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Holographic effective field theories: a case study

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Abstract

The identification of the low-energy effective field theory associated with a given microscopic strongly interacting theory constitutes a fundamental problem in theoretical physics, which is particularly hard when the theory is not sufficiently constrained by symmetries. Recently, a new approach has been proposed, which addresses this problem for a large class of four-dimensional minimally supersymmetric strongly coupled superconformal field theories, admitting a dual weakly coupled holographic description in string theory. This approach provides a precise prescription for the holographic derivation of the associated effective field theories. The aim of the thesis is to further explore this approach by focusing on a specific model, whose effective field theory has not been investigated so far. *(modificare abstract alla fine del lavoro.)*

Contents

Chapter 1

Introduction

Strongly-coupled quantum field theories represent canonical examples of physical systems whose study is extremely challenging. Even the question of the mere existence of any interacting QFT in four dimensions from a formal standpoint has not been settled. In addition to this, strong couplings are not amenable to the tools of perturbation theory. The interest in this class of theories stems actually from practical considerations - many of them represent realistic models for physical phenomena, e.g. the theory of strong interaction.

A subset of questions concerns whether a given strongly-interacting theory is described at low energy by an effective local field theory, and if so, what are its degrees of freedom and their precise dynamics. Often, part of the structure of the effective theory is constrained by symmetries, but no general method exists to fix it completely. Recently[9], a novel approach for determining the effective Lagrangian was introduced that makes use of tools from an apparently unrelated area of physics: string theory.

It's remarkable that string theory was originally conceived as a description of hadronic physics, so a low-energy effective theory for what ultimately turned out to be a gauge theory, QCD. When string theory was found to have unsuitable qualities for this application, it was replaced by the full theory of quantum chromodynamics - however it also proved to be effective for solving a seemingly unrelated problem of fundamental physics: quantizing gravity. Since then, string theory blossomed into a vast and rich field reaching into numerous areas of mathematics and physics, and of course a candidate for a "Theory of Everything" describing the entirety of fundamental physics.

Among the most unexpected discoveries in strings, made decades after their conception, is a series of unusual exact equivalences between string theories set in particular effectively five-dimensional backgrounds and four-dimensional gauge QFTs. More generally, one finds families of exact equivalences between local quantum field theories and +1-dimensional theories containing gravity, which are termed "holographic". This explains the original partial success of strings in modeling strong interactions, assuming that some or perhaps most gauge theories have or can be approximated as having a holographic description as a *five* dimensional theory involving strings.

This roundtrip has therefore brought strings back to strongly-coupled gauge theories. Various aspects, qualitative and most importantly quantitative, of QFTs can be studied directly by means of their holographic string dual, a gravitational theory, if they have one. What has been done in this particular case is the degrees of freedom and the Lagrangian for the effective low-energy theory for a class of gauge theories with holographic duals have been identified by expanding the supergravity action on the dual bulk. Interestingly, these are somewhat special in that they are models of minimal supersymmetry ($\mathcal{N} = 1$), which makes for more realistic but by converse less constrained theories than typical holographic field theories, with higher supersymmetry.

In this work, we will specialize this construction to a specific field theory, the $Y^{2,0}$ theory, a strongly-coupled superconformal quiver theory for which we will therefore fix the exact effective Lagrangian, entirely through the geometry of the relative string background.

This thesis will be structured as follows. We will first provide a general introduction to IIB superstring theory, D-brane stacks on cones and the resulting gauge field theories, and holography. Then, we will summarize the relevant results and techniques from [9]. Finally, we will present a complete parametrization of the geometry of the $Y^{2,0}$ theory and will apply those results and techniques to identify the exact effective Lagrangian of the field theory.

Chapter 2

IIB superstrings and branes

Intro

2.1 Superstring theory

String theory either does not admit a nonperturbative Lagrangian formulation, or this formulation is unknown. An action functional can only be written upon choosing a perturbative vacuum; since we anticipate a string theory must include gravity, a choice of vacuum will also require a choice of background metric - in the simplest case Minkowski spacetime. The configuration of a string moving in this spacetime (the target M) is then given by specifying the two-dimensional submanifold (the worldsheet W_1) it traces in it[1], the worldsheet. In essence, this coincides with providing an embedding of the worldsheet

$$X^\mu(\tau, \sigma) : W_1 \to M \tag{2.1}$$

then quotiented under diffeomorphisms of the coordinates τ, σ on W_1.

[1]We note the worldsheet is just the obvious generalization of the concept of worldline of a particle to the case of 1-dimensional strings.

With a given choice of background metric the most natural action for a string is the Nambu-Goto action, the worldsheet area:

$$S_{NG}[X] = -T \int_{W^1} d\,\mathrm{vol}_h = -T \int_{W^1} d^2\sigma \sqrt{-h} \tag{2.2}$$

where $h_{ab} = \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b} G_{\mu\nu}$ is the induced metric on the worldsheet from the target space metric $G_{\mu\nu}$ under the embedding given by X^μ. T instead is a dimensionful constant called the string tension; in fact it is the only free parameter in string theory. We will also often refer to the entirely equivalent quantity α', the Regge slope. *check convenzioni!*

$$T = \frac{1}{2\pi\alpha'} \tag{2.3}$$

The Nambu-Goto action is very difficult (if not impossible) to quantize. It proves much easier to switch to the classically equivalent Polyakov action:

$$S_B[X, g] = -\frac{T}{2} \int_{W^1} d^2\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X_\nu G_{\mu\nu} \tag{2.4}$$

where now g_{ab} is an independent auxiliary field, not the induced metric from the X^μ. The equivalence is readily shown by computing the classical equation of motion for g_{ab} and substituting back into S_B to recover S_{NG}.

There are essentially two[2] different sensible choices for the topology of W_1: either a cylinder, with σ being the periodic variable running around, or a strip, so that σ is limited to an interval $[0, \sigma_1]$. These are respectively the closed and open string. The former is always a closed loop at any given instant in time. The open string instead has two endpoints for which we have to fix boundary conditions. One could choose between either Neumann boundary conditions, meaning

[2]We ignore the question of orientability of the worldsheet, not important for our purposes.

$$\left.\frac{\partial X^\mu}{\partial \sigma}\right|_{\sigma=0,\sigma_1} = 0 \tag{2.5}$$

which is just the constraint that no momentum flows out of the string endpoints, or Dirichlet boundary conditions, which fix

$$X^\mu|_{\sigma=0} = X_0^\mu, \qquad\qquad X^\mu|_{\sigma=\sigma_1} = X_{\sigma_1}^\mu \tag{2.6}$$

where X_0^μ, $X_{\sigma_1}^\mu$ are constants, essentially forcing the string endpoints to a specific spacetime point. In general one could mix $p+1$ Neumann conditions and $D-p-1$ Dirichlet conditions for different values of μ, so that the endpoints are constrained to a p-dimensional submanifold in space, and can move freely within it. Dirichlet conditions evidently break the symmetries of the target spacetime (Poincaré if we choose a Minkowski background) as they specify a preferential frame and submanifold; this symmetry will be recovered when it is recognized that the p-dimensional submanifold to which open strings attach is actually a dynamical object, a Dp-brane (D alluding to Dirichlet). We will return to D-branes after studying the string spectrum.

The Polyakov action displays invariance under worldsheet diffemorphisms

$$\sigma_a \to \sigma'_a(\sigma_a) \tag{2.7}$$

and Weyl transformations:

$$g_{ab} \to e^{\phi(\sigma)} g_{ab} \tag{2.8}$$

and thus perturbative string theory is naturally a two-dimensional conformal field theory. These symmetries must be quotiented out someway on quantization. The most straightforward way is to eliminate them by fixing a particular gauge and then quantizing (canonical quantization). The three symmetry generators can kill the three degrees of freedom in the metric to fix it to the 2D Minkowski: $g_{ab} = \eta_{ab}$. We get

$$S_B = -\frac{T}{2}\int d^2\sigma \partial_a X^\mu \partial^a X_\mu \tag{2.9}$$

where indices are raised with η^{ab}.

The theory described so far would be what is known as bosonic string theory. There are two issues with bosonic strings: the first is the presence of tachyons in both the open and closed string spectrum, i.e. some string modes will have a negative mass squared, signaling an instability of our choice of perturbative vacuum. The second is that, as the name suggests, there are exclusively bosons in the spectrum, which makes it unsuitable at the very least for phenomenological application. A modification to include supersymmetry can be performed to solve both of these issues and produces a set of string theories with fermions and without tachyons, the superstrings. We will in particular sketch the path from bosonic string theory to the so-called type II superstrings, two slightly different theories IIA and IIB.

There are at least two different approaches to introducing supersymmetry into a string theory. The path followed by the RNS (Ramond-Neveu-Schwarz) formalism is to impose SUSY at the worldsheet level; explicitly, adding fermions ψ^μ to act as superpartners to the bosons X^μ. We follow the derivation in [2]. The action is extended to

$$S = S_B + S_F = -\frac{T}{2}\int d^2\sigma \; \partial_a X^\mu \partial^a X_\mu + \bar{\psi}^\mu \rho^a \partial_a \psi_\mu \tag{2.10}$$

where the $\rho^{1,2}$ are two-dimensional gamma matrices satisfying the Clifford algebra

$$\left\{\rho^a, \rho^b\right\} = 2\eta^{ab} \tag{2.11}$$

The spinors' equation of motion, the Dirac equation, is actually the Weyl condition in two dimension. This brings the real degrees of freedom in the spinor for each μ from 4 to 2. Recalling that in (2 mod 8) dimensions there exist Weyl-Majorana spinors satisfying both the Weyl and Majorana conditions, imposing the latter on ψ halves again the on-shell polarizations to 1. Thus we have a match between bosonic and fermionic degrees of freedom. It can be proven *(ref)* the theory above is indeed worldsheet supersymmetric.

To quantize canonically, we introduce canonical commutation/anticommutation relations:

$$[X^\mu(\sigma), X^\nu(\sigma')] = \eta^{\mu\nu}\delta^2(\sigma - \sigma') \quad \{\psi^\mu(\sigma), \psi^\nu(\sigma')\} = \eta^{\mu\nu}\delta^2(\sigma - \sigma') \quad (2.12)$$

Note the X^0 and ψ^0 would create negative norm states, but these modes are eliminated by resorting to superconformal invariance. Classically this symmetry imposes the stress-energy tensor $T^{\mu\nu}$ and the supercurrent J^a_α vanish; imposing that in the quantum theory they annihilate physical states yields the restriction that removes the longitudinal ghosts from the spectrum. These take the name of super-Virasoro constrain.

Then the procedure for building the string spectrum is to expand the classical solutions in terms of Fourier modes, identify creators and destructors, and then select the states of the Fock basis that satisfy the super-Virasoro constraints. There are various ways to proceed at this point; the simplest and perhaps the most inelegant is light-cone quantization, which we will refer to. Essentially, a subset of $D-2$ transverse directions $i = 2, \ldots, D-1$ are selected and only X^i and ψ^i are made to correspond to operators, and the remaining longitudinal fields are to be determined from the former by the classical constraints. This procedure renders Lorentz-invariance non-manifest - we will verify indeed it is actually recovered a posteriori. A different common approach, more rigorous, takes the form of a BRST quantization, akin to that employed in Yang-Mills theories, by exchanging superconformal gauge invariance with the introduction of a series of Faddev-Popov ghosts. The final quantum theories are identical and the choice of quantization is conventional.

Boundary conditions for ψ^μ for an open string can actually be satisfied in two different by imposing periodicity or antiperiodicity, giving rise to the NS (Neveu-Schwarz) and R (Ramond) sectors, built over two grounds $|0\rangle_{NS}$ and $|0\rangle_{R}$. Closed strings have four: $|0\rangle_{NS-NS}$, $|0\rangle_{R-R}$, $|0\rangle_{R-NS}$, $|0\rangle_{NS-R}$ corresponding with different choice periodicity conditions for left and right-movers.

2.1.1 Open strings

It can be shown that while the NS ground $|0\rangle_{NS}$ is unique, and thus a spacetime scalar, $|0\rangle_{R}$ is eight-fold degenerate and this 8-plet transforms under the spinor representation of transverse $SO(8)$ - in other words, it is a spacetime spinor. In particular, it is a chiral Weyl-Majorana spinor, so it can be taken to be either of positive or negative chirality, choices we will denote as $|+\rangle_{R}^{a}$, $|-\rangle_{R}^{\dot{a}}$, with $a, \dot{a} = 1, \ldots, 8$ the spinor index.

The spectrum is built by acting on one of the grounds with bosonic and fermionic creators, to obtain states of higher and higher mass. For the NS sector, there are bosonic creators $a_n^{i\dagger}$ ($n \geq 1$) and fermionic $b_r^{i\dagger}$ (r positive half-integer), and the mass of the excited string is given by:

$$\alpha' M^2 = \sum_{n=1}^{\infty} n\, a_n^{i\dagger} a_n^i + \sum_{r=1/2}^{\infty} r b_r^{i\dagger} b_r^i - \frac{1}{2} \tag{2.13}$$

while for the R sector the fermionic creators are replaced by the integer-indexed $d_n^{\dagger i}$:

$$\alpha' M^2 = \sum_{n=1}^{\infty} \left(n\, a_n^{i\dagger} a_n^i + n d_n^{i\dagger} d_n^i \right) \tag{2.14}$$

The i indices here are target spacetime transverse indices, $i = 1, \cdots, 8$. Therefore each creator increases the spacetime spin of the string by one unit. We conclude the NS sector contains only spacetime bosons, and the R sector only spacetime fermions.

The constant shift of $-1/2$ in (2.13) (and of 0 in (2.14)) actually results from an ordering ambiguity of the creators and destructors $a_n^\dagger$, a_n (omitting spacetime indices for now) which introduces an arbitrary constant shift in the Hamiltonian upon quantization. If we want a consistent quantum theory with Lorentz invariance, this shift in the NS sector is fixed to $-1/2$; this is because for example the state

$$b^{i\dagger}_{1/2}\left|0\right\rangle_{NS} \tag{2.15}$$

is a vector with $(D-2)$ polarizations, and thus must be massless. However, it can be shown that this shift is also related to the number of spacetime dimensions D, so that a condition restricting D to a particular value can be found. An intuitive and perhaps heuristic explanation of this fact is as follows: the energy shift should be equal to the sum of the zero-point energies (ZPE) of the infinite harmonic oscillators, bosonic for $n = 1, 2, \ldots$ and fermionic for $r = 1/2, 3/2, \ldots$ We recall bosonic / fermionic QHOs have Hamiltonians

$$H = \omega\left(a^\dagger a + \frac{1}{2}\right) \qquad\qquad H = \omega\left(b^\dagger b - \frac{1}{2}\right) \tag{2.16}$$

Then for a given value of i the sum of all the ZPEs of the oscillators in the NS sector would be

$$E_0 = \sum_{n\in\mathbb{N}}\frac{n}{2} - \sum_{r\in\mathbb{N}+\frac{1}{2}}\frac{r}{2} = \frac{1}{4}\sum_{m\in\mathbb{N}}(-1)^m m \tag{2.17}$$

The sum $-1+2-3+\ldots$ is evidently divergent; we assume it is admissible to replace it with its ζ-regularized value[3] of $-1/4$. Therefore the ZPE per transverse direction is $E_0 = -1/16$; insisting the total ZPE is equal to $-1/2$

[3] The evaluation by analytic continuation of the divergent series is easily computed by considering $1 - 2x + 3x^2 - \ldots = (1+x)^{-2}$, converging for $|x| < 1$. Setting $x = 1$, right on the edge of the disk of convergence, yields the desired result.

results in

$$-\frac{D-2}{16} = -\frac{1}{2} \Rightarrow D = 10 \tag{2.18}$$

While this argument is not completely rigorous, it correctly identifies the existence and value of the so called critical dimension $D = 10$ for superstring theories. In a more formal setting (e.g., in BRST quantization) it can be shown that the classical Weyl symmetry of the action is spoiled by quantization and a conformal anomaly arises; this anomaly can be proven to cancel[4] only for $D = 10$.

It may worry that the mass-shell formula above assigns a negative mass-squared to the NS ground, which is therefore a tachyon. In addition, it is the *only* tachyon, meaning this theory is not spacetime supersymmetric. We will see in the next section how this state is actually removed and target supersymmetry recovered. For now, we note the only massless states are

$$b^{i\dagger}_{1/2}\left|0\right\rangle_{NS} \qquad \left|+\right\rangle_{NS} \tag{2.19}$$

while the rest of the tower of states have string scale-large $\sim (\alpha')^{-1/2}$ masses. Our interest in massless modes stems from the fact that in a low-energy ($\alpha' \to 0$) the strings can be approximated as pointlike particles (as their typical size $l_s \sim \sqrt{\alpha'}$) and their quantum theory as the corresponding field theory, with a field for each massless string mode, as the massive modes have decoupled. Such an effective low-energy theory will be described in section 2.3.

The first of the two states in (2.19) is a massless spin-1 boson, so it must be a photon associated with a $U(1)$ gauge theory. The latter is its spin-1/2 superpartner, a photino. It must be noted that what was described up to now holds for the directions in which the string endpoints are free to move, hence

[4]it's possible to have the conformal anomaly cancelled by the introduction of additional fields if $D < 10$, resulting in non-critical string theories. These have properties that make them unsuitable for our applications, however, and we will ignore them.

those for which Neumann conditions are imposed. As we have anticipated open strings in general end on Dp-branes and $D-p-1$ directions are actually constrained by Dirichlet conditions so as to keep the endpoints on the brane; the conclusion is the gauge interaction this massless string mode mediates is actually confined to the p-dimensional volume of the brane.

2.1.2 GSO projection

The construction above does not define a consistent theory. This is in part because it is not spacetime supersymmetric, an essential requirement considering that, as will be seen shortly, the closed string spectrum includes a gravitino (a massless spin-3/2 state) which must be associated with local supersymmetry. A procedure known as the Gliozzi, Scherk, Olive (GSO) projection solves this issue and in addition also eliminates the tachyonic state $|0\rangle_{NS}$, to end up with a consistent quantum theory.

The following operator is introduced, acting on the NS sector as

$$G = (-1)^{1+\sum_r b_r^{i\dagger} b_r^i} = (-1)^{\hat{F}+1} \tag{2.20}$$

and on the R sector as

$$G = \Gamma_{11}(-1)^{\sum_r d_r^{i\dagger} d_r^i} = \Gamma_{11}(-1)^{\hat{F}} \tag{2.21}$$

$\hat{F}$ is the worldsheet fermion number, and $\Gamma_{11} = \Gamma_0 \cdots \Gamma_9$ gives the chirality of the state.

Then the spectrum is projected into the $G = 1$ subspace for the NS sector, and into $G = \pm 1$ (either choice works) for the R sector. These two choices correspond to keeping either $|+\rangle_R^a$ or $|-\rangle_R^{\dot{a}}$ respectively and discarding the other.

When amputated with this precise prescription the spectrum is found to be spacetime supersymmetric. The scalar tachyon $|0\rangle_{NS}$ in particular is eliminated, being G-odd.

Commento sulle altre 3 superstringhe

2.1.3 Closed strings

The closed string spectrum, in somewhat poetic language, is the "square" of the open string spectrum. As seen before the choice can be made for either NS or R boundary conditions separately for left-movers and right-movers, giving four sectors. The GSO projection is performed separately on left and right movers, so that one is presented with the choice of the relative chirality of the two projections and so of the R grounds. These two possibilities will actually result in two different string theories. Choosing opposite chiralities gives type IIA strings, whose massless spectrum is given by

$$\tilde{b}^{i\dagger}_{1/2}\,|0\rangle_{NS} \otimes b^{j\dagger}_{1/2}\,|0\rangle_{NS} \tag{2.22}$$
$$\tilde{b}^{i\dagger}_{1/2}\,|0\rangle_{NS} \otimes |+\rangle_{R} \tag{2.23}$$
$$|-\rangle_{R} \otimes b^{j\dagger}_{1/2}\,|0\rangle_{NS} \tag{2.24}$$
$$|-\rangle_{R} \otimes |+\rangle_{R} \tag{2.25}$$

(the $\sim$ distinguishes creators/destructor for left movers from right movers). And IIB strings arise from equal chiralities:

$$\tilde{b}^{i\dagger}_{1/2}\,|0\rangle_{NS} \otimes b^{j\dagger}_{1/2}\,|0\rangle_{NS} \tag{2.26}$$
$$\tilde{b}^{i\dagger}_{1/2}\,|0\rangle_{NS} \otimes |+\rangle_{R} \tag{2.27}$$
$$|+\rangle_{R} \otimes b^{j\dagger}_{1/2}\,|0\rangle_{NS} \tag{2.28}$$
$$|+\rangle_{R} \otimes |+\rangle_{R} \tag{2.29}$$

So the massless spectrum is composed of 4 sectors of 64 physical states, two of them bosonic (NSNS, and RR) and the other fermionic (RNS and NSR). Massless states will correspond to fields in the supergravity approximation, in which the massive modes of the string decouple and the string theory is well described by the corresponding variety of 10D supergravity.

