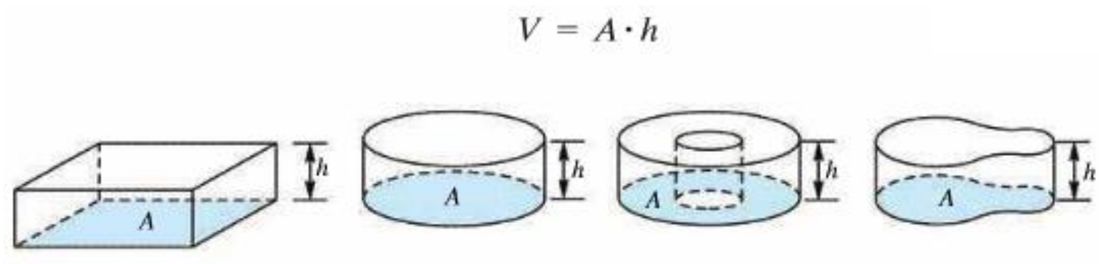
<u>Volumes</u>

There are two types of volumes that we are going to look at in this class;

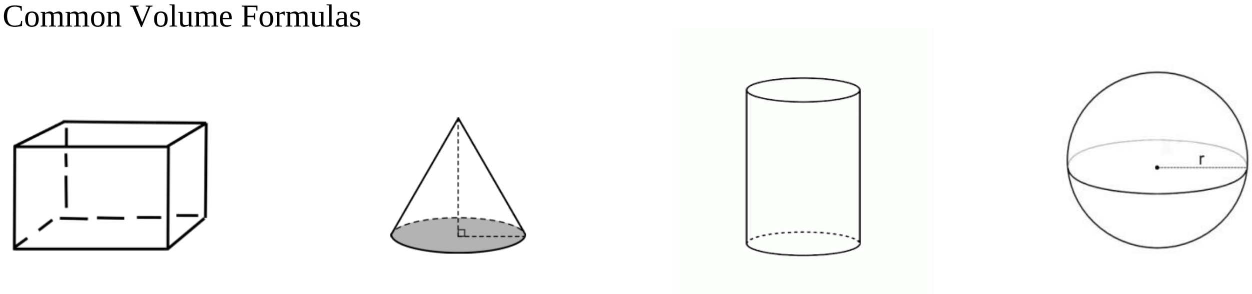
1. Volumes of Revolution
2. Volumes of Known Cross Sections

<u>Common Volumes</u> (Uniform Cross – Sectional Area)

The volume of a geometric solid with uniform cross – sectional area is defined as the area of the base multiplied by the height.



Common Volume Formulas



$V = lwh$ $\qquad$ $V = \frac{1}{3}\pi r^2 h$ $\qquad$ $V = \pi r^2 h$ $\qquad$ $V = \frac{4}{3}\pi r^3$

* Volumes of simple shapes, such as regular, straight-edged, and circular shapes can be calculated using already derived formulas. If we are not given such a shape, but instead a more complicated formula, these volumes can be calculated using integration

Important Definitions

1. <u>Cross Section</u> – the intersection of a 3-D solid with a plane

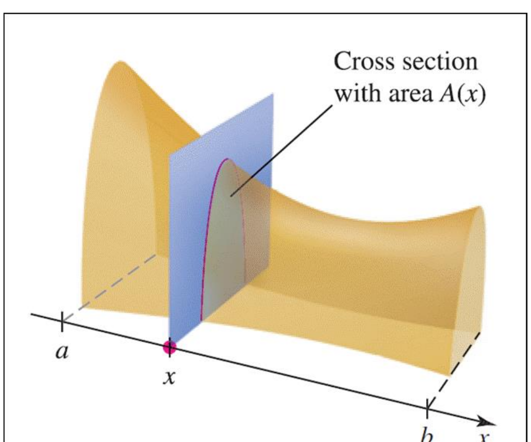
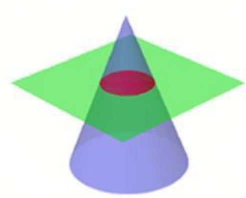
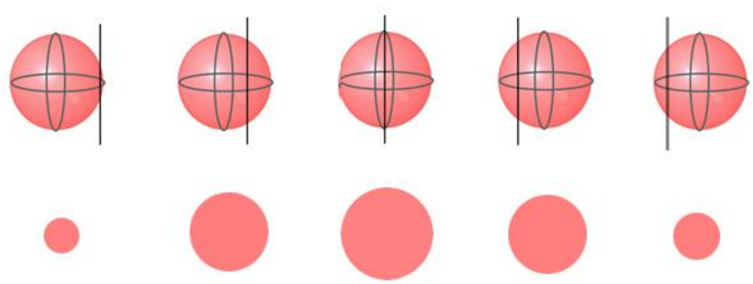


