L'Hopital's Rule

Evaluate the following Limits.

1. $\lim\limits_{x \to 1} \dfrac{x^9 - 1}{x^5 - 1}$

2. $\lim\limits_{x \to 0} \dfrac{x + \tan x}{\sin x}$

3. $\lim\limits_{x \to \infty} \dfrac{x}{(\ln x)^3}$

4. $\lim\limits_{x \to \infty} x^3 e^{-x^2}$

5. $\lim\limits_{x \to 1} \left(\dfrac{1}{\ln x} - \dfrac{1}{x-1} \right)$

6. $\lim\limits_{t \to 0} \dfrac{e^{3t} - 1}{t}$

7. $\lim\limits_{t \to 0} \dfrac{e^t - 1}{t^3}$

8. $\lim\limits_{x \to \infty} x \left(e^{\frac{1}{x}} - 1 \right)$

9. $\lim\limits_{x \to 0} \dfrac{xe^{3x} - x}{1 - \cos(2x)}$

10. $\lim\limits_{x \to 0} \dfrac{\sin^{-1} x}{x}$

11. $\lim\limits_{x \to 0} \dfrac{\ln(\cos x)}{x^2}$

12. $\lim\limits_{x \to \infty} x \tan\left(\dfrac{1}{x} \right)$

13. $\lim\limits_{x \to 0} \dfrac{5^x - 3^x}{x}$

14. $\lim\limits_{x \to 0} \dfrac{\sqrt{1-x} - \sqrt{1-x^2}}{x}$

15. $\lim\limits_{x \to \frac{\pi}{2}^+} \dfrac{1 - \sin x}{\cos x}$

16. $\lim\limits_{x \to 0} (\csc x - \cot x)$

Answers:

1. $\dfrac{9}{5}$ 2. 2 3. ∞ 4. 0 5. $\dfrac{1}{2}$ 6. 3 7. ∞

8. 1 9. $\dfrac{3}{2}$ 10. 1 11. $-\dfrac{1}{2}$ 12. 1 13. $\ln 5 - \ln 3 = \ln\left(\dfrac{5}{3}\right)$

14. $-\dfrac{1}{2}$ 15. 0 16. 0

(1) $\lim\limits_{x\to 0}\dfrac{xe^{3x}-x}{1-\cos(2x)}$

(2) $\lim\limits_{x\to +\infty}\dfrac{x}{(\ln x)^3}$

(3) $\lim\limits_{x\to 0}[\ln(1-\cos x)-\ln(x^2)]$

(4) $\lim\limits_{x\to +\infty}\dfrac{\cos\left(\frac{1}{x}\right)-1}{\cos\left(\frac{2}{x}\right)-1}$

(5) $\lim\limits_{x\to 0}\dfrac{\sqrt{1-x}-\sqrt{1-x^2}}{x}$

(6) $\lim\limits_{x\to 1}\dfrac{5x^4-7x^3+x^2-x+2}{3x^4-8x^3+6x^2-1}$

(7) $\lim\limits_{x\to 0}\dfrac{9-\sqrt{81-5x}}{x}$

(8) $\lim\limits_{x\to +\infty}\dfrac{\ln\left(x^3+2\right)}{\ln\left(5x^3-1\right)}$

(9) $\lim\limits_{x\to 0}\dfrac{\sin(4x)-2\sin(2x)}{x^3}$

(10) $\lim\limits_{x\to 0}\left[\dfrac{1}{\sin x}-\dfrac{1}{x}\right]$

(11) $\lim\limits_{x\to +\infty}x\left(e^{1/x}-1\right)$

(12) $\lim\limits_{x\to 0^+}\sqrt[3]{x}\ln x$

(13) $\lim\limits_{x\to 0}\dfrac{\ln\left(\frac{2x+1}{5x+1}\right)}{x}$

(14) $\lim\limits_{x\to 0}\dfrac{\ln(\cos x)}{x^2}$

(15) $\lim\limits_{x\to +\infty}\left(\ln(e^x+1)-x\right)$

Answers

1. $\frac{3}{2}$ 2. $+\infty$ 3. $-\ln 2$ 4. $\frac{1}{4}$ 5. $-\frac{1}{2}$ 6. $\frac{20}{0}\Rightarrow$ limit does not exist.

