Phys210: Mathematical Methods in Physics II

Homework 2

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Policies

- Please adhere to the *academic integrity* rules: see my explanations here for further details!
- For the overall grading scheme or any other course-related details, see the syllabus.
- Non-graded question(s) (if any) are for your own practice!
- Unless stated otherwise, you are expected to show your derivation of the results.
- The homework is due March 22$^{\text{th}}$ 2024, 23:59 TSI.

(1) Problem One *(2.5 points)*

We have seen in class that not all vector spaces have a built-in function that converts a vector into a scalar; in such vector spaces, it is impossible to define the angle between vectors or even the magnitude of the vector itself. We have also seen that there exists special kind of vector spaces, called *normed vector spaces*: in such systems, we have a function called *norm* (denoted as $\|\cdot\|$) which satisfies the following:

$$F :: \{\mathbb{R}, \mathbb{C}\} \tag{1.1a}$$

$$V :: \texttt{Vector Space over } F \tag{1.1b}$$

$$\|\cdot\| :: V \to \mathbb{R} \tag{1.1c}$$

$$(\forall v \in V)[\|v\| \neq 0] \vee [v = 0] \tag{1.1d}$$

$$(\forall v \in V)(\forall s \in F)\|s \odot v\| = |s|.\|v\| \tag{1.1e}$$

$$(\forall v, w \in V)\|v \oplus w\| \leq \|v\| + \|w\| \tag{1.1f}$$

We then call $(V, \|\cdot\|)$ a *normed vector space*.

Consider the following functions:

$$\alpha, \beta, \gamma, \lambda, \rho :: V \to \mathbb{R} \tag{1.2a}$$

$$\alpha = a^i e_i \to \left(\sum_{j=1}^{n} a_j^2 \right)^{1/2} \tag{1.2b}$$

$$\beta = a^i e_i \to \left(\sum_{j=1}^{n} a_j^2 \right)^{1/4} \tag{1.2c}$$

$$\gamma = a^i e_i \to \left(\sum_{j=1}^{n} a_j^2 \right) \tag{1.2d}$$

$$\lambda = a^i e_i \to \left(\sum_{j=1}^{n} a_j \right) \tag{1.2e}$$

$$\rho = a^i e_i \to \left(\sum_{j=1}^{\lfloor n/2 \rfloor} a_j - \sum_{j=1+\lfloor n/2 \rfloor}^{n} a_j \right) \tag{1.2f}$$

where e_i are basis vectors of the n-dimensional vector space V and where we are using Einstein's summation convention (i.e. repeated indices are summed over). Also, remember that $\lfloor x \rfloor$ is the value of the *floor* function evaluated at x. Assuming that V is a vector field over real numbers $\mathbb{R}$, determine the subset of $\left\{(V, \alpha), (V, \beta), (V, \gamma), (V, \lambda), (V, \rho)\right\}$ that are normed vector spaces.

(2) Problem Two *(3.5 points)*

We discussed in class that the so-called inner product vector spaces are vector spaces with the additional operation $\langle ., . \rangle$ (called inner product) such that

$$\begin{aligned} F \in& \{\mathbb{R}, \mathbb{C}\} \\ V ::& \texttt{Vector Space over } F \\ \langle ., . \rangle ::& (V, V) \to F \end{aligned} \tag{2.1}$$

By definition, it has to satisfy the following properties:

- $(\forall v, w \in V)\ \langle v, w \rangle = \langle w, v \rangle^*$ (conjugate symmetry)
- $(\forall u, v, w \in V)(\forall a, b \in F)\ \langle au + bv, w \rangle = a \langle u, w \rangle + b \langle v, w \rangle$
- $(\forall v \in V \backslash \{0\})\ \langle v, v \rangle > 0$
- $\langle 0, 0 \rangle = 0$

Consider a two dimensional vector space over complex numbers (denote $\mathcal{V}$), and take $e_{1,2}$ as the basis elements. Consider the inner product defined as

$$\left\langle a^i e_i, b^j e_j \right\rangle = a^i (b^j)^* \delta_{i,j} \tag{2.2}$$

where δ is the Kronecker-delta function, and a^* denotes the complex conjugation of a.

