DERIVATIVES OF INVERSE FUNCTIONS

1. Let $g(x) = x^5 + 3x - 2$, and let g^{-1} denote the inverse of g. Then $\left(g^{-1}\right)'(2)$ is equal to:

A) $\frac{1}{83}$ B) $\frac{1}{8}$ C) 1 D) 8 E) 83

2. Let $f(x) = \frac{1}{4}x^3 + x - 1$, and let f^{-1} denote the inverse of f. Then $\left(f^{-1}\right)'(3)$ is equal to:

A) $\frac{4}{31}$ B) $\frac{1}{6}$ C) $\frac{1}{4}$ D) $\frac{1}{3}$ E) $\frac{35}{4}$

3. Let $f(x) = x^5 + x$. Find the value of $\frac{d}{dx}f^{-1}(x)$ at $x = 2$.

A) 81 B) 6 C) $\frac{1}{81}$ D) $\frac{1}{6}$ E) $-\frac{1}{6}$

4. Let $h(x) = x^3 + 2x - 1$, and let h^{-1} denote the inverse of h. Then $\left(h^{-1}\right)'(2)$ is equal to:

A) 14 B) 5 C) $\frac{1}{5}$ D) $\frac{1}{9}$ E) $\frac{1}{14}$

5. If $f(4) = 5$ and $f'(4) = \frac{2}{3}$, then calculate: $\left(f^{-1}\right)'(5)$.

A) $\frac{1}{5}$ B) $\frac{1}{4}$ C) $\frac{2}{3}$ D) $\frac{3}{2}$ E) 4

6. If $g(7) = 3$ *and* $g'(3) = \frac{5}{6}$ *and* $g'(7) = \frac{3}{4}$, then $\left(g^{-1}\right)'(3) = ?$

A) $\frac{3}{4}$ B) $\frac{5}{6}$ C) $\frac{6}{5}$ D) $\frac{4}{3}$ E) $\frac{1}{7}$

7. Let $f(x) = 3x^4 + x$ and let g be the inverse function of f. What is the value of $g'(2)$?

A) $\frac{1}{2}$ B) $-\frac{1}{11}$ C) $\frac{1}{11}$ D) $\frac{1}{13}$ E) $-\frac{1}{13}$

8. Let $f(x) = x^5 + 1$ and let g be the inverse function of f. What is the value of $g'(0)$?

A) -1 B) $\frac{1}{5}$ C) 1 D) $g'(0)$ does not exist

E) $g'(0)$ cannot be determined from the given information

9. The following table shows the values of differentiable functions f and g.

x	f	f'	g	g'
1	2	$\frac{1}{2}$	-3	5
2	3	1	0	4
3	4	2	2	3
4	6	4	3	$\frac{1}{2}$

If $H(x) = f^{-1}(x)$, then $H'(3)$ equals

A) $-\frac{1}{16}$ B) $-\frac{1}{8}$ C) $-\frac{1}{2}$ D) $\frac{1}{2}$ E) 1

10. The following table shows the values of differentiable functions f and g.

x	f	f'	g	g'
-1	$\frac{1}{2}$	-2	-1	6
0	1	-1	3	5
1	2	3	4	4
2	3	4	7	3

If $H(x) = f^{-1}(x)$, then $H'(1)$ equals

A) 1 B) 0 C) *undefined* D) $\frac{1}{2}$ E) -1

11. Let $f(x) = x^3 - \frac{4}{x}$ and let f^{-1} denote the inverse of f. Then $\left(f^{-1}\right)'(6)$ is equal to:

A) $\frac{1}{13}$ B) $\frac{1}{6}$ C) $\frac{1}{12}$ D) $-\frac{1}{12}$ E) $-\frac{1}{6}$

12. Let $h(x) = \sqrt{x-4}$ and let h^{-1} denote the inverse of h. Then $\left(h^{-1}\right)'(2)$ is equal to:

A) $\frac{1}{4}$ B) $\frac{1}{8}$ C) 8 D) 4 E) $\frac{1}{2}$

13. Let $f(x) = \sin x,\ -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, and let g be the inverse function of f. What is the value of $g'\left(\frac{1}{2}\right)$?