7. $\frac{5}{18}$ 8. 1 9. -8 10. 0 11. 1 12. 0 13. -3

14. $-\frac{1}{2}$ 15. 0

L'Hopital's Rule for Indeterminate Forms - Homework

Basic Problems - calculate your answers and check on your calculators

1. $\lim_{x \to -1} \frac{2x^2 - x - 3}{x+1}$

2. $\lim_{x \to 3} \frac{x^2 - x - 6}{x - 3}$

3. $\lim_{x \to 3} \frac{x^2 - x - 3}{x - 2}$

4. $\lim_{x \to 1} \frac{x - 1}{\sqrt{x^2 + 3} - 2}$

5. $\lim_{r \to 1} \frac{1 - r^3}{2 - \sqrt{r^2 + 3}}$

6. $\lim_{x \to 1} \frac{x^3 - 3x + 2}{x^3 - x^2 - x + 1}$

7. $\lim_{x \to 1} \frac{1 - x + \ln x}{x^3 - 3x + 2}$

8. $\lim_{x \to 0} \frac{x}{1 - e^x}$

9. $\lim_{x\to 0} \frac{x\cos x - \sin x}{x}$

10. $\lim_{x\to 0} \frac{\sin 2x \ \tan x}{3x}$

11. $\lim_{x\to 0} \frac{\sin 2x + \tan x}{6x}$

12. $\lim_{x\to 0} \frac{\tan x - x}{x - \sin x}$

13. $\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$

14. $\lim_{x\to 0} \frac{10^{2x} - 2x - 10^{-2x}}{10^{2x} - 10^{-2x}}$

15. $\lim_{x\to \infty} \frac{x^2 - 1}{4x^2 + x}$

16. $\lim_{x\to \infty} \frac{2x^2 + 4x - 7}{x^3 + 3x^2 - 5}$

17. $\lim_{x\to\infty}\frac{x^3}{e^x}$

18. $\lim_{x\to\infty}\frac{\sqrt{x^2-1}}{2x+1}$

19. $\lim_{x\to-\infty}\frac{x^2}{x+1}$

20. $\lim_{x\to 0}\frac{\ln x}{\frac{1}{x}}$

21. $\lim_{x\to\infty}\frac{e^x}{\ln x}$

22. $\lim_{x\to\infty}\frac{\sqrt{x}}{e^x}$

23. $\lim_{x\to\infty}x^2e^{-3x}$

24. $\lim_{x\to\frac{\pi}{2}}\frac{\sec x}{\sec^2 3x}$

25. $\lim_{x\to 0^+}\left(x^2\ln x\right)$

26. $\lim_{x\to 0}\left(\csc x-\cot x\right)$

27. $\lim_{x\to 0}\left[\frac{1}{\sin x}-\frac{1}{x}\right]$

28. $\lim_{x\to 1}\left[\frac{1}{\ln x}-\frac{1}{x-1}\right]$

More advanced L'Hopital's Rule Problems - Calculate your answers and check on your calculators

29. $\lim_{x\to 0} x^{\left(x^2\right)}$

30. $\lim_{x\to 0}(1+x)^{1/x}$

31. $\lim_{x\to 0^+}\left(\frac{1}{x}\right)^{\sin x}$

32. $\lim_{x\to \pi/2^-}(\tan x)^{\cos x}$

Limits of the type $\frac{0}{\infty}$, $\frac{\infty}{0}$, 0^{∞}, $\infty \bullet \infty$, $\infty + \infty$ are **not** indeterminate forms. Find the following by inspection:

33. $\lim_{x\to 0^+}\frac{x}{\ln x}$

33. $\lim_{x\to\left(\pi/2\right)^-}(\cos x)^{\tan x}$

34. $\lim_{x\to\left(\pi/2\right)^-}\left(\frac{2}{\pi-2x}+\tan x\right)$

L'Hopital's Rule for Indeterminate Forms - Homework

Basic Problems - calculate your answers and check on your calculators

1. $$\lim_{x\to -1}\frac{2x^2-x-3}{x+1}$$
$$\lim_{x\to -1}\frac{4x-1}{1}=-5$$

2. $$\lim_{x\to 3}\frac{x^2-x-6}{x-3}$$
$$\lim_{x\to 3}\frac{2x-1}{1}=5$$

3. $$\lim_{x\to 3}\frac{x^2-x-3}{x-2}=3$$

4. $$\lim_{x\to 1}\frac{x-1}{\sqrt{x^2+3}-2}$$
$$\lim_{x\to 1}\frac{1}{\frac{2x}{2\sqrt{x^2+3}}}=\lim_{x\to 1}\frac{2\sqrt{x^2+3}}{2x}=2$$

5. $$\lim_{r\to 1}\frac{1-r^3}{2-\sqrt{r^2+3}}$$
$$\lim_{r\to 1}\frac{-3r^2}{-\frac{2r}{2\sqrt{r^2+3}}}=\lim_{r\to 1}3r\sqrt{r^2+3}=6$$

