Introduction to String Theory

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Note To The Reader

These notes are a work in progress and will be updated very frequently. The updated notes can be accessed at the link

https://ranveer14.github.io/String_Theory_notes.pdf

These notes are based on several references, but mostly I am following Polchinki's two volumes on string theory and Lüst and Theisen's Basic concepts of string theory. David Tong's lecture notes are also a great addition to the selection of topics discussed in these notes. Depending on the topics, I have also included discussions from standard references. For example, in conformal field theory, I have included topics from the Yellow book.

If you find typos/corrections, please send them to my email ranveersfl@gmail.com.

Thank you
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Chapter 1

The Free Relativistic Particle

Let $\mathbb{R}^{1,D-1}$ denote the $D-$dimensional spacetime. We denote a typical vector $X^\mu \in \mathbb{R}^{1,D-1}$ by

$$X^\mu = (X^0, X^i)$$

where $(X^i) \in \mathbb{R}^{D-1}$. We sometimes write $X^\mu = (X^0, \vec{X})$. Our signature for the Minkowski space is

$$\eta_{\mu\nu} = \text{diag}(-1, 1, \ldots, 1).$$

We use natural units $\hbar = c = 1$. Consider a free particle moving in $D-$dimensional spacetime. We want to describe its dynamics. We want our theory to be relativistic, meaning that the theory that we develop must be invariant under Lorentz transformation.

1.1 The Action of a Free Relativistic Particle

We begin by writing down a Lorentz invariant action for the free particle. The most natural choice for a Lorentz invariant action is choosing a Lorentz scalar and the cannonical choice is to chose the length of the world line traced by the particle in spacetime. Put

$$S = \int dt\ L = -m \int dt \sqrt{1 - \dot{\vec{X}}^2}, \quad X^0 = t, \dot{\vec{X}} = \frac{d\vec{X}}{dt}, \tag{1.1.1}$$

where m is a parameter which we will identify with the mass of the particle. We can now compute the conjugate momentum of the system in the usual way. We have

$$P^i = \frac{\delta S}{\delta \dot{X}^i} = \frac{-m(2\dot{X}^i)}{-2\sqrt{1 - \dot{\vec{X}}^2}} = \frac{m\dot{X}^i}{\sqrt{1 - \dot{\vec{X}}^2}}.$$

The Hamiltonian of the system is given by

$$H = \vec{P} \cdot \dot{\vec{X}} - L = \frac{m\dot{\vec{X}}^2}{\sqrt{1 - \dot{\vec{X}}^2}} + m\sqrt{1 - \dot{\vec{X}}^2} = \frac{m}{\sqrt{1 - \dot{\vec{X}}^2}},$$

which we recognise as the usual relativistic energy of a free particle and hence m is identified as the mass of the particle. Note that the action is not manifestly Lorentz invariant as we are treating the first component of the spacetime vector differently from the remaining components. But we want an action which is manifestly Lorentz invariant. One way to obtain such an action is to promote t to be an independent variable and then parametrize the spacetime coordinates by some other parameter say τ. So put $t = X^0$ and parametrize $X^\mu = (X^0, X^i)$ as

$$X^\mu = X^\mu(\tau).$$

By a simple application of chain rule, we have

$$dt = \frac{dX^0}{d\tau} d\tau.$$

The action can then be written as

$$\begin{aligned} S = -m \int d\tau \sqrt{\left(\frac{dX^0}{d\tau}\right)^2 - \left(\frac{d\vec{X}}{dt}\right)^2 \left(\frac{dX^0}{d\tau}\right)^2} &= -m \int d\tau \sqrt{-\left[-\left(\frac{dX^0}{d\tau}\right)^2 + \left(\frac{d\vec{X}}{d\tau}\right)^2\right]} \\ &= -m \int d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}}. \end{aligned}$$

Remark 1.1.1. It seems that we have added a new degree of freedom to our system, namely X^0. Later we will see that this is not the case as our system will have *reparametrization invariance* also called *diffeomorphism invariance* which will make one of the degree of freedom redundant.

It is now clear that S can be interpreted as the length of the worldline it traces in spacetime.

1.2 Symmetries of the Action

Let us now look at the symmetries of our system:

1.2.1 Poincaré invariance

This is a manifest global symmetry of the system.

$$X^\mu \to \widetilde{X}^\mu = \Lambda^\mu{}_\nu X^\nu + \xi^\mu, \quad \Lambda^\mu{}_\nu \in \mathrm{SO}(1, D-1), \xi^\mu \in \mathbb{R}^{1,D-1},$$

where $\mathrm{SO}(1, D-1)$ denotes the Lorentz group. We have

$$\begin{aligned} \frac{d\widetilde{X}^\mu}{d\tau}\frac{d\widetilde{X}_\mu}{d\tau} = \eta_{\mu\nu}\frac{d\widetilde{X}^\mu}{d\tau}\frac{d\widetilde{X}^\nu}{d\tau} = \eta_{\mu\nu}\Lambda^\mu{}_\rho\frac{dX^\rho}{d\tau}\Lambda^\nu{}_\sigma\frac{dX^\sigma}{d\tau} = \left(\Lambda^\mu{}_\rho\eta_{\mu\nu}\Lambda^\nu{}_\sigma\right)\frac{dX^\rho}{d\tau}\frac{dX^\sigma}{d\tau} &= \eta_{\rho\sigma}\frac{dX^\rho}{d\tau}\frac{dX^\sigma}{d\tau} \\ &= \frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}, \end{aligned}$$

where we used the property of Lorentz transformations

$$\Lambda^T \eta \Lambda = \eta. \tag{1.2.1}$$

This implies that

$$\widetilde{S} = -m \int d\tau \sqrt{-\frac{d\widetilde{X}^\mu}{d\tau}\frac{d\widetilde{X}_\mu}{d\tau}} = -m \int d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}} = S.$$

We could have directly concluded this by the fact that the action is the length of a curve and hence a Lorentz scalar. So it does not transform under Lorentz transformations.

1.2.2 Diffeomorphism Invariance

We can reparametrize the world line by changing the parameter τ:

$$\tau \to \widetilde{\tau} = \widetilde{\tau}(\tau).$$

where $\widetilde{\tau}(\tau)$ is a monotonic function[1] of τ. The integration measure changes according to the usual change of variable rule. Next under reparametrization we have

$$\widetilde{X}^\mu(\widetilde{\tau}(\tau)) = X^\mu(\tau).$$

Hence we see that the transformed action is

$$S = -m \int d\widetilde{\tau} \left|\frac{d\tau}{d\widetilde{\tau}}\right| \sqrt{-\frac{d\widetilde{X}^\mu}{d\widetilde{\tau}}\frac{d\widetilde{X}_\mu}{d\widetilde{\tau}}\left(\frac{d\widetilde{\tau}}{d\tau}\right)^2} = -m \int d\widetilde{\tau} \left|\frac{d\tau}{d\widetilde{\tau}}\frac{d\widetilde{\tau}}{d\tau}\right| \sqrt{-\frac{d\widetilde{X}^\mu}{d\widetilde{\tau}}\frac{d\widetilde{X}_\mu}{d\widetilde{\tau}}} = \widetilde{S}.$$

This is a local symmetry of the theory - a gauge symmetry as it depends on the the local coordinates of the spacetime. It is also a continuous symmetry of the action. As is well know, gauge symmetries are not really symmetries in the sense that we do not have an associated conserved charge, rather it is a redundancy in the description of the theory which we need to fix when we go to quantum theory by a process called gauge fixing. We now return to the resolution of Remark 1.1.1. Since the time component of spacetime vector is monotonically increasing, we can reparametrize the worldline in such a way that

$$\widetilde{\tau} = X^0(\tau) = t.$$

Fixing the redundancy of the system we get back to our original action. This shows that we have not increased the number of degrees of freedom of the theory by introducing a parameter.

[1] monotonicity is a techinical requirement for reparametrization. Basically what we need is that as we increase τ, we should traverse the worldline in one given direction and not flip between positions. Generally, $\widetilde{\tau}$ is assumed to be increasing so that we travel the worldline in the same direction as in the original parametrization.

1.3 Quantisation

We will now try to quantise the system. We will illustrate four different methods of quantisation, each with its own advantage. This will help us when we go to the string action.

1.3.1 First Method

We quantise our original action in (1.1.1) directly using the Dirac prescription. The conjugate momentum and the Hamiltonian was calculated to be

$$P^i = \frac{m\dot{X}^i}{\sqrt{1-\dot{\vec{X}}^2}}, \quad H = \frac{m}{\sqrt{1-\dot{\vec{X}}^2}}.$$

where the dot represents derivative with respect to t. We promote the fields to operators with the standard substitution $P^i = -i\partial_i$ where $\partial_i = \frac{\partial}{\partial X^i}$ and introduce the wavefunction which satisfies the Schrödinger equation with the above Hamiltonian. Let $\phi(t, X^i)$ be the wavefunction. Then the Schroödinger equation is given by

$$i\frac{\partial\phi}{\partial t} = H\phi.$$

This implies that

$$-\frac{\partial^2\phi}{\partial t^2} = H^2\phi.$$

Next, one can easily check that

$$H^2 = \vec{P}^2 + m^2.$$

Thus the Schroödinger equation becomes

$$-\frac{\partial^2\phi}{\partial t^2} = (-\partial_i^2 + m^2)\phi$$

which implies

$$(\partial_\mu\partial^\mu - m^2)\phi = 0. \tag{1.3.1}$$

We can now solve (1.3.1) and get all the quantum dynamics of the system.

Remark 1.3.1. We can recognise (1.3.1) with the Klein-Gordan equation in field theory. There is one crucial difference in our case and the field theory Klein-Gordan equation. In field theory we quantise quantum fields while in our case (relativistic quantum mechanics), we quantise wavefunctions.

1.3.2 Second Method

We will now denote the τ derivative by dot. That is

$$\dot{X}^\mu = \frac{dX^\mu}{d\tau}.$$

Momentum conjugate to X^μ is

$$P^\mu = \frac{\delta S}{\delta \dot{X}^\mu} = \frac{m\dot{X}^\mu}{\sqrt{-\dot{X}^\mu \dot{X}_\mu}}.$$

One easily sees that

$$P^\mu P_\mu + m^2 = 0. \tag{1.3.2}$$

Eq. (1.3.1) is a constraint. Note that we have not yet appealed to the equation of motion of the action to derive Eq. (1.3.1). Such constraints which follow directly from the definition of the conjugate momenta are called *primary constraints*. The number of primary constraints in a system is equal to the number of zero eigenvalues of the Hessian matrix

$$\frac{\partial P^\mu}{\partial \dot{X}^\nu} = \frac{\partial^2 L}{\partial \dot{X}^\mu \dot{X}^\nu}.$$

Note that by the Inverse Function Theorem we need that all eigenvalues of $\frac{\partial P^\mu}{\partial \dot{X}^\nu}$ be nonzero if we want to express P^μ as a function of $\dot{X}^\mu$. Hence in a system with primary constraint, we cannot express P^μ as functions of $\dot{X}^\mu$.

Remark 1.3.2. Any system with "τ"−reparametrization invariance has primary constraints.

The Hamiltonian for the system is

$$H = P^\mu \dot{X}_\mu - L = \frac{m\dot{X}^\mu \dot{X}_\mu}{\sqrt{-\dot{X}^\nu \dot{X}_\nu}} + m\sqrt{-\dot{X}^\nu \dot{X}_\nu} = 0.$$

This is not surprising. Vanishing Hamiltonian signals that nothing changes if we pick another parametrization. To quantise the system, we follow Dirac prescription. We promote the fields to operators and the constraint to an operator equation and demand that the wavefunction $|psi(X)$ satisfy the operator equation:

$$(P^\mu P_\mu + m^2)\Psi(X) = 0.$$

The Schrödinger equation is

$$i\frac{\partial \Psi}{\partial \tau} = H\Psi = 0.$$

This simply implies that the wavefunction does not depend on the parametrization - something that we expected. After the standard substitution

$$P^\mu = -i\partial_\mu, \quad \text{where } \partial_\mu = \frac{\partial}{\partial X^\mu},$$

the operator equation (1.3.1) becomes

$$(\partial_\mu \partial^\mu - m^2)\Psi = 0. \tag{1.3.3}$$

This is the same equation that we got in the first method. Hence we again get the same dynamics.

1.3.3 Third Method - Introducing Einbein

Note that both the equivalent actions above have squareroots which makes it difficult to quantise when we go to path-integral quantisation. So, we somehow want to get rid of the squareroot. Moreover the two previous actions cannot be generalised to massless particles due to the m factor in front of the action. Both these problems can be fixed on the expense of introducing another auxiliary field - an einbein in the action which will be fixed by its equation of motion in the classical theory. To be more precise, consider the action

$$S_e = \frac{1}{2}\int d\tau \left(\frac{\dot{X}^\mu \dot{X}_\mu}{e} - em^2\right),$$

where $e = e(\tau)$ is the auxiliary einbein field. Varying the action with respect to e gives

$$\delta S = \frac{1}{2}\int d\tau \left(-\frac{\dot{X}^\mu \dot{X}_\mu}{e^2} - m^2\right)\delta e.$$

Thus $\delta S = 0$ implies

$$e = \frac{\sqrt{-\dot{X}^\mu \dot{X}_\mu}}{m}. \tag{1.3.4}$$

Not that the equation of motion of e is an algebraic equation and hence the field e is not dynamical. If we now plug the expression for e from (1.3.4) in the action S_e we get

$$\begin{aligned} S_e = \frac{1}{2}\int d\tau \left(-m\frac{\dot{X}^\mu \dot{X}_\mu}{\sqrt{-\dot{X}^\mu \dot{X}_\mu}} - m^2\frac{\sqrt{-\dot{X}^\mu \dot{X}_\mu}}{m}\right) &= -\frac{2m}{2}\int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu} \\ &= S. \end{aligned}$$

So both the actions are really the same. Thus the two actions are equivalent classically and give the same dynamics. We now want to quantise this action. The conjugate momentum corresponding to e is

$$P_e = \frac{\partial L}{\partial \dot{e}} = 0.$$

The momentum conjugate to X^μ is

$$P^\mu = \frac{\partial L}{\partial \dot{X}_\mu} = \frac{2}{2e}\dot{X}^\mu \implies \dot{X}^\mu = eP^\mu. \tag{1.3.5}$$

The Hamiltonian of the system is given by

$$H = \dot{X}^\mu P_\mu - L = eP^\mu P_\mu - \frac{m}{2e}e^2 P^\mu P_\mu + \frac{m^2}{2}e = \frac{e}{2}(P^\mu P_\mu + m^2),$$

where we used (1.3.5). The Poisson bracket

$$\{P_e, H\}_{P.B.} = \frac{\partial P_e}{\partial P_e}\frac{\partial H}{\partial e} = \frac{1}{2}(P^\mu P_\mu + m^2).$$

But since $P_e = 0$, we get

$$H = \frac{e}{2}(P^\mu P_\mu + m^2) = 0. \tag{1.3.6}$$

The next step in the quantisation process is to promote fields to operators and use the standard operator substitution for P^μ. Suppose the wavefunction of the system is $\psi = \psi(X, e)$. Then the operator eqaution corresponding to $P_e = 0$ implies

$$-i\frac{\partial \psi}{\partial e} = 0.$$

This means that the wavefunction is independent of the einbein - again something that we expected physically. The operator equation corresponding to (1.3.6) gives

$$(P^\mu P_\mu + m^2)\psi = 0 \implies (\partial_\mu \partial^\mu - m^2)\psi = 0.$$

Thus we get the same quantum dynamics as in the previous two methods.

1.3.4 Fourth Method - Gauge Fixing

We begin by observing that S_e has diffeomorphism symmetry. Indeed if we choose another parametrization $\widetilde{\tau} = \widetilde{\tau}(\tau)$, then

$$X^\mu(\tau) \to \widetilde{X}^\mu(\widetilde{\tau}(\tau)) = X^\mu(\tau)$$

and

$$\frac{\partial X^\mu}{\partial \tau} = \frac{\partial \widetilde{X}^\mu}{\partial \widetilde{\tau}}\frac{\partial \widetilde{\tau}}{\partial \tau}$$

and using (1.3.4)

$$e(\tau) = \sqrt{-\frac{\partial \widetilde{X}^\mu}{\partial \tau}\frac{\partial \widetilde{X}_\mu}{\partial \tau}} = \sqrt{-\frac{\partial \widetilde{X}^\mu}{\partial \widetilde{\tau}}\frac{\partial \widetilde{X}_\mu}{\partial \widetilde{\tau}}}\left|\frac{\partial \widetilde{\tau}}{\partial \tau}\right| = \widetilde{e}(\widetilde{\tau})\left|\frac{\partial \widetilde{\tau}}{\partial \tau}\right|.$$

Thus we see that

$$\begin{aligned}
S_e &= \frac{1}{2}\int d\widetilde{\tau}\left|\frac{\partial \tau}{\partial \widetilde{\tau}}\right|\left(-\frac{\partial \widetilde{X}^\mu}{\partial \widetilde{\tau}}\frac{\partial \widetilde{X}_\mu}{\partial \widetilde{\tau}}\left(\frac{\partial \widetilde{\tau}}{\partial \tau}\right)^2\frac{1}{\widetilde{e}(\widetilde{\tau})}\left|\frac{\partial \widetilde{\tau}}{\partial \tau}\right|^{-1} - m^2\widetilde{e}(\widetilde{\tau})\left|\frac{\partial \widetilde{\tau}}{\partial \tau}\right|\right) \\
&= \frac{1}{2}\int d\widetilde{\tau}\left(-\frac{\partial \widetilde{X}^\mu}{\partial \widetilde{\tau}}\frac{\partial \widetilde{X}_\mu}{\partial \widetilde{\tau}}\frac{1}{\widetilde{e}(\widetilde{\tau})} - m^2\widetilde{e}(\widetilde{\tau})\right) \\
&= \widetilde{S}_e.
\end{aligned}$$

So before we go on quantising the system, we will fix a gauge. We choose a reparametrization $\widetilde{\tau}(\tau)$ such that

$$\widetilde{e}(\widetilde{\tau}) = 1.$$

With this gauge choice, when we go to the quantum theory, we will have to take care of the equation of motion of the einbein and impose it as operator equation with the chosen gauge. We follow the standard procedure of quatization by promoting fields to operators. The equation of motion for e with the above gauge choice becomes:

$$\widetilde{e}(\widetilde{\tau})^2 = -\frac{1}{m^2}\frac{\partial \widetilde{X}^\mu}{\partial \widetilde{\tau}}\frac{\partial \widetilde{X}_\mu}{\partial \widetilde{\tau}} = 1.$$

Using (1.3.5), the equation of motion of einbein becomes

$$P^\mu P_\mu + m^2 = 0. \tag{1.3.7}$$

With the chosen gauge, the action becomes

$$S = \frac{1}{2}\int d\tau \left(\dot{X}^\mu \dot{X}_\mu - m^2\right),$$

where we removed the tildes for brevity. Using (1.3.6) and the gauge choice along with (1.3.7), the Hamiltonian is given by

$$H = \frac{1}{2}(P^\mu P_\mu + m^2) = 0.$$

With the standard substitution for the momentum operator $P_\mu = -i\partial_\mu$, the wavefunction ψ of the system satisfies

$$(\partial_\mu \partial^\mu - m^2)\psi = 0.$$

Hence we again get the same dynamics.

Chapter 2

The Relativistic String

We now want to write an action of the a free relativistic string - the fundamental objects in string theory. As we discussed in the previous chapter, we need to start with a Lorentz invariant action. Since the string is a two dimensional object, it traces a surface called the *worldsheet* in the spacetime. The most natural choice of the action would then be the surface area of the worldsheet traced by the string. We begin by deriving the action of the relativistic string.

2.1 Nambu-Goto Action

The surface traced by the string can be parametrized by two parameters (σ, τ). Let the worldsheet coordinates be $X^\mu(\sigma, \tau)$. To calculate the area of the worldsheet, we will use the worldsheet coordinates. Infinitesimal change in the parameters σ and τ along the worldsheet coordinates is

$$\delta\sigma = \frac{\partial X^\mu}{\partial \sigma} d\sigma, \quad \delta\tau = \frac{\partial X^\mu}{\partial \tau} d\tau.$$

Note that the area of the parallelogram determined by two vectors $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\begin{aligned} \|\mathbf{A}\|\|\mathbf{B}\| \sin\theta = \|\mathbf{A}\|\|\mathbf{B}\|\sqrt{1-\cos^2\theta} &= \sqrt{\mathbf{A}^2\mathbf{B}^2 - \frac{(\mathbf{A}\cdot\mathbf{B})^2}{\mathbf{A}^2\mathbf{B}^2}\mathbf{A}^2\mathbf{B}^2} \\ &= \sqrt{(\mathbf{A}\cdot\mathbf{A})(\mathbf{B}\cdot\mathbf{B}) - (\mathbf{A}\cdot\mathbf{B})^2} \\ &= \left(\det\begin{bmatrix} \mathbf{A}\cdot\mathbf{A} & \mathbf{A}\cdot\mathbf{B} \\ \mathbf{A}\cdot\mathbf{B} & \mathbf{B}\cdot\mathbf{B}\end{bmatrix}\right)^{\frac{1}{2}}, \end{aligned}$$

where $\|\mathbf{A}\|^2 = \mathbf{A}\cdot\mathbf{A} = \mathbf{A}^2$. So the infinitesimal area of the parallelogram on the worldsheet determined by the vectors $\delta\sigma$ and $\delta\tau$ is

$$d\text{Area} = \left[-\det\left(\partial_\alpha X^\mu \partial_\beta X_\mu\right)\right]^{\frac{1}{2}}, \qquad \boldsymbol{\sigma} = \sigma^\alpha \equiv (\sigma, \tau), \quad \alpha = 1, 2.$$

The minus sign indicates the fact that one of the vectors is timelike ($X^2 < 0$). The Nambu-Goto action is then defined by

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_M d\sigma d\tau \sqrt{-\det\left(\partial_\alpha X^\mu \partial_\beta X_\mu\right)}, \tag{2.1.1}$$

where M is the surface traced by the string, α' is called the *Regge slope*. The reason for this name will be evident in later chapters. We often write

$$h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu.$$

The action can then be written as

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_M d\sigma d\tau \mathcal{L}_{NG}, \quad \mathcal{L}_{NG} = \left[-\det(h_{\alpha\beta})\right]^{\frac{1}{2}}.$$

The worldsheet is in general a curved manifold embedded in spacetine. In the language of differential geometry, $h_{\alpha\beta}$ is called the Pullback metric from the ambient spacetine. The factor of $\frac{1}{2\pi\alpha'}$ can be interpreted as string tension.

2.1.1 Symmetries of the Nambu-Goto action

S_{NG} has global symmetries as well as local symmetries. Let us look at them more closely.

Reparametrization Invariance

If we choose another parametrization for the worldsheet $\widetilde{\tau}(\sigma,\tau), \widetilde{\sigma}(\sigma,\tau)$ then the Jacobian of the variable change is

$$J = \det\begin{bmatrix} \frac{\partial\widetilde{\sigma}}{\partial\sigma} & \frac{\partial\widetilde{\sigma}}{\partial\tau} \\ \frac{\partial\widetilde{\tau}}{\partial\sigma} & \frac{\partial\widetilde{\tau}}{\partial\tau} \end{bmatrix}$$

and the worldsheet coordinates changes as

$$\frac{\partial X^\mu}{\partial\sigma^\alpha} = \frac{\partial\widetilde{X}^\mu}{\partial\widetilde{\sigma}^\beta}\frac{\partial\widetilde{\sigma}^\beta}{\partial\sigma^\alpha}.$$

This gives

$$h_{\alpha\beta} = \frac{\partial\widetilde{X}^\mu}{\partial\widetilde{\sigma}^\gamma}\frac{\partial\widetilde{X}_\mu}{\partial\widetilde{\sigma}^\delta}\frac{\partial\widetilde{\sigma}^\gamma}{\partial\sigma^\alpha}\frac{\partial\widetilde{\sigma}^\delta}{\partial\sigma^\beta} = \widetilde{h}_{\alpha\beta}\frac{\partial\widetilde{\sigma}^\gamma}{\partial\sigma^\alpha}\frac{\partial\widetilde{\sigma}^\delta}{\partial\sigma^\beta}.$$

Thus we have

$$\det\left(h_{\alpha\beta}\right) = \det\left(\widetilde{h}_{\alpha\beta}\right)J^2,$$

where we used the fact that $J = \det\left(\frac{\partial\widetilde{\sigma}^\alpha}{\partial\sigma^\beta}\right)$. Plugging everything in the action, we see that

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_M d\widetilde{\sigma}d\widetilde{\tau}J^{-1}\left(-\det\left(\widetilde{h}_{\alpha\beta}\right)\right)^{\frac{1}{2}}J = \widetilde{S}_{NG}.$$

Reparametrization invariance is also called *diffeomorphism* invariance and is a *gauge symmetry* of the action. We can write the infinitesimal version of the reparametrization as follows:

$$\sigma^\alpha \to \widetilde{\sigma}^\alpha = \sigma^\alpha + \xi^\alpha + O(\xi^2). \tag{2.1.2}$$

Under this change in parameter we have

$$X^\mu(\sigma^\alpha) \to \widetilde{X}^\mu(\widetilde{\sigma}^\alpha) = X^\mu(\sigma^\alpha).$$

We have

$$\widetilde{X}^\mu(\widetilde{\sigma}^\alpha) = X^\mu(\sigma^\alpha) = X^\mu(\widetilde{\sigma}^\alpha - \xi^\alpha) = X^\mu(\widetilde{\sigma}^\alpha) - \xi^\alpha \partial_\alpha X^\mu,$$

where we used Taylor's theorem. This gives

$$\delta X^\mu = \widetilde{X}^\mu(\widetilde{\sigma}^\alpha) - X^\mu(\widetilde{\sigma}^\alpha) = -\xi^\alpha \partial_\alpha X^\mu \tag{2.1.3}$$

Poincaré Invariance

The worldsheet coordinates transform under the Poincaré transformation as follows:

$$X^\mu \to \widetilde{X}^\mu = \Lambda^\mu{}_\nu X^\nu + c^\mu, \tag{2.1.4}$$

where $\Lambda^\mu{}_\nu$ is a Lorentz transformation and c^μ is a constant vector. This is a manifest symmetry of the action. Poincaré invariance is a global symmetry of the action. The infinitesimal version is often calculated in a first course in quantum field theory. We will record it here for later use.

$$\delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu, \quad (a_{\mu\nu} = -a_{\nu\mu})\,. \tag{2.1.5}$$

2.1.2 Equations of Motion

We begin by expanding out the determinant in the action. We get

$$\det(h_{\alpha\beta}) = \det(\partial_\alpha X^\mu \partial_\beta X_\mu) = X'^2 \dot{X}^2 - (X' \cdot \dot{X})^2,$$

where

$$X' = \frac{\partial X}{\partial \sigma} \quad ; \quad \dot{X} = \frac{\partial X}{\partial \tau} \quad \& \quad X^2 = X^\mu X_\mu.$$

So we have

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\sigma d\tau \left[-X'^2 \dot{X}^2 + \left(X' \cdot \dot{X} \right)^2 \right]^{\frac{1}{2}},$$

where we have written $X \equiv X^\mu$. The conjugate momenta are given by

$$\Pi^\tau_\mu = \frac{\partial L_{NG}}{\partial \dot{X}^\mu} = -\frac{1}{2\pi\alpha'} \left[\frac{\left(\dot{X} \cdot X' \right) X'_\mu - (X'^2)\, X_\mu}{\sqrt{\left(X' \cdot \dot{X} \right)^2 - (X'^2 \dot{X}^2)}} \right]$$

$$\Pi^\sigma_\mu = \frac{2\mathcal{L}_{NG}}{\partial X'^\mu} = -\frac{1}{2\pi\alpha'} \left[\frac{\left(\dot{X} \cdot X' \right) \dot{X}_\mu - (X'^2)\, \dot{X}_\mu}{\sqrt{\left(X' \cdot \dot{X} \right)^2 - \left(X'^2 \dot{X}^2 \right)}} \right].$$

Observe that

$$\frac{\partial^2 \mathcal{L}_{NG}}{\partial \dot{X}^\mu \partial \dot{X}^\nu} \cdot \dot{X}^\nu = \frac{\partial \Pi^\tau_\mu}{\partial \dot{X}^\nu} \dot{X}^\nu = 0,$$
$$\frac{\partial^2 \mathcal{L}_{NG}}{\partial \dot{X}^\mu \partial \dot{X}^\nu} \cdot X'^\nu = \frac{\partial \Pi^\tau_\mu}{\partial \dot{X}^\nu} X'^\nu = 0.$$

So the Hessian $\frac{\partial^2 \mathcal{L}_{NG}}{\partial \dot{x}^\mu \partial \dot{x}^\nu}$ has two zero eigenvalues with eigenvectors $\dot{X}^\mu, X'^\mu$ must have two constraints. We can check that

$$\Pi^\tau_\mu X'^\mu = 0, \quad \Pi^\tau_\mu \Pi^{\tau\mu} + \frac{1}{4\pi^2 \alpha'^2} X'^\mu X'_\mu = 0. \tag{2.1.6}$$

These are one set of constraints. Another set of constraints arise from the fact that

$$\frac{\partial^2 \mathcal{L}_{NG}}{\partial X'^\mu \partial X'^2} \dot{X}^\nu = 0 \quad \text{and} \quad \frac{\partial^2 \mathcal{L}_{NG}}{\partial X'^\mu \partial X'^\nu} X'^\nu = 0.$$

The resulting constraints are

$$\Pi^\sigma_\mu \dot{X}^\mu = 0, \quad \Pi^\sigma_\mu \Pi^{\sigma\mu} + \frac{1}{4\pi^2 \alpha^2} \dot{X}^\mu \dot{X}_\mu = 0. \tag{2.1.7}$$

The Hamiltorian

$$\mathcal{H}^\sigma = \Pi^\sigma_\mu X'^\mu - \mathcal{L}_{NG} = 0; \quad \& \quad \mathcal{H}^\tau = \Pi^\tau_\mu \dot{X}^\mu - \mathcal{L}_{NG} = 0.$$

So the dynamics is determined by constraints. The equation of motion is given by

$$\frac{\partial \Pi^\tau_\mu}{\partial \tau} + \frac{\partial \Pi^\sigma_\mu}{\partial \sigma} = 0. \tag{2.1.8}$$

We can also write the equation of motion in another way. Recall that

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int_M d\sigma d\tau \sqrt{-h}; \quad h = \det h_{\alpha\beta}.$$

From general relativity we have

$$\delta\sqrt{-h} = \frac{1}{2}\sqrt{-h} h^\alpha \beta \delta h_{\alpha\beta}$$

So

$$\frac{\delta \mathcal{L}_{NG}}{\delta \left(\partial_\alpha X^\mu\right)} = -\frac{1}{2\pi\alpha'} \left(\frac{1}{2} \sqrt{-h} h^{\alpha\beta} \left(2 \partial_\beta X_\mu\right) \right)$$

So equation of motion is

$$\partial_\alpha \left(\frac{\partial \mathcal{L}_{NG}}{\partial \left(\partial_\alpha X^\mu\right)} \right) = 0$$

which gives

$$\partial_\alpha \left(\sqrt{-h} h^{\alpha\beta} \left(\partial_\beta X_\mu\right) \right) = 0.$$

2.2 The Polyakov Action

The final goal of studying string action is to quantise the action and analyse the spectrum that we obtain. The first challenge that we face when we try to quantise the Nambu-Goto action is the squareroot in the action. It is generally tricky to quantise such complicated actions when we go to path integral quantisation. This is why we will use the fourth method of quantisation introduced in Chapter 1. To this end, consider the following action:

$$S_p = -\frac{1}{4\pi\alpha'}\int_M d\sigma d\tau \sqrt{-g}g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu, \tag{2.2.1}$$

where $g = \det(g^{\alpha\beta})$ and $g^{\alpha\beta}$ is an auxiliary background field which plays the role of the einbein in the fourth method of quantisation. This action is called the Polyakov action. The auxiliary field $g_{\alpha\beta}$ is a dynamical metric on the world-sheet with Lorentzian signature $(-,+)$. Thus the action S_P can be viewed as a bunch of scalar fields $X^\mu(\sigma,\tau)$ coupled to a $2d$ gravity theory.

