Phys209: Mathematical Methods in Physics I

Homework 3

Soner Albayrak[†]

[†]*Middle East Technical University, Ankara 06800, Turkey*

Policies

- Please adhere to the *academic integrity* rules: see my explanations here for further details!
- For the overall grading scheme or any other course-related details, see the syllabus.
- Non-graded question(s) are for your own practice!
- The homework is due October 27th 2023, 23:59 TSI.

We have seen in class that *a linear ordinary differential equation with constant coefficients* for the unknown function f can be solved by the following procedure:

1. Construct the characteristic equation and find its roots
2. For any root r_i with multiplicity $1+\alpha_i$, the corresponding solution is $\left(\sum_{k=0}^{\alpha_i} c_{ik}x^k\right)e^{r_i x}$
3. The sum of these solutions are called *the homogeneous solution*: $f_h(x)=\sum_i\left[\left(\sum_{k=0}^{\alpha_i} c_{ik}x^k\right)e^{r_i x}\right]$
4. Find the impulse response $i(x)$. To do this, consider the given differential equation with its non-homogeneous piece replaced by *Dirac-delta generalized function* $\delta(x)$, $\left(a_n\frac{\mathrm{d}^n}{\mathrm{d}x^n}+\cdots+a_1\frac{\mathrm{d}}{\mathrm{d}x}+a_0\right)i(x)=\delta(x)$, and then take its Laplace transform.
5. Use the conditions $i(0)=i'(0)=\cdots=i^{(k)}(0)=0$. We then end up with $I(s)=\dfrac{1}{a_ns^n+\ldots a_1s+a_0}$. Rewrite this in the form $I(s)=\dfrac{b_{11}}{s-r_1}+\dfrac{b_{12}}{(s-r_1)^2}+\cdots+\dfrac{b_{21}}{s-r_2}+\ldots$, which indicates $i(x)=b_{11}e^{r_1x}+b_{12}xe^{r_1x}+\cdots+b_{21}e^{r_2x}+\ldots$.
6. Take the convolution of the impulse response with the nonhomogeneous piece $h(x)$: this gives the particular solution, i.e. $f_p(x)=\int_0^\infty h(x-y)i(y)dy$.
7. The full solution is the summation of the homogeneous and the particular pieces:

$$\left(a_n\frac{\mathrm{d}^n}{\mathrm{d}x^n}+\cdots+a_1\frac{\mathrm{d}}{\mathrm{d}x}+a_0\right)f(x)=h(x)$$

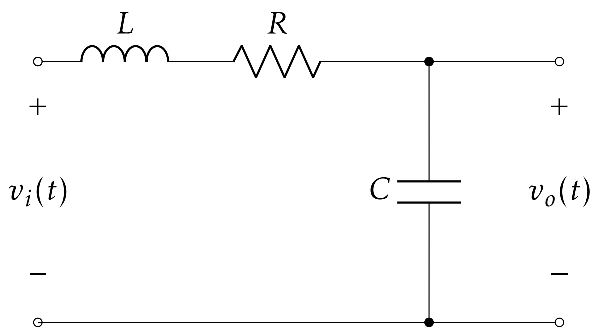
$$\Downarrow$$

$$f(x)=\sum_i\left[\left(\sum_{k=0}^{\alpha_i} c_{ik}x^k\right)e^{r_i x}\right]+\int_0^\infty h(x-y)i(y)dy \quad (0.1)$$

8. The variables c_{ik} are undetermined as it is: one can go ahead and fix them if $\{f(0), f'(0), \dots\}$ (or something similar) is provided.

Let us try to put this procedure into practice in this homework!

(1) Problem One *(6 points)*



State equations

$$v_R(t) = R\, i(t) \tag{1.1a}$$

$$v_C'(t) = \frac{1}{C}\, i(t) \tag{1.1b}$$

$$v_L(t) = L\, i'(t) \tag{1.1c}$$

Kirchhoff's Voltage Law

$$v_i(t) = v_L(t) + v_R(t) + v_C(t) \tag{1.2}$$

Consider the electric circuit above. We assume that the given state equations hold and the operational conditions are sufficient to assume that the parameters R, C, and L are time-independent. We also assume that Kirchhoff's Voltage Law holds. Finally, $v_i(t)$ and $v_o(t) = v_C(t)$ will be referred as input and output voltages respectively.

