Mathematical Methods in Physics II

Homework IV

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Problem 1

Solution. (a) The expression of the gradient of f_1 is given by

$$\text{grad}(f_1) = \left\langle \frac{\partial f_1}{\partial x_1}, \frac{\partial f_1}{\partial x_2}, \ldots, \frac{\partial f_1}{\partial x_7}, \right\rangle$$

Since f_1 is a function of x_3 and x_5 only, the only non-zero partial derivatives are $\dfrac{\partial f_1}{\partial x_3}$ and $\dfrac{\partial f_1}{\partial x_5}$. Let's evaluate these partial derivatives:

$$\frac{\partial}{\partial x_3}\left(x_3^2 x_5\right) = 2x_3x_5$$
$$\frac{\partial}{\partial x_5}\left(x_3^2 x_5\right) = x_3^2$$

Hence, $\text{grad}(f_1) = \langle 0, 0, 2x_3x_5, 0, x_3^2, 0, 0\rangle$.

Let's apply the same procedure to f_2:

$$\frac{\partial}{\partial x_2}\left(\frac{x_2^2 x_5}{1+x_6}\right) = \frac{2x_2x_5}{1+x_6}$$
$$\frac{\partial}{\partial x_5}\left(\frac{x_2^2 x_5}{1+x_6}\right) = \frac{x_2^2}{1+x_6}$$
$$\frac{\partial}{\partial x_6}\left(\frac{x_2^2 x_5}{1+x_6}\right) = -\frac{x_2^2 x_5}{(1+x_6)^2}$$

Hence, $\text{grad}(f_2) = \left\langle 0, \frac{2x_2x_5}{1+x_6}, 0, 0, \frac{x_2^2}{1+x_6}, -\frac{x_2^2x_5}{(1+x_6)^2}, 0\right\rangle$.

And finally, let's evaluate the gradient of f_3:

$$\frac{\partial}{\partial x_1}\left(\frac{x_7^2 x_1}{1+x_4}\cos(x_1)\right) = \frac{x_7^2}{1+x_4}\cos(x_1) - \frac{x_7^2x_1}{1+x_4}\sin(x_1)$$
$$\frac{\partial}{\partial x_4}\left(\frac{x_7^2 x_1}{1+x_4}\cos(x_1)\right) = -\frac{x_7^2x_1}{(1+x_4)^2}\cos(x_1)$$
$$\frac{\partial}{\partial x_7}\left(\frac{x_7^2 x_1}{1+x_4}\cos(x_1)\right) = \frac{2x_7x_1}{1+x_4}\cos(x_1)$$

Hence, $\text{grad}(f_3) = \left\langle \frac{x_7^2}{1+x_4}\cos(x_1) - \frac{x_7^2x_1}{1+x_4}\sin(x_1), 0, 0, -\frac{x_7^2x_1}{(1+x_4)^2}\cos(x_1), 0, 0, \frac{2x_7x_1}{1+x_4}\cos(x_1)\right\rangle$.

(b) The expression of the divergence of g is given by

$$\text{div}(g) = \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}$$

Let's evaluate $g(-1)$:

$$g(-1) = (x^{-1} + y)\frac{\partial}{\partial x} - (x + y^{-1})\frac{\partial}{\partial y}$$

The partial derivatives are:

$$\frac{\partial}{\partial x}\left(x^{-1} + y\right) = -\frac{1}{x^2}$$
$$\frac{\partial}{\partial y}\left(-\left(x + y^{-1}\right)\right) = \frac{1}{y^2}$$

Hence, $\operatorname{div}(g(-1)) = -\dfrac{1}{x^2} + \dfrac{1}{y^2}$.

Let's evaluate $g(0)$:

$$g(0) = 2\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}$$

The partial derivatives are:

$$\frac{\partial}{\partial x}(2) = 0$$
$$\frac{\partial}{\partial y}(-2) = 0$$

Hence, $\operatorname{div}(g(0)) = 0$.

Let's evaluate $g(1)$:

$$g(1) = (x + y^{-1})\frac{\partial}{\partial x} - (x^{-1} + y)\frac{\partial}{\partial y}$$

The partial derivatives are:

$$\frac{\partial}{\partial x}\left(x + y^{-1}\right) = 1$$
$$\frac{\partial}{\partial y}\left(-\left(x^{-1} + y\right)\right) = -1$$

Hence, $\operatorname{div}(g(1)) = 1 - 1 = 0$.

(c) The expression of the curl of h is given by

$$\operatorname{curl}(h) = \left(\frac{\partial h_y}{\partial x} - \frac{\partial h_x}{\partial y}\right)\frac{\partial}{\partial z}$$

Let's evaluate $h(-1)$:

$$h(-1) = \left(x^{-1} + y\right)\frac{\partial}{\partial x} - (x + y^{-1})\frac{\partial}{\partial y}$$

The partial derivatives are:

$$\frac{\partial}{\partial x}\left(-\left(x + y^{-1}\right)\right) = -1$$
$$\frac{\partial}{\partial y}\left(x^{-1} + y\right) = 1$$

Hence, $\operatorname{curl}(h(-1)) = -2\dfrac{\partial}{\partial z}$.

Let's evaluate $h(0)$:

$$h(0) = 2\frac{\partial}{\partial x} - 2\frac{\partial}{\partial y}$$

The partial derivatives are:

$$\frac{\partial}{\partial x}(-2) = 0$$
$$\frac{\partial}{\partial y}(2) = 0$$

Hence, $\operatorname{curl}(h(0)) = 0$.

Let's evaluate $h(1)$:

$$h(1) = \left(x + y^{-1}\right)\frac{\partial}{\partial x} - (x^{-1} + y)\frac{\partial}{\partial y}$$

The partial derivatives are:

$$\frac{\partial}{\partial x}\left(-(x^{-1} + y)\right) = \frac{1}{x^2}$$
$$\frac{\partial}{\partial y}\left(x + y^{-1}\right) = -\frac{1}{y^2}$$

Hence, $\mathrm{curl}(h(1)) = \left(\frac{1}{x^2} + \frac{1}{y^2}\right)\frac{\partial}{\partial z}$. ■