2.2.1 Equivalence of S_P and S_{NG}

Let us find the equations of motion of $g_{\alpha\beta}$. Varying S_P with respect to $g_{\alpha\beta}$ gives two terms. We get

$$\delta S_P = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \left[\sqrt{-g}\delta g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X\mu - \frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}g^{ab}\partial_a X^\mu \partial_b X\mu\right].$$

So

$$\delta S_P = 0 \implies \sqrt{-g}\delta g^{\alpha\beta}\left(\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}g_{\alpha\beta}g^{ab}\partial_a X^\mu \partial_b X_\mu\right) = 0.$$

Here we used

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta} = \frac{1}{2}\sqrt{-g}g^{\alpha\beta}\delta g_{\alpha\beta}.$$

So the equation of motion of $g_{\alpha\beta}$ is

$$\partial\alpha X^\mu \partial\beta X_\mu = \frac{1}{2}g_{\alpha\beta}g^{ab}\partial_a X^\mu \partial_b X_\mu.$$

Or

$$\partial_\alpha X^\mu \partial_\beta X_\mu = \frac{1}{2}g_{\alpha\beta}\partial^c X^\mu \partial_c X_\mu. \tag{2.2.2}$$

Taking determinant both sides we get

$$\det\left(\partial_\alpha X^\mu \partial_\beta X_\mu\right) = \det\left(\frac{1}{2}g_{\alpha\beta}\partial^c X^\mu \partial_c X_\mu\right).$$

Since $g_{\alpha\beta}$ is 2×2, we get

$$\det\left(\partial\alpha X^\mu \partial_\beta X_\mu\right) = \frac{1}{4}\left(\partial^c X^\mu \partial_c X_\mu\right)^2 g$$
$$\implies \sqrt{-\det\left(\partial_\alpha X^\mu \partial_\beta X_\mu\right)} = \frac{1}{2}\sqrt{-g}\left(\partial^c X^\mu \partial_c X_\mu\right).$$

Substituting this in S_P gives S_{NG}. Thus we see that S_P and S_{NG} are equivalent classically. These two actions presumably gives same quantum dynamics but a rigorous proof is lacking. Indeed path-integral quantisation of S_{NG} is rather difficult to perform due to squareroot and manipulating it to get results involves similar tricks as we have used in the $S_{NG} \longrightarrow S_P$ transition.

2.2.2 Equation of Motion

Let us vary the action with respect to X^μ with $\delta X^\mu(\sigma, \tau_0) = \delta X^\mu(\sigma, \tau_1) = 0$ for some initial and final value τ_0, τ_1 respectively of the parameter τ. Assuming that the string length is ℓ, we have

$$
\begin{aligned}
\delta S_P &= -\frac{1}{4\pi\alpha'}\int_{\tau_0}^{\tau_1} d\tau \int_0^\ell d\sigma \left[2\partial_\alpha X^\mu \left(\partial^\alpha \delta X_\mu\right)\right] \\
&= -\frac{1}{2\pi\alpha'}\int_{\tau_0}^{\tau_1} d\tau \int_0^\ell d\sigma \left[\partial_\alpha \left(\partial^\alpha X^\mu \delta X_\mu\right) - \left(\partial_\alpha \partial^\alpha X^\mu\right)\delta X_\mu\right] \\
&= \frac{1}{2\pi\alpha'}\int_{\tau_0}^{\tau_1} d\tau \int_0^\ell d\sigma \left(\partial_\alpha \partial^\alpha X^\mu\right)\delta X_\mu - \frac{1}{2\pi\alpha'}\int_{\tau_0}^{\tau_1} d\tau \int_0^\ell d\sigma \left(\partial_\sigma \left(\partial^\sigma X^\mu \delta X_\mu\right) - \partial_\tau \left(\partial^\tau X^\mu \delta X_\mu\right)\right) \\
&= \frac{1}{2\pi\alpha}\int_{\tau_0}^{\tau_1} d\tau \int_0^\ell d\sigma \left(\partial_\alpha \partial^\alpha X^\mu\right)\delta X_\mu + \underbrace{\frac{1}{2\pi\alpha'}\int_0^\ell d\sigma \left(\partial^\tau X^\mu\right)\delta X_\mu \Big|_{\tau_0}^{\tau_1}}_{=0 \text{ as } \delta X^\mu(\sigma,\tau_0) = \delta X^\mu(\sigma,\tau_1) = 0} \\
&\qquad + \frac{1}{2\pi\alpha'}\int_{\tau_0}^{\tau_1} d\tau \underbrace{\left(\partial^\sigma X^\mu \delta X_\mu\right)\Big|_0^\ell}_{\text{surface term}}.
\end{aligned}
$$

To get the equations of motion, we need the surface term to go to zero. Physically we distinguish between two cases - the closed string and the open string. We will deal with the two cases separately.

Closed Strings

We normalise the string length so that $\ell = 2\pi$. Closed string then means that the ends of the string are joined together in a smooth fashion to form a loop. This means that $X^\mu(\sigma, \tau)$ are periodic in σ with period 2π :

$$X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau).$$

This implies that $\delta X_\mu(0,\tau) = \delta X_\mu(2\pi,\tau) = 0$. Thus the equation of motion for closed strings is

$$\partial_\alpha \partial^\alpha X^\mu = 0. \tag{2.2.3}$$

Open Strings

The ends of open string are free and so we need to impose boundary conditions on the ends of the open string. We normalise the length of the string to $\ell = \pi$. We impose the boundary condition such that the surface term vanishes. There are three ways for this to happen - atleast one of the two $\partial^\sigma X^\mu$ and δX_μ or the combination $\partial^\sigma X^\mu \delta X_\mu$ must be zero at $\sigma = 0$ and $\sigma = \pi$. Hence we have three different bondary condition:

1. **Dirichlet boundary condition:** $\delta X_\mu = 0 \quad \text{at } \sigma = 0, \pi.$
2. **Neumann boundary condition:** $\partial_\sigma X_\mu = 0$ at $\sigma = 0, \pi$.
3. **Robin boundary condition:** $\partial_\sigma X_\mu \delta X^\mu = 0$ at $\sigma = 0, \pi$.

The first two boundary conditions have been studied in detail in literature and we will also analyse each boundary condition along with mixed boundary condition in detail as we progress in our study.

2.2.3 Symmetries of S_P

As with S_{NG}, we can directly read off two obvious symmetries of S_P:

Reparametrization Invariance

If we transform the parameters as $\sigma^\alpha \longrightarrow \widetilde{\sigma}^\alpha = \widetilde{\sigma}^\alpha(\boldsymbol{\sigma})$ then the scalar fields X^μ transform as

$$X^\mu(\sigma,\tau) \longrightarrow \widetilde{X}^\mu\left(\widetilde{\sigma}^\alpha\right) = X^\mu\left(\sigma^\alpha\right)$$

and the world-sheet metric $g_{\alpha\beta}$ transforms in the usual way

$$g_{\alpha\beta} \longrightarrow \widetilde{g}_{\alpha\beta}\left(\tilde{\sigma}^\alpha\right) = \frac{\partial \sigma^\gamma}{\partial \widetilde{\sigma}^\alpha} \frac{\partial \sigma^\delta}{\partial \widetilde{\sigma} \beta} g_{\gamma\delta}(\sigma).$$

We can find the infinitesimal transformation under $\sigma^\alpha \longrightarrow \sigma^\alpha = \sigma^\alpha - \eta^\alpha$, where η^α is small, using Lie derivative. Indeed under infinitesimal transformation

$$\delta g_{\alpha\beta} = \mathcal{L}_\eta g_{\alpha\beta} = \nabla_\alpha \eta_\beta + \nabla_\beta \eta_\alpha,$$

where ∇_α is the Levi-Civita covariant derivative with the usual Levi-Civita connection

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}\left(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\rho\gamma}\right)$$

Also $\sqrt{-g}$ changes as $\delta(\sqrt{-g}) = \partial_\alpha\left(\eta^\alpha \sqrt{-g}\right)$. The Polyakov action S_P is easily seen to be invariant under reparametrizations. This is a gauge symmetry of the action.

Poincaré Invariance

This is a global symmetry of the action.

$$X^\mu \longrightarrow \widetilde{X}^\mu = \Lambda^\mu{}_\nu X^\nu + \xi^\mu$$

for some constant ξ^μ. The infinitesimal version of this transformation is given in Subsection 2.1.1.

Weyl Invariance

There is another gauge invariance called Weyl symmetry. Under this $X^\mu \longrightarrow X^\mu$ and the metric transforms as

$$g_{\alpha\beta} \longrightarrow \widetilde{g}_{\alpha\beta} = \Omega^2(\boldsymbol{\sigma}) g_{\alpha\beta}$$

or infinitesimally if $\Omega^2(\boldsymbol{\sigma}) = e^{2\phi(\boldsymbol{\sigma})}$ then

$$\delta g_{\alpha\beta} = 2\phi(\boldsymbol{\sigma}) g_{\alpha\beta}.$$

To see that this is a symmetry of the action, note that $\sqrt{-g} \longrightarrow \Omega^2(\boldsymbol{\sigma})\sqrt{-g}$ as

$$\det\left(\Omega^2 g_{\alpha\beta}\right) = \Omega^4(\boldsymbol{\sigma}) \det\left(g_{\alpha\beta}\right)$$

and $g^{\alpha\beta} \longrightarrow (\Omega(\boldsymbol{\sigma}))^{-2} g^{\alpha\beta}$. Thus factors from $\sqrt{-g}$ and $g^{\alpha\beta}$ cancel.

Remark 2.2.1. Weyl transformation is not a coordinate transformation. Rather it is a local change of scale under which the theory is invariant. More precisely, this scale change preserves angles between as the metric transforms conformally.

Remark 2.2.2. Weyl transformation is unique to two dimensions since $\sqrt{-g}g^{\alpha\beta}$ remain invariant under $g_{\alpha\beta} \longrightarrow \Omega^2 g_{\alpha\beta}$ only in two dimensions.

Chapter 3

The Closed String

In Chapter 2, we found the correct action for relativistic strings namely the Polyakov action. We also found the equations of motion arising from the action and depending on the type of string - open or closed, we imposed boundary conditions. In this chapter, we will solve the classical equations of motion for the closed string and also quantise the theory using two different procedures. We will also analyse the closed string spectrum.

3.1 The Closed Classical String

As we saw in the previous chapter, the Polyakov action has two gauge symmetries. Hence to find the equations of motion, we first need to fix a gauge. This means that we should make an appropriate choice of the background metric using our gauge symmetries.

3.1.1 Fixing a Gauge

We have two diffeomorphism invariance namely for σ, τ and three independent metric components. Write

$$g_{\alpha\beta} = \begin{pmatrix} g_{\sigma\sigma} & g_{\sigma\tau} \\ g_{\tau\sigma} & g_{\tau\tau} \end{pmatrix} \qquad \text{then} \qquad g_{\sigma\tau} = g_{\tau\sigma}.$$

Now since $g_{\alpha\beta}$ has signature $(-,+)$, at least locally one out of $g_{\sigma\sigma}$ and $g_{\tau\tau}$ must be positive. Under diffeomorphism we have

$$g_{\alpha\beta} \longrightarrow \widetilde{g}_{\alpha\beta} = \frac{\partial \sigma^\gamma}{\partial \widetilde{\sigma}^\alpha} \frac{\partial \sigma^\delta}{\partial \widetilde{\sigma}^\beta} g_{\gamma\delta}.$$

This gives

$$\widetilde{g}_{\sigma\sigma} = \left(\frac{\partial \sigma}{\partial \widetilde{\sigma}}\right)^2 g_{\sigma\sigma} + \left(\frac{\partial \tau}{\partial \widetilde{\sigma}}\right)^2 g_{\tau\tau} + 2\frac{\partial \sigma}{\partial \widetilde{\sigma}}\frac{\partial \tau}{\partial \widetilde{\sigma}} g_{\sigma\tau}$$
$$\widetilde{g}_{\tau\tau} = \left(\frac{\partial \sigma}{\partial \widetilde{\tau}}\right)^2 g_{\sigma\sigma} + \left(\frac{\partial \tau}{\partial \widetilde{\tau}}\right)^2 g_{\tau\tau} + 2\frac{\partial \sigma}{\partial \widetilde{\tau}}\frac{\partial \tau}{\partial \widetilde{\tau}} g_{\sigma\tau}$$

and

$$\widetilde{g}_{\sigma\tau} = \widetilde{g}_{\tau\sigma} = \frac{\partial\sigma}{\partial\widetilde{\sigma}}\frac{\partial\sigma}{\partial\widetilde{\tau}}g_{\sigma\sigma} + \frac{\partial\tau}{\partial\widetilde{\sigma}}\frac{\partial\tau}{\partial\widetilde{\tau}}g_{\tau\tau} + \frac{\partial\tau}{\partial\widetilde{\sigma}}\frac{\partial\sigma}{\partial\widetilde{\tau}}g_{\tau\sigma} + \frac{\partial\sigma}{\partial\widetilde{\sigma}}\frac{\partial\tau}{\partial\widetilde{\tau}}g_{\sigma\tau}.$$

Now suppose in a neighbourhood of (σ, τ), $g_{\tau\tau} > 0$ then we put $\widetilde{g}_{\sigma\tau} = \widetilde{g}_{\tau\sigma} = 0$ and $\widetilde{g}_{\sigma\sigma} = -g_{\tau\tau}$. Thus we have a system of two first order partial differential equations to solve for two function $\widetilde{\sigma}(\sigma, \tau)$ and $\widetilde{\tau}(\sigma, \tau)$ that is we need to solve for $\widetilde{\sigma}(\sigma, \tau)$ and $\widetilde{\tau}(\sigma, \tau)$ from

$$\begin{gathered}
\left(\frac{\partial\sigma}{\partial\widetilde{\sigma}}\right)^2 g_{\gamma\sigma} + \left(\frac{\partial\tau}{\partial\widetilde{\sigma}}\right)^2 g_{\tau\tau} + 2\frac{\partial\sigma}{\partial\widetilde{\sigma}}\frac{\partial\tau}{\partial\widetilde{\sigma}}g_{\sigma\tau} = -g_{\tau\tau} \\
\frac{\partial\sigma}{\partial\widetilde{\sigma}}\frac{\partial\sigma}{\partial\widetilde{\tau}}g_{\sigma\sigma} + \frac{\partial\tau}{\partial\widetilde{\sigma}}\frac{\partial\tau}{\partial\widetilde{\tau}}g_{\tau\tau} + \frac{\partial\tau}{\partial\widetilde{\sigma}}\frac{\partial\sigma}{\partial\widetilde{\tau}}g_{\tau\sigma} + \frac{\partial\sigma}{\partial\widetilde{\sigma}}\frac{\partial\tau}{\partial\widetilde{\tau}}g_{\sigma\tau} = 0.
\end{gathered}$$

Solution to this exists atleast locally by Cauchy-Kowalevski theorem since the coefficient functions are real analytic. Thus we have transformed $g_{\alpha\beta}$ to $g_{\tau\tau}\eta_{\alpha\beta}$ using the two diffeomorphisms. Since $g_{\tau\tau} = e^{\phi(\boldsymbol{\sigma})}$ for some function ϕ, thus we now use Weyl rescaling to transform

$$g_{\alpha\beta} \longrightarrow e^{-\phi(\boldsymbol{\sigma})}g_{\alpha\beta} = \eta_{\alpha\beta}.$$

This gauge is called *Conformal gauge.*

Remark 3.1.1. Any $2d$ metric can be made flat using Wely invariance: Suppose $g'_{\alpha\beta} = e^{\phi(\boldsymbol{\sigma})}g_{\alpha\beta}$ then one can easily check that

$$\sqrt{-g'}R' = \sqrt{g}\left(R - \nabla^2\phi\right).$$

If we choose ϕ such that $\nabla^2\phi = R$ then $R' = 0$. But in $2d$ vanishing Ricci scalar implies that Riemann curvature tensor is zero since in $2d$ one can show that

$$R_{\alpha\beta\gamma\delta} = \frac{R}{2}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right).$$

Hence the metric is flat.

Remark 3.1.2. Can the world-sheet metric be made flat globally? Depends on the topology of the space. Locally the metric can be made flat using the three gauge symmetries. Suppose we could extend this locally flat metric to whole worldsheet. This means that the whole worldsheet is covered by a coordinate chart which is flat. This in turn means that the Ricci scalar identically vanishes on the worldsheet. Topologically since in $2d$, the Euler characteristic χ of a manifold satisfies

$$\chi \propto \int_M R.$$

Thus a necessary condition of the extension to be possible is that $\chi = 0$.

We have fixed a gauge. Now we need to find the equation of motion of $g_{\alpha\beta}$ and impose it as a constraint on the classical system after substituting $g_{\alpha\beta} = \eta_{\alpha\beta}$. We have already calculated the equation of motion in Subsection 2.2.1 but we can recast it in terms of energy momentum tensor which is often more useful. We begin by writing the gauge fixed action:

$$S_P^{gf} = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \partial_\alpha X^\mu \partial^\alpha X_\mu. \tag{3.1.1}$$

The equation of motion for X^μ is

$$\partial^\alpha \partial_\alpha X^\mu = 0. \tag{3.1.2}$$

Next we have

$$\frac{\delta S}{\delta g^{\alpha\beta}} = -\frac{1}{4\pi\alpha'}\left[-\frac{\sqrt{-g}}{2} g_{\alpha\beta}\partial_c X^\mu \partial^c X_\mu + \sqrt{-g}\partial_\alpha X^\mu \partial_\beta X_\mu\right].$$

Define the energy momentum tensor[1] by

$$T_{\alpha\beta} = -4\pi\alpha' \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\alpha\beta}}.$$

We get

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} g_{\alpha\beta}\partial_c X^\mu \partial^c X_\mu.$$

So that

$$T_{\alpha\beta}\big|_{g_{\alpha\beta}=\eta_{\alpha\beta}} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\partial_c X^\mu \partial^c X_\mu.$$

The equation of motion for $g_{\alpha\beta}$ was

$$\partial_\alpha X^\mu \partial_\beta X_\mu = \frac{1}{2} g_{\alpha\beta}\partial_c X^\mu \partial' X_\mu.$$

So our constraint is $T_{\alpha\beta} = 0$. Written in terms of components:

$$T_{01} = \dot{X}^\mu X'_\mu = 0 \quad T_{11} = T_{00} = \dot{X}^2 - \frac{1}{2}\left(-\left(-\dot{X}^2 + X'^2\right)\right) = \frac{1}{2}\left(\dot{X}^2 + X'^2\right).$$

So we have to impose two constraints

$$\dot{X}^\mu X'_\mu = 0, \quad \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0. \tag{3.1.3}$$

So the equation of motion is a wave equation along with the two constraints. We will now solve it.

[1] note that this is not the usual definition of energy momentum tensor. In general relativity (GR) we have different normalisation. In GR the energy momentum tensor is given by $T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\alpha\beta}}$.

3.1.2 Solving the Equation of Motion: Mode Expansion

We will use the lightcone coordinates to solve the equations of motion (3.1.2) subject to constraints (3.1.3). Introduce the lightcone coordinates

$$\sigma^{\pm} = \tau \pm \sigma,$$

then

$$\partial_+ = \partial_\tau + \partial_\sigma, \quad \partial_- = \partial_\tau - \partial_\sigma.$$

With this, the equation of motion $\partial_\alpha \left(\partial^\alpha X^\mu\right) = 0$ reduces to

$$\partial_+ \partial_- X^\mu = 0. \tag{3.1.4}$$

Indeed we have

$$\partial_+\partial_- X^\mu = \partial_+ \left(\partial_\tau X^\mu - \partial_\sigma X^\mu\right) = \partial_{\tau\tau} X^\mu - \partial_{\tau\sigma} X^\mu - \partial_{\sigma\tau} X^\mu - \partial_{\sigma\sigma} X^\mu = 0.$$

The most general solution to $\partial_+\partial_- X^\mu = 0$ is given by

$$X^\mu(\sigma, \tau) = X_L^\mu(\sigma+) + X_R^\mu(\sigma^-) \tag{3.1.5}$$

for arbitrary functions X_L and X_R. For closed strings, we have the periodicity condition

$$X^\mu(\sigma + 2\pi, \tau) = \quad X^\mu(\sigma, \tau).$$

This implies that X^μ can be written as a Fourier series. More precisely, we have

$$\begin{aligned} X_L^\mu\left(\sigma^+\right) &= \frac{x^\mu}{2} + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \widetilde{\alpha}_n^\mu e^{-in\sigma^+} \\ X_R^\mu\left(\sigma^-\right) &= \frac{x^\mu}{2} + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\sigma^-}. \end{aligned} \tag{3.1.6}$$

The functions X_L^μ are called left movers and X_R^μ are called right movers.

Remark 3.1.3. 1. The factors $\alpha', \frac{1}{n}$ have been chosen for convenience when we quantise the system.

2. X_L^μ and X_R^μ are not periodic due to the linear term σ^+, σ^- but the combination $X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right)$ is periodic as σ cancels from the combination $\sigma^+ + \sigma^- = 2\tau$.

3. The quantities x^μ and p^μ are the position and momentum of the center of mass of the string. We will prove this explicitly. Observe that for the Polyakov action,

$$\Pi_\mu^\tau = \frac{\partial \mathcal{L}_P}{\partial \dot{X}^\mu} = -\frac{1}{4\pi\alpha'} \frac{\partial}{\partial \dot{X}^\mu} \left[-\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu\right] = \frac{1}{2\pi\alpha'} \dot{X}_\mu.$$

So

$$P^\mu = \int_0^{2\pi} d\sigma \frac{1}{2\pi\alpha'} \dot{X}^\mu = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \dot{X}_L^\mu \left(\sigma^+\right) + \dot{X}_R^\mu \left(\sigma^-\right) = \frac{1}{2\pi\alpha'} 2\pi\alpha' p^\mu = p^\mu,$$

and

$$q^\mu = \frac{1}{2\pi} \int_0^{2\pi} d\sigma X^\mu = \frac{1}{2\pi} \int_0^{2\pi} d\sigma X_L^\mu \left(\sigma^+\right) + X_R^\mu \left(\sigma^-\right) = \frac{1}{2\pi} \left[2\pi x^\mu + 2\pi\alpha' p^\mu \tau\right] = x^\mu + \alpha' p^\mu \tau.$$

So we see that p^μ is indeed the momentum and x^μ is the position of center of mass of the string at $\tau = 0$.

4. The coordinate functions X^μ is real. So $(X_L^\mu)^\star = X_L^\mu$ and $(X_R^\mu)^\star = X_R^\mu$. This means that the coefficients α_n^μ and $\widetilde{\alpha}_n^\mu$ satisfy

$$(\alpha_n^\mu)^\star = \alpha_{-n}^\mu \quad \text{and} \quad (\widetilde{\alpha}_n^\mu)^\star = \alpha_{-n}^\mu \quad \forall n \in \mathbb{Z}\backslash\{0\}.$$

Recall that we had two constraints

$$\dot{X}^\mu X'_\mu = 0 \quad \text{and} \quad \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0.$$

In Light-cone coordinates, these transform to

$$\begin{aligned}
&\left(\frac{\partial_+ + \partial_-}{2}\right) X^\mu \left(\frac{\partial_+ - \partial_-}{2}\right) X_\mu = 0 \\
\implies &\left(\partial_+ X^\mu + \partial_- X^\mu\right)\left(\partial_+ X_\mu - \partial_- X_\mu\right) = 0 \\
\implies &\left(\partial_+ X^\mu\right)^2 - \left(\partial_- X^\mu\right)^2 = 0 \\
\implies &\left(\partial_+ X^\mu\right)^2 = \left(\partial_- X^\mu\right)^2.
\end{aligned}$$

The second constrain becomes

$$\begin{aligned}
&\left(\left(\frac{\partial_+ + \partial_-}{2}\right) X^\mu\right)^2 + \left(\left(\frac{\partial_+ - \partial_-}{2}\right) X^\mu\right)^2 = 0 \\
\implies &\left(\partial_+ X^\mu\right)^2 + \left(\partial_- X^\mu\right)^2 + 2\partial_+ X^\mu \partial_- X_\mu + \left(\partial_+ X^\mu\right)^2 + \left(\partial_- X^\mu\right)^2 - 2\partial_+ X^\mu \partial_- X_\mu = 0 \\
\implies &\left(\partial_+ X^\mu\right)^2 + \left(\partial_- X^\mu\right)^2 = 0.
\end{aligned}$$

Combining these two we get the constraint

$$\left(\partial_+ X^\mu\right)^2 = 0 = \left(\partial_- X^\mu\right). \tag{3.1.7}$$

We now impose this constraint on the Fourier modes. We have

$$\begin{aligned}
\partial_- X^\mu = \partial_- X_R^\mu &= \frac{\alpha' p^\mu}{2} + \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}\backslash\{0\}} \alpha_n^\mu e^{-in\sigma^-} \\
&= \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \alpha_n^\mu e^{-in\sigma^-},
\end{aligned}$$

where we have defined

$$\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu.$$

The constraint $(\partial_- X^\mu)^2 = 0$ gives

$$\left(\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_{n\mu} e^{-in\sigma^-}\right)\left(\sqrt{\frac{\alpha'}{2}} \sum_{m\in\mathbb{Z}} \alpha_m^\mu e^{-im\sigma^-}\right) = \frac{\alpha'}{2} \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{n-k}\right) e^{-in\sigma^-} = 0$$

where we used Cauchy product formula and $\boldsymbol{\alpha}_k \equiv \alpha_k^\mu$. If we define

$$L_n := \frac{1}{2} \sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{n-k}, \tag{3.1.8}$$

then the constraint becomes $L_n = 0$ for every $n \in \mathbb{Z}$. Similarly the constraint $(\partial_+ X^\mu)^2 = 0$ gives $\widetilde{L}_n = 0$ for every $n \in \mathbb{Z}$ where

$$\widetilde{L}_n := \frac{1}{2} \sum_{k\in\mathbb{Z}} \widetilde{\boldsymbol{\alpha}}_k \cdot \widetilde{\boldsymbol{\alpha}}_{n-k}, \tag{3.1.9}$$

and $\widetilde{\alpha}_0^\mu = \alpha_0^\mu$. The quantities L_n and $\widetilde{L}_n$ are called *Virasoro generators*. The constraints $L_0 = 0 = \widetilde{L}_0$ are particularly interesting as they contain information about the physical degrees of freedom of the string - the string momentum. We have

$$L_0 = \frac{1}{2} \sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{-k}, \quad \widetilde{L}_0 = \frac{1}{2} \sum_{k\in\mathbb{Z}} \widetilde{\boldsymbol{\alpha}}_k \cdot \widetilde{\boldsymbol{\alpha}}_{-k}.$$

In relativistic mechanics, we know that

$$p^\mu p_\mu = -M^2$$

where M is the rest mass of the particle. Since

$$p^\mu p_\mu = \frac{2}{\alpha'} \boldsymbol{\alpha}_0^2 = \frac{2}{\alpha'} \widetilde{\boldsymbol{\alpha}}_0^2,$$

we see that the constraints $L_0 = \widetilde{L}_0 = 0$ implies

$$\frac{1}{2} \sum_{n\neq 0} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n - \frac{\alpha'}{4} M^2 = 0 = \frac{1}{2} \sum_{n\neq 0} \widetilde{\boldsymbol{\alpha}}_{-n} \cdot \widetilde{\boldsymbol{\alpha}}_n - \frac{\alpha'}{4} M^2.$$

This gives

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n = \frac{4}{\alpha'} \sum_{n>0} \widetilde{\boldsymbol{\alpha}}_{-n} \cdot \widetilde{\boldsymbol{\alpha}}_n. \tag{3.1.10}$$

This is called the *level matching condition* and will be crucial when we analyse the spectrum of the quantised theory.

