WIGNER'S CLASSIFICATION OF REPRESENTATIONS OF THE POINCARE GROUP

ABSTRACT

These are a set of notes discussing the seminal work of Wigner which classified all unitary representations of the Poincaré group. We begin discussing the basic background on Lie theory and representation. We assume familiarity with manifold theory and basic representation theory. I have not included proofs of the mathematical theorems to manage the flow of understanding.

LIE GROUPS

Definition: A set G is called a Lie group if G is a smooth manifold endowed with a group structure such that the multiplicat-ion map

$$\mu : G \times G \longrightarrow G$$
$$(g_1, g_2) \longmapsto g_1 \cdot g_2$$

and the inverse map

$$i : G \longrightarrow G$$
$$g \longmapsto g^{-1}$$

are smooth maps between manifolds.

Examples: (1) The real/complex general linear group $GL(n,\mathbb{R}) / GL(n,\mathbb{C})$ is the set of all invertible $n \times n$ matrix with real/complex entries. It is clearly a group. It can be shown that $GL(n,\mathbb{C})$, $GL(n,\mathbb{R})$ are smooth manifolds of real dimension $2n^2$ & n^2 resp.

(2) The real/complex special linear groups $SL(n,\mathbb{R}) / SL(n,\mathbb{C})$ are Lie groups of dimension (real) n^2-1 & $2n^2-1$ respectively.

(3) The orthogonal group $O(n)$ is the set of all $n \times n$ real matri-ces satisfying $A^T A = I$. $O(n)$ is a Lie group of dimension $\frac{n^2-n}{2}$.

Definition (Lie subgroup): A subset H of a Lie group G is called a Lie subgroup if:

(1) H is a subgroup of G. (algebraic subgroup)

(2) H is an immersed submanifold of G via the inclusion map.

(3) the operations on H are smooth.

Let us digress a bit to discuss about submanifolds: Mainly there are three types of submanifolds discussed in books:

(1) Regular submanifolds: Let M be a smooth manifold and $S \subset M$ be a subset of M with subspace topology. S is called a regular submanifold of dimension k if for every point $p \in S$, there is a chart $(U, \phi) = (U, x^1, \dots, x^n)$ around p such that $U \cap S$ is defined by the vanishing of $n-k$ coordinates. By rearranging, we may assume that $x^{k+1}, x^{k+2}, \dots, x^n$ vanish on $U \cap S$.

(2) Immersed submanifold: Recall that a smooth map $f: M \longrightarrow N$ of manifolds is called an immersion at p if the differential at p

$$f_{*,p} : T_p M \longrightarrow T_{f(p)} N$$

of f is injective. f is said to be an immersion if it is an immersion at every point $p \in M$. Now give $f(M) \subset N$, the topology and differentiable structure inherited from f i.e. $f(U) \subset f(M)$ is open in $f(M)$ iff $U \subset M$ is open. One can also check that if $\{(U_\alpha, \phi_\alpha)\}$ is an atlas on M then $\{(f(U_\alpha), \phi_\alpha \circ f^{-1})\}$. We have assumed that f is one-one immersion so that $\phi_\alpha \circ f^{-1}$ makes sense. Then $f: M \to f(M)$ is automatically a diffeomorphism. With this atlast $f(M) \subset N$ is a manifold which we called immersed submanifold.

(3) Embedded submanifold: Recall that $f: M \to N$ is called an embedding if f is smooth, a one-one immersion & $f(M)$ with subspace topology is homeomorphic to M under f. It can be proved that if f is an embedding then $f(M) \subset N$ is a regular submanifold of N. In this case we call $f(M)$ an embedded submanifold of N.

Moreover we can show that if $N \subset M$ is a regular submanifold then the inclusion map $\iota: N \hookrightarrow M$ is an embedding. Now Let us come back to Lie theory. We demand that $H \subset G$ be an immersed submanifold, the reason will be clear when we study Lie algebras. We have the following proposition:

Proposition: Let G be a Lie group & $H < G$ be a subgroup. If H is a regular submanifold of G then H is a Lie subgroup of G.

Remark: Since $\iota : H \hookrightarrow G$ is an embedding if H is a regular submanifold thus, the Lie subgroups as in above proposition are called embedded Lie subgroups.

A particularly simple criteria to check if a given subgroup is a Lie subgroup is the closed subgroup theorem.

Theorem (Closed subgroup theorem): Let $H \subset G$ be a subgroup of the Lie group. If H is closed in the subspace topology in G then H is a Lie subgroup of G.

Using this theorem, we define matrix Lie groups without going to manifold theory.

Definition (Matrix Lie group): A subgroup $G < GL(n, \mathbb{C})$ is called a matrix Lie group if given any sequence $\{A_m\}_{m=1}^{\infty} \subset G$ we have that either

$$\lim_{m \to \infty} A_m = A \in G \quad \text{or} \quad G \notin GL(n, \mathbb{C}).$$

Remark: Here the convergence is component-wise. Moreover the condition on G exactly means that G is closed in $GL(n,\mathbb{C})$ in the subspace topology. Note that $GL(n,\mathbb{C})$ is trivially a matrix Lie group. Since $M(n,\mathbb{C})$ – the set of all $n\times n$ complex matrices is not even a group (inverses of many elements does not exist) so we consider matrix Lie groups to be subgroups of $GL(n,\mathbb{C})$. It turns out that most of the interesting examples of Lie groups are matrix Lie groups but it is not true in general.

Examples: $GL(n,\mathbb{C})$, $GL(n,\mathbb{R})$, $SL(n,\mathbb{C})$, $O(n)$, the special orthogonal group $SO(n)=\{A\in O(n) : \det(A)=1\}$, the unitary group $U(n)=\{A\in GL(n,\mathbb{C}) : A^{\dagger}A=I\}$, $SU(n)=\{A\in U(n) : \det(A)=1\}$, the complex orthogonal groups $O(n,\mathbb{C})=\{A\in GL(n,\mathbb{C}) : A^{T}A=I\}$ & the special complex orthogonal group $SO(n,\mathbb{C})=\{A\in O(n,\mathbb{C}) : \det(A)=1\}$.

