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Holographic effective field theories: a case study

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Abstract

The identification of the low-energy effective field theory associated with a given microscopic strongly interacting theory constitutes a fundamental problem in theoretical physics, which is particularly hard when the theory is not sufficiently constrained by symmetries. Recently, a new approach has been proposed, which addresses this problem for a large class of four-dimensional minimally supersymmetric strongly coupled superconformal field theories, admitting a dual weakly coupled holographic description in string theory. This approach provides a precise prescription for the holographic derivation of the associated effective field theories. The aim of the thesis is to further explore this approach by focusing on a specific model, whose effective field theory has not been investigated so far. *(modificare abstract alla fine del lavoro.)*

Contents

Chapter 1

Introduction

Strongly-coupled quantum field theories represent canonical examples of physical systems whose study is extremely challenging. Even the question of the mere existence of any interacting QFT in four dimensions from a formal standpoint has not been settled. In addition to this, strong couplings are not amenable to the tools of perturbation theory. The interest in this class of theories stems actually from practical considerations - many of them represent realistic models for physical phenomena, e.g. the theory of strong interaction.

A subset of questions concerns whether a given strongly-interacting theory is described at low energy by an effective local field theory, and if so, what are its degrees of freedom and their precise dynamics. Often, part of the structure of the effective theory is constrained by symmetries, but no general method exists to fix it completely. Recently[4], a novel approach for determining the effective Lagrangian was introduced that makes use of tools from an apparently unrelated area of physics: string theory.

It's remarkable that string theory was originally conceived as a description of hadronic physics, so a low-energy effective theory for what ultimately turned out to be a gauge theory, QCD. When string theory was found to have unsuitable qualities for this application, it was replaced by the full theory of quantum chromodynamics - however it also proved to be effective for solving a seemingly unrelated problem of fundamental physics: quantizing gravity.

Since then, string theory blossomed into a vast and rich field reaching into numerous areas of mathematics and physics, and of course a candidate for a "Theory of Everything" describing the entirety of fundamental physics.

Among the most unexpected discoveries in strings, made decades after their conception, is a series of unusual exact equivalences between string theories set in particular effectively five-dimensional backgrounds and four-dimensional gauge QFTs. More generally, one finds families of exact equivalences between local quantum field theories and +1-dimensional theories containing gravity, which are termed "holographic". This explains the original partial success of strings in modeling strong interactions, assuming that some or perhaps most gauge theories have or can be approximated as having a holographic description as a *five* dimensional theory involving strings.

This roundtrip has therefore brought strings back to strongly-coupled gauge theories. Various aspects, qualitative and most importantly quantitative, of QFTs can be studied directly by means of their holographic string dual, a gravitational theory, if they have one. What has been done in this particular case is the degrees of freedom and the Lagrangian for the effective low-energy theory for a class of gauge theories with holographic duals have been identified by expanding the supergravity action on the dual bulk. Interestingly, these are somewhat special in that they are models of minimal supersymmetry ($\mathcal{N} = 1$), which makes for more realistic but by converse less constrained theories than typical holographic field theories, with higher supersymmetry.

In this work, we will specialize this construction to a specific field theory, the $Y^{2,0}$ theory, a strongly-coupled superconformal quiver theory for which we will therefore fix the exact effective Lagrangian, entirely through the geometry of the relative string background.

This thesis will be structured as follows. We will first provide a general introduction to IIB superstring theory, D-brane stacks on cones and the resulting gauge field theories, and holography. Then, we will summarize the relevant results and techniques from [4]. Finally, we will present a complete parametrization of the geometry of the $Y^{2,0}$ theory and will apply those results and techniques to identify the exact effective Lagrangian of the field theory.

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Chapter 2

IIB superstrings and branes

2.1 Superstring theory

String theory either does not admit a nonperturbative Lagrangian formulation, or this formulation is unknown. An action functional can only be written upon choosing a perturbative vacuum; since we anticipate a string theory must include gravity, a choice of vacuum will also require a choice of background metric - in the simplest case Minkowski spacetime. With this choice the action for a string in the simplest case of bosonic string theory is the Polyakov action:

$$S_B = -\frac{T}{2} \int d^2\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X_\mu \tag{2.1}$$

where the D fields X^μ describe the embedding of the string's worldsheet in the D-dimensional target spacetime, and the integral is performed over the worldsheet coordinates $\sigma^a = (\tau, \sigma)$. The X^μ are of course scalars from the point of view of the worldsheet. The auxilary field g_{ab} is a metric on the worldsheet. The action displays worldsheet diffeomorphism and Weyl invariance, and thus perturbative string theory is naturally a two-dimensional conformal field theory. These symmetries must be quotiented out someway on quantization. The most straightforward way is to eliminate them by fixing a particular gauge and then quantizing (canonical quantization). The three symmetry generators can kill the three degrees of freedom in the metric to fix it to the 2D Minkowski: $g_{ab} = \eta_{ab}$. We get

$$S_B = -\frac{1}{2\pi}\int d^2\sigma \partial_a X^\mu \partial^a X_\mu \tag{2.2}$$

where indices are raised with η^{ab}.

There are at least two different approaches to introducing supersymmetry into a string theory. The path followed by the RNS (Ramond-Neveu-Schwarz) formalism is to impose SUSY at the worldsheet level; explicitly, adding fermions ψ^μ to act as superpartners to the bosons X^μ. The action is extended to

$$S = S_B + S_F = -\frac{1}{2\pi}\int d^2\sigma \partial_a X^\mu \partial^a X_\mu + \bar{\psi}^\mu \rho^a \partial_a \psi_\mu \tag{2.3}$$

The spinors' equation of motion, the Dirac equation, is actually the Weyl condition in two dimension. This brings the real degrees of freedom in the spinor for each μ from 4 to 2. Recalling that in (2 mod 8) dimensions there exist Weyl-Majorana spinors satisfying both the Weyl and Majorana conditions, imposing the latter on ψ halves again the on-shell polarizations to 1. Thus we have a match between bosonic and fermionic degrees of freedom. It can be proven the theory above is indeed worldsheet supersymmetric.

To quantize canonically, we introduce canonical commutation/anticommutation relations:

$$[X^\mu(\sigma), X^\nu(\sigma')] = \eta^{\mu\nu}\delta^2(\sigma-\sigma') \quad \{\psi^\mu(\sigma), \psi^\nu(\sigma')\} = \eta^{\mu\nu}\delta^2(\sigma-\sigma') \tag{2.4}$$

Note the X^0 and ψ^0 would create negative norm states, but these modes are eliminated by resorting to superconformal invariance. Classically this symmetry imposes the stress-energy tensor $T^{\mu\nu}$ and the supercurrent J^a_α vanish; imposing that in the quantum theory they annihilate physical states yields the restriction that removes the longitudinal ghosts from the spectrum. These take the name of super-Virasoro constraints.

Then the procedure for building the string spectrum is to expand the classical solutions in terms of Fourier modes, identify creators and destructors, and then select the states of the Fock basis that satisfy the super-Virasoro constraints.

Boundary conditions for ψ^μ for an open string can actually be satisfied in two different by imposing periodicity or antiperiodicity, giving rise to the NS (Neveu-Schwarz) and R (Ramond) sectors, built over two grounds $|0\rangle_{NS}$ and $|0\rangle_R$. Closed strings have four: $|0\rangle_{NS-NS}$, $|0\rangle_{R-R}$, $|0\rangle_{R-NS}$, $|0\rangle_{NS-R}$ corresponding with different choice periodicity conditions for left and right-movers.

2.1.1 Open strings

It can be shown that while the NS ground $|0\rangle_{NS}$ is unique, and thus a spacetime scalar, $|0\rangle_R$ is eight-fold degenerate and this 8-plet transforms under the spinor representation of transverse $SO(8)$ - in other words, it's a spacetime spinor. In particular, it's a chiral Weyl-Majorana spinor, so it can be taken to be either of positive or negative chirality, choices we will denote as $|+\rangle_R$, $|-\rangle_R$.

The spectrum is built by acting on one of the grounds with bosonic and fermionic creators, to obtain states of higher and higher mass. For the NS sector, there are bosonic creators $a_n^{i\dagger}$ ($n \geq 1$) and fermionic $b_r^{i\dagger}$ (r positive half-integer), and the mass of the excited string is given by:

$$\alpha' M^2 = \sum_{n=1}^{\infty} n\, a_n^{i\dagger} a_n^i + \sum_{r=1/2}^{\infty} r b_r^{i\dagger} b_r^i - \frac{1}{2} \tag{2.5}$$

while for the R sector the fermionic creators are replaced by the integer-indexed $d_n^{\dagger i}$:

$$\alpha' M^2 = \sum_{n=1}^{\infty} \left(n\, a_n^{i\dagger} a_n^i + n d_n^{i\dagger} d_n^i \right) \tag{2.6}$$

The i indices here are target spacetime transverse indices, $i = 1, \cdots, 8$.

Therefore each creator increases the spin of the string by one unit.

It's worrying that the mass-shell formula above assigns a negative mass-squared to the NS ground, which is therefore a tachyon. In addition, it's the *only* tachyon, meaning this theory is not spacetime supersymmetric. We will see in the next section how this state is actually removed and target supersymmetry recovered. For now, we note the only massless states are

$$b_1^{i\dagger}\left|0\right\rangle_{NS} \qquad \left|+\right\rangle_{NS} \tag{2.7}$$

while the rest of the tower of states have Planck-large $\sim (\alpha')^{-1/2}$ masses. The former state is a massless spin-1 boson, so it must be a photon associated with a $U(1)$ gauge theory. The latter is its spin-1/2 superpartner, a photino.

2.1.2 GSO projection

The construction above does not define a consistent theory. This is in part because it's not spacetime supersymmetric, an essential requirement considering that, as will be seen shortly, the closed string spectrum includes a gravitino (a massless spin-3/2 state) which must be associated with local supersymmetry. A procedure known as the Gliozzi, Scherk, Olive (GSO) projection solves this issue and in addition also eliminates the tachyonic state $\left|0\right\rangle_{NS}$.

The following operator is introduced, acting on the NS sector as

$$G = (-1)^{1+\sum_r b_r^{i\dagger} b_r^i} = (-1)^{\hat{F}+1} \tag{2.8}$$

and on the R sector as

$$G = \Gamma_{11}(-1)^{\sum_r d_r^{i\dagger} d_r^i} = \Gamma_{11}(-1)^{\hat{F}} \tag{2.9}$$

$\hat{F}$ is the worldsheet fermion number, and $\Gamma_{11} = \Gamma_0 \cdots \Gamma_9$ gives the chirality of the state.

Then the spectrum is projected into the $G = 1$ subspace for the NS sector, and into $G = \pm 1$ (either choice works) for the R sector. These two choices correspond to keeping either $|+\rangle_R$ or $|-\rangle_R$ respectively and discarding the other.

When amputated with this precise prescription the spectrum is found to be spacetime supersymmetric. The scalar tachyon $|0\rangle_{NS}$ in particular is eliminated, being G-odd.

2.1.3 Closed strings

The closed string spectrum, in somewhat poetic language, is the "square" of the open string spectrum. As seen before the choice can be made for either NS or R boundary conditions separately for left-movers and right-movers, giving four sectors. The GSO projection is performed separately on left and right movers, so that one is presented with the choice of the relative chirality of the two projections and so of the R grounds. These two possibilities will actually result in two different string theories. Choosing opposite chiralities gives type IIA strings, whose massless spectrum is given by

$$\tilde{b}^{i\dagger}_{1/2}\,|0\rangle_{NS} \otimes b^{j\dagger}_{1/2}\,|0\rangle_{NS} \tag{2.10}$$
$$\tilde{b}^{i\dagger}_{1/2}\,|0\rangle_{NS} \otimes |+\rangle_R \tag{2.11}$$
$$|-\rangle_R \otimes b^{j\dagger}_{1/2}\,|0\rangle_{NS} \tag{2.12}$$
$$|-\rangle_R \otimes |+\rangle_R \tag{2.13}$$

(the $\sim$ distinguishes creators/destructor for left movers from right movers). And IIB strings arise from equal chiralities:

$$\tilde{b}^{i\dagger}_{1/2}\,|0\rangle_{NS} \otimes b^{j\dagger}_{1/2}\,|0\rangle_{NS} \tag{2.14}$$
$$\tilde{b}^{i\dagger}_{1/2}\,|0\rangle_{NS} \otimes |+\rangle_R \tag{2.15}$$
$$|+\rangle_R \otimes b^{j\dagger}_{1/2}\,|0\rangle_{NS} \tag{2.16}$$
$$|+\rangle_R \otimes |+\rangle_R \tag{2.17}$$

So the massless spectrum is composed of 4 sectors of 64 physical states, two of them bosonic (NSNS, and RR) and the other fermionic (RNS and NSR). Massless states will correspond to fields in the supergravity approximation, in which the massive modes of the string decouple and the string theory is well described by the corresponding variety of 10D supergravity.

