\} \xrightarrow{\mathrm{egFunc1}} \{0\} \tag{1.10}$$

Likewise, the following function

$$\mathrm{egFunc2} :: \mathbb{C} \to \mathbb{C} \tag{1.11a}$$
$$\mathrm{egFunc2} = x \to \left(x^3 - \pi x^2 - 6x^2 + 6\pi x + 8x - 8\pi\right) \tag{1.11b}$$

has the map

$$\{2, 4, \pi\} \xrightarrow{\text{egFunc2}} \{0\} \tag{1.12}$$

Both of these functions map a particular set to $\{0\}$; in fact, the first set needs not be a finite set. For example, it is actually an infinite set for the *cosine function*, please fill in ??? below:

$$\{???\} \xrightarrow{\cos} \{0\} \tag{1.13}$$

Answer:

$$\left\{\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\right\} \xrightarrow{\cos} \{0\} \tag{1.14}$$

The input set that gets mapped to $\{0\}$ by a transformation is called *the kernel of the transformation*. In vector spaces, the linear transformations are implemented by matrices, and the kernel of such linear transformations are equivalent to the *nullspace of the transformation matrix*. In contrast, the analogous transformations are implemented by differential operators on the space of functions, and the kernel of the differential operator actually forms the *homogeneous solution*. We can summarize this paragraph with the following table:

Input	**Transformation**	**Output**	**Kernel**
Sets	Function application	$\{0\}$	Zeros of function
Vectors	Matrix multiplication	$\vec{0}$	Null space of the matrix
Functions	Differential operator	zero	Homogeneous solutions

In the table above, zero is the *zero function* defined as follows:

$$\text{zero} :: \mathbb{C} \to \mathbb{C} \tag{1.15a}$$

$$\text{zero} = x \to 0 \tag{1.15b}$$

Clearly, this function takes anything to 0, i.e.

$$\text{zero}(2) = 0\ , \quad \text{zero}(\pi) = 0\ , \quad \text{zero}(i) = 0\ , \quad \text{zero}(-1.2) = 0\ , \quad \ldots \tag{1.16}$$

To find the *kernel* of a given differential operator $\mathcal{D}$, we simply solve for the function f which satisfies

$$\mathcal{D} \cdot f :: \mathbb{C} \to \mathbb{C} \tag{1.17a}$$

$$\mathcal{D} \cdot f = \text{zero} \tag{1.17b}$$

What would be the type and result for the kernel of the differential operator $\left(\frac{\mathrm{d}}{\mathrm{d}x} - a\right)$:

$$\ker\left(\frac{\mathrm{d}}{\mathrm{d}x} - a\right) :: ??? \tag{1.18a}$$

$$\ker\left(\frac{\mathrm{d}}{\mathrm{d}x} - a\right) = ??? \tag{1.18b}$$

where ker denotes the kernel; for example,

$$\ker\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) :: \{\mathbb{C} \to \mathbb{C}\} \tag{1.19a}$$

$$\ker\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) = \{x \to \text{constant}\} \tag{1.19b}$$

Hint: Note that the kernel of a differential operator is a list *of functions, hence the kernel above has the type* $\{\mathbb{C} \to \mathbb{C}\}$, not $\mathbb{C} \to \mathbb{C}$.

Answer:

$$\ker\left(\frac{\mathrm{d}}{\mathrm{d}x} - a\right) :: \{\mathbb{C} \to \mathbb{C}\} \tag{1.20a}$$

$$\ker\left(\frac{\mathrm{d}}{\mathrm{d}x} - a\right) = \{x \to 0, x \to e^{ax}\} \tag{1.20b}$$

(1.4) (0.8pt)

We have seen in class that the way to solve for the homogeneous equation for the *linear ordinary differential equations with constant coefficients* is to solve the *characteristic equation*, and then exponentiate the roots. For instance,

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - a\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - b\right) f(x) = 0 \quad \Rightarrow \quad f(x) = c_1 e^{ax} + c_2 e^{bx} \tag{1.21}$$

where the roots $r_{1,2}$ of the characteristic equation $(r-a)(r-b) = 0$ are a, b.

With this same method, solve the differential equation:

$$\left(\frac{\mathrm{d}^5}{\mathrm{d}x^5} - 3\frac{\mathrm{d}^4}{\mathrm{d}x^4} - 23\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 51\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 94\frac{\mathrm{d}}{\mathrm{d}x} - 120\right) f(x) =??? \tag{1.22}$$

Hint: When you construct the characteristic equation, you will observe that it is a 5^{th} order polynomial. Abel–Ruffini theorem guarantees that we cannot find an algebraic solution for such a polynomial for arbitrary coefficients; however, this particular polynomial will have nice roots as I choose the coefficients deliberately. Nevertheless, it would be hard for you to find the roots by pen and paper, so let me give you two of the roots: $r = 5$ and $r = -4$. You can now in principle solve for the other three roots by using cubic equation *(see wikipedia). But even that is extremely hard to do with pen and paper, so I'll give you one more root: $r = 3$. Now you can find the other two roots easily :)*

