Mathematical Methods in Physics II

Lecture Notes

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Preface

These lecture notes were prepared for the course *PHYS210 Mathematical Methods in Physics II* at Middle East Technical University, Ankara, Turkey. The course is intended for second-year undergraduate students in the Department of Physics. The notes are based on the lectures given by Prof. Soner Albayrak in the Spring semester of the 2023–2024 academic year.

The course is a continuation of *Mathematical Methods in Physics I*, which is a prerequisite for this course. The main topics covered in this course include vector analysis, complex analysis, and partial differential equations. The notes are intended to be a useful resource for students taking the course, as well as for anyone interested in learning about the mathematical methods used in physics. If you have noticed any errors or have any suggestions for improvement, please feel free to contact me.

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Contents

Preface i

Contents ii

List of Figures iii

List of Tables iv

I Vector Analysis 5

1 February 19 6
- 1.1 Course Information 6
- 1.2 Introduction 6
- 1.3 Sets 7
  - 1.3.1 Set Comprehension 7
  - 1.3.2 Quantifiers 8
- 1.4 Groups 9

2 February 23 11
- 2.1 Propositional Logic 11
- 2.2 Groups 11
  - 2.2.1 Basic Definitions 11
  - 2.2.2 Commutative Groups 12
- 2.3 Rings 13
- 2.4 Division Rings 14

3 March 1 16
- 3.1 Linear Spaces 18
- 3.2 Linear Algebra 19
- 3.3 Lie Algebra 20

4 March 4 21
- 4.1 Basis of Vector Spaces 21
- 4.2 Normed Vector Space 23
- 4.3 Inner Product Vector Space 24

List of Figures

2.1 A diagram of the relationships between different algebraic structures. 15

4.1 S 22

List of Tables

Vector Analysis

1

February 19

1.1 Course Information

- **Instructor:** Prof. Soner Albayrak
- **Syllabus:** https://soneralbayrak.com/teaching/Phys210

The course will cover the following topics:

1. Basics of vector spaces.
2. Differentiation in vector spaces.
3. Integration in vector spaces.
4. Curvilinear coordinate systems.
5. Basis of complex calculus.
6. Complex analysis.

The first four topics are necessary because vectors (and their generalization tensors) form the way to describe physics objectively. The last two topics are necessary because nature speaks to us in complex numbers.

1.2 Introduction

Schrödinger's equation is given by

$$i\frac{\mathrm{d}}{\mathrm{d}t}\Psi = \hat{H}\Psi,$$

where $\hat{H}$ is the Hamiltonian operator, and Ψ is the wave function. In a letter to physicist Hendrik Lorentz on June 6, 1926, Schrödinger wrote, "What is unpleasant here, and indeed directly to be objected to, is the use of complex numbers. Ψ [the wave function] is surely fundamentally a real function." This example shows that complex numbers are necessary to describe nature.

Loosely speaking, **vectors** are elements of a **vector space**. A vector space is a structure that is built on fields. A field, on the other hand, is a structure built on groups. Finally, a group is a structure built on sets which is a familiar concept. Vectors will be in great use in this course, and will be defined in a more general way than the usual primitive definition of vectors used earlier in physics courses.

1.3 Sets

Definition 1.1 — Set. A **set** is a collection of distinct elements, also called members. The set is denoted by curly braces, and the elements are separated by commas. The set is denoted by a capital letter, and the elements are denoted by lowercase letters. The set is defined by the following notation:

$$A = \{a, b, c, \dots\}.$$

Here, A is the set of elements a, b, c, and so on.

When defining sets, we will make use of the **type notation**:

$$A :: \texttt{Set},$$
$$A = \{1, 2, 3\}.$$

The first line means "A is of type `Set`", and the second line means "A is the set of 1, 2, 3".

1.3.1 Set Comprehension

A set can be directly defined by listing its elements, which is called the **roster method**. For example,

$$A = \{1, 2, 3\}.$$

If the elements of the set form a regular pattern, an ellipsis can be used to represent the pattern. For example,

$$A = \{1, 2, 3, \dots\}.$$

The drawback of the roster method is that it is not possible to list all the elements of an infinite set, and the use of ellipsis might lead to ambiguity since it is not clear whether the set is finite or infinite. Hence, we will use the **set comprehension** to define sets.

Definition 1.2 — Set Comprehension. A **set comprehension** is a method of defining a set by specifying a property that the elements of the set must satisfy. The set is defined by the following notation:

$$A = \{x \in \texttt{Domain} \mid \texttt{Condition}\},$$

where x is the element of the set A, and the `Domain` is the set of all possible values of x. The `Condition` (or sometimes the `Rule` or `Predicate`) is a statement that is either true or false for each $x \in \texttt{Domain}$. Note that a colon is sometimes used instead of the vertical bar to separate the domain and the condition.

Example 1.1 Let's define the set of all odd integers using set comprehension:

$$A = \{x \in \mathbb{Z} \mid (\exists n \in \mathbb{Z})[a = 2n + 1]\}.$$

The expression $(\exists n \in \mathbb{Z})[a = 2n + 1]$ is read as "there exists an integer n such that a is equal to $2n + 1$".

