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Holographic effective field theories: a case study

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Anno accademico 2015/2016
bozza compilata November 14, 2016

Abstract

The identification of the low-energy effective field theory associated with a given microscopic strongly interacting theory constitutes a fundamental problem in theoretical physics, which is particularly hard when the theory is not sufficiently constrained by symmetries. Recently, a new approach has been proposed, which addresses this problem for a large class of four-dimensional minimally supersymmetric strongly coupled superconformal field theories, admitting a dual weakly coupled holographic description in string theory. This approach provides a precise prescription for the holographic derivation of the associated effective field theories. The aim of the thesis is to further explore this approach by focusing on a specific model, whose effective field theory has not been investigated so far. *(modificare abstract alla fine del lavoro.)*

Contents

Chapter 1

Introduction

Strongly-coupled quantum field theories represent canonical examples of physical systems whose study is extremely challenging. Even the question of the mere existence of any interacting QFT in four dimensions from a formal standpoint has not been settled. In addition to this, strong couplings are not amenable to the tools of perturbation theory. The interest in this class of theories stems actually from practical considerations - many of them represent realistic models for physical phenomena, e.g. the theory of strong interaction.

A subset of questions concerns whether a given strongly-interacting theory is described at low energy by an effective local field theory, and if so, what are its degrees of freedom and their precise dynamics. Often, part of the structure of the effective theory is constrained by symmetries, but no general method exists to fix it completely. Recently[21], a novel approach for determining the effective Lagrangian was introduced that makes use of tools from an apparently unrelated area of physics: string theory.

It is remarkable that string theory was originally conceived as a description of hadronic physics, so a low-energy effective theory for what ultimately turned out to be a gauge theory, QCD. When string theory was found to have unsuitable qualities for this application, it was replaced by the theory of quantum chromodynamics - however it also proved to be effective for solving a seemingly unrelated problem of fundamental physics: quantizing gravity. Since then, string theory blossomed into a vast and rich field reaching into numerous areas of mathematics and physics, and of course a candidate for a "Theory of Everything" describing the entirety of fundamental physics.

Among the most unexpected discoveries in strings, made decades after their conception, is a series of unusual exact equivalences between string theories set in particular ten-dimensional backgrounds and four-dimensional gauge QFTs. More generally, one finds families of exact equivalences between local quantum field theories and higher-dimensional theories containing gravity, which are termed "holographic". This explains the original partial success of strings in modeling strong interactions, assuming that some or perhaps most gauge theories have or can be approximated as having a holographic description as a ten dimensional theory involving strings. This roundtrip has therefore brought strings back to strongly-coupled gauge theories. Various aspects, qualitative and most importantly quantitative, of QFTs can be studied directly by means of their holographic string dual, a gravitational theory, if it exists.

The first and most important case of such a duality[17] equates IIB superstring theory set on the background

$$\mathrm{AdS}_5 \times \mathbb{S}^5 \tag{1.1}$$

(the "bulk") with maximally supersymmetric Yang-Mills theory on $\mathbb{R}^{1,3}$ (the "boundary"), which is a conformal field theory. The denomination of "AdS/CFT" (anti-de Sitter / conformal field theory) correspondence for holographic dualities stems from this (even though cases either with no AdS geometry or not conformal are known). One direction in which to generalize this construction is to replace $\mathbb{S}^5$ with other compact 5-manifolds Y_5. This yields dualities involving more complex and interesting field theories, less constrained by symmetries; therefore this thesis will be focused on this class of correspondences.

Actually, since string theory in general is very challenging to study, AdS/CFT only becomes truly useful in terms of describing the dynamics of the CFT if string theory can be approximated by a weakly coupled effective field theory of its own, supergravity. This limit corresponds to the CFT being strongly-coupled and having a large number of colours. Therefore the regime accessible through holography is precisely the strongly-coupled region where the gauge theory would be normally impossible to investigate.

Recently, a novel approach was introduced[21] for determining the effective theory to such duals of $\mathrm{AdS}_5 \times Y_5$. A procedure for identifying the degrees of freedom and the Lagrangian for the effective low-energy theory for a class of gauge theories with $\mathrm{AdS}_5 \times Y_5$ holographic duals is provided, by expanding the supergravity action on the dual bulk geometry. Interestingly, these are somewhat special in that they include models of minimal supersymmetry ($\mathcal{N} = 1$), which makes for more realistic but by converse less constrained theories than typical holographic field theories, with higher supersymmetry. The ability to pinpoint the exact effective Lagrangian is then particularly noteworthy.

The original contribution in this work is the specialization of this construction to a specific field theory, the $Y^{2,0}$ theory, a strongly-coupled superconformal quiver theory for which we will therefore fix the exact effective Lagrangian, entirely through the geometry of the relative string background. This will require necessarily the determination of the general Calabi-Yau deformation of the background in complex coordinates, which was not known previously.

This thesis will be structured as follows. We will first provide a general introduction to IIB superstring theory, D-brane stacks on cones and the resulting gauge field theories, and holography. Then, we will summarize the relevant results and techniques from [21]. Finally, we will present a complete parametrization of the geometry of the $Y^{2,0}$ theory and will apply those results and techniques to identify the exact effective Lagrangian of the field theory.

Chapter 2

IIB superstrings and branes

String theories are quantum theories involving 1-dimensional dynamical objects, the fundamental strings. Therefore, they can be regarded as a generalization of systems of 0-dimensional quantum particles, such as quantum field theories. This apparently innocuous modification results in amazing depth and complexity of the structure of the resulting theories. The celebrated inclusion of Einstein gravity is only a small part of the large diversity of phenomena.

The downside is a significant difficulty in probing the general structure of these theories. Only the behaviour in particular regimes is known, while the overall theory interpolating between these limiting cases is mostly unknown, or at the very least impossible to formulate. Consequently, we will only attempt an introduction to the aspects of string theory relevant to our purposes, namely when strings can be approximated by an effective field theory (and which one), and stable non-perturbative states known as D-branes.

Our interest will be directed towards string theories involving supersymmetry, and in particular a particular variety known as type IIB string theory, set in ten dimensions, which will always be involved in the holographic dualities we will study. In fact, IIB will be both the background theory employed to construct the holographic duality, and one side of the duality itself. Thus, we now provide a summary starting from perturbative superstring theory to present some general aspects of type-IIB strings, including the corresponding effective field theory and D-branes.

2.1 Superstring theory

String theory either does not admit a nonperturbative Lagrangian formulation, or this formulation is unknown. An action functional can only be written upon choosing a perturbative vacuum; since we anticipate a string theory must include gravity, a choice of vacuum will also require a choice of background metric - in the simplest case Minkowski spacetime. The configuration of a string moving in this spacetime (the target M) is then given by specifying the two-dimensional submanifold (the worldsheet W_1) it traces in it[1], the worldsheet. In essence, this coincides with providing an embedding of the worldsheet

$$X^{\mu}(\tau,\sigma) : W_1 \to M \tag{2.1}$$

then quotiented under diffeomorphisms of the coordinates τ, σ on W_1.

With a given choice of background metric the most natural action for a string is the Nambu-Goto action, the worldsheet area:

$$S_{NG}[X] = -T \int_{W^1} d\,\mathrm{vol}_h = -T \int_{W^1} d^2\sigma \sqrt{-h} \tag{2.2}$$

where $h_{ab} = \frac{\partial x^{\mu}}{\partial \sigma^a} \frac{\partial x^{\nu}}{\partial \sigma^b} G_{\mu\nu}$ is the induced metric on the worldsheet from the target space metric $G_{\mu\nu}$ under the embedding given by X^{μ}. T instead is a dimensionful constant called the string tension; in fact it is the only free parameter in string theory. We will also often refer to the entirely equivalent quantity α', the Regge slope. *check convenzioni!*

$$T = \frac{1}{2\pi\alpha'} \tag{2.3}$$

The Nambu-Goto action is very difficult (if not impossible) to quantize. It proves much easier to switch to the classically equivalent Polyakov action:

$$S_B[X, g] = -\frac{T}{2} \int_{W^1} d^2\sigma \sqrt{-g} g^{ab} \partial_a X^{\mu} \partial_b X_{\nu} G_{\mu\nu} \tag{2.4}$$

[1] We note the worldsheet is just the obvious generalization of the concept of worldline of a particle to the case of 1-dimensional strings.

where now g_{ab} is an independent auxiliary field, not the induced metric from the X^μ. The equivalence is readily shown by computing the classical equation of motion for g_{ab} and substituting back into S_B to recover S_{NG}.

There are essentially two[2] different sensible choices for the topology of W_1: either a cylinder, with σ being the periodic variable running around, or a strip, so that σ is limited to an interval $[0, \sigma_1]$. These are respectively the closed and open string. The former is always a closed loop at any given instant in time. The open string instead has two endpoints for which we have to fix boundary conditions. One could choose between either Neumann boundary conditions, meaning

$$\left.\frac{\partial X^\mu}{\partial \sigma}\right|_{\sigma=0,\sigma_1} = 0 \tag{2.5}$$

which is just the constraint that no momentum flows out of the string endpoints, or Dirichlet boundary conditions, which fix

$$X^\mu|_{\sigma=0} = X_0^\mu, \qquad\qquad X^\mu|_{\sigma=\sigma_1} = X_{\sigma_1}^\mu \tag{2.6}$$

where X_0^μ, $X_{\sigma_1}^\mu$ are constants, essentially forcing the string endpoints to a specific spacetime point. In general one could mix $p+1$ Neumann conditions and $D-p-1$ Dirichlet conditions for different values of μ, so that the endpoints are constrained to a p-dimensional submanifold in space, and can move freely within it. Dirichlet conditions evidently break the symmetries of the target spacetime (Poincaré if we choose a Minkowski background) as they specify a preferential frame and submanifold; this symmetry will be recovered when it is recognized that the p-dimensional submanifold to which open strings attach is actually a dynamical object, a Dp-brane (D alluding to Dirichlet). We will return to D-branes after studying the string spectrum.

The Polyakov action displays invariance under worldsheet diffemorphisms

$$\sigma_a \to \sigma'_a(\sigma_a) \tag{2.7}$$

[2]We ignore the question of orientability of the worldsheet, not important for our purposes.

and Weyl transformations:

$$g_{ab} \rightarrow e^{\phi(\sigma)} g_{ab} \tag{2.8}$$

and thus perturbative string theory is naturally a two-dimensional conformal field theory. These symmetries must be quotiented out someway on quantization. The most straightforward way is to eliminate them by fixing a particular gauge and then quantizing (canonical quantization). The three symmetry generators can kill the three degrees of freedom in the metric to fix it to the 2D Minkowski: $g_{ab} = \eta_{ab}$. We get

$$S_B = -\frac{T}{2} \int d^2\sigma \partial_a X^\mu \partial^a X_\mu \tag{2.9}$$

where indices are raised with η^{ab}.

The theory described so far would be what is known as bosonic string theory. There are two issues with bosonic strings: the first is the presence of tachyons in both the open and closed string spectrum, i.e. some string modes will have a negative mass squared, signaling an instability of our choice of perturbative vacuum. The second is that, as the name suggests, there are exclusively bosons in the spectrum, which makes it unsuitable at the very least for phenomenological application. A modification to include supersymmetry can be performed to solve both of these issues and produces a set of string theories with fermions and without tachyons, the superstrings. We will in particular sketch the path from bosonic string theory to the so-called type II superstrings, two slightly different theories IIA and IIB.

There are at least two different approaches to introducing supersymmetry into a string theory. The path followed by the RNS (Ramond-Neveu-Schwarz) formalism is to impose SUSY at the worldsheet level; explicitly, adding fermions ψ^μ to act as superpartners to the bosons X^μ. We follow the derivation in [4]. The action is extended to

$$S = S_B + S_F = -\frac{T}{2} \int d^2\sigma \; \partial_a X^\mu \partial^a X_\mu + \bar{\psi}^\mu \rho^a \partial_a \psi_\mu \tag{2.10}$$

where the $\rho^{1,2}$ are two-dimensional gamma matrices satisfying the Clifford algebra

$$\left\{\rho^a, \rho^b\right\} = 2\eta^{ab} \tag{2.11}$$

The spinors' equation of motion, the Dirac equation, is actually the Weyl condition in two dimension. This brings the real degrees of freedom in the spinor for each μ from 4 to 2. Recalling that in (2 mod 8) dimensions there exist Weyl-Majorana spinors satisfying both the Weyl and Majorana conditions, imposing the latter on ψ halves again the on-shell polarizations to 1. Thus we have a match between bosonic and fermionic degrees of freedom. It can be proven*(ref)* the theory above is indeed worldsheet supersymmetric.

To quantize canonically, we introduce canonical commutation/anticommutation relations:

$$[X^\mu(\sigma), X^\nu(\sigma')] = \eta^{\mu\nu}\delta^2(\sigma - \sigma') \quad \{\psi^\mu(\sigma), \psi^\nu(\sigma')\} = \eta^{\mu\nu}\delta^2(\sigma - \sigma') \tag{2.12}$$

Note the X^0 and ψ^0 would create negative norm states, but these modes are eliminated by resorting to superconformal invariance. Classically this symmetry imposes the stress-energy tensor $T^{\mu\nu}$ and the supercurrent J^a_α vanish; imposing that in the quantum theory they annihilate physical states yields the restriction that removes the longitudinal ghosts from the spectrum. These take the name of super-Virasoro constrain.

Then the procedure for building the string spectrum is to expand the classical solutions in terms of Fourier modes, identify creators and destructors, and then select the states of the Fock basis that satisfy the super-Virasoro constraints. There are various ways to proceed at this point; the simplest and perhaps the most inelegant is light-cone quantization, which we will refer to. Essentially, a subset of $D-2$ transverse directions $i = 2, \dots, D-1$ are selected and only X^i and ψ^i are made to correspond to operators, and the remaining longitudinal fields are to be determined from the former by the classical constraints. This procedure renders Lorentz-invariance non-manifest - we will verify indeed it is actually recovered a posteriori. A different common approach, more rigorous, takes the form of a BRST quantization, akin to that employed in Yang-Mills theories, by exchanging superconformal gauge invariance with the introduction of a series of Faddev-Popov ghosts. The final quantum theories are identical and the choice of quantization is conventional.

Boundary conditions for ψ^μ for an open string can actually be satisfied in two different by imposing periodicity or antiperiodicity, giving rise to the NS (Neveu-Schwarz) and R (Ramond) sectors, built over two grounds $|0\rangle_{NS}$ and $|0\rangle_R$. Closed strings have four: $|0\rangle_{NS-NS}$, $|0\rangle_{R-R}$, $|0\rangle_{R-NS}$, $|0\rangle_{NS-R}$ corresponding with different choice periodicity conditions for left and right-movers.

2.1.1 Open strings

It can be shown that while the NS ground $|0\rangle_{NS}$ is unique, and thus a spacetime scalar, $|0\rangle_R$ is eight-fold degenerate and this 8-plet transforms under the spinor representation of transverse $SO(8)$ - in other words, it is a spacetime spinor. In particular, it is a chiral Weyl-Majorana spinor, so it can be taken to be either of positive or negative chirality, choices we will denote as $|+\rangle_R^a$, $|-\rangle_R^{\dot{a}}$, with $a, \dot{a} = 1, \ldots, 8$ the spinor index.

The spectrum is built by acting on one of the grounds with bosonic and fermionic creators, to obtain states of higher and higher mass. For the NS sector, there are bosonic creators $a_n^{i\dagger}$ ($n \geq 1$) and fermionic $b_r^{i\dagger}$ (r positive half-integer), and the mass of the excited string is given by:

$$\alpha' M^2 = \sum_{n=1}^{\infty} n\, a_n^{i\dagger} a_n^i + \sum_{r=1/2}^{\infty} r b_r^{i\dagger} b_r^i - \frac{1}{2} \tag{2.13}$$

while for the R sector the fermionic creators are replaced by the integer-indexed $d_n^{\dagger i}$:

$$\alpha' M^2 = \sum_{n=1}^{\infty} \left(n\, a_n^{i\dagger} a_n^i + n d_n^{i\dagger} d_n^i \right) \tag{2.14}$$

The i indices here are target spacetime transverse indices, $i = 1, \cdots, 8$. Therefore each creator increases the spacetime spin of the string by one unit. We conclude the NS sector contains only spacetime bosons, and the R sector only spacetime fermions.

The constant shift of $-1/2$ in (2.13) (and of 0 in (2.14)) actually results from an ordering ambiguity of the creators and destructors $a_n^\dagger$, a_n (omitting spacetime indices for now) which introduces an arbitrary constant shift in the Hamiltonian upon quantization. If we want a consistent quantum theory with Lorentz invariance, this shift in the NS sector is fixed to $-1/2$; this is because for example the state

$$b_{1/2}^{i\dagger} \left|0\right\rangle_{NS} \tag{2.15}$$

is a vector with $(D-2)$ polarizations, and thus must be massless. However, it can be shown that this shift is also related to the number of spacetime dimensions D, so that a condition restricting D to a particular value can be found. An intuitive and perhaps heuristic explanation of this fact is as follows: the energy shift should be equal to the sum of the zero-point energies (ZPE) of the infinite harmonic oscillators, bosonic for $n = 1, 2, \dots$ and fermionic for $r = 1/2, 3/2, \dots$ We recall bosonic / fermionic QHOs have Hamiltonians

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right) \qquad\qquad H = \omega\left(b^\dagger b - \frac{1}{2}\right) \tag{2.16}$$

Then for a given value of i the sum of all the ZPEs of the oscillators in the NS sector would be

$$E_0 = \sum_{n\in\mathbb{N}} \frac{n}{2} - \sum_{r\in\mathbb{N}+\frac{1}{2}} \frac{r}{2} = \frac{1}{4}\sum_{m\in\mathbb{N}} (-1)^m m \tag{2.17}$$

The sum $-1+2-3+\dots$ is evidently divergent; we assume it is admissible to replace it with its ζ-regularized value[3] of $-1/4$. Therefore the ZPE per transverse direction is $E_0 = -1/16$; insisting the total ZPE is equal to $-1/2$ results in

$$-\frac{D-2}{16} = -\frac{1}{2} \Rightarrow D = 10 \tag{2.18}$$

[3]The evaluation by analytic continuation of the divergent series is easily computed by considering $1 - 2x + 3x^2 - \dots = (1+x)^{-2}$, converging for $|x| < 1$. Setting $x = 1$, right on the edge of the disk of convergence, yields the desired result.

While this argument is not completely rigorous, it correctly identifies the existence and value of the so called critical dimension $D = 10$ for superstring theories. In a more formal setting (e.g., in BRST quantization) it can be shown that the classical Weyl symmetry of the action is spoiled by quantization and a conformal anomaly arises; this anomaly can be proven to cancel[4] only for $D = 10$.

It may worry that the mass-shell formula above assigns a negative mass-squared to the NS ground, which is therefore a tachyon. In addition, it is the *only* tachyon, meaning this theory is not spacetime supersymmetric. We will see in the next section how this state is actually removed and target supersymmetry recovered. For now, we note the only massless states are

$$b^{i\dagger}_{1/2} |0\rangle_{NS} \qquad |+\rangle_{NS} \tag{2.19}$$

while the rest of the tower of states have string scale-large $\sim (\alpha')^{-1/2}$ masses. Our interest in massless modes stems from the fact that in a low-energy ($\alpha' \to 0$) the strings can be approximated as pointlike particles (as their typical size $l_s \sim \sqrt{\alpha'}$) and their quantum theory as the corresponding field theory, with a field for each massless string mode, as the massive modes have decoupled. Such an effective low-energy theory will be described in section 2.3.

The first of the two states in (2.19) is a massless spin-1 boson, so it must be a photon associated with a $U(1)$ gauge theory. The latter is its spin-1/2 superpartner, a photino. It must be noted that what was described up to now holds for the directions in which the string endpoints are free to move, hence those for which Neumann conditions are imposed. As we have anticipated open strings in general end on Dp-branes and $D - p - 1$ directions are actually constrained by Dirichlet conditions so as to keep the endpoints on the brane; the conclusion is the gauge interaction this massless string mode mediates is actually confined to the p-dimensional volume of the brane.

[4]it is possible to have the conformal anomaly cancelled by the introduction of additional fields if $D < 10$, resulting in non-critical string theories. These have properties that make them unsuitable for our applications, however, and we will ignore them.

2.1.2 GSO projection

The construction above does not define a consistent theory. This is in part because it is not spacetime supersymmetric, an essential requirement considering that, as will be seen shortly, the closed string spectrum includes a gravitino (a massless spin-3/2 state) which must be associated with local supersymmetry. A procedure known as the Gliozzi, Scherk, Olive (GSO) projection solves this issue and in addition also eliminates the tachyonic state $|0\rangle_{NS}$, to end up with a consistent quantum theory.

The following operator is introduced, acting on the NS sector as

$$G = (-1)^{1+\sum_r b_r^{i\dagger} b_r^i} = (-1)^{\hat{F}+1} \tag{2.20}$$

and on the R sector as

$$G = \Gamma_{11}(-1)^{\sum_r d_r^{i\dagger} d_r^i} = \Gamma_{11}(-1)^{\hat{F}} \tag{2.21}$$

$\hat{F}$ is the worldsheet fermion number, and $\Gamma_{11} = \Gamma_0 \cdots \Gamma_9$ gives the chirality of the state.

Then the spectrum is projected into the $G = 1$ subspace for the NS sector, and into $G = \pm 1$ (either choice works) for the R sector. These two choices correspond to keeping either $|+\rangle_R^a$ or $|-\rangle_R^{\dot{a}}$ respectively and discarding the other.

When amputated with this precise prescription the spectrum is found to be spacetime supersymmetric. The scalar tachyon $|0\rangle_{NS}$ in particular is eliminated, being G-odd.

2.1.3 Closed strings

The closed string spectrum, in somewhat poetic language, is the "square" of the open string spectrum. On closed strings, excitations are allowed to move in either clockwise or counterclockwise direction along the string, forming a left-moving and a right-moving spectrum[5]. As seen before the choice can

[5] In the open string, the boundary conditions fixed the right-movers in terms of the left-movers (or vice versa) so that they were not independent.

be made for either NS or R boundary conditions, and this can be performed separately for left-movers and right-movers, giving four sectors. The GSO projection is performed separately on left and right movers, so that one is presented with the choice of the relative chirality of the two projections and so of the R grounds. These two possibilities will actually result in two different string theories. Choosing opposite chiralities gives type IIA strings, whose massless spectrum is given by

$$\tilde{b}^{i\dagger}_{1/2}\,|0\rangle_{NS} \otimes b^{j\dagger}_{1/2}\,|0\rangle_{NS} \tag{2.22}$$

$$\tilde{b}^{i\dagger}_{1/2}\,|0\rangle_{NS} \otimes |+\rangle^{b}_{R} \tag{2.23}$$

$$|-\rangle^{\dot{a}}_{R} \otimes b^{j\dagger}_{1/2}\,|0\rangle_{NS} \tag{2.24}$$

$$|-\rangle^{\dot{a}}_{R} \otimes |+\rangle^{b}_{R} \tag{2.25}$$

(the $\sim$ distinguishes creators/destructor for left movers from right movers). And IIB strings arise from equal chiralities:

$$\tilde{b}^{i\dagger}_{1/2}\,|0\rangle_{NS} \otimes b^{j\dagger}_{1/2}\,|0\rangle_{NS} \tag{2.26}$$

$$\tilde{b}^{i\dagger}_{1/2}\,|0\rangle_{NS} \otimes |+\rangle^{b}_{R} \tag{2.27}$$

$$|+\rangle^{a}_{R} \otimes b^{j\dagger}_{1/2}\,|0\rangle_{NS} \tag{2.28}$$

$$|+\rangle^{a}_{R} \otimes |+\rangle^{b}_{R} \tag{2.29}$$

So the massless spectrum is composed of 4 sectors of 64 physical states, two of them bosonic (NSNS, and RR) and the other fermionic (RNS and NSR). Massless states will correspond to fields in the supergravity approximation, in which the massive modes of the string decouple and the string theory is well described by the corresponding variety of 10D supergravity.

To present these states in a less opaque manner, consider for example the NSNS mode (2.22), (2.26), equal in both theories. This is a tensor H^{ij}; we can switch the transverse indices i, j to Lorentz indices $\mu\nu$ provided we keep memory of the transversality constraint. Such a general tensor can be decomposed into Lorentz irreps as a symmetric traceless, antisymmetric and trace parts, as

$$H^{\mu\nu} = G^{(\mu\nu)} + B^{[\mu\nu]} + const?\eta^{\mu\nu}\phi \tag{2.30}$$

The first of these is a transverse, massless, symmetric traceless rank-2 tensor, which means it must be a graviton. The presence of a field describing variations in the target spacetime metric from the chosen background metric signals that in general string theory will contain gravity. So in general string theory is a consistent theory of quantum gravity. We will describe the rest of the massless states better in the next sections.

The type II theories defined above are not the only consistent superstring theories. Three additional target-supersymmetric theories can be defined: type I, heterotic $SO(32)$, and heterotic $E_8 \times E_8$, all set in 10 dimensions, for a total of five superstring theories. Our interest will be mainly focused on the type II theories, however.

A relevant point is that type II strings do not actually feature stable "free" open strings with all Neumann boundary conditions. Open strings can only appear attached to D-branes. (The discussion of section 2.1.1 was therefore mostly intended as preparatory to the introduction of the closed string spectrum). This means for example the closed-string spectrum above is sufficient to encompass the dynamics in the absence of D-branes, and that type II supergravity will result only from the massless closed string modes above.

2.1.4 Background fields, string coupling and loop expansion

It was already hinted that the massless string modes we found should give rise to fields in some limit. In particular the NS-NS closed string ground (2.26) is a spacetime rank-2 tensor, which can be split into symmetric, antisymmetric and trace parts.

Non-zero values of these fields could be incorporated back into the background the perturbative string is based on. In fact, since the symmetric tensor field above is actually the graviton, we already have: the Polyakov action (**??**) already includes a coupling of the string to the target background metric $G_{\mu\nu}$. This is just the background value of the graviton.

The antisymmetric NS-NS $B_{\mu\nu}$ field (equivalently, a 2-form $B_2 = \frac{1}{2}B_{\mu\nu}dx^\mu\wedge$

dx^ν), called the Kalb-Ramond potential, is instead coupled to the fundamental string in a way that resembles the generalization of the coupling of a particle to the EM potential; a background value of $B_{\mu\nu}$ would result in the addition to the action of a term

$$S_B = \frac{T}{2}\int_{W_1} d^2\sigma \varepsilon^{\alpha\beta} B_{\mu\nu}(X)\partial_\alpha X^\mu \partial_\beta X^\nu \propto \int_{W_1} B_2 \tag{2.31}$$

this will result ultimately in a stringy generalization of electrodynamics with strings coupling to a field strength 3-form $H_3 = dB_2$.

Finally, strings will also couple to the last NS-NS closed string field, the trace scalar ϕ called the dilaton. The coupling to a background dilaton is of the form

$$S_\phi = \frac{1}{4\pi}\int_{W_1} d^2\sigma \sqrt{g}\, R^{(2)}[g]\, \phi(X) \tag{2.32}$$

where $R^{(2)}[g]$ is the 2D Ricci scalar associated to g. Note that in the presence of a constant background dilaton $\phi(X) = \phi$, the integral above is simply the Euler characteristic $\chi(W_1)$, an integer topological invariant, by the Gauss-Bonnet theorem. χ has a simple expression in terms of the number of handles (the genus h), the number of boundaries n_b and the number of cross-caps n_c of the surface:

$$\chi = 2 - 2h - n_b - n_c \tag{2.33}$$

To understand the physical content of this contribution, we consider the simple case of orientable closed strings, and we imagine computing the amplitude of a process involving n external string states. Since only closed strings are involved, the only boundaries are the $n_b = n$ boundaries at infinity of the asymptotic states, and thus n_b is constant. Therefore the action from a constant dilaton background reduces to

$$S_\phi = \phi\chi = \phi(2 - n - 2h) = \text{const} - 2h\phi \tag{2.34}$$

and the Euclidean path integral for the amplitude would take the form

$$A = \int DXD\psi Dg \; e^{-S_{\mathrm{P}}} e^{-2\phi\, h[g]} \tag{2.35}$$

apart from normalization, and the path-integral is over worldsheets that match the external states. The integral over metric structures $\int Dg$ splits into disconnected components indexed by the genus, so that

$$A = \sum_{h=0}^{\infty} A_h = \sum_{h=0}^{\infty} (e^{\phi})^{-2h} \int_h DXD\psi Dg \; e^{-S_P} \tag{2.36}$$

This is a loop expansion, since the genus h counts the number of virtual string loops; however it is also a perturbative expansion in the string coupling $g_s := e^{\phi}$, giving the strength of a cubic string interaction "vertex". Therefore, we come to understand that the applicability of the perturbative string theory, including what has been introduced in this chapter so far, rests on the smallness of this string coupling g_s. It is worth of notice that this coupling is not an external dimensionless parameter of the theory (since string theory has none) but rather is related to the expectation value of a scalar field.

