MATH 229 Worksheet
Integrals using substitution

A

Integrate

1. $\int (5x+4)^5\,dx$
2. $\int 3t^2(t^3+4)^5\,dt$
3. $\int \sqrt{4x-5}\,dx$
4. $\int t^2(t^3+4)^{-1/2}\,dt$
5. $\int \cos(2x+1)\,dx$
6. $\int \sin^{10} x\cos x\,dx$
7. $\int \frac{\sin x}{(\cos x)^5}\,dx$
8. $\int \frac{(\sqrt{x}-1)^2}{\sqrt{x}}\,dx$
9. $\int \sqrt{x^3+x^2}(3x^2+2x)\,dx$
10. $\int_{-1}^{1} \frac{x+1}{(x^2+2x+2)^3}\,dx$
11. $\int_0^{\pi} \cos x\sqrt{\sin x}\,dx$
12. $\int (x+1)\sin(x^2+2x+3)\,dx$
13. $\int \left(1+\frac{1}{t}\right)^3 \frac{1}{t^2}\,dt$
14. $\int_{-1}^{1} x^2\sqrt{x^3+1}\,dx$
15. $\int \frac{2}{\sqrt{3x-7}}\,dx$
16. $\int_1^4 \frac{1}{\sqrt{x}(\sqrt{x}+1)^2}\,dx$
17. $\int_0^1 \frac{x}{\sqrt{x+1}}\,dx$
18. $\int x\sqrt{2x+1}\,dx$
19. $\int \sqrt{x}\sqrt{x\sqrt{x}+1}\,dx$
20. $\int x^3\sqrt{x^2-1}\,dx$
21. $\int (x^2+1)\sqrt{x-2}\,dx$
22. $\int \frac{x^2+2x}{x^2+2x+1}\,dx$
23. $\int \frac{1}{x^2+6x+9}\,dx$
24. $\int \frac{\sec^2 x}{(1+\tan x)^3}\,dx$
25. $\int \frac{\sin x}{(2+3\cos x)^2}\,dx$
26. $\int x\tan(x^2)\sec(x^2)\,dx$
27. $\int 3t^3(t^2+4)^5\,dt$
28. $\int (\tan 2x+\cot 2x)^2\,dx$

Answers:

1. $\frac{1}{30}(5x+4)^6+C$
2. $\frac{1}{6}(t^3+4)^6+C$
3. $\frac{1}{6}(4x-5)^{3/2}+C$
4. $\frac{2}{3}(t^3+4)^{1/2}+C$
5. $\frac{1}{2}\sin(2x+1)+C$
6. $\frac{1}{11}\sin^{11} x+C$
7. $\frac{1}{4}(\cos x)^{-4}+C$
8. $\frac{2}{3}x^{3/2}-2x+2x^{1/2}+C$
9. $\frac{2}{3}(x^3+x^2)^{3/2}+C$
10. $-\frac{1}{4}(x^2+2x+2)^{-2}\Big|_{-1}^{1}=\frac{6}{25}$
11. $\frac{2}{3}(\sin x)^{3/2}\Big|_0^{\pi}=0$
12. $-\frac{1}{2}\cos(x^2+2x+3)+C$
13. $-\frac{1}{4}\left(1+\frac{1}{t}\right)^4+C$
14. $\frac{2}{9}(x^3+1)^{3/2}\Big|_{-1}^{1}=\frac{4\sqrt{2}}{9}$
15. $\frac{4}{3}(3x-7)^{1/2}+c$
16. $-2(\sqrt{x}+1)^{-1}\Big|_1^4=\frac{1}{3}$
17. $\frac{2}{3}(x+1)^{3/2}-2(x+1)^{1/2}\Big|_0^1=\frac{4-2\sqrt{2}}{3}$
18. $\frac{1}{10}(2x+1)^{5/2}-\frac{1}{6}(2x+1)^{3/2}+C$
19. $\frac{4}{9}(x^{3/2}+1)^{3/2}+C$
20. $\frac{1}{5}(x^2-1)^{5/2}+\frac{1}{3}(x^2-1)^{3/2}+C$
21. $\frac{2}{7}(x-2)^{\frac{7}{2}}+\frac{8}{5}(x-2)^{\frac{5}{2}}+\frac{10}{3}(x-2)^{\frac{3}{2}}+C$
22. $x+(x+1)^{-1}+C$
23. $-(x+3)^{-1}+C$
24. $-\frac{1}{2}(1+\tan x)^{-2}+C$
25. $\frac{1}{3}(2+3\cos x)^{-1}+C$
26. $\frac{1}{2}\sec(x^2)+C$
27. $\frac{3}{14}(t^2+4)^7-(t^2+4)^6+C$
28. $\frac{1}{2}\tan 2x-\frac{1}{2}\cot 2x+C$