A) $\frac{2}{\sqrt{3}}$ B) $\frac{\sqrt{3}}{2}$ C) $\frac{1}{2}$ D) 2 E) $\frac{6}{\pi}$

14. Let $g(x) = 2x^3 - x^2 + 1$. Find the value of $\frac{d}{dx}g^{-1}(x)$ at x = 13.

A) $\frac{1}{20}$ B) $\frac{1}{13}$ C) 13 D) 20 E) $\frac{1}{5}$

15. If $h(2) = -3$, $h'(2) = \frac{1}{4}$, $h'(-3) = \frac{1}{3}$. Compute $\left(h^{-1}\right)'(-3)$

A) $\frac{1}{4}$ B) $\frac{1}{3}$ C) 4 D) 3 E) $-\frac{1}{3}$

Answers

1. B	2. C	3. D	4. C	5. D	6. D
7. B	8. B	9. E	10. E	11. A	12. D
13. A	14. A	15. C			

Derivatives of Trig Inverse

7. Find the derivative of each of the following functions.

(a) $f(x)=\sin^{-1}2x$ (b) $f(x)=\tan^{-1}(3x)$ (c) $y=\sin^{-1}\left(\frac{1}{3}x\right)$

(d) $y=\tan^{-1}\left(\frac{3}{4}x\right)$ (e) $f(x)=6\sin^{-1}(4x)$ (f) $f(x)=-5\tan^{-1}(7x)$

(g) $y=\tan^{-1}\left(e^{2x}\right)$ (h) $y=e^{\tan^{-1}(11x)}$ (i) $f(x)=-9\sin^{-1}\left(x^4\right)$

(j) $f(x)=\sin^{-1}\left(\sqrt{x}\right)$ (k) $y=\tan^{-1}\left(x^2+2\right)$ (l) $y=4\sin^{-1}(3x-1)$

(m) $f(x)=\tan^{-1}(\sin x)$ (n) $f(x)=\left[\sin^{-1}x\right]^3$ (o) $y=3x^5\sin^{-1}\left(x^5\right)$

(p) $y=x^2\tan^{-1}\left(\frac{1}{x^2}\right)$ (q) $y=\sin^{-1}(\cos x)$ (r) $f(x)=\sin^{-1}\left(\sqrt{1-x^2}\right)$

Answers

7. (a) $\frac{2}{\sqrt{1-4x^2}}$ (b) $\frac{3}{1+9x^2}$ (c) $\frac{1}{\sqrt{9-x^2}}$ (d) $\frac{12}{16+9x^2}$ (e) $\frac{24}{\sqrt{1-16x^2}}$ (f) $\frac{-35}{1+49x^2}$

(g) $\frac{2e^{2x}}{1+e^{4x}}$ (h) $\frac{11e^{\tan^{-1}11x}}{1+121x^2}$ (i) $\frac{-36x^3}{\sqrt{1-x^8}}$ (j) $\frac{1}{2\sqrt{x}\sqrt{1-x}}$ (k) $\frac{2x}{5+4x^2+x^4}$ (l) $\frac{12}{\sqrt{6x-9x^2}}$ (m) $\frac{\cos x}{1+\sin^2 x}$

(n) $\frac{3\left(\sin^{-1}x\right)^2}{\sqrt{1-x^2}}$ (o) $15x^4\sin^{-1}\left(x^5\right)+\frac{15x^9}{\sqrt{1-x^{10}}}$ (p) $2x\tan^{-1}\left(x^{-2}\right)-\frac{2x^{-1}}{1+x^{-4}}$ (q) $\frac{-\sin x}{|\sin x|}$ (r) $\frac{-x}{|x|\sqrt{1-x^2}}$

Inverse Functions and Their Derivatives

Recall: In general if $y = f(x)$ and f is one – to – one, then $f^{-1}(x)$ denotes the inverse of f, and

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Note: "x" is really the y-value for some x-value for $f(x)$

Ex: If $f(2) = 5$ then $f^{-1}(5) = 2$

Trig Inverse Derivative Formulas:

$$\frac{d}{dx}\left(\sin^{-1} f(x)\right) = \frac{1}{\sqrt{1-[f(x)]^2}} \cdot f'(x) \qquad \frac{d}{dx}\left(\cos^{-1} f(x)\right) = -\frac{1}{\sqrt{1-[f(x)]^2}} \cdot f'(x)$$

$$\frac{d}{dx}\left(\tan^{-1} f(x)\right) = \frac{1}{1+[f(x)]^2} \cdot f'(x) \qquad \frac{d}{dx}\left(\sec^{-1} f(x)\right) = \frac{1}{|f(x)|\sqrt{[f(x)]^2-1}} \cdot f'(x)$$

$$\frac{d}{dx}\left(\csc^{-1} f(x)\right) = -\frac{1}{|f(x)|\sqrt{[f(x)]^2-1}} \cdot f'(x) \qquad \frac{d}{dx}\left(\cot^{-1} f(x)\right) = -\frac{1}{1+[f(x)]^2} \cdot f'(x)$$