3.2 Quantisation of Closed String

There are two ways to quantise the Polyakov action. One is the cannonical quantisation using Dirac's presciption. The other is Feynman's path integral quantisation. The cannonical quantisation procedure involves two ways as we are dealing with a gauge theory:

- **Covariant quantisation:** Change cannonical Poisson brackets to commutators and impose the constraint obtained by fixing a gauge as an operator equation to be satisfied by the states X^μ which are now operators. This method is manifestly Lorentz invariant but gives rise to negative norm states called *ghosts*. These decouple from the theory in the critical dimension $D = 26$.

- **Lightcone quantisation:** In this method we first solve the constraints to classify all classically distinct states and then we quantise the physical states. We break Lorentz invariance in the process and later obtain the same critical dimension $D = 26$ to ensure Lorentz invariance.

We will look at both of these quantisation scheme in detail now.

3.3 Covariant Quantisation

We have D scalar fields $X^\mu, \mu = 0, 1, \ldots, D-1$ and two constraints

$$\dot{X}^\mu X'_\mu = 0 \quad \text{and} \quad \dot{X}^2 + X'^2 = 0.$$

3.3.1 Poisson Brackets

Let us begin by computing the classical Poisson brackets.

(i) Equal τ Poisson bracket $\{X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)\}_{P.B.} = 0.$

Proof. For Polyakov action, we have $\Pi^\tau_\mu \sim \dot{X}_\mu$. We will use the notation $\Pi_\mu := \Pi^\tau_\mu$ everywhere unless stated explicitly. Thus this P.B. is obvious. □

(ii) Equal τ Poisson bracket $\{\Pi^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)\}_{P.B.} = 0.$

Proof. Obvious from the fact that $\Pi^\tau_\mu \sim \dot{X}_\mu$. □

(iii) Equal τ Poisson bracket $\{X^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)\}_{P.B.} = \eta^{\mu\nu}\delta(\sigma - \sigma').$

Proof. By definition

$$\begin{aligned}
\{X^{\mu}(\sigma,\tau),\Pi^{\nu}(\sigma',\tau)\}_{P.B.} &= \eta^{\rho\lambda}\frac{\partial X^{\mu}(\sigma,\tau)}{\partial X^{\rho}(\sigma,\tau)}\frac{\partial \Pi^{\nu}(\sigma',\tau)}{\partial \Pi^{\lambda}(\sigma,\tau)} \\
&= \eta^{\rho\lambda}\delta^{\mu}_{\rho}\delta^{\nu}_{\lambda}\delta(\sigma-\sigma') \\
&= \eta^{\mu\nu}\delta(\sigma-\sigma')
\end{aligned}$$

□

From these Poisson brackets, we can easily calculate the Poisson brackets for $x^{\mu}, p^{\mu}, \alpha^{\mu}_{n}, \widetilde{\alpha}^{\mu}_{n}$ We have

$$\begin{aligned}
&\{x^{\mu},p^{\nu}\}_{P.B.} = \eta^{\mu\nu}, \quad \{\widetilde{\alpha}^{\mu}_{m},\alpha^{\nu}_{n}\}_{P.B.} = 0 \\
&\{\alpha^{\mu}_{m},\alpha^{\nu}_{n}\}_{P.B.} = \{\widetilde{\alpha}^{\mu}_{m},\widetilde{\alpha}^{\nu}_{n}\} = -im\eta^{\mu\nu}\delta_{m+n,0}.
\end{aligned} \tag{3.3.1}$$

Using these Poisson brackets, we can get a algebra satisfied by the Virasoro generators.

Lemma 3.3.1. *The Virasoro generators satisfy the Virasoro algebra also called Witt algebra:*

$$\{L_n,L_m\}_{P.B.} = 2(m-n)L_{m+n}, \quad \left\{\widetilde{L}_n,\widetilde{L}_m\right\}_{P.B.} = i(m-n)\widetilde{L}_{n+m}, \quad \left\{\widetilde{L}_n,L_m\right\}_{P.B.} = 0. \tag{3.3.2}$$

Proof. We have

$$\begin{aligned}
\{L_n,L_m\}_{P.B.} &= \left\{\sum_{l\in\mathbb{Z}}\boldsymbol{\alpha}_{n-l}\cdot\boldsymbol{\alpha}_{l}, \sum_{k\in\mathbb{Z}}\boldsymbol{\alpha}_{m-k}\cdot\boldsymbol{\alpha}_{k}\right\}_{P.B.} \\
&= \sum_{l,k\in\mathbb{Z}}\left\{\eta_{\mu\nu}\alpha^{\mu}_{n-l}\alpha^{v}_{l},\eta_{\rho\sigma}\alpha^{\rho}_{m-k}\alpha^{\sigma}_{k}\right\}_{P.B.}
\end{aligned}$$

Using

$$\begin{aligned}
\{AB,CD\}_{P.B.} &= \{A,CD\}_{P.B.}B + A\{B,CD\}_{P.B.} \\
&= C\{A,D\}_{P.B.}B + \{A,C\}_{P.B.}DB + AC\{B,D\}_{P.B.} + A\{B,C\}_{P.B.}D,
\end{aligned}$$

we get

$$\begin{aligned}
\{L_n,L_m\}_{P.B.} = \sum_{l,k\in\mathbb{Z}}&\eta_{\mu\nu}\eta_{\rho\sigma}\big[\alpha^{\mu}_{m-k}\left\{\alpha^{\mu}_{n-l},\alpha^{\sigma}_{k}\right\}_{P.B.}\alpha^{\nu}_{l} + \left\{\alpha^{\mu}_{n-l},a^{\rho}_{m-k}\right\}_{P.B.}\alpha^{\nu}_{l}\alpha^{\sigma}_{k} \\
&\qquad + \alpha^{\mu}_{n-l}\alpha^{\rho}_{m-k}\left\{\alpha^{v}_{k},\alpha^{\sigma}_{k}\right\}_{P.B.} + \alpha^{\mu}_{n-k}\left\{\alpha^{\nu}_{l},\alpha^{\rho}_{m-k}\right\}_{P.B.}\alpha^{\sigma}_{k}\big] \\
= \sum_{l,k\in\mathbb{Z}}&\eta_{\mu\nu}\eta_{\rho\sigma}\big[-\alpha^{\rho}_{m-k}\eta^{\mu\sigma}i(n-l)\delta_{n-l+k,0}\alpha^{\nu}_{l} - i(n-l)\eta^{\mu\rho}\delta_{n-l+m-k,0}\alpha^{\nu}_{l}\alpha^{\sigma}_{k} \\
&\qquad - il\delta_{l+k,0}\eta^{\nu\sigma}\alpha^{\mu}_{n-l}\alpha^{\rho}_{m-k} - il\eta^{\nu\rho}\delta_{l+m-k,0}\alpha^{\mu}_{n-k}\alpha^{\sigma}_{k}\big] \\
= -i\sum_{k\in\mathbb{Z}}&\big[\eta_{\nu\rho}\alpha^{\rho}_{m-k}\alpha^{\nu}_{n+k}k + \eta_{\nu\sigma}\alpha^{\nu}_{n+m-k}\alpha^{\sigma}_{k}(k-m)+ \\
&\qquad + \eta_{\mu\rho}\alpha^{\mu}_{n+k}\alpha^{\rho}_{m-k}k + \eta_{\mu\sigma}(k-m)\alpha^{\mu}_{n+m-k}\alpha^{\sigma}_{k}\big],
\end{aligned}$$

where we used (3.3.1). Replacing $m-k$ by k in first and k by $k-n$ in third sum we get

$$
\begin{aligned}
\{L_n, L_m\}_{P.B.} &= -i\sum_{k\in\mathbb{Z}} \left[\eta_{\nu\rho}\alpha^{\rho}_{k}\alpha^{\nu}_{n+m-k}(m-k) + \eta_{\nu\sigma}\alpha^{\nu}_{n+m-k}\alpha^{\sigma}_{k}(k-m)+\right.\\
&\qquad\qquad\left. + \eta_{\mu\rho}\alpha^{\mu}_{k}\alpha^{\rho}_{m+n-k}(n-k) + \eta_{\mu\sigma}(k-m)\alpha^{\mu}_{n+m-k}\alpha^{\sigma}_{k}\right]\\
&= -i\sum_{k\in\mathbb{Z}} \eta_{\mu\nu}\alpha^{\mu}_{n+m-k}\alpha^{\nu}_{k}(n-m)\\
&= i(m-n)L_{m+n}.
\end{aligned}
$$

Similarly, we get all other Poisson brackets. □

3.3.2 Cannonical Commutation Relations

Following the usual way, promote the scalar fields X^μ to operator valued fields and impose the cannonical commutation relation following the rule:

$$
\{\cdot,\cdot\}_{P.B.} = \frac{1}{i}[\cdot,\cdot].
$$

Using the Poisson brackets for X^μ, Π^μ, we get the following commutation relations:

$$
\begin{gathered}
[X^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)] = i\eta^{\mu\nu}\delta(\sigma-\sigma')\\
[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = 0 = [X^\mu(\sigma,\tau), \Pi^\nu(\sigma',\tau)].
\end{gathered}
$$

For the Fourier modes, using (3.3.1), we get

$$
\begin{gathered}
[x^\mu, p^\nu] = i\eta^{\mu\nu}\\
[\alpha^\mu_n, \alpha^\nu_m] = m\eta^{\mu\nu}\delta_{m+n,0} = [\widetilde{\alpha}^\mu_m, \widetilde{\alpha}^\nu_n]
\end{gathered}
\tag{3.3.3}
$$

and all other combinations are zero. These commutation relations are similar to those of creation and annihilation operators. Indeed if we define

$$
a^\mu_n = \frac{1}{\sqrt{n}}\alpha^\mu_n \quad , \quad (a^\mu_n)^\dagger = \frac{1}{\sqrt{n}}\alpha^\mu_{-n} \quad , \quad n>0,
$$

then we will get the usual commutation relations:

$$
\left[a^\mu_n, (a^\mu_m)^\dagger\right] = i\delta_{nm}.
$$

Similarly we can put

$$
\widetilde{a}^\mu_n = \frac{1}{\sqrt{n}}\widetilde{\alpha}^\mu_n \quad , \quad (\widetilde{a}^\mu_n)^\dagger = \frac{1}{\sqrt{n}}\widetilde{\alpha}^\mu_{-n} \quad , \quad n>0,
$$

then we get the commutation relations:

$$
\left[\widetilde{a}^\mu_n, (\widetilde{a}^\mu_m)^\dagger\right] = i\delta_{nm}.
$$

So for every scalar field X^μ, $\mu = 0,1,\ldots,D-1$ we have two family of creation and annihilation operators corresponding to the Left movers and the right movers.

Remark 3.3.2. We cannot directly get the commutation relations satisfied by the Virasoro generators from the Virasoro algebra. In subsequent sections, we will calculate the quantum algebra of Virasoro generators from the commutation relations of the Fourier modes. As we will discover later, the quantum algebra of Virasoro generators has an extra *central charge* term and hence the quantum algebra is the central extension of the Witt algebra. We will see that this is related related to the fact the Weyl symmetry which is a symmetry of the classical action does not survive quantisation.

3.3.3 Constructing the Fock Space

We will now construct the Fock space of the theory. We begin by constructing the ground state. We now have the creation and annilation operators to define the vacuum of the theory Denote it by $|0\rangle$. Then we demand:

$$a_n^\mu|0\rangle = 0 = \widetilde{a}_n^\mu|0\rangle \quad \text{ for } \mu = 0, 1, \ldots, D-1;\ n > 0.$$

Note that this condition alone does not uniquely fix the ground state. This is because, the ground state here is quite different from the one in field theory in the sense that there is a string specified by the center of mass position x^μ and momentum p^μ. So we denote the ground state by $|0; p^\mu\rangle$ which now has the property that

$$\widehat{p}^\mu|0;p^\mu\rangle = p^\mu\,|0;p^\mu\rangle\,, \tag{3.3.4}$$

where p^μ is the momentum of the string. So the ground state of the theory is now defined by

$$\alpha_n^\mu|0;p^\mu\rangle = 0 = \widetilde{\alpha}_n^\mu|0;p^\mu\rangle \quad \text{ for } \mu = 0, 1, \ldots, D-1;\ n > 0 \tag{3.3.5}$$

and (3.3.4). A general excitation of the string is

$$\left(\alpha_{-1}^{\mu_1}\right)^{n_{\mu_1}}\left(\alpha_{-2}^{\mu_2}\right)^{n_{\mu_2}}\cdots\left(\widetilde{\alpha}_{-1}^{\nu_1}\right)^{n_{\nu_1}}\left(\widetilde{\alpha}_{-2}^{\nu_2}\right)^{n_{\nu_2}}\cdots|0;p^\mu\rangle.$$

Each excited state has interpretation of a particle. Hence we have infinitely many species of particles in this theory.

3.3.4 Ghosts

We immediately come across a problem. The theory has negative norm states – the so called ghost states[2]. Since $\eta^{00} = -1 < 0$ we have

$$\begin{aligned}\left[\alpha_n^0, \alpha_{-n}^0\right] &= \left[\alpha_n^0, \left(\alpha_{-n}^0\right)^\dagger\right] = -n \quad \text{ and}\\ \left[\widetilde{\alpha}_n, \tilde{\alpha}_n^0\right] &= \left[\widetilde{\alpha}_n^0, \left(\tilde{\alpha}_{-n}^0\right)^\dagger\right] = -n\end{aligned}$$

[2]these are different from Fadeev-Popov ghosts

Consider states of the form $|\psi\rangle = \alpha^0_{-m}|0;p^\mu\rangle$ for $m>0$. For these states we have

$$\begin{aligned}\langle\psi \mid \psi\rangle &= \left\langle p^\mu;0\left|\left(\alpha^0_{-m}\right)^\dagger\alpha^0_{-m}\right|0;p^\mu\right\rangle \\ &= \left\langle p^\mu;0\left|\alpha^0_m\alpha^0_{-m}\right|0;p^\mu\right\rangle \\ &= \left\langle p^\mu;0\left|-m+\alpha^0_{-m}\alpha^0_m\right|0;p^\mu\right\rangle \\ &= -m\left\langle p^\mu;0\mid 0;p^\mu\right\rangle + \left\langle p^\mu;0\left|\left(\alpha^0_m\right)^\dagger\alpha^0_m\right|0;p^\mu\right\rangle \\ &= -m<0.\end{aligned}$$

Ghosts are problematic because these are in contradiction to the probabilistic interpretation of norm in Quantum mechanics. Our only hope is to apply the constraints and hope that these ghosts decouple from our theory. That is indeed the case when we fix the dimension of spacetime to be 26.

3.3.5 Normal Ordering and the Quantum Virasoro Algebra

As discussed in the previous section, the constraints in terms of Fourier modes is given by the vanishing of the Virasoro generators. But now, the Fourier modes are no more scalar valued functions but are operators on the Hilbert space. From the commutation relations (3.3.3), we see that the Virasoro generators

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}}\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{n-k} \quad \text{and} \quad \widetilde{L}_n = \frac{1}{2}\sum_{k\in\mathbb{Z}}\widetilde{\boldsymbol{\alpha}}_k\cdot\widetilde{\boldsymbol{\alpha}}_{n-k}$$

can be defined unambiguously for $n\neq 0$ as $\alpha^\mu_k,\alpha^\nu_{n-k}$ and the respective tildes commute for $n\neq 0$. For $n=0$, we have

$$L_0 = \frac{1}{2}\sum_{k\in\mathbb{Z}}\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{-k}, \quad \widetilde{L}_0 = \frac{1}{2}\sum_{k\in\mathbb{Z}}\widetilde{\boldsymbol{\alpha}}_k\cdot\widetilde{\boldsymbol{\alpha}}_{-k},$$

but since $\alpha^\mu_k,\alpha^\nu_{-k}$ and the respective tildes do not commute, the definition of L_0 and $\widetilde{L}_0$ is ambiguous in the quantum theory. We need to pick an ordering convention to define L_0 and $\widetilde{L}_0$. The natural choice is the *normal ordering* – we put annihilation operators $\alpha^\mu_n,\ \ n>0$ to the right of creation operator $\alpha^\mu_n,\ \ n<0$. With this choice of normal ordering, we put

$$:\!L_0\!: \,= \frac{1}{2}\sum_{k\in\mathbb{Z}}:\!\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{-k}\!: \,= \sum_{k=1}^{\infty}\boldsymbol{\alpha}_{-k}\cdot\boldsymbol{\alpha}_k+\frac{1}{2}\boldsymbol{\alpha}_0^2, \quad :\!\widetilde{L}_0\!: \,= \frac{1}{2}\sum_{k\in\mathbb{Z}}:\!\widetilde{\boldsymbol{\alpha}}_k\cdot\widetilde{\boldsymbol{\alpha}}_{-k}\!: \,= \sum_{k=1}^{\infty}\widetilde{\boldsymbol{\alpha}}_{-k}\cdot\widetilde{\boldsymbol{\alpha}}_k+\frac{1}{2}\widetilde{\boldsymbol{\alpha}}_0^2.$$

Under the commutation relation on $\alpha^\mu_n,\widetilde{\alpha}^\mu_n$ and the choice of normal ordering, we can calculate the quantum Virasoro algebra. We will show that

$$[L_n,L_m] = (n-m)L_{n+m} + \frac{c}{12}\left(n\left(n^2-1\right)\right)\delta_{m+n,0}.$$

where c is called the central charge. Recall that the classical Poisson bracket is

$$\{L_n, L_m\}_{P.B.} = i(m-n)L_{m+n}.$$

This extra term is due to *conformal anomaly* which is due to the breaking of Weyl symmetry in the quantum theory. As we will prove later, the expectation value of the trace of the energy momentum tensor $\langle T^\alpha_{\ \alpha}\rangle \propto R$ where R is the Ricci scalar. The nonvanishing of the trace implies that the Weyl symmetry is broken and hence we have the conformal anomaly. Hence the quantum Virasoro algebra is the central extension of the Witt algebra. We will not define this term precisely here. From now on, we will omit the colons in the Virasoro generators but they are assumed to be normal ordered. We begin by proving a lemma.

Lemma 3.3.3. *For any $m, n \in \mathbb{Z}$, we have*

$$[\alpha^\mu_m, L_n] = m\alpha^\mu_{m+n}, \quad \left[\widetilde{\alpha}^\mu_m, \widetilde{L}_n\right] = m\widetilde{\alpha}^\mu_{m+n}.$$

Proof. With the choice normal ordering we have

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} :\boldsymbol{\alpha}_{n-k}\cdot\boldsymbol{\alpha_k}:.$$

So we have

$$[\alpha^\mu_m, L_n] = \frac{1}{2}\sum_{k\in\mathbb{Z}} [\alpha^\mu_m : \boldsymbol{\alpha}_{n-k}\cdot\boldsymbol{\alpha}_k]$$

Now using $[A, BC] = [A, B]C + B[A, C]$ we get

$$\begin{aligned}
[\alpha^\mu_m, L_n] &= \frac{1}{2}\sum_{k\in\mathbb{Z}} \left\{\eta_{\rho\sigma}{:}\alpha^\rho_{n-k}\left[\alpha^\mu_m, \alpha^\sigma_k\right]{:} + \eta_{\rho\sigma}{:}\left[\alpha^\mu_m, \alpha^\rho_{n-k}\right]\alpha^\rho_k{:}\right\} \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}} \left\{\eta_{\rho\sigma}\left(\alpha^\rho_{n-k} m\eta^{\mu\sigma}\delta_{m+k,0} + \alpha^\sigma_k\eta^{\mu\rho} m\delta_{m+n-k,0}\right)\right\} \\
&= \frac{1}{2}\left\{\eta^\mu_\sigma\alpha^p_{n+m} m + n^\mu_\sigma\alpha^\sigma_{m+n} - m\right\} \\
&= m\alpha^\mu_{m+n}.
\end{aligned}$$

The proof for the tildes is identical. $\square$

Theorem 3.3.4. *For any $m, n \in \mathbb{Z}$, we have*

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}\left(n\left(n^2-1\right)\right)\delta_{m+n,0}, \\
\left[\widetilde{L}_n, \widetilde{L}_m\right] &= (n-m)\widetilde{L}_{n+m} + \frac{c}{12}\left(n\left(n^2-1\right)\right)\delta_{m+n,0}.
\end{aligned}$$

Proof. We have

$$
\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\sum_{k\in\mathbb{Z}}[:\boldsymbol{\alpha}_{m-k}\cdot\boldsymbol{\alpha}_k:, L_n] \\
&= \frac{1}{2}\sum_{k\leq 0}[\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m-k}, L_n] + \frac{1}{2}\sum_{k=1}^{\infty}[\boldsymbol{\alpha}_{m-k}\cdot\boldsymbol{\alpha}_k, L_n] \\
&= \frac{1}{2}\sum_{k\leq 0}\boldsymbol{\alpha}_k\cdot[\boldsymbol{\alpha}_{m-k}, L_n] + [\boldsymbol{\alpha}_k, L_n]\cdot\boldsymbol{\alpha}_{m-k} \\
&+ \frac{1}{2}\sum_{k\geq 1}\boldsymbol{\alpha}_{m-k}\cdot[\boldsymbol{\alpha}_k, L_n] + [\boldsymbol{\alpha}_{m-k}, L_n]\cdot\boldsymbol{\alpha}_k \\
&= \frac{1}{2}\sum_{k\leq 0}\left\{(m-k)\boldsymbol{\alpha}_\kappa\cdot\boldsymbol{\alpha}_{m+n-k} + k\boldsymbol{\alpha}_{n+k}\cdot\boldsymbol{\alpha}_{m-k}\right\} \\
&+ \frac{1}{2}\sum_{k\geq 1}\left\{k\boldsymbol{\alpha}_{m-k}\cdot\boldsymbol{\alpha}_{n+k} + (m-k)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k\right\},
\end{aligned}
$$

where in the second line we broke the normal ordering in the two sums and used Lemma 3.3.3 in last line. We now shift the second and third sum by n i,e. substitute $n+k$ by k. We get

$$
\begin{aligned}
[L_m, L_n] =& \frac{1}{2}\sum_{k\leq 0}(m-k)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} + \frac{1}{2}\sum_{k\leq n}(k-n)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} \\
&+ \frac{1}{2}\sum_{k\geq n+1}(k-n)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k + \frac{1}{2}\sum_{k\geq 1}(m-k)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k.
\end{aligned}
$$

Now we need to normal order the second and the third sum. If $n>0$ then the second sum is not normal ordered and if $n\leq 0$ then the third sum is not normal ordered. We will assume $n>0$ and proceed. One can get the result for $n\leq 0$ case using the same process. Breaking the second and fourth sum at 0 and $n+1$ respectively, we get

$$
\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\left[\sum_{k\leq 0}(m-k)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} + \sum_{k\leq 0}(k-n)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} + \sum_{k=1}^{n}(k-n)d_k\cdot\boldsymbol{\alpha}_{m+n-k}\right. \\
&\left. + \sum_{k\geq n+1}(k-n)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k + \sum_{k=1}^{n}(m-k)\delta_{n+n-k}\cdot\boldsymbol{\alpha}_k + \sum_{k\geq n+1}(m-k)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k\right] \\
&= \frac{1}{2}\left[\sum_{k\leq 0}(m-n)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} + \sum_{k\geq n+1}(m-n)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k\right. \\
&\left. + \sum_{k=1}^{n}(k-n)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} + \sum_{k=1}^{n}(m-k)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k\right]
\end{aligned}
$$

We will now use $\left[\boldsymbol{\alpha}_k^\mu, \boldsymbol{\alpha}_{m+n-k}^\nu\right] = \eta^{\mu\nu} k \delta_{m+n,0}$ to normal order the third sum. We get

$$
\begin{aligned}
[L_m, L_n] = \frac{1}{2}\Bigg[&\sum_{k\leq 0}(m-n)\boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{m+n-k} + \sum_{k\geq n+1}(m-n)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k \\
&+ \sum_{k=1}^{n}(k-n)\left(\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k + \eta_\mu^\mu k\delta_{m+n,0}\right) + \sum_{k=1}^{n}(m-k)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k\Bigg] \\
= \frac{1}{2}\Bigg[&\sum_{k\leq 0}(m-n)\boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{m+n-k} + \sum_{k\geq 1}(m-n)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k \\
&+ \sum_{k=1}^{n}(k-n)k \underbrace{n_\mu^\mu}_{D} \delta_{m+n,0}\Bigg] \\
&= (m-n)\frac{1}{2}\sum_{k\in\mathbb{Z}} {:}\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k}{:} + \frac{D}{2}\delta_{m+n,0}\sum_{k=1}^{n}\left(k^2 - nk\right) \\
&= (m-n)L_{m+n} + \frac{D}{2}\delta_{m+n,0}\left(\frac{n(n+1)(2n+1)}{6} - n\frac{n(n+1)}{2}\right) \\
&= (m-n)L_{m+n} + \frac{D}{2}\delta_{m+n,0}\left(\frac{n(n+1)}{2}\left(\frac{2n+1}{3} - n\right)\right) \\
&= (m-n)L_{m+n} + \frac{D}{2}\delta_{m+n,0}\frac{n(n+1)}{2}\left(\frac{1-n}{3}\right) \\
&= (m-n)L_{m+n} + \frac{D}{2}\delta_{m+n,0}\frac{n\left(1-n^2\right)}{6} \\
&= (m-n)L_{m+n} + \frac{D}{12}m\left(m^2-1\right)\delta_{m+n,0},
\end{aligned}
$$

where we replaced n by $-m$ in last step. The proof for the tildes is identical. □

Remark 3.3.5. We can also derive the structure of the central charge term by using Jacobi identity of the Lie bracket. We will rederive this algebra using the tools of conformal field theory.

Remark 3.3.6. In case of only free Bosonic fields, $c = \eta_\mu{}^\mu = D$ i,e. each scalar field contributes one unit to central charge. When we will rederive this algebra using conformal field theory and quantise the string using path integral, we will calculate the contribution of Fadeev-Popov ghosts to the central charge.

3.3.6 Imposing the Constraints

Recall that the constraints are $L_n = 0 = \widetilde{L}_n$ but this cannot be directly imposed on the Hilbert space of the theory. Indeed if $|\phi\rangle$ is any quantum mechanical state then for any

$n \in \mathbb{Z}$

$$0=\langle\phi\,|[L_n,L_{-n}]|\,\phi\rangle=2n\,\langle\phi\,|L_0|\,\phi\rangle+\frac{c}{12}n\left(n^2-1\right)\langle\phi\mid\phi\rangle$$

which does not hold if $n \neq 0, \pm 1$. So we cannot impose $L_n|\phi\rangle = 0$ for all n. So the alternative method of imposing the constraint would be to demand that the positive modes annihilate the physical states of the theory:

$$L_n|\text{phys}\rangle = 0, \quad \widetilde{L}_n|\text{phys}\rangle = 0, \quad n > 0, \tag{3.3.6}$$

where $|\text{phys}\rangle$ are the physical states of the theory. This way of imposing the constraints is equivalent to requiring that the matrix elements of all L_n (and the tildes) for $n \neq 0$ vanish. Indeed, we easily see that $L_n^\dagger = L_{-n}$ for $n \neq 0$, thus

$$\langle\text{phys}'|L_n|\text{phys}\rangle = 0, \quad \langle\text{phys}'|L_n|\text{phys}\rangle = 0, \quad \forall n.$$

We are left with imposing the constraint for L_0 and $\widetilde{L}_0$. Recall that we have an ordering ambiguity in defining L_0 and $\widetilde{L}_0$. We now define them using the normal ordering convention we have chosen and impose the constraints $L_0 = 0 = \widetilde{L}_0$ by shifting them by a constant which we will determine later:

$$(L_0 - a)|\text{phys}\rangle = 0 = \left(\widetilde{L}_0 - a\right)|\text{phys}\rangle \quad \text{(Mass-shell condition)}. \tag{3.3.7}$$

The constant a is called the *normal ordering constant*. In the classical theory, we saw that the constraints $L_0 = 0 = \widetilde{L}_0$ gave us the level matching condition. We want to understand its quantum version. Noting that

$$\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu = \widetilde{\alpha}_0^\mu, \qquad \text{and} \quad p^\mu p_\mu = -M^2,$$

we see that (3.3.7) can be written as

$$\begin{gathered}\left(N - \frac{\alpha'}{4}M^2 - a\right)|\text{phys}\rangle = 0\\ \left(\widetilde{N} - \frac{\alpha'}{4}M^2 - a\right)|\text{phys}\rangle = 0,\end{gathered}$$

where

$$N = \sum_{k=1}^{\infty} \boldsymbol{\alpha}_{-k}\cdot\boldsymbol{\alpha}_k \quad \text{and} \quad \widetilde{N} = \sum_{k=1}^{\infty} \widetilde{\boldsymbol{\alpha}}_{-k}\cdot\widetilde{\boldsymbol{\alpha}}_k \tag{3.3.8}$$

are the number operators. Thus the *quantum level matching condition* is

$$M^2 = \frac{4}{\alpha'}(N - a) = \frac{4}{\alpha'}(\widetilde{N} - a). \tag{3.3.9}$$

Since the number operator gives the number of excitations of the string, we see that the number of left-moving and right moving excitations are equal. Thus quantum level matching condition imply equal number of left-moving and right-moving modes.