Integration by Parts

B

I. Evaluate each integral using integration by parts with the indicated choices of u and dv.

1. $\int x\ln(x)dx$; $u=\ln(x),\quad dv=xdx$

2. $\int x\cos x\,dx$; $u=x,\quad dv=\cos x dx$

3. $\int x\cdot 2^x\,dx$; $u=x,\quad dv=2^x dx$

4. $\int \sqrt{x}\ln(x)dx$; $u=\ln(x),\quad dv=\sqrt{x}dx$

II. Evaluate the following integrals

5. $\int xe^{-x}\,dx$

6. $\int x\sin(3x)dx$

7. $\int (\ln x)^2\,dx$

8. $\int \sin^{-1}(x)dx$

9. $\int x^2\cos(3x)dx$

10. $\int \frac{x^2}{e^{2x}}dx$

11. $\int x^2e^{5x}\,dx$

12. $\int (x-3)\sec^2 x\,dx$

ANSWERS: 1.

$$\int x\ln(x)dx=\frac{1}{2}x^2\ln x-\int\frac{1}{2}xdx$$
$$=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+C$$

2.

$$\int x\cos x\,dx=x\sin x-\int\sin x dx$$
$$=x\sin x+\cos x+C$$

3.

$$\int x\cdot 2^x\,dx=\frac{x\cdot 2^x}{\ln 2}-\int\frac{2^x}{\ln 2}dx$$
$$=\frac{x\cdot 2^x}{\ln 2}-\frac{2^x}{(\ln 2)^2}+C$$

4.

$$\int\sqrt{x}\ln(x)dx=\frac{2}{3}x^{\frac{3}{2}}\ln x-\int\frac{2}{3}x^{\frac{1}{2}}dx$$
$$=\frac{2}{3}x^{\frac{3}{2}}\ln x-\frac{4}{9}x^{\frac{3}{2}}+C$$

5.

$$\int xe^{-x}\,dx=-xe^{-x}+\int e^{-x}dx$$
$$=-xe^{-x}-e^{-x}+C$$

6.

$$\int x\sin(3x)dx=-\frac{1}{3}x\cos(3x)+\frac{1}{3}\int\cos(3x)dx$$
$$=-\frac{1}{3}x\cos(3x)+\frac{1}{9}\sin(3x)+C$$

7.

$$\int(\ln x)^2\,dx=x(\ln x)^2-2\int\ln x\,dx$$
$$=x(\ln x)^2-2\left[x\ln x-\int dx\right]$$
$$=x(\ln x)^2-2x\ln x+2x+C$$

8.

$$\int\sin^{-1}(x)dx=x\sin^{-1}(x)-\int\frac{x}{\sqrt{1-x^2}}dx$$
use a u – substitution
$$=x\sin^{-1}(x)+\sqrt{1-x^2}+C$$

9.

$$\int x^2\cos(3x)dx=\frac{x^2\sin(3x)}{3}-\int\frac{2x\sin(3x)}{3}dx$$
use IBP again
$$=\frac{x^2\sin(3x)}{3}-\left[-\frac{2x\cos(3x)}{9}-\int\frac{2\cos(3x)}{9}dx\right]$$
$$=\frac{x^2\sin(3x)}{3}+\frac{2x\cos(3x)}{9}-\frac{2\sin(3x)}{27}+C$$