Examples:

I. Find $\frac{dy}{dx}$ for the following.

1. $y = \tan^{-1}(x)$
2. $y = 2\cos^{-1}(x)$
3. $y = \sin^{-1}(2x)$
4. $y = \sec^{-1}(3x)$
5. $y = \tan^{-1}(x^3)$
4. $y = \sin^{-1}\left(\frac{3}{t^2}\right)$

II. The position of a particle is given by $x(t) = 4\sin^{-1}\left(\frac{\sqrt{t}}{2}\right)$, determine the velocity of the particle at t = 3.

(Calculator)

III. Write the equation of the line tangent to $y = \tan^{-1}(x^2)$ at x = 1. (No Calculator)

IV. Find the acceleration of a particle at $t = \frac{1}{2}$ if the position function is $x(t) = \cos^{-1}(x)$. (Calculator)

V. Let f be defined by the function $f(x) = x^5 + 2x^3 + x - 1$

1. Find $f(1)$ and $f'(1)$
2. Find $f^{-1}(3)$ and $\left(f^{-1}\right)'(3)$

VI. If $g(3)=1$, $g(7)=3$, $g'(3)=\frac{5}{6}$, and $g'(7)=\frac{3}{4}$, then $\left(g^{-1}\right)'(3)=$?

VII. Let $f(x)=\sin x$, $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$, and let g be the inverse function of f. What is the value of $g'\left(\frac{1}{2}\right)$?

VIII. Let $f(x)=x^5+1$ and let g be the inverse of f.

1. What is the value of $g'(0)$?
2. What is the value of $g'(1)$?

IX. The following table shows the values of differentiable functions f and g.

x	f	f'	g	g'
1	2	$\frac{1}{2}$	-3	5
2	3	-2	0	4
3	4	2	2	3
4	6	4	3	$\frac{1}{2}$

1. If $H(x)=f^{-1}(x)$, then $H'(3)=$?
2. If $H(x)=f^{-1}(x)$, then $H'(2)=$?
3. If $H(x)=f^{-1}(x)$, then $H'(4)=$?
4. Determine the equation of the tangent line to the curve of $H(x)$ at $x=4$
5. If $J(x)=g^{-1}(x)$, then $J'(3)=$?
6. If $J(x)=g^{-1}(x)$, then $J'(0)=$?
7. Determine the equation of the tangent line to the curve of $J(x)$ at $x=2$
8. Determine the equation of the normal line to the curve of $J(x)$ at $x=3$
9. If $T(x)=g^{-1}[f(x)]$, then $T'(2)=$?

Answers:

I 1. $\frac{dy}{dx} = \frac{1}{1+x^2}$ 2. $\frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$ 3. $\frac{dy}{dx} = \frac{2}{\sqrt{1-4x^2}}$

4. $\frac{dy}{dx} = \frac{3}{|3x|\sqrt{9x^2-1}}$ 5. $\frac{dy}{dx} = \frac{3x^2}{1+x^6}$ 6. $\frac{dy}{dx} = \frac{1}{\sqrt{1-(\frac{3}{t^2})^2}} \cdot (-6t^{-3})$