2.2 Type II supergravity and D-brane content

At scales much lower than the Planck scale (equivalently: when the curvature radii are $\gg$ than the string size), all massive modes of a string theory decouple and a good description is given by an effective field theory comprising only the massless excitation. Since the string length goes to zero in this limit strings in massless states are essentially pointlike and the quantum theory will correspond to a local quantum field theory.

The effective field theories of the five superstring theories are the five supergravity (SUGRA) theories in 10 dimensions. The name of each SUGRA coincides with that of the superstring theory it's the effective theory of (e.g., IIB SUGRA is the effective theory of IIB superstrings). Supergravities are supersymmetric theories containing general relativity. Just like Einstein gravity, they are nonrenormalizable, reflecting their origin as effective theories. As field theories, they are considerably simpler than general strings to find background solutions to; therefore we will make extensive use of the supergravity approximation in the context of holography.

$10D$ SUGRAs are perhaps easier to introduce starting instead from the unique $11D$ SUGRA. The field content of $11D$ SUGRA is as follows (number of physical polarizations in parentheses):

- graviton g_{MN} (44)
- 3-form A_3 (84)

- Majorana gravitino ψ_M (128)

As required by supersymmetry, the number of on-shell boson and fermion states are equal. These states form an irreducible supermultiplet, a gravity multiplet.
Upon dimensional reduction on a circle, in $10D$ these fields decompose into those of type IIA SUGRA:

- graviton $g_{\mu\nu}$ (35), Kalb-Ramond 2-form B_2 (28), dilaton ϕ (1)
- 1-form A_1 (8), 3-form A_3 (56)
- two Weyl-Majorana gravitinos of opposite chirality ψ_μ (56 each), two Weyl-Majorana dilatinos of opposite chirality λ (8 each)

These are respectively the NSNS, RR, and NSR + RNS massless modes.

Obviously, again we find that the total bosonic states are $35+28+1+8+56 = 128$ and the fermions $2 \cdot (56+8) = 128$. This is a theory with $\mathcal{N} = (1,1)$ SUSY, meaning there's two Weyl-Majorana (we recall again the existence of Weyl-Majorana fermions in $D = 10$) SUSY generators of opposite chirality.

We will mainly be interested, however, in type IIB SUGRA, which is not obtainable from dimensional reduction, but rather is the T-dual of type IIA. The field content is as follows:

- graviton $g_{\mu\nu}$ (35), Kalb-Ramond 2-form B_2 (28), dilaton ϕ (1)
- 0-form A_0 (1), 2-form A_2 (28), 4-form A_4 with self-dual field strength (35)
- two Weyl-Majorana gravitinos of equal chirality ψ_μ (56 each), two Weyl-Majorana dilatinos of equal chirality λ (8 each)

Again, these are respectively the NSNS and RR bosons, and the NSR+RNS fermions. IIB SUGRA has $\mathcal{N} = (2,0)$ supersymmetry.

This net of relationships between SUGRAs in 10 and 11 dimension is actually the effective limit of dualities between string/M-theories of which these SUGRAs are effective field theories. The relevant part of the scheme is as follows:

$$\begin{tikzcd}
\text{"M-theory"} \arrow[r, "\text{dim. red. on } \mathbb{S}^1"] \arrow[d, "\text{eff. th.}"] & \text{IIA strings} \arrow[r, "\text{T-duality}"] \arrow[d, "\text{eff.th.}"] & \text{IIB strings} \arrow[d, "\text{eff.th.}"] \\
\text{11D SUGRA} \arrow[r, "\text{dim. red. on } \mathbb{S}^1"] & \text{IIA SUGRA} \arrow[r, "\text{T-duality}"] & \text{IIB SUGRA}
\end{tikzcd}$$

In both IIA and IIB, the RR sector admits the following gauge transformations:

$$B_2 \to B_2 + d\Lambda_1 \qquad A_p \to A_p + d\Lambda_{p-1} - H_3 \wedge \Lambda_{p-3} \tag{2.18}$$

for any set of arbitrary k-forms Λ_p, leaving invariant the field strengths:

$$\begin{aligned} H_3 &:= dB_2 \\ F_{p+1} &:= dA_p + H_3 \wedge A_{p-2} \end{aligned} \tag{2.19}$$

Where A_p with $p < 0$ is set to 0. Now, the RR potential A_p obviously couples to $D(p-1)$-branes by an interaction term which is the integral of A_p over the worldvolume; this is an electric coupling of the $D(p-1)$-brane to F_{p+1}. The coupling however could also be magnetic, electric-magnetic duality being implemented in general through Hodge duality. We define F_p for additional values of p through

$$F_{9-p} = \widetilde{\star} F_{p+1} \tag{2.20}$$

note that for the IIB F_5 this is actually a constraint. The new field strengths can then be locally trivialized as of 2.19 and so we end up with a complete set of potentials $A_0, \dots A_8$ for IIB and $A_1 \dots A_9$ for IIA. The duality between potentials would be given by $A_p \leftrightarrow A_{8-p}$, and if $D(p-1)$-branes couple electrically to A_p, then $D(7-p)$-branes couple magnetically to it, that is to say electrically to A_{8-p}.

Therefore, the magnetic dual to a Dp-brane is a $D(6-p)$-brane.

2.3 Action functional for IIB SUGRA

There is a considerable obstacle to a covariant (i.e. explictly supersymmetric) formulation of type IIB supergravity in the self-duality constraint for the field strength 5-form $\tilde{F}_5$. We will take the common path of formulating the Lagrangian theory ignoring the constraint (and thus in excess of bosonic polarizations with respect to an explicity supersymmetric theory) and then imposing self-duality by hand after deriving the equations of motion. Therefore the action will not be supersymmetric itself, while the Euler-Lagrange equations augmented with the constraint will be.

Actually, for the purpose of building classical solutions, where spinor fields vanish anyway, the fermionic sector of the action will not be important. The bosonic sector is as such:

$$S_B = S_{NS} + S_R + S_{CS}$$

where S_{NS} is the action relevant to the fields originally from the superstring NS-NS sector:

$$S_{NS} = \frac{1}{2\kappa^2} \int d^{10}x\sqrt{-g}\, e^{-2\phi} \left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2 \right)$$

Then S_R is for R-R fields, essentially just kinetic terms for the A forms:

$$S_R = -\frac{1}{4\kappa^2} \int d^{10}x\sqrt{-g} \left(|F_1|^2 + |\tilde{F}_3|^2 + |\tilde{F}_5|^2 \right)$$

And finally we supplement with a Chern-Simons type term:

$$S_{CS} = -\frac{1}{4\kappa^2} \int A_4 \wedge H_3 \wedge F_3$$

note the untilded F_3. This is evidently a purely topological term.

2.4 D-brane action

D-branes appear as nonperturbative objects in string theories. They are themselves dynamical and the dynamics are modeled in the string perturbative regime by an action functional[2]. To formulate the action, we introduce coordinates σ^a on the $(p+1)$-dimensional worldvolume W and functions $X^\mu(\sigma^a)$ describing the embedding of W in spacetime.

Open strings can exist with their endpoints on D-branes. With a single D-brane, the massless open string modes with endpoints on the brane include a $U(1)$ gauge field A, with field strength F.

The bosonic part of the Dp-brane action is:

$$S_{Dp} = -\,\mu_{Dp} \int_W d^{p+1}\sigma e^{-\phi} \sqrt{-\det\left(X^*(g - B_2) - 2\pi\alpha' F\right)} \tag{2.21}$$

$$+\,\mu_{Dp} \int_W \left[X^* \left(\sum_k C_k \right) \wedge e^{2\pi\alpha' F - B_2} \wedge (1 + \mathcal{O}(R^2)) \right]_{p+1} \tag{2.22}$$

Where

$$\mu_{Dp} = \alpha'^{-\frac{p+1}{2}} (2\pi)^{-p} \tag{2.23}$$

The first line 2.21 is the Dirac-Born-Infeld action and generalizes the Nambu-Goto action; the notation $X^*(T)$ denotes the pull-back of a spacetime tensor to the worldsheet. For example, if $B = F = 0$, $X^*(g)$ is the induced metric $h_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$. Setting only $B = 0, \phi = \text{const}$ and expanding S_{DBI} in powers of α':

$$S_{DBI} = -\frac{\mu_{Dp}}{g_S} \int_W d^{p+1}\sigma \sqrt{-h} + \frac{\alpha'^{-(p-3)/2}}{4 g_s (2\pi)^{p-2}} \int_W d^{p+1}\sigma \sqrt{-g} F_{\mu\nu} F^{\mu\nu} + \dots \tag{2.24}$$

the first term is the direct generalization of the Nambu-Goto action, allowing us to identify the Dp-brane tension $T_{Dp} = \frac{\mu_{Dp}}{g_S}$. The second is a Yang-Mills action for the $U(1)$ gauge field, restricted to the worldvolume.

The second line 2.22 is a Chern-Simons type term coupling the brane to the RR potentials. The sum over k only spans odd or even respectively for IIA or IIB, and the $[\,]_{p+1}$ notation means the $p+1$-form component must be selected so as to define a meaningful integral. We note that in vanishing B_2 and curvature, and expanding in F, the physical interpretation becomes less obscure:

$$S_{CS} = \mu_P \int_W C_{p+1} + \mu_P(2\pi\alpha') \int_W C_{p-1} \wedge F + \mathcal{O}(F^2) \tag{2.25}$$

so that there is a direct, standard coupling of the C_{p+1} potential to the Dp-brane at the zeroth order in F. Higher order terms mean a coupling with the lower RR potentials and are due to nontrivial F configurations which induce lower-dimensional D-brane charges localized inside the Dp-brane.
We touch briefly upon the easy generalization of the above action to the case of N coincident Dp-branes, a "stack". The salient point is the extension of the gauge group from $U(1)$ to $U(N)$. The gauge bosons in the adjoint representation with indices $i\bar{j}$ come from massless modes of open strings stretching between brane i and brane $\bar{j}$. Essentially, the F^2 term (and higher) in 2.24 must be supplemented with gauge traces.

Chapter 3

D3-brane stacks on Calabi-Yau cones

Perhaps the most essential ingredient for the conception of the idea of holography was the fact that coincident D3-branes (a "stack") naturally feature a 4D gauge theory on their world-volume, where the 4D fields emerge from the modes of open strings stretching between them. In the simplest and most famous example, a stack of N D3-branes is placed in otherwise Minkowski $\mathbb{R}^{1,9}$; the corresponding field theory is the maximally supersymmetric Yang-Mills in four dimensions (SYM4).

Setting the stack on a different background geometry instead gives rise to a large family of different field theories; a particularly interesting subset is given by spacetimes of the form:

$$M = \mathbb{R}^{1,3} \times X_6 \tag{3.1}$$

where the $\mathbb{R}^{1,3}$ is parallel to the branes (and must be identified with the field theory spacetime) and X_6 is a 6-dimensional Calabi-Yau cone over a compact 5-fold base Y_5. In this language, the SYM4 example above corresponds to $X_6 = \mathbb{R}^6 = \mathbb{C}^3$, which is (trivially) a cone over $\mathbb{S}^5$. This is the only case where X_6 turns out to be smooth; in general it will feature a conical singularity in the origin. Other choices for the base will typically yield theories with reduced (even minimal) supersymmetry, which are considerably more challenging to study.

The resulting gauge group will be $U(N)^g$, where $g = 1+b_2(X_6)+b_4(X_6)$, and the theory will be populated by chiral fields in "bifundamental" representations, i.e. with an index in the fundamental of one $U(N)$ and a second in the antifundamental. These sort of theories are termed quiver gauge theories and they can be encoded in a quiver diagram, where $U(N)$ factors are denoted by nodes and bifundamental fields as directed arrows stretching between two nodes.

