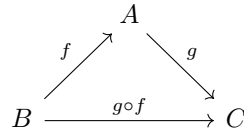


# 1 Categories

**Definition 1.** A category  $C$  consists of objects  $obj(C)$  and arrows  $hom(C)$  which satisfy

Categories focus on the relation between themselves, the arrows are the important parts. Here is an example category showing function composition with a commutative diagram.



And a quick check of why these are called commutative diagrams

$$h \circ (g \circ f) = (h \circ g) \circ f$$

*Remark.* For any category  $C$  there is always an identity arrow  $1_C$  though it would clutter diagrams if it were written every time.  $C \ni id_C$

*Proposition 1.* There is always one unique identity homomorphism.  $\exists! 1_A : A \rightarrow A$

*Proof.* Assume there are two unique identity morphisms from category  $A$ ,  $1$  and  $1'$  as shown in the diagram below.

$$A \begin{array}{c} \xrightarrow{1} \\ \xrightarrow{1'} \end{array} A$$

Then composing these two homomorphisms makes a contradiction.

$$1 = 1 \circ 1' = 1'$$

□

*Remark.* A homomorphism moving between a category and itself is also known as an endomorphism.

**Definition 2.** Small and Locally small categories Let  $C$  be a category

- if all  $hom(C)$ 's together form a set, the category is small
- if  $hom$  are all sets, the category is locally small

Some examples of categories:

- SET - The category of all sets with mappings between them is locally small but not small

$$\mathcal{P}(\mathbb{X}) = \{A \subseteq \mathbb{X}\} = 2^{\mathbb{X}}$$

$B^A ::$  All functions from  $A \rightarrow B$

- Grp - An object is a group and a map  $G \rightarrow H$  is a group homomorphism
- Ab - Abelian groups under homomorphism
- Top - Topological Spaces with continuous maps
- Vect - An object is a vector space and a map  $V \rightarrow W$  is a linear map

**Definition 3.** Let  $A$  and  $B$  be objects in a category. Then a map  $f : A \rightarrow B$  is an isomorphism if there is a map  $f^{-1} : B \rightarrow A$  (the inverse of  $f$ ) such that  $f^{-1} \circ f = Id_A$  and  $f \circ f^{-1} = Id_B$ .

If there exists an isomorphism between  $A$  and  $B$ , we say that  $A$  and  $B$  are isomorphic and write  $A \cong B$ .

*Proposition 2.* In Set, a map is an isomorphism iff it is a bijection. Two sets are isomorphic iff they have the same cardinality.

## 2 Abstract Structures

### 2.1 Epis and Monos

**Definition 4.** In any category  $C$ , an arrow

$$f : A \rightarrow B$$

is called a *monomorphism* if given any  $g, h : C \rightarrow A$ ,  $gh = fh$  implies  $g = h$

*epimorphism* if given any  $i, j : B \rightarrow D$ ,  $if = jf$  implies  $i = j$

Remember having a left inverse is monic and having a right inverse is epic. Having both makes the mapping an isomorphism. In SETS, the converse of the previous is also true: every mono-epi is iso; but this is not true in the general case.

This definition of monomorphism is the category theory equivalent to injective and this definition of epimorphism is the surjective translation.

*Proposition 3.* A function  $f : A \rightarrow B$  between sets is monic just in case it is injective.

**Definition 5.** Product

In any category  $C$ , a product diagram for the objects  $A$  and  $B$  consists of an object  $P$  and arrows satisfying the universal mapping property: There is some  $u : X \rightarrow U$  such that  $x_1 = p_1u$  and  $x_2 = p_2u$ . Given any  $v : X \rightarrow U$ , if  $p_1v = x_1$  and  $p_2v = x_2$  then  $v = u$ .

An example: Let us consider the category of types of the simply typed  $\lambda$ -calculus. The  $\lambda$ -calculus is a formalism for the specification and manipulation of functions, based on the notions of "binding variables" and function evaluation. The relation  $a \sim b$  (usually called  $\beta\eta$ -equivalence) on terms is defined to be

the equivalence relation generated by the equations, and the remaining bound variables:

$$\lambda x.b = \lambda y.b[y/x](\text{noyinb})$$

The category of types  $C(\lambda)$  is now defined as follows:

- Objects: the types
- Arrows  $A \rightarrow B$ : closed terms  $c : A \rightarrow B$ , identified if  $c \equiv c'$ ,
- Identities  $1_A = \lambda x.x(\text{where } x : A)$
- Composition  $c \circ b = \lambda x.c(bx)$ .