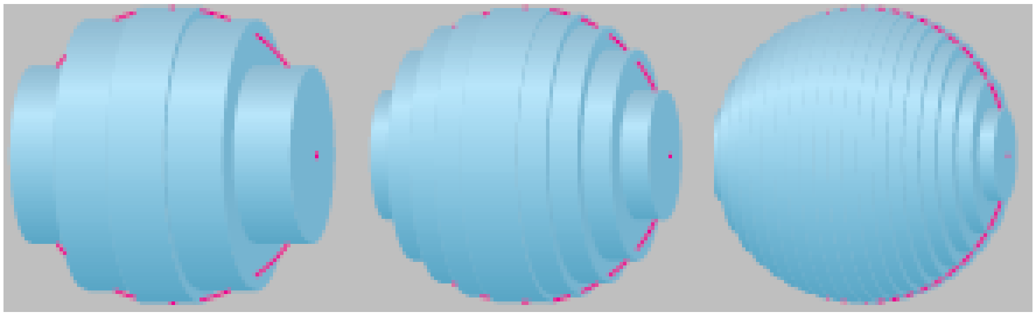
FIGURE 6.22





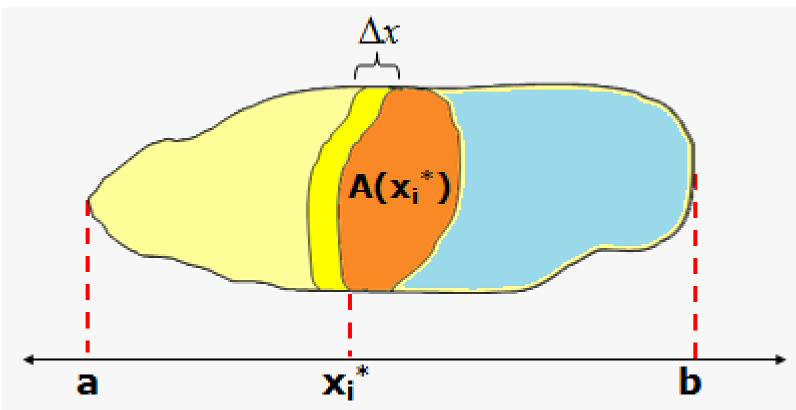
2. Volume – if we take all of the cross sectional areas, A(x), and add them up, then we create a volume.

The volume created with a finite number of cross sections will be an approximation rather than exact.



5 Cross Sections   10 Cross Sections   20 Cross Sections

To calculate the exact volume of a solid we would have to break the solid up into infinitely many cross sectional areas and add them up to obtain a volume



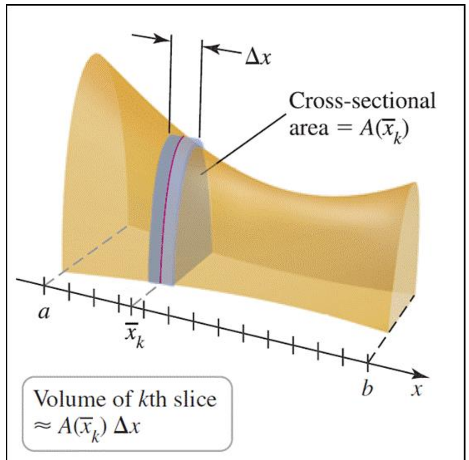


FIGURE 6.23

Volume = (Area of the cross section)(Height/Thickness)

The volume of one of the slabs would approximately be

$$V_{slice} \approx A(x_i)\Delta x$$

Adding up the volume of all of the slabs, we get an approximation of the total volume

$$V \approx \sum_{i=1}^{n} A(x_i)\Delta x$$

The approximation for the volume becomes better as the number of slabs increases $(n \to \infty)$. Each slab will become thinner and thinner, therefore we define the volume as:

