String Theory Notes

$\eta_{\mu\nu} = (-1, 1, 1, \dots, 1)$ – $\mathbb{R}^{1,D-1}$ Minkowski metric.

$\hbar = 1 = c$.

Free relativistic particle:

The most simple action that we can consider is

$$S = -m \int dt \left(1 - \dot{\vec{x}}^2\right)^{1/2}$$

where our frame is $X^\mu = (t, \vec{x})$.

This action gives rise to conjugate momenta (conjugate to $\vec{x}$)

$$\vec{p} = \frac{\delta S}{\delta \dot{\vec{x}}} = \frac{-m\left(2\dot{\vec{x}}\right)}{-2\sqrt{1 - \dot{\vec{x}}^2}} = \frac{m\dot{\vec{x}}}{\sqrt{1 - \dot{\vec{x}}^2}}$$

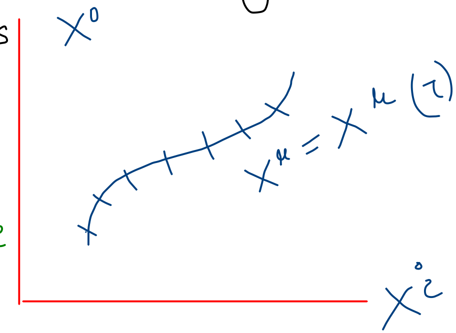
The Hamiltonian of the system is

$$H = \vec{p} \cdot \dot{\vec{x}} - L = \frac{m\dot{\vec{x}} \cdot \dot{\vec{x}}}{\sqrt{1 - \dot{\vec{x}}^2}} + m\sqrt{1 - \dot{\vec{x}}^2}$$

$$= \frac{m\dot{\vec{x}}^2 + m - m\dot{\vec{x}}^2}{\sqrt{1 - \dot{\vec{x}}^2}} = \frac{m}{\sqrt{1 - \dot{\vec{x}}^2}}$$

So the total energy of the system is $E = \dfrac{m}{\sqrt{1 - \dot{\vec{x}}^2}}$

The action is no manifestly Lorentz invariant. We want Lorentz invariant action. One way to do this is to promote t to a variable and parametrize the frame $X^\mu = (X^0, X^i)$ by another parameter τ. Thus we write $X^\mu = X^\mu(\tau)$. It seems that we are adding a new degree of freedom but this is not the case as we will see that our system will have reparametrization invariance also called diffeomorphism invariance.



We can write the action as

$$S = -m \int d\tau \sqrt{-\frac{dx^\mu}{d\tau}\frac{dX_\mu}{d\tau}} \quad \text{as} \quad dt = \frac{dx^0}{d\tau}\, d\tau$$

so that

$$dt\sqrt{1-\dot{x}^2} = d\tau \sqrt{\left(\frac{dx^0}{d\tau}\right)^2 - \left(\frac{dx^i}{dt}\right)^2\left(\frac{dx^0}{d\tau}\right)^2}$$

$$= d\tau \sqrt{-\left[-\left(\frac{dX^0}{d\tau}\right)^2 + \left(\frac{dx^i}{d\tau}\right)^2\right]} = d\tau\sqrt{\frac{-dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}}$$

Now the Poincarè invariance is manifest:

Under $X^\mu \longrightarrow \tilde{X}^\mu = \Lambda^\mu{}_\nu X^\nu + C^\mu$ for some constant vector C^μ

$$\frac{d\tilde{X}^\mu}{d\tau}\frac{d\tilde{X}_\mu}{d\tau} = \frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau} \qquad (\because\ \Lambda\eta\Lambda^T = \eta)$$

We can interpret the action as simply the length of world line traced by the particle. So it is manifestly lorentz invariant

We now have another symmetry of this action. Consider the reparametrization $\tilde{\tau} = \tilde{\tau}(\tau)$

where $\tilde{\tau}$ is a monotonic function of τ. (to avoid self-intersections).

then $d\tau = \frac{d\tau}{d\tilde{\tau}} d\tilde{\tau}$.

Moreover under a reparametrization $\tilde{X}^\mu(\tilde{\tau}(\tau)) = X^\mu(\tau)$

so $\frac{d\tilde{X}^\mu}{d\tilde{\tau}} = \frac{dX^\mu}{d\tau}\frac{d\tau}{d\tilde{\tau}}$.

So the action is
$$\tilde{S} = -m\int d\tilde{\tau}\underbrace{\left|\frac{d\tau}{d\tilde{\tau}}\right|}_{\text{Jacobian of variable change}}\sqrt{-\frac{d\tilde{X}^\mu}{d\tilde{\tau}}\frac{d\tilde{X}_\mu}{d\tilde{\tau}}}\left|\frac{d\tilde{\tau}}{d\tau}\right| = S$$

If we had taken our reparametrization to be increasing then no modulus had appeared/appealed. Since X^0 is a monotonic function ($t \rightarrow$) so we can use diffeomorphism invariance to put $\tilde{\tau} = X^0(\tau) \equiv t$ & we could get back to original action.

Quantisation of the Free relativistic Particle:

We will look at four ways of quantising the free particle.

First method: The action is
$$S = -m\int d\tau\sqrt{-\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}}$$

Momentum conjugate to X^μ is

$$p^\mu = \frac{\delta S}{\delta \dot{X}^\mu} = \frac{m\dot{X}^\mu}{\sqrt{-\dot{X}^\mu \dot{X}_\mu}} \qquad \left\{ \dot{X}^\mu = \frac{dX^\mu}{d\tau} \right.$$

One easily sees that $\boxed{p^\mu p_\mu + m^2 = 0}$

This is a constraint. These type of constraint which follow directly from the definition of conjugate momenta are called primary constraints. They are equal to the number of zero eigenvalues of $\frac{\partial p^\mu}{\partial \dot{X}^\nu} = \frac{\partial^2 \mathcal{L}}{\partial \dot{X}^\nu \partial \dot{X}^\mu}$. Note that by IFT we need that $\frac{\partial p^\mu}{\partial \dot{X}^\nu}$ have all non-zero eigenvalues to express p^μ uniquely in terms of $\dot{X}^\nu$ & X^ν. So in system with primary constants, we cannot express p^μ in terms of $\dot{x}^\nu$ & x^ν.

Note: Any system with "τ"-reparametrization invariance has primary constraints.

The Hamiltonian for the free relativistic particle is

$$H = p^\mu \dot{x}_\mu - L = \frac{m \dot{x}^\mu \dot{x}_\mu}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}} + m\sqrt{-\dot{x}^\nu \dot{x}_\nu}$$

$$= \frac{m \dot{x}^\mu \dot{x}_\mu - m \dot{x}_\nu \dot{x}^\nu}{\sqrt{-\dot{x}^\nu \dot{x}_\nu}} = 0.$$

This is not surprising. This only says that nothing changes as you pick another parametrization. To quantise this system, we promote the constraint $p^\mu p_\mu + m^2 = 0$ to an operator equation to be satisfies by our wave function. The Schrödinger equation is

$$i\frac{\partial \psi}{\partial t} = H\psi = 0 \quad \text{as } H = 0 \Rightarrow \psi = \psi(x^\mu)$$

i.e ψ does not depend on τ. (physically sound).

The constraint becomes (after the standard substitution):

$$p_\mu = -i\frac{\partial}{\partial x^\mu}, \text{ we get } \boxed{(\partial_\mu \partial^\mu - m^2)\psi = 0}$$

We can now solve this equation and get all the dynamics.

Second method: We could have directly quantised our original action as follows: $x = (t, x^i)$

$$S = -m\int dt \sqrt{1 - \dot{\vec{x}}^2} \qquad \dot{\vec{x}}^i = \frac{dx^i}{dt}$$

Conjugate momenta $p^i = \frac{m\dot{x}^i}{\sqrt{1-\dot{x}^{i2}}}$, Hamiltonian is given by

$$H = p^i \dot{x}_i - L = \frac{m\dot{x}^{i2}}{\sqrt{1-\dot{x}^{i2}}} + m\sqrt{1-\dot{x}^{i2}} = \frac{m}{\sqrt{1-\dot{x}^{i2}}} \neq 0$$

Let $\phi(x^0, x^i)$ be the wave function. The schrödinger equation is given by $i\frac{\partial \phi}{\partial t} = H\phi$.

Observe that $p^{i2} = \frac{m^2 \dot{x}^{i2}}{1-\dot{x}^{i2}}$ & $p^{i2} + m^2 = \frac{m^2\dot{x}^{i2}}{1-\dot{x}^{i2}} + m^2 = \frac{m^2}{1-\dot{x}^{i2}}$

so that $H = \sqrt{p^{i2}+m^2}$

Quantisation prescription says $p^i \longrightarrow -i\partial_i$. We get

$$i\frac{\partial\phi}{\partial t} = \sqrt{-\partial_i^2 + m^2}\ \phi$$

Squarring both sides we get $-\frac{\partial^2\phi}{\partial t^2} = (-\partial_i^2 + m^2)\phi$

$$\Rightarrow \boxed{(-\partial^\mu\partial_\mu + m^2)\phi = 0}$$

so we get same dynamics.

Note that in both cases we get equations similar to Klein-Gordan equation. But the interpretation is different. In RQM we quantise wave function & in field theory we quantise scalar fields.

Third method: Introducing Einbein: (Trick to remove $\sqrt{\cdot}$)

Let e be an arbitrary parameter. Write the following action:

$$S = \frac{1}{2}\int d\tau \left(\frac{\dot{x}^\mu \dot{x}_\mu}{e} - e m^2 \right)$$

We claim that this action is equivalent to our original action.

E.O.M of e is: $\frac{\delta S}{\delta e} = 0 \Rightarrow \frac{1}{2}\int d\tau \left(-\frac{\dot{x}^\mu \dot{x}_\mu}{e^2} - m^2 \right) = 0 \Rightarrow$

$e = \sqrt{-\dot{x}_\mu \dot{x}^\mu}/m$ (note that E.O.M of e is algebraic, so e is not a dynamical variable.)

Substituting this in above action, we get

$$S = \frac{1}{2}\int d\tau \left(\frac{m \dot{x}^\mu \dot{x}_\mu}{\sqrt{-\dot{x}^\mu \dot{x}_\mu}} - \frac{m^2}{m}\sqrt{-\dot{x}_\mu \dot{x}^\mu} \right) = \frac{m}{2}\int d\tau \left(-2\sqrt{-\dot{x}^\mu \dot{x}_\mu} \right)$$

$$= -m \int d\tau \sqrt{-\dot{x}^\mu \dot{x}_\mu}$$

So both the actions are really the same. The conjugate momenta for e is

$$P_e = \frac{\partial L}{\partial \dot{e}} = 0$$

Next $p^\mu = \frac{2}{2e}\dot{x}^\mu \Rightarrow \dot{x}^\mu = e p^\mu$. So Hamiltonian of the system is

$$H = \dot{x}^\mu p_\mu - L = e p^\mu p_\mu - \frac{1}{2e} e^2 p^\mu p_\mu + \frac{m^2 e}{2}$$

$$\Rightarrow H = \frac{e}{2} p^\mu p_\mu + \frac{m^2 e}{2} = \frac{e}{2}\left(p^\mu p_\mu + m^2\right)$$

Now $\{P_e, H\}_{P.B} = \frac{\partial P_e}{\partial P_e}\frac{\partial H}{\partial e} = \frac{1}{2}\left(p^\mu p_\mu + m^2\right)$ (only surviving term)

so $p^\mu p_\mu + m^2 = 0$ as $P_e = 0$.

Let the wave function be $\psi(x^\mu, e)$ then $P_e = 0 \Rightarrow -i\frac{\partial}{\partial e}\psi(x^\mu, e) = 0$

$\Rightarrow \psi(x^\mu, e) = \psi(x^\mu)$.

Next $p^\mu p_\mu + m^2 = 0$ gives $(\partial^\mu \partial_\mu - m^2)\psi = 0$

We get the same dynamics.

Fourth Method : We will first check that our action with Einbein has diffeomorphism invariance. Suppose $\tilde{\tau} = \tilde{\tau}(\tau)$ then

$\tilde{X}(\tilde{\tau}(\tau)) = X(\tau)$ and $\tilde{e}(\tilde{\tau}(\tau)) = \sqrt{-\frac{d\tilde{X}^\mu}{d\tilde{\tau}}\frac{d\tilde{X}_\mu}{d\tilde{\tau}}}\left|\frac{d\tilde{\tau}}{d\tau}\right|$

so we get
$$\hat{S} = \frac{1}{2}\int d\tilde{\tau} \underbrace{\left|\frac{d\tau}{d\tilde{\tau}}\right|}_{>0} \left[\frac{d\tilde{X}^\mu}{d\tilde{\tau}}\frac{d\tilde{X}_\mu}{d\tilde{\tau}} \cdot \frac{d\tilde{\tau}}{d\tau}, \frac{1}{\tilde{e}(\tilde{\tau})}\left(\frac{d\tilde{\tau}}{d\tau}\right)^{-1} - m^2\tilde{e}(\tilde{\tau})\frac{d\tilde{\tau}}{de}\right]$$

$$= \frac{1}{2}\int d\tilde{\tau}\left(\frac{\dot{\tilde{X}}^\mu \dot{\tilde{X}}_\mu}{\tilde{e}} - m^2\tilde{e}\right) = S.$$

Thus we can choose $\tilde{\tau} = \tilde{\tau}(\tau)$ such that $\tilde{e}(\tilde{\tau}(\tau)) = 1$.

With this gauge choice, EOM for e becomes:

$$\tilde{e}^2 = -\frac{\dot{\tilde{X}}^\mu \dot{\tilde{X}}_\mu}{m^2} = L \quad \Rightarrow \quad \dot{\tilde{X}}^\mu \dot{\tilde{X}}_\mu + m^2 = 0 .$$

$$\Rightarrow \quad \tilde{p}^\mu \tilde{p}_\mu + m^2 = 0 \quad \text{as} \quad \dot{\tilde{X}}^\mu = \tilde{e}\, p^\mu$$

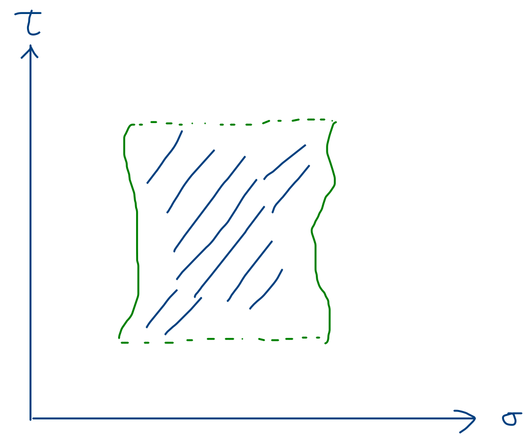
Action now becomes $S = \frac{1}{2}\int d\tau \left(\dot{X}^\mu \dot{X}_\mu - m^2 \right)$ (removed $\sim$)

so $p^\mu = \dot{X}^\mu$ so that the Hamiltonian is

$H = \frac{p^\mu p_\mu + m^2}{2} = 0$ due to the gauge choice.

So we again get same dynamics.

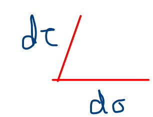
The String Action:



The simplest Lorentz invariant quantity for a string is the area of the surface it traces. So the simplest action would be its area.

Nambu-Goto action:

The surface traced by the string can be parametrized by two parameters (σ, τ). The world-sheet coordinates be $X^\mu(\sigma, \tau)$. To calculate the area of the world sheet, we will use world sheet coordinates.



Infinitesimal change in parameters σ & τ along the worldsheet coordinates is

$$\delta\sigma = \frac{dX^\mu}{d\sigma}\, d\sigma \quad \& \quad \delta\tau = \frac{dX^\mu}{d\tau}\, d\tau$$

The area of parallelogram determined by vectors $\vec{A}$ & $\vec{B}$ is

$$AB\sin\theta = AB\sqrt{1-\cos^2\theta} = \sqrt{A^2B^2 - \frac{(\vec{A}\cdot\vec{B})^2}{A^2B^2}A^2B^2} = \sqrt{(\vec{A}\cdot\vec{A})(\vec{B}\cdot\vec{B}) - (\vec{A}\cdot\vec{B})^2}$$

$$= \left(\det\begin{bmatrix} A\cdot A & A\cdot B \\ A\cdot B & B\cdot B \end{bmatrix}\right)^{1/2}$$

so the infinitesimal area on string world-sheet is

$$dArea = \left(\partial_\alpha X^\mu \partial_\beta X_\mu\right)^{1/2} \qquad \sigma^\alpha \equiv (\sigma, \tau) \qquad \alpha = 1, 2$$

The Nambu-Goto string action is defined as

$$\boxed{S_{NG} = -\frac{1}{2\pi\alpha'}\int_M d\sigma\, d\tau \sqrt{-\det\left(\partial_\alpha X^\mu \partial_\beta X_\mu\right)}}$$

where M is the world-sheet, α' is called Regge slope. The $-$ve sign in the square root is due to the fact that one of the two vectors is time like ($ds^2 < 0$)

We often write $h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu$

so that $S_{NG} = -\frac{1}{2\pi\alpha'}\int_M d\sigma\, d\tau\, \mathcal{L}_{NG}$, $\mathcal{L}_{NG} = -\det(h_{\alpha\beta})^{1/2}$.