**Definition 6.** A category  $C$  is said to have all finite products if it has a terminal object and all binary products (and therewith products of any finite cardinality). The category  $C$  has all (small) products if every set of objects in  $C$  has a product.

**Definition 7.** Slice Category Let  $C$  be a category, and  $I$  be a  $C$ -object. Then the category  $C/I$ , the slice category over  $I$ , has the following data.

- The objects are pairs  $(A, f)$  where  $A$  is an object in  $C$  and  $f : A \rightarrow I$  is an arrow.
- An arrow from  $(A, f)$  to  $(B, g)$  is an arrow  $j : A \rightarrow B$  such that  $g \circ j = f$  in  $C$
- The identity arrow on  $(A, f)$  is the arrow  $1_A : A \rightarrow A$ .
- Given arrows  $j : (A, f) \rightarrow (B, g)$  and  $k : (B, g) \rightarrow (C, h)$ , their composition  $k \circ j : (A, f) \rightarrow (C, h)$  is the arrow  $k \circ j : A \rightarrow C$ .

**Definition 8.** An object  $0$  is an initial object if for every object  $A$ , there is a unique map  $0 \rightarrow A$

*Proposition 4.* Initial and terminal objects are unique up to isomorphism.

*Proof.* Suppose that  $0$  and  $0'$  are both terminal or initial objects in some category  $C$ ; this diagram states that  $0$  and  $0'$  are uniquely isomorphic.

For terminal objects, apply the previous to  $C^{op}$ .  $\square$

**Definition 9.** Disjoint Union The disjoint union of two sets  $A$  and  $B$  is the set

$$A \sqcup B = \{(0, a) : a \in A\} \cup \{(1, b) : b \in B\}.$$

**Definition 10.** Coproduct Let  $A$  and  $B$  be objects in a category. Then a sum (or coproduct) of  $A$  and  $B$  is an object  $A+B$  together with maps  $i_0 : A \rightarrow A+B$  and  $i_1 : B \rightarrow A+B$  such that whenever we have an object  $C$  and maps  $f_0 : A \rightarrow C$  and  $f_1 : B \rightarrow C$ , there is a unique map  $f : A+B \rightarrow C$  such that  $f \circ i_0 = f_0$  and  $f \circ i_1 = f_1$

$$\begin{array}{ccccc} A & \xrightarrow{i_0} & A+B & \xleftarrow{i_1} & B \\ & \searrow f_0 & & \swarrow f_1 & \\ & & C & & \end{array}$$

*Theorem 2.1.* Let  $A$  and  $B$  be objects and let  $A \rightarrow [i_0]P \leftarrow [i_1]B$  and  $A \rightarrow [j_0]Q \leftarrow [j_1]B$  be two sums of  $A$  and  $B$ . Then there exists a unique isomorphism  $f : P \rightarrow Q$  such that the following diagram commutes:

$$\begin{array}{ccc}
 & P & \\
 i_0 \nearrow & \downarrow f & \nwarrow i_1 \\
 A & & B \\
 j_0 \searrow & & \swarrow j_1 \\
 & Q &
 \end{array}$$

### 3 Duality

*Proposition 5.* Formal Duality For any sentence  $\Sigma$  in the language of category theory, if  $\Sigma$  follows from the axioms for categories, then so does its dual  $\Sigma^*$ :

$$CT \Rightarrow \Sigma \text{ implies } CT \Rightarrow \Sigma^*$$

Taking a diagram to illustrate, if this is a statement  $\Sigma$

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 & \searrow g \circ f & \downarrow g \\
 & & C
 \end{array}$$

then this is the dual statement  $\Sigma^*$

$$\begin{array}{ccc}
 A & \xleftarrow{f} & B \\
 & \swarrow f \circ g & \uparrow g \\
 & & C
 \end{array}$$

Note how close this is to the idea of an opposite category  $C^{op}$ .

*Proposition 6.* Conceptual duality For any statement  $\Sigma$  about categories, if  $\Sigma$  holds for all categories, then so does the dual statement  $\Sigma^*$ .

## 4 Equalizers and Coequalizers

### 4.1 Equalizers

*Proposition 7.* In any category, if  $e : E \rightarrow A$  is an equalizer of some pair of arrows, then  $e$  is monic.