3.3.7 The No Ghost Theorem

As we discussed in previous sections, our theory has ghosts. We also have a free parameter – the normal ordering constant to fix. It turns out that we can prove a no ghost theorem:

Theorem 3.3.7. (No ghost theorem) *The ghosts decouple in the critical dimension $D = 26$ and with $a = 1$.*

To prove this theorem we follow the following strategy: The normal ordering constant a and the dimension D are not arbitrary in the Quantum theory. For some values of a and D, negative norm states are part of the physical Hilbert space while at other values of a and D, the physical Hilbert space is positive definite. The transition then occurs at the value of a and D where zero norm states become physical. Our strategy will be to find that value of a D where zero norm states become physical – the so called *spurious states.*

Spurious States

Definition 3.3.8. A state $|\psi\rangle$ is called spurious if it satisfies the mass-shell condition and is orthogonal to all physical states. That is

$$(L_0 - a)\,|\psi\rangle = 0 \quad \text{and} \quad \langle\phi \mid \psi\rangle = 0 \quad \forall \quad |\phi\rangle \text{ physical .}$$

Lemma 3.3.9. *A general spurious state is of the form*

$$|\psi\rangle = \sum_{n=1}^{\infty} L_{-n}\,|\chi_n\rangle$$

where $|\chi_n\rangle$ are some states satisfying the modified mass-shell condition

$$(L_0 - a + n)\,|\chi_n\rangle = 0, \quad \forall n \geq 1.$$

Proof. By definition, we have

$$\langle\phi \mid \psi\rangle = 0 \quad \forall \quad |\phi\rangle \quad \text{physical.}$$

We know that

$$L_n|\phi\rangle = 0 \quad \forall n > 0$$

Thus we can write

$$|\psi\rangle = \sum_{n=1}^{\infty} L_{-n}\,|\chi_n\rangle \qquad \Big(\text{since} \quad L_{-n}^{\dagger} = L_n\Big)$$

for some state $|\chi_n\rangle$. Mass-shell condition implies

$$(L_0 - a)\,|\psi\rangle = 0$$
$$\Rightarrow \quad \sum_{n=1}^{\infty} (L_0 L_{-n} - aL_n)\,|\chi_n\rangle = 0.$$

By quantum Virasoro algebra $L_0 L_{-n} = L_{-n} L_0 + nL_{-n}$. We get

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left(L_{-n} L_0 + n - aL_{-n}\right) |\chi_n\rangle = 0 \\
\Rightarrow \quad &\sum_{n=1}^{\infty} L_{-n} \left(L_0 - a + n\right) |\chi_n\rangle = 0 \\
\Rightarrow \quad &\left(L_0 - a + n\right) |\chi_n\rangle = 0 \quad \forall \quad n > 0.
\end{aligned}$$

□

Definition 3.3.10. The states $|\chi_n\rangle$ satisfying $\left(L_0 - a + n\right) |\chi_n\rangle = 0$ are called level n states.

Lemma 3.3.11. *Any spurious state $|\psi\rangle$ can be written as*

$$|\psi\rangle = L_{-1} |\chi_1\rangle + L_{-2} |\chi_2\rangle$$

where $|\chi_1\rangle$ and $|\chi_2\rangle$ are level 1 level 2 states i,e. they satisfy

$$\left(L_0 - a + 1\right) |\chi_1\rangle = 0 \quad \textit{and} \quad \left(L_0 - a + 2\right) |\chi_2\rangle = 0.$$

Proof. By Lemma 3.3.9, we have

$$|\psi\rangle = \sum_{n=1}^{\infty} L_{-n} |\chi_n\rangle ,$$

where $\left(L_0 - a + n\right) |\chi_n\rangle = 0$. We will show that $L_{-n}\chi_n$ can be written as $L_{-1} |\chi_1\rangle + L_{-2} |\chi_2\rangle$ for some level 1 and level 2 states $|\chi_1\rangle$ and $|\chi_2\rangle$ respectively for all $n \geq 3$. Let us begin with the base case. Note that by quantum Virasoro algebra, we have

$$[L_{-L}, L_{-2}] = (-1+2)L_{-2-1} + 0 = L_{-3}.$$

Thus

$$L_{-3} |\chi_3\rangle = [L_{-1}, L_2] |\chi_3\rangle = L_{-1} \left(L - 2 |\chi_3\rangle\right) + L_2 \left(-\chi_{-1} \left(\chi_3\right)\right).$$

Take $|\chi_1\rangle = L_2 |\chi_3\rangle$ and $|\chi_2\rangle = -\chi_1 |\chi_3\rangle$. It remains to show that $|\chi_1\rangle$ and $|\chi_2\rangle$ are Level 1 and level 2 states respectively. Indeed since $\left(L_0 - a + 3\right) |\chi_3\rangle = 0$, we have

$$\begin{aligned}
\left(L_0 - a + 1\right) L_{-2} |\chi_3\rangle &= \left(L_0 L - 2 + L - 2(-a+1)\right) |\chi_3\rangle \\
&= \left(L_{-2} L_0 + 2L_{-2} + L_{-2}(-a+1)\right) (\chi_3\rangle \\
&= L_2 \left(L_0 - a + 3\right) |\chi_3\rangle \\
&= 0,
\end{aligned}$$

where we used the quantum Virasoro algebra: $L_0L_{-2} = L_{-2}L_0 + 2L_{-2}$. Similarly we have

$$\begin{aligned}(L_0 - a + 2)\left(-L_{-1}\left|\chi_3\right\rangle\right) &= -\left(L_{-1}L_0 + L_{-1}(-a+2)\right)\left|\chi_3\right\rangle \\ &= -\left(L_0L_{-1} + L_{-1} + L_{-1}(-a+2)\right)\left|\chi_3\right\rangle \\ &= -L_{-L}\left(L_0 - a + 3\right)\left|\chi_3\right\rangle \\ &= 0.\end{aligned}$$

For any n, we assume that $L_{-n+1}\left|\chi_{n-1}\right\rangle$ can be written as $L_{-1}\left|\chi_1\right\rangle + L_{-2}\left|\chi_2\right\rangle$. Then since

$$L_{-n} = \frac{1}{n}\left[L_{-1}, L_{-n+1}\right],$$

so that

$$L_n\left|\chi_n\right\rangle = \frac{1}{n}L_{-1}\left(L_{-n+1}\left|\chi_n\right\rangle\right) + \frac{1}{n}L_{-n+1}\left(-L_{-1}\left|\chi_n\right\rangle\right).$$

Following similar method as in the base case, we can show that $-\frac{1}{n}L_{-1}|\chi_n\rangle$ is a level $n-1$ state. Indeed observe that

$$\begin{aligned}-\frac{1}{n}\left(L_0 - a + n - 1\right)L_{-1}\left|\chi_n\right\rangle &= -\frac{1}{n}\left(L_{-1}L_0 + L_{-1} + L_1(-a+n-1)\right)\left|\chi_n\right\rangle \\ &= -\frac{1}{n}L_{-1}\left(L_0 - a + n\right)\left|\chi_n\right\rangle \\ &= 0\end{aligned}$$

So using induction hypothesis, we get

$$L_{-n}\left|\chi_n\right\rangle = L_{-1}\left(\frac{1}{n}L_{n+1}\left|\chi_n\right\rangle\right) + L_{-1}\left|\widetilde{\chi}_1\right\rangle + L_{-2}\left|\widetilde{\chi}_2\right\rangle$$

for some level 1 state $\left|\widetilde{\chi}_1\right\rangle$ a level 2 state $\left|\widetilde{\chi}_2\right\rangle$. It is also clear that $\frac{1}{n}L_{n+1}\left|\chi_n\right\rangle$ is a level 1 state. Thus define

$$\left|\widehat{\chi}_1\right\rangle = \left|\widetilde{\chi}_1\right\rangle + \frac{1}{n}L_{n+1}\left|\chi_n\right\rangle \quad \left|\widehat{\chi}_2\right\rangle = \left|\widetilde{\chi}_2\right\rangle$$

so that

$$L_{-n}\left|\chi_n\right\rangle = L_{-1}\left|\widehat{\chi}_1\right\rangle + L_{-2}\left|\widehat{\chi}_2\right\rangle$$

where $\widehat{\chi}_1$ and $\widehat{\chi}_2$ are level 1 and level 2 states respectively. □

Norm Zero States

Note that the spurious states are orthogonal to all physical states. Thus if we require the spurious states themselves to be physical we must have

$$\langle\psi \mid \psi\rangle = 0.$$

Thus all physical spurious states are norm-zero states. We will now find values of a and D so that all spurious states become physical. Any spurious state decouples from all physical process as they have zero norm.

Physical Spurious States

In view of Lemma 3.3.11, it is sufficient to find the values of a and D such that the spurious states $L_{-1}|\chi_1\rangle$ and $L_{-2}|\chi_2\rangle$ become physical where $|\chi_1\rangle$ and $|\chi_2\rangle$ are level 1 and level 2 states respectively.

Theorem 3.3.12. *Let $|\chi_1\rangle$ be a level 1 state satisfying $L_m|\chi_1\rangle = 0$ for all $m > 0$.Then the spurious state $|\psi\rangle = L_{-1}|\chi_1\rangle$ is physical if and only if $a = 1$.*

Proof. ($\Longrightarrow$) Suppose $L_{-1}|\chi_1\rangle$ is physical. Then $L_1 L_{-1}|\chi_1\rangle = 0$ as physical states $|\phi\rangle$ satisfy $L_m|\phi\rangle = 0$ for all $m > 0$. We get

$$\begin{aligned} L_1 L_{-1}|\chi_1\rangle &= (L_1 L_1 + 2L_0)|\chi_1\rangle \\ &= 2L_0|\chi_1\rangle \\ &= 2(a-1)|\chi_1\rangle, \end{aligned}$$

since $|\chi_1\rangle$ is a level 1 state satisfying $(L_0 - a + 1)|\chi_1\rangle = 0$. Thus $L_1 L_{-1}|\chi_1\rangle = 0 \Rightarrow a = 1$.
($\Longleftarrow$) If $a = 1$ then backtracking above steps we get $L_1 L_{-1}|\chi_1\rangle = 0$. To check that $L_m L_{-1}|\chi_1\rangle = 0$, we proceed inductively. We have the base case. Next

$$L_m L_{-1}|x_1\rangle = L_{-1}L_m|\chi_1\rangle + (m+1)L_{m-1}|\chi_1\rangle = 0,$$

since $L_m|\chi_1\rangle = 0$ by assumption and $L_{m-1}|\chi_1\rangle = 0$ by induction hypothesis. Next thing to check is

$$(L_0 - a)\, L_{-1}|\chi_1\rangle = 0, \quad (a = 1).$$

Indeed

$$\begin{aligned} L_0 L_{-1}|\chi_1\rangle &= L_{-1}L_0|\chi_1\rangle + L_{-1}|\chi_1\rangle \\ &= 0 + L_{-1}|\chi_1\rangle, \end{aligned}$$

since $(L_0 - a + 1)|\chi_1\rangle = L_0|\chi_1\rangle = 0$. ☐

Next we look at level 2 spurious states. A general level 2 spurious state is

$$|\psi\rangle = (L_2 + \gamma L_1 L_1)|\chi_2\rangle.$$

We will show that $|\psi\rangle$ is physical if and only if $\gamma = \frac{3}{2}$ and $D = 26$.

Theorem 3.3.13. *Let $|\chi_2\rangle$ be a level 2 state satisfying $L_m|\chi_2\rangle = 0$ for all $m > 0$. Then the spurious state $|\psi\rangle = (L_{-2} + \gamma L_{-1}[_{-1})|\chi_2\rangle$ is physical if and only if $\gamma = \frac{3}{2}$ and $D = 26$.*

Proof. ($\Longrightarrow$) Suppose $|\psi\rangle$ is physical, then we must have $\langle\psi|\psi\rangle = 0$ since $|\psi\rangle$ is spurious. Next we demand $L_m|\psi\rangle = 0$ for all $m > 0$. In particular $L_1|\psi\rangle = 0$. We have

$$\begin{aligned} &(L_1 L_{-2} + \gamma L_\perp L_{-1} L_{-1})|\chi_2\rangle = 0 \\ &\Longrightarrow\ (L_{-2}L_1 + 3L_{-1} + \gamma L_{-1}L_1 L_{-1} + 2\gamma L_0 L_{-1})|\chi_2\rangle = 0 \\ &\Longrightarrow\ (L_1 + 3L_{-1} + \gamma L_{-1}L_{-1}L_1 + 2\gamma L_{-1}L_0 + 2\gamma L_{-1}L_0 + 2\gamma L_{-1})|\chi_2\rangle = 0 \\ &\Longrightarrow\ L_{-1}(3 + 4\gamma L_0 + 2\gamma)|\chi_2\rangle = 0, \end{aligned}$$

where we used $L_1|\chi_2\rangle = 0$. Now since $L_0|\chi_2\rangle = -|\chi_2\rangle$, we get

$$\begin{aligned}
&L_{-1}(3 + y(-1)\gamma + 2\gamma)\,|\chi_2\rangle = 0\\
&\implies (3 - 2\gamma)L_{-1}\,|\chi_2\rangle = 0\\
&\implies \gamma = \frac{3}{2}
\end{aligned}$$

So

$$|\psi\rangle = \left(L_{-2} + \frac{3}{2}L_{-1}L_{-1}\right)|x_2\rangle\,.$$

Next we impose $L_2|\psi\rangle = 0$. We have

$$\begin{aligned}
&L_2\left(L_{-2} + \frac{3}{2}L_{-1}L_1\right)|\chi_2\rangle = 0\\
&\implies \left[L_2, L_{-2} + \frac{3}{2}L_{-1}L_{-1}\right]|\chi_2\rangle + \left(L_{-2} + \frac{3}{2}L_{-1}L_{-L}\right)L_2\,|\chi_2\rangle = 0\\
&\implies \left(4L_0 + \frac{c}{12}2(3)\delta_{0,0} + \frac{3}{2}\left[L_2, L_1L_{-1}\right]\right)|\chi_2\rangle = 0,
\end{aligned}$$

where we used our assumption that $L_2\,|\chi_2\rangle = 0$. Now since

$$\begin{aligned}
[L_2,\ L_{-1}L_{-1}] &= [L_2, L_1]\,L_1 + L_{-1}\,[L_2, L_1]\\
&= 3L_1L_{-1} + 3L_{-1}L_1\\
&= 3\,(L_{-1}L_1 + 2L_0) + 3L_{-1}L_1\\
&= 6L_{-1}L_1 + 6L_0.
\end{aligned}$$

So we have

$$\begin{aligned}
&\left(4L_0 + \frac{c}{2} + \frac{3}{2}\,(6L_{-1}L_1 + 6L_0)\right)|\chi_2\rangle = 0\\
&\implies \left(13L_0 + 9L_{-1}L_1 + \frac{c}{2}\right)|\chi_2\rangle = 0\\
&\implies c = 26,
\end{aligned}$$

where we used $L_1|\chi_2\rangle = 0$ and $L_0|\chi_2\rangle = -|\chi_2\rangle$. In free Bosonic string theory, we know that $c = \eta_\mu{}^\mu = D$, so $D = 26$.
($\Longleftarrow$) Assuming $D = 26$, $\gamma = \frac{3}{2}$, we can show that $L_1|\psi\rangle = 0$ and $L_2|\psi\rangle = 0$ back tracking the steps. For $m \geq 3$, it is easily proved using induction as in the proof of Theorem 3.3.12.

Finally we need to show that $(L_0 - 1)\,|\psi\rangle = 0$. To see that this is true, observe that

$$\begin{aligned}
(L_0 - 1)\left(L_{-2} + \frac{3}{2}L_{-1}L_1\right)|\chi_2\rangle &= \left(L_0L_{-2} + \frac{3}{2}L_0L_{-1}L_{-1}\right)|\chi_2\rangle - |\psi\rangle \\
&= \left(L_{-2}L_0 + 2L_2 + \frac{3}{2}L_{-1}L_0L_1 + \frac{3}{2}L_{-1}L_{-1}\right)|\chi_2\rangle - |\psi\rangle \\
&= \left(L_2(-1) + 2L_{-2} + \frac{3}{2}L_{-1}L_{-1}L_0 + \frac{3}{2}L_{-1}L_{-1} + \frac{3}{2}L_{-1}L_1\right)|\chi_2\rangle - |\psi\rangle \\
&= \left(L_{-2} - \frac{3}{2}L_{-1}L_{-1} + \frac{3}{2}L_{-1}L_{-1} + \frac{3}{2}L_{-1}L_{-1}\right)|\chi_2\rangle - |\psi\rangle \\
&= 0,
\end{aligned}$$

where we used $L_0\,|\chi_2\rangle = -\,|\chi_2\rangle$. □

Thus we have shown that infinite classes of spurious states of zero norm appear in our theory when $D = 26$ and $a = 1$. Thus we have determined the boundary where positive norm states turn into negative norm states. Thus for these values of a and D, the ghosts decouple from the theory as infinitely many zero norm states appear in our theory. There are *non-critical string theories* free of ghosts for $a \leq 1$ and $D \leq 25$ but we will not pursue it here. We conclude the covariant quantisation of closed strings with the result that the spectrum is well defined and ghost free in the critical dimension. We will arrive at the same result in the next section using another quantisation scheme.

3.4 Lightcone Quantisation

In lightcone quantisation, we begin by solving the constraints first and separating the physical degrees of freedom. Before we begin, let us discuss about reparametrizations, conformal transformations and Weyl rescaling.

Given any reparametrization of the worldsheet, it corresponds to choosing a different coordinate chart for the manifold. This has no physical consequence as all points, curves remain same on the manifold (worldsheet). Thus any diffeomorphism automatically preserves circular and hyperbolic angles. On the other hand coordinate transformations which transform the metric as

$$g_{\mu\nu} \longrightarrow \Omega^2(\sigma)g_{\mu\nu}(\sigma)$$

are called conformal transformations. These transformations preserve angles (circular as well as hyperbolic) Another version of Conformal transformations are maps between manifolds. Let (M, g) and $(N, \widetilde{g})$ be Riemannian manifolds and $\varphi : M \longrightarrow N$ be a smooth map. Then φ is said to be a conformal map if the pullback $\varphi^*\widetilde{g} = \Omega^2 g$ for some smooth function Ω. Writing $x' = \varphi(x)$ we see that

$$\widetilde{g}_{\mu\nu}\,(x')\,\frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} = \Omega^2(x)g_{\rho\sigma}.$$

Thus angles are preserved. In particular if $M = N$ and $\widetilde{g} = g$ then

$$g_{\mu\nu}\left(x'\right)\frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}} = \Omega^2(x)g_{\rho\sigma}$$

which are usual conformal transformations. Weyl rescalings on the other hand, are completely different. They are not coordinate transformations. These do not act on the parametrizations but act on the metric. Since the metric is only scaled thus angles are preserved.

3.4.1 Residual Gauge Freedom: Lightcone Gauge

We have already fixed a gauge i,e. chosen two reparametrizations and used Weyl rescaling to fix the metric to $\eta_{\mu\nu}$. But we have some residual gauge symmetry. Indeed consider a reparametrization $\sigma^\alpha \longrightarrow \widetilde{\sigma}^\alpha = \widetilde{\sigma}^\alpha(\boldsymbol{\sigma})$ such that the metric changes by $\eta_{\alpha\beta} \longrightarrow \widetilde{\eta}_{\mu\beta} = \Omega^2(\boldsymbol{\sigma})\eta_{\mu\nu}$. This can then be undone by a Weyl rescaling ($d\sigma d\tau$ introduces a Jacobian in the action which again ensures that the resulting action is Weyl invariant and we can again use Weyl rescaling). These are exactly the conformal transformations. Thus we see that

"diffeomorphisms = conformal × Weyl".

Next we need to find all such transformation. We will do this in the so called *lightcone coordinates*. Introduce

$$\sigma^{\pm} = \tau \pm \sigma \implies \tau = \frac{\sigma^+ + \sigma^-}{2}, \quad \sigma = \frac{\sigma^+ - \sigma^-}{2}.$$

The metric is given by

$$\begin{aligned}
ds^2 = -d\tau^2 + d\sigma^2 &= -\frac{1}{4}\left(d\sigma^+ + d\sigma^-\right)^2 + \frac{1}{4}\left(d\sigma^+ - d\sigma^-\right)^2 \\
&= -\frac{1}{4}d\sigma^{+2} - \frac{1}{4}d\sigma^{-2} - \frac{1}{2}d\sigma^+ d\sigma^- + \frac{1}{4}d\sigma^{+2} + \frac{1}{4}d\sigma^{-2} - \frac{1}{2}d\sigma^+ d\sigma^- \\
&= -d\sigma^+ d\sigma^-.
\end{aligned}$$

So a reparametrization $\sigma^+ \longrightarrow \widetilde{\sigma}+(\sigma^+)$ and $\sigma^- \longrightarrow \widetilde{\sigma}^-(\sigma^-)$, ds^2 simply changes by scaling. Indeed

$$ds^2 = -\frac{\partial\sigma^+}{\partial\widetilde{\sigma}^+}d\widetilde{\sigma}^+\frac{\partial\sigma^-}{\partial\widetilde{\sigma}^-}d\sigma^- = -\frac{\partial\sigma^+}{\partial\tilde{\sigma}^+}\frac{\partial\sigma^-}{\partial\widetilde{\sigma}}d\tilde{\sigma}^+ d\widetilde{\sigma}.$$

Note that the reparametrizations are single variable. We would like to fix the remnant gauge. The choice that we will make here is called *lightcone gauge*. Introduce

$$X = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{D-1}\right).$$

Such a choice breaks Lorentz invariance in classical as well as quantum theory as we have picked a special time and space part while Lorentz transformations mixes space and time coordinates. So when we quantise our system, we will look for conditions that restores the Lorentz invariance. It is now easy to see that

$$ds^2 = -2dX^+dX^- + \sum_{i=1}^{D-2} \left(dX^i\right)^2.$$

So the metric $n_{++} = 0 = n_{--}$ and $n_{+-} = n_{-+} = -1$ and $\quad \eta_{ii} = 1 \quad \forall \quad \dot{i} = 1, 2, \ldots, D-2$ and all other elements vanish. So any vector $A^\mu = (A^+, A^-, A^i)$ is lowered as

$$A_\mu = (-A_-, -A_+, \quad A_i)$$

and the dot product is

$$A^\mu B_\mu = -A^+B_- - A^-B_+ + A^iB^i.$$

Solution of the equation of motion is

$$X^+ = X_L^+\left(\sigma^+\right) + X_R^+\left(\sigma^-\right).$$

To see this, note that

$$X^\mu = X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right)$$

so that

$$\begin{aligned} X^+ &= \frac{1}{\sqrt{2}}\left(X^0 + X^{D-1}\right) = \frac{1}{\sqrt{2}}\left[X_L^0\left(\sigma^+\right) + X_L^{D-1}\left(\sigma^+\right) + X_R^0\left(\sigma^-\right) + X_R^{D-L}\left(\sigma^-\right)\right] \\ &= X_L^+\left(\sigma^+\right) + X_R^+\left(\sigma^-\right). \end{aligned}$$

We now fix our gauge. Note that X^+ satisfies the wave equation $\partial_+\partial_-X^+ = 0$. Now note that a reparametrization $\widetilde{\sigma}^+ = \widetilde{\sigma}^+\left(\sigma^+\right)$ and $\widetilde{\sigma}^- = \widetilde{\sigma}^-\left(\sigma^-\right)$ corresponds to

$$\widetilde{\tau} = \frac{\sigma^+ + \sigma^-}{2}, \quad \widetilde{\sigma} = \frac{\widetilde{\sigma}^+ - \widetilde{\sigma}^-}{2}.$$

But $\widetilde{\tau}$ has to satisfy $\partial_+\partial_-\widetilde{\tau} = 0$. So we can choose

$$\widetilde{\tau} = \frac{X^+}{\alpha' p^+} - x^+.$$

This is called *lightcone gauge*. The coordinate X^- still satisfies the wave equation

$$\partial_+\partial_-X^- = 0$$

The usual solution is

$$X^- = X_L^-\left(\sigma^+\right) + X_R\left(\sigma^-\right).$$

Let us look at the constraints in lightcone gauge. We had the constraint $(\partial_+X)^2 = 0 = (\partial_-X)^2$ with $X = (X^+, X^-, X^0)$. So we get

$$\begin{aligned}(\partial_+X)^2 &= -2\partial_+X^-\partial_+X^+ + \sum_{i=1}^{D-2}\left(\partial_+X^i\right)^2 \\ (\partial_-X)^2 &= -2\partial_-X^-\partial_-X^+ + \sum_{i=1}^{D-2}\left(\partial_-X^i\right)^2.\end{aligned}$$

Since

$$\partial_+X^+ = \frac{\alpha' p^+}{2} = \partial_-X^+ \quad \left(\text{as} \quad \tau = \frac{\sigma^+ + \sigma^-}{2}\right),$$

the constraints $(\partial_+X)^2 = 0 = (\partial_-X)^2$ gives

$$\begin{aligned}\partial_+X^- &= \frac{1}{\alpha' p^+}\sum_{i=1}^{D-2}\left(\partial_+X^i\right)^2 \\ \partial_-X^- &= \frac{1}{\alpha' p^+}\sum_{i=1}^{D-2}\left(\partial_-X^2\right)^2.\end{aligned} \tag{3.4.1}$$

Thus we see that in lightcone gauge the $D-2$ scalar fields determine X^- upto an additive constant coming from integration. Indeed we see that if we write the mode expansion of $X^-_{L/R}$

$$\begin{aligned}X_L^-\left(\sigma^+\right) &= \frac{1}{2}x^- + \frac{\alpha'}{2}p^-\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}_n^- e^{-in\sigma^+} \\ X_R^-\left(\sigma^-\right) &= \frac{1}{2}x^- + \frac{\alpha'}{2}p^-\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^- e^{-in\sigma^-}\end{aligned}$$

then x^- is coming as the integration constant while all other terms p^- and $\widetilde{\alpha}_n^-, \alpha_n^-$ is determined in terms of $\widetilde{\alpha}_n^i, \alpha_n^i$ and p^+. Indeed if we write

$$\begin{aligned}\partial_+X_L^- &= \sqrt{\frac{\alpha}{2}}\sum_{n\in\mathbb{Z}}\widetilde{\alpha}_n^- e^{in\sigma^+} \quad \text{with} \quad \widetilde{\alpha}_0^- = \sqrt{\frac{\alpha'}{2}}p^- \\ \partial_-X_R^- &= \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}^{n}\alpha_n e^{-in\sigma^-} \quad \text{with} \quad \alpha_0^- = \sqrt{\frac{\alpha'}{2}}p^-.\end{aligned}$$

Then substituting $\left(\partial_+X\right)^2$ using Fourier modes of X^i in (3.4.1), we get by comparing coefficients of $e^{-in\sigma^\pm}$ that

$$\begin{aligned}\alpha_n^- &= \frac{1}{\sqrt{2\alpha'}}\frac{1}{p^+}\sum_{m\in\mathbb{Z}}\sum_{i=1}^{D-1}\alpha_{n-m}^i\alpha_m^i \\ \widetilde{\alpha}_n^- &= \frac{1}{\sqrt{2\alpha'}}\frac{1}{p^+}\sum_{m\in\mathbb{Z}}\sum_{i=1}^{D-2}\widetilde{\alpha}_{n-m}^i\widetilde{\alpha}_m^i.\end{aligned}$$

For $n=0$, we get two expressions for p^-:

$$\frac{\alpha' p^-}{2} = \frac{1}{2p^+}\sum_{i=1}^{D-2}\left(\frac{\alpha'}{2}p^ip^i + \sum_{n\neq 0}\alpha^i_{-n}\alpha^i_n\right) = \frac{1}{2p^+}\sum_{i=1}^{D-2}\left(\frac{\alpha'}{2}p^ip^i + \sum_{n\neq 0}\widetilde{\alpha}^i_{-n}\widetilde{\alpha}^i_n\right).$$

Using $p^\mu = (p^+, p^-, p^i)$ we see that

$$M^2 = -p^\mu p_\mu = 2p^+p^- - \sum_{i=1}^{D-2} p^ip^i.$$

Using the above equality for $\frac{\alpha' p^-}{2}$ above we get

$$M^2 = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\widetilde{\alpha}^i_{-n}\widetilde{\alpha}^i_n,$$

where we used

$$\sum_{n\neq 0}{}' \alpha^i_{-n}\alpha^i_n = 2\sum_{n>0}{}'\alpha^i_{-n}\alpha^i_n.$$

The oscillators $\alpha^i_n, \widetilde{\alpha}^i_n$ are called *transverse oscillators*. These are physical excitations in the sense that knowing α^i_n and $\widetilde{\alpha}'_n$ determines all other modes. Thus the most general classical solution can be determined in terms of $2(D-2)$ oscillator modes $\alpha^i_n, \widetilde{\alpha}^i_n$ and a bunch of zero modes $p^\pm, p^i, x^\pm$.

3.4.2 Quantisation

The usual way of quantisation is to compute the classical Poisson brackets and use Dirac prescription. As we did in covariant quantisation, using the Poisson brackets, the following commutation relations are obvious:

$$\begin{aligned} \left[x^i, p^j\right] = i\delta^{ij}, \quad \left[x^-, p^+\right] = -i, \quad \left[x^+, p^-\right] = -i \\ \left[\alpha^i_n, \alpha^j_m\right] = n\delta^{ij}\delta_{m+n,0} = \left[\widetilde{\alpha}^i_n, \widetilde{\alpha}^j_m\right]. \end{aligned} \tag{3.4.2}$$

The ground state is again $|0;p^\mu\rangle$ with $|0\rangle$ being the string. To build the Fock space, we impose

$$\widehat{p}^\mu\,|0;p^\mu\rangle = p^\mu\,|0;p^\mu\rangle\,, \quad \widetilde{\alpha}^i_n\,|0;p^\mu\rangle = 0 = \alpha^i_n\,|0;p^\mu\rangle\,, \quad \forall \quad n>0 \quad \mu = 1,2,\cdots, D-1.$$

We act with $\alpha^i_{-n}, \widetilde{\alpha}^i_{-n}, n>0$ to build the Fock space. Notice that i runs only over spatial index $i = 1, 2\cdots, D-1$, so the theory does not have ghosts by construction. Its time to impose the constraints. As we had in covariant quantisation, level matching with normal ordering implies

$$M^2 = \frac{4}{\alpha'}(N-a) = \frac{4}{\alpha'}(\widetilde{N}-a),$$

where now the number operators are

$$N = \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n\neq 0}\alpha_n^i\alpha_n^i \quad \text{and} \quad \widetilde{N} = \frac{1}{2}\sum_{i=1}^{D-1}\sum_{n\neq 0}\widetilde{\alpha}_n^i\widetilde{\alpha}_n^i$$

and a is again the normal ordering constant which we again fix by requiring that the spectrum be Lorentz invariant. Note that

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{D-1}\sum_{n\neq 0}\alpha_{-n}^i\alpha_n^i &= \frac{1}{2}\sum_i\left[\sum_{n>0}\alpha_{-n}^i\alpha_n^i + \sum_{n<0}\alpha_{-n}^i\alpha_n^i\right] \\
&= \frac{1}{2}\sum_i\sum_{n>0}\alpha_{-n}^i\alpha_n^i + \frac{1}{2}\sum_i\left[\sum_{n<0}\alpha_n^i\alpha_{-n}^i - n\right] \\
&= \sum_i\sum_{n>0}\alpha_{-n}^i\alpha_n^i + \frac{D-2}{2}\sum_{n>0}n,
\end{aligned}$$

where we used the commutator $\left[\alpha_n^i, \alpha_{-n}^i\right] = n$. The last sum is divergent but we need to extract physics out of this divergence. The result is the appearance of *Casimir force*. We will do this in two ways.