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i)\Delta x$$

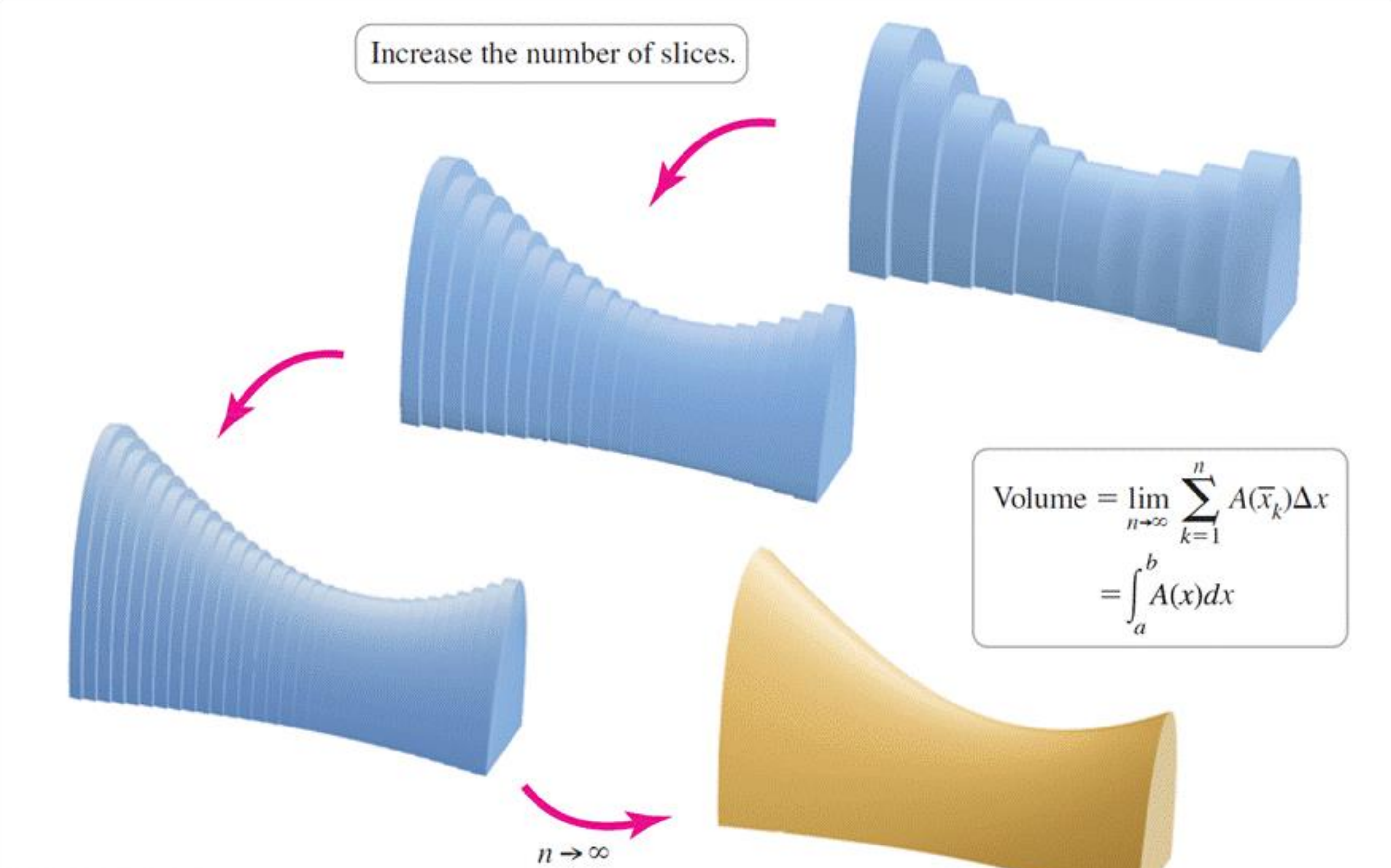


FIGURE 6.24

Recall, from areas and definite integrals, that we can represent the infinite sum as a definite integral which gives us the following definition

Definition of Volume:

Let S be a solid that lies between x = a and x = b. If the cross-sectional area of the solid that is perpendicular to the x – axis is A(x), where A is a continuous function, then the volume of the solid is:

$$V = \int_a^b A(x)dx$$

where: *A(x)* is the area of the cross section, in terms of x, obtained by slicing the solid perpendicular to the x-axis

Note: If our cross section is perpendicular to the y-axis then the volume is $V = \int_c^d A(y)dy$

Types of Volumes

1. Volumes of Revolution

- A volume/solid can be created by rotating a plane area about a fixed axis.

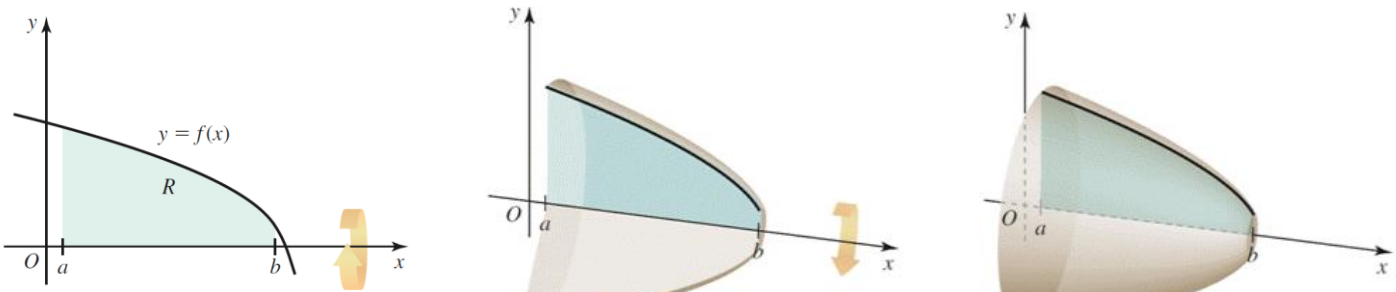
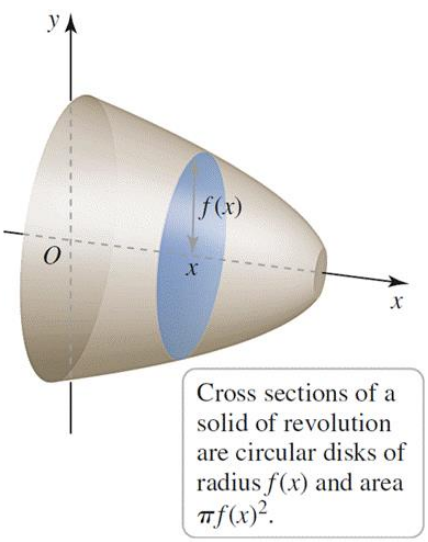


FIGURE 6.26

- Since the solid is obtained by rotating a plane about an axis, the cross-section that is created is a circular disk with radius y = f(x), that is perpendicular to the axis of rotation. This is known as the disk method.



Ex: Find the volume of the solid obtained by rotating the region bounded by $y = x^{\frac{3}{2}}$, x = 4, x = 0 and y = 0, about the x - axis.

If the plane/region is rotated about the y – axis the cross – sections will be circular disks perpendicular to the y – axis. Therefore the volume will be given by:

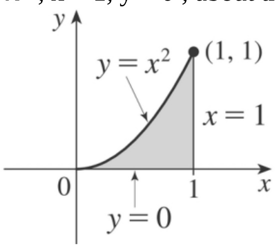
$$V = \int_c^d A(y)dy$$

Ex: Find the volume of revolution of the solid obtained by rotating the region bounded by $y = x^2$, y = 4, x = 0, and x = 2 about the y – axis.

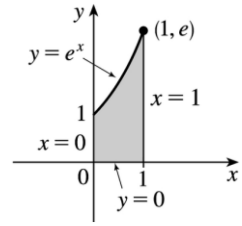
Volumes of Revolution Practice

Find the volume of the solid obtained by rotating the region bounded by the given curves about the specific line.
Show the setup of the cross section area, the setup of the integral and your calculation.
Questions marked with a * are calculator questions.

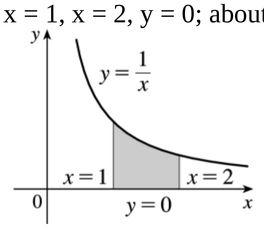
1. $y = x^2$, x = 1, y = 0 ; about the x – axis.



