1 Introduction

Definition 1. A category C consists of objects $obj(C)$ and arrows $hom(C)$ which satisfy

Categories focus on the relation between themselves, the arrows are the important parts. Here is an example category showing function composition with a commutative diagram.

$$\begin{tikzcd}
 & A \arrow[dr, "g"] & \\
B \arrow[ur, "f"] \arrow[rr, "g \circ f"] & & C
\end{tikzcd}$$

And a quick check of why these are called commutative diagrams

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Remark. For any category C there is always an identity arrow 1_C though it would clutter diagrams if it were written every time. $\begin{tikzcd} C \arrow[loop right, "id_C"] \end{tikzcd}$

Proposition 1. There is always one unique identity homomorphism. $\exists! 1_A : A \to A$

Proof. Assume there are two unique identity morphisms from category A, 1 and $1'$ as shown in the diagram below.

$$\begin{tikzcd}
A \arrow[r, shift left, "1"] \arrow[r, shift right, "1'"'] & A
\end{tikzcd}$$

Then composing these two homomorphisms makes a contradiction.

$$1 = 1 \circ 1' = 1'$$

□

Remark. A homomorphism moving between a category and itself is also know as an endomorphism.

Definition 2. Small and Locally small categories Let C be a category

- if all $hom(C)$'s together form a set, the category is small
- if hom are all sets, the category is locally small

Some examples of categories:

- SET - The category of all sets with mappings between them is locally small but not small
$$\mathcal{P}(\mathbb{X}) = \{A \subseteq \mathbb{X}\} = 2^{\mathbb{X}}$$
B^A :: All functions from $A \to B$

- Grp - An object is a group and a map $G \to H$ is a group homomorphism
- Vect - An object is a vector space and a map $V \to W$ is a linear map

Definition 3. Let A and B be objects in a category. Them a map $f : A \to B$ is an isomorphism is the is a map $f^{-1} : B \to A$ (the inverse of f) such that $f^{-1} \circ f = Id_A$ and $f \circ f^{-1} = Id_B$.

If there exists an isomorphism between A and B, we say that A and B are isomorphic and write $A \cong B$.

Proposition 2. In Set, a map is an isomorphism iff it is a bijection. Two sets are isomorphic off they have the same cardinality.

2 Epis and Monos

Definition 4. In any category C, an arrow

$$f : A \to B$$

is called a *monomorphism* if given any $g, h : C \to A$, $gh = fh$ implies $g = h$

epimorphism if given any $i, j : B \to D, if = jf$ implies $i = j$

Having a left inverse is monic and having a right inverse is epic. Having both makes the mapping an isomorphism. In SETS, the converse of the previous is also true: every mono-epi is iso; but this is not true in the general case.

This definition of monomorphism is the category theory equivalent to injective and this definition of epimorphism is the surjective translation.

Proposition 3. A *function* $f : A \to B$ between sets is monic just in case it is injective.

Definition 5. An object 0 is an initial object if for every object A, there is a unique map $0 \to A$

Proposition 4. Initial and terminal objects are unique up to isomorphism.

Proof. Suppose that 0 and $0'$ are both terminal or initial objects in some category C; this diagram states that 0 and $0'$ are uniquely isomorphic.

For terminal objects, apply the previous to C^{op}. □

Definition 6. Disjoint Union The disjoint union of two sets A and B is the set

$$A \sqcup B = (0, a) : a \in A \cup (1, b) : b \in B.$$

Definition 7. Coproduct Let A and B be objects in a category. Then a sum (or coproduct) of A and B is an object $A+B$ together with maps $i_0 : A \to A+B$ and $i_1 : B \to A+B$ such that whenever we have an object C and maps $f_0 : A \to C$ and $f_1 : B \to C$, there is a unique map $f : A+B \to C$ such that $f_0 = fi_0$ and $f_1 = fi_1$

$$\begin{tikzcd}
A \arrow[r, "i_0"] \arrow[dr, "f_0"'] & A+B & B \arrow[l, "i_1"'] \arrow[dl, "f_1"] \\
& C &
\end{tikzcd}$$

Theorem 2.1. Let A and B be objects and let $A \to [''i_0'']P \leftarrow [''i_1'']B$ and $A \to [''j_0'']Q \leftarrow [''j_1'']B$ be two sums of A and B. Then there exists a unique isomorphism $f : P \to Q$ such that the following diagram commutes:

$$\begin{tikzcd}
& P \arrow[dd, "f"] & \\
A \arrow[ur, "i_0"] \arrow[dr, "j_0"'] & & B \arrow[ul, "i_1"'] \arrow[dl, "j_1"] \\
& Q &
\end{tikzcd}$$

Definition 8. Product

In any category C, a product diagram for the objects A and B consists of an object P and arrows satisfying the universal mapping property: There is some $u : X \to U$ such that $x_1 = p_1 u$ and $x_2 = p_2 u$. Given any $v : X \to U$, if $p_1 v = x_1$ and $p_2 v = x_2$ then $v = u$.