We will be in particular interested in the moduli spaces of these theories, so the spaces of distinct vacua. Because of supersymmetry, the quantum moduli space will often coincide with the classical one, which is the locus of the F-flatness condition:

$$F^i = \frac{\partial W}{\partial \phi_i} = 0 \tag{3.2}$$

where $W(\phi_i)$ is the superpotential function of the chiral fields ϕ_i, and the D-flatness condition:

$$D^a_{U(N)^g} = -\sum_i \phi_i^\dagger T^a \phi_i = 0 \tag{3.3}$$

where T^a are the gauge generators. (The $U(N)^g$ subscript indicates the index a spans over all generators of the g factors of $U(N)$). The space $\mathcal{M}$ of simultaneous solution of the F and D-flatness conditions will be a complex manifold.

A subspace of $\mathcal{M}$ is given by the so-called mesonic moduli space $\mathcal{M}_m$. Points of $\mathcal{M}_m$ will correspond to the position of the N branes on the background cone - therefore $\dim_{\mathbb{C}} \mathcal{M}_m = 3N$. The moduli space is not exhausted in the purely mesonic directions though; to investigate the remaining "baryonic" directions we first anticipate we will be mainly concerned with the IR limit, in which our theories will flow to (super) conformal field theories ((S)CFTs). In the IR limit, the abelian $U(1)$ factor in each $U(N)$ node "freeze" and become global baryonic symmetries. Therefore their D-flatness condition is relaxed and one is left with only the D-term for the $SU(N)^g$ part. So

$$D^a_{SU(N)^g} = 0 \qquad\qquad D^i_{U(1)^g} = V^i \tag{3.4}$$

$(i = 1, \cdots, g)$. V^i are classically functions of the fields and in the quantum version will be gauge-invariant operators. Their g VEVs $\langle V^i \rangle =: \xi^i$ will parametrize the missing flat directions of moduli space. To be precise, however, since the overall trace $U(1)$ (generated by the sum of the generators of the g abelian trace factors) is completely decoupled, we have to impose $\sum \xi^i = 0$. Therefore that there are really only $g - 1$ baryonic moduli, corresponding to $g(N^2 - 1) + 1$ independent D-flatness conditions.

Thus we conclude $\dim \mathcal{M} = 3N + g - 1$. While the $3N$ mesonic directions have a direct geometrical interpretation as D3-brane movement, the baryonic directions correspond in terms of the superstring description to deformations of the X_6 background metric itself - generally resulting in a resolution of the conical singularity.

3.1 Brane stack in $\mathbb{C}^3$ and $\mathcal{N} = 4$ super-Yang-Mills

If $X_6 = \mathbb{C}^3$, the branes are invariant under half of the $16 \times 2 = 32$ IIB supercharges. The only possibility for a 4D theory to have 16 supercharges is to be an $\mathcal{N} = 4$, superconformal field theory[1]. Moreover, the theory features gluons as the massless spin-1 modes for the sector of strings stretching between brane i and brane j so that the gauge group is $U(N)$, as seen in 2.4. The information that the theory is a $U(N)$ gauge theory and is maximally supersymmetric is enough to uniquely fix it.

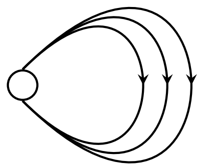
In $\mathcal{N} = 1$ language (which we employ even though the model has $\mathcal{N} = 4$) the

[1]Indeed, the number of supercharges is 2 for the components of a 4D Majorana spinor, times $\mathcal{N}$, times a factor of 2 since the superconformal algebra has twice the supercharges of the usual SUSY algebra.

theory describes the dynamics of $U(N)$ gauge vector supermultiplets A_μ and three complex chiral superfields $(X^a)_{ij}$, $a = 1, 2, 3$ in the adjoint of the gauge group (we will frequently omit gauge indices). These are nothing else than the parametrization of the D3-branes' position in $\mathbb{C}^3$ and therefore transform in the fundamental of $SU(3)$. The superpotential is the only one allowed by gauge and $SU(3)$ invariance:

$$W(X) = \epsilon_{abc}\,\mathrm{Tr}(X^a X^b X^c) \tag{3.5}$$

and the quiver diagram is quite simple:



nota sul moduli space triviale

3.2 The conifold and the Klebanov-Witten model

In [3] the case of X_6 being the conifold was studied. The conifold is a specific Calabi-Yau 3-cone defined for example as the following variety in $\mathbb{C}^4$:

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \tag{3.6}$$

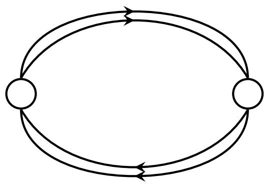
The base can be found by quotienting by dilations $z_i \to \lambda z_i$ (with $\lambda \in \mathbb{R}_+$) and turns out to be the homogeneous space $SO(4)/U(1) = SU(2) \times$

$SU(2)/U(1)$, where the $U(1)$ is a diagonal subgroup generated by, say, $T^3_L + T^3_R$. We will therefore have $SU(2) \times SU(2)$ as part of the isometry group of both Y_5 and X_6, and thus will also appear as a global symmetry of the wordvolume theory. An equivalent description of the topology of the conifold is as a $U(1)$ bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1$; in these terms the metric on the base that makes the cone Calabi-Yau is

$$ds_5^2 = \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6}(d\Omega_1^2 + d\Omega_2^2) \tag{3.7}$$

where $\Omega_i^2 = d\theta_i^2 + \sin\theta_i^2 d\phi_i^2$ is the metric on the $\mathbb{C}P^1_i$, and ψ is the fibral coordinate with period 4π.

The corresponding gauge field theory on the worldvolume is a $U(N) \times U(N)$ field theory featuring two chiral doublets A_i, B_j with $i, j = 1, 2$ transforming in opposite bifundamentals, that is A_i in $(N, \bar{N})$ and B_j in $(\bar{N}, N)$. Or more succintly, this can be depicted in a quiver diagram:



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The i and j indices, instead, are acted upon respectively by the global left and right $SU(2)$ symmetries. Finally, A and B have R-charge $1/2$. The symmetries and R-charges fix the form of the superpotential:

$$W = \frac{\lambda}{2}\epsilon^{ij}\epsilon^{kl}\,\mathrm{Tr}\,(A_i B_k A_j B_l) \tag{3.8}$$

While this theory won't be in general superconformal, unlike the $\mathcal{N} = 4$ SYM seen before, it will flow through renormalization in the IR to a conformal submanifold in the space of couplings (λ, g_1, g_2), the locus where the β functions for these three couplings vanish. It turns out these three conditions are all equivalent. In particular, requiring $\beta_{g_1} = 0$ and making use of the NSVZ expression for the β function of a supersymmetric gauge theory, this unique condition is equivalent to

$$3T[\mathrm{Adj}] - \sum_i T[R_i](1 - 2\gamma_i) = 0 \tag{3.9}$$

where $T[R]$ is the Dynkin invariant of representation R, the sum is over charged fields and γ_i is the anomalous dimension[2]. When evaluating this, care should be taken with the fact that A_i and B_j have a $U(N)_2$ index which is uncharged under $U(N)_1$ and must be summed over. This gives, also noting $\gamma_{A_1} = \gamma_{A_2}$ and the same for B because of the global symmetry:

$$\gamma_A + \gamma_B + \frac{1}{2} = 0 \tag{3.10}$$

Being $\gamma_{A,B}$ functions of the couplings, this equation defines a critical 2-surface in parameter space. We note this equation is consistent with the relationship $\frac{3}{2}R - 1 = \gamma$ between R-charge and the anomalous dimension of an operator in a SCFT, with the given assignment of R-charges.
descrizione moduli space

3.3 The $Y^{(2,0)}$ orbifold theory

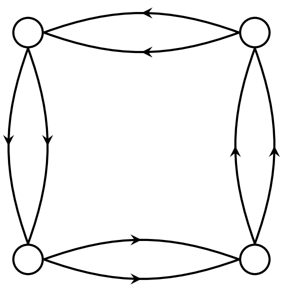
The same construction on a $\mathbb{Z}_2$ orbifold of the conifold yields a quiver gauge theory which will be the main interest of this work. The geometry of the base of the cone is very simply introduced in polar coordinates as

$$ds_5^2 = \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6}(d\Omega_1^2 + d\Omega_2^2) \tag{3.11}$$

i.e., exactly the same metric in form as the conifold, but with ψ now with period 2π. This background and the resulting worldvolume field theory are just one entry $Y^{2,0}$ of an infinite class $Y^{p,q}$ of examples introduced in [1].

The quiver diagram "splits" to yield four doublets of bifundamental chiral fields stretching in a square between four nodes:

[2]Note we use the definition $\gamma = -\frac{1}{2}\frac{d\ln Z}{d\ln\mu}$, where $\sqrt{Z}$ renormalizes the field.



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and the superpotential can be shown to have the form

$$W = \lambda \epsilon^{ij} \epsilon^{kl} \operatorname{Tr} (A_i B_k C_j D_l) \tag{3.12}$$

from which it's clear that the $SU(2) \times SU(2)$ isometry of the cone, corresponding to a global symmetry of the field theory, must now act with the left factor on A_i and C_i, and the right on B_i and D_i. This time three of the four gauge β functions are independent:

$$\gamma_A + \gamma_D + \frac{1}{2} = 0 \tag{3.13}$$

$$\gamma_B + \gamma_A + \frac{1}{2} = 0 \tag{3.14}$$

$$\gamma_C + \gamma_B + \frac{1}{2} = 0 \tag{3.15}$$

$$\tag{3.16}$$

$\beta_\lambda = 0$ is also not independent. At any superconformal point, $\frac{3}{2}R - 1 = \gamma$, so that the condition that W be scale invariant, which is equivalent to it having R-charge 2, becomes

$$2 = R_W = R_A + R_B + R_C + R_D \Rightarrow \gamma_A + \gamma_B + \gamma_C + \gamma_D + 1 = 0 \tag{3.17}$$

which is indeed equivalent to the above system. Three independent equations in a five-parameter space define, again, a critical 2-submanifold.

modulispace

Chapter 4

Holography

In the previous chapter we explained how the dynamics of brane stacks, in particular D3-branes in type IIB, are described by gauge field theory on their worldvolumes. It's however important to note that parallel to this "open string" picture of the brane stack system there is also a dual description in terms of the curved spacetimes generated by their mass. Insisting these two viewpoints are equivalent, one is able to deduce an exact correspondence between the gauge theory and string theory on the near-horizon geometry.

This kind of duality is exotic as it connects a local field theory in four dimensions with an essentially five-dimensional string (and so, inherently gravitational) theory through a perfect mapping. It is reasonable in fact to identify the spacetime of the field theory with the conformal boundary of the higher-dimensional gravitational background it's dual to (the bulk), for reasons we will clarify - so that in more colloquial language the dynamics in the bulk are "encoded" in the screen at infinity, hence the adjective "holographic" for this sort of correspondences.

Explicit holographic correspondences are not only interesting by themselves as elegant structures; they're also extremely practical tools for studying the theories involved on both sides. It's certainly very attractive for the purpose of quantum gravity or the definition of string theory - non-local theories without action functionals - if these situations happen to be equivalent to a local quantum field theory.

However in this work our interest will be focused on the opposite direction, investigating the dynamics of the field theory by exploiting the dual gravitational system. The power of holographic dualities lies in the fact that they map the strongly-coupled regime for the field theory to the regime where the bulk dynamics can be approximated by supergravity. The traditionally untreatable strong coupling region for some gauge QFTs in four dimensions can then be probed by studying the relatively tamer dynamics of a smooth dual spacetime.

4.1 Maldacena duality

We now consider the IIB supergravity solution modeling the spacetime created by a system of D3-branes in a background $\mathbb{R}^{1,9}$. This is given by

$$ds^2 = H^{-1/2} dx_\mu dx^\mu + H^{1/2}(dr^2 + r^2 d\Omega_5^2) \tag{4.1}$$

$$e^\Phi = \text{const} =: g_s \tag{4.2}$$

$$F_5 = dH^{-1} \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{4.3}$$

$$H(r) = 1 + \left(\frac{R}{r}\right)^4 \tag{4.4}$$

where x^μ, $\mu = 0, \ldots, 3$ are coordinates parallel to the brane stack and $d\Omega_5$ is the standard metric on $\mathbb{S}^5$.

The curvature radius R is given by

$$R^4 = 4\pi g_s N \alpha'^2 \tag{4.5}$$

where N is the number of D3-branes in the stack.

note on the throat.