10.

$$\int x^2e^{-2x}\,dx=-\frac{1}{2}x^2e^{-2x}+\int xe^{-2x}dx$$
use IBP again
$$=-\frac{1}{2}x^2e^{-2x}+\left[-\frac{1}{2}xe^{-2x}-\int-\frac{1}{2}e^{-2x}dx\right]$$
$$=-\frac{1}{2}x^2e^{-2x}-\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}+C$$

11.

$$\int x^2e^{5x}\,dx=\frac{1}{5}x^2e^{5x}-\int\frac{2}{5}xe^{5x}dx$$
use IBP again
$$=\frac{1}{5}x^2e^{5x}-\left[\frac{2}{25}xe^{5x}-\int\frac{2}{25}e^{5x}dx\right]$$
$$=\frac{1}{5}x^2e^{5x}-\frac{2}{25}xe^{5x}+\frac{2}{125}e^{5x}+C$$

12

$$\int(x-3)\sec^2 x\,dx=\int x\sec^2 xdx-3\int\sec^2 xdx$$
$$=x\tan x-\int\tan xdx-3\int\sec^2 xdx$$
$$=x\tan x-\ln|\sec x|-3\tan x+C$$

Integration by Parts: Part 2

C

I. Evaluate the following integrals using the " table " method.

1. $\int x\cos(2x)dx$
2. $\int x^2e^x dx$
3. $\int (x^3+2x)e^{2x}dx$
4. $\int x^2\cos x dx$

II. Evaluate the following integrals.

1. $\int e^{3x}\cos(2x)dx$
2. $\int \sin(\ln x)dx$
3. $\int \cos(\sqrt{x})dx$
4. $\int e^{\sqrt{x}}dx$

Answers:

I.

1. $\frac{1}{2}x\sin(2x)+\frac{1}{4}\cos(2x)+C$
2. $x^2e^x-2xe^x+2e^x+C$
3. $\frac{x^3+2x}{2}e^{2x}-\frac{3x^2+2}{4}e^{2x}+\frac{6x}{8}e^{2x}-\frac{6}{16}e^{2x}+C$
4. $x^2\sin x+2x\cos x-2\sin x+C$

II.

1. $\frac{3}{13}e^{3x}\cos(2x)+\frac{2}{13}e^{3x}\sin(2x)+C$
2. $\frac{1}{2}x\sin(\ln x)-\frac{1}{2}x\cos(\ln x)+C$
3. $2\sqrt{x}\sin(\sqrt{x})+2\cos(\sqrt{x})+C$
4. $2\sqrt{x}e^{\sqrt{x}}-2e^{\sqrt{x}}+C$

Integration with Rational Functions – Part 1

I. Simplify the following polynomials using long division.

1. $\dfrac{x^2-7x-11}{x-8}$ 2. $\dfrac{x^2-28}{x-5}$ 3. $\dfrac{2x^3+4x^2-5}{x+3}$ 4. $\dfrac{3x^3+4x+11}{x^2-3x+2}$

II. Complete the square for the following quadratics.

1. $x^2+4x+11$ 2. $x^2-12x+41$ 3. $x^2-5x+10$ 4. $3x^2+6x+4$

III. Integrate the following rational functions.

1. $\int \dfrac{1}{x^2+4}\,dx$ 2. $\int \dfrac{7}{x^2+5}\,dx$ 3. $\int \dfrac{1}{9x^2+1}\,dx$ 4. $\int \dfrac{2}{x-3}\,dx$ 5. $\int \dfrac{2x}{x-3}\,dx$

6. $\int \dfrac{x}{x^2+4}\,dx$ 7. $\int \dfrac{x-1}{x^2+4}\,dx$ 8. $\int \dfrac{x^2-x+7}{x^2+4}\,dx$ 9. $\int \dfrac{2x^3-9x^2+14x-8}{x-3}\,dx$

10. $\int \dfrac{x^3+3x^2-10}{x^2+1}\,dx$ 11. $\int \dfrac{1}{x^2+2x+5}\,dx$ 12. $\int \dfrac{x+3}{x^2+2x+10}\,dx$