1. Show that this definition of the inner product satisfies all necessary conditions.
2. Determine the value k for which $(\mathcal{V}, f)$ would be a normed vector space with $f = v \to \langle v, v \rangle^k$.
3. If we defined $\left\langle a^i e_i, b^j e_j \right\rangle = b^i (a^j)^* \delta_{i,j}$, this would not be an inner-product vector space. Why?
4. Define *angle* between two vectors as follows:

$$\begin{aligned} \mathsf{angle} ::& (\mathcal{V}, \mathcal{V}) \to \mathcal{V} \\ \mathsf{angle} =& (v, w) \to \arccos\left(\frac{\mathrm{Re}(\langle v, w \rangle)}{\sqrt{\langle v, v \rangle \langle w, w \rangle}} \right) \end{aligned} \tag{2.3}$$

   Calculate $\mathsf{angle}(e_1 + ie_2, e_1 - ie_2)$ and $\mathsf{angle}(e_1 + ie_2, e_1 - ie_2)$.

5. Observe that a two dimensional real vector can be parameterized as $v = r\cos(\theta)e_1 + r\sin(\theta)e_2$ for $(r, \theta) :: (\mathbb{R}^+, [0, 2\pi))$. With a little bit computation, we can easily show that

$$\mathsf{angle}(r_1\cos(\theta_1)e_1 + r_1\sin(\theta_1)e_2\ ,\ r_2\cos(\theta_2)e_1 + r_2\sin(\theta_2)e_2) = \theta_1 - \theta_2 \tag{2.4}$$

The generalization of such a parametrization of a vector to vector spaces over *complex* field is

$$c = re^{i\alpha}\cos(\theta)e_1 + re^{i\beta}\sin(\theta)e_2 \tag{2.5}$$

for $(r, \theta, \alpha, \beta) :: (\mathbb{R}^+, [0, 2\pi), [0, 2\pi), [0, 2\pi))$. *Show that the angle between two vectors over complex numbers is given as*

$$\begin{aligned}&\mathsf{angle}\Big(r_1 e^{i\alpha_1}\cos(\theta_1)e_1 + r_1 e^{i\beta_1}\sin(\theta_1)e_2\ ,\ r_2 e^{i\alpha_2}\cos(\theta_2)e_1 + r_2 e^{i\beta_2}\sin(\theta_2)e_2\Big)\\ &= \arccos\Big(\cos(\theta_1)\cos(\theta_2)\cos(\alpha_1 - \alpha_2) + \sin(\theta_1)\sin(\theta_2)\cos(\beta_1 - \beta_2)\Big)\end{aligned} \tag{2.6}$$

(3) Problem Three *(not graded)*

Mathematica can be utilized to check the claims in the previous question. For instance, following code computes the angle given in (2.6):

```
With[{
        v=a{E^(I b) Cos[d],E^(I c) Sin[d]},
        w=e{E^(I f) Cos[h],E^(I g) Sin[h]},
        constraints=And[Element[a|e,PositiveReals],And@@(Function[0<=#<2
            Pi]/@{b,c,d,f,g,h})]
},
FullSimplify[ArcCos[Re[v.Conjugate[w]]/Sqrt[v.Conjugate[v] w.Conjugate[w
    ]]]],constraints]
]
```

(4) Problem Four *(2 points)*

We have defined tensor and exterior products in the class, i.e.

$$\begin{aligned}&V :: \text{Vector Space}\\ &\otimes :: (V, V) \to V \otimes V\\ &\wedge :: (V, V) \to V \wedge V\end{aligned} \tag{4.1}$$

where these *bilinear* and *associative* operations generate new objects out of the elements of the vector space V. Consider the spaces

$$\begin{aligned} T^2 V &= V \otimes V \\ \Lambda^3 V &= V \wedge V \wedge V \end{aligned} \tag{4.2}$$

Are $T^2 V$ and $\Lambda^3 V$ vector spaces? If so, show that they satisfy the necessary conditions! If not, show how they fail to satisfy the conditions. If they are vector spaces, what would be their basis vectors if the basis vectors of V is the set $\{e_1, \ldots, e_n\}$?