This system (IIB string theory on the metric 4.1) must then be equivalent to the stack of D3-branes in the background Minkowski, taking into account both open and closed string interactions. The action is schematically:

$$S = \frac{1}{g_s} \int d^4x F^2 + \frac{1}{\alpha'^4} \int d^{10}x \sqrt{g} R e^{-2\phi} + O(\alpha') + \dots \tag{4.6}$$

(where, we recall, $g^2_{YM} \sim g_s$). The first two terms are respectively the actions for SYM and free IIB SUGRA in the Minkowski background; the following terms, with higher powers of α', establish the coupling between these two systems. It's clear that in the limit $\alpha' \to 0$ the free SUGRA part decouples from the SYM.

We repeat this decoupling limit for the black 3-brane metric. If $\alpha' \to 0$, so does R, and effectively the metric seems to converge to flat spacetime. We have however to take into consideration the throat described before. *decoupling limit*

4.2 Features of AdS/CFT

relazione fra on-shell nel bulk e off-shell nel boundary; operator-state e le funzioni di partizione; massa e scaling dimension; rinormalizzazione e dimensione extra

4.3 Large N limit

We will now clarify what is meant by large N limit for a Yang-Mills theory.

The Lagrangian is

$$\mathcal{L} = \mathrm{Tr}\left(F^2\right) + \dots$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i g_{YM}[A_\mu, A_\nu]$ and $\dots$ can include fields in the fundamental, adjoint, bifundamental, etc. These will all of course be representable as object with a certain number of colour indices (and symmetries between them).

We can modify the standard Feynman prescription for pictorially representing amplitudes to get a "double line" or "ribbon" representation in which each colour index is carried by a line. For example, the gluon self energy diagram becomes as such:

INSERIRE DIAGRAMMA

colour indices i, $\bar{i}$, j, $\bar{j} = 1, \dots, N$ are fixed, while k must be summed over. Also, the amplitude has two three-gluon vertices, each carrying a factor of g_{YM}^2, for an overall factor of $g_{YM}^2 N$.

It's easy to convince oneself that as long as we restrict to planar diagrams, that is diagrams that can be drawn on the plane (or more precisely the sphere), adding one strip will always introduce exactly one additional loop and two additional vertices, again carrying a factor of $g_{YM}^2 N$. The combination $\lambda := g_{YM}^2 N$ is the 't Hooft coupling, and is better suited to represent the strength of the gauge interaction than g_{YM} if we are to modify the number of colours.

So the 't Hooft large N limit is defined as:

$$N \to \infty, \quad \text{but keeping } \lambda \text{ fixed} \tag{4.7}$$

A useful rescaling of the fields shifts all the g_{YM} dependence of the Lagrangian to a factor in front:

$$\mathcal{L} = \frac{1}{g_{YM}^2} \left(\mathrm{Tr}\, F^2 + \dots \right) \tag{4.8}$$

so that now all types of vertices bring $g_{YM}^2 = \lambda/N$ and propagators bring $1/g_{YM}^2 = N/\lambda$.

We extend to nonplanar graphs by noting these can always be drawn on some

Riemann surface of genus g, and, since they induce triangular tilings of said surface, the famous formula for the Euler characteristic holds:

$$F - V + E = \chi = 2 - 2g$$

F, V, E being the number of faces, vertices, edges respectively. Now each face (loop) carries a factor of N, each vertex a factor of λ/N, and each edge N/λ, so that the total contribution is

$$\lambda^{E-V} N^{F-V+E} = \lambda^{E-V} N^{2-2g}$$

so that at fixed λ, an expansion in N (or better $1/N$) is a genus expansion reminiscent of the loop expansion in perturbative string theory. This for example means that the free energy admits a power expansion in $1/N$:

$$F = \sum_{g=0}^{\infty} f_g(\lambda) N^{2-2g} \tag{4.9}$$

One could be perplexed by the N^2 divergence of the genus zero contribution. This is not problematic however; it's an artifact of the rescaling 4.8 which makes the Lagrangian itself diverge as $g_{YM}^{-2} \operatorname{Tr} F^2 \sim N/\lambda \cdot N$, since the trace of a matrix in the adjoint scales as N.

4.4 AdS/CFT over a cone

As seen above, the original motivation for the AdS/CFT conjecture is the identification of a system of N coincident D3-branes in a $\mathbb{M}^{10}$ Minkowski background and the corresponding 3-brane supergravity solution. In an appropriate low-energy limit a system of closed IIB strings on flat spacetime decouples in both pictures, suggesting it should be conjectured that the remaining parts are equivalent. These are respectively $\mathcal{N} = 4$, $SU(N)$ SYM on $\mathbb{M}^4$ and IIB strings on $AdS_5 \times S_5$.

We repeat this reasoning, but in the more interesting case where the background for the D3-branes is generalized as $\mathbb{M}^4 \times X_6$, where X_6 is a cone over a base 5-manifold Y_5. We anticipate the bulk dual in this case is IIB strings over $AdS_5 \times Y_5$. By X_6 being a cone over Y_5 it is meant that the metric on it is

$$ds_6^2 = dr^2 + r^2 ds_5^2 \tag{4.10}$$

where of course ds_5^2 is the metric on Y_5. If $Y_5 = \mathbb{S}^5$ with the unit round metric then the cone is $X_6 = \mathbb{R}^6$ and one returns to the flat case.

For this to be a string background, X_6 should be Ricci-flat. This is equivalent to Y_5 being Einstein of positive curvature. ds_6^2 is conformally equivalent to the canonical metric on a cylinder over Y_5, as evidenced by the reparametrization $\phi = \ln r$:

$$ds_6^2 = e^{2\phi}\left(d\phi^2 + ds_5^2\right) \tag{4.11}$$

Recalling the transformation law of the Ricci tensor in n dimensions under conformal rescalings:

$$R'_{ij} = R_{ij} - (n-2)\left(\nabla_i\partial_j\phi - \partial_i\phi\partial_j\phi\right) + \left(\nabla^2\phi - (n-2)\nabla_k\phi\nabla^k\phi\right) g_{ij} \tag{4.12}$$

And noting that for the cylinder the restriction of R_{ij} to Y_5 indices gives Y_5's own Ricci tensor $R_{ij}^{(5)}$, we obtain

$$R_{ij}^{(5)} = 4g_{ij}^{(5)} \tag{4.13}$$

proving Y_5 is Einstein.

We are also interested in X_6 being Calabi-Yau, that is being Kähler with holonomy $\subset SU(3)$. We define Y_5 to be Sasaki-Einstein iff the corresponding cone is Calabi-Yau. The complex structure on the cone induces a vector field on the base, the Reeb vector:

$$\xi := J(r\partial_r) \tag{4.14}$$

where J is the complex structure on the cone and ξ is to be thought of as restricted to, say, $r = 1 \cong Y_5$; this is a Killing vector on the base, inducing a 1-dimensional foliation. The dual form, $\theta = g_{ij}\xi^i dx^j$, is a contact form for the base, contact meaning the 2-form on the cone

$$\omega = t^2 d\theta + t dt \wedge \theta \tag{4.15}$$

is symplectic. This is of course the symplectic form associated to the hermitian structure.

After placing 3-branes in this $X_6 \times \mathbb{R}^4$ background, parallel to the Minkowski, the resulting geometry from their backreaction is:

$$ds^2 = H^{-1/2}(r, y)\, dx \cdot dx + H^{1/2}(r, y)\, ds_6^2 \tag{4.16}$$

già fatto! ref ref ref Where $x^{0,\dots,3}$ are coordinates parallel to the brane stack, $dx \cdot dx = -(dx^0)^2 + (dx^i)^2$, r is the radial coordinate and the remaining $y^{1,\dots,5}$ parametrize the cone's base Y_5. This is a simple generalization of the well-known black 3-brane solution by substitution of $\mathbb{S}^5$ with Y_5.

Ricci-flatness implies the function H is harmonic: $\nabla H(r) = 0$. The linearity of this equation arises from the fact that D-branes are BPS states, corresponding in the gravitational picture to extremal p-branes; these notably do not interact mutually.

If the branes are coincident, the corresponding harmonic potential is

$$H(r) = 1 + \frac{R^4}{r^4} \qquad\qquad R^4 = 4\pi g_s N \alpha'^2 \tag{4.17}$$

The near-horizon limit $(r \to 0)$ in that case can be read immediately:

$$ds^2 = \frac{dx \cdot dx + dz^2}{z^2} + ds_5^2 \tag{4.18}$$

where $z := 1/r$; this is evidently the product metric on $AdS_5 \times Y_5$, where of AdS_5 we're only considering the Poincaré patch.

We note that the introduction of a conical singularity results in reduced supersymmetry. Unbroken SUSY generators are identified from the Killing spinor equation:

$$\left(\partial_\mu + \frac{1}{4}\omega_{\mu\alpha\beta}\Gamma^{\alpha\beta}\right)\eta = 0 \tag{4.19}$$

Explicitly for the cone metric 4.10:

$$\left(\partial_i + \frac{1}{4}\omega_{ijk}\Gamma^{jk} + \frac{1}{2}\Gamma^r_i\right)\eta = 0 \tag{4.20}$$

this is, as expected, coincident with the (Y_5 sector of) Killing spinor equation for the backreacted $AdS_5 \times Y_5$ geometry, including also the effect of F_5. This is to show there is a match between the unbroken SUSYs in the bulk theory and in the boundary.

If the cone is of holonomy $SU(n)$, this will result in a reduction of supersymmetries by a factor of 2^{1-n} with respect to $\mathbb{M}^{10}$ Minkowski. In particular, if X_6 is Calabi-Yau, then the $32 = 16 \times 2$ fermionic generators of IIB SUGRA are reduced to $32 \times 2^{-2} = 8$, which means the SCFT in $4D$ has $\mathcal{N} = 1$ (in contrast to the usual SUSY algebra, the $\mathcal{N} = 1$ $4D$ superconformal group has both 4 supertranslations and 4 additional fermionic superconformal generators). If, instead, we were to consider the more restrictive case of manifolds of $SU(2)$ holonomy, there would be 16 unbroken supersymmetries signaling an $\mathcal{N} = 2$ dual SCFT.

4.5 The Klebanov-Witten model

A well-known specific example of SCFT holographically dual to D-branes in a Calabi-Yau cone has been introduced in [3]. In this case the base of the cone is the manifold $T^{1,1} = (SU(2) \times SU(2))/U(1)$, where $U(1) \subset SU(2)_L \times SU(2)_R$ is generated by $\sigma^3_L + \sigma^3_R$.

We give a characterization of the cone X_6 over $T^{1,1}$ as a submanifold of $\mathbb{C}^4 \ni (A^1, A^2, B^1, B^2)$ given by

$$|A^1|^2 + |A^2|^2 - |B^3|^2 - |B^4|^2 = 0 \tag{4.21}$$

quotiented by $U(1)$ acting on A^i with charge 1 and B^i with charge -1. This makes the $SU(2) \times SU(2) \approx SO(4)$ symmetry manifest with the two copies of $SU(2)$ acting respectively only on A^i and B^i.

The holographic dual theory to a stack of N D3-brane moving in the background given by $\mathbb{R}^4$ times the cone over $T^{1,1}$ is found to be an $\mathcal{N} = 1$ superconformal quiver gauge theory with gauge group $SU(N) \times SU(N)$, with chiral superfields A^i, B^i $(i = 1, 2)$ transforming respectively in the bifundamentals $(\mathbf{N}, \bar{\mathbf{N}})$, $(\bar{\mathbf{N}}, \mathbf{N})$. The $SU(2) \times SU(2)$ isometry in the bulk is here implemented as a "flavour" global symmetry acting separately on A^i and B^i. Moreover, the diagonal $U(1)$ reappears as a baryonic symmetry where again A^i has charge 1 and B^i charge -1.

relazione fra i campi della CFT e la posizione delle D-brane, moduli space mesonico e totale, un po' di più sul KW

Chapter 5

Holographic effective field theories

In this chapter, we present the technique and results introduce in [4] to find the effective theory for strongly-interacting CFTs with holographic duals. Instead of repeating the arguments presented therein, we'll strive to provide an intuitive summary of the concepts involved.

Since we are considering strongly-interacting quantum field theories with minimal supersymmetry, the problem of identifying the low-energy effective field theory directly is generally untractable. However, as we have seen, the strong-coupling regime for the CFT corresponds to effectiveness of the supergravity approximation on the holographically dual string side. Therefore, the low-energy dynamics of the dual system can in principle be read and the resulting theory will coincide with the effective theory for the original QFT. Having been obtained by passing through the holographic dual, these will be termed holographic effective field theories (HEFTs).