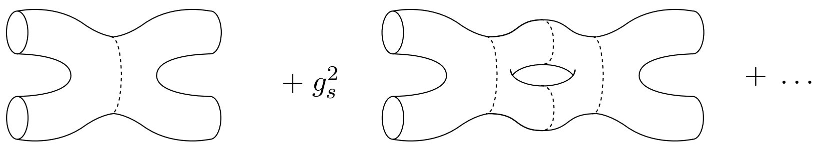


Figure 2.1: First terms in the loop expansion of a closed string four-point function. The worldsheets have been cut into "pair of pants" surfaces to count string interactions.

The perturbation series (2.36) is the stringy analogue of the QFT sum over Feynman diagrams. Its power comes from the fact that the genus-h contribution involves the calculation of a single diagram - the multiplicity of inequivalent Feynman graphs of a field theory (growing as $\sim e^h$) is then interpreted as the various inequivalent ways in which the unique worldsheet topology of genus h can degenerate to a diagram with pointlike particles and interactions as the string length is sent to zero.

Open strings interactions are instead controlled by a different coupling g_o. Consider the addition of an open string loop. This introduces two open string vertices and thus should result in a g_o^2 suppression. However, this

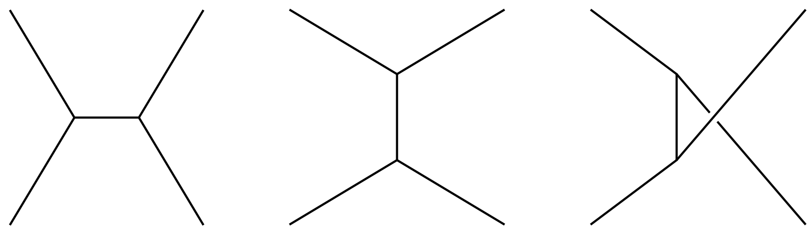


Figure 2.2: The possible Feynman graphs from degeneration of the genus-0 worldsheet from figure 2.1

operation results in the addition of a boundary and no change in genus, so $\Delta\chi = -1$. Therefore the suppression is also $(e^{\phi})^{-\Delta\chi} = g_s$ and we find that

$$g_o^2 \sim g_s \tag{2.37}$$

2.2 Type II supergravity and D-brane content

At energy scales much lower than the string scale $(\alpha')^{-1/2}$, equivalently $\alpha' \to 0$, all massive modes of a string theory decouple and a good description is given by an effective field theory comprising only the massless excitation. Since the string length goes to zero in this limit strings in massless states are essentially pointlike and the quantum theory will correspond to a local quantum field theory.

The effective field theories of the five superstring theories are the five supergravity (SUGRA) theories in 10 dimensions. The name of each SUGRA coincides with that of the superstring theory it is the effective theory of (e.g., IIB SUGRA is the effective theory of IIB superstrings). Supergravities are supersymmetric theories containing general relativity, and are obtained by extending local Poincaré invariance to include local supersymmetry. Just like Einstein gravity, they are nonrenormalizable, reflecting their origin as effective theories. As field theories, they are considerably simpler than general strings to find background solutions to; therefore we will make extensive use of the supergravity approximation in the context of holography.

10D SUGRAs are perhaps easier to introduce starting instead from the unique 11D SUGRA. The field content of 11D SUGRA is as follows (we also note the number of physical polarizations):

Bosons	Graviton	$g_{(M,N)}$	44
	3-form	$A_3 = \frac{1}{3!}A_{MNL}dx^M \wedge dx^N \wedge dx^L$	84
Fermions	Gravitino	Majorana ψ_M	128

As required by supersymmetry, the number of on-shell boson and fermion states are equal. These states form an irreducible supermultiplet, a gravity multiplet.

Upon dimensional reduction on a circle, in 10D these fields decompose into those of type IIA SUGRA:

NSNS	Graviton	$g_{(\mu,\nu)}$	35
	Kalb-Ramond	$B_2 = \frac{1}{2}B_{\mu\nu}dx^\mu \wedge dx^\nu$	28
	Dilaton	ϕ	1
RR	1-form	$A_1 = A_\mu dx^\mu$	8
	3-form	$A_3 = \frac{1}{3!}A_{\mu\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho$	56
NSR, RNS	Two gravitinos	Weyl-Majorana $\psi_\mu^{(L)}$, $\psi_\mu^{(R)}$	$56+56$
	Two dilatinos	Weyl-Majorana $\lambda^{(L)}$, $\lambda^{(R)}$	$8+8$

where we have matched the fields with the massless modes of the four IIA string sectors. In fact, these fields are just the ground states defined in (**??**), decomposed in irreducible representations of $SO(8)$. For example, the NS-NS ground $G_{\mu\nu} = \tilde{b}^{\mu\dagger}_{1/2}\,|0\rangle_{NS} \otimes b^{\nu\dagger}_{1/2}\,|0\rangle_{NS}$ (reintroducing unphysical polarizations) is a spacetime rank-2 tensor, decomposable as a symmetric form, an antisymmetric form, and a trace, as in

$$\mathbf{8}_V \otimes \mathbf{8}_V = \mathbf{35} + \mathbf{28} + \mathbf{1} \tag{2.38}$$

These are respectively the graviton, the Kalb-Ramond field, and the dilaton. The fields from the other sectors result accordingly from the decompostions:

$$NSR \qquad \mathbf{8}_V \otimes \mathbf{8}_R = \mathbf{56} + \mathbf{8}_L \tag{2.39}$$

$$RNS \qquad \mathbf{8}_L \otimes \mathbf{8}_V = \mathbf{56} + \mathbf{8}_R \tag{2.40}$$

$$RR \qquad \mathbf{8}_L \otimes \mathbf{8}_R = \mathbf{56} + \mathbf{8}_V \tag{2.41}$$

where the three inequivalent $SO(8)$ irreps $\mathbf{8}_V$, $\mathbf{8}_L$, $\mathbf{8}_R$ are respectively the vector and negative and positive chirality Weyl-Majorana representations.

We note the two gravitinos and dilatinos are of opposite chirality, as do the RNS and NSR sectors of IIA superstrings.

Obviously, again we find that the total bosonic states are $35+28+1+8+56 = 128$ and the fermions $2\cdot(56+8) = 128$. There is therefore a match both with the number of degrees of freedom of 11D SUGRA and with supersymmetry. IIA SUGRA in particular is a theory with $\mathcal{N} = (1,1)$ SUSY, meaning there are two Weyl-Majorana SUSY generators of opposite chirality.

We will mainly be interested, however, in type IIB SUGRA, which is not obtainable from dimensional reduction, and is the effective field theory for IIB strings. The field content is as follows:

NSNS	Graviton	$g_{(\mu,\nu)}$	35
	Kalb-Ramond	$B_2 = \frac{1}{2} B_{\mu\nu}\, dx^\mu \wedge dx^\nu$	28
	Dilaton	ϕ	1
RR	0-form	A_0	1
	2-form	$A_2 = \frac{1}{2!} A_{\mu\nu}\, dx^\mu \wedge dx^\nu$	28
	4-form	$A_4 = \frac{1}{4!} A_{\mu\nu\rho\sigma}\, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$	35
NSR, RNS	Two gravitinos	Weyl-Majorana $\psi_\mu^{(1)}$, $\psi_\mu^{(2)}$	$56+56$
	Two dilatinos	Weyl-Majorana $\lambda^{(1)}$, $\lambda^{(2)}$	$8+8$

The 4-form has its physical polarizations halved from $\binom{8}{4} = 70$ to 35 by the introduction of a constraint we will specify shortly.

IIB SUGRA has $\mathcal{N} = (2,0)$ supersymmetry, with two equal-chirality Weyl-Majorana generators. It is therefore a chiral theory. While IIA is non-chiral and thus automatically anomaly free, IIB as a QFT has the potential to develop a gravitational anomaly due to the chiral fermionic sector. It is however found that the total anomaly miraculously cancels with the listed field content *reference per questo?*. This cancellation is obvious in light of the fact that IIB SUGRA is the effective field theory to IIB superstrings, which are not just free from anomalies but from all UV divergences.

2.2.1 Gauge invariance of RR fields

In both IIA and IIB, the RR sector admits the following gauge transformations:

$$B_2 \to B_2 + d\Lambda_1 \qquad A_p \to A_p + d\Lambda_{p-1} - H_3 \wedge \Lambda_{p-3} \tag{2.42}$$

for any set of arbitrary p-forms Λ_p, leaving invariant the field strengths:

$$\begin{aligned} H_3 &:= dB_2 \\ F_{p+1} &:= dA_p - H_3 \wedge A_{p-2} \end{aligned} \tag{2.43}$$

Where A_p, Λ_p with $p < 0$ is set to 0. Now, intuitively, the RR form A_p couples to $(p-1)$-dimensional objects ($(p-1)$-branes) through an interaction term of the type

$$S_{int} = \int_{W_{p-1}} A_p \tag{2.44}$$

integrated over the p-dimensional worldvolume W_{p-1}, a sensible generalization of the coupling of the EM potential to a charged particle. In particular, as it was already established that it is possible for open strings to end on D-branes, we would like to investigate for which values of p a certain string theory admits stable D$(p-1)$-branes - certainly, if they have a conserved charge under a gauge field then they are protected from decay. Therefore the set of RR fields in a superstring theory determines the list of stable D-brane dimensionalities.

The above is an electric coupling of the $(p-1)$-brane to F_{p+1}. The coupling however could also be magnetic, electric-magnetic duality being implemented in general through Hodge duality. We define F_p for additional values of $p > 5$ (odd for IIA, even for IIB) through

$$F_p = \widetilde{\star} F_{10-p} \tag{2.45}$$

(Note that for the IIB F_5 this would be actually a constraint, precisely the one that acts on A_4 to reduce its degrees of freedom; $F_5 = *F_5$ however is not the exact form of the constraint, as will be clarified in section 2.3). The new field strengths can then be locally trivialized as of (2.43) and so we end up with a complete set of potentials $A_0, \ldots A_8$ for IIB and $A_1 \ldots A_9$ for IIA. The duality between potentials would act as $A_p \leftrightarrow A_{8-p}$, and if $D(p-1)$-branes couple electrically to A_p, then $D(7-p)$-branes couple magnetically to it, that is to say electrically to A_{8-p}.

Therefore, the magnetic dual to a Dp-brane is a $D(6-p)$-brane. So the definitive list of stable D-branes in type II string theories along with the RR fields they are charge under is given by:

D(−1)	A_0	F_1
D0	A_1	F_2
D1	A_2	F_3
D2	A_3	F_4
D3	A_4	F_5
D4	A_5	F_6
D5	A_6	F_7
D6	A_7	F_8
D7	A_8	F_9
D8	A_9	F_{10}
D9	A_{10}	0

where shaded entries are for IIB, and unshaded for IIA, and arrows represent electric-magnetic duality. We comment on a few apparent anomalies.

- The IIB D(−1)-brane would have a 0-dimensional worldvolume, hence a single event. These type of branes are therefore actually instantons. They couple to the A_0 potential, which has axionic character.

- IIA D8-branes and the A_9 potentials do not have duals, yet D8-branes can be shown to exist and to couple to the F_{10} field strength, as was first noted in [27]. However the action $\int F_{10} \wedge \star F_{10}$ implies $d \star F_{10} = 0$ which means F_{10} is a constant. Therefore there are no additional physical degrees of freedom from A_9.

- Space-filling D9 branes can be introduced in IIB superstrings, but the equation of motion of the A_{10} form implies only special arrangements of D9s and anti-D9s are allowed - this was first appreciated in [28].

2.3 Action functional for IIB SUGRA

There is a considerable obstacle to a covariant (i.e. explictly supersymmetric) formulation of type IIB supergravity in the self-duality constraint for the field strength 5-form F_5. We will take the common path of formulating the Lagrangian theory ignoring the constraint (and thus in excess of bosonic polarizations with respect to an explicity supersymmetric theory) and then imposing self-duality by hand after deriving the equations of motion. Therefore the action will not be supersymmetric itself, while the Euler-Lagrange equations augmented with the constraint will be. This issue becomes very problematic on quantization, but, as we will prove, classical supergravity will be sufficient for our purposes.

Actually, for the purpose of building classical solutions, where spinor fields vanish anyway, the fermionic sector of the action will not be important. After introducing the "string length" ℓ_s through[6]

$$\ell_s := 2\pi\sqrt{\alpha'} \tag{2.46}$$

the bosonic sector is as such:

$$S_{\mathrm{IIB},B} = S_{NS} + S_R + S_{CS} \tag{2.47}$$

where S_{NS} is the action relevant to the fields originally from the superstring NS-NS sector:

$$S_{NS} = \frac{2\pi}{\ell_s^8} \int d^{10}x\sqrt{-g}\, e^{-2\phi} \left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2 \right) \tag{2.48}$$

where the p-form norm is $|\omega|^2 = \omega\wedge\star\omega$. Then S_R is for R-R fields, essentially just kinetic terms for the A forms:

[6] Care should be taken with inequivalent convention with the definition of the string length in the literature. For the purpose of this work we will refer to this definition.

$$S_R = -\frac{2\pi}{\ell_s^8} \int d^{10}x\sqrt{-g}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right) \tag{2.49}$$

$$\tilde{F}_3 := F_3 - A_0 H_3 \tag{2.50}$$

$$\tilde{F}_5 := F_5 - \frac{1}{2}A_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 \tag{2.51}$$

And finally we supplement with a Chern-Simons type term:

$$S_{CS} = -\frac{2\pi}{\ell_s^8} \int A_4 \wedge H_3 \wedge F_3 \tag{2.52}$$

note the untilded F_3. This is evidently a purely topological term.

The self-duality constraint is then imposed in terms of the modified field strength $\tilde{F}_5$

$$\tilde{F}_5 = *\tilde{F}_5 \tag{2.53}$$

The action presented above is in what is known as the "string frame". It becomes convenient in many occasion to switch to an alternative formulation through a field redefinition, to move to the "Einstein frame". The change is

$$g_{\mu\nu}^{EF} = e^{-\phi/2} g_{\mu\nu}^{SF} \tag{2.54}$$

and the action in terms of the new metric is[4]

$$\begin{aligned}\left(\frac{\ell_s^8}{2\pi}\right) S^{EF} &= \int d^{10}x\sqrt{-g}\left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{-\phi}|H_3|^2\right) \\ &- \frac{1}{2}\int d^{10}x\sqrt{-g}\left(e^{2\phi}|F_1|^2 + e^{\phi}|\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right) \\ &- \frac{1}{2}\int A_4 \wedge H_3 \wedge F_3\end{aligned} \tag{2.55}$$

The advantage of this picture is the canonical Einstein-Hilbert and dilaton terms; by converse exponential couplings of the dilaton with the form fields are introduced.

2.4 D-branes

2.4.1 D-brane action

D-branes appear as nonperturbative objects in string theories. They are themselves dynamical and the dynamics are modeled in the string perturbative regime by an action functional[13]. To formulate the action, we introduce coordinates σ^a on the $(p+1)$-dimensional worldvolume W_p and functions $X^\mu(\sigma^a)$ describing the embedding of W_p in spacetime as $\iota : \sigma^\alpha \mapsto X^\mu(\sigma^a)$. It is then tempting to choose as bosonic action the obvious generalization of the Nambu-Goto action:

$$S_{Dp} = -\mu_{Dp} \int_{W_p} d^{p+1}\sigma \sqrt{-\det \iota^*(G)} = -\mu_{Dp} \,\mathrm{vol}\, W_p \tag{2.56}$$

the notation $\iota^*(T)$ denotes the pull-back of a spacetime tensor to the worldsheet. For example, $\iota^*(G)$ is the induced metric $h_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$. μ_{Dp} would be the D-brane tension. The insight of (2.56) is correct, but incomplete; firstly it does not include a coupling with possible background fields, but most importantly it does not account for all of the open string modes living on the worldvolume.

In fact, if the quantization procedure we performed for the open string is repeated with the endpoints constrained to a Dp-brane (i.e., with $9-p$ Dirichlet and $p+1$ Neumann conditions) then one finds among the massless bosonic states both scalar (from the worldvolume point of view) fields $X^{p+1,\dots,9}$ which indeed correspond to motion of the D-brane in the transverse directions, but also, as we have seen, a massless vector potential mediating a $U(1)$ gauge theory confined to the D-brane volume. Therefore, it is directly from the modes of open strings that it is possible to deduce the brane is itself a dynamical object, as its position is related to VEVs of open-string scalars. Enforcing this idea that the D-brane dynamics should be encoded in those of the open-string modes, D-branes should always host at least a $U(1)$ gauge theory on them.

The bosonic part of the Dp-brane action is then found to be

$$S_{Dp} = -\,\mu_{Dp} \int_W d^{p+1}\sigma e^{-\phi} \sqrt{-\det\left(\iota^*(g - B_2) - 2\pi\alpha' F\right)} \tag{2.57}$$

$$+\,\mu_{Dp} \int_W \left[\iota^* \left(\sum_k A_k\right) \wedge e^{2\pi\alpha' F - B_2} \wedge (1 + \mathcal{O}(R^2))\right]_{p+1} \tag{2.58}$$

Where μ_{Dp} can be fixed as

$$\mu_{Dp} = \alpha'^{-\frac{p+1}{2}}(2\pi)^{-p} \tag{2.59}$$

and F is the field-strength 2-form of the $U(1)$ gauge theory.

The first line (2.57) is the Dirac-Born-Infeld action and generalizes the Nambu-Goto action; Setting only $B = 0, \phi =$ const and expanding S_{DBI} in powers of α':

$$S_{DBI} = -\frac{\mu_{Dp}}{g_s} \int_W d^{p+1}\sigma\sqrt{-h} \;+\; \frac{\alpha'^{-(p-3)/2}}{4g_s(2\pi)^{p-2}} \int_W d^{p+1}\sigma\sqrt{-h}\, F_{\mu\nu}F^{\mu\nu} \;+\ldots \tag{2.60}$$

the first term is the direct generalization of the Nambu-Goto action, allowing us to identify the Dp-brane tension $T_{Dp} = \frac{\mu_{Dp}}{g_s}$. The second is a Yang-Mills action for the $U(1)$ gauge field, restricted to the worldvolume.

It becomes clear D-branes carry a mass per unit p-volume $T_{D_p} \sim g_s^{-1}$ and are thus nonperturbative in terms of the string coupling. Note also the Maxwell action is weighted by g_s^{-1}, in agreement with what previously found for the relation between open and closed string couplings: $g_{YM} \sim g_0 \sim g_s^{1/2}$. To be more exact, comparing with the canonical Maxwell action one can deduce

$$g_{YM}^2 = 2\pi g_s \tag{2.61}$$

The second line (2.58) is a Chern-Simons type term coupling the brane to the RR potentials. The sum over k only spans odd or even respectively for IIA or IIB, and the $[\,]_{p+1}$ notation means the $p+1$-form component must be selected so as to define a meaningful integral. We note that in vanishing curvature, and expanding in $2\pi\alpha' F - B_2$, the physical interpretation becomes less obscure:

$$S_{CS} = \mu_P \int_W A_{p+1} + \mu_P \int_W A_{p-1} \wedge (2\pi\alpha' F - B_2) + \mathcal{O}(F^2) \tag{2.62}$$

so that there is a direct, standard coupling of the A_{p+1} potential to the Dp-brane at the zeroth order in F. A Dp-brane is therefore also understood as a localized charge for the F_{p+2} field. Higher order terms mean a coupling with the lower RR potentials and are due to nontrivial F configurations which induce lower-dimensional D-brane charges localized inside the Dp-brane.

We touch briefly upon the easy generalization of the above action to the case of N coincident Dp-branes, a "stack". We imagine first taking 2 separated parallel D-branes (1 and 2) and bringing them closer together. Normally, open string modes stretching from 1 to 2 are suppressed by the increase in mass squared due to the minimum elastic energy to span the inter-brane distance. In fact, the open string spectrum is found to be identical except for the constant shift in the mass-shell condition[4]:

$$\Delta M^2 = T^2 \sum_{i=p+1}^{9} (X_1^i - X_2^i)^2 \tag{2.63}$$

If then the branes are brought to coincide, this shift vanishes and one must admit an additional $1 \to 2$ sector of massless states has been created. In general, with N coincident Dp-branes there will be an $N \times N$ matrix of massless sectors indexed by $a, b = 1, \ldots, N$ marking the starting and ending brane (Chan-Paton indices). This means then a matrix of gauge vectors A_{ab} generating $U(N)$ gauge transformations; this $U(N)$ group is nothing else than transformations mixing the N identical, coincident D-branes with eachother. Therefore they act on Chan-Paton indices in the defining representation.

Then, it is clear the fields $\Phi^i_{ab} := (X^i_a - X^i_b)$, parametrizing relative D-brane transverse position, transform in the adjoint of the gauge group. The action of separating again the D-branes is then interpreted in this point of view as a Higgsing of the gauge group by these scalars; when the stack of N branes splits into two groups of N_1 and N_2 branes which are separated, this corresponds to the relevant Φ fields acquiring a VEV, breaking part of the gauge group to the corresponding group for two separate stacks

$$U(N) \to U(N_1) \times U(N_2) \tag{2.64}$$

and the gauge fields corresponding to the broken generators, which gain a mass through the Higgs mechanism, are in fact modes of strings stretching between the two stacks, so that the Higgsed mass can also be viewed as the elastic energy from (2.63).

The salient point in any case is the extension of the gauge group from $U(1)$ to $U(N)$. Essentially, the F^2 term (and higher) in (2.60) must be supplemented with gauge traces.

2.4.2 D-branes as supergravity solitons

As D-brane carry mass, a dual description of D-branes in terms of the warped spacetime they produce should be possible. These spacetimes indeed appear as solitonic solutions in supergravity generalizing the usual four-dimensional black hole metrics, under the name of p-branes or black branes. In particular, as Dp-branes are (for certain values of p) stable objects, stabilized by F_{p+2} charge, their supergravity description must be a zero-temperature state, unable to lose mass to Hawking radiation, and thus extremal.

These solutions are constructed in complete analogy to the derivations of usual general relativity black holes, by inserting an ansatz with the desired symmetries into the supergravity action, and in fact are direct generalizations of the extremal Reissner-Nordström hole. We provide a succint review of the derivation and result, following for example [16]. Since the p-brane is stabilized by its charge under the A_{p+1} potential or the F_{p+2} field strength, it can be assumed all the other form fields vanish, including H_3. Moreover the fermion fields vanish for classical solutions. Therefore the Einstein frame IIB action reduces to

$$S_{\text{IIB}} = \frac{2\pi}{\ell_s^8} \int \sqrt{-g} \left(R - \frac{1}{2}(d\phi)^2 - \frac{1}{2\eta} e^{\frac{3-p}{2}\phi} |F_{p+2}|^2 \right) \tag{2.65}$$

if $p = 3$, the self-duality constraint $F_5 = \tilde{F}_5$ is to be imposed after finding the Euler-Lagrange equations, and $\eta = 2$; otherwise $\eta = 1$.

It is clear one can assume the black p-brane has $\left(\mathbb{R}^{p+1} \rtimes SO(1,p)\right) \times SO(9-p)$ symmetry, so that we can introduce a set of longitudinal coordinates $x^{0,\ldots,p}$ and transverse coordinates $y^{p+1,\ldots,10}$ such that the dependence of all fields components in this coordinates is reduced to the single radial transverse variable $r^2 = \vec{y} \cdot \vec{y}$. The most general Einstein frame metric with these symmetries is then

$$ds^2 = H_p^{-1/2} dx \cdot dx + H_p^{1/2} dy \cdot dy \tag{2.66}$$

where the warp factor $H_p(r)$ is a function of r only, and $dx \cdot dx$ and $dy \cdot dy = dr^2 + r^2 d\Omega_{8-p}$ are respectively the Minkowski and Euclidean metrics. Analogously, the dilaton ϕ is also a function of r and the form potential must take the form

$$A_{012\ldots p} = A(r) \tag{2.67}$$

After variation of the action (2.65) and insertion of the described ansatz in the resulting equations of motion (plus simplifications in the $p = 3$ case since $|F_5|^2 = F_5 \wedge *F_5 = F_5 \wedge F_5 = 0$), one is left with a differential equation for H_p. It is found, remarkably, that (for $r > 0$)

$$\nabla^2 H_p = 0 \tag{2.68}$$

where ∇^2 is the *flat* space Laplacian, hence a linear equation. This traces back to the fact that multiple extremal black holes are non-interacting, as the gravitational and electrostatic forces cancel - the same holds for D-branes in supergravity. This makes it possible to construct exact solutions with multiple branes by superposition. In any case, the linear equation has (taking into account the boundary condition of asymptotic flatness $H_p(\infty) = 1$) a simple solution as (for $p < 7$):

$$H_p(r) = 1 + \left(\frac{R_p}{r}\right)^{7-p} \tag{2.69}$$

for some R_p to be determined. Exploiting the equations of motion one finds also

$$e^{\phi}(r) = g_s H_p(r)^{(3-p)/4} \tag{2.70}$$

where $g_s = e^{\phi(\infty)}$ is the background value of the string coupling, and

$$A_{01\dots p} = H_p^{-1} - 1 \tag{2.71}$$

$$\Rightarrow F_{p+1} = d(H_p^{-1}) \wedge dx^0 \wedge \dots \wedge dx^p \tag{2.72}$$

which completes the specification of the class of solutions. Now R_p is fixed by matching the gravitational flux to the mass (per unit longitudinal volume) of the D-brane. In fact, we are free to consider a set of N coincident Dp-branes, a "stack", an easy generalization which will prove very important in the context of AdS/CFT. The exact dependence is

$$(R_p)^{7-p} = \left((4\pi)^{\frac{5-p}{2}} \Gamma\left(\frac{7-p}{2}\right)\right) \alpha'^{\frac{7-p}{2}} g_s N \tag{2.73}$$

While this result was derived from the IIB action and so for $p = 1, 3, 5$, it actually applies identically for IIA $p = 0, 2, 4, 6$ branes[7]. We single out from this the self-dual case of the D3 brane which displays a uniform value for the dilaton. In general instead from (2.70) and the warp factor (2.69) we have the near-horizon behaviour

$$e^{4\phi} \sim \left(\frac{r}{R}\right)^{(3-p)(p-7)} \tag{2.74}$$

which means the local string coupling diverges for $p < 3$ and vanishes for $p > 3$. $p = 3$ acts as a middle ground between these cases.

[7]It should be remarked that similar D7 and D8 solutions are possible, but that the behaviour of the warp factor is respectively logarithmic and linear in r, and so asymptotic flatness cannot be enforced.

Chapter 3

D3-brane stacks on Calabi-Yau cones

One of the most essential ingredients for the conception of the idea of holography was the fact that a stack of coincident D3-branes naturally features a 4D gauge theory on their world-volume, where the 4D fields emerge from the modes of open strings stretching between them. In the simplest and most famous example, a stack of N D3-branes is placed in otherwise Minkowski $\mathbb{R}^{1,9}$; the corresponding field theory is the maximally supersymmetric Yang-Mills in four dimensions (SYM4).

Setting the stack on a different background geometry instead gives rise to a large family of different field theories; a particularly interesting subset is given by spacetimes of the form

$$M = \mathbb{R}^{1,3} \times X_6 \, , \tag{3.1}$$

where the $\mathbb{R}^{1,3}$ is parallel to the branes (and must be identified with the field theory spacetime) and X_6 is a 6-dimensional Calabi-Yau cone over a compact 5-fold base Y_5. By X_6 being a cone it is meant there exists a conical radial coordinate r such that the metric on X_6 is of the form

$$ds^2 = dr^2 + r^2 ds_5^2 \tag{3.2}$$

With ds_5^2 the metric on Y_5.

In this language, the SYM4 example above corresponds to $X_6 = \mathbb{R}^6 = \mathbb{C}^3$, which is (trivially) a cone over $Y_5 = \mathbb{S}^5$. This is the only case where X_6 turns out to be smooth; in general it will feature a conical singularity in the origin. The motivation for jumping so hastily to a generalization as radical as a singular spacetime, instead of other smooth spacetimes, is that the latter will not actually introduce any novel features. As will be clarified in a holographic context (see for example section 4.4), the flow towards the IR of the field theory will actually correspond to "zooming in" on the D-branes, and any smooth spacetime will converge to flat $\mathbb{R}^6$ in this limit. Only a genuine singularity is going to introduce any novel behaviour in the IR field theory, and conical defects are a well-known example of singularities on which string theory is known to be formulable[**?**].

Non-trivial choices for the base will typically yield theories with reduced (even minimal) supersymmetry, which are considerably more challenging to study.

In this chapter, we will first describe some general features of the theories resulting from the placement of D3-branes on these conical backgrounds. Then, we will concentrate in particular on a specific chain of field theories starting from the simplest example of SYM4 and ending up on the $Y^{2,0}$ theory, the study of which is the main objective of this work.

3.1 Superconformal field theory

We now provide a short introduction to 4D conformal field theories, their supersymmetric variants, and the relevant terminology.

Consider flat spacetime of dimension d and signature p, q, with $p + q = d$. Take a coordinate chart x^μ in which the metric takes the standard form $\eta_{\mu\nu}$. Conformal transformations are defined to be diffeomorphisms $x^\mu \to x'^\mu$ which leave the metric unchanged in form up to an x dependent scalar function (a conformal factor):

$$g'_{\mu\nu}(x') = \Omega(x)\eta_{\mu\nu}\,. \tag{3.3}$$

These maps form evidently a group, which is known as the conformal group

$CO(p,q)$. We will mainly be interested in $CO(1,d-1)$, but most of what we will now show applies in general signatures.