We will now define generalised orthogonal group which will be useful for physics application: define the bilinear form, for $x,y\in\mathbb{R}^{n+k}$,

$$[x,y]_{n+k}=\sum_{i=1}^{n}x_iy_i-\sum_{j=n+1}^{k}x_jy_j.$$

Let $O(n,k)$ be the set of matrices $A \in GL(n+k, \mathbb{R})$ which preserve this bilinear form:

$$[Ax, Ay]_{n,k} = [x,y]_{n,k} \qquad \forall\ x,y \in \mathbb{R}^{n+k}.$$

$O(n,k)$ is called the generalised orthogonal group. If we write

$$I_{n,k} = \mathrm{diag}(\underbrace{1,1,\dots,1}_{n}, \underbrace{-1,-1,\dots,-1}_{k})$$

then we have that

$$O(n,k) = \{A \in GL(n+k, \mathbb{R}) : A^T I_{n,k} A = I_{n,k}\}.$$

Lorentz-group: $O(1,3)$ is called the Lorentz group. In this case $I_{1,3} = \eta_{\mu\nu}$ & any Lorentz matrix $\Lambda^\mu{}_\nu$ satisfies

$$\Lambda^T \eta \Lambda = \eta.$$

We will now define the symplectic groups. Let $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ where I is $n \times n$-identity matrix. Then we define

$$Sp(n, \mathbb{R}) = \{A \in GL(2n, \mathbb{R}) : A^T J A = J\}$$

$$Sp(n, \mathbb{C}) = \{A \in GL(2n, \mathbb{C}) : A^T J A = J\}$$

$$Sp(n) = Sp(n, \mathbb{C}) \cap U(2n).$$

It turns out that $\det(A) = 1 \ \forall\ A \in Sp(n, \mathbb{C})$.

We have two more important examples. The Euclidean group $E(n)$ the set of all transformations of $\mathbb{R}^n$ which preserve the distance. It can be shown that

$$E(n) = O(n) \ltimes \mathbb{R}^n \quad (\text{semi-direct product})$$

where the homomorphism $\varphi : O(n) \longrightarrow \mathrm{Aut}(\mathbb{R}^n)$ is given by matrix multiplication.

Aside (semi-direct product): Given two groups G, H, we can construct the (external) semi-direct product with respect to a homomorphism $\varphi : G \to \mathrm{Aut}(H)$ as follows:

The underlying set is $G \times H$ & the operation is

$$\cdot : (G \ltimes_\varphi H) \times (G \ltimes_\varphi H) \longrightarrow G \ltimes_\varphi H$$

$$((g_1, h_1), (g_2, h_2)) \longmapsto (g_1 g_2, h_1 \cdot \varphi(g_1)(h_2)).$$

In case of $E(n)$, $(\mathbb{R}^n, +)$ is a group thus the operation on $E(n)$ is

$$((R_1, x_1), (R_2, x_2)) = (R_1 R_2, x_1 + R_1 x_2).$$

Similarly, the Poincaré group is defined as

$$P(n,1) = O(1,n) \ltimes \mathbb{R}^{1+n}$$

where again the homomorphism $\varphi : O(1,n) \longrightarrow \mathrm{Aut}(\mathbb{R}^{1+n})$ is the same. This concludes the list of our examples.

Definition: A Lie group homomorphism is a smooth map $\phi: G \to H$ which is a group homomorphism.

Note that a Lie group homomorphism lifts to a map of tangent spaces. In particular if e_G & e_H are identity elements then $\phi_{*,e_G}: T_{e_G}G \to T_{e_H}H$ is a linear maps of vector spaces. We will be mostly be dealing with matrix Lie groups. So it is necessary to define matrix exponential. For $X \in M(n, \mathbb{C})$, define

$$e^X = \exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

This infinite series can be shown to converge.

Theorem: Let $X, Y \in M(n, \mathbb{C})$. Then

(i) $e^0 = I$

(ii) $(e^X)^\dagger = e^{X^\dagger}$

(iii) $(e^X)^{-1} = e^{-X}$

(iv) $e^{(\alpha+\beta)X} = e^{\alpha X} e^{\beta X}$ $\forall \alpha, \beta \in \mathbb{C}$.

(v) $e^X e^Y = \left(\exp X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \dots\right)$

where $[X,Y] = XY - YX$

In particular if $[X,Y]=0$ then
$e^{X+Y} = e^X e^Y$. This is called Baker-Campbell-Haussdorff Formula.

(vi) For any $C \in GL(n,\mathbb{C})$,
$$Ce^X C^{-1} = e^{CXC^{-1}}.$$

(vii) $\det(e^X) = e^{\operatorname{tr} X}$. In particular $e^X \in GL(n,\mathbb{C})$ $\forall X \in M(n,\mathbb{C})$.

(viii) $\exp : \mathbb{R} \longrightarrow GL(n,\mathbb{C})$, $t \longmapsto e^{tX}$ is a smooth map of manifolds and
$$\frac{d}{dt} e^{tX} = Xe^{tX}.$$

Exponential gives a nice way to construct curves in Lie groups. In particular for any $A \in GL(n,\mathbb{C})$, the map $c : \mathbb{R} \to GL(n,\mathbb{C})$ defined by $t \longmapsto Ae^{tX}$ is a smooth map with $c(0) = A$ & $c'(0) = AX$. Thus we can easily characterise tangent vectors of a Lie group at a point using these curves.

Note: A Lie group is in particular a topological space. So we can talk about a Lie group being compact, connected, path connected or simply connected.

LIE ALGEBRA

Definition: A Lie algebra is a vector space $\mathfrak{g}$ together with a bracket $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ which is anticommutative bilinear map and satisfies the Jacobi identity.

Definition: A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a subspace which is closed under bracket i.e $[x,y] \in \mathfrak{h}$ $\forall\ x,y \in \mathfrak{h}$. An ideal $\mathfrak{I} \subset \mathfrak{g}$ is a subspace which satisfies $[x,y] \in \mathfrak{I}$ $\forall\ x \in \mathfrak{I}$, $y \in \mathfrak{g}$.

Remark: An ideal is automatically a subalgebra. A Lie algebra is called abelian if $[x,y] = 0$ $\forall\ x,y$. otherwise it is called non-abelian.

Definition: A map $\varphi : \mathfrak{g} \longrightarrow \mathfrak{h}$ of Lie algebra is a linear map such that $\varphi([x,y]) = [\varphi(x), \varphi(y)]$ $\forall\ x,y \in \mathfrak{g}$.

Definition: A Lie algebra is called simple if the only ideals of $\mathfrak{g}$ are $\mathfrak{g}$ and $\{0\}$ and it is nonabelian.

Definition: A Lie algebra is called semi-simple if it is isomorphic to direct sum of simple Lie algebras.

