Quantum particle on a circle

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November 27, 2017

Divergent oscillations

$$\mathcal{A} = \sum_{\phi(t)} e^{iF[\phi(t)]}$$

Quantum observables are oscillatory sums[1], need **regularization**.

[1] politically correct language for "divergent"

Wick rotation

Standard answer: tilt head $\pi/2$.

$$t = i\eta$$

$$F[\phi(i\eta)] = i\mathcal{F}[\varphi(\eta)]$$

$$\mathcal{A}(it) = \sum_{\varphi(\eta)} e^{-\mathcal{F}[\varphi(i\eta)]}$$

Now they converge. Then hopefully get back to real axis through analytic continuation.

QM on circles

Place free quantum particle on a circle.

$$x \sim x + 1\,, \qquad \psi(x+1,t) = \psi(x,t)$$

$$i\frac{\partial}{\partial t}\psi(x,t) = -\frac{\Delta}{2}\psi(x,t) \qquad \partial_x\psi(x+1,t) = \partial_x\psi(x,t)$$

Initial condition is a position (generalized) eigenstate

$$\psi(x,0) = \delta(x)$$

$\psi(x,t)$ is the **propagator** or **fundamental solution**.

Unwrapping the circle

On $\mathbb{R}$, the propagator is innocuous enough:

$$G_{\mathbb{R}}(x,t) = \frac{1}{\sqrt{2\pi i t}} \exp\left(\frac{ix^2}{2t}\right)$$

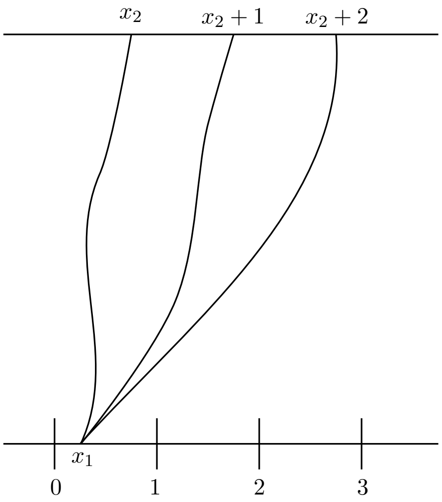
But $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, therefore

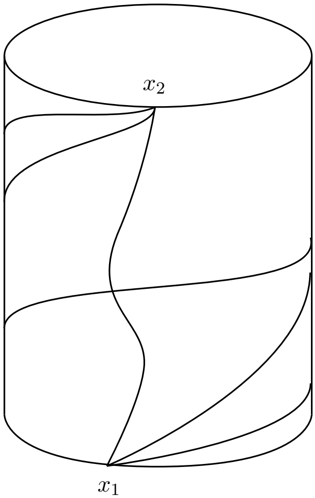
$$G(x,t) = \sum_{n=-\infty}^{\infty} G_{\mathbb{R}}(x+n,t)$$

$$= \exp\left(\frac{ix^2}{2t}\right) \frac{1}{\sqrt{2\pi i t}} \sum_{n=-\infty}^{\infty} \exp\left(\frac{in^2}{2t} + \frac{ixn}{t}\right)$$

Simply a **method of images**!

Unwrapping the circle II





A pleasant surprise

$$G(x,t) = \exp\left(\frac{ix^2}{2t}\right)\frac{1}{\sqrt{2\pi it}}\sum_{n=-\infty}^{\infty}\exp\left(\frac{in^2}{2t}+\frac{ixn}{t}\right)$$

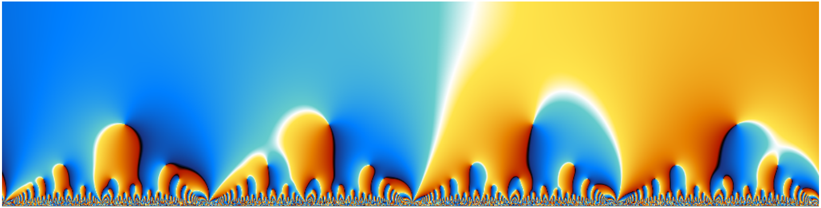
Change variables $\tau := 2\pi t$, $z := x$, and use Poisson resummation:

$$G(z,\tau) = \sum_{n=-\infty}^{\infty}\exp\bigl(\pi i n^2\tau + 2\pi i n z\bigr)$$

It's the Jacobi ϑ function!