Answers:

I. 1. $x+1-\dfrac{3}{x-8}$ 2. $x+5-\dfrac{3}{x-5}$ 3. $2x^2-2x+6-\dfrac{23}{x+3}$ 4. $3x+9+\dfrac{25x-7}{x^2-3x+2}$

II. 1. $(x+2)^2+7$ 2. $(x-6)^2+5$ 3. $\left(x-\dfrac{5}{2}\right)^2+\dfrac{15}{4}$ 4. $3(x+1)^2+1$

III. 1. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right)+C$ 2. $\dfrac{7}{\sqrt{5}}\tan^{-1}\left(\dfrac{x}{\sqrt{5}}\right)+C$ 3. $\dfrac{1}{3}\tan^{-1}(3x)+C$ 4. $2\ln|x-3|+C$

5. $2x+6\ln|x-3|+C$ 6. $\dfrac{1}{2}\ln|x^2+4|+C$ 7. $\dfrac{1}{2}\ln|x^2+4|-\dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right)+C$

8. $x-\dfrac{1}{2}\ln|x^2+4|+\dfrac{3}{2}\tan^{-1}\left(\dfrac{x}{2}\right)+C$ 9. $\dfrac{2}{3}x^3-\dfrac{3}{2}x^2+5x+7\ln|x-3|+C$

10. $\dfrac{1}{2}x^2+3x-\dfrac{1}{2}\ln|x^2+1|-13\tan^{-1}(x)+C$ 11. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x+1}{2}\right)+C$

12. $\dfrac{1}{2}\ln|x^2+2x+10|+\dfrac{2}{3}\tan^{-1}\left(\dfrac{x+1}{3}\right)+C$

Partial Fraction Decomposition

F

Part I: Two Linear Factors with no ~~repetition~~ repetition

1. Find the values of the constants A and B in each identity.

a $\dfrac{2}{(x+1)(x+3)} \equiv \dfrac{A}{x+1} + \dfrac{B}{x+3}$

b $\dfrac{x-3}{x(x-1)} \equiv \dfrac{A}{x} + \dfrac{B}{x-1}$

c $\dfrac{x+1}{(x-3)(x-5)} \equiv \dfrac{A}{x-3} + \dfrac{B}{x-5}$

d $\dfrac{x+10}{(1+x)(2-x)} \equiv \dfrac{A}{1+x} + \dfrac{B}{2-x}$

e $\dfrac{4x-1}{x^2+x-2} \equiv \dfrac{A}{x+2} + \dfrac{B}{x-1}$

f $\dfrac{x-9}{x^2-4x+3} \equiv \dfrac{A}{x-1} + \dfrac{B}{x-3}$

2. Express in partial fractions

a $\dfrac{8}{(x-1)(x+3)}$

b $\dfrac{x-1}{(x+2)(x+3)}$

c $\dfrac{10x}{(x+4)(x-1)}$

d $\dfrac{5x+7}{x^2+x}$

e $\dfrac{x+2}{x^2-5x+4}$

f $\dfrac{4x+6}{x^2-9}$

g $\dfrac{3x+2}{x^2-2x-24}$

h $\dfrac{38-x}{12-x-x^2}$

i $\dfrac{4x-5}{(2x+1)(x-3)}$

j $\dfrac{1-3x}{(3x+4)(2x+1)}$

k $\dfrac{x+1}{x-3x^2}$

l $\dfrac{5}{2x^2+3x-2}$

m $\dfrac{2(x+5)}{8x^2+10x-3}$

n $\dfrac{3x-7}{x^2-2x-3}$

o $\dfrac{1-3x}{1-x-2x^2}$

Part II: Multiple Linear Terms with Repetition

Find the values of the constants A, B and C in each identity.

