1 Introduction

Definition 1. A category C consists of objects $obj(C)$ and arrows $hom(C)$ which satisfy

Categories focus on the relation between themselves, the arrows are the important parts. Here is an example category showing function composition with a commutative diagram.

$$\begin{tikzcd} & A \arrow[dr, "g"] & \\ B \arrow[ur, "f"] \arrow[rr, "g \circ f"] & & C \end{tikzcd}$$

And a quick check of why these are called commutative diagrams

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Remark. For any category C there is always an identity arrow 1_C though it would clutter diagrams if it were written every time. $C \circlearrowleft id_C$

Proposition 1. There is always one unique identity homomorphism. $\exists! 1_A : A \to A$

Proof. Assume there are two unique identity morphisms from category A, 1 and $1'$ as shown in the diagram below.

$$\begin{tikzcd} A \arrow[r, shift left, "1"] \arrow[r, shift right, "1'"'] & A \end{tikzcd}$$

Then composing these two homomorphisms makes a contradiction.

$$1 = 1 \circ 1' = 1'$$

$\square$

Remark. A homomorphism moving between a category and itself is also know as an endomorphism.

Definition 2. Small and Locally small categories Let C be a category

- if all $hom(C)$'s together form a set, the category is small
- if hom are all sets, the category is locally small

Some examples of categories:

- SET - The category of all sets with mappings between them is locally small but not small
  $$\mathcal{P}(\mathbb{X}) = \{A \subseteq \mathbb{X}\} = 2^{\mathbb{X}}$$
  B^A :: All functions from $A \to B$

- Grp - An object is a group and a map $G \to H$ is a group homomorphism
- Vect - An object is a vector space and a map $V \to W$ is a linear map

Definition 3. Let A and B be objects in a category. Them a map $f : A \to B$ is an isomorphism is the is a map $f^{-1} : B \to A$ (the inverse of f) such that $f^{-1} \circ f = Id_A$ and $f \circ f^{-1} = Id_B$.

If there exists an isomorphism between A and B, we say that A and B are isomorphic and write $A \cong B$.

Proposition 2. In Set, a map is an isomorphism iff it is a bijection. Two sets are isomorphic off they have the same cardinality.

2 Epis and monos

Definition 4. In any category C, an arrow

$$f : A \to B$$

is called a *monomorphism* if given any $g, h : C \to A$, $gh = fh$ implies $g = h$

epimorphism if given any $i, j : B \to D, if = jf$ implies $i = j$

Having a left inverse is monic and having a right inverse is epic. Having both makes the mapping an isomorphism. In SETS, the converse of the previous is also true: every mono-epi is iso; but this is not true in the general case.

This definition of monomorphism is the category theory equivalent to injective and this definition of epimorphism is the surjective translation.

Proposition 3. A *function* $f : A \to B$ between sets is monic just in case it is injective.

Definition 5. An object 0 is an initial object if for every object A, there is a unique map $0 \to A$

Definition 6. Disjoint Union The disjoint union of two sets A and B is the set

$$A \sqcup B = (0, a) : a \in A \cup (1, b) : b \in B.$$

Definition 7. Coproduct Let A and B be objects in a category. Then a sum (or coproduct) of A and B is an object $A+B$ together with maps $i_0 : A \to A+B$ and $i_1 : B \to A+B$ such that whenever we have an object C and maps $f_0 : A \to C$ and $f_1 : B \to C$, there is a unique map $f : A+B \to C$ such that $f_0 = fi_0$ and $f_1 = fi_1$

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[dr, "f_0"'] & A+B & B \arrow[l, "i_1"'] \arrow[dl, "f_1"] \\
 & C &
\end{tikzcd}$$

Theorem 2.1. Let A and B be objects and lot $A \rightarrow$ ["i_0"]$P \leftarrow$ ["i_1"]B and $A \rightarrow$ ["j_0"]$Q \leftarrow$ ["j_1"]B be two sums of A and B. Then there exists a unique isomorphism $f : P \rightarrow Q$ such that the following diagram commutes:

$$
\begin{tikzcd}
& P \arrow[dd, "f"] & \\
A \arrow[ur, "i_0"] \arrow[dr, "j_0"] & & B \arrow[ul, "i_1"'] \arrow[dl, "j_1"] \\
& Q &
\end{tikzcd}
$$