In $d>2$, the conformal group will turn out to be a finite-dimensional Lie group, of which we specify now the connected component. An obvious subgroup is maps that leave the metric unchanged, so Poincaré transformations, with generators P_μ and $J_{\mu\nu}$. A second easy guess is the subgroup with constant conformal factors, that is scale transformations or dilations

$$x^\mu \to \lambda x^\mu \qquad\qquad \eta_{\mu\nu} \to \lambda^{-2}\eta_{\mu\nu} \tag{3.4}$$

whose generator is called D. To generate the whole conformal group a final class of transformations must be introduced, special conformal transformations, generated by K_μ and with finite action[1]

$$x^\mu \to \frac{x^\mu - b^\mu x^2}{1-2b^\nu x_\nu + b^2x^2} \tag{3.5}$$

Together, P_μ, $J_{\mu\nu}$, D and K_μ generate the connected component of the conformal group in d dimensions. The extension of the Poincaré algebra to the conformal one is characterized by the following additional commutators (using hermitian generators)

$$[J_{\mu\nu}, K_\rho] = 2i\eta_{\rho[\mu}K_{\nu]} \tag{3.6}$$
$$[J_{\mu\nu}, D] = 0 \tag{3.7}$$
$$[D, P_\mu] = iP_\mu \tag{3.8}$$
$$[D, K_\mu] = -iK_\mu \tag{3.9}$$

Equations (3.6) and (3.7) just confirm K_ρ is a vector and D is a scalar. (3.8) and (3.9) instead state that P_μ and K_μ are respectively raising and lowering operators for D. It is worth of notice that this group $CO(1,d-1)$ is actually $SO(2,d)$, the Lorentz group in mixed signature $(2,d)$. This can be shown

[1] It should be noted special conformal transformations are not well-defined on $\mathbb{R}^{p,q}$, as the denominator can vanish. Indeed, these more naturally act on the conformal compactification $\overline{\mathbb{R}^{p,q}}$, including points at infinity.

by combining the generators in

$$J_{MN} = \begin{pmatrix} J_{\mu\nu} & (K_\mu - P_\mu)/2 & -(K_\mu + P_\mu)/2 \\ (P_\mu - K_\mu)/2 & 0 & D \\ (K_\mu + P_\mu)/2 & -D & 0 \end{pmatrix} \tag{3.10}$$

and then it can be verified that J_{MN} satisfy the algebra of $\mathfrak{so}(2, D)$. This equivalence will be relevant when we will introduce AdS/CFT, since $SO(2, D)$ is also the isometry group of AdS_{D+1}.

A quantum field theory which has the conformal group as symmetries is called a conformal field theory (CFT). In such a theory, particles lie in irreducible representations of the conformal group; since the mass P^2 is not a Casimir for the whole group, it becomes useful to replace it with more relevant quantum numbers. Consider the dilation operator: in the quantum theory it will be represented by

$$D = -i(x^\mu \partial_\mu + \Delta) \tag{3.11}$$

where Δ gives the intrinsic scaling dimension of a field, which will in general transform as $\phi(x) \to \lambda^\Delta \phi(\lambda x)$. Δ is therefore a good quantum number. Considering the role of P and K as ladder operators, changing the conformal dimension by ± 1, we can deduce states will come in multiplets of ever-increasing dimension $\Delta_{(0)} + n$, $n \geq 0$, and that the lowest-dimension state will be annihilated by K_μ. Fields in the kernel of K_μ will be called primary, and others, obtained by applying powers of P_μ (hence, derivatives) will be called descendants.

A primary field is then identified by its conformal dimension and its representation under the Lorentz group, so, now specializing to $D = 4$, by quantum numbers (Δ, j_L, j_R). We recall Lorentz irreps are indexed by two half-integers (j_L, j_R), for example: $(0, 0)$ is a scalar, $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ are left/right Weyl spinors, $(\frac{1}{2}, \frac{1}{2})$ is a vector, and so on.

A classically conformal field theory very often fails to be conformal when quantized. This happens because the dilation symmetry is anomalous. Classical scale invariance clearly implies all couplings are adimensional; in the quantum theory these couplings g^i will run under renormalization with a corresponding β function, as in

$$\frac{dg^i}{d\ln\mu} =: \beta^i(g)\,. \tag{3.12}$$

The dependency of the running coupling on the energy scale, or equivalently the creation of a mass scale by dimensional transmutation, means the conformal symmetry is spoiled[2]. This happens for example in quantum chromodynamics, a classically conformal theory with a scale anomaly giving rise to the Λ_{QCD} mass scale, or quantum electrodynamics where the scale is at the Landau pole. Since the Noether current corresponding to dilations is the trace of the energy-momentum tensor, the anomaly will be detectable by the appearence of a nonzero matrix element $\langle T^{\mu}{}_{\mu}\rangle \propto \beta(g) \neq 0$.

Only if all the β functions vanish identically, i.e. if the theory is finite, is quantum conformal invariance guaranteed. We will encounter an example of such a theory in section 3.3. Otherwise the theory will only be conformal for specific values of the g^i at which all the β functions vanish, that is to say at fixed points. In general a quantum field theory will flow under renormalization from a non-conformal point towards an attracting IR fixed submanifold, the locus of $\left\{\beta^i(g) = 0\right\}$, called the conformal manifold.

An important point is that after the theory has regained its classical conformal symmetry after converging through RG flow to an IR fixed point, the quantum scaling dimensions Δ of operators will not coincide with the original value they had in the classical theory, the canonical dimension Δ_0. They will be modified by quantum corrections that add an anomalous dimension

$$\Delta = \Delta_0 + \gamma(g_*)\,, \quad \gamma(g) = -\frac{1}{2}\frac{d\ln Z}{d\ln\mu}, \tag{3.13}$$

where $\sqrt{Z}$ renormalizes the wavefunction[3], and g_* are the values of the

[2]We are here using conformal and scale (i.e. dilation) invariance interchangeably, but they are not identical. Conformal symmetry obviously includes dilations, but scale invariance + Poincaré does not generate the whole conformal group, as special conformal transformations are independent. Scale invariant but not conformal theories are known explicitly[3], but they are rare. We will work with the assumption dilation-invariant $\Rightarrow$ conformal.

[3]It should be noted some authors prefer to define $\gamma = -\frac{d\ln Z}{d\ln\mu}$. In addition, Z is generally a matrix that mixes different fields together under RG flow; however in all of the cases considered in this work this can be ignored, since only fields in the same gauge representation and with the same spin could ever mix, and those will always turn out to

couplings at the conformal fixed point.

Having introduced the extension of the Poincaré group to $SO(2,4)$, we would like to press this further to include supersymmetry. Supersymmetry is implemented by adding $\mathcal{N} \leq 4$ Weyl supercharges Q^A, $\overline{Q}_A$ $(A = 1, \ldots, \mathcal{N})$ to generate the super-Poincaré supergroup $\mathrm{ISO}(1,3|\mathcal{N})$. The superconformal group $SO(2,4|\mathcal{N})$ is then the minimal supergroup containing both. The first important feature is that a second set of supercharges S_A, $\overline{S}^A$ must be introduced to close the algebra, since

$$[K_\mu, Q^A] = -\sigma_\mu \overline{S}^A\,, \qquad\qquad [P_\mu, S_A] = \overline{Q}_A \overline{\sigma}_\mu\,; \tag{3.14}$$

so that superconformal symmetry $SO(2,4|\mathcal{N})$ involves twice as many supercharges as normal supersymmetry for a given $\mathcal{N}$. Another relevant excerpt from the table of commutators (which we do not reproduce in full) states Q^A and S_A are also ladder operators for dilations,

$$[D, Q^A] = \frac{i}{2} Q^A\,, \qquad\qquad [D, S_A] = -\frac{i}{2} S_A\,, \tag{3.15}$$

raising and lowering the dimension Δ by $\pm 1/2$. In a superconformal field theory (SCFT) we then expect multiplets of dimension $\Delta = \Delta_0 + \frac{n}{2}$. Primary operators must now be annihilated by both K_μ and S_A, and are classified again by dimension and spin (Δ, j_L, j_R) but also by the $U(1) \times SU(\mathcal{N})$ R-symmetry quantum numbers $(R, \mathbf{r})$ ($\mathbf{r}$ denoting a generic irrep of $SU(\mathcal{N})$). Then, by acting with the raising operator Q^A charges one can reconstruct a finite-dimensional supermultiplet, as in normal supersymmetry. Instead, powers of P_μ reconstruct the infinite ladder of derivatives forming an infinite-dimensional representation of the conformal group; these can be recombined into a field by Taylor expansion. In conclusion, an infinite-dimensional representation of the superconformal group can be organized into a superfield

$$\Phi_{\ldots}(x^\mu, \theta^A, \overline{\theta}^A) \tag{3.16}$$

be connected by a flavour symmetry.

where ... stands for Lorentz and R-charge-$SU(\mathcal{N})$ indices for the primary.

Actually, not all values for Δ are allowed in a quantum theory. Imposing physical states have non-negative norm (i.e., unitarity) results in lower bounds for the quantum scaling dimension[23],

$$\Delta \geq f(j_1, j_2)\,; \tag{3.17}$$

moreover, since violation of the bound results in negative norms, by continuity when it is saturated zero-norm states appear. This correspond in general to a shortening of the multiplet, which becomes constrained to be annihilated by a polynomial of P_μ. For example, for scalar fields

$$\Delta \geq 1 \tag{3.18}$$

and $\Delta = 1$ iff $\partial_\mu \partial^\mu \Phi = 0$, that is, Φ is free. In the spin-1 case, $\Delta \geq 3$ and equality holds only if the field is a conserved current ($\partial_\mu J^\mu = 0$); for spin-2 $\Delta \geq 4$ and $\Delta = 4$ only if $\partial_\mu T^{\mu\nu} = 0$, that is to say $T^{\mu\nu}$ must be the stress-energy tensor. This result implies in particular that conserved operators must have fixed, canonical dimension and so are not renormalized.

In $\mathcal{N} \geq 1$ SCFTs, more interesting unitarity bounds can be introduced by extending the above reasoning to include superconformal symmetries. Introducing the $U(1)$ R-charge symmetry (and normalizing such that $R_Q = 1$), one is led to bounds of the type

$$\Delta \geq f(j_1, j_2, R) \tag{3.19}$$

depending also on the R-charge of the superfield. Saturation corresponds to the appearance of ghosts and a shortening of the multiplet, which is annihilated by a polynomial of P_μ and Q. In particular, for scalars

$$\Delta \geq \frac{3}{2} R \tag{3.20}$$

and equality holds iff $\overline{D}_{\dot\alpha} \Phi = 0$, i.e. if the superfield is chiral. Thus, chiral fields will satisfy

$$\Delta = \frac{3}{2}R\,. \tag{3.21}$$

3.2 Features of D3-brane on Calabi-Yau cones

We consider a stack of N D3-branes on a ten-dimensional background of the form $\mathbb{R}^{1,3} \times X_6$. The branes are parallel to the $\mathbb{R}^{1,3}$ (which can be identified with the worldvolume) and are essentially points from the point of view of the 6D manifold X_6. Since, as it was anticipated, there is an interest in having the D-branes probe a conical singularity, we choose X_6 to be a cone, in the sense that $X_6 = \mathbb{R}_+ \times Y_5$ and

$$ds_6^2 = dr^2 + r^2 ds_5^2 \tag{3.22}$$

If $Y_5 = \mathbb{S}^5$ with the unit round metric then the cone is $X_6 = \mathbb{R}^6$ and one returns to the flat case. Therefore we include this as a trivial example of a cone.

In addition, we must require that the cone be Ricci-flat, so that it satisfies the supergravity equations of motion in vacuum. This is equivalent to Y_5 being Einstein of positive curvature, as we now show. ds_6^2 is conformally equivalent to the canonical metric on a cylinder over Y_5, as evidenced by the reparametrization $\phi = \ln r$:

$$ds_6^2 = e^{2\phi}\left(d\phi^2 + ds_5^2\right)\;; \tag{3.23}$$

recalling the transformation law of the Ricci tensor in n dimensions under conformal rescalings:

$$\begin{aligned} R'_{ij} = R_{ij} &- (n-2)\left(\nabla_i\partial_j\phi - \partial_i\phi\partial_j\phi\right) \\ &+ \left(\nabla^2\phi - (n-2)\nabla_k\phi\nabla^k\phi\right) g_{ij}\,, \end{aligned} \tag{3.24}$$

and noting that for the cylinder (which has a product metric) the restriction of R_{ij} to Y_5 indices gives Y_5's own Ricci tensor $R^{(5)}_{ij}$, we obtain

$$R_{ij}^{(5)} = 4g_{ij}^{(5)}\,. \tag{3.25}$$

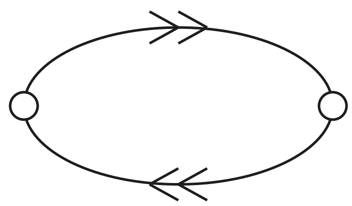
A manifold with $R_{ij} = \Lambda g_{ij}$, with Λ a constant, is called Einstein.

Also, we require that X_6 be Kähler, with the Ricci-flat metric g_{ij} being the Kähler metric. This restrictive property is necessary[14] for the field theory to have at least $\mathcal{N} = 1$ supersymmetry.

Indeed, bein Kähler and Ricci-flat implies that X_6 is a Calabi-Yau manifold, thus of restricted holonomy $\subset SU(3)$. More restricted holonomy results in enhanced supersymmetry; in particular if the holonomy group is contained in $SU(2)$ then the gauge theory will have $\mathcal{N} = 2$, and if the holonomy is trivial (i.e. $X_6 = \mathbb{R}^6$) then the supersymmetry will be maximal, $\mathcal{N} = 4$. Intuitively, this is because supersymmetries of X_6 carry over as rigid supersymmetries of the field theory; this fact will be explained more rigorously in the context of holography, however. An Einstein manifold Y_5 such that the corresponding cone X_6 is Calabi-Yau is called Sasaki-Einstein.

Independently of the background, theories resulting from D3-brane stacks will always be gauge theories, as gluons mode will always be present. In particular, the gauge group will be a product of $U(N)$ factors (nodes); the number of gauge factors is related to the topology of the cone as it is its Euler characteristic χ.*ref?*

In addition, the theory will be populated by chiral fields in "bifundamental" representations, i.e. with an index in the fundamental of one $U(N)$ node and a second in the antifundamental of another. These sort of theories are termed quiver gauge theories and they can be encoded in a quiver diagram, where $U(N)$ factors are denoted by nodes and bifundamental fields as directed arrows stretching between two nodes. As an example, we present the quiver diagram for one of the CFTs we will introduce in this chapter, the Klebanov Witten theory:



The diagram has a left and right node signaling each a $U(N)$ gauge factor, so the gauge group is $SU(N) \times SU(N)$. The two arrows moving from left to right represent two different chiral fields A_1 and A_2 both in the $(\overline{\mathbf{N}}, \mathbf{N})$ gauge representation, while the lower arrows are two other fields B_1 and B_2 transforming as $(\mathbf{N}, \overline{\mathbf{N}})$.

Let us review briefly the structure of the action of an $\mathcal{N} \geq 1$ gauge theory; we follow [31]. There is a gauge vector superfield V (corresponding to an on-shell multiplet (A_μ, λ)) with values in the algebra (that is $V = T^a V_a$ with V_a in the adjoint), with an associated field-strength

$$W_\alpha = -\frac{1}{4}\overline{D}_{\dot{\alpha}}\overline{D}^{\dot{\alpha}} D_\alpha V \tag{3.26}$$

and the dynamics of the free vector are given by the Lagrangian

$$\mathcal{L}_{\mathrm{SYM}} = \frac{1}{4}\int d^2\theta\ \mathrm{Tr}\, W^\alpha W_\alpha + \mathrm{h.\,c.} \tag{3.27}$$

In addition, one can also include chiral superfields $\Phi_I = (\varphi_I, \psi_I)$ charged under the gauge group. These will have a kinetic term

$$\mathcal{L}_\Phi = \int d^4\theta\, \Phi_I^\dagger e^{gV} \Phi_I \tag{3.28}$$

which is a correction of the canonical $\int d^4\theta\, \Phi_I^\dagger \Phi_I$ to implement gauge invariance. g is the gauge coupling; note this means that one should really introduce separate gauge fields for each simple factor in the gauge group, as each one will have an independent coupling. Finally, one is free to add an interaction superpotential for the chiral fields:

$$\mathcal{L}_{\mathrm{int}} = \int d^2\theta\, W(\Phi) + \mathrm{h.\,c.} \tag{3.29}$$

provided W is a holomorphic, gauge invariant combination of the Φ_I.

Actually, it still possible to add another term consistent with the symmetries to the Lagrangian. In the non-supersymmetric case, the Maxwell action $\frac{1}{2g_{YM}^2}\int \mathrm{Tr}\, F \wedge *F$ can be supplemented by a topological term $\frac{\theta}{8\pi^2}\int \mathrm{Tr}\, F \wedge F$ fronted by a theta angle θ. The same can be done in an $\mathcal{N} = 1$ gauge theory by replacing the kinetic lagrangian (3.27) with

$$\mathcal{L}_{SYM} = \frac{1}{32\pi} \operatorname{Im} \left(\tau \int d^2\theta \operatorname{Tr} W^\alpha W_\alpha \right) \tag{3.30}$$

after having introduced the complexified coupling

$$\tau := \frac{\theta}{2\pi} + \frac{4\pi i}{g^2} \tag{3.31}$$

3.2.1 Renormalization and supersymmetric beta functions

In the study of D3-brane quiver theories we will need to identify their conformal manifolds as defined in 3.1, or less ambitiously just its dimension. This is the number of marginal directions, deformations of the theory that preserve its superconformal invariance. In general we will have a space of parameters $(g_1, g_2, \ldots, g_\chi, \lambda_1, \ldots, \lambda_k)$ including gauge and superpotential couplings, and the conformal manifold will be the submanifolds of values of these couplings for which $\beta_{g_1} = \ldots = \beta_{g_\chi} = \beta_{\lambda_1} = \ldots = \beta_{\lambda_k} = 0$.

Thankfully, the renormalization structure of supersymmetric theories is in general vastly simplified with respect to the general case. In the case of $\mathcal{N} \geq 1$ supersymmetry, there is a remarkably simple formula for the gauge beta functions, connecting them to the anomalous dimensions of the fields:

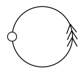
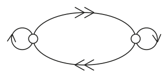
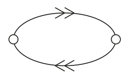
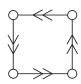
$$\beta(g_a) = -\frac{g_a^3}{16\pi^2} \frac{3T[\text{Adj}] - \sum_i T[R_i](1 - 2\gamma_i)}{1 - N g_a^2/8\pi^2} \,. \tag{3.32}$$

$T[R]$ is the Dynkin index of the representation R of the gauge group. The sum is over the chiral fields charged under the gauge group, and their representations and anomalous dimensions R_i, γ_i. This is known as the Novikov-Shifman-Vainshtein-Zakharov (NSVZ) β function, and is known to be correct to all orders in perturbation theory[9]. A further simplifying step will be possible because in addition to supersymmetry the theory has superconformal invariance, as the anomalous dimension $\gamma = \Delta - 1$ will be connected to the R-charge at conformal points according to (3.21).

For what concerns instead the β functions for superpotential couplings, it is easy to see the scale-invariance of the action corresponds to an exact

R-charge of 2 for the superpotential. Therefore the vanishing of these β functions is equivalent to imposing the sum of R-charges of the entering chiral fields is 2.

We now are ready to begin our investigation of the following specific chain of theories:

Theory	$\mathcal{N}$	Quiver diagram
$\mathcal{N} = 4$ Super-Yang-Mills	4	
	$\downarrow \mathbb{Z}_2$ orbifold	
$\mathbb{C}^3/\mathbb{Z}_2$	2	
	$\downarrow$ mass deformation	
Klebanov-Witten	1	
	$\downarrow \mathbb{Z}_2$ orbifold	
$Y^{2,0}$	1	

The transformations turning each theory into the next will be explained progressively.

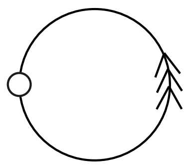
3.3 Brane stack in $\mathbb{C}^3$ and $\mathcal{N} = 4$ super-Yang-Mills

If $X_6 = \mathbb{C}^3$, the branes are invariant under half of the $16 \times 2 = 32$ IIB supercharges. This implies $\mathcal{N} = 4$ for the field theory. Moreover, the theory features gluons as the massless spin-1 modes for the sector of strings stretching between brane i and brane j so that the gauge group is $U(N)$, as seen in 2.4.1. The information that the theory is a $U(N)$ gauge theory and is maximally supersymmetric is enough to uniquely fix it. Actually, since our main interest is in the IR limit, the $U(1)$ gauge factor decouples as we have explained in general, and the group is reduced to $SU(N)$.

In $\mathcal{N} = 1$ language (which we employ even though the model has $\mathcal{N} = 4$) the theory describes the dynamics of an $U(N)$ gauge vector supermultiplet V_μ and three complex chiral superfields $(X^a)_{i\dot{j}}$, $a = 1, 2, 3$ in the adjoint of the gauge group (we will frequently omit gauge indices). These are nothing else than the parametrization of the D3-branes' position in $\mathbb{C}^3$ and therefore transform in the fundamental of $SU(3)$. The superpotential is the only one allowed by gauge and $SU(3)$ invariance,

$$W(X) = g\epsilon_{abc} \operatorname{Tr}(X^a X^b X^c) \, ; \tag{3.33}$$

note the coupling g is also the coupling of the gauge group, not an indipendent parameter. Since the theory has a single $SU(N)$ factor in the gauge group, and three fields in the adjoint (which is trivially a bifundamental with the same node at the two ends), the quiver diagram is rather simple:



Note that the obvious global symmetries, the $U(1)$ R-charge for $\mathcal{N} = 1$ and the $SU(3)$ flavour symmetry acting on the X^a are actually just a subgroup of a $SU(4)$ R-symmetry group for the hidden $\mathcal{N} = 4$ supersymmetry. To make this manifest, split the $\mathcal{N} = 1$ superfields as $V \to (A_\mu, \lambda)$ and $X^a \to (\phi^a, \psi^a)$, and regroup the fields as

$$\lambda^\alpha := (\lambda, \psi^1, \psi^2, \psi^3) \, , \quad \phi^i =: \varphi^i + i\varphi^{i+3} \, , \tag{3.34}$$

then $(A_\mu, \lambda^\alpha, \varphi^i)$ form an $\mathcal{N} = 4$ vector supermultiplet, the components transforming respectively as **1**, **4**, **6** under R $SU(4)$, and in the adjoint of gauge $SU(N)$.

3.3.1 Marginal deformations

$\mathcal{N} = 4$ (maximal) supersymmetry is extremely constraining, and indeed a non-renormalization theorem (first proven nonperturbatively in [29]) states the theory is exactly finite. No divergences means the β function (one for the only gauge coupling) vanishes identically, and the theory is always superconformal. Indeed, there is a set of additional 16 supercharges beyond the standard ones. These do *not* however find a direct correspondence as supersymmetries of the D3-brane system, but only of the "near-horizon" ($\alpha' \to 0$) geometry of the brane stack.

However, one can also discuss the vanishing of the beta function in $\mathcal{N} = 1$ language as a trivial example. We note $SU(3)$ flavour symmetry imposes that the R-charges and anomalous dimensions of the three chiral fields are equal. Then, for the superpotential to be scale invariant we need R_W to be 2, so

$$R_W = 2 = R + R + R \Rightarrow R = \frac{2}{3} \Rightarrow \gamma = 0\,, \tag{3.35}$$

where (3.21) was used; thus, the chiral fields have canonical dimension. Then the NVSZ beta function for the gauge coupling reads

$$\beta_\tau \propto 3N - 3N(1 - 2\gamma) = 3N - 3N = 0\,. \tag{3.36}$$

So, this vanishes identically, for all values of the coupling τ. Thus the conformal manifold of such a theory is minimal: there is a single marginal deformation corresponding to changing τ, and the theory is a SCFT for every value of this unique parameter.

3.3.2 Moduli space

One could wonder instead about the moduli space $\mathcal{M}$ of the theory. This is the space of inequivalent vacua; classically, one plugs uniform vacuum expectation values X^a for the chiral fields into the action to get an effective potential:

$$\mathcal{V}(X^a) = \frac{\partial W^\dagger}{\partial X_a^\dagger}\frac{\partial W}{\partial X^a} =: F^{a\dagger} F_a\,, \tag{3.37}$$

which is then minimized. Since $\mathcal{V} \geq 0$ and it equals zero at the special point $X^a = 0$, its minima will coincide with its zeroes, which occurr when all of the F^a vanish:

$$F_a := \frac{\partial W}{\partial X^a} \tag{3.38}$$

F_a is known as an F-term and the conditition (3.38) is known as an F-flatness condition.

The moduli space of SYM4 is then very easy to describe. The F-term conditions simply read:

$$\frac{\partial W}{\partial X^a} \propto \varepsilon_{abc} X^b X^c = 0\,, \tag{3.39}$$

so that the space of solution is given by (VEVs of) X^a that commute with eachother as $N \times N$ matrices. Therefore, they can be simultaneously diagonalized by a gauge transformation, and the $3N$ eigenvalues x_I^a, $I = 1, \ldots, N$ are completely free. In fact, these coincide directly with the coordinates of the N D3-brane on $\mathbb{C}^3$. The moduli space is $\mathbb{C}^{3N}$, or to be precise, taking into account brane indistinguishability (i.e., the residual permutation gauge symmetry after diagonalization of the X^a),

$$\mathcal{M} = \mathrm{Sym}^N \mathbb{C}^3\,. \tag{3.40}$$

In fact, this structure will always be present in D3-brane theories. Moduli space will always include a $\mathcal{M}_{\mathrm{mes}} \subset \mathcal{M}$ describing the motion of the D-branes on X_6, and it will be true in general that $\mathcal{M}_{\mathrm{mes}} = \mathrm{Sym}^N X_6$. However, $\mathcal{M}_{\mathrm{mes}}$ will not comprise the totality of moduli space and in more complex theories additional directions to $\mathcal{M}$ will arise.

3.4 $\mathbb{C}^3/\mathbb{Z}_2$ orbifold

We now move to a less trivial case by performing an orbifold of the background. Essentially, we act on $\mathbb{C}^3$ as such

$$(z^1, z^2, z^3) \mapsto (-z^1, -z^2, z^3) \tag{3.41}$$

and quotient under this $\mathbb{Z}_2$ group. This yields a Calabi-Yau manifold with a conical singularity in $z^1 = z^2 = 0$. Equivalently, if the background is presented in polar coordinates as $\mathbb{R}_+ \times \mathbb{S}^5$, the orbifold produces $\mathbb{R}_+ \times (\mathbb{S}^5/\mathbb{Z}_2)$. Our interested is then for the worldvolume theory of N D3-branes placed in this singular background.

To investigate this, we consider $\tilde{N} = 2N$ D3-branes on $\mathbb{C}^3$, producing SYM4 as seen in the previous section, then we act on the A_μ and X^a fields with the $\mathbb{Z}_2$ action and select the subset of invariant fields - these will be the degrees of freedom of the orbifold theory.

We need to specify an action of $\mathbb{Z}_2$ on the gauge indices. The following choice is convenient: act on an object in the fundamental $\tilde{\mathbf{N}} = \mathbf{N} \oplus \mathbf{N}$ as 1 on the first factor and -1 on the second. This means, for example, that having decomposed A_μ in subrepresentations its transformation under $\mathbb{Z}_2$ is

$$A_\mu = \begin{pmatrix} A^{0,0} & A^{1,0} \\ A^{0,1} & A^{1,1} \end{pmatrix} \mapsto \begin{pmatrix} A^{0,0} & -A^{1,0} \\ -A^{0,1} & A^{1,1} \end{pmatrix}, \tag{3.42}$$

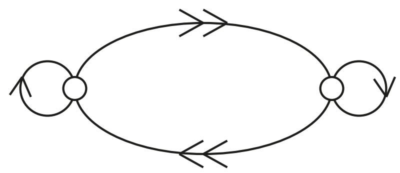
where $A^{i,j}$ are $N \times N$ matrices. We see the surviving gauge fields are $A^{0,0}$ and $A^{1,1}$, adjoint for the $U(N) \times U(N)$ subgroup of $U(2\tilde{N})$. The new gauge group is therefore $U(N) \times U(N)$.

The same holds for X^3, which becomes $\Phi := (X^3)^{0,0}$ and $\tilde{\Phi} := (X^3)^{1,1}$, two chiral fields each in the adjoint of one of the $U(N)$ factors.