LIE ALGEBRA OF A LIE GROUP

There is a natural "Lie bracket" on the space of vector fields of a Lie group G. Using this bracket, there is a natural bracket operation defined on the tangent space $T_e G$ which turns it into a Lie algebra. One can characterise $T_e G$ using the exponential map when considering matrix Lie group. Recall that if M is a manifold and $X \in T_p M$ then there is a smooth curve $c : (-\epsilon, \epsilon) \longrightarrow M$ such that $c(0) = p$ & $c'(0) = X$. Now we can reverse the definition to define the Lie algebra of a Lie group G. Since $c : \mathbb{R} \longrightarrow GL(n, \mathbb{C})$ defined by $c(t) = e^{tX}$ is a curve in $GL(n, \mathbb{C})$ starting at I, we can define Lie algebra of a matrix Lie group G as follows:

Definition: The Lie algebra $\mathfrak{g}$ of a matrix Lie group G is defined as

$$\mathfrak{g} = \{X \in M(n, \mathbb{C}) : e^{tX} \in G \ \forall \ t \in \mathbb{R}\}$$

In physics, one usually considers the map $t \longmapsto e^{itX}$. So the computations usually differ by a factor of i.

Theorem: Let G be a matrix Lie group. Then the bracket operation on its Lie algebra induced from the Lie-bracket of vector fields is the usual commutator of matrices.

Examples: Calculating Lie algebra directly from definition is usually not difficult.

Lie groups	Lie algebra
$GL(n,\mathbb{C})$	$gl(n,\mathbb{C}) = M(n,\mathbb{C})$
$GL(n,\mathbb{R})$	$gl(n,\mathbb{R}) = M(n,\mathbb{R})$
$SL(n,\mathbb{C})$	$sl(n,\mathbb{C}) = \{A \in M(n,\mathbb{C}) : tr(A) = 0\}$
$SL(n,\mathbb{R})$	$sl(n,\mathbb{R}) = \{A \in M(n,\mathbb{R}) : tr(A) = 0\}$
$U(n)$	$u(n) = \{A \in M(n,\mathbb{C}) : A^{\dagger} = -A\}$
$SU(n)$	$su(n) = \{A \in M(n,\mathbb{C}) : A^{\dagger} = -A,\ tr(A) = 0\}$
$O(n)$	$o(n) = \{A \in M(n,\mathbb{R}) : A^{tr} = A\}$
$SO(n)$	$so(n) = o(n)$
$O(n,k)$	$o(n,k) = \{A \in M(n+k,\mathbb{R}) : I_{n,k} A^{tr} I_{n,k} = -A\}$
$SO(n,k)$	$so(n,k) = o(n,k)$.
$Sp(n,\mathbb{C})$	$sp(n,\mathbb{C}) = \{A \in M(2n,\mathbb{C}) : \Omega A^{tr} \Omega = A\}$
$Sp(n,\mathbb{R})$	$sp(n,\mathbb{R}) = \{A \in M(2n,\mathbb{R}) : \Omega A^{tr} \Omega = A\}$
$Sp(n)$	$sp(n) = sp(n,\mathbb{C}) \cap u(n)$.

where $\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

Theorem: Let $\phi : G \longrightarrow H$ be a Lie group homomorphism then the differential $\phi_{*,e} : T_eG \longrightarrow T_eH$ is a Lie algebra homomorphism. In particular if ϕ is a diffeomorphism then $\phi_{*,e}$ is an isomorphism of vector spaces.

Thus isomorphic Lie groups have isomorphic Lie algebra. The converse may not be true.

BASIC REPRESENTATION THEORY

Definition: Let V be a finite dimensional vector space. Let $GL(V)$ denote the set of all invertible linear maps from V to V. & Let $End(V) =: gl(V)$ denote the space of all linear operators on V. One can see that $GL(V) \cong GL(n,\mathbb{C})$ & $gl(V) \cong M(n,\mathbb{C})$ so that $GL(V)$ can be thought of as a Lie group and $gl(V)$ as a Lie algebra with bracket $[X,Y] = XY - YX$.

Definition: Let V be a finite-dimensional vector space over $\mathbb{F} = \mathbb{C}/\mathbb{R}$.

(i) A representation (rep) (Π, V) of a Lie group G is a Lie group homomorphism $\Pi : G \longrightarrow GL(V)$

(ii) A representation (π, V) of a Lie algebra g is a Lie algebra homomorphism $\pi : g \longrightarrow gl(V)$.

If π or Π is injective the representation is said to be faithful.

Remark: Since $\Pi(g)$, $\pi(x)$ are operators on V, we often say (Π, V) to be a representation acting on V.

Definition: Let (Π, V) be a representation of a Lie group G acting on V. A subspace $W \subset V$ is said to be invariant subspace if $\Pi(g)w \in W$ $\forall\, g \in G$ & $w \in W$. The rep. (Π, V) is said to be irreducible representation (irrep) if only invariant subspace of V are $\{0\}$ and V. Analogous for Lie algebra.

Definition: Let (Π_1, V_1) & (Π_2, V_2) be reps. of a Lie group G. A linear map $\phi : V_1 \to V_2$ is called an intertwiner if

$$\phi(\Pi_1(g)v) = \Pi_2(g)(\phi(v)) \quad \forall\ g \in G \ \&\ v \in V_1.$$

Intertwiner for Lie algebra is defined analogously. If ϕ is an isomorphism of vector spaces then (Π_1, V_1) & (Π_2, V_2) are said to be isomorphic reps.

Given a rep (Π, V) of a Lie group G, the differential at identity will give a rep. of the associated Lie algebra since Lie algebra of $GL(V)$ is $gl(V)$. Infact we have a stronger statement in case of matrix Lie groups.

Theorem: Let (Π, V) be a rep. of a matrix Lie group G. Then there exists a unique rep (π, V) of the associated Lie algebra $\mathfrak{g}$ such that

$$\Pi(e^X) = e^{\pi(X)} \quad \forall\ X \in \mathfrak{g}.$$

Moreover Π is irreducible iff π is irreducible. & if Π_1, Π_2 are any two Lie group reps with corresponding Lie alg. reps π_1, π_2 then $\Pi_1 \cong \Pi_2$ iff $\pi_1 \cong \pi_2$.

Of particular interest to us will be unitary representations.

Definition: Let V be a finite-dimensional inner product space. A rep. (Π, V)

is called a unitary rep. if $\Pi(g)$ is a unitary operator $\forall g \in G$.
We can construct new representations from old using direct sum and tensor product.

