Quantum particle on a circle

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Divergent oscillations

$$\mathcal{A} = \sum_{\phi(t)} e^{iF[\phi(t)]}$$

Quantum observables are oscillatory sums[1], need **regularization**.

[1] politically correct language for "divergent"

Wick rotation

Standard answer: tilt head $\pi/2$.

$$t = i\tau$$

$$F[\phi(i\tau)] = i\mathcal{F}[\varphi(\tau)]$$

$$\mathcal{A}(it) = \sum_{\varphi(\tau)} e^{-\mathcal{F}[\varphi(\tau)]}$$

Now they converge. Then hopefully get back to real axis through analytic continuation.

QM on circles

Place free quantum particle on a circle.

$$x \sim x + 1\,, \qquad \psi(x+1,t) = \psi(x,t)$$

$$i\frac{\partial}{\partial t}\psi(x,t) = -\frac{\nabla}{2}\psi(x,t) \qquad \partial_x\psi(x+1,t) = \partial_x\psi(x,t)$$

Initial condition is a position (generalized) eigenstate

$$\psi(x,0) = \delta(x)$$

$\psi(x,t)$ is the **propagator** or **fundamental solution**.

Unwrapping the circle

On $\mathbb{R}$, the propagator is innocuous enough:

$$G_{\mathbb{R}}(x,t) = \frac{1}{\sqrt{2\pi i t}} \exp\left(\frac{ix^2}{2t}\right)$$

But $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, therefore

$$G(x,t) = \sum_{n=-\infty}^{\infty} G_{\mathbb{R}}(x+n,t)$$

$$= \exp\left(\frac{ix^2}{2t}\right) \frac{1}{\sqrt{2\pi i t}} \sum_{n=-\infty}^{\infty} \exp\left(\frac{in^2}{2t} + \frac{ixn}{t}\right)$$

Simply a method of images!

A pleasant surprise

$$G(x,t) = \exp\left(\frac{ix^2}{2t}\right)\frac{1}{\sqrt{2\pi i t}}\sum_{n=-\infty}^{\infty}\exp\left(\frac{in^2}{2t}+\frac{ixn}{t}\right)$$

Change variables $\tau := 2\pi t$, $z := x$, and use Poisson resummation:

$$G(z,\tau) = \sum_{n=-\infty}^{\infty}\exp\bigl(\pi i n^2\tau + 2\pi i n z\bigr)$$

It's the Jacobi ϑ function!

A less pleasant surprise