1. Given the following context-free grammar G, use set notation to define what is L(G)?

1.1) $S \longrightarrow aSB \mid aB$
$B \longrightarrow bb \mid b$
L(G) = {a^x b^y | y ≥ x and y>0, x>0}

1.2) $S \longrightarrow aaSB \mid \varepsilon$
$B \longrightarrow bB \mid b$
L(G) = {a^x b^y | x is multiple of 2 and x ≥ 0, y ≥ 0}

1.3) $S \longrightarrow aSbS \mid aS \mid \varepsilon$
L(G) = {a^x b^y {a,b}* | x ≥ 0, y ≥ 0}

1.4) $S \longrightarrow aS \mid bA \mid \varepsilon$
$A \longrightarrow bA \mid aS \mid \varepsilon$
L(G) = {any string with a and b}

2. For each part below, give the context-free grammar that generates the language specified.

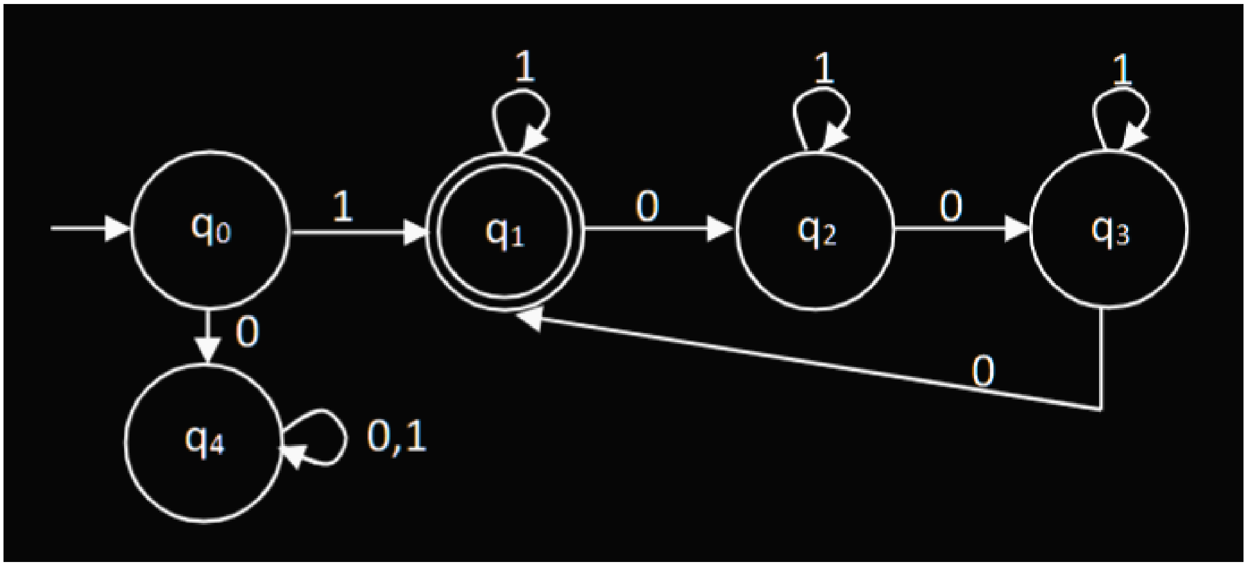
2.1) L1 = {a^n b^m c^i | 0 ≤ n + m ≤ i } and Σ = {a, b, c}
X => aXc | bXc | Yc | ε
Y => Yc | ε

2.2) L2 = {a^m b^n| 0 ≤ n ≤ m ≤ 3n } and Σ = {a, b}
X => aaaXb | aaXb | aXb | ε

2.3) L3 = {a^n b^m c^i | n + m + i is odd } and Σ = {a, b, c}. (Hint: consider the cases where n, m, i are all odd, or two of them are even, one is odd, etc)
S => A | B | C
A => XaYbZc | XbYaZc | XcYbZa
B => XaYbZc | XbYaZc | XcYbZa
C => XaYbZc | XbYaZc | XcYbZa
X => Xa | ε
Y => Yb | ε
Z => Zc | ε

2.4) L4 = {All odd length strings over Σ = {a, b}, *where the string have the same symbol in the first and middle position*}
X => aYa | aYb | bZa | bZb | b | a
Y => aYa | aYb | bYa | bYb | a
Z => aZa | aZb | bZa | bZb | b

3. Convert the DFA below into an equivalent CFG using the procedure discussed in class. Show both your non-simplified and simplified CFGs. What language is described by this CFG?