UV Cut-off $\varepsilon \ll 1$

Write

$$\sum_{n>0} n \longrightarrow \sum_{n>0} ne^{-\varepsilon n} = -\frac{\partial}{\partial\varepsilon}\sum_{n>0}e^{-\varepsilon n} = -\frac{\partial}{\partial\varepsilon}\left[\left(L - e^{-\varepsilon}\right)^{-1}\right].$$

Now

$$\begin{aligned}
-\frac{\partial}{\partial\varepsilon}\left[\frac{1}{1-e^{-\varepsilon}}\right] = \frac{e^{-\varepsilon}}{(1-e^{-\varepsilon})^2} &= \frac{\left(1-\varepsilon+\frac{\varepsilon^2}{2}+O\left(\varepsilon^3\right)\right)}{\left(1-1+\varepsilon-\frac{\varepsilon^2}{2}+\cdots\right)^2} \\
&= \frac{\left(1-\varepsilon+\frac{\varepsilon^2}{2}+O\left(\varepsilon^3\right)\right)}{\varepsilon^2\left(1-\frac{\varepsilon}{2}+\ldots\right)^2} \\
&= \frac{1}{\varepsilon^2}\left(t-\varepsilon+\frac{\varepsilon^2}{2}+O\left(\varepsilon^3\right)\right)\left(1+2\frac{\varepsilon}{2}-2\frac{\varepsilon^2}{3!}+\frac{3}{4}\varepsilon^2+O\left(\varepsilon^3\right)\right) \\
&= \frac{1}{\varepsilon^2}\left(1+\frac{\varepsilon^2}{2}-\varepsilon^2-\frac{2}{6}\varepsilon^2+\frac{3}{4}\varepsilon^2+O\left(\varepsilon^3\right)\right) \\
&= \frac{1}{\varepsilon^2}-\frac{1}{12}+O(\varepsilon).
\end{aligned}$$

The $\frac{1}{\varepsilon^2}$ must be renormalised away. After renormalising and taking $\varepsilon \to 0$, we get the odd result

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}. \tag{3.4.3}$$

Zeta Function Regularisation

The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}.$$

The series defining $\zeta(s)$ converges absolutely and uniformly on compact subsets of the half plane $\{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$ and hence $\zeta(s)$ is holomorphic on this half plane. Moreover, the Riemann zeta function admits a unique analytic continuation to the whole $s-$plane. To be precise, Riemann in 1859 proved the following integral representation of the Riemann zeta function:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \frac{1}{s(s-1)} + \int_1^{\infty} W(x)\left(x^{s/2} + x^{\frac{1-s}{2}}\right)\frac{dx}{x}, \tag{3.4.4}$$

where

$$W(x) = \sum_{n=1}^{\infty} e^{n^2\pi x}$$

and $\Gamma(s)$ is the gamma function. The integral on the right hand side of (3.4.5) converges for all $\mathbb{C}$. So this integral gives an analytic continuation of $\zeta(s)$. Indeed putting

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

we see that $\xi(s)$ is an entire function and satisfies

$$\xi(1-s) = \xi(s).$$

From the fact that $\xi(s)$ is entire, we see that $\zeta(s)$ (analytically continued) has simple zeros at $s = -2n,\ n \in \mathbb{N}$ corresponding to poles of $\Gamma\left(\frac{s}{2}\right)$.[3] Now at $s = -1$, we have that $\xi(-1) = \xi(2)$. This implies

$$\begin{aligned} & 2\pi^{1/2}\Gamma\left(-\frac{1}{2}\right)\zeta(-1) = 2\pi^{-1}\Gamma(1)\zeta(2) \\ \Rightarrow \quad & \pi^{1/2}\left(-\frac{1}{2}\right)\sqrt{\pi}\zeta(-1) = \pi^{-1}\frac{\pi^2}{6} \\ \Rightarrow\ & \zeta(-1) = -\frac{1}{12}. \end{aligned}$$

So we see that both of the computation gives same result.

[3] These are called the *trivial zeros* of $\zeta(s)$. The *Riemann hypothesis* says that all other *non trivial zeros* of $\zeta(s)$ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$. This is still an open problem and a million dollar problem announced by the Clay Mathematical Institute.

3.4.3 String Spectrum

With the above regularisation, the level matching condition becomes

$$M^2 = \frac{4}{\alpha'}\left[\sum_{i=1}^{D-2}\sum_{n>0}\alpha^2_{-n}\alpha^i_n - \frac{D-2}{24}\right] = \frac{4}{\alpha'}\left[:N: - \frac{D-2}{24}\right]$$
$$= \frac{4}{\alpha'}\left[\sum_{i=1}^{D-2}\sum_{n>0}\widetilde{\alpha}^i_{-n}\widetilde{\alpha}^i_n - \frac{D-2}{24}\right] = \frac{4}{\alpha'}\left[:\widetilde{N}: - \frac{D-2}{24}\right].$$

We also identify the normal ordering constant as

$$a = \frac{D-2}{24}.$$

Let us look at the ground state $|0;p^\mu\rangle$. By our definition of vacuum $|0\rangle$, we have

$$:N:|0;p^\mu\rangle = 0 = :\widetilde{N}:|0;p^\mu\rangle\,.$$

So Level matching gives

$$M^2 = -\frac{D-2}{6\alpha'} < 0.$$

These are particles weth negative mass-squared. These are called *Tachyons*. These are a problem in Bosonic string theory. But when we study superstring theory where we include Fermionic fields on the worldsheet, then these states automatically vanish. Now let us look at excited states. First excited state is obtained by acting α^i_{-1} and $\widetilde{\alpha}^i_{-1}$. To see this observe that for $n > 0$,

$$\begin{aligned}
N\alpha^j_{-n}|0;p^\mu\rangle &= \sum_{i=1}^{D-1}\sum_{k=1}^{\infty}\alpha^i_{-k}\alpha^i_k\alpha^j_{-n}|0;p^\mu\rangle \\
&= \left[\sum_i\sum_{k=1}^{\infty}\alpha^i_{-k}\alpha^j_{-n}\alpha^i_k + k\delta^{ij}\delta_{k-n,0}\alpha^i_{-k}\right]|0;p^\mu\rangle \\
&= n\alpha_{-n}|0;p^\mu\rangle\,.
\end{aligned}$$

So α^i_{-1} and $\widetilde{\alpha}^i_{-1}$ give first excited states. Thus level matching requires us to act α^i_{-1} and $\widetilde{\alpha}_{-1}$ together. So the first excited states are $\alpha^i_{-1}\widetilde{\alpha}^j_{-1}|0;p^\mu\rangle$. Mass of each of these states is

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{D-2}{24}\right). \tag{3.4.5}$$

3.4.4 Fixing Lorentz Invariance

Our states are labelled by indices $i, j = 1, 2, \ldots, D-1$ and hence these transform as vectors with respect to the group $SO(D-2) \hookrightarrow SO(1, D-1)$ where $SO(1, D-1)$ is the full Lorentz group. But finally we want our states to fit into some representation of the Lorentz group $S0(1, D-1)$. Here we invoke Wigner's classification of representations of Poincaré group. From the discussion in Appendix A[4], we see that if we want our states $\alpha^i_{-1}\widetilde{\alpha}^j_{-1}\left|0;p^\mu\right\rangle$ to transform as some representation of the Lorentz group, then these states must be massless as these states fit into the representation of the little group $SO(D-2)$ which is the little group corresponding to massless representation. Thus (3.4.5) implies that $D = 26$ which also gives $a = 1$. Thus we have recovered the critical dimension by requiring that the first excited state be representations of the Lorentz group. We still need to make sure that the higher excited states also transform as some representations of the Lorentz invariant and we now have no choice other than to hope that with the values of a and D that we have chosen, we somehow manage to embed the higher excited states into the representation of Lorentz group. This is indeed the case. We will show this for the second excited state but one can check that the all higher excited states fit into some massive representation of the Lorentz group. We first note from (3.4.5) that all higher excited states are massive with the values of D that we have chosen. So by Wigner's classification, all these states must fit into some representation of $SO(D-1)$ as the little group for massive representations is $SO(D-1)$. For $N = \widetilde{N} = 2$, the states are

$$\begin{array}{ll} \alpha^i_{-1}\alpha^j_{-1}\left|0;p^\mu\right\rangle, \alpha^i_{-2}\left|0;p^\mu\right\rangle & \text{– Right moving} \\ \widetilde{\alpha}^i_{-1}\widetilde{\alpha}^j_{-1}\left|0;p^\mu\right\rangle, \widetilde{\alpha}^i_{-2}\left|0;p^\mu\right\rangle & \text{– Left moving.} \end{array}$$

Since $\alpha^i_{-1}, \alpha^j_{-1}$ commute, in the right moving sector there are a total of

$$\begin{aligned} \frac{1}{2}(D-2)(D-1) + (D-2) &= (D-2)\left(\frac{D-1+2}{2}\right) \\ &= \frac{(D-2)(D+1)}{2} \\ &= \frac{1}{2}D(D-1) - 1 \end{aligned}$$

states. These easily fit into the symmetric traceless representation of $SO(D-1)$. Infact one can prove that all higher excited states fit into some representation of $SO(D-1)$. Hence we have recovered Lorentz invariance by fixing the dimension of spacetime.

There is one other way to explicitly check that we have recovered Lorentz invariance: We compute the conserved charges and currents corresponding to the global Poincaré symmetry $X^\mu \longrightarrow \Lambda^\mu{}_\nu X^\nu + C^\mu$ of the action and require that they satisfy Poincaré algebra. Let us begin with translations $X^\mu \longrightarrow X^\mu + C^\mu$. One can compute the Noether current. It turns out to be:

$$P^\alpha_\mu = \frac{1}{2\pi\alpha'}\partial^\alpha X_\mu.$$

[4] I recommend going through Appendix A to understand the uses of Wigner's classification in string theory context.

It is easy to see that $\partial_\alpha P^\alpha_\mu = 0$ as $\partial_\alpha \partial^\alpha X_\mu = 0$ on-shell. Next the Noether charge corresponding to Lorentz transformation $X^\mu \longrightarrow \Lambda^\mu{}_\nu X^\nu$ is

$$J^\alpha_{\mu\nu} = P^\alpha_\mu X_\nu - P^\alpha_\nu X_\mu.$$

We can again check that $\partial_\alpha J^\alpha_{\mu\nu} = 0$. Indeed we have

$$\begin{aligned}
\partial_\alpha J^\alpha_{\mu\nu} &= \left(\partial_\alpha P^\alpha_\mu\right) X_\nu + P^\alpha_\mu \partial_\alpha X_\nu - \left(\partial_\alpha P^\alpha_\nu\right) X_\mu - P^\alpha_\nu \partial_\alpha X_\mu \\
&= P^\alpha_\mu \partial_\alpha X_\nu - P^\alpha_\nu \partial_\alpha X_\mu \\
&= \frac{1}{2\pi\alpha'} \left(\partial^\alpha X_\mu \partial_\alpha X_\nu - \partial^\alpha X_\nu \partial_\alpha X_\mu\right) \\
&= 0.
\end{aligned}$$

The conserved charges corresponding to J^τ_μ is

$$M_{\mu\nu} = \int_0^\pi d\sigma J^\tau_{\mu\nu}.$$

Now using the mode expansion for X^μ we get

$$\begin{aligned}
M^{\mu\nu} = \int_0^\pi d\sigma \left(X^\mu \Pi^\nu - X^\nu \Pi^\mu\right) &= \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\right) \\
&= l^{\mu\nu} + E^{\mu\nu} + \widetilde{E}^{\mu\nu},
\end{aligned}$$

where

$$\begin{aligned}
l^{\mu\nu} &= x^\mu p^\nu - x^\nu p^\mu, \\
E^{\mu\nu} &= -i \sum_{n=1}^\infty \frac{1}{n} \left(\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_n \alpha^\mu_n\right), \\
\widetilde{E}^{\mu\nu} &= -i \sum_{n=1}^\infty \frac{1}{n} \left(\widetilde{\alpha}^\mu_{-n} \widetilde{\alpha}^\nu_n - \widetilde{\alpha}^\nu_{-n} \widetilde{\alpha}^\mu_n\right).
\end{aligned}$$

The first piece $l^{\mu\nu}$ is the orbital angular momentum of the string while the other two pieces arise from excited states. Classically, one can check that the Poisson bracket for $M_{\mu\nu}$ satisfies Lorentz algebra. In covariant quantisation, it is easy to check that $M_{\mu\nu}$ satisfies the Lorentz algebra but in lightcone quantisation, things are not so clear. In lightcone gauge, we must be able to produce the Lorentz algebra i,e.

$$\left[M^{\rho\sigma}, M^{\tau\nu}\right] = \eta^{\sigma\tau} M^{\rho\nu} - \eta^{\rho\tau} M^{\sigma\nu} + \eta^{\rho\nu} M^{\sigma\tau} - \eta^{\sigma\nu} M^{\rho\tau}.$$

The only bracket which is non trivial is $[M^{i-}, M^{j-}] = 0$. This commutator involves p^- and α^-_n which has been fixed in lightcone gauge in terms of other transverse oscillators. A messy calculation gives

$$\begin{aligned}
\left[M^{i-}, M^{j-}\right] = &\frac{2}{(p^+)^2} \sum_{n>0} \left(\left[\frac{D-2}{24} - 1\right] n + \frac{1}{n}\left[a - \frac{D-2}{24}\right]\right) \left(\alpha^i_{-n} \alpha^j_n - \alpha^j_{-n} \alpha^i_n\right) \\
&+ + \frac{2}{(p^+)^2} \sum_{n>0} \left(\left[\frac{D-2}{24} - 1\right] n + \frac{1}{n}\left[a - \frac{D-2}{24}\right]\right) \left(\widetilde{\alpha}^i_{-n} \widetilde{\alpha}^j_n - \widetilde{\alpha}^j_{-n} \widetilde{\alpha}^i_n\right),
\end{aligned}$$

which is 0 if and only if $a = 1$ and $D = 26$. This is consistent with our earlier derivation of the critical dimension.

3.4.5 First String Excitation

The first excited states are massless representations of the little group $SO(D-2)$. There are a total $(D-2)^2$ particles in the first excitation. So we want to get the irreducible representations of $SO(D-2)$ of dimension $(D-2)^2$ so that each irreducible factor would correspond to an elementary particle by Wigner's proposal. Using the method of Young Tableau, we can prove that the tensorial representation of $SO(D-2)$ of dimension $(D-2)^2$ consists of three irreducible parts:

$$\begin{array}{llll} & \textbf{Traceless symmetric} \oplus & \textbf{Antisymmetric} \oplus & \textbf{Trace (Scalar)} \\ \text{Dim:} & \dfrac{(D-2)(D-1)}{2} - 1 & \dfrac{(D-2)(D-3)}{2} & 1 \end{array}$$

Following the usual method of constructing field theory from representations, we attach a tensor field to each of these representation. We get three particles.

1. $G_{\mu\nu}(X)$: the traceless symmetric tensor field which we will identify with *graviton*.
2. $B_{\mu\nu}(X)$: the antisymmetric tensor field. This is sometimes called the *Kalb-Ramond field.*
3. $\Phi(X)$: the trace part of the tensor representations. This scalar field is called the *dilaton.*

To see that these fields arise in our theory, we decompose the first excited state as follows:

$$\alpha_1^i \widetilde{\alpha}_{-1}^j \left|0;p^\mu\right\rangle = \underbrace{\left(\alpha_{-1}^{(i}\widetilde{\alpha}_{-1}^{j)} - \frac{1}{D-2}\delta^{ij}\alpha_{-1}^k\widetilde{\alpha}_{-1}^k\right)\left|0;p^\mu\right\rangle}_{\text{symmetric traceless}} + \underbrace{\alpha_{-1}^{[i}\widetilde{\alpha}_{-1}^{j]}\left|0;p^\mu\right\rangle}_{\text{antisymmetric}} + \frac{1}{D-2}\underbrace{\delta^{ij}\alpha_{-1}^k\widetilde{\alpha}_{-1}^k\left|0;p^\mu\right\rangle}_{\text{trace}},$$

where $(,)$ and $[,]$ are the symmetrized and antisymmetrized indices. The traceless symmetric field $G_{\mu\nu}$ is particularly interesting as it represents massless spin 2 particle. We will identify this field with the metric of spacetime, the graviton because Weinberg in [DOI: 10.1103/PhysRev.138.B988] showed that any interacting theory of massless spin 2 particle is Einstein's gravity. Later we will explicitly derive Einstein's field equations from this field.

Chapter 4

Open Strings and D-Branes

In the previous chapter, we quantised the closed string and found that the spectrum contains three particles including the graviton. In this chapter, we will quantise the open strings with different boundary conditions.

4.1 Solving the Equations of Motion

We have already found the equations of motion of the open string subject to three different boundary conditions in Subsection 2.2.2. As already mentioned in Subsection 2.2.2, we will normalise the length of the string so that $\sigma \in [0, \pi)$. We will now solve the equations of motion for the first two boundary conditions.

4.1.1 Neumann Boundary Condition at Both Ends (NN)

This means that

$$\partial_\sigma X^\mu = 0 \quad \text{for} \quad \sigma = 0, \pi.$$

Since the equation of motion is

$$\partial_\alpha \partial^\alpha X^\mu = 0,$$

we again have

$$X^\mu(\sigma, \tau) = X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right)$$

with

$$X_L^\mu\left(\sigma^+\right) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \widetilde{\alpha}_n^\mu e^{-in\sigma^+}$$

and

$$X_R^\mu\left(\sigma^-\right) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\sigma^-}.$$

Now $\sigma = \frac{\sigma^+ - \sigma^-}{2}$, so $\sigma = 0 \Rightarrow \sigma^+ = \widetilde{\sigma} = \tau/2$. Since

$$\partial_\sigma X^\mu = \frac{1}{2}\left(\partial_+ X^\mu - \partial_- X^\mu\right),$$

the condition $\partial_\sigma X^\mu = 0$ implies $\partial_+ X^\mu = \partial_- X^\mu$. Using the Fourier expansion above, we get

$$\alpha' p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \widetilde{\alpha}_n^\mu e^{-in\sigma^+} \Bigg|_{\sigma=0,\pi} = \alpha\prime p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n^\mu e^{-in\sigma^-} \Bigg|_{\sigma=0,\pi}.$$

At $\sigma = 0$, we get

$$\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left(\widetilde{\alpha}_n^\mu - \alpha_n^k\right) e^{-n\pi\tau_2} = 0 \quad \Rightarrow \quad \widetilde{\alpha}_n^\mu = \alpha_n^\mu \quad \forall\ n \neq 0.$$

So we have

$$X^\mu = x^\mu + 2p^\mu \alpha' \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau}\left(e^{-in\sigma} + e^{in\sigma}\right).$$

This gives

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma). \tag{4.1.1}$$

We can check that the boundary condition at $\sigma = \pi$ is automatically satisfied. Again we can check that x^μ le p^μ are center of mass position and momentum of the string. Constraints are

$$\left(\partial_+ X\right)^2 = 0 = \left(\partial_- X\right)^2.$$

With the given Fourier expansion, we still have the same classical constraints

$$L_n = 0 \text{ where } L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} \boldsymbol{\alpha}_{n-k} \cdot \boldsymbol{\alpha}_k \quad \forall n \in \mathbb{Z}$$

where now $\alpha_0^\mu = \sqrt{2\alpha'} p^\mu$. The Poisson bracket for α_n^μ are still the same.

$$\{\alpha_m^\mu, \alpha_n^\nu\}_{P.B.} = -im\eta^{\mu\nu}\delta_{m+n,0}, \quad \{x^\mu, p^\nu\}_{P.B.} = \eta^{\mu\nu}. \tag{4.1.2}$$

Virasoro algebra is also the same

$$\{L_m, L_n\}_{P.B.} = -i(m-n)L_{m+n}.$$

The Poisson bracket for the Fourier modes and the Virasoro generators remain the same.

4.1.2 Dirichlet Boundary Condition at Both Ends (DD)

We impose $\delta X^\mu = 0$ at $\sigma = 0, \pi$. This means that $\dot{X}^\mu = 0$ at $\sigma = 0, \pi \quad \forall\ \tau$. Suppose $X^\mu(0,\tau) = x_0^\mu$ and $X^\mu(\pi,\tau) = x_1^\mu$. The constraint is the same. We can still write

$$X^\mu = X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right).$$

where

$$\begin{aligned} X_L^\mu\left(\sigma^+\right) &= \frac{1}{2}x^\mu + p^\mu\alpha'\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}_n^\mu e^{-in\sigma^+} \\ X_R^\mu\left(\sigma^-\right) &= \frac{1}{2}x^\mu + p^\mu\alpha'\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}. \end{aligned}$$

But the boundary condition implies

$$\begin{aligned} X^\mu(0,\tau) = x_0^\mu \Rightarrow \quad & x^\mu + 2p^\mu\alpha'\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\widetilde{\alpha}_n^\mu + \alpha_n^\mu\right)e^{-in\tau} \\ & \Rightarrow p^\mu = 0, \quad x^\mu = x_0^\mu \quad \text{and} \quad \widetilde{\alpha}_n = -\alpha_n^\mu \end{aligned}$$

But the second condition $X^\mu(\pi,\tau) = x_1^\mu$ is not satisfied. Thus the general solution must have the form

$$X^\mu = x_0^\mu + \frac{x_1^\mu - x_0^\mu}{\pi}\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\sin(n\sigma). \tag{4.1.3}$$

This is gotten by assuming the forms of X_L and X_R as

$$\begin{aligned} X_L^\mu\left(\sigma^+\right) &= \frac{1}{2}x^\mu + p^\mu\alpha'\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}_n^\mu e^{-in\sigma^+} \\ X_R^\mu\left(\sigma^-\right) &= \frac{1}{2}x^\mu - p^\mu\alpha'\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}, \end{aligned}$$

so that

$$X^\mu(\sigma,\tau) = x^\mu + 2\alpha' p^\mu\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\widetilde{\alpha}_n^\mu e^{in\sigma^+} + \alpha_n^\mu e^{-in\sigma^-}\right).$$

Then $X^\mu(0,\tau) = x_0^\mu \Rightarrow x^\mu = x_0^\mu \quad$ and $\quad \widetilde{\alpha}_n^\mu = -\alpha_n^\mu$ and $X^\mu(\pi,\tau) = x_1^\mu$ implies

$$\begin{aligned} & x_0^\mu + 2\alpha' p^\mu\pi + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\widetilde{\alpha}_n^\mu e^{-in\pi} - \widetilde{\alpha}_n^\mu e^{in\pi}\right)e^{-in\tau} = x_1^\mu \\ & \Rightarrow x_0^\mu + 2\alpha' p^\mu\pi = x_1^\mu \\ & \Rightarrow 2\alpha' p^\mu = \frac{x_1^\mu - x_0^\mu}{\pi}. \end{aligned}$$

There is no center of mass momentum and the center of mass position is $\frac{x_0^\mu + x_1^\mu}{2}$ as is easily computed:

$$q^\mu = \frac{L}{\pi}\int_0^\pi d\sigma x^\mu(\sigma,\tau) = x_0^\mu + \frac{1}{\pi}\frac{x_1^\mu - x_0^\mu}{\pi}\frac{1}{2}\pi^2 + 0 = \frac{x_0^\mu + x_1^\mu}{2}.$$

Next, we find the classical constraints in terms of Fourier modes. The constraints are

$$\left(\partial_+ X^\mu\right)^2 = 0 = \left(\partial_- X^\mu\right)^2.$$

We have

$$\begin{aligned}
\partial_+ X^\mu &= \frac{x_1^\mu - x_0^\mu}{\pi}\frac{1}{2} + \sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^\mu \frac{1}{2in}\partial_+\left(e^{-in(\tau-\sigma)} - e^{-in(\tau+\sigma)}\right)\\
&= \frac{x_1^\mu - x_0^\mu}{\pi}\frac{1}{2} + \sqrt{\frac{\alpha}{2}}\sum_{n\neq 0}\alpha_n^\mu e^{-in\sigma^+}\\
&= \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in\sigma^+},
\end{aligned}$$

where

$$\alpha_0^\mu = \frac{1}{\sqrt{2\alpha'}}\frac{x_1^\mu - x_0^\mu}{\pi}.$$

Similarly

$$\partial_- X^\mu = -\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\alpha_n^\mu e^{-in\sigma^-},$$

with same $\boldsymbol{\alpha}_0$. Thus constraints are

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}}\boldsymbol{\alpha}_{n-k}\cdot\boldsymbol{\alpha}_k = 0 \quad \forall n\in\mathbb{Z}.$$

All Poisson brackets remain the same.

4.1.3 Neumann at $\sigma = 0$ and Dirichlet at $\sigma = \pi$ (ND)

This means

$$\partial_\sigma X^\mu = 0 \text{ at } \sigma = 0,\tau \text{ and } X^\mu = x^\mu \text{ at } \sigma = \pi,\tau.$$

As usual

$$X^\mu = X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right).$$

where

$$X_L^\mu\left(\sigma^+\right) = \frac{1}{2}x^\mu + p^\mu\alpha'\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}_n^\mu e^{-in\sigma^+}$$

$$X_R^\mu\left(\sigma^-\right) = \frac{1}{2}x^\mu + p^\mu\alpha'\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}.$$

The condition $\partial_\sigma X^\mu = 0$ at $\sigma = 0 \Rightarrow \alpha_n^\mu = \widetilde{\alpha}_n^\mu$ as in previous case. Next

$$X^\mu = x^\mu \text{ at } \sigma = \pi \quad \Rightarrow p^\mu = 0$$

$$i\sqrt{\frac{\alpha'}{2}}2\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\pi) = 0 \quad \forall \quad \tau.$$

This is possible only if $\cos(n\pi) = 0 \quad \forall n \Rightarrow n \in \mathbb{Z} + \frac{1}{2}$. So the sum must actually run over half integers. So we get

$$X^\mu(\sigma,\tau) = x^\mu + i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma). \tag{4.1.4}$$

One can again show that the oscillators, which are now half integral, satisfy the same Poisson bracket. It is easy to check that

$$\partial_\pm X^\mu = \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\alpha_n^\mu e^{-in\sigma^\pm} \Rightarrow (\partial_\pm X^\mu)^2 = \alpha'\frac{1}{2}\sum_{n\in\frac{1}{2}\mathbb{Z}}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\boldsymbol{\alpha}_{n-r}\cdot\boldsymbol{\alpha}_r e^{-in\sigma^\pm},$$

so that the classical constraints are again the same with the same expression for the Virasoro generators:

$$L_n = \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\boldsymbol{\alpha}_{n-r}\cdot\boldsymbol{\alpha}_r = 0 \quad \forall\ n \in \frac{1}{2}\mathbb{Z}.$$

4.1.4 Dirichlet at $\sigma = 0$ and Neumann at $\sigma = \pi$ (DN)

Following similar process as in Subsection 4.1.3, we get that

$$X^\mu(\sigma,\tau) = x^\mu + \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\sin(n\sigma), \tag{4.1.5}$$

and

$$\partial_\pm X^\mu = \pm\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\alpha_n^\mu e^{-in\sigma^\pm}.$$

This gives us the same classical constraints. The Poisson bracket also remains the same.

4.1.5 NN for $0 \leq \mu \leq p$ and DD for $p+1 \leq \mu \leq D-1$: D-Branes

This means that

$$\partial_\sigma X^a = 0 \text{ for } a = 0,\cdots,p \text{ at } \sigma = 0,\pi$$
$$X^I(0,\tau) = c^I, \quad X^I(\pi,\tau) = d^I \quad \text{for } I = p+1,\cdots,D-1.$$

This fixes the endpoints of the string in the $D-p-1$ directions and hence is constrained to move in the $(p+1)-$dimensional hypersurface. This hypersurface is usually called a *Dp-Brane*. So a $D0$-brane is a particle, a $D1$-brane is itself a string, a $D2$-brane is a membrane and so on. In particular if $p = D-1$ then we get to NN case which means all space is a D-brane, that is we get space filling D-brane. Combining Fourier modes of NN and DD conditions, we get

$$\begin{aligned} X^\mu(\sigma,\tau) &= x^\mu + 2p^\mu\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma), \mu = 0,1,\cdots,p \\ X^\mu(\sigma,\tau) &= c^\mu + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu \sin(n\sigma), \quad \mu = p+1,\cdots,D-1. \end{aligned} \tag{4.1.6}$$

One can also work out the Poisson bracket and show that they remain the same.

4.2 Quantisation

We can again quantise the open string in the cannonical way or using path integral. Here we will discuss the cannonical quantisation. As usual, it can be done in two ways. We will quickly discuss covariant quantisation but the lighcone quantisation will be discussed in some detail.

