Fundamental Theorem of Calculus Part 1 and 2 Review

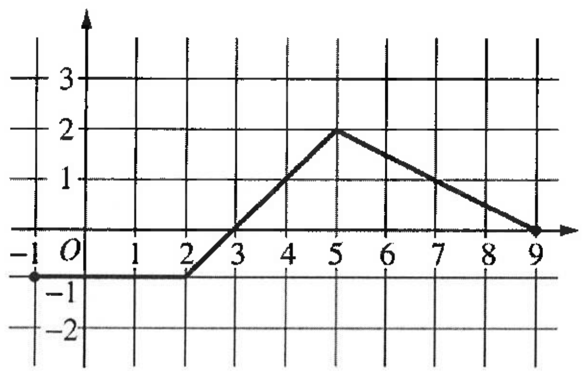
Exam Date: February 25/26, 2021 (FTC Part 1 and Part 2 Quiz on February 23/24, 2021)

Fundament Theorem of Calculus Part 1

I. Compute $\frac{dy}{dx}$ for the following (No Calculator)

1. $y=\int_0^x t^2 dt$
2. $y=\int_x^5 \cos(m^3)dm$
3. $y=\int_1^{x^4} t\sin(2t)dt$
4. $y=\int_1^x \sqrt{t}\,dt-\int_4^x \sqrt{t}dt$
5. $y=\left(\int_0^x \cos t\,dt\right)^3$
6. $\int_3^x y(t)dt=(9x+2)(\sqrt[4]{x+1})$

II. The graph of the piecewise linear function f is shown below. Let h be the function given by $h(x)=\int_2^x f(t)dt$ (No Calculator)



Graph of f

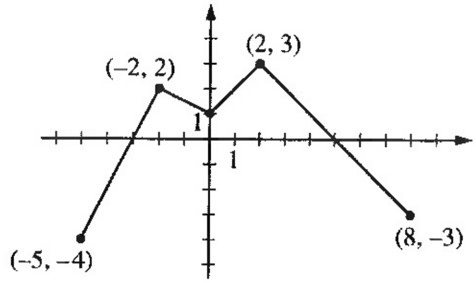
1. On which interval(s) is h decreasing? Justify.
2. Determine the absolute maximum value and the absolute minimum value of h on the interval $-1\le x\le 9$.
3. On which interval(s) is the graph of h concave down? Justify.
4. Evaluate $\lim_{n\to\infty}\sum_{k=1}^{n} f\left(4+\frac{3k}{n}\right)\left(\frac{3}{n}\right)$

III. Determine the equation of the tangent line to the curve $g(x)=4-\int_2^x e^{t^3}\,dt$ at x = 2. (No Calculator)

IV. Give that $f(x)=2x-\int_0^x te^t\,dt$, on which interval(s) is f increasing and concave down. (Calculator)

V. Given that $g(x)=\int_{-3}^{x}(t^2-4t+3)\,dt$, determine the interval(s) where g is decreasing and concave down. Justify your answer. (No Calculator)

VI.

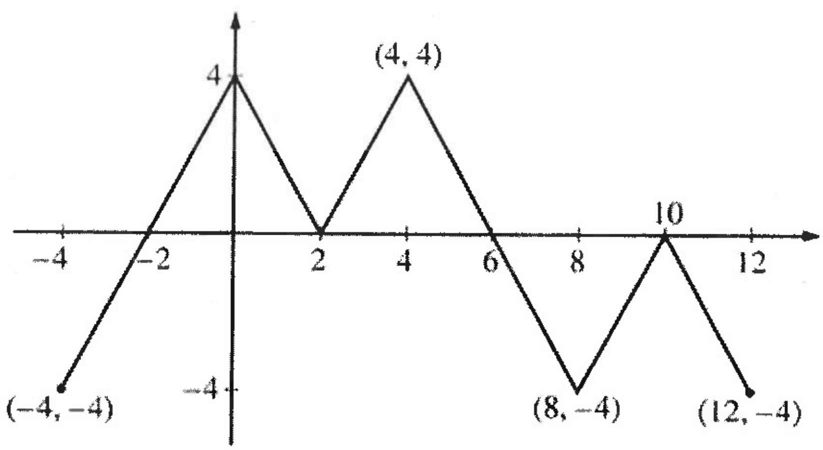


Graph of f

The continuous function f is defined on the interval $-5 \le x \le 8$. The graph of f, which consists of four line segments, is shown in the figure above. Let g be the function given by $g(x) = 2 + \int_2^x f(t)\,dt$

1. Determine the value of g(0) and g(8)
2. Determine the value of g'(5) and g"(5)
3. Determine the interval(s) where the graph of g is decreasing and concave up. Justify.