(1.1) (0.6pt)

Given the information above, construct the differential equation that describes this system. *Hint: construct it for the current $i(t)$.*

Answer: Consider the following:

$$\begin{aligned} Li''(t) =& v_L'(t) && (1.3a)\\ =& v_i'(t) - v_R'(t) - v_C'(t) && (1.3b)\\ =& v_i'(t) - Ri'(t) - \frac{1}{C} i(t) && (1.3c)\end{aligned}$$

hence

$$\left(L\frac{\mathrm{d}^2}{\mathrm{d}t^2} + R\frac{\mathrm{d}}{\mathrm{d}t} + \frac{1}{C}\right) i(t) = v_i'(t) \tag{1.4}$$

(1.2) (0.3pt)

In the previous part, you should have found a differential equation of the form

$$\left(a_2\frac{\mathrm{d}^2}{\mathrm{d}t^2} + a_1\frac{\mathrm{d}}{\mathrm{d}t} + a_0\right) f(t) = h(t) \tag{1.5}$$

for some a_i, $f(t)$, and $h(t)$. What is the characteristic equation for this differential equation? Please give the answer *in terms of* a_0, a_1, *and* a_2!

Answer:

$$a_2 r^2 + a_1 r + a_0 = 0 \tag{1.6}$$

(1.3) (0.3pt)

What are the roots of this characteristic equation? Please give the answer in terms of R, L, and C.

Answer:

$$r_\pm = \frac{-R \pm \sqrt{R^2 - 4\frac{L}{C}}}{2L} \tag{1.7}$$

(1.4) (0.3pt)

The response of a *second-order linear ordinary differential equation with constant coefficients* can be described by two parameters: ***the natural frequency ω*** and ***the damping factor β***. For this system, we can define them as

$$\omega = \frac{1}{\sqrt{CL}} \tag{1.8a}$$

$$\beta = \frac{R}{2L} \tag{1.8b}$$

Rewrite the roots in terms of these parameters.

Answer:

$$r_\pm = -\beta \pm \sqrt{\beta^2 - \omega^2} \tag{1.9}$$

(1.5) (0.3pt)

Assume that $\beta \neq \omega$. What is the homogeneous solution to this system?

Answer: If $\beta \neq \omega$, then the roots are distinct, meaning that we can write down the homogeneous solution as

$$i_h(t) = c_+ \exp\left(\left[-\beta + \sqrt{\beta^2 - \omega^2}\right] t\right) + c_- \exp\left(\left[-\beta - \sqrt{\beta^2 - \omega^2}\right] t\right) \tag{1.10}$$

for the undetermined coefficients $c_\pm$.

(1.6) (0.3pt)

Assume that $\beta = \omega$. What is the homogeneous solution to this system?

Answer: If $\beta = \omega$, then the roots are same, meaning that we can write down the homogeneous solution as

$$i_h(t) = (c_1 + c_2 t) \exp(-\beta t) \tag{1.11}$$

for the undetermined coefficients $c_{1,2}$.

(1.7) (0.3pt)

Consider the differential equation in (1.5) with the source term $h(t)$ given by *Dirac-delta generalized function*, i.e. $h(t) = \delta(t)$. As we discussed in the class, $\delta(t)$ is *not* a well-defined function: there are mathematical ways to define it rigorously as a generalized function (also called *distribution*), but we do not need to know them. In fact, for our purposes, what is needed is merely the fact that $\delta(x)$ goes to 1 under Laplace transformation.

With that in mind, consider the following system

$$\left(a_2 \frac{\mathrm{d}^2}{\mathrm{d}t^2} + a_1 \frac{\mathrm{d}}{\mathrm{d}t} + a_0\right) \mathbb{i}(t) = \delta(t)\,, \qquad \mathbb{i}(0) = \mathbb{i}'(0) = 0 \tag{1.12}$$

Here, $\mathbb{i}(t)$ is called *the impulse response* of the system. We will compute it now.

Denote the Laplace transform of the impulse response as $\mathbb{I}(s)$. Find it in terms of a_0, a_1, and a_2.

Answer: In Laplace domain, the given system becomes

$$\left(a_2 s^2 + a_1 s + a_0\right) \mathbb{I}(s) = 1 \tag{1.13}$$

hence

$$\mathbb{I}(s) = \frac{1}{a_2 s^2 + a_1 s + a_0} \tag{1.14}$$

(1.8) (0.6pt)

Rewrite the result of the previous section in terms L, β, and ω. Then message the expression for two different cases ($\beta \neq \omega$ and $\beta = \omega$) so that it takes the form $\frac{\cdots}{s-r_1} + \frac{\cdots}{s-r_2}$ for $\omega \neq \beta$ and $\frac{\cdots}{s-r} + \frac{\cdots}{(s-r)^2}$ for $\beta = \omega$.