For X^1, X^2 instead one has to take into account both the action on the gauge indices and the direct action on the $\mathbb{C}^3$ geometry. With $i = 1, 2$, the overall action is

$$X^i = \begin{pmatrix} (X^i)^{0,0} & (X^i)^{1,0} \\ (X^i)^{1,0} & (X^i)^{1,1} \end{pmatrix} \mapsto \begin{pmatrix} -(X^i)^{0,0} & (X^i)^{1,0} \\ (X^i)^{1,0} & -(X^i)^{1,1} \end{pmatrix}, \tag{3.43}$$

and one ends up with two pairs of chiral fields in bifundamentals, $A^i := (X^i)^{0,1}$, $B^i := (X^i)^{1,0}$, in representations $(\mathbf{N}, \overline{\mathbf{N}})$ and $(\overline{\mathbf{N}}, \mathbf{N})$ of $U(N) \times U(N)$ respectively. The structure of the theory can be more elegantly presented in a quiver diagram:



The superpotential is just given by restriction of the SYM4 potential (3.33) to the surviving fields; after some algebra

$$W = \mu \left(\operatorname{Tr} \Phi(A_1 B_1 + A_2 B_2) + \operatorname{Tr} \tilde{\Phi}(B_1 A_1 + B_2 A_2) \right) \tag{3.44}$$

It is clear at this point that, given the introduction of an asymmetry between X^3 and $X^{1,2}$, the $\mathcal{N} = 4$ R-symmetry $SU(4)$ is broken and so is maximal supersymmetry. The orbifold theory has indeed $\mathcal{N} = 2$ supersymmetry[10].

We will not concentrate on the details of this $\mathbb{C}^3/\mathbb{Z}_2$ theory; we use it as a stepping stone from SYM4 to the Klebanov-Witten theory, which we will introduce in the next section as a deformation of the orbifold theory.

3.5 The conifold and the Klebanov-Witten model

In [14] the case of X_6 being the conifold was studied. The conifold is a specific Calabi-Yau 3-cone defined for example as the following variety in $\mathbb{C}^4$:

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\,, \tag{3.45}$$

with the Kähler structure inherited from the standard one on $\mathbb{C}^4$, or, after a simple change of variables:

$$uv - xy = 0\,. \tag{3.46}$$

The base can be found by quotienting by dilations $z_i \to \lambda z_i$ (with $\lambda \in \mathbb{R}_+$) and turns out to be the homogeneous space $SO(4)/U(1) = SU(2) \times SU(2)/U(1)$, where the $U(1)$ is a diagonal subgroup generated by, say, $T^3_L + T^3_R$. Topologically, this is $\mathbb{S}^2 \times \mathbb{S}^3$. We will therefore have $SU(2) \times SU(2)$ as part of the isometry group of both Y_5 and X_6, and thus will also appear as a global symmetry of the wordvolume theory. An equivalent description of the topology of the conifold is as a $U(1)$ bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1 \cong \mathbb{S}^2 \times \mathbb{S}^2$; in these terms the metric on this cone is

$$\begin{aligned} ds_6^2 &= dr^2 + r^2 ds_5^2 \\ ds_5^2 &= \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6}(d\Omega_1^2 + d\Omega_2^2) \end{aligned} \tag{3.47}$$

where $\Omega_i^2 = d\theta_i^2 + \sin\theta_i^2 d\phi_i^2$ is the metric on the $\mathbb{C}P^1_i$, and ψ is the fibral coordinate with period 4π.

The corresponding field theory (which we will call the Klebanov-Witten model), however, can also be found by applying a particular modification to the orbifold theory of the previous section[14]. Let us derive the form of this SCFT in this way, and then show how the above geometry reappears from the field theory side. The modification is the addition of a relevant term to the Lagrangian, a mass for the Φ, $\tilde{\Phi}$ adjoint fields

$$\mathcal{M} = \frac{M}{2}\left(\mathrm{Tr}\,\Phi^2 - \mathrm{Tr}\,\tilde{\Phi}^2\right), \tag{3.48}$$

thus providing a possible UV completion for the $\mathbb{C}^3/\mathbb{Z}_2$ theory. These fields are then eliminated using the resulting classical equations of motion. For example,

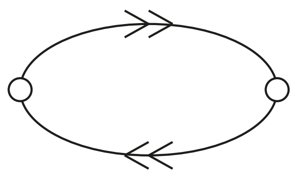
$$\frac{\partial(\mathcal{M} + W)}{\partial\Phi} = 0 \Rightarrow \Phi = -\frac{\mu}{M}\left(A_1 B_1 + A_2 B_2\right) \tag{3.49}$$

and analogously for $\tilde{\Phi}$. Finally, these are substituted back into the action, reducing the chiral fields to just A_i and B_i and the superpotential to

$$W = \frac{\lambda}{2}\epsilon^{ij}\epsilon^{kl}\,\mathrm{Tr}\left(A_i B_k A_j B_l\right) \tag{3.50}$$

having defined $\lambda := \frac{\mu^2}{2M}$. More formally, it is argued that after the introduction of the relevant term (3.48) the $\mathbb{C}^3/\mathbb{Z}_2$ theory flows in the IR to the Klebanov-Witten model.

Therefore, the worldvolume gauge theory is a $U(N) \times U(N)$ field theory featuring two chiral doublets A_i, B_j with $i, j = 1, 2$ transforming in opposite bifundamentals, that is A_i in $(N, \bar{N})$ and B_j in $(\bar{N}, N)$. The quiver diagram is simpler:



The i and j indices, instead, are acted upon respectively by the global left and right $SU(2)$ symmetries. From the form of the superpotential and these symmetries, furthermore, it is possible to deduce the R-charges of A_i and B_i are $1/2$, implying a dimension of $\Delta = 3/4$ when the theory is conformal. A non-canonical scaling dimension is a symptom that the CFT will always be strongly-coupled.

It is also clear the supersymmetry has been broken further to $\mathcal{N} = 1$, since (3.48) is not $\mathcal{N} = 2$ invariant. The case at hand is thus a four-dimensional strongly-coupled field theory with minimal supersymmetry.

3.5.1 Marginal deformations

The Klebanov-Witten theory will not be in general superconformal; it will flow through renormalization in the IR to a conformal submanifold in the space of couplings $(\lambda, \tau_1, \tau_2)$, the locus where the β functions for these three couplings vanish. It turns out these three conditions are all equivalent. In particular, requiring either $\beta_{\tau_1} = 0$ or $\beta_{\tau_2} = 0$ and making use of the NSZV (3.32) this unique condition is equivalent to

$$3T[\mathrm{Adj}] - \sum_i T[R_i](1 - 2\gamma_i) = 0\,. \tag{3.51}$$

When evaluating this, care should be taken with the fact that A_i and B_j have a $U(N)_2$ index which is uncharged under $U(N)_1$ and must be summed over. Noting $\gamma_{A_1} = \gamma_{A_2}$ and $\gamma_{B_1} = \gamma_{B_2}$ because of the global flavour $SU(2) \times SU(2)$ symmetry, this gives

$$\gamma_A + \gamma_B + \frac{1}{2} = 0\,. \tag{3.52}$$

Being $\gamma_{A,B}$ functions of the couplings, this equation defines a critical 2-surface in parameter space. Switching to R-charges, this means

$$R_A + R_B = 1\,, \tag{3.53}$$

but this is also the equation for the vanishing of the β function for the superpotential coupling λ, since $R_W = 2$ is just

$$R_A + R_B + R_A + R_B = 2\,. \tag{3.54}$$

Thus, since the three conditions are equivalent, the final conformal manifold is the locus of a single equation in three variables $(\lambda, \tau_1, \tau_2)$ and is then two-dimensional. Therefore there will be two marginal deformations of the theory.

3.5.2 Global $U(1)$

Let us comment briefly on the whereabouts of the abelian gauge factors of $U(N) \times U(N)$ under RG flow towards the infrared. Denoting as T^1 and T^2 the generators of the two trace $U(1)$ of the left and right nodes, it is clear $T^1 + T^2$ is the overall completely decoupled $U(1)_{\text{trace}}$ that we can safely ignore. The only remaining abelian factor is generated by $B = T^1 - T^2$. This symmetry is non-anomalous; since it is an abelian gauge interaction with massless charged matter, under RG flow towards the infrared its coupling vanishes and thus freezes into a rigid $U(1)_{\text{baryonic}}$ in the IR. We call this charge, corresponding to the symmetry

$$A_i \to e^{i\theta} A_i\,, \quad B_i \to e^{-i\theta} B_i \tag{3.55}$$

a baryon number. This quantum number will become important in classifying gauge invariant operators, as we will see shortly.

3.5.3 Moduli space

Position in moduli space $\mathcal{M}$ should be parametrized by the expectation values of gauge-invariant operator (hadrons). This should be obtained, as it was sketched for SYM4, by considering the locus of minima of the effective potential $\mathcal{V}$ in the space of VEVs of the chiral fields, and then quotienting by complexified gauge transformations. Taking for the moment a general $\mathcal{N} = 1$ theory with gauge generators T^a and chiral fields ϕ^i, the effective potential can be found to be

$$\mathcal{V}(\phi) \;=\; \frac{g_a^2}{2}(\phi^\dagger T^a \phi)^2 + \frac{\partial W^\dagger}{\partial \phi_i^\dagger}\frac{\partial W}{\partial \phi^i} \;=:\; \frac{g_a^2}{2}(D^a)^2 + F_i^\dagger F^i \tag{3.56}$$

We have defined the F-terms and the D-terms:

$$F^i = \frac{\partial W}{\partial \phi^i}\,, \tag{3.57}$$

$$D^a = \sum_i \phi_i^\dagger T^a \phi_i\,. \tag{3.58}$$

Again, $\mathcal{V}$ will be minimum when $\mathcal{V} = 0$, thus when the F-term and D-term vanish, conditions we will call F-flatness and D-flatness. The space of simultaneous solutions to $F^i = D^a = 0$, quotiented by gauge transformations, will be moduli space $\mathcal{M}$.

Specializing to the Klebanov-Witten model, $D^a = 0$ will hold only with a spanning over generators of $SU(N) \times SU(N)$, not involving the abelian factors $U(1)_1 \times U(1)_2 = U(1)_{\text{trace}} \times U(1)_{\text{baryon}}$, since according to our discussion in section 3.5.2 these do not survive as gauge symmetries in the IR. The D-term of $U(1)_{\text{trace}}$ has no significance, since this is completely decoupled and vanishes identically. The baryon D-term instead gets relaxed

$$D^{U(1)_{\text{baryon}}} = \xi \tag{3.59}$$

where ξ is an arbitrary parameter. Actually, ξ is a modulus, parametrizing a direction in $\mathcal{M}$ itself. Indeed, it is the only additional direction in $\mathcal{M}$ to those along $\mathcal{M}_{\mathrm{mes}}$, which is a strict subset of the KW moduli space. Let us determine $\mathcal{M}$, identify $\mathcal{M}_{\mathrm{mes}}$ and verify these claims. The vanishing of the F-terms (3.57) and D-terms (3.58), specialized to the particular case, are

$$\epsilon^{ij} A_i B_a A_j = \epsilon^{ij} B_i A_a B_j = 0 \tag{3.60}$$

$$A_i A_i^\dagger - B_i B_i^\dagger = A_i^\dagger A_i - B_i^\dagger B_i = \xi \mathbb{1} \tag{3.61}$$

the equation have to be understood to hold for VEVs. Note the first and second D-term condition are respectively from the left and right gauge group.

Consider the subset of $\mathcal{M}$ with $\xi = 0$; our claim is this is precisely $\mathcal{M}_{\mathrm{mes}}$. To reinterpret the F-flatness condition, we introduce the four matrices $\Phi_{ij} = A_i B_j$ and note

$$[\Phi_{ij}, \Phi_{jk}] = 0 \tag{3.62}$$

$$\Phi_{11}\Phi_{22} = \Phi_{21}\Phi_{12} \tag{3.63}$$

which can be immediately checked to follow from the vanishing of the F-term. Since these commute, they can be simultaneously diagonalized and their N eigenvalues (one for each brane) satisfy the conifold's equation (3.46):

$$\phi^I_{11}\phi^I_{22} = \phi^I_{21}\phi^I_{12} \tag{3.64}$$

so these quite literally parametrize the motion of the N D3-branes on the background cone. These actually determine the VEVs of mesonic operators, mesons[4] being generated by prototipical trace operators:

[4]The meson/baryon terminology is meant to be a direct generalization of the concept of QCD hadrons. A QCD meson is to a first approximation built from two quarks as a symmetric product $|M\rangle \sim \delta^i_j\,|q^i \overline{q}_j\rangle + \mathcal{O}(gluons)$, while a baryon is $|B\rangle \sim \epsilon_{ijk}\,|q^i q^j q^k\rangle$. In general, $SU(N)$-singlets can be built by contracting gauge indices either with the symmetrical δ^i_j or the antisymmetric $\epsilon_{a_1 \ldots a_N}$.

$$M_{(ab\ldots),(ij\ldots)} = \mathrm{Tr}\left((A_a B_i)(A_b B_j)\cdots\right) . \tag{3.65}$$

This justifies the terminology of "mesonic moduli space" for the submanifold $\mathcal{M}_{\mathrm{mes}}$ of $\mathcal{M}$ they describe.

Note mesons are built by tracing over closed loops in the quiver diagram to make a gauge-invariant operator. All of these operators are actually expressable as products of Φ matrices, and as it was shown, only 3 out of 4 of those are independent. In the end, there are (accounting for gauge indices) $3N$ independent mesons whose VEVs parametrize mesonic moduli space, coincident with the $\mathrm{Sym}^N C$, where C is the conifold.

Operators of non-zero baryon number can also be constructed by using the antisymmetric invariant gauge tensor $\epsilon^{a_1 \cdots a_N}$, as such:

$$\mathcal{B}^A_{[k]} = \epsilon^{a_1 \ldots a_N} \epsilon_{b_1 \ldots b_n} (A_1)^{b_1}_{a_1} \ldots (A_1)^{a_k}_{b_k} (A_2)^{a_{k+1}}_{b_{k+1}} \ldots (A_2)^{b_N}_{a_N} \tag{3.66}$$

where we have displayed gauge indices on the A fields. There are only $N+1$ different assignment for the $SU(2)$ indices because of antisymmetry, so that there are $N+1$ fundamental baryons of the form of $\mathcal{B}^A$. The same could be done by swapping the two gauge groups and using B fields, to get additional $N+1$ $\mathcal{B}^B$ baryons. These all have baryon number N under $U(1)_{\mathrm{baryonic}}$ while mesons have baryon number 0.

All gauge-invariant operators in the theory are built out of these fundamental mesons, fundamental baryons, and their respective antiparticles (made out of the conjugate fields $A^\dagger$, $B^\dagger$. However, as we have anticipated, we only expect $g-1=1$ baryonic VEV to be independent. This VEV will be associated with the resolution of the cone singularity into a $\mathbb{CP}^1 \cong \mathbb{S}^2$, and will essentially coincide with ξ. To see an example of this deformation of the background geometry, let us set ξ to a constant nonzero value. Then hypothesizing for simplicity that the A_1, A_2, B_1, B_2 matrices commute, applying F-term conditions we get that each set of eigenvalues satisfies

$$a_1/a_2 = b_1/b_2 \tag{3.67}$$

So that a_i and b_i are proportional vectors of $\mathbb{C}^2$, therefore

$$a_i = ae^{i\theta_A} n_i, \qquad be^{i\theta_B} n_i \tag{3.68}$$

where a, b are real and n_i belongs to a $\mathbb{CP}^1$. The phases are cancelled by modding gauge invariance, and a and b then are involved in the D-term:

$$a^2 - b^2 = \xi \tag{3.69}$$

so that essentially our mesonic VEVs are composed of N copies of one non-compact radial coordinate (say, a) and a point on $\mathbb{CP}^1$. This means the conical singularity has disappeared to be replaced by a two-cycle on which the branes can move.

A more thorough investigation of the geometry probed by these mesonic directions when ξ has a non-zero value would show the D3-branes are moving on the geometry of the resolved conifold, the metric[25]

$$ds_6^2 = \kappa^{-1} dr^2 + \frac{1}{9}\kappa r^2 (d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6} r^2 d\Omega_1^2 + \frac{1}{6}(r^2 + a^2) d\Omega_2^2 \tag{3.70}$$

$$\kappa(r) = \frac{r^2 + 9a^2}{r^2 + 6a^2} \tag{3.71}$$

parametrized by a modulus a. Turning on the ξ modulus corresponds to increasing a (in fact, the two are proproportional[21]), and to the blow up of the two-sphere at $r = 0$. Instead, for $a = 0$ one recovers the singular conifold (3.47).

Thus, we have uncovered the following structure for $\mathcal{M}$. It is a $(3N + 1)$-dimensional[5] complex manifold on which one can define the coordinate ξ, a baryonic modulus. The $3N$-submanifold at $\xi = 0$ is $\mathcal{M}_{\text{mes}}$, which is equal to the symmetric product of N copies of the conifold and is parametrized by $3N$ complex moduli, VEVs of particular combinations of mesonic operators, that can be identified with the positions of the threebranes. If instead one consider a constant but nonzero value $\xi = \xi^*$ for the baryonic modulus,

[5] There is an obvious dimensional mismatch in that the complex modulus ξ corresponds to a real parameter a in the metric. It turns out that when the real part of a modulus maps to a Kähler deformation, the imaginary part describes instead a modulus for the A_4 Ramond-Ramond form; we will explain more in chapter 5

the same mesonic directions define a submanifold with the structure of the symmetric product of N copies of the *resolved* conifold (3.70).

The explicit form of the baryon generating this deformation in terms of the fundamental hadrons is very challenging to determine[11]; fortunately, we will not need it for our purposes. In any case, all of this information will be clarified in the context of holography.

3.6 The $Y^{2,0}$ orbifold theory

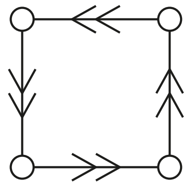
The same construction on a $\mathbb{Z}_2$ orbifold of the conifold yields a quiver gauge theory which will be the main interest of this work. This theory has a few interesting additional features with respect to the Klebanov-Witten model while remaining relatively simple, thus being an optimal candidate for the investigation of its effective low-energy theory, which we will perform in chapter 6. It is, like the conifold theory, an $\mathcal{N} = 1$ SCFT but boasts three baryonic moduli, of which one has no clear geometric interpretation, and two anomalous $U(1)$s. We now describe this theory and extract these features.

The geometry of the base of the cone is very simply introduced in polar coordinates as

$$\begin{aligned} ds_6^2 &= dr^2 + r^2 ds_5^2 \\ ds_5^2 &= \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6}(d\Omega_1^2 + d\Omega_2^2) \end{aligned} \tag{3.72}$$

i.e., exactly the same metric in form as the conifold, but with ψ now with period 2π. This background and the resulting worldvolume field theory are just one entry $Y^{2,0}$ of an infinite class $Y^{p,q}$ of examples introduced in [5]. (To be precise, we will use $Y^{2,0}$ to refer to the 5-dimensional base (3.72), following [18], and $X^{2,0}$ for the corresponding cone).

Following an identical procedure to that performed for the $\mathbb{Z}_2$ orbifold of SYM4, we can deduce the quiver diagram splits to yield four doublets A_i, B_i, C_i, D_i of bifundamental chiral fields stretching in a square between four nodes:



so that the gauge group is $SU(N)^4$. The superpotential can be again obtained by truncation, yielding

$$W = \lambda \epsilon^{ij} \epsilon^{kl} \operatorname{Tr} (A_i B_k C_j D_l) \tag{3.73}$$

from which it is clear that the $SU(2) \times SU(2)$ isometry of the cone, corresponding to a global flavour symmetry of the field theory, must now act as

$$\begin{aligned} A_i &\in (\mathbf{2}, \mathbf{1}) \\ B_i &\in (\mathbf{1}, \mathbf{2}) \\ C_i &\in (\mathbf{2}, \mathbf{1}) \\ D_i &\in (\mathbf{1}, \mathbf{2}) \end{aligned} \tag{3.74}$$

Note (3.73) is also the only possible superpotential allowed by this symmetry[6], so that the theory can also be constructed by starting with the flavour symmetry as an assumption.

3.6.1 Global $U(1)$s

The novelty in this model is the appearance of anomalous $U(1)$s. In detail, of the four gauge $U(1)$s, one (the symmetric sum) is the overall trace and decouples completely. It turns out that the three remaining abelian factors are partitioned into one non-anomalous baryonic $U(1)$ (which is inherited directly from the KW model) and two new anomalous $U(1)$s. The non-anomalous baryon number acts on A_i, C_i with charge $+1$ and B_i, D_i with charge -1, a fact obvious from orbifold of the KW baryon number. The action of the two anomalous $U(1)$s has an arbitrariness as they can obviously be shifted by the trace or non-anomalous charge; an example of charge assignments could be

charges of A, B, C, D	$U(1)$
$(+1,+1,+1,+1)$	Trace (decoupled)
$(+1,-1,+1,-1)$	Non-anomalous baryon number
$(+1,+1,-1,-1)$	Anomalous
$(+1,+1,+1,-1)$	Anomalous

We have already commented on how the non-anomalous abelian factor becomes a rigid global symmetry flowing down to the IR. However, the anomaly of the other two abelian gauge symmetries is worrying as it would make the theory inconsistent. Actually, it turns out these anomalies are cancelled at the level of the supergravity background by the contribution of axionic fields charged under the anomalous $U(1)$s; in particular the $X^{2,0}$ cone has a 2-cycle S and a 4-cycle Q such that the integrals

$$\phi_1 = \int_S A_2 \quad \phi_2 = \int_Q A_4 \tag{3.75}$$

constitute dynamical four-dimensional fields $\phi_1(x^\mu)$, $\phi_2(x^\mu)$ which cancel the anomaly, in a form of the Green-Schwarz mechanism[6]. However, these fields transform as axions under the two anomalous $U(1)$, and so break the corresponding symmetry in a Stückelberg mechanism, thus the corresponding photons become massive.

3.6.2 Marginal deformations

We turn to the study of marginal deformations. Again, the flavour symmetry (3.74) guarantees $\gamma_{A_1} = \gamma_{A_2} = \gamma_A$ and so on. This time three of the four gauge β functions are independent:

[6]This is actually expected, as string theory is quantum consistent and no configuration in it can display anomalies.

$$\begin{aligned} \beta_1 = 0 \quad &\Rightarrow \quad \gamma_A + \gamma_D + \frac{1}{2} = 0\,, \\ \beta_2 = 0 \quad &\Rightarrow \quad \gamma_B + \gamma_A + \frac{1}{2} = 0\,, \\ \beta_3 = 0 \quad &\Rightarrow \quad \gamma_C + \gamma_B + \frac{1}{2} = 0\,. \end{aligned} \tag{3.76}$$

In addition, $\beta_\lambda = 0$ is also not independent: at any superconformal point, $\frac{3}{2}R - 1 = \gamma$, so that the condition that W be scale invariant, which is equivalent to it having R-charge 2, becomes

$$\begin{aligned} 2 = R_W &= R_A + R_B + R_C + R_D \\ &\Rightarrow \gamma_A + \gamma_B + \gamma_C + \gamma_D + 1 = 0 \end{aligned} \tag{3.77}$$

which is indeed equivalent to the above system. Three independent equations in the five-parameter space of $(\tau_1, \tau_2, \tau_3, \tau_4, \lambda)$ define, again, a critical 2-submanifold.

3.6.3 Moduli space

Turning to the investigation of the moduli space, the F-term condition for the given superpotential read

$$\begin{aligned} A_\alpha B_\sigma C_\beta \epsilon^{\alpha\beta} &= 0 \\ B_\alpha C_\sigma D_\beta \epsilon^{\alpha\beta} &= 0 \\ C_\alpha D_\sigma A_\beta \epsilon^{\alpha\beta} &= 0 \\ D_\alpha A_\sigma B_\beta \epsilon^{\alpha\beta} &= 0 \end{aligned} \tag{3.78}$$

For what concerns the D-terms, as in the case of the conifold we only need to consider those relative to the true gauge symmetries of the IR theory, so those for $SU(N)^4$. As we have seen in the previous section, the $U(1)$ factors either become rigid or broken. Accordingly, the abelian D-terms get relaxed into four arbitrary parameters:

$$D^{U(1)_i} = \xi^i \,, \quad i = 1, 2, 3, 4 \tag{3.79}$$

Combining this information, the vanishing of the D-term takes the form

$$\begin{aligned} A_i A_i^\dagger - B_i B_i^\dagger &= \xi_1 \mathbb{1} \\ B_i B_i^\dagger - C_i C_i^\dagger &= \xi_2 \mathbb{1} \\ C_i C_i^\dagger - D_i D_i^\dagger &= \xi_3 \mathbb{1} \\ D_i D_i^\dagger - A_i A_i^\dagger &= \xi_4 \mathbb{1} \end{aligned} \tag{3.80}$$

with the constraint $\sum_i \xi_i = 0$ (obvious by summing the four equations (3.80), and clear since it corresponds to the trace $U(1)$). Thus ξ^i include 3 independent baryonic moduli.

We already expect that the solutions to (3.78) and (3.80) (modulo gauge symmetry) should be parametrized by $3N$ mesonic operators measuring the positions of the branes and 3 baryonic operators associated with the ξ^i, and that for $\xi^i = 0$ we recover $\mathcal{M}_{\text{mes}}$ corresponding to the motion of the branes on the singular $X^{2,0}$ (3.72). Instead, for any generic non-zero constant values of ξ^i, the $3N$ mesons should describe the motion of N D-branes on the general resolution of the cone.

Let us investigate the geometry of the latter by taking ξ^i to have generic non-zero values (with $\sum_i \xi = 0$). Again we make the simplifying assumption the eight A, B, C, D matrices commute and can be simultaneously diagonalized. So, for each of the N rows of corresponding eigenvalues F-flatness (3.78) reads:

$$a_1/a_2 = c_1/c_2 \qquad\qquad b_1/b_2 = d_1/d_2 \tag{3.81}$$

thus again $a_\alpha \propto c_\alpha$ and $b_\alpha \propto d_\alpha$, so we can "projectivize":

$$\begin{aligned} a_\alpha = a\, e^{i\theta_A} n_\alpha \quad b_\alpha = b\, e^{i\theta_B} n_\alpha \\ c_\alpha = c\, e^{i\theta_C} m_\alpha \quad d_\alpha = d\, e^{i\theta_D} m_\alpha \end{aligned} \tag{3.82}$$

and again the phases are modded out by gauge symmetry, and the a, b, c, d real numbers are reduced to a single coordinate (schematically r^2) by the three independent D-flatness conditions. Therefore the resolved geometry of the singularity is now $\mathbb{S}^2 \times \mathbb{S}^2$, parametrized by the (n^I_α, m^I_α) $(I = 1, \ldots, N)$ coordinates of the N D3-branes. Thus again for generic values of the ξ_i the structure of mesonic moduli space encodes the information that the branes are moving on a resolved version of $X^{2,0}$, where the singularity has been smoothed out into an $\mathbb{S}^2 \times \mathbb{S}^2$.

More specifically, there are actually two moduli describing deformations of the background cone[6]. One considers the general Calabi-Yau deformation of $Y^{2,0}$, which is given by[26]

$$\begin{aligned} ds^2 = \kappa^{-1}(r)dr^2 + \frac{1}{9}\kappa(r)r^2\left(d\psi + \cos\theta_1 d\varphi_1 + \cos\theta_2 d\varphi_2\right)^2 \\ +\frac{1}{6}r^2 d\Omega_1^2 + \frac{1}{6}(r^2 + a^2)d\Omega_2^2\,, \end{aligned} \tag{3.83}$$

$$\kappa(r) = \frac{1 + \frac{9a^2}{r^2} - \frac{b^6}{r^6}}{1 + \frac{6a^2}{r^2}}\,. \tag{3.84}$$

It can be seen that the singular $Y^{2,0}$ cone is obtained as the two moduli a, b vanish. Turning on the b modulus the volume of the four-dimensional base $\mathbb{S}^2 \times \mathbb{S}^2$ is increased (one speaks of a blow up) and the singular tip[7] of the cone is resolved as an $\mathbb{S}^2 \times \mathbb{S}^2$. The modulus a instead controls the relative volume of the two spheres. Therefore there are also specific values of (a, b) (or, equivalently, of the ξ_i) such that we have a “partial” resolution where only one of the spheres is blown up and branes move on an $\mathbb{S}^2$; this is the modulus inherited from the conifold.

In this case, the presence of $g - 1 = 3$ independent ξ parameters (matching with three independent baryons) should perplex, as we have just seen the resolutions are parametrized by *two* moduli. In fact, the general Calabi-Yau deformation of the $Y^{2,0}$ metric will indeed depend on two moduli. In this case, the third modulus is not interpretable as due to deformation of the

[7] This does not actually lie at $r = 0$ because of the way the r coordinate is defined, rather it corresponds to $\kappa(r) = 0$; we will clarify this in our study of the $Y^{2,0}$ resolved geometry in chapter 6

background metric, but is actually connected to the moduli of IIB two-form fields. We will review this fact in a holographic context.

