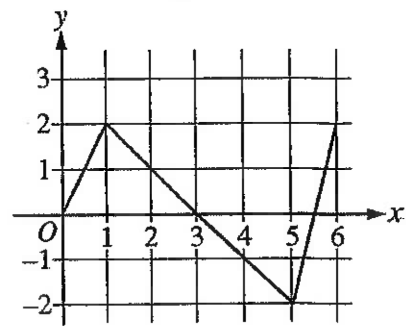
AP CALCULUS – FUNDAMENTAL THEOREM PART 1

Notes/Examples

MULTIPLE CHOICE – NON CALCULATOR

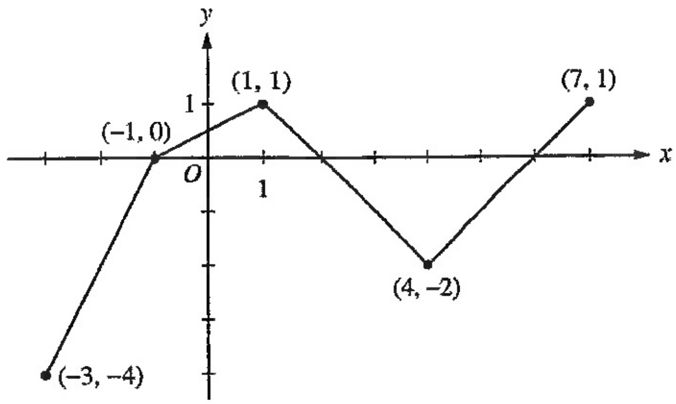
1. Determine the equation of the tangent line to the graph of $y = 3 - \int_{-1}^{x} e^{-t^3}\,dt$ at $x = -1$.

2. For $0 \le x \le 6$, the graph of f is a piecewise linear as shown above. If $g(x) = \int_{1}^{x} f(t)\,dt$, on which open interval(s) is the graph of g decreasing?



Graph of f

3. Let f be the function defined by $f(x) = \int_{0}^{x^2} (2t - 8)\,dt$. On which interval(s) is the graph of f increasing and concave up?



Graph of f

Let f be a continuous function defined on $[-3, 7]$ whose graph, consisting of four line segments, is given above. Let g be the function given by $g(x) = \int_{-1}^{x} f(t)\,dt$.

1. Determine the values of g(2), g(5), and g(−3)

2. Determine the values of g'(2) and g''(2)

3. Determine the critical numbers of g.

4. Determine the x-coordinate of any relative max/min points for g. Justify.

5. Determine where g is increasing

6. Determine where g is concave down

7. Determine the x –coordinate of any point of inflection for g. Explain

8. Determine the equation of the tangent line to g at $x = 1$.

9. Determine the absolute maximum value of g on the interval [–3, 7].

10. Determine the average rate of change of f on the interval [-3, 7]. Does the Mean Value Theorem applied in the interval [-3, 7] guarantee a value of c, for $-3 < x < 7$, such that $f'(c)$ is equal to this average rate of change? Why or why not?

Fundamental Theorem of Calculus: Part 1

Practice A

I. Compute the first derivative for the following functions.

1. $g(x)=\int_0^x \frac{1}{t^3+1}\,dt$
2. $g(x)=\int_0^x \left(2+t^4\right)^5\,dt$
3. $g(y)=\int_2^y t^2\sin t\,dt$
4. $g(r)=\int_0^r \sqrt{x^2+4}\,dx$
5. $F(x)=\int_x^\pi \sqrt{1+\sec t}\,dt$
6. $G(x)=\int_x^1 \cos\sqrt{t}\,dt$
7. $h(x)=\int_2^{1/x} \sin^4 t\,dt$
8. $h(x)=\int_0^{x^2} \sqrt{1+r^3}\,dr$
9. $y=\int_2^{\tan x} \sqrt{t+\sqrt{t}}\,dt$
10. $y=\int_1^{\cos x} \left(1+v^2\right)^{10}\,dv$
11. $y=\int_{1-3x}^1 \frac{u^3}{1+u^2}\,du$
12. $y=\int_{1/x^2}^0 \sin^3 t\,dt$
13. $g(x)=\int_{2x}^{x^3} \frac{1}{t^3+1}\,dt$
14. $g(x)=\int_{\sin x}^{\sqrt{x}} \sqrt{t^4+1}\,dt$