2.1.4 Background fields, string coupling and loop expansion

It was already hinted that the massless string modes we found should give rise to fields in some limit. In particular the NS-NS closed string ground (2.26) is a spacetime rank-2 tensor, which can be split into symmetric, antisymmetric and trace parts.

Non-zero values of these fields could be incorporated back into the background the perturbative string is based on. In fact, since the symmetric tensor field above is actually the graviton, we already have: the Polyakov action (**??**) already includes a coupling of the string to the target background metric $G_{\mu\nu}$. This is just the background value of the graviton.

The antisymmetric NS-NS $B_{\mu\nu}$ field (equivalently, a 2-form $B_2 = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$), called the Kalb-Ramond potential, is instead coupled to the fundamental string in a way that resembles the generalization of the coupling of a particle to the EM potential; a background value of $B_{\mu\nu}$ would result in the addition to the action of a term

$$S_B = \frac{T}{2} \int_{W_1} d^2\sigma \varepsilon^{\alpha\beta} B_{\mu\nu}(X) \partial_\alpha X^\mu \partial_\beta X^\nu \propto \int_{W_1} B_2 \qquad (2.30)$$

this will result ultimately in a stringy generalization of electrodynamics with strings coupling to a field strength 3-form $H_3 = dB_2$.

Finally, strings will also couple to the last NS-NS closed string field, the trace scalar ϕ called the dilaton. The coupling to a background dilaton is of the form

$$S_\phi = \frac{1}{4\pi}\int_{W_1} d^2\sigma\sqrt{g}\,R^{(2)}[g]\,\phi(X) \tag{2.31}$$

where $R^{(2)}[g]$ is the 2D Ricci scalar associated to g. Note that in the presence of a constant background dilaton $\phi(X) = \phi$, the integral above is simply the Euler characteristic $\chi(W_1)$, an integer topological invariant, by the Gauss-Bonnet theorem. χ has a simple expression in terms of the number of handles (the genus h), the number of boundaries n_b and the number of cross-caps n_c of the surface:

$$\chi = 2 - 2h - n_b - n_c \tag{2.32}$$

To understand the physical content of this contribution, we consider the simple case of orientable closed strings, and we imagine computing the amplitude of a process involving n external string states. Since only closed strings are involved, the only boundaries are the $n_b = n$ boundaries at infinity of the asymptotic states, and thus n_b is constant. Therefore the action from a constant dilaton background reduces to

$$S_\phi = \phi\chi = \phi(2 - n - 2h) = \text{const} - 2h\phi \tag{2.33}$$

and the Euclidean path integral for the amplitude would take the form

$$A = \int DX\,D\psi Dg\; e^{-S_{\mathrm{P}}}e^{-2\phi\, h[g]} \tag{2.34}$$

apart from normalization, and the path-integral is over worldsheets that match the external states. The integral over metric structures $\int Dg$ splits into disconnected components indexed by the genus, so that

$$A = \sum_{h=0}^{\infty} A_h = \sum_{h=0}^{\infty} (e^{\phi})^{-2h} \int_h DXD\psi Dg\, e^{-S_P} \tag{2.35}$$

This is a loop expansion, since the genus h counts the number of virtual string loops; however it is also a perturbative expansion in the string coupling $g_s := e^{\phi}$, giving the strength of a cubic string interaction "vertex". Therefore, we come to understand that the applicability of the perturbative string theory, including what has been introduced in this chapter so far, rests on the smallness of this string coupling g_s. It's worth of notice that this coupling is not an external dimensionless parameter of the theory (since string theory has none) but rather is related to the expectation value of a scalar field.

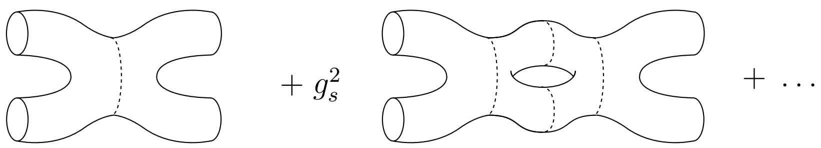


Figure 2.1: First terms in the loop expansion of a closed string four-point function. The worldsheets have been cut into "pair of pants" surfaces to count string interactions.

The perturbation series (2.35) is the stringy analogue of the QFT sum over Feynman diagrams. Its power comes from the fact that the genus-h contribution involves the calculation of a single diagram - the multiplicity of inequivalent Feynman graphs of a field theory (growing as $\sim e^h$) is then interpreted as the various inequivalent ways in which the unique worldsheet topology of genus h can degenerate to a diagram with pointlike particles and interactions as the string length is sent to zero.

Open strings interactions are instead controlled by a different coupling g_o. Consider the addition of an open string loop. This introduces two open string vertices and thus should result in a g_o^2 suppression. However, this operation results in the addition of a boundary and no change in genus, so $\Delta\chi = -1$. Therefore the suppression is also $(e^{\phi})^{-\Delta\chi} = g_s$ and we find that

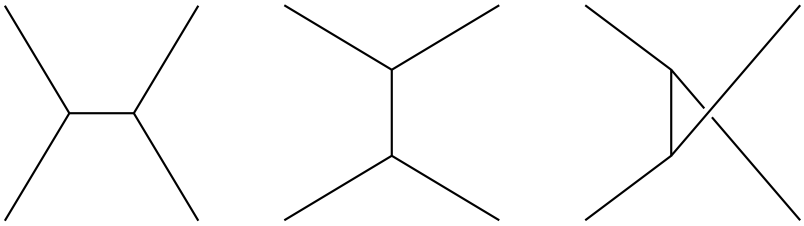


Figure 2.2: The possible Feynman graphs from degeneration of the genus-0 worldsheet from figure 2.1

$$g_o^2 \sim g_s \tag{2.36}$$

2.2 Type II supergravity and D-brane content

At energy scales much lower than the string scale $(\alpha')^{-1/2}$, equivalently $\alpha' \to 0$, all massive modes of a string theory decouple and a good description is given by an effective field theory comprising only the massless excitation. Since the string length goes to zero in this limit strings in massless states are essentially pointlike and the quantum theory will correspond to a local quantum field theory.

The effective field theories of the five superstring theories are the five supergravity (SUGRA) theories in 10 dimensions. The name of each SUGRA coincides with that of the superstring theory it is the effective theory of (e.g., IIB SUGRA is the effective theory of IIB superstrings). Supergravities are supersymmetric theories containing general relativity, and are obtained by extending local Poincaré invariance to include local supersymmetry. Just like Einstein gravity, they are nonrenormalizable, reflecting their origin as effective theories. As field theories, they are considerably simpler than general strings to find background solutions to; therefore we will make extensive use of the supergravity approximation in the context of holography.

10D SUGRAs are perhaps easier to introduce starting instead from the unique 11D SUGRA. The field content of 11D SUGRA is as follows (we also note the number of physical polarizations):

Bosons	Graviton	$g_{(M,N)}$	44
	3-form	$A_3 = \frac{1}{3!} A_{MNL} dx^M \wedge dx^N \wedge dx^L$	84
Fermions	Gravitino	Majorana ψ_M	128

As required by supersymmetry, the number of on-shell boson and fermion states are equal. These states form an irreducible supermultiplet, a gravity multiplet.

Upon dimensional reduction on a circle, in 10D these fields decompose into those of type IIA SUGRA:

NSNS	Graviton	$g_{(\mu,\nu)}$	35
	Kalb-Ramond 2-form	$B_2 = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$	28
	Dilaton	ϕ	1
RR	1-form	$A_1 = A_\mu dx^\mu$	8
	3-form	$A_3 = \frac{1}{3!} A_{\mu\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho$	56
NSR, RNS	Two gravitinos	Weyl-Majorana $\psi_\mu^{(L)}$, $\psi_\mu^{(R)}$	$56+56$
	Two dilatinos	Weyl-Majorana $\lambda^{(L)}$, $\lambda^{(R)}$	$8+8$

where we have matched the fields with the massless modes of the four IIA string sectors. In fact, these fields are just the ground states defined in (**??**), decomposed in irreducible representations of $SO(8)$. For example, the NS-NS ground $G_{\mu\nu} = \tilde{b}^{\mu\dagger}_{1/2} \left|0\right\rangle_{NS} \otimes b^{\nu\dagger}_{1/2} \left|0\right\rangle_{NS}$ (reintroducing unphysical polarizations) is a spacetime rank-2 tensor, decomposable as a symmetric form, an antisymmetric form, and a trace, as in

$$\mathbf{8}_V \otimes \mathbf{8}_V = \mathbf{35} + \mathbf{28} + \mathbf{1} \tag{2.37}$$

These are respectively the graviton, the Kalb-Ramond field, and the dilaton. The fields from the other sectors result accordingly from the decompostions:

$$NSR \qquad \mathbf{8}_V \otimes \mathbf{8}_R = \mathbf{56} + \mathbf{8}_L \tag{2.38}$$

$$RNS \qquad \mathbf{8}_L \otimes \mathbf{8}_V = \mathbf{56} + \mathbf{8}_R \tag{2.39}$$

$$RR \qquad \mathbf{8}_L \otimes \mathbf{8}_R = \mathbf{56} + \mathbf{8}_V \tag{2.40}$$

where the three inequivalent $SO(8)$ irreps $\mathbf{8}_V$, $\mathbf{8}_L$, $\mathbf{8}_R$ are respectively the vector and negative and positive chirality Weyl-Majorana representations.

We note the two gravitinos and dilatinos are of opposite chirality, as do the RNS and NSR sectors of IIA superstrings.

Obviously, again we find that the total bosonic states are $35+28+1+8+56 = 128$ and the fermions $2 \cdot (56+8) = 128$. There is therefore a match both with the number of degrees of freedom of 11D SUGRA and with supersymmetry. IIA SUGRA in particular is a theory with $\mathcal{N} = (1,1)$ SUSY, meaning there are two Weyl-Majorana SUSY generators of opposite chirality.

We will mainly be interested, however, in type IIB SUGRA, which is not obtainable from dimensional reduction, and is the effective field theory for IIB strings. The field content is as follows:

NSNS	Graviton	$g_{(\mu,\nu)}$	35
	Kalb-Ramond 2-form	$B_2 = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$	28
	Dilaton	ϕ	1
RR	0-form	$A_0 = A_\mu dx^\mu$	1
	2-form	$A_2 = \frac{1}{2!} A_{\mu\nu}\, dx^\mu \wedge dx^\nu$	28
	4-form (with constraint)	$A_4 = \frac{1}{4!} A_{\mu\nu\rho\sigma}\, dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma$	35
NSR, RNS	Two gravitinos	Weyl-Majorana $\psi_\mu^{(1)}$, $\psi_\mu^{(2)}$	$56+56$
	Two dilatinos	Weyl-Majorana $\lambda^{(1)}$, $\lambda^{(2)}$	$8+8$

The 4-form has its physical polarizations halved from $\binom{8}{4} = 70$ to 35 by the introduction of a constraint we will specify shortly.

IIB SUGRA has $\mathcal{N} = (2,0)$ supersymmetry, with two equal-chirality Weyl-Majorana generators. It is therefore a chiral theory. While IIA is non-chiral and thus automatically anomaly free, IIB as a QFT has the potential to develop a gravitational anomaly due to the chiral fermionic sector. It is however found that the total anomaly miraculously cancels with the listed field content *reference per questo?*. This cancellation is obvious in light of the fact that IIB SUGRA is the effective field theory to IIB superstrings, which are not just free from anomalies but from all UV divergences.

introdurre T and S-duality?

This net of relationships between SUGRAs in 10 and 11 dimension is actually the effective limit of dualities between string/M-theories of which these SUGRAs are effective field theories. The relevant part of the scheme is as follows:

$$\begin{array}{ccccc}
\text{"M-theory"} & \xrightarrow{\text{dim. red. on } \mathbb{S}^1} & \text{IIA strings} & \xrightarrow{\text{T-duality}} & \text{IIB strings} \\
\Big\downarrow{\scriptstyle \text{eff. th.}} & & \Big\downarrow{\scriptstyle \text{eff.th.}} & & \Big\downarrow{\scriptstyle \text{eff.th.}} \\
\text{11D SUGRA} & \xrightarrow{\text{dim. red. on } \mathbb{S}^1} & \text{IIA SUGRA} & \xrightarrow{\text{T-duality}} & \text{IIB SUGRA}
\end{array}$$

2.2.1 Gauge invariance of RR fields

In both IIA and IIB, the RR sector admits the following gauge transformations:

$$B_2 \to B_2 + d\Lambda_1 \qquad\qquad A_p \to A_p + d\Lambda_{p-1} - H_3 \wedge \Lambda_{p-3} \tag{2.41}$$

for any set of arbitrary p-forms Λ_p, leaving invariant the field strengths:

$$\begin{aligned} H_3 &:= dB_2 \\ F_{p+1} &:= dA_p - H_3 \wedge A_{p-2} \end{aligned} \tag{2.42}$$

Where A_p, Λ_p with $p < 0$ is set to 0. Now, intuitively, the RR form A_p couples to $(p-1)$-dimensional objects ($(p-1)$-branes) through an interaction term of the type

$$S_{int} = \int_{W_{p-1}} A_p \tag{2.43}$$

integrated over the p-dimensional worldvolume W_{p-1}, a sensible generalization of the coupling of the EM potential to a charged particle. In particular, as it was already established that it is possible for open strings to end on D-branes, we would like to investigate for which values of p a certain string theory admits stable D$(p-1)$-branes - certainly, if they have a conserved charge under a gauge field then they are protected from decay. Therefore the set of RR fields in a superstring theory determines the list of stable D-brane dimensionalities.

The above is an electric coupling of the $(p-1)$-brane to F_{p+1}. The coupling however could also be magnetic, electric-magnetic duality being implemented in general through Hodge duality. We define F_p for additional values of $p > 5$ (odd for IIA, even for IIB) through

$$F_p = \widetilde{\star} F_{10-p} \tag{2.44}$$

(Note that for the IIB F_5 this would be actually a constraint, precisely the one that acts on A_4 to reduce its degrees of freedom; $F_5 = *F_5$ however is not the exact form of the constraint, as will be clarified in section 2.3). The new field strengths can then be locally trivialized as of (2.42) and so we end up with a complete set of potentials $A_0, \ldots A_8$ for IIB and $A_1 \ldots A_9$ for IIA. The duality between potentials would act as $A_p \leftrightarrow A_{8-p}$, and if $D(p-1)$-branes couple electrically to A_p, then $D(7-p)$-branes couple magnetically to it, that is to say electrically to A_{8-p}.

Therefore, the magnetic dual to a Dp-brane is a $D(6-p)$-brane. So the definitive list of stable D-branes in type II string theories along with the RR fields they are charge under is given by:

D(−1)	A_0	F_1
D0	A_1	F_2
D1	A_2	F_3
D2	A_3	F_4
D3	A_4	F_5
D4	A_5	F_6
D5	A_6	F_7
D6	A_7	F_8
D7	A_8	F_9
D8	A_9	F_{10}
D9	A_{10}	0

where shaded entries are for IIB, and unshaded for IIA, and arrows represent electric-magnetic duality. We comment on a few apparent anomalies.

- The IIB D(−1)-brane would have a 0-dimensional worldvolume, hence a single event. These type of branes are therefore actually instantons. They couple to the A_0 potential, which has axionic character.
- IIA D8-branes and the A_9 potentials do not have duals, yet D8-branes can be shown to exist and to couple to the F_{10} field strength, as was first noted in [11]. However the action $\int F_{10} \wedge \star F_{10}$ implies $d\star F_{10} = 0$ which means F_{10} is a constant. Therefore there are no additional physical degrees of freedom from A_9.
- Space-filling D9 branes can be introduced in IIB superstrings, but the equation of motion of the A_{10} form implies only special arrangements of D9s and anti-D9s are allowed - this was first appreciated in [12].

2.3 Action functional for IIB SUGRA

There is a considerable obstacle to a covariant (i.e. explictly supersymmetric) formulation of type IIB supergravity in the self-duality constraint for the field strength 5-form F_5. We will take the common path of formulating the Lagrangian theory ignoring the constraint (and thus in excess of bosonic polarizations with respect to an explicity supersymmetric theory) and then imposing self-duality by hand after deriving the equations of motion. Therefore the action will not be supersymmetric itself, while the Euler-Lagrange equations augmented with the constraint will be.

Actually, for the purpose of building classical solutions, where spinor fields vanish anyway, the fermionic sector of the action will not be important. After introducing the "string length" ℓ_s through[5]

$$\ell_s := 2\pi\sqrt{\alpha'} \tag{2.45}$$

the bosonic sector is as such:

$$S_{\mathrm{IIB},B} = S_{NS} + S_R + S_{CS} \tag{2.46}$$

where S_{NS} is the action relevant to the fields originally from the superstring NS-NS sector:

$$S_{NS} = \frac{2\pi}{\ell_s^8}\int d^{10}x\sqrt{-g}\,e^{-2\phi}\left(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2\right) \tag{2.47}$$

where the p-form norm is $|\omega|^2 = \omega\wedge\star\omega$. Then S_R is for R-R fields, essentially just kinetic terms for the A forms:

[5]Care should be taken with inequivalent convention with the definition of the string length in the literature. For the purpose of this work we will refer to this definition.

$$S_R = -\frac{2\pi}{\ell_s^8} \int d^{10}x\sqrt{-g} \left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2 \right) \tag{2.48}$$

$$\tilde{F}_3 := F_3 - A_0 H_3 \tag{2.49}$$

$$\tilde{F}_5 := F_5 - \frac{1}{2}A_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 \tag{2.50}$$

And finally we supplement with a Chern-Simons type term:

$$S_{CS} = -\frac{2\pi}{\ell_s^8} \int A_4 \wedge H_3 \wedge F_3 \tag{2.51}$$

note the untilded F_3. This is evidently a purely topological term.

The self-duality constraint is then imposed in terms of the modified field strength $\tilde{F}_5$

$$\tilde{F}_5 = *\tilde{F}_5 \tag{2.52}$$

The action presented above is in what is known as the "string frame". It becomes convenient in many occasion to switch to an alternative formulation through a field redefinition, to move to the "Einstein frame". The change is

$$g^{EF}_{\mu\nu} = e^{-\phi/2} g^{SF}_{\mu\nu} \tag{2.53}$$

and the action in terms of the new metric is[2]

$$\begin{aligned} \left(\frac{\ell_s^8}{2\pi}\right) S^{EF} = & \int d^{10}x\sqrt{-g} \left(R - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}e^{-\phi}|H_3|^2 \right) \\ & - \frac{1}{2}\int d^{10}x\sqrt{-g} \left(e^{2\phi}|F_1|^2 + e^{\phi}|\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2 \right) \\ & - \frac{1}{2}\int A_4 \wedge H_3 \wedge F_3 \end{aligned} \tag{2.54}$$

The advantage of this picture is the canonical Einstein-Hilbert and dilaton terms; by converse exponential couplings of the dilaton with the form fields are introduced.