*Proof.* Consider the diagram

$$\begin{array}{ccccc} E & \xrightarrow{e} & A & \xrightleftharpoons[g]{f} & B \\ \uparrow x & & \nearrow z & & \\ Z & & & & \end{array}$$

in which we assume  $e$  is the equalizer of  $f$  and  $g$ . Supposing  $ex = ey$ , we want to show  $x = y$ . Put  $z = ex = ey$ . Then  $fz = fex = gex = gz$ , so there is a unique  $u : Z \rightarrow E$  such that  $eu = z$ . So from  $ex = z$  follows that  $x = u = y$ .  $\square$

In SETS, the equalizer would just be the set  $x \in A \mid f(x) = g(x)$ .

Suppose  $f, g : R^2 \rightarrow R$  where  $f(x, y) = x^2 + y^2$  and  $g = 1$ . We take the equalizer, say in TOP, which is the subspace  $S = (x, y) \in R^2 \mid x^2 + y^2 = 1 \rightarrow R^2$  which is the unit circle in the plane!

Awodey: In abelian groups though, using the fact that

$$f(x) = g(x)$$

iff

$$(f - g)(x) = 0$$

we know that the equalizer of  $f$  and  $g$  is the same as that of the homomorphism  $(f - g)$  and the zero homomorphism  $0 : A \rightarrow B$ , so it suffices to consider equalizers of the special form  $A(h, 0) \rightarrowtail A$  for arbitrary homomorphisms  $h : A \rightarrow B$ . This subgroup of  $A$  is the *kernel*.

Cook: In abelian groups:  $G \xrightleftharpoons[f]{Hom \phi} H$

$$E = \{g \in G \mid \phi(g) = f(g)\} = \{g \in G \mid \phi(g) = 1_{+1}\}$$

Is the kernel of a homomorphism by definition, also equalizers don't have to exist.

## 4.2 Coequalizers

$$\begin{array}{ccccc} A & \rightrightarrows & B & \xrightarrow{c} & Q \\ & & \searrow z & & \downarrow u \\ & & & & Z \end{array}$$

This is the weakest equivalence relation that forces  $f(a)$  relates  $g(a) \forall a \in A$

## 5 Limits and Colimits

## 6 Cones

**Definition 11.** A limit for a diagram  $D : J \rightarrow C$  is a terminal object in  $\text{Cone}(D)$ . A finite limit is a limit for a diagram on a finite index category  $J$ .

A cone is a universal pullback.

A category has all finite limits iff it has finite products and equalizers.

*Proof.* Take a finite diagram

$$D : J \rightarrow C$$

Consider first the product

$$\prod_{i \in J_0} D_i$$

which has correct projections  $p_j : \prod_{i \in J_0} D_i \rightarrow D_j$  □

## 7 Pullbacks and Pushins

$$\begin{array}{ccc} A \times_X B & \xrightarrow{\pi_2} & B \\ \downarrow \pi_1 & & \downarrow g \\ A & \xrightarrow{f} & X \end{array}$$
 Pullback of arrows shown is the same as their product in  
 the slice category  $C/X$

## 8 Exponentials

## 9 Exponential

**Definition 12.** For a category  $C$  with binary products, an exponential  $C^B$  is associated with two objects and an evaluation arrow  $\epsilon : A \times B \rightarrow C$  if

$$A \times B \xrightarrow{f} C$$

then there exists a unique

$$A \times B \xrightarrow{\tilde{f}} C^B$$

where  $\tilde{f}$  is the transpose of  $f$  such that

$$\begin{array}{ccc} C^B \times B & \xrightarrow{\epsilon} & C \\ \tilde{f} \times 1_B \uparrow & \nearrow f & \\ A \times B & & \end{array} \quad \text{and} \quad \begin{array}{ccc} C^B & & \\ g \uparrow & & \\ A & & \end{array}$$

Check the transpose of the transpose is the thing...

$$\begin{aligned}\varepsilon \circ f \times 1_B &= \tilde{f} \\ \tilde{\tilde{f}} &= \varepsilon \circ (\tilde{f} \times 1_B) = f \\ \bar{g} &= \varepsilon \circ (g \times 1_B) \therefore \tilde{\tilde{g}} = g\end{aligned}$$

In set:

$$\begin{aligned}C^B &= \{f : B \rightarrow C\} \\ \varepsilon : C^B \times B &\rightarrow C \\ \varepsilon(f, b) &= f(b)\end{aligned}$$

So...

$$\begin{aligned}f : A \times B &\rightarrow C \\ f(a, b) &= c \\ f(\tilde{a}) &= f(a, *) : B \rightarrow C \\ g : A &\rightarrow C^B \\ \bar{g} : A \times B &\rightarrow C \\ \bar{g}(a, b) &= (g(a))(b)\end{aligned}$$

This is just currying!