1.3.2 Quantifiers

Definition 1.3 — Universal and Existential Quantifiers. A **universal quantifier** is a symbol that denotes "for all" or "for each". The symbol is $\forall$. An **existential quantifier** is a symbol that denotes "there exists" or "for some". The symbol is $\exists$.

We can use the quantifiers to define sets using set comprehension:

$$\{f(x) \mid (\exists y \in \texttt{Domain})[y = f(x) \land \texttt{Condition}]\}.$$

The expression $(\exists y \in \texttt{Domain})[y = f(x) \land \texttt{Condition}]$ is read as "there exists a y in the domain such that y is equal to $f(x)$ and the condition is satisfied".

Example 1.2 Let's consider the following statements:

$$\begin{aligned}
&(\exists n \in \mathbb{N})[n = 2] = \texttt{True},\\
&(\forall n \in \mathbb{R})[n = n^2] = \texttt{False},\\
&(\forall n \in \mathbb{Z})(\exists x \in \mathbb{Z})[x = 2n] = \texttt{True},\\
&(\exists x \in \mathbb{Z})(\forall n \in \mathbb{Z})[x = 2n] = \texttt{False}.
\end{aligned}$$

The first statement is true because there exists a natural number n such that $n = 2$. The second statement is false because for all real numbers n, $n \neq n^2$. The third statement is true because for all integers n, there exists an integer x such that $x = 2n$. The fourth statement is false because there exists an integer x such that for all integers n, $x \neq 2n$.

As you can see from the last two examples, the order of quantifiers is important.

Definition 1.4 — Predicate. A **predicate** is a function with the codomain $\{\texttt{True}, \texttt{False}\}$:

$$\begin{aligned}
&f :: \texttt{Predicate},\\
&f = A \to \{\texttt{True}, \texttt{False}\}.
\end{aligned}$$

The codomain is also called the **truth set** or the **Boolean set**.

Let's define a predicate g that takes an integer a and returns `True` if a is odd, and `False` otherwise.

$$\begin{aligned}
&g :: \texttt{Predicate},\\
&g = a \to (\exists n \in \mathbb{Z})[a = 2n + 1].
\end{aligned}$$

Then, $g(3) = \texttt{True}$, and $g(4) = \texttt{False}$. We can define the set of all odd integers using the predicate g:

$$\{n \in \mathbb{Z} \mid g(n)\} = \{\ldots, -3, -1, 1, 3, \ldots\}.$$

Here, we are using the predicate to avoid the use of the ellipsis.

Example 1.3 Let's define the set of all multiples of 3 using a predicate:

$$\{n \in \mathbb{Z} \mid \alpha(n)\} = \{\ldots, 3, 6, 9, 12, \ldots\},$$

where

$$\alpha :: \texttt{Predicate},$$
$$\alpha = x \to (\exists n \in \mathbb{Z})[x = 3n].$$

Write about type theory.

1.4 Groups

If we impose the existence of two functions o and i on a set, we get a group.

Definition 1.5 — Group. A **group** is a set S such that

$$S :: \texttt{Set},$$
$$o :: (S, S) \to S,$$
$$i :: S \to S,$$

where o is called the **group operation**, and i is called the **inverse operation**.

The group and inverse operations have the following notation:

$$o(x, y) = x * y,$$
$$i(x) = x^{-1}.$$

where $*$ is the group operation, and x^{-1} is the inverse of x.

Several conditions must be satisfied for a set to be a group. These conditions are called the **group axioms**.

Theorem 1.1 — Group Axioms. Let S be a set with a group operation o and an inverse operation i. Then, S is a group if the following conditions are satisfied:

1. $(\exists e \in S)\ (\forall s \in S)\ e * s = s * e = s$ (the existence of an identity element e).
2. $(\forall s \in S)\ (\exists s^{-1} \in S)\ s * s^{-1} = s^{-1} * s = e$ (the existence of an inverse element).
3. $(\forall a, b, c \in S)\ (a * b) * c = a * (b * c)$ (the associativity property).

A group is denoted by the following notation:

$$\texttt{Group} = (\texttt{Set}, \texttt{GroupOperation}).$$

Integers under arithmetic addition form a group. The group operation is addition, and the inverse operation is negation:

$$\texttt{Group} = (\mathbb{Z}, +),$$
$$\texttt{GroupOperation} = (x, y) \to x + y,$$
$$\texttt{InverseOperation} = x \to -x.$$

Proof. The group axioms are satisfied:

$$(\exists e \in \mathbb{Z})(\forall s \in \mathbb{Z})\ e + s = s + e = s,$$
$$(\forall s \in \mathbb{Z})(\exists s^{-1} \in \mathbb{Z})\ s + s^{-1} = s^{-1} + s = e,$$
$$(\forall a, b, c \in \mathbb{Z})(a + b) + c = a + (b + c),$$

where $e = 0$ is the identity element, and $s^{-1} = -s$ is the inverse element. ■

Integers under arithmetic multiplication do not form a group because the inverse operation is not defined for all elements of the set. For example, the inverse of 2 is $\frac{1}{2}$, which is not an integer. Real numbers under the arithmetic multiplication, on the other hand, form a group.