6. $$\lim_{x\to 1}\frac{x^3-3x+2}{x^3-x^2-x+1}$$
$$\lim_{x\to 1}\frac{3x^2-3}{3x^2-2x-1}$$
$$\lim_{x\to 1}\frac{6x}{6x-2}=\frac{3}{2}$$

7. $$\lim_{x\to 1}\frac{1-x+\ln x}{x^3-3x+2}$$
$$\lim_{x\to 1}\frac{-1+1/x}{3x^2-3}=\lim_{x\to 1}\frac{-1/x^2}{6x}=\frac{-1}{6}$$

8. $$\lim_{x\to 0}\frac{x}{1-e^x}$$
$$\lim_{x\to 0}\frac{1}{-e^x}=-1$$

9. $$\lim_{x\to 0}\frac{x\cos x-\sin x}{x}$$
$$\lim_{x\to 0}\frac{x(-\sin x)+\cos x-\cos x}{1}=0$$

10. $$\lim_{x\to 0}\frac{\sin 2x\ \tan x}{3x}$$
$$\lim_{x\to 0}\frac{\sin 2x\ \sec^2 x+2\tan x\cos 2x}{3}=0$$

11. $$\lim_{x\to 0}\frac{\sin 2x+\tan x}{6x}$$
$$\lim_{x\to 0}\frac{2\cos 2x+\sec^2 x}{6}=\frac{1}{2}$$

12. $$\lim_{x\to 0}\frac{\tan x-x}{x-\sin x}$$
$$\lim_{x\to 0}\frac{\sec^2 x-1}{1-\cos x}$$
$$\lim_{x\to 0}\frac{2\sec^2 x\tan x}{\sin x}$$
$$\lim_{x\to 0}\frac{2\sec^2 x\sec^2 x+4\sec^2 x\tan^2 x}{\cos x}=2$$

13. $$\lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}$$
$$\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos x}$$
$$\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}$$
$$\lim_{x\to 0}\frac{e^x+e^{-x}}{\cos x}=2$$

14. $$\lim_{x\to 0}\frac{10^{2x}-2x-10^{-2x}}{10^{2x}-10^{-2x}}$$
$$\lim_{x\to 0}\frac{2\ln 10\cdot 10^{2x}-2+2\ln 10\cdot 10^{-2x}}{2\ln 10\cdot 10^{2x}+2\ln 10\cdot 10^{-2x}}$$
$$\frac{4\ln 10-2}{4\ln 10}=\frac{2\ln 10-1}{2\ln 10}$$

15. $$\lim_{x\to\infty}\frac{x^2-1}{4x^2+x}=\frac{1}{4}$$

16. $$\lim_{x\to\infty}\frac{2x^2+4x-7}{x^3+3x^2-5}=0$$

17. $\boxed{\lim_{x\to\infty}\frac{x^3}{e^x}=\lim_{x\to\infty}\frac{3x^2}{e^x}=\lim_{x\to\infty}\frac{6x}{e^x}=\lim_{x\to\infty}\frac{6}{e^x}=0}$

18. $\boxed{\lim_{x\to\infty}\frac{\sqrt{x^2-1}}{2x+1}=\frac{1}{2}}$ (Note: L'Hopital won't work)

19. $\boxed{\lim_{x\to-\infty}\frac{\sqrt{x^2-1}}{2x+1}=\frac{1}{2}\lim_{x\to-\infty}\frac{x^2}{x+1}=-\infty}$

20. $\boxed{\lim_{x\to 0}\frac{\ln x}{\frac{1}{x}}=\lim_{x\to 0}\frac{\frac{1}{x}}{\frac{-1}{x^2}}=\lim_{x\to 0}-x=0}$

21. $\boxed{\lim_{x\to\infty}\frac{e^x}{\ln x}=\lim_{x\to\infty}\frac{e^x}{1/x}=\lim_{x\to\infty}xe^x=\infty}$

22. $\boxed{\lim_{x\to\infty}\frac{\sqrt{x}}{e^x}=\lim_{x\to\infty}\frac{1}{2\sqrt{x}e^x}=0}$

23.
$$\lim_{x\to\infty}x^2e^{-3x}=\lim_{x\to\infty}\frac{x^2}{e^{3x}}$$
$$\lim_{x\to\infty}\frac{2x}{3e^{3x}}=\lim_{x\to\infty}\frac{2}{9e^{3x}}=0$$

24.
$$\lim_{x\to\frac{\pi}{2}}\frac{\sec x}{\sec^2 3x}=\lim_{x\to\frac{\pi}{2}}\frac{\cos^2 3x}{\cos x}$$
$$\lim_{x\to\frac{\pi}{2}}\frac{-6\cos 3x(-\sin 3x)}{-\sin x}=0$$

