Inverse Functions and Differential Equations Review

Exam Date: Monday May 10, 2021/Tuesday May 11, 2021

Inverse Functions

I. Find $\frac{dy}{dx}$ for the following. (No Calculator)

1. $y = \tan^{-1}(x)$
2. $y = 2\cos^{-1}(x)$
3. $y = \sin^{-1}(2x)$
4. $y = e^x \cos^{-1}(3x)$
5. $y = \tan^{-1}(x^3)$
4. $y = \sin^{-1}\left(\frac{3\ln x}{x^2}\right)$

II. Write the equation of the line tangent to $y = \tan^{-1}(x^2)$ at x = 1. (No Calculator)

III. Find the acceleration of a particle at $t = \frac{1}{2}$ if the position function is $x(t) = \cos^{-1}(x)$. (Calculator)

IV. Let f be defined by the function $f(x) = x^5 + 2x^3 + x - 1$ (No Calculator)

1. Show that f is one to one.
2. Find $f(1)$ and $f'(1)$
3. Find $f^{-1}(3)$ and $(f^{-1})'(3)$

V. If $g(3) = 1$, $g(7) = 3$, $g'(3) = \frac{5}{6}$, and $g'(7) = \frac{3}{4}$, then $(g^{-1})'(3) = ?$ (No Calculator)

VI. Let $f(x) = 1 - x^5$ and let g be the inverse of f. (No Calculator)

1. Show that f is one to one.
2. Determine the inverse function $f^{-1}(x)$
3. Determine the value of $g'(0)$.
4. Determine the value of $g'(2)$.

VII. The following table shows the values of differentiable functions f and g. (No Calculator)

x	f	f'	g	g'
1	2	$\frac{1}{2}$	- 3	5
2	3	-2	0	4
3	4	2	2	3
4	6	4	3	$\frac{1}{2}$

1. If $H(x) = f^{-1}(x)$, then $H'(3) = ?$
2. If $J(x) = g^{-1}(x)$, then $J'(3) = ?$

Differential Equations

I. Write an expression for $y = f(x)$ given the following information. Note, you will have to determine the value of C first.

1. $y^2 = x^2 + C \quad ; f(1) = 3$

2. $-\dfrac{1}{y} = -\dfrac{x^4}{4} + C \quad ; f(-1) = 2$

3. $\dfrac{y^2}{2} = -x^2 + C \quad ; f(1) = -1$

4. $-(y-1)^{-1} = \dfrac{1}{\pi}\sin(\pi x) + C \quad ; f(1) = 0$

5. $\ln|y-1| = -\dfrac{1}{x} + C \quad ; f(2) = 0$

II. Solve the following differential equations

1. $\dfrac{dy}{dx} = \dfrac{x}{y} \;;\; y(1) = -2$

2. $\dfrac{dy}{dx} = -\dfrac{x}{y} \;;\; y(4) = 3$

3. $\dfrac{dy}{dx} = \dfrac{y}{x} \;;\; y(2) = 2$

4. $\dfrac{dy}{dx} = 2xy \;;\; y(0) = -3$

5. $\dfrac{dy}{dx} = (y+5)(x+2) \;;\; y(0) = -1$

6. $\dfrac{dy}{dx} = \cos^2 y \;;\; y(0) = 0$

7. $\dfrac{dy}{dx} = (\cos x)e^{y+\sin x} \;;\; y(0) = 0$

8. $\dfrac{dy}{dx} = e^{x-y} \;;\; y(0) = 2$

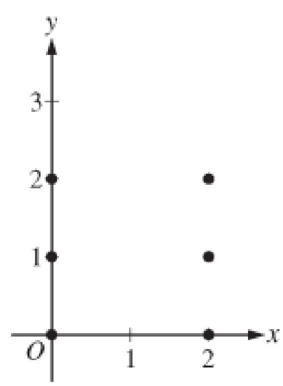
9. $\dfrac{dy}{dx} = -2xy^2 \;;\; y(1) = \dfrac{1}{4}$

III. Show that the function $y = 3e^{-4x}$ is a solution to the differential equation $y' + 4y = 0$.

IV. For what value(s) of a is the function $y = e^{ax}$ a solution to the differential equation $y'' - y' - 6y = 0$?

V. Consider the differential equation $\dfrac{dy}{dx} = \dfrac{y^2}{x-1}$

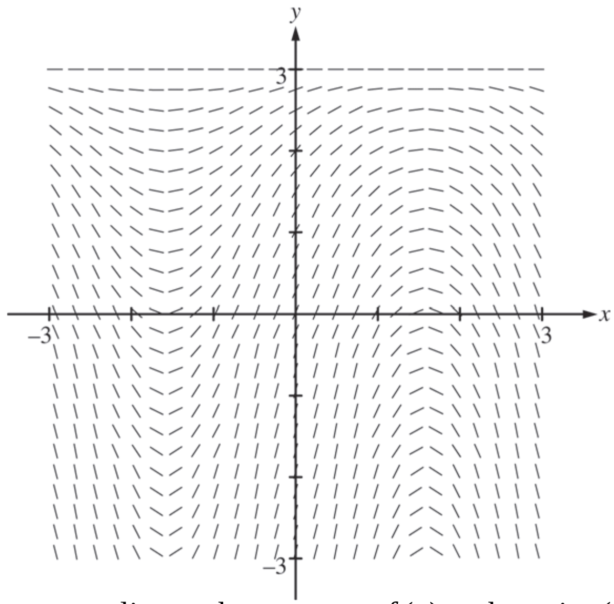
1. On the axis provided, sketch a slope field for the given differential equation at the six points indicated.