4.2.1 Covariant Quantisation

Using the classical Poisson brackets (4.1.2), we impose the commutation relations

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [\alpha_n^\mu, \alpha_m^\nu] = n\delta_{m+n,0}\eta^{\mu\nu}, \tag{4.2.1}$$

with all others being zero. Construct the Fock space as usual from ground state $|0;p^\mu\rangle$. We will again encounter ghosts which we can again get rid of by choosing a and D as in closed string case by the same spurious state analysis. Infact in open string case we only have one set of Virasoro generators

$$L_n = \sum_r \boldsymbol{\alpha}_{n-r}\cdot\boldsymbol{\alpha}_r,$$

where the summation index and mode index run over integers or half-integers depending on boundary conditions whether NN, DD or DN, ND. The quantum Virasoro algebra is again the same i,e. the central extension of the Witt algebra. Thus the constraints are again imposed as

$$\begin{aligned} L_n|\text{phys}\rangle &= 0 \\ (L_0 - a)\,|\text{phys}\rangle &= 0 \end{aligned}$$

where a is the normal ordering constant. The number operator is

$$N = \sum_{n=1}^{\infty}\left(\alpha_{-n}^\mu\alpha_{\mu,n} + \alpha_{-n}^i\alpha_{i,n}\right) + \sum_{r\in\mathbb{N}_0+\frac{1}{2}}\alpha_{-r}^a\alpha_{a,r},$$

where μ denotes NN direction, i denotes DD direction and a denotes the DN and ND directions and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Again using the spurious state discussion we have $a = 1, D = 26$ for our spectrum to be ghost free. Lorentz invariance is manifest and the the normal ordering constant drops out of any expressions involving angular momentum.

4.2.2 Lightcone Quantisation

As usual, we go to lightcone gauge by introducing

$$X^{\pm} = \frac{1}{\sqrt{2}} \left(X^0 \pm X^p \right)$$

and choosing $X^+ = 2\alpha' p^+ \tau$. It is easy to see that $X^{\pm}$ has to satisfy Neumann boundary condition (due to τ in X^+). The X^+ oscillators are all zero except the zero mode

$$\alpha_0^+ = \sqrt{2\alpha'} p^+.$$

As in closed string case, the oscillators of X^- is determined by the transverse oscillators upto a constant x^-. Let us now impose the commutation relations

$$\begin{aligned} \left[q^-, p^+ \right] &= -i, \left[q^i, pj \right] = i\delta^{ij} \\ \left[\alpha_n^i, \alpha_m^j \right] &= n\delta^{ij} \delta_{n+m,0}. \end{aligned} \tag{4.2.2}$$

We can now construct the Fock space from vacuum $|0; p^\mu\rangle$ by acting $\alpha_m^i,\ m < 0$ on $|0; p^\mu\rangle$. The spectrum is manifestly ghost free. Let us look at the ordering ambiguity. We have

$$\begin{aligned} L_0 - \alpha_0^2 = \sum_{n \neq 0} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n &= \sum_{n>0} \boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_n + \sum_{n<0}^{0} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n \\ &= \sum_{n>0} \boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_n + \sum_{n<0} \left(\boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_n - n(D-2) \right) \\ &= 2 \left(\sum_{n>0} \boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_n + \frac{D-2}{2} \sum_{n>0} n \right). \end{aligned}$$

Now the sum above can go over integer or half-integer depending on NN, DD or ND, DN boundary conditions. In integral case, the last term is regularised using zeta function:

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}.$$

If the sum goes over half integers then the last term is regularised using Hurwitz zeta function. The last sum can be written as

$$\sum_{n \in \mathbb{N}_0 + \frac{1}{2}} n = \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right).$$

The Hurwitz zeta function is defined as

$$\zeta(s,q) = \sum_{n=0}^{\infty} \frac{1}{(n+q)^s}$$

and is holomorphic for $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(q) > 0$. It can be analytically continued to the whole s-plane for a given value of q in the domain of definition but we do not need the complete sophisticated machinery here rather a simple trick here will do the job. We note that

$$\begin{aligned}
\zeta\left(s, \frac{1}{2}\right) &= 2^s \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} \\
&= 2^s \left[\zeta(s) - \sum_{n=1}^{\infty} \frac{1}{(2n)^s}\right] \\
&= 2^s \left(\zeta(s) - 2^{-s}\zeta(s)\right) \\
&= (2^s - 1)\zeta(s).
\end{aligned}$$

Thus, using the analytic continuation of the Riemann zeta function, we get

$$\zeta\left(-1, \frac{1}{2}\right) = -\frac{1}{2}\zeta(-1) = \frac{1}{24}.$$

We have five boundary conditions. In general we can have a mix of all those boundary conditions. Let i_1 be NN and DD directions and i_2 be ND and DN directions. Then we have

$$L_0 - \alpha_0^2 = 2\sum_{n=1}^{\infty} \alpha_{-n}^{i_1}\alpha_{i_1,n} + 2\sum_{n\in\mathbb{N}_0+\frac{1}{2}} \alpha_{-n}^{i_2}\alpha_{i_2,n} + D_1\left(-\frac{1}{12}\right) + D_2\left(\frac{1}{24}\right)$$

where $D_1 + D_2 = D - 2$ where D_1 denotes the total number of NN and DD directions and D_2 denotes the total number of DN and ND directions. We recognise the last two constant terms as the contribution to normal ordering constant. In terms of the number operator, we have

$$\begin{aligned}
L_0 - a &= \alpha_0^2 + 2\sum_{n=1}^{\infty} \alpha_{-n}^{i_1}\alpha_{i_1,n} + 2\sum_{n\in\mathbb{N}_0+\frac{1}{2}} \alpha_{-n}^{i_2}\alpha_{i_2,n} + D_1\left(-\frac{1}{12}\right) + D_2\left(\frac{1}{24}\right) \\
&= \alpha_0^2 + 2N + D_1\left(-\frac{1}{12}\right) + D_2\left(\frac{1}{24}\right).
\end{aligned}$$

If we consider the first string excitation $\alpha_{-n}^{i}|0;p^\mu\rangle$ where $n=1$ if $i=i_1$ and $n=\frac{1}{2}$ if $i=i_2$ then

$$N\alpha_{-n}^{i}\left|0;p^\mu\right\rangle = n\alpha_{-n}^{i}\left|0;p^\mu\right\rangle.$$

Next the mass-spectrum is calculated using the constraint

$$(L_0 - a)\,|\text{phys}\rangle = 0.$$

We now need to find an expression for α_0^2. Indeed, note that in the DD directions,

$$\alpha_0^\mu = \frac{1}{\sqrt{2\alpha'}}\frac{x_1^\mu - x_0^\mu}{\pi} = \frac{\Delta X^\mu}{\sqrt{2\alpha'}\pi}, \quad \Delta X^\mu := x_1^\mu - x_0^\mu,$$

where ΔX is the string length in the DD direction. In the NN direction,

$$\alpha_0^\mu = \sqrt{2\alpha'}p^\mu,$$

and there are no zero modes in ND and DN directions. This implies that

$$\alpha_0^2 = 2\alpha' p^2 + \left(\frac{\Delta X}{\sqrt{2\alpha'}\pi}\right)^2 = -2\alpha' M^2 + 2\alpha'\left(\frac{\Delta X}{2\alpha'\pi}\right)^2.$$

Using the expression for $(L_0 - a)$ from previous calculation, we get

$$\begin{aligned} & 2N - \frac{D_1}{12} + \frac{D_2}{24} - 2\alpha' M^2 + 2\alpha'\left(\frac{\Delta X}{2\pi\alpha'}\right)^2 = 0 \\ \implies \quad & \alpha' M^2 = N - \frac{D-2}{24} + \frac{D_2}{16} + 2\alpha'\left(\frac{\Delta X}{2\pi\alpha'}\right)^2, \end{aligned}$$

where N is the number of states in the physical excitation. Thus we see that any physical excitation has to satisfy the above mass-shell condition. Let us explore the origin of the extra term ΔX. Note that we have

$$\alpha_0^2 = -2\alpha' M^2 + 2\alpha'\left(\frac{\Delta X}{2\alpha'\pi}\right)^2.$$

The extra term has natural physical interpretation: it is the mass of the string stretched between two branes.

4.2.3 String Spectrum

Let us start with NN boundary conditions. The ground state is $|0; p^\mu\rangle$ and the mass spectrum gives

$$M^2 = -\frac{D-2}{24\alpha'} < 0.$$

So the ground state is Tachyonic. The first excited state is

$$\alpha_{-1}^i\,|0; p^\mu\rangle$$

which transforms as a vector representation of $SO(D-2)$. Again Wigner's theorem implies that this state is a massless representation. Thus we get

$$D = 1 - \frac{p-2}{24} \Rightarrow D = 26.$$

We can go on constructing the higher excited states and show that $D = 26$ forces all of them to be massive representations of the Lorentz group. At level n, the mass spectrum is

$$\alpha' M^2 = n - 1$$

and at level n, the representation includes a symmetric tensor of rank n (this comes from Young Tableau method which we shall not describe here). This state corresponds to the maximum spin n of this excitation. Let us pause and prove this. For each spin component, we will produce a level N state and show that its spin eigenvalue corresponding to the particular component is N. To make this explicit, we first recall the spin generators

$$E^{\mu\nu} = -i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha^{\mu}_{-n}\alpha^{\nu}_{n} - \alpha^{\nu}_{n}\alpha^{\mu}_{n}\right), \quad \widetilde{E}^{\mu\nu} = -i \sum_{n=1}^{\infty} \frac{1}{n} \left(\widetilde{\alpha}^{\mu}_{-n}\widetilde{\alpha}^{\nu}_{n} - \widetilde{\alpha}^{\nu}_{n}\widetilde{\alpha}^{\mu}_{n}\right).$$

We will distinguish between open and closed strings. In lightcone quantisation, the relevant spin generators are E^{ij} and $\widetilde{E}_{ij}$ for $1 \leq i, j \leq D-2$. In closed string case, the state corresponding to the spin component E^{ij} and $\widetilde{E}^{ij}$ is given by

$$\Omega^{ij} = \left(\alpha^{i}_{-1} + i\alpha^{j}_{-1}\right)^{N} \left(\widetilde{\alpha}^{i}_{-1} + i\widetilde{\alpha}^{j}_{-1}\right)^{N} |0; p^{\mu}\rangle .$$

Now observe that

$$\begin{aligned}
E^{ij}\Omega^{ij} &= \left(\widetilde{\alpha}^{i}_{-1} + i\widetilde{\alpha}^{j}_{-1}\right)^{N} (-i) \sum_{n=1} \frac{1}{n} \left(\alpha^{i}_{-n}\alpha^{j}_{n} - \alpha^{j}_{-n}\alpha^{i}_{n}\right) \left(\alpha^{i}_{-1} + i\alpha^{i}_{-1}\right)^{N} |0; p^{\mu}\rangle \\
&= \left(\widetilde{\alpha}^{i}_{-1} + i\widetilde{\alpha}^{j}_{-1}\right)^{N} (-i) \left(\alpha^{i}_{-1}\alpha^{j}_{1} - \alpha^{j}_{-1}\alpha^{i}_{1}\right) \left(\alpha^{i}_{-1} + i\alpha^{j}_{-1}\right)^{N} |0; p^{\mu}\rangle ,
\end{aligned}$$

where we used the fact that $\left(\alpha^{i}_{-n}\alpha^{j}_{n} - \alpha^{j}_{-n}\alpha^{i}_{n}\right)$ commutes with $\alpha^{i}_{-1} + i\alpha^{j}_{-1} \quad \forall n > 1$. Now we have

$$\begin{aligned}
\left[\alpha^{i}_{-1}\alpha^{j}_{1} - \alpha^{j}_{-1}\alpha^{i}_{1},\ \alpha^{i}_{-1} + i\alpha^{j}_{-1}\right] &= \alpha^{i}_{-1}\left[\alpha^{j}_{1}, \alpha^{i}_{-1}\right] + i\alpha^{i}_{-1}\left[\alpha^{j}_{1}, \alpha^{j}_{-1}\right] - \alpha^{j}_{-1}\left[\alpha^{i}_{1},\ \alpha^{i}_{-1}\right] - i\alpha^{j}_{-1}\left[\alpha^{i}_{1}, \alpha^{j}_{-1}\right] \\
&= \alpha^{i}_{-1}\delta^{ji} + i\alpha^{i}_{-1} - \alpha^{j}_{-1} - i\alpha^{j}_{-1}\delta^{ij} \\
&= \begin{cases} i\left(\alpha^{i}_{-1} + i\alpha^{j}_{-1}\right) & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}
\end{aligned}$$

So assuming $i \neq j$, we get

$$\begin{aligned}
E^{ij}\Omega^{ij} &= \left(\widetilde{\alpha}^{i}_{-1} + i\widetilde{\alpha}^{j}_{-1}\right)^{N} (-i) \left(\alpha^{i}_{-1}\alpha^{j}_{1} - \alpha^{j}_{-1}\alpha^{i}_{1}\right) \left(\alpha^{i}_{-1} + i\alpha^{j}_{-1}\right)^{N} |0; p^{\mu}\rangle \\
&= \left(\widetilde{\alpha}^{i}_{-1} + i\widetilde{\alpha}^{j}_{-1}\right)^{N} (-i)(iN) \left(\alpha^{i}_{-1} + i\alpha^{j}_{-1}\right)^{N} |0; p^{\mu}\rangle \\
&= N\Omega^{ij}.
\end{aligned}$$

Similarly $\widetilde{E}^{ij}\Omega^{ij} = N\Omega^{ij}$. In case of open strings with NN boundary conditions:

$$E^{ij}\left(\alpha_{-1}^{i} + i\alpha_{-1}^{j}\right)^{N}|0;p^{\mu}\rangle = N\left(\alpha_{-1}^{i} + 2\alpha_{-1}^{j}\right)|0;p^{\mu}\rangle\,.$$

Thus the maximum spin at each level is

$$J_{\max} = n.$$

Hence we have

$$J_{\max} = \alpha' M^{2} + 1.$$

If we plot $J_{\max}$ verses M^2 at each level, we get a straight line with slope α'. This is why α' is called the *Regge slope*. All states at a given level satisfy

$$J_{\max} \leq \alpha' M^{2} + 1,$$

and since J and M^2 are quantised, all states lie on straight lines with the Tachyon lying on the leading trajectory. These lines are called *Regge trajectories*. Regge trajectories are observed in nature both for Mesons and baryons.

We now consider Dp branes i,e. NN boundary conditions in $p+1$ directions and DD boundary conditions in $D-p-1$ direction. There are two cases to distinguish.

One Dp Brane

In this case, we have

$$X^{\mu}(0,\tau) = c^{I} = X^{\mu}(\pi,\tau) \quad \mu = p+1,\ldots,D-1.$$

Thus the ends of the string are constrained to lie on one Dp brane. The ground state is now defined by

$$\alpha_{n}^{i}|0;p^{\mu}\rangle = 0, \quad n>0, \quad i = 1,2,\cdots p-1, p+1,\cdots, D-1.$$

Note that the string momentum p^{μ} is actually only in $p+1$ directions. The $SO(1,D-1)$ Lorentz group is broken into $SO(1,p)\times SO(D-P-1)$. Again Lorentz invariance requires $D=26$ and $a=1$ as we can readily see by looking at the mass spectrum of first excited state. To be explicit, the first excited states are $\alpha_{-1}^{i}|0;p^{\mu}\rangle$ for $i=1,2,\ldots,p-1$ which transforms as a vector representation of $SO(1,p)$. As the first excited state has maximum spin 1, these states represent gauge fields as is known from quantum field theory. We introduce a gauge field $A_i,\ i=0,\ldots,p$ and its quanta represents spin 1 photons. The other oscillators are

$$\alpha_{-1}^{I}|0;p^{\mu}\rangle\,, \quad I = p+1,\ldots,D-1.$$

These transform as scalar representations of $SO(1,p)$ and hence we introduce $D-p-1$ scalar fields ϕ^I. These ϕ^I have physical interpretation of fluctuations of the Dp brane. This suggests that Dp branes are themselves dynamical as we will see later. Although ϕ^I transform as scalars under the $SO(1,p)$ Lorentz group of the Dp brane they transform as vectors as representations of the $SO(D-p-1)$ rotation group. This appears as a global symmetry of the brane world volume. One can also consider ϕ^I as the Goldstone Bosons associated to the spontaneously broken translational symmetry.

Two Dp Branes: String Stretched Between Two Branes

In this case $X^\mu(0,\sigma) = X^\mu(\pi,\sigma), \quad \mu = p+1,\cdots,D-1$. We get a shift in mass spectrum:

$$\alpha' M^2 = N - \frac{D-2}{24} + \alpha' \left(\frac{x_1^\mu - x_0^\mu}{2\alpha'\pi}\right)^2 .$$

Thus the states $\alpha_{-1}^i \left|\Delta x^\pm, p^i\right\rangle$ are no longer massless. In general we can stack N such Dp branes on top of each other and denote the massless vector excitation as

$$\alpha_{-1}^i |k,\ell,p^i\rangle$$

where k,ℓ are labels which encode the Dp branes on which the endpoints of the string end. These are called *Chan-Paton labels*. The resulting N^2 states can be embedded in an $N\times N$ matrix and expanded in a complete set of $N\times N$ matrices

$$\left|k,\ell;p^i\right\rangle = \lambda_{k\ell}^a |a;p^i\rangle, \quad a\in\left\{1,\cdots,N^2\right\},$$

where $\lambda_{k\ell}^a$ are called *Chan-Paton factors*. The resulting fields T_ℓ^k, $\left(\phi^I\right)_\ell^k$ and $(A^a)_\ell^k$ can be fit into Hermitian matrices. The diagonal fields arise from strings ending on same brane. We will later see that $(A^a)_\ell^k$ are identified with $U(N)$ Yang-Mills gauge Bosons and $\left(\phi^I\right)_\ell^k$ transform in the adjoint representation of $U(N)$.

4.3 Discrete Diffeomorphisms: Oriented verses Nonoriented Strings

Until now, we dealt with oriented string theories that is we have not considered reparametrizations of the form

$$\begin{aligned}\sigma &\to \sigma' = \pi - \sigma\\ \tau &\to \tau' = \tau.\end{aligned}$$

Such a reparametrization respects the periodicity of closed strings and maps the two ends of an open string to each other and reverses the orientation $d\sigma\wedge d\tau$ of the worldsheet[1]. The above discrete diffeomorphism can be implemented by a unitary operator Ω:

$$\Omega X^\mu(\sigma,\tau)\Omega^{-1} = X^\mu(\pi-\sigma,\tau).$$

Since the same operation twice is trivial we demand $\Omega^2=1$. So that the only eigenvalues of Ω can be ±1. These actions can be expressed in terms of the oscillators. For closed string we substitute

$$X^\mu(\sigma,\tau) = x^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\alpha_n^\mu e^{-in\sigma^-} + \widetilde{\alpha}_n^\mu e^{-in\sigma^+}\right).$$

[1] orientation of a manifold can be defined in terms of a top form on the manifold.

Also the length of the string was normalised to 2π, so that the condition now becomes

$$\Omega X^\mu(\sigma,\tau)\Omega^{-1} = X^\mu(2\pi-\sigma,\tau).$$

This gives

$$\Omega\alpha_n^\mu\Omega^{-1} = \widetilde{\alpha}_n^\mu, \quad \Omega\widetilde{\alpha}_n^\mu\Omega^{-1} = \alpha_n^\mu.$$

For open string 6 , we need to differentiate between different boundary conditions. Using similar calculation as for closed strings, we get the following:

- NN boundary condition: $\Omega\alpha_n^\mu\Omega^{-1} = (-1)^n\alpha_n^\mu$.
- DD boundary condition: $\Omega\alpha_n^\mu\Omega^{-1} = (-1)^{n+1}\alpha_n^\mu \quad \Omega x_{0,1}^\mu\Omega^{-1} = x_{1,0}^\mu$.
- ND-DN boundary condition: $\Omega\alpha_{n+\frac{1}{2}}^{\mu,ND}\Omega^{-1} = i(-1)^n\alpha_{n+\frac{1}{2}}^{\mu,DN}$.

We need to fix the action of Ω on the ground state. It turns out that $\Omega|0;p^\mu\rangle$ is determined upto a sign which is fixed by the so called *Tadpole cancellation* (will be investigated later). For closed strings, the unoriented string spectrum must be invariant under left moving - right moving sector exchange. This means that of the three massless fields, only graviton and dilaton survives. This is called the *restricted Shapiro-Virasoro model* and the oriented one is called the *extended Shapiro-Virasoro model*.

Let us now turn to the open strings. If Ω acts on the ground state with plus sign, then the unoriented open string spectrum with NN (respectively DD) boundary condition must consist of even (respectively odd) level number. For $2N$ branes stacked on top of each other, one must also consider the action of Ω on the Chan-Paton factors. Since Ω changes orientations $\left(\Omega x_{1,0}^\mu\Omega^{-1} = x_{0,1}^\mu\right)$ we have

$$\Omega\,|k,\ell;p^\mu\rangle = |\ell,k,p^\mu\rangle$$

at massless vector level. This means we only have $N(2N-1)$ (symmetric) surviving Chan-Paton factors. Thus we get a massless vector of a $SO(2N)\subset U(2N)$ gauge theory. If Ω acts with negative sign, the Chan-Paton labels are antisymmetrized and we get a massless vector of a $Sp(2N)\subset U(2N)$ gauge theory where $Sp(2N)$ is the symplectic group of rank N defined as follows:

$$Sp(2N) = \left\{M\in GL(2N,\mathbb{R}) : MJM^T = J\right\}$$

where

$$J = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

where $\mathbb{I}$ is $N\times N$ identity matrix. This is because the dimension of the antisymmetric representation is

$$(2N)^2 - \frac{2N(2N-1)}{2} = (2N)\left[\frac{4N-(2N-1)}{2}\right] = N(2N+1)$$

which is equal to the real dimension of $Sp(2N)$.

Chapter 5

Conformal Field Theory

In this chapter, we will review conformal transformations and conformal group in detail. We will describe the conformal transformations in N-dimensional case but later specify to two dimensions which is relevant to string theory.

5.1 Conformal Transformations

We have already looked at conformal transformations Let us recall the definition:

Definition 5.1.1. Let (M, g) and $(N, \widetilde{g})$ be Riemannian (or pseudo-Riemmanian) manifolds and $\varphi : U \longrightarrow V$ be a smooth map where $U \subset M, V \subset N$ are open sets. Then φ is called a conformal map if the pullback φ^* of φ satisfies

$$\varphi^* \widetilde{g} = \Omega^2 g, \tag{5.1.1}$$

where $\Omega \in C^\infty(M)$ is called the *scale factor*. Here $C^\infty(M)$ denotes the space of smooth functions $f : M \longrightarrow \mathbb{R}$.

Suppose M is m dimensional and M is n dimensional. Let $(x^0, \ldots, x^{m-1})$ and $(y^0, \ldots, y^{n-1})$ be chart maps on U and V respectively[1]. Let the components of the metric tensor be given by

$$g_{\mu\nu}(x) := g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right), \quad \widetilde{g}_{\mu\nu}(y) := \widetilde{g}\left(\frac{\partial}{\partial y^\mu}, \frac{\partial}{\partial y^\nu}\right)$$

Then writing $x'^\mu = y^\mu \circ \varphi$ and using the definition of pullback, (5.1.1) becomes

$$\widetilde{g}_{\rho\sigma}(x') \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} = \Omega^2(x) g_{\mu\nu}(x), \quad x' = \varphi(x). \tag{5.1.2}$$

Conformal transformations between distinct Riemannian manifolds are important for many

[1] these charts may not cover all of U and V.

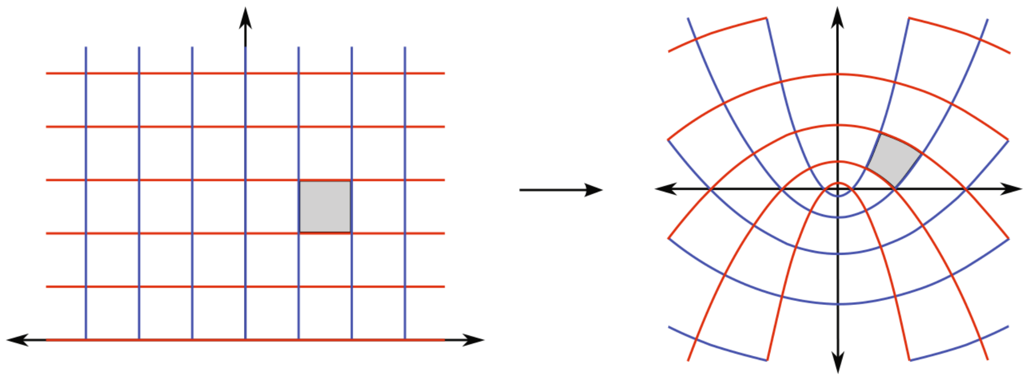

Figure 5.1: Conformal transformation in two dimensions. It is clearly visible that this transformation preserves angles.

applications but we will analyse the case when $M = N = \mathbb{R}^{1,D-1}$ and $g_{\mu\nu} = \widetilde{g}_{\mu\nu} = \eta_{\mu\nu}$. In this case, a conformal transformation is just a spacetime transformation which only scales the metric. It is also clear that the set of all conformal transformations forms a group under composition of maps. We denote this group by $\mathrm{Conf}(\mathbb{R}^{1,D-1})$. We would like to determine this group for various spacetime dimensions. It turns out to be the Lorentz group for dimensions greater than 3. In dimension 2, it is a bit more complicated and turns out to be infinite dimensional if we look at local conformal transformations which we will differentiate from the global conformal transformations in a precise sense. The global conformal transformations coincide with the local conformal transformations in $D \geq 3$. We begin by analysing infinitesimal conformal transformations which helps us get the Lie algebra of $\mathrm{Conf}(\mathbb{R}^{1,D-1})$ immediately.

5.1.1 Infinitesimal Conformal Transformations

We will now concentrate on infinitesimal spacetime transformations and find conditions on the infinitesimal parameter so that it is a conformal transformation. To this end, consider the local infinitesimal coordinate transformation

$$x^\mu \longrightarrow x'^\mu = x^\mu + \varepsilon^\mu(x) + O(\varepsilon^2). \tag{5.1.3}$$

Under this coordinate transformation, we have

$$\begin{aligned}
\eta_{\rho\sigma}\frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} &= \eta_{\rho\sigma}\left(\delta^\rho_\mu + \frac{\partial \varepsilon^\rho}{\partial x^\mu} + O\left(\varepsilon^2\right)\right)\left(\delta^\sigma_\nu + \frac{\partial \varepsilon^\sigma}{\partial x^\nu} + O\left(\varepsilon^2\right)\right) \\
&= \eta_{\mu\nu} + \eta_{\mu\sigma}\frac{\partial \varepsilon^\sigma}{\partial x^\nu} + \eta_{\rho\nu}\frac{\partial \varepsilon^\rho}{\partial x^\mu} + O\left(\varepsilon^2\right) \\
&= \eta_{\mu\nu} + \left(\frac{\partial \varepsilon_\mu}{\partial x^\nu} + \frac{\partial \varepsilon_\nu}{\partial x^\mu}\right) + O\left(\varepsilon^2\right),
\end{aligned}$$

where in the last step, we used

$$\frac{\partial \varepsilon_\nu}{\partial x^\mu} = \frac{\partial \eta_{\mu\nu}\varepsilon^\mu}{\partial x^\mu} = \eta_{\mu\nu}\frac{\partial \varepsilon^\nu}{\partial x^\mu}.$$

If we demand that this infinitesimal transformation be a conformal transformation, then we must have

$$\eta_{\mu\nu} + \partial_\mu \varepsilon_\nu + \partial_\nu + O(\varepsilon^2) = \Omega^2 \eta_{\mu\nu}\varepsilon_\mu$$

which implies that

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = f(x)\eta_{\mu\nu} \tag{5.1.4}$$

for some function f. To determine the function f in terms of ε, we contract (5.1.4) with $\eta^{\mu\nu}$. We get

$$\begin{aligned}
&\eta^{\mu\nu}\left(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu\right) = f(x)\eta^{\mu\nu}\eta_{\mu\nu} \\
&\implies 2\partial^\mu \varepsilon_\mu = f(x) D \\
&\implies f(x) = \frac{2}{D}(\partial \cdot \varepsilon).
\end{aligned}$$

Plugging this expression in (5.1.4), we get

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{D}(\partial \cdot \varepsilon)\eta_{\mu\nu}. \tag{5.1.5}$$

The scale factor upto linear order in ε is given by

$$\Omega^2(x) = 1 + \frac{2}{D}(\partial \cdot \varepsilon) + O(\varepsilon^2).$$

We now derive several relations which will be useful in later computations. Taking partial derivative ∂^ν of (5.1.5), we obtain

$$\begin{aligned}
\partial^\nu\left(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu\right) &= \frac{2}{D}\partial^\nu(\partial \cdot \epsilon)\eta_{\mu v} \\
\implies \partial_\mu(\partial \cdot \varepsilon) + \Box \varepsilon_\mu &= \frac{2}{D}\partial_\mu(\partial \cdot \varepsilon),
\end{aligned}$$

where $\Box = \partial^\mu \partial_\mu$. Further taking partial derivative ∂_ν of above equation, we get

$$\partial_\mu \partial_\nu(\partial \cdot \varepsilon) + \Box \partial_\nu \varepsilon_\mu = \frac{2}{D}\partial_\mu \partial_\nu(\partial \cdot \varepsilon). \tag{5.1.6}$$

Interchanging $\mu \leftrightarrow \nu$ in (5.1.6) and adding to the same equation, we obtain

$$\begin{aligned}
&(\partial_\mu \partial_\nu(\partial \cdot \varepsilon) + \Box(\partial_\nu \varepsilon_\mu)) + (\partial_\nu \partial_\mu(\partial \cdot \varepsilon) + \Box(\partial_\mu \varepsilon_\nu)) = \frac{4}{D}\partial_\mu \partial_\nu(\partial \cdot \varepsilon) \\
&\implies 2\partial_\mu \partial_\nu(\partial \cdot \varepsilon) + \Box(\partial_\nu \varepsilon_\mu + \partial_\mu \varepsilon_\nu) = \frac{4}{D}\partial_\mu \partial_\nu(\partial \cdot \varepsilon) \\
&\implies 2\partial_\mu \partial_\nu(\partial \cdot \varepsilon) + \Box\left(\frac{2}{D}(\partial \cdot \varepsilon)\eta_{\mu\nu}\right) = \frac{4}{D}\partial_\mu \partial_\nu(\partial \cdot \varepsilon) \\
&\implies (\eta_{\mu\nu}\Box + (D-2)\partial_\mu \partial_\nu)(\partial \cdot \varepsilon) = 0,
\end{aligned}$$

where we used (5.1.5) in the last step. Finally contracting this equation with $\eta^{\mu\nu}$, we get

$$(D-1)\Box(\partial\cdot\varepsilon)=0 \tag{5.1.7}$$

We now derive another equation for later use. Taking derivatives ∂_ρ of (5.1.5) and permuting indices we get

$$\begin{aligned}
\partial_\rho\partial_\mu\varepsilon_\nu+\partial_\rho\partial_\nu\varepsilon_\mu &= \frac{2}{D}\eta_{\mu\nu}\partial_\rho(\partial\cdot\varepsilon)\\
\partial_\nu\partial_\rho\varepsilon_\mu+\partial_\mu\partial_\rho\varepsilon_\nu &= \frac{2}{D}\eta_{\rho\mu}\partial_\nu(\partial\cdot\varepsilon)\\
\partial_\mu\partial_\nu\varepsilon_\rho+\partial_\nu\partial_\mu\varepsilon_\rho &= \frac{2}{D}\eta_{\nu\rho}\partial_\mu(\partial\cdot\varepsilon).
\end{aligned}$$

Adding the last two equations and subtracting the first gives

$$2\partial_\mu\partial_\nu\varepsilon_\rho=\frac{2}{D}\left(-\eta_{\mu\nu}\partial_\rho+\eta_{\rho\mu}\partial_\nu+\eta_{\nu\rho}\partial_\mu\right)(\partial\cdot\varepsilon). \tag{5.1.8}$$

5.2 Conformal Group in $D\geq 3$

Let us first define the *global* conformal group and its algebra.