Answer: We have already found the roots previously, thus

$$\mathbb{I}(s) = \begin{cases} \dfrac{1}{L(s-r_+)(s-r_-)} & \beta \neq \omega \\ \dfrac{1}{L(s+\beta)^2} & \beta = \omega \end{cases} \tag{1.15}$$

where $r_\pm = -\beta \pm \sqrt{\beta^2 - \omega^2}$. These can be rewritten as

$$\mathbb{I}(s) = \begin{cases} \dfrac{1}{2L\sqrt{\beta^2-\omega^2}} \dfrac{1}{s-r_+} - \dfrac{1}{2L\sqrt{\beta^2-\omega^2}} \dfrac{1}{s-r_-} & \beta \neq \omega \\ \dfrac{1}{L} \dfrac{1}{(s+\beta)^2} & \beta = \omega \end{cases} \tag{1.16}$$

(1.9) (0.5pt)

We know that the functions e^{at} and ta^{at} has the following mappings under Laplace transformation:

$$e^{at} \xrightarrow{\text{Laplace transform}} \frac{1}{s-a} \tag{1.17a}$$

$$te^{at} \xrightarrow{\text{Laplace transform}} \frac{1}{(s-a)^2} \tag{1.17b}$$

Use this information to find the impulse response $\mathbb{i}(t)$ for both $\omega = \beta$ and $\omega \neq \beta$ cases.

Answer:

$$\mathbb{i}(t) = \begin{cases} \dfrac{e^{-\beta t}\sinh\left(\sqrt{\beta^2-\omega^2}\,t\right)}{L\sqrt{\beta^2-\omega^2}} & \beta \neq \omega \\ \dfrac{te^{-\beta t}}{L} & \beta = \omega \end{cases} \tag{1.18}$$

where sine hyperbolic is defined via the relation $2\sinh(x) = e^x - e^{-x}$.

(1.10) (1pt)

Using the the result of the previous section, write down the particular solution (for both $\beta = \omega$ and $\beta \neq \omega$) for the current $i(t)$ in terms of the *derivative of the input voltage*.

Answer: We have found that the differential equation for the current $i(t)$ is

$$\left(L\frac{\mathrm{d}^2}{\mathrm{d}t^2} + R\frac{\mathrm{d}}{\mathrm{d}t} + \frac{1}{C}\right) i(t) = v_i'(t) \tag{1.19}$$

whereas the impulse response for the system

$$\left(a_2\frac{\mathrm{d}^2}{\mathrm{d}t^2} + a_1\frac{\mathrm{d}}{\mathrm{d}t} + a_0\right) \mathbb{i}(t) = \delta(t)\,, \qquad \mathbb{i}(0) = \mathbb{i}'(0) = 0 \tag{1.20}$$

is given as

$$\mathbb{i}(t) = \begin{cases} \dfrac{e^{-\beta t}\sinh\left(\sqrt{\beta^2-\omega^2}\,t\right)}{L\sqrt{\beta^2-\omega^2}} & \beta \neq \omega \\ \dfrac{te^{-\beta t}}{L} & \beta = \omega \end{cases} \tag{1.21}$$

Thus, we can immediately write down the particular solution $i_p(t) = \int\limits_0^\infty v_i'(t-\tau)i(\tau)d\tau$ as

$$i_p(t) = \begin{cases} \displaystyle\int\limits_0^\infty v_i'(t-\tau)\frac{e^{-\beta\tau}\sinh\left(\sqrt{\beta^2-\omega^2}\,\tau\right)}{L\sqrt{\beta^2-\omega^2}}d\tau & \beta \neq \omega \\ \displaystyle\int\limits_0^\infty v_i'(t-\tau)\frac{\tau e^{-\beta\tau}}{L}d\tau & \beta = \omega \end{cases} \tag{1.22}$$

(1.11) (1.5pt)

Assume that the system initially had zero stored energy on the capacitor and zero energy stored in the inductor at $t = 0$, hence $i(0) = i'(0) = 0$. If we start an input voltage at $t = 0$ of the form

$$v_i(t) = \begin{cases} -\dfrac{\cos(ft)}{f} & t > 0 \\ 0 & t < 0 \end{cases} \tag{1.23}$$

what would be the output voltage as the function of time $t > 0$? Assume that $\beta \neq \omega$!

Hint: You may leave the result with TWO implicit integrations, i.e. in the form $v_o(t) = \int d\rho \int d\tau(\ldots)$ *for some dummy integration variables* τ *and* ρ.