Definition: Let (Π_1, V_1), (Π_2, V_2) be reps of a Lie group G then $(\Pi_1 \oplus \Pi_2, V_1 \oplus V_2)$ – called the direct sum and defined by
$$(\Pi_1 \oplus \Pi_2)(g)(v_1, v_2) = (\Pi_1(g) v_1, \Pi_2(g) v_2).$$
Similarly we can define direct sum of reps. of Lie algebra (π_1, V_1) & (π_2, V_2). In contrast, tensor product of reps. is defined differently for Lie group and Lie algebra. We define $(\Pi_1 \otimes \Pi_2, V_1 \otimes V_2)$ & $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$ by
$$(\Pi_1 \otimes \Pi_2)(g)(v_1 \otimes v_2) = \Pi_1(g)(v_1) \otimes \Pi_2(g)(v_2)$$
&
$$(\pi_1 \otimes \pi_2)(g)(v_1 \otimes v_2) = \pi_1(g)(v_1) \otimes v_2 + v_1 \otimes \pi_2(g)(v_2).$$

Definition: A rep. acting on V is said to be unitrizable if there is an inner product on V such that it becomes unitary rep.

Definition: A rep. is said to be completely reducible if it is isomorphic to the direct sum of irrep.

Theorem: (i) Every unitary rep. is completely reducible.
(ii) Let (π, V) be a rep. of a compact Lie group G. Then (π, V) is unitrizable.

Theorem: Let G be a non-compact connected Lie group with the Lie algebra being simple. Then any unitary reps of G is not finite dimensional.

Theorem: The generalised orthogonal group $O(n,k)$ is non-compact with simple Lie algebra for $n+k>2$ & $nk\neq 0$.

This theorem says that Lorentz group is non-compact with simple Lie algebra. Then classifying its unitary reps. is going to be difficult as it is infinite dimensional. Sometimes it is easier to work with the reps. of Lie algebra. The next theorem says that if G is a simply-connected Lie group then it is enough.

Theorem: Let G and H be matrix Lie groups with Lie algebra $\mathfrak{g}$ and $\mathfrak{h}$. Let $\phi: \mathfrak{g}\to\mathfrak{h}$ be a Lie algebra homomorphism. If G is simply-connected then there exists a unique Lie group homomorphism $\Phi: G\longrightarrow H$ such that $\Phi(e^X)=e^{\phi(X)}$. $\forall\ X\in\mathfrak{g}$. In particular if ϕ is an isomorphism then so is Φ.

It turns out that this not a big obstacle. Given any Lie group, one can consider its universal cover which is simply connected. Moreover the Lie algebra of the universal covering group is isomorphic to the Lie algebra of the original Lie group. Let us formalise this a bit.

Definition: Let G be a path connected & locally path connected Lie group. A universal cover of G is a simply connected Lie group H together with a Lie group homomorphism $\Phi: H \to G$ such the associated Lie algebra homomorphism $\phi: \mathfrak{h} \to \mathfrak{g}$ is an isomorphism. The map Φ is called the covering map.

Remark: It can be shown that every connected Lie group admits a universal cover. But it is not true that every connected matrix Lie group has a universal cover which is a matrix Lie group. For example the universal cover of $SL(2,\mathbb{R})$ is the metaplectic group which is not a matrix Lie group. For the next theorem we assume that the matrix Lie group G has universal cover $\tilde{G}$ which is also a matrix Lie group.

Theorem: Let G be a connected matrix Lie group with universal cover $\tilde{G}$. We think of G & $\tilde{G}$ having same Lie algebra $\mathfrak{g}$ (upto isomorphism). If H is a matrix Lie group with Lie algebra $\mathfrak{h}$ and $\phi: \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism then $\exists$ a Lie group homomorphism $\Phi: \tilde{G} \to H$ such that

$$\Phi(e^X) = e^{\phi(X)} \quad \forall \quad X \in \mathfrak{g}.$$

Remark: A theorem guarentees that every subalgebra of a Lie algebra

corresponds to a unique connected Lie subgroup of the correspondi-
-ng Lie group. In particular if V is a finite-dimensional vector space then $gl(V)$ is the Lie algebra of $GL(V)$. If $\pi : g \longrightarrow gl(V)$ is a reps. of Lie algebra g of G then π lifts to a rep $\Pi : \tilde{G} \longrightarrow GL(V)$ of the universal cover $\tilde{G}$. Restricting $\Pi|_G$ gives the corresponding representation of G.

MACKEY MACHINE

We want to classify all unitary irrep. of the Poincare group. As described in previous section, this cannot be done directly as the Poincare group is non-compact and not simply connected. So our strategy will be to pass on to the universal cover & classify all unitary irreps. and come back to Poincare group. Mackey's theory comes in handy at this point. It reduces the problem of describing irreps. of G to the calculation of certain orbits and stabilisers & describe irreps. of stabilisers. This is called the method of induced reps. Let us begin by making some definitions.

Definition: Let G be a Lie group and H a Lie subgroup. Suppose (π, W) is a representation of H. Then the induced representation denoted by $\mathrm{Ind}_G^H(\pi)$ acts on the vector space of functions:

$$\mathrm{Ind}_G^H(W) := \{\phi : G \to W \mid \phi(hg) = \pi(h)\phi(g), \forall h \in H, g \in G, \phi \text{ smooth}\}$$

and G acts on $\mathrm{Ind}_G^H(W)$ as follows:

$$\left(\mathrm{Ind}_G^H(\pi)(g)(\phi)\right)(g') = \phi(g'g).$$

To check that $\mathrm{Ind}_G^H(\pi)(g)(\phi) \in \mathrm{Ind}_G^H(W)$, we need to check two things. Firstly

$$\left(\mathrm{Ind}_G^H(\pi)(g)(\phi)\right)(hg') = \phi(hg'g) = \pi(h)\,\phi(g'g)$$

$$= \pi(h)\left(\mathrm{Ind}_G^H(\pi)(g)(\phi)\right)(g').$$

Clearly $\mathrm{Ind}_G^H(\pi)(g)(\phi)$ is smooth. So we indeed have a representation. Let us now go to semi-direct products. Let $G = H \ltimes_\phi N$ where N is an abelian Lie group. Since we will consider these kinds of semi-direct product, the next theorem is relavant.

Theorem: Let A be an abelian group. Then all irreducible rep. of A are 1-dimensional. Moreover, the irreducible characters of A form a group $\widehat{A} := \mathrm{Hom}(A, \mathbb{C}^*)$ – the group of all group homomorphisms from A to $\mathbb{C}^*$.