VII.



Graph of f

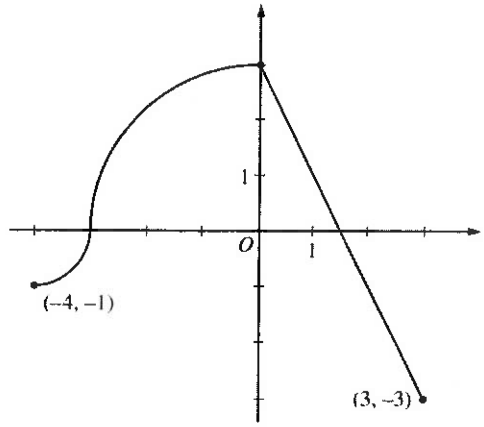
The figure above shows the graph of the piecewise-linear function f. For $-4 \le x \le 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.

(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.

(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \le x \le 12$. Justify your answers.

(d) For $-4 \le x \le 12$, find all intervals for which $g(x) \le 0$.



Graph of f'

VIII. The function f is defined and differentiable on the closed interval $-4 \le x \le 3$ and satisfies $f(0) = -2$. The graph of f', the derivative of f, consists of two quarter circles and one line segment, as shown in the figure above.

a) Determine the value of $f(3)$ and $f(-3)$.

b) On what interval(s) is f increasing? Justify your answer.

c) One what interval(s) is the function f decreasing and concave up? Explain.

d) State the critical numbers for f on $-4 < x < 3$. For each critical number, determine if it is a relative maximum, relative minimum, or neither for the function f. Justify.

e) Determine the absolute maximum value for the function f on the interval $-4 \le x \le 3$. Justify.

f) Must there be a value c such that $f''(c) = -\frac{2}{7}$ in the interval $-4 < x < 3$? Explain your reasoning.

g) Determine the meaning and value of $\frac{1}{7}\int_{-4}^{3} f'(x)\,dx$.

h) Let $g(x) = e^{f(x)}$. Determine the equation of the tangent line to the curve at $x = 2$.

i) Evaluate $\int_{-1}^{0} f''(3x)\,dx$.

j) Evaluate $\int_{-3}^{0} (5 - f'(x))\,dx$.

t (hours)	0	2	5	7	8
$E(t)$ (number of entries)	0	400	1300	2100	2300

IX. A zoo sponsored a one–day contest to name a new baby elephant. Zoo visitors deposited entries in a special box between noon ($t = 0$) and 8 P.M. ($t = 8$). The number of entries in the box t hours after noon is modeled by a differentiable function E for $0 \le t \le 8$. Values of $E(t)$, the number of entries, at various times t are shown in the table above.

a) Use the data in the table to approximate the rate, in entries per hour, at which entries were being deposited at $t = 6$.

b) For $5 < t < 7$, must there be at least one time t such that $E(t) = 1750$? Justify your answer.

c) For $2 < t < 5$, must there be a time t such that $E'(t) = 300$? Justify your answer.

d) Explain the meaning of $\frac{1}{8}\int_0^8 E(t)\,dt$ in the context of the problem using the correct units.

e) Approximate the value of $\frac{1}{8}\int_0^8 E(t)\,dt$, including units, using:

- Right Riemann Sum with 4 subintervals.
- Left Riemann Sum with 4 subintervals.
- Midpoint Riemann Sum with 2 subintervals.
- Trapezoidal Riemann Sum with 2 subintervals.

f) If the graph of E is increasing on the interval $0 < t < 8$, which of the calculations in part (e) is

i) an overestimate? Explain

ii) an underestimate? Explain

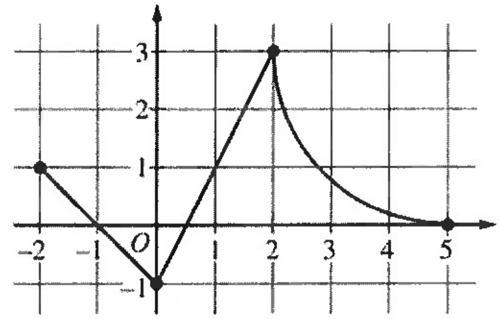
g) Explain the meaning of $\frac{1}{8}\int_0^8 E'(t)\,dt$ in the context of the problem using the correct units. Calculate the value of $\frac{1}{8}\int_0^8 E'(t)\,dt$ including units.