II. Compute the second derivative for the following functions.

1. $g(x)=\int_0^x \left(t^4-6\right)dt$
2. $g(x)=\int_0^x \frac{t}{t^3+1}\,dt$
3. $g(x)=\int_x^6 \sqrt[3]{t^2+1}\,dt$
4. $h(x)=\int_0^{x^2} t\sin t\,dt$
5. $h(x)=\int_{2x}^{x^3} \left(u^2+1\right)^5\,du$

Answers:

I. 1. $g'(x)=\frac{1}{x^3+1}$

2. $g'(x)=\left(2+x^4\right)^5$

3. $g'(y)=y^2\sin y$

4. $g'(r)=\sqrt{r^2+4}$

5. $F'(x)=-\sqrt{1+\sec x}$

6. $G'(x)=-\cos\sqrt{x}$

7. $h'(x)=\sin^4\left(\frac{1}{x}\right)\left(-x^{-2}\right)$

8. $h'(x)=2x\sqrt{1+x^6}$

9. $y'=\sec^2 x\sqrt{\tan x+\sqrt{\tan x}}$

10. $y'=\left(1+\cos^2 x\right)^{10}\left(-\sin x\right)$

11. $y'=-\frac{(1-3x)^3}{1+(1-3x)^2}(-3)$

12. $y'=-\sin^3\left(\frac{1}{x^2}\right)\left(-2x^{-3}\right)$

13. $g'(x)=\frac{3x^2}{x^9+1}-\frac{2}{8x^3+1}$

14. $g'(x)=\frac{1}{2}x^{\frac{-1}{2}}\sqrt{x^2+1}-\sqrt{\sin^4 x+1}(\cos x)$

II. 1. $g'(x)=x^4-6$

$g''(x)=4x^3$

2. $g'(x)=\frac{x}{x^3+1}$

$g''(x)=\frac{1\left(x^3+1\right)-3x^2(x)}{\left(x^3+1\right)^2}$

3. $g'(x)=-\sqrt[3]{x^2+1}$

$g''(x)=-\frac{1}{3}\left(x^2+1\right)^{\frac{-2}{3}}(2x)$

4. $h'(x)=x^2\sin\left(x^2\right)(2x)=2x^3\sin\left(x^2\right)$

$h''(x)=6x^2\sin\left(x^2\right)+\cos\left(x^2\right)(2x)\left(2x^3\right)$

5. $h'(x)=\left(x^6+1\right)^5\left(3x^2\right)-\left(4x^2+1\right)^5(2)$

$h''(x)=6x\left(x^6+1\right)^5+5\left(x^6+1\right)^4\left(6x^5\right)\left(3x^2\right)-10\left(4x^2+1\right)^4(8x)$

Fundamental Theorem of Calculus: Part 1

Practice B

Multiple Choice (NON – CALCULATOR) – Choose the best answer for each of the following

1. If $F(x)=\int_0^x e^{-t^2}dt$, then $F'(x)=$

(A) $2xe^{-x^2}$ (B) $2xe^{-x^2}$ (C) $e^{-x^2}-1$ (D) e^{-x^2}

2. If $F(x)=\int_0^x \sqrt{t^3+1}\,dt$, then $F'(2)=$

(A) 2 (B) 3 (C) 9 (D) 18

3. If $f(x)=\int_0^x (t^2+1)dt$, which of the following is false.

(A) $f(0)=0$ (B) $f(1)>0$ (C) $f'(0)=0$ (D) $f'(1)=2$

4. $\frac{d}{dx}\left(\int_0^{3x}\sqrt{1+u^2}\,du\right)=$

(A) $3\sqrt{1+9x^2}$ (B) $3\sqrt{1+3x}$ (C) $\sqrt{1+9x^2}$ (D) $\sqrt{1+x^2}$

5. $\frac{d}{dx}\left(\int_x^3 \cos(2\pi t)dt\right)=$

(A) 0 (B) $\cos(2\pi x)$ (C) $-\cos(2\pi x)$ (D) $-2\pi\cos(2\pi x)$

6. Let $f(x)=\int_{-2}^x (3t-12)dt$. On which interval below is f increasing?

(A) $(-\infty,4)$ only (B) $(4,\infty)$ only (C) $(3,\infty)$ only (D) $(-\infty,0)$ only

