0.1 Exponentials

Definition 1. For a category C with binary products, an exponential C^B is associated with two objects and an evaluation arrow $\epsilon : A \times B \to$ if

$A \times B \xrightarrow{f} C$

then there exists a unique

$A \times B \xrightarrow{\tilde{f}} C^B$

where $\tilde{f}$ is the transpose of f such that

$$\begin{tikzcd} C^B \times B \arrow[r, "\varepsilon"] & C & & C^B \\ A \times B \arrow[u, "\tilde{f} \times 1_B"] \arrow[ur, "f"] & & \text{and} & A \arrow[u, "g"'] \end{tikzcd}$$

Check the transpose of the transpose is the thing...

$$\varepsilon \circ f \times 1_B = \tilde{f}$$

$$\bar{\tilde{f}} = \varepsilon \circ (\tilde{f} \times 1_B) = f$$

$$\bar{g} = \varepsilon \circ (g \times 1_B) \therefore \tilde{\bar{g}} = g$$

In set:

$$C^B = \{f : B \to C\}$$

$$\varepsilon : C^B \times B \to C$$

$$\varepsilon(f, b) = f(b)$$

So...

$$f : A \times B \to C$$

$$f(a, b) = c$$

$$f\tilde{(a)} = f(a, *) : B \to C$$

$$g : A \to C^B$$

$$\bar{g} : A \times B \to C$$

$$\bar{g}(a, b) = (g(a))(b)$$

This is just currying!

0.2 More Exponentials

Another way to write $A \times B \to C$ is as $A \to C^B$, meaning $Hom(A \times B, C) \cong Hom(A, C^B)$ is natural. This is an example of a left adjoint $(- \times B)$ and a right adjoint $(-^B)$.

TODO: Example 6.6 with graphs pg 124

$-^A : C \to C$ is a functor. For a functor we need to know what it does with objects and with morphisms.

Define $\beta : B^A \to C^A$ and $\varepsilon :< TODO >$. then $B \longmapsto B^A \qquad B \xrightarrow{\beta} C \qquad B^A \xrightarrow{\beta \circ \varepsilon} C^A$