February 23

2.1 Propositional Logic

$$\texttt{Boolean} = \{\texttt{True}, \texttt{False}\}$$

$$\texttt{True} :: \texttt{Boolean},$$
$$\texttt{False} :: \texttt{Boolean}$$

Quantifiers: $\forall, \exists, \exists!$
Statements: $(\forall n \in \mathbb{Z})(\exists k \in \mathbb{Z})[k = 3n] = \texttt{True} :: \texttt{Boolean}$
Predicates: Functions with codomain `Boolean`

Example 2.1

$$(\exists n \in \mathbb{Z})[0 \cdot n] = 0 = \texttt{True},$$
$$(\exists! n \in \mathbb{Z})[0 \cdot n] = 0 = \texttt{False}$$

Example 2.2

$$f :: Z \to \texttt{Boolean},$$
$$f = x \to (\exists k \in \mathbb{Z})[x = 2k].$$

It follows that $f(1) = \texttt{False}$, $f(2) = \texttt{True}$, $f(3) = \texttt{False}$.

As we stated before, we can define sets using set comprehension. For example, the set of even numbers can be defined as follows

$$S_{even} = \{x \in \mathbb{Z} \mid f(x)\}$$

2.2 Groups

2.2.1 Basic Definitions

A **group operation**, or **binary operation** or **law of composition** on a set S is a function $S \times S \to S$ that assigns to each pair $(a, b) \in S \times S$ a unique element $a \circ b \in S$, called the composition of a and b. Let's denote the function as o, and express the definition symbolically as follows:

$$o :: S \times S \to S,$$
$$o(a, b) = a \circ b.$$

Remark 2.1 The operation o is called **binary** because it takes two arguments. The symbols "$*$" and "$\bullet$" are often used to denote the group operation.

An **inverse operation** on a set S is a function $S \to S$ that assigns to each element $s \in S$ a unique element $s^{-1} \in S$, called the inverse of s. Let's denote the function as i, and express the definition symbolically as follows:

$$i :: S \to S,$$
$$i(s) = s^{-1}.$$

A **group** is a set S endowed with a group operation o and an inverse operation i that satisfies certain rules, called the **group axioms**.

Theorem 2.1 — Group Axioms. Let S be a non-empty set, endowed with a group (or *binary*) operation o. Then (S, o) is a group if and only if the following three axioms are satisfied:

1. The group operation is *associative*, that is,

$$(\forall a, b, c \in S)\ a \circ (b \circ c) = (a \circ b) \circ c.$$

2. There exists an *identity element* e (or e_S), that is,

$$(\exists e \in S)(\forall s \in S)\ e \circ s = s \circ e = s.$$

3. Every element has an *inverse*, that is,

$$(\forall s \in S)\ s \circ s^{-1} = s^{-1} \circ s = e.$$

2.2.2 Commutative Groups

A **commutative group** or **abelian group** is a group in which the group operation is commutative, that is, for all $a, b \in S$, we have $a \circ b = b \circ a$. In such a case, the "product" is often called the **addition** and is denoted by the symbol "$+$". The identity element is denoted by "0", and the inverse of s is denoted by "$-s$". The *commutative* group axioms have the following form:

$$(\forall a, b, c \in S)\ a + (b + c) = (a + b) + c,$$
$$(\forall s \in S)\ 0 + s = s + 0 = s,$$
$$(\forall s \in S)\ s + (-s) = (-s) + s = 0,$$
$$(\forall a, b \in S)\ a + b = b + a.$$

Example 2.3 Let's consider the set of classrooms:

$$\begin{aligned} S &:: \texttt{Set}, \\ S &= \{\texttt{P1}, \texttt{P2}, \texttt{P3}\}, \\ <> &:: (S, S) \to S. \end{aligned}$$

Assume that $<>$ is the operation of combining two classrooms:

$$\begin{aligned} &\texttt{P1} <> \texttt{P1} = \texttt{P3}, \\ &\texttt{P2} <> \texttt{P2} = \texttt{P2}, \\ &\texttt{P3} <> \texttt{P3} = \texttt{P1}, \\ &\texttt{P1} <> \texttt{P2} = \texttt{P2} <> \texttt{P1} = \texttt{P1}, \\ &\texttt{P1} <> \texttt{P3} = \texttt{P3} <> \texttt{P1} = \texttt{P2}, \\ &\texttt{P2} <> \texttt{P3} = \texttt{P3} <> \texttt{P2} = \texttt{P3}. \end{aligned}$$

Determine whether $(S, <>)$ is a group or not.