7. Let $f(x)=\int_{-2}^{x^2-4x} e^t\,dt$. At which value of x does $f(x)$ have a minimum.

(A) -2 (B) 2 (C) 0 (D) For no values of x

8. Let $f(x)=\int_0^{x^2}(t-4)dt$. How many critical numbers does the function f have?

(A) 0 (B) 1 (C) 2 (D) 3

9. Let $h(x)=\int_0^x (t^2-4t)dt$. For which value of x is $h''(x)=0$?

(A) 0 (B) 1 (C) 2 (D) 3

10. If $g(x)=\int_1^x e^{-2t}dt$, then which of the following is the correct arrangement for $g(1)$, $g'(1)$, and $g''(1)$.

(A) $g(1)<g'(1)<g''(1)$

(B) $g'(1)<g(1)<g''(1)$

(C) $g''(1)<g'(1)<g(1)$

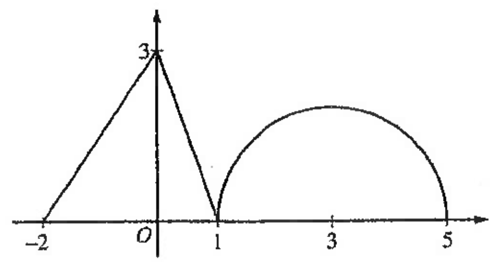
(D) $g''(1)<g(1)<g'(1)$

11. Let f be the function defined by $f(x)=\int_0^x (2t^3-15t^2+36t)dt$. On which of the following intervals is the graph of f concave down?

(A) $(-\infty,2)$ only (B) $(2,3)$ only (C) $(3,\infty)$ only (D) $(-\infty,0)$ only

USE THE FOLLOWING INFORMATION TO ANSWER QUESTIONS 12, 13 AND 14

The graph of the function f shown below consists of two line segments and a semicircle. Let g be defined by $g(x) = \int_0^x f(t)dt$.



Graph of f

12. What is the value of $g(5)$?

(A) 0 (B) 2π (C) $1.5 + 2\pi$ (D) $4.5 + 2\pi$

13. What is the value of $g(-2)$?

(A) -4.5 (B) -3 (C) 1.5 (D) 4.5

14. What is the value of $g'(3)$?

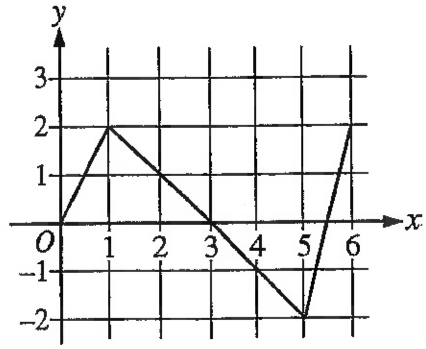
(A) -4.5 (B) -3 (C) 0 (D) 2

15. If $w(x) = \int_4^{2x} \sqrt{t^2 - t}\, dt$, then $w'(2) =$

(A) 0 (B) $2\sqrt{2}$ (C) $\sqrt{12}$ (D) $2\sqrt{12}$

USE THE FOLLOWING INFORMATION TO ANSWER QUESTIONS 16, 17 AND 18

The graph of the function f shown below consists of three line segments. Let h be defined by $h(x)=\int_0^x f(t)dt$.



16. What is the value of $h(3)$?

(A) 0 (B) 3 (C) 4 (D) 6

17. What is the value of $h'(3)$?

(A) 0 (B) 3 (C) 4 (D) 6

18. What is the value of $h''(3)$?

(A) -3 (B) -1 (C) 0 (D) 2

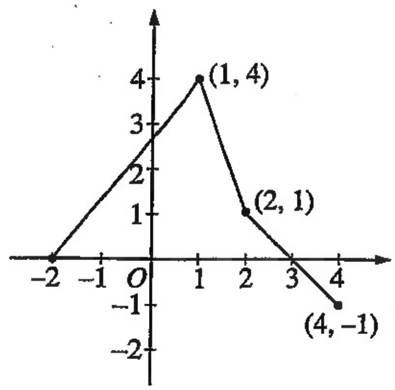
19. If $f(x)=\int_0^x (t^2+1)dt$, for which value of x does $f'(x)=f''(x)$