The world-sheet is in general a curved manifold embedded in spacetime. In the language of differential geometry, $h_{\alpha\beta}$ is called the Pullback metric from the ambient spacetime.

The factor $-\frac{1}{2\pi\alpha'}$ can be interpreted as string tension. We will see this later. Let us look at the symmetries of S_{NG}.

Symmetries of S_{NG}:

(i) Reparametrization invariance: If we choose another parametrization $\tilde{\tau} = \tilde{\tau}(\sigma,\tau)$ & $\tilde{\sigma} = \tilde{\sigma}(\sigma,\tau)$ then the Jacobian of variable change will be

$$J = \det \begin{bmatrix} \frac{\partial\tilde{\sigma}}{\partial\sigma} & \frac{\partial\tilde{\sigma}}{\partial\tau} \\ \frac{\partial\tilde{\tau}}{\partial\sigma} & \frac{\partial\tilde{\tau}}{\partial\tau} \end{bmatrix}$$

and the world-sheet coordinates change as

$$\frac{\partial X^\mu}{\partial\sigma^\alpha} = \frac{\partial\tilde{X}^\mu}{\partial\tilde{\sigma}^\beta}\frac{\partial\tilde{\sigma}^\beta}{\partial\sigma^\alpha}$$

so

$$h_{\alpha\beta} = \frac{\partial\tilde{X}^\mu}{\partial\tilde{\sigma}^\gamma}\frac{\partial\tilde{X}_\mu}{\partial\tilde{\sigma}^\delta}\frac{\partial\tilde{\sigma}^\gamma}{\partial\sigma^\alpha}\frac{\partial\tilde{\sigma}^\delta}{\partial\sigma^\beta}$$

so $\det(h_{\alpha\beta}) = \det(\tilde{h}_{\alpha\beta})\,J^2$

since $J = \det\left(\frac{\partial\tilde{\sigma}^\alpha}{\partial\sigma^\beta}\right)$. So we get

$$\tilde{S}_{NG} = -\frac{1}{2\pi\alpha'}\int_M d\tilde{\sigma}\,d\tilde{\tau}\,|J|^{-1}\left(-\det(h_{\alpha\beta})\right)^{1/2}\cdot|J| = S_{NG}.$$

This is a Gauge-symmetry of the action.

(ii) Poincare invariance : The transformation $X^\mu \longrightarrow \tilde{X}^\mu = \Lambda^\mu{}_\nu X^\nu + \xi^\mu$ is a manifest symmetry of S_{NG}. This is a global symmetry of the action.

Equations of Motion : Note that expanding out the determinant, we get

$$\det(h_{\alpha\beta}) = \det(\partial_\alpha X^\mu \partial_\beta X_\mu) = X'^2 \dot{X}^2 - (X' \cdot \dot{X})^2$$

where $X' = \frac{\partial X^\mu}{\partial \sigma}$ & $\dot{X} = \frac{\partial X}{\partial \tau}$ & $X^2 = X^\mu X_\mu$.

so $$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\sigma \, d\tau \left[- X'^2 \dot{X}^2 + (X' \cdot \dot{X})^2 \right]^{1/2}$$

where we have written $X \equiv X^\mu$.

The conjugate momenta are

$$\Pi^\tau_\mu = \frac{\partial \mathcal{L}_{NG}}{\partial \dot{X}^\mu} = -\frac{1}{2\pi\alpha'} \left[\frac{(\dot{X} \cdot X') X'_\mu - (X'^2) \dot{X}_\mu}{\sqrt{(X' \cdot \dot{X})^2 - X'^2 \dot{X}^2}} \right]$$

& $$\Pi^\sigma_\mu = \frac{\partial \mathcal{L}_{NG}}{\partial X'^\mu} = -\frac{1}{2\pi\alpha'} \left[\frac{(\dot{X} \cdot X') \dot{X}_\mu - (X'^2) \dot{X}_\mu}{\sqrt{(X' \cdot \dot{X})^2 - (X'^2 \dot{X}^2)}} \right]$$

Observe that $\frac{\partial^2 \mathcal{L}_{NG}}{\partial \dot{X}^\mu \partial \dot{X}^\nu} \cdot \dot{X}^\nu = \frac{\partial \Pi^\tau_\mu}{\partial \dot{X}^\nu} \dot{X}^\nu = 0$

& $$\frac{\partial^2 \mathcal{L}_{NG}}{\partial \dot{X}^\mu \partial \dot{X}^\nu} X'^\nu = \frac{\partial \Pi^\tau_\mu}{\partial \dot{X}^\nu} X'^\nu = 0$$

So the Hessian $\frac{\partial^2 \mathcal{L}_{NG}}{\partial \dot{X}^\mu \partial \dot{X}^\nu}$ has two zero eigenvalues with eigenvectors $\dot{X}^\mu$, X'^μ. So we must have two constraints. We can check that

$$\boxed{\Pi^\tau_\mu X'^\mu = 0\,, \qquad \Pi^\tau_\mu \Pi^{\tau\mu} + \frac{1}{4\pi^2\alpha'^2} X'^\mu X'_\mu = 0}$$

These are one set of constraints. Another set of constraints arise from the fact that

$$\frac{\partial^2 \mathcal{L}_{NG}}{\partial X'^\mu \partial X'^\nu} \dot{X}^\nu = 0 \quad \text{and} \quad \frac{\partial^2 \mathcal{L}_{NG}}{\partial X'^\mu \partial X'^\nu} X'^\nu = 0$$

The resulting constraints are

$$\boxed{\Pi^\sigma_\mu \dot{X}^\mu = 0\,, \qquad \Pi^\sigma_\mu \Pi^{\sigma\mu} + \frac{1}{4\pi^2\alpha'^2} \dot{X}^\mu \dot{X}_\mu = 0}$$

The Hamiltonian,

$$H^\sigma = \Pi^\sigma_\mu X'^\mu - \mathcal{L}_{NG} = 0 \quad \& \quad H^\tau = \Pi^\tau_\mu \dot{X}^\mu - \mathcal{L}_{NG} = 0$$

so the dynamics is determined by constraints.

The equation of motion is given by (Euler-Lagrange equations)

$$\boxed{\frac{\partial \Pi^{\tau}_{\mu}}{\partial \tau} + \frac{\partial \Pi^{\sigma}_{\mu}}{\partial \sigma} = 0}$$

We can also write the equation of motion in another way. Recall that

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int_M d\sigma\, d\tau \sqrt{-h} \qquad h = \det h_{\alpha\beta}.$$

From general relativity $\delta\sqrt{-h} = \frac{1}{2}\sqrt{-h}\, h^{\alpha\beta}\, \delta h_{\alpha\beta}$

so $$\frac{\delta \mathcal{L}_{NG}}{\delta(\partial_\alpha X^\mu)} = -\frac{1}{2\pi\alpha'}\left(\frac{1}{2}\sqrt{-h}\, h^{\alpha\beta}\left(2\,\partial_\beta X_\mu\right)\right)$$

so $$E.O.M = \partial_\alpha\left(\frac{\partial \mathcal{L}_{NG}}{\partial(\partial_\alpha X^\mu)}\right) = 0$$ which gives

$$\boxed{\partial_\alpha\left(\sqrt{-h}\, h^{\alpha\beta}\left(\partial_\beta X_\mu\right)\right) = 0}$$

This is still the same equation.

Polyakov Action:

Introduce another field $g^{\alpha\beta}$ as follows:

$$\boxed{S_P = -\frac{1}{4\pi\alpha'} \int_M d\sigma\, d\tau \sqrt{-g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X_\mu}$$

where $g = \det g_{\alpha\beta}$.

This is called the Polyakov action. The auxillary field $g_{\alpha\beta}$ is a dynamical metric on the world-sheet with Lorentzian signature $(-,+)$. Thus the action S_P can be viewed as a bunch of scalar fields $X^\mu(\sigma,\tau)$ coupled to a 2d gravity theory.

Equivalence of S_P & S_{NG}:

Let us find the EOM of $g_{\alpha\beta}$. Varying S_P w.r.t $g_{\alpha\beta}$ gives two terms. We get

$$\delta S_P = -\frac{1}{4\pi\alpha'}\int d\sigma\, d\tau\left[\sqrt{-g}\,\delta g^{\alpha\beta}\,\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\,\delta g^{\alpha\beta} g^{ab}\partial_a X^\mu \partial_b X_\mu\right]$$

$$\text{so } \delta S_P = 0 \Rightarrow \sqrt{-g}\,\delta g^{\alpha\beta}\left(\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} g_{\alpha\beta}\, g^{ab}\partial_a X^\mu \partial_b X_\mu\right) = 0$$

Here we used $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{\alpha\beta}\,\delta g^{\alpha\beta} = \frac{1}{2}\sqrt{-g}\, g^{\alpha\beta}\,\delta g_{\alpha\beta}$

so the equation of motion of $g_{\alpha\beta}$ is

$$\partial_\alpha X^\mu \partial_\beta X_\mu = \frac{1}{2} g_{\alpha\beta}\, g^{ab}\partial_a X^\mu \partial_b X_\mu$$

or

$$\boxed{\partial_\alpha X^\mu \partial_\beta X_\mu = \frac{1}{2} g_{\alpha\beta}\,\partial^c X^\mu \partial_c X_\mu}$$

Taking determinant both sides we get

$$\det\left(\partial_\alpha X^\mu \partial_\beta X_\mu\right) = \det\left(\frac{1}{2} g_{\alpha\beta}\,\partial^c X^\mu \partial_c X_\mu\right)$$

Since g is 2×2 we get

$$\det(\partial_\alpha X^\mu \partial_\beta X_\mu) = \frac{1}{4}(\partial^c X^\mu \partial_c X_\mu)^2 g.$$

$$\Rightarrow \sqrt{-\det(\partial_\alpha X^\mu \partial_\beta X_\mu)} = \frac{1}{2}\sqrt{-g}\,(\partial^c X^\mu \partial_c X_\mu)$$

Substituting this in S_P gives S_{NG}.

Thus we see that S_P and S_{NG} are equivalent classically. These two actions presumably gives same quatum dynamics but a rigorous proof is lacking. Indeed path-integral quantisation of S_{NG} is rather difficult to perform due to squareroot & manipulating it to get results involves similar tricks as we have used in the $S_{NG} \longrightarrow S_P$ transition.

E.O.M of X^μ:

Directly using Euler-Lagrange equation gives the E.O.M of X^μ:

$$\boxed{\partial_\alpha\left(\sqrt{-g}\, g^{\alpha\beta}\, \partial_\beta X_\mu\right) = 0}$$

This is same as the E.O.M in Nambu-Goto action except that the induced metric is now replaced by the Einbein metric.

Symmetries of S_P: As with S_{NG}, we can directly read of two obvious symmetries of S_P:

(i) **Reparametrization invariance:** If we transform the parameters $\sigma^\alpha \to \tilde{\sigma}^\alpha = \tilde{\sigma}^\alpha(\sigma)$. then the scalar fields X^μ transform as

$$X^\mu(\sigma, \tau) \longrightarrow \tilde{X}^\mu(\tilde{\sigma}^\alpha) = X^\mu(\sigma^\alpha).$$

and the world-sheet metric $g_{\alpha\beta}$ transforms in the usual way

$$g_{\alpha\beta} \to \tilde{g}_{\alpha\beta}(\tilde{\sigma}^\alpha) = \frac{\partial \sigma^\gamma}{\partial \tilde{\sigma}^\alpha} \frac{\partial \sigma^\delta}{\partial \tilde{\sigma}^\beta} g_{\gamma\delta}(\sigma).$$

We can find the infinitesimal transformation under $\sigma^\alpha \longrightarrow \tilde{\sigma}^\alpha = \sigma^\alpha - \eta^\alpha$ where η^α is small using Lie derivative. Indeed under this infinitesimal tranformation

$$\delta g_{\alpha\beta} = \mathcal{L}_\eta g = \nabla_\alpha \eta_\beta + \nabla_\beta \eta_\alpha$$

where ∇_α is the Levi-Civita covariant derivative with the usual Levi civita connection

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}).$$

Also $\sqrt{-g}$ changes as $\delta(\sqrt{-g}) = \partial_\alpha(\eta^\alpha \sqrt{-g})$.

S_P is easily seen to be invariant under reparametrizations.

Poincare invariance: This is a global symmetry.

$$X^\mu \longrightarrow \tilde{X}^\mu = \Lambda^\mu_{\ \nu} X^\nu + \xi^\mu \quad \text{for some constant } \xi^\mu$$

Weyl Invariance: There is another gauge invariance called Weyl symmetry. Under this $X^\mu \longrightarrow X^\mu$ & the metric transforms as

$$g_{\alpha\beta} \longrightarrow \tilde{g}_{\alpha\beta} = \Omega^2(\sigma)\, g_{\alpha\beta}$$

or infinitesimally if $\Omega^2(\sigma) = e^{2\phi(\sigma)}$ then

$$\delta g_{\alpha\beta} = 2\phi(\sigma)\, g_{\alpha\beta}.$$

To see that this is a symmetry of the action, note that $\sqrt{-g} \longrightarrow \Omega^2(\sigma)\sqrt{-g}$ as $\det(\Omega^2 g_{\alpha\beta}) = \Omega^4(\sigma)\det(g_{\alpha\beta})$.
& $g^{\alpha\beta} \longrightarrow (\Omega(\sigma))^{-2} g^{\alpha\beta}$. Thus factors from $\sqrt{-g}$ & $g^{\alpha\beta}$ cancel.

Note: Weyl transformation is not a coordinate transformation. Rather it is a local change of scale under which the theory is invariant. More precisely, this scale change preserves angles between as the metric transforms conformally.

Important note: Weyl transformation is unique to 2-dimensions.
i,e $\sqrt{-g}\, g^{\alpha\beta}$ remain invariant under $g_{\alpha\beta} \longrightarrow \Omega^2 g_{\alpha\beta}$ only in 2-dimensions.

Boundary conditions: Boundary condition on the string depends on whether it is an open string or closed string.

(i) **Open string:** We have parametrized the string to have length 2π.

$$0 \leq \sigma \leq 2\pi, \qquad -\infty < \tau < \infty$$

There are two types of boundary conditions on the ends of open strings: Dirichlet & Neumann. We will study them thoroughly when we study open strings.

(ii) **Closed strings:** The ends of the string join to form a loop in this case. So $0 \leq \sigma \leq 2\pi$ & $-\infty < \tau < \infty$. We must have

$$X(\sigma, \tau) = X(\sigma + 2\pi, \tau) \quad \forall \quad 0 \leq \sigma \leq 2\pi$$

We first discuss closed strings:

Closed strings:

To begin with, we need to fix a gauge. $g_{\alpha\beta}$ has three independent component. We have 2 diffeomorphism invariance (σ, τ). and three independent metric component. Write

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix} \quad \text{then } g_{01} = g_{10}.$$

Now since $g_{\alpha\beta}$ has signature $(-,+)$ at least locally one out of g_{11} & g_{00} is +ve. (Why?). Under diffeomorphisms, we see that

$$g_{\alpha\beta} \longrightarrow \tilde{g}_{\alpha\beta} = \frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^\delta}{\partial\tilde{\sigma}^\beta} g_{\gamma\delta}$$

This gives

$$\tilde{g}_{00} = \left(\frac{\partial\sigma}{\partial\tilde{\sigma}}\right)^2 g_{\sigma\sigma} + \left(\frac{\partial\tau}{\partial\tilde{\sigma}}\right)^2 g_{\tau\tau} + 2\frac{\partial\sigma}{\partial\tilde{\sigma}}\frac{\partial\tau}{\partial\tilde{\sigma}} g_{\sigma\tau} \qquad \left(\tilde{\sigma}^\alpha = (\tilde{\sigma},\tilde{\tau})\right)$$

$$\tilde{g}_{11} = \left(\frac{\partial\sigma}{\partial\tilde{\tau}}\right)^2 g_{\sigma\sigma} + \left(\frac{\partial\tau}{\partial\tilde{\tau}}\right)^2 g_{\tau\tau} + 2\frac{\partial\sigma}{\partial\tilde{\tau}}\frac{\partial\tau}{\partial\tilde{\tau}} g_{\sigma\tau}$$

$$\tilde{g}_{01} = \tilde{g}_{10} = \frac{\partial\sigma}{\partial\tilde{\sigma}}\frac{\partial\sigma}{\partial\tilde{\tau}} g_{\sigma\sigma} + \frac{\partial\tau}{\partial\tilde{\sigma}}\frac{\partial\tau}{\partial\tilde{\tau}} g_{\tau\tau} + \frac{\partial\tau}{\partial\tilde{\sigma}}\frac{\partial\sigma}{\partial\tilde{\tau}} g_{\tau\sigma} + \frac{\partial\sigma}{\partial\tilde{\sigma}}\frac{\partial\tau}{\partial\tilde{\tau}} g_{\sigma\tau}.$$

Now suppose in a nbhd of (σ,τ), $g_{11} > 0$ then we put

$$\tilde{g}_{01} = \tilde{g}_{10} = 0 \quad \&$$

$$\tilde{g}_{00} = -g_{11}.$$

Thus we have a system of two first order P.D.E to solve for two function $\tilde{\sigma}(\sigma,\tau)$ & $\tilde{\tau}(\sigma,\tau)$. ie we need to solve for $\tilde{\sigma}(\sigma,\tau)$ & $\tilde{\tau}(\sigma,\tau)$ from

$$\left(\frac{\partial\sigma}{\partial\tilde{\sigma}}\right)^2 g_{\sigma\sigma} + \left(\frac{\partial\tau}{\partial\tilde{\sigma}}\right)^2 g_{\tau\tau} + 2\frac{\partial\sigma}{\partial\tilde{\sigma}}\frac{\partial\tau}{\partial\tilde{\sigma}} g_{\sigma\tau} = -g_{11}$$

$$\frac{\partial\sigma}{\partial\tilde{\sigma}}\frac{\partial\sigma}{\partial\tilde{\tau}} g_{\sigma\sigma} + \frac{\partial\tau}{\partial\tilde{\sigma}}\frac{\partial\tau}{\partial\tilde{\tau}} g_{\tau\tau} + \frac{\partial\tau}{\partial\tilde{\sigma}}\frac{\partial\sigma}{\partial\tilde{\tau}} g_{\tau\sigma} + \frac{\partial\sigma}{\partial\tilde{\sigma}}\frac{\partial\tau}{\partial\tilde{\tau}} g_{\sigma\tau} = 0$$

Solution to this exists atleast locally (Cauchy - Kowalevski theorem) since the coefficient functions are real analytic.

https://encyclopediaofmath.org/wiki/Cauchy-Kovalevskaya_theorem

Thus we have transformed $g_{\alpha\beta}$ to $g_{11}\eta_{\alpha\beta}$ using the two diffeomorphisms. Since $g_{11} = e^{\phi(\sigma)}$ thus we now use Weyl rescaling to transform $g_{\alpha\beta} \longrightarrow e^{-\phi(\sigma)} g_{\alpha\beta} = \eta_{\alpha\beta}$.