X.

x	0	25	30	50
$f(x)$	4	6	8	12

The values of a continuous function f for selected values of x are given in the table above. What is the value of the left Riemann sum approximation to $\int_0^{50} f(x)\,dx$ using the subintervals $[0, 25]$, $[25, 30]$, and $[30, 50]$?

XI.



Graph of f

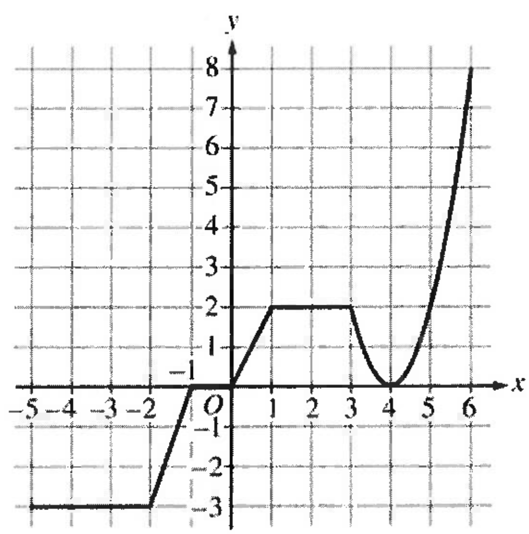
The continuous function f is defined on the closed interval $-6 \le x \le 5$. The figure above shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3, 3-\sqrt{5})$ is on the graph of f.

(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

(b) Evaluate $\int_{3}^{5} (2f'(x) + 4)\,dx$.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\,dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

XII.



Graph of g

The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \le x < 3$, and $g(x) = 2(x-4)^2$ for $3 \le x \le 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$?

(b) Evaluate $\int_{1}^{6} g(x)\,dx$.

(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

(d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer.

Answers:

FTC 1

I. 1. $\frac{dy}{dx} = x^2$ 2. $\frac{dy}{dx} = -\cos(x^3)$ 3. $\frac{dy}{dx} = 4x^7\sin(2x^4)$ 4. $\frac{dy}{dx} = 0$

5. $\frac{dy}{dx} = 3\sin^2 x\cos x$ 6. $\frac{dy}{dx} = \frac{9}{4}(x+1)^{\frac{-3}{4}} - \frac{3}{16}(x+1)^{\frac{-7}{4}}(9x+2) + \frac{9}{4}(x+1)^{\frac{-3}{4}}$

II. 1. $-1 < x < 3$ since $h' = f < 0$ 2. Abs min $h(3) = -0.5$ Abs max $h(9) = 5.5$

3. h is concave down on the interval $5 < x < 9$ since $h' = f$ is decreasing on the interval $5 < x < 9$

4. $\int_4^7 f(x)\,dx = 4.5$

III. $y - 4 = -e^8(x-2)$ IV. (-1.000, 0.852) V. (1, 2)

Since f' is positive and decreasing

VI. 1. $g(0) = -2$ $g(8) = 2$ 2. $g'(5) = f(5) = 0$ $g''(5) = f'(5) = -1$

3. g is decreasing and concave up on the interval (-5, -3) since $g' = f$ is negative and increasing on (-5, -3)

VII. (a) The function g has neither a relative minimum nor a relative maximum at $x = 10$ since $g'(x) = f(x)$ and $f(x) \le 0$ for $8 \le x \le 12$.

(b) The graph of g has a point of inflection at $x = 4$ since $g'(x) = f(x)$ is increasing for $2 \le x \le 4$ and decreasing for $4 \le x \le 8$.

(c) $g'(x) = f(x)$ changes sign only at $x = -2$ and $x = 6$.

x	$g(x)$
-4	-4
-2	-8
6	8
12	-4

On the interval $-4 \le x \le 12$, the absolute minimum value is $g(-2) = -8$ and the absolute maximum value is $g(6) = 8$.

(d) $g(x) \le 0$ for $-4 \le x \le 2$ and $10 \le x \le 12$.