3. **a** $\dfrac{8x+14}{(x-2)(x+1)(x+3)} \equiv \dfrac{A}{x-2} + \dfrac{B}{x+1} + \dfrac{C}{x+3}$

b $\dfrac{2x^2-6x+20}{(x+1)(x+2)(x-6)} \equiv \dfrac{A}{x+1} + \dfrac{B}{x+2} + \dfrac{C}{x-6}$

c $\dfrac{9x-14}{(x+4)(x-1)^2} \equiv \dfrac{A}{x+4} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$

4. Express in partial fractions

a $\dfrac{2x^2+4}{x(x-1)(x-4)}$

b $\dfrac{9}{(x-2)(x+1)^2}$

c $\dfrac{x^2+11x-21}{(2x+1)(x-2)(x-3)}$

d $\dfrac{10x+9}{(x-4)(x+3)^2}$

e $\dfrac{x^2+4x+5}{(x+1)(x+2)^2}$

f $\dfrac{16-2x}{(x-3)(x^2-4)}$

g $\dfrac{2-9x}{(x-3)(2x-1)^2}$

h $\dfrac{3+24x-4x^2}{(x+1)(x-4)^2}$

i $\dfrac{9x^2-2x-12}{x^3+x^2-6x}$

Part III: Long Division First

Express in partial fractions

5. **a** $\dfrac{x^2+3}{(x-3)(x+1)}$

b $\dfrac{x^3-3x^2-x+2}{x^2-4}$

c $\dfrac{2x^2+7x}{x^2+6x+8}$

Part IV: Quadratic Factors

Express the following as a sum of partial fractions

6. a) $\dfrac{x^2-3x-7}{(x^2+x+2)(2x-1)}$ b) $\dfrac{13}{(2x+3)(x^2+1)}$ c) $\dfrac{x}{(x^2-x+1)(3x-2)}$

Answers:

1a) $\frac{1}{x+1} - \frac{1}{x+3}$ b) $\frac{3}{x} - \frac{2}{x-1}$ c) $\frac{-2}{x-3} + \frac{3}{x-5}$ d) $\frac{3}{1+x} + \frac{4}{2-x}$

e) $\frac{3}{x+2} + \frac{1}{x-1}$ f) $\frac{4}{x-1} - \frac{3}{x-3}$

2a) $\frac{2}{x-1} - \frac{2}{x+3}$ b) $\frac{4}{x+3} - \frac{3}{x+2}$ c) $\frac{8}{x+4} + \frac{2}{x-1}$ d) $\frac{7}{x} - \frac{2}{x+1}$

e) $\frac{2}{x-4} - \frac{1}{x-1}$ f) $\frac{1}{x+3} + \frac{3}{x-3}$ g) $\frac{2}{x-6} + \frac{1}{x+4}$ h) $\frac{6}{4+x} + \frac{5}{3-x}$

i) $\frac{2}{2x+1} + \frac{1}{x-3}$ j) $\frac{1}{2x+1} - \frac{3}{3x+4}$ k) $\frac{1}{x} + \frac{4}{1-3x}$ l) $\frac{2}{2x-1} - \frac{1}{x+2}$

m) $\frac{3}{4x-1} - \frac{1}{2x+3}$ n) $\frac{\frac{5}{2}}{x+1} + \frac{\frac{1}{2}}{x-3}$ o) $\frac{\frac{4}{3}}{1+x} - \frac{\frac{1}{3}}{1-2x}$

3a) $\frac{2}{x-2} - \frac{1}{x+1} - \frac{1}{x+3}$ b) $\frac{-4}{x+1} + \frac{5}{x+2} + \frac{1}{x-6}$ c) $\frac{-2}{x+4} + \frac{2}{x-1} - \frac{1}{(x-1)^2}$

4a) $\frac{1}{x} - \frac{2}{x-1} + \frac{3}{x-4}$ b) $\frac{1}{x-2} - \frac{1}{x+1} - \frac{3}{(x+1)^2}$ c) $\frac{3}{x-3} - \frac{3}{2x+1} - \frac{1}{x-2}$

d) $\frac{1}{x-4} - \frac{1}{x+3} + \frac{3}{(x+3)^2}$ e) $\frac{2}{x+1} - \frac{1}{x+2} - \frac{1}{(x+2)^2}$ f) $\frac{2}{x-3} + \frac{1}{x+2} - \frac{3}{x-2}$

g) $-\frac{1}{x-3} + \frac{2}{2x+1} + \frac{1}{(2x-1)^2}$ h) $-\frac{1}{x+1} - \frac{3}{x-4} + \frac{7}{(x-4)^2}$ i) $\frac{2}{x} + \frac{5}{x+3} + \frac{2}{x-2}$