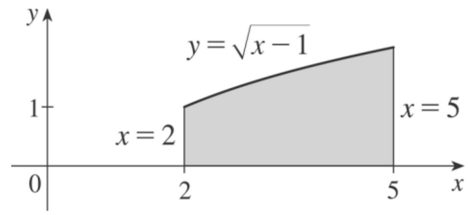
2. $y = e^x$, x = 0, x = 1, y = 0; about the x – axis



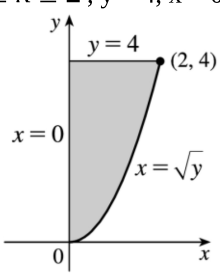
3. $y = \dfrac{1}{x}$, x = 1, x = 2, y = 0; about the x – axis



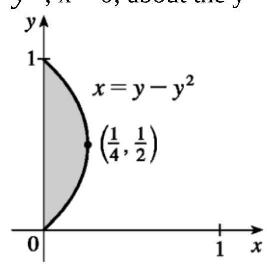
4. $y = \sqrt{x-1}$, x = 2, x = 5, y = 0 about the x - axis



5. $y = x^2$, $0 \le x \le 2$, y = 4, x = 0; about the y – axis



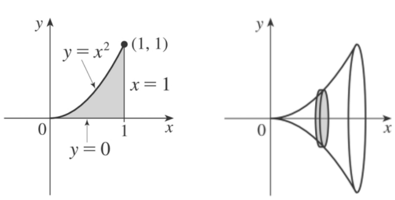
6.* $x = y - y^2$, x = 0; about the y – axis



<u>Answers:</u>

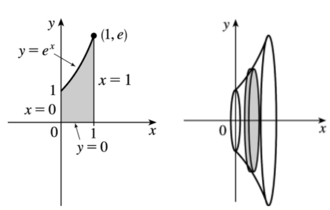
1. $A(x) = \pi\left[x^2\right]^2$

$$V = \int_0^1 \pi\left[x^2\right]^2 dx = \frac{\pi}{5}$$



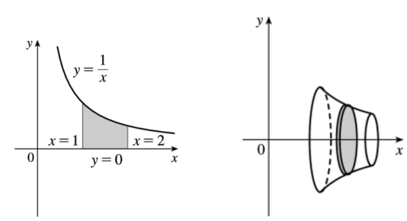
2. $A(x) = \pi\left[e^x\right]^2$

$$V = \int_0^1 \pi e^{2x} dx = \frac{\pi}{2}\left(e^2 - 1\right)$$



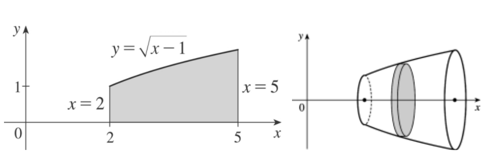
3. $A(x) = \pi\left[\dfrac{1}{x}\right]^2$

$$V = \int_1^2 \pi\left[\frac{1}{x}\right]^2 dx = \frac{\pi}{2}$$



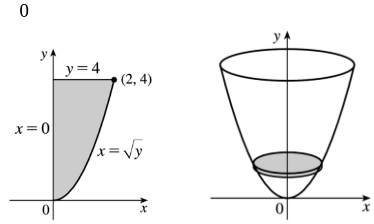
4. $A(x) = \pi\left[\sqrt{x-1}\right]^2$

$$V = \int_2^5 \pi(x-1)dx = \frac{15}{2}\pi$$



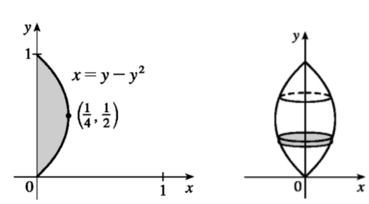
5. $A(y) = \pi\left[\sqrt{y}\right]^2$

$$V = \int_0^4 \pi y dy = 8\pi$$



6. $A(y) = \pi\left[y - y^2\right]^2$

$$V = \int_0^1 \pi\left[y - y^2\right]^2 dy = 0.104 \text{ or } 0.105$$



We can also create a solid of revolution by rotating a plane area created by two curves about a fixed axis.

If we look at the cross – section we get the following.

Similarly, if we are rotating an area bounded by two curves about the y – axis we get washers perpendicular to the y – axis.

Ex: Calculate the volume of the following solids obtained by rotating the plane area about the given axis.

1. The region bounded by $y = x^2 + 1$ and $y = 3 - x^2$ about the x – axis

2. The region bounded by $y = x^2$ and $y = 2x$:

a) about the x – axis

b) about the y – axis

We can also rotate a plane region about a ling that is parallel to the x – axis and parallel to the y – axis.

- If the axis of rotation is parallel to the x – axis (y = line), then our cross – sections will be perpendicular to the x – axis. As a result we integrate with respect to x.
- If the axis of rotation is parallel to the y – axis (x = line), then our cross – sections will be perpendicular to the y – axis. As a result we integrate with respect to y.

(In general, draw the picture to determine the outer radius and the inner radius)

Or use the following table:

Parallel to the x – axis	Parallel to the y – axis
- below the region will be addition - above the region will be subtraction	- left of the region will be addition - right of the region will be subtraction

Ex: Determine the volume of the solid obtained by rotating the region in Quadrant I bounded by $y = \sqrt[3]{x}$ and $y = \frac{1}{4}x$ about the:

a) x – axis

b) y – axis

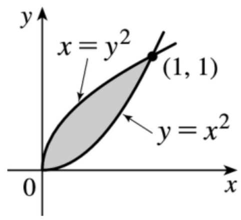
c) line y = – 3

d) x = 8

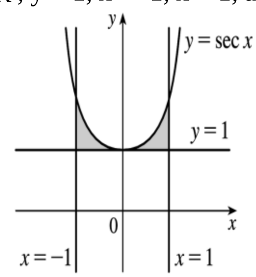
Volumes of Revolution Assignment

Find the volume of the solid obtained by rotating the region bounded by the given curves about the specific line.
Show the setup of the cross section area, the setup of the integral and your calculation.
Questions marked with a * are calculator questions.