For completeness we adapt the construction of hadronic operators. We note fundamental mesons are now built using $ABCD$ loops (omitting $SU(2)^2$ indices):

$$M = \mathrm{Tr}\left((ABCD)(ABCD)\ldots\right) \tag{3.85}$$

and four classes of fundamental baryons can be introduced as before

$$\mathcal{B}^A = \epsilon^{a_1 \ldots a_n} \epsilon_{b_1 \ldots b_n} A^{b_1}_{a_1} \ldots A^{b_n}_{a_n} \tag{3.86}$$

and this can be repeated for $\mathcal{B}^B$, $\mathcal{B}^C$, $\mathcal{B}^D$. Similarly to the previous case, we expect only three baryons to be truly indepedent and a suitable triple of combinations should generate the three aforementioned flat shifts.

3.7 Towards the general case

The properties introduced with the previous examples of field theories can be reexamined for the general case of a theory arising from D3-branes on $\mathbb{R}^{1,3} \times X_6$.

For example, we can provide a more systematic description of moduli space, so the space of distinct vacua. Because of supersymmetry, the quantum moduli space will often coincide with the classical one, which is intuitively the space of minima of the potential quotiented by gauge transformations. As (see (3.56))

$$\mathcal{V} = \frac{g_a^2}{2}(D^a)^2 + F_i^\dagger F^i \tag{3.87}$$

the minima occur when the VEVs of the chiral fields ϕ_i satisfy both the F-flatness conditions

$$F^i = \frac{\partial W}{\partial \phi_i} = 0 \tag{3.88}$$

and the D-flatness conditions

$$D^a = -\sum_i \phi_i^\dagger T^a \phi_i = 0 \tag{3.89}$$

where T^a are the gauge generators. The space $\mathcal{M}$ of solutions of both F and D-flatness conditions (modulo gauge transformations) will be a complex manifold, the moduli space.

A subspace of $\mathcal{M}$ is given by the so-called mesonic moduli space $\mathcal{M}_{\text{mes}}$. Points of $\mathcal{M}_{\text{mes}}$ will correspond to the position of the N branes on the background cone - therefore $\dim_{\mathbb{C}} \mathcal{M}_{\text{mes}} = 3N$. In fact, $\mathcal{M}_{\text{mes}} = \text{Sym}^N X_6$, where the symmetric product has to be taken because of brane indistinguishability.

As it was seen in the explicit examples, however, in general the moduli space is not exhausted in the purely mesonic directions; the additional "baryonic" directions appear from the relaxation of the D-terms corresponding to the abelian factors $U(1)^\chi$, because these do not appear as gauged symmetries in the IR theory.

Curiously, the reason for this can be different for each $U(1)$. The situation is as follows:

- Part of the $U(1)$ factors are non-anomalous, and under RG flow their coupling vanishes and become rigid global baryonic symmetries.
- The rest of the $U(1)$ are actually anomalous, with the anomaly arising from $U(1)$-$SU(N)_i$-$SU(N)_j$ triangle diagrams. The anomaly is actually cancelled by a supergravity axionic field as explained in section 3.5.2, and the associated photon is made massive by a Stückelberg mechanism[19].

In a holographic context we will be able to provide an explanation of the cancellation of this anomaly and also relate the number of anomalous and non-anomalous $U(1)$s to the topology of the cone; for now we are satisfied with knowing their D-flatness condition is relaxed and one is left with only the D-term for the $SU(N)^\chi$ part. So

$$D^a_{SU(N)^\chi} = 0 \qquad\qquad D^{U(1)_i} = V^i \tag{3.90}$$

($i = 1, \cdots, \chi$). V^i are classically functions of the fields and in the quantum version will be gauge-invariant operators. Their g VEVs $\langle V^i \rangle =: \xi^i$ will parametrize the missing flat directions of moduli space. To be precise, however, since the overall trace $U(1)$ (generated by the sum of the generators of the g abelian trace factors) is completely decoupled, and the relative D-term $D^{U(1)_1} + \ldots D^{U(1)_\chi}$ vanishes identically, we have to impose $\sum \xi^i = 0$. Therefore that there are really only $\chi - 1$ baryonic moduli, and only $\chi(N^2 - 1) + 1$ independent remaining D-flatness conditions.

Thus we conclude $\dim \mathcal{M} = 3N + g - 1$. While the $3N$ mesonic directions have a direct geometrical interpretation as D3-brane movement, the baryonic directions correspond in terms of the superstring description to deformations of the X_6 background metric itself - generally resulting in a resolution of the conical singularity.

A different kind of feature is the presence of marginal deformations of the theory. As explained, the quantization of classically conformal field theories produces quantum systems that are only conformal in a conformal submanifold of the space of parameters. Therefore there will be a set of deformations of the couplings of the field theory (gauge and superpotential) that keeps the theory conformal, and the number will equate the dimension of the conformal manifold. These marginal deformations thus parametrize motion along the conformal variety.

We have already seen in practice how the counting of marginal directions is very easy for this type of superconformal quiver theories thanks to the NSVZ β function (**??**) and the dimension/R-charge relationship (3.21) for chiral operators. However, it turns out to be non-trivial to guess how many of these conditions are actually independent in the general case. Thus, no "fits-all" procedure for counting marginal deformations is known; we just limit ourselves to stating the conjectural relationship with the topology of the cone:

$$\#\text{ of marginal deformations} = b_3(Y_5) + 1\,, \tag{3.91}$$

which has a clearer interpretation in holography, but no general proof purely from the field theory side. We note (3.91) holds for our list of examples: $b_3(\mathbb{S}^5) = 0$ and SYM4 has 1 marginal deformations, $b_3(\mathbb{S}^2 \times \mathbb{S}^3) = b_3(\mathbb{S}^2 \times \mathbb{S}^3/\mathbb{Z}_2) = 1$ and indeed both the KW and the $Y^{2,0}$ models have 2 marginal directions.

Finally, we recap all information acquired for our chain of example theories:

Theory	Y_5	gauge group	non-an. $U(1)$s	an. $U(1)$s	$\dim \mathcal{M}$
SYM4	$\mathbb{S}^5$	$SU(N)$	0	0	$3N$
KW	$\mathbb{S}^2 \times \mathbb{S}^3$	$SU(N)^2$	1	0	$3N+1$
$Y^{2,0}$	$\mathbb{S}^2 \times \mathbb{S}^3/\mathbb{Z}_2$	$SU(N)^4$	1	2	$3N+3$

Chapter 4

Holography

In the previous chapter we explained how the dynamics of brane stacks, in particular D3-branes in type IIB, are described by gauge field theory on their worldvolumes. It is however important to note that parallel to this open string picture of the brane stack system there is also a dual description in terms of the curved spacetimes generated by their mass and Ramond-Ramond charge. Insisting these two viewpoints are equivalent, one is able to deduce an exact correspondence between the gauge theory and string theory on a specific background geometry.

This kind of duality is exotic as it connects a local field theory in four dimensions with a ten-dimensional string (and so, inherently gravitational) theory through a perfect mapping. It is reasonable in fact to identify the spacetime of the field theory with the conformal boundary of the higher-dimensional dual gravitational background (the bulk), for reasons we will clarify - so that in more colloquial language the dynamics in the bulk are "encoded" in the screen at infinity, hence the adjective "holographic" for this sort of correspondences.

Explicit holographic correspondences are not only interesting by themselves as elegant structures; they are also extremely practical tools for studying the theories involved on both sides. It is certainly very attractive for the purpose of quantum gravity or the definition of string theory - non-local theories without action functionals - if these situations happen to be equivalent to a local quantum field theory.

However in this work our interest will be focused on the opposite direction, investigating the dynamics of the field theory by exploiting the dual gravitational system. The power of holographic dualities lies in the fact that they map the strongly-coupled regime for the field theory to the regime where the bulk dynamics can be approximated by supergravity. The traditionally untreatable strong coupling region for some gauge QFTs in four dimensions can then be probed by studying the relatively tamer dynamics of a smooth dual spacetime.

4.1 Maldacena duality

We introduce now the simplest and most celebrated example of holographic correspondence[17]. Consider the IIB supergravity solution for the warped spacetime created by a system of D3-branes in a background $\mathbb{R}^{1,9}$. We specialize the solution of section 2.4.2 to $p = 3$:

$$ds^2 = H^{-1/2} dx_\mu dx^\mu + H^{1/2}(dr^2 + r^2 d\Omega_5^2)\,, \tag{4.1}$$

$$e^\Phi = \text{const} =: g_s\,, \tag{4.2}$$

$$F_5 = dH^{-1} \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3\,, \tag{4.3}$$

$$H(r) = 1 + \left(\frac{R}{r}\right)^4\,, \tag{4.4}$$

where x^μ, $\mu = 0, \ldots, 3$ are coordinates parallel to the brane stack and $d\Omega_5$ is the standard metric on $\mathbb{S}^5$.

The curvature radius R is given by (see (2.73))

$$R^4 = 4\pi g_s N \alpha'^2 = 2 g_{YM}^2 N \alpha'^2\,, \tag{4.5}$$

where N is the number of D3-branes in the stack, and (2.61) was used.

We note an important peculiarity of this metric as opposed to analogous solution for the Dp-brane with $p \neq 3$: there exists a horizon at $r = 0$, but this horizon is an infinite distance away, namely

$$\int_{\varepsilon}^{r'} \left(1 + (R/r)^4\right)^{1/4} dr \sim -\ln \varepsilon \tag{4.6}$$

therefore the "near-horizon" ($r << R$) geometry takes the form of an infinitely long "throat". This should be compared, just to make an example, to the Schwarzschild solution where the horizon is at a finite distance from any given point in the exterior. This throat feature will play a crucial role in the AdS/CFT correspondence, as will be seen shortly.

We insist this system (IIB string theory on the metric 4.1) is equivalent to the stack of D3-branes in the background Minkowski, taking into account both open and closed string interactions. This system will include the dynamics of the $\mathbb{R}^{1,9}$ background, the D-branes, and brane-background interactions. We are now set to show that in the low-energy $\alpha' \to 0$ limit the interactions are suppressed and the branes and the background decouple; we follow [2]. We study such a picture in the regime in which it admits the Lagrangian description, with the hope that an analogous argument could apply in the general case. The three components become then three terms in an effective action

$$S = S_{D3} + S_{bg} + S_{int} \, . \tag{4.7}$$

S_{bg} is simply the string-frame IIB supergravity action (2.47) based on a Minkowski background. Let us examine just the graviton and dilaton section as an example. Linearizing the graviton as $g = \eta + \kappa h$, with $\frac{1}{2\kappa^2} = \frac{2\pi}{\ell_s^8}$, this is expanded as

$$S_{bg}^{[g,\phi]} = -\frac{1}{2\kappa^2} \int d^{10}x \, \sqrt{-g} \, e^{-2\phi} (R + 4\partial_\mu \phi \partial^\mu \phi) \tag{4.8}$$

$$= -\frac{1}{2} \int d^{10}x \, \partial_\mu h \partial^\mu h + \mathcal{O}(\kappa) \, ; \tag{4.9}$$

the normalization for h was chosen to guarantee a canonical kinetic term for the graviton independent of α'. However, the kinetic action for the dilaton is still $\mathcal{O}(\kappa) = \mathcal{O}(\alpha'^2)$; it too can be rescaled as $e^\phi \to g_s e^{\kappa\varphi}$ to make it canonical. This procedure can be continued for the rest of the IIB fields.

S_{D3} and S_{int} instead both come from an expansion of the Dirac-Born-Infeld action in the background fields (h, φ, ...). The zeroeth order part is the pure D3-brane action (see (2.60))

$$S_{D3} = -\frac{1}{g_s \alpha'^2 (2\pi)^3} \int d^4x \sqrt{-\iota^*(g)} + \frac{1}{4(2\pi) g_s} \int d^4x \sqrt{-\iota^*(g)} \,\mathrm{Tr}\, F^2 + \mathcal{O}(\alpha'^2) \tag{4.10}$$

or, if we define the transverse position fields $\Phi^i := X^i/(2\pi\alpha')$:

$$= -\frac{1}{g_{YM}^2} \int d^4x \left(\frac{1}{4} \mathrm{Tr}\, F^2 + \frac{1}{2} \eta^{\mu\nu} \partial_\mu \Phi^i \partial_\nu \Phi_i \right) + \mathcal{O}(\alpha') \,, \tag{4.11}$$

i.e., the bosonic part of the action for $\mathcal{N} = 4$ SYM.

All higher-order terms in (h,φ,...) comprise brane-background interactions. For example, the first term in S_{int} is a dilaton-gluon-gluon vertex:

$$S_{int} = -\frac{1}{8\pi g_s} \int d^4x (\kappa\varphi) \,\mathrm{Tr}\, F^2 + \dots \tag{4.12}$$

and since all IIB fields are proportional to κ after the rescaling, S_{int} will be at least $\mathcal{O}(\alpha')$, and will vanish in the low-energy limit.

Therefore, to conclude: in the $\alpha \to 0$ limit, the D3-brane stack is comprised of two decoupled systems. One is IIB supergravity on flat $\mathbb{R}^{1,9}$ spacetime, and the other is $\mathcal{N} = 4$ super-Yang-Mills

We repeat this decoupling limit for the black three-brane metric(4.1). If $\alpha' \to 0$, so does R, and effectively the metric seems to converge to flat spacetime. We have however to take into consideration the throat described before. The throat shrinks as $R^2 \sim \alpha' \to 0$; to maintain focus on it as we lower the Regge slope we need to rescale our r coordinate. We can introduce $\phi = r/\alpha'$ and keep ϕ fixed as α' goes to zero. With this choice the metric actually reduces to

$$ds^2 = \alpha' \left(\frac{\phi^2}{R^2} dx^\mu dx_\mu + \frac{R^2}{\phi^2} d\phi^2 + R^2 d\Omega_5^2 \right) \tag{4.13}$$

which is actually the metric for the product space $AdS_5 \times \mathbb{S}^5$. However, this is no evidence that this near-horizon geometry will be relevant to the low-energy limit; in fact, since $ds^2 \sim \alpha'$, we would expect string states deep in the throat ($r \ll R$) to be very energetic and decouple in the $\alpha' \to 0$ limit. Actually, one has to take into account the redshift factor. A string state at a radius $r \ll R$ with fixed energy E_r is measured by an observer at infinity to have a redshifted energy

$$E_\infty = H^{-1/4}(r) E_r \sim \frac{r}{R} E_r \tag{4.14}$$

so that $E_\infty \sim r$ and there are low-energy in the region $r \ll R$. This establishes two regions of low-energy states, an $r \gg R$ Minkowski background and the $r \ll R$ throat, separated by an intermediate barrier of high-energy states mediating throat-background interactions. As $\alpha' \to 0$, the barrier rises and the two systems decouple.

Therefore under $\alpha' \to 0$ also on the black brane side we have a decoupling of two systems, IIB on Minkowski, and IIB on the near-horizon geometry.

The essential point is that if the two pictures are to be equivalent, and so SYM4 plus decoupled IIB on Minkowski is to be equal to IIB on $\mathrm{AdS}_5 \times \mathbb{S}^5$ plus decoupled IIB on Minkowski, then intuitively one expects to be able to "factor away" the decoupled theory from both sides and obtain an equivalence between the gauge theory and the gravitational theory on $\mathrm{AdS}_5 \times \mathbb{S}^5$. The conclusion of this intuition, in varying degrees of strength, forms the core of the AdS/CFT conjecture, though the details of this equivalence still have to be specified.

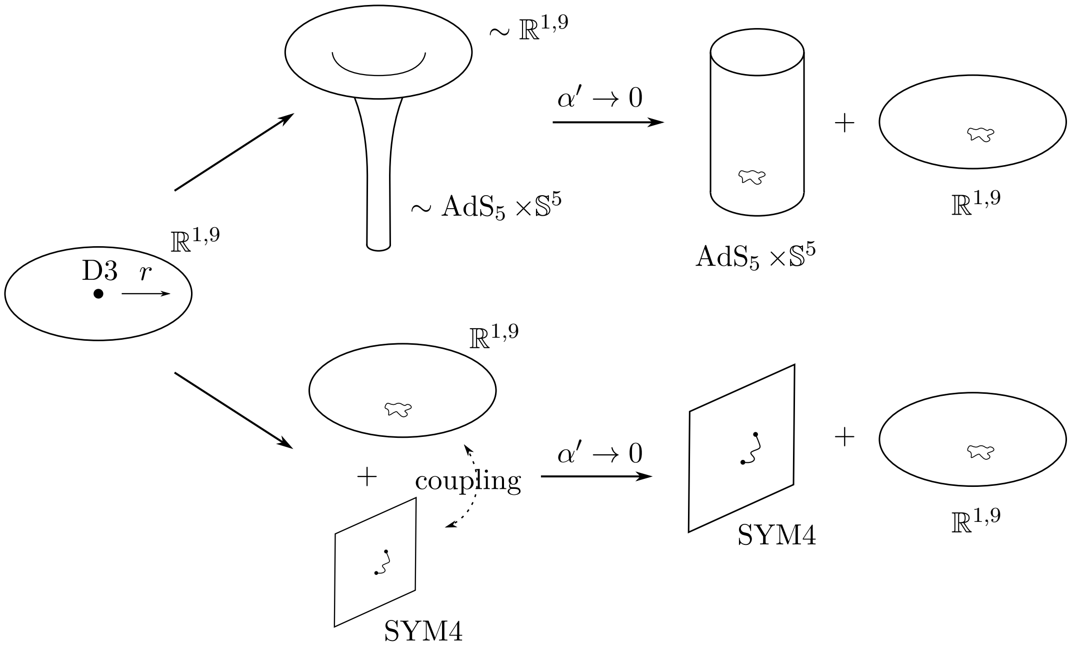


Figure 4.1: Scheme of the two interpretations of the D3 stack system and the decoupling limit.

For example, in the above decoupling argument, by itself not particularly rigorous, we approximated string theory as its effective field theory, two-derivative classical supergravity. This means respectively that $g_s \ll 1$, so that string loops and thus quantum corrections are small, and the curvature is larger than the string scale: $R \gg \sqrt{\alpha'}$. Let us reinterpret these limits in the CFT side.

Weak string coupling $g_s \ll 1$ maps to a "large N" limit. More precisely, keeping constant the combination

$$\lambda := g_{YM}^2 N \,, \tag{4.15}$$

known as the 't Hooft coupling, and then sending $N \to \infty$ (so that $g_{YM} \to 0$ accordingly) is equivalent to having $g_s \ll 1$, since

$$4\pi g_s = \frac{\lambda}{N} \,. \tag{4.16}$$

Weak string coupling amounts to a suppression of string loops and effectiveness of the string perturbative expansion. A more surprising conclusion concerns however the limit of large $\lambda \gg 1$, since

$$\frac{R^2}{\alpha'} = \sqrt{\lambda} \tag{4.17}$$

so that large coupling means a large $\mathbb{S}^5$ radius in units of $\sqrt{\alpha'}$ (or the string length ℓ_s) - a low-energy limit. Equivalently, choosing units that fix the value of R, we have $\alpha' \to 0$. Thus, the dual physics is well described by IIB supergravity instead of the full string theory.

Combining both results, a large N, strong-coupling gauge theory will be dual to weakly-coupled supergravity. The strong/weak interchange is what earns the correspondence the title of "duality". In the next section the connotation of "holographic" will also be justified.

In any case, the weakest form of the AdS/CFT conjecture limits itself to this regime, and states SYM4 in the large N, strong-coupling limit is equivalent to classical IIB supergravity on $\mathrm{AdS}_5 \times \mathbb{S}^5$. A stronger form relaxes the $\lambda \to \infty$ limit and requires the equivalence of SYM4 in the large N limit, for any value of λ, to classical (i.e., weakly-coupled) IIB string theory on $\mathrm{AdS}_5 \times \mathbb{S}^5$. Finally, the strongest conjecture relates SYM4 with IIB strings on $\mathrm{AdS}_5 \times \mathbb{S}^5$ exactly, for all values of the couplings.

Strength	Bulk	Boundary
Weakest	$g_s \ll 1, \alpha' \ll 1$ (classical SUGRA)	$N \gg 1, \lambda \gg 1$
Stronger	$g_s \ll 1$, any α' (classical string theory)	$N \gg 1$, any λ
Strongest	any g_s, α' (string theory)	any N, λ

All of these correspondences do not have a formal proof, but rather a large volume of evidence for the equivalence in the form of nontrivial checks, with the weakest form enjoying the greatest certainty. It is understood that difficulty in verifying the conjecture in the general case is just a reflection of the difficulty of defining string theory non-perturbatively; in any case, we will only make use of the weakest form, which we will assume as true.

We add that the correspondence $g_{YM}^2 \sim g_s = e^{\phi}$ can be extended slightly to include complexified couplings. It is possible to show that the theta angle θ of the gauge theory is holographically dual (up to constants) to the RR form A_0. In fact, if one defines the complex axio-dilaton scalar

$$\tau_{\rm IIB} = A_0 + i e^{-\phi} \tag{4.18}$$

and the complex gauge coupling

$$\tau_{\rm YM} = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2}\,, \tag{4.19}$$

then these are holographically dual:

$$\tau_{\rm IIB} = \tau_{\rm YM}\,; \tag{4.20}$$

and the string perturbative regime is $\operatorname{Im}\tau \gg 1$. We will omit the subscripts on τ from now on in light of this correspondence and its generalizations.

4.1.1 Comment on multi-brane decoupling

We clarify that the decoupling / near-horizon limit we implemented must be generalized carefully to the case of non-coincident branes, since we will be particularly interested in those kind of configurations. As seen in section 2.4.2, it is easy to construct multi-centre solutions for the warped geometry created by multiple, non-coincident branes by simple superposition; the decoupling just described for N coincident branes can be adapted to this case too. If two branes are a distance Δr apart, there will be massive Higgses from open strings stretching between them, with masses of the order of the string tension times the distance: $m \sim \Delta r/\alpha'$. We would like to keep this masses constant. The obvious choice is then to rescale the brane r positions as $\phi_i = r_i/\alpha'$ and keep those constant as we zoom in. This is what we will refer to as the near-horizon limit in the general case and the resulting geometry as the near-horizon warped geometry.

4.2 Large N limit

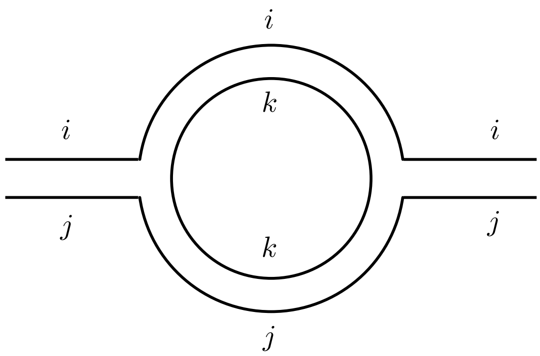
It was just seen how the weak-coupling regime for the string theory maps to a "large N" limit on the gauge theory side. How this limit is understood has to be explained more carefully.

Consider a gauge theory of the type considered in chapter 3, with an $SU(N)^g$ gauge group. The bosonic part of the Lagrangian is

$$\mathcal{L} = \mathrm{Tr}\left(F^2\right) + \dots \tag{4.21}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i g_{YM}[A_\mu, A_\nu]$ and $\dots$ can include fields in adjoint and bifundamental representations[1], and all irreps obtainable by tensoring these. Ultimately, all fields will be representable as objects with a certain number of colour indices, at most related by symmetries in those indices.

We can modify the standard Feynman prescription for pictorially representing amplitudes to get a "double line" or "ribbon" representation in which each colour index is carried by a line. For example, the gluon self energy diagram becomes as such:



colour indices i, $\bar{\imath}$, j, $\bar{\jmath} = 1, \dots, N$ are fixed, while k must be summed over. The amplitude has two three-gluon vertices, each carrying a factor of g_{YM}^2, for an overall factor of $g_{YM}^2 N$.

It iss easy to convince oneself that as long as we restrict to planar diagrams, that is diagrams that can be drawn on the plane (or more precisely the sphere), adding one strip will always introduce exactly one additional

[1]The following construction can be extended to include particles in the (anti-)fundamental, but the details are different and this case is not relevant to our interests.

loop and two additional vertices, again carrying a factor of $g_{YM}^2 N$. The combination $\lambda := g_{YM}^2 N$ is the 't Hooft coupling, and is better suited to represent the strength of the gauge interaction than g_{YM} if we are to modify the number of colours.

Then, the 't Hooft large N limit is defined as:

$$N \to \infty, \quad \text{but keeping } \lambda \text{ fixed.} \tag{4.22}$$

Equivalently, keeping λ constant and sending $g_{YM} \to 0$.

A useful rescaling of the fields shifts all the g_{YM} dependence of the Lagrangian to a factor in front:

$$\mathcal{L} = \frac{1}{g_{YM}^2} \left(\operatorname{Tr} F^2 + \ldots \right) \tag{4.23}$$

so that now all types of vertices bring $g_{YM}^2 = \lambda/N$ and propagators bring $1/g_{YM}^2 = N/\lambda$.

We extend to nonplanar graphs by noting these can always be drawn on some Riemann surface of genus h, and, since they induce triangular tilings of said surface, the famous formula for the Euler characteristic holds:

$$F - V + E = \chi = 2 - 2h \tag{4.24}$$

F, V, E being the number of faces, vertices, edges respectively. Now each face (loop) carries a factor of N, each vertex a factor of λ/N, and each edge N/λ, so that the total contribution is

$$\lambda^{E-V} N^{F-V+E} = \lambda^{E-V} N^{2-2g} \tag{4.25}$$

so that at fixed λ, an expansion in N (or better $1/N$) is a genus expansion reminiscent of the loop expansion in perturbative string theory. This for example means that the free energy admits a power expansion in $1/N$:

$$F = \sum_{g=0}^{\infty} f_g(\lambda) N^{2-2g} \,. \tag{4.26}$$

In conclusion, the 't Hooft limit results in suppression of the higher-genus contributions by powers of N^{-2} with respect to the planar diagrams[2], and is thus also known as the planar limit. It is remarkable that this genus expansion parallels that of the perturbative string theory, and that their regimes of effectiveness in holographic dualities coincide.

4.3 Symmetries

If there is indeed an equivalence between an $\mathcal{N} = 4$ superconformal theory in $\mathbb{R}^{1,3}$ and IIB string theory (or just supergravity) on $\mathrm{AdS}_5 \times \mathbb{S}^5$, then verifying a match between the global symmetries of both theories would be a first test of the correspondence.

The bosonic part of the superconformal group is composed by the conformal group $SO(2,4)$ (or more precisely the fundamental cover $SU(2,2)$) and the R-charge group $SU(4)$. These are also evidently the isometry groups of AdS_5 (see appendix A.1), and of $\mathbb{S}^5$ (which is $SO(6)$, double covered by $SU(4)$).

In terms of supersymmetries, the bulk dual arises from a stack of D3-brane, which preserve 16 of the 32 IIB supercharges. These map directly to the 16 Q^A supercharges of $\mathcal{N} = 4$ supersymmetry. Actually, however, the near-horizon geometry ($\mathrm{AdS}_5 \times \mathbb{S}^5$ plus the self-dual F_5 configuration (4.3)) has *more* supersymmetry than the D3-branes themselves; it is actually is maximally supersymmetric, preserving all 32 supercharges of IIB strings[3]. This provides the additional 16 supercharges to pair with the S^A superconformal generators.

There is therefore a perfect match between the symmetries on both sides.

[2]One could be perplexed by the N^2 divergence of the genus zero contribution. This is not problematic however; it is an artifact of the rescaling 4.23 which makes the Lagrangian itself diverge as $g_{YM}^{-2} \operatorname{Tr} F^2 \sim N/\lambda \cdot N$, since the trace of a matrix in the adjoint scales as N.

[3]This fact can be proven in various way, but the most elegant perhaps is to recognize[22] $\mathrm{AdS}_5 \times \mathbb{S}^5$ can be written as the bosonic part of the homogeneous superspace $\dfrac{SO(2,4|4)}{SO(1,4) \times SO(5)}$ so that $SO(2,4|4)$ acts as an isometry supergroup.

Because of the limpid correspondence of the 4-dimensional Poincaré group with the obvious subgroup of $SO(2,4)$, it's immediate the duality must map local CFT objects at boundary coordinates x^μ to objects in the bulk computed at the same coordinates x^μ (in the Poincaré chart). This fixes the interpretation of the four x^μ coordinates in the duality; the extra five (the radial r and those on $\mathbb{S}^5$) remain to be understood.

If the field theory R-symmetry equals the $SO(6)$ isometries of the 5-sphere, it must be that whatever shape the correspondence takes on, it maps objects that transform identically under this unique group. Thus it is sensible to decompose bulk fields into irreducible representations of it[32]. This is done by decomposing the ten-dimensional IIB fields into spherical harmonics to yield a tower of five-dimensional fields; for example for a scalar field:

$$\phi(r, x^\mu, y^i) = \sum_{\mathbf{R},i} \phi^i_{\mathbf{R}}(r, x^\mu) Y^{\mathbf{R}}_i(y^i)\,. \tag{4.27}$$

The sum is over the irreps $\mathbf{R}$ of $SO(6)$ ($\mathbf{R} = \mathbf{1}, \mathbf{6}, \ldots$)[4] while the index $i = 1, \ldots, \dim \mathbf{R}$ transforms under $\mathbf{R}$; the spherical harmonics have been arranged into irreps themselves so that $(Y^{\mathbf{R}}_1, \ldots, Y^{\mathbf{R}}_{\dim \mathbf{R}})$ also lies in $\mathbf{R}$. The five-dimensional theory describing these fields on AdS_5 is found by inserting these modes into the IIB supergravity action. At the quadratic level, most of these fields $\phi_{\mathbf{R}}$ will acquire a mass in the AdS_5 sense - even though the ten dimensional supergravity fields were all massless.

