Mathematical Methods in Physics I
Homework 6

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November 17, 2023

1 Problem One

1.1

Solution. To reduce the order of the given differential equation, let's take $f(x) = g(x)f_1(x)$:

$$f(x) = \frac{g(x)}{x^k} \tag{1.1}$$

$$f'(x) = \frac{g'(x)}{x^k} - \frac{kg(x)}{x^{k+1}} \tag{1.2}$$

$$f''(x) = \frac{g''(x)}{x^k} - \frac{2kg'(x)}{x^{k+1}} + \frac{k(k+1)g(x)}{x^{k+2}} \tag{1.3}$$

$$f'''(x) = \frac{g'''(x)}{x^k} - \frac{kg''(x)}{x^{k+1}} - \frac{2kg''(x)}{x^{k+1}} + \frac{3k(k+1)g'(x)}{x^{k+2}} - \frac{k(k+1)(k+2)g(x)}{x^{k+3}} \tag{1.4}$$

If we substitute these into the equation, we obtain:

$$x^2\left(\frac{g'''(x)}{x^k} - \frac{kg''(x)}{x^{k+1}} - \frac{2kg''(x)}{x^{k+1}} + \frac{3k(k+1)g'(x)}{x^{k+2}} - \frac{k(k+1)(k+2)g(x)}{x^{k+3}}\right) + \tag{1.5}$$

$$+\frac{(4+x)x}{2}\left(\frac{g''(x)}{x^k} - \frac{2kg'(x)}{x^{k+1}} + \frac{k(k+1)g(x)}{x^{k+2}}\right) - \tag{1.6}$$

$$-\frac{(4-x)}{2}\left(\frac{g'(x)}{x^k} - \frac{kg(x)}{x^{k+1}}\right) - \frac{g(x)}{2x^k} = 0 \tag{1.7}$$

If we collect the terms with $g(x)$, we obtain:

$$-\frac{k(k+1)(k+2)}{x^{k+1}} + \frac{(4+x)}{2}\frac{k(k+1)}{x^{k+1}} + \frac{(4-x)}{2}\frac{k}{x^{k+1}} - \frac{1}{2x^k} = 0 \tag{1.8}$$

If we simplify the expression, we obtain:

$$2k^3 + (2-x)k^2 - 4k + x = 0 \tag{1.9}$$

Or:

$$(k-1)(2k^2 + 4k - xk - x) = 0 \tag{1.10}$$

So $k_1 = 1$. The other two solutions are found as follows:

$$2k^2 + (4-x)k - x = 0 \tag{1.11}$$
$$D = (4-x)^2 - 4(2)(-x) \tag{1.12}$$
$$D = x^2 + 16 \tag{1.13}$$
$$k_{2,3} = \frac{x}{4} - 1 \pm \frac{\sqrt{x^2+16}}{4} \tag{1.14}$$

1.2

Solution. If we substitute $k = 1$ into the equation, we obtain:

$$x^2 \left(\frac{g'''(x)}{x} - \frac{(1)g''(x)}{x^2} - \frac{2(1)g''(x)}{x^2} + \frac{3(1)(2)g'(x)}{x^3} - \frac{1(2)(3)g(x)}{x^4} \right) + \tag{1.15}$$
$$+ \frac{(4+x)x}{2} \left(\frac{g''(x)}{x} - \frac{2(1)g'(x)}{x^2} + \frac{1(2)g(x)}{x^3} \right) - \tag{1.16}$$
$$- \frac{(4-x)}{2} \left(\frac{g'(x)}{x} - \frac{(1)g(x)}{x^2} \right) - \frac{g(x)}{2x} = 0 \tag{1.17}$$

If we simplify the expression, we obtain:

$$2xg'''(x) + (x-2)g''(x) - g'(x) = 0 \tag{1.18}$$

1.3

Solution. If we insert $h(x) = g'(x)$ into the last equation, we obtain:

$$2xh''(x) + (x-2)h'(x) - h(x) = 0 \tag{1.19}$$

Let's expand the given expression:

$$\alpha(x)h'(x) + h(x) + \beta(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[\alpha(x)h'(x) + h(x)\right] = 0 \tag{1.20}$$
$$\alpha(x)h'(x) + h(x) + \beta(x)\left[\alpha'(x)h'(x) + \alpha(x)h''(x) + h'(x)\right] = 0 \tag{1.21}$$
$$\alpha(x)h'(x) + h(x) + \beta(x)\alpha'(x)h'(x) + \beta(x)\alpha(x)h''(x) + \beta(x)h'(x) = 0 \tag{1.22}$$
$$\beta(x)\alpha(x)h''(x) + (\alpha(x) + \alpha'(x)\beta(x) + \beta(x))h'(x) + h(x) = 0 \tag{1.23}$$
$$\tag{1.24}$$

