Definition 1. A category C consists of objects $obj(C)$ and arrows $hom(C)$.

Categories focus on the relation between themselves, the arrows are the important parts. Here is an example category showing function composition with a commutative diagram.

$$\begin{tikzcd} & A \arrow[dr, "g"] & \\ B \arrow[ur, "f"] \arrow[rr, "g \circ f"] & & C \end{tikzcd}$$

And a quick check of why these are called commutative diagrams

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Remark. For any category C there is always an identity arrow 1_C though it would clutter diagrams if it were written every time. $C \arrow[loop right, "id_C"]$

Proposition 1. There is always one unique identity homomorphism. $\exists! 1_A : A \to A$

Proof. Assume there are two unique identity morphisms from category A, 1 and $1'$ as shown in the diagram below.

$$\begin{tikzcd} A \arrow[r, shift left, red, "1"] \arrow[r, shift right, blue, "1'"'] & A \end{tikzcd}$$

Then composing these two homomorphisms makes a contradiction.

$$1 = 1 \circ 1' = 1'$$

□

Remark. A homomorphism moving between a category and itself is also know as an endomorphism.

Definition 2. Small and Locally small categories Let C be a category

- if all $hom(C)$'s together form a set, the category is small
- if hom are all sets, the category is locally small

Some examples of categories:

- SET - The category of all sets with mappings between them is locally small but not small
$$\mathcal{P}(\mathbb{X}) = \{A \subseteq \mathbb{X}\} = 2^{\mathbb{X}}$$
B^A :: All functions from $A \to B$
- Grp - An object is a group and a map $G \to H$ is a group homomorphism

- Ab - Abelian groups under homomorphism
- Top - Topological Spaces with continuous maps
- Vect - An object is a vector space and a map $V \to W$ is a linear map

Definition 3. Let A and B be objects in a category. Them a map $f : A \to B$ is an isomorphism is the is a map $f^{-1} : B \to A$ (the inverse of f) such that $f^{-1} \circ f = Id_A$ and $f \circ f^{-1} = Id_B$.

If there exists an isomorphism between A and B, we say that A and B are isomorphic and write $A \cong B$.

Proposition 2. In Set, a map is an isomorphism iff it is a bijection. Two sets are isomorphic off they have the same cardinality.