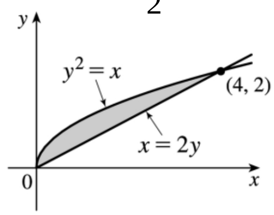
7. $y = x^2$, $x = y^2$, about the x – axis



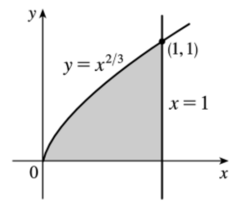
8.* $y = \sec x$, y = 1, x = -1, x = 1, about the x – axis



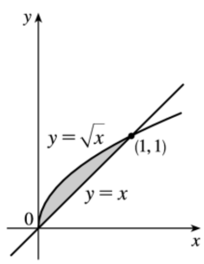
9.* $y = \sqrt{x}$, $y = \dfrac{1}{2}x$; about the y – axis



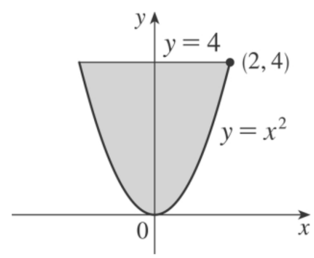
10. $y = x^{\frac{2}{3}}$, x = 1, y = 0; about the y – axis



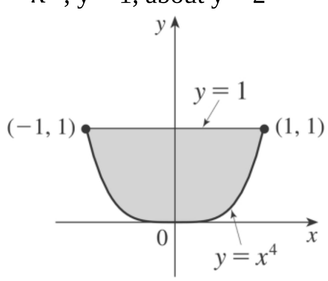
11.* $y = x$, $y = \sqrt{x}$, about y = 1



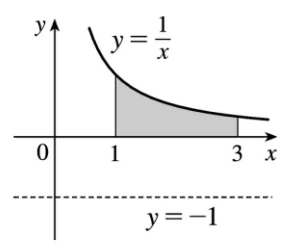
12.* above $y = x^2$, below y = 4, about y = 4



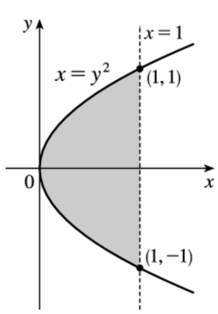
13.* $y = x^4$, y = 1, about y = 2



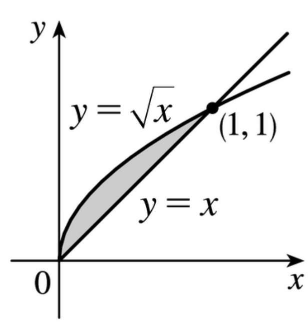
14.* $y = \dfrac{1}{x^2}$, y = 0, x = 1, x = 3; about y = -1



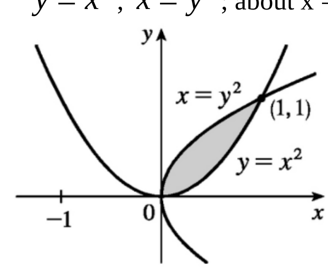
15. $x = y^2$, x = 1, about x = 1



16.* $y = x$, $y = \sqrt{x}$; about x = 2

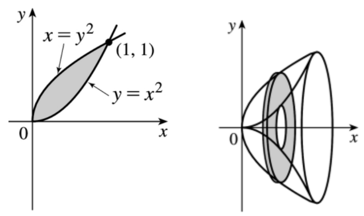


17.* $y = x^2$, $x = y^2$; about x = -1



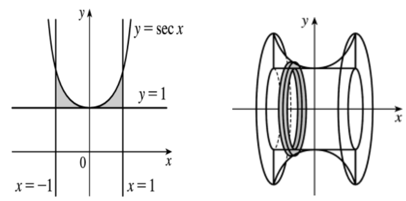
7. $A(x)=\pi\left[\left(\sqrt{x}\right)^2-\left(x^2\right)^2\right]$

$$V=\int_0^1 \pi\left[\left(\sqrt{x}\right)^2-\left(x^2\right)^2\right]dx=\frac{3}{10}\pi$$



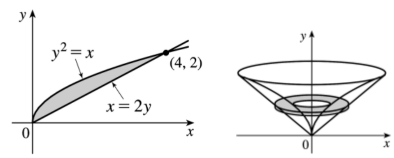
8. $A(x)=\pi\left[(\sec x)^2-(1)^2\right]$

$$V=\int_{-1}^1 \pi\left[(\sec x)^2-(1)^2\right]dx=3.502$$



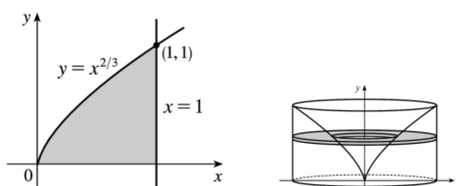
9. $A(y)=\pi\left[(2y)^2-\left(y^2\right)^2\right]$

$$V=\int_0^2 \pi\left[(2y)^2-\left(y^2\right)^2\right]dy=13.404$$



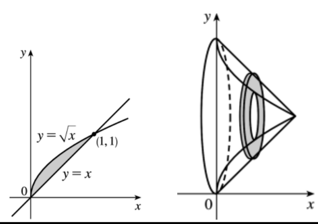
10. $A(y)=\pi\left[(1)^2-\left(y^{\frac{3}{2}}\right)^2\right]$

$$V=\int_0^1 \pi\left[(1)^2-\left(y^{\frac{3}{2}}\right)^2\right]dy=\frac{3\pi}{4}$$