Solution. It follows by inspection that $(S, <>)$ satisfies the first group axiom since

$$\texttt{P1} <> (\texttt{P2} <> \texttt{P3}) = \texttt{P1} <> \texttt{P3} = \texttt{P2}$$

is equal to

$$(\texttt{P1} <> \texttt{P2}) <> \texttt{P3} = \texttt{P1} <> \texttt{P3} = \texttt{P2}.$$

It also follows by inspection that $(S, <>)$ satisfies the second group axiom, where $\texttt{P2}$ is the identity element:

$$\begin{aligned} &\texttt{P1} <> \texttt{P2} = \texttt{P2} <> \texttt{P1} = \texttt{P1}, \\ &\texttt{P3} <> \texttt{P2} = \texttt{P2} <> \texttt{P3} = \texttt{P3}, \\ &\texttt{P2} <> \texttt{P2} = \texttt{P2}. \end{aligned}$$

Finally, it follows by inspection that every element has an inverse:

$$\begin{aligned} &\texttt{P1} <> \texttt{P3} = \texttt{P3} <> \texttt{P1} = \texttt{P2}, \\ &\texttt{P2} <> \texttt{P2} = \texttt{P2} \end{aligned}$$

Hence, $(S, <>)$ satisfies the third group axiom. Therefore, $(S, <>)$ is a group. Moreover, $(S, <>)$ is a commutative group since the group operation $<>$ is commutative. ■

Some of the most important examples of groups are $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, and $(\mathbb{C}, +)$. All of them are commutative groups. $(\mathbb{Z}, \times)$ is not a group because it does not satisfy the third group axiom. $(\mathbb{R}, \times)$ is not a group because it does not satisfy the second group axiom. However, $(\mathbb{R}^+, \times)$ and $(\mathbb{R} \setminus \{0\}, \times)$ are groups.

2.3 Rings

Let's introduce another algebraic structure called a **ring**. First, we define the **multiplication** operation on a set S, denoted by the symbol "$\cdot$", as a function $S \times S \to S$ that assigns to each pair $(a, b) \in S \times S$ a unique element $a \cdot b \in S$.

Definition 2.1 A **ring** is a set S endowed with two operations, called **addition** and **multiplication** that satisfy **ring axioms**.

Theorem 2.2 — Ring Axioms. Let S be a non-empty set, endowed with two operations, *addition* and *multiplication*. Then $(S, +, \cdot)$ is a ring if and only if the following three axioms are satisfied:

1. The set $(S, +)$ is a *commutative group*.
2. The multiplication operation is *associative*, that is,
$$(\forall a, b, c \in S)\ a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$
3. The multiplication operation is *distributive* with respect to the addition operation, that is,
$$(\forall a, b, c \in S)\ a \cdot (b + c) = a \cdot b + a \cdot c, \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

Example 2.4 We have already seen that $(\mathbb{Z}, +)$ is a commutative group. Check if $(\mathbb{Z}, +, \cdot)$ is a ring.

Solution. The first axiom is satisfied. The second axiom is satisfied because multiplication is associative:
$$(\forall a, b, c \in \mathbb{Z})\ a \cdot (b \cdot c) = a \cdot b \cdot c.$$

The third axiom is satisfied because multiplication is distributive with respect to addition:
$$(\forall a, b, c \in \mathbb{Z})\ a \cdot (b + c) = a \cdot b + a \cdot c, \quad (a + b) \cdot c = a \cdot c + b \cdot c.$$

Therefore, $(\mathbb{Z}, +, \cdot)$ is a ring. ■

2.4 Division Rings

A **division ring** is a ring in which every non-zero element has a multiplicative inverse. A **skew field** is a division ring in which the multiplication operation is not necessarily commutative.

Definition 2.2 A **skew field** is a set S endowed with two operations, **addition** and **multiplication**, that satisfy **skew field axioms**.

Theorem 2.3 — Skew Field Axioms. Let S be a non-empty set, endowed with two operations, *addition* and *multiplication*. Then $(S, +, \cdot)$ is a skew field if and only if the following two axioms are satisfied:

1. The set $(S, +)$ is a *commutative group*.

2. Every non-zero element has a *multiplicative inverse*, that is,

$$(\forall s \in S \setminus \{0\})(\exists s^{-1} \in S \setminus \{0\})\ s \cdot s^{-1} = s^{-1} \cdot s = 1.$$

Consider the ring $(\mathbb{Z}, +, \cdot)$. The set $(\mathbb{Z}, +)$ is a commutative group. However, the set $(\mathbb{Z} \setminus \{0\}, \cdot)$ is not a group because not every non-zero element has a multiplicative inverse. For example, there is no integer x such that $2 \cdot x = 1$. Therefore, $(\mathbb{Z}, +, \cdot)$ is not a skew field. The ring $(\mathbb{R}, +, \cdot)$ is a skew field. The ring $(\mathbb{C}, +, \cdot)$ is also a skew field.

Example 2.5 Determine whether $(M_{2\times 2}(\mathbb{R}), +, \cdot)$ is a skew field or not.

Proof. The set $(M_{2\times 2}(\mathbb{R}), +)$ is a commutative group. Let's consider the set $(M_{2\times 2}(\mathbb{R}) \setminus \{0\}, \cdot)$. Not every non-zero element has a multiplicative inverse. For example, the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

does not have a multiplicative inverse. Therefore, $(M_{2\times 2}(\mathbb{R}), +, \cdot)$ is not a skew field. ∎

A division ring that is multiplicative commutative is called a **field**. The ring $(\mathbb{R}, +, \cdot)$ is a field. The ring $(\mathbb{C}, +, \cdot)$ is also a field.