A less pleasant surprise

$\vartheta(z,\tau)$ is defined and holomorphic for $\operatorname{Im}\tau > 0$, but **time is real**!



Is there any hope of making sense of $\vartheta(z,\tau)$ for real periods?

Set $z = 0$ for now.

(physically: final point = starting point)

The propagator exists

$\vartheta(z,\tau)$ for real τ **is** well defined, but as a tempered distribution. Indeed, for Schwartz $\phi(\tau)$:

$$\int d\tau \phi(\tau)\vartheta(0,\tau) = \sum_{n=-\infty}^{\infty} \int d\tau \phi(\tau) e^{i\pi n^2 \tau} = \sum_{n=-\infty}^{\infty} \hat{\phi}(-\pi n^2) \in \mathbb{C}$$

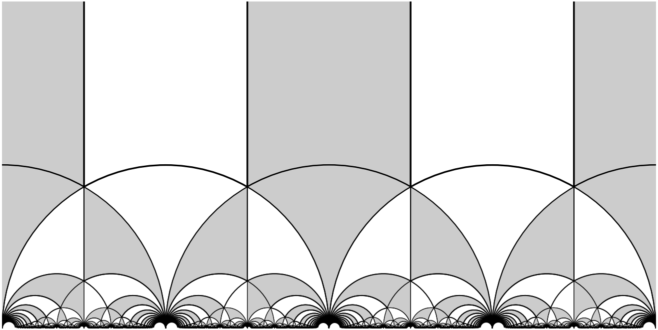
However, *what* distribution is it? It is neither induced by an L^1_{loc} function, nor does it have any part with single point support (δ-like).

The even modular group

$\vartheta(0,\tau)$ has remarkable transformation properties under modular transformations.

$$\vartheta(0,\tau+2)=\vartheta(0,\tau)\,,\quad \vartheta\left(0,-\frac{1}{\tau}\right)=\sqrt{-i\tau}\,\vartheta(0,\tau)$$

All rational times p/q can be mapped back to either 1 or 0 depending on the parity of pq.



Self-similarity

Self-similar structure:

The behaviour around $\tau = 0$ is repeated at all even rationals. The behaviour around $\tau = 1$ is repeated at all odd rationals.

Even rationals

$\vartheta(\tau)$ is "asymptotic" to $1/\sqrt{\tau}$ for $\tau \to 0$. Normal analytic methods however fail. We test with shrinking gaussian test functions:

$$\langle g_\sigma(\tau), \vartheta \rangle \sim \left\langle g_\sigma(\tau), \tau^{-1/2} \right\rangle$$

$$\int d\tau\, \vartheta(\tau) \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\tau^2}{2\sigma^2}} = \sum_{n=-\infty}^{\infty} \exp\left(-\pi^2\sigma^2 n^4/2\right)$$

$$\sim \frac{2^{1/4}}{\sqrt{\pi\sigma}} \int d\xi e^{-\xi^4} = \frac{2^{-3/4}}{\sqrt{\pi\sigma}} \Gamma\left(\frac{1}{4}\right)$$

$$\int d\tau\, \frac{1}{\sqrt{\tau}} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\tau^2}{2\sigma^2}} = \frac{2^{-3/4}}{\sqrt{\pi\sigma}} \Gamma\left(\frac{1}{4}\right)$$

Modular symmetry $\Rightarrow$ there is a $(\Delta\tau)^{-1/2}$ singularity near all even p/q. In particular we find:

$$\vartheta(\tau) \sim \frac{1}{\sqrt{p - q\tau}}$$

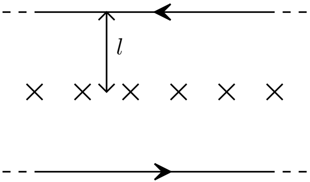
$\theta(\tau)$ has a **dense** set of singularities! They are rescaled by $1/\sqrt{q}$.