Introdurre le equazioni del moto generali? Non le userò mai in forma generale

2.4 D-branes

2.4.1 D-brane action

D-branes appear as nonperturbative objects in string theories. They are themselves dynamical and the dynamics are modeled in the string perturbative regime by an action functional[5]. To formulate the action, we introduce coordinates σ^a on the $(p+1)$-dimensional worldvolume W_p and functions $X^\mu(\sigma^a)$ describing the embedding of W_p in spacetime as $\iota : \sigma^\alpha \mapsto X^\mu(\sigma^a)$. It's then tempting to choose as bosonic action the obvious generalization of the Nambu-Goto action:

$$S_{Dp} = -\mu_{Dp} \int_{W_p} d^{p+1}\sigma \sqrt{-\det \iota^*(G)} = -\mu_{Dp} \operatorname{vol} W_p \tag{2.55}$$

the notation $\iota^*(T)$ denotes the pull-back of a spacetime tensor to the worldsheet. For example, $\iota^*(G)$ is the induced metric $h_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$. μ_{Dp} would be the D-brane tension. The insight of (2.55) is correct, but incomplete; firstly it does not include a coupling with possible background fields, but most importantly it does not account for all of the open string modes living on the worldvolume.

In fact, if the quantization procedure we performed for the open string is repeated with the endpoints constrained to a Dp-brane (i.e., with $9-p$ Dirichlet and $p+1$ Neumann conditions) then one finds among the massless bosonic states both scalar (from the worldvolume point of view) fields $X^{p+1,\ldots,9}$ which indeed correspond to motion of the D-brane in the transverse directions, but also, as we have seen, a massless vector potential mediating a $U(1)$ gauge theory confined to the D-brane volume. Therefore, it is directly from the modes of open strings that it is possible to deduce the brane is itself a dynamical object, as its position is related to VEVs of open-string scalars. Enforcing this idea that the D-brane dynamics should be encoded in those of the open-string modes, D-branes should always host at least a $U(1)$ gauge theory on them.

The bosonic part of the Dp-brane action is then found to be

$$S_{Dp} = -\,\mu_{Dp} \int_W d^{p+1}\sigma e^{-\phi} \sqrt{-\det\left(\iota^*(g - B_2) - 2\pi\alpha' F\right)} \tag{2.56}$$

$$+\,\mu_{Dp} \int_W \left[\iota^* \left(\sum_k A_k \right) \wedge e^{2\pi\alpha' F - B_2} \wedge \left(1 + \mathcal{O}(R^2)\right) \right]_{p+1} \tag{2.57}$$

Where μ_{Dp} can be fixed as

$$\mu_{Dp} = \alpha'^{\,-\frac{p+1}{2}} (2\pi)^{-p} \tag{2.58}$$

and F is the field-strength 2-form of the $U(1)$ gauge theory.

The first line (2.56) is the Dirac-Born-Infeld action and generalizes the Nambu-Goto action; Setting only $B = 0, \phi =$ const and expanding S_{DBI} in powers of α':

$$S_{DBI} = -\frac{\mu_{Dp}}{g_S} \int_W d^{p+1}\sigma \sqrt{-h} \;+\; \frac{\alpha'^{-(p-3)/2}}{4 g_s (2\pi)^{p-2}} \int_W d^{p+1}\sigma \sqrt{-h}\, F_{\mu\nu} F^{\mu\nu} \;+ \dots \tag{2.59}$$

the first term is the direct generalization of the Nambu-Goto action, allowing us to identify the Dp-brane tension $T_{Dp} = \frac{\mu_{Dp}}{g_S}$. The second is a Yang-Mills action for the $U(1)$ gauge field, restricted to the worldvolume.

It becomes clear D-branes carry a mass per unit p-volume $T_{D_p} \sim g_s^{-1}$ and are thus nonperturbative in terms of the string coupling. Note also the Maxwell action is weighted by g_s^{-1}, in agreement with what previously found for the relation between open and closed string couplings: $g_{YM} \sim g_0 \sim g_s^{1/2}$.

The second line (2.57) is a Chern-Simons type term coupling the brane to the RR potentials. The sum over k only spans odd or even respectively for IIA or IIB, and the $[\,]_{p+1}$ notation means the $p+1$-form component must be selected so as to define a meaningful integral. We note that in vanishing curvature, and expanding in $2\pi\alpha' F - B_2$, the physical interpretation becomes less obscure:

$$S_{CS} = \mu_P \int_W A_{p+1} + \mu_P \int_W A_{p-1} \wedge (2\pi\alpha' F - B_2) + \mathcal{O}(F^2) \tag{2.60}$$

so that there is a direct, standard coupling of the A_{p+1} potential to the Dp-brane at the zeroth order in F. A Dp-brane is therefore also understood as a localized charge for the F_{p+2} field. Higher order terms mean a coupling with the lower RR potentials and are due to nontrivial F configurations which induce lower-dimensional D-brane charges localized inside the Dp-brane.

We touch briefly upon the easy generalization of the above action to the case of N coincident Dp-branes, a "stack". We imagine first taking 2 separated parallel D-branes (1 and 2) and bringing them closer together. Normally, open string modes stretching from 1 to 2 are suppressed by the increase in mass squared due to the minimum elastic energy to span the inter-brane distance. In fact, the open string spectrum is found to be identical except for the constant shift in the mass-shell condition[2]:

$$\Delta M^2 = T^2 \sum_{i=p+1}^{9} (X_1^i - X_2^i)^2 \tag{2.61}$$

If then the branes are brought to coincide, this shift vanishes and one must admit an additional $1 \to 2$ sector of massless states has been created. In general, with N coincident Dp-branes there will be an $N \times N$ matrix of massless sectors indexed by $a, b = 1, \ldots, N$ marking the starting and ending brane (Chan-Paton indices). This means then a matrix of gauge vectors A_{ab} generating $U(N)$ gauge transformations; this $U(N)$ group is nothing else than transformations mixing the N identical, coincident D-branes with eachother. Therefore they act on Chan-Paton indices in the defining representation.

Then, it's clear the fields $\Phi^i_{ab} := (X^i_a - X^i_b)$, parametrizing relative D-brane transverse position, transform in the adjoint of the gauge group. The action of separating again the D-branes is then interpreted in this point of view as a Higgsing of the gauge group by these scalars; when the stack of N branes splits into two groups of N_1 and N_2 branes which are separated, this corresponds to the relevant Φ fields acquiring a VEV, breaking part of the gauge group to the corresponding group for two separate stacks

$$U(N) \to U(N_1) \times U(N_2) \tag{2.62}$$

and the gauge fields corresponding to the broken generators, which gain a mass through the Higgs mechanism, are in fact modes of strings stretching between the two stacks, so that the Higgsed mass can also be viewed as the elastic energy from (2.61).

The salient point in any case is the extension of the gauge group from $U(1)$ to $U(N)$. Essentially, the F^2 term (and higher) in (2.59) must be supplemented with gauge traces.

2.4.2 D-branes as supergravity solitons

As D-brane carry mass, a dual description of D-branes in terms of the warped spacetime they produce should be possible. These spacetimes indeed appear as solitonic solutions in supergravity generalizing the usual four-dimensional black hole metrics, under the name of p-branes or black branes. In particular, as Dp-branes are (for certain values of p) stable objects, stabilized by F_{p+2} charge, their supergravity description must be a zero-temperature state, unable to lose mass to Hawking radiation, and thus extremal.

These solutions are constructed in complete analogy to the derivations of usual general relativity black holes, by inserting an ansatz with the desired symmetries into the supergravity action, and in fact are direct generalizations of the extremal Reissner-Nordström hole. We provide a succint review of the derivation and result, following for example [7]. Since the p-brane is stabilized by its charge under the A_{p+1} potential or the F_{p+2} field strength, it can be assumed all the other form fields vanish, including H_3. Moreover the fermion fields vanish for classical solutions. Therefore the Einstein frame IIB action reduces to

$$S_{\mathrm{IIB}} = \frac{2\pi}{\ell_s^8} \int \sqrt{-g} \left(R - \frac{1}{2}(d\phi)^2 - \frac{1}{2\eta} e^{\frac{3-p}{2}\phi} |F_{p+2}|^2 \right) \tag{2.63}$$

if $p = 3$, the self-duality constraint $F_5 = \tilde{F}_5$ is to be imposed after finding the Euler-Lagrange equations, and $\eta = 2$; otherwise $\eta = 1$.

It's clear one can assume the black p-brane has $(\mathbb{R}^{p+1} \rtimes SO(1,p)) \times SO(9-p)$ symmetry, so that we can introduce a set of longitudinal coordinates $x^{0,\ldots,p}$ and transverse coordinates $y^{p+1,\ldots,10}$ such that the dependence of all fields components in this coordinates is reduced to the single radial transverse variable $r^2 = \vec{y} \cdot \vec{y}$. The most general Einstein frame metric with these symmetries is then

$$ds^2 = H_p^{-1/2} dx \cdot dx + H_p^{1/2} dy \cdot dy \tag{2.64}$$

where the warp factor $H_p(r)$ is a function of r only, and $dx \cdot dx$ and $dy \cdot dy = dr^2 + r^2 d\Omega_{8-p}$ are respectively the Minkowski and Euclidean metrics. Analogously, the dilaton ϕ is also a function of r and the form potential must take the form

$$A_{012\ldots p} = A(r) \tag{2.65}$$

After variation of the action (2.63) and insertion of the described ansatz in the resulting equations of motion (plus simplifications in the $p = 3$ case since

$|F_5|^2 = F_5 \wedge *F_5 = F_5 \wedge F_5 = 0$), one is left with a differential equation for H_p. It is found, remarkably, that (for $r > 0$)

$$\nabla^2 H_p = 0 \tag{2.66}$$

where ∇^2 is the *flat* space Laplacian, hence a linear equation. This traces back to the fact that multiple extremal black holes are non-interacting, as the gravitational and electrostatic forces cancel - the same holds for D-branes in supergravity. This makes it possible to construct exact solutions with multiple branes by superposition. In any case, the linear equation has (taking into account the boundary condition of asymptotic flatness $H_p(\infty) = 1$) a simple solution as (for $p < 7$):

$$H_p(r) = 1 + \left(\frac{R_p}{r}\right)^{7-p} \tag{2.67}$$

for some R_p to be determined. Exploiting the equations of motion one finds also

$$e^{\phi}(r) = g_s H_p(r)^{(3-p)/4} \tag{2.68}$$

where $g_s = e^{\phi(\infty)}$ is the background value of the string coupling, and

$$A_{01\ldots p} = H_p^{-1} - 1 \tag{2.69}$$

$$\Rightarrow F_{p+1} = d(H_p^{-1}) \wedge dx^0 \wedge \ldots \wedge dx^p \tag{2.70}$$

which completes the specification of the class of solutions. Now R_p is fixed by matching the gravitational flux to the mass (per unit longitudinal volume) of the D-brane. In fact, we are free to consider a set of N coincident Dp-branes, a "stack", an easy generalization which will prove very important in the context of AdS/CFT. The exact dependence is

$$(R_p)^{7-p} = \left((4\pi)^{\frac{5-p}{2}}\Gamma\left(\frac{7-p}{2}\right)\right)\alpha'^{\frac{7-p}{2}}g_sN \tag{2.71}$$

While this result was derived from the IIB action and so for $p = 1, 3, 5$, it actually applies identically for IIA $p = 0, 2, 4, 6$ branes[6]. We single out from this the self-dual case of the D3 brane which displays a uniform value for the dilaton. In general instead from (2.68) and the warp factor (2.67) we have the near-horizon behaviour

$$e^{4\phi} \sim \left(\frac{r}{R}\right)^{(3-p)(p-7)} \tag{2.72}$$

which means the local string coupling diverges for $p < 3$ and vanishes for $p > 3$. $p = 3$ acts as a middle ground between these cases.

[6]It should be remarked that similar D7 and D8 solutions are possible, but that the behaviour of the warp factor is respectively logarithmic and linear in r, and so asymptotic flatness cannot be enforced.

Chapter 3

D3-brane stacks on Calabi-Yau cones

Perhaps the most essential ingredient for the conception of the idea of holography was the fact that coincident D3-branes (a "stack") naturally feature a 4D gauge theory on their world-volume, where the 4D fields emerge from the modes of open strings stretching between them. In the simplest and most famous example, a stack of N D3-branes is placed in otherwise Minkowski $\mathbb{R}^{1,9}$; the corresponding field theory is the maximally supersymmetric Yang-Mills in four dimensions (SYM4).

Setting the stack on a different background geometry instead gives rise to a large family of different field theories; a particularly interesting subset is given by spacetimes of the form:

$$M = \mathbb{R}^{1,3} \times X_6 \tag{3.1}$$

where the $\mathbb{R}^{1,3}$ is parallel to the branes (and must be identified with the field theory spacetime) and X_6 is a 6-dimensional Calabi-Yau cone over a compact 5-fold base Y_5. By X_6 being a cone it is meant there exists a conical radial coordinate r such that the metric on X_6 is of the form

$$ds^2 = dr^2 + r^2 ds_5^2 \tag{3.2}$$

In this language, the SYM4 example above corresponds to $X_6 = \mathbb{R}^6 = \mathbb{C}^3$, which is (trivially) a cone over $\mathbb{S}^5$. This is the only case where X_6 turns out to be smooth; in general it will feature a conical singularity in the origin. Other choices for the base will typically yield theories with reduced (even minimal) supersymmetry, which are considerably more challenging to study.

3.1 Superconformal field theory

We now provide a short introduction to 4D conformal field theories, their supersymmetric variants, and the relevant terminology.

For a given d-dimensional spacetime with metric $g_{\mu\nu}$, conformal transformations are defined to be diffeomorphisms $x^\mu \to x'^\mu$ which leave the metric unchanged in form up to an x dependent scalar function (a conformal factor):

$$ds^2 \to ds'^2 = \Omega(x) ds^2, \tag{3.3}$$

or, equivalently:

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x). \tag{3.4}$$

The group of these transformations is known as the conformal group associated with the metric. Our case of interest is flat spacetime[1], $g_{\mu\nu} = \eta_{\mu\nu}$, and the corresponding group is commonly known as *the* conformal group.

In $d > 2$, the conformal group will turn out to be a finite-dimensional Lie group, of which we specify now the connected component. An obvious sub-

[1]to be precise, the conformal group is clearly equal for two metrics that are conformally equivalent ($g_{\mu\nu} = e^{\phi(x)} h_{\mu\nu}$), so that the group essentially depends only on the conformal class of the manifold. Therefore, what we will find for $\mathbb{R}^{1,D-1}$ will apply equally to all conformally flat metrics.

group is maps that leave the metric unchanged, so Poincaré transformations, with generators P_μ and $J_{\mu\nu}$. A second easy guess is the subgroup with constant conformal factors, that is scale transformations or dilations

$$x^\mu \to \lambda x^\mu \qquad\qquad \eta_{\mu\nu} \to \lambda^{-2}\eta_{\mu\nu} \tag{3.5}$$

whose generator is called D. To generate the whole conformal group a final class of transformations must be introduced, special conformal transformations, generated by K_μ and with finite action

$$x^\mu \to b_\mu x^2 - 2x_\mu(b_\nu x^\nu). \tag{3.6}$$

Together, P_μ, $J_{\mu\nu}$, D and K_μ generate the connected component of the conformal group in D dimensions. The extension of the Poincaré algebra to the conformal one is characterized by the following additional commutators (using hermitian generators)

$$[J_{\mu\nu}, K_\rho] = 2i\eta_{\rho[\mu}K_{\nu]} \tag{3.7}$$
$$[J_{\mu\nu}, D] = 0 \tag{3.8}$$
$$[D, P_\mu] = iP_\mu \tag{3.9}$$
$$[D, K_\mu] = -iK_\mu \tag{3.10}$$

Equations (3.7) and (3.8) just confirm K_ρ is a vector and D is a scalar. (3.9) and (3.10) instead state that P_μ and K_μ are respectively raising and lowering operators for D. It is definitely worth of notice that this group is actually $SO(2, D)$, the Lorentz group in mixed signature $(2, D)$. This can be shown by combining the generators in

$$J_{MN} = \begin{pmatrix} J_{\mu\nu} & (K_\mu - P_\mu)/2 & -(K_\mu + P_\mu)/2 \\ (P_\mu - K_\mu)/2 & 0 & D \\ (K_\mu + P_\mu)/2 & -D & 0 \end{pmatrix} \tag{3.11}$$

and then it can be verified that J_{MN} satisfy the algebra of $\mathfrak{so}(2, D)$. This equivalence will be relevant when we will introduce AdS/CFT, since $SO(2, D)$ is also the isometry group of AdS_{D+1}.

A quantum field theory which has the conformal group as symmetries is called a conformal field theory (CFT). In such a theory, particles lie in irreducible representations of the conformal group; since the mass P^2 is not a Casimir for the whole group, it becomes useful to replace it with more relevant quantum numbers. Consider the dilation operator: in the quantum theory it will be represented by

$$D = -i(x^\mu \partial_\mu + \Delta) \tag{3.12}$$

where Δ gives the intrinsic scaling dimension of a field, which will in general transform as $\phi(x) \to \lambda^\Delta \phi(\lambda x)$. Δ is therefore a good quantum number. Considering the role of P and K as ladder operators, changing the conformal dimension by ± 1, we can deduce states will come in multiplets of ever-increasing dimension $\Delta_{(0)} + n$, $n \geq 0$, and that the lowest-dimension state will be annihilated by K_μ. Fields in the kernel of K_μ will be called primary, and others, obtained by applying powers of P_μ (hence, derivatives) will be called descendants.

A primary field is then identified by its conformal dimension and its representation under the Lorentz group, so, now specializing to $D = 4$, by quantum numbers (Δ, j_L, j_R).

A classically conformal field theory very often fails to be conformal when quantized. This happens because the dilation symmetry is anomalous. Classical scale invariance clearly implies all couplings are adimensional; in the quantum theory these couplings g^i will run under renormalization with a corresponding β function, as in

$$\frac{dg^i}{d\ln\mu} =: \beta^i(g)\,. \tag{3.13}$$

The dependency of the running coupling on the energy scale, or equivalently the creation of a mass scale by dimensional transmutation, means the conformal symmetry is spoiled[2]. This happens for example in quantum chromodynamics, a classically conformal theory with a scale anomaly giving rise to the Λ_{QCD} mass scale, or quantum electrodynamics where the scale is at the Landau pole. Since the Nöther current corresponding to dilations is the trace of the energy-momentum tensor, the anomaly will be detectable by the appearence of a nonzero matrix element $\langle T^\mu{}_\mu\rangle \propto \beta(g) \neq 0$.

Only if all the β functions vanish identically, i.e. if the theory is finite, is quantum conformal invariance guaranteed. We will encounter an example of such a theory in section 3.3. Otherwise the theory will only be conformal for specific values of the g^i at which all the β functions vanish, that is to say at fixed points. In general a quantum field theory will flow under renormalization from a non-conformal point towards an attracting IR fixed submanifold, the locus of $\{\beta^i(g) = 0\}$, called the conformal manifold.

An important point is that after the theory has regained its classical conformal symmetry after converging through RG flow to an IR fixed point, the quantum scaling dimensions Δ of operators will not coincide with the original value they had in the classical theory, the canonical dimension Δ_0. They will be modified by quantum corrections that add an anomalous dimension

$$\Delta = \Delta_0 + \gamma(g*)\,, \quad \gamma(g) = -\frac{1}{2}\frac{d\ln Z}{d\ln\mu}, \tag{3.14}$$

[2] We are here using conformal and scale (i.e. dilation) invariance interchangeably, but they are not identical. Conformal symmetry obviously includes dilations, but scale invariance + Poincaré does not generate the whole conformal group, as special conformal transformations are independent. Scale invariant but not conformal theories are known explicitly[1], but they are rare. We will work with the assumption dilation-invariant $\Rightarrow$ conformal.

where $\sqrt{Z}$ renormalizes the wavefunction[3], and $g*$ are the values of the couplings at the conformal fixed point.