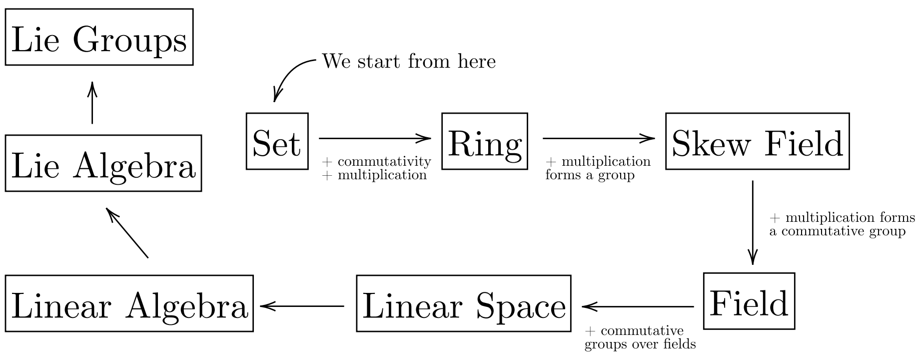


Figure 2.1: A diagram of the relationships between different algebraic structures.

March 1

$$
\begin{aligned}
S &:: \texttt{Set}, \\
f &:: (S, S) \to S \\
(S, f) &:: \texttt{Group (assuming some condition)} \\
g &:: (S, S) \to S, \\
(S, f, g) &:: Ring, SkewField, Field(dependingonpropertiesoffandg), \\
(R, +, \cdot) &:: Field
\end{aligned}
$$

We can create sets of *functions*. For example,

$$\{x \to 2x, y \to y^2 + 1\}$$

is a set of two functions.

Let's consider a function p:

$$
\begin{aligned}
p &:: \mathbb{R} \to (\mathbb{R} \to \mathbb{R}), \\
p &= a \to (x \to (a + x))
\end{aligned}
$$

For instance,

$$
\begin{aligned}
p(1) &:: \mathbb{R} \to \mathbb{R}, \\
p(1) &= x \to (1 + x), \\
p(1)(2) &:: \mathbb{R}, \\
p(1)(2) &= 1 + 2 = 3.
\end{aligned}
$$

In a more general form,

$$
\begin{aligned}
p(a) &= x \to (a + x), \\
p(a)(b) &= a + b.
\end{aligned}
$$

Let's consider the following set:

$$
\begin{aligned}
A &= \{p(a) \mid a \in \mathbb{R}\}, \\
A &= \{x \to x + 1, x \to x + \sqrt{2}, x \to x + \pi, \ldots\}.
\end{aligned}
$$

Let's consider the following function:

$$\oplus :: (\mathbb{R} \to \mathbb{R}, \mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R}),$$

Consider the following:

$$(\mathbb{Z}, +) :: \texttt{Group},$$
$$+ :: (\mathbb{Z}, \mathbb{Z}) \to \mathbb{Z}.$$

Then:

$$\oplus :: (\mathbb{R} \to \mathbb{R}, \mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R}),$$
$$p(a) \oplus p(b) = p(a+b).$$

It follows that:

$$(A, \oplus) :: \texttt{Commutative group}$$

Consider the following:

$$(x \to x+3) \oplus (x \to x - \pi) = (x \to x + 3 - \pi).$$

The identity element is $p(0)$:

$$p(0) \oplus p(a) = p(a) \oplus p(0) = p(a).$$

The inverse element is simply $p(-a)$:

$$p(a) \oplus p(-a) = p(-a) \oplus p(a) = p(0).$$

The inverse of the function $x \to x + 2$ is $x \to x - 2$. Let's turn this into a ring:

$$\otimes :: (\mathbb{R} \to \mathbb{R}, \mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R}),$$

If we need $(A, \oplus, \otimes)$ to be a ring:

$$p(a) \otimes (p(b) \oplus p(c)) = (p(a) \otimes p(b)) \oplus (p(a) \otimes p(c)),$$
$$(p(a) \oplus p(b)) \otimes p(c) = (p(a) \otimes p(c)) \oplus (p(b) \otimes p(c)).$$

Consider the following:

$$p(a) \oplus p(b) = p(a+b),$$
$$p(a) \otimes p(b) = p(a \cdot b).$$

Then the whole things becomes a field:

$$(A, \oplus, \otimes) :: \texttt{Field},$$

where each element is a function.

$$(A, \oplus) :: \texttt{Commutative group},$$
$$(A \setminus \{p(0)\}, \otimes) :: \texttt{Commutative group}.$$

$$(x \to x+3) \oplus (x \to x+5) = (x \to x+8),$$
$$(x \to x+3) \otimes (x \to x+5) = (x \to x+15).$$