Odd rationals

For $\tau \to 1$, $\vartheta \to 0$, and all derivatives $\frac{\mathrm{d}^k}{\mathrm{d}\tau^k}\vartheta \to 0$, again as a distributional limit:

$$\left\langle g_\sigma(\tau-1), \frac{\mathrm{d}^k}{\mathrm{d}\tau^k}\vartheta \right\rangle \to 0\,, \quad \sigma \to 0\,,\ k = 0, 1, 2, \ldots$$

$$\propto \sum_{n=-\infty}^{\infty} n^{2k}(-1)^n e^{-\pi^2\sigma^2 n^4/2} = \frac{1}{2\pi i}\oint \frac{\pi}{\sin \pi z} z^{2k} \exp\!\left(-\pi^2\sigma^2 n^4\right) dz$$

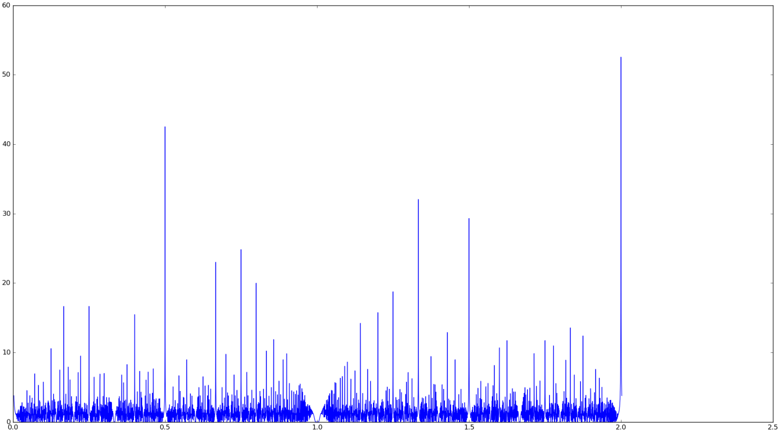


Send $\sigma \to 0$, but keep $l\sqrt{\sigma}$ constant.

Modular symmetry $\Rightarrow$ there is a "flat point" at all odd p/q.
$\vartheta(\tau)$ has a **dense** set of zeroes!

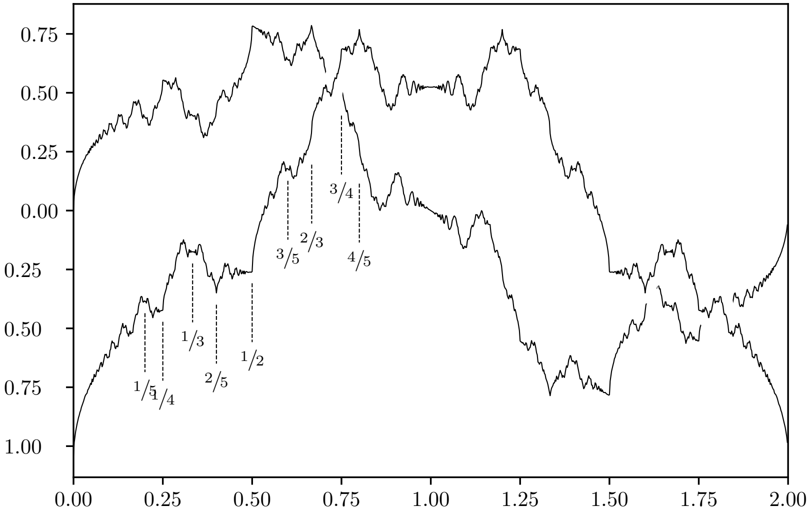
Plot of $\vartheta(\tau)$?

Plotting $\vartheta(\tau)$ doesn't really work...



Plot $B(\tau) = \int d\tau \vartheta(\tau)$

Plot distributional antiderivative $B(\tau) := \sum_{n=-\infty}^{\infty} \frac{2}{i\pi} \frac{e^{i\pi n^2 \tau}}{n^2}$



Fix $\tau = p/q$, study z dependence.

$$\vartheta\left(z, \frac{p}{q}\right) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \frac{p}{q} + 2\pi i n z}$$

Trick: $n = aq + b$, $a \in \mathbb{Z}$, $b = 0, \ldots, q-1$

$$\vartheta\left(z, \frac{p}{q}\right) = \frac{1}{\sqrt{q}} \sum_{c=0}^{q-1} \phi_{p,q,c}\, \delta\left(z - \frac{c+\Delta}{q}\right)$$

The propagator is a "**fractional comb**" of q equally-spaced δ-functions.