Having introduced the extension of the Poincaré group to $SO(2,4)$, we would like to press this further to include supersymmetry. The latter is implemented by adding $\mathcal{N} \leq 4$ Weyl supercharges Q^A, $\overline{Q}_A$ ($A = 1, \dots, \mathcal{N}$) to generate the super-Poincaré supergroup ISO$(1,3|\mathcal{N})$. The superconformal group $SO(2,4|\mathcal{N})$ is the minimal supergroup containing both. The first important feature is that a second set of supercharges S_A, $\overline{S}^A$ must be introduced to close the algebra, since

$$[K_\mu, Q^A] = -\sigma_\mu \overline{S}^A\,, \qquad\qquad [P_\mu, S_A] = \overline{Q}_A \overline{\sigma}_\mu\,; \tag{3.15}$$

so that superconformal symmetry $SO(2,4|\mathcal{N})$ involves twice as many supercharges as normal supersymmetry for a given $\mathcal{N}$. Another relevant excerpt from the table of commutators (which we do not reproduce in full) states Q^A and S_A are also ladder operators for dilations,

$$[D, Q^A] = \frac{i}{2} Q^A\,, \qquad\qquad [D, S_A] = -\frac{i}{2} S_A\,, \tag{3.16}$$

raising and lowering the dimension Δ by $\pm 1/2$. In a superconformal field theory (SCFT) we then expect multiplets of dimension $\Delta = \Delta_0 + \frac{n}{2}$. Primary operators must now be annihilated by both K_μ and S_A, and are classified again by dimension and spin (Δ, j_L, j_R) but also by the $U(1) \times SU(\mathcal{N})$ R-symmetry quantum numbers $(R, \mathbf{r})$ ($\mathbf{r}$ denoting a generic irrep of $SU(\mathcal{N})$). Then, by acting with the raising operator Q^A charges one can reconstruct a finite-dimensional supermultiplet, as in normal supersymmetry. Instead, powers of P_μ reconstruct the infinite ladder of derivatives forming an infinite representation of the conformal group; these can be recombined into a field by Taylor expansion. In conclusion, an infinite representation of the

[3]It should be noted some authors prefer to define $\gamma = -\frac{d \ln Z}{d \ln \mu}$.

superconformal group is given by a superfield

$$\Phi_{\dots}(x^\mu, \theta^A, \overline{\theta}^A) \tag{3.17}$$

where ... stands for Lorentz and R-charge-$SU(\mathcal{N})$ indices for the primary.

unitarity bounds & R-charge ¡-¿ dimension

3.2 General features of D3-brane worldvolume theories

The resulting gauge group will be $U(N)^g$, where $g = 1 + b_2(X_6) + b_4(X_6)$, and the theory will be populated by chiral fields in "bifundamental" representations, i.e. with an index in the fundamental of one $U(N)$ and a second in the antifundamental. These sort of theories are termed quiver gauge theories and they can be encoded in a quiver diagram, where $U(N)$ factors are denoted by nodes and bifundamental fields as directed arrows stretching between two nodes.

We will be in particular interested in the moduli spaces of these theories, so the spaces of distinct vacua. Because of supersymmetry, the quantum moduli space will often coincide with the classical one, which is the locus of the F-flatness condition:

$$F^i = \frac{\partial W}{\partial \phi_i} = 0 \tag{3.18}$$

where $W(\phi_i)$ is the superpotential function of the chiral fields ϕ_i, and the D-flatness condition:

$$D^a_{U(N)^g} = -\sum_i \phi_i^\dagger T^a \phi_i = 0 \tag{3.19}$$

where T^a are the gauge generators. (The $U(N)^g$ subscript indicates the index a spans over all generators of the g factors of $U(N)$). The space $\mathcal{M}$ of simultaneous solution of the F and D-flatness conditions will be a complex manifold.

A subspace of $\mathcal{M}$ is given by the so-called mesonic moduli space $\mathcal{M}_m$. Points of $\mathcal{M}_m$ will correspond to the position of the N branes on the background cone - therefore $\dim_{\mathbb{C}} \mathcal{M}_m = 3N$. The moduli space is not exhausted in the purely mesonic directions though; to investigate the remaining "baryonic" directions we first anticipate we will be mainly concerned with the IR limit, in which our theories will flow to (super) conformal field theories ((S)CFTs). In the IR limit, the abelian $U(1)$ factor in each $U(N)$ node "freeze" and become global baryonic symmetries. Therefore their D-flatness condition is relaxed and one is left with only the D-term for the $SU(N)^g$ part. So

$$D^a_{SU(N)^g} = 0 \qquad\qquad D^i_{U(1)^g} = V^i \tag{3.20}$$

($i = 1, \cdots, g$). V^i are classically functions of the fields and in the quantum version will be gauge-invariant operators. Their g VEVs $\langle V^i \rangle =: \xi^i$ will parametrize the missing flat directions of moduli space. To be precise, however, since the overall trace $U(1)$ (generated by the sum of the generators of the g abelian trace factors) is completely decoupled, we have to impose $\sum \xi^i = 0$. Therefore that there are really only $g-1$ baryonic moduli, corresponding to $g(N^2-1)+1$ independent D-flatness conditions.

Thus we conclude $\dim \mathcal{M} = 3N + g - 1$. While the $3N$ mesonic directions have a direct geometrical interpretation as D3-brane movement, the baryonic directions correspond in terms of the superstring description to deformations of the X_6 background metric itself - generally resulting in a resolution of the conical singularity.

spostare le cose dall'intro

3.3 Brane stack in $\mathbb{C}^3$ and $\mathcal{N}=4$ super-Yang-Mills

If $X_6 = \mathbb{C}^3$, the branes are invariant under half of the $16 \times 2 = 32$ IIB supercharges. The only possibility for a 4D theory to have 16 supercharges is to be an $\mathcal{N}=4$, superconformal field theory[4]. Moreover, the theory features gluons as the massless spin-1 modes for the sector of strings stretching between brane i and brane j so that the gauge group is $U(N)$, as seen in 2.4.1. The information that the theory is a $U(N)$ gauge theory and is maximally supersymmetric is enough to uniquely fix it.

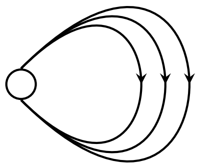
In $\mathcal{N}=1$ language (which we employ even though the model has $\mathcal{N}=4$) the theory describes the dynamics of $U(N)$ gauge vector supermultiplets A_μ and three complex chiral superfields $(X^a)_{i\dot{j}}$, $a=1,2,3$ in the adjoint of the gauge group (we will frequently omit gauge indices). These are nothing else than the parametrization of the D3-branes' position in $\mathbb{C}^3$ and therefore transform in the fundamental of $SU(3)$. The superpotential is the only one allowed by gauge and $SU(3)$ invariance:

$$W(X) = \epsilon_{abc} \operatorname{Tr}(X^a X^b X^c) \tag{3.21}$$

and the quiver diagram is quite simple:

nota sul moduli space triviale

[4]Indeed, the number of supercharges is 2 for the components of a 4D Majorana spinor, times $\mathcal{N}$, times a factor of 2 since the superconformal algebra has twice the supercharges of the usual SUSY algebra.



3.4 The conifold and the Klebanov-Witten model

In [6] the case of X_6 being the conifold was studied. The conifold is a specific Calabi-Yau 3-cone defined for example as the following variety in $\mathbb{C}^4$:

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \tag{3.22}$$

or, after a simple change of variables:

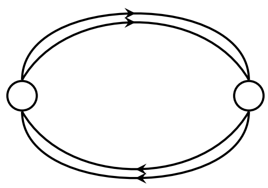
$$uv - xy = 0 \tag{3.23}$$

The base can be found by quotienting by dilations $z_i \to \lambda z_i$ (with $\lambda \in \mathbb{R}_+$) and turns out to be the homogeneous space $SO(4)/U(1) = SU(2) \times SU(2)/U(1)$, where the $U(1)$ is a diagonal subgroup generated by, say, $T_L^3 + T_R^3$. We will therefore have $SU(2) \times SU(2)$ as part of the isometry group of both Y_5 and X_6, and thus will also appear as a global symmetry of the wordvolume theory. An equivalent description of the topology of the conifold is as a $U(1)$ bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1$; in these terms the metric on the base that makes the cone Calabi-Yau is

$$ds_5^2 = \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6}(d\Omega_1^2 + d\Omega_2^2) \tag{3.24}$$

where $\Omega_i^2 = d\theta_i^2 + \sin\theta_i^2 d\phi_i^2$ is the metric on the $\mathbb{C}P_i^1$, and ψ is the fibral coordinate with period 4π.

The corresponding gauge field theory on the worldvolume is a $U(N) \times U(N)$ field theory featuring two chiral doublets A_i, B_j with $i, j = 1, 2$ transforming in opposite bifundamentals, that is A_i in $(N, \bar{N})$ and B_j in $(\bar{N}, N)$. Or more succintly, this can be depicted in a quiver diagram:



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The i and j indices, instead, are acted upon respectively by the global left and right $SU(2)$ symmetries. Finally, A and B have R-charge 1/2. The symmetries and R-charges fix the form of the superpotential:

$$W = \frac{\lambda}{2}\epsilon^{ij}\epsilon^{kl}\,\mathrm{Tr}\,(A_i B_k A_j B_l) \tag{3.25}$$

While this theory won't be in general superconformal, unlike the $\mathcal{N} = 4$ SYM seen before, it will flow through renormalization in the IR to a conformal submanifold in the space of couplings (λ, g_1, g_2), the locus where the β functions for these three couplings vanish. It turns out these three conditions are all equivalent. In particular, requiring $\beta_{g_1} = 0$ and making use of the NSVZ expression for the β function of a supersymmetric gauge theory, this unique condition is equivalent to

$$3T[\text{Adj}] - \sum_i T[R_i](1 - 2\gamma_i) = 0 \tag{3.26}$$

where $T[R]$ is the Dynkin invariant of representation R, the sum is over charged fields and γ_i is the anomalous dimension[5]. When evaluating this, care should be taken with the fact that A_i and B_j have a $U(N)_2$ index which is uncharged under $U(N)_1$ and must be summed over. This gives, also noting $\gamma_{A_1} = \gamma_{A_2}$ and the same for B because of the global symmetry:

$$\gamma_A + \gamma_B + \frac{1}{2} = 0 \tag{3.27}$$

Being $\gamma_{A,B}$ functions of the couplings, this equation defines a critical 2-surface in parameter space. We note this equation is consistent with the relationship $\frac{3}{2}R - 1 = \gamma$ between R-charge and the anomalous dimension of an operator in a SCFT, with the given assignment of R-charges.

Position in moduli space $\mathcal{M}$ should be parametrized by the expectation values of gauge-invariant operator (hadrons). In particular, the classical moduli space is given by the F-term and D-term conditions, which specialized to the particular case are

$$\epsilon^{ij} A_i B_a A_j = \epsilon^{ij} B_i A_a B_j = 0 \tag{3.28}$$

$$A_i A_i^\dagger - B_i B_i^\dagger = A_i^\dagger A_i - B_i^\dagger B_i = \xi \mathbb{1} \tag{3.29}$$

the equation have to be understood to hold for VEVs. Note the first and second D-term condition are respectively from the left and right gauge group. We set temporarily $\xi = 0$ to study the mesonic moduli space $\mathcal{M}_{\text{mes}}$. To reinterpret the F-flatness condition, we introduce the four matrices $\Phi_{ij} = A_i B_j$ and note

[5]Note we use the definition $\gamma = -\frac{1}{2}\frac{d\ln Z}{d\ln\mu}$, where $\sqrt{Z}$ renormalizes the field.

$$[\Phi_{ij}, \Phi_{jk}] = 0 \tag{3.30}$$

$$\Phi_{11}\Phi_{22} = \Phi_{21}\Phi_{12} \tag{3.31}$$

which can be immediately checked to follow from the vanishing of the F-term. Since these commute, they can be simultaneously diagonalized and their N eigenvalues (one for each brane) satisfy the conifold's equation:

$$\phi^I_{11}\phi^I_{22} = \phi^I_{21}\phi^I_{12} \tag{3.32}$$

so these quite literally parametrize the motion of the N D3-branes on the background cone. These are actually VEVs of mesonic operators, mesons being generated by prototipical trace operators:

$$M_{(ab\ldots),(ij\ldots)} = \mathrm{Tr}\left((A_a B_i)(A_b B_j)\cdots\right) \tag{3.33}$$

(Note mesons are built by tracing over closed loops in the quiver diagram to make a gauge-invariant operator). All of these operators are actually expressable as products of Φ matrices, and as we've seen, only 3 out of 4 of those are independent. In the end, there are (accounting for gauge indices) $3N$ independent mesons whose VEVs parametrize mesonic moduli space, coincident with the $\mathrm{Sym}^N C$, where C is the conifold.

Operators of non-zero baryon number can also be constructed by using the antisymmetric invariant gauge tensor $\epsilon^{a_1 \ldots a_n}$, as such:

$$B^A = \epsilon^{a_1\ldots a_n}\epsilon_{b_1\ldots b_n} A^{b_1}_{a_1}\ldots A^{b_n}_{a_n} \tag{3.34}$$

where the indices on the A fields are gauge indices and we have omitted the $SU(2)$ indices. There are only $N+1$ different assignment for the $SU(2)$ indices anyway, so that there are $N+1$ fundamental baryons of the form of

B^A. The same could be done by swapping the two gauge groups and using B fields, to get additional $N+1$ B^B baryons. These all have baryon number N.

All gauge-invariant operators in the theory are built out of these fundamental mesons, fundamental baryons, and their respective antiparticles (made out of the conjugate fields $A^\dagger$, $B^\dagger$. However, as we have anticipated, we only expect $g-1=1$ baryonic VEV to be independent. This VEV will be associated with the resolution of the cone singularity into a $\mathbb{CP}^1$, and will essentially coincide with ξ. To see an example of this deformation of the background geometry, let us set ξ to a constant nonzero value. Then hypothesizing the A_1, A_2, B_1, B_2 matrices commute, applying F-term conditions we get that each set of eigenvalues satisfies

$$a_1/a_2 = b_1/b_2 \tag{3.35}$$

So that a_i and b_i are proportional vectors of $\mathbb{C}^2$, therefore

$$a_i = ae^{i\theta_A}n_i, \qquad be^{i\theta_B}n_i \tag{3.36}$$

where a, b are real and n_i belongs to a $\mathbb{CP}^1$. The phases are cancelled by modding gauge invariance, and a and b then are involved in the D-term:

$$a^2 - b^2 = \xi \tag{3.37}$$

so that essentially our mesonic VEVs are composed of N copies of one non-compact radial coordinate (say, a) and a point on $\mathbb{CP}^1$. This means the conical singularity has disappeared to be replaced by a two-cycle on which the branes can move.

The explicit form of the baryon generating this deformation in terms of the fundamental hadrons is very challenging to determine[4]; fortunately, we will not need it for our purposes. In any case, all of this information will be clarified in the context of holography.

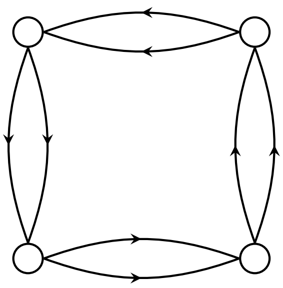
3.5 The $Y^{(2,0)}$ orbifold theory

The same construction on a $\mathbb{Z}_2$ orbifold of the conifold yields a quiver gauge theory which will be the main interest of this work. The geometry of the base of the cone is very simply introduced in polar coordinates as

$$ds_5^2 = \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 + \frac{1}{6}(d\Omega_1^2 + d\Omega_2^2) \tag{3.38}$$

i.e., exactly the same metric in form as the conifold, but with ψ now with period 2π. This background and the resulting worldvolume field theory are just one entry $Y^{2,0}$ of an infinite class $Y^{p,q}$ of examples introduced in [3].

The quiver diagram "splits" to yield four doublets of bifundamental chiral fields stretching in a square between four nodes:



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and the superpotential can be shown to have the form

$$W = \lambda \epsilon^{ij} \epsilon^{kl} \operatorname{Tr}(A_i B_k C_j D_l) \tag{3.39}$$

from which it's clear that the $SU(2) \times SU(2)$ isometry of the cone, corresponding to a global symmetry of the field theory, must now act with the left factor on A_i and C_i, and the right on B_i and D_i. This time three of the four gauge β functions are independent:

$$\gamma_A + \gamma_D + \frac{1}{2} = 0 \tag{3.40}$$
$$\gamma_B + \gamma_A + \frac{1}{2} = 0 \tag{3.41}$$
$$\gamma_C + \gamma_B + \frac{1}{2} = 0 \tag{3.42}$$
$$\tag{3.43}$$

$\beta_\lambda = 0$ is also not independent. At any superconformal point, $\frac{3}{2}R - 1 = \gamma$, so that the condition that W be scale invariant, which is equivalent to it having R-charge 2, becomes

$$2 = R_W = R_A + R_B + R_C + R_D \Rightarrow \gamma_A + \gamma_B + \gamma_C + \gamma_D + 1 = 0 \tag{3.44}$$

which is indeed equivalent to the above system. Three independent equations in a five-parameter space define, again, a critical 2-submanifold.

Turning to the investigation of the moduli space, the F-term condition for the given superpotential read

$$\begin{aligned} A_\alpha B_\sigma C_\beta \epsilon^{\alpha\beta} &= 0 \\ B_\alpha C_\sigma D_\beta \epsilon^{\alpha\beta} &= 0 \\ C_\alpha D_\sigma A_\beta \epsilon^{\alpha\beta} &= 0 \\ D_\alpha A_\sigma B_\beta \epsilon^{\alpha\beta} &= 0 \end{aligned} \tag{3.45}$$

while the vanishing of the D-term takes the form

$$\begin{aligned} A_i A_i^\dagger - B_i B_i^\dagger &= \xi_1 \mathbb{1} \\ B_i B_i^\dagger - C_i C_i^\dagger &= \xi_2 \mathbb{1} \\ C_i C_i^\dagger - D_i D_i^\dagger &= \xi_3 \mathbb{1} \\ D_i D_i^\dagger - A_i A_i^\dagger &= \xi_4 \mathbb{1} \end{aligned} \tag{3.46}$$

with the constraint $\sum_i \xi_i = 0$ (obvious by summing the four equations). To sketch out mesonic moduli space, again we make the simplifying assumption the eight A, B, C, D matrices commute and can be simultaneously diagonalized. So, for each of the N rows of corresponding eigenvalues,

$$a_1/a_2 = c_1/c_2 \qquad\qquad b_1/b_2 = d_1/d_2 \tag{3.47}$$

thus again $a_\alpha \propto c_\alpha$ and $b_\alpha \propto d_\alpha$, so we can “projectivize”:

$$a_\alpha = a\, e^{i\theta_A} n_\alpha \qquad\qquad b_\alpha = b\, e^{i\theta_B} n_\alpha \tag{3.48}$$

$$c_\alpha = c\, e^{i\theta_C} m_\alpha \qquad\qquad d_\alpha = d\, e^{i\theta_D} m_\alpha \tag{3.49}$$

and again the phases are modded out by gauge symmetry, and the a, b, c, d real numbers are reduced to a single coordinate (schematically r^2) by the three independent D-flatness conditions. Therefore the resolved geometry of the singularity is now $\mathbb{CP}^1 \times \mathbb{CP}^1$, with a clear correspondence with the explicit metric of the resolved $Y^{2,0}$.

In this case, the presence of $g - 1 = 3$ independent ξ parameters (matching with three independent baryons) should perplex, as the resolution of the singularity should be controlled by the *two* volumes of the spheres. In fact, the general Calabi-Yau deformation of the $Y^{2,0}$ metric will indeed depend on two moduli. In this case, the third modulus is not interpretable as due to deformation of the background metric, but is actually connected to the flux of IIB two-form fields. We will review this fact in a holographic context.

For completeness we adapt the construction of hadronic operators. We note fundamental mesons are now built using $ABCD$ loops (omitting $SU(2)^2$ indices):

$$M = \mathrm{Tr}\left((ABCD)(ABCD)\dots\right) \tag{3.50}$$

and four classes of fundamental baryons can be introduced as before

$$B^A = \epsilon^{a_1\dots a_n}\epsilon_{b_1\dots b_n}A^{b_1}_{a_1}\dots A^{b_n}_{a_n} \tag{3.51}$$

and this can be repeated for B, C, D. Similarly to the previous case, we expect only three baryons to be truly indepedent and a suitable triple of combinations should generate the three aforementioned flat shifts.

Chapter 4

Holography

In the previous chapter we explained how the dynamics of brane stacks, in particular D3-branes in type IIB, are described by gauge field theory on their worldvolumes. It's however important to note that parallel to this "open string" picture of the brane stack system there is also a dual description in terms of the curved spacetimes generated by their mass. Insisting these two viewpoints are equivalent, one is able to deduce an exact correspondence between the gauge theory and string theory on the near-horizon geometry.