VIII. a) $\int_0^3 f'(x)dx = f(3) - f(0)$ $\int_{-3}^0 f'(x)dx = f(0) - f(-3)$

$f(3) = -2$ $f(-3) = -2 - \frac{9\pi}{4}$

b) f is increasing on $\left(-3, \frac{3}{2}\right)$ since

$f' > 0$ on $\left(-3, \frac{3}{2}\right)$.

c) f is decreasing and concave up on $(-4,-3)$ since $f'(x)<0$ and $f'(x)$ is increasing on that interval.

d) f has critical numbers at $x=-3$ and $x=\frac{3}{2}$ since $f'(x)=0$ at $x=-3$ and $x=\frac{3}{2}$. f has a relative minimum at $x=-3$ since $f'(x)$ changes from negative to positive at $x=-3$. f has a relative maximum at $x=\frac{3}{2}$ since $f'(x)$ changes from positive to negative at $x=\frac{3}{2}$.

e) EVT application. f will have an absolute max at either the endpoints $x=-4$, or $x=3$, or at the critical numbers of f $x=-3$, or $x=\frac{3}{2}$.

Endpoints:

$$f(3)=-2$$

$$\int_{-4}^{0} f'(x)dx = f(0)-f(-4)$$

$$f(-4)=-2-2\pi$$

Critical Numbers

$$f(-3)=-2-\frac{9\pi}{4}$$

$$\int_{0}^{\frac{3}{2}} f'(x)dx = f\left(\frac{3}{2}\right)-f(0)$$

$$f\left(\frac{3}{2}\right)=\frac{1}{4}$$

Therefore, by the EVT f has its absolute maximum at $f\left(\frac{3}{2}\right)=\frac{1}{4}$ on the interval $[-4, 3]$.

f) No, since f' is not differentiable on the interval $-4<x<3$, then the Mean Value Theorem does not guarantee that there is a value of c in $-4<x<3$ such that $f''(3)=\frac{f'(3)-f'(-4)}{3-(-4)}=-\frac{2}{7}$.

g) $\frac{1}{7}\int_{-4}^{3} f'(x)dx$ represents the average value of f' on the interval $[-4, 3]$.

$$\frac{1}{7}\int_{-4}^{3} f'(x)dx=\frac{1}{7}\left[-\frac{\pi}{4}+\frac{9\pi}{4}+\frac{9}{4}-\frac{9}{4}\right]=\frac{2\pi}{7}$$

h)

$$g(x)=e^{f(x)}$$
$$g'(x)=e^{f(x)}\cdot f'(x)$$
$$g'(2)=e^{f(2)}\cdot f'(2)$$
$$g'(2)=e^{0}\cdot(-1)=-1$$

$$g(x)=e^{f(x)}$$
$$g(2)=e^{f(2)}$$
$$g(2)=e^{0}$$
$$g(2)=1$$

$$\int_{0}^{2} f'(x)dx = f(2)-f(0)$$
$$f(2)=0$$

equation $y-1=-1(x-2)$

i) $\int_{-1}^{0} f''(3x)\,dx$ using u = 3x

$$= \int_{-3}^{0} \frac{1}{3} f''(u)\,du$$

$$= \frac{1}{3} f'(u)\Big]_{-3}^{0}$$

$$= \frac{1}{3}\left[f'(0) - f'(3)\right]$$

$$= 1$$

j) $\int_{-3}^{0} (5 - f'(x))\,dx$

$$= \int_{-3}^{0} 5\,dx - \int_{-3}^{0} f'(x)\,dx$$

$$= 5\,\Big]_{-3}^{0} - \left[\frac{9\pi}{4}\right]$$

$$= 5(0) - 5(-3) - \frac{9\pi}{4}$$

$$= 15 - \frac{9\pi}{4}$$

IX. a) $E'(6) \approx \dfrac{E(7) - E(5)}{7 - 5} = 400$ entries/hour

b) Since E(t) is differentiable, then E(t) is also a continuous function on $0 \le t \le 8$ and $E(5) = 1300$ and $E(7) = 2100$, then by the Intermediate Value Theorem there must be at least one value t, on $5 < t < 7$ such that $E(t) = 1750$.

c) Since E(t) is a differentiable function and $\dfrac{E(5) - E(2)}{5 - 2} = 300$, then by the Mean Value Theorem, there must be a value t, in $2 < t < 5$ such that $E'(t) = \dfrac{E(5) - E(2)}{5 - 2} = 300$.