This gauge is called Conformal guage.

Warning: We have been able to do this only locally.

Any 2d metric can be made flat using Weyl invariance:

Suppose $g'_{\alpha\beta} = e^{\phi(\sigma)} g_{\alpha\beta}$

then one can easily check that

$$\sqrt{-g'}\,R' = \sqrt{g}\left(R - \nabla^2\phi\right)$$

If we choose ϕ such that $\nabla^2\phi = R$ then $R' = 0$. But in 2d- vanishing Ricci scalar implies that Riemann curvature tensor is zero. Since in 2d - one can show that

$$R_{\alpha\beta\gamma\delta} = \frac{R}{2}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right).$$

Thus the metric is flat.

Can the world-sheet metric be made flat globally?

Depends on the topology of the space. Locally the metric can be made flat using the three gauge symmetries. Suppose we could extend this locally flat metric to whole world-sheet. This means that the whole-worldsheet is covered by a coordinate chart which is flat. This in turn means that the Ricci scalar identically vanishes on the worldsheet. Topologically since in 2d, the Euler-characteristic χ of a manifold satisfies

$$\chi \propto \int_M R$$

thus a necessary condition of the extension to be possible is that $\chi = 0$.

Equations of motion:

Fixing the Conformal gauge, the action has the form:

$$S_P = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \, \partial_\alpha X^\mu \partial^\alpha X_\mu \qquad \left(\sqrt{-\eta} = 1\right)$$

From here, one readily reads of the EOM for X^μ:

$$\boxed{\partial^\alpha \partial_\alpha X^\mu = 0}$$

Since we have fixed a gauge, we need to impose the gauge-fixed EOM for $g_{\alpha\beta}$ as constraint.

We have already calculated the E.O.M for $g_{\alpha\beta}$. More precisely

$$\frac{\delta S}{\delta g^{\alpha\beta}} = -\frac{1}{4\pi\alpha'}\left[-\frac{\sqrt{-g}}{2} g_{\alpha\beta} \partial_c X^\mu \partial^c X_\mu + \sqrt{-g}\, \partial_\alpha X^\mu \partial_\beta X_\mu\right]$$

Define the stress-energy tensor (changing normalisation)

$$T_{\alpha\beta} = -4\pi\alpha' \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\alpha\beta}}$$

$\left(\text{Usual definition in GR is } T_{\alpha\beta} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g_{\alpha\beta}}\right)$

We get $\quad T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} g_{\alpha\beta} \partial_c X^\mu \partial^c X_\mu$

so $\quad T_{\alpha\beta}\Big|_{g_{\alpha\beta}=\eta_{\alpha\beta}} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} \eta_{\alpha\beta} \partial_c X^\mu \partial^c X_\mu$

E.O.M for $g_{\alpha\beta}$ was $\quad \partial_\alpha X^\mu \partial_\beta X_\mu = \frac{1}{2} g_{\alpha\beta} \partial_c X^\mu \partial^c X_\mu$

so our constraint is $\quad T_{\alpha\beta} = 0$

Written in terms of components:

$$T_{01} = \dot{X}^{\mu} X'_{\mu} = 0 \,, \qquad T_{11} = T_{00} = \dot{X}^2 - \frac{1}{2}\left(-\left(-\dot{X}^2 + X'^2\right)\right)$$

$$= \frac{1}{2}\left(\dot{X}^2 + X'^2\right)$$

So we have to impose two constraints:

$$\boxed{\dot{X}^{\mu} X'_{\mu} = 0 \,, \qquad \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0}$$

So the EOM is a wave equation along with two constraints. We will now solve it.

Solution of EOM : Mode expansions:

We will use the so called light-cone coordinates to solve the system

Put

$$\boxed{\sigma^{\pm} = \tau \pm \sigma}$$

then

$$\boxed{\partial_+ = \partial_\tau + \partial_\sigma \,, \qquad \partial_- = \partial_\tau - \partial_\sigma}$$

With this, the EOM $\partial_\alpha\left(\partial^\alpha X^\mu\right) = 0$ reduces to

$$\boxed{\partial_+ \partial_- X^\mu = 0}$$

Indeed we have

$$\partial_+ \partial_- X^\mu = \partial_+\left(\partial_\tau X^\mu - \partial_\sigma X^\mu\right) = \partial_{\tau\tau} X^\mu - \partial_{\tau\sigma} X^\mu - \partial_{\sigma\tau} X^\mu - \partial_{\sigma\sigma} X^\mu = 0$$

$$\Rightarrow \quad \partial_+\partial_- X^\mu = (\partial_{\tau\tau} - \partial_{\sigma\sigma}) X^\mu = 0$$

i.e $$\partial_\alpha(\partial^\alpha X^\mu) = 0$$

The most general solution $\partial_+\partial_- X^\mu = 0$ is given by

$$X^\mu(\sigma,\tau) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$$

for arbitrary functions X_L & X_R.

For closed strings, we have the periodicity condition

$$X^\mu(\sigma+2\pi,\tau) = X^\mu(\sigma,\tau).$$

Thus X^μ has Fourier expansion of the form

$$X_L^\mu(\sigma^+) = \frac{x^\mu}{2} + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0}' \frac{1}{n} \tilde{\alpha}_n^\mu e^{-in\sigma^+}$$

$$X_R^\mu(\sigma^-) = \frac{x^\mu}{2} + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0}' \frac{1}{n} \alpha_n^\mu e^{-in\sigma^-}$$

X_L^μ are called left movers and X_R^μ are called right movers.

Several remarks are in order:

(1) The factors α', $\frac{1}{n}$ are for convenience when we quantise the system

(2) X_L^μ & X_R^μ are not periodic due to the linear term σ^+, σ^- but the combination $X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$ is periodic as σ cancels from the combination $\sigma^+ + \sigma^- = 2\tau$.

(3) The quantities x^μ & p^μ are the position and momentum of the center of mass of the string. We will prove this explicitly. Observe that for the Polyakov action,

$$\Pi^\tau_\mu = \frac{\partial \mathcal{L}_P}{\partial \dot{X}^\mu} = -\frac{1}{4\pi\alpha'}\frac{\partial}{\partial \dot{X}^\mu}\left[-\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu\right]$$

$$= \frac{1}{2\pi\alpha'}\dot{X}_\mu$$

so $$p^\mu = \int_0^{2\pi} d\sigma \frac{1}{2\pi\alpha'}\dot{X}^\mu = \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\, \dot{X}^\mu_L(\sigma^+) + \dot{X}^\mu_R(\sigma^-)$$

$$= \frac{1}{2\pi\alpha'} 2\pi\alpha' p^\mu = p^\mu$$

& $$q^\mu = \frac{1}{2\pi}\int_0^{2\pi} d\sigma\, X^\mu = \frac{1}{2\pi}\int_0^{2\pi} d\sigma\, X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$$

$$= \frac{1}{2\pi}\left[2\pi x^\mu + 2\pi\alpha' p^\mu \tau\right] = x^\mu + \alpha' p^\mu \tau$$

So we see that p^μ is indeed the momentum and x^μ is the position of C.O.M of the string.

(4) X^μ is real. So $(X^\mu_L)^* = X^\mu_L$ and $(X^\mu_R)^* = X^\mu_R$

This means that the coefficients α^μ_n & $\tilde{\alpha}^\mu_n$ satisfy

$$(\alpha_n^\mu)^* = \alpha_{-n}^\mu \quad \& \quad (\tilde{\alpha}_n^\mu)^* = \alpha_{-n}^\mu \qquad \forall\ n \in \mathbb{Z}\setminus\{0\}.$$

Recall that we had two constraints:

$$\dot{X}^\mu X'_\mu = 0 \quad \text{and} \quad \frac{1}{2}(\dot{X}^2 + X'^2) = 0$$

In Light-cone coordinates, these transform to:

$$\left(\frac{\partial_+ + \partial_-}{2}\right)X^\mu \left(\frac{\partial_+ - \partial_-}{2}\right)X_\mu = 0$$

$$\Rightarrow \quad (\partial_+ X^\mu + \partial_- X^\mu)(\partial_+ X_\mu - \partial_- X_\mu) = 0$$

$$\Rightarrow \quad (\partial_+ X^\mu)^2 - (\partial_- X^\mu)^2 = 0$$

$$\Rightarrow \quad (\partial_+ X^\mu)^2 = (\partial_- X^\mu)^2 \qquad \text{———} \quad (*)$$

Second constraint implies $\left(\left(\frac{\partial_+ + \partial_-}{2}\right)X^\mu\right)^2 + \left(\left(\frac{\partial_+ - \partial_-}{2}\right)X^\mu\right)^2 = 0$

$$\Rightarrow \quad (\partial_+ X^\mu)^2 + (\partial_- X^\mu)^2 + \cancel{2\partial_+ X^\mu \partial_- X_\mu} + (\partial_+ X^\mu)^2 + (\partial_- X^\mu)^2 - \cancel{2\partial_+ X^\mu \partial_- X_\mu} = 0$$

$$\Rightarrow \quad (\partial_+ X^\mu)^2 + (\partial_- X^\mu)^2 = 0 \qquad \text{———} \quad (**)$$

Hence from $(*)$ & $(**)$ we get

$$\boxed{(\partial_+ X^\mu)^2 = 0 = (\partial_- X^\mu)^2}$$

These constraints will give conditions on x^μ and p^μ.
Let us calculate $\partial_- X^\mu$.

$$\partial_- X^\mu = \partial_- X^\mu_R = \frac{\alpha'}{2} p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}^*} \alpha_n^\mu e^{-in\sigma^-} \qquad \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$$

$$= \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu e^{-in\sigma^-}$$

where we have defined $\boxed{\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu}$.
Now the constraint $(\partial_- X^\mu)^2 = 0$ gives

$$\left(\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu e^{-in\sigma^-} \right) \left(\sqrt{\frac{\alpha'}{2}} \sum_{m \in \mathbb{Z}} \alpha_m^\mu e^{-im\sigma^-} \right)$$

$$= \frac{\alpha'}{2} \sum_{n \in \mathbb{Z}}' \left(\sum_{k \in \mathbb{Z}}' (\alpha_k)_\mu (\alpha^\mu_{n-k}) \right) e^{-in\sigma^-} = 0$$

where we used Cauchy-product formula.
We get

$$\alpha \sum_{n \in \mathbb{Z}} L_n e^{-in\sigma^-} = 0 \qquad \text{where we defined}$$

$$\boxed{L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}}{}' \alpha^{\mu}_{n-k}\,\alpha_{k\mu}.}$$

Thus the constraint gives $\boxed{L_n = 0 \quad \forall\ n\in\mathbb{Z}.}$

Similarly we can calculate $\partial_+ X^{\mu} = \partial_+ X^{\mu}_L$

Defining $\boxed{\tilde{\alpha}^{\mu}_0 = \sqrt{\frac{\alpha'}{2}}\, p^{\mu}}$

we see that this constraint implies $\boxed{\tilde{L}_n = 0 \quad \forall\ n\in\mathbb{Z}.}$

where $\boxed{\tilde{L}_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{\alpha}^{\mu}_{n-k}(\tilde{\alpha}_k)_{\mu}}$

The constraints L_0 & $\tilde{L}_0$:

Note that

$$L_0 = \sqrt{\frac{\alpha'}{2}}\sum_{k\in\mathbb{Z}} \alpha^{\mu}_{-k}\alpha_{k\mu} \quad \& \quad \tilde{L}_0 = \sqrt{\frac{\alpha'}{2}}\sum_{k\in\mathbb{Z}}{}' \tilde{\alpha}^{\mu}_{-k}\tilde{\alpha}_{k\mu}.$$

In relativistic mechanics, we know that

$$p^\mu p_\mu = -M^2$$

where M is the rest mass of the particle. Since L_0 & $\tilde{L}_0$ contains $p^\mu p_\mu$ so it has some special features.

Since

$$p^\mu p_\mu = \frac{2}{\alpha'} \alpha_0^\mu \alpha_{0\mu} = \frac{2}{\alpha} \tilde{\alpha}_0^\mu \tilde{\alpha}_{0\mu}$$

we see that the constraints $L_0 = \tilde{L}_0 = 0$ implies

$$\frac{1}{2} \sum_{n \neq 0} \alpha_{-n}^\mu \alpha_{n\mu} - \frac{\alpha'}{4} M^2 = 0$$

&

$$\frac{1}{2} \sum_{n \neq 0}{}' \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_{n\mu} - \frac{\alpha'}{4} M^2 = 0$$

Thus

$$\boxed{M^2 = \frac{4}{\alpha'} \sum_{n>0}{}' \alpha_{-n}^\mu \alpha_{n\mu} = \frac{4}{\alpha'} \sum_{n>0}{}' \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_{n\mu}}$$

where we used the fact that $\sum_{n\neq 0}{}' \alpha_{-n}^\mu \alpha_{n\mu} = 2 \sum_{n>0}{}' \alpha_{-n}^\mu \alpha_{n\mu}$

Thus the constaint implies that we have two expression for mass in terms of α_n^μ & $\tilde{\alpha}_n^\mu$. This is called Level matching.

Quantisation of string :

There are two ways to quantise the strings. One is the cannonical quantisation using Dirac's prescription. The other is Feynman's path integral quantisation. The cannonical quantisation procedure involves two ways as we are dealing with a gauge theory :

(i) Covariant quantisation : Change cannonical Poisson brackets to commutator and impose the constraint obtained by fixing a gauge as an operator equation to be satisfied by the states X^μ which are now operators. This method is manifestly Lorentz invariant but gives rise to -ve-norm states called ghosts. These decouple from the theory in the critical dimension $D=26$.

(ii) Lightcone quantisation : In this method we first solve the constraint to classify all classically distinct states and then we quantise the physical states. We break Lorentz invariance in the process and later obtain the same critical dimension $D=26$ to ensure Lorentz invariance.

We will look at both of these quantisation scheme. As stated, we will do all analysis for closed string and then deal with open strings seperately.

Covariant Quantisation:

We have D scalar fields X^μ, $\mu = 0, 1, \ldots, D-1$ and two constraint-s.

$$\dot{X}^\mu X'_\mu = 0 \quad \text{and} \quad \dot{X}^2 + X'^2 = 0.$$

Let us begin by calculating the classical Poisson brackets:

(i) Equal τ- Poisson bracket $\boxed{\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\}_{P.B.} = 0}$

For Polyakov action, we have $\Pi^\tau_\mu \sim \dot{X}_\mu$. We will use Π_μ for Π^τ_μ everywhere unless stated explicitly. Thus this P.B. is obvious.

(ii) Equal τ- Poisson bracket $\boxed{\{\Pi^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)\}_{P.B.} = 0}$

Obvious from the fact that $\Pi_\mu \sim \dot{X}_\mu$.