Definition 5.2.1. The conformal group is the group consisting of *globally* defined, invertible and finite conformal transformations, that is conformal diffeomorphisms.

Definition 5.2.2. The conformal algebra is the Lie algebra corresponding to the conformal group.

To find the conformal group, we first work out the infinitesimal conformal transformation and then obtain the finite conformal transformation by exponentiating the infinitesimal ones.

5.2.1 Infinitesimal Conformal Transformations: $D\geq 3$

We begin by observing that (5.1.7) constraints $\varepsilon(x)$ to be atmost quadratic in x. Thus the most general form of the local infinitesimal parameter $\varepsilon(x)$ is

$$\varepsilon_\mu(x)=a_\mu+b_{\mu\nu}x^\nu+c_{\mu\nu\rho}x^\nu x^\rho, \tag{5.2.1}$$

where $a_\mu, b_{\mu\nu}, c_{\mu\nu\rho}\ll 1$ are constants and $c_{\mu\nu\rho}$ is symmetric in $\nu,\rho:\ c_{\mu\nu\rho}=c_{\mu\rho\nu}$. Now since the condition for conformal transformation is encoded only in the constants appearing in (5.1.7) and these conditions should not depend on the spacetime point, we can analyse the conditions on the constants order by order.

(i) The constant term a_μ (Translation): This term is not constrained by (5.1.5). This corresponds to spacetime translation, for which the generator[2] is $P_\mu=-i\partial_\mu$ as is well known.

[2]see Appendix B for details on symmetry generators and the explicit calculations of the generators of the conformal algebra.

(ii) The linear term $b_{\mu\nu}$ (Dilatation and Lorentz transformation): Plugging the expression in (5.2.1) upto linear term into (5.1.5), we obtain

$$b_{\nu\mu} + b_{\mu\nu} = \frac{2}{D}\left(\eta^{\rho\sigma} b_{\sigma\rho}\right)\eta_{\mu\nu}.$$

Now if we split $b_{\mu\nu}$ into symmetric and antisymmetric part as

$$b_{\mu\nu} = \frac{b_{\mu\nu} + b_{\nu\mu}}{2} + \frac{b_{\mu\nu} - b_{\nu\mu}}{2}$$

then the above equation implies that the symmetric part is proportional to $\eta_{\mu\nu}$. Thus we see that $b_{\mu\nu}$ can be split in the following way

$$b_{\mu\nu} = \alpha\eta_{\mu\nu} + m_{\mu\nu}$$

where $m_{\mu\nu} = -m_{\nu\mu}$. If we consider the symmetric term $\alpha\eta_{\mu\nu}$ alone, then we get the transformation $x'^{\mu} = (1+\alpha)x^{\mu}$ which describes infinitesimal scale transformations also called *dilatation*. The generator corresponding to this transformation is $D = -ix^{\mu}\partial_{\mu}$.

The antisymmetric part $m_{\mu\nu}$ corresponds to infinitesimal rotations $x'^{\mu} = \left(\delta^{\mu}_{\nu} + m^{\mu}{}_{\nu}\right)x^{\nu}$ with generator being the angular momentum operator $M_{\mu\nu} = i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right)$.

(iii) The quadratic term $c_{\mu\nu\rho}$ (Special conformal transforamtion): Plugging the expression in (5.2.1) into (5.1.8), we get

$$\begin{aligned}
2\partial_{\mu}\partial_{\nu}c_{\rho\sigma\lambda}x^{\sigma}x^{\lambda} &= \frac{2}{D}\left(-\eta_{\mu\nu}\partial_{\rho} + \eta_{\rho\mu}\partial_{\nu} + \eta_{\nu\rho}\partial_{\mu}\right)\left(\eta^{\mu\nu}\partial_{\nu}(b_{\mu\rho}x^{\rho} + c_{\mu\sigma\lambda}x^{\sigma}x^{\lambda})\right)\\
\implies 2c_{\rho\mu\nu} &= \frac{1}{D}\left(-\eta_{\mu\nu}\partial_{\rho} + \eta_{\rho\mu}\partial_{\nu} + \eta_{\nu\rho}\partial_{\mu}\right)\left(b^{\mu}{}_{\mu} + 2c^{\mu}{}_{\mu\lambda}x^{\lambda}\right)\\
\implies 2c_{\rho\mu\nu} &= \frac{2}{D}\left(-\eta_{\mu\nu}c^{\sigma}{}_{\sigma\rho} + \eta_{\rho\mu}c^{\sigma}{}_{\sigma\nu} + \eta_{\nu\rho}c^{\sigma}{}_{\sigma\mu}\right)
\end{aligned}$$

Thus we have

$$c_{\mu\nu\rho} = \eta_{\mu\rho}b_{\nu} + \eta_{\mu\nu}b_{\rho} - \eta_{\nu\rho}b_{\mu} \quad \text{with} \quad b_{\mu} = \frac{1}{D}c^{\rho}{}_{\rho\mu}.$$

Thus the infinitesimal parameter is

$$\begin{aligned}
\varepsilon_{\mu}(x) &= (\eta_{\mu\rho}b_{\nu} + \eta_{\mu\nu}b_{\rho} - \eta_{\nu\rho}b_{\mu})x^{\nu}x^{\rho}\\
&= 2(b\cdot x)x_{\mu} + (x\cdot x)b_{\mu}.
\end{aligned}$$

The resulting transformations are called *Special Conformal Transformations* (SCT) which infinitesimally is given by:

$$x'^{\mu} = x^{\mu} + 2(x\cdot b)x^{\mu} - (x\cdot x)b^{\mu}. \tag{5.2.2}$$

The corresponding generator is written as

$$K_{\mu} = -i\left(2x_{\mu}x^{\nu}\partial_{\nu} - (x\cdot x)\partial_{\mu}\right).$$

So we have four infinitesimal transformations:

- Infinitesimal translation $x'^{\mu} = x^{\mu} + a^{\mu},\ a^{\mu} \ll 1$ with generator $P_{\mu} = -i\partial_{\mu}$.
- Infinitesimal dilatation $x'^{\mu} = (1+\alpha)x^{\mu},\ \alpha \ll 1$ with generator $D = ix^{\mu}\partial_{\mu}$.
- Lorentz transformation $x'^{\mu} = m^{\mu}{}_{\nu}x^{\nu},\ m_{\mu\nu} = -m_{\nu\mu},\ m_{\mu\nu} \ll 1$ with generator $M_{\mu\nu} = i\left(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}\right)$.
- Special conformal transformations $x'^{\mu} = x^{\mu}+2(x\cdot b)x^{\mu}-(x\cdot x)b^{\mu},\ b^{\mu} \ll 1$ with generator $K_{\mu} = -i\left(2x_{\mu}x^{\nu}\partial_{\nu} - (x\cdot x)\partial_{\mu}\right)$.

5.2.2 Finite Conformal Transformations: $D \geq 3$

To get finite conformal transformations, we need to exponentiate the generators with finite parameters.

(i) Translations: let a^{μ} be a finite translation. Then it is implemented on spacetime by the operator

$$T(a) := \exp\left(ia^{\mu}P_{\mu}\right).$$

Thus finite translations are given by

$$\begin{aligned} T(a)x^{\mu} &= \exp\left(ia^{\nu}P_{\nu}\right)x^{\mu} \\ &= \left(1 + a^{\nu}\partial_{\nu} + \frac{1}{2!}a^{\nu}a^{\rho}\partial_{\nu}\partial_{\rho} + \ldots\right)x^{\mu} \\ &= x^{\mu} + a^{\nu}\delta^{\mu}{}_{\nu} + 0 \\ &= x^{\mu} + a^{\mu}, \end{aligned}$$

as expected.

(ii) Dilatation: let α be a finite dilatation parameter. It is implemented on spacetime by the operator

$$S(\alpha) := \exp\left(i\alpha D\right).$$

Thus finite dilatation is given by

$$\begin{aligned} S(\alpha)x^{\mu} &= \exp\left(i\alpha D\right)x^{\mu} \\ &= \left(1 + \alpha x^{\nu}\partial_{\nu} + \frac{1}{2!}\alpha^2 x^{\nu}\partial_{\nu}x^{\rho}\partial_{\rho} + \ldots\right)x^{\mu} \\ &= x^{\mu} + \alpha x^{\nu}\delta^{\mu}{}_{\nu} + \frac{1}{2!}\alpha^2 x^{\nu}\partial_{\nu}(x^{\rho}\delta^{\mu}{}_{\rho}) + \ldots \\ &= x^{\mu} + \alpha x^{\mu} + \frac{1}{2!}\alpha^2 x^{\mu} + \ldots \\ &= e^{\alpha}x^{\mu}. \end{aligned}$$

(iii) Lorentz transformation: let $\omega_{\mu\nu}$ be finite rotation and boost parameters with $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Then it is clear that $\exp(i\omega^{\mu\nu}M_{\mu\nu}) \in \mathrm{SO}(1, D-1)$. Then on spacetime, finite Lorentz transformation is implemented by the Lorentz transformation operator

$$\Lambda^{\mu}{}_{\nu} := \left(\exp(i\omega^{\rho\lambda}M_{\rho\lambda})\right)^{\mu}{}_{\nu},$$

and on spacetime it acts in the usual way.

(iv) Special conformal transformation: let b^μ be a finite SCT parameter. To get finite SCT transformation on spacetime, we need to compute $\exp(ib^\rho K_\rho)x^\mu$ by expanding the exponential. Observe that

$$ib^\rho K_\rho x^\mu = (2(b\cdot x)x^\rho - (x\cdot x)b^\rho)\,\partial_\rho x^\mu = 2(b\cdot x)x^\mu - (x\cdot x)b^\mu.$$

Next we have

$$\begin{aligned}(ib^\rho K_\rho)^2 x^\mu &= (2(b\cdot x)x^\rho - (x\cdot x)b^\rho)\,\partial_\rho(2(b\cdot x)x^\mu - (x\cdot x)b^\mu)\\ &= (2(b\cdot x)x^\rho - (x\cdot x)b^\rho)\,(2b_\rho x^\mu + 2(b\cdot x)\delta^\mu{}_\rho - 2x_\rho b^\mu)\\ &= 2(4(b\cdot x)^2x^\mu - 2(b\cdot x)(x\cdot x)b^\mu - (x\cdot x)(b\cdot b)x^\mu)\end{aligned}$$

We can go on computing higher powers of $ib^\rho K_\rho$. Adding up, we obtain

$$\begin{aligned}\exp(ib^\rho K_\rho)x^\mu &= x^\mu + 2(b\cdot x)x^\mu - (x\cdot x)b^\mu + \frac{1}{2!}2(4(b\cdot x)^2x^\mu - 2(b\cdot x)(x\cdot x)b^\mu\\ &\quad - (x\cdot x)(b\cdot b)x^\mu) + \dots\\ &= (x^\mu - (x\cdot x)b^\mu) + 2(b\cdot x)x^\mu - 2(b\cdot x)(x\cdot x)b^\mu - (x\cdot x)(b\cdot b)x^\mu\\ &\quad + (b\cdot b)(x\cdot x)^2b^\mu - (b\cdot b)(x\cdot x)^2b^\mu + 4(b\cdot x)^2x^\mu + \dots\\ &= (x^\mu - (x\cdot x)b^\mu) + (x^\mu - (x\cdot x)b^\mu)(2(b\cdot x) - (b\cdot b)(x\cdot x)) + \dots\\ &= (x^\mu - (x\cdot x)b^\mu)(1 + (2(b\cdot x) - (b\cdot b)(x\cdot x)) + \dots)\\ &= \frac{x^\mu - (x\cdot x)b^\mu}{1 - 2(b\cdot x) + (b\cdot b)(x\cdot x)}\end{aligned}$$

We thus have the action of all finite conformal transformations on spacetime. We list them in the table below. It is clear that the metric remains invariant under translation and Lorentz

Transformations		Generators
translation	$x'^\mu = x^\mu + a^\mu$	$P_\mu = -i\partial_\mu$
dilatation	$x'^\mu = \alpha x^\mu$	$D = -ix^\mu\partial_\mu$
rotation	$x'^\mu = \Lambda^\mu{}_\nu x^\nu$	$M_{\mu\nu} = i\,(x_\mu\partial_\nu - x_\nu\partial_\mu)$
SCT	$x'^\mu = \frac{x^\mu-(x\cdot x)b^\mu}{1-2(b\cdot x)+(b\cdot b)(x\cdot x)}$	$K_\mu = -i\,(2x_\mu x^\nu\partial_\nu - (x\cdot x)\partial_\mu)$

Table 5.1: Global conformal transformations in $D \geq 3$

transformation. Under dilatation, the scale factor is $\Omega^2(x) = \alpha^2$. Under SCT, the metric scales non trivially. Indeed using SCT transformation given in Table 5.1 and (5.1.2), one can show that the scale factor is

$$\Omega^2(x) = (1 - 2(b\cdot x) + (b\cdot b)(x\cdot x))^2.$$

We know the geometrical meaning of three of the transformations that we have got above namely translation, dilatation and Lorentz transformation. Let us see what SCT means geometrically. First observe that

$$\frac{x'^\mu}{x'\cdot x'} = \frac{x^\mu}{x\cdot x} - b^\mu.$$

Indeed we have

$$\begin{aligned}
\frac{x'^\mu}{x'\cdot x'} &= \frac{x^\mu - (x\cdot x)b^\mu}{1-2(b\cdot x)+(b\cdot b)(x\cdot x)}\frac{(1-2(b\cdot x)+(b\cdot b)(x\cdot x))^2}{(x^\mu-(x\cdot x)b^\mu)(x_\mu-(x\cdot x)b_\mu)}\\
&= \frac{x^\mu - (x\cdot x)b^\mu}{1-2(b\cdot x)+(b\cdot b)(x\cdot x)}\frac{(1-2(b\cdot x)+(b\cdot b)(x\cdot x))^2}{[(x\cdot x)-2(x\cdot x)(b\cdot x)+(x\cdot x)(b\cdot b)]}\\
&= \frac{x^\mu}{x\cdot x} - b^\mu.
\end{aligned}$$

This suggests that SCT corresponds to inversion followed by translation followed by inversion. Moreover SCT is not defined globally. In particular, the transformation blows up at

$$x^\mu = \frac{b^\mu}{b\cdot b} \in \mathbb{R}^{1,D-1}$$

because the denominator $1 - 2(b\cdot x) + (b\cdot b)(x\cdot x) = 0$ at this point. Thus to define SCT globally, we need to compactify the Minkowski space by including the point at infinity by a construction in topology called one point compactification[3]. We will see this construction explicitly in two dimensional case where the one point compactification is explicitly known namely the Riemann sphere.

5.2.3 The Conformal Group and its Algebra

We begin by describing the algebra of conformal transformation generators.

Proposition 5.2.3. *The generators $P_\mu, D, L_{\mu\nu}, K_\mu$ of conformal transformations satisfy the following algebra:*

$$\begin{aligned}
[D, P_\mu] &= iP_\mu\\
[D, K_\mu] &= -iK_\mu\\
[K_\mu, P_\nu] &= 2i\left(\eta_{\mu\nu}D - M_{\mu\nu}\right)\\
[K_\rho, M_{\mu\nu}] &= i\left(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\right)\\
[P_\rho, M_{\mu\nu}] &= i\left(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\right)\\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}\right)
\end{aligned} \tag{5.2.3}$$

[3]see Munkres topology for precise formulation

Proof. The last two Lie brackets are standard Lorentz algebra proved in preliminary quantum field theory course and hence we omit it here. We will prove the first Lie bracket relation and the rest is similar. Suppose f is a test function. Then we have

$$\begin{aligned}[D, P_\mu]f &= (-ix^\nu\partial_\nu)(-i\partial_\mu)f - (-i\partial_\mu)(-ix^\nu\partial_\nu)f \\ &= -x^\nu\partial_\nu\partial_\mu f + x^\nu\partial_\mu\partial_\nu f + \delta^\nu{}_\mu\partial_\nu f \\ &= i(-i\partial_\mu)f \\ &= iP_\mu f.\end{aligned}$$

□

We easily see that the Lorentz algebra is a subalgebra of the conformal algebra. Moreover we know that the Lorentz algebra is $D(D-1)/2$ dimensional. The dimension of the conformal algebra is thus

$$\begin{aligned}&1 \text{ dilatation } + D \text{ translations } + D \text{ special conformal} \\ &\qquad + \frac{D(D-1)}{2} \text{ Lorentz } = \frac{(D+2)(D+1)}{2} \text{ generators.}\end{aligned}$$

To identify this algebra with standard Lie algebra, let us consider certain linear combinations of the conformal generators. Define

$$\begin{aligned}&J_{\mu\nu} = M_{\mu\nu}, \quad \mu,\nu = 0,\ldots,D-1 \\ &J_{-1\mu} = \frac{1}{2}\left(P_\mu - K_\mu\right), \quad \mu = 0,\ldots,D-1 \\ &J_{-1D} = D, \quad J_{D\mu} = \frac{1}{2}\left(P_\mu + K_\mu\right), \quad \mu = 0,\ldots,D-1.\end{aligned}$$

Moreover we define $J_{mn} = J_{nm}$ for $n, m = -1, 0, 1, \ldots, D-1, D$.

Proposition 5.2.4. *The generators J_{ab} satisfy the following Lie bracket relation:*

$$[J_{mn}, J_{rs}] = i\left(\eta_{ms}J_{nr} + \eta_{nr}J_{ms} - \eta_{mr}J_{ns} - \eta_{ns}J_{mr}\right),$$

where $\eta_{mn} = \mathrm{diag}(-1, \underbrace{-1, 1, \ldots, 1}_{m,n=0,\ldots,D-1}, 1)$

Proof. For $n, m = 0, \ldots, D-1$, the relation is immediate from the Lorentz algebra. We check the Lie bracket of $J_{-1\mu}$ and $J_{D\mu}$. We have

$$\begin{aligned}[J_{-1\mu}, J_{D\nu}] &= \frac{1}{4}[P_\mu - K_\mu, P_\nu + K_\nu] \\ &= \frac{1}{4}\left([P_\mu, K_\nu] - [K_\mu, P_\nu]\right) \\ &= \frac{1}{4}\left[-2i(\eta_{\nu\mu}D - M_{\nu\mu}) - 2i(\eta_{\mu\nu}D - M_{\mu\nu})\right] \\ &= -i\eta_{\mu\nu}D = -i\eta_{\mu\nu}J_{-1D},\end{aligned}$$

where we used (5.2.3) and the antisymmetry of $M_{\mu\nu}$. The right hand side of the algebra is

$$i\left(\eta_{-1\nu}J_{\mu D}+\eta_{\mu D}J_{-1\nu}-\eta_{-1D}J_{\mu\nu s}-\eta_{\mu\nu}J_{-1D}\right)=-i\eta_{\mu\nu}J_{-1D}.$$

Other brackets are similar. $\square$

To identify the conformal algebra with standard Lie algebra, we define the generalised orthogonal group.

Definition 5.2.5. (Generalised orthogonal group) Let n, k be two positive integers. Define a bilinear form $B:\mathbb{R}^{n+k}\times\mathbb{R}^{n+k}\longrightarrow\mathbb{R}$ by

$$B(\mathbf{x},\mathbf{y}):=-\sum_{i=1}^{n}x_iy_i+\sum_{j=1}^{k}x_{n+j}y_{n+j},$$

where $\mathbf{x}=(x_1,\ldots,x_n,\ldots x_{n+k}), \mathbf{y}=(y_1,\ldots,y_n,\ldots y_{n+k})\in\mathbb{R}^{n+k}$. Define the set $\mathrm{O}(n,k)$ as the set of matrices which preserve the bilinear form B:

$$\mathrm{O}(n,k):=\{A\in\mathrm{GL}(n+k,\mathbb{R})\mid B(A\mathbf{x},A\mathbf{y})=B(\mathbf{x},\mathbf{y}),\ \forall\ \mathbf{x},\mathbf{y}\in\mathbb{R}^{n+k}\},$$

where $\mathrm{GL}(n+k,\mathbb{R})$ denotes the set of all invertible real matrices of size $(n+k)\times(n+k)$. If we write

$$\mathbb{1}_{n,k}:=\begin{pmatrix}-\mathbb{1}_n & 0\\ 0 & \mathbb{1}_k\end{pmatrix}$$

where $\mathbb{1}_n$ is the $n\times n$ identity matrix, then it is easy to see that

$$\mathrm{O}(n,k)=\left\{A\in\mathrm{GL}(n+k,\mathbb{R})\mid A^T\mathbb{1}_{n,k}A=\mathbb{1}_{n,k}\right\}.$$

$\mathrm{O}(n,k)$ is called the generalised orthogonal group. We also define

$$\mathrm{SO}(n,k):=\{A\in\mathrm{O}(n,k)\mid \det A=1\}.$$

Thus, we identify the conformal algebra with the Lie algebra $\mathfrak{so}(2,D-1)$ of $\mathrm{SO}(2,D-1)$. In general if $\mathbb{R}^{p,q}$ denotes the Minkowski space with metric $\mathbb{1}_{p,q}$, then following the same procedure, we can get the conformal algebra and the conformal group of $\mathbb{R}^{p,q}$. Thus we have the following theorem.

Theorem 5.2.6. *For Minkowski space $\mathbb{R}^{p,q}$ with dimension $D=p+q\geq 3$, the conformal group is* $\mathrm{SO}(p+1,q+1)$.

5.3 Conformal Group in $D=2$

We work with Euclidean metric but everything can be formulated in Lorentzian signature equally well.

5.3.1 Local Conformal Transformations

In two dimensions, let (z^0, z^1) be the coordinates on the plane. Under a spacetime transformation $z^\mu \longrightarrow w^\mu(x)$ the metric tensor transforms as

$$g^{\mu\nu} \longrightarrow \widetilde{g}^{\mu\nu}(w) = \left(\frac{\partial w^\mu}{\partial z^\alpha}\right)\left(\frac{\partial w^\nu}{\partial z^\beta}\right) g^{\alpha\beta}$$

Since $g'_{\mu\nu}(w) = \Omega^2(x) g_{\mu\nu}(z)$ under conformal transformations, thus for various μ, ν we get

$$\begin{aligned}
\Omega^2(x) &= \left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2, \quad \mu,\nu = 0,0 \\
\Omega^2(x) &= \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2, \quad \mu,\nu = 1 \\
0 &= \frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1}, \quad (\mu,\nu) = (1,0),(0,1).
\end{aligned}$$

Thus we conclude that

$$\begin{aligned}
\left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 &= \left(\frac{\partial w^1}{\partial z^0}\right)^2 + \left(\frac{\partial w^1}{\partial z^1}\right)^2 \\
\frac{\partial w^0}{\partial z^0}\frac{\partial w^1}{\partial z^0} + \frac{\partial w^0}{\partial z^1}\frac{\partial w^1}{\partial z^1} &= 0.
\end{aligned} \tag{5.3.1}$$

Second equation of (5.3.1) gives

$$\frac{\frac{\partial w^0}{\partial z^0}}{\frac{\partial w^1}{\partial z^1}} = -\frac{\frac{\partial w^0}{\partial z^1}}{\frac{\partial w^1}{\partial z^0}} = \lambda \implies \frac{\partial w^0}{\partial z^0} = \lambda\frac{\partial w^1}{\partial z^1}, \quad \frac{\partial w^0}{\partial z^1} = -\lambda\frac{\partial w^1}{\partial z^0}.$$

Substituting this in first equation of (5.3.1), we get

$$\left(\frac{\partial w^0}{\partial z^0}\right)^2 + \left(\frac{\partial w^0}{\partial z^1}\right)^2 = \lambda^2\left(\frac{\partial w^0}{\partial z^1}\right)^2 + \left(\frac{\partial w^0}{\partial z^0}\right)^2 \implies \lambda^2 = 1.$$

Thus we obtain two other conditions which are independently equivalent to (5.3.1):

$$\frac{\partial w^1}{\partial z^0} = \frac{\partial w^0}{\partial z^1}, \quad \frac{\partial w^0}{\partial z^0} = -\frac{\partial w^1}{\partial z^1} \tag{5.3.2}$$

or

$$\frac{\partial w^1}{\partial z^0} = -\frac{\partial w^0}{\partial z^1}, \quad \frac{\partial w^0}{\partial z^0} = \frac{\partial w^1}{\partial z^1} \tag{5.3.3}$$

(5.3.2) resembles the Cauchy-Riemann equations for holomorphic functions. On the other hand, we define antiholomorphic functions using (5.3.3). To make this explicit, we make a transition to complex coordinates using the change of coordinates given below:

$$\begin{aligned} z &= z^0 + iz^1, \quad \bar{z} = z^0 - iz^1, \\ z^0 &= \frac{1}{2}(z+\bar{z}), \quad z^1 = \frac{1}{2i}(z-\bar{z}). \\ \partial_z &= \frac{1}{2}\left(\partial_0 - i\partial_1\right), \quad \partial_0 = \partial_z + \partial_{\bar{z}}. \\ \partial_{\bar{z}} &= \frac{1}{2}\left(\partial_0 + i\partial_1\right), \quad \partial_1 = i\left(\partial_z - \partial_{\bar{z}}\right). \end{aligned} \tag{5.3.4}$$

In terms of the coordinates z and $\bar{z}$, we have

$$ds^2 = (dz^0)^2 + (dz^1)^2 = \frac{1}{4}(dz + d\bar{z})^2 - \frac{1}{4}(dz - d\bar{z})^2 = dzd\bar{z}.$$

So in the metric tensor in complex coordinates is

$$g^{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad g_{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix},$$

where the index μ, ν run over $z, \bar{z}$. With this notation, if we define

$$w(z,\bar{z}) = w^0(z,\bar{z}) + iw^1(z,\bar{z}),$$

then (5.3.2) implies that $\partial_{\bar{z}} w(z,\bar{z}) = 0$ and (5.3.3) implies that $\partial_z \bar{w}(z,\bar{z}) = 0$. This means that that the function $w(z)$ and $\bar{w}(\bar{z})$ are holomorphic in some open set of the complex plane. Thus conformal transformation in two dimension amounts to a *holomorphic* change of coordinates:

$$z \longrightarrow z' = f(z), \quad \bar{z} \longrightarrow \bar{f}(\bar{z}),$$

where the two transformations result from the two equations (5.3.2) and (5.3.3) but in both cases the change of coordinates is holomorphic. Conversely, if we have a transformation $z \longrightarrow f(z)$ for a holomorphic function f in some open set of the complex plane, then the Euclidean metric $dzd\bar{z}$ on[4] $\mathbb{C}$ transforms as

$$dzd\bar{z} \longrightarrow \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}} dzd\bar{z},$$

from which we see that the metric transforms conformally with scale factor $\left|\frac{\partial f}{\partial z}\right|^2$. An important point to note is that we require holomorphicity only in some open set, which means that the conformal transformations we have obtained are *local*. Thus we have proved the following theorem.

Theorem 5.3.1. *The group of local conformal transformations in dimension two is isomorphic to the group of all holomorphic functions*[5] *in some open set on the complex plane and hence is infinite dimensional.*

[4]induced from the Euclidean metric $dx^2 + dy^2$ on $\mathbb{R}^2$.

[5]the set of all holomorphic maps is a group under usual composition of maps.