## 9.1 More Exponentials

Another way to write  $A \times B \rightarrow C$  is as  $A \rightarrow C^B$ , meaning  $Hom(A \times B, C) \cong Hom(A, C^B)$  is natural. This is an example of a left adjoint  $(- \times B)$  and a right adjoint  $(-^B)$ .

TODO: Example 6.6 with graphs pg 124

$-^A : C \rightarrow C$  is a functor. For a functor we need to know what it does with objects and with morphisms.

$$\text{Define } \beta : B^A \rightarrow C^A \text{ and } \varepsilon : < TODO >. \text{ then } B \longmapsto B^A \quad B \xrightarrow{\beta} C \quad B^A \xrightarrow{\beta \circ \varepsilon} C^A$$

## 10 Cartesian Closed Categories

Cartesian Closed Categories - Finite exponentials and finite products.

Exponential object - Functors from  $\mathcal{A}$  to  $\mathcal{B}$   $\mathcal{B}^{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{B}$

In the category of posets, which are Transitive and anti-symmetric, the objects are just sets  $X$  and the morphisms are defined as

meaning that if there is a morphism between two objects, there is only one.

## 11 Naturality

**Definition 13.** Functors map between categories while respecting composition and identity.

$$F : C \rightarrow D$$

$$F : \text{obj}(C) \rightarrow \text{obj}(D)$$

$$F : \text{hom}(C) \rightarrow \text{hom}(D)$$

$$A \xrightarrow{f} B$$

$$F(A) \xrightarrow{F(f)} F(B)$$

## 12 Natural Transformations

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \downarrow \eta \\ \xrightarrow{G} \end{array} \mathcal{D}$$

$\eta$  is a natural transformation  $\forall C \in \text{obj}(\mathcal{C})$  we have a morphism

$$F(C) \xrightarrow{\eta_C} G(C)$$

such that given  $f : A \rightarrow B$  in  $\mathcal{C}$

$$\begin{array}{ccc} F(A) & \xrightarrow{\eta_A} & G(A) \\ \downarrow F(f) & & \downarrow G(f) \\ F(B) & \xrightarrow{\eta_B} & G(B) \end{array}$$

We also have the identity:  $\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \downarrow 1_F \\ \xrightarrow{F} \end{array} \mathcal{D}$

Composition and commutativity work here too!

Exponential of Categories of Categories:  $\mathcal{D}^{\mathcal{C}}$  -wow-

### 12.1 Representable Functors

Let  $\mathcal{C}$  be a locally small category,  $C \in \text{obj}(\mathcal{C})$  and  $\text{Hom}_{\mathcal{C}}(C, -) : \mathcal{C} \rightarrow \text{Set}$ . If the morphisms are set function, this will be a faithful (injective) mapping.

Contravariant of a representable functor  $\text{Hom}_{\mathcal{C}}(-, C) : \mathcal{C}^{op} \rightarrow \text{Set}$

Ring of continuous functions  $C(X)$  -wow-  $f : X \rightarrow R$  | *discontinuous*



## 13 Misc

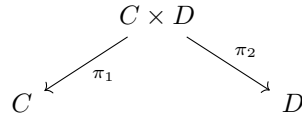
## 14 Misc

Every topology provides a complete heyting algebra in the form of its open set lattice.

### 14.1 Constructions on Categories

$C \times D$  has elements  $(A, B)$ , we define  $f : A \rightarrow C$   $g : B \rightarrow D$ ,  $h : A \times B \rightarrow C \times D$  or  $h : (f, g)$

Projections



$$\pi_1(1_A, 1_B) = 1_A = 1_{\pi_1(A, B)}$$

Opposite Category For any category  $C$ , the opposite category is notated  $C^{op}$ . It is the same but with the arrows reversed.

Op is its own inverse

$$(C^{op})^{op} = C$$

Op preserves products, functors, and slices

$$(C \times D)^{op} = C^{op} \times D^{op}$$

$$(F(C, D))^{op} = F(C^{op}, D^{op})$$

Arrow Category

**Definition 14.** Monoids One object category, group without inverses Natural Numbers with addition