In practice, it's found that for any given point in the longitudinal coordinates $x^{0,\dots,3}$ the transverse supergravity configuration will belong to a manifold of different supergravity vacua, and that this manifold is finite-dimensional, in the sense that there is only a finite number of moduli parametrizing deformations of the vacuum configurations. This moduli space coincides of course

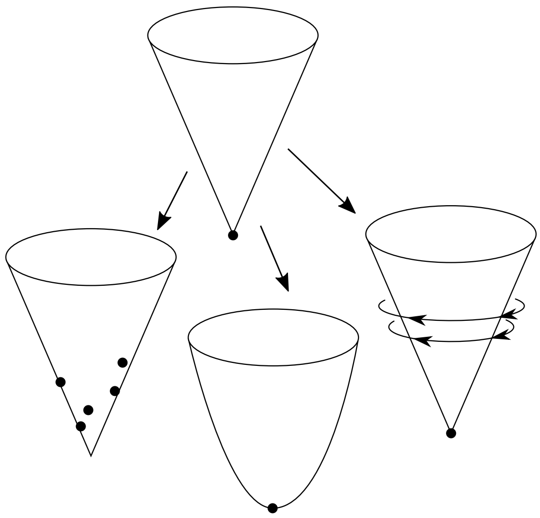

Figure 5.1: Schematic depiction of the three type of moduli of the dual vacuum: motion of D3-branes, deformation of Kähler structure of the background, generally involving resolution of the conical singularity, and field fluxes.

with the field theory moduli space.

A first class of moduli are given by deformations of the dual geometry. These include Kähler moduli of the *background* cone in which the branes are placed in, and then the position of the D3-branes themselves on that background - which manifests as a deformation in the resulting warped geometry. Another class instead will be given by the moduli corresponding to the deformations of the B_2 and R-R fields of IIB supergravity; while these would be full fields defined on the six-dimensional background, gauge invariance will result in only a finite number of topological invariants of the field configuration to be physical.

In short, there will be a finite number of flat directions parametrizing moduli space, and each of these moduli will result in a corresponding scalar field when we extend these deformations to depend on the longitudinal point. Reintroducing $\mathcal{N} = 1$ supersymmetry, these will be the lowest components of chiral supermultiplets which will exhaust the degrees of freedom of the low-energy effective field theory.

Then, expanding the supergravity action in these modes the action governing these chiral fields can be found. This is nothing else than the explicit form of the effective theory for our original strongly-interacting theory.

5.1 Topology *titolo!*

in

First of all, we take as an assumption that the third Betti number of the cone vanishes:

$$b_3(X) = 0 \tag{5.1}$$

It can be proven from Myers' theorem that Y_5 being Sasaki-Einstein means the following Betti numbers vanish:

$$b_1(Y) = b_4(Y) = 0 \tag{5.2}$$

It's also possible to prove that

$$b_1(X) = b_5(X) = b_6(X) = 0 \tag{5.3}$$

The vanishing of the odd Betti numbers for X_6 means Dp-branes with $p = 1, 3, 5$ cannot be wrapped around nontrivial p-cycles.

We recall the long sequence involving relative homology groups:

$$\ldots \to H^{i-1}(Y) \to H^i(X,Y) \to H^i(X) \to H^i(Y) \to H^{i+1}(X,Y) \to \ldots \tag{5.4}$$

where $H^i(X, Y; \mathbb{R})$ is the relative homology group - closed k-forms on X vanishing on Y modulo exact forms with the same property - and when the $;\mathbb{R}$ is omitted we implicitly mean the base field is $\mathbb{R}$. We cut the sequence short by setting $i = 2$ and noting $H^1(Y) = 0$ as of 5.2 and $H^3(X, Y) \subset H^3(X) = 0$

as of 5.1; the short exact sequence is

$$0 \to H^2(X, Y) \to H^2(X) \to H^2(Y) \to 0 \tag{5.5}$$

Implying $H^2(X) = H^2(Y) \oplus H^2(X, Y)$. Applying Poincaré duality on the two components and counting dimensions gives

$$b_2(X) = b_3(Y) + b_4(X) \tag{5.6}$$

5.2 Kähler moduli

We now consider the moduli describing the deformation of the Kähler (so, geometric) structure of the background. Since $b_3(X) = 0$ by hypothesis, the complex structure is rigid. There are instead moduli for the Kähler form J; in particular we know from *citazione teoremi esistenza* that every cohomology class $[J]$ of $H^2(X)$ contains a single representative Ricci-flat Kähler form J, so that $H^2(X)$ is the moduli space for the Kähler structure. We can expand the cohomology class as

$$[J] = v^a[\omega_a] \tag{5.7}$$

with $[\omega_a]$ being a basis for the integral cohomology $H^2(X; \mathbb{Z})$, as the latter modulo torsion is a lattice sitting in $H^2(X; \mathbb{R})$. This means

$$\delta[J] = \delta v^a[\omega_a] \tag{5.8}$$

meaning there exist representatives in the classes such that the equation without square brackets holds. Since small variations of the Kähler form must be $(1, 1)$ harmonic forms *reference*, we then know there exist $(1, 1)$ harmonic representatives ω_a for the aforementioned basis of classes. Returning to 5.7 we can rewrite it as

$$J - v^a\omega_a \in [0] \tag{5.9}$$

But for the LHS to belong to the zero class just means to be exact. Therefore

$$J = J_0 + v^a \omega_a \tag{5.10}$$

with J_0 being exact and $(1,1)$. Note the linearity of this parametrization is an illusion of notation: the condition $\Delta\omega_a = 0$ depends on the metric and so on both J_0 and v^a.

It is then useful to decompose this set of $b_2(X)$ harmonic forms according to identity 5.6 into $b_3(Y)$ noncompact elements $\tilde{\omega}_\beta$ and $b_4(X)$ normalizable forms $\hat{\omega}_\alpha$. By "normalizable" it's meant the hatted forms have finite norm according to the product

$$\int_X \omega_a \wedge \star\omega_b =: \mathcal{M}_{ab} \tag{5.11}$$

while the other $b_3(Y)$ don't. They are however all normalizable according to the "warped" product

$$\int_X e^{-4A} \omega_a \wedge \star\omega_b =: \mathcal{G}_{ab} \tag{5.12}$$

where the factor e^{-4A}, as will be explained later, is the warp factor resulting from the backreaction of the D-branes. More intuitively, in our particular case of X being (asymptotically) a cone, this means that $||\hat{\omega}_\alpha||^2$ must drop at least as fast as r^{-8}, while $||\tilde{\omega}_\beta||^2$ will go as r^{-4}.

In any local chart the ω_a forms will be generated by potentials κ_a as $\omega_a = i\partial\bar{\partial}\kappa_a$ just like J is generated by the Kähler potential $J = i\partial\bar{\partial}k_0$. This means in particular κ_a will coincide with $\frac{\partial k_0}{\partial v^a}$ up to a $z_i, \bar{z}_i$-independent piece, that is a function of the $\{v^a\}$ only. To fix this arbitrariness, the κ_a potentials are required to satisfy

$$\frac{\partial \kappa_a}{\partial v^a} \sim r^{-k} \qquad , k \geq 2 \tag{5.13}$$

so that they are determined up to a constant. This asymptotic condition must be enforced for the following analysis to be meaningful.

5.3 Remaining moduli

We are now in position to classify flat deformations of the axio-dilaton τ and the 2-forms C_2 and B_2, which we compose into a single complex 2-form $C_2 - \tau B_2$, plus τ itself. The former field's flat deformation will be generated by cohomology classes of $H_2(X)$, so in practice the harmonic forms ω_a found above can be used as a basis. Therefore the following decomposition is possible:

$$C_2 - \tau B_2 = l_s^2 \left(\beta^\alpha \hat{\omega}_\alpha + \lambda^\beta \tilde{\omega}_\beta\right) \tag{5.14}$$

The $b_4(X)$ moduli β^a weighing the compact forms $\hat{\omega}_\alpha$ will result in dynamical chiral fields in the HEFT. Instead, the $b_3(Y)$ λ^β moduli, to which we add one complex modulus for τ, parametrize deformations which will turn out to be non-dynamical. The reason is precisely that the kinetic matrix for these fields will turn out to be $\mathcal{M}_{ab}$, which is only finite for the normalizable forms.

Then, an obvious set of moduli z_I^i ($I = 1, \cdots, N$, $i = 1, 2, 3$) are to be introduced to parametrize the motion of the D3-branes on the background. As was hinted before, these alone are coordinates for the submanifold $\mathcal{M}_{\text{mes}}$ in $\mathcal{M}$, which to be precise should be quotiented by permutation of the branes. Therefore, $\mathcal{M}_{\text{mes}}$ based at any given point of moduli space is the symmetric product of N copies of the background geometry, as it is at that particular point.

There is a final class of flat shifts that should be considered, those of the C_4 potential. These moduli should (very schematically) better be thought of as paired with the v^a to form complex moduli. In the end, it turns out it's not really necessary to study the C_4 moduli explicitly for the purpose of finding the HEFT.

5.4 Chiral fields

Finally, we have to introduce the chiral fields corresponding to the dynamical moduli. We use the moduli β^α and z_I^i directly as the lowest component of the corresponding superfield, while to $v^a = (\hat{v}^\alpha, \tilde{v}^\beta)$ it is useful to associate fields $\rho^a = (\hat{\rho}_\alpha, \tilde{\rho}_\beta)$, obtained by a sort of Legendre transform:

$$\operatorname{Re}\rho_a = \frac{1}{2}\sum_I \kappa_a(z_I, \bar{z}_I; v) - \frac{1}{2\operatorname{Im}\tau} I_{a\alpha\beta} \operatorname{Im}\beta^\alpha \operatorname{Im}\beta^\beta - \frac{1}{\operatorname{Im}\tau} I_{a\alpha\sigma} \operatorname{Im}\beta^\alpha \operatorname{Im}\lambda^\sigma \tag{5.15}$$

where the κ_a are the potentials for the ω_a forms as defined before. The imaginary part instead as expected is schematically dual to the C_4 moduli; the explicit form of $\operatorname{Im}\rho_a$ is not necessary for our purposes.

To wrap up, the dynamical chiral fields in the effective field theory are *(meglio tabella?)*

- $b_4(X_6)$ fields $\hat{\rho}_\alpha$ parametrizing normalizable Kähler deformations,
- $b_3(Y_5)$ fields $\tilde{\rho}_\beta$ parametrizing non-normalizable (but warp-normalizable) Kähler deformations,
- $b_4(X_6)$ fields β_α parametrizing $C_2 - \tau B_2$ deformations,
- $3N$ fields z^i_I, parametrizing D3-brane positions.

plus the following non-dynamical marginal parameters:

- $b_3(Y_5)$ parameters λ for $C_2 - \tau B_2$ deformations,
- 1 parameter τ, the axio-dilaton

5.5 Effective action

azione efficace calcolata in [4]; forse anche la derivazione?

5.6 Example: the Klebanov-Witten HEFT

As an example, we now summarize a direct application of this method to the conifold theory described in 3.2.

It's essential for the metric for the background to be presented in complex coordinates for the construction above to be applicable. We exploit the fact that at any generic point of moduli space *except* the origin, where the background is the singular conifold, the geometry is that of a smooth complex 3-fold describable as the total space of the sum of the tautological bundle on a $\mathbb{CP}^1$ with itself: $\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)$. The base $\mathbb{CP}^1 \sim \mathbb{S}^2$ of the bundle is the "resolution" of the conical singularity. We use this to build a chart on the resolved conifold by extending a stereographic chart on the base to get the complex coordinates $z^i = (\lambda, U, Y)$, the first stereographic for the base and the latter two fibral.

With this presentation, it's clear that the only 2-cycle of X_6 is the base $\mathbb{S}^2$, so that $b_2(X) = 1$. Moreover, we already know that as a real cone the conifold has base $SU(2) \times SU(2)/U(1) \approx \mathbb{S}^{\nvDash} \times \mathbb{S}^{\nVDash}$, and that the resolution cannot really change the topology of the base at infinity, so that the only 3-cycle in Y_5 is the $\mathbb{S}^3$ and $b_3(Y) = 1$. So, according to 5.6, $b_4(X_6) = 0$. This will mean, according to the previous identification, that we will have a single $\tilde{v} =: v$ modulus parametrizing a non-normalizable Kähler deformation and dual to a single chiral field $\tilde{\rho} =: \rho$, no β fields, and two non-dynamical parametres λ and τ.