Recall that the composition law on $H \ltimes_\phi N$ is

$$(h_1, n_1) \cdot (h_2, n_2) = (h_1 h_2, n_1 \phi(h_1)(n_2)).$$

Usually the homomorphism $\phi : H \longrightarrow \mathrm{Aut}(N)$ is omitted so that we can write $\phi(h)(n) = h(n)$. The inverse of (h, n) is $(h^{-1}, h^{-1}(n^{-1}))$ & identity is (e_H, e_N). We often say that H acts on N via automorphisms.

Lemma: $G = H \ltimes N$ acts naturally on N as $(h, n)(n') = n . h(n')$.

We now turn to unitary irreps. of the semi-direct product. Observe if H acts on N via automorphisms then it induces a natural action of H on $\hat{N}$ as follows:

$$(h \cdot \chi)(n) = \chi(h^{-1}(n)).$$

Let orbit of this action be denoted by O_i with representatives χ_i. Also let $H_i = H_{\chi_i}$ be the stabiliser of χ_i.

Definition: By a section for the H-action on $\hat{N}$ we mean a subset $S \subset \hat{N}$ which intersects every H-orbit in precisely one-point. We shall call a section σ-compact if it is a countable union of compact subsets of $\hat{N}$.

$\chi \in \hat{N}$ can be thought of a rep. of $H_\chi \ltimes N$: $\chi(h,n) = \chi(n)$.

We now have the following important result due to Mackey.

Theorem (Mackey): Let $G = H \ltimes N$ where N is abelian & the action of H on N allows a σ-compact section. Let $\chi \in \hat{N}$ & ξ be a unitary irrep. of H_χ, the stabiliser subgroup of χ. Then the rep. $\mathrm{Ind}_{H_\chi \ltimes N}^{H \ltimes N}(\xi \otimes \chi)$ is a unitary irrep. of G. Moreover every unitary irrep. of G is of this form. Furthermore $\mathrm{Ind}_{H_\chi \ltimes N}^{H \ltimes N}(\xi \otimes \chi) \cong \mathrm{Ind}_{H_{\chi'} \ltimes N}^{H \ltimes N}(\xi' \otimes \chi')$ iff. $\exists\, g \in G$ such that $g \cdot \chi = \chi'$ & $\xi \cong \xi' \circ C_g$ where C_g is the conjugate representation of G on itself.

Mackey's theorem says that to find all unitary irreps of $G = H \ltimes N$, find all orbits of H-action on $\widehat{N}$ & stabiliser of a representative from each orbit, and find all unitary irreps. of the subgroup $H_\chi \ltimes N$ and induce. We will see that this will simplify matters a lot in case of Poincaré group.

UNITARY IRREDUCIBLE REPRESENTATIONS OF THE POINCARE GROUP

As we noted earlier, the Poincaré group is the semi-direct product of generalised orthogonal group and translation group. In physics literature, it is usually denoted by

$$IO(D-1,1) = O(D-1,1) \ltimes \mathbb{R}^{D-1,1}$$

But we will be interested in the connected component of this group which is connected to identity – the space of inhomogenous orthochronous Lorentz transformations:

$$ISO(D-1,1) = SO(D-1,1)_{I} \ltimes \mathbb{R}^{D-1,1}$$

where

$$SO(D-1,1)_{I} = \{\Lambda \in SO(D-1,1) : \Lambda^{0}_{0} \geq 0\}$$

is the component connected to identity. Clearly $\mathbb{R}^{D-1,1}$ is an abelian Lie group, so we are in the setting of Mackey's theorem.

Definition: The set of generators $\{X_i\}$ of a Lie algebra g is the set of elements of g such that the smallest subalgebra containing $\{X_i\}$ is g.

The Lie algebra of $ISO(D-1,1)$ is denoted by $iso(D-1,1)$ &

it can be shown that $iso(D-1,1)$ is generated by $\{M_{\mu\nu}, P_\rho\}$ and the generators satisfy: (μ, ν, ρ run over $0, 1, \dots, D-1$)

$$i[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\sigma\mu} M_{\rho\nu} + \eta_{\sigma\nu} M_{\rho\mu}$$
$$i[P_\mu, M_{\rho\sigma}] = \eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho,$$
$$i[P_\mu, P_\rho] = 0.$$

Two subalgebras of $iso(D-1,1)$ are clearly visible: The Lorentz algebra generated by $\{M_{\mu\nu}\}$ and the subalgebra $\mathbb{R}^D$ generated by $\{P_\mu\}$. The second commutator says that $\mathbb{R}^D$ is an ideal of $iso(D-1,1)$. In fact it can be shown that

$$iso(D-1,1) = so(D-1,1) \oplus \mathbb{R}^D$$

where $\oplus$ is the semi-direct sum which we will not define here & hence will not be used. Let us now consider the Minkowski space.

Definition: By a pseudo-Riemannian manifold (M, β), we mean a smooth manifold M endowed with a pseudo-Riemannian metric β. If $Diff(M)$ is the diffeomorphism group of M then we define the automorphism group of M by

$$Aut(M) = \{\phi \in Diff(M) : \phi^* \beta = \beta\}$$

where ϕ^* is the pullback of ϕ.

Projective representations:

Let $|\Psi\rangle$ be a state in Hilbert state $\mathcal{H}$. Note that any two states $|\Psi\rangle$ & $|\Phi\rangle$ which are non-zero and related by

$$|\Psi\rangle = \lambda|\Phi\rangle \qquad \lambda \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}. \quad \text{——— } (1)$$

are the same quantum mechanical states. So it is pertinent to consider the quotient space of $\mathcal{H}^* = \mathcal{H}\setminus\{0\}$ as

$$\mathbb{P}(\mathcal{H}) := \mathcal{H}^*/\sim \qquad \text{where } |\Psi\rangle \sim |\Phi\rangle \text{ iff } (1) \text{ is true.}$$

$\mathbb{P}(\mathcal{H})$ is called the projectivised Hilbert space.

Recall that the probability amplitute of transition from $|\Psi\rangle$ to $|\Phi\rangle$ is given by

$$p(|\Psi\rangle, |\Phi\rangle) = \frac{\langle\Psi|\Phi\rangle}{\langle\Psi|\Psi\rangle\langle\Phi|\Phi\rangle}$$

In the quotient topology on $\mathbb{P}(\mathcal{H})$, p induces a continuous map on $\mathbb{P}(\mathcal{H})$ (Quotient topology theorem) which we denote by $\tilde{p}$. A homeomorphism $T: \mathbb{P}(\mathcal{H}) \longrightarrow \mathbb{P}(\mathcal{H})$ satisfying

$$\tilde{p}(T[\Psi], T[\Phi]) = \tilde{p}(|\Psi\rangle, |\Phi\rangle)$$

where $[\Psi]$ & $[\Phi]$ are equivalence classes in $\mathbb{P}(\mathcal{H})$, is called a projective automorphism. The set of all such maps denoted by $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$ is a group called projective automorphism group. The action of this group on $\mathbb{P}(\mathcal{H})$ leaves transition probabilities invariant

We have the following theorem concerning the symmetry group of the Minkowski space.