(iii) Equal τ- Poisson bracket $\boxed{\{X^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)\}_{P.B.} = \eta^{\mu\nu}\delta(\sigma-\sigma')}$

By definition,

$$\begin{aligned}
\{X^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)\}_{P.B.} &= \eta^{\rho\sigma} \frac{\partial X^\mu(\sigma,\tau)}{\partial X^\rho(\sigma,\tau)} \frac{\partial \Pi^\nu(\sigma',\tau)}{\partial \Pi^\sigma(\sigma,\tau)} \\
&= \eta^{\rho\sigma} \delta^\mu_\rho \delta^\nu_\sigma \, \delta(\sigma-\sigma') \\
&= \eta^{\mu\nu}\delta(\sigma-\sigma').
\end{aligned}$$

From these Poisson brackets, we can easily calculate the Poisson brackets for x^μ, p^μ, α^μ_n, $\tilde{\alpha}^\mu_n$. Indeed we have

$$\boxed{\{x^\mu, p^\nu\}_{P.B.} = \eta^{\mu\nu}, \quad \{\tilde{\alpha}^\mu_m, \alpha^\nu_n\}_{P.B.} = 0}$$

and $\boxed{\{\alpha^\mu_m, \alpha^\nu_n\}_{P.B.} = \{\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n\} = -im\,\eta^{\mu\nu}\,\delta_{m+n,0}}$

Using these, Poisson brackets, we can get a algebra satisfied by the Virasoro generators. We have

$$\{L_n, L_m\}_{P.B.} = \Big\{ {\sum_{l\in\mathbb{Z}}}' \alpha_{n-l}\cdot\alpha_l \, , \, {\sum_{k\in\mathbb{Z}}}' \alpha_{m-k}\cdot\alpha_k \Big\}_{P.B.}$$

Here $\alpha_{n-l}\cdot\alpha_l = \alpha^\mu_{n-l}\,\alpha_{l\mu}$. So we have

$$\{L_n, L_m\}_{P.B} = {\sum_{l,k\in\mathbb{Z}}}' \left\{ \eta_{\mu\nu}\alpha^\mu_{n-l}\,\alpha^\nu_l \, , \, \eta_{\rho\sigma}\,\alpha^\rho_{m-k}\,\alpha^\sigma_k \right\}_{P.B.}$$

Using $\{AB, CD\}_{P.B.} = \{A, CD\}_{P.B.} B + A\{B, CD\}_{P.B.}$

$$= C\{A, D\}_{P.B.} B + \{A, C\}_{P.B} DB + AC\{B, D\}_{P.B.} + A\{B, C\}_{P.B.} D$$

we get

$$\{L_n, L_m\}_{P.B.} = {\sum_{l,m\in\mathbb{Z}}}' \eta_{\mu\nu}\eta_{\rho\sigma}\Big[\alpha^\rho_{m-k}\{\alpha^\mu_{n-l}, \alpha^\sigma_k\}_{P.B}\,\alpha^\nu_l + \{\alpha^\mu_{n-l}, \alpha^\rho_{m-k}\}_{P.B}\,\alpha^\nu_l\,\alpha^\sigma_k + \alpha^\mu_{n-l}\,\alpha^\rho_{m-k}\{\alpha^\nu_l, \alpha^\sigma_k\}_{P.B} + \alpha^\mu_{n-l}\{\alpha^\nu_l, \alpha^\rho_{m-k}\}_{P.B}\,\alpha^\sigma_k\Big]$$

$$= \sum_{l,k\in\mathbb{Z}} \eta_{\mu\nu}\eta_{\rho\sigma}\Big[-\alpha^{\rho}_{m-k}\,\eta^{\mu\sigma}\, i(n-l)\,\delta_{n-l+k,0}\,\alpha^{\nu}_{l} - i(n-l)\,\eta^{\mu\rho}\,\delta_{n-l+m-k,0}\,\alpha^{\nu}_{l}\alpha^{\sigma}_{k} - i l\,\delta_{l+k,0}\,\eta^{\nu\sigma}\alpha^{\mu}_{n-l}\alpha^{\rho}_{m-k} - i l\,\eta^{\nu\rho}\delta_{l+m-k,0}\,\alpha^{\mu}_{n-l}\,\alpha^{\sigma}_{k}\Big]$$

$$= -i\sum_{k\in\mathbb{Z}}\Big[\eta_{\nu\rho}\,\alpha^{\rho}_{m-k}\alpha^{\nu}_{n+k}(-k) + \eta_{\nu\sigma}\,\alpha^{\nu}_{n+m-k}\,\alpha^{\sigma}_{k}(k-m) + \eta_{\mu\rho}\,\alpha^{\mu}_{n+k}\,\alpha^{\rho}_{m-k}(-k) + \eta_{\mu\sigma}(k-m)\,\alpha^{\mu}_{n+m-k}\,\alpha^{\sigma}_{k}\Big]$$

Replacing $m-k$ by k in first & k by $k-n$ in third sum we get

$$= -i\sum_{k\in\mathbb{Z}}\Big[\eta_{\nu\rho}\,\cancel{\alpha^{\nu}_{n+m-k}\,\alpha^{\rho}_{k}}(m-k) + \eta_{\nu\sigma}\,\alpha^{\nu}_{n+m-k}\,\alpha^{\sigma}_{k}(k-m) + \eta_{\mu\rho}\,\alpha^{\mu}_{k}\,\alpha^{\rho}_{n+m-k}(n-k) + \eta_{\mu\rho}\,(k-m)\,\cancel{\alpha^{\mu}_{n+m-k}}\,\alpha^{\sigma}_{k}\Big]$$

$$= -i\sum_{k\in\mathbb{Z}}\eta_{\mu\nu}\,\alpha^{\mu}_{n+m-k}\,\alpha^{\nu}_{k}\,(n-m) = i(m-n)\,L_{m+n}.$$

So we have $\{L_n, L_m\}_{P.B.} = i(m-n)\,L_{m+n}.$

Similarly we have:

$$\{\tilde{L}_n, \tilde{L}_m\}_{P.B.} = i(m-n)\,\tilde{L}_{n+m}, \qquad \{\tilde{L}_n, L_m\}_{P.B.} = 0$$

This algebra is called the Virasoro algebra in physics and the Witt algebra in mathematics.

Following the usual way, promote the scalar fields X^μ to operator valued fields and impose the cannonical commutation relation following the rule:

$$\{\ ,\ \}_{P.B.} = \frac{1}{i}[\ ,\ ]$$

We get the following commutation relations:

$$[X^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)] = i\eta^{\mu\nu}\delta(\sigma-\sigma')$$

$$[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = 0 = [\Pi^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)]$$

For the Fourier modes, we get the commutation relations:

$$[x^\mu, p^\nu] = i\eta^{\mu\nu},$$

$$[\alpha^\mu_n, \alpha^\nu_m] = m\eta^{\mu\nu}\delta_{m+n,0} = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n]$$

and all others 0. These commutation relations are similar to those of creation and annihilation operator. Indeed, if we define

$$a^\mu_n = \frac{1}{\sqrt{n}}\alpha^\mu_n, \quad (a^\mu_n)^\dagger = \frac{1}{\sqrt{n}}\alpha^\mu_{-n}, \quad n>0$$

Then we have the usual commutation relation:

$$[a^\mu_n, (a^\mu_m)^\dagger] = i\delta_{nm}$$

Similarly, we can put

$$\tilde{a}^\mu_n = \frac{1}{\sqrt{n}} \tilde{\alpha}^\mu_n \,, \qquad (\tilde{a}^\mu_n)^\dagger = \tilde{\alpha}^\mu_{-n} \,, \qquad n > 0$$

Then we will have

$$[\tilde{a}^\mu_n, (\tilde{a}^\mu_m)^\dagger] = i\delta_{nm}$$

So for every scalar field X^μ, $\mu = 0, 1, \dots, D-1$ we have two family of creation and annihilation operators corresponding to the left movers and the right movers.

Ground state of the theory:

We now have the creation and annilation operators to define the vacuum of the theory. Denote it by $|0\rangle$. Then we demand:

$$a^\mu_n |0\rangle = 0 = \tilde{a}^\mu_n |0\rangle \quad \text{for} \quad n > 0.$$

Note that this condition alone does not uniquely fix the ground state. This is because, the ground state here is quite different from the one in field theory in the sense that there is a string specified by the center of mass position x^μ and momentum p^μ. So the ground state is a tensor of the vacuum $|0\rangle$ with the wavefunction $\Psi(p)$ of the string. So we denote the ground state by $|0;p^\mu\rangle$. The state $|0;p^\mu\rangle$ now has the property that

$$\hat{p}^\mu|0;p^\mu\rangle = p^\mu|0;p^\mu\rangle$$

where p^μ is the momentum of the string.
A general excitation of the string is

$$(\alpha_{-1}^{\mu_1})^{n_{\mu_1}}(\alpha_{-2}^{\mu_2})^{n_{\mu_2}}\cdots(\tilde{\alpha}_{-1}^{\nu_1})^{n_{\nu_1}}(\tilde{\alpha}_{-2}^{\nu_2})^{n_{\nu_2}}\;|0;p^\mu\rangle$$

Each excited state has interpretation of a particle. Hence we have infinitely many species of particles in this theory.

Ghosts:

Since $\eta^{00}=-1$, we have

$$[\alpha_n^0, \alpha_{-n}^0] = [\alpha_n^0, (\alpha_{-n}^0)^\dagger] = -n \quad \text{and}$$

$$[\tilde{\alpha}_n^0, \tilde{\alpha}_{-n}^0] = [\tilde{\alpha}_n^0, (\tilde{\alpha}_{-n}^0)^\dagger] = -n$$

Consider states of the form $|\psi\rangle = \alpha_{-m}^0|0;p^\mu\rangle$ for $m>0$.

For these states we have

$$\begin{aligned}\langle\psi|\psi\rangle &= \langle p^\mu;0|(\alpha^0_{-m})^\dagger \alpha^0_{-m}|0;p^\mu\rangle \\ &= \langle p^\mu;0|\alpha^0_m \alpha^0_{-m}|0;p^\mu\rangle \\ &= \langle p^\mu;0|-m+\alpha^0_{-m}\alpha^0_m|0;p^\mu\rangle \\ &= -m\langle p^\mu;0|0;p^\mu\rangle + \langle p^\mu;0|(\alpha^0_{-m})^\dagger \alpha^0_m|0;p^\mu\rangle \\ &= -m<0.\end{aligned}$$

Such states $|\psi\rangle$ with -ve norm are called Ghosts (different from Fadeev-Popov Ghosts). These states are problematic because these are in contradiction to the probabilistic interpretation of norm in Quantum mechanics. Our only hope is to apply the constraints and hope that these ghosts decouple from our theory. That is indeed the case as we will see now.

Imposing the Constraints:

As discussed previously, the constraints in terms of Fourier modes are:

$$L_n = \sum_{k\in\mathbb{Z}} \alpha^\mu_{k-n}\alpha_{\mu k} = 0 \ , \qquad \tilde{L}_n = \sum_{k\in\mathbb{Z}} \tilde{\alpha}^\mu_{n-k}\tilde{\alpha}_{\mu k} = 0 \qquad \forall\, n\in\mathbb{Z}.$$

These constraints are not imposed directly on the Hilbert space.

Rather we require that the matrix elements of the operators $L_n, \tilde{L}_n$ with respect to the physical states of the theory vanish. That is (This will be clear when we calculate the algebra of $\hat{L}_n$)

$$\langle phys'|L_n|phys\rangle = 0, \quad \langle phys'|\tilde{L}_n|phys\rangle = 0 \quad \forall\, n \in \mathbb{Z}.$$

One can easily check that $L_n^\dagger = L_{-n}$ & $\hat{L}_n = \tilde{L}^\dagger_{-n}$. Indeed we have

$$L_n^\dagger = \sum_{k\in\mathbb{Z}} (\alpha^\mu_{n-k}\,\alpha_{\mu k})^\dagger = \sum_{k\in\mathbb{Z}} \alpha_{-\mu k}\,\alpha^\mu_{k-n} = L_{-n}$$

where we got the last equality by replacing $k-n$ by $-(n+k)$. Similarly we can prove $\tilde{L}_n^\dagger = \tilde{L}_{-n}$.

$$L_n|phys\rangle = 0 \quad \& \quad \tilde{L}_n|phys\rangle = 0 \quad \forall\, n>0$$

But these does not include L_0 & $\tilde{L}_0$. There is an inherent problem in imposing the constraint $L_0, \tilde{L}_0$ in quantum theory. For $n\neq 0$, note that $[\alpha^\mu_{n-k}, \alpha_{\mu k}] = 0$ & $[\tilde{\alpha}^\mu_{n-k}, \tilde{\alpha}_{\mu k}] = 0$. So it does not matter in which order we put $\alpha^\mu_{n-k}\cdot\alpha_{\mu k}$ in the definition of L_n & similarly in $\hat{L}_n$ for $n\neq 0$. But for $n=0$, there is an ambiguity in the ordering of $\alpha^\mu_{-k}\,\alpha_{\mu k}$ as these do not commute. Changing the order of these operators gives extra constants in the quantum theory.

Let us look at this more carefully. Let us choose a particular normal ordering. We put annihilation operators α^μ_n $n>0$ to the right of creation operator α^μ_n $n<0$. With this choice, we have

$$:L_0: = \frac{1}{2}\sum_{k\in\mathbb{Z}}{}' :\alpha^\mu_{-k}\,\alpha_{\mu k}: = \sum_{k=1}^{\infty}{}' \alpha^\mu_{-k}\,\alpha_{\mu k} + \frac{1}{2}\alpha_0^2$$

$$\& \quad :\tilde{L}_0: = \sum_{k=1}^{\infty}{}' \tilde{\alpha}^\mu_{-k}\,\tilde{\alpha}_{\mu k} + \frac{1}{2}\tilde{\alpha}_0^2 \qquad :\;: \text{ - normal ordering.}$$

When we impose the constraint $L_0 = 0 = \tilde{L}_0$, the choice of ordering comes into picture:

$$(L_0 - a)|phys\rangle = 0 \quad \& \quad (\tilde{L}_0 - a)|phys\rangle = 0$$

for some constant a. Here the constant "a" is called the normal ordering constant. From classical perspective, we saw that $L_0, \tilde{L}_0$ determine the mass of the string. To be precise, we had:

$$M^2 = \frac{4}{\alpha'}\sum_{k=1}^{\infty}{}' \alpha^\mu_{k}\,\alpha_{-\mu k} = \frac{4}{\alpha'}\sum_{k\,1}^{\infty}{}' \tilde{\alpha}^\mu_{k}\,\tilde{\alpha}_{-\mu k}$$

But since $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}\, p^\mu = \tilde{\alpha}_0^\mu$, & $p^\mu p_\mu = -M^2$

we see that the constraint

$$(L_0 - a)|phys\rangle = 0 = (\tilde{L}_0 - a)|phys\rangle$$

implies

$$\left(N - \frac{\alpha'}{4}M^2 - a\right)|phys\rangle = 0 \quad \&$$

$$\left(\tilde{N} - \frac{\alpha'}{4}M^2 - a\right)|phys\rangle = 0$$

where

$$N = \sum_{k=1}^{\infty} \alpha_{-k}^\mu \alpha_{\mu k} \quad \& \quad \tilde{N} = \sum_{k=1}^{\infty} \tilde{\alpha}_{-k}^\mu \tilde{\alpha}_{\mu k}$$

are number operators of left-moving & right moving modes.

Then the previous equations give

$$M = \frac{4}{\alpha'}(N - a) = \frac{4}{\alpha'}(\tilde{N} - a).$$

Since the number operator gives the number of excitations of the string, we see that the number of left-moving and right moving excitations are equal. Thus level matching condition in quantum theory imply equal number of left-moving and right-moving modes.

Under the commutation relation on α_n^μ, $\tilde{\alpha}_n^\mu$ and the choice of normal ordering, we can calulate the quantum version of the Virasoro algebra. A very careful computation shows that

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(n(n^2-1)) \delta_{m+n,0}$$

where c is called the central charge.
Recall that the classical Poisson braeket is

$$\{L_n, L_m\}_{P.B.} = i(m-n) L_{m+n}$$

The quantum Virasoro algebra is the so called central-extension of the Witt algebra.

The anomaly between the classical & quantum bracket is the so called conformal anomaly and it is due to fact that quantum theory breaks Weyl invariance as can be inferred from $\langle T^{\alpha}_{\alpha}\rangle \propto R$ (proof later). Non-vanishing trace implies Weyl symmetry breaking. In case of only free Bosonic fields, $c = \eta^{\mu}_{\mu} = D$ i.e each scalar field contributes one unit to central charge. We will rederive this algebra using Conformal field theory and quantise the string using path integral. Then we will calculate the contribution of Fadeev-Popov ghosts to the central charge. This anomaly requires us to impose the constants not directly as $L_n = 0 = \tilde{L}_n$ but to require the vanishing of matrix elements. Indeed if $|\phi\rangle$ is any quantum mechanical state then for any $n \in \mathbb{Z}$

$$0 = \langle \phi | [L_n, L_{-n}] | \phi \rangle = 2n \langle \phi | L_0 | \phi \rangle + \frac{c}{12} n(n^2-1) \langle \phi | \phi \rangle$$

which does not hold if $n \neq 0$.