A = Q0
B = Q1
C = Q2
D = Q3
E = Q4

Unsimplified:

- **A -> 1B | 0E**
- **B -> 1B | 0C | e**
- **C -> 1C | 0D**
- **D -> 1D | 0B**
- **E -> 0E | 1E**

Simplified:

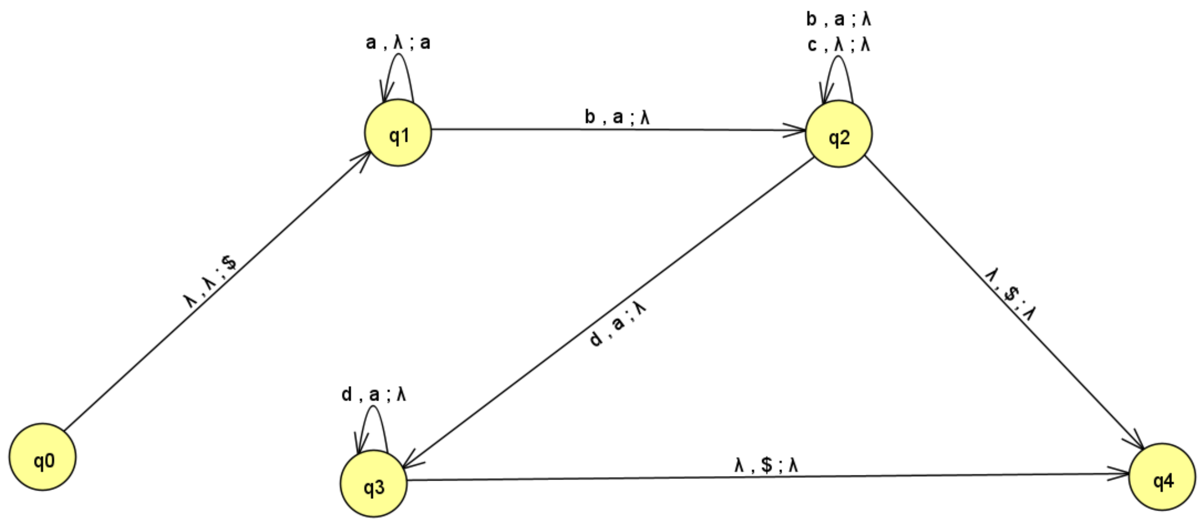
- **A -> 1B | 1**
- **B -> 1B | 0C | 1**
- **C -> 1C | 0D**
- **D -> 1D | 0B | 0**

Set Notation:

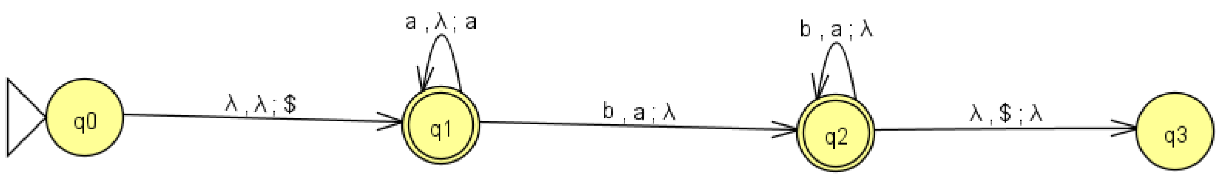
- **L = {x | x starts with the character one and the number of zeros is divisible by three}**

4. For each part below, draw the state diagram of a nondeterministic pushdown automaton that recognizes the language specified. 15 points

