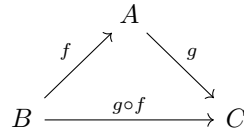


**Definition 1.** A category  $C$  consists of objects  $obj(C)$  and arrows  $hom(C)$

Categories focus on the relation between themselves, the arrows are the important parts. Here is an example category showing function composition with a commutative diagram.



And a quick check of why these are called commutative diagrams

$$h \circ (g \circ f) = (h \circ g) \circ f$$

*Remark.* For any category  $C$  there is always an identity arrow  $1_C$  though it would clutter diagrams if it were written every time.  $C \curvearrowright id_C$

*Proposition 1.* There is always one unique identity homomorphism.  $\exists! 1_A : A \rightarrow A$

*Proof.* Assume there are two unique identity morphisms from category  $A$ ,  $1$  and  $1'$  as shown in the diagram below.

$$A \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{1'} \end{array} A$$

Then composing these two homomorphisms makes a contradiction.

$$1 = 1 \circ 1' = 1'$$

□

*Remark.* A homomorphism moving between a category and itself is also known as an endomorphism.

**Definition 2.** Small and Locally small categories Let  $C$  be a category

- if all  $hom(C)$ 's together form a set, the category is small
- if  $hom$  are all sets, the category is locally small

Some examples of categories:

- SET - The category of all sets with mappings between them is locally small but not small

$$\mathcal{P}(\mathbb{X}) = \{A \subseteq \mathbb{X}\} = 2^{\mathbb{X}}$$

$B^A$  :: All functions from  $A \rightarrow B$

- Grp - An object is a group and a map  $G \rightarrow H$  is a group homomorphism

- Vect - An object is a vector space and a map  $V \rightarrow W$  is a linear map

**Definition 3.** Let  $A$  and  $B$  be objects in a category. Then a map  $f : A \rightarrow B$  is an isomorphism if there is a map  $f^{-1} : B \rightarrow A$  (the inverse of  $f$ ) such that  $f^{-1} \circ f = Id_A$  and  $f \circ f^{-1} = Id_B$ .

If there exists an isomorphism between  $A$  and  $B$ , we say that  $A$  and  $B$  are isomorphic and write  $A \cong B$ .

*Proposition 2.* In Set, a map is an isomorphism iff it is a bijection. Two sets are isomorphic iff they have the same cardinality.

**Definition 4.** An object  $0$  is an initial object if for every object  $A$ , there is a unique map  $0 \rightarrow A$

**Definition 5.** Disjoint Union The disjoint union of two sets  $A$  and  $B$  is the set

$$A \sqcup B = \{(0, a) : a \in A\} \cup \{(1, b) : b \in B\}.$$

**Definition 6** (Coproduct). Let  $A$  and  $B$  be objects in a category. Then a sum (or coproduct) of  $A$  and  $B$  is an object  $A+B$  together with maps  $i_0 : A \rightarrow A+B$  and  $i_1 : B \rightarrow A+B$  such that whenever we have an object  $C$  and maps  $f_0 : A \rightarrow C$  and  $f_1 : B \rightarrow C$ , there is a unique map  $f : A+B \rightarrow C$  such that  $f \circ i_0 = f_0$  and  $f \circ i_1 = f_1$

$$\begin{array}{ccccc} A & \xrightarrow{i_0} & A+B & \xleftarrow{i_1} & B \\ & \searrow f_0 & & \swarrow f_1 & \\ & & C & & \end{array}$$

*Theorem 0.1.* Let  $A$  and  $B$  be objects and let  $A \rightarrow [i_0]P \leftarrow [i_1]B$  and  $A \rightarrow [j_0]Q \leftarrow [j_1]B$  be two sums of  $A$  and  $B$ . Then there exists a unique isomorphism  $f : P \rightarrow Q$  such that the following diagram commutes:

$$\begin{array}{ccccc} & & P & & \\ & \nearrow i_0 & \downarrow f & \nwarrow i_1 & \\ A & & & & B \\ & \searrow j_0 & & \swarrow j_1 & \\ & & Q & & \end{array}$$