This kind of duality is exotic as it connects a local field theory in four dimensions with an essentially five-dimensional string (and so, inherently gravitational) theory through a perfect mapping. It is reasonable in fact to identify the spacetime of the field theory with the conformal boundary of the higher-dimensional gravitational background it's dual to (the bulk), for reasons we will clarify - so that in more colloquial language the dynamics in the bulk are "encoded" in the screen at infinity, hence the adjective "holographic" for this sort of correspondences.

Explicit holographic correspondences are not only interesting by themselves as elegant structures; they're also extremely practical tools for studying the theories involved on both sides. It's certainly very attractive for the purpose of quantum gravity or the definition of string theory - non-local theories without action functionals - if these situations happen to be equivalent to a local quantum field theory.

However in this work our interest will be focused on the opposite direction, investigating the dynamics of the field theory by exploiting the dual gravitational system. The power of holographic dualities lies in the fact that they map the strongly-coupled regime for the field theory to the regime where the bulk dynamics can be approximated by supergravity. The traditionally untreatable strong coupling region for some gauge QFTs in four dimensions can then be probed by studying the relatively tamer dynamics of a smooth dual spacetime.

4.1 Large N limit

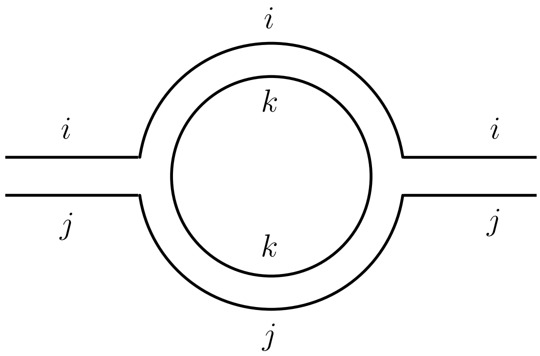
We will now clarify what is meant by large N limit for a Yang-Mills theory.

The Lagrangian is

$$\mathcal{L} = \mathrm{Tr}\left(F^2\right) + \ldots$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig_{YM}[A_\mu, A_\nu]$ and $\ldots$ can include fields in the fundamental, adjoint, bifundamental, etc. These will all of course be representable as object with a certain number of colour indices (and symmetries between them).

We can modify the standard Feynman prescription for pictorially representing amplitudes to get a "double line" or "ribbon" representation in which each colour index is carried by a line. For example, the gluon self energy diagram becomes as such:



colour indices i, $\bar{i}$, j, $\bar{j} = 1, \ldots, N$ are fixed, while k must be summed over. Also, the amplitude has two three-gluon vertices, each carrying a factor of g_{YM}^2, for an overall factor of $g_{YM}^2 N$.

It's easy to convince oneself that as long as we restrict to planar diagrams, that is diagrams that can be drawn on the plane (or more precisely the sphere), adding one strip will always introduce exactly one additional loop and two additional vertices, again carrying a factor of $g_{YM}^2 N$. The combination $\lambda := g_{YM}^2 N$ is the 't Hooft coupling, and is better suited to represent the strength of the gauge interaction than g_{YM} if we are to modify the number of colours.

So the 't Hooft large N limit is defined as:

$$N \to \infty, \quad \text{but keeping } \lambda \text{ fixed} \tag{4.1}$$

A useful rescaling of the fields shifts all the g_{YM} dependence of the Lagrangian to a factor in front:

$$\mathcal{L} = \frac{1}{g_{YM}^2} \left(\operatorname{Tr} F^2 + \ldots \right) \tag{4.2}$$

so that now all types of vertices bring $g_{YM}^2 = \lambda/N$ and propagators bring $1/g_{YM}^2 = N/\lambda$.

We extend to nonplanar graphs by noting these can always be drawn on some Riemann surface of genus g, and, since they induce triangular tilings of said surface, the famous formula for the Euler characteristic holds:

$$F - V + E = \chi = 2 - 2g$$

F, V, E being the number of faces, vertices, edges respectively. Now each face (loop) carries a factor of N, each vertex a factor of λ/N, and each edge N/λ, so that the total contribution is

$$\lambda^{E-V} N^{F-V+E} = \lambda^{E-V} N^{2-2g}$$

so that at fixed λ, an expansion in N (or better $1/N$) is a genus expansion reminiscent of the loop expansion in perturbative string theory. This for example means that the free energy admits a power expansion in $1/N$:

$$F = \sum_{g=0}^{\infty} f_g(\lambda) N^{2-2g} \tag{4.3}$$

One could be perplexed by the N^2 divergence of the genus zero contribution. This is not problematic however; it's an artifact of the rescaling 4.2 which makes the Lagrangian itself diverge as $g_{YM}^{-2} \operatorname{Tr} F^2 \sim N/\lambda \cdot N$, since the trace of a matrix in the adjoint scales as N.

4.2 Maldacena duality

We now consider the IIB supergravity solution modeling the spacetime created by a system of D3-branes in a background $\mathbb{R}^{1,9}$. This is given by

$$ds^2 = H^{-1/2} dx_\mu dx^\mu + H^{1/2}(dr^2 + r^2 d\Omega_5^2) \tag{4.4}$$

$$e^\Phi = \text{const} =: g_s \tag{4.5}$$

$$F_5 = dH^{-1} \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{4.6}$$

$$H(r) = 1 + \left(\frac{R}{r}\right)^4 \tag{4.7}$$

where x^μ, $\mu = 0, \dots, 3$ are coordinates parallel to the brane stack and $d\Omega_5$ is the standard metric on $\mathbb{S}^5$.

The curvature radius R is given by

$$R^4 = 4\pi g_s N \alpha'^2 = g_{YM}^2 N \alpha'^2 \tag{4.8}$$

where N is the number of D3-branes in the stack.

We note an important peculiarity of this metric as opposed to analogous solution for the Dp-brane with $p \neq 3$: there exists a horizon at $r = 0$, but this horizon is an infinite distance away, namely

$$\int_\varepsilon^{r'} \left(1 + (R/r)^4\right)^{1/4} dr \sim -\ln \varepsilon \tag{4.9}$$

therefore the "near-horizon" ($r << R$) geometry takes the form of an infinitely long "throat". This should be compared, just to make an example, to the Schwarzschild solution where the horizon is at a finite distance from any given point in the exterior. This throat feature will play a crucial role for the AdS/CFT correspondence, as will be seen shortly.

This system (IIB string theory on the metric 4.4) must then be equivalent to the stack of D3-branes in the background Minkowski, taking into account both open and closed string interactions. The action is schematically:

$$S = \frac{1}{g_s}\int d^4x F^2 + \frac{1}{\alpha'^4}\int d^{10}x\sqrt{g}Re^{-2\phi} + O(\alpha') + \dots \tag{4.10}$$

(since, we recall, $g_{YM}^2 \sim g_s$). The first two terms are respectively the actions for SYM and free IIB SUGRA in the Minkowski background; the following terms, with higher powers of α', establish the coupling between these two systems. It's clear that in the limit $\alpha' \to 0$ the free SUGRA part decouples from the SYM.

We repeat this decoupling limit for the black 3-brane metric. If $\alpha' \to 0$, so does R, and effectively the metric seems to converge to flat spacetime. We have however to take into consideration the throat described before. The throat shrinks as $R^2 \sim \alpha' \to 0$; to maintain focus on it as we lower the Regge slope we need to rescale our r coordinate. We can introduce $\phi = r/\alpha'$ and keep ϕ fixed as α' goes to zero. With this choice the metric actually reduces to

$$ds^2 = \alpha'\left(\frac{\phi^2}{R^2}dx^\mu dx_\mu + \frac{R^2}{\phi^2}d\phi^2 + R^2 d\Omega_5^2\right) \tag{4.11}$$

which is actually the metric for the product space $AdS_5 \times \mathbb{S}^5$. Therefore under $\alpha' \to 0$ also on the black brane side we have a decoupling of two systems, IIB on Minkowski, and IIB on the near-horizon geometry.

The essential point is that if the two pictures are to be equivalent, and so SYM4 plus decoupled IIB on Minkowski is to be equal to IIB on $AdS_5 \times \mathbb{S}^5$ plus decoupled IIB on Minkowski, then intuitively one expects to be able to "subtract off" the decoupled theory from both sides and obtain an equivalence between the gauge theory and the gravitational theory on $AdS_5 \times \mathbb{S}^5$.

This intuition turns out to be correct, though the details of this equivalence have to be specified. In any case, this is the simplest (and most famous) case of an AdS/CFT correspondence.

We clarify that the decoupling / near-horizon limit we implemented must be generalized carefully to the case of non-coincident branes, since we will be particularly interested in those kind of configurations. If two branes are a distance Δr apart, there will be massive Higgses from open strings stretching between them, with masses of the order of the string tension times the distance: $m \sim \Delta r/\alpha'$. We would like to keep this masses constant. The obvious choice is then to rescale the brane r positions as $\phi_i = r_i/\alpha'$ and keep those constant as we zoom in. This is what we will refer to as the near-horizon limit in the general case and the resulting geometry as the near-horizon warped geometry.

The relevance of the large N limit to the correspondence is evident when it's translated to a limit in the AdS side. Keeping $\lambda = g_{YM}^2 N$ constant and sending $N \gg 1$, considering that

$$4\pi g_s = \frac{\lambda}{N} \tag{4.12}$$

is equivalent to having $g_s \ll 1$. Weak string coupling amounts to a suppression of string loops and effectiveness of the perturbative expansion. A more surprising conclusion concerns however the limit of large 't Hooft coupling $\lambda \gg 1$, since

$$\frac{R^2}{\alpha'} = \sqrt{\lambda} \tag{4.13}$$

so that large coupling means a large $\mathbb{S}^5$ radius. This implies two desirable suppressions: of the massive Kaluza-Klein-like modes wrapping around the $\mathbb{S}^5$, and of terms with more than two derivatives in the effective action. In essence, the dual physics is well described by IIB supergravity, a field theory, instead of full string theory.

Combining both results, a large N, strong-coupling gauge theory will be dual to weakly-coupled supergravity. The strong/weak interchange is what earns the correspondence the title of "duality". In the next section the connotation of "holographic" will also be justified.

4.3 Features of AdS/CFT

Having ascertained that a correspondence of some form exists, one would then seek a more precise description of how the mapping between the four-dimensional gauge theory (the boundary) and the supergravity side (the bulk) is structured. In general, one has what is called an operator-state correspondence: operators in the boundary are associated by a holographic dictionary to "states", or classical solutions in the bulk. More precisely, consider a 4D local operator field $\hat{\phi}(x)$. The generating functional for its correlation functions is given by coupling it to a current $h(x)$:

$$e^{W[h]} = \frac{1}{Z_{\text{CFT}}} \int D\psi e^{iS_{\text{CFT}} + \int d^4x h(x) \hat{\phi}(x)} \tag{4.14}$$

then, the source $h(x)$ is viewed as the limit as $r \to \infty$ of a five dimensional field configuration $h_5(x, r)$, given by solving the equations of motion from S_{AdS} with $h(x)$ as a boundary condition. The correspondence is between the "off-shell" boundary operator $\int d^4x \hat{\phi}(x)$ and the "on-shell" bulk configuration $h_5(x, r)$ - or between the fields $\hat{\phi}(x)$ and h_5, and states that the generating functional above is equal to the bulk action computed on the specific classical solution $h(x, r)$:

$$e^{W[h]} = \langle e^{-\int d^4x h \hat{\phi}(x)} \rangle_{\text{CFT}} = e^{iS_{AdS}[h_5]} \tag{4.15}$$

Therefore, correlation functions for the strongly-coupled CFT can be calculated entirely through the weakly-coupled, two-derivatives bulk action.

One may then wonder about the interpretation one should employ for the fifth extra dimension in the bulk from the CFT perspective. A tentative identification comes from the fact that the AdS metric is invariant under dilations

$$x^\mu \to \lambda x^\mu, \qquad z \to \lambda z \tag{4.16}$$

Since $1/z$ scales like an energy, it could be paired holographically with the boundary energy scale. This turns out to be correct in the sense of renormalization: probing AdS at large distances, closer to the boundary at infinity (as we did before by coupling $\hat{\phi}(x)$ with the value at infinity of a 5D field) coincides with probing the microscopical, UV theory. Moving inwards, operators at larger values of z equate probing the theory at a lower energy scale $\mu \sim 1/z$, up until the horizon which is identified with the IR. The fifth dimension is to be roughly identified with the renormalization flow of the field theory.

Therefore it might be useful to think of the microscopic field theory with no degrees of freedom integrated out, the UV limit, as being somewhat literally located at the conformal boundary at infinity of the AdS dual. Hence the "boundary"/"bulk" terminology, and since all of the physics in the 5D gravitational theory are encoded in a codimension-1 "screen" at infinity, one speaks of holography, in analogy with the real-life technique of encoding three-dimensional objects in a two-dimensional hologram.

In light of the above energy-r relationship, our pairing of field configurations $h_5(x,z)$ with boundary operators has to be corrected. If $h_5(x,z)$ is asymptotically constant as $z \to 0$ so that the limit $h_5 \to h$ is well-defined, it must be that the corresponding dual operator does not scale under dilations, so that its conformal dimension $\Delta = 0$. The extension to CFT operators with arbitrary scaling dimensions is then realized by adding the possibility of $h_5(x, z)$ diverging as a power of z as

$$h_5(x, z) \to z^{\Delta} h(x) \tag{4.17}$$

so that $h(x)$ can then be coupled as source to an operator of conformal dimension Δ. This in turn will induce a dependency of Δ on the mass of the dual bulk field. As an example, let's take a scalar field in AdS_5 minimally coupled to the graviton:

$$S \propto d^4x \, dz \, \sqrt{g} \left(g^{mn} \partial_m h_5 \partial_n h_5 + m^2 h_5^2 \right) \tag{4.18}$$

so that the classical equation of motion is (ignoring x dependency, since it does not affect this argument):

$$\partial_z \left(z^{-3} \partial_z h_5 \right) = z^{-5} R^2 m^2 h_5 \tag{4.19}$$

plugging in a power law $h_5 = hz^{\Delta}$ yields the conformal dimension-mass relation:

$$\Delta(\Delta - 4) = R^2 m^2 \tag{4.20}$$

where only solutions $\Delta \geq 1$ are to be considered[1]. It would seem as if the smallest possible dimension of a boundary operator is 4, however one should take into account the fact that the hyperbolic curvature of AdS produces an effective confining potential that allows particles with $m^2 < 0$ to be stable. It can be shown that the Breitenlohner-Freedman bound holds for stable scalar fields

$$m^2 R^2 \geq -4 \tag{4.21}$$

[1]The bound $\Delta \geq 1$ for a 4D CFT is implied by unitarity.

so that it is possible to reach down to the unitarity limit at $\Delta = 1$.

This was for scalar fields. Higher-spin fields will be dual to operators of the same spin on the boundary and the $m - \Delta$ equation will be modified. For example, $(\frac{1}{2}, \frac{1}{2})$ vectors will have

$$R^2 m^2 = (\Delta - 1)(\Delta - 3) \tag{4.22}$$

while $(1, 1)$ symmetric tensor will instead have the same equation as scalars:

$$R^2 m^2 = \Delta(\Delta - 4) \tag{4.23}$$

The relevance of this is that the operators coupled to massless spin 1 or 2 bosons must be conserved currents by gauge invariance, and so the operators dual to a massless bulk photon or graviton are a conserved vector current and the stress-energy tensor, respectively. The anomalous dimensions of these conserved currents must vanish, and indeed $\Delta_J = 3$ and $\Delta_T = 4$ are canonical.

4.4 AdS/CFT over a cone

As seen above, the original motivation for the AdS/CFT conjecture is the identification of a system of N coincident D3-branes in a $\mathbb{M}^{10}$ Minkowski background and the corresponding 3-brane supergravity solution. In an appropriate low-energy limit a system of closed IIB strings on flat spacetime decouples in both pictures, suggesting it should be conjectured that the remaining parts are equivalent. These are respectively $\mathcal{N} = 4$, $SU(N)$ SYM on $\mathbb{M}^4$ and IIB strings on $AdS_5 \times S_5$.

We repeat this reasoning, but in the more interesting case where the background for the D3-branes is generalized as $\mathbb{M}^4 \times X_6$, where X_6 is a cone over a base 5-manifold Y_5. We anticipate the bulk dual in this case is IIB strings over $AdS_5 \times Y_5$. By X_6 being a cone over Y_5 it is meant that the metric on it is

$$ds_6^2 = dr^2 + r^2 ds_5^2 \tag{4.24}$$

where of course ds_5^2 is the metric on Y_5. If $Y_5 = \mathbb{S}^5$ with the unit round metric then the cone is $X_6 = \mathbb{R}^6$ and one returns to the flat case.

For this to be a string background, X_6 should be Ricci-flat. This is equivalent to Y_5 being Einstein of positive curvature. ds_6^2 is conformally equivalent to the canonical metric on a cylinder over Y_5, as evidenced by the reparametrization $\phi = \ln r$:

$$ds_6^2 = e^{2\phi}\left(d\phi^2 + ds_5^2\right) \tag{4.25}$$

Recalling the transformation law of the Ricci tensor in n dimensions under conformal rescalings:

$$R'_{ij} = R_{ij} - (n-2)\left(\nabla_i \partial_j \phi - \partial_i \phi \partial_j \phi\right) + \left(\nabla^2 \phi - (n-2)\nabla_k \phi \nabla^k \phi\right) g_{ij} \tag{4.26}$$

And noting that for the cylinder the restriction of R_{ij} to Y_5 indices gives Y_5's own Ricci tensor $R^{(5)}_{ij}$, we obtain

$$R^{(5)}_{ij} = 4 g^{(5)}_{ij} \tag{4.27}$$

proving Y_5 is Einstein.

We are also interested in X_6 being Calabi-Yau, that is being Kähler with holonomy $\subset SU(3)$. We define Y_5 to be Sasaki-Einstein iff the corresponding cone is Calabi-Yau. The complex structure on the cone induces a vector field on the base, the Reeb vector:

$$\xi := J(r\partial_r) \tag{4.28}$$

where J is the complex structure on the cone and ξ is to be thought of as restricted to, say, $r = 1 \cong Y_5$; this is a Killing vector on the base, inducing a 1-dimensional foliation. The dual form, $\theta = g_{ij}\xi^i dx^j$, is a contact form for the base, contact meaning the 2-form on the cone

$$\omega = t^2 d\theta + t dt \wedge \theta \tag{4.29}$$

is symplectic. This is of course the symplectic form associated to the hermitian structure.

After placing 3-branes in this $X_6 \times \mathbb{R}^4$ background, parallel to the Minkowski, the resulting geometry from their backreaction is:

$$ds^2 = H^{-1/2}(r,y)\, dx \cdot dx + H^{1/2}(r,y)\, ds_6^2 \tag{4.30}$$

già fatto! ref ref ref Where $x^{0,\dots,3}$ are coordinates parallel to the brane stack, $dx \cdot dx = -(dx^0)^2 + (dx^i)^2$, r is the radial coordinate and the remaining $y^{1,\dots,5}$ parametrize the cone's base Y_5. This is a simple generalization of the well-known black 3-brane solution by substitution of $\mathbb{S}^5$ with Y_5.

Ricci-flatness implies the function H is harmonic: $\nabla H(r) = 0$. The linearity of this equation arises from the fact that D-branes are BPS states, corresponding in the gravitational picture to extremal p-branes; these notably do not interact mutually.

If the branes are coincident, the corresponding harmonic potential is

$$H(r) = 1 + \frac{R^4}{r^4} \qquad\qquad R^4 = 4\pi g_s N \alpha'^2 \tag{4.31}$$

The near-horizon limit ($r \to 0$) in that case can be read immediately:

$$ds^2 = \frac{dx \cdot dx + dz^2}{z^2} + ds_5^2 \tag{4.32}$$

where $z := 1/r$; this is evidently the product metric on $AdS_5 \times Y_5$, where of AdS_5 we're only considering the Poincaré patch.