3.1 Linear Spaces

A linear, or *vector* space over a field F is a triplet $(V, \oplus, \odot)$, where:

$$\begin{aligned} V &:: \texttt{Set}, \\ \oplus &:: (V, V) \to V, \\ \odot &:: (F, V) \to V, \\ (F = (S, +, \odot)) &:: \texttt{Field}. \end{aligned}$$

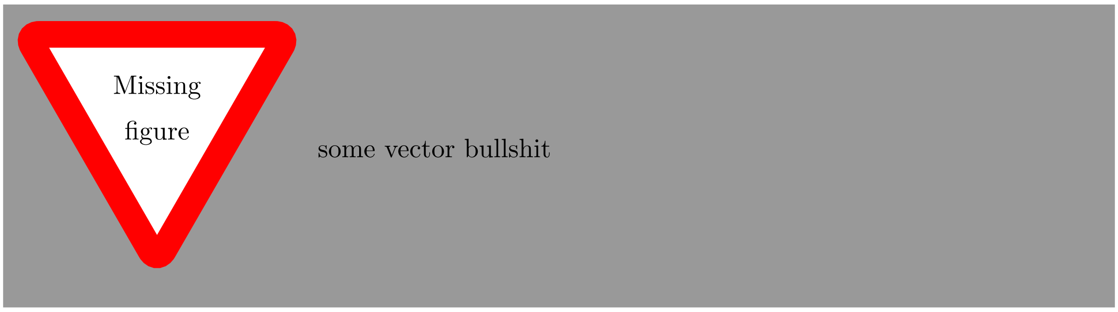
The following conditions must be satisfied:

1. $(V, \oplus) :: \texttt{Commutative group}$.
2. $(\forall v \in V) \quad 1\, v = v$.
3. $(\forall v \in V)(\forall s \in S)\, s \odot v \in V$.
4. $(\forall v \in V)(\forall a, b \in S)\, (a \cdot b) \odot v = a \odot (b \odot v)$.
5. $(\forall v \in V)(\forall a, b \in S)\, (a + b) \odot v = (a \odot v) \oplus (b \odot v)$.
6. $(\forall v, w \in V)(\forall s \in S)\, s \odot (v \oplus w) = (s \odot v) \oplus (s \odot w)$.

Example 3.1 Let's take $F = (\mathbb{R}, +, \cdot)$, and $V = \{(a, b) \mid a, b \in \mathbb{R}\}$, and:

$$\begin{aligned} (a, b) \oplus (c, d) &= (a + c, b + d), \\ s \odot (a, b) &= (s \cdot a, s \cdot b). \end{aligned}$$

which are a definition. This set with these operations is a linear space over $\mathbb{R}$: $(V, \oplus, \odot)$.



$$s \odot (a, b) = (s \cdot a, s \cdot b),$$

Let's go through the conditions:

1. The identity element is $e = (0, 0)$, and inverse of (a, b) is $(-a, -b)$.
2. $1 \odot (a, b) = (1 \cdot a, 1 \cdot b) = (a, b)$.
3. $s \in \mathbb{R}, (a, b) \in V, s \odot (a, b) = (s \cdot a, s \cdot b) \in V$.
4. If you multiply two scalars and then multiply the vector, it's the same as multiplying multiplying the first scalar and then multiplying the vector by the second scalar.

Example 3.2 Consider the triplet $(V, \Box, \odot)$. We will try to write that as a vector space over the field F, where $F = (A, \oplus, \otimes)$, where $A = \{p(a) \mid a \in \mathbb{R}\}$, and:

$$\begin{aligned} p &:: \mathbb{R} \to (\mathbb{R} \to \mathbb{R}), \\ p(a) &= x \to (x \to x + a). \end{aligned}$$

We define the field operations as:

$$\begin{aligned} p(a) \oplus p(b) &= p(a + b), \\ p(a) \otimes p(b) &= p(a \cdot b). \end{aligned}$$

We will be using four different symbols:

$$\begin{aligned} \Box &\text{is the vector addition,} \\ \odot &\text{is the scalar multiplication,} \\ \oplus &\text{is the field addition,} \\ \otimes &\text{is the field multiplication.} \end{aligned}$$

Consider the folllowing:

$$\begin{aligned} V &= \{(a, b, c) \mid a, b, c \in \mathbb{R}\} \\ (a, b, c)\Box(d, e, f) &= (a + d, b + e, c + f) \\ p(a) \odot (b, c, d) &= (a \cdot b, a \cdot c, a \cdot d) \end{aligned}$$

3.2 Linear Algebra

A linear algebra, or *vector* algebra over a field F is a quadruplet $(V, \oplus, \odot, \otimes)$, where:

$$\begin{aligned} V &:: \texttt{Set} \\ \oplus &:: (V, V) \to V \\ \odot &:: (F, V) \to V \\ \otimes &:: (V, V) \to V \end{aligned}$$

The following conditions must be satisfied:

1. $(V, \oplus, \otimes) :: \texttt{Ring}$
2. $(V, \oplus, \odot) :: \texttt{Linear space}$
3. $(\forall s \in S)(\forall v, w \in V)\, s \odot (v \otimes w) = (s \odot v) \otimes w$