5a) $1 + \frac{3}{x-3} - \frac{1}{x+1}$ b) $x - 3 + \frac{4}{x+2} - \frac{1}{x-2}$ c) $2 - \frac{3}{x+2} - \frac{2}{x+4}$

6a) $\frac{2x+1}{x^2+x+2} - \frac{3}{2x-1}$ b) $\frac{4}{2x+3} - \frac{2x-3}{x^2+1}$ c) $\frac{-\frac{2}{7}x + \frac{3}{7}}{x^2-x+1} + \frac{\frac{6}{7}}{3x-2}$

F

Express the following in Partial Fractions

1. $\frac{3x}{(x+2)(x-1)}$
2. $\frac{2x-1}{(x+1)(x-2)(x+3)}$
3. $\frac{2x}{x^2-25}$
4. $\frac{4}{x(x^2+4)}$
5. $\frac{3x^2+2x}{(x+2)(x^2+4)}$
6. $\frac{x^2+1}{x(x^2-1)}$
7. $\frac{2}{(x-1)^2(x+1)}$
8. $\frac{x^2+3x}{x^2-4}$
9. $\frac{5x-3}{(x-2)(x-3)^2}$
10. $\frac{x^4+1}{x^3+2x}$
11. $\frac{1}{x(x-1)(x+1)}$
12. $\frac{16}{(x-1)^2(x+1)^3}$

Answers:

1. $\frac{2}{x+2}+\frac{1}{x-1}$
2. $\frac{1}{2(x+1)}+\frac{1}{5(x-2)}-\frac{7}{10(x+3)}$
3. $\frac{1}{x-5}+\frac{1}{x+5}$
4. $\frac{1}{x}-\frac{x}{x^2+4}$
5. $\frac{1}{x+2}+\frac{2x-2}{x^2+4}$
6. $\frac{1}{x+1}+\frac{1}{x-1}-\frac{1}{x}$
7. $\frac{1}{(x-1)^2}-\frac{1}{2(x-1)}+\frac{1}{2(x+1)}$
8. $1+\frac{5}{2(x-2)}+\frac{1}{2(x+2)}$
9. $\frac{12}{(x-3)^2}-\frac{7}{x-3}+\frac{7}{x-2}$
10. $x+\frac{1}{2x}-\frac{5x}{2(x^2+2)}$
11. $\frac{1}{2(x-1)}+\frac{1}{2(x+1)}-\frac{1}{x}$
12. $\frac{2}{(x-1)^2}-\frac{3}{x-1}+\frac{4}{(x+1)^3}+\frac{4}{(x+1)^2}+\frac{3}{x+1}$

Integration by Partial Fractions

9

Evaluate the following integrals

1. $\int \frac{1}{x^2-4}dx$
2. $\int \frac{2x+3}{x^2-9}dx$
3. $\int \frac{2-x}{x^2+5x}dx$
4. $\int \frac{x^2-1}{x^2-16}dx$
5. $\int \frac{x^4+x^3+x^2+1}{x^2+x-2}dx$
6. $\int \frac{x^2+x-1}{x^3-x}dx$
7. $\int \frac{x+7}{x^2(x+2)}dx$
8. $\int \frac{x^5+1}{x^4+2x^3}dx$
9. $\int \frac{1}{x^4-16}dx$
10. $\int \frac{\cos x}{\sin^3 x+\sin x}dx$

Answers:

1. $-\frac{1}{4}\ln|x+2|+\frac{1}{4}\ln|x-2|+C$
2. $\frac{1}{2}\ln|x+3|+\frac{3}{2}\ln|x-3|+C$
3. $\frac{2}{5}\ln|x|-\frac{7}{5}\ln|x+5|+C$
4. $x-\frac{15}{8}\ln|x+4|+\frac{15}{8}\ln|x-4|+C$
5. $\frac{x^3}{3}+3x-\frac{13}{3}\ln|x+2|+\frac{4}{3}\ln|x-1|+C$
6. $\ln|x|-\frac{1}{2}\ln|x+1|+\frac{1}{2}\ln|x-1|+C$
7. $-\frac{5}{4}\ln|x|-\frac{7}{2}x^{-1}+\frac{5}{4}\ln|x+2|+C$
8. $\frac{x^2}{2}-2x+\frac{1}{8}\ln|x|+\frac{1}{4}x^{-1}-\frac{1}{4}x^{-2}+\frac{31}{8}\ln|x+2|+C$
9. $-\frac{1}{32}\ln|x+2|+\frac{1}{32}\ln|x-2|-\frac{1}{16}\tan^{-1}\left(\frac{x}{2}\right)+C$
10. $u=\sin x$, $du=\cos x dx$, $\ln|\sin x|-\frac{1}{2}\ln|\sin^2 x+1|+C$

TRIGONOMETRY IDENTITIES REVIEW

Identities

1. Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

2. Reciprocal Identities

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sin\theta = \frac{1}{\csc\theta} \qquad \tan\theta = \frac{1}{\cot\theta}$$

3. The Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

4. Double Angle Identities

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$

Trigonometric Identities Review

1. If $\sin\theta = \frac{4}{5}$ in QI determine

   a) cos θ b) tan θ c) sin 2θ d) θ

2. If $\sec\alpha = \frac{\sqrt{5}}{2}$ in QI, determine the value of the following

   a) sin α b) tan α c) cos α d) sin 2α e) α

3. If $\sin\theta = \frac{x}{4}$ and θ is in quadrant I, than calculate the following in terms of x.

   a) tan θ b) sin2θ c) sec θ d) θ

4. If $\tan\theta = \frac{x}{3}$ and θ is in quadrant I, than calculate the following in terms of x.

   a) sin θ b) cos θ c) θ d) sec θ

   e) $\ln|\sec\theta + \tan\theta|$ f) $\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta$ g) $\ln|\sec\theta| + \tan\theta + \theta$

ANSWERS:

1.a) $\frac{3}{5}$ b) $\frac{4}{3}$ c) $2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)=\frac{24}{25}$ d) $\theta=\sin^{-1}\left(\frac{4}{5}\right)$

2. a) $\frac{1}{\sqrt{5}}$ b) $\frac{1}{2}$ c) $\frac{2}{\sqrt{5}}$ d) $2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)=\frac{24}{25}$ e) $\alpha=\sec^{-1}\left(\frac{\sqrt{5}}{2}\right)$

3. a) $\frac{x}{\sqrt{16-x^2}}$ b) $2\left(\frac{x}{4}\right)\left(\frac{\sqrt{16-x^2}}{4}\right)=\frac{x\sqrt{16-x^2}}{8}$ c) $\frac{4}{\sqrt{16-x^2}}$ d) $\theta=\sin^{-1}\left(\frac{x}{4}\right)$

4. a) $\frac{x}{\sqrt{x^2+9}}$ b) $\frac{3}{\sqrt{x^2+9}}$ c) $\theta=\tan^{-1}\left(\frac{x}{3}\right)$ d) $\frac{\sqrt{x^2+9}}{3}$

e) $\ln\left|\frac{\sqrt{x^2+9}}{3}+\frac{x}{3}\right|$ f) $\frac{1}{2}\tan^{-1}\left(\frac{x}{3}\right)+\frac{1}{4}\left[2\left(\frac{x}{\sqrt{x^2+9}}\right)\left(\frac{3}{\sqrt{x^2+9}}\right)\right]$
$=\frac{1}{2}\tan^{-1}\left(\frac{x}{3}\right)+\frac{3x}{2\left(x^2+9\right)}$

g) $\ln\left|\frac{\sqrt{x^2+9}}{3}\right|+\frac{x}{3}+\tan^{-1}\left(\frac{x}{3}\right)$