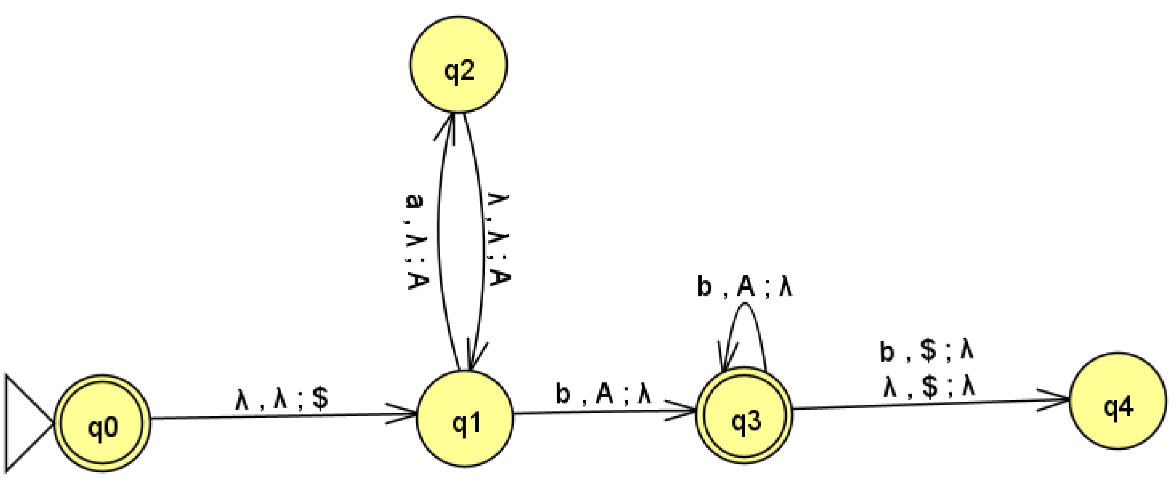
4.1) L5 = {a^i b^j c^k d^l | $l = i - j$ and $i, j, k, l \geq 0$}



4.2) L6 = $\{a^i b^j \mid i \neq j\}$



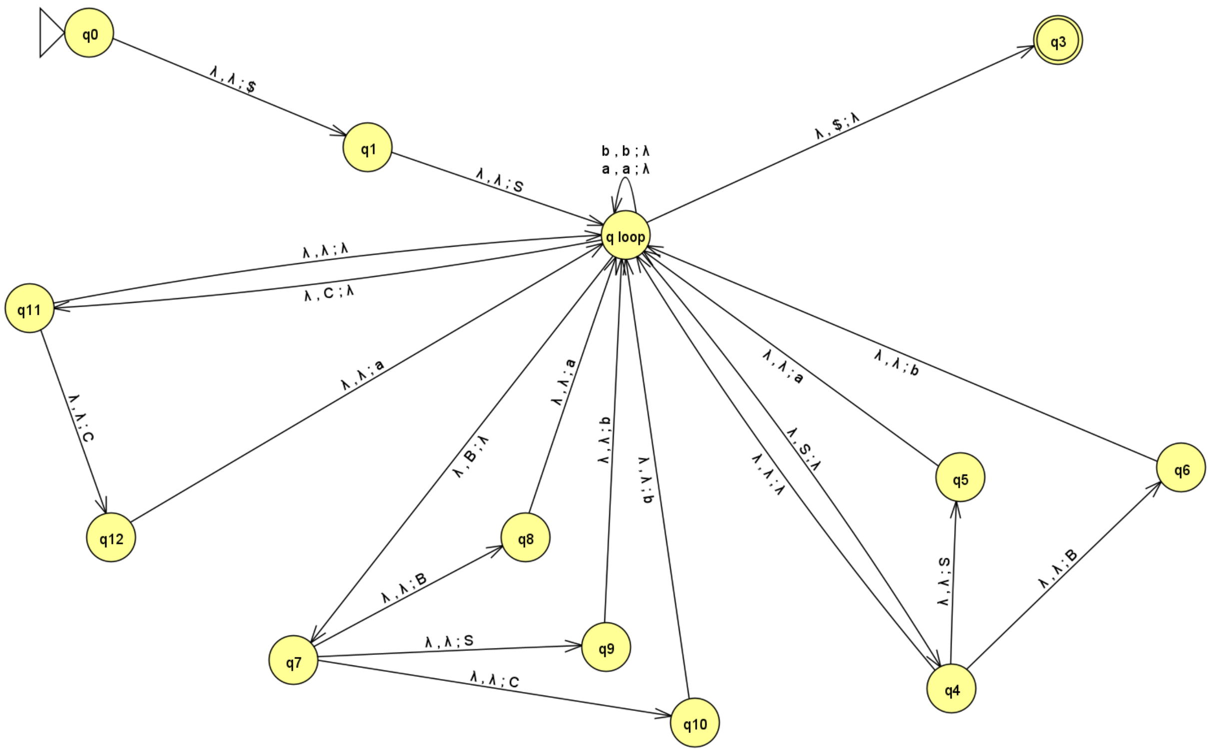
4.3) L7 = $\{a^i b^j \mid 0 \leq i \leq j \leq 2i\}$



