1 Categories

Definition 1. A category C consists of objects $obj(C)$ and arrows $hom(C)$.

Categories focus on the relation between themselves, the arrows are the important parts. Here is an example category showing function composition with a commutative diagram.

$$\begin{tikzcd} & A \arrow[dr, "g"] & \\ B \arrow[ur, "f"] \arrow[rr, "g \circ f"] & & C \end{tikzcd}$$

And a quick check of why these are called commutative diagrams

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Remark. For any category C there is always an identity arrow 1_C though it would clutter diagrams if it were written every time. $C \circlearrowleft id_C$

Proposition 1. There is always one unique identity homomorphism. $\exists! 1_A : A \to A$

Proof. Assume there are two unique identity morphisms from category A, 1 and $1'$ as shown in the diagram below.

$$\begin{tikzcd} A \arrow[r, shift left, "1"] \arrow[r, shift right, "1'"'] & A \end{tikzcd}$$

Then composing these two homomorphisms makes a contradiction.

$$1 = 1 \circ 1' = 1'$$

□

Remark. A homomorphism moving between a category and itself is also know as an endomorphism.

Definition 2. Small and Locally small categories Let C be a category

- if all $hom(C)$'s together form a set, the category is small
- if hom are all sets, the category is locally small

Some examples of categories:

- SET - The category of all sets with mappings between them is locally small but not small
$$\mathcal{P}(\mathbb{X}) = \{A \subseteq \mathbb{X}\} = 2^{\mathbb{X}}$$
B^A :: All functions from $A \to B$

- Grp - An object is a group and a map $G \to H$ is a group homomorphism
- Ab - Abelian groups under homomorphism
- Top - Topological Spaces with continuous maps
- Vect - An object is a vector space and a map $V \to W$ is a linear map

Definition 3. Let A and B be objects in a category. Them a map $f : A \to B$ is an isomorphism is the is a map $f^{-1} : B \to A$ (the inverse of f) such that $f^{-1} \circ f = Id_A$ and $f \circ f^{-1} = Id_B$.

If there exists an isomorphism between A and B, we say that A and B are isomorphic and write $A \cong B$.

Proposition 2. In Set, a map is an isomorphism iff it is a bijection. Two sets are isomorphic off they have the same cardinality.

2 Abstract Structures

2.1 Epis and Monos

Definition 4. In any category C, an arrow

$$f : A \to B$$

is called a *monomorphism* if given any $g, h : C \to A$, $gh = fh$ implies $g = h$

epimorphism if given any $i, j : B \to D, if = jf$ implies $i = j$

Remember having a left inverse is monic and having a right inverse is epic. Having both makes the mapping an isomorphism. In SETS, the converse of the previous is also true: every mono-epi is iso; but this is not true in the general case.

This definition of monomorphism is the category theory equivalent to injective and this definition of epimorphism is the surjective translation.

Proposition 3. A *function* $f : A \to B$ between sets is monic just in case it is injective.

Definition 5. Product

In any category C, a product diagram for the objects A and B consists of an object P and arrows satisfying the universal mapping property: There is some $u : X \to U$ such that $x_1 = p_1 u$ and $x_2 = p_2 u$. Given any $v : X \to U$, if $p_1 v = x_1$ and $p_2 v = x_2$ then $v = u$.

An example: Let us consider the category of types of the simply typed λ-calculus. The λ -calculus is a formalism for the specification and manipulation of functions, based on the notions of "binding variables" and function evaluation. The relation a b (usually called $\beta\eta$ -equivalence) on terms is defined to be the equivalence relation generated by the equations, and the remaining bound variables:

$$\lambda x.b = \lambda y.b[y/x](noyinb)$$

The category of types $C(\lambda)$ is now defined as follows:

- Objects: the types
- Arrows $A \to B$: closed terms $c : A \to B$, identified if $c\ c'$,
- Identities $1_A = \lambda x.x(wherex : A)$
- Composition $c \circ b = \lambda x.c(bx)$.

Definition 6. A category C is said to have all finite products if it has a terminal object and all binary products (and therewith products of any finite cardinality). The category C has all (small) products if every set of objects in C has a product.

Definition 7. Slice Category Let C be a category, and I be a C-object. Then the category C/I, the slice category over I, has the following data.

- The objects are pairs (A, f) where A is an object in C and $f : A \to I$ is an arrow.
- An arrow from (A, f) to (B, g) is an arrow $j : A \to B$ such that $g \circ j = f$ in C
- The identity arrow on (A, f) is the arrow $1_A : A \to A$.
- Given arrows $j : (A, f) \to (B, g)$ and $k : (B, g) \to (C, h)$, their composition $k \circ j : (A, f) \to (C, h)$ is the arrow $k \circ j : A \to C$.

Definition 8. An object 0 is an initial object if for every object A, there is a unique map $0 \to A$

Proposition 4. Initial and terminal objects are unique up to isomorphism.

Proof. Suppose that 0 and $0'$ are both terminal or initial objects in some category C; this diagram states that 0 and $0'$ are uniquely isomorphic.