If we multiply the expression by -1, we obtain:

$$-\beta(x)\alpha(x)h''(x) - (\alpha(x) + \alpha'(x)\beta(x) + \beta(x))h'(x) - h(x) = 0 \tag{1.25}$$
$$\tag{1.26}$$

It follows that:

$$-\beta(x)\alpha(x) = 2x \tag{1.27}$$
$$\tag{1.28}$$

And:

$$\alpha(x) + \alpha'(x)\beta(x) + \beta(x) = 2 - x \tag{1.29}$$

It follows, from inspection, that $\alpha(x) = 2$ and $\beta(x) = -x$.

1.4

Solution. Let's solve the given differential equation:

$$2h'(x) + h(x) = 0 \tag{1.30}$$

The characteristic equation is:

$$2r + 1 = 0 \tag{1.31}$$

$$r = -\frac{1}{2} \tag{1.32}$$

So the general solution is:

$$h(x) = a_1 \mathrm{e}^{-\frac{x}{2}} \tag{1.33}$$

1.5

Solution. Assume that $h(x) = h_1(x)j(x) = a_1\mathrm{e}^{-\frac{x}{2}}j(x)$. Then:

$$h'(x) = a_1\mathrm{e}^{-\frac{x}{2}}j'(x) - \frac{a_1}{2}\mathrm{e}^{-\frac{x}{2}}j(x) \tag{1.34}$$

$$h''(x) = a_1\mathrm{e}^{-\frac{x}{2}}j''(x) - \frac{a_1}{2}\mathrm{e}^{-\frac{x}{2}}j'(x) - \frac{a_1}{2}\mathrm{e}^{-\frac{x}{2}}j'(x) + \frac{a_1}{4}\mathrm{e}^{-\frac{x}{2}}j(x) \tag{1.35}$$

If we substitute these into the differential equation, we obtain:

$$\begin{aligned} &2x\left(a_1\mathrm{e}^{-\frac{x}{2}}j''(x) - a_1\mathrm{e}^{-\frac{x}{2}}j'(x) + \frac{a_1}{4}\mathrm{e}^{-\frac{x}{2}}j(x)\right) + \\ &+ (x-2)\left(a_1\mathrm{e}^{-\frac{x}{2}}j'(x) - \frac{a_1}{2}\mathrm{e}^{-\frac{x}{2}}j(x)\right) - a_1\mathrm{e}^{-\frac{x}{2}}j(x) = 0 \end{aligned} \tag{1.36}$$

If we simplify this expression, we obtain:

$$j''(x) - \frac{1}{x}j'(x) - \frac{1}{2}j'(x) = 0 \tag{1.37}$$

Assume that $r(x) = j'(x)$:

$$r'(x) - \frac{1}{x}r(x) - \frac{1}{2}r(x) = 0 \tag{1.38}$$

$$r'(x) - r(x)\left(\frac{1}{x} + \frac{1}{2}\right) = 0 \tag{1.39}$$

$$\frac{r'(x)}{r(x)} = \frac{1}{x} + \frac{1}{2} \tag{1.40}$$

If we integrate both sides, we obtain:

$$\ln r(x) = \ln x + \frac{x}{2} + a_2 \tag{1.41}$$

The general solution is:

$$r(x) = a_3 x \mathrm{e}^{\frac{x}{2}} \tag{1.42}$$

So $j(x)$ is:

$$j(x) = \int a_3 x \mathrm{e}^{\frac{x}{2}} \,\mathrm{d}x = a_3 \left(\mathrm{e}^{\frac{x}{2}} (2x - 4)\right) + a_4 \tag{1.43}$$

If we substitue $j(x)$ into $h(x)$, we obtain:

$$h(x) = a_1 \mathrm{e}^{-\frac{x}{2}} \left(a_3 \left(\mathrm{e}^{\frac{x}{2}} (2x - 4)\right) + a_4\right) \tag{1.44}$$

$$= 2a_1 a_3 (x - 2) + a_1 a_4 \mathrm{e}^{-\frac{x}{2}} \tag{1.45}$$

Let $2a_1a_3 = c_1$ and $a_4 = c_2$. Then:

$$h(x) = c_1(x - 2) + c_2 h_1(x) \tag{1.46}$$

So $a = -2$.