

# **Mathematical Methods in Physics I**

Lecture Notes

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These lecture notes are based on the lectures of Prof. Dr. Soner Albayrak given in the course *Mathematical Methods in Physics I* at the Middle East Technical University in the Fall semester of the 2023-2024 academic year. I would like to thank Prof. Dr. Soner Albayrak for his excellent and insightful lectures, which inspired me to delve deeper into the subject and write these lecture notes. I hope these lecture notes will be useful to the students taking the course and to anyone interested in the subject.

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# 1. Preliminaries

## 1.1 Functions

A **function** is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output. An example is the function that relates each real number  $x$  to its square  $x^2$ . The output of a function  $f$  corresponding to an input  $x$  is denoted by  $f(x)$  (read “ $f$  of  $x$ ”). In this example, if the input is 3, then the output is 9, and we may write  $f(3) = 9$ . The input variable(s) are sometimes referred to as the argument(s) of the function.

**Definition 1.1** Let  $X$  and  $Y$  be nonempty sets. A function  $f$  from  $X$  to  $Y$  is a relation  $f \subseteq X \times Y$  such that for every  $x \in X$  there is exactly one  $y \in Y$  with  $(x, y) \in f$ . We write  $f : X \rightarrow Y$  to indicate that  $f$  is a function from  $X$  to  $Y$ .

 Functions are sometimes referred to as **mappings** or **transformations**.

## 1.2 Type Notation



## 2. Linear Equations with Constant Coefficients

### 2.1 Laplace Transform

#### 2.1.1 Integral Transform

An **integral transform** is a type of transform that maps a function from its original function space into another function space via integration, where some of the properties of the original function might be more easily characterized and manipulated than in the original function space. The transformed function can generally be mapped back to the original function space using the inverse transform.

Integral transforms are closely related to the area of mathematics, called **operational**, or **symbolic analysis**. Its essence is to convert a given function  $f(t)$  of a real (or complex) variable  $t$  to a function  $F(s)$  of a complex variable  $s$  using integral transformations. It is used to solve non-homogeneous linear differential equations and to analyze properties of linear dynamic systems, like electric circuits. It allows the conversion of the problem of finding solutions to differential equations into the simpler problem of finding solutions to algebraic equations.

**Definition 2.1 — Integral transform.** An integral transform is an operator  $\mathfrak{T}$  that maps a function  $f$  to another one:

$$\mathfrak{T} : (\mathbb{C} \rightarrow \mathbb{C}) \rightarrow (\mathbb{C} \rightarrow \mathbb{C}) \quad (2.1)$$

$$\mathfrak{T} = (x \rightarrow f(x)) \rightarrow \left( s \rightarrow \int_a^b K(x, t) f(t) dt \right) \quad (2.2)$$

where  $K(x, t)$  is a given function of two variables called the **kernel** of the transform.

The choice of the kernel determines the properties of the integral transform. For example, the Laplace transform is an integral transform with the kernel  $e^{-st}$ , where  $s$  is a complex number.

#### 2.1.2 Definition

**Definition 2.2 — Laplace Transform.** The Laplace transform  $\mathcal{L}$  is defined as:

$$\mathcal{L} :: (\mathbb{C} \rightarrow \mathbb{C}) \rightarrow (\mathbb{C} \rightarrow \mathbb{C}), \quad (2.3)$$

$$\mathcal{L} = (t \rightarrow f(t)) \rightarrow \left( s \rightarrow \int_0^\infty e^{-st} f(t) dt \right), \quad (2.4)$$

where  $t$  is a real or complex variable and  $s$  is a complex variable.

Several conditions should be satisfied for the Laplace transform to exist. The following theorem states the conditions for the existence of the Laplace transform.

**Theorem 2.1 — Existence of the Laplace transform.** The function  $f(t)$  should satisfy the following conditions:

1.  $f(t)$  is piecewise continuous on  $[0, \infty)$ ;
2.  $f(t)$  is of exponential order, i.e. there exist constants  $M$  and  $a$  such that  $|f(t)| \leq Me^{at}$  for all  $t \geq 0$ ;
3.  $f(t) \equiv 0$  for  $t < 0$ .

■ **Example 2.1** Find the Laplace transform of the function  $f(t) = 1$ .

**Solution.** Let's apply the definition of the Laplace transform:

$$\begin{aligned} \mathcal{L}[f(t)](s) &= \int_0^\infty e^{-st} f(t) dt \\ &= \int_0^\infty e^{-st} dt \\ &= \left[ -\frac{1}{s} e^{-st} \right]_0^\infty \\ &= \left( -\frac{1}{s} e^{-s\infty} \right) - \left( -\frac{1}{s} e^{-s0} \right) \\ &= \frac{1}{s} \end{aligned} \quad (2.5)$$

■

### 2.1.3 Properties

Time domain	Laplace s-domain	Region of convergence
1	$\frac{1}{s}$	$s > 0$
$\delta(t)$	1	all $s$
$e^{at}$	$\frac{1}{s-a}$	$s > a$
$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$	$s > 0$
$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$	$s > 0$

Table 2.1: Elementary Laplace transforms.