For terminal objects, apply the previous to C^{op}. □

Definition 9. Disjoint Union The disjoint union of two sets A and B is the set

$$A \sqcup B = (0, a) : a \in A \cup (1, b) : b \in B.$$

Definition 10. Coproduct Let A and B be objects in a category. Then a sum (or coproduct) of A and B is an object $A+B$ together with maps $i_0 : A \to A+B$ and $i_1 : B \to A + B$ such that whenever we have an object C and maps $f_0 : A \to C$ and $f_1 : B \to C$, there is a unique map $f : A + B \to C$ such that $f_0 = fi_0$ and $f_1 = fi_1$

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[dr, "f_0"'] & A + B & B \arrow[l, "i_1"'] \arrow[dl, "f_1"] \\ & C & \end{tikzcd}$$

Theorem 2.1. Let A and B be objects and let $A \to ["i_0"]P \leftarrow ["i_1"]B$ and $A \to ["j_0"]Q \leftarrow ["j_1"]B$ be two sums of A and B. Then there exists a unique isomorphism $f : P \to Q$ such that the following diagram commutes:

$$
\begin{tikzcd}
 & P \arrow[dd, "f"] & \\
A \arrow[ur, "i_0"] \arrow[dr, "j_0"] & & B \arrow[ul, "i_1"'] \arrow[dl, "j_1"] \\
 & Q &
\end{tikzcd}
$$

3 Duality

3.1 Duality

Proposition 5. Formal Duality For any sentence Σ in the language of category theory, if Σ follows from the axioms for categories, then so does its dual Σ^*:

$$CT \Rightarrow \Sigma implies CT \Rightarrow \Sigma^*$$

Taking a diagram to illustrate, if this is a statement Σ

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g \circ f"] & B \arrow[d, "g"] \\
 & C
\end{tikzcd}
$$

then this is the dual statement Σ^*

$$
\begin{tikzcd}
A & B \arrow[l, "f"'] \\
 & C \arrow[u, "g"'] \arrow[ul, "f \circ g"]
\end{tikzcd}
$$

Note how close this is to the idea of an opposite category C^{op}.

Proposition 6. Conceptual duality For any statement Σ about categories, if Σ holds for all categories, then so does the dual statement Σ^*.

3.2 Equalizers and Coequalizers

3.2.1 Equalizers

Proposition 7. In any category, if $e : E \to A$ is an equalizer of some pair of arrows, then e is monic.

Proof. Consider the diagram

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[u, shift left, "x"] \arrow[u, shift right, "y"'] \arrow[ur, "z"'] & & \end{tikzcd}$$

in which we assume e is the equalizer of f and g. Supposing $ex = ey$, we want to show $x = y$. Put $z = ex = ey$. Then $fz = fex = gex = gz$, so there is a *unique* $u : Z \to E$ such that $eu = z$. So from $ex = z$ follows that $x = u = y$. □

In SETS, the equalizer would just be the set $x \in A|f(x) = g(x)$.

Suppose $f, g : R^2 \to R$ where $f(x, y) = x^2 + y^2$ and $g = 1$. We take the equalizer, say in TOP, which is the subspace $S = (x, y) \in R^2|X^2 + y^2 = 1 \to R^2$ which is the unit circle in the plane!

Awodey: In abelian groups though, using the fact that

$$f(x) = g(x)$$

iff

$$(f - g)(x) = 0$$

we know that the equalizer of f and g is the same as that of the homomorphism $(f - g)$ and the zero homomorphism $0 : A \to B$, so it suffices to consider equalizers of the special form $A(h, 0) \rightarrowtail A$ for arbitrary homomorphisms $h : A \to B$. This subgroup of A sis the *kernel*.

Cook: In abelian groups: $G \overset{Hom\phi}{\underset{f}{\rightrightarrows}} H$

$$E = \{g \in G|\phi(g) = f(g)\} = \{g \in G|\phi(g) = 1_{+1}\}$$

Is the kernel of a homomorphism by definition, also equalizers don't have to exist.

3.2.2 Coequalizers

$$\begin{tikzcd} A \arrow[r, shift left] \arrow[r, shift right] & B \arrow[r, two heads, "c"] \arrow[dr, "z"'] & Q \arrow[d, dashed, "u"] \\ & & Z \end{tikzcd}$$

This is the weakest equivalence relation that forces $f(a)$ relates $g(a) \forall a \in A$

4 Limits and Colimits

4.1 Cones

Definition 11. A limit for a diagram $D : J \to C$ is a terminal object in Cone(D). A finite limit is a limit for a diagram on a finite index category J.

A cone is a universal pullback.

A category has all finite limits iff it has finite products and equalizers.

Proof. Take a finite diagram

$$D : J \to C$$

.

