0.1 Equalizers and Coequalizers

0.1.1 Equalizers

Proposition 1. In any category, if $e : E \to A$ is an equalizer of some pair of arrows, then e is monic.

Proof. Consider the diagram

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[u, shift left, "x"] \arrow[u, shift right, "y"'] \arrow[ur, "z"'] & & \end{tikzcd}$$

in which we assume e is the equalizer of f and g. Supposing $ex = ey$, we want to show $x = y$. Put $z = ex = ey$. Then $fz = fex = gex = gz$, so there is a *unique* $u : Z \to E$ such that $eu = z$. So from $ex = z$ follows that $x = u = y$.

□

In SETS, the equalizer would just be the set $x \in A | f(x) = g(x)$.

Suppose $f, g : R^2 \to R$ where $f(x, y) = x^2 + y^2$ and $g = 1$. We take the equalizer, say in TOP, which is the subspace $S = (x, y) \in R^2 | X^2 + y^2 = 1 \to R^2$ which is the unit circle in the plane!

Awodey: In abelian groups though, using the fact that

$$f(x) = g(x)$$

iff

$$(f - g)(x) = 0$$

we know that the equalizer of f and g is the same as that of the homomorphism $(f - g)$ and the zero homomorphism $0 : A \to B$, so it suffices to consider equalizers of the special form $A(h, 0) \rightarrowtail A$ for arbitrary homomorphisms $h : A \to B$. This subgroup of A sis the *kernel*.

Cook: In abelian groups: $G \underset{f}{\overset{Hom\phi}{\rightrightarrows}} H$

$$E = \{g \in G | \phi(g) = f(g)\} = \{g \in G | \phi(g) = 1_{+1}\}$$

Is the kernel of a homomorphism by definition, also equalizers don't have to exist.

0.1.2 Coequalizers

$$\begin{tikzcd} A \arrow[r, shift left] \arrow[r, shift right] & B \arrow[r, two heads, "c"] \arrow[dr, "z"'] & Q \arrow[d, dashed, "u"] \\ & & Z \end{tikzcd}$$

This is the weakest equivalence relation that forces $f(a)$ relates $g(a) \forall a \in A$