### 2.1.4 Application to Differential Equations

**Definition 2.3 — Convolution integral.** The **convolution integral** of two functions  $f(t)$  and  $g(t)$  is defined as:

$$(f * g)(t) = \int_{-\infty}^\infty f(\tau) g(t - \tau) d\tau \quad (2.6)$$

The Laplace transform can be used to solve differential equations. Let's consider the following differential equation:

$$\frac{dy}{dt} + ay = b(t) \quad (2.7)$$

where  $a$  is a constant and  $b(t)$  is a function of  $t$ . We can apply the Laplace transform to both sides of the equation:

$$\begin{aligned} \mathcal{L} \left[ \frac{dy}{dt} + ay \right] &= \mathcal{L} [b(t)] \\ \mathcal{L} \left[ \frac{dy}{dt} \right] + a\mathcal{L} [y] &= \mathcal{L} [b(t)] \\ s\mathcal{L} [y] - y(0) + a\mathcal{L} [y] &= \mathcal{L} [b(t)] \\ \mathcal{L} [y] (s + a) &= \mathcal{L} [b(t)] + y(0) \\ \mathcal{L} [y] &= \frac{\mathcal{L} [b(t)] + y(0)}{s + a} \end{aligned} \quad (2.8)$$

Hence, we have:

$$y(t) = \mathcal{L}^{-1} \left[ \frac{\mathcal{L} [b(t)] + y(0)}{s + a} \right] \quad (2.9)$$

where  $\mathcal{L}^{-1}$  is the inverse Laplace transform.



## 3. Linear Homogeneous Equations with Functional Coefficients

### 3.1 Reparametrization

Some differential equations can be reparametrized to a simpler form by making a substitution. For example, consider the following general differential equation:

$$\left( \frac{d^2}{dx^2} + p(x) \frac{d}{dx} + q(x) \right) f(x) = 0 \quad (3.1)$$

Let's make the substitution  $x = u$ , where  $u$  is a function of  $x$ :

$$\frac{d}{dx} = \frac{du}{dx} \frac{d}{du} = u' \frac{d}{du} \quad (3.2)$$

And the second derivative:

$$\begin{aligned} \frac{d^2}{dx^2} &= \frac{d}{dx} \left( \frac{d}{dx} \right) \\ &= \frac{d}{dx} \left( u' \frac{d}{du} \right) \\ &= u'' \frac{d}{du} + (u')^2 \frac{d^2}{du^2} \end{aligned} \quad (3.3)$$

Hence, the differential equation becomes:

$$\left( u'' \frac{d}{du} + (u')^2 \frac{d^2}{du^2} + p(x) u' \frac{d}{du} + q(x) \right) f(u) = 0 \quad (3.4)$$

If we divide both sides by  $(u')^2$ , we get:

$$\left( \frac{d^2}{du^2} + \frac{u'' + u'p(x)}{(u')^2} \frac{d}{du} + \frac{q(x)}{(u')^2} \right) f(u) = 0 \quad (3.5)$$

For the equation to be a constant coefficient one,  $(u'' + u'p(x))/(u')^2$  and  $q(x)/(u')^2$  must be constants. Let's choose the latter to be equal to 1. Therefore, we have:

$$u(x) = \int \sqrt{q(x)} dx \quad (3.6)$$

In conclusion, a differential equation can be reparametrized if  $u(x) = \int \sqrt{q(x)} dx$  and  $(u'' + u'p(x))/(u')^2$  is a constant.



Note that  $\frac{q(x)}{(u')^2}$  can be any constant, not just 1. We chose it to be 1 for simplicity.

## 3.2 Exact Differential Equations

Some differential equations are exact, i.e. if they are of the form:

$$\left( p_n(x) \frac{d^n}{dx^n} + \dots + p_1(x) \frac{d}{dx} + p_0(x) \right) f(x) = 0 \quad (3.7)$$

For some functions  $p_i(x)$ ,  $i = 0, \dots, n$ , then they can be rewritten as:

$$\frac{d}{dx} \left[ \left( q_{n-1} \frac{d^{n-1}}{dx^{n-1}} + \dots + q_1 \frac{d}{dx} + q_0(x) \right) f(x) \right] = 0 \quad (3.8)$$

For some functions  $q_i(x)$ ,  $i = 0, \dots, n-1$ . Let's consider a second-order differential equation and derive the conditions for the exactness of the equation:

$$\left( p_2(x) \frac{d^2}{dx^2} + p_1(x) \frac{d}{dx} + p_0(x) \right) f(x) = 0 \quad (3.9)$$