We note that the introduction of a conical singularity results in reduced supersymmetry. Unbroken SUSY generators are identified from the Killing spinor equation:

$$\left(\partial_\mu + \frac{1}{4}\omega_{\mu\alpha\beta}\Gamma^{\alpha\beta}\right)\eta = 0 \tag{4.33}$$

Explicitly for the cone metric 4.24:

$$\left(\partial_i + \frac{1}{4}\omega_{ijk}\Gamma^{jk} + \frac{1}{2}\Gamma^r_i\right)\eta = 0 \tag{4.34}$$

this is, as expected, coincident with the (Y_5 sector of) Killing spinor equation for the backreacted $AdS_5 \times Y_5$ geometry, including also the effect of F_5. This is to show there is a match between the unbroken SUSYs in the bulk theory and in the boundary.

If the cone is of holonomy $SU(n)$, this will result in a reduction of supersymmetries by a factor of 2^{1-n} with respect to $\mathbb{M}^{10}$ Minkowski. In particular, if X_6 is Calabi-Yau, then the $32 = 16 \times 2$ fermionic generators of IIB SUGRA are reduced to $32 \times 2^{-2} = 8$, which means the SCFT in $4D$ has $\mathcal{N} = 1$ (in contrast to the usual SUSY algebra, the $\mathcal{N} = 1$ $4D$ superconformal group has both 4 supertranslations and 4 additional fermionic superconformal generators). If, instead, we were to consider the more restrictive case of manifolds of $SU(2)$ holonomy, there would be 16 unbroken supersymmetries signaling an $\mathcal{N} = 2$ dual SCFT.

Chapter 5

Holographic effective field theories

In this chapter, we present the technique and results introduced in [9] to find the effective theory for strongly-interacting CFTs with holographic duals. Instead of repeating the arguments presented therein, we'll strive to provide an intuitive summary of the concepts involved.

Since we are considering strongly-interacting quantum field theories with minimal supersymmetry, the problem of identifying the low-energy effective field theory directly is generally untractable. However, as we have seen, the strong-coupling regime for the CFT corresponds to effectiveness of the supergravity approximation on the holographically dual string side. Therefore, the low-energy dynamics of the dual system can in principle be read and the resulting theory will coincide with the effective theory for the original QFT. Having been obtained by passing through the holographic dual, these will be termed holographic effective field theories (HEFTs).

In practice, it's found that for any given point in the longitudinal coordinates $x^{0,\dots,3}$ the transverse supergravity configuration will belong to a manifold of different supergravity vacua, and that this manifold is finite-dimensional, in

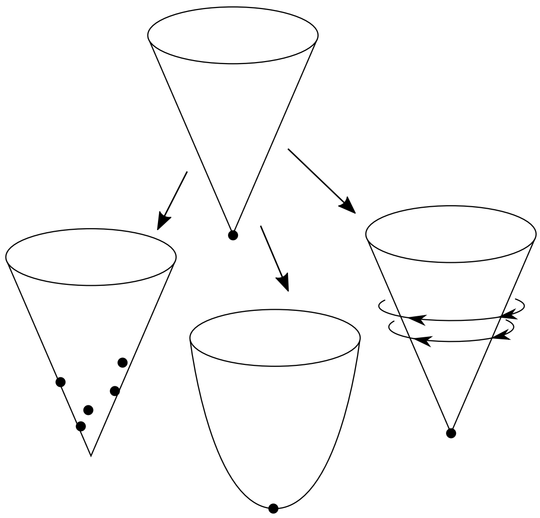


Figure 5.1: Schematic depiction of the three type of moduli of the dual vacuum: motion of D3-branes, deformation of Kähler structure of the background, generally involving resolution of the conical singularity, and field fluxes.

the sense that there is only a finite number of moduli parametrizing deformations of the vacuum configurations. This moduli space coincides of course with the field theory moduli space.

A first class of moduli are given by deformations of the dual geometry. These include Kähler moduli of the *background* cone in which the branes are placed in, and then the position of the D3-branes themselves on that background - which manifests as a deformation in the resulting warped geometry. Another class instead will be given by the moduli corresponding to the deformations of the B_2 and R-R fields of IIB supergravity; while these would be full fields defined on the six-dimensional background, gauge invariance will result in only a finite number of topological invariants of the field configuration to be physical.

In short, there will be a finite number of flat directions parametrizing moduli space, and each of these moduli will result in a corresponding scalar field when we extend these deformations to depend on the longitudinal point.

Reintroducing $\mathcal{N} = 1$ supersymmetry, these will be the lowest components of chiral supermultiplets which will exhaust the degrees of freedom of the low-energy effective field theory.

Then, expanding the supergravity action in these modes the action governing these chiral fields can be found. This is nothing else than the explicit form of the effective theory for our original strongly-interacting theory.

5.1 Topology of X_6

It's necessary to quickly introduce a certain fact about the topology of X_6 for us to distinguish between normalizable and non-normalizable Kähler deformations. First of all, we take as an assumption that the third Betti number of the cone vanishes:

$$b_3(X) = 0 \tag{5.1}$$

Moreover, it can be proven from Myers' theorem that Y_5 being Sasaki-Einstein means the following Betti numbers vanish:

$$b_1(Y) = b_4(Y) = 0 \tag{5.2}$$

We recall the long sequence involving relative homology groups:

$$\ldots \to H^{i-1}(Y) \to H^i(X,Y) \to H^i(X) \to H^i(Y) \to H^{i+1}(X,Y) \to \ldots \tag{5.3}$$

where $H^i(X,Y;\mathbb{R})$ is the relative homology group - closed k-forms on X vanishing on Y modulo exact forms with the same property - and when the $;\mathbb{R}$ is omitted we implicitly mean the base field is $\mathbb{R}$. We cut the sequence short by setting $i = 2$ and noting $H^1(Y) = 0$ as of 5.2 and $H^3(X,Y) \subset H^3(X) = 0$

as of 5.1; the short exact sequence is

$$0 \to H^2(X,Y) \to H^2(X) \to H^2(Y) \to 0 \tag{5.4}$$

Implying $H^2(X) = H^2(Y) \oplus H^2(X,Y)$. Applying Poincaré duality[1] on the two components and counting dimensions gives

$$b_2(X) = b_3(Y) + b_4(X) \tag{5.5}$$

This identity will be necessary in the decomposition of harmonic 2-forms.

5.2 Kähler moduli

We now consider the moduli describing the deformation of the Kähler (so, geometric) structure of the background. Since $b_3(X) = 0$ by hypothesis, the complex structure is rigid. There are instead moduli for the Kähler form J; in particular we know from *citazione teoremi esistenza* that every cohomology class $[J]$ of $H^2(X)$ contains a single representative Ricci-flat Kähler form J, so that $H^2(X)$ is the moduli space for the Kähler structure. We can expand the cohomology class as

$$[J] = v^a[\omega_a] \tag{5.6}$$

with $[\omega_a]$ being a basis for the integral cohomology $H^2(X;\mathbb{Z})$, as the latter modulo torsion is a lattice sitting in $H^2(X;\mathbb{R})$. This means

$$\delta[J] = \delta v^a[\omega_a] \tag{5.7}$$

meaning there exist representatives in the classes such that the equation

[1] We note in the case of non-compact manifolds, such as X_6, Poincaré-Lefschetz duality is actually an isometry between $H^k(X)$ and $H_{6-k}(X,Y)$, instead of $H_{6-k}(X)$.

without square brackets holds. Since small variations of the Kähler form must be $(1,1)$ harmonic forms *reference*, we then know there exist $(1,1)$ harmonic representatives ω_a for the aforementioned basis of classes. Returning to 5.6 we can rewrite it as

$$J - v^a \omega_a \in [0] \tag{5.8}$$

But for the LHS to belong to the zero class just means to be exact. Therefore

$$J = J_0 + v^a \omega_a \tag{5.9}$$

with J_0 being exact and $(1,1)$. Note the linearity of this parametrization is an illusion of notation: the condition $\Delta\omega_a = 0$ depends on the metric and so on the v^a.

It is then useful to decompose this set of $b_2(X)$ harmonic forms according to identity 5.5 into $b_3(Y)$ noncompact elements $\tilde{\omega}_\beta$ and $b_4(X)$ normalizable forms $\hat{\omega}_\alpha$. By "normalizable" it's meant the hatted forms have finite norm according to the product

$$\int_X \omega_a \wedge \star\omega_b =: \mathcal{M}_{ab} \tag{5.10}$$

while the other $b_3(Y)$ don't. They are however all normalizable according to the "warped" product

$$\int_X e^{-4A} \omega_a \wedge \star\omega_b =: \mathcal{G}_{ab} \tag{5.11}$$

where the factor e^{-4A}, as will be explained later, is the warp factor resulting from the backreaction of the D-branes. More intuitively, in our particular case of X being (asymptotically) a cone, this means that $||\hat{\omega}_\alpha||^2$ must drop at least as fast as r^{-8}, while $||\tilde{\omega}_\beta||^2$ will go as r^{-4}.

In any local chart the ω_a forms will be generated by potentials κ_a as $\omega_a = i\partial\bar{\partial}\kappa_a$ just like J is generated by the Kähler potential $J = i\partial\bar{\partial}k$. This means in particular κ_a will coincide with $\frac{\partial k}{\partial v^a}$ up to a $z_i, \bar{z}_i$-independent piece, that is a function of the $\{v^a\}$ only. To fix this arbitrariness, the κ_a potentials are required to satisfy

$$\frac{\partial \kappa_a}{\partial v^a} \sim r^{-k} \qquad , k \geq 2 \tag{5.12}$$

so that they are determined up to a constant. This asymptotic condition must be enforced for the following analysis to be meaningful.

5.3 Remaining moduli

We are now in position to classify flat deformations of the axio-dilaton τ and the 2-forms C_2 and B_2, which we compose into a single complex 2-form $C_2 - \tau B_2$, plus τ itself. The former field's flat deformation will be generated by cohomology classes of $H_2(X)$, so in practice the harmonic forms ω_a found above can be used as a basis. Therefore the following decomposition is possible:

$$C_2 - \tau B_2 = l_s^2 \left(\beta^\alpha \hat{\omega}_\alpha + \lambda^\beta \tilde{\omega}_\beta\right) \tag{5.13}$$

The $b_4(X)$ moduli β^α weighing the compact forms $\hat{\omega}_\alpha$ will result in dynamical chiral fields in the HEFT. Instead, the $b_3(Y)$ λ^β moduli, to which we add one complex modulus for τ, parametrize deformations which will turn out to be non-dynamical. The reason is precisely that the kinetic matrix for these fields will turn out to be the inverse of $\mathcal{M}_{ab}$, which is only finite for the normalizable forms.

Then, an obvious set of moduli z_I^i ($I = 1, \cdots, N$, $i = 1, 2, 3$) are to be introduced to parametrize the motion of the D3-branes on the background. As was hinted before, these alone are coordinates for the submanifold $\mathcal{M}_{\text{mes}}$ in $\mathcal{M}$, which to be precise should be quotiented by permutation of the branes. Therefore, $\mathcal{M}_{\text{mes}}$ based at any given point of moduli space is the symmetric product of N copies of the background geometry, as it for those particular values of the Kähler moduli.

There is a final class of flat shifts that should be considered, those of the C_4 potential. These moduli should (very schematically) better be thought of as paired with the v^a to form complex moduli. In the end, it turns out it's not really necessary to study the C_4 moduli explicitly for the purpose of finding the HEFT.

5.4 Chiral fields

Finally, we have to introduce the chiral fields corresponding to the dynamical moduli. We use the moduli β^α and z_I^i directly as the lowest component of the corresponding superfield, while to $v^a = (\hat{v}^\alpha, \tilde{v}^\beta)$ it is useful to associate fields $\rho^a = (\hat{\rho}_\alpha, \tilde{\rho}_\beta)$, obtained by a particular transformation:

$$\operatorname{Re}\rho_a = \frac{1}{2}\sum_I \kappa_a(z_I, \bar{z}_I; v) - \frac{1}{2\operatorname{Im}\tau} I_{a\alpha\beta}\operatorname{Im}\beta^\alpha \operatorname{Im}\beta^\beta - \frac{1}{\operatorname{Im}\tau} I_{a\alpha\sigma}\operatorname{Im}\beta^\alpha \operatorname{Im}\lambda^\sigma \tag{5.14}$$

where the κ_a are the potentials for the ω_a forms as defined in 5.2. The imaginary part instead as expected is related to the C_4 moduli; the explicit form of $\operatorname{Im}\rho_a$ is not necessary for our purposes.

To wrap up, the dynamical chiral fields in the effective field theory are

$b_4(X_6)$	$\hat{\rho}_\alpha$	norm. Kähler and C_4 deformations
$b_3(Y_5)$	$\tilde{\rho}_\sigma$	warp norm. Kähler and C_4 deformations
$b_4(X_6)$	β_α	norm. $C_2 - \tau B_2$ deformations
$3N$	z_I^i	D3-brane positions

plus the following non-dynamical marginal parameters:

$b_3(Y_5)$	λ_σ	warp norm. $C_2 - \tau B_2$ deformations
1	τ	axio-dilaton

The identification of the meaning of these fields and parameters in terms of the four-dimensional side is as follows. The z_I^i correspond clearly to the $3N$ independent mesons of the CFT, as it was already established. The ρ and β fields instead are the independent baryons. Their total number is

$$2b_4(X) + b_3(Y) = b_2(X) + b_4(X) = g - 1 \tag{5.15}$$

where we've used respectively identity 5.5 and the expression **??** for the number of gauge groups. There is therefore a match with the number of baryonic moduli of the CFT as it was determined by solving the F- and D-term conditions.

λ_σ and τ instead are dual to the marginal deformations of the CFT, that is of the gauge and superpotential couplings. The $b_3(Y) + 1$ such marginal moduli correspond directly to the $b_3(Y) + 1$ marginal couplings we *finire*

5.5 Effective action

It now remains to specify the dynamics of these chiral fields through an effective action. A way to proceed is to start with the similar case of compactifications[8], where the Calabi-Yau transverse space X_6 is actually compact, unlike the asymptotically conical noncompact 6-folds encountered until now. That is, N D3-branes are placed on the background metric

$$ds_{10}^2 = l_S^2(ds_4^2 + ds_X^2) \tag{5.16}$$

where $\int_X d\,\mathrm{vol}_X = V_0$ is the unwarped volume of the compact space, and as a result the geometry is warped into

$$ds_{10}^2 = l_S^2(e^{2A}ds_4^2 + e^{-2A}ds_6^2) \tag{5.17}$$

$$\nabla e^{-4A}(z^i) = \star\, l_S^4 \sum_I \delta^6(z^i - z_I^i) \ + \ (\text{fluxes}\ldots) \tag{5.18}$$

with warped volume $\int_X e^{-4A} d\,\mathrm{vol}_X =: V_w =: aV_0$. a is termed a universal modulus. In [8] it is argued that the low-energy effective four-dimensional theory resulting from such a compactification is actually a superconformal supergravity theory, and that on the basis of these symmetries the total superpotential is forced into a remarkably simple form:

$$K = -3\ln(4\pi V_w) = -3\ln(4\pi V_0 a) \tag{5.19}$$

The effective Lagrangian is then obtainable directly by differentiating the Kähler potential with respect to the moduli of the compactification, provided the dependency of the universal modulus from the latter is known. Just like in the non-compact case, it's possible to expand variations of the Kähler form in a basis of harmonic forms as $J = J_0 + v^a\omega_a$, and the z_I^i moduli also reappear identically, mapping the positions of the D3-branes in the compact dimensions rather than the noncompact resolved cone. The β and λ - type moduli are set aside temporarily. The Lagrangian is composed of a chiral part for the chiral fields corresponding to the moduli plus a set of N fully decoupled $U(1)$ super-QED sectors due to the open string modes on each isolated D3-brane (as we're in a generic point of $\mathcal{M}$ where no branes coincide). The bosonic part of the chiral sector is

$$\mathcal{L} = -\pi \tilde{\mathcal{G}}^{ab} \nabla \rho_a \wedge \star \nabla \bar{\rho}_b - 2\pi \sum_I J_{i\bar{\jmath}} dz_I^i d\bar{z}_I^{\bar{\jmath}} \tag{5.20}$$

where also in this case the ρ_a fields are obtained by a Legendre transform of the v^a Kähler moduli:

$$\operatorname{Re}\rho = \frac{1}{2} a I_{abc} v^b v^c + \sum_I \kappa(z_I, \bar{z}_I; v) + \text{ (fluxes} \ldots) \tag{5.21}$$

and the kinetic matrix and connection are given by

$$\tilde{\mathcal{G}}^{ab} = \frac{1}{2V_0 a} v^a v^b - \mathcal{G}^{ab} \tag{5.22}$$

$$\nabla \rho_a = d\rho_a + \sum_I \mathcal{A}_{ai}^I dz_I^i, \qquad \mathcal{A}_{ai}^I = \frac{\partial \kappa_a}{\partial z_i^I} \tag{5.23}$$

(we recall the matrix of warped products $\mathcal{G}_{ab}$ is as defined in 5.11).

This result is then first "decompactified" by taking the limit $V_0 \to \infty$, so as to extend to the case of non-compact X_6 as in our case. Then a second limit, $a \to 0$, is considered, equivalent to the near-horizon limit we require to get into the holographic regime. Under this limit, $\operatorname{Re}\rho(v)$ turns into the form 5.14 we introduced earlier, $\tilde{\mathcal{G}}_{ab}$ reduces to $\mathcal{G}_{ab}$, and the Lagrangian simplifies to:

$$\mathcal{L}_{\rho,z} = -\pi \mathcal{G}^{ab} \nabla \rho_a \wedge \star \nabla \rho_b - 2\pi \sum_I J_{i\bar{\jmath}} dz^i d\bar{z}^{\bar{\jmath}} \tag{5.24}$$

It can happen that in decompactification some of the ω_a will have turned into non-warp-normalizable forms for which $\mathcal{G}_{ab}$ diverges. In that case, the kinetic term $\mathcal{G}^{ab}$ for the corresponding ρ fields vanishes, so that the latter disappear from the dynamics. Therefore it is understood that the a, b indices span only over warp-normalizable Kähler moduli of the non-compact X_6.

Finally, one adds back the C_2, B_2 moduli by again expanding $C_2 - \tau B_2$ in the basis of ω_a forms to obtain the $\beta^\alpha, \lambda^\sigma$ fields as explained before; then it's shown that they contribute to the action with a kinetic matrix proportional to the *unwarped* product matrix $\mathcal{M}_{ab}$ defined in 5.10. This means that only the β^α fields, modulating unwarped-normalizable flat shifts are dynamical fields. The final chiral Lagrangian is:

$$\mathcal{L}_{\text{chiral}} = \mathcal{L}_{\rho,z} - \frac{\pi}{\operatorname{Im}\tau}\mathcal{M}_{\alpha\beta}\, d\beta^\alpha \wedge \star d\bar{\beta}^\beta \tag{5.25}$$

where one must also take into account the fact that the β fields couple to the ρ fields through an addition to the latter's covariant derivative, whose final form is

$$\nabla\rho_a = d\rho_a - \mathcal{A}^I_{ai} dz^i_I - \frac{i}{\operatorname{Im}\tau}\left(I_{a\alpha\beta}\operatorname{Im}\beta^\beta + I_{a\alpha\sigma}\operatorname{Im}\lambda^\sigma\right) d\beta^\alpha \tag{5.26}$$

With this information, it is now possible to compute the HEFT corresponding to a given conical background provided the Kähler form for the generic Calabi-Yau deformation is known in complex coordinates. The following identity for $\mathcal{G}_{ab}$ can be proven[9] and is a useful computational shortcut:

$$\mathcal{G}_{ab} = -\frac{\partial\kappa_a}{\partial v^b} \tag{5.27}$$

Therefore, to outline a systematic procedure: once the general Kähler potential and Kähler form are known, derivatives with respect to the moduli yield the κ potentials, from which one computes the $\mathcal{G}$ matrix and the $\mathcal{A}$ connection; the intersection numbers and in particular the $\mathcal{M}_{\alpha\beta}$ matrix are instead invariants independent of the moduli and are readily evaluated on geometrical grounds.

5.6 Example: the Klebanov-Witten HEFT

As an example, we now summarize a direct application of this method to the conifold theory described in 3.4.