Example 3.3 If $F = (\mathbb{R}, +, \cdot)$, and $V = \{(a, b, c) \mid a, b, c \in \mathbb{R}\}$:

$$(a, b, c) \otimes (d, e, f) = (bf - ec, -af + cd, ae - bd), \quad \text{cross product}$$

$$(v_1 \otimes v_2)_i = \sum_{j,k} \epsilon_{ijk} v_{1j} v_{2k}$$

check whether the cross product has been written correctly

3.3 Lie Algebra

A Lie algebra is a linear algebra where $\otimes$ is anti-symmetric: $v_1 \otimes v_2 = -v_2 \otimes v_1$:

$$v_1 \otimes v_2 = v_3$$
$$\otimes :: (V, V) \to V$$

$[v_1, v_2] = v_3$ is called the *commutator* of v_1 and v_2:

$$[v_1, v_2] = v_1 \otimes v_2 - v_2 \otimes v_1$$

If you take a commutator like $[v_1, [v_2, v_3]] + [v_2, [v_3, v_1]] + [v_3, [v_1, v_2]] = 0$,, this is called the *Jacobi identity*. Lie algebra: $[v_1, v_2] = v_3$. If we would expand this:

$$[e_i, e_j] = \sum_k c_{ij}^k e_k, \quad i, j, k = 1, \quad N$$

the c_{ij}^k are called the *structure constants* of the Lie algebra. What are the structure constants of the cross product?

$$[\imath, \jmath] = \kappa, \quad \imath = e_1, \quad \jmath = e_2, \quad \kappa = e_3$$
$$[\imath, \kappa] = -\jmath, \quad c_{12}^3 = 1, c_{21}^3 = -1, c_{12}^1 = c_{21}^1 = c_{12}^2 = c_{21}^2 = 0$$
$$[e_1, e_2] = c_{12}^1 e_1 + c_{12}^2 e_2 + c_{12}^3 e_3$$

March 4

Suppose F is a field:

$$(F = (S, +, \cdot)) :: \texttt{Field}$$

Consider a linear space:

$$\begin{aligned}
&S :: \texttt{Set (scalars)},\\
&V :: \texttt{Set (vectors)},\\
&\oplus :: (V, V) \to V\,\texttt{(vector addition)},\\
&\odot :: (S, V) \to V\,\texttt{(scalar multiplication of a vector)}
\end{aligned}$$

Then, we can consider the following:

$$\begin{aligned}
(V, \oplus, \odot) :: &\texttt{ Vector space over the field } F\\
&\texttt{ Linear space over the field } F
\end{aligned}$$

4.1 Basis of Vector Spaces

Let $B \subset V :: \texttt{Set}$. B is called a basis of V if the following conditions hold:

1. $(\forall k \in \{1, 2, \ldots, |B|\})(\forall e_1, e_2, \ldots, e_k \in B)(\forall c_1, c_2, \ldots, c_k \in S)[c_1 = c_2 = \ldots = c_k = 0] \vee [(c_1 \odot e_1) \oplus (c_2 \odot e_2) \oplus \ldots \oplus (c_k \odot e_k) \neq 0]$

2. $(\forall v \in V)(\exists! a_1, a_2, \ldots, a_{|B|} \in S) v = a_1 e_1 + a_2 e_2 + \ldots + a_{|B|} e_{|B|}$
   $v = (a_1 \odot e_1) \oplus (a_2 \odot e_2) \oplus \ldots \oplus (a_{|B|} \odot e_{|B|})$
   $|B| < \infty \Rightarrow |B| = \dim B = \dim V$
   $|B| = \infty \Rightarrow \dim V = \infty$

Suppose we have a basis:

$$B = \{i, j, k\} \quad v = 2i + 3j$$

$$v = \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} i \\ j \\ k \end{bmatrix}$$

Hence, we can say

$$(2, 0, 1) :: \texttt{Scalars}$$

$$(2i + k) :: \texttt{Vector}$$

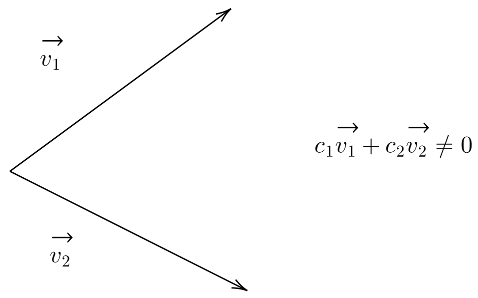


Figure 4.1: S

Remark 4.1 In physics, we always deal with normed vector spaces.

Suppose that $B = \{e_1, e_2, \ldots, e_n\}$ is a basis, where $v \in V, a \in S$:

$$v = (a_1 \odot e_1) \oplus (a_2 \odot e_2) \oplus \ldots \oplus (a_n \odot e_n)$$

It can be written as:

$$v = a_1 e_1 + a_2 e_2 + \ldots + a_n e_n$$

or

$$v = \sum_{i=1}^{n} a_i e_i$$

There is a concept called the *Einstein summation convention*:

$$v = a^i e_i$$

Here, i is called a *reapeated index*.