The Kähler modulus v is identified with the volume of the base. Therefore, all Calabi-Yau deformations of the conifold are a one-parameter family and the singular conifold itself lies at the origin, $v = 0$. The Kähler potential for X_6 in the stereographic coordinates is detailed for example in [5], and takes the form:

$$k(z, \bar{z}; v) = \frac{1}{2} \int_0^{s^2} d \ln x \; \gamma(x; v) \; + \frac{v}{2\pi} \ln(1 + |\chi|^2) \tag{5.16}$$

where $s^2 = (1 + |\chi|^2)(|U|^2 + |Y|^2)$ and γ must satisfy the following for the metric to be Ricci-flat:

$$\gamma^3 + \frac{3v}{2\pi} \gamma^2 - x^2 = 0 \tag{5.17}$$

The potential κ generating the unique harmonic form $\tilde{\omega}$ is given simply by derivative of k with respect to the modulus, plus a v-dependent piece fixed by the asymptotic condition 5.13:

$$\kappa = -\frac{1}{4}\int_0^{s^2} d\ln x\, \frac{\gamma}{\pi\gamma + v} + \frac{1}{2\pi}\ln(1+|\chi|^2) - \frac{3}{8\pi}\ln v \tag{5.18}$$

This in turn defines the relationship between ρ and v as

$$\mathrm{Re}\,\rho = -\frac{1}{8}\sum_I \int_0^{s_I^2} d\ln x\, \frac{\gamma}{\pi\gamma + v} + \frac{1}{4\pi}\sum_I \ln(1+|\chi_I|^2) - \frac{3N}{16\pi}\ln v \tag{5.19}$$

from which one can readily find the 1×1 $\mathcal{G}$ matrix:

$$\mathcal{G} = -\frac{\partial\rho}{\partial v} = \frac{3}{16\pi}\sum_I (v + \pi\gamma)^{-1} \tag{5.20}$$

All ingredients for writing down the HEFT are now available. The chiral part of the bosonic lagrangian for the effective low-energy theory of the KW model is given by

$$\mathcal{L} = -\pi\mathcal{G}^{-1}\nabla\rho \wedge \star\nabla\bar{\rho} - 2\pi\sum_I J_{i\bar{j}}\, dz_I^i \wedge \star d\bar{z}_I^{\bar{j}} \tag{5.21}$$

where of course $J_{i\bar{j}} = \partial_i\bar{\partial}_j k(z,\bar{z};v)$ is the metric tensor of the resolved conifold. The expressions here are deceivingly simple: both $\mathcal{G}$ and J depend on the modulus v which must be understood as a function of the chiral field ρ by inversion of the Legendre transform. The connection for the covariant derivative, being a simple derivative of the Kähler potential, can also be explicited:

$$\mathcal{A}_i dz^i = (4v + 4\pi\gamma)^{-1}\left(\frac{2v + \pi\gamma}{\pi(1+|\chi|^2)}\bar{\chi}d\chi - \frac{\gamma\left(\bar{U}dU + \bar{Y}dY\right)}{|U|^2 + |Y|^2}\right) \tag{5.22}$$

Chapter 6

The $Y^{(2,0)}$ HEFT

6.1 Kähler form

The general Calabi-Yau deformation of the $Y^{2,0}$ cone is already well-known in real coordinates as:

$$ds^2 = \kappa^{-1}(r)dr^2 + \frac{1}{9}\kappa(r)r^2(d\psi + \cos\theta_L d\phi_L + \cos\theta_R d\phi_R)^2 + \frac{1}{6}r^2 d\Omega_L^2 + \frac{1}{6}(r^2 + a^2)d\Omega_R^2 \tag{6.1}$$

$$\kappa(r) = \frac{1 + \frac{9a^2}{r^2} - \frac{b^6}{r^6}}{1 + \frac{6a^2}{r^2}} \tag{6.2}$$

with a, b the two unique real moduli. The topology is that of an $\mathbb{R}^2$ bundle over $\mathbb{S}^2 \times \mathbb{S}^2$.

For the purpose of building the effective theory, however, this metric must be rewritten in a complex chart. To that end, we try to find the general CY metric on a $\mathbb{C} \to \mathbb{CP}^1 \times \mathbb{CP}^1$ bundle; on the spheres of the base we take the round metric, given by the Kähler forms j^L and j^R. It's easy to verify explicitly that, given any set of complex coordinates on the base (y_L, y_R),

$$j^L \wedge j^R = e^{-\Lambda k} dy^L \wedge dy^R \wedge d\bar{y}^L \wedge d\bar{y}^R \tag{6.3}$$

with $k = k^L + k^R$ the total base potential, and for some Λ depending on the overall size of the spheres (for unit radius, $\Lambda = 1$).

We also introduce the function t of the fibral coordinate ζ as

$$t = |\zeta|^2 e^{\Lambda k} \tag{6.4}$$

We then start from the following ansatz for the Kähler potential:

$$k_X = f(t) + \alpha k^L + \tilde{\alpha} k^R \tag{6.5}$$

where $\alpha, \tilde{\alpha}$, controlling the volume at $t = 0$ of the base 2-spheres, parametrize the Ricci-flat Kähler resolutions of the cone. We are now set to prove that there is always an $f(t; \alpha, \tilde{\alpha})$ that makes the metric Ricci-flat.

The corresponding Kähler form is straightforward:

$$J = A^L j^L + A^R j^R + ie^{\Lambda k}(f' + tf'')(d\zeta + \Lambda\zeta\partial k) \wedge (\text{c.c.}) \tag{6.6}$$

with $A^L = \alpha + \Lambda t f'(t)$ and $A^R = \tilde{\alpha} + \Lambda t f'(t)$. This is more simply $J = J_M + M e^3 \wedge \bar{e}^{\bar{3}}$, where J_M is the purely basal part, $e^3 = d\zeta + \Lambda\zeta\partial k$ and M is a scalar factor. The volume form is then clearly

$$J \wedge J \wedge J = 3A^L A^R M\, j^1 \wedge j^2 \wedge e^3 \wedge \bar{e}^{\bar{3}} \tag{6.7}$$

as all other terms in the cube vanish. Since the volume form is $\sqrt{\det g}\, d\Omega \wedge \bar{\Omega}$, with $\Omega = d\zeta \wedge dy^L \wedge dy^R$, and the Ricci tensor for a Kähler space is proportional to $\partial\bar{\partial} \ln \det g$, then the condition for Ricci-flatness is equivalent to the prefactor of $\Omega \wedge \bar{\Omega}$ in $J \wedge J \wedge J$ being constant, that is to say

$$(\alpha + \Lambda t f')(\tilde{\alpha} + \Lambda t f') \frac{d}{dt}(\Lambda t f') = c \tag{6.8}$$

or, having defined $y = \Lambda t f'$,

$$(\alpha + y)(\tilde{\alpha} + y) y' = c \tag{6.9}$$

Since $f(t)$ must be regular as $t = 0$, and $f' = \frac{y}{\Lambda t}$, it must be that y goes to zero at least as fast as t as $t \to 0$; this condition eliminates the freedom from the constant of integration for equation 6.8. The constant c on the other hand can be readily reabsorbed into a t rescaling. Therefore there should be a unique y (and so a unique f up to unconsequential constant shifts) that gives a Ricci-flat metric. Let us see this explicitly: we integrate 6.8 to obtain

$$\frac{y^3}{3} + \frac{\alpha+\tilde{\alpha}}{2}y^2 + \alpha\tilde{\alpha}y = ct + d \tag{6.10}$$

And then the regularity condition $y(0) = 0$ is satisfied with $d = 0$, and this cubic equation for y is immediately seen to have one single real solution for any positive values of α^i, c.

Before exhibiting the explicit form of $y(t; \alpha, \tilde{\alpha})$, let us express the Kähler form in terms of y and show it's actually equal to the real-coordinate metric 6.1. We have

$$J = (\alpha + y)j^1 + (\tilde{\alpha} + y)j^2 + \frac{ie^{\Lambda k}}{\Lambda}y' e^3 \wedge \bar{e}^{\bar{3}} \tag{6.11}$$

$$= (\alpha + y)j^1 + (\tilde{\alpha} + y)j^2 + \frac{ie^{\Lambda k}\, c}{\Lambda(\alpha + y)(\tilde{\alpha} + y)} e^3 \wedge \bar{e}^{\bar{3}} \tag{6.12}$$

Now, we parametrize the fiber as $\zeta = e^{-\Lambda k/2}t^{1/2}e^{i\psi}$, and the 2-spheres with spherical coordinates θ_i, ϕ_i which fixes $\Lambda = 1$. Then the metric corresponding to J is

$$ds^2 = A^L d\Omega_L^2 + A^R d\Omega_R^2 + \frac{y'}{t}\left(\frac{dt^2}{4} + t^2(d\psi + \sigma)^2\right) \tag{6.13}$$

Where $\sigma = -i\frac{\Lambda}{2}(\partial k - \bar{\partial}k)$. But the $t - \psi$ part is simply

$$ds^2 = \frac{1}{4y't}dy^2 + (y't)(d\psi + \sigma)^2 \tag{6.14}$$

Exploiting both 6.8 and its integrated form 6.10 we rewrite

$$y't = \frac{1}{A^L A^R}\left(\frac{y^3}{3} + \frac{\alpha+\tilde{\alpha}}{2}y^2 + \alpha\tilde{\alpha}y\right) \tag{6.15}$$

$$= 3cr^2 \frac{1 + \frac{3}{2}\frac{\tilde{\alpha}-\alpha}{r^2} + \frac{\alpha^2(\alpha-3\tilde{\alpha})}{2r^6}}{1 + \frac{\tilde{\alpha}-\alpha}{r^2}} \tag{6.16}$$

$$= 3cr^2\kappa(r) \tag{6.17}$$

provided we make the identifications

$$a^2 = \frac{1}{6}(\tilde{\alpha}-\alpha) \qquad b^6 = \frac{\alpha^2(3\tilde{\alpha}-\alpha)}{2} \tag{6.18}$$

The final coordinate change to the r coordinate is then given by $r^2 = A^L = y + \alpha$ - note this renders the inherent symmetry between the left and right 2-cycles non-manifest. The resulting metric, after taking $c = 1/3$, is precisely 6.1. Thus, as the latter is the most general Calabi-Yau deformation of the $Y^{2,0}$ cone, we have to conclude that the two-parameter family of metrics 6.11 in complex coordinates coincides with it.

Now we're left with solving for the explicit form of y. Switching temporarily to $z = y + (\alpha+\tilde{\alpha})/2$ equation 6.10 is brought into depressed form:

$$z^3 - \frac{3}{4}(\alpha-\tilde{\alpha})^2 = ct + D \tag{6.19}$$

Where

$$D = \frac{1}{12}(-\alpha^3 + 3\alpha^2\tilde{\alpha} + 3\alpha\tilde{\alpha} - \tilde{\alpha}^3) = \frac{b^6 - 36a^6}{3} \tag{6.20}$$

So that the explicit solution for y is

$$z = |\alpha-\tilde{\alpha}|C_{1/3}\left(12\frac{ct+D}{|\alpha-\tilde{\alpha}|^3}\right) \tag{6.21}$$

$$y = z - \frac{\alpha+\tilde{\alpha}}{2} \tag{6.22}$$

where we defined the function $C_{1/3} = \mathrm{ch}(1/3\ \ \mathrm{ch}^{-1}(x))$; that 6.22 solves

6.19 can be readily verified by means of the trigonometric identity $\mathrm{ch}(3x) = 4\,\mathrm{ch}^3(x) - 3\,\mathrm{ch}(x)$.

Fixing $c = 1/3$ for future convenience and introducing the notation $\delta = \alpha - \tilde{\alpha}$, $\sigma = \alpha + \tilde{\alpha}$, the Kähler form is explicitly given by

$$J(\sigma, \delta) = \left(z + \frac{\delta}{2}\right) j^1 + \left(z - \frac{\delta}{2}\right) j^2 + ie^k z' e^3 \wedge \bar{e}^{\bar{3}} \tag{6.23}$$

$$z(t; \sigma, \delta) = \delta\, C_{1/3}\left(\delta^{-3}\left(4t + \frac{\sigma(3\delta^2 - \sigma^2)}{2}\right)\right) \tag{6.24}$$

or equivalently, in terms of the y function:

$$J(\sigma, \delta) = (y + \alpha)j^1 + (y + \tilde{\alpha})j^2 + ie^k y' e^3 \wedge \bar{e}^{\bar{3}} \tag{6.25}$$

Unfortunately, a closed-form expression for the Kähler potential seems impossible. The function $f(t)$ can nevertheless be written in integral form, as such:

$$f(t; \sigma, \delta) = \int_0^t d\ln t' y(t') \tag{6.26}$$

6.2 Kähler moduli

We will construct the two harmonic forms $\hat{\omega}$ and $\tilde{\omega}$ by differentiating J directly with respect to the relevant moduli.

