1 Epis and Monos

Definition 1. In any category C, an arrow

$$f : A \to B$$

is called a *monomorphism* if given any $g, h : C \to A$, $gh = fh$ implies $g = h$

epimorphism if given any $i, j : B \to D, if = jf$ implies $i = j$

Remember having a left inverse is monic and having a right inverse is epic. Having both makes the mapping an isomorphism. In SETS, the converse of the previous is also true: every mono-epi is iso; but this is not true in the general case.

This definition of monomorphism is the category theory equivalent to injective and this definition of epimorphism is the surjective translation.

Proposition 1. A *function* $f : A \to B$ between sets is monic just in case it is injective.

Definition 2. Product

In any category C, a product diagram for the objects A and B consists of an object P and arrows satisfying the universal mapping property: There is some $u : X \to U$ such that $x_1 = p_1 u$ and $x_2 = p_2 u$. Given any $v : X \to U$, if $p_1 v = x_1$ and $p_2 v = x_2$ then $v = u$.

An example: Let us consider the category of types of the simply typed λ-calculus. The λ -calculus is a formalism for the specfication and manipulation of functions, based on the notions of "binding variables" and function evaluation. The relation a b (usually called $\beta\eta$ -equivalence) on terms is defined to be the equivalence relation generated by the equations, and the remaining bound variables:

$$\lambda x.b = \lambda y.b[y/x](noyinb)$$

The category of types $C(\lambda)$ is now defined as follows:

- Objects: the types
- Arrows $A \to B$: closed terms $c : A \to B$, identified if c c',
- Identities $1_A = \lambda x.x(wherex : A)$
- Composition $c \circ b = \lambda x.c(bx)$.

Definition 3. A category C is said to have all finite products if it has a terminal object and all binary products (and therewith products of any finite cardinality). The category C has all (small) products if every set of objects in C has a product.

Definition 4. Slice Category Let C be a category, and I be a C-object. Then the category C/I, the slice category over I, has the following data.

- The objects are pairs (A, f) where A is an object in C and $f : A \to I$ is an arrow.
- An arrow from (A, f) to (B, g) is an arrow $j : A \to B$ such that $g \circ j = f$ in C
- The identity arrow on (A, f) is the arrow $1_A : A \to A$.
- Given arrows $j : (A, f) \to (B, g)$ and $k : (B, g) \to (C, h)$, their compostion $k \circ j : (A, f) \to (C, h)$ is the arrow $k \circ j : A \to C$.