Therefore, we can establish that boundary objects at position x^μ and in the R-symmetry representation $\mathbf{R}$ will be somehow related in holography to a five-dimensional bulk field $\phi^i_{\mathbf{R}}(r, x^\mu)$, which is one component in the Fourier expansion of $\phi(r, x^\mu, y^i)$ on $\mathbb{S}^5$.

Now, we are interested in the actual prescription for pairing bulk and boundary objects, and in the meaning of the remaining extra dimension r.

[4]When repeating this argument for spinor fields, the universal cover Spin(6) = $SU(4)$ should rather be considered, so that all representations of $SU(4)$ ($\mathbf{1}, \mathbf{4}, \mathbf{6}, \ldots$) appear.

4.4 Features of AdS/CFT

Having ascertained that a correspondence of some form exists, one would then seek a more precise description of how the mapping between the four-dimensional gauge theory (the boundary) and the supergravity side (the bulk) is structured. In general, one has what is called an operator-state correspondence: operators in the boundary are associated by a holographic dictionary to "states", or classical solutions in the bulk. More precisely, consider a 4D local operator field $\hat{\phi}(x)$. The generating functional for its correlation functions is given by coupling it to a current $h(x)$:

$$e^{W[h]} = \frac{1}{Z_{\text{CFT}}} \int D\psi e^{iS_{\text{CFT}} + \int d^4x h(x)\hat{\phi}(x)} \tag{4.28}$$

then, the source $h(x)$ is viewed as the limit as $r \to \infty$ of a five dimensional field configuration $h_5(x, r)$, given by solving the equations of motion from S_{AdS} with $h(x)$ as a boundary condition. The correspondence is between the "off-shell" boundary operator $\hat{\phi}(x)$ and the "on-shell" bulk field $h_5(x, r)$, and states[32] that the generating functional above is equal to the bulk action computed on the specific classical solution $h(x, r)$:

$$e^{W[h]} = \langle e^{-\int d^4x h\hat{\phi}(x)} \rangle_{\text{CFT}} = e^{iS_{AdS}[h_5]} \tag{4.29}$$

Therefore, correlation functions for the strongly-coupled CFT can be calculated entirely through the weakly-coupled, two-derivatives bulk action.

One may then wonder about the interpretation one should employ for the fifth extra dimension in the bulk from the CFT perspective. A tentative identification comes from the fact that the AdS metric is invariant under dilations

$$x^\mu \to \lambda x^\mu, \qquad z \to \lambda z \tag{4.30}$$

Since $1/z$ scales like an energy, it could be paired holographically with the boundary energy scale. This turns out to be correct in the sense of renormalization: probing AdS at large distances, closer to the boundary at infinity (as we did before by coupling $\hat{\phi}(x)$ with the value at infinity of a 5D field) coincides with probing the microscopical, UV theory. Moving inwards, operators at larger values of z equate probing the theory at a lower energy scale $\mu \sim 1/z$, up until the horizon which is identified with the IR. The fifth dimension is to be roughly identified with the renormalization flow of the field theory.

Therefore it might be useful to think of the UV microscopic field theory as being someway literally located at the conformal boundary at infinity of the AdS dual. Moving inwards, the highest-momentum modes get integrated out and the theory flows towards the infrared. Hence the "boundary"/"bulk" terminology, and since all of the physics in the 5D gravitational theory are encoded in a codimension-1 "screen" at infinity, one speaks of holography, in analogy with the real-life technique of encoding three-dimensional objects in a two-dimensional hologram.

This concludes our dictionary for the correspondence of bulk and boundary dimensions:

symmetries		CFT	$\mathrm{AdS}_5 \times \mathbb{S}^5$
$SO(2,4)$	$SO(1,3)$	x^μ	x^μ
	dilations	renormalization scale	z
$SO(6)$		R-symmetry indices	y^i

In light of the above energy-r relationship, our pairing of field configurations $h_5(x,z)$ with boundary operators has to be corrected. If $h_5(x,z)$ is asymptotically constant as $z \to 0$ so that the limit $h_5 \to h$ is well-defined, it must be that the corresponding dual operator does not scale under dilations, so that its conformal dimension $\Delta = 0$. The extension to CFT operators with arbitrary scaling dimensions is then realized by adding the possibility of $h_5(x,z)$ diverging as a power of z as

$$h_5(x,z) \to z^{\Delta} h(x) \tag{4.31}$$

so that $h(x)$ can then be coupled as source to an operator of conformal dimension Δ. This in turn will induce a dependency of Δ on the mass of the dual bulk field. As an example, let us take a scalar field in AdS_5 minimally coupled to the graviton:

$$S \propto \int d^4x\, dz\, \sqrt{g} \left(g^{mn} \partial_m h_5 \partial_n h_5 + m^2 h_5^2 \right) , \tag{4.32}$$

so that the classical equation of motion is (ignoring x dependency, since it does not affect this argument)

$$\partial_z \left(z^{-3} \partial_z h_5 \right) = z^{-5} R^2 m^2 h_5 . \tag{4.33}$$

Plugging in a power law $h_5 = hz^{\Delta}$ yields the conformal dimension-mass relation for a scalar:

$$\Delta(\Delta - 4) = R^2 m^2 . \tag{4.34}$$

So that a scalar field of mass m^2 can only be dual to a boundary operator with dimension $\Delta_{\pm} = 2 \pm \sqrt{4 + R^2 m^2}$, where Δ_- needs to be excluded whenever it violates the unitarity bound $\Delta \leq 1$ (see (3.18)).

It would seem as if the smallest possible dimension of a boundary operator is 4, however one should take into account the fact that the hyperbolic curvature of AdS space produces an effective confining potential that allows particles with $m^2 < 0$ to be stable. It can be shown[1] that the Breitenlohner-Freedman bound holds for stable scalar fields

$$m^2 R^2 \geq -4 \tag{4.35}$$

so that it is possible to reach down to the unitarity limit at $\Delta = 1$. For the mass range $-4 < m^2 R^2 < -3$, both Δ_+ and Δ_- are viable dimensions for a dual operator; in [15] it is argued that both choices are implemented in distinct holographic dualities, and that Δ_- should not be excluded.

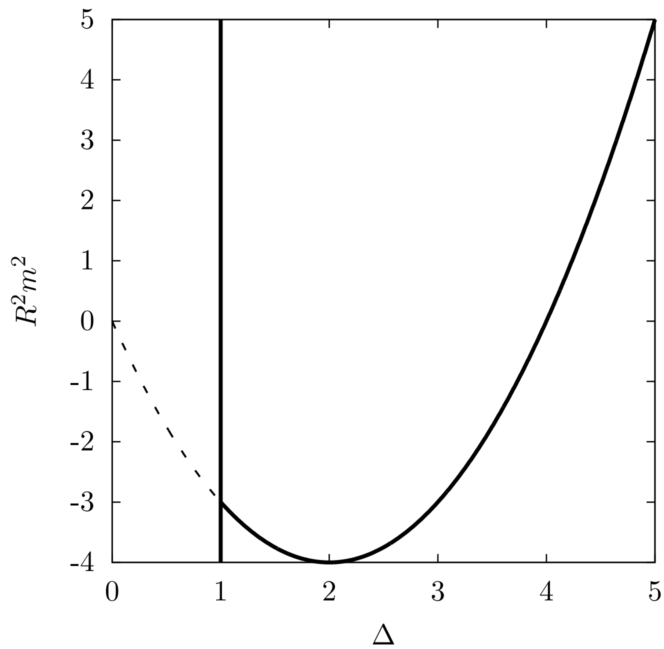


Figure 4.2: Scaling dimensions Δ corresponding to a given bulk mass for a scalar field, alongside the unitarity bound $\Delta \geq 1$.

This was for scalar fields. Higher-spin fields will be dual to operators of the same spin on the boundary and the m-Δ equation will be modified[34]. For example, $(\frac{1}{2}, \frac{1}{2})$ vectors will have

$$R^2m^2 = (\Delta - 1)(\Delta - 3) \tag{4.36}$$

while $(1, 1)$ symmetric tensor will instead have the same equation as scalars:

$$R^2m^2 = \Delta(\Delta - 4) \tag{4.37}$$

The relevance of this is that the operators coupled to massless spin 1 or 2 bosons must be conserved currents by gauge invariance, and so the operators dual to a massless bulk photon or graviton are a conserved vector current and the stress-energy tensor, respectively. The anomalous dimensions of these conserved currents must vanish, and indeed $\Delta_J = 3$ and $\Delta_T = 4$ are canonical.

4.5 AdS/CFT over a cone

As seen above, the original motivation for the AdS/CFT conjecture is the identification of a system of N coincident D3-branes in a $\mathbb{R}^{1,9}$ Minkowski background and the corresponding 3-brane supergravity solution. In an appropriate low-energy limit a system of closed IIB strings on flat spacetime decouples in both pictures, suggesting it should be conjectured that the remaining parts are equivalent. These are respectively $\mathcal{N} = 4$, $SU(N)$ SYM on $\mathbb{R}^{1,3}$ and IIB strings on $\mathrm{AdS}_5 \times \mathbb{S}_5$.

We repeat this reasoning, but in the more interesting case where the background for the D3-branes is generalized as $\mathbb{R}^{1,3} \times X_6$, where X_6 is a cone over a base 5-manifold Y_5. Our intention is to build holographic dualities for the larger set of theories described in chapter 3. We anticipate the bulk dual in this case is IIB strings over $\mathrm{AdS}_5 \times Y_5$.

After placing 3-branes in this $\mathbb{R}^{1,3} \times X_6$ background, parallel to the Minkowski, the resulting geometry from their backreaction is:

$$ds^2 = H^{-1/2}(r, y)\, dx_\mu dx^\mu + H^{1/2}(r, y)\, ds_6^2 \tag{4.38}$$

Where $x^{0,\dots,3}$ are coordinates parallel to the brane stack, r is the radial coordinate and the remaining $y^{1,\dots,5}$ parametrize the cone's base Y_5. This is a simple generalization of the flat-background 3-brane solution of 2.4.2, by substitution of $\mathbb{S}^5$ with Y_5. The above form can be argued for purely in terms of the $SO(1, 3)$ symmetry acting on the x^μ; the symmetry of the transverse dimensions is instead in general broken unless the branes lie exactly on the singularity, therefore the warp factor will depend also on y^i.

The equations of motion implies the function H is harmonic $\nabla H = 0$ for $r > 0$, identically to the flat-space case, since the branes are again extremal states.

If the branes are coincident and on the singularity, the corresponding harmonic potential is

$$H(r) = 1 + \frac{R^4}{r^4}\,, \quad R^4 = 4\pi g_s N \alpha'^2\,. \tag{4.39}$$

The near-horizon limit ($r \to 0$) in that case can be read immediately:

$$ds^2 = \frac{dx_\mu dx^\mu + dz^2}{z^2} + ds_5^2 \tag{4.40}$$

where $z := 1/r$; this is evidently the product metric on $\mathrm{AdS}_5 \times Y_5$.

Therefore, if it is possible to adapt the original argument for AdS/CFT for this situation, the existence a holographic duality can be established between the D3-brane worldvolume theory at the conical singularity and IIB strings (or, less ambitiously, supergravity) on $\mathrm{AdS}_5 \times Y_5$.

Actually, we restrict to X_6 being Calabi-Yau, that is being Kähler with holonomy $\subset SU(3)$, because as detailed in section 3.2 this guarantees at least $\mathcal{N} \geq 1$; equivalently, we only consider Sasaki-Einstein 5-folds for bases. A rough explanation of the holographic interpretation of the connection between special holonomy and supersymmetries is as follows: the holonomy of X_6 will determine how many independent Killing spinors are on X_6 and thus the background $\mathbb{R}^{1,3} \times X_6$ - in particular it will result in a reduction of the number of supersymmetries of the background from $\mathbb{R}^{1,3} \times \mathbb{R}^6$ as

holonomy		SUSY charges
trivial ($X_6 = \mathbb{R}^6$)	$\longrightarrow$	32
$SU(2)$	$\longrightarrow$	16
$SU(3)$	$\longrightarrow$	8

However, it can also be shown[14] that the Killing spinor equation on $\mathbb{R}^{1,3} \times X_6$ is actually equivalent to the Killing spinor equation on $\mathrm{AdS}_5 \times Y_5$, including the effect of the F_5 five-form. Therefore the $\mathrm{AdS}_5 \times Y_5$ dual has as many supersymmetries as the background $\mathbb{R}^{1,3} \times X_6$. Note that, again, this is only true for the near-horizon geometry; the brane themselves break half of the supersymmetries of the background. Consequently, the field theory acquires this enhanced supersymmetry only in the IR limit. There, it flows to a superconformal field theory, with $4\mathcal{N} + 4\mathcal{N} = 8\mathcal{N}$ supercharges. Matching with the supercharges count of the dual theory, the supersymmetry is respectively $\mathcal{N} = 4, 2, 1$. In particular, only $X_6 = \mathbb{R}^6$ has $\mathcal{N} = 4$; every other choice of cone will result in a boundary theory with reduced supersymmetry.

Finally, we also require X_6 to be a proper supergravity background, which would be stable without the introductions of the D-branes. This means in particular it should satisfy the vacuum Einstein field equations, and thus should have $R_{ab} = 0$. A conjecture of Calabi, proven in [33], states that Calabi-Yau manifolds admit a Ricci-flat metric compatible with the Kähler structure[5]. Thus ds_6^2 can always be taken as equal to this unique metric, and ds_5^2 is Sasaki-Einstein.

4.5.1 Moduli and marginal directions

On the quiver theory side, it was already seen how generalizing $\mathbb{S}^5 \to Y_5$ introduces novel features such as baryonic moduli and marginal deformations in addition to the coupling τ. Holographically, we see this as possible deformations of the $\mathrm{AdS}_5 \times Y_5$ IIB background. For example, it was already seen that turning on mesonic moduli corresponds to motion of the D3-branes on X_6, from, say, $z_I^i = 0$ ($z^{1,2,3}$ are a complex chart on X_6) to some generic positions $z_I^i = (r_I, y_I^i)$; in that case the bulk dual has the warped geometry (4.38) with warp factor given by the solution of[21]

$$\nabla H(z) = \sum_I \delta^6(z - z_I) \tag{4.41}$$

the kind of multicentre solutions we have anticipated in section2.4.2. This is a possible deformation of the original IIB supergravity background, and these $3N$ moduli in the supergravity solution are clearly holographically paired with the $3N$ field theory mesonic moduli. While this first match is promising, it's not particularly interesting nor is it special to conical X_6; they are of course also present in the Maldacena case. The core feature in the moduli space of conical CFTs is rather the appearance of $\chi - 1$ baryonic moduli[6]; these are dual to deformations of the various form fields of IIB supergravity: ϕ, B_2, $A_{0,2,4}$, and of course the Kähler form J. These are only made possible by a non-trivial topology of X_6: the introduction of homology k-cycles in the warped geometry allows for the supergravity forms to have non-zero integrals on them.

Let us sketch for example the case of the Klebanov Witten model, which

[5] more precisely, this means that if the cone is Calabi-Yau with Kähler form J, then the cohomology class $[J] \in H_2(X_6)$ contains another J' whose associated metric is Ricci-flat.

[6] We recall χ is the number of nodes in the gauge group $G = SU(N)^\chi$, and also the Euler characteristic of X_6.

by the above argument should be dual in the large N, strongly-coupled limit to IIB supergravity on $\mathrm{AdS}_5 \times T^{1,1}$. This has the topology of $\mathbb{S}^2 \times \mathbb{S}^3$[7]. The gauge group is $SU(N) \times SU(N)$ and so there is $2 - 1 = 1$ baryonic modulus. We have already seen that the turning on of this modulus corresponds to a Kähler deformation of the conical background $X_6 = \mathbb{R}_+ \times T^{1,1}$ with a blowing up of a two-cycle S, which would otherwise collapsed in the singularity, to result in a regular geometry. On the supergravity dual, set on the warped geometry, this will be reflected in a modulus for the Kähler 2-form J wrapping on the two-cycle S, so a quantity of the type

$$\int_S J = \mathrm{vol}\, S\,. \tag{4.42}$$

Other supergravity form integrals on S will map to the marginal deformations of the KW theory. There are two, since the conformal manifold is a surface in (g_1, g_2, λ) space. It can be shown[2] that the IIB string coupling (itself a IIB modulus, being essentially the VEV of the scalar dilaton) are dual to a symmetric combination of the gauge couplings

$$\frac{4\pi}{g_1^2} + \frac{4\pi}{g_2^2} = e^{-\phi}\,, \tag{4.43}$$

while the antisymmetric part corresponds to the integral of the Kalb-Ramond form over S:

$$\frac{4\pi}{g_1^2} - \frac{4\pi}{g_2^2} = e^{-\phi}\left(-1 + \frac{1}{2\pi^2\alpha'}\int_S B_2\right)\,. \tag{4.44}$$

This can actually be extended to complexified couplings as such

$$\tau_1 + \tau_2 \quad \longleftrightarrow \quad \tau\,, \tag{4.45}$$

$$\tau_1 - \tau_2 \quad \longleftrightarrow \quad \int_S (A_2 - \tau B_2)\,. \tag{4.46}$$

Thus, new features in moduli and marginal couplings on the KW model with respect to SYM4 are due to the appearance of a single new two-cycle S in the cone. The precise relationship between the number of k-homology cycles of X_6 (Betti numbers) and moduli and marginal deformations in the general case will be established in the next chapter.

Chapter 5

Holographic effective field theories

In this chapter, we present the technique and results introduced in [21] to find the effective theory for strongly-interacting CFTs with holographic duals. Instead of repeating the arguments presented therein, we will strive to provide an intuitive summary of the concepts involved.

Since we are considering strongly-interacting quantum field theories with minimal supersymmetry, the problem of identifying the low-energy effective field theory directly is generally untractable. However, as we have seen, the strong-coupling regime for the CFT corresponds to effectiveness of the supergravity approximation on the holographically dual string side. Therefore, the low-energy dynamics of the dual system can in principle be read and the resulting theory will coincide with the effective theory for the original QFT. Having been obtained by passing through the holographic dual, these will be termed holographic effective field theories (HEFTs).

In practice, it is found that for any given point in the longitudinal coordinates $x^{0,\dots,3}$ the transverse supergravity configuration will belong to a manifold of different supergravity vacua, and that this manifold is finite-dimensional, in the sense that there is only a finite number of moduli parametrizing deformations of the vacuum configurations. This moduli space coincides of course with the field theory moduli space.

A first class of moduli are given by deformations of the dual geometry. These

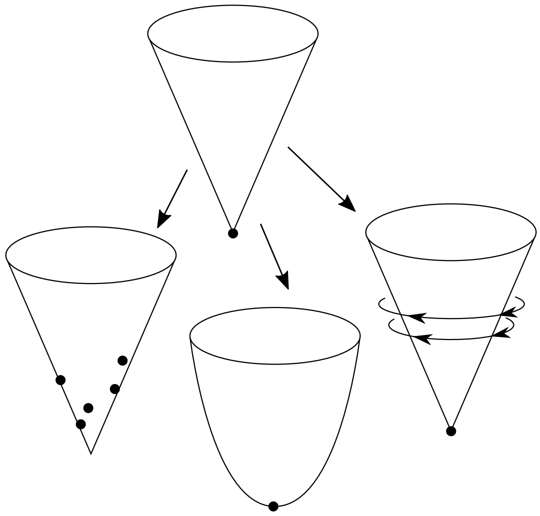


Figure 5.1: Schematic depiction of the three type of moduli of the dual vacuum: motion of D3-branes, deformation of Kähler structure of the background, generally involving resolution of the conical singularity, and field fluxes.

include Kähler moduli of the *background* cone in which the branes are placed in, and then the position of the D3-branes themselves on that background - which manifests as a deformation in the resulting warped geometry. Another class instead will be given by the moduli corresponding to the deformations of the B_2 and R-R fields of IIB supergravity; while these would be full fields defined on the six-dimensional background, gauge invariance will result in only a finite number of topological invariants of the field configuration to be physical.

In short, there will be a finite number of flat directions parametrizing moduli space, and each of these moduli will result in a corresponding scalar field when we extend these deformations to depend on the longitudinal point. Reintroducing $\mathcal{N} = 1$ supersymmetry, these will be the lowest components of chiral supermultiplets which will exhaust the degrees of freedom of the low-energy effective field theory.

Then, expanding the supergravity action in these modes the action governing these chiral fields can be found. This is nothing else than the explicit form of the effective theory for our original strongly-interacting theory.

5.1 Topology of X_6

It is necessary to quickly introduce a certain fact about the topology of X_6 for us to distinguish between normalizable and non-normalizable Kähler deformations. First of all, we take as an assumption that the third Betti number of the cone vanishes:

$$b_3(X) = 0 \tag{5.1}$$

Moreover, it can be proven[30] from Myers' theorem[24] that Y_5 being Sasaki-Einstein means the following Betti numbers vanish[1]:

$$b_1(Y) = b_4(Y) = 0 \tag{5.2}$$

We recall the long sequence involving relative homology groups:

$$\ldots \to H^{i-1}(Y) \to H^i(X,Y) \to H^i(X) \to H^i(Y) \to H^{i+1}(X,Y) \to \ldots \tag{5.3}$$

where $H^i(X,Y;\mathbb{R})$ is the relative homology group - closed k-forms on X vanishing on Y modulo exact forms with the same property - and when the $;\mathbb{R}$ is omitted we implicitly mean the base field is $\mathbb{R}$. We cut the sequence short by setting $i = 2$ and noting $H^1(Y) = 0$ as of 5.2 and $H^3(X,Y) \subset H^3(X) = 0$ as of 5.1; the short exact sequence is

$$0 \to H^2(X,Y) \to H^2(X) \to H^2(Y) \to 0 \tag{5.4}$$

Implying $H^2(X) = H^2(Y) \oplus H^2(X,Y)$. Applying Poincaré duality[2] on the two components and counting dimensions gives

$$b_2(X) = b_3(Y) + b_4(X) \tag{5.5}$$

[1]More in detail, Myers' theorem implies $\pi_1(Y_5)$ is a finite group. Thus the abelianization $H^1(Y)$ is also finite, and the rank $b_1(Y)$ must vanish. By Poincaré duality $b_4(Y) = 0$ follows.

[2]We note in the case of non-compact manifolds, such as X_6, Poincaré-Lefschetz duality is actually an isometry between $H^k(X)$ and $H_{6-k}(X,Y)$, instead of $H_{6-k}(X)$.

This identity will be necessary in the decomposition of harmonic 2-forms.

5.2 Kähler moduli

We now consider the moduli describing the deformation of the Kähler (so, geometric) structure of the background. Since $b_3(X) = 0$ by hypothesis, the complex structure is rigid. There are instead moduli for the Kähler form J; in particular it can be proven[12] that every cohomology class $[J]$ of $H^2(X)$ contains a single representative Ricci-flat Kähler form J, so that $H^2(X)$ is the moduli space for the Kähler structure. We can expand the cohomology class as

$$[J] = v^a[\omega_a] \tag{5.6}$$

with $[\omega_a]$ being a basis for the integral cohomology $H^2(X;\mathbb{Z})$, as the latter modulo torsion is a lattice sitting in $H^2(X;\mathbb{R})$. This means

$$\delta[J] = \delta v^a[\omega_a] \tag{5.7}$$

meaning there exist representatives in the classes such that the equation without square brackets holds. Since small variations of the Kähler form must be $(1,1)$ harmonic forms [8], we then know there exist $(1,1)$ harmonic representatives ω_a for the aforementioned basis of classes. Returning to 5.6 we can rewrite it as

$$J - v^a\omega_a \in [0] \tag{5.8}$$

But for the LHS to belong to the zero class just means to be exact. Therefore

$$J = J_0 + v^a\omega_a \tag{5.9}$$

with J_0 being exact and $(1,1)$. Note the linearity of this parametrization is an illusion of notation: the condition $\Delta\omega_a = 0$ depends on the metric and so on the v^a.

It is then useful to decompose this set of $b_2(X)$ harmonic forms according to identity 5.5 into $b_3(Y)$ noncompact elements $\tilde{\omega}_\beta$ and $b_4(X)$ normalizable forms $\hat{\omega}_\alpha$. By "normalizable" it is meant the hatted forms have finite norm according to the product

$$\int_X \omega_a \wedge \star\omega_b =: \mathcal{M}_{ab} \tag{5.10}$$

while the other $b_3(Y)$ don't. They are however all normalizable according to the "warped" product

$$\int_X e^{-4A}\omega_a \wedge \star\omega_b =: \mathcal{G}_{ab} \tag{5.11}$$

where the factor e^{-4A}, as will be explained later, is the warp factor resulting from the backreaction of the D-branes. More intuitively, in our particular case of X being (asymptotically) a cone, this means that $||\hat{\omega}_\alpha||^2$ must drop at least as fast as r^{-8}, while $||\tilde{\omega}_\beta||^2$ will go as r^{-4}.

In any local chart the ω_a forms will be generated by potentials κ_a as $\omega_a = i\partial\bar{\partial}\kappa_a$ just like J is generated by the Kähler potential $J = i\partial\bar{\partial}k$. This means in particular κ_a will coincide with $\frac{\partial k}{\partial v^a}$ up to a $z_i, \bar{z}_i$-independent piece, that is a function of the $\{v^a\}$ only. To fix this arbitrariness, the κ_a potentials are required to satisfy

$$\frac{\partial \kappa_a}{\partial v^a} \sim r^{-k}\,, \qquad k \geq 2 \tag{5.12}$$

so that they are determined up to a constant. This asymptotic condition must be enforced for the following analysis to be meaningful.

5.3 Remaining moduli

We are now in position to classify flat deformations of the axio-dilaton τ and the 2-forms A_2 and B_2, which we compose into a single complex 2-form $A_2 - \tau B_2$, plus τ itself. The former field's flat deformation will be generated by cohomology classes of $H_2(X)$, so in practice the harmonic forms ω_a found above can be used as a basis. Therefore the following decomposition is possible:

$$A_2 - \tau B_2 = l_s^2 \left(\beta^\alpha \hat{\omega}_\alpha + \lambda^\beta \tilde{\omega}_\beta \right) \tag{5.13}$$

The $b_4(X)$ moduli β^α weighing the compact forms $\hat{\omega}_\alpha$ will result in dynamical chiral fields in the HEFT. Instead, the $b_3(Y)$ λ^β moduli, to which we add one complex modulus for τ, parametrize deformations which will turn out to be non-dynamical. The reason is precisely that the kinetic matrix for these fields will turn out to be the inverse of $\mathcal{M}_{ab}$, which is only finite for the normalizable form.

Then, an obvious set of moduli z_I^i ($I = 1, \cdots, N,\ i = 1, 2, 3$) are to be introduced to parametrize the motion of the D3-branes on the background. As was hinted before, these alone are coordinates for the submanifold $\mathcal{M}_{\text{mes}}$ in $\mathcal{M}$, which to be precise should be quotiented by permutation of the branes. Therefore, $\mathcal{M}_{\text{mes}}$ based at any given point of moduli space is the symmetric product of N copies of the background geometry, as it for those particular values of the Kähler moduli.

There is a final class of flat shifts that should be considered, those of the A_4 potential. These moduli should (very schematically) better be thought of as paired with the v^a to form complex moduli. In the end, it turns out it is not really necessary to study the A_4 moduli explicitly for the purpose of finding the HEFT.

5.4 Chiral fields

Finally, we have to introduce the chiral fields corresponding to the dynamical moduli. We use the moduli β^α and z_I^i directly as the lowest component of the corresponding superfield, while to $v^a = (\hat{v}^\alpha, \tilde{v}^\beta)$ it is useful to associate fields $\rho^a = (\hat{\rho}_\alpha, \tilde{\rho}_\beta)$, obtained by a particular transformation:

$$\operatorname{Re} \rho_a = \frac{1}{2} \sum_I \kappa_a(z_I, \bar{z}_I; v) - \frac{1}{2 \operatorname{Im} \tau} I_{a\alpha\beta} \operatorname{Im} \beta^\alpha \operatorname{Im} \beta^\beta - \frac{1}{\operatorname{Im} \tau} I_{a\alpha\sigma} \operatorname{Im} \beta^\alpha \operatorname{Im} \lambda^\sigma\,, \tag{5.14}$$

where the κ_a are the potentials for the ω_a forms as defined in 5.2. The imaginary part instead as expected is related to the A_4 moduli; the explicit form of $\operatorname{Im}\rho_a$ is not necessary for our purposes.