The most we can do is require

$$L_n |phys\rangle = 0 \qquad \forall\ n > 0$$

$$(L_0 + a) |phys\rangle = 0 \qquad \text{(Mass-shell condition)}$$

Similar condition for $\tilde{L}_n$ & $\tilde{L}_0$.

We will give a proof of the commutation relation for Virasoro generators now.

Lemma 1: For any $m, n \in \mathbb{Z}$ we have

$$[\alpha^\mu_m, L_n] = m\,\alpha^\mu_{m+n}$$

Proof: With the choice normal ordering we have

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} :\alpha_{n-k}\cdot\alpha_k:$$

So we have

$$[\alpha^\mu_m, L_n] = \frac{1}{2}\sum_{k\in\mathbb{Z}} [\alpha^\mu_m, :\alpha_{n-k}\cdot\alpha_k:]$$

Now using $[A, BC] = [A,B]C + B[A,C]$ we get

$$[\alpha^\mu_m, L_n] = \frac{1}{2}\sum_{k\in\mathbb{Z}}\left\{\eta_{\rho\sigma}:\alpha^\rho_{n-k}[\alpha^\mu_m, \alpha^\sigma_k]: + \eta_{\rho\sigma}:[\alpha^\mu_m, \alpha^\rho_{n-k}]\alpha^\sigma_k\right\}$$

$$= \frac{1}{2}\sum_{k\in\mathbb{Z}}\left\{\eta_{\rho\sigma}\left(\alpha^\rho_{n-k}\, m\,\eta^{\mu\sigma}\delta_{m+k,0} + \alpha^\sigma_k\,\eta^{\mu\rho} m\,\delta_{m+n-k,0}\right)\right\}$$

$$= \frac{1}{2}\left\{\eta^\mu_\sigma\,\alpha^\rho_{n+m}\,m + \eta^\mu_\sigma\,\alpha^\sigma_{m+n}\cdot m\right\} = m\,\alpha^\mu_{m+n}.$$

$\blacksquare$

Theorem: For any $m, n \in \mathbb{Z}$, we have

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{D}{12} m(m^2-1)\, \delta_{m+n,0}.$$

Proof: We have

$$[L_m, L_n] = \frac{1}{2} \sum_{k \in \mathbb{Z}} [:\alpha_{m-k} \cdot \alpha_k:, L_n]$$

$$= \frac{1}{2} \sum_{k=-\infty}^{0} [\alpha_k \cdot \alpha_{m-k}, L_n] + \frac{1}{2} \sum_{k=1}^{\infty} [\alpha_{m-k} \cdot \alpha_k, L_n]$$

$$= \frac{1}{2} \sum_{k \leq 0} \alpha_k \cdot [\alpha_{m-k}, L_n] + [\alpha_k, L_n] \cdot \alpha_{m-k} +$$

$$\frac{1}{2} \sum_{k \geq 1} \alpha_{m-k} \cdot [\alpha_k, L_n] + [\alpha_{m-k}, L_n] \cdot \alpha_k$$

$$= \frac{1}{2} \sum_{k \leq 0} \left\{ (m-k)\, \alpha_k \cdot \alpha_{m+n-k} + k\, \alpha_{n+k} \cdot \alpha_{m-k} \right\} +$$

$$\frac{1}{2} \sum_{k \geq 1} \left\{ k\, \alpha_{m-k} \cdot \alpha_{n+k} + (m-k)\, \alpha_{m+n-k} \cdot \alpha_k \right\}$$

where in the second line we broke the normal ordering in two sums & used Lemma 1 in last line. We now shift the second and third sum by n i.e substitute n+k by k. We get

$$[L_m, L_n] = \frac{1}{2}\sum_{k\le 0}{}' (m-k)\,\alpha_k \cdot \alpha_{m+n-k} + \frac{1}{2}\sum_{k\le n}{}' (k-n)\,\alpha_k \cdot \alpha_{m+n-k}$$
$$\frac{1}{2}\sum_{k\ge n+1}{}' (k-n)\,\alpha_{m+n-k} \cdot \alpha_k + \frac{1}{2}\sum_{k\ge 1} (m-k)\,\alpha_{m+n-k} \cdot \alpha_k$$

Now we need to normal order second & third sum. If $n>0$ then second sum is not normal ordered & if $n\le 0$ then third sum is not normal ordered. Let us assume $n>0$ & $n\le 0$ is also dealt with similarly. Breaking second and fourth sum at 0 & $n+1$ respectively we get

$$[L_m, L_n] = \frac{1}{2}\Bigg[\sum_{k\le 0}(m-k)\,\alpha_k \cdot \alpha_{m+n-k} + \sum_{k\le 0}{}' (k-n)\,\alpha_k \cdot \alpha_{m+n-k} + \sum_{k=1}^{n}{}' (k-n)\,\alpha_k \cdot \alpha_{m+n-k}$$
$$+ \sum_{k\ge n+1}{}' (k-n)\,\alpha_{m+n-k} \cdot \alpha_k + \sum_{k=1}^{n}{}' (m-k)\,\alpha_{m+n-k} \cdot \alpha_k + \sum_{k\ge n+1}{}' (m-k)\,\alpha_{m+n-k} \cdot \alpha_k\Bigg]$$

$$= \frac{1}{2}\Bigg[\sum_{k\le 0}{}' (m-n)\,\alpha_k\cdot\alpha_{m+n-k} + \sum_{k\ge n+1}{}' (m-n)\,\alpha_{m+n-k}\cdot\alpha_k \, +$$

$$\sum_{k=1}^{n}{}'' (k-n)\,\alpha_k\cdot\alpha_{m+n-k} + \sum_{k=1}^{n} (m-k)\,\alpha_{m+n-k}\cdot\alpha_k\Bigg]$$

We will now use $[\alpha^\mu_k , \alpha^\nu_{m+n-k}] = \eta^{\mu\nu} k\, \delta_{m+n}$ to normal order third sum. We get

$$[L_m, L_n] = \frac{1}{2}\Bigg[\sum_{k\le 0}{}' (m-n)\,\alpha_k\cdot\alpha_{m+n-k} + \sum_{k\ge n+1}{}' (m-n)\,\alpha_{m+n-k}\cdot\alpha_k \, +$$

$$\sum_{k=1}^{n} (k-n)\left(\alpha_{m+n-k}\cdot\alpha_k + \eta^\mu_\mu\, k\,\delta_{m+n}\right) + \sum_{k=1}^{n} (m-k)\,\alpha_{m+n-k}\cdot\alpha_k\Bigg]$$

$$= \frac{1}{2}\Bigg[\sum_{k\le 0}{}' (m-n)\,\alpha_k\cdot\alpha_{m+n-k} + \sum_{k\ge 1}{}' (m-n)\,\alpha_{m+n-k}\cdot\alpha_k \, +$$

$$\sum_{k=1}^{n} (k-n)\,k\,\underbrace{\eta^\mu_\mu}_{D}\,\delta_{m+n}\Bigg]$$

$$= (m-n)\,\frac{1}{2}\sum_{k\in\mathbb{Z}}{}' :\alpha_k\cdot\alpha_{m+n-k}: + \frac{D\,\delta_{m+n}}{2}\sum_{k=1}^{n}{}' (k^2 - nk)$$

$$= (m-n)L_{m+n} + \frac{D}{2}\delta_{m+n}\left(\frac{n(n+1)(2n+1)}{6} - n\,\frac{n(n+1)}{2}\right)$$

$$= (m-n)L_{m+n} + \frac{D}{2}\delta_{m+n}\left(\frac{n(n+1)}{2}\left(\frac{2n+1}{3} - n\right)\right)$$

$$= (m-n)L_{m+n} + \frac{D}{2}\delta_{m+n}\,\frac{n(n+1)}{2}\left(\frac{1-n}{3}\right)$$

$$= (m-n)L_{m+n} + \frac{D}{2}\delta_{m+n}\,\frac{n(1-n^2)}{6}$$

$$= (m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\delta_{m+n}. \qquad (\text{replace } n \text{ by } -m) \quad \blacksquare$$

We can also derive the structure of the central charge term by using Jacobi identity of the Lie bracket. We will rederive this algebra using the tools of conformal field theory.

We have two issues to resolve: decouple ghosts and fix the normal-ordering constant. It turns out that we can prove a no-ghost theorem:

No-ghost theorem: The ghosts decouple in the critical dimension $D = 26$ and with $a = 1$.

To prove this theorem we follow the follows strategy: The normal ordering constant a & the dimension D are not arbitrary in the Quantum theory. For some values of a & D negative norm states are part of the physical Hilbert space while at other values of a & D, the physical Hilbert space is positive definite. The transition then occurs at the value of a & D where zero norm states become physical. Our strategy will be to find that value of a & D where zero-norm states become physical - the so called spurious states.

Spurious states: A state $|\psi\rangle$ is called spurious if it satisfies the mass-shell condition and is orthogonal to all physical states - i.e

$$(L_0 - a)|\psi\rangle = 0 \quad \& \quad \langle\phi|\psi\rangle = 0 \quad \forall \quad |\phi\rangle \text{ physical.}$$

Lemma 1: A general spurious state is of the form

$$|\psi\rangle = \sum_{n=1}^{\infty} L_{-n}|\chi_n\rangle$$

where $|\chi_n\rangle$ are some states satisfying the modified mass-shell condition:

$$(L_0 - a + n)|\chi_n\rangle = 0 \qquad \forall\ n \geq 1.$$

Proof: By definition we have

$$\langle\phi|\psi\rangle = 0 \qquad \forall\ |\phi\rangle \text{ physical.}$$

We know that

$$L_n|\phi\rangle = 0 \qquad \forall\ n > 0.$$

Thus we can write

$$|\psi\rangle = \sum_{n=1}^{\infty} L_{-n}|\chi_n\rangle \qquad (\text{since } L_{-n}^{\dagger} = L_n)$$

for some state $|\chi_n\rangle$.

Mass-shell condition implies

$$(L_0 - a)|\psi\rangle = 0$$

$$\Rightarrow \sum_{n=1}^{\infty}(L_0 L_{-n} - a L_{-n})|\chi_n\rangle = 0$$

By Virasoro algebra $L_0 L_{-n} = L_{-n} L_0 + n L_{-n}$. We get

$$\sum_{n=1}^{\infty} \left(L_{-n}L_0 + n - aL_{-n}\right)|\chi_n\rangle = 0$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} L_{-n}\left(L_0 - a + n\right)|\chi_n\rangle = 0$$

$$\Rightarrow \quad (L_0 - a + n)|\chi_n\rangle = 0 \qquad \forall \; n > 0 \qquad \blacksquare$$

Note: The states $|\chi_n\rangle$ satisfying $(L_0 - a + n)|\chi_n\rangle = 0$ are called level n states.

Lemma 2: Any spurious state $|\psi\rangle$ can be written as

$$|\psi\rangle = L_{-1}|\chi_1\rangle + L_{-2}|\chi_2\rangle$$

where $|\chi_1\rangle$ & $|\chi_2\rangle$ are level 1 & level 2 states satisfying

$$(L_0 - a + 1)|\chi_1\rangle = 0 \quad \& \quad (L_0 - a + 2)|\chi_2\rangle = 0.$$

Proof: By Lemma 1, we have that

$$|\psi\rangle = \sum_{n=1}^{\infty} L_{-n}|\chi_n\rangle$$

where $(L_0 - a + n)|\chi_n\rangle = 0$. We will show that $L_{-n}|\chi_n\rangle$ can written as $L_{-1}|\chi_1\rangle + L_{-2}|\chi_2\rangle$ for some level 1 & level 2 states $|\chi_1\rangle$ & $|\chi_2\rangle$ respectively for all $n \geq 3$. Let us begin with the base case. Note that by Virasoro algebra we have

$$L_{-3} = [L_{-1}, L_{-2}] = (-1+2)L_{-2-1} + 0 = L_{-3}$$

Thus

$$L_{-3}|\chi_3\rangle = [L_{-1}, L_{-2}]|\chi_3\rangle = L_{-1}(L_{-2}|\chi_3\rangle) + L_{-2}(-\chi_{-1}|\chi_3\rangle)$$

Take $|\chi_1\rangle = L_{-2}|\chi_3\rangle$ & $|\chi_2\rangle = -\chi_{-1}|\chi_3\rangle$.

It remains to show that $|\chi_1\rangle$ & $|\chi_2\rangle$ are level 1 & level 2 states respectively. Indeed since

$$(L_0 - a + 3)|\chi_3\rangle = 0,$$ we have

$$\begin{aligned}
(L_0 - a + 1)L_{-2}|\chi_3\rangle &= (L_0L_{-2} + L_{-2}(-a+1))|\chi_3\rangle \\
&= (L_{-2}L_0 + 2L_{-2} + L_{-2}(-a+1))|\chi_3\rangle \\
&= L_{-2}(L_0 - a + 3)|\chi_3\rangle \\
&= 0.
\end{aligned}$$

where we used the Virasoro algebra: $L_0L_{-2} = L_{-2}L_0 + 2L_{-2}$.

Similarly we have

$$\begin{aligned}
(L_0 - a + 2)(-L_{-1}|\chi_3\rangle) &= -(L_{-1}L_0 + L_{-1}(-a+2))|\chi_3\rangle \\
&= -(L_0L_{-1} + L_{-1} + L_{-1}(-a+2))|\chi_3\rangle
\end{aligned}$$

$$= -L_{-1}(L_0 - a + 3)|\chi_3\rangle$$
$$= 0.$$

For any n, we assume that $L_{-n+1}|\chi_{n-1}\rangle$ can be written as $L_{-1}|\chi_1\rangle + L_{-2}|\chi_2\rangle$. Then since

$$L_{-n} = \frac{1}{n}[L_{-1}, L_{-n+1}]$$

so that

$$L_{-n}|\chi_n\rangle = \frac{1}{n} L_{-1}\left(L_{-n+1}|\chi_n\rangle\right) + \frac{1}{n} L_{-n+1}\Big(\underbrace{-L_{-1}|\chi_n\rangle}_{|\chi_{n-1}\rangle}\Big).$$

Following similar method as in base case we can show that $-\frac{1}{n}L_{-1}|\chi_n\rangle$ is a level $n-1$ state. Indeed observe that

$$-\frac{1}{n}(L_0 - a + n - 1)L_{-1}|\chi_n\rangle = -\frac{1}{n}\left(L_{-1}L_0 + L_{-1} + L_{-1}(-a+n-1)\right)|\chi_n\rangle$$
$$= -\frac{1}{n}L_{-1}(L_0 - a + n)|\chi_n\rangle$$
$$= 0.$$

So using induction hypothesis we get

$$L_{-n}|\chi_n\rangle = L_{-1}\left(\frac{1}{n}L_{-n+1}|\chi_n\rangle\right) + L_{-1}|\tilde{\chi}_1\rangle + L_{-2}|\tilde{\chi}_2\rangle$$

for some level 1 state $|\tilde{\chi}_1\rangle$ & level 2 state $|\tilde{\chi}_2\rangle$.

It is also clear that $\frac{1}{n} L_{-n+1} |\chi_n\rangle$ is a level 1 state. Thus define

$$|\chi_1\rangle = |\tilde{\chi}_1\rangle + \frac{1}{n} L_{-n+1} |\chi_n\rangle$$

$$\& \quad |\chi_2\rangle = |\tilde{\chi}_2\rangle \quad \text{so that}$$

$$L_{-n} |\chi_n\rangle = L_{-1} |\chi_1\rangle + L_{-2} |\chi_2\rangle .$$

where $|\chi_1\rangle$ & $|\chi_2\rangle$ are level 1 & level 2 states respectively. ▨

Norm-zero states: Note that the spurious states are orthogonal to all physical states. Thus if we require the spurious states themselves to be physical we must have

$$\langle\psi|\psi\rangle = 0$$

Thus all physical spurious states are norm-zero states. We will now find values of a & D so that all spurious states become physical.

Note: Any spurious state decouple from all physical process since their norm is zero.

Physical spurious states: It suffices to find constraint on a & D so that spurious states $L_{-1}|\chi_1\rangle$ & $L_{-2}|\chi_2\rangle$ become physical where $|\chi_1\rangle$ & $|\chi_2\rangle$ are level 1 & level 2 states resp.

Theorem: Let $|\chi_1\rangle$ be a level 1 state satisfying $L_m|\chi_1\rangle = 0$ $\forall\, m>0$. Then the spurious state $|\psi\rangle = L_{-1}|\chi_1\rangle$ is physical if and only if $a=1$.

Proof: ($\Rightarrow$) Suppose $L_{-1}|\chi_1\rangle$ is physical. Then $L_1 L_{-1}|\chi_1\rangle = 0$ as physical states satisfy $L_m|\phi\rangle = 0$ $\forall\, m>0$.