All forms obtained in this way from J are primitive: in the dy^1, dy^2, e^3 basis the Kahler form is $\mathrm{diag}(A^L j^1_{1\bar{1}}, A^R j^2_{2\bar{2}}, ie^k y')$ so the contraction of J with any derivative $\partial_x J$ of it with respect to a parameter is

$$J^{a\bar{b}}\partial_x J_{a\bar{b}} = \mathrm{Tr}\left((J_{a\bar{b}})^{-1}\partial_x J_{a\bar{b}}\right) = A^{-1}\partial_x A + \tilde{A}^{-1}\partial_x \tilde{A} + (y')^{-1}\partial_x y' = \partial_x \ln\left(A\tilde{A}y'\right) \tag{6.27}$$

which vanishes thanks to the Ricci-flatness equation.
It's easy to see that the modulus $\hat{v}$ relative to $\hat{\omega}$ must be proportional to

the sum σ of the basal volumes, since the corresponding harmonic form $\frac{\partial J}{\partial \sigma}$ is normalizable. To show this, we consider the asymptotic behaviour of the Kähler form as $t \to \infty$. Defined

$$T := \left(4t + \frac{\sigma(3\delta^2 - \sigma^2)}{2}\right) \tag{6.28}$$

we have

$$z \sim T^{1/3} \qquad z' \sim T^{-2/3} \tag{6.29}$$

$$\frac{\partial z}{\partial \sigma} \sim \frac{\partial T}{\partial \sigma} T^{-2/3} \qquad \frac{\partial z'}{\partial \sigma} \sim \frac{\partial T}{\partial \sigma} T^{-5/3} \tag{6.30}$$

where we have omitted constant factors. Therefore

$$J \sim \left(T^{1/3} + \frac{\delta}{2}\right) j^1 + \left(T^{1/3} - \frac{\delta}{2}\right) j^2 + ie^k T^{-2/3} e^3 \wedge \bar{e}^{\bar{3}} \tag{6.31}$$

and

$$\frac{\partial J}{\partial \sigma} \sim \frac{\partial T}{\partial \sigma}\left(T^{-2/3} j^1 + T^{-2/3} j^2 + ie^k T^{-5/3} e^3 \wedge \bar{e}^{\bar{3}}\right) \tag{6.32}$$

so that the norm $\left\|\frac{\partial J}{\partial \sigma}\right\|^2 = \frac{\partial J}{\partial \sigma} \wedge \star \frac{\partial J}{\partial \sigma} \sim t^{-2} \sim r^{-12}$, which is integrable[1].
By contrast, the remaining harmonic form must be nonrenormalizable. For example, differentiating with respect to δ, we obtain

$$\frac{\partial J}{\partial \delta} \sim j^1 - j^2 + ie^k \frac{\partial T}{\partial \delta} T^{-4/3} e^3 \bar{e}^{\bar{3}} \tag{6.33}$$

with norm $\left\|\frac{\partial J}{\partial \delta}\right\| \sim t^{-2/3} \sim r^{-4}$, not integrable.

We now consider the two two-cycles C^1, C^2 given by the two basal $\mathbb{CP}^1$ respectively. We note that

$$\int_{C^1} \frac{\partial J}{\partial \alpha} = \int_{C^1} \left(\left.\frac{\partial A}{\partial \alpha}\right|_{t=0} j^1\right) = \int_{C^1} j^1 = 4\pi \tag{6.34}$$

[1] Asymptotically, as $t \gg \alpha, \tilde{\alpha}$, the metric reduces to the sharp cone $dr^2 + r^5 ds_5^2$, and in this regime $t \propto r^6$; therefore a function on X is integrable if it decays faster than $r^6 \sim t^1$.

where we exploited the fact that $y(t = 0) = 0$, as it's clear from 6.10. Identically one shows $\int_{C^1} \frac{\partial J}{\partial \alpha} = \int_{C^1} \frac{\partial J}{\partial \tilde{\alpha}} = 0$ and $\int_{C^2} \frac{\partial J}{\partial \tilde{\alpha}} = 4\pi$. These are also the intersection number of C^L, C^R with the Poincaré dual 4-cycles of these forms; since the two-cycles form a basis, a 4-dual with the same intersection numbers will necessarily belong the dual class. We consider the (noncompact) 4-cycles D^1, D^2 given respectively by the fibres of C^1, C^2. Since

$$D^i \cdot C^j = \epsilon^{ij} \tag{6.35}$$

then we can identify the Poincaré duals:

$$-\frac{1}{4\pi}\frac{\partial J}{\partial \alpha} \leftrightarrow D^2 \qquad\qquad -\frac{1}{4\pi}\frac{\partial J}{\partial \tilde{\alpha}} \leftrightarrow D^1 \tag{6.36}$$

Then a useful normalization for $\hat{\omega}$ would be

$$\hat{\omega} = \omega_1 = \frac{1}{2\pi}\left(\frac{\partial J}{\partial \sigma}\right) = \frac{\partial J}{\partial \hat{v}} \qquad\qquad \hat{v} := 2\pi\sigma \tag{6.37}$$

which makes it so $\int_{C^i} \omega = 2$ is integer. The dual to $\hat{\omega}$ is $-2(D^1 + D^2) =: E$; it's easy to show this is actually the base $\mathbb{CP}^1 \times \mathbb{CP}^1$.
Similarly, we choose

$$\tilde{\omega} = \omega_2 = \frac{1}{2\pi}\left(\frac{\partial J}{\partial \delta}\right) = \frac{\partial J}{\partial \tilde{v}} \qquad\qquad \tilde{v} := 2\pi\delta \tag{6.38}$$

dual to the 4-cycle $F = 2(D^1 - D^2)$.

This choice for the harmonic 2-forms and the moduli v^a allows for easy computation of the intersection numbers:

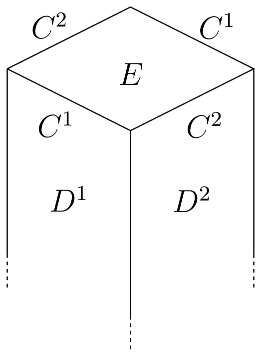

Figure 6.1: Schematic representation of the manifold X as a line bundle, with the relevant 2- and 4-cycles.

$$I_0 = \int \hat{\omega} \wedge \hat{\omega} \wedge \hat{\omega} = E \cdot E \cdot E \tag{6.39}$$

$$I_1 = \int \hat{\omega} \wedge \hat{\omega} \wedge \tilde{\omega} = E \cdot E \cdot F \tag{6.40}$$

$$I_2 = \int \hat{\omega} \wedge \tilde{\omega} \wedge \tilde{\omega} = E \cdot F \cdot F \tag{6.41}$$

To evaluate these expression, we first note that $E \cap D^i = C^i$, and that[2] $E \cdot C^i = -2$.

$$I_0 = E \cdot E \cdot E = -2E \cdot E \cdot (D^1 + D^2) = -2E \cdot (C^1 + C^2) = 8 \tag{6.42}$$

$$I_1 = E \cdot E \cdot F = 0 \tag{6.43}$$

$$I_2 = E \cdot F \cdot F = -8 \tag{6.44}$$

6.3 Chiral fields and effective Lagrangian

The HEFT will feature the following chiral fields:

[2]This intersection can be computed as follows. We represent E with the set $\{\zeta = 0\}$, and C^1 with the set of points with $y^2 = 0$, and $\zeta = y^1$ if $|y^1| < 1$, $\zeta = 1/(y^1)^*$ if $|y^1| > 1$ ($y \in \bar{\mathbb{C}}$). With this embedding the cycles are in general position and the intersection is given by the two points $y^2 = 0 = \zeta$, $y^1 = 0, \infty$.

$$
\begin{array}{r|ll}
z_I^i & = (y_I^1, y_I^2, \zeta_I) & \text{D3-brane positions on } X \\
\hat{\rho} = \rho_1 & \text{dual to } \hat{v} & \text{4-cycle blowup deformation of } X \\
\tilde{\rho} = \rho_2 & \text{dual to } \tilde{v} & \text{2-cycle blowup deformation of } X \\
\beta & & \text{axio-dilaton}
\end{array}
$$

and the following non-dynamical chiral parameter:

$$
\begin{array}{r|l}
\lambda & \text{nonrenormalizable axio-dilaton deformation}
\end{array}
$$

The chiral fields $\rho_a = (\hat{\rho}, \tilde{\rho})$ are related to the moduli $v_a = (\hat{v}, \tilde{v})$ by a type of Legendre transform; we will only need the precise form of the real part of $\rho_a(v_a)$:

$$
\operatorname{Re}\hat{\rho} = \frac{1}{2}\sum_I \hat{\kappa}(z_I, \bar{z}_I; v) - \frac{1}{2\operatorname{Im}\tau} I_0 (\operatorname{Im}\beta)^2 - \frac{1}{\operatorname{Im}\tau} I_1 \operatorname{Im}\beta \operatorname{Im}\lambda \tag{6.45}
$$

$$
\operatorname{Re}\tilde{\rho} = \frac{1}{2}\sum_I \tilde{\kappa}(z_I, \bar{z}_I; v) - \frac{1}{2\operatorname{Im}\tau} I_1 (\operatorname{Im}\beta)^2 - \frac{1}{\operatorname{Im}\tau} I_2 \operatorname{Im}\beta \operatorname{Im}\lambda \tag{6.46}
$$

where $\kappa_a(z_I, \bar{z}_I; v) = (\hat{\kappa}, \tilde{\kappa})$ are defined as the potentials that generate the ω_a, as in

$$
\omega_a = i\partial\bar{\partial}\kappa_a \tag{6.47}
$$

and also satisfy the following condition:

$$
\frac{\partial \kappa_a}{\partial v_a} \sim r^{-k} \sim t^{-k/6}, \quad k \geq 2 \tag{6.48}
$$

We are now able to present the bosonic part of the effective Lagrangian. There is first of all a decoupled sector of N copies of $U(1)$ SYMs (assuming we are in a generic point where no z_I coincide). Then, the rest of the bosonic effective Lagrangian describes the chiral fields listed above:

$$\mathcal{L}_{\text{chiral}} = -\pi \mathcal{G}^{ab} \nabla\rho_a \wedge \star\nabla\bar{\rho}_b - 2\pi \sum_I J_{i\bar{j}} dz^i d\bar{z}^{\bar{j}} - \frac{\pi\mathcal{M}}{\operatorname{Im}\tau} d\beta \wedge \star d\bar{\beta} \tag{6.49}$$

where the kinetic factors are computable as follows:

$$\mathcal{G}_{ab} = \int_X e^{-4A}\omega_a \wedge \star\omega_b = -\int_X e^{-4A} J \wedge \omega_a \wedge \omega_b = -\frac{\partial \operatorname{Re}\rho_a}{\partial v_b} \tag{6.50}$$

$$\mathcal{M} = \int_X \hat{\omega} \wedge \star\hat{\omega} = -\int J \wedge \hat{\omega} \wedge \hat{\omega} = -\hat{v} I_0 \tag{6.51}$$

($\mathcal{G}^{ab}$ being the inverse matrix of $\mathcal{G}_{ab}$) and the covariant derivative ∇ is

$$\nabla = d - \mathcal{A}^I_i dz^i_I - \frac{i}{\operatorname{Im}\tau}(I_0 \operatorname{Im}\beta + I_1\lambda) d\beta \tag{6.52}$$

$$\mathcal{A}^I_i = \frac{\partial\hat{\kappa}(z_I, \bar{z}_I; v)}{\partial z^i_I} \tag{6.53}$$

To compute these coefficients $\mathcal{G}, \mathcal{M}, \mathcal{A}$ we first find the form of the κ potentials.

