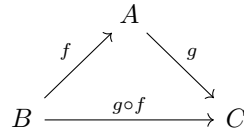


Definition 1. A category consists of objects $obj(C)$ and arrows $hom(C)$

Categories focus on the relation between themselves, the arrows are the important parts. Here is an example category showing function composition with a commutative diagram.



And a quick check of why these are called commutative diagrams

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Remark. For any category C there is always an identity arrow 1_C though it would clutter diagrams if it were written every time. $C \curvearrowright id_C$

There is always one unique identity homomorphism. $\exists! 1_A : A \rightarrow A$

Proof. Assume there are two unique identity morphisms from category A , 1 and $1'$ as shown in the diagram below.

$$A \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{1'} \end{array} A$$

Then composing these two homomorphisms makes a contradiction.

$$1 = 1 \circ 1' = 1'$$

□

Remark. A homomorphism moving between a category and itself is also known as an endomorphism.

Definition 2. Small and Locally small categories

Let C be a category

- if all $hom(C)$'s together form a set, the category is small
- if hom are all sets, the category is locally small

Some examples of categories:

- SET - The category of all sets with mappings between them is locally small but not small

$$\mathcal{P}(\mathbb{X}) = \{A \subseteq \mathbb{X}\} = 2^{\mathbb{X}}$$

B^A :: All functions from $A \rightarrow B$

- Grp - An object is a group and a map $G \rightarrow H$ is a group homomorphism

- Vect - An object is a vector space and a map $V \rightarrow W$ is a linear map

Definition 3. Let A and B be objects in a category. Then a map $f : A \rightarrow B$ is an isomorphism if there is a map $f^{-1} : B \rightarrow A$ (the inverse of f) such that $f^{-1} \circ f = Id_A$ and $f \circ f^{-1} = Id_B$.

If there exists an isomorphism between A and B , we say that A and B are isomorphic and write $A \cong B$.

Proposition 1. In Set, a map is an isomorphism iff it is a bijection. Two sets are isomorphic iff they have the same cardinality.

Definition 4. An object 0 is an initial object if for every object A , there is a unique map $0 \rightarrow A$

Definition 5. The disjoint union of two sets A and B is the set $A \sqcup B = \{(0, a) : a \in A\} \cup \{(1, b) : b \in B\}$.

Definition 6 (Coproduct). Let A and B be objects in a category. Then a sum (or coproduct) of A and B is an object $A + B$ together with maps $i_0 : A \rightarrow A + B$ and $i_1 : B \rightarrow A + B$ such that whenever we have an object C and maps $f_0 : A \rightarrow C$ and $f_1 : B \rightarrow C$, there is a unique map $f : A + B \rightarrow C$ such that $f \circ i_0 = f_0$ and $f \circ i_1 = f_1$

$$\begin{array}{ccccc} A & \xrightarrow{i_0} & A + B & \xleftarrow{i_1} & B \\ & \searrow f_0 & & \swarrow f_1 & \\ & & C & & \end{array}$$

Theorem 0.1. Let A and B be objects and let $A \rightarrow [i_0]P \leftarrow [i_1]B$ and $A \rightarrow [j_0]Q \leftarrow [j_1]B$ be two sums of A and B . Then there exists a unique isomorphism $f : P \rightarrow Q$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & P & & \\ & \nearrow i_0 & \downarrow f & \nwarrow i_1 & \\ A & & & & B \\ & \searrow j_0 & & \swarrow j_1 & \\ & & Q & & \end{array}$$