Definition 1. A category C consists of objects $obj(C)$ and arrows $hom(C)$

Categories focus on the relation between themselves, the arrows are the important parts. Here is an example category showing function composition with a commutative diagram.

$$
\begin{tikzcd}
 & A \arrow[dr, "g"] & \\
B \arrow[ur, "f"] \arrow[rr, "g \circ f"] & & C
\end{tikzcd}
$$

And a quick check of why these are called commutative diagrams

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Remark. For any category C there is always an identity arrow 1_C though it would clutter diagrams if it were written every time. $\begin{tikzcd} C \arrow[loop right, "id_C"] \end{tikzcd}$

Proposition 1. There is always one unique identity homomorphism. $\exists ! 1_A : A \to A$

Proof. Assume there are two unique identity morphisms from category A, 1 and $1'$ as shown in the diagram below.

$$
\begin{tikzcd}
A \arrow[r, shift left, "1"] \arrow[r, shift right, "1'"'] & A
\end{tikzcd}
$$

Then composing these two homomorphisms makes a contradiction.

$$1 = 1 \circ 1' = 1'$$

□

Remark. A homomorphism moving between a category and itself is also know as an endomorphism.

Definition 2. Small and Locally small categories Let C be a category

- if all $hom(C)$'s together form a set, the category is small
- if hom are all sets, the category is locally small

Some examples of categories:

- SET - The category of all sets with mappings between them is locally small but not small
$$\mathcal{P}(\mathbb{X}) = \{A \subseteq \mathbb{X}\} = 2^{\mathbb{X}}$$
B^A :: All functions from $A \to B$
- Grp - An object is a group and a map $G \to H$ is a group homomorphism

- Vect - An object is a vector space and a map $V \to W$ is a linear map

Definition 3. Let A and B be objects in a category. Them a map $f : A \to B$ is an isomorphism is the is a map $f^{-1} : B \to A$ (the inverse of f) such that $f^{-1} \circ f = Id_A$ and $f \circ f^{-1} = Id_B$.

If there exists an isomorphism between A and B, we say that A and B are isomorphic and write $A \cong B$.

Proposition 2. In Set, a map is an isomorphism iff it is a bijection. Two sets are isomorphic off they have the same cardinality.

Definition 4. An object 0 is an initial object if for every object A, there is a unique map $0 \to A$

Definition 5. Disjoint Union The disjoint union of two sets A and B is the set

$$A \sqcup B = (0, a) : a \in A \cup (1, b) : b \in B.$$

Definition 6 (Coproduct)**.** Let A and B be objects in a category. Then a sum (or coproduct) of A and B is an object $A+B$ together with maps $i_0 : A \to A+B$ and $i_1 : B \to A+B$ such that whenever we have an object C and maps $f_0 : A \to C$ and $f_1 : B \to C$, there is a unique map $f : A+B \to C$ such that $f_0 = fi_0$ and $f_1 = fi_1$

$$\begin{tikzcd} A \arrow[r, "i_0"] \arrow[dr, "f_0"'] & A+B & B \arrow[l, "i_1"'] \arrow[dl, "f_1"] \\ & C & \end{tikzcd}$$

Theorem 0.1. Let A and B be objects and lot $A \to [\text{"}i_0\text{"}]P \leftarrow [\text{"}i_1\text{"}]B$ and $A \to [\text{"}j_0\text{"}]Q \leftarrow [\text{"}j_1\text{"}]B$ be two sums of A and B. Then there exists a unique isomorphism $f : P \to Q$ such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dd, "f"] & \\ A \arrow[ur, "i_0"] \arrow[dr, "j_0"'] & & B \arrow[ul, "i_1"'] \arrow[dl, "j_1"] \\ & Q & \end{tikzcd}$$