Its exact form is, therefore:

$$\frac{d}{dx} \left( q_1(x) \frac{d}{dx} + q_0(x) \right) f(x) = 0 \quad (3.10)$$

First, we expand the expression:

$$\begin{aligned} \frac{d}{dx} (q_1(x) f'(x) + q_0(x) f(x)) &= 0 \\ q_1(x) f''(x) + q_1'(x) f'(x) + q_0'(x) f(x) + q_0(x) f'(x) &= 0 \\ q_1(x) f''(x) + (q_1'(x) + q_0(x)) f'(x) + q_0'(x) f(x) &= 0 \end{aligned} \quad (3.11)$$

It follows that  $p_2(x) = q_1(x)$ ,  $p_1(x) = q_1'(x) + q_0(x)$ , and  $p_0(x) = q_0'(x)$ . Hence, the condition for the exactness of the equation is:

$$p_2''(x) - p_1'(x) + p_0(x) = 0 \quad (3.12)$$

Conditions for higher-order differential equations can be derived similarly. For example, the condition for the exactness of a third-order differential equation is:

$$p_3'''(x) - p_2''(x) + p_1'(x) - p_0(x) = 0 \quad (3.13)$$

## 3.3 Reduction of Order

Some differential equations can be solved by reducing the order of the equation if one or more solutions are known. It follows that an  $n$  order differential equation can be reduced to an  $(n - k)$  order differential equation if  $k$  solutions are known.

■ **Example 3.1** Rewrite the following differential equation as a first-order differential equation, given that  $f_1(x) = e^x$  is a solution:

$$f(x) = g(x) f_1(x) = g(x) e^x \quad (3.14)$$

It follows that:

$$\begin{aligned}\frac{d}{dx}f(x) &= \frac{d}{dx}(g(x)e^x) \\ &= g'(x)e^x + g(x)e^x\end{aligned}\tag{3.15}$$

$$\begin{aligned}\frac{d^2}{dx^2}f(x) &= \frac{d}{dx}\left(\frac{d}{dx}f(x)\right) \\ &= \frac{d}{dx}(g'(x)e^x + g(x)e^x) \\ &= g''(x)e^x + 2g'(x)e^x + g(x)e^x\end{aligned}\tag{3.16}$$

If we substitute the derived expressions into the differential equation, we obtain:

$$\begin{aligned}(x-1)(g''(x)e^x + 2g'(x)e^x + g(x)e^x) - x(g'(x)e^x + g(x)e^x) + g(x)e^x &= 0 \\ (x-1)(g''(x) + 2g'(x) + g(x)) - x(g'(x) + g(x)) + g(x) &= 0 \\ xg''(x) + 2xg'(x) + xg(x) - g''(x) - 2g'(x) - g(x) - g'(x) - xg(x) + g(x) &= 0 \\ (x-1)g''(x) + (2x-3)g'(x) &= 0\end{aligned}\tag{3.17}$$

If  $g'(x) = h(x)$ , then:

$$(x-1)h'(x) + (2x-3)h(x) = 0\tag{3.18}$$

■

## 3.4 Levi-Civita Symbol

Levi-Civita symbol is a mathematical symbol used in multilinear algebra and tensor calculus. It is also known as the permutation symbol, antisymmetric symbol, or alternating symbol. It is commonly denoted as  $\epsilon_{ijk}$  or  $\varepsilon_{ijk}$ , where  $i, j, k$  are the indices. It is defined as:

**Definition 3.1 — Levi-Civita symbol.**

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3) \\ -1 & \text{if } (i, j, k) \text{ is an odd permutation of } (1, 2, 3) \\ 0 & \text{if two indices are equal} \end{cases}\tag{3.19}$$

We can define the function  $\det$  using the Levi-Civita symbol:

**Definition 3.2 — Determinant.**

$$\det :: \mathfrak{M}_{n \times n}(\mathbb{C}) \rightarrow \mathbb{C}\tag{3.20}$$

$$\det = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \rightarrow \sum\tag{3.21}$$

## 3.5 Series Solutions

Some differential equations cannot be solved using the methods we have discussed so far. In this case, we can use power series to find solutions to the differential equations.

**Definition 3.3** A **power series** is a series of the form:

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots \quad (3.22)$$

where  $a_n$  are the coefficients of the series,  $x_0$  is the center of the series, and  $x$  is the variable.

## 3.6 Fourier Analysis

### 3.6.1 Fourier Series

### 3.6.2 Fourier Transform

**Fourier transform (FT)** is an integral transform that decomposes a function into its constituent frequencies. The output of the transform is a complex-valued function of frequency. The Fourier transform is not limited to functions of time, but to have a unified language, the domain of the original function is commonly referred to as the time domain.