Theorem: Let $\mathbb{R}^{1,D-1}$ be the Minkowski space endowed with the Minkowki metric. Then the automorphism group of $\mathbb{R}^{1,D-1}$ is the Poincarè group.

Now consider a particle in the Minkowski space $\mathbb{R}^{1,D-1}$. The symmetry group of this space is precisely the Poincarè group which we denote by $\mathcal{P}$. Let two observers O and O' related by $\Lambda \in \mathcal{P}$ measure the quantum mechanical particle. In general there measurement result will reveal different states say $[\Psi]$ and $[\Psi']$ respectively. Thus physically one expects that transition probabilities in O & O' be same. This means that the two states must be related by some projective automorphism: $[\Psi] = T_\Lambda[\Psi']$ for some $T_\Lambda \in \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$. If $O = O'$ then $\Lambda = \mathrm{Id}$ & we should have $T_\Lambda = T_{\mathrm{Id}} \doteq \mathrm{Id} \in \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$. Lastly if a third observer O'' related to O' by Γ measures the state then we must impose $T_\Lambda \circ T_\Gamma = T_{\Lambda\Gamma}$. Thus the change of frame induces a representation $\Pi : \mathcal{P} \longrightarrow \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$. This is called the projective representation.

Elementary particles:

The representation $(\Pi, \mathcal{H})$ of the Poincaré group is called irreducible if the only non-trivial closed invariant subspace is $\mathcal{H}$. That is $\Pi(\mathcal{P})(V) \subset V$ for some closed subspace V of $\mathcal{H}$ if and only if $V = \mathcal{H}$. The closed condition is technical: We want the invariant subspace to be a Hilbert space in its own right which is not automatically true in infinite dimensional Hilbert space unless the subspace is closed.

Wigner suggested that the irreducible projective representations of the Poincaré group correspond to elementary particles. with in the quantum system under consideration. Wigner's argument was as follows: An elementary particle in a quantum mechanical system is a vector in $\mathbb{P}(\mathcal{H})$. As discussed, different observers will see different vectors in $\mathbb{P}(\mathcal{H})$ corresponding to the elementary particle. All these vectors must be related by some projective automorphism. The set of all these vectors constitute a $\mathcal{P}$-invariant subspace of $\mathbb{P}(\mathcal{H})$ and hence we obtain a subrepresent-

-ation of $(\Pi, \mathcal{H})$. This subrepresentation can be thought of as a subsystem which is elementary if it is irreducible (otherwise it will have more smaller subsystems). This reduces the problem of determining all relativistic free particles in Minkowski space time to the mathematical task of finding all irreducible projective representations of the Poincarè group.

Using Mackey Machine:

Recall that the Poincaré group is (the relevant one)

$$\mathcal{P} = SO(1, D-1)_I \ltimes \mathbb{R}^{1,D-1}$$

Since $\mathbb{R}^{1,D-1}$ is an abelian group, we are in the setting of Mackey's theorem. The first step to classify all unitary irreps of $\mathcal{P}$ is to go to the universal cover of $\mathcal{P}$ and then calculate the orbits and stabilisers. To make this precise, recall that the double cover of $SO(1,D-1)_I$ was denoted by $Spin(1,D-1)$. Let Φ be the covering map. Then we can consider the semi-direct product

$$Spin(1,D-1) \ltimes \mathbb{R}^{1,D-1}$$

where $Spin(1,D-1)$ acts on $\mathbb{R}^{1,D-1}$ via the covering map i.e for $A \in Spin(1, D-1)$ & $v \in \mathbb{R}^{1,D-1}$

$$A \cdot v := \Phi(A)\, v.$$

So we have got a simply connected & connected semi-direct product to study the unitary irreps. The usefulness of going to the universal cover here lies in the following theorem:

Theorem: Every irreducible unitary representation of $Spin(1, D-1) \ltimes \mathbb{R}^{1,D-1}$ in a complex Hilbert space $\mathcal{H}$ naturally induces an irreducible projective representation of $\mathcal{P}$ in $\mathcal{H}$. This sets up a bijective correspondence between the irreducible projective reps of $\mathcal{P}$ and unitary irreps of $Spin(1, D-1) \ltimes \mathbb{R}^{1,D-1}$.

We stress that this is not true in general. Now the next task is to find orbits & stabilisers. We first reduce this to a simpler problem.

Theorem: The map $T : \mathbb{R}^{1,D-1} \longrightarrow \widehat{\mathbb{R}}^{1,D-1}$ given by $T(v)(x) = e^{i\beta(v,x)}$ where $\beta(v,x) = v \cdot x$ in Minkowski metric, is an isomorphism. Moreover $v \in \mathbb{R}^{1,D-1}$ is $Spin(1, D-1)$-stable iff $\chi = T(v)$ is $Spin(1, D-1)$-stable in the action introduced in previous section.

This theorem says that if $\chi \in \widehat{\mathbb{R}}^{1,D-1}$ then $\exists\, v \in \mathbb{R}^{1,D-1}$ such that $T(v) = \chi$ and that for $A \in Spin(1, D-1)$

$$A \cdot v = \Phi(A) v = v \quad \Leftrightarrow \quad A \cdot \chi = \chi$$

where $(A \cdot \chi)(x) = \chi(A^{-1} \cdot x) = \chi(\Phi(A)^{-1} x)$.

Now since Φ is surjective by definition, the problem reduces to finding the orbits & stabilisers of $SO(1, D-1)_I$ — action on $\mathbb{R}^{1,D-1}$. The computation of orbit is a bit technical, so we just mention the result here for the ease of understanding. The orbits are labelled by $c \in \mathbb{R}$. Put $\beta(v,v) = c$.

(i) $\beta(v,v) = v^2 = m^2 > 0$. The orbit is a one-sheeted hyperboloid and the stabiliser is $Spin(1, D-2) \hookrightarrow Spin(1, D-1)$.

