Propositional Logic

Discrete Mathematics
Andrei Bulatov

What is Logic?

"Computer science is a mere continuation of logic by other means"

Georg Gottlob

"Contrariwise", continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic"

Lewis Carroll, *Through the Looking Glass*

"Logic celebrates the unity of a pathological masculine self-identity that cannot listen and recognizes only negation but not difference"

Andrea Nye, *Words of Power*

Use of Logic

- In mathematics and rhetoric:
    - Give precise meaning to statements.
    - Distinguish between valid and invalid arguments.
    - Provide rules of `correct' reasoning.

Natural language can be very ambiguous

`If you do your homework, then you'll get to watch the game.'

$\wr\wr$

`If you don't do your homework, then you will not get to watch ...'

`You do your homework, or you'll fail the exam.'

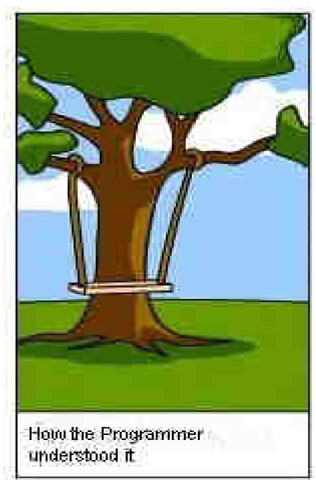
$\parallel$

`If you don't do your homework, then you'll fail the exam.'

Use of Logic (cntd)

- In computing:
  - Derive new data / knowledge from existing facts
  - Design of computer circuits.
  - Construction of computer programs.
  - Verification of correctness of programs and circuit design.
  - Specification

What the customer really needed



How the Programmer understood it

What the customer got

Statements (Propositions)

- Propositional logic deals with statements and their truth values
- A statement is a declarative sentence that can be true or false
- Truth values are TRUTH (T or 1) and FALSE (F or 0).
- Examples:
  - $1 + 1 = 2$ (statement, T)
  - The moon is made of cheese (statement, F)
  - Go home! (not statement, imperative)
  - What a beautiful garden! (not statement, exclamation)
  - $y + 1 = 2$ (not statement, uncertain)
  - The God exists (statement, ?)

Compound Statements

- Simplest statements are called primitive statements

  We shall use propositional variables to denote primitive statements, *p, q, r*, …

- We cannot decide the truth value of a primitive statement. This is not what logic does.

- Instead we combine primitive statements by means of logic connectives into compound statements or formulas and look how the truth value of a compound statement depends on the truth values of the primitive statements it includes.

  We will denote compound statements by Φ, Ψ, …

Logic Connectives

- negation (not, $\neg$) `It is not true that at least one politician was honest'
- conjunction (and, $\wedge$) `In this room there is a lady, and in the other room there is a tiger'
- disjunction (or, $\vee$) `Margaret Mitchell wrote `Gone with the Wind or I am going home'
- implication (if…, then…, $\rightarrow$) `If there is a tiger in this room, then there is no lady there'
- exclusive or (either …, or …, $\oplus$) `There is either a tiger in this room, or a lady'
- biconditional (equivalence) (if and only if, $\leftrightarrow$)
  `There is a lady in this room if and only if there is a tiger in the other room'

Truth Tables

Truth table is a way to specify the exact dependence of the truth value of a compound statement through the values of primitive statements involved

truth values of primitive statements (propositional variables)

truth value of compound statements (formulas)

p	q	Φ
0	0	0
0	1	1
…	…	…

Truth Tables of Connectives (Negation and Conjunction)

- Negation

p	$\neg$p
F (0)	T (1)
T (1)	F (0)

unary connective

`Today is Friday' $\Rightarrow$ `Today is not Friday'

- Conjunction

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

binary connective

`Today is Friday'
`It is raining' $\Rightarrow$

`Today is Friday and it is raining'

Truth Tables of Connectives (Disjunction)

- Disjunction – inclusive \`or’

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

\`Students who have taken calculus can take this course’ $\Rightarrow$
\`Students who have taken computing can take this course’

\`Students who have taken calculus or computing can take this course’

Be careful with \`or’ constructions in natural languages!

\`You do your homework, or you’ll fail the exam.’

\`Today is Friday or Saturday’

Truth Tables of Connectives (Exclusive or)

- Exclusive `or’

One of the statements is true but not both

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

`You can follow the rules or be disqualified.’

`Natalie will arrive today or Natalie will not arrive at all.’

Truth Tables of Connectives (Implication)

- Implication

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

Note that logical (*material*) implication does not assume any causal connection.

`**If** black is white, **then** we live in Antarctic.'

'**If** pigs fly, **then** Paris is in France.'

Implication as a Promise

- Implication can be thought of as a promise, and it is true if the promise is kept

  `**If** I am elected, **then** I will lower taxes'

  - He is not elected and taxes are not lowered — promise kept!
  - He is not elected and taxes are lowered — promise kept!
  - He is elected, but (=and) taxes are not lowered — promise broken!
  - He is elected and taxes are lowered — promise kept!

Playing with Implication

- Parts of implication

$$p \rightarrow q$$

hypothesis
antecedent
premise

conclusion
consequence

`**if** p, **then** q'

`**if** p, q'

`p **is sufficient for** q'

`q **if** p'

`q **when** p'

`q **unless** $\neg$p'

`p **implies** q'

`p **only if** q'

`q **whenever** p'

`q **follows from** p'

`a **sufficient condition for** q is p'

`**a necessary condition** for p is q'

Playing with Implication (cntd)

- Converse, contrapositive, and inverse

$p \to q$ `The home team wins whenever it is raining'
(`If it is raining then the home team wins')

- Converse $q \to p$
  `If the home team wins, then it is raining'
- Contrapositive $\neg q \to \neg p$
  `If the home team does not win, then it is not raining'
- Inverse $\neg p \to \neg q$
  `If it is not raining, then the home team does not win'

Truth Tables of Connectives (Biconditional)

- Biconditional or Equivalence

  One of the statements is true if and only if the other is true

p	q	$p \leftrightarrow q$
0	0	1
0	1	0
1	0	0
1	1	1

`You can take the flight if and only if you buy a ticket.'

Example

`You can access the Internet from campus if you are a computer science major or if you are not a freshman.'

p - `you can access the Internet from campus'

q - `you are a computer science major'

r - `you are a freshman'

Homework

Exercises from the Book:

No. 1 ,3, 4, 8a, 8c (page 54)