Example 4.1

$$\begin{aligned} i &= 1, 2, \\ k &= 1, 2, 3, 4 \\ x &= a_i b_k c^{ik} \end{aligned}$$

In Einstein summation convention, we can write:

$$x = a_1 b_1 c^{11} + a_1 b_2 c^{12} + a_1 b_3 c^{13} + a_1 b_4 c^{14} + a_2 b_1 c^{21} + a_2 b_2 c^{22} + a_2 b_3 c^{23} + a_2 b_4 c^{24}$$

Example 4.2

$$a = \begin{bmatrix} 1 & 0 & x \end{bmatrix}$$
$$b = \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}$$
$$a^i b_i = a = \begin{bmatrix} 1 & 0 & x \end{bmatrix} \times \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix} = 2 + x$$

$$a = \{a^1, a^2, \ldots, a^n\}$$
$$e = \{e_1, e_2, \ldots, e_n\}$$

Hence, v is:

$$v = a^i e_i = \begin{bmatrix} a^1 & a^2 & \ldots & a^n \end{bmatrix} \times \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}$$

Consider:

$$S = \{\texttt{plus}(a) \mid a \in \mathbb{R}\},$$
$$\texttt{plus} = a \to (x \to (a + x))$$

$$V = \{a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} \mid a, b, c \in \mathbb{R}\}$$
$$v_1 = \frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}$$
$$v_2 = -2\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$
$$v_1 + v_2 = -\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}$$

4.2 Normed Vector Space

Consider an arbitrary vector space V over a field F. A norm on V is a function:

$$\texttt{norm} :: V \to \mathbb{R},$$
$$\texttt{norm} = v \to ||v||$$

A vector space V qualified with a norm is called a normed vector space if the following conditions hold:

1. $(\forall v \in V) \quad (||v|| \neq 0) \vee (v = 0)$
2. $(\forall v \in V)(\forall s \in S) \quad ||s \odot v|| = |s| \cdot ||v||$
3. $(\forall v, w \in V) \quad ||v \oplus w|| \leq ||v|| + ||w||$, which is called the *triangle inequality*

Example 4.3 Consider the following:

$$V = \{ai + bj \mid a, b \in \mathbb{R}\},$$
$$|| \cdot || = v \to (||v|| = a^2 + 2b^2)$$

Hence,

$$||i + 2j|| = 9$$

If in addition, we have $(|| \cdot || :: V \to \mathbb{R})$, normed vector space V.

$$||0 \odot v|| = |0| \cdot ||v|| = 0 \texttt{by the second condition},$$
$$||0|| = 0,$$
$$||v \oplus (-v)|| \leq ||v|| + ||(-v)||,$$
$$||0|| \leq ||v|| + |-1| \cdot ||v|| \Rightarrow \frac{1}{2}||0|| \leq ||v|| \Rightarrow 0 \leq ||v||$$

Now, we can define angles between vectors:

$$\texttt{angle} :: (V, V) \to \mathbb{R},$$
$$\texttt{angle} = (v, w) \to \arccos \frac{||v||^2 + ||w||^2 - ||v - w||^2}{2||v|| \cdot ||w||}$$

Distance between functions:

$$\texttt{distance} :: (V, V) \to \mathbb{R},$$
$$\texttt{distance} = (v, w) \to ||v - w||$$

4.3 Inner Product Vector Space

$$< \cdot, \cdot > :: (V, V) \to F, \quad \texttt{for a vector space over a field } F = \mathbb{R}, \mathbb{C}$$

if the following are true:

1. $(\forall v_1, v_2 \in V) < v_1, v_2 >= (< v_2, v_1 >)^*$ *conjugate symmetry*
2. $(\forall v_1, v_2, v_3 \in V)(\forall V a_1, a_2 \in S) < a_1 v_1 + a_2 v_2, v_3 >= a_1 < v_1, v_3 > + a_2 < v_2, v_3 >$ *linearity in the first argument*
3. $[(\forall v \in V \setminus 0) < v, v >> 0] \wedge [< 0, 0 >= 0]$

Some bs about introducing stuff

$$\texttt{angle} = (v, w) \to \arccos \frac{||v||^2 + ||w||^2 - ||v - w||^2}{2||v|| \cdot ||w||}$$

The angle becomes:

$$\texttt{angle} = (v, w) \to \arccos \frac{< v, v > + < w, w > - < v - w, v - w >}{2\sqrt{< v, v >}\sqrt{< w, w >}}$$

$$\begin{aligned}
< v - w, v - w > &=< v, v - w > - < w, v - w > \\
&= (< v - w, v >)^* - (< v - w, w >)^* \\
&= (< v, v > - < w, v > - < v, w > + < w, w >)^* \\
&=< v, v > + < w, w > - < v, w > - < v, w >^*
\end{aligned}$$

Then, we can write:

$$\texttt{angle} = (v, w) \to \arccos \frac{\Re(< v, w >)}{2\sqrt{< v, v >}\sqrt{< w, w >}}$$