To wrap up, the dynamical chiral fields in the effective field theory are

$b_4(X_6)$	$\hat{\rho}_\alpha$	norm. Kähler and A_4 deformations
$b_3(Y_5)$	$\tilde{\rho}_\sigma$	warp norm. Kähler and A_4 deformations
$b_4(X_6)$	β_α	norm. $A_2 - \tau B_2$ deformations
$3N$	z_I^i	D3-brane positions

plus the following non-dynamical marginal parameters:

$b_3(Y_5)$	λ_σ	warp norm. $A_2 - \tau B_2$ deformations
1	τ	axio-dilaton

The identification of the meaning of these fields and parameters in terms of the four-dimensional side is as follows. The z_I^i correspond clearly to the $3N$ independent mesons of the CFT, as it was already established. The ρ and β fields instead are the independent baryons. Their total number is

$$2b_4(X) + b_3(Y) = b_2(X) + b_4(X) = g - 1 \tag{5.15}$$

where we've used respectively identity 5.5 and the fact that the number of gauge groups is $\chi(X_6) = \sum_k b_k(-1)^k = 1 + b_2(X) + b_4(X)$. There is therefore a match with the number of baryonic moduli of the CFT as it was determined by solving the F- and D-term conditions.

λ_σ and τ instead are dual to the marginal deformations of the CFT, that is of the gauge and superpotential couplings. The $b_3(Y) + 1$ such marginal moduli correspond directly to the $b_3(Y) + 1$ marginal couplings that were found in the study of the conformal manifold of the quiver theory. (Thus, this constitutes the bulk-side proof of the conjecture (3.91)). A general feature is that the axio-dilaton τ will always be paired with the symmetric combination of gauge couplings:

$$\tau \leftrightarrow \tau_1 + \tau_2 + \ldots + \tau_\chi\,, \tag{5.16}$$

while the $b_3(Y)$ parameters λ_σ will be dual to the other $b_3(Y)$ independent combinations that generate marginal deformations.

5.5 Effective action

It now remains to specify the dynamics of these chiral fields through an effective action. A way to proceed is to start with the similar case of compactifications[20], where the Calabi-Yau transverse space X_6 is actually compact, unlike the asymptotically conical noncompact 6-folds encountered until now. That is, N D3-branes are placed on the background metric

$$ds_{10}^2 = l_S^2(ds_4^2 + ds_X^2) \tag{5.17}$$

where $\int_X d\,\mathrm{vol}_X = V_0$ is the unwarped volume of the compact space ($d\,\mathrm{vol}_X$ is the volume form associated with ds_6^2), and as a result the geometry is warped into

$$ds_{10}^2 = l_S^2(e^{2A}ds_4^2 + e^{-2A}ds_6^2)\,, \tag{5.18}$$

$$\nabla e^{-4A}(z^i) = \star\, l_S^4 \sum_I \delta^6(z^i - z_I^i) \ + \ (\text{fluxes}\ldots) \tag{5.19}$$

with warped volume $\int_X e^{-4A} d\,\mathrm{vol}_X =: V_w =: aV_0$. a is known as the universal modulus. In [20] it is argued that the low-energy effective four-dimensional theory resulting from such a compactification can be formulated as a superconformal supergravity theory, and that the total Kähler potential takes a remarkably simple form:

$$K = -3\ln(4\pi V_w) = -3\ln(4\pi V_0 a)\,. \tag{5.20}$$

The effective Lagrangian is then obtainable directly by differentiating the Kähler potential with respect to the moduli of the compactification, provided the dependency of the universal modulus from the latter is known. Just like in the non-compact case, it s possible to expand variations of the Kähler form in a basis of harmonic forms as $J = v^a\omega_a$, and the z_I^i moduli also reappear identically, mapping the positions of the D3-branes in the compact dimensions rather than the noncompact resolved cone. The β and λ - type moduli are set aside temporarily. The Lagrangian is composed by a chiral part for the chiral fields corresponding to the moduli plus a set of N fully decoupled $U(1)$ super-QED sectors due to the open string modes on each isolated D3-brane (as we're in a generic point of $\mathcal{M}$ where no branes coincide). The bosonic part of the chiral sector is

$$\mathcal{L} = -\pi\tilde{\mathcal{G}}^{ab}\nabla\rho_a \wedge \star\nabla\bar{\rho}_b - 2\pi\sum_I J_{i\bar{\jmath}}dz_I^i d\bar{z}_I^{\bar{\jmath}} \tag{5.21}$$

where also in this case the ρ_a fields are obtained by a transform of the v^a Kähler moduli similar to (5.14):

$$\operatorname{Re}\rho = \frac{1}{2}aI_{abc}v^b v^c + \sum_I \kappa(z_I, \bar{z}_I; v) + \text{(fluxes}\ldots) \,, \tag{5.22}$$

and the kinetic matrix and connection are given by

$$\tilde{\mathcal{G}}^{ab} = \frac{1}{2V_0 a}v^a v^b - \mathcal{G}^{ab} \tag{5.23}$$

$$\nabla\rho_a = d\rho_a + \sum_I \mathcal{A}_{ai}^I dz_I^i, \qquad \mathcal{A}_{ai}^I = \frac{\partial\kappa_a}{\partial z_i^I} \tag{5.24}$$

(we recall the matrix of warped products $\mathcal{G}_{ab}$ is as defined in 5.11).

This result is then first "decompactified" by taking the limit $V_0 \to \infty$, so as to extend to the case of non-compact X_6 as in our case. Then a second limit, $a \to 0$, is considered, equivalent to the near-horizon limit we require to get into the holographic regime. Under this limit, $\operatorname{Re}\rho(v)$ turns into the form 5.14 we introduced earlier, $\tilde{\mathcal{G}}_{ab}$ reduces to $\mathcal{G}_{ab}$, and the Lagrangian simplifies to:

$$\mathcal{L}_{\rho,z} = -\pi\mathcal{G}^{ab}\nabla\rho_a \wedge \star\nabla\rho_b - 2\pi\sum_I J_{i\bar{\jmath}}dz^i d\bar{z}^{\bar{\jmath}} \tag{5.25}$$

It can happen that in decompactification some of the ω_a will have turned into non-warp-normalizable forms for which $\mathcal{G}_{ab}$ diverges. In that case, the kinetic term $\mathcal{G}^{ab}$ for the corresponding ρ fields vanishes, so that the latter disappear from the dynamics[3]. Therefore it is understood that the a, b indices span only over warp-normalizable Kähler moduli of the non-compact X_6.

Finally, one adds back the A_2, B_2 moduli by again expanding $A_2 - \tau B_2$ in the basis of ω_a forms to obtain the $\beta^\alpha, \lambda^\sigma$ fields as explained before; then it is shown that they contribute to the action with a kinetic matrix proportional to the *unwarped* product matrix $\mathcal{M}_{ab}$ defined in 5.10. This means that only the β^α fields, modulating unwarped-normalizable flat shifts are dynamical fields. The final chiral Lagrangian is:

$$\mathcal{L}_{\text{chiral}} = \mathcal{L}_{\rho,z} - \frac{\pi}{\operatorname{Im}\tau}\mathcal{M}_{\alpha\beta}\, d\beta^\alpha \wedge \star d\bar{\beta}^\beta \tag{5.26}$$

where one must also take into account the fact that the β fields couple to the ρ fields through an addition to the latter's covariant derivative, whose final form is

$$\nabla\rho_a = d\rho_a - \mathcal{A}^I_{ai} dz^i_I - \frac{i}{\operatorname{Im}\tau}\left(I_{a\alpha\beta}\operatorname{Im}\beta^\beta + I_{a\alpha\sigma}\operatorname{Im}\lambda^\sigma\right) d\beta^\alpha \tag{5.27}$$

With this information, it is now possible to compute the HEFT corresponding to a given conical background provided the Kähler form for the generic Calabi-Yau deformation is known in complex coordinates. The following identity for $\mathcal{G}_{ab}$ can be proven[21] and is a useful computational shortcut:

$$\mathcal{G}_{ab} = -\frac{\partial \kappa_a}{\partial v^b} \tag{5.28}$$

Therefore, to outline a systematic procedure: once the general Kähler potential and Kähler form are known, derivatives with respect to the moduli yield the κ potentials, from which one computes the $\mathcal{G}$ matrix and the $\mathcal{A}$ connection; the intersection numbers and in particular the $\mathcal{M}_{\alpha\beta}$ matrix are instead invariants independent of the moduli and are readily evaluated on geometrical grounds.

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[3] This is not completely rigorous, as we would expect fields that decouple to have *divergent*, not vanishing, kinetic terms. The decoupling can be proven[21] by switching to the equivalent formulation with linear multiplets, performing the limit, and switching back to chiral fields.

5.6 Example: the Klebanov-Witten HEFT

As an example, we now summarize a direct application of this method to the conifold theory described in 3.5.

It is essential for the metric for the background to be presented in complex coordinates for the construction above to be applicable. We exploit the fact that at any generic point of moduli space *except* the origin, where the background is the singular conifold, the geometry is that of a smooth complex 3-fold describable as the total space of the sum of the tautological bundle on a $\mathbb{CP}^1$ with itself: $\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)$. The base $\mathbb{CP}^1 \cong \mathbb{S}^2$ of the bundle is the "resolution" of the conical singularity. We use this to build a chart on the resolved conifold by extending a stereographic chart on the base to get the complex coordinates $z^i = (\lambda, U, Y)$, the first stereographic for the base and the latter two fibral.

With this presentation, it is clear that the only 2-cycle of X_6 is the base $\mathbb{S}^2$, so that $b_2(X) = 1$. Moreover, we already know that as a real cone the conifold has base $SU(2) \times SU(2)/U(1) \cong \mathbb{S}^2 \times \mathbb{S}^3$, and that the resolution cannot really change the topology of the base at infinity, so that the only 3-cycle in Y_5 is the $\mathbb{S}^3$ and $b_3(Y) = 1$. So, according to 5.5, $b_4(X_6) = 0$. This will mean, according to the previous identification, that we will have a single $\tilde{v} =: v$ modulus parametrizing a non-normalizable Kähler deformation and dual to a single chiral field $\tilde{\rho} =: \rho$, no β fields, and two non-dynamical parametres λ and τ.

The Kähler modulus v is identified with the volume of the base. Therefore, all Calabi-Yau deformations of the conifold are a one-parameter family and the singular conifold itself lies at the origin, $v = 0$. The Kähler potential for X_6 in stereographic coordinates is detailed for example in [25], and takes the form:

$$k(z, \bar{z}; v) = \frac{1}{2} \int_0^{s^2} d \ln x \; \gamma(x; v) \; + \frac{v}{2\pi} \ln(1 + |\chi|^2) \tag{5.29}$$

where $s^2 = (1 + |\chi|^2)(|U|^2 + |Y|^2)$ and γ must satisfy the following for the metric to be Ricci-flat:

$$\gamma^3 + \frac{3v}{2\pi}\gamma^2 - x^2 = 0 \tag{5.30}$$

The potential κ generating the unique harmonic form $\tilde{\omega}$ is given simply by derivative of k with respect to the modulus, plus a v-dependent piece fixed by the asymptotic condition 5.12:

$$\kappa = -\frac{1}{4}\int_0^{s^2} d\ln x \, \frac{\gamma}{\pi\gamma + v} + \frac{1}{2\pi}\ln(1+|\chi|^2) - \frac{3}{8\pi}\ln v \tag{5.31}$$

This in turn defines the relationship between ρ and v as

$$\mathrm{Re}\,\rho = -\frac{1}{8}\sum_I \int_0^{s_I^2} d\ln x \, \frac{\gamma}{\pi\gamma + v} + \frac{1}{4\pi}\sum_I \ln(1+|\chi_I|^2) - \frac{3N}{16\pi}\ln v \tag{5.32}$$

from which one can readily find the 1×1 $\mathcal{G}$ matrix:

$$\mathcal{G} = -\frac{\partial\rho}{\partial v} = \frac{3}{16\pi}\sum_I (v + \pi\gamma)^{-1} \tag{5.33}$$

All ingredients for writing down the HEFT are now available. The chiral part of the bosonic lagrangian for the effective low-energy theory of the KW model is given by

$$\mathcal{L} = -\pi\mathcal{G}^{-1}\nabla\rho \wedge \star\nabla\bar{\rho} - 2\pi\sum_I J_{i\bar{j}}dz_I^i \wedge \star d\bar{z}_I^{\bar{j}} \tag{5.34}$$

where of course $J_{i\bar{j}} = \partial_i\bar{\partial}_j k(z,\bar{z};v)$ is the metric tensor of the resolved conifold. The expressions here are deceivingly simple: both $\mathcal{G}$ and J depend on the modulus v which must be understood as a function of the chiral field ρ by inversion of the Legendre transform. The connection for the covariant derivative, being a simple derivative of the Kähler potential, can also be explicited:

$$\mathcal{A}_i dz^i = (4v + 4\pi\gamma)^{-1} \left(\frac{2v + \pi\gamma}{\pi(1 + |\chi|^2)} \bar{\chi} d\chi - \frac{\gamma \left(\bar{U} dU + \bar{Y} dY \right)}{|U|^2 + |Y|^2} \right) \tag{5.35}$$

Chapter 6

The $Y^{2,0}$ HEFT

Having secured the tools required, we are now ready to take on the holographic effective theory of the $Y^{2,0}$ theory introduced in 3.6, to which this thesis is dedicated, and whose low-energy effective dynamics had not been investigated so far.

Exploiting the architecture established in the previous chapter, we will identify the fields and parameters on the gravitational dual which enter in the low-energy description, and determine the exact form of the effective action for this strongly-coupled, minimally supersymmetric gauge theory.

The bulk of this calculation turns out to be occupied by the determination of the Ricci-flat metric of the general Calabi-Yau deformation of the $X^{2,0}$ cone over the Sasaki-Einstein base $Y^{2,0}$; the deformations are in this case parametrized by two moduli, measuring the volumes of a 2-cycle and a 4-cycle respectively. Due to the introduction of this 4-cycle blowup (ultimately arising from the $\mathbb{Z}_2$ orbifold) the metric is quite more complicated than the deformation of the conifold, which only featured a 2-cycle blowup. As part of our original contributions we will thus present therefore our determination of the deformed metric in complex coordinates for generic values of the two moduli.

6.1 General properties

The features of the SCFT analyzed in 3.6 can be rederived holographically. First of all, the homology of the singular cone will allow for the counting of supergravity moduli and marginal deformations that can be seen to match with those found from the field theory side. We recall the metric on the $X^{2,0}$ cone is

$$ds_6^2 = dr^2 + r^2 ds_5^2 \tag{6.1}$$

$$ds_5^2 = \frac{1}{9}\left(d\psi + \sum_{i=1,2} \cos\theta_i d\phi_i\right)^2 + \frac{1}{6}\sum_{i=1,2}\left(d\theta_i^2 + \cos^2\theta_i d\phi_i^2\right) \tag{6.2}$$

$$\psi \in [0, 2\pi] \tag{6.3}$$

Since the conifold base had topology $\mathbb{S}^2 \times \mathbb{S}^3$[7], this base will be $Y^{2,0} \cong (\mathbb{S}^2 \times \mathbb{S}^3)/\mathbb{Z}_2$. The new Betti numbers of the cone are easily read from the fact that the generic deformation of $X^{2,0}$ is a bundle $\mathbb{C} \to \mathbb{CP}^1 \times \mathbb{CP}^1$, which we will prove in the next section. First, we have $b_3(Y^{2,0}) = b_3(\mathbb{S}^2 \times \mathbb{S}^3) = 1$, while $b_4(X^{2,0}) = b_4(\mathbb{CP}^1 \times \mathbb{CP}^1) = 1$. Finally, using (5.5), $b_2(X_6) = 2$. To summarize:

$$b_2(X_6) = 2, \quad b_4(X_6) = 1, \quad b_3(Y_5) = 1\,. \tag{6.4}$$

According to the discussion of section 5.2, there will be $b_2(X) = 2$ independent harmonic forms ω_a from the two 2-cohomology classes, and these will parametrize deformations of the Kähler form:

$$J = J_0 + v^a \omega_a\,, \tag{6.5}$$

with two associated moduli v^a. These forms will be divided in $b_3(Y) = 1$ "non-compact" form $\tilde{\omega}$, and $b_4(X) = 1$ "compact" form $\hat{\omega}$. The latter, associated with the blow-up of the 4-cycle, is the novel feature with respect to the Klebanov-Witten model, which only featured $\tilde{\omega}$.

In addition, ω_a will generate $A_2 - \tau B_2$ deformations:

$$(A_2 - \tau B_2) = l_s^2 \left(\beta\,\hat{\omega} + \lambda\,\tilde{\omega}\right) . \tag{6.6}$$

However, only the renormalizable deformation will yield a dynamical field β, as shown in section 5.3. The parameter λ will be a marginal parameter, alongside the axio-dilaton τ.

Once the $v^a = (\hat{v}, \tilde{v})$ Kähler moduli have been transformed into the $\rho^a = (\hat{\rho}, \tilde{\rho})$ fields according to (5.14), it is now possible to match the counting of moduli and marginal parameters between the bulk and the $Y^{2,0}$ CFT as following:

	$\mathrm{AdS}_5 \times Y_5$	CFT
dynamic moduli	z_I^i	$3N$ mesons
	$\hat{\rho}$	$\chi - 1 = 3$ baryons
	$\tilde{\rho}$	
	β	
marginal parameters	τ	2 marginal deformations
	λ	

Since the dynamical fields of the effective theory have been identified and matched, the explicit construction of the ω_a forms will allow for the specification of their dynamics. In practice, this requires determining the metric in complex coordinates of the general Calabi-Yau deformation of the $X^{2,0}$ cone. This metric is known in real, “polar” coordinates (see (3.83)) but a diffeomorphism to a chart compatible with the Kähler structure is not. In the next section, we present our solution to this problem in the form of an explicit expression for the Calabi-Yau metric in a set of complex coordinates, and verify the match with the known real parametrization.

6.2 Kähler form

The metric of the general Calabi-Yau deformation of the $X^{2,0}$ cone is already well-known in real coordinates as (repeating (3.83))

$$\begin{aligned} ds^2 \;=\; & \kappa^{-1}(r)\, dr^2 \;+\; \frac{1}{9}\kappa(r) r^2 (d\psi + \cos\theta_L d\phi_L + \cos\theta_R d\phi_R)^2 \\ & + \frac{1}{6} r^2 d\Omega_L^2 \;+\; \frac{1}{6}(r^2 + a^2) d\Omega_R^2 \,, \end{aligned} \tag{6.7}$$

$$\kappa(r) = \frac{1 + \dfrac{9a^2}{r^2} - \dfrac{b^6}{r^6}}{1 + \dfrac{6a^2}{r^2}} \,, \tag{6.8}$$

with a, b the two unique real moduli. The topology is that of an $\mathbb{R}^2$ bundle over $\mathbb{S}^2 \times \mathbb{S}^2$. We take it as an assumption that this matches with the complex structure associated to the Kähler form so that this is the total space of a $\mathbb{C}$ bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$ - we will confirm this a posteriori when we'll provide the complex-coordinates expression and show it agrees with the real form.

With this assumption, we search for the general CY metric on a $\mathbb{C} \to \mathbb{CP}^1 \times \mathbb{CP}^1$ bundle; on the spheres of the base we take the round metric, given by the Kähler forms j^L and j^R generated by Kähler potentials[1] k^L, k^R. We thus, having chosen any local complex charts y_L, y_R on the two basal spheres (e.g.: stereographic) and a fibral coordinate ζ, have a local chart $z^i = (\zeta, y_L, y_R)$ for the bundle. It is easy to verify explicitly that, given any set of complex coordinates on the base (y_L, y_R),

$$j^L \wedge j^R = e^{-\Lambda(k_L + k_R)} \left(dy^L \wedge dy^R \wedge d\bar{y}^L \wedge d\bar{y}^R \right) \,, \tag{6.9}$$

for some Λ depending on the overall size of the spheres (for the unit sphere, $\Lambda = 1$).

We also introduce the radial coordinate $t(\zeta, y_L, y_R)$ as

$$t := |\zeta|^2 e^{\Lambda k} \,. \tag{6.10}$$

We then propose the following ansatz for the Kähler potential:

[1] Attention should be paid to the fact that the potentials are different for different coordinates.

$$k_X = f(t) + \alpha k^L + \tilde{\alpha} k^R \tag{6.11}$$

where the moduli $\alpha, \tilde{\alpha}$, controlling the volume at $t = 0$ of the base 2-spheres, should parametrize the Ricci-flat Kähler resolutions of the cone. We now prove that there is always an $f(t; \alpha, \tilde{\alpha})$ that makes the metric Ricci-flat.

The Kähler form generated by (6.11) is straightforward:

$$\begin{aligned} J =& \ (\alpha + \Lambda t f')j^L + \ (\tilde{\alpha} + \Lambda t f')j^R \\ &+ i e^{\Lambda(k_L + k_R)}(f' + t f'')\,(d\zeta + \Lambda \zeta \partial k) \wedge (\text{c.c.}) \end{aligned} \tag{6.12}$$

The form $J = J_{i\bar{\jmath}}\, dz^i \wedge d\bar{z}^{\bar{\jmath}}$ defines a metric $ds^2 = J_{i\bar{\jmath}}\, dz^i \otimes d\bar{z}^{\bar{\jmath}}$, or equivalently $g_{i\bar{\jmath}} = J_{i\bar{\jmath}}$. We would like to identify the condition on $f(t)$ for such a metric to be Ricci-flat. In a Kähler manifold, and in a given local chart, the Ricci form is given by*ref*

$$R = i\partial\bar{\partial} \log \sqrt{\det g}\,, \tag{6.13}$$

so that Ricci-flatness, $R = 0$, means $\det g = \det J_{i\bar{\jmath}}$ is a constant. However, note the volume form induced by the metric would be

$$d\,\mathrm{vol}_{X_6} = \sqrt{\det g}\ \left(\Omega \wedge \overline{\Omega}\right) \tag{6.14}$$

with $\Omega = d\zeta \wedge dy^L \wedge dy^R$ the holomorphic three-form on X_6. However, for a Kähler 3-fold one has the equivalent expression in terms of the Kähler form:

$$3!\, d\,\mathrm{vol}_{X_6} = J \wedge J \wedge J \tag{6.15}$$

Having defined $e^3 := d\zeta + \Lambda\zeta\partial(k_L + k_R)$, this is

$$= 3(\alpha + \Lambda t f')(\tilde{\alpha} + \Lambda t f') i e^{\Lambda(k_L + k_R)}\left(f' + t f''\right)\left(j^1 \wedge j^2 \wedge e^3 \wedge \bar{e}^{\bar{3}}\right) \tag{6.16}$$

and using (6.9):

$$= ((\alpha + \Lambda t f'))((\tilde{\alpha} + \Lambda t f')) \left(f' + t f''\right) \Omega \wedge \overline{\Omega} \tag{6.17}$$

By comparing (6.17) with expression (6.14) for the volume form, we have to deduce that Ricci-flatness is equivalent to the following expression being constant:

$$(\alpha + \Lambda t f')(\widetilde{\alpha} + \Lambda t f') \frac{d}{dt}(\Lambda t f') =: c \tag{6.18}$$

or, having defined $y(t) := \Lambda t f'(t)$,

$$(\alpha + y)\,(\widetilde{\alpha} + y)\,y' = c\,. \tag{6.19}$$

This differential equation (second order in $f(t)$) is thus the condition for the metric resulting from the ansatz (6.11) to be Ricci-flat, and therefore a supergravity solution.

Since $f(t)$ must be regular as $t = 0$, and $f' = \frac{y}{\Lambda t}$, it must be that y goes to zero at least as fast as t as $t \to 0$; this condition eliminates the freedom from the constant of integration for equation (6.19). The constant c on the other hand can be readily reabsorbed into a t rescaling. Therefore there should be a unique y (and so a unique f up to unconsequential constant shifts) that gives a Ricci-flat metric. In appendix A.3 we prove that the solution is indeed unique, and the solution to (6.19) is given by

$$y(t; \alpha, \widetilde{\alpha}) = |\alpha - \widetilde{\alpha}| \cosh\left(\frac{1}{3}\cosh^{-1}\left(12\frac{ct + D}{|\alpha - \widetilde{\alpha}|^3}\right)\right) - \frac{\alpha + \widetilde{\alpha}}{2}\,, \tag{6.20}$$

$$D := \frac{1}{12}(-\alpha^3 + 3\alpha^2\tilde{\alpha} + 3\alpha\tilde{\alpha} - \tilde{\alpha}^3)\,. \tag{6.21}$$

While essential for our derivation of the HEFT, the explicit form above of $y(t; \alpha, \tilde{\alpha})$ is not necessary to verify this metric matches with the real-coordinate form (6.7): let us express the Kähler form in terms of y and show it actually coincides with the latter. We have

$$J = (\alpha + y)j^1 + (\tilde{\alpha} + y)j^2 + \frac{ie^{\Lambda k}}{\Lambda} y' e^3 \wedge \bar{e}^{\bar{3}} \tag{6.22}$$

$$= (\alpha + y)j^1 + (\tilde{\alpha} + y)j^2 + \frac{ie^{\Lambda k}\, c}{\Lambda(\alpha + y)(\tilde{\alpha} + y)} e^3 \wedge \bar{e}^{\bar{3}} \tag{6.23}$$

Now, we parametrize the fiber as

$$\zeta = e^{-\Lambda k/2}\sqrt{t}\, e^{i\psi}\,, \tag{6.24}$$

and the 2-spheres with spherical coordinates θ_i, ϕ_i which fixes $\Lambda = 1$. Then the metric corresponding to J is

$$ds^2 = (\alpha + y)\, d\Omega_L^2 + (\tilde{\alpha} + y)\, d\Omega_R^2 + \frac{y'}{t}\left(\frac{dt^2}{4} + t^2(d\psi + \sigma)^2\right) \tag{6.25}$$

Where $\sigma := -i\frac{\Lambda}{2}(\partial k - \bar{\partial} k) = \sum_i \cos\theta_i d\phi_i$. But the $t - \psi$ part is simply

$$ds^2 = \frac{1}{4y't}dy^2 + (y't)(d\psi + \sigma)^2 \tag{6.26}$$

Exploiting both the Ricci-flatness condition (6.19) and its integrated form (A.10) we rewrite the expression $y't$ as

$$y'\,t = \left(\frac{1}{(\alpha + y)(\tilde{\alpha} + y)}\right)\left(\frac{y^3}{3} + \frac{\alpha + \tilde{\alpha}}{2}y^2 + \alpha\tilde{\alpha}y\right) \tag{6.27}$$

$$= 3c\,r^2\left(1 + \frac{3}{2}\frac{\tilde{\alpha} - \alpha}{r^2} + \alpha^2(\alpha - 3\tilde{\alpha})2r^6\right)\Big/\left(1 + \frac{\tilde{\alpha} - \alpha}{r^2}\right) \tag{6.28}$$

$$= 3c\,r^2\left(1 + \frac{9a^2}{r^2} - \frac{b^6}{r^6}\right)\Big/\left(1 + \frac{6a^2}{r^2}\right) \tag{6.29}$$

$$= 3c\,r^2\kappa(r) \tag{6.30}$$

provided we make the identifications

$$a^2 = \frac{1}{6}(\tilde{\alpha} - \alpha) \qquad\qquad b^6 = \frac{\alpha^2(3\tilde{\alpha} - \alpha)}{2} \tag{6.31}$$

The final coordinate change to the (asymptotically) conical r coordinate is then given by $r^2 := y + \alpha$ - note this renders the inherent symmetry between the left and right 2-cycles non-manifest[2]. The resulting metric, after taking $c = 1/3$, is precisely the known real-coordinate metric (6.7). Thus, as the latter is the most general Calabi-Yau deformation of the $X^{2,0}$ cone, we have to conclude that the two-parameter family of metrics (6.22) in complex coordinates coincides with it.

Using the freedom to scale t to fix $c = 1/3$ for future convenience and introducing the notation

$$\delta := \alpha - \tilde{\alpha}\,, \quad \sigma := \alpha + \tilde{\alpha}\,, \tag{6.32}$$

the Kähler form is explicitly given by

$$J(\sigma,\delta) = \left(z + \frac{\delta}{2}\right) j^1 + \left(z - \frac{\delta}{2}\right) j^2 + ie^k z' e^3 \wedge \bar{e}^{\bar{3}} \tag{6.33}$$

$$z(t;\sigma,\delta) = \delta\ \cosh\left(\frac{1}{3}\cosh^{-1}\left(\delta^{-3}\left(4t + \frac{\sigma(3\delta^2 - \sigma^2)}{2}\right)\right)\right)\,. \tag{6.34}$$

Or, equivalently, in terms of the y function:

$$J(\sigma,\delta) = (y+\alpha) j^1 + (y+\tilde{\alpha}) j^2 + ie^k y' e^3 \wedge \bar{e}^{\bar{3}} \tag{6.35}$$

Unfortunately, a closed-form expression for the Kähler potential seems impossible. The function $f(t)$ can nevertheless be written in integral form, as such:

$$f(t;\sigma,\delta) = \int_0^t d\ln t' y(t') \tag{6.36}$$

6.3 Kähler moduli

As we have seen, a very convenient basis of moduli for the Kähler structure is given by the sum and difference σ and δ of the volume of the base

[2]Clearly, we could swap α and $\tilde{\alpha}$ in all of the above definitions, with no consequence.

spheres defined in (6.32), up to a normalization which we will shortly determine. These correspond respectively to blowing up the two basal 2-cycles (which equates to a blowup of the product 4-cycle of the base) and to an antisymmetric blowing and shrinking of the two $\mathbb{S}^2$ (a blowup of the difference 2-cycle). We'll examine this geometric structure in more detail in this section.