(A) 0 (B) 1 (C) 2 (D) 3

Answers:

1. D	2. B	3. C	4. A	5. C	6. B	7. B	8. D
9. C	10. D	11. B	12. C	13. B	14. D	15. D	16. B
17. A	18. B	19. B					

Practice C



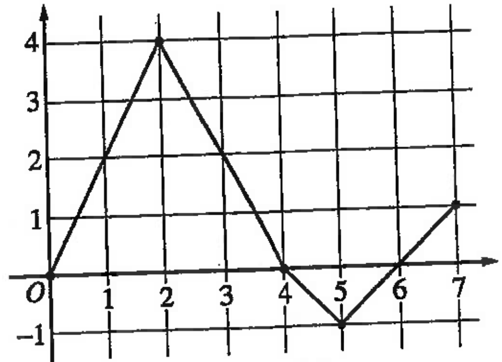
1. The graph of the function f, consisting of three line segments, is given above. Let $g(x) = \int_1^x f(t)\,dt$.

(a) Compute $g(4)$ and $g(-2)$.

(b) Find the instantaneous rate of change of g, with respect to x, at $x = 1$.

(c) Find the absolute minimum value of g on the closed interval $[-2, 4]$. Justify your answer.

(d) The second derivative of g is not defined at $x = 1$ and $x = 2$. How many of these values are x-coordinates of points of inflection of the graph of g? Justify your answer.



Graph of f

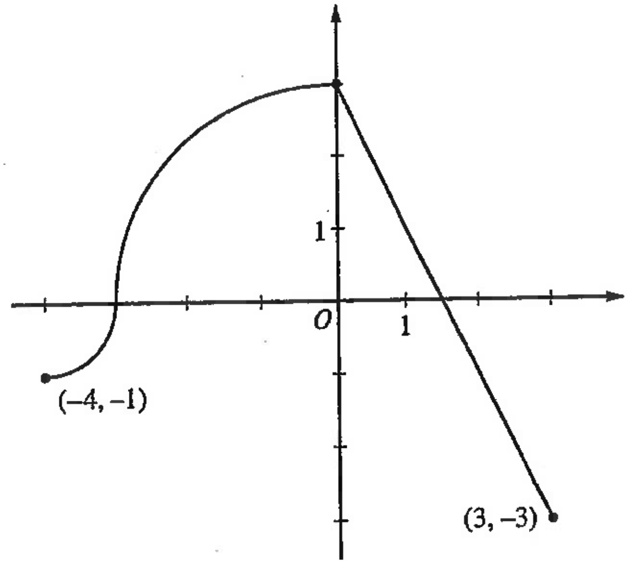
2. Let f be a function defined on the closed interval $[0, 7]$. The graph of f, consisting of four line segments, is shown above. Let g be the function given by $g(x) = \int_2^x f(t)\,dt$.

(a) Find $g(3)$, $g'(3)$, and $g''(3)$.

(b) Find the average rate of change of g on the interval $0 \le x \le 3$.

(c) For how many values c, where $0 < c < 3$, is $g'(c)$ equal to the average rate found in part (b) ? Explain your reasoning.

(d) Find the x-coordinate of each point of inflection of the graph of g on the interval $0 < x < 7$. Justify your answer.



Graph of f

34. The continuous function f is defined on the interval $-4 \le x \le 3$. The graph of f consists of two quarter circles and one line segment, as shown in the figure above. Let $g(x) = 2x + \int_0^x f(t)\,dt$.

(a) Find $g(-3)$. Find $g'(x)$ and evaluate $g'(-3)$.

(b) Determine the x-coordinate of the point at which g has an absolute maximum on the interval $-4 \le x \le 3$. Justify your answer.

(c) Find all values of x on the interval $-4 < x < 3$ for which the graph of g has a point of inflection. Give a reason for your answer.

(d) Find the average rate of change of f on the interval $-4 \le x \le 3$. There is no point c, $-4 < c < 3$, for which $f'(c)$ is equal to that average rate of change. Explain why this statement does not contradict the Mean Value Theorem.