An example: Let us consider the category of types of the simply typed λ-calculus. The λ -calculus is a formalism for the specfication and manipulation of functions, based on the notions of "binding variables" and function evaluation. The relation $a\ b$ (usually called $\beta\eta$ -equivalence) on terms is defined to be the equivalence relation generated by the equations, and the remaining bound variables:

$$\lambda x.b = \lambda y.b[y/x](noyinb)$$

The category of types $C(\lambda)$ is now defined as follows:

- Objects: the types
- Arrows $A \to B$: closed terms $c : A \to B$, identified if $c\ c'$,
- Identities $1_A = \lambda x.x(wherex : A)$
- Composition $c \circ b = \lambda x.c(bx)$.

Definition 9. A category C is said to have all finite products if it has a terminal object and all binary products (and therewith products of any finite cardinality). The category C has all (small) products if every set of objects in C has a product.

Definition 10. Slice Category Let C be a category, and I be a C-object. Then the category C/I, the slice category over I, has the following data.

- The objects are pairs (A, f) where A is an object in C and $f : A \to I$ is an arrow.
- An arrow from (A, f) to (B, g) is an arrow $j : A \to B$ such that $g \circ j = f$ in C
- The identity arrow on (A, f) is the arrow $1_A : A \to A$.
- Given arrows $j : (A, f) \to (B, g)$ and $k : (B, g) \to (C, h)$, their compostion $k \circ j : (A, f) \to (C, h)$ is the arrow $k \circ j : A \to C$.

Proposition 5. Formal Duality For any sentence Σ in the language of category theory, if Σ follows from the axioms for categories, then so does its dual Σ^*:

$$CT \Rightarrow \Sigma implies CT \Rightarrow \Sigma^*$$

Taking a diagram to illustrate, if this is a statement Σ

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

then this is the dual statement Σ^*

$$\begin{tikzcd} A & B \arrow[l, "f"'] \\ & C \arrow[ul, "f \circ g"] \arrow[u, "g"'] \end{tikzcd}$$

Note how close this is to the idea of an opposite category C^{op}.

Proposition 6. Conceptual duality For any statement Σ about categories, if Σ holds for all categories, then so does the dual statement Σ^*.

3 Equalizers and Coqualizers

3.1 Equalizers

Proposition 7. In any category, if $e : E \to A$ is an eualzire of some pair of arrows, then e is monic.

Proof. Consider the diagram

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[u, shift left, "x"] \arrow[u, shift right, "y"'] \arrow[ur, "z"'] & & \end{tikzcd}$$

in which we assume e is the equalizer of f and g. Supposing $ex = ey$, we want to show $x = y$. Put $z = ex = ey$. Then $fz = fex = gex = gz$, so there is a *unique* $u : Z \to E$ such that $eu = z$. So from $ex = z$ th follows that $x = u = y$. $\square$

In SETS, the qualizer would just be the set $x \in A | f(x) = g(x)$.

Awodey: In abelian groups though, using the fact that

$$f(x) = g(x)$$

iff

$$(f - g)(x) = 0$$

we know that the equalizer of f and g is the same as that of the homomorphism $(f - g)$ and the zero homomorphism $0 : A \to B$, so it suffices to consider equalizers of the special form $A(h, 0) \rightarrowtail A$ for arbitrary homomorphisms $h : A \to B$. This subgroup of A sis the *kernel*.

Cook: In abelian groups: $G \overset{Hom\phi}{\underset{f}{\rightrightarrows}} H$

$$E = \{g \in G | \phi(g) = f(g)\} = \{g \in G | \phi(g) = 1_{+1)}\}$$

Is the kernel by definition, also equalizers don't have to exist.

3.2 Equalizers

4 Equalizers and Coqualizers

4.1 Equalizers

Proposition 8. In any category, if $e : E \to A$ is an eualizer of some pair of arrows, then e is monic.

Proof. Consider the diagram

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
Z \arrow[u, shift left, "x"] \arrow[u, shift right, "y"'] \arrow[ur, "z"'] & &
\end{tikzcd}$$

in which we assume e is the equalizer of f and g. Supposing $ex = ey$, we want to show $x = y$. Put $z = ex = ey$. Then $fz = fex = gex = gz$, so there is a *unique* $u : Z \to E$ such that $eu = z$. So from $ex = z$ th follows that $x = u = y$. $\square$

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4.2 Coequalizers

$$\begin{tikzcd} A \arrow[r, shift left] \arrow[r, shift right] & B \arrow[r, two heads, "c"] \arrow[dr, "z"'] & Q \arrow[d, dashed, "u"] \\ & & Z \end{tikzcd}$$

This is the weakest equivalence relation that forces $f(a)$ relates $g(a) \forall a \in A$