5. Convert the following CFG into a nondeterministic PDA using the procedure discussed in class. 10 points

$$S \longrightarrow aS \mid bB \mid \varepsilon$$
$$B \longrightarrow aB \mid bS \mid bC$$

$C \longrightarrow aC \mid \varepsilon$



6. Convert the following context free grammar into Chomsky normal form. Show all required steps to receive full points. 10 points

$S \longrightarrow aTb \mid ab$
$T \longrightarrow ST \mid \varepsilon$

Converting to CNF:

- **Make a new start state**
  - **X -> S**
  - S **->** aTb **|** ab
  - T **->** ST **|** ε
- **Find Nullable Vars and Remove epsilons**
  - **X -> S |** ε
  - S **->** aTb **|** ab
  - T **->** ST **| S**
- **Eliminate unit rules**
  - **X ->** aTb **|** ab **|** ε
  - S **->** aTb **|** ab
  - T **->** ST **|** aTb **|** ab
- **Ensure RHS is all Vars, or Single Terminals**
  - **X -> ATA | AB |** ε
  - S **-> ATB | AB**

- T **-> ST | ATB | AB**
- **A -> a**
- **B -> b**

- **Reduce RHS so only 2 Vars**
  - **X -> JA | AB |** ε
  - S **-> JB | AB**
  - T **-> ST | JB | AB**
  - **A -> a**
  - **B -> b**
  - **J -> AT**

7. For each part, show that the language is context-free, or use the pumping lemma to show that the language is non-context-free. You must justify your answers to receive full credit.

7.1) L8 = $\{\omega x | \omega, x \in \{0, 1\}*$ and ω contains x as substring }

- **In order too show that this language is in fact, context-free, we need to show a context-free grammar that generates it. Now, we can construct a context-free grammar as follows:**
  - **S → 0S | 1S | A**
  - **A → 0A | 1A | xA | ε**

- **In this context-free grammar, S generates the substring ω, and A generates the substring x. The productions for S allow us to construct any combination of 0s, 1s, and x's where ω contains x as a substring. Therefore, we can make the conclusion that L8 is context-free language.**

7.2) L9 = $\{ai\ bj\ ck \mid i \geq j$ and $i \leq k$ and $j \leq k$, where $i, j, k \geq 0\}$

- **In order to determine whether L9 is context-free, we can attempt to apply the pumping lemma for context-free languages. The pumping lemma states that for any context-free language, there exists a constant 'p' that any string in the language with a length of 'p' or more can be separated into three parts, u, v, & w. These three parts satisfy certain conditions. If we can find a string that is in the language that doesn't satisfy these conditions, then the language is not context-free.**

- **Let's assume that L9 is context-free and choose a string from the language, let's say s = a^p b^p c^p. According to the pumping lemma, we can split s into u, v, and w such that:**
  - **uvwxy is in L9.**
  - **|vwx| ≤ p.**
  - **|vx| ≥ 1.**
  - **For all i ≥ 0, u(v^i)w(x^i)y is in L9.**

- **Now, we can consider the conditions:**

- **$s = a^p b^p c^p$ is in L9 because $p \geq p$ and $p \leq p$.**

- **We have $|vwx| \leq p$, which means that v and x together can't contain all three types of characters at the same time because their combined length can be at most 'p'. Because of this, either v and x contain only one type of character or two types of characters. Also, we know that $|vx| \geq 1$. Therefore, it's possible that v and x contain only a's or both a's and b's. But, if we choose i = 2, we will end up with more a's than b's and c's, which violates the condition of L9 which is $i \geq j$ and $i \leq k$ and $j \leq k$, where $i, j, k \geq 0$. For i = 2, $u(v^2)w(x^2)y$ would result in $a^{(p+k)} b^{(p)} c^{(p)}$, where k is the length of v and x. Since $p+k > p$, we have more a's than b's and c's, which doesn't satisfy the conditions of L9. Due to this,we can say that L9 is not context-free. So, L8 is a context-free language and L9 is not a context-free language.**