Trigonometric Integrals

I

I. Evaluate the following integrals

1. $\int \cos^2 x\,dx$ 2. $\int \sin^3 x\,dx$ 3. $\int \sin^4 x\,dx$ 4. $\int \sin^3 x\cos^2 x\,dx$

5. $\int \sin^6 x\cos^3 x\,dx$ 6. $\int (1+\cos x)^2\,dx$

II. Evaluate the following integrals.

1. $\int \sec^4 x\tan^4 x\,dx$ 2. $\int \tan^2 x\,dx$ 3. $\int \tan^5 x\sec x\,dx$ 4. $\int \sec^2 x\tan x\,dx$

5. $\int \sec^6 x\tan^5 x\,dx$ 6. $\int \sec^4 x\,dx$ 7. $\int \tan^3 x\sec^4 x\,dx$

Answers

I. 1. $\frac{1}{2}x+\frac{1}{4}\sin(2x)+C$ 2. $\frac{\cos^3 x}{3}-\cos x+C$ 3. $\frac{3}{8}x-\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+C$

4. $\frac{\cos^5 x}{5}-\frac{\cos^3 x}{3}+C$ 5. $\frac{\sin^7 x}{7}-\frac{\sin^9 x}{9}+C$ 6. $\frac{3}{2}x+\frac{1}{4}\sin(2x)+2\sin x+C$

II. 1. $\frac{1}{7}\tan^7 x+\frac{1}{5}\tan^5 x+C$ 2. $\tan x-x+C$ 3. $\sec x-\frac{2}{3}\sec^3 x+\frac{1}{5}\sec^5 x+C$

4. $\frac{1}{2}\tan^2 x+C$ or $\frac{1}{2}\sec^2 x+C$ 5. $\frac{1}{10}\sec^{10} x-\frac{1}{4}\sec^8 x+\frac{1}{6}\sec^6 x+C$ *or* $\frac{1}{10}\tan^{10} x+\frac{1}{4}\tan^8 x+\frac{1}{6}\tan^6 x+C$

6. $\frac{1}{3}\tan^3 x+\tan x+C$ 7. $\frac{1}{6}\tan^6 x+\frac{1}{4}\tan^4 x+C$

Trigonometric Substitution

Evaluate the following integrals.

1. $\int \sqrt{9-x^2}\,dx$ 2. $\int \frac{\sqrt{x^2-1}}{x}\,dx$ 3. $\int \frac{1}{\sqrt{x^2+1}}\,dx$

Answers:

1. $\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)+\frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right)+C$ 2. $\sqrt{x^2-1}-\sec^{-1}x+C$ 3. $\ln\left|\sqrt{x^2+1}+x\right|+C$

Integration by Trigonometric Substitution: Part 2

Evaluate the following integrals

1. $\int \frac{dx}{x^2\sqrt{x^2-16}}$ 2. $\int \frac{1}{\sqrt{x^2+25}}\,dx$ 3. $\int x^3\sqrt{4-x^2}\,dx$

4. $\int \frac{x^2}{\sqrt{36-x^2}}\,dx$ 5. $\int \frac{\sqrt{25x^2-1}}{x^2}\,dx$

ANSWERS:

1. $\frac{\sqrt{x^2-16}}{16x}+C$ 2. $\ln\left|\sqrt{x^2+25}+\frac{x}{5}\right|+C$ 3. $-32\left[\frac{1}{3}\left[\frac{\sqrt{4-x^2}}{2}\right]^3-\frac{1}{5}\left[\frac{\sqrt{4-x^2}}{2}\right]^5\right]+C$

4. $36\left[\frac{1}{2}\sin^{-1}\left(\frac{x}{6}\right)-\frac{1}{2}\left(\frac{6}{x}\right)\left(\frac{\sqrt{36-x^2}}{6}\right)\right]+C$

5. *First let* $u=5x;\ \frac{1}{5}du=dx$

$$-\frac{\sqrt{25x^2-1}}{x}+5\ln\left|5x+\sqrt{25x^2-1}\right|+C$$