25.
$$\lim_{x\to 0^+}\left(x^2\ln x\right)$$
$$\lim_{x\to 0^+}\frac{\ln x}{1/x^2}=\lim_{x\to 0^+}\frac{1/x}{-1/x^3}$$
$$\lim_{x\to 0^+}\left(-x^2\right)=0$$

26.
$$\lim_{x\to 0}(\csc x-\cot x)=\lim_{x\to 0}\left(\frac{1-\cos x}{\sin x}\right)$$
$$\lim_{x\to 0}\left(\frac{\sin x}{\cos x}\right)=0$$

27. $$\lim_{x\to 0}\left[\frac{1}{\sin x}-\frac{1}{x}\right]=\lim_{x\to 0}\left[\frac{x-\sin x}{x\sin x}\right]$$
$$\lim_{x\to 0}\left[\frac{1-\cos x}{x\cos x+\sin x}\right]$$
$$\lim_{x\to 0}\left[\frac{-\sin x}{x(-\sin x)+\cos x+\cos x}\right]=0$$

28. $$\lim_{x\to 1}\left[\frac{1}{\ln x}-\frac{1}{x-1}\right]=\lim_{x\to 1}\left[\frac{x-1-\ln x}{(x-1)\ln x}\right]$$
$$\lim_{x\to 1}\left[\frac{1-1/x}{\frac{(x-1)}{x}+\ln x}\right]=\lim_{x\to 1}\left[\frac{x-1}{x-1+\ln x}\right]$$
$$\lim_{x\to 1}\left[\frac{1}{1+1/x}\right]=\frac{1}{2}$$

More advanced L'Hopital's Rule Problems - Calculate your answers and check on your calculators

29. $$\lim_{x\to 0} x^{(x^2)} \Rightarrow y=x^{(x^2)}$$
$$\ln y = x^2\ln x$$
$$\lim_{x\to 0}\frac{\ln x}{1/x^2}=\lim_{x\to 0}\frac{1/x}{-1/x^3}=0$$
$$\ln y=0\Rightarrow y=1$$
Technically, this is a right-sided limit

30. $$\lim_{x\to 0}(1+x)^{1/x}\Rightarrow y=(1+x)^{1/x}$$
$$\ln y=\frac{\ln(1+x)}{x}$$
$$\lim_{x\to\infty}\frac{\ln(1+x)}{x}=\lim_{x\to\infty}\frac{1}{1+x}=0$$
$$\ln y=0\Rightarrow y=e$$

31. $$\lim_{x\to 0^+}\left(\frac{1}{x}\right)^{\sin x}\Rightarrow y=\left(\frac{1}{x}\right)^{\sin x}$$
$$\ln y=\sin x\ln\left(\frac{1}{x}\right)=-\sin x\ln x$$
$$\lim_{x\to 0^+}\frac{-\ln x}{1/\sin x}=\lim_{x\to 0^+}\frac{-1/x}{-\cos x/\sin^2 x}$$
$$\lim_{x\to 0^+}\frac{\sin^2 x}{x\cos x}=\lim_{x\to 0^+}\frac{2\sin x\cos x}{x(-\sin x)+\cos x}=0$$
$$\ln y=0\Rightarrow y=1$$

32. $$\lim_{x\to \pi/2^-}(\tan x)^{\cos x}\Rightarrow y=(\tan x)^{\cos x}$$
$$\ln y=\cos x\ln\tan x$$
$$\lim_{x\to \pi/2^-}\frac{\ln\tan x}{1/\cos x}=\lim_{x\to \pi/2^-}\frac{\sec^2 x/\tan x}{\sin x/\cos^2 x}$$
$$\lim_{x\to \pi/2^-}\frac{\sec^2 x}{\sin}\cdot\frac{\cos^2 x}{\tan x}=\lim_{x\to \pi/2^-}\frac{\cos x}{\sin^2 x}=0$$
$$\ln y=0\Rightarrow y=1$$

Limits of the type $\frac{0}{\infty}$, $\frac{\infty}{0}$, 0^{∞}, $\infty\bullet\infty$, $\infty+\infty$ are **not** indeterminate forms. Find the following by inspection:

33. $$\lim_{x\to 0^+}\frac{x}{\ln x}=0$$

33. $$\lim_{x\to(\pi/2)^-}(\cos x)^{\tan x}=0$$

34. $$\lim_{x\to(\pi/2)^-}\left(\frac{2}{\pi-2x}+\tan x\right)=\infty$$