It's essential for the metric for the background to be presented in complex coordinates for the construction above to be applicable. We exploit the fact that at any generic point of moduli space *except* the origin, where the background is the singular conifold, the geometry is that of a smooth complex 3-fold describable as the total space of the sum of the tautological bundle on a $\mathbb{CP}^1$ with itself: $\mathcal{O}_{\mathbb{CP}^1}(-1) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)$. The base $\mathbb{CP}^1 \cong \mathbb{S}^2$ of the bundle is the "resolution" of the conical singularity. We use this to build a chart on the resolved conifold by extending a stereographic chart on the base to get the complex coordinates $z^i = (\lambda, U, Y)$, the first stereographic for the base and the latter two fibral.

With this presentation, it's clear that the only 2-cycle of X_6 is the base $\mathbb{S}^2$, so that $b_2(X) = 1$. Moreover, we already know that as a real cone the conifold has base $SU(2) \times SU(2)/U(1) \cong \mathbb{S}^2 \times \mathbb{S}^3$, and that the resolution cannot really change the topology of the base at infinity, so that the only 3-cycle in Y_5 is the $\mathbb{S}^3$ and $b_3(Y) = 1$. So, according to 5.5, $b_4(X_6) = 0$. This will mean, according to the previous identification, that we will have a single $\tilde{v} =: v$ modulus parametrizing a non-normalizable Kähler deformation and dual to a single chiral field $\tilde{\rho} =: \rho$, no β fields, and two non-dynamical parametres λ and τ.

The Kähler modulus v is identified with the volume of the base. Therefore, all Calabi-Yau deformations of the conifold are a one-parameter family and the singular conifold itself lies at the origin, $v = 0$. The Kähler potential for X_6 in stereographic coordinates is detailed for example in [10], and takes the form:

$$k(z, \bar{z}; v) = \frac{1}{2} \int_0^{s^2} d \ln x \; \gamma(x; v) \; + \frac{v}{2\pi} \ln(1 + |\chi|^2) \tag{5.28}$$

where $s^2 = (1 + |\chi|^2)(|U|^2 + |Y|^2)$ and γ must satisfy the following for the metric to be Ricci-flat:

$$\gamma^3 + \frac{3v}{2\pi}\gamma^2 - x^2 = 0 \tag{5.29}$$

The potential κ generating the unique harmonic form $\tilde{\omega}$ is given simply by derivative of k with respect to the modulus, plus a v-dependent piece fixed by the asymptotic condition 5.12:

$$\kappa = -\frac{1}{4} \int_0^{s^2} d \ln x \; \frac{\gamma}{\pi\gamma + v} \; + \frac{1}{2\pi} \ln(1 + |\chi|^2) - \frac{3}{8\pi} \ln v \tag{5.30}$$

This in turn defines the relationship between ρ and v as

$$\mathrm{Re}\,\rho = -\frac{1}{8} \sum_I \int_0^{s_I^2} d \ln x \; \frac{\gamma}{\pi\gamma + v} + \frac{1}{4\pi} \sum_I \ln(1 + |\chi_I|^2) - \frac{3N}{16\pi} \ln v \tag{5.31}$$

from which one can readily find the 1×1 $\mathcal{G}$ matrix:

$$\mathcal{G} = -\frac{\partial \rho}{\partial v} = \frac{3}{16\pi} \sum_I (v + \pi\gamma)^{-1} \tag{5.32}$$

All ingredients for writing down the HEFT are now available. The chiral part of the bosonic lagrangian for the effective low-energy theory of the KW model is given by

$$\mathcal{L} = -\pi \mathcal{G}^{-1} \nabla\rho \wedge \star \nabla\bar{\rho} - 2\pi \sum_I J_{i\bar{j}} dz_I^i \wedge \star d\bar{z}_I^{\bar{j}} \tag{5.33}$$

where of course $J_{i\bar{j}} = \partial_i \bar{\partial}_j k(z, \bar{z}; v)$ is the metric tensor of the resolved conifold. The expressions here are deceivingly simple: both $\mathcal{G}$ and J depend on the modulus v which must be understood as a function of the chiral field ρ by inversion of the Legendre transform. The connection for the covariant derivative, being a simple derivative of the Kähler potential, can also be explicited:

$$\mathcal{A}_i dz^i = (4v + 4\pi\gamma)^{-1} \left(\frac{2v + \pi\gamma}{\pi(1 + |\chi|^2)} \bar{\chi} d\chi - \frac{\gamma \left(\bar{U} dU + \bar{Y} dY \right)}{|U|^2 + |Y|^2} \right) \tag{5.34}$$

Chapter 6

The $Y^{(2,0)}$ HEFT

Having secured the tools required, we are now ready to take on the holographic effective theory of the $Y^{2,0}$ theory introduced in 3.5. To this end, we will first derive the form of the general Kähler metric of a Calabi-Yau deformation of the $Y^{2,0}$ cone in complex coordinates.

6.1 Kähler form

The metric of the general Calabi-Yau deformation of the $Y^{2,0}$ cone is already well-known in real coordinates as:

$$ds^2 = \kappa^{-1}(r)dr^2 + \frac{1}{9}\kappa(r)r^2\left(d\psi + \cos\theta_L d\phi_L + \cos\theta_R d\phi_R\right)^2 \\ + \frac{1}{6}r^2 d\Omega_L^2 + \frac{1}{6}(r^2 + a^2)d\Omega_R^2 \quad (6.1)$$

$$\kappa(r) = \frac{1 + \frac{9a^2}{r^2} - \frac{b^6}{r^6}}{1 + \frac{6a^2}{r^2}} \quad (6.2)$$

with a, b the two unique real moduli. The topology is that of an $\mathbb{R}^2$ bundle over $\mathbb{S}^2 \times \mathbb{S}^2$. We take it as an assumption that this matches with the complex structure associated to the Kähler form so that this is the total space of a $\mathbb{C}$ bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$ - we will confirm this a posteriori when we'll provide the complex-coordinates expression and show it agrees with the real form.

With this assumption, we search for the general CY metric on a $\mathbb{C} \to \mathbb{CP}^1 \times \mathbb{CP}^1$ bundle; on the spheres of the base we take the round metric, given by the Kähler forms j^L and j^R. It's easy to verify explicitly that, given any set of complex coordinates on the base (y_L, y_R),

$$j^L \wedge j^R = e^{-\Lambda k} dy^L \wedge dy^R \wedge d\bar{y}^L \wedge d\bar{y}^R \tag{6.3}$$

with $k = k^L + k^R$ the total base potential, and for some Λ depending on the overall size of the spheres (for *standard, $R = 1/2$?*, $\Lambda = 1$).

we also introduce the function t of the fibral coordinate ζ as

$$t = |\zeta|^2 e^{\Lambda k} \tag{6.4}$$

We then start from the following ansatz for the Kähler potential

$$k_X = f(t) + \alpha k^L + \tilde{\alpha} k^R \tag{6.5}$$

where $\alpha, \tilde{\alpha}$, controlling the volume at $t = 0$ of the base 2-spheres, should parametrize the Ricci-flat Kähler resolutions of the cone. We now prove that there is always an $f(t; \alpha, \tilde{\alpha})$ that makes the metric Ricci-flat.

The corresponding Kähler form is straightforward:

$$J = A^L(t) j^L + A^R(t) j^R + i e^{\Lambda k}(f' + t f'')(d\zeta + \Lambda \zeta \partial k) \wedge (\text{c.c.}) \tag{6.6}$$

with $A^L = \alpha + \Lambda t f'(t)$ and $A^R = \tilde{\alpha} + \Lambda t f'(t)$. This is more simply $J = J_M + M e^3 \wedge \bar{e}^{\bar{3}}$, where J_M is the purely basal part, $e^3 = d\zeta + \Lambda\zeta\partial k$ and M is a scalar factor. The volume form is then clearly

$$3! d\,\mathrm{vol}_X = J \wedge J \wedge J = 3A^L A^R M \left(j^1 \wedge j^2 \wedge e^3 \wedge \bar{e}^{\bar{3}} \right) \tag{6.7}$$

as all other terms in the cube vanish. Since the volume form is $\sqrt{\det g}\, d\Omega \wedge \bar{\Omega}$, with $\Omega = d\zeta \wedge dy^L \wedge dy^R$, and the Ricci tensor for a Kähler space is proportional to $\partial\bar{\partial} \ln \det g$, then the condition for Ricci-flatness is equivalent to the prefactor of $\Omega \wedge \bar{\Omega}$ in $J \wedge J \wedge J$ being constant, that is to say

$$A^L(t) A^R(t)(f' + t f'') = (\alpha + \Lambda t f')(\tilde{\alpha} + \Lambda t f') \frac{d}{dt}(\Lambda t f') =: c \tag{6.8}$$

or, having defined $y := \Lambda t f'$,

$$(\alpha + y)(\tilde{\alpha} + y) y' = c \tag{6.9}$$

Since $f(t)$ must be regular as $t = 0$, and $f' = \frac{y}{\Lambda t}$, it must be that y goes to zero at least as fast as t as $t \to 0$; this condition eliminates the freedom from the constant of integration for equation 6.8. The constant c on the other hand can be readily reabsorbed into a t rescaling. Therefore there should be a unique y (and so a unique f up to unconsequential constant shifts) that gives a Ricci-flat metric. Let us see this explicitly: we integrate 6.8 to obtain

$$\frac{y^3}{3} + \frac{\alpha + \tilde{\alpha}}{2} y^2 + \alpha\tilde{\alpha} y = ct + d \tag{6.10}$$

And then the regularity condition $y(0) = 0$ is satisfied with $d = 0$, and this cubic equation for y is immediately seen to have one single real solution for any positive values of α^i, c.

Before exhibiting the explicit form of $y(t; \alpha, \tilde{\alpha})$, let us express the Kähler form in terms of y and show it actually coincides with the real-coordinate metric 6.1. We have

$$J = (\alpha + y)j^1 + (\tilde{\alpha} + y)j^2 + \frac{ie^{\Lambda k}}{\Lambda} y' e^3 \wedge \bar{e}^{\bar{3}} \tag{6.11}$$

$$= (\alpha + y)j^1 + (\tilde{\alpha} + y)j^2 + \frac{ie^{\Lambda k}\, c}{\Lambda(\alpha + y)(\tilde{\alpha} + y)} e^3 \wedge \bar{e}^{\bar{3}} \tag{6.12}$$

Now, we parametrize the fiber as $\zeta = e^{-\Lambda k/2} t^{1/2} e^{i\psi}$, and the 2-spheres with spherical coordinates θ_i, ϕ_i which fixes $\Lambda = 1$. Then the metric corresponding to J is

$$ds^2 = A^L d\Omega_L^2 + A^R d\Omega_R^2 + \frac{y'}{t}\left(\frac{dt^2}{4} + t^2(d\psi + \sigma)^2\right) \tag{6.13}$$

Where $\sigma = -i\frac{\Lambda}{2}(\partial k - \bar{\partial} k)$. But the $t - \psi$ part is simply

$$ds^2 = \frac{1}{4y't} dy^2 + (y't)(d\psi + \sigma)^2 \tag{6.14}$$

Exploiting both 6.8 and its integrated form 6.10 we rewrite

$$y't = \frac{1}{A^L A^R}\left(\frac{y^3}{3} + \frac{\alpha + \tilde{\alpha}}{2} y^2 + \alpha\tilde{\alpha} y\right) \tag{6.15}$$

$$= 3cr^2 \frac{1 + \frac{3}{2}\frac{\tilde{\alpha} - \alpha}{r^2} + \frac{\alpha^2(\alpha - 3\tilde{\alpha})}{2r^6}}{1 + \frac{\tilde{\alpha} - \alpha}{r^2}} \tag{6.16}$$

$$= 3cr^2 \kappa(r) \tag{6.17}$$

provided we make the identifications

$$a^2 = \frac{1}{6}(\tilde{\alpha} - \alpha) \qquad\qquad b^6 = \frac{\alpha^2(3\tilde{\alpha} - \alpha)}{2} \tag{6.18}$$

The final coordinate change to the (asymptotically) conical r coordinate is then given by $r^2 = A^L = y + \alpha$ - note this renders the inherent symmetry between the left and right 2-cycles non-manifest[1]. The resulting metric, after taking $c = 1/3$, is precisely 6.1. Thus, as the latter is the most general Calabi-Yau deformation of the $Y^{2,0}$ cone, we have to conclude that the two-parameter family of metrics 6.11 in complex coordinates coincides with it.

Now we're left with solving for the explicit form of y. Switching temporarily to $z = y + (\alpha + \tilde{\alpha})/2$ equation 6.10 is brought into depressed form:

$$z^3 - \frac{3}{4}(\alpha - \tilde{\alpha})^2 = ct + D \tag{6.19}$$

Where

$$D = \frac{1}{12}(-\alpha^3 + 3\alpha^2\tilde{\alpha} + 3\alpha\tilde{\alpha} - \tilde{\alpha}^3) = \frac{b^6 - 36a^6}{3} \tag{6.20}$$

So that the explicit solution for y is

$$z = |\alpha - \tilde{\alpha}| C_{1/3}\left(12\frac{ct + D}{|\alpha - \tilde{\alpha}|^3}\right) \tag{6.21}$$

$$y = z - \frac{\alpha + \tilde{\alpha}}{2} \tag{6.22}$$

where we defined the function $C_{1/3} = \mathrm{ch}(1/3\ \ \mathrm{ch}^{-1}(x))$; that 6.22 solves 6.19 can be readily verified by means of the trigonometric identity $\mathrm{ch}(3x) = 4\,\mathrm{ch}^3(x) - 3\,\mathrm{ch}(x)$.

[1] Clearly, we could swap α and $\tilde{\alpha}$ in all of the above definitions, with no consequence.

Using the freedom to scale t to fix $c = 1/3$ for future convenience and introducing the notation $\delta = \alpha - \tilde{\alpha}$, $\sigma = \alpha + \tilde{\alpha}$, the Kähler form is explicitly given by

$$J(\sigma, \delta) = \left(z + \frac{\delta}{2}\right) j^1 + \left(z - \frac{\delta}{2}\right) j^2 + ie^k z' e^3 \wedge \bar{e}^{\bar{3}} \tag{6.23}$$

$$z(t; \sigma, \delta) = \delta\, C_{1/3}\left(\delta^{-3}\left(4t + \frac{\sigma(3\delta^2 - \sigma^2)}{2}\right)\right) \tag{6.24}$$

or equivalently, in terms of the y function:

$$J(\sigma, \delta) = (y + \alpha) j^1 + (y + \tilde{\alpha}) j^2 + ie^k y' e^3 \wedge \bar{e}^{\bar{3}} \tag{6.25}$$

Unfortunately, a closed-form expression for the Kähler potential seems impossible. The function $f(t)$ can nevertheless be written in integral form, as such:

$$f(t; \sigma, \delta) = \int_0^t d \ln t' y(t') \tag{6.26}$$

6.2 Kähler moduli

As we have seen, a very convenient basis of moduli for the Kähler structure is given by σ and δ, up to a normalization which we will shortly determine. These correspond respectively to blowing up the two basal 2-cycles (which equates to a blowup of the product 4-cycle of the base) and to an antisymmetric blowing and shrinking of the two $\mathbb{CP}^1$ (a blowup of the difference 2-cycle). We'll examine this geometric structure in more detail in this section.

We will construct the two harmonic forms $\hat{\omega}$ and $\tilde{\omega}$ by differentiating J directly with respect to the relevant moduli.

È utile mostrare che siano primitive? As an aside, we check all forms obtained in this way from J are primitive: in the dy^1, dy^2, e^3 basis the Kahler form is $\mathrm{diag}(A^L j^1_{1\bar{1}}, A^R j^2_{2\bar{2}}, ie^k y')$ so the contraction of J with any derivative $\partial_x J$ of it with respect to a parameter is

$$\begin{aligned} J^{a\bar{b}}\partial_x J_{a\bar{b}} = \mathrm{Tr}\left((J_{a\bar{b}})^{-1}\partial_x J_{a\bar{b}}\right) = A^{-1}\partial_x A + \tilde{A}^{-1}\partial_x \tilde{A} + (y')^{-1}\partial_x y' \\ = \partial_x \ln\left(A\tilde{A}y'\right) \end{aligned} \tag{6.27}$$

which vanishes thanks to the Ricci-flatness equation.

It's easy to see that the modulus $\hat{v}$ relative to $\hat{\omega}$ must be proportional to the sum σ of the basal volumes, since the corresponding harmonic form $\frac{\partial J}{\partial \sigma}$ is normalizable. To show this, we consider the asymptotic behaviour of the Kähler form as $t \to \infty$. Defined

$$T := \left(4t + \frac{\sigma(3\delta^2 - \sigma^2)}{2}\right) \tag{6.28}$$

we have

$$z \sim T^{1/3} \qquad z' \sim T^{-2/3} \tag{6.29}$$

$$\frac{\partial z}{\partial \sigma} \sim \frac{\partial T}{\partial \sigma} T^{-2/3} \qquad \frac{\partial z'}{\partial \sigma} \sim \frac{\partial T}{\partial \sigma} T^{-5/3} \tag{6.30}$$

where we have omitted constant factors. Therefore

$$J \sim \left(T^{1/3} + \frac{\delta}{2}\right) j^1 + \left(T^{1/3} - \frac{\delta}{2}\right) j^2 + ie^k T^{-2/3} e^3 \wedge \bar{e}^{\bar{3}} \tag{6.31}$$

and

$$\frac{\partial J}{\partial \sigma} \sim \frac{\partial T}{\partial \sigma}\left(T^{-2/3} j^1 + T^{-2/3} j^2 + i e^k T^{-5/3} e^3 \wedge \bar{e}^{\bar{3}}\right) \tag{6.32}$$

so that the norm goes as $\left\|\frac{\partial J}{\partial \sigma}\right\|^2 = \frac{\partial J}{\partial \sigma} \wedge \star \frac{\partial J}{\partial \sigma} \sim t^{-2} \sim r^{-12}$, which is integrable[2]. By contrast, the remaining harmonic form must be nonrenormalizable. For example, differentiating with respect to δ, we obtain

$$\frac{\partial J}{\partial \delta} \sim j^1 - j^2 + i e^k \frac{\partial T}{\partial \delta} T^{-4/3} e^3 \bar{e}^{\bar{3}} \tag{6.33}$$

with norm $\left\|\frac{\partial J}{\partial \delta}\right\| \sim t^{-2/3} \sim r^{-4}$, not integrable. We note however that it is still warp-integrable, according to our general discussion in 5.2.

We now consider the two homology 2-cycles C^1, C^2 given by the two basal $\mathbb{CP}^1$ respectively. We note that

$$\int_{C^1} \frac{\partial J}{\partial \alpha} = \int_{C^1} \left(\left.\frac{\partial A}{\partial \alpha}\right|_{t=0} j^1 \right) = \int_{C^1} j^1 = 4\pi \tag{6.34}$$

where we've exploited the fact that $y(t=0) = 0$, as it's clear from 6.10. Identically one shows $\int_{C^1} \frac{\partial J}{\partial \alpha} = \int_{C^1} \frac{\partial J}{\partial \tilde{\alpha}} = 0$ and $\int_{C^2} \frac{\partial J}{\partial \tilde{\alpha}} = 4\pi$. These are also the intersection number of C^L, C^R with the Poincaré dual 4-cycles of these forms; since the two-cycles form a basis, a 4-dual with the same intersection numbers will necessarily belong the dual class. We consider the (noncompact) 4-cycles D^1, D^2 given respectively by the fibres of C^1, C^2. Since

$$D^i \cdot C^j = \epsilon^{ij} \tag{6.35}$$

then we can identify the Poincaré duals:

[2] Asymptotically, as $t \gg \alpha, \tilde{\alpha}$, the metric reduces to the sharp cone $dr^2 + r^5 ds_5^2$, and in this regime $t \propto r^6$; therefore a function on X is integrable if it decays faster than $r^{-6} \sim t^{-1}$.