Consider first the product

$$\prod_{i \in J_0} D_i$$

which has correct projections $p_j : \prod_{i \in J_0} D_i \to D_j$ $\square$

4.2 Pullbacks and Pushins

$$\begin{tikzcd} A \times_X B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & X \end{tikzcd}$$

Pullback of arrows shown is the same as their product in the slice category C/X

5 Exponentials

5.1 Exponentials

Definition 12. For a category C with binary products, an exponential C^B is associated with two objects and an evaluation arrow $\epsilon : A \times B \to$ if

$A \times B \xrightarrow{f} C$

then there exists a unique

$A \times B \xrightarrow{\tilde{f}} C^B$

where $\tilde{f}$ is the transpose of f such that

$$\begin{tikzcd} C^B \times B \arrow[r, "\varepsilon"] & C \\ A \times B \arrow[u, "\tilde{f} \times 1_B"] \arrow[ur, "f"] & \end{tikzcd} \quad and \quad \begin{tikzcd} C^B \\ A \arrow[u, "g"] \end{tikzcd}$$

Check the transpose of the transpose is the thing...

$$\varepsilon \circ f \times 1_B = \tilde{f}$$

$$\bar{\tilde{f}} = \varepsilon \circ (\tilde{f} \times 1_B) = f$$

$$\bar{g} = \varepsilon \circ (g \times 1_B) \therefore \tilde{\bar{g}} = g$$

In set:

$$C^B = \{f : B \to C\}$$

$$\varepsilon : C^B \times B \to C$$

$$\varepsilon(f, b) = f(b)$$

So...

$$f : A \times B \to C$$

$$f(a, b) = c$$

$$f(\tilde{a}) = f(a, *) : B \to C$$

$$g : A \to C^B$$

$$\bar{g} : A \times B \to C$$

$$\bar{g}(a, b) = (g(a))(b)$$

This is just currying!

5.2 More Exponentials

Another way to write $A \times B \to C$ is as $A \to C^B$, meaning $Hom(A \times B, C) \cong Hom(A, C^B)$ is natural. This is an example of a left adjoint $(- \times B)$ and a right adjoint $(-^B)$.

TODO: Example 6.6 with graphs pg 124

$-^A : C \to C$ is a functor. For a functor we need to know what it does with objects and with morphisms.

Define $\beta : B^A \to C^A$ and $\varepsilon :< TODO >$. then $B \longmapsto B^A$ $\quad B \xrightarrow{\beta} C \quad B^A \xrightarrow{\beta \circ \varepsilon} C^A$

6 Cartesian Closed Categories

Cartesian Closed Categories - Finite exponentials and finite products.

Exponential object - Functors from $\mathcal{A}$ to $\mathcal{B}$ $\mathcal{B}^{\mathcal{A}} F : \mathcal{A} \to \mathcal{B}$

In the category of posets, which are Transitive and anti-symmetric, the objects are just sets X and the morphisms are defined as

meaning that if there is a morphism between two objects, there is only one.

7 Naturality

7.1 Functors

Definition 13. Functors map between categories while respecting composition and identity.

$$F : C \to D$$
$$F : obj(C) \to obj(D)$$
$$F : hom(C) \to hom(D)$$

$$A \xrightarrow{f} B$$

$$F(A) \xrightarrow{F(f)} F(B)$$

7.2 Natural Transformations

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=L}] & \mathcal{D} \arrow[from=U, to=L, Rightarrow, "\eta"] \end{tikzcd}$$

η is a natural transformation $\forall C \in obj(\mathcal{C})$ we have a morphism

$$F(C) \xrightarrow{\eta_C} G(C)$$

such that given $f : A \to B$ in $\mathcal{C}$

$$\begin{tikzcd} F(A) \arrow[r, "\eta_A"] \arrow[d, "F(f)"] & G(A) \arrow[d, "G(f)"] \\ F(B) \arrow[r, "\eta_B"] & G(B) \end{tikzcd}$$

We also have the identity: $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "F"'{name=L}] & \mathcal{D} \arrow[from=U, to=L, Rightarrow, "1_F"] \end{tikzcd}$

Composition and communitivity work here too!

Exponential of Categories of Categories: $\mathcal{D}^{\mathcal{C}}$ -wow-

7.3 Representable Functors

Let $\mathcal{C}$ be a locally small category, $C \in obj(\mathcal{C})$ and $Hom_{\mathcal{C}}(C, -) : \mathcal{C} \to Set$. If the morphisms are set function, this will be a faithful (injective) mapping.

Contravariant of a representable functor $Hom_{\mathcal{C}}(-, C) : C^{op} \to Set$

Ring of continuous functions $C(X)$ -wow- $f : X \to R | fiscontinuous$

8 Misc

Every topology provides a complete Heyting algebra in the form of its open set lattice.

8.1 Constructions on Categories

$C \times D$ has elements (A, B), we define $f : A \to C$ $g : B \to D$, $h : A \times B \to C \times D$ or $h : (f, g)$

Projections

$$\begin{tikzcd} & C \times D \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ C & & D \end{tikzcd}$$

$\pi_1(1_A, 1_B) = 1_A = 1_{\pi_1(A,B)}$

Opposite Category For any category C, the opposite category is notated C^{op}. It is the same but with the arrows reversed.

Op is its own inverse

$$(C^{op})^{op} = C$$

Op preservers products, functors, and slices

$$(C \times D)^{op} = C^{op} \times D^{op}$$

$$(F(C, D))^{op} = F(C^{op}, C^{op}$$

Arrow Category

Definition 14. Monoids One object category, group without inverses Natural Numbers with addition