(ii) $\beta(v,v) = v^2 = 0$. The orbit is a cone with vertex at origin — the lightcone. The stability group is the Euclidean group

$$E(D-2) = SO(D-2) \ltimes \mathbb{R}^{D-2}.$$

This is the isometry group of $\mathbb{R}^{D-2}$ (with Euclidean metric).

(iii) $\beta(v,v) = v^2 = -m^2 < 0$. The orbit is a two sheeted hyperboloid corresponding to $v^0 > 0$ or $v^0 < 0$. The stability group is $Spin(D-1)$.

Note that $SO(1, D-1)_I$ also preserves the sign of v^0 if $v^2 \le 0$.

Gender	orbit	Stability group	Unitary irrep.
$v^2 = -m^2$	Mass-shell	$Spin(D-1)$	Massive
$v^2 = m^2$	Hyperboloid	$Spin(1, D-2)$	Tachyonic
$v^2 = 0$	Light cone	$E(D-2)$	Massless
$v = 0$	origin	$Spin(1, D-1)$	zero momentum

Here $Spin(D-1)$ is the compact, connected and simply connected double cover of $SO(D-1)$. We now need to classify all unitary irreps. of the stability groups. For physical applications, Tachyonic reps. are irrelevant. So we only have to find unitary irreps. of the remaining three. Next, for $v=0$, the stability group is $Spin(1, D-1)$ which is simply connected but not compact with simple Lie algebra & hence its unitary irreps. are all infinite dimensional. Hence analysing its irreps. is hard and we shall not pursue it any further. We are left with two cases - the massive & mass-less cases. $Spin(D-1)$ is compact, simply connected, so all its unitary irreps. are finite dimensional. Thus we can classify its unitary irreps. by going to its Lie algebra and using weights and roots. We omit all details of this classifica-

-tion as it deserves another set of discussions altogether. The last remaining case is $v^2 = 0$ whose stability group is $E(D-2) = SO(D-2) \ltimes \mathbb{R}^{D-2}$. We can again use Mackey Machine here to get all unitary irreps. The $SO(D-2)$-action breaks $\mathbb{R}^{D-2}$ into two orbits: (i) A sphere of radius $\mu^2 > 0$ (ii) origin, and the corresponding stability groups are $SO(D-3)$ and $SO(D-2)$ respectively which are compact & hence their representations can be easily studied. The unitary irrep. corresponding to sphere orbit is called "Infinite spin" and that correspon-ding to the origin orbit is called "Helicity." We will again not go into the details.

D = 4 case:

The D=4 case is standard for quantum field theory and Partic-le physics applications. In this case the massive and massless representations are well known:

(i) $v^2 = -m^2 < 0$, the stability group is $Spin(3) \cong SU(2)$. $SU(2)$ is a compact, simply connected Lie group. All its unitary irreps. are labelled by $s \in \frac{1}{2}\mathbb{N} \cup \{0\}$ called "spin" of particle. The representais realised on $\mathbb{C}^{2s+1}$ and hence are $2s+1$ dimensional.

(ii) $\vartheta^2 = 0$. The stability subgroup is $SO(2) \ltimes \mathbb{R}^2$. The orbits of $SO(2)$-action is (i) a circle of radius $r>0$ with stability group $SO(1)$ and (ii) the origin with stability group $SO(2)$. The unitary irreps. of the abelian group $SO(2)$ are all one-dimensional. They are labelled by $n \in \mathbb{Z}$. It is customary to write $n = 2s$ with $s \in \frac{1}{2}\mathbb{Z}$. $|s|$ is called the spin of the representation and the sign of s is called the polarisation.

The mass-squared parameter:

The orbits are labelled by m^2 which has a natural physical interpretation of mass squared. We will investigate this in this section. To do so, we first introduce the universal enveloping algebra and Casimir element.

Definition: An algebra A is a vector space over a field K along with a multiplication map $\mu : A \times A \longrightarrow A$, $(a,b) \mapsto \mu(a,b) = a \cdot b$ satisfying:

$$\left.\begin{array}{ll} \text{(i) (Associativity)}: & a\cdot(b\cdot c) = (a\cdot b)\cdot c \\ \text{(ii) (Distributivity)}: & a\cdot(b+c) = a\cdot b + a\cdot c \end{array}\right\} a, b \in A$$

$$\text{(iii) (Homogeneity)}: \quad r(a\cdot b) = (ra)\cdot b = a\cdot(rb) \quad \} \; r \in K.$$

Due the associativity, we sometimes call A an associative algebra. The universal enveloping algebra is defined by the universal property as below:

Definition/Theorem: Let g be a Lie algebra, then there exists an associative algebra $\mathcal{U}(g)$ with identity together with a linear map $i: g \longrightarrow \mathcal{U}(g)$ such that the following property holds:

(1) $\forall\ X, Y \in g$, we have

$$i([X,Y]) = i(X)i(Y) - i(Y)i(X).$$

(2) $\mathcal{U}(g)$ is generated by the elements $i(X)$ for $X \in g$.

(3) (Universal property): If A is an associative algebra with identity and map $j: g \longrightarrow A$ (linear) satisfying (1) then there exists a unique algebra homomorphism $\phi: \mathcal{U}(g) \longrightarrow A$ with $\phi(1) = 1$ and $\phi(i(X)) = j(X)\ \forall\ X \in g$. A pair $(\mathcal{U}(g), i)$ with the above three properties is called the universal enveloping algebra.

The following proposition will be useful. Let $\mathrm{End}(V)$ denote the vector space of linear operators from V to V. It is easy to show that $\mathrm{End}(V)$ is an associative algebra under composition.

Proposition: If $\pi: g \longrightarrow \mathrm{End}(V)$ is a representation of a Lie algebra g (possibly infinite dimensional), there is a unique algebra homomorphism $\tilde{\pi} : \mathcal{U}(g) \longrightarrow \mathrm{End}(V)$ such that $\tilde{\pi}(1) = I$ and $\tilde{\pi}(X) = \pi(X)$ $\forall$ $X \in g \subset \mathcal{U}(g)$.

A well known theorem (Poincarè - Birkhoff - Witt theorem) says that the map $i : g \longrightarrow \mathcal{U}(g)$ from the universal property is injective. if g is a finite dimensional. Moreover if $X_1, \ldots, X_n$ is a basis of g then the elements of the form

$$i(X_1)^{k_1} \ldots i(X_n)^{k_n}$$

spans $\mathcal{U}(g)$ and is linearly independent.

Definition: Let g be a Lie algebra with the universal enveloping algebra $\mathcal{U}(g)$. The Casimir elements of g are homogenous polynom-ials in the generators of g which commute with all elements of g & hence with all elements of $\mathcal{U}(g)$ & hence lie in the center $Z(\mathcal{U}(g))$ of $\mathcal{U}(g)$.