2. Determine the equation of the tangent line to the curve f at the point (2, 3).
3. Using your equation from question 2, approximate the value of $f(1.9)$.
4. Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(2) = 3$.

VI. Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. Let y = *f* (x) be the particular solution to the differential equation with the initial condition *f* (0) = 1. The function *f* is defined for all real numbers.

1. A portion of the slope field of the differential equation is given below. Sketch the solution curve though the point (0, 1).



2. Determine the equation of the tangent line to the curve y = *f* (x) at the point (0, 1). Use your equation to approximate *f* (0.1)

3. Find the particular solution y = *f* (x) to the given differential equation with initial condition *f* (0) = 1.

VII. A tank initially holds 100 gallons of brine solution containing 1 pound of salt. At $t = 0$ another brine solution containing 1 pound of salt per gallon is poured into the tank at a constant rate of 3 gallons per minute. While the second solution is being added to the tank, the well-stirred mixture leaves the tank at the same rate. The function Q represents the amount of salt in the tank in pounds at some time t, where t is measured in minutes. The function Q is modeled by the differential equation $\frac{dQ}{dt} = 0.03(100 - Q)$.

(a) Determine the function Q as a function of t.

(b) Determine the value of $\lim_{t \to \infty} Q(t)$.

(c) At what time t does the tank have 50 pounds of salt in it?

VIII. The function G, measured in degrees Celsius, is used to model the internal temperature of a baked potato t minutes after the potato is removed from the oven. The function G satisfies the differential equation $\frac{dG}{dt} = -(G - 27)^{2/3}$. If $G(0) = 91$, determine the function $G(t)$.

Answers: Inverses

I. 1. $\frac{dy}{dx}=\frac{1}{1+x^2}$ 2. $\frac{dy}{dx}=\frac{-2}{\sqrt{1-x^2}}$ 3. $\frac{dy}{dx}=\frac{2}{\sqrt{1-4x^2}}$ 4.
$\frac{dy}{dx}=e^x\cos^{-1}(3x)-\frac{3e^x}{\sqrt{1-9x^2}}$

5. $\frac{dy}{dx}=\frac{3x^2}{1+x^6}$ 6. $\frac{dy}{dx}=\frac{1}{\sqrt{1-\left[\frac{3\ln x}{x^2}\right]^2}}\left[\frac{\left(\frac{3}{x}\right)x^2-2x(3\ln x)}{x^4}\right]$

II. $y-\frac{\pi}{4}=1(x-1)$ III. $a(0.5)=x''(0.5)=-0.769$

IV. 1. Since $f'(x)>0$ for all x, then f is strictly increasing, which means the function f is one to one.

2. $f(1)=3$ $f'(1)=12$ 3. $f^{-1}(3)=1$ $(f^{-1})'(3)=\frac{1}{12}$

V. $(g^{-1})'(3)=\frac{4}{3}$

VI. 1. Since $f'(x)<0$ for all x, then f is strictly decreasing, which means the function f is one to one.

2. $f^{-1}(x)=\sqrt[5]{1-x}$ 3. $g'(0)=-\frac{1}{5}$ 4. $g'(2)=-\frac{1}{5}$

VII. 1. $H'(3)=-\frac{1}{2}$ 2. $J'(3)=2$

Answers: Differential Equations

I. 1. $C=8$, $y=\sqrt{x^2+8}$ 2. $C=-\frac{1}{4}$, $y=\frac{4}{x^4+1}$ 3. $C=\frac{3}{2}$, $y=-\sqrt{3-2x^2}$ 4. $C=1$, $y=\frac{\sin(\pi x)}{\sin(\pi x)+\pi}$

5. $C=\frac{1}{2}$, $y=1-e^{\left(\frac{1}{2}-\frac{1}{x}\right)}$

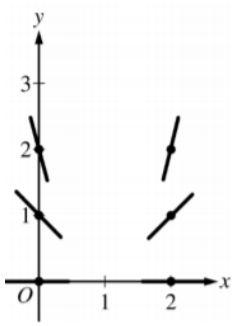
II. 1. $y=-\sqrt{x^2+3}$ 2. $y=\sqrt{25-x^2}$ 3. $y=|x|$ 4. $y=-3e^{x^2}$

5. $y = 4e^{\left(\frac{1}{2}x^2+2x\right)} - 5$ 6. $y = \tan^{-1}(x)$ 7. $y = -\ln\left(2 - e^{\sin x}\right)$ 8. $y = \ln\left(e^x + e^2 - 1\right)$

9. $y = \dfrac{1}{x^2+3}$

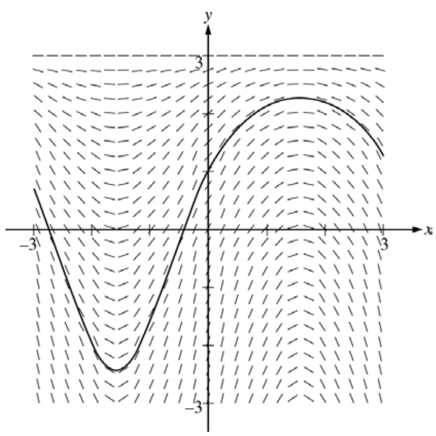
III. show   IV. $a = -2$, $a = 3$

V. 1.



2. $y - 3 = 9(x-2)$ 3. $f(2.1) \approx 3.9$ 4. $y = \dfrac{3}{1 - 3\ln(x-1)}$

VI. 1.



2. $y = 2x + 1$; $f(0.1) \approx 1.2$ 3. $y = -2e^{-\sin x} + 3$

VII. (a) $Q(t) = 100 - 99e^{-0.03t}$ (b) 100   (c) $t = 22.769$ min

VIII. $G(t) = 27 + \left(\dfrac{12 - t}{3}\right)^3$