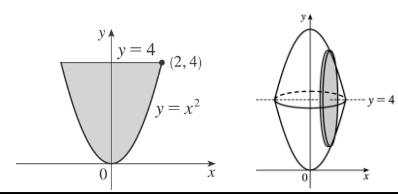
11. $A(x)=\pi\left[(1-x)^2-\left(1-\sqrt{x}\right)^2\right]$

$$V=\int_0^1 \pi\left[(1-x)^2-\left(1-\sqrt{x}\right)^2\right]dx=0.523$$



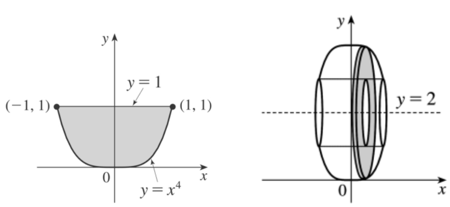
12. $A(x)=\pi\left[4-x^2\right]^2$

$$V=\int_{-2}^2 \pi\left[4-x^2\right]^2dx=107.233$$



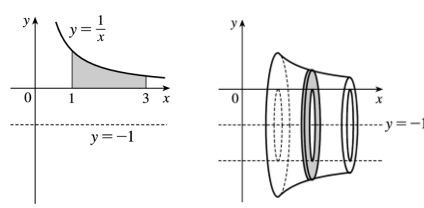
13. $A(x)=\pi\left[\left(2-x^4\right)^2-(2-1)^2\right]$

$$V=\int_{-1}^1 \pi\left[\left(2-x^4\right)^2-(2-1)^2\right]dx=14.521$$



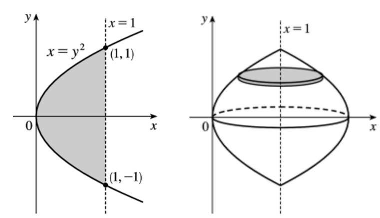
14. $A(x)=\pi\left[\left(1+\frac{1}{x^2}\right)^2-(1+0)^2\right]$

$$V=\int_1^3 \pi\left[\left(1+\frac{1}{x^2}\right)^2-(1+0)^2\right]dx=5.197$$



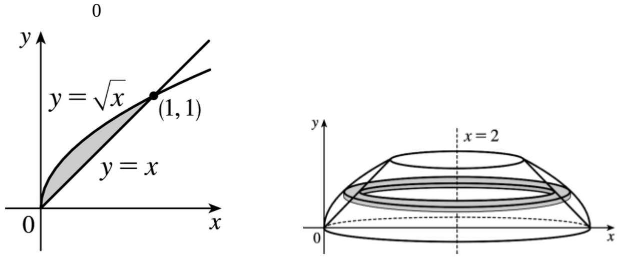
15. $A(y)=\pi\left[1-y^2\right]^2$

$$V=\int_{-1}^1 \pi\left[1-y^2\right]^2dy=\frac{16\pi}{15}$$



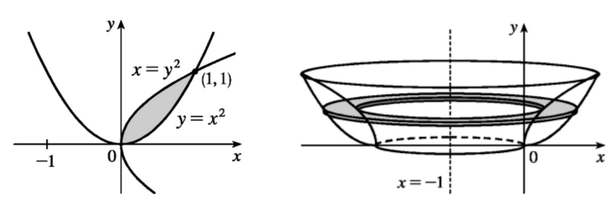
16. $A(y)=\pi\left[\left(2-y^2\right)^2-(2-y)^2\right]$

$$V=\int_0^1 \pi\left[\left(2-y^2\right)^2-(2-y)^2\right]dy=1.676$$



17. $A(y)=\pi\left[\left(1+\sqrt{y}\right)^2-\left(1+y^2\right)^2\right]$

$$V=\int_0^1 \pi\left[\left(1+\sqrt{y}\right)^2-\left(1+y^2\right)^2\right]dy=3.037$$



Calculating Volumes of Known Cross – Sections

In order to determine volumes of known cross – sections, rather than rotations, we must remember that

$$V=\int_a^b A(x)dx \qquad\qquad V=\int_c^d A(y)dy$$

where: *A(x)* is the area of the cross section perpendicular to the x – axis

where: *A(y)* is the area of the cross section perpendicular to the y – axis

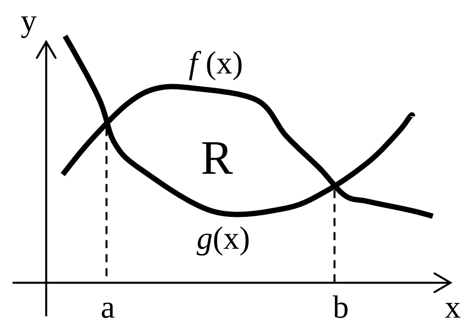
Sample Areas [A(x) / A(y)]

- rectangle / square $\rightarrow A(x)=l(x)\cdot w(x)$ or $A(x)=b(x)\cdot h(x)$
- triangle $\rightarrow A(x)=\dfrac{b(x)\cdot h(x)}{2}$
- semi – circles $\rightarrow A(x)=\dfrac{\pi}{2}[r(x)]^2$

**NOTE: If the cross – section is perpendicular to the x – axis, we integrate with respect to x
If the cross – section is perpendicular to the y – axis, we integrate with respect to y

In General:

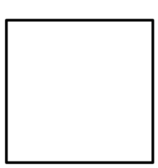
If the base area is a region R bounded by two curves $y=f(x)$ and $y=g(x)$ from x = a to x = b, then we can calculate the volume of the known cross – sections the following ways