Theorem: Let g be a Lie algebra. For any irreducible representation of g, the Casimirs of g act as scalars.

For the Lie algebra $so(1, D-1)$ of the Lorentz group, the quadratic Casimir elements are

$$\mathcal{C}_2(so(1,D-1)) = \frac{1}{2} M^{\mu\nu} M_{\mu\nu}$$

where we have summation over μ,ν as follows:

$$M^{\mu\nu}M_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}M^{\mu\nu}M^{\rho\sigma} \qquad \text{(sum over repeated index)}$$

where $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{dia}(-1, \underbrace{1, 1, \ldots 1}_{D-1})$ is the Minkowki metric.

For the Poincarè algebra $iso(1, D-1)$, the quadratic Casimir element is

$$C_2(iso(1, D-1)) = P^\mu P_\mu = P^2.$$

and the quartic Casimir elements are

$$C_4(iso(1, D-1)) = \frac{1}{2}P^2 M_{\mu\nu}M^{\mu\nu} + M_{\mu\rho}P^\rho M^{\mu\sigma}P_\sigma$$

Suppose now that Π is a representation of the Poincarè group on a Hilbert space $\mathcal{H}$. Recall that it induces a representation π of the Lie algebra on $\mathcal{H}$. Moreover recall that

$$\Pi(e^{iX}) = e^{i\pi(X)} \qquad \forall\, X \text{ in the Lie algebra.}$$

(We used the physicits convention of putting an i). In particular for $P^\mu \in iso(1, D-1)$ we have

$$\Pi(e^{iP_\mu})|\psi\rangle = e^{i\pi(P_\mu)}|\psi\rangle = \Pi(T_\mu)|\psi\rangle$$

where T_μ is unit translation. The last equality comes from the fact that P_μ generates translations in spacetime. We now apply Mackey's theorem to the universal cover of the Poincaré group, i.e to $G = Spin(1, D-1) \ltimes \mathbb{R}^{1,D-1}$. Any unitary irrep. of G is of induced type. Thus we can identify $\mathcal{H}$ with the space of continuous functions on G with some transformation property w.r.t the stabiliser subgroup:

$$\mathcal{H} = \mathrm{Ind}_{G_\chi}^{G}(\xi \otimes \chi) = \{ \phi : G \to V_\xi \mid \phi((h,n)g) = \chi(n)\xi(h)\phi(g) ; \phi \text{ is continuous} \}$$

where $G_\chi = Spin(1, D-1)_\chi \ltimes \mathbb{R}^{1,D-1}$ where $Spin(1, D-1)_\chi$ is the stabiliser of χ & (ξ, V_ξ) is a representation of $Spin(1, D-1)_\chi$.

Theorem: Let $\Phi : G \to H$ be a Lie group homomorphism and $\phi : g \to h$ be the corresponding Lie algebra homomorphism. Then

(i) $\Phi(e^X) = e^{\phi(X)} \quad \forall X \in g$.

(ii) $\phi(AXA^{-1}) = \Phi(A)\phi(X)\Phi(A)^{-1} \quad \forall X \in g, A \in G$.

(iii) $\phi([X, Y]) = [\phi(X), \phi(Y)] \quad \forall X, Y \in g$

(iv) $\phi(X) = \frac{d}{dt}\Phi(e^{tX})\Big|_{t=0} \quad \forall X \in g$.

Using the (iv) of previous theorem,

$$(\pi(P_\mu)\phi)(g) = \left[\left(\frac{d}{dt}\,\Pi(e^{tP_\mu})\Big|_{t=0}\right)\phi\right](g)$$

$$= \frac{d}{dt}\left(\phi(g\cdot e^{tP_\mu})\right)\Big|_{t=0}$$

where we used the fact that evaluation of ϕ at g is linear and hence commutes with time derivative. Now by the property of universal enveloping algebra, the representation π lifts to a homomorphism of associative algebras

$$\tilde{\pi} : \mathcal{U}(g) \longrightarrow \mathrm{End}(\mathcal{H}).$$

Then for $X \in g$

$$(\pi(X)\phi)(e) = \frac{d}{dt}\,\phi(e\,\exp(tX))\Big|_{t=0} = \frac{d}{dt}\,\phi(ext(tX)\,e)\Big|_{t=0}$$

$$= \frac{d}{dt}\,\chi(\exp(tX))\,\xi(e)\phi(e)\Big|_{t=0} = \frac{d}{dt}\,\chi(\exp(tX))\,\phi(e)$$

$$= \chi_*(X)\,\phi(e)$$

where χ_* is the Lie algebra homomorphism corresponding to the Lie group homomorphism χ. Using these we have

$$\pi(P^\mu P_\mu) = -\pi(P^0)^2 + \sum_{i=1}^{D-2} \pi(P^i)^2$$

$$(\pi(p^\mu p_\mu)\phi)(e) = \left(-\chi_*(p^0)^2 + \sum_{i=1}^{D-2} \chi_*(p^i)^2\right)\phi(e)$$

But now by the theorem on Casimirs, since $(\pi, \mathcal{H})$ is an irreducible rep, thus $p^\mu p_\mu$ acts as scalar. So that

$$\pi(p^\mu p_\mu)\phi = \lambda\phi$$

So we evaluate it at identity to get λ. Thus above relation gives

$$\lambda = -\chi_*(p^0)^2 + \sum_{i=1}^{D-1} \chi_*(p^i)^2$$

Now recall that for a character χ, $\exists\ v \in \mathbb{R}^{1,D-1}$ such that

$$\chi(\omega) = e^{i\beta(v,\omega)}$$

so $$\chi_*(X) = i\beta(v, X)$$

so that $\chi_*(p^\mu) = i\beta(v, p^\mu)$. Note that $p^\mu \in \mathbb{R}^D$ is just the μth basis vector multiplied by $-i$ (due to i in e^{ip^μ}). Thus $i\beta(v, p^\mu) = v^\mu$. So

$$\lambda = -(v^0)^2 + \sum_{i=1}^{D-1}(v^i)^2 = \beta(v,v) = m^2$$

Thus $\tilde{\pi}(p^\mu p_\mu) = m^2$. In physics literature, $\tilde{\pi}(p^\mu p_\mu)$ is simply denoted by $p^\mu p_\mu$ & we identify

$$p^\mu p_\mu = m^2$$

which we recognise as the total relativistic energy. Thus m is identify with the mass of the particle.