We get

$$\begin{aligned} L_1 L_{-1}|\chi_1\rangle &= (L_{-1}L_1 + 2L_0)|\chi_1\rangle \\ &= 2L_0|\chi_1\rangle \\ &= 2(a-1)|\chi_1\rangle \end{aligned}$$

since $|\chi_1\rangle$ is a level 1 state satisfying $(L_0 - a + 1)|\chi_1\rangle = 0$

Thus $L_1 L_{-1}|\chi_1\rangle = 0 \Rightarrow a=1$.

($\Leftarrow$) If $a=1$ then backtracking above steps we get $L_1 L_{-1}|\chi_1\rangle = 0$.

To check that $L_m L_{-1}|\chi_1\rangle = 0$ we proceed inductively. We have the base case. Next

$$\begin{aligned} L_m L_{-1}|\chi_1\rangle &= L_{-1}\underbrace{L_m|\chi_1\rangle}_{=0 \text{ by assumption}} + \underbrace{(m+1)L_{m-1}|\chi_1\rangle}_{=0 \text{ induction hypothesis.}} \\ &= 0 \end{aligned}$$

Next thing to check is

$$(L_0 - a) L_{-1}|\chi_1\rangle = 0 \qquad (a=1)$$

Indeed $\quad L_0 L_{-1}|\chi_1\rangle = L_{-1} L_0|\chi_1\rangle + L_{-1}|\chi_1\rangle$

$$= 0 + L_{-1}|\chi_1\rangle$$

since $\quad (L_0 - a+1)|\chi_1\rangle = L_0|\chi_1\rangle = 0.$ $\blacksquare$

Next we look at level 2 spurious states. A general level 2 spurious state is

$$|\psi\rangle = (L_{-2} + \gamma L_{-1} L_{-1})|\chi_2\rangle .$$

We will show that $|\psi\rangle$ is physical if $\gamma = 3/2$ & $D = 26$.

Theorem: Let $|\chi_2\rangle$ be a Level 2 state satisfying $L_m|\chi_2\rangle = 0$ $\forall\, m>0$. Then the spurious state $|\psi\rangle = (L_{-2} + \gamma L_{-1} L_{-1})|\chi_2\rangle$ is physical if and only if $\gamma = 3/2$ & $D = 26$.

Proof: $(\Rightarrow)$ Suppose $|\psi\rangle$ is physical. Then we must have

$$\langle\psi|\psi\rangle = 0$$

This is automatically satisfied irrespective of γ as $L_1|\chi_2\rangle = 0$.

Next we demand $L_m|\psi\rangle = 0 \quad \forall\, m>0$.

In particular

$$L_1|\psi\rangle = 0$$

$$\Rightarrow \left(L_1 L_{-2} + \gamma L_1 L_{-1} L_{-1}\right)|\chi_2\rangle = 0$$

$$\Rightarrow \left(L_{-2} L_1 + 3L_{-1} + \gamma L_{-1} L_1 L_{-1} + 2\gamma L_0 L_{-1}\right)|\chi_2\rangle = 0$$

$$\Rightarrow \left(L_{-2} L_1 + 3L_{-1} + \gamma L_{-1} L_{-1} L_1 + 2\gamma L_{-1} L_0 + 2\gamma L_{-1} L_0 + 2\gamma L_{-1}\right)|\chi_2\rangle = 0$$

$$\Rightarrow L_{-1}\left(3 + 4\gamma L_0 + 2\gamma\right)|\chi_2\rangle = 0$$

where we used $L_1|\chi_2\rangle = 0$. we get $\left\{ L_0|\chi_2\rangle = -|\chi_2\rangle \text{ as } a = 1 \right\}$

$$L_{-1}\left(3 + 4(-1)\gamma + 2\gamma\right)|\chi_2\rangle = 0$$

$$\Rightarrow (3 - 2\gamma) L_{-1}|\chi_2\rangle = 0$$

$$\Rightarrow \gamma = \frac{3}{2}.$$

so $|\psi\rangle = \left(L_{-2} + \frac{3}{2} L_{-1} L_{-1}\right)|\chi_2\rangle$.

Next we impose $L_2|\psi\rangle = 0$

$$L_2\left(L_{-2} + \frac{3}{2} L_{-1} L_{-1}\right)|\chi_2\rangle = 0$$

$$\Rightarrow \left[L_2, L_{-2} + \frac{3}{2} L_{-1} L_{-1}\right]|\chi_2\rangle + \left(L_{-2} + \frac{3}{2} L_{-1} L_{-1}\right)\underbrace{L_2|\chi_2\rangle}_{=0} = 0$$

by assumption

$$\Rightarrow \left(4L_0 + \frac{c}{12} 2(3)\delta_{0,0} + \frac{3}{2}\left[L_2, L_{-1} L_{-1}\right]\right)|\chi_2\rangle = 0$$

Now $[L_2, L_{-1}L_{-1}] = [L_2, L_{-1}]L_{-1} + L_{-1}[L_2, L_{-1}]$

$$= 3L_1L_{-1} + 3L_{-1}L_1$$

$$= 3(L_{-1}L_1 + 2L_0) + 3L_{-1}L_1$$

$$= 6L_{-1}L_1 + 6L_0$$

So we have

$$\left(4L_0 + \frac{c}{2} + \frac{3}{2}(6L_{-1}L_1 + 6L_0)\right)|\chi_2\rangle = 0$$

$$\Rightarrow \left(13L_0 + 9L_{-1}L_1 + \frac{c}{2}\right)|\chi_2\rangle = 0$$

$$\Rightarrow c = 26.$$

In free Bosonic string theory, we know that $c = \eta^\mu_\mu = D$. So $D = 26$.

($\Leftarrow$) Assuming $D = 26$, $\gamma = 3/2$, we can show that $L_1|\psi\rangle = 0$ & $L_2|\psi\rangle = 0$ back tracking the steps. For $m \geq 3$, we again follow the same procedure as in previous theorem. Finally we need to show that $(L_0 - 1)|\psi\rangle = 0$

To see that this is true, observe that

$$(L_0 - 1)\left(L_{-2} + \frac{3}{2} L_{-1} L_{-1}\right)|\chi_2\rangle$$

$$= \left(L_0 L_{-2} + \frac{3}{2} L_0 L_{-1} L_{-1}\right)|\chi_2\rangle - |\psi\rangle$$

$$= \left(L_{-2} L_0 + 2 L_{-2} + \frac{3}{2} L_{-1} L_0 L_{-1} + \frac{3}{2} L_{-1} L_{-1}\right)|\chi_2\rangle - |\psi\rangle$$

$$= \left(L_{-2}(-1) + 2 L_{-2} + \frac{3}{2} L_{-1} L_{-1} L_0 + \frac{3}{2} L_{-1} L_{-1} + \frac{3}{2} L_{-1} L_{-1}\right)|\chi_2\rangle - |\psi\rangle$$

$$= \left(L_{-2} \cancel{- \frac{3}{2} L_{-1} L_{-1}} \cancel{+ \frac{3}{2} L_{-1} L_{-1}} + \frac{3}{2} L_{-1} L_{-1}\right)|\chi_2\rangle - |\psi\rangle$$

$$= |\psi\rangle - |\psi\rangle$$

$$= 0.$$

We used $L_0 |\chi_2\rangle = -|\chi_2\rangle$. $\blacksquare$

Thus we have shown that infinite classes of spurious states of zero norm appear in our theory when $D = 26$ & $a = 1$. Thus we have determined the boundary where +ve norm states turn into −ve norm states. There are non-critical string theories free of ghosts for $a \leq 1$ & $D \leq 25$.

Light-cone quantisation:

In light-cone quantisation, we begin by solving the constraints first and seperating the physical degrees of freedom. Before, we begin, lets discuss about reparametrizations, conformal transformations & Weyl rescaling.

Given any reparametrization of the world sheet, it corresponds to choosing a different coordinate chart for the manifold. This has no physical consequence as all points, curves remain same on the manifold (world-sheet). Thus any diffeomorphism automatically preserves circular & hyperbolic angles. On the other hand coordinate transformations which transform the metric as follows:

$$g_{\mu\nu} \longrightarrow \Omega^2(\sigma)\, g_{\mu\nu}(\sigma)$$

are called conformal transformations. These transformations preserve angles (circular as well as hyperbolic). Another version of Conformal transformations are maps between manifolds. Let (M, g) & $(N, \tilde{g})$ be Riemannian manifolds & $\varphi : M \longrightarrow N$ be a smooth map. Then φ is said to be a conformal map if the pullback $\varphi^* \tilde{g} = \Omega^2 g$ for some $\Omega \in C^\infty(M)$. When we write out these in

Writing $x' = \varphi(x)$ we see that

$$\tilde{g}_{\mu\nu}(x') \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} = \Omega^2(x)\, g_{\rho\sigma}$$

Thus angles are preserved. In particular if $M = N$ & $\tilde{g} = g$ then

$$g_{\mu\nu}(x') \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} = \Omega^2(x)\, g_{\rho\sigma}$$

which are the usual conformal transformation.
Weyl rescalings, on the other hand, are completely different. They are not coordinate transformations. These do not act on the parametri-zations but act on the metric. Since the metric is only scaled thus angles are preserved.

Now, we have already fixed a gauge i,e chosen two reparametrizations and used Weyl rescaling to fix the metric to $\eta_{\alpha\beta}$. But we have some residual gauge symmetry. Indeed consider a reparametrization $\sigma^{\alpha} \to \tilde{\sigma}^{\alpha} = \tilde{\sigma}^{\alpha}(\sigma)$ such that the metric changes by $\eta_{\alpha\beta} \longrightarrow \tilde{\eta}_{\alpha\beta} = \Omega^2(\sigma)\eta_{\alpha\beta}$. This can then be undone by a Weyl rescaling. ($d\sigma d\tau$ introduces a Jacobian in the action which again ensures Weyl scaling). These are exactly the conformal transformations. Thus we see that

"diffeomorphisms = Conformal × Weyl.

Next we need to find all such transformation. We will do this in Lightcone coordinates.

$$\sigma^{\pm} = \tau \pm \sigma \qquad \tau = \frac{\sigma^+ + \sigma^-}{2}, \qquad \sigma = \frac{\sigma^+ - \sigma^-}{2}$$

$$ds^2 = -d\tau^2 + d\sigma^2 = -\frac{1}{4}\left(d\sigma^+ + d\sigma^-\right)^2 + \frac{1}{4}\left(d\sigma^+ - d\sigma^-\right)^2$$

$$= -\cancel{\frac{1}{4}d\sigma^{+2}} - \cancel{\frac{1}{4}d\sigma^{-2}} - \frac{1}{2}d\sigma^+ d\sigma^- + \cancel{\frac{1}{4}d\sigma^{+2}} + \cancel{\frac{1}{4}d\sigma^{-2}} - \frac{1}{2}d\sigma^+ d\sigma^-$$

$$= -d\sigma^+ d\sigma^-$$

so a reparametrization $\sigma^+ \longrightarrow \tilde{\sigma}^+(\sigma^+)$ & $\sigma^- \longrightarrow \tilde{\sigma}^-(\sigma^-)$ ds^2 simply changes by scaling. Indeed

$$ds^2 = -\frac{\partial \sigma^+}{\partial \tilde{\sigma}^+} d\tilde{\sigma}^+ \frac{\partial \sigma^-}{\partial \tilde{\sigma}^-} d\tilde{\sigma}^- = -\frac{\partial \sigma^+}{\partial \tilde{\sigma}^+}\frac{\partial \sigma^-}{\partial \tilde{\sigma}^-} d\tilde{\sigma}^+ d\tilde{\sigma}^-.$$

Note that the reparametrizations are single variable.

Lightcone gauge:

We would like to fix the remanant gauge choice. The choice that we will make here is called Lightcone gauge.

Introduce

$$X^{\pm} = \frac{1}{\sqrt{2}} \left(X^0 \pm X^{D-1} \right)$$

Such a choice breaks Lorentz invariance in classical as well as quantum theory as we have picked a special time & space part. So when we quantise our system, we will look for conditions that preserve Lorentz invariance.

It is now easy to see that

$$ds^2 = -2\, dX^+ dX^- + \sum_{i=1}^{D-2} (dX^i)^2 .$$

So the metric $\eta_{++} = 0 = \eta_{--}$ & $\eta_{+-} = \eta_{-+} = -1$.

& $\eta_{ii} = 1 \quad \forall\ i = 1, 2, \ldots, D-2$ & all other elements vanish.

So any vector $A^\mu = (A^+, A^-, A^i)$ is Lowered as

$$A_\mu = (-A_-, -A_+, A_i).$$

and the dot product is

$$A^\mu B_\mu = -A^+ B_- - A^- B_+ + A^i B^i .$$

Solution of the equation of motion is

$$X^+ = X_L^+(\sigma^+) + X_R^+(\sigma^-)$$

To see this note that

$$X^\mu = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$$

so $X^+ = \frac{1}{\sqrt{2}}\left(X^0 + X^{D-1}\right) = \frac{1}{\sqrt{2}}\left(X^0_L(\sigma^+) + X^{D-1}_L(\sigma^+) + X^0_R(\sigma^-) + X^{D-1}_R(\sigma^-)\right)$

$$= X^+_L(\sigma^+) + X^+_R(\sigma^-)$$

We now fix our gauge. Note that X^+ satisfies the wave equation $\partial_+\partial_- X^+ = 0$. Now note that a reparametrization $\tilde{\sigma}^+ = \tilde{\sigma}^+(\sigma^+)$ & $\tilde{\sigma}^- = \tilde{\sigma}^-(\sigma^-)$ corresponds to

$$\tilde{\tau} = \frac{\tilde{\sigma}^+ + \tilde{\sigma}^-}{2}, \qquad \tilde{\sigma} = \frac{\tilde{\sigma}^+ - \tilde{\sigma}^-}{2}$$

But $\tilde{\tau}$ has to satisfy $\partial_+\partial_-\tilde{\tau} = 0$. So we can choose

$$\tilde{\tau} = \frac{X^+}{\alpha' p^+} - x^+$$

so that $X^+ = x^+ + \alpha' p^+ \tilde{\tau}$.

This is called the Lightcone gauge

X^- still satisfies the wave equation

$$\partial_+\partial_- X^- = 0$$

The usual solution is

$$X^- = X^-_L(\sigma^+) + X^-_R(\sigma^-).$$

Let us look at the constraints in Lightcone gauge. We had the constraint

$$(\partial_+ X)^2 = 0 = (\partial_- X)^2$$

$$X = (X^+, X^-, X^i)$$

so $$(\partial_+ X)^2 = -2\,\partial_+ X^- \partial_+ X^+ + \sum_{i=1}^{D-2} (\partial_+ X^i)^2$$

& $$(\partial_- X)^2 = -2\,\partial_- X^- \partial_- X^+ + \sum_{i=1}^{D-2} (\partial_- X^i)^2$$

since $$\partial_+ X^+ = \frac{\alpha' p^+}{2} = \partial_- X^+ \qquad \left(\tau = \frac{\sigma^+ + \sigma^-}{2}\right)$$

the constraints $(\partial_+ X)^2 = 0 = (\partial_- X)^2$ gives

$$\left.\begin{aligned} \partial_+ X^- &= \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} (\partial_+ X^i)^2 \\ \partial_- X^- &= \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} (\partial_- X^i)^2 \end{aligned}\right\} \qquad (*)$$

Thus we see that in Lightcone gauge the D-2 scalar fields determine X^- upto an additive constant coming from integration. Indeed we see that if we write the mode expansion of $X^-_{L/R}$

$$X_L^-(\sigma^+) = \frac{1}{2}x^- + \frac{\alpha'}{2}p^-\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^- e^{-in\sigma^+}$$

$$X_R^-(\sigma^-) = \frac{1}{2}x^- + \frac{\alpha'}{2}p^-\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\sigma^-}$$

then x^- is coming as the integration constant while all other terms p^- & $\tilde{\alpha}_n^-$, α_n^- is determined in terms of $\tilde{\alpha}_n^i$, α_n^i & p^+. Indeed if we write

$$\partial_+ X_L^- = \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\tilde{\alpha}_n^- e^{-in\sigma^+} \quad \text{with } \tilde{\alpha}_0^- = \sqrt{\frac{\alpha'}{2}}\,p^-$$

& $$\partial_- X_R^- = \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\alpha_n^- e^{-in\sigma^-} \quad \text{with } \alpha_0^- = \sqrt{\frac{\alpha'}{2}}\,p^-$$

then substituting $(\partial_+ X^i)^2$ using Fourier modes of X^i in (*) we get by compairing coefficients of $e^{-in\sigma^\pm}$ that

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}}\frac{1}{p^+}\sum_{m\in\mathbb{Z}}\sum_{i=1}^{D-1}\alpha_{n-m}^i\alpha_m^i$$

$$\tilde{\alpha}^-_n = \frac{1}{\sqrt{2\alpha'}}\frac{1}{p^+}\sum_{m\in\mathbb{Z}}{}' \sum_{i=1}^{D-2}{}' \tilde{\alpha}^i_{n-m}\tilde{\alpha}^i_m .$$

For $n=0$, we get two expressions for p^-

$$\frac{\alpha' p^-}{2} = \frac{1}{2p^+}\sum_{i=1}^{D-2}{}'\left(\frac{\alpha'}{2}p^i p^i + \sum_{n\neq 0}\alpha^i_{-n}\alpha^i_n\right) = \frac{1}{2p^+}\sum_{i=1}^{D-2}{}'\left(\frac{\alpha'}{2}p^i p^i + \sum_{n\neq 0}{}'\tilde{\alpha}^i_{-n}\alpha^i_n\right)$$

Using $p^\mu = (p^+, p^-, p^i)$ we get the classical level matching condition: $p^\mu p_\mu = -M^2$

so $M^2 = -p^\mu p_\mu = 2p^+p^- - \sum_{i=1}^{D-1}{}' p^i p^i$.