Different Cross – Sectional Areas Perpendicular to the *x* – axis

Squares $\left(A=l\cdot w \quad or \quad A=s^2\right)$

$$A(x)=[f(x)-g(x)]^2$$

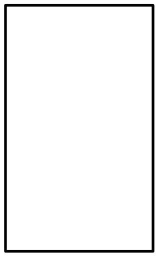


$f(x)-g(x)$

$$V=\int_a^b [f(x)-g(x)]^2\,dx$$

$f(x)-g(x)$

Rectangles $(A = b \cdot h)$



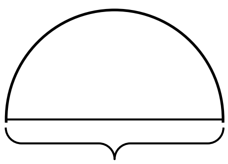
$c\left[f(x) - g(x)\right]$

$f(x) - g(x)$

$$A(x) = c\left[f(x) - g(x)\right]^2$$

$$V = c\int_a^b \left[f(x) - g(x)\right]^2 dx$$

Semi – Circles $\left(A = \frac{\pi}{2} r^2\right)$



$f(x) - g(x)$ = diameter

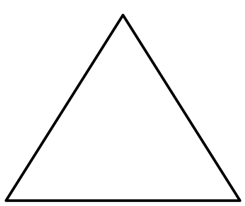
$\frac{f(x) - g(x)}{2} = radius$

$$A(x) = \frac{\pi}{2}\left[\frac{f(x) - g(x)}{2}\right]^2$$

$$V = \int_a^b \frac{\pi}{2}\left[\frac{f(x) - g(x)}{2}\right]^2 dx$$

$$V = \frac{\pi}{8}\int_a^b \left[f(x) - g(x)\right]^2 dx$$

Equilateral Triangles $\left(A = \frac{\sqrt{3}}{4} s^2\right)$; s = one of the side lengths of the triangle



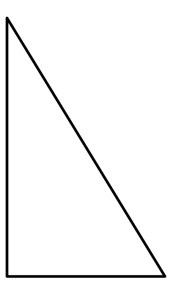
$f(x) - g(x)$

$$A(x) = \frac{\sqrt{3}}{4}\left[f(x) - g(x)\right]^2$$

$$V = \int_a^b \frac{\sqrt{3}}{4}\left[f(x) - g(x)\right]^2 dx$$

Right Triangles $\left(A = \frac{b \cdot h}{2}\right)$

$c\left[f(x) - g(x)\right]$



$f(x) - g(x)$

$$A(x) = \frac{c}{2}\left[f(x) - g(x)\right]^2$$

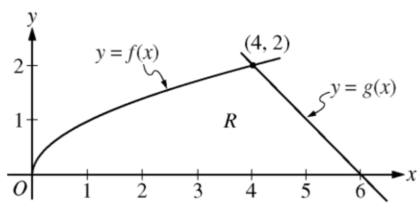
$$V = \int_a^b \frac{c}{2}\left[f(x) - g(x)\right]^2 dx$$

Ex: Find the volume of the solid generated when cross – sections perpendicular to the x – axis form squares and have a base solid bounded by the curves $y = \sqrt{x} + 4$ and the line x = 4 in QI.

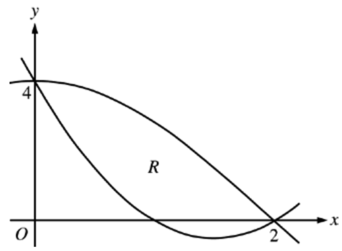
Ex: Find the volume of the solid generated when cross – sections perpendicular to the y – axis form squares and have a base solid bounded by the curves $y = \sqrt{x} + 4$ and the line x = 4 in QI.

Ex: The region bounded by $y = e^{\frac{x}{2}}$ and $y = -x^2 + 5x + 1$ is the base of a solid. For this solid the cross – sections perpendicular to the x – axis are semi – circles. Find the volume of this solid.

Ex: Let R be the region in the first quadrant enclosed by the graphs of $f(x) = \sqrt{x}$ and $g(x) = 6 - x$, as shown in the figure below.



Write, but do not solve, an integral expression for the volume of the solid generated when cross – sections perpendicular to the x – axis form right isosceles triangles whose leg lies in the region R formed by $f(x) = \sqrt{x}$ and $g(x) = 6 - x$.



Let $f(x) = 2x^2 - 6x + 4$ and $g(x) = 4\cos\left(\frac{\pi}{4}x\right)$. Let R be the region bounded by the graphs of f and g, as shown in the figure above.

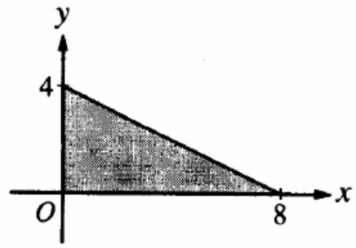
(a) Find the area of R.

(b) Write, but do not evaluate, an integral expression that gives the volume of the solid generated when R is rotated about the horizontal line $y = 4$.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x – axis is a square. Write, but do not evaluate, an integral expression that gives the volume of this sold.

AP CALCULUS – Solids of Known Cross Sections

1. The base of a solid is the region in the first quadrant bounded by the *x*-axis, the *y*-axis, and the line $x+2y=8$, as shown in the figure. What is the volume of the solid if: (Calculator)
   a) The cross sections of the solid perpendicular to the x – axis are semicircles?
   b) The cross sections of the solid perpendicular to the y – axis are squares?