$\Delta := (-1)^{pq}/2$

Phases $\phi_{p,q,c}$ are quadratic Gauss sums!

$$\phi_{p,q,c} = \sqrt{q} \sum_{b=0}^{q-1} (e^{i\pi/q})^{pb^2+2bc+2\Delta b}$$

No general closed form.

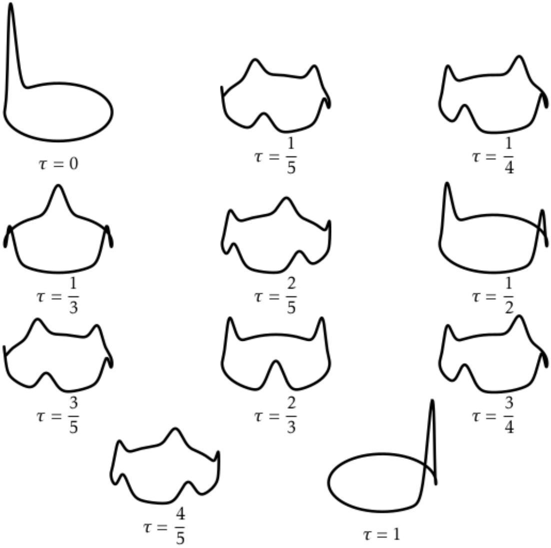
Back to physics

Periodicity $\vartheta(z, \tau + 2) = \vartheta(z, \tau)$ is expected physically: **quantum revival**.

If $E_n = m_n E$, $m_n \in \mathbb{Z}$, then state is periodic in time:

$$\Psi(t + 2\pi/E) = \Psi(t)$$

However at $\tau \in \mathbb{Q}$ initial state is "cloned" into q reduced and shifted copies: **fractional revival**



What is the physical origin of fractional revivals?

Heuristic reasoning: apply saddle point to path integral:

$$\mathcal{A} = \int D\phi e^{iS[\phi_{cl}]}(1 + \text{oscillatory} \ldots)$$

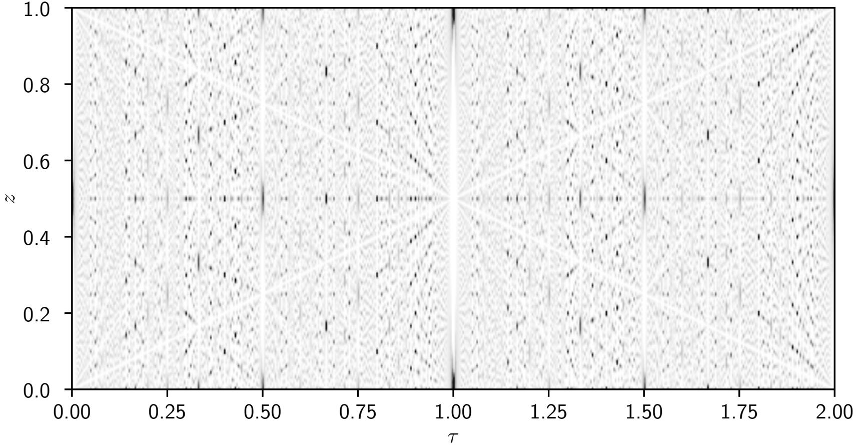
But if there are infinite ϕ_{cl}, $\mathcal{A}$ would diverge...

Reconvergence of classical solutions, a **caustic**, generates a δ-function singularity in an amplitude (simply resonance!).

When $2\pi t$ is rational, then an infinite number of classical solutions starting at $z = 0$:

$$z_v(t) = vt\,; \quad v = 2\pi \frac{n}{p}$$

will reconverge at the q points 0, $1/q$, $2/q$, $\ldots$, $(q-1)/q$.


1.0
0.8
0.6
0.4
0.2
0.0
z
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
2.00
τ