Using the above equality for $\frac{\alpha' p^-}{2}$ we get

$$M^2 = \frac{4}{\alpha'}\sum_{i=1}^{D-2}{}'\sum_{n>0}{}'\alpha^i_{-n}\alpha^i_n = \frac{4}{\alpha'}\sum_{i=1}^{D-2}{}'\sum_{n>0}{}'\tilde{\alpha}^i_{-n}\alpha^i_n$$

where we use $\displaystyle\sum_{n\neq 0}{}'\alpha^i_{-n}\alpha^i_n = 2\sum_{n>0}{}'\alpha^i_{-n}\alpha^i_n$.

The oscillators $\alpha^i_n, \tilde{\alpha}^i_n$ are called transerve oscillators. These are physical excitations in the sense that knowing α^i_n & $\tilde{\alpha}^i_n$ determines all other modes. Thus the most general classical solution can be determined in terms of $2(D-2)$ oscillator modes α_n^i, $\tilde{\alpha}_n^i$ & a bunch of zero modes $p^\pm$, p^i, $x^\pm$.

Quantisation:

The usual way of quantisation is to compute the classical Poisson brackets and use Dirac prescription. As we did in covariant quantisation, using the Poisson brakets, the following commutation relations are obvious:

$$[x^i, p^j] = i\,\delta^{ij}, \qquad [x^-, p^+] = -i, \qquad [x^+, p^-] = -i$$

$$[\alpha_n^i, \alpha_m^j] = n\,\delta^{ij}\,\delta_{m+n,0} = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j]$$

The ground state is again $|0;p^\mu\rangle$ with $|0\rangle$ being the vacuum of the theory & $|p^\mu\rangle$ the initial state of the string. To build the Fock space, we impose

$$\hat{p}^\mu\,|0;p^\mu\rangle = p^\mu\,|0;p^\mu\rangle, \qquad \tilde{\alpha}_n^i\,|0;p^\mu\rangle = 0 = \alpha_n^i\,|0;p^\mu\rangle$$

$$\forall \quad n>0 \quad \& \quad i = 1,2,\dots,D-1$$

We act with α_{-n}^i, $\tilde{\alpha}_{-n}^i$, $n>0$ to build the Fock space. Notice that i runs only over spatial index $i = 1,\dots,D-1$, so the theory doesnot have ghosts by construction. Its time to impose the constraints.

As we had in covariant quantisation, Level matching with normal ordering implies

$$M^2 = \frac{4}{\alpha'}(N - a) = \frac{4}{\alpha'}(\tilde{N} - a)$$

where now the number operators are

$$N = \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n\neq 0} \alpha^i_{-n}\alpha^i_n \quad \& \quad \tilde{N} = \frac{1}{2}\sum_{i=1}^{D-1}\sum_{n\neq 0} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n$$

& a is again the normal ordering constant. We need to determine it.

Note that

$$\frac{1}{2}\sum_{i=1}^{D-1}\sum_{n\neq 0}\alpha^i_{-n}\alpha^i_n = \frac{1}{2}\sum_i\left[\sum_{n>0}\alpha^i_{-n}\alpha^i_n + \sum_{n<0}\alpha^i_{-n}\alpha^i_n\right]$$

$$= \frac{1}{2}\sum_i\sum_{n>0}\alpha^i_{-n}\alpha^i_n + \frac{1}{2}\sum_i\left[\sum_{n<0}\alpha^i_n\alpha^i_{-n} - n\right]$$

$$= \sum_i\sum_{n>0}\alpha^i_{-n}\alpha^i_n + \frac{D-2}{2}\sum_{n>0} n$$

where we used the commutor $[\alpha^i_n, \alpha^j_{-n}] = n$

The sum is divergent but we need to extract physics out of this divergence. The result is the appearence of Casimir force. To do this, there are two ways:

(i) UV cut-off $\epsilon << 1$.

Write
$$\sum_{n>0}' n \longrightarrow \sum_{n>0} n e^{-\epsilon n} = -\frac{\partial}{\partial \epsilon} \sum_{n>0}' e^{-\epsilon n}$$
$$= -\frac{\partial}{\partial \epsilon}\left[\left(1-e^{-\epsilon}\right)^{-1}\right]$$

Now
$$-\frac{\partial}{\partial \epsilon}\left[\frac{1}{1-e^{-\epsilon}}\right] = \frac{e^{-\epsilon}}{\left(1-e^{-\epsilon}\right)^2} = \frac{\left(1-\epsilon+\frac{\epsilon^2}{2}+O(\epsilon^3)\right)}{\left(1-1+\epsilon-\frac{\epsilon^2}{2}+\dots\right)^2}$$

$$= \frac{\left(1-\epsilon+\frac{\epsilon^2}{2}+O(\epsilon^3)\right)}{\epsilon^2\left(1-\frac{\epsilon}{2}+\dots\right)^2} = \frac{1}{\epsilon^2}\left(1-\epsilon+\frac{\epsilon^2}{2}+O(\epsilon^3)\right)\left(1+2\frac{\epsilon}{2}-2\frac{\epsilon^2}{3!}+\frac{3}{4}\epsilon^2+O(\epsilon^3)\right)$$

$$= \frac{1}{\epsilon^2}\left(1+\frac{\epsilon^2}{2}-\epsilon^2-\frac{2}{6}\epsilon^2+\frac{3}{4}\epsilon^2+O(\epsilon^3)\right)$$
$$= \frac{1}{\epsilon^2}-\frac{1}{12}+O(\epsilon)$$

The $\frac{1}{\epsilon^2}$ must be renormalised away. After renormalising, we are left with the odd result:

$$\sum_{n>0} n = -\frac{1}{12}.$$

(ii) Zeta-function regularisation:

We have

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad s \in \mathbb{C}$$

$\zeta(s)$ defines a holomorphic function on the half plane $\{s \in \mathbb{C} \mid Re(s) > 1\}$ as the sum coverges absolutely & uniformly on every compact subsets of this half plane. $\zeta(s)$ admits a unique analytic continuation to the whole s-plane with a simple pole at $s=1$. More precisely, we have an integral representation of $\zeta(s)$ as follows:

$$\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} W(x) \left(x^{s/2} + x^{\frac{1-s}{2}} \right) \frac{dx}{x}$$

where $W(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x} < \infty$. Since $W(x) = O(e^{-\pi x})$ so the integral on R.H.S converges $\forall\, s \in \mathbb{C}$. So this integral gives an analytic continuation of $\zeta(s)$ Indeed putting

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

we easily see that $\xi(s)$ is entire and satisfies

$$\xi(s) = \xi(1-s) \quad \forall \quad s \in \mathbb{C}.$$

From the fact that $\xi(s)$ is entire we see that $\zeta(s)$ (analytically continued) has simple zeros at $s=-2n$ $n \in \mathbb{N}$ corresponding to poles of $\Gamma\left(\frac{s}{2}\right)$.

Now at $s=-1$, we have that

$$\xi(-1) = \xi(2)$$

i,e

$$\cancel{2}\pi^{1/2}\Gamma\left(-\frac{1}{2}\right)\zeta(-1) = \cancel{2}\pi^{-1}\Gamma(1)\zeta(2)$$

$$\Rightarrow \quad \pi^{1/2}\left(-\frac{1}{2}\right)\sqrt{\pi}\,\zeta(-1) = \pi^{-1}\frac{\pi^2}{6}$$

$$\Rightarrow \quad \boxed{\zeta(-1) = -\frac{1}{12}.}$$

So we see that both of the computation gives same result. Thus the level matching condition becomes

$$M^2 = \frac{4}{\alpha'}\left[\sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n - \frac{D-2}{24}\right] = \frac{4}{\alpha'}\left[:N: - \frac{D-2}{24}\right]$$

$$= \frac{4}{\alpha'}\left[\sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n - \frac{D-2}{24}\right] = \frac{4}{\alpha'}\left[:\tilde{N}: - \frac{D-2}{24}\right].$$

String spectrum:

Lets look at the ground state $|0;p^\mu\rangle$. By our definition of vacuum $|0\rangle$,

$$:N:|0;p^\mu\rangle = 0 = :\tilde{N}:|0;p^\mu\rangle.$$

So level matching gives

$$M^2 = -\frac{D-2}{6\alpha'} < 0$$

These are particles with negative mass-squared. These are called Tachyons. These are a problem in Bosonic string theory. But when we study Superstring theory where we include Fermionic fields on the worldsheet, then these states automatically vanish.

Now lets look at excited states:

First excited state is obtained by acting α^i_{-1} & $\tilde{\alpha}^j_{-1}$

To see this observe that

$$N\,\alpha^j_{-n}|0;p^\mu\rangle = \sum_{i=1}^{D-1}\sum_{k=1}^{\infty}{}' \alpha^i_{-k}\alpha^i_k\alpha^j_{-n}|0;p^\mu\rangle \qquad n>0$$

$$= \left[\sum_i{}' \sum_{k=1}^{\infty}{}' \alpha^i_{-k}\alpha^j_{-n}\alpha^i_k + k\,\delta^{ij}\delta_{k-n,0}\,\alpha^i_{-k}\right]|0;p^\mu\rangle$$

$$= n\,\alpha^j_{-n}|0;p^\mu\rangle .$$

so α^i_{-1} & $\tilde{\alpha}^i_{-1}$ give first excited states. Thus level matching requires us to act α^i_{-1} & $\tilde{\alpha}^j_{-1}$ together. So consider

$$\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0;p^\mu\rangle .$$

Mass of each of these states is

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{D-2}{24}\right).$$

Problem with Lorentz invariance:

Our states are labelled by indices $i,j = 1,2,\cdots,D-1$ & hence these transform as vectors w.r.t the group $SO(D-2) \hookrightarrow$

$SO(1,D-1)$ — the full Lorentz group. But finally we want our states to fit into some representation of the Lorentz group $SO(1,D-1)$. To proceed we digress to recall Wigner's classification of the representations of the Poincare group.

Projective representations:

Let $|\Psi\rangle$ be a state in Hilbert state $\mathcal{H}$. Note that any two states $|\Psi\rangle$ & $|\Phi\rangle$ which are non-zero and related by

$$|\Psi\rangle = \lambda |\Phi\rangle \qquad \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}. \qquad (1)$$

are the same quantum mechanical states. So it is pertinent to consider the quotient space of $\mathcal{H}^* = \mathcal{H} \setminus \{0\}$ as

$$\mathbb{P}(\mathcal{H}) := \mathcal{H}^*/\sim \qquad \text{where} \quad |\Psi\rangle \sim |\Phi\rangle \ \text{iff (1) is true.}$$

$\mathbb{P}(\mathcal{H})$ is called the projectivised Hilbert space.

Recall that the probability amplitute of transition from $|\Psi\rangle$ to $|\Phi\rangle$ is given by

$$p(|\Psi\rangle, |\Phi\rangle) = \frac{\langle \Psi | \Phi \rangle}{\langle \Psi | \Psi \rangle \langle \Phi | \Phi \rangle}$$

In the quotient topology on $\mathbb{P}(\mathcal{H})$, p induces a continuous map on $\mathbb{P}(\mathcal{H})$ (Quotient topology theorem) which we denote by $\tilde{p}$. A homeomorphism $T : \mathbb{P}(\mathcal{H}) \longrightarrow \mathbb{P}(\mathcal{H})$ satisfying

$$\tilde{p}(T[\Psi], T[\Phi]) = \tilde{p}(|\Psi\rangle, |\Phi\rangle)$$

where $[\Psi]$ & $[\Phi]$ are equivalence classes in $\mathbb{P}(\mathcal{H})$, is called a projective automorphism. The set of all such maps denoted by $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$ is a group called projective automorphism group. The action of this group on $\mathbb{P}(\mathcal{H})$ leaves transition probabilities invariant. Now consider a particle in the Minkowski space $\mathbb{R}^{1,D-1}$. The symmetry group of this space is precisely the Poincarè group. which we denote by $\mathcal{P}$. Let two observers O and O' related by $\Lambda \in \mathcal{P}$ measure the quantum mechanical particle. In general there measurement result will reveal different states say $[\Psi]$ and $[\Psi']$ respectively. Thus physically one expects that transition probabilities in O & O' be same. This means that the two states must be related by some projective automorphism: $[\Psi] = T_\Lambda[\Psi']$ for some $T_\Lambda \in \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$. If $O = O'$ then $\Lambda = \mathrm{Id}$ & we should have $T_\Lambda = T_{\mathrm{Id}} \doteq \mathrm{Id} \in \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$. Lastly if a third observer O'' related to O' by Γ measures the state then we must impose $T_\Lambda \circ T_\Gamma = T_{\Lambda\Gamma}$. Thus the change of frame induces a representation $\Pi : \mathcal{P} \longrightarrow \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$. This is called the projective representation.

Elementary particles:

The representation $(\Pi, \mathcal{H})$ of the Poincaré group is called irreducible if the only non-trivial closed invariant subspace is $\mathcal{H}$. That is $\Pi(\mathcal{P})(V) \subset V$ for some closed subspace V of $\mathcal{H}$ if and only if $V = \mathcal{H}$. The closed condition is technical: We want the invariant subspace to be a Hilbert space in its own right which is not automatically true in infinite dimensional Hilbert space unless the subspace is closed.

Wigner suggested that the irreducible projective representations of the Poincaré group correspond to elementary particles. within the quantum system under consideration. Wigner's argument was as follows: An elementary particle in a quantum mechanical system is a vector in $\mathbb{P}(\mathcal{H})$. As discussed, different observers will see different vectors in $\mathbb{P}(\mathcal{H})$ corresponding to the elementary particle. All these vectors must be related by some projective automorphism. The set of all these vectors constitute a $\mathcal{P}$-invariant subspace of $\mathbb{P}(\mathcal{H})$ and hence we obtain a subrepresent-

-ation of $(\Pi, \mathcal{H})$. This subrepresentation can be thought of as a subsystem which is elementary if it is irreducible (otherwise it will have more smaller subsystems). This reduces the problem of determining all relativistic free particles in Minkowski space time to the mathematical task of finding all irreducible projective representations of the Poincarè group.

Projective representations of Poincarè group:

Let us now take a look at the Poincarè group more closely. The physically relevant Poincare group is the semi-direct product of the orthochronous proper Lorentz group and the abelian Translation group. i,e

$$\mathcal{P} := ISO(1, D-1) = SO(1,D-1)_I \ltimes \mathbb{R}^{1,D-1}$$

where $SO(1,D-1)_I$ is the connected component of identity in the Lorentz group. The Poincarè algebra is generated by the generators of translations & Lorentz transformations denoted by P^μ, $M^{\mu\nu}$ respectively. They satisfy

$$i[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\sigma\mu} M_{\rho\nu} + \eta_{\sigma\nu} M_{\rho\mu}$$

$$i[P_\mu, M_{\rho\sigma}] = \eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho,$$

$$i[P_\mu, P_\rho] = 0.$$

The third commutator says that P_μ commutes among themselves. So we start with states in $\mathbb{P}(\mathcal{H})$ which are simultaneous eigenvectors of P^μ. We label all other degrees of freedom by σ. We have

$$P^\mu \psi_{q,\sigma} = q^\mu \psi_{q,\sigma}$$

Note that infinitesimal translations are represented by $U = \mathbb{1} - i P^\mu \epsilon_\mu$ and repeating this, we obtain finite translations

$$U(\mathbb{1}, a) = e^{-i P^\mu a_\mu}$$

so that

$$U(\mathbb{1}, a) \psi_{q,\sigma} = e^{-i q \cdot a} \psi_{q,\sigma}.$$

These $U(\mathbb{1},a)$ are the projective representations of the translation part of the Poincaré group. Usually the physical requirement restricts U to be unitary which restricts P^μ to be Hermitian.