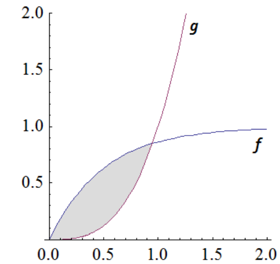


2. The region bounded by the graph of $y=2x-x^2$ and the *x*-axis is the base of a solid. For this solid, each cross section perpendicular to the *x*-axis is an equilateral triangle. What is the volume of the solid? (Calculator)

3. The region in Quadrant I bounded by the graph of $f(x)=1-e^{-x}$ and $g(x)=x^3$.

   a. If the region is the base of the solid, find the volume of the solid if each cross section perpendicular to the *x*-axis is a semi-circle. (Calculator)

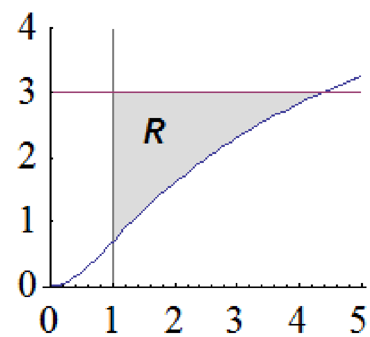
   b. Find the volume of the solid if the region bounded by *f* and *g* are rotated about the line y = 2. (Calculator)



4. Let *R* be the region in Quadrant I bounded by the graph of $y=e^x$, the *y*-axis, and the horizontal line $y=4$. (Calculator)
   a. Find the area of *R*.
   b. The region *R* is the base of a solid. For this solid, each cross section **perpendicular to the *y*-axis** is a square. Find the volume of this solid.

5. Let *R* be the region bounded by the graph of $y=\ln\left(x^2+1\right)$, the horizontal line $y=3$, and the vertical line $x=1$, as shown in the figure to the right. (Calculator)

   a. Find the area of *R*.
   b. The region *R* is the base of a solid. For this solid, each cross section perpendicular to the *x*-axis is a triangle with height equal to twice the length of the base. Find the volume of this solid.
   c. Another solid whose base is also the region *R*. For this solid, each cross section perpendicular to the *x*-axis is a semicircle. Find the volume of this solid.



6. The region in Quadrant I bounded by the graphs of $y=\tan^{-1}x$ and $y=\frac{1}{2}x$ is the base of a solid. For this solid, each cross section **perpendicular to the *y*-axis** is a rectangle with height four times the width. Find the volume of this solid. (Calculator)

<u>Answers:</u>

1. a) Cross – Section => Semi – Circle $A(x)=\frac{\pi}{2}[r(x)]^2$

$$A(x)=\frac{\pi}{2}\left[\frac{-\frac{1}{2}x+4}{2}\right]^2$$

$$V=\int_0^8 \frac{\pi}{2}[-0.25x+2]^2dx=16.755$$

b) Cross – Section => Square $A(y)=l(y)\cdot w(y)$

$$A(y)=[8-2y][8-2y]$$
$$A(y)=[8-2y]^2$$

$$V=\int_0^4 [8-2y]^2dy=85.333$$

2. Cross – Section => Equilateral Triangle $A(x) = \frac{\sqrt{3}}{4}[s(x)]^2$

$$A(x) = \frac{\sqrt{3}}{4}\left[2x - x^2\right]^2 \qquad V = \int_0^2 \frac{\sqrt{3}}{4}\left[2x - x^2\right]^2 dx = 0.461 \text{ or } 0.462$$

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3. a) Cross – Section => Semi – Circle $A(x) = \frac{\pi}{2}[r(x)]^2$

$$A(x) = \frac{\pi}{2}\left[\frac{1 - e^{-x} - x^3}{2}\right]^2$$

$$V = \int_0^{0.82515547} \frac{\pi}{2}\left[\frac{1 - e^{-x} - x^3}{2}\right]^2 dx = 0.012 \text{ or } 0.013$$

b) Cross – Section => Washer

$$A(x) = \pi\left[\left(2 - x^3\right)^2 - \left(2 - \left(1 - e^{-x}\right)\right)^2\right]$$

$$V = \int_0^{0.82515547} \pi\left[\left(2 - x^3\right)^2 - \left(2 - \left(1 - e^{-x}\right)\right)^2\right] dx = 1.638$$

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4. a)

$$A = \int_0^{1.3862944} \left[4 - e^x\right] dx = 2.545$$

b) Cross – Section => Square $A(y) = l(y) \cdot w(y)$

$$A(y) = [\ln y][\ln y]$$

$$V = \int_1^4 [\ln y]^2 dy = 2.596 \text{ or } 2.597$$

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5. a) $A = \int_1^{4.3686997} \left[3 - \ln\left(x^2 + 1\right)\right] dx = 3.309 \text{ or } 3.310$

b) Cross – Section => Triangle $A(x) = \frac{b(x) \cdot h(x)}{2}$

$$A(x) = \frac{2\left(3 - \ln\left(x^2 + 1\right)\right) \cdot \left(3 - \ln\left(x^2 + 1\right)\right)}{2}$$

$$V = \int_1^{4.3686997} \left[3 - \ln\left(x^2 + 1\right)\right]^2 dx = 4.721 \text{ or } 4.722$$

c) Cross – Section => Semi – Circle $A(x) = \frac{\pi}{2}[r(x)]^2$

$$A(x) = \frac{\pi}{2}\left[\frac{3 - \ln\left(x^2 + 1\right)}{2}\right]^2$$

$$V = \int_1^{4.3686997} \frac{\pi}{2}\left[\frac{3 - \ln\left(x^2 + 1\right)}{2}\right]^2 dx = 1.854$$

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6. $y = \tan^{-1} x$ $\qquad y = \frac{1}{2}x$

$x = \tan y$ $\qquad x = 2y$

Cross – Section => Rectangle $A(y) = b(y) \cdot h(y)$

$$A(y) = 4[2y - \tan y] \cdot [2y - \tan y]$$

$$V = \int_0^{1.1655612} 4[2y - \tan y]^2 dy = 0.769 \text{ or } 0.770$$