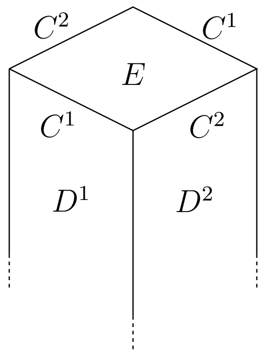


Figure 6.1: Schematic representation of the manifold X as a line bundle, with the relevant 2- and 4-cycles.

$$-\frac{1}{4\pi}\frac{\partial J}{\partial \alpha} \leftrightarrow D^2 \qquad\qquad -\frac{1}{4\pi}\frac{\partial J}{\partial \tilde{\alpha}} \leftrightarrow D^1 \tag{6.36}$$

Then a useful normalization for $\hat{\omega}$ would be

$$\hat{\omega} = \omega_1 = \frac{1}{2\pi}\left(\frac{\partial J}{\partial \sigma}\right) = \frac{\partial J}{\partial \hat{v}} \qquad\qquad \hat{v} := 2\pi\sigma \tag{6.37}$$

which makes it so $\int_{C^i}\omega = 2$ is integer. The dual to $\hat{\omega}$ is $-2(D^1 + D^2) =: E$; it's easy to show this is actually the base $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Similarly, we choose

$$\tilde{\omega} = \omega_2 = \frac{1}{2\pi}\left(\frac{\partial J}{\partial \delta}\right) = \frac{\partial J}{\partial \tilde{v}} \qquad\qquad \tilde{v} := 2\pi\delta \tag{6.38}$$

dual to the 4-cycle $F = 2(D^1 - D^2)$.

This choice for the harmonic 2-forms and the moduli v^a allows for easy computation of the intersection numbers:

$$I_0 = \int \hat{\omega} \wedge \hat{\omega} \wedge \hat{\omega} = E \cdot E \cdot E \tag{6.39}$$

$$I_1 = \int \hat{\omega} \wedge \hat{\omega} \wedge \tilde{\omega} = E \cdot E \cdot F \tag{6.40}$$

$$I_2 = \int \hat{\omega} \wedge \tilde{\omega} \wedge \tilde{\omega} = E \cdot F \cdot F \tag{6.41}$$

To evaluate these expression, we first note that $E \cap D^i = C^i$, and that[3] $E \cdot C^i = -2$.

$$I_0 = E \cdot E \cdot E = -2E \cdot E \cdot (D^1 + D^2) = -2E \cdot (C^1 + C^2) = 8 \tag{6.42}$$

$$I_1 = E \cdot E \cdot F = 0 \tag{6.43}$$

$$I_2 = E \cdot F \cdot F = -8 \tag{6.44}$$

6.3 Chiral fields and effective Lagrangian

The HEFT will feature the following chiral fields:

z_I^i	$= (y_I^1, y_I^2, \zeta_I)$	D3-brane positions on X
$\hat{\rho} = \rho_1$	dual to $\hat{v}$	4-cycle blowup deformation of X
$\tilde{\rho} = \rho_2$	dual to $\tilde{v}$	2-cycle blowup deformation of X
β		$C_2 - \tau B_2$ normalizable deformation

and the following non-dynamical chiral parameters:

[3] This intersection can be computed as follows. We represent E with the set $\{\zeta = 0\}$, and C^1 with the set of points with $y^2 = 0$, and $\zeta = y^1$ if $|y^1| < 1$, $\zeta = 1/(y^1)^*$ if $|y^1| > 1$ ($y \in \bar{\mathbb{C}}$). With this embedding the cycles are in general position and the intersection is given by the two points $y^2 = 0 = \zeta$, $y^1 = 0, \infty$.

λ	$C_2 - \tau B_2$ nonrenormalizable deformation
τ	axio-dilaton

The chiral fields $\rho_a = (\hat{\rho}, \tilde{\rho})$ are related to the moduli $v_a = (\hat{v}, \tilde{v})$ by the Legendre transform described in the previous chapter; as anticipated we will only need to specialize the precise form of the real part of $\rho_a(v_a)$:

$$\begin{aligned} \operatorname{Re}\hat{\rho} &= \frac{1}{2}\sum_I \hat{\kappa}(z_I, \bar{z}_I; v) - \frac{1}{2\operatorname{Im}\tau} I_0 (\operatorname{Im}\beta)^2 - \frac{1}{\operatorname{Im}\tau} I_1 \operatorname{Im}\beta \operatorname{Im}\lambda \\ &= \frac{1}{2}\sum_I \hat{\kappa}(z_I, \bar{z}_I; v) - \frac{4}{\operatorname{Im}\tau} (\operatorname{Im}\beta)^2 \end{aligned} \tag{6.45}$$

$$\begin{aligned} \operatorname{Re}\tilde{\rho} &= \frac{1}{2}\sum_I \tilde{\kappa}(z_I, \bar{z}_I; v) - \frac{1}{2\operatorname{Im}\tau} I_1 (\operatorname{Im}\beta)^2 - \frac{1}{\operatorname{Im}\tau} I_2 \operatorname{Im}\beta \operatorname{Im}\lambda \\ &= \frac{1}{2}\sum_I \hat{\kappa}(z_I, \bar{z}_I; v) + \frac{8}{\operatorname{Im}\tau} \operatorname{Im}\lambda \operatorname{Im}\beta \end{aligned} \tag{6.46}$$

where $\kappa_a(z_I, \bar{z}_I; v) = (\hat{\kappa}, \tilde{\kappa})$ are defined as the potentials that generate the ω_a, as in

$$\omega_a = i\partial\bar{\partial}\kappa_a \tag{6.47}$$

and also satisfy the following condition:

$$\frac{\partial \kappa_a}{\partial v_a} \sim r^{-k} \sim t^{-k/6}, \quad k \geq 2 \tag{6.48}$$

We are now able to present the bosonic part of the effective Lagrangian. There is first of all a decoupled sector of N copies of $U(1)$ SYMs (assuming we are in a generic point where no z_I coincide). Then, the rest of the bosonic effective Lagrangian describes the chiral fields listed above:

$$\mathcal{L}_{\text{chiral}} = -\pi\mathcal{G}^{ab}\nabla\rho_a \wedge \star\nabla\bar{\rho}_b - 2\pi\sum_I J_{i\bar{j}}dz^i d\bar{z}^{\bar{j}} - \frac{\pi\mathcal{M}}{\operatorname{Im}\tau}d\beta\wedge\star d\bar{\beta} \tag{6.49}$$

where the kinetic factors are computable as follows (using 5.27):

$$\mathcal{G}_{ab} = \int_X e^{-4A}\omega_a\wedge\star\omega_b = -\int_X e^{-4A}J\wedge\omega_a\wedge\omega_b = -\frac{\partial\operatorname{Re}\rho_a}{\partial v_b} \tag{6.50}$$

$$\mathcal{M} = \int_X \hat{\omega}\wedge\star\hat{\omega} = -\int J\wedge\hat{\omega}\wedge\hat{\omega} = -\hat{v}I_0 = 8\hat{v} \tag{6.51}$$

($\mathcal{G}^{ab}$ being of course the inverse matrix of $\mathcal{G}_{ab}$) and the covariant derivative ∇ is

$$\nabla\hat{\rho} = d\hat{\rho} - \mathcal{A}^I_{1i}dz^i_I - \frac{8i}{\operatorname{Im}\tau}\operatorname{Im}\beta\, d\beta \tag{6.52}$$

$$\nabla\tilde{\rho} = d\tilde{\rho} - \mathcal{A}^I_{2i}dz^i_I + \frac{8i}{\operatorname{Im}\tau}\operatorname{Im}\lambda\, d\beta \tag{6.53}$$

$$\mathcal{A}^I_{ai} = \frac{\partial\kappa_a(z_I,\bar{z}_I;v)}{\partial z^i_I} \tag{6.54}$$

To compute the coefficients $\mathcal{G},\mathcal{M},\mathcal{A}$ we first determine the form of the κ potentials.

6.4 κ potentials

In accord to what was discussed in 5.2, since $\frac{\partial k_X}{\partial v^a}$ generates $\frac{\partial J}{\partial v^a} = \omega_a$, it must be that $\kappa_a = \frac{\partial k_0}{\partial v^a} + h(v)$ with $h(v)$ an arbitrary function of the moduli which would then be fixed as to satisfy the condition 6.48 (up to an additive constant). However, as will be seen shortly, $\frac{\partial k_0}{\partial v^a}$ itself satisfies the asymptotic condition, so that $h(v)$ is actually a constant, which we will omit.

Recalling $k_0 = f(t) + \frac{\sigma+\delta}{2}k^L + \frac{\sigma-\delta}{2}k^R$, and $f(t) = \int_0^t d\,\ln(t')y(t')$, we find

$$2\pi\hat{\kappa}(t;\sigma,\delta) = \frac{\partial k_X}{\partial\sigma} = \left(\int d\,\ln t'\frac{\partial y}{\partial\sigma}\right) + \frac{1}{2}k^L + \frac{1}{2}k^R \tag{6.55}$$

$$2\pi\tilde{\kappa}(t;\sigma,\delta) = \frac{\partial k_X}{\partial\delta} = \left(\int d\,\ln t'\frac{\partial y}{\partial\delta}\right) + \frac{1}{2}k^L - \frac{1}{2}k^R \tag{6.56}$$

so that the derivatives of the κ potentials become

$$\frac{\partial\kappa_a}{\partial v^b} = \frac{\partial^2 k_X}{\partial v^a\partial v^b} = \frac{1}{4\pi^2}\int_0^t d\,\ln(t')\begin{pmatrix} \frac{\partial^2 y}{\partial\sigma^2} & \frac{\partial^2 y}{\partial\sigma\partial\delta} \\ \\ \frac{\partial^2 y}{\partial\sigma\partial\delta} & \frac{\partial^2 y}{\partial\delta^2}\end{pmatrix}_{ab} \tag{6.57}$$

The explicit forms of the second derivatives of the y function, rather convoluted, are listed in appendix A.3. It's clear they have at most asymptotic behaviour $\sim t^{-2/3}$, which will be the same as that of their $\int d\,\ln t'$, so that the κ_a defined above satisfy 6.48 and no addition of a function of the moduli $h(v)$ is necessary.

Then, this allows immediately for the computation of the $\mathcal{G}_{ab}$ matrix:

$$\mathcal{G}_{ab} = -\frac{\partial\,\mathrm{Re}\,\rho_a}{\partial v^b} = -\sum_I \frac{\partial\kappa_a(z_I,\bar{z}_I;v)}{\partial v^b} = -\frac{1}{4\pi^2}\sum_I\int_0^{t_I} d\,\ln t'\frac{\partial^2 y}{\partial v^a\partial v^b}(t';v) \tag{6.58}$$

again resting on the explicit form of the second derivatives of y. The integrals are not solvable in closed form. The matrix will always be invertible and its inverse $\mathcal{G}^{ab}$ is the kinetic matrix for the ρ fields.

The connection $\mathcal{A}^I_{ai}$ instead can be found more explicitly. We treat the $z^3_I = \zeta_I$ and $z^{1,2}_I = y^{1,2}$ cases separately.

$$\mathcal{A}^i_{aI} = \frac{\partial^2 k_0}{\partial\zeta_I \partial v^a} = \frac{\partial^2 f}{\partial\zeta_I \partial v^a} \tag{6.59}$$

but, recalling $t = |\zeta|^2 e^k$, $\frac{\partial f(t)}{\partial\zeta} = \bar{\zeta} e^k f'(t) = \bar{\zeta} e^k y(t)/t = (\bar{\zeta})^{-1} y(t)$ so that this is simply

$$= \bar{\zeta}^{-1} \frac{\partial y}{\partial v^a} \tag{6.60}$$

and

$$\mathcal{A}^3_{1I} = \frac{1}{2\pi} \bar{\zeta}^{-1} \frac{\partial y}{\partial\sigma} \tag{6.61}$$

$$\mathcal{A}^3_{2I} = \frac{1}{2\pi} \bar{\zeta}^{-1} \frac{\partial y}{\partial\delta} \tag{6.62}$$

The $i = 1, 2$ components, instead, are

$$\mathcal{A}^i_{aI} = \frac{\partial^2 k_X}{\partial y^i_I \partial v^a} = \frac{\partial^2 (\alpha^i k^i)}{\partial y^i_I \partial v^a} = \frac{\partial\alpha^i}{\partial v^a} \frac{\partial k^i}{\partial y^i_I} \tag{6.63}$$

(no summation on i is implied), so essentially:

$$\mathcal{A}^1_{1I} = \mathcal{A}^1_{2I} = \frac{1}{4\pi} \frac{\partial k^1}{\partial y^1} \tag{6.64}$$

$$\mathcal{A}^2_{1I} = -\mathcal{A}^2_{2I} = \frac{1}{4\pi} \frac{\partial k^2}{\partial y^2} \tag{6.65}$$

$$\tag{6.66}$$

Appendix A

Appendix

A.1 AdS space

Anti-de Sitter n-space is best understood as the Lorentzian analogue of hyperbolic n-space. It can be built by considering the following locus in the mixed-signature space $\mathbb{R}^{2,n-1}$:

$$x^\mu x_\mu = -(t^1)^2 - (t^2)^2 + \sum_{i=1}^{n-1} (x^i)^2 = -R^2 \tag{A.1}$$

which is reminiscent of the embedding of hyperbolic n-space in $\mathbb{R}^{1,n}$:

$$x^\mu x_\mu = -t^2 + \sum_{i=1}^{n} (x^i)^2 = -R^2 \tag{A.2}$$

Equation A.1 is explicitly preserved by $SO(2, n-1)$, and this group acts transitively on it, so that the locus inherits a Lorentzian metric from the ambient Minkowski space with that same symmetry group. This means the locus is a maximally symmetric space, having the same number of symmetries as $\mathbb{R}^{1,n-1}$ since $\dim SO(2, n-1) = \dim(\mathbb{R}^n \rtimes SO(1, n))$. (To press on with the analogy, in the Riemannian case $\mathbb{H}^n$ has the same number of Killing vectors as $\mathbb{R}^n$ since $\dim SO(1,n) = \dim (\mathbb{R}^n \rtimes SO(n)))$.

The locus has constant negative scalar curvature (using S for the Ricci scalar to avoid confusion with the R radius introduced above):

$$S = -\frac{n(n-1)}{R^2} \tag{A.3}$$

However, the locus built above is not suitable to be used as a spacetime for a reasonable physical theory, as it contains closed timelike curves (CTCs), signaling a pathological causal structure. An example of CTC is the unit circle in the t^1t^2 plane. It's possible however to consider the covering space of the locus, which will be what we will refer to as anti-de Sitter n-space, AdS$_n$. The covering space is again a maximally symmetric space, but it's now simply-connected and CTC-free.

AdS, similarly to dS, admits multiple useful coordinate charts. The Poincaré chart is the analogue of the Poincaré half plane model, and the metric is:

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + dx^\mu dx_\mu\right) \tag{A.4}$$

where $z > 0$, $x^\mu \in \mathbb{R}^{1,n-2}$, and $dx^\mu dx_\mu$ is the standard metric on $\mathbb{R}^{1,n-2}$. The Poincaré chart, unlike the Riemannian case, is not global and only maps a particular wedge of the full AdS. A global chart would be given by the following coordinates, accordingly called global coordinates or cylindrical coordinates:

$$ds^2 = R^2\left(-\cosh^2\chi\, d\tau^2 + d\chi^2 + \sinh^2\chi\, d\Omega^2\right) \tag{A.5}$$

With $d\Omega^2$ the line element on $\mathbb{S}^{n-2}$. Note that constant τ slices are copies of $\mathbb{H}^{n-1}$. Remapping the radial coordinate as $d\chi = d\rho/\cos\rho$ to a finite range $(0 \leq \rho \leq \pi/2)$ this can also be rewritten as

$$ds^2 = R^2 \frac{1}{\cos^2 \rho} \left(-dt^2 + d\rho^2 + \sin^2 \rho d\Omega^2\right) \tag{A.6}$$

A.2 Conformal boundary and symmetries

The last set of coordinates A.6 are a starting point for building the Penrose diagram of AdS. For fixed Ω_i the t,ρ part of the metric is sent to the flat metric by multiplication with the conformal factor $\cos^2 \rho$. AdS is thus represented as an infinite solid cylinder.

We can read the induced topology and metric on the boundary, with the caveat that the conformal factor was arbitrary (provided it was such the metric did not diverge), and thus the boundary's metric will be defined up to a conformal rescaling - we can only identify a natural conformal class for the boundary. This will prove to have physical relevance as possible holographic duals will be conformal.

The topology of the boundary is therefore $\mathbb{S}^{n-2} \times \mathbb{R}$ and a representative of the conformal class is given by setting $\rho = \pi/2$:

$$ds^2 = dt^2 - d\Omega^2 \tag{A.7}$$

which is a Lorentzian metric. The conformal boundary of AdS is itself a spacetime; this is a nontrivial fact which has to be compared with the other constant-curvature manifolds of the same signature: the boundary of Minkowski space $\mathbb{R}^{1,n-1}$ has a vanishing (null) metric, being composed of null past and future, while the positive curvature case, de Sitter, has two spacelike boundaries in the infinite past and future. The relevance of this for the realization of holography should be evident. Only the negative curvature case seems to be able to naturally incorporate a Lorentzian structure on the boundary.

It will be much more useful for the application to holography to consider the boundary in the form it comes out from the Poincaré patch. This is located at $z = 0$ and is only a part of the full boundary. Taking the metric A.4 and factor a conformal z^2 we just obtain

$$ds^2 = x^\mu x_\mu \tag{A.8}$$

that is, the boundary is (locally) Minkowski $(n-2)$-space. This will be our preferential choice of representative metric.

We now turn to the description of the interplay between the bulk's and the boundary's symmetries. Essentially, isometries of AdS will induce conformal transformations on its boundary. As we've seen through its construction, the isometry group of AdS is $SO(2, n-1)$, this also coincides with the conformal group on $\mathbb{R}^{1,n-2}$.

+altre banalità di geometria

A.3 Derivatives of y

We list the explicit derivatives of the $y(t)$ function required for the formulation of the HEFT. We recall y is

$$y(t; \sigma, \delta) = \delta\, C_{1/3}\left(\delta^{-3} T\right) - \frac{\sigma}{2} \tag{A.9}$$

where $T := 4t + \frac{\sigma(3\delta^2 - \sigma^2)}{2}$ and $C_{1/3}(x) = \cosh\left(\frac{1}{3}\cosh^{-1}(x)\right)$. We will make use in the following table of the notation C, C', C'', ... to refer to the zeroth, first, second, ... derivatives of the $C_{1/3}$ function evaluated always at $\delta^{-3}T$.

The first derivatives are:

$$\frac{\partial y}{\partial \sigma} = \delta^{-2}\frac{\partial T}{\partial \sigma}C' - \frac{1}{2} \tag{A.10}$$

$$\frac{\partial y}{\partial \delta} = C + \left(-3\delta^{-3}T + \delta^{-2}\frac{\partial T}{\partial \delta}\right)C' \tag{A.11}$$

And the second derivatives:

$$\frac{\partial^2 y}{\partial \sigma^2} = \delta^{-2}\left(\frac{\partial^2 T}{\partial \sigma^2}C' + \left(\frac{\partial T}{\partial \sigma}\right)^2 \delta^{-3}C''\right) \tag{A.12}$$

$$\frac{\partial^2 y}{\partial \delta \partial \sigma} = \left(-2\delta^{-3}\frac{\partial T}{\partial \sigma} + \delta^{-2}\frac{\partial^2 T}{\partial \sigma \partial \delta}\right)C' + \delta^{-2}\frac{\partial T}{\partial \sigma}\left(-3\delta^{-4}T + \delta^{-3}\frac{\partial T}{\partial \delta}\right)C'' \tag{A.13}$$

$$\frac{\partial^2 y}{\partial \delta^2} = \left(-4\delta^{-3}\frac{\partial T}{\partial \delta} + \delta^{-2}\frac{\partial^2 T}{\partial \delta^2}\right)C' + \delta^{-1}\left(-3\delta^{-3}T + \delta^{-2}\frac{\partial T}{\partial \delta}\right)C'' \tag{A.14}$$

The asymptotic behaviour can be read easily by noting $T \sim t$, $C \sim t^{1/3}$, $C' \sim t^{-2/3}$, $C'' \sim t^{-5/3}$, and that derivatives of T do not depend on t.

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