For many functions of practical interest one can define an operation that reverses this: the inverse Fourier transformation, also called Fourier synthesis, of a frequency domain representation combines the contributions of all the different frequencies to recover the original function of time. The Fourier transform is named in honor of Joseph Fourier, who introduced the transform in his study of heat transfer, where Gaussian functions appear as solutions of the heat equation.

**Definition 3.4 — Fourier transform.** The Fourier transform of a function  $f(x)$ , denoted by  $\mathcal{F}_t[f(t)](k)$ , is defined as:

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ikt} dt \quad (3.23)$$

Here,  $k$  is the **frequency** in Hertz of the function  $f(t)$ ,  $t$  is time in seconds, and  $\hat{f}(k) = \mathcal{F}_t[f(t)](k)$  is called the *forward* ( $-i$ ) Fourier transform of  $f(t)$ .

**Definition 3.5 — Inverse Fourier transform.** The *inverse* ( $+i$ ) Fourier transform of a function  $f(t)$ , denoted by  $\mathcal{F}_t^{-1}[f(t)](t)$ , is defined as:

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(k)e^{2\pi ikt} dk \quad (3.24)$$

There are several conventions for defining the Fourier transform of a function, due to possible differences in the sign of the exponential and the normalization factor which may put a factor of  $\frac{1}{2\pi}$  or  $\frac{1}{\sqrt{2\pi}}$  in front of the integral. The definition above is the most common one and is used mostly in engineering textbooks and literature.

Some authors, especially us, physicists, write the Fourier transform in terms of **angular frequency**  $\omega$  instead of frequency  $k$ , where  $\omega = 2\pi k$ . In this case, the Fourier transform and the inverse Fourier transform are defined as:

**Definition 3.6 — Fourier transform in terms of angular frequency.**

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx \quad (3.25)$$

**Definition 3.7 — Inverse Fourier transform in terms of angular frequency.**

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega x} d\omega \quad (3.26)$$

Here,  $\omega$  is in radians per second. As you might have noticed, the inverse Fourier transform is defined with a normalization factor of  $\frac{1}{2\pi}$ , while the forward Fourier transform is not. You should be careful with these conventions when using properties of the Fourier transform and solving differential equations.

**R** In the course, Prof. Dr. Soner Albayrak uses the latter convention with a decision to denote angular frequency as  $k$ , which might be confusing. However, it is crucial to orient yourself with the normalization factors and the signs of the exponential, not the notation of the (angular) frequency. In these lecture notes, we will follow the convention of the course.

In solving problems, we will be using the definition of Dirac delta function:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk \quad (3.27)$$

Note that the Dirac delta function is not a function in the usual sense, but a distribution. The Dirac delta function is a linear function that maps every function  $f$  to its value at 0. In other words, the Dirac delta function is the identity element of the convolution algebra. One of the important properties of the Dirac delta function is that it is even, i.e.  $\delta(-x) = \delta(x)$ .

■ **Example 3.2** Find the Fourier transform of the function  $f(x) = \cos(x)$ .

Let's apply the definition of the Fourier transform:

$$\begin{aligned} \hat{f}(k) &= \int_{\mathbb{R}} f(t) e^{-ikx} dx \\ &= \int_{\mathbb{R}} \cos(x) e^{-ikx} dx \\ &= \int_{\mathbb{R}} \frac{e^{ix} + e^{-ix}}{2} e^{-ikx} dx \\ &= \frac{1}{2} \int_{\mathbb{R}} e^{ix(1-k)} dx + \frac{1}{2} \int_{\mathbb{R}} e^{-ix(1+k)} dx \end{aligned} \quad (3.28)$$

Using the Equation 3.27, we can write:

$$\begin{aligned} \hat{f}(k) &= \frac{1}{2} \int_{\mathbb{R}} e^{ix(1-k)} dx + \frac{1}{2} \int_{\mathbb{R}} e^{-ix(1+k)} dx \\ &= \frac{1}{2} \delta(1-k) + \frac{1}{2} \delta(1+k) \\ &= \frac{1}{2} \delta(k-1) + \frac{1}{2} \delta(k+1) \end{aligned} \quad (3.29)$$

■



## 4. Systems of First-Order Differential Equations

Many problems in physics and engineering can be formulated as systems of first-order differential equations. For example, the motion of a particle in three-dimensional space can be described by a system of three first-order differential equations, or current in an electric circuit can be described by a system of first-order differential equations. In this chapter, we will discuss how to solve systems of first-order differential equations.

### 4.1 Introduction

Suppose that we are given a system of  $n$  first-order differential equations:

$$a_0 \frac{dy_0}{dx} + a_1 \frac{dy_1}{dx} + \cdots + a_{n-1} \frac{dy_{n-1}}{dx} = b(x) \quad (4.1)$$



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