Answer: We have found the homogeneous and particular solutions above, so the full solution reads

$$i(t) = c_+ \exp\left(\left[-\beta + \sqrt{\beta^2 - \omega^2}\right] t\right) + c_- \exp\left(\left[-\beta - \sqrt{\beta^2 - \omega^2}\right] t\right) + \int_0^t \sin(f(t-\tau)) \frac{e^{-\beta\tau} \sinh\left(\sqrt{\beta^2 - \omega^2}\tau\right)}{L\sqrt{\beta^2 - \omega^2}} d\tau \quad (1.24)$$

At $t = 0$, the particular solution drops, so $i(0) = 0$ yields $c_+ + c_- = 0$. Taking the derivative and then applying $i'(0) = 0$ yields $-\beta(c_+ + c_-) + \sqrt{\beta^2 - \omega^2}(c_+ - c_-) = 0$. Thus, we have $c_+ = c_- = 0$, hence

$$i(t) = \int_0^t \sin(f(t-\tau)) \frac{e^{-\beta\tau} \sinh\left(\sqrt{\beta^2 - \omega^2}\tau\right)}{L\sqrt{\beta^2 - \omega^2}} d\tau \quad (1.25)$$

As the output voltage is the voltage of the capacitor, the state equation along with the initial condition for $i(t)$ yields

$$v_o(t) = \int_0^t \frac{i(\rho)}{C} d\rho \quad (1.26)$$

hence

$$v_o(t) = \int_0^t d\rho \int_0^\rho d\tau \sin(f(\rho - \tau)) \frac{\omega^2 e^{-\beta\tau} \sinh\left(\sqrt{\beta^2 - \omega^2}\tau\right)}{\sqrt{\beta^2 - \omega^2}} \quad (1.27)$$

where we also used $CL = \omega^{-2}$.

(2) Problem Two *(not graded)*

The previous question can be simply solved with the following Mathematica code:

```
With[{
(*Start with the differential equation, along with initial conditions*)
theSystem = {
        l i''[t] + r i'[t] + 1/c i[t] == Sin[f t], i[0] == 0, i'[0] == 0
        }
},
        RightComposition[
        (*Solve the equation and get the current:*)
        i[t] /. DSolve[#, i[t], t][[1]] &,
        (*Divide by capacitance for later convenience,
        and then switch to \[Beta] and c*)
        (#/c) /. {r -> 2 l \[Beta] , c -> 1/(\[Omega]^2 l)} &,
        (*Simplify the expression*)
        FullSimplify[#, Element[\[Beta] | \[Omega] | f | l,
            PositiveReals]] &,
        (*change to the dummy variable \[Tau]*)
        # /. t -> \[Tau] &,
        (*integrate*)
        Integrate[#, {\[Tau], 0, t}] &,
        (*Simplify the result*)
        FullSimplify[#, Element[\[Beta] | \[Omega] | f | l,
            PositiveReals]] &
        ][theSystem]
]
```

This gives the full result for $v_0(t)$.

Although the result is quite long, one can draw interesting conclusions from it. For instance, the following code analyzes $\beta = 0$ case if it is run immediately after the code above:

```
FullSimplify[% /. \[Beta] -> 0,
        Element[\[Omega] | f | l, PositiveReals]
]
```

which yields the interesting result

$$v_0(t)\Big|_{\beta=0} = \frac{f^2(-\cos(t\omega)) + f^2 + \omega^2\cos(f t) - \omega^2}{f^3 - f\omega^2} \tag{2.1}$$

We see that the output voltage only oscillates: its magnitude *does not increase or decrease with time*. However, the situation changes if we *tune* our input frequency f to the natural frequency ω of the system. With the code

```
Limit[(f^2 - \[Omega]^2 + \[Omega]^2 Cos[f t] - f^2 Cos[t \[Omega]])/(
        f^3 - f \[Omega]^2), f -> \[Omega]
]
```

we find that

$$\lim_{f\to\omega} v_0(t)\Big|_{\beta=0} = -\frac{t\omega\sin(t\omega) + 2\cos(t\omega) - 2}{2\omega} \tag{2.2}$$

Now, the magnitude of the voltage *grows* with time t: The output becomes **unbounded**!

It is mathematically fine to get these expressions, but in physical systems, unbounded output is a sign for danger. It shows that the system *cannot shake the energy pumped into it*, meaning that it will become too much energetic and break at one point. For an electrical system, this simply means that the components will burn and the system will get damaged. In mechanical systems, this indicates that a mechanical failure will happen! In civil engineering, a bridge excited with something (such as a wind) at the same frequency with its natural frequency ω will collapse. This is why engineers have to design system with natural frequencies ω such that the input frequencies to those systems do not coincide with ω. The natural frequency ω is also called *resonance frequency*, as the system resonates with the input. Resonances are also important in Physics; for instance, high energy physicists discover particles by analyzing the *resonances* in scattering experiments, and then by deriving their masses from the resonance frequencies.