II $v(3) = x'(3) = 66.159$

III $y = \tan^{-1}(x^2)$

$f(1) = \tan^{-1}(1)$

$f(1) = \frac{\pi}{4}$

$f'(x) = \frac{1}{1+x^4} \cdot 2x$

$f'(1) = \frac{2}{1+1} = 1$

$\boxed{y - \frac{\pi}{4} = 1(x-1)}$

IV $a(0.5) = v'(0.5) = x''(0.5)$

$x(t) = \cos^{-1} x$

$v(t) = \frac{-1}{\sqrt{1-x^2}} = -1(1-x^2)^{-\frac{1}{2}}$

$a(t) = \frac{1}{2}(1-x^2)^{-\frac{3}{2}}(-2x)$

$a(0.5) = -0.769$

V. $f(x) = x^5 + 2x^3 + x - 1$

$\boxed{f(1) = 3}$; $f'(x) = 5x^4 + 6x^2 + 1$

$\boxed{f(1) = 12}$

$\boxed{f^{-1}(3) = 1}$; $(f^{-1})'(3) = \frac{1}{f'[f^{-1}(3)]}$

$= \frac{1}{f'(1)}$

$\boxed{= \frac{1}{12}}$

VI. $(g^{-1})'(3) = \frac{1}{g'[g^{-1}(3)]}$

$= \frac{1}{g'(7)}$

$= \frac{1}{\frac{3}{4}}$

$\boxed{= \frac{4}{3}}$

VII. $y = \sin x$

$\sin^{-1} y = x$

$y = \sin^{-1} x$

$g(x) = \sin^{-1} x$

$g'(x) = \frac{1}{\sqrt{1-x^2}}$

$g'(\frac{1}{2}) = \frac{1}{\sqrt{1-(\frac{1}{2})^2}}$

$g'(\frac{1}{2}) = \frac{1}{\sqrt{\frac{3}{4}}}$

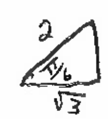
$g'(\frac{1}{2}) = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

OR

$f(x) = \sin x$; $g = f^{-1}$

$g'(\frac{1}{2}) = (f^{-1})'(\frac{1}{2}) = \frac{1}{f'[f^{-1}(\frac{1}{2})]}$

$* f^{-1}(\frac{1}{2}) \iff \frac{1}{2} = \sin x$



$x = \frac{\pi}{6}$

$f'(x) = \cos$

$g'(\frac{1}{2}) = \frac{1}{f'(\frac{\pi}{6})}$

$= \frac{1}{\cos(\frac{\pi}{6})}$

$= \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$

VIII. $f(x) = x^5 + 1 \quad ; \quad g = f^{-1}$

1. $g'(0) = (f^{-1})'(0) = \dfrac{1}{f'[f^{-1}(0)]}$

$f^{-1}(0) \Leftrightarrow 0 = x^5 + 1$

$x^5 = -1$

$x = -1$

$g'(0) = \dfrac{1}{f'(-1)} \quad ; \quad f'(x) = 5x^4$

$= \dfrac{1}{5(-1)^4}$

$\boxed{g'(0) = \dfrac{1}{5}}$

2. $g'(1) = (f^{-1})'(1) = \dfrac{1}{f'[f^{-1}(1)]}$

$f^{-1}(1) \Leftrightarrow 1 = x^5 + 1$

$x^5 = 0$

$x = 0$

$g'(1) = \dfrac{1}{f'(0)}$

$= \dfrac{1}{5(0)^4}$

$\boxed{g'(1) \text{ DNE}}$

IX. 1. $H'(3) = \dfrac{1}{f'[f^{-1}(3)]}$

$= \dfrac{1}{f'(2)}$

$\boxed{H'(3) = \dfrac{1}{-2}}$

2. $H'(2) = \dfrac{1}{f'[f^{-1}(2)]}$

$= \dfrac{1}{f'(1)}$

$= \dfrac{1}{\frac{1}{2}}$

$\boxed{H'(2) = 2}$

3. $H'(4) = \dfrac{1}{f'[f^{-1}(4)]}$

$= \dfrac{1}{f'(3)}$

$\boxed{H'(4) = \dfrac{1}{2}}$

4. $H(4) = 3$

$H'(4) = \frac{1}{2}$

$\boxed{y - 3 = \frac{1}{2}(x - 4)}$

5. $J'(3) = \dfrac{1}{g'[g^{-1}(3)]}$

$= \dfrac{1}{g'(4)}$

$= \dfrac{1}{\frac{1}{2}}$

$\boxed{J'(3) = 2}$

6. $J'(0) = \dfrac{1}{g'[g^{-1}(0)]}$

$= \dfrac{1}{g'(2)}$

$\boxed{J'(0) = \dfrac{1}{4}}$

7. $J(2) = 3$

$J'(2) = \dfrac{1}{g'[g^{-1}(2)]}$

$= \dfrac{1}{g'(3)}$

$= \dfrac{1}{3}$

$\boxed{y - 3 = \frac{1}{3}(x - 2)}$

8. $J(3) = 4$

$J'(3) = 2$

$\boxed{y - 4 = 2(x - 3)}$

9. $T'(x) = \dfrac{1}{g'[g^{-1}(f(x))]} \cdot f'(x$

$T'(2) = \dfrac{1}{g'[g^{-1}(f(2))]} \cdot f'($

$T'(2) = \dfrac{1}{g'[g^{-1}(3)]} \cdot (-2) = 2(-2$

$\boxed{= -4}$