d) $\frac{1}{8}\int_{0}^{8} E(t)\,dt$ represents the average number of entries put into the box over the time interval $0 \le t \le 8$ hours.

e) RRS $\quad \frac{1}{8}\int_{0}^{8} E(t)\,dt \approx \frac{1}{8}\left[2(400) + 3(1300) + 2(2100) + 1(2300)\right] = 1400 \qquad$ 1400 entries

LRS $\quad \frac{1}{8}\int_{0}^{8} E(t)\,dt \approx \frac{1}{8}\left[2(0) + 3(400) + 2(1300) + 1(2100)\right] = 737.5 \qquad$ 738 entries

MRS $\quad \frac{1}{8}\int_{0}^{8} E(t)\,dt \approx \frac{1}{8}\left[5(400) + 3(2100)\right] = 1037.5 \qquad$ 1038 entries

TRS $\quad \frac{1}{8}\int_{0}^{8} E(t)\,dt \approx \frac{1}{8}\left[\frac{5(0 + 1300)}{2} + \frac{3(1300 + 2300)}{2}\right] = 1081.25 \qquad$ 1081 entries

f) The right Riemann sum is an over approximation, since E is increasing on the interval $0 < t < 8$
The left Riemann sum is an under approximation, since E is increasing on the interval $0 < t < 8$

g) $\frac{1}{8}\int_0^8 E'(t)\,dt$ represents the average rate that the entries are put into the box in entries/hour over the time interval $0 \le t \le 8$ hours.

$$\begin{aligned}
&\frac{1}{8}\int_0^8 E'(t)\,dt \\
&= \frac{1}{8}\left[E(8) - E(0)\right] \\
&= \frac{1}{8}\left[2300 - 0\right] \\
&= 287.5 \quad \textit{entries / hour}
\end{aligned}$$

X. 290

XI. (a) $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$

$$\Rightarrow 7 = \int_{-6}^{-2} f(x)\,dx + 2 + \left(9 - \frac{9\pi}{4}\right)$$

$$\Rightarrow \int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4$$

(b) $\int_3^5 (2f'(x) + 4)\,dx = 2\int_3^5 f'(x)\,dx + \int_3^5 4\,dx$

$$\begin{aligned}
&= 2(f(5) - f(3)) + 4(5 - 3) \\
&= 2\left(0 - (3 - \sqrt{5})\right) + 8 \\
&= 2(-3 + \sqrt{5}) + 8 = 2 + 2\sqrt{5}
\end{aligned}$$

— OR —

$$\begin{aligned}
\int_3^5 (2f'(x) + 4)\,dx &= \left[2f(x) + 4x\right]_{x=3}^{x=5} \\
&= (2f(5) + 20) - (2f(3) + 12) \\
&= (2 \cdot 0 + 20) - \left(2(3 - \sqrt{5}) + 12\right) \\
&= 2 + 2\sqrt{5}
\end{aligned}$$

(c) $g'(x) = f(x) = 0 \Rightarrow x = -1,\ x = \frac{1}{2},\ x = 5$

x	$g(x)$
-2	0
-1	$\frac{1}{2}$
$\frac{1}{2}$	$-\frac{1}{4}$
5	$11 - \frac{9\pi}{4}$

On the interval $-2 \le x \le 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$.

XII. (a) $f(-5) = f(1) + \int_1^{-5} g(x)\,dx = f(1) - \int_{-5}^{1} g(x)\,dx$

$$= 3 - \left(-9 - \frac{3}{2} + 1\right) = 3 - \left(-\frac{19}{2}\right) = \frac{25}{2}$$

(b) $\int_1^6 g(x)\,dx = \int_1^3 g(x)\,dx + \int_3^6 g(x)\,dx$

$$\begin{aligned}
&= \int_1^3 2\,dx + \int_3^6 2(x-4)^2\,dx \\
&= 4 + \left[\frac{2}{3}(x-4)^3\right]_{x=3}^{x=6} = 4 + \frac{16}{3} - \left(-\frac{2}{3}\right) = 10
\end{aligned}$$

(c) The graph of f is increasing and concave up on $0 < x < 1$ and $4 < x < 6$ because $f'(x) = g(x) > 0$ and $f'(x) = g(x)$ is increasing on those intervals.

(d) The graph of f has a point of inflection at $x = 4$ because $f'(x) = g(x)$ changes from decreasing to increasing at $x = 4$.