We will construct the two harmonic forms $\widehat{\omega}$ and $\widetilde{\omega}$ by differentiating J directly with respect to the relevant moduli, since (see (5.9))

$$J = J_0 + \widehat{v}\,\widehat{\omega} + \widetilde{v}\,\widetilde{\omega} \tag{6.37}$$

A basis $(\widehat{\omega}, \widetilde{\omega})$ of harmonic two-forms will allow us to also parametrize deformations of the A_2 and B_2 fields according to (5.13).

As it is clear from our general discussion, we expect to be able to choose $(\widehat{\omega}, \widetilde{\omega})$ such that $\widehat{\omega}$ is normalizable in the sense of (5.10) - this would be the form generating the blowup of the 4-cycle. The other form $\widetilde{\omega}$ will be non-renormalizable and will correspond to a 2-cycle blowup, the same that was already present in the Klebanov-Witten theory. It is easy to see that the modulus $\widehat{v}$ relative to $\widehat{\omega}$ must be proportional to the sum σ of the basal volumes, since the corresponding harmonic form $\frac{\partial J}{\partial \sigma}$ is normalizable. To show this, we consider the asymptotic behaviour of the Kähler form as $t \to \infty$. Defined

$$T := \left(4t + \frac{\sigma(3\delta^2 - \sigma^2)}{2}\right) \tag{6.38}$$

we have

$$z \sim T^{1/3} \qquad\qquad z' \sim T^{-2/3} \tag{6.39}$$

$$\frac{\partial z}{\partial \sigma} \sim \frac{\partial T}{\partial \sigma} T^{-2/3} \qquad\qquad \frac{\partial z'}{\partial \sigma} \sim \frac{\partial T}{\partial \sigma} T^{-5/3} \tag{6.40}$$

where we have omitted constant factors. Therefore

$$J \sim \left(T^{1/3} + \frac{\delta}{2}\right) j^1 + \left(T^{1/3} - \frac{\delta}{2}\right) j^2 + i e^k T^{-2/3} e^3 \wedge \bar{e}^{\bar{3}} \tag{6.41}$$

and

$$\frac{\partial J}{\partial \sigma} \sim \frac{\partial T}{\partial \sigma}\left(T^{-2/3}j^1 + T^{-2/3}j^2 + ie^k T^{-5/3}e^3 \wedge \bar{e}^{\bar{3}}\right) \tag{6.42}$$

so that the norm goes as $\left\|\frac{\partial J}{\partial \sigma}\right\|^2 = \frac{\partial J}{\partial \sigma} \wedge \star \frac{\partial J}{\partial \sigma} \sim t^{-2} \sim r^{-12}$, which is integrable[3].

By contrast, the remaining harmonic form must be nonrenormalizable. For example, differentiating with respect to δ, we obtain

$$\frac{\partial J}{\partial \delta} \sim j^1 - j^2 + ie^k \frac{\partial T}{\partial \delta} T^{-4/3} e^3 \bar{e}^{\bar{3}} \tag{6.43}$$

with norm $\left\|\frac{\partial J}{\partial \delta}\right\| \sim t^{-2/3} \sim r^{-4}$, not integrable. We note however that it is still warp-integrable, according to our general discussion in 5.2.

Having ascertained we would like $\widehat{v} \propto \sigma$ and $\widetilde{v} \propto \delta$, one would like to also fix the right normalization for the Kähler moduli. For this, note that[4] , denoted C^1, C^2 the basis 2-spheres, and $\alpha^i = (\alpha, \widetilde{\alpha})$,

$$\int_{C^i} \frac{\partial J}{\partial \alpha^j} = 4\pi\, \delta_{ij} \tag{6.44}$$

These are also the intersection number of C^1, C^2 with the Poincaré dual 4-cycles of these forms $\frac{\partial J}{\partial \alpha^i}$; since the C^i form a basis, a pair 4-cycles with the same intersection numbers will necessarily belong the dual classes. We consider the (noncompact) 4-cycles D^1, D^2 given respectively by the fibres of C^1, C^2, recalling the cone is a $\mathbb{C} \to \mathbb{CP}^1 \times \mathbb{CP}^1$ bundle. Since it is easily seen that

$$D^i \cdot C^j = \epsilon^{ij}\,, \tag{6.45}$$

then we can identify the Poincaré duals:

[3]Asymptotically, as $t \gg \alpha, \tilde{\alpha}$, the metric reduces to the sharp cone $dr^2 + r^5 ds_5^2$, and in this regime $t \propto r^6$; therefore a function on X is integrable if it decays faster than $r^{-6} \sim t^{-1}$.

[4]We note that $\int_{C^1} \frac{\partial J}{\partial \alpha} = \int_{C^1} \left.\frac{\partial (y+\alpha)}{\partial \alpha}\right|_{t=0} j^1 = \int_{C^1} j^1 = 4\pi$, where we've exploited the fact that $y(t=0) = 0$, as it is clear from (A.10). The other three cases are identical.

$$-\frac{1}{4\pi}\frac{\partial J}{\partial \alpha} \longleftrightarrow D^2 \tag{6.46}$$

$$-\frac{1}{4\pi}\frac{\partial J}{\partial \widetilde{\alpha}} \longleftrightarrow D^1 . \tag{6.47}$$

Then a useful normalization for $\widehat{\omega}$ would be

$$\widehat{\omega} = \omega_1 = \frac{1}{2\pi}\left(\frac{\partial J}{\partial \sigma}\right) = \frac{\partial J}{\partial \widehat{v}}\,, \qquad \text{with } \widehat{v} := 2\pi\sigma \tag{6.48}$$

which makes it so $\int_{C^i}\widehat{\omega} = 2$ is integer. The dual to $\widehat{\omega}$ is $-2(D^1 + D^2) =: E$; it is easy to show this is actually the base $\mathbb{CP}^1 \times \mathbb{CP}^1$. Thus, not only does $\widehat{\omega}$ generate the blowup of the 4-cycle made by the product of the two basal spheres $E = \mathbb{S}^2 \times \mathbb{S}^2$ (and $\widehat{v}$ parametrizes its volume), they are actually Poincaré dual.

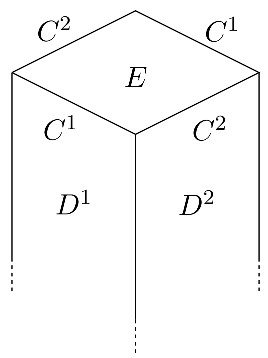

Figure 6.1: Schematic representation of the resolved $X^{2,0}$ as a line bundle, with the relevant 2- and 4-cycles.

Similarly, we choose

$$\widetilde{\omega} = \omega_2 = \frac{1}{2\pi}\left(\frac{\partial J}{\partial \delta}\right) = \frac{\partial J}{\partial \widetilde{v}} \qquad \widetilde{v} := 2\pi\delta \tag{6.49}$$

dual to the 4-cycle $F = 2(D^1 - D^2)$. Actually, there is an inherent arbitrariness in the non-renormalizable form $\widetilde{\omega}$ as it could be shifted by an arbitrary multiple of the normalizable form $\widehat{\omega}$. Still, we stand by choice (6.49) for $\widetilde{v}$

and $\widetilde{\omega}$ as it proves to be as convenient as possible for explicit calculations.

This choice for the harmonic 2-forms and the moduli v^a allows for easy computation of the intersection numbers:

$$I_0 = \int \widehat{\omega} \wedge \widehat{\omega} \wedge \widehat{\omega} = E \cdot E \cdot E = 8 \tag{6.50}$$

$$I_1 = \int \widehat{\omega} \wedge \widehat{\omega} \wedge \widetilde{\omega} = E \cdot E \cdot F = 0 \tag{6.51}$$

$$I_2 = \int \widehat{\omega} \wedge \widetilde{\omega} \wedge \widetilde{\omega} = E \cdot F \cdot F = -8 \tag{6.52}$$

6.4 Chiral fields and effective Lagrangian

The HEFT will feature the following chiral fields:

z_I^i	$= (y_I^1, y_I^2, \zeta_I)$	D3-brane positions on X
$\widehat{\rho} = \rho_1$	related to $\widehat{v}$	4-cycle blowup deformation of X
$\widetilde{\rho} = \rho_2$	related to $\widetilde{v}$	2-cycle blowup deformation of X
β		$C_2 - \tau B_2$ normalizable deformation

and the following non-dynamical chiral parameters:

λ	$C_2 - \tau B_2$ non-renormalizable deformation
τ	axio-dilaton

We can match these directly with the field theory objects discovered in section 3.6. z_I^i map directly with the $3N$ mesons parametrizing $\mathcal{M}_{\mathrm{mes}}$. $\widehat{\rho}$, $\widetilde{\rho}$ match with the two baryons generating the resolution of the cone as in 3.6.3, while β is the third "non-geometric" baryonic modulus.

The chiral fields $\rho_a = (\widehat{\rho}, \widetilde{\rho})$ are related to the moduli $v_a = (\widehat{v}, \widetilde{v})$ by the transform (5.14) described in the previous chapter; as anticipated we will only need to specialize the precise form of the real part of $\rho_a(v_a)$:

$$
\begin{aligned}
\operatorname{Re}\widehat{\rho} &= \frac{1}{2}\sum_I \widehat{\kappa}(z_I,\bar{z}_I;v) - \frac{1}{2\operatorname{Im}\tau}I_0(\operatorname{Im}\beta)^2 - \frac{1}{\operatorname{Im}\tau}I_1\operatorname{Im}\beta\operatorname{Im}\lambda \\
&= \frac{1}{2}\sum_I \widehat{\kappa}(z_I,\bar{z}_I;v) - \frac{4}{\operatorname{Im}\tau}(\operatorname{Im}\beta)^2
\end{aligned}
\tag{6.53}
$$

$$
\begin{aligned}
\operatorname{Re}\widetilde{\rho} &= \frac{1}{2}\sum_I \widetilde{\kappa}(z_I,\bar{z}_I;v) - \frac{1}{2\operatorname{Im}\tau}I_1(\operatorname{Im}\beta)^2 - \frac{1}{\operatorname{Im}\tau}I_2\operatorname{Im}\beta\operatorname{Im}\lambda \\
&= \frac{1}{2}\sum_I \widehat{\kappa}(z_I,\bar{z}_I;v) + \frac{8}{\operatorname{Im}\tau}\operatorname{Im}\lambda\operatorname{Im}\beta
\end{aligned}
\tag{6.54}
$$

where $\kappa_a(z_I,\bar{z}_I;v) = (\widehat{\kappa},\widetilde{\kappa})$ are defined as the potentials that generate the ω_a, as in

$$
\omega_a = i\partial\bar{\partial}\kappa_a \tag{6.55}
$$

and also satisfy the following condition:

$$
\frac{\partial\kappa_a}{\partial v_a} \sim r^{-k} \sim t^{-k/6}, \quad k \geq 2 \tag{6.56}
$$

We are now able to present the bosonic part of the effective Lagrangian. This holds in a generic moduli space point where no D3-branes coincide, therefore there is first of all a decoupled sector of N copies of $U(1)$ SYMs, the normal abelian gauge theory each D-brane hosts. Then, the rest of the bosonic effective Lagrangian describes the chiral fields listed above:

$$
\mathcal{L}_{\text{chiral}} = -\pi\mathcal{G}^{ab}\nabla\rho_a\wedge\star\nabla\bar{\rho}_b - 2\pi\sum_I J_{i\bar{j}}dz^i d\bar{z}^{\bar{j}} - \frac{\pi\mathcal{M}}{\operatorname{Im}\tau}d\beta\wedge\star d\bar{\beta} \tag{6.57}
$$

where the kinetic factors are computable as follows (using (5.28)):

$$
\mathcal{G}_{ab} = \int_X e^{-4A}\omega_a\wedge\star\omega_b = -\int_X e^{-4A}J\wedge\omega_a\wedge\omega_b = -\frac{\partial\operatorname{Re}\rho_a}{\partial v_b} \tag{6.58}
$$

$$\mathcal{M} = \int_X \widehat{\omega} \wedge \star\widehat{\omega} = -\int J \wedge \widehat{\omega} \wedge \widehat{\omega} = -\widehat{v} I_0 = 8\widehat{v} \tag{6.59}$$

($\mathcal{G}^{ab}$ being of course the inverse matrix of $\mathcal{G}_{ab}$) and the covariant derivative ∇ is

$$\nabla\widehat{\rho} = d\widehat{\rho} - \mathcal{A}^I_{1i} dz^i_I - \frac{8i}{\operatorname{Im}\tau} \operatorname{Im}\beta \, d\beta \tag{6.60}$$

$$\nabla\widetilde{\rho} = d\widetilde{\rho} - \mathcal{A}^I_{2i} dz^i_I + \frac{8i}{\operatorname{Im}\tau} \operatorname{Im}\lambda \, d\beta \tag{6.61}$$

$$\mathcal{A}^I_{ai} = \frac{\partial\kappa_a(z_I, \bar{z}_I; v)}{\partial z^i_I} \tag{6.62}$$

To compute the coefficients $\mathcal{G}$,$\mathcal{A}$ we first determine the form of the κ potentials.

6.5 κ potentials

In accord to what was discussed in 5.2, since $\frac{\partial k_X}{\partial v^a}$ generates $\frac{\partial J}{\partial v^a} = \omega_a$, it must be that $\kappa_a = \frac{\partial k_0}{\partial v^a} + h(v)$ with $h(v)$ an arbitrary function of the moduli which would then be fixed as to satisfy the condition (6.56) (up to an additive constant). However, as will be seen shortly, $\frac{\partial k_0}{\partial v^a}$ itself satisfies the asymptotic condition, so that $h(v)$ is actually a constant, which we will omit.

Recalling $k_0 = f(t) + \frac{\sigma+\delta}{2} k^L + \frac{\sigma-\delta}{2} k^R$, and $f(t) = \int_0^t d\,\ln(t')y(t')$, we find

$$2\pi\widehat{\kappa}(t;\sigma,\delta) = \frac{\partial k_X}{\partial\sigma} = \left(\int d\,\ln t' \frac{\partial y}{\partial\sigma}\right) + \frac{1}{2}k^L + \frac{1}{2}k^R \tag{6.63}$$

$$2\pi\widetilde{\kappa}(t;\sigma,\delta) = \frac{\partial k_X}{\partial\delta} = \left(\int d\,\ln t' \frac{\partial y}{\partial\delta}\right) + \frac{1}{2}k^L - \frac{1}{2}k^R \tag{6.64}$$

so that the derivatives of the κ potentials become

$$\frac{\partial\kappa_a}{\partial v^b} = \frac{\partial^2 k_X}{\partial v^a \partial v^b} = \frac{1}{4\pi^2}\int_0^t d\,\ln(t')\begin{pmatrix} \frac{\partial^2 y}{\partial\sigma^2} & \frac{\partial^2 y}{\partial\sigma\partial\delta} \\ \\ \frac{\partial^2 y}{\partial\sigma\partial\delta} & \frac{\partial^2 y}{\partial\delta^2} \end{pmatrix}_{ab} \tag{6.65}$$

The explicit forms of the second derivatives of the y function, rather convoluted, are listed in appendix A.4. It is clear they have at most asymptotic behaviour $\sim t^{-2/3}$, which will be the same as that of their $\int d\,\ln t'$, so that the κ_a defined above satisfy (6.56) and no addition of a function of the moduli $h(v)$ is necessary.

Then, this allows immediately for the computation of the $\mathcal{G}_{ab}$ matrix:

$$\mathcal{G}_{ab} = -\frac{\partial\,\mathrm{Re}\,\rho_a}{\partial v^b} = -\sum_I \frac{\partial\kappa_a(z_I,\bar{z}_I;v)}{\partial v^b} = -\frac{1}{4\pi^2}\sum_I\int_0^{t_I} d\,\ln t'\frac{\partial^2 y}{\partial v^a\partial v^b}(t';v) \tag{6.66}$$

again resting on the explicit form of the second derivatives of y. The integrals are not solvable in closed form. The matrix will always be invertible and its inverse $\mathcal{G}^{ab}$ is the kinetic matrix for the ρ fields.

The connection $\mathcal{A}^I_{ai}$ instead can be found more explicitly. We treat the $z^3_I = \zeta_I$ and $z^{1,2}_I = y^{1,2}$ cases separately.

$$\mathcal{A}^i_{aI} = \frac{\partial^2 k_0}{\partial\zeta_I\partial v^a} = \frac{\partial^2 f}{\partial\zeta_I\partial v^a} \tag{6.67}$$

but, recalling $t = |\zeta|^2 e^k$, $\frac{\partial f(t)}{\partial\zeta} = \bar{\zeta}e^k f'(t) = \bar{\zeta}e^k y(t)/t = (\zeta)^{-1}y(t)$ so that this is simply

$$= \bar{\zeta}^{-1}\frac{\partial y}{\partial v^a} \tag{6.68}$$

and

$$\mathcal{A}^3_{1I} = \frac{1}{2\pi}\bar{\zeta}^{-1}\frac{\partial y}{\partial \sigma} \tag{6.69}$$

$$\mathcal{A}^3_{2I} = \frac{1}{2\pi}\bar{\zeta}^{-1}\frac{\partial y}{\partial \delta} \tag{6.70}$$

The $i = 1, 2$ components, instead, are

$$\mathcal{A}^i_{aI} = \frac{\partial^2 k_X}{\partial y^i_I \partial v^a} = \frac{\partial^2(\alpha^i k^i)}{\partial y^i_I \partial v^a} = \frac{\partial \alpha^i}{\partial v^a}\frac{\partial k^i}{\partial y^i_I} \tag{6.71}$$

(no summation on i is implied), so essentially:

$$\mathcal{A}^1_{1I} = \mathcal{A}^1_{2I} = \frac{1}{4\pi}\frac{\partial k^1}{\partial y^1} \tag{6.72}$$

$$\mathcal{A}^2_{1I} = -\mathcal{A}^2_{2I} = \frac{1}{4\pi}\frac{\partial k^2}{\partial y^2} \tag{6.73}$$

$$\tag{6.74}$$

6.6 Conclusions

We summarize the results obtained. The HEFT for the $Y^{2,0}$ model will be an $\mathcal{N} = 1$ field theory, with chiral superfields $\widehat{\rho}$, $\widetilde{\rho}$, β, and Lagrangian given by (6.57). The kinetic matrices $\mathcal{G}$, $\mathcal{M}$, and connection $\mathcal{A}$ are given respectively in (6.66), (6.59), (6.70) and (6.71). These quantities are expressed in terms of the y function and its derivatives with respect to the moduli; these are listed in appendix A.4.

Qualche parola in più?

Appendix A

Appendix

A.1 AdS space

Anti-de Sitter n-space is best understood as the Lorentzian analogue of hyperbolic n-space. It can be built by considering the following locus in the mixed-signature space $\mathbb{R}^{2,n-1}$:

$$x^\mu x_\mu = -(t^1)^2 - (t^2)^2 + \sum_{i=1}^{n-1} (x^i)^2 = -R^2 \tag{A.1}$$

which is reminiscent of the embedding of hyperbolic n-space in $\mathbb{R}^{1,n}$:

$$x^\mu x_\mu = -t^2 + \sum_{i=1}^{n} (x^i)^2 = -R^2 \tag{A.2}$$

Equation A.1 is explicitly preserved by $SO(2, n-1)$, and this group acts transitively on it, so that the locus inherits a Lorentzian metric from the ambient Minkowski space with that same symmetry group. This means the locus is a maximally symmetric space, having the same number of symmetries as $\mathbb{R}^{1,n-1}$ since $\dim SO(2, n-1) = \dim(\mathbb{R}^n \rtimes SO(1,n))$. (To press on with the analogy, in the Riemannian case $\mathbb{H}^n$ has the same number of Killing vectors as $\mathbb{R}^n$ since $\dim SO(1,n) = \dim(\mathbb{R}^n \rtimes SO(n))$).

The locus has constant negative scalar curvature (using S for the Ricci scalar to avoid confusion with the R radius introduced above):

$$S = -\frac{n(n-1)}{R^2} \tag{A.3}$$

However, the locus built above is not suitable to be used as a spacetime for a reasonable physical theory, as it contains closed timelike curves (CTCs), signaling a pathological causal structure. An example of CTC is the unit circle in the t^1t^2 plane. It is possible however to consider the covering space of the locus, which will be what we will refer to as anti-de Sitter n-space, AdS$_n$. The covering space is again a maximally symmetric space, but it is now simply-connected and CTC-free.

AdS, similarly to dS, admits multiple useful coordinate charts. The Poincaré chart is the analogue of the Poincaré half plane model, and the metric is:

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + dx^\mu dx_\mu\right) \tag{A.4}$$

where $z > 0$, $x^\mu \in \mathbb{R}^{1,n-2}$, and $dx^\mu dx_\mu$ is the standard metric on $\mathbb{R}^{1,n-2}$. The Poincaré chart, unlike the Riemannian case, is not global and only maps a particular wedge of the full AdS. A global chart would be given by the following coordinates, accordingly called global coordinates or cylindrical coordinates:

$$ds^2 = R^2\left(-\cosh^2\chi\, d\tau^2 + d\chi^2 + \sinh^2\chi\, d\Omega^2\right) \tag{A.5}$$

With $d\Omega^2$ the line element on $\mathbb{S}^{n-2}$. Note that constant τ slices are copies of $\mathbb{H}^{n-1}$. Remapping the radial coordinate as $d\chi = d\rho/\cos\rho$ to a finite range $(0 \leq \rho \leq \pi/2)$ this can also be rewritten as

$$ds^2 = R^2\frac{1}{\cos^2\rho}\left(-dt^2 + d\rho^2 + \sin^2\rho d\Omega^2\right) \tag{A.6}$$

A.2 Conformal boundary and symmetries

The last set of coordinates A.6 are a starting point for building the Penrose diagram of AdS. For fixed Ω_i the t,ρ part of the metric is sent to the flat metric by multiplication with the conformal factor $\cos^2\rho$. AdS is thus represented as an infinite solid cylinder.

We can read the induced topology and metric on the boundary, with the caveat that the conformal factor was arbitrary (provided it was such the metric did not diverge), and thus the boundary's metric will be defined up to a conformal rescaling - we can only identify a natural conformal class for the boundary. This will prove to have physical relevance as possible holographic duals will be conformal.

The topology of the boundary is therefore $\mathbb{S}^{n-2} \times \mathbb{R}$ and a representative of the conformal class is given by setting $\rho = \pi/2$:

$$ds^2 = dt^2 - d\Omega^2 \tag{A.7}$$

which is a Lorentzian metric. The conformal boundary of AdS is itself a spacetime; this is a nontrivial fact which has to be compared with the other constant-curvature manifolds of the same signature: the boundary of Minkowski space $\mathbb{R}^{1,n-1}$ has a vanishing (null) metric, being composed of null past and future, while the positive curvature case, de Sitter, has two spacelike boundaries in the infinite past and future. The relevance of this for the realization of holography should be evident. Only the negative curvature case seems to be able to naturally incorporate a Lorentzian structure on the boundary.

It will be much more useful for the application to holography to consider the boundary in the form it comes out from the Poincaré patch. This is located at $z = 0$ and is only a part of the full boundary. Taking the metric A.4 and factor a conformal z^2 we just obtain

$$ds^2 = x^\mu x_\mu \tag{A.8}$$

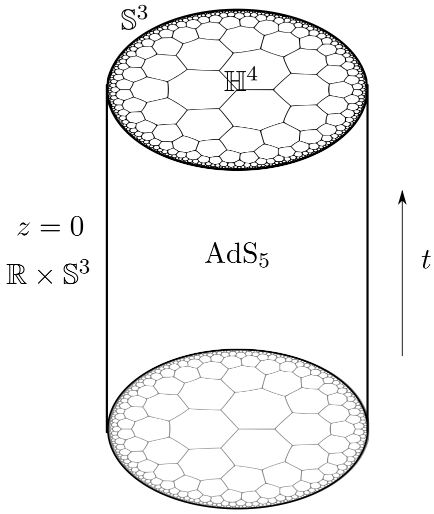

Figure A.1: Penrose diagram of AdS_5. The tiling of the hyperbolic plane represents the fact that the constant-time slices of AdS_5 are hyperbolic 4-space.

that is, the boundary is (locally) Minkowski $(n-2)$-space. This will be our preferential choice of representative metric.

We now turn to the description of the interplay between the bulk's and the boundary's symmetries. Essentially, isometries of AdS will induce conformal transformations on its boundary. As we've seen through its construction, the isometry group of AdS is $SO(2, n-1)$, this also coincides with the conformal group on $\mathbb{R}^{1,n-2}$.

+altre banalità di geometria

A.3 Solution of Ricci-flatness equation

We have seen how Ricci-flatness for the $Y^{2,0}$ reduces to the equation (6.19)

$$(\alpha + y)(\widetilde{\alpha} + y)y' = c \tag{A.9}$$

which we now provide an explicit solution to. We integrate (A.9) to obtain

$$\frac{y^3}{3} + \frac{\alpha + \widetilde{\alpha}}{2} y^2 + \alpha\widetilde{\alpha} y = ct + d \tag{A.10}$$

And then the regularity condition $y(0) = 0$ is satisfied with $d = 0$, and this cubic equation for y

$$\frac{y^3}{3} + \frac{\alpha + \widetilde{\alpha}}{2} y^2 + (\alpha\widetilde{\alpha})\, y = ct \tag{A.11}$$

is immediately seen to have one single real solution for any positive values of α, $\tilde{\alpha}$, c.

Now we're left with solving for the explicit form of y. Switching temporarily to $z = y + (\alpha + \tilde{\alpha})/2$ equation (A.10) is brought into the depressed form

$$z^3 - \frac{3}{4}(\alpha - \tilde{\alpha})^2 z = ct + D\,, \tag{A.12}$$

where

$$D := \frac{1}{12}(-\alpha^3 + 3\alpha^2\tilde{\alpha} + 3\alpha\tilde{\alpha} - \tilde{\alpha}^3) = \frac{b^6 - 36a^6}{3}\,, \tag{A.13}$$

so that the explicit solution for z and y is

$$z = |\alpha - \tilde{\alpha}| \cosh\left(\frac{1}{3}\cosh^{-1}\left(12\frac{ct + D}{|\alpha - \tilde{\alpha}|^3}\right)\right)\,, \tag{A.14}$$

$$y = z - \frac{\alpha + \tilde{\alpha}}{2}\,. \tag{A.15}$$

That (A.14) solves (A.12) can be readily verified by means of the trigonometric identity $\cosh(3x) = 4\cosh^3(x) - 3\cosh(x)$.

A.4 Derivatives of y

We list the explicit derivatives of the $y(t)$ function required for the formulation of the HEFT. We recall y is

$$y(t;\sigma,\delta) = \delta\, C_{1/3}\left(\delta^{-3}T\right) - \frac{\sigma}{2} \tag{A.16}$$

where $T := 4t + \frac{\sigma(3\delta^2-\sigma^2)}{2}$ and $C_{1/3}(x) = \cosh\left(\frac{1}{3}\cosh^{-1}(x)\right)$. We will make use in the following table of the notation C, C', C'', ... to refer to the zeroth, first, second, ... derivatives of the $C_{1/3}$ function evaluated always at $\delta^{-3}T$.

The first derivatives are:

$$\frac{\partial y}{\partial \sigma} = \delta^{-2}\frac{\partial T}{\partial \sigma}C' - \frac{1}{2} \tag{A.17}$$

$$\frac{\partial y}{\partial \delta} = C + \left(-3\delta^{-3}T + \delta^{-2}\frac{\partial T}{\partial \delta}\right)C' \tag{A.18}$$

And the second derivatives:

$$\frac{\partial^2 y}{\partial \sigma^2} = \delta^{-2}\left(\frac{\partial^2 T}{\partial \sigma^2}C' + \left(\frac{\partial T}{\partial \sigma}\right)^2 \delta^{-3}C''\right) \tag{A.19}$$

$$\frac{\partial^2 y}{\partial \delta \partial \sigma} = \left(-2\delta^{-3}\frac{\partial T}{\partial \sigma} + \delta^{-2}\frac{\partial^2 T}{\partial \sigma \partial \delta}\right)C' + \delta^{-2}\frac{\partial T}{\partial \sigma}\left(-3\delta^{-4}T + \delta^{-3}\frac{\partial T}{\partial \delta}\right)C'' \tag{A.20}$$

$$\frac{\partial^2 y}{\partial \delta^2} = \left(-4\delta^{-3}\frac{\partial T}{\partial \delta} + \delta^{-2}\frac{\partial^2 T}{\partial \delta^2}\right)C' + \delta^{-1}\left(-3\delta^{-3}T + \delta^{-2}\frac{\partial T}{\partial \delta}\right)C'' \tag{A.21}$$

The asymptotic behaviour can be read easily by noting $T \sim t$, $C \sim t^{1/3}$, $C' \sim t^{-2/3}$, $C'' \sim t^{-5/3}$, and that derivatives of T do not depend on t.

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