6.4 κ potentials

Since $\frac{\partial k_X}{\partial v^a}$ generates $\frac{\partial J}{\partial v^a} = \omega_a$, it must be that $\kappa_a = \frac{\partial k_X}{\partial v^a} + h(v)$ with $h(v)$ an arbitrary function of the moduli which would then be fixed as to satisfy the condition 6.48 (up to an additive constant). However, as will be seen shortly, $\frac{\partial k_X}{\partial v^a}$ itself satisfies the asymptotic condition, so that $h(v)$ is actually a constant, which we will omit.

Recalling $k_X = f(t) + \frac{\sigma+\delta}{2}k^1 + \frac{\sigma-\delta}{2}k^2$, and $f(t) = \int_0^t d\ \ln(t')y(t')$, we find

$$2\pi\hat{\kappa} = \frac{\partial k_X}{\partial\sigma} = \left(\int d\ \ln t' \frac{\partial y}{\partial\sigma}\right) + \frac{1}{2}k^1 + \frac{1}{2}k^2 \tag{6.54}$$

$$2\pi\tilde{\kappa} = \frac{\partial k_X}{\partial\delta} = \left(\int d\ \ln t' \frac{\partial y}{\partial\delta}\right) + \frac{1}{2}k^1 - \frac{1}{2}k^2 \tag{6.55}$$

so that the derivatives of the κ potentials become

$$\frac{\partial \kappa_a}{\partial v^b} = \frac{\partial^2 k_X}{\partial v^a \partial v^b} = \frac{1}{4\pi^2}\int_0^t d\,\ln(t') \begin{pmatrix} \frac{\partial^2 y}{\partial \sigma^2} & \frac{\partial^2 y}{\partial \sigma \partial \delta} \\ \\ \frac{\partial^2 y}{\partial \sigma \partial \delta} & \frac{\partial^2 y}{\partial \delta^2} \end{pmatrix}_{ab} \tag{6.56}$$

The explicit forms of the second derivatives of the y function, rather convoluted, are listed in 6.5. It's clear they have at most asymptotic behaviour $\sim t^{-2/3}$, which will be the same as that of their $\int d\,\ln t'$, so that the κ_a defined above satisfy 6.48 and no addition of a function of the moduli $h(v)$ is necessary.

Then, this allows immediately for the computation of the $\mathcal{G}_{ab}$ matrix:

$$\mathcal{G}_{ab} = -\frac{\partial\,\mathrm{Re}\,\rho_a}{\partial v^b} = -\sum_I \frac{\partial \kappa_a(z_I, \bar{z}_I; v)}{\partial v^b} = -\frac{1}{4\pi^2}\sum_I \int_0^{t_I} d\,\ln t' \frac{\partial^2 y}{\partial v^a \partial v^b}(t'; v) \tag{6.57}$$

again resting on the explicit form of the second derivatives of y. The integrals are not solvable in closed form. The matrix will always be invertible and its inverse $\mathcal{G}^{ab}$ is the kinetic matrix for the ρ fields.

The connection $\mathcal{A}^I_{ai}$ instead can be found more explicitly. We treat the $z_I^3 = \zeta_I$ and $z_I^{1,2} = y^{1,2}$ cases separately.

$$\mathcal{A}^i_{aI} = \frac{\partial^2 k_X}{\partial \zeta_I \partial v^a} = \frac{\partial^2 f}{\partial \zeta_I \partial v^a} \tag{6.58}$$

but, recalling $t = |\zeta|^2 e^k$, $\frac{\partial f(t)}{\partial \zeta} = \bar{\zeta} e^k f'(t) = \bar{\zeta} e^k y(t)/t = (\bar{\zeta})^{-1} y(t)$ so that this is simply

$$= \bar{\zeta}^{-1} \frac{\partial y}{\partial v^a} \tag{6.59}$$

and

$$\mathcal{A}^3_{1I} = \frac{1}{2\pi}\bar{\zeta}^{-1}\frac{\partial y}{\partial \sigma} \tag{6.60}$$

$$\mathcal{A}^3_{2I} = \frac{1}{2\pi}\bar{\zeta}^{-1}\frac{\partial y}{\partial \delta} \tag{6.61}$$

The $i = 1, 2$ components, instead, are

$$\mathcal{A}^i_{aI} = \frac{\partial^2 k_X}{\partial y^i_I \partial v^a} = \frac{\partial^2(\alpha^i k^i)}{\partial y^i_I \partial v^a} = \frac{\partial \alpha^i}{\partial v^a}\frac{\partial k^i}{\partial y^i_I} \tag{6.62}$$

(no summation on i is implied), so essentially:

$$\mathcal{A}^1_{1I} = \mathcal{A}^1_{2I} = \frac{1}{4\pi}\frac{\partial k^1}{\partial y^1} \tag{6.63}$$

$$\mathcal{A}^2_{1I} = -\mathcal{A}^2_{2I} = \frac{1}{4\pi}\frac{\partial k^2}{\partial y^2} \tag{6.64}$$

$$\tag{6.65}$$

6.5 Derivatives of y

We list the explicit derivatives of the $y(t)$ function required for the formulation of the HEFT. We recall y is

$$y(t; \sigma, \delta) = \delta\, C_{1/3}\left(\delta^{-3}T\right) - \frac{\sigma}{2} \tag{6.66}$$

where $T := 4t + \frac{\sigma(3\delta^2-\sigma^2)}{2}$ and $C_{1/3}(x) = \cosh\left(\frac{1}{3}\cosh^{-1}(x)\right)$. We will make use in the following table of the notation C, C', C'', ... to refer to the zeroth, first, second, ... derivatives of the $C_{1/3}$ function evaluated always at $\delta^{-3}T$.

The first derivatives are:

$$\frac{\partial y}{\partial \sigma} = \delta^{-2}\frac{\partial T}{\partial \sigma}C' - \frac{1}{2} \tag{6.67}$$

$$\frac{\partial y}{\partial \delta} = C + \left(-3\delta^{-3}T + \delta^{-2}\frac{\partial T}{\partial \delta}\right)C' \tag{6.68}$$

And the second derivatives:

$$\frac{\partial^2 y}{\partial \sigma^2} = \delta^{-2}\left(\frac{\partial^2 T}{\partial \sigma^2}C' + \left(\frac{\partial T}{\partial \sigma}\right)^2 \delta^{-3}C''\right) \tag{6.69}$$

$$\frac{\partial^2 y}{\partial \delta \partial \sigma} = \left(-2\delta^{-3}\frac{\partial T}{\partial \sigma} + \delta^{-2}\frac{\partial^2 T}{\partial \sigma \partial \delta}\right)C' + \delta^{-2}\frac{\partial T}{\partial \sigma}\left(-3\delta^{-4}T + \delta^{-3}\frac{\partial T}{\partial \delta}\right)C'' \tag{6.70}$$

$$\frac{\partial^2 y}{\partial \delta^2} = \left(-4\delta^{-3}\frac{\partial T}{\partial \delta} + \delta^{-2}\frac{\partial^2 T}{\partial \delta^2}\right)C' + \delta^{-1}\left(-3\delta^{-3}T + \delta^{-2}\frac{\partial T}{\partial \delta}\right)C'' \tag{6.71}$$

The asymptotic behaviour can be read easily by noting $T \sim t$, $C \sim t^{1/3}$, $C' \sim t^{-2/3}$, $C'' \sim t^{-5/3}$, and that derivatives of T do not depend on t.

Appendix A

Appendix

A.1 AdS space

Anti-de Sitter n-space is best understood as the Lorentzian analogue of hyperbolic n-space. It can be built by considering the following locus in the mixed-signature space $\mathbb{R}^{2,n-1}$:

$$x^\mu x_\mu = -(t^1)^2 - (t^2)^2 + \sum_{i=1}^{n-1} (x^i)^2 = -R^2 \tag{A.1}$$

which is reminiscent of the embedding of hyperbolic n-space in $\mathbb{R}^{1,n}$:

$$x^\mu x_\mu = -t^2 + \sum_{i=1}^{n} (x^i)^2 = -R^2 \tag{A.2}$$

Equation A.1 is explicitly preserved by $SO(2, n-1)$, and this group acts transitively on it, so that the locus inherits a Lorentzian metric from the ambient Minkowski space with that same symmetry group. This means the locus is a maximally symmetric space, having the same number of symmetries as $\mathbb{R}^{1,n-1}$ since $\dim SO(2, n-1) = \dim (\mathbb{R}^n \rtimes SO(1, n))$. (To press on with the analogy, in the Riemannian case $\mathbb{H}^n$ has the same number of Killing vectors as $\mathbb{R}^n$ since $\dim SO(1, n) = \dim (\mathbb{R}^n \rtimes SO(n))$).

The locus has constant negative scalar curvature (using S for the Ricci scalar to avoid confusion with the R radius introduced above):

$$S = -\frac{n(n-1)}{R^2} \tag{A.3}$$

However, the locus built above is not suitable to be used as a spacetime for a reasonable physical theory, as it contains closed timelike curves (CTCs), signaling a pathological causal structure. An example of CTC is the unit circle in the t^1t^2 plane. It's possible however to consider the covering space of the locus, which will be what we will refer to as anti-de Sitter n-space, AdS$_n$. The covering space is again a maximally symmetric space, but it's now simply-connected and CTC-free.

AdS, similarly to dS, admits multiple useful coordinate charts. The Poincaré chart is the analogue of the Poincaré half plane model, and the metric is:

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + dx^\mu dx_\mu\right) \tag{A.4}$$

where $z > 0$, $x^\mu \in \mathbb{R}^{1,n-2}$, and $dx^\mu dx_\mu$ is the standard metric on $\mathbb{R}^{1,n-2}$. The Poincaré chart, unlike the Riemannian case, is not global and only maps a particular wedge of the full AdS. A global chart would be given by the following coordinates, accordingly called global coordinates or cylindrical coordinates:

$$ds^2 = R^2\left(-\cosh^2\chi\, d\tau^2 + d\chi^2 + \sinh^2\chi\, d\Omega^2\right) \tag{A.5}$$

With $d\Omega^2$ the line element on $\mathbb{S}^{n-2}$. Note that constant τ slices are copies of $\mathbb{H}^{n-1}$. Remapping the radial coordinate as $d\chi = d\rho/\cos\rho$ to a finite range $(0 \leq \rho \leq \pi/2)$ this can also be rewritten as

$$ds^2 = R^2\frac{1}{\cos^2\rho}\left(-dt^2 + d\rho^2 + \sin^2\rho d\Omega^2\right) \tag{A.6}$$

A.2 Conformal boundary and symmetries

The last set of coordinates A.6 are a starting point for building the Penrose diagram of AdS. For fixed Ω_i the t,ρ part of the metric is sent to the flat metric by multiplication with the conformal factor $\cos^2\rho$. AdS is thus represented as an infinite solid cylinder.

We can read the induced topology and metric on the boundary, with the caveat that the conformal factor was arbitrary (provided it was such the metric did not diverge), and thus the boundary's metric will be defined up to a conformal rescaling - we can only identify a natural conformal class for the boundary. This will prove to have physical relevance as possible holographic duals will be conformal.

The topology of the boundary is therefore $\mathbb{S}^{n-2} \times \mathbb{R}$ and a representative of the conformal class is given by setting $\rho = \pi/2$:

$$ds^2 = dt^2 - d\Omega^2 \tag{A.7}$$

which is a Lorentzian metric. The conformal boundary of AdS is itself a spacetime; this is a nontrivial fact which has to be compared with the other constant-curvature manifolds of the same signature: the boundary of Minkowski space $\mathbb{R}^{1,n-1}$ has a vanishing (null) metric, being composed of null past and future, while the positive curvature case, de Sitter, has two spacelike boundaries in the infinite past and future. The relevance of this for the realization of holography should be evident. Only the negative curvature case seems to be able to naturally incorporate a Lorentzian structure on the boundary.

It will be much more useful for the application to holography to consider the boundary in the form it comes out from the Poincaré patch. This is located at $z = 0$ and is only a part of the full boundary. Taking the metric A.4 and factor a conformal z^2 we just obtain

$$ds^2 = x^\mu x_\mu \tag{A.8}$$

that is, the boundary is (locally) Minkowski $(n-2)$-space. This will be our preferential choice of representative metric.

We now turn to the description of the interplay between the bulk's and the boundary's symmetries. Essentially, isometries of AdS will induce conformal transformations on its boundary. As we've seen through its construction, the isometry group of AdS is $SO(2, n-1)$, this also coincides with the conformal group on $\mathbb{R}^{1,n-2}$.

+altre banalità di geometria

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