0.1 Exponentials

Definition 1. For a category C with binary products, an exponential C^B is associated with two objects and an evaluation arrow $\epsilon : A \times B \to$ if

$$A \times B \xrightarrow{\ f\ } C$$

then there exists a unique

$$A \times B \xrightarrow{\ \tilde{f}\ } C^B$$

where $\tilde{f}$ is the transpose of f such that

$$
\begin{array}{ccc}
C^B \times B \xrightarrow{\ \varepsilon\ } C & & C^B \\
{\scriptstyle \tilde{f} \times 1_B} \uparrow \quad {\scriptstyle f} \nearrow & and & {\scriptstyle g} \uparrow \\
A \times B & & A
\end{array}
$$

Check the transpose of the transpose is the thing...

$$\varepsilon \circ f \times 1_B = \tilde{f}$$
$$\bar{\tilde{f}} = \varepsilon \circ (\tilde{f} \times 1_B) = f$$
$$\bar{g} = \varepsilon \circ (g \times 1_B) \therefore \tilde{\bar{g}} = g$$

In set:

$$C^B = \{ f : B \to C \}$$
$$\varepsilon : C^B \times B \to C$$
$$\varepsilon(f, b) = f(b)$$

So...

$$f : A \times B \to C$$
$$f(a, b) = c$$
$$\tilde{f}(a) = f(a, *) : B \to C$$
$$g : A \to C^B$$
$$\bar{g} : A \times B \to C$$
$$\bar{g}(a, b) = (g(a))(b)$$

This is just currying!

0.2 More Exponentials

Another way to write $A \times B \to C$ is as $A \to C^B$, meaning $Hom(A \times B, C) \cong Hom(A, C^B)$ is natural. This is an example of a left adjoint $(- \times B)$ and a right adjoint $(-^B)$.

   TODO: Example 6.6 with graphs pg 124

   $-^A : C \to C$ is a functor. For a functor we need to know what it does with objects and with morphisms.

   Define $\beta : B^A \to C^A$ and $\varepsilon :< TODO >$. then $\ B \longmapsto B^A \qquad\quad B \xrightarrow{\ \beta\ } C \qquad\quad B^A \xrightarrow{\ \beta \circ \varepsilon\ } C^A$

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