Answers:

1 a) $g(4) = \int_1^4 f(t)\,dt$

$= \frac{4+1}{2} + \frac{1}{2} - \frac{1}{2}$

$= \frac{5}{2}$

$g(-2) = \int_1^{-2} f(t)\,dt$

$= -\int_{-2}^{1} f(t)\,dt$

$= -\frac{(3)(4)}{2}$

$= -6$

(1 mark each)

b) $g'(x) = f(x)$

$g'(1) = f(1) = 4$

(one mark)

c) g has a critical # at $x = 3$ since $g'(x) = f(x)$ changes sign. So the min will occur at the end points or crit#

$g(4) = \frac{5}{2}$; $g(-2) = -6$

$g(3) = \int_1^3 f(t)\,dt = \frac{5}{2} + \frac{1}{2} = 3$

min occurs at $x = -2$ and the min value is -6.

(3 marks)

d) Inflection points of g happen when $g''(x) = f'(x) = 0$ or DNE and changes sign. Possible inflection points are $x = 2$ and $x = 1$

on $(-2, 1)$ $g''(x) = f'(x) > 0$

on $(1, 2)$ $g''(x) = f'(x) < 0$

on $(2, 4)$ $g''(x) = f'(x) < 0$

Therefore g has a point of inflection at $(1, g(1)) = (1, 0)$ since $g''(x)$ DNE and changes sign at $x = 1$.

(3 marks)

2 a) $g(3) = \int_2^3 f(t)\,dt = \frac{1(4+2)}{2} = 3$

$g'(3) = f(3) = 2$

$g''(3) = f'(3) = \frac{0-4}{4-2} = -2$

(3 marks total)

b) $\frac{g(3) - g(0)}{3 - 0} = \frac{1}{3}\int_0^3 f(t)\,dt$

$= \frac{1}{3}\left[\frac{2 \cdot 4}{2} + \frac{1(4+2)}{2}\right]$

$= \frac{7}{3}$

(2 marks)

c) $g'(c) = \frac{7}{3}$

$* \; g'(x) = f(x)$

$f(x) = \frac{7}{3} \approx 2.\overline{3}$

This happens twice between $x = 0$ and $x = 3$ since the graph of $f(x)$ intersects the line $y = \frac{7}{3}$ twice.

(2 marks)

d) Inflection points of g occur when $g''(x) = f'(x) = 0$ or DNE and change sign.

Therefore the points of infl. occur at $x = 2$ and $x = 5$, since $g''(x) = f'(x)$ changes from pos to neg at $x = 2$ and $g''(x) = f'(x)$ changes from neg to pos at $x = 5$.

(2 marks)

3 a) $g(-3) = 2(-3) + \int_0^{-3} f(t)\,dt$

$g(-3) = -6 - \int_{-3}^{0} f(t)\,dt$

$g(-3) = -6 - \left[\frac{1}{4}\pi(3)^2\right]$

$g(-3) = -6 - \frac{9\pi}{4}$

$g'(x) = 2 + f(x)$

$g'(-3) = 2 + f(-3)$

$g'(-3) = 2$

(3 marks)

b) Abs max at endpoints or critical #'s

crit #'s $g'(x) = 0$ or DNE

$g'(x) = 2 + f(x) = 0$

$2 + f(x) = 0$

$f(x) = -2$

This occurs at $x = \frac{5}{2}$

$g(-4) = 2(-4) + \int_0^{-4} f(t)\,dt$

$g(-4) = -8 - 2\pi$

$g(3) = 2(3) + \int_0^3 f(t)\,dt$

$g(3) = 6$

$g\left(\frac{5}{2}\right) = 2\left(\frac{5}{2}\right) + \int_0^{\frac{5}{2}} f(t)\,dt$

$g\left(\frac{5}{2}\right) = 5 + \frac{5}{4}$

$= \frac{25}{4}$ abs max at $x = \frac{5}{2}$ (3 marks)

c) point of inflection when $g''(x) = f'(x) = 0$ or DNE and changes sign.

g has a point of inflection at $x = 0$ since $g''(x) = f'(x)$ DNE and changes sign at $x = 0$. (1 mark)

d) $\frac{f(3) - f(-4)}{3 - (-4)} = -\frac{2}{7}$

This does not contradict the MVT since f must be continuous and differentiable for the MVT to apply. Since f is not differentiable at $x = -3$ and $x = 0$, the MVT does not apply and the statement is not a contradiction.

(2 marks)