Answer: The characteristic equation is

$$r^5 - 3r^4 - 23r^3 + 51r^2 + 94r - 120 = 0 \tag{1.23}$$

which can be brought to the form

$$(r-5)(r+4)(r-3)(r+2)(r-1) = 0 \tag{1.24}$$

hence

$$f(x) = c_1 e^{5x} + c_2 e^{-4x} + c_3 e^{3x} + c_4 e^{-2x} + c_5 e^{x} \tag{1.25}$$

(1.5) Not graded

Following two-line code solves the previous problem:

```
DSolve[f'''''[x] - 3 f''''[x] - 23 f'''[x] + 51 f''[x] + 94 f'[x] -
120 f[x] == 0, f[x], x]
```

```
{{f[x] -> E^(-4 x) C[1] + E^(-2 x) C[2] + E^x C[3] + E^(3 x) C[4]
    + E^(5 x) C[5]}}
```

Likewise, we can factorize any given polynomial, which vastly simplifies our computations in daily life:

```
r^5 - 3 r^4 - 23 r^3 + 51 r^2 + 94 r - 120 // Factor
```

```
(-5 + r) (-3 + r) (-1 + r) (2 + r) (4 + r)
```

(2) Problem Two *(2 points)*

In class, we have talked about the integral transformations. They are higher order functions that transforms a function from "one function domain" to another function in "another function domain", i.e.

$$\mathrm{IT} :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{2.1a}$$

$$\mathrm{IT} = (x \to f(x)) \to \left(s \to \int_{\alpha}^{\beta} K(x,s) f(x) dx \right) \tag{2.1b}$$

where the integration range and the kernel of the integral transformation (i.e. $K(x,s)$) are fixed by the chosen transformation. For instance, a special integral transformation called *Laplace transformation* (about which we have talked a lot this week) takes the form

$$\mathcal{L} :: (\mathbb{C} \to \mathbb{C}) \to (\mathbb{C} \to \mathbb{C}) \tag{2.2a}$$

$$\mathcal{L} = (x \to f(x)) \to \left(s \to \int_{0}^{\infty} e^{-xs} f(x) dx \right) \tag{2.2b}$$

Let us play with this transformation a little bit.

(2.1) (2pt)

Consider the constant function defined as follows:

$$\mathrm{const}_a :: \mathbb{C} \to \mathbb{C} \tag{2.3a}$$

$$\mathrm{const}_a = x \to a \tag{2.3b}$$

which satisfies $\mathrm{const}_a(x) = a$ for any x. The Laplace transform of this function is given as

$$\mathcal{L} \cdot \mathrm{const}_a :: \mathbb{C} \to \mathbb{C} \tag{2.4a}$$

$$\mathcal{L} \cdot \mathrm{const}_a = s \to \frac{a}{s} \tag{2.4b}$$

hence $(\mathcal{L} \cdot \mathrm{const}_a)(s) = \frac{a}{s}$. We derived this by using the identity $\int_{0}^{\infty} e^{-xs} a dx = \frac{a}{s}$.

Let us consider a more general function:

$$f_{n,a} :: \mathbb{C} \to \mathbb{C} \tag{2.5a}$$
$$f_{n,a} = x \to x^n e^{ax} \tag{2.5b}$$

What is its Laplace transform, i.e.

$$\mathcal{L} \cdot f_{n,a} :: \mathbb{C} \to \mathbb{C} \tag{2.6a}$$
$$\mathcal{L} \cdot f_{n,a} = ??? \tag{2.6b}$$

Derive the result step by step. You may assume $s > 0$ in the Laplace domain, but do not assume $n \in \mathbb{Z}$. *Hint: See the definition of Gamma function.*

Answer:

$$(\mathcal{L} \cdot f_{n,a})(s) = \int_0^\infty e^{-sx} x^n e^{ax} dx \tag{2.7}$$

Shift the dummy variable:

$$(\mathcal{L} \cdot f_{n,a})(s + a) = \int_0^\infty e^{-sx} x^n dx \tag{2.8}$$

Define $x = y/s$ and change the integration variable:

$$(\mathcal{L} \cdot f_{n,a})(s + a) = \frac{1}{s^{n+1}} \int_0^\infty e^{-y} y^n dy \tag{2.9}$$

The integral is *definition* of the gamma function:

$$(\mathcal{L} \cdot f_{n,a})(s + a) = \frac{\Gamma(n+1)}{s^{n+1}} \tag{2.10}$$

Shift the dummy variable back:

$$(\mathcal{L} \cdot f_{n,a})(s) = \frac{\Gamma(n+1)}{(s-a)^{n+1}} \tag{2.11}$$

Therefore:

$$\mathcal{L} \cdot f_{n,a} :: \mathbb{C} \to \mathbb{C} \tag{2.12a}$$

$$\mathcal{L} \cdot f_{n,a} = s \to \frac{\Gamma(n+1)}{(s-a)^{n+1}} \tag{2.12b}$$

(2.2) Not graded

Following code solves the previous problem:

```
LaplaceTransform[x^n Exp[a x], x, s]
```

```
(-a + s)^(-1 - n) Gamma[1 + n]
```