Recall that

$$(\Lambda, a)\cdot(\Lambda', a') = (\Lambda\Lambda', a' + \Lambda a) \quad \text{in} \quad \mathcal{P}.$$

$$\& \quad (\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda a)$$

An infinitesimal Poincare transformation with parameters ω, ϵ is unitarily represented as

$$U(1+\omega, \epsilon) = \mathbb{1} + \frac{i}{2}\omega_{\mu\nu} M^{\mu\nu} - \epsilon_\mu P^\mu + \ldots$$

So for a general $\Lambda \in SO(1, D-1)$ we have

$$U(\Lambda, a)\, U(1+\omega, \epsilon)\, U(\Lambda, a)^{-1} = U(\Lambda(1+\omega)\Lambda^{-1}, \Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)$$

Using infinitesimal version upto linear order in ω & ϵ, we get

$$U(\Lambda, a)\left[\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} - \epsilon_\mu P^\mu\right] U(\Lambda, a)^{-1} =$$

$$\mathbb{1} + \frac{i}{2}(\Lambda\omega\Lambda^{-1})_{\mu\nu} M^{\mu\nu} - (\Lambda\epsilon - \Lambda\omega\Lambda^{-1}a)_\mu P^\mu$$

Comparing coefficients of $\omega_{\mu\nu}$ & ϵ_μ we get

$$\boxed{\begin{aligned} & U(\Lambda, a) M^{\mu\nu} U(\Lambda, a)^{-1} = (\Lambda^{-1})^\mu{}_\lambda (\Lambda^{-1})^\nu{}_\rho (M^{\lambda\rho} - a^\lambda P^\rho + a^\rho P^\lambda) \\ \& \ & U(\Lambda, a) P^\rho U(\Lambda, a)^{-1} = (\Lambda^{-1})^\rho{}_\mu P^\mu. \end{aligned}}$$

Our aim now is to find the projective representation of the Lorentz part of the Poincarè group. Indeed if $U(\Lambda, 0) \equiv U(\Lambda)$ is such a representation then

$$P^\mu U(\Lambda)\Psi_{p,\sigma} = U(\Lambda)\, U(\Lambda)^{-1} P^\mu U(\Lambda)\, \Psi_{p,\sigma}$$

$$= U(\Lambda)\, \Lambda^{\mu}{}_{\nu}\, P^{\nu}\, \Psi_{p,\sigma}$$

$$= (\Lambda p)\, U(\Lambda) \Psi_{p,\sigma}$$

So we must have

$$U(\Lambda) \Psi_{p,\sigma} = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda, p)\, \Psi_{\Lambda p, \sigma'}$$

In general this representation is reducible. Since this is unitary rep., a theorem in rep theory say that it is completely reducible i.e it can be written as direct sum of irreducible reps with respect to invariant subspace of eigenvectors of P^{μ} with eigenvalue Λp. Our goal is to classify all such irreps. To do so, we first calculate the orbit of action of Lorentz group on $\mathbb{R}^{1,D-1}$. It is clear that $SO(1,D-1)_I$ fixes p^2 $\forall\, p \in \mathbb{R}^{1,D-1}$ but when $p^2 \leq 0$ then it also fixes the sign of p^0. Accordingly we get the following orbits:

(i) $p^2 = m^2 > 0$ - one sheeted hyperboloid.

(ii) $p^2 = -m^2 < 0$ - two sheeted hyperboloid corresponding to $p^0 > 0$ or $p^0 < 0$.

(iii) $p^2 = 0$ - a cone with vertex at origin.

Now given any p^μ, one can choose (depending on the orbit of p^μ) a standard q^μ such that

$$p^\mu = L^\mu{}_\nu(p) q^\nu$$

where $L^\mu{}_\nu \in SO(1, D-1)_I$. By above discussion

$$\Psi_{p,\sigma} = N(p) U(L^\mu{}_\nu(p)) \Psi_{q,\sigma}$$

where $N(p)$ is some normalisation factor. Now for any $\Lambda \in SO(1, D-1)_I$ we have

$$U(\Lambda) \Psi_{p,\sigma} = N(p)\, U(\Lambda) U(L(p)) \Psi_{q,\sigma}$$

$$= N(p)\, U(L(\Lambda p))\, U(L^{-1}(\Lambda p) \Lambda L(p)) \Psi_{q,\sigma}$$

where we used property of group representations. Note that

$$L^{-1}(\Lambda p) \Lambda L(p) q = L^{-1}(\Lambda p) \Lambda p = q.$$

The set of all such elements of $\mathcal{P}$ is called the stability group of q or the little group. For any two elements W & $\overline{W}$ in the little group

$$U(W) \Psi_{q,\sigma} = \sum_{\sigma'} D^q_{\sigma,\sigma'}(W) \Psi_{q,\sigma'}$$

& $$U(\overline{W} W) \Psi_{q,\sigma} = \sum_{\sigma'} D^q_{\sigma,\sigma'}(W) \sum_{\sigma''} D^q_{\sigma',\sigma''}(\overline{W}) \Psi_{q,\sigma''}$$

$$= \sum_{\sigma',\sigma''} D^q_{\sigma,\sigma'}(W)\, D^q_{\sigma',\sigma''}(\bar{W})\, \Psi_{q,\sigma''}$$

$$= \sum_{\sigma''} D^q_{\sigma,\sigma''}(\bar{W} W)\, \Psi_{q,\sigma''}$$

where $\quad D^q_{\sigma,\sigma''}(\bar{W} W) = \sum_{\sigma'} D^q_{\sigma,\sigma'}(W)\, D^q_{\sigma',\sigma''}(\bar{W}).$

✓ Thus we see that $D^q(W)$ is a representation of the little group. So $W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p)$ we have

$$U(W(\Lambda,p))\, \Psi_{q,\sigma} = \sum_{\sigma'} D_{\sigma,\sigma'}(W(\Lambda, p))\, \Psi_{q,\sigma'}$$

So that

$$U(\Lambda)\, \Psi_{p,\sigma} = N(p) \sum_{\sigma'} D_{\sigma,\sigma'}(W(\Lambda,p))\, U(L(\Lambda p))\, \Psi_{q,\sigma'}$$

$$= \frac{N(p)}{N(\Lambda p)} \sum_{\sigma'} D_{\sigma,\sigma'}(W(\Lambda,p))\, \Psi_{\Lambda p,\sigma'}$$

So apart from the normalisation factor, the problem of finding unitary irrep. of Poincaré group has been reduced to finding unitary irreps of the little group depending on the orbit. So we first find little group corresponding to each orbit.

(i) $q^2 = m^2 \geq 0$. By going to rest frame, we can set q^μ to

$$q^\mu = (0, 0, \cdots, 0, m).$$

Looking at the form of this vector, we can see that the little group is $SO(1, D-2)_I \hookrightarrow SO(1, D-1)_I$.

(ii) $q^2 = -m^2 \leq 0$. By going to rest frame, we can take q^μ to be

$$q^\mu = (m, \vec{0}).$$

The little group is $SO(D-1)$.

(iii) $q^2 = 0$. It is not so obvious. Although it turns out to be the Euclidean group $E(D-2) = SO(D-2) \ltimes \mathbb{R}^{D-2}$. This is the isometry group of $\mathbb{R}^{D-2}$ with the standard Euclidean metric.

Gender	orbit	Little group	Unitary irrep.
$q^2 = -m^2$	Mass-shell	$SO(D-1)$	Massive
$q^2 = m^2$	Hyperboloid	$SO(1, D-2)_I$	Tachyonic
$q^2 = 0$	Light cone	$E(D-2)$	Massless
$q^\mu = 0$	origin	$\mathcal{P}$	zero momentum

In $q^2=0$, one case is $q^\mu=0$ whose stabiliser is the whole Poincaré group $\mathcal{P}$.

Physically, Tachyonic representations are not accepted. So we will only deal with the other two. One can use the method of induced representations to find all irreducible reps. of the Euclidean group. The idea is to go to the Lie algebra of $E(D-2)$ and identify the "translations" generators and repeat the procedure. The upshot of this computation is that we get two orbits and the corresponding little groups are called short little group. The corresponding unitary irreducible reps. are labeled as helicity and infinite spin. The analogue of the Lorentz group here is obviously $SO(D-2)$. The short Little group corresponding to infinite spin is $SO(D-3)$ & the short little group corresponding to helicity is $SO(D-2)$.

Next one can use Young Tableau to embed the irrep. of the Little groups in all cases into tensorial reps. It turns out for $SO(D-2)$ the tensorial rep. of dimension $(D-2)^2$ consists of three parts irreducible parts:

$$\text{Traceless symmetric} \oplus \text{Antisymmetric} \oplus \text{Trace (Scalar)}$$

$$\text{Dim:} \quad \frac{(D-2)(D-1)}{2}-1 \qquad\qquad \frac{(D-2)(D-3)}{2} \qquad\qquad 1$$

First String excitation:

The first excited state of string is a set of $(D-2)^2$ particle states

$$\alpha^i_{-1}\,\alpha^j_{-1}\,|0;p^\mu\rangle$$

i.e a tensor product of reps. of $SO(D-2)$. Thus if we want to embed these states into a rep. of $SO(1,D-1)_I \ltimes \mathbb{R}^{1,D-1}$, these have to be massless reps. This means that

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{D-2}{24}\right) = 0$$

$$\Rightarrow \quad \boxed{D = 26.}$$

This also fixes normal ordering constant to be 1. This is consistent with our discussion of spurious states. Let us look at the tensorial reps. of the first string excitation:

$$G_{\mu\nu}(X), \qquad B_{\mu\nu}(X), \qquad \Phi(X)$$

$G_{\mu\nu}(X)$ → Graviton (Symmetric traceless)

$B_{\mu\nu}(X)$ → Kalb-Ramond field (Antisymmetric)

$\Phi(X)$ → Dilaton (Trace)

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Look at $G_{\mu\nu}$. It is symmetric traceless rank 2 tensor. Let us look at the state:

$$\alpha^i_{-1}\alpha^j_{-1}|0;p^\mu\rangle = \underbrace{\left(\alpha^{(i}_{-1}\tilde{\alpha}^{j)}_{-1} - \frac{1}{d-2}\delta^{ij}\alpha^k_{-1}\tilde{\alpha}^k_{-1}\right)}_{\text{symmetric traceless}}|0;p^\mu\rangle + \overbrace{\alpha^{[i}_{-1}\tilde{\alpha}^{j]}_{-1}|0;p^\mu\rangle}^{\text{Antisymmetric.}}$$

$$+ \underbrace{\frac{1}{d-2}\delta^{ij}\alpha^k_{-1}\tilde{\alpha}^k_{-1}|0;p^\mu\rangle}_{\text{trace}}$$

where $(\,,\,)$ & $[\,,\,]$ are the symmetrized and antisymmetrized indices. Thus the first excited state decomposes into three pieces giving rise to three fields. $G_{\mu\nu}$ is particu--lary interesting as it represents spin 2 massless field. We will identify with the metric of spacetime, the Graviton because Weinberg in [DOI: 10.1103/PhysRev.138.B988] showed that any interacting theory of Massless spin 2 particle is Einstein's gravity. Later we will explicitly derive Einstein's field equations from this field.

Higher Excited states:

All higher excited states are massive with our choice of D & a as can be seen from mass spectrum

$$M^2 = \frac{4}{\alpha'}(N-1) = \frac{4}{\alpha'}(\tilde{N}-a)$$

So by Wigner's classification theorem, all of these must fit into representation of $SO(D-1)$. Let us check this for $N=\tilde{N}=2$. These states are:

$$\alpha^i_{-1}\alpha^j_{-1}|0;p^\mu\rangle, \quad \alpha^i_{-2}|0;p^\mu\rangle \qquad \text{- right moving}$$
$$\tilde{\alpha}^i_{-1}\tilde{\alpha}^j_{-1}|0;p^\mu\rangle, \quad \tilde{\alpha}^i_{-2}|0;p^\mu\rangle \qquad \text{- Left moving.}$$

Since α^i_{-1}, α^j_{-1} commutes, in the right moving sector there are a total of

$$\frac{1}{2}(D-2)(D-1)+(D-2) = (D-2)\left(\frac{D-1+2}{2}\right)$$

$$= \frac{(D-2)(D+1)}{2} = \frac{1}{2}D(D-1)-1$$

These easily fit into the symmetric traceless representation of $SO(D-1)$. Infact one can prove that all higher excited states fit into $SO(D-1)$ representation.

Lorentz invariance:

In Lightcone quantisation, Lorentz invariance is not manifest. Infact we had choose $a=1$ & $D=26$ to ensure Lorentz invariance. This can be made concrete as follows: We compute the conserved charges and currents corresponding to the global Poincare symmetry

$X^\mu \longrightarrow \Lambda^\mu_{\ \nu} X^\nu + c^\mu$ of the action and require that they satisfy Poincarè algebra. Let us begin with translations $X^\mu \longrightarrow X^\mu + c^\mu$. One can compute the Noether current. It turns out to be:

$$P^\alpha_\mu = \frac{1}{2\pi\alpha'} \partial^\alpha X_\mu .$$

It is easy to see that $\partial_\alpha P^\alpha_\mu = 0$ as $\partial_\alpha \partial^\alpha X_\mu = 0$ on-shell. Next, the Noether charge corresponding to Lorentz transformation $X^\mu \longrightarrow \Lambda^\mu_{\ \nu} X^\nu$ is:

$$J^\alpha_{\mu\nu} = P^\alpha_\mu X_\nu - P^\alpha_\nu X_\mu .$$

We can again check that $\partial_\alpha J^\alpha_{\mu\nu} = 0$. Indeed

$$\begin{aligned}
\partial_\alpha J^\alpha_{\mu\nu} &= (\partial_\alpha P^\alpha_{\mu\nu}) X_\nu + P^\alpha_\mu \partial_\alpha X_\nu - (\partial_\alpha P^\alpha_\nu) X_\mu - P^\alpha_\nu \partial_\alpha X_\mu \\
&= P^\alpha_\mu \partial_\alpha X_\nu - P^\alpha_\nu \partial_\alpha X_\mu \\
&= \frac{1}{2\pi\alpha'} \left(\partial^\alpha X_\mu \partial_\alpha X_\nu - \partial^\alpha X_\nu \partial_\alpha X_\mu \right) \\
&= 0 .
\end{aligned}$$

The conserved charges corresponding to $J^{\tau}_{\mu\nu}$ is

$$M_{\mu\nu} = \int_0^{\pi} d\sigma \, J^{\tau}_{\mu\nu}$$

Now using the mode expansion for X^{μ} we get

$$M^{\mu\nu} = \int_0^{\pi} d\sigma \left(X^{\mu}\Pi^{\nu} - X^{\nu}\Pi^{\mu}\right) = \frac{1}{2\pi\alpha'}\int_0^{\pi} d\sigma \left(X^{\mu}\dot{X}^{\nu} - X^{\nu}\dot{X}^{\mu}\right)$$

$$= l^{\mu\nu} + E^{\mu\nu} + \tilde{E}^{\mu\nu}$$

where $l^{\mu\nu} = x^{\mu}p^{\nu} - x^{\nu}p^{\mu}$,

$$E^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha^{\mu}_{-n}\alpha^{\nu}_{n} - \alpha^{\nu}_{-n}\alpha^{\mu}_{n}\right), \quad \tilde{E}^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\tilde{\alpha}^{\mu}_{-n}\tilde{\alpha}^{\nu}_{n} - \tilde{\alpha}^{\nu}_{-n}\tilde{\alpha}^{\mu}_{n}\right)$$

The first piece $l^{\mu\nu}$ is the orbital angular momentum of the string while the other two pieces arise from excited states. Classically, one can check that the Poisson bracket for $M_{\mu\nu}$ satisfies Lorentz algebra. In covariant quantisation, it is easy to check that $M_{\mu\nu}$ satisfies Lorentz algebra but in Lightcone gauge things are not so clear. In lightcone gauge, we must be able to produce Lorentz algebra. The only bracket which is non-trivial is

$$[M^{i-}, M^{j-}] = 0.$$

This commutator involves p^- & α_n^- which has been fixed in Lightcone guage in terms of other transverse oscillators. A messy calculation gives

$$[M^{i-}, M^{j-}] = \frac{2}{(p^+)^2} \sum_{n>0}' \left(\left[\frac{D-2}{24} - 1 \right] n + \frac{1}{n}\left[a - \frac{D-2}{24} \right] \right) \left(\alpha_{-n}^i \alpha_n^j - \alpha_{-n}^j \alpha_n^i \right)$$

$$+ \frac{2}{(p^+)^2} \sum_{n>0}' \left(\left[\frac{D-2}{24} - 1 \right] n + \frac{1}{n}\left[a - \frac{D-2}{24} \right] \right) \left(\tilde{\alpha}_{-n}^i \tilde{\alpha}_n^j - \tilde{\alpha}_{-n}^j \tilde{\alpha}_n^i \right)$$

which is 0 if & only if $a = 1$ & $D = 26$. This is consistent with our earlier derivation of critical dimension.

Open strings and D-branes:

pop up because it cancels from both parts. Now we have

$$J^{ij}\left(\alpha_{-1}^{(i}\tilde{\alpha}_{-1}^{j)} - \frac{1}{d-2}\delta^{ij}\alpha_{-1}^{k}\tilde{\alpha}_{-1}^{k}\right)|0;p^{\mu}\rangle$$

$$= \frac{1}{2}$$

y

generator and check the spin of these parts. Indeed using expression for an ular momentum we have

When we promote these to operators, normal ordering does not