Proof. The isomorphism is explicit from our discussion above. The dimensionality follows from the fact that the set of all holomorphic functions is infinite dimensional. To see this, note that any complex function f holomorphic in some open set admits a Laurent expansion:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}.$$

Thus we need an infinite number of parameters, namely the coefficients in the Laurent expansion to specify a holomorphic function. $\square$

5.3.2 Infinitesimal Generators: The Witt Algebra

Any infinitesimal conformal transformation can be written as

$$z \longrightarrow z' = z + \varepsilon(z), \quad \bar{z} \longrightarrow \bar{z}' = \bar{z} + \bar{\varepsilon}(\bar{z}),$$

where $|\varepsilon(z)| \ll 1$. Since $\varepsilon(z)$ and $\bar{\varepsilon}(\bar{z})$ are holomorphic in some open set, we can write its Laurent expansion around 0:

$$\begin{aligned} z' &= z + \varepsilon(z) = z + \sum_{n\in\mathbb{Z}} \varepsilon_n \left(-z^{n+1}\right) \\ \bar{z}' &= \bar{z} + \bar{\varepsilon}(\bar{z}) = \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\varepsilon}_n \left(-\bar{z}^{n+1}\right) \end{aligned}$$

where the infinitesimal parameters ε_n and $\bar{\varepsilon}_n$ are constants defining the Laurent expansion. Let l_n and $\bar{l}_n$ be the generators corresponding to the transformation $z \longrightarrow z - \varepsilon_n z^{n+1}$ and $\bar{z} \longrightarrow \bar{z} - \bar{\varepsilon}_n \bar{z}^{n+1}$ respectively. Then we have[6]

$$l_n = -z^{n+1}\partial_z \quad \text{and} \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{5.3.5}$$

Thus we have infinite number of generators for infinitesimal conformal transformations in two dimensions. Thus we conclude that local conformal transformation is infinite dimensional.

We now calculate the algebra of the infinitesimal generators. We have

$$\begin{aligned} [l_m, l_n] &= z^{m+1}\partial_z \left(z^{n+1}\partial_z\right) - z^{n+1}\partial_z \left(z^{m+1}\partial_z\right) \\ &= (n+1)z^{m+n+1}\partial_z - (m+1)z^{m+n+1}\partial_z \\ &= -(m-n)z^{m+n+1}\partial_z \\ &= (m-n)l_{m+n}. \end{aligned}$$

Similarly we have

$$\left[\bar{l}_m, \bar{l}_n\right] = (m-n)\bar{l}_{m+n},$$

[6] see Appendix B for explicit calculations

and as expected

$$\left[l_m, \bar{l}_n\right] = 0.$$

Thus the algebra of infinitesimal generators is

$$[l_m, l_n] = (m-n)l_{m+n}, \quad \left[\bar{l}_m, \bar{l}_n\right] = (m-n)\bar{l}_{m+n}, \quad \left[l_m, \bar{l}_n\right] = 0.$$

The commutation relation satisfied by the generator l_n and $\bar{l}_n$ is called Witt algebra. Thus the algebra of infinitesimal generators of conformal transformations in two dimensions consists of two copies of the Witt algebra as subalgebras.

Remark 5.3.2. We may identify this algebra as the classical Virasoro generators that we obtained when we imposed the classical constraints on the string. We also saw that the quantum Virasoro algebra involved an additional term called the central charge. We will rederive the quantum Virasoro algebra in next section.

5.3.3 The Global Conformal Group

We now try to extract the subalgebra of the infinitesimal conformal algebra which is globally defined. We first analyse the generators l_n. Observe that these generators are not defined at $z=0$. Thus we need to include the point at infinity to the complex plane and consider the Riemann sphere $\mathbb{C} \cup \{\infty\} \cong S^2$. Even if we consider the Riemann sphere then also not all z=generators are well defined. For example the generators $l_n = -z^{n+1}\partial_z$ is non singular at $z=0$ only for $n \geq -1$. The other problematic point is ∞. To understand the behaviour of l_n at ∞, we make a change of variable $z \longrightarrow -1/\omega$ and then study the limit $\omega \to 0$. The generators transform as

$$l_n = -z^{n+1}\partial_z \longrightarrow -\left(-\frac{1}{\omega}\right)^{n-1}\partial_\omega.$$

From this expression, we see that these generators are non singular at infinity only for $n \leq 1$. Thus we see that only three generators are globally defined namely $\{l_{-1}, l_0, l_1\}$. Thus we have proved the following theorem.

Theorem 5.3.3. *The global conformal group of the Riemann sphere $\mathbb{C} \cup \{\infty\} \cong S^2$ is three dimensional and is generated by l_{-1}, l_0, l_1 which satisfies the Witt algebra.*

To identify the global conformal group, we will analyse the transformations generated by the generators l_{-1}, l_0, l_1.

It is clear that l_{-1} generates translations[7] $z \longrightarrow z + a$. It is also clear that l_0 generators dilatation[8] $z \longrightarrow \alpha z$. We are left with l_0. This corresponds to SCT. Let us work out the explicit transformation. We have

$$\exp(cl_1)z = \left(\sum_{n=0}^{\infty} \frac{(cl_1)^2}{n!}\right) z.$$

[7] compare the generator l_{-1} with the momentum operator P_μ.

[8] compare l_0 with D.

Observe that

$$l_1 z = -z^2 \partial_z z = -z^2.$$

By induction, we see that

$$l_1^n z = (-1)^n n! z^{n+1}.$$

Thus we have that

$$\exp(cl_1)z = \sum_{n=0}^{\infty} \frac{(-c)^n n! z^{n+1}}{n!} = z \sum_{n=0}^{\infty} (-cz)^n = \frac{z}{cz+1}.$$

In total, a combination of l_{-1}, l_0, l_1 produces the following transformation:

$$z \longrightarrow \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0,$$

where the last condition is required for invertibility of the map. We can rescale the complex numbers a, b, c, d such that $ad - bc = 1$. Now we can identify each such map with the matrix

$$\frac{az+b}{cz+d} \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

A straightforward calculation calculation shows that composition of two such maps corresponds to matrix multiplication of the corresponding matrices. Moreover observe that the matrices A and $-A$ produce the same conformal transformation. Hence we have proved the following theorem,

Theorem 5.3.4. *The global conformal group of the Riemann sphere $\mathbb{C} \cup \{\infty\} \cong S^2$ is isomorphic to $\mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$ where $\mathrm{SL}(2, \mathbb{C})$ denotes the group of 2×2 complex matrices with determinant 1.*

The Virasoro Algebra

Recall that the classical constraints we obtained when we quantised the Polyakov string action in cannonical formalism satisfies the Witt algebra. Whereas in the quantum theory, we got a nontrivial central term in the quantum Virasoro algebra. Here we rederive the quantum Virasoro algebra which is the so called central extension of the Witt algebra.

Roughly speaking, a central extension by $\mathbb{C}$ of a Lie algebra $\mathfrak{g}$ is $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}$ and is characterised by the Lie bracket

$$\begin{aligned}
&[\widetilde{X}, \widetilde{Y}]_{\widetilde{\mathfrak{g}}} = [X, Y]_{\mathfrak{g}} + cp(X, Y), \quad \widetilde{X}, \widetilde{Y} \in \widetilde{\mathfrak{g}}, \\
&[\widetilde{X}, c]_{\widetilde{\mathfrak{g}}} = 0, \\
&[c, c]_{\widetilde{\mathfrak{g}}} = 0, \quad c \in \mathbb{C},
\end{aligned}$$

where $p : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C}$ is a bilinear map.

Let $L_n,\ n \in \mathbb{Z}$ denote the elements of the central extension of the Witt algebra. Then by definition, we have

$$[L_m, L_n] = (m-n)L_{m+n} + cp(m,n)$$

We now determine $p(m,n)$ in three steps:

Step 1: $p(m,n) = -p(n,m)$ and we can assume $p(1,-1) = 0$ and $p(n,0) = 0$.

Proof. Since the Lie bracket is antisymmetric, we obtain the first assertion. Next, without the loss of generality, we can assume that $p(1,-1) = 0$ and $p(n,0) = 0$. If not then we can make the following redefinition:

$$\begin{aligned} \widehat{L}_n &= L_n + \frac{cp(n,0)}{n}, \quad n \neq 0 \\ \widehat{L}_0 &= L_0 + \frac{cp(1,-1)}{2}. \end{aligned}$$

We can check that with this redefinition, we have $p(1,-1) = 0$ and $p(n,0) = 0$. Indeed, for the modified generators we have

$$\begin{aligned} \left[\widehat{L}_n, \widehat{L}_0\right] &= nL_n + cp(n,0) = n\widehat{L}_n, \\ \left[\widehat{L}_1, \widehat{L}_{-1}\right] &= 2L_0 + cp(1,-1) = 2\widehat{L}_0 \end{aligned}$$

□

Step 2: $p(n,m) = 0$ for $n \neq -m$.

Proof. To prove this, we begin by observing that Jacobi identity gives

$$[[L_m, L_n], L_0] + [[L_n, L_0], L_m] + [[L_0, L_m], L_n] = 0.$$

Using the characterisation of the central extension and the fact that the Lie bracket of the Witt algebra also satisfies Jacobi identity, we get

$$\begin{aligned} (m-n)cp(m+n,0) + ncp(n,m) - mcp(m,n) = 0 \\ \implies (m+n)p(n,m) = 0, \end{aligned}$$

where we used results of step 1. The result is now immediate. □

We are now left with the only non-vanishing central extensions $p(n,-n)$ for $|n| \geq 2$.

Step 3: $p(n,-n) = \frac{1}{12}(n^3 - n)$.

Proof. Again by Jacobi identity, we have

$$[[L_{-n+1}, L_n], L_{-1}] + [[L_n, L_{-1}], L_{-n+1}] + [[L_{-1}, L_{-n+1}], L_n] = 0.$$

Again proceeding as in Step 2, we obtain

$$(-2n+1)cp(1,-1) + (n+1)cp(n-1,-n+1) + (n-2)cp(-n,n) = 0.$$

Using $p(1,-1) = 0$, we obtain the recursion relation

$$p(n,-n) = \left(\frac{n+1}{n-2}\right) p(n-1,-n+1), \quad |n| \geq 3.$$

Thus we are free to choose $p(2,-2)$ to solve the recursion. We choose $p(2,-2)$ for later suitability. We get

$$\begin{aligned} p(n,-n) &= \frac{1}{2}\left(\frac{n+1}{n-2}\right)\left(\frac{n+1}{n-2}\right)\cdots\left(\frac{4}{3}\right) \\ &= \frac{1}{2}\binom{n+1}{3} \\ &= \frac{1}{12}(n+1)n(n-1) \\ &= \frac{1}{12}(n^3-n). \end{aligned}$$

□

Thus we see that the central extension of the Witt algebra satisfies the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\left(n\left(n^2-1\right)\right)\delta_{m+n,0}.$$

Similar algebra is satisfies by the central extension of the generators $\bar{l}_n$.

Remark 5.3.5. For the Minkowski metric, we can perform a similar analysis. To do so, we define the lightcone coordinates $u = -t + x$ and $v = t + x$ where t denotes the time direction and x the space direction. The metric becomes

$$ds^2 = -dt^2 + dx^2 = dudv$$

and conformal transformations are given by $u \mapsto f(u)$ and $v \mapsto g(v)$ which gives

$$ds'^2 = \frac{\partial f}{\partial u}\frac{\partial g}{\partial v}dudv.$$

Thus we see that again the Lie algebra of infinitesimal generators is infinite dimensional.

5.4 Primary Fields

We begin by discussing the transformation of fields under conformal transformation. This requires us to investigate representations of the conformal algebra.

5.4.1 Representation of the Conformal Group in D Dimensions

Let $\Phi(x)$ be a multicomponent classical field. We want to find representations of the conformal group and its action on the field Φ. We separately analyse $D \geq 3$ and $D = 2$ case.

Dimension $D \geq 3$

We use a cute little trick for this calculation. We begin by computing the generators which relate the transformed field to the original field at $x = 0$. We do this computation for the generators which keep the origin invariant. Then we use the translation generator to get the generator at any arbitrary spacetime point. Since Lorentz transformations, dilatations and special conformal transformations preserve the origin, we start by writing

$$\begin{aligned}
&M_{\mu\nu}\Phi(0) = S_{\mu\nu}\Phi(0) \\
&K_\mu\Phi(0) = \kappa_\mu\Phi(0) \\
&D\Phi(0) = \widetilde{\Delta}\Phi(0),
\end{aligned}$$

where $S_{\mu\nu}, \widetilde{\Delta}$ and κ_μ are the operators associated to the representation Φ corresponding to Lorentz transformation, dilatation and SCT respectively. Now recall that under translation $x \longrightarrow x + a$,

$$\Phi'(x') = \Phi(x) \implies \Phi'(x) = \Phi(x - a) \implies e^{ia^\mu P_\mu}\Phi(x) = \Phi\left(e^{-ia^\mu P_\mu}x\right).$$

This implies that

$$e^{-ix^\mu P_\mu}\Phi(0) = \Phi(x).$$

Now we have

$$M_{\mu\nu}\Phi(x) = M_{\mu\nu}e^{-ix^\lambda P_\lambda}\Phi(0) = e^{-ix^\lambda P_\lambda}\left[e^{ix^\lambda P_\lambda}M_{\mu\nu}e^{-ix^\lambda P_\lambda}\right]\Phi(0).$$

Now by first equation of (A.3.1), we have

$$e^{ix^\lambda P_\lambda}M_{\mu\nu}e^{-ix^\lambda P_\lambda} = M_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu.$$

Thus we have

$$\begin{aligned}
M_{\mu\nu}\Phi(x) &= e^{-ix^\lambda P_\lambda}\left[M_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu\right]\Phi(0) \\
&= e^{-ix^\lambda P_\lambda}M_{\mu\nu}\Phi(0) - (x_\mu P_\nu - x_\nu P_\mu)e^{-ix^\lambda P_\lambda}\Phi(0) \\
&= e^{-ix^\lambda P_\lambda}S_{\mu\nu}\Phi(0) - (x_\mu P_\nu - x_\nu P_\mu)\Phi(x)
\end{aligned}$$

Thus we conclude that

$$\begin{aligned}
&P_\mu = -i\partial_\mu \Phi(x) \\
&M_{\mu\nu}\Phi(x) = S_{\mu\nu}\Phi(x) + i(x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi(x).
\end{aligned} \tag{5.4.1}$$

Instead of using (A.3.1) to evaluate $e^{ix^\lambda P_\lambda} M_{\mu\nu} e^{-ix^\lambda P_\lambda}$, one could have used the conformal algebra and the Hausdorff formula:

$$e^{-A}Be^{A} = B + [B,A] + \frac{1}{2!}[[B,A],A] + \frac{1}{3!}[[[B,A],A],A] + \cdots$$

for operators A and B. Indeed it will be useful for later computations.

We see that the operators $S_{\mu\nu}, \widetilde{\Delta}$ and κ_μ must satisfy the conformal algebra:

$$\begin{aligned}
&[\widetilde{\Delta}, S_{\mu\nu}] = 0 \\
&[\widetilde{\Delta}, \kappa_\mu] = -i\kappa_\mu \\
&[\kappa_\nu, \kappa_\mu] = 0 \\
&[\kappa_\rho, S_{\mu\nu}] = i\left(\eta_{\rho\mu}\kappa_\nu - \eta_{\rho\nu}\kappa_\mu\right) \\
&[S_{\mu\nu}, S_{\rho\sigma}] = i\left(\eta_{\nu\rho}S_{\mu\sigma} + \eta_{\mu\sigma}S_{\nu\rho} - \eta_{\mu\rho}S_{\nu\sigma} - \eta_{\nu\sigma}S_{\mu\rho}\right).
\end{aligned} \tag{5.4.2}$$

Using (5.2.3) and the Hausdorff formula, we have

$$\begin{aligned}
&e^{ix^\rho P_\rho} D e^{-ix^\rho P_\rho} = D + x^\nu P_\nu \\
&e^{ix^\rho P_\rho} K_\mu e^{-ix^\rho P_\rho} = K_\mu + 2x_\mu D - 2x^\nu M_{\mu\nu} + 2x_\mu\left(x^\nu P_\nu\right) - x^2 P_\mu.
\end{aligned}$$

This gives us the transformation of the field $\Phi(x)$ under dilatations and SCT:

$$\begin{aligned}
&D\Phi(x) = \left(-ix^\nu\partial_\nu + \widetilde{\Delta}\right)\Phi(x) \\
&K_\mu\Phi(x) = \left(\kappa_\mu + 2x_\mu\widetilde{\Delta} - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu\partial_\nu + ix^2\partial_\mu\right)\Phi(x).
\end{aligned} \tag{5.4.3}$$

We now know how the field $\Phi(x)$ transforms under all generators of the conformal algebra. Let us assume that $(S_{\mu\nu}, \Phi)$ furnishes an *irreducible* representation of the Lorentz algebra. The following theorem will be crucial.

Theorem 5.4.1. (Schur's Lemma) *Let Π be an irreducible complex representation of a Lie group G. If A is in the center of G, then $\Pi(A) = \lambda I$, for some $\lambda \in \mathbb{C}$. Similarly, if π is an irreducible complex representation of a Lie algebra $\mathfrak{g}$ and if $[X,Y] = 0$ for every $Y \in \mathfrak{g}$, then $\pi(X) = \lambda I$.*

Since $\widetilde{\Delta}$ commutes with $S_{\mu\nu}$, thus it must act as a multiple of identity on Φ. From (B.2.3), it is clear that

$$\widetilde{\Delta} = -i\Delta,$$

where Δ is the scaling dimension of the field Φ. This obviates the fact that $\widetilde{\Delta}$ is not Hermitian.

Next, observe that since $\widetilde{\Delta}$ is a multiple of identity, it commutes with every other generator, in particular the generator of SCT. Thus the algebra of these generators in (5.4.2) implies that $\kappa_\mu = 0$. This is a crucial result:

the generators of SCT act trivially on fields Φ if they belong to irreducible representation of the Lorentz algebra.

Now we want to get the transformation of field Φ under finite conformal transformation. To do this we employ the fact that a Lie algebra representation gives rise to a Lie group representation via the exponential map. To make this precise, let $\Phi_\alpha(x)$ transform in the irreducible representation of the Lorentz algebra. Then under the conformal transformation $x \longrightarrow x'$ with parameter $a_\mu, \lambda, \omega_{\mu\nu}, b_\mu$ corresponding to translation, dilatation, rotation and SCT respectively, the field $\Phi_\alpha(x)$ transforms as

$$\Phi_\alpha(x) \longrightarrow \Phi'_\alpha(x) = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/D} [\exp(i\omega^{\mu\nu} S_{\mu\nu})]_{\alpha\beta}\, \Phi_\beta(\Lambda^{-1}x),$$

where Λ is the Lorentz transformation acting on spacetime with parameters $\omega_{\mu\nu}$. To prove this, observe that the Jacobian for a general conformal transformation (excluding SCT as its generator acts trivially so that SCT acts as identity) is given by λ^D. Moreover by assumption $\Phi(\lambda x) = \lambda^{-\Delta}\Phi(x)$. The transformation is now immediate.

In particular, for spinless field ϕ i.e. $S_{\mu\nu}\phi = 0$, the transformation is

$$\phi(x) \longrightarrow \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/D} \phi(x). \tag{5.4.4}$$

Definition 5.4.2. A field $\phi(x)$ transforming as in (5.4.4) under global conformal transformations is called a *quasi primary* field with scaling dimension Δ. A field which is not quasi primary is called *secondary*.

Dimension $D = 2$

We have seen that the conformal algebra of the plane parametrized by (x^0, x^1) is most conveniently expressed in terms of Witt algebra generators which in turn are expressed in terms of the complexified coordinates $z = x^0 + ix^1$ and $\bar{z} = x^0 - ix^1$. In what follows, we will consider z and $\bar{z}$ to be two independent complex variables but we also keep in mind that at the end of the calculation, we have to identify $\bar{z}$ with the complex conjugate of z. With this understanding, the fields ϕ on the plane transform to field on the four real dimensional $\mathbb{C}^2$ via the *complexification* $\mathbb{R}^2 \longrightarrow \mathbb{C}^2$:

$$\phi\left(x^0, x^1\right) \longrightarrow \phi(z, \bar{z}),$$

where $(x^0, x^1) \in \mathbb{R}^2$ and $(z, \bar{z}) \in \mathbb{C}^2$. In two dimensions we already saw that the local conformal group is infinite dimensional, so we have two different definitions based on the transformation of the fields as we will see in a moment.

We know that the global conformal group of two dimensional Euclidean spacetime is generated by l_{-1}, l_0, l_1, so we work with a basis of eigenstates of the operators l_0 and $\bar{l}_0$. Let the corresponding eigenvalues be h and $\bar{h}$. These are known as the *conformal weights* of the state. Since $l_0 + \bar{l}_0$ and $i\left(l_0 - \bar{l}_0\right)$ are identified with the generators of dilatations and rotations (see Table 5.1 for the generators), the scaling dimension Δ and the spin s of the state are given by

$$\Delta = h + \bar{h}, \quad s = h - \bar{h}. \tag{5.4.5}$$

Definition 5.4.3. (i) Fields only depending on z, i.e. $\phi(z)$, are called *chiral fields* or *holomorphic* fields and fields $\phi(\bar{z})$ only depending on $\bar{z}$ are called *anti chiral* or *anti holomorphic* fields.

(ii) A field $\phi(z, \bar{z})$ which transforms under dilatations $z \longmapsto \lambda z$ according to

$$\phi(z, \bar{z}) \longmapsto \phi'(z, \bar{z}) = \lambda^h \bar{\lambda}^{\bar{h}} \phi(\lambda z, \bar{\lambda}\bar{z}),$$

is said to have *conformal dimensions* $(h, \bar{h})$.

(iii) A field which transforms under conformal transformations $z \longmapsto f(z)$ according to

$$\phi(z, \bar{z}) \longmapsto \phi'(z, \bar{z}) = \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z}))$$

is called a *primary field of conformal dimension* $(h, \bar{h})$.

(iv) A field ϕ which transforms as a primary field only for global conformal transformations $f \in \mathrm{SL}(2, \mathbb{C})/\mathbb{Z}_2$ is called a *quasi primary field*.

(v) A primary field is always quasi primary but the converse is not true. A field in a CFT which is neither primary nor quasi primary is called *secondary fields*.

We now find the infinitesimal version of transformation of primary fields. To this end, consider the infinitesimal conformal transformation $f(z) = z + \varepsilon(z)$ with $\varepsilon(z) \ll 1$. Up to first order in $\varepsilon(z)$, we have

$$\begin{aligned} \left(\frac{\partial f}{\partial z}\right)^h &= 1 + h\partial_z \varepsilon(z) + O\left(\varepsilon^2\right), \\ \phi(z + \varepsilon(z), \bar{z}) &= \phi(z, \bar{z}) + \varepsilon(z)\partial_z \phi(z, \bar{z}) + O\left(\varepsilon^2\right). \end{aligned}$$

Using these expressions, we see that a primary field with conformal dimensions $h, \bar{h}$ transforms as

$$\phi(z, \bar{z}) \longmapsto \phi(z, \bar{z}) + \left(h\partial_z \varepsilon + \varepsilon \partial_z + \bar{h}\partial_{\bar{z}}\bar{\varepsilon} + \bar{\varepsilon}\partial_{\bar{z}}\right) \phi(z, \bar{z}).$$

Thus under infinitesimal conformal transformation, a primary field transforms as

$$\delta_{\varepsilon, \bar{\varepsilon}} \phi(z, \bar{z}) = \left(h\partial_z \varepsilon + \varepsilon \partial_z + \bar{h}\partial_{\bar{z}}\bar{\varepsilon} + \bar{\varepsilon}\partial_{\bar{z}}\right) \phi(z, \bar{z}). \tag{5.4.6}$$

5.5 Consequences of Conformal Invariance: Classical Aspects

A theory of fields invariant under conformal transformations is called a conformal field theory (CFT)[9]. We have seen in previous sections that the global conformal group in various dimensions includes translations, Lorentz transformations, SCT and dilatations. Invariance under conformal transformations has many consequences. We will analyse the classical aspects of a CFT.

5.5.1 Translation Invariance: Energy-Momentum Tensor

Recall that by Noether's theorem (see Appendix B), there is a classical conserved current corresponding to every classical continuous symmetry of the action. The current corresponding to translation invariance is called the energy momentum tensor. Suppose that a classical theory of fields Φ with Lagrangian $\mathcal{L}$ is invariant under infinitesimal translation $x \longrightarrow x + \varepsilon(x)$. Then by (B.2.4), the energy momentum tensor[10] is given by

$$T^{\mu}_{\ \nu} := j^{\mu}_{\ \nu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)} \partial_\nu \Phi - \eta^{\mu}_{\ \nu} \mathcal{L}. \tag{5.5.1}$$

An alternative way of deriving the energy momentum tensor is the following: consider the same theory but now let the background metric be dynamical $\eta_{\mu\nu} \longrightarrow g_{\mu\nu}$. Then translation invariance of the action may be thought of as a diffeomorphism. Under such a transformation, the metric transforms as

$$\begin{aligned}
\widetilde{g}_{\mu\nu} &= \frac{\partial x^\alpha}{\partial x'} \frac{\partial x^\beta}{\partial x'} g_{\alpha\beta} \\
&= \left(\delta^\alpha_\mu - \partial_\mu \varepsilon^\alpha\right)\left(\delta^\beta_\nu - \partial_\nu \varepsilon^\beta\right) g_{\alpha\beta} \\
&= g_{\mu\nu} - \left(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu\right).
\end{aligned}$$

By (B.2.5), the action varies as

$$\begin{aligned}
\delta S &= \int d^D x T^{\mu\nu} \partial_\mu \varepsilon_\nu \\
&= \frac{1}{2} \int d^D x T^{\mu\nu} \left(\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu\right),
\end{aligned}$$

where we used the fact that the energy momentum tensor is symmetric[11]. Thus we have

$$\delta S = -\frac{1}{2} \int d^D x T^{\mu\nu} \delta g_{\mu\nu}.$$

[9] see Appendix B for precise definitions of symmetries

[10] the counting index a in the current as in (B.2.4) is now a spacetime index because of the spacetime index in the transformation parameter $\varepsilon(x)$.

[11] in general, the energy momentum tensor may not be symmetric. But one can show that it can always be made symmetric by adding the divergence of an antisymmetric tensor which neither affects the conservation of current nor the Ward identities. The new energy momentum tensor is called the Belinfante tensor. See Subsection 5.5.2 for the details of the construction.

This shows that the energy momentum tensor is given by

$$T^{\mu\nu} = -2\frac{\delta S}{\delta g_{\mu\nu}}.$$

In string theory, we usually choose a different normalisation and define the energy momentum to be

$$T_{\mu\nu} = -\frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}, \tag{5.5.2}$$

where g denotes the determinant of the metric. If the space is flat, we evaluate $T_{\mu\nu}$ on $g_{\mu\nu} = \eta_{\mu\nu}$ and the resulting expression obeys $\partial^\alpha T_{\alpha\beta} = 0$. In general, the energy momentum tensor is covariantly conserved,

$$\nabla^\mu T_{\mu\nu} = 0.$$

Remark 5.5.1. The energy momentum tensor in string theory differs from that in usual QFT by a factor of 2π when the background metric is flat. We will keep this in mind and when we apply CFT to string theory, we will drop any extra factor of 2π that appear. We will mention this whenever we do so.

We now prove a typical consequence of conformal invariance.

Theorem 5.5.2. *In a CFT, the trace of the energy momentum tensor vanishes.*

Proof. Since the theory is invariant under dilatation, this, let us vary the action with respect to an infinitesimal dilatation $x \longrightarrow x' = (1+\alpha)x$. We have

$$\delta g_{\mu\nu} = \alpha g_{\mu\nu}.$$

The action varies as

$$\delta S = \int d^D x \frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} = -\frac{1}{4\pi}\int d^D x \sqrt{g}\alpha T^\mu_{\ \mu}.$$

Since dilatations are symmetry of the theory, $\delta S = 0$ which implies

$$T^\mu_{\ \mu} = 0.$$

□

Remark 5.5.3. In a conformal field theory, vanishing trace of the energy momentum tensor is a typical feature, but as it turns out, this does not hold at quantum level in general, for example, in Yang-Mills theory it does not hold. In 2 dimensional CFT, it holds at quantum level only when the metric is flat. For curved background, we get an anomaly called the conformal anomaly which was mentioned in Subsection 3.3.5.

Energy Momentum Tensor in Two Euclidean Dimensions

We make the change of variables from real to complex as given in (5.3.4). Then using the transformation of the energy momentum tensor

$$T'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu} T_{\alpha\beta}.$$

Using $x^0 = \frac{1}{2}(z+\bar{z})$ and $x^1 = \frac{1}{2i}(z-\bar{z})$, it is straightforward to work out the components (we have removed the primes)

$$\begin{aligned} T_{zz} &= \frac{1}{4}\left(T_{00} - 2iT_{10} - T_{11}\right), \\ T_{\bar{z}\bar{z}} &= \frac{1}{4}\left(T_{00} + 2iT_{10} - T_{11}\right) \\ T_{z\bar{z}} &= T_{\bar{z}z} = \frac{1}{4}\left(T_{00} + T_{11}\right) = \frac{1}{4}T^\mu_{\ \mu} = 0, \end{aligned} \tag{5.5.3}$$

where we used the fact that $T^\mu_{\ \mu} = 0$. Indeed, tracelessness gives

$$T_{zz} = \frac{1}{2}\left(T_{00} - iT_{10}\right), \quad T_{\bar{z}\bar{z}} = \frac{1}{2}\left(T_{00} + iT_{10}\right).$$

Now using translation invariance $\partial_\mu T^{\mu\nu} = 0$, we get

$$\partial_0 T_{00} + \partial_1 T_{10} = 0, \quad \partial_0 T_{01} + \partial_1 T_{11} = 0, \tag{5.5.4}$$

from which it follows that

$$\partial_{\bar{z}} T_{zz} = \frac{1}{4}\left(\partial_0 + i\partial_1\right)\left(T_{00} - iT_{10}\right) = \frac{1}{4}(\partial_0 T_{00} + \partial_1 T_{10} + i\partial_1 \underbrace{T_{00}}_{=-T_{11}} - i\partial_0 \underbrace{T_{10}}_{=T_{01}}) = 0,$$

where we used (5.5.4) and $T^\mu_{\ \mu} = 0$. Similarly, one can show that $\partial_z T_{\bar{z}\bar{z}} = 0$. Thus we have the following result:

Theorem 5.5.4. *The two non-vanishing components of the energy momentum tensor in two dimensions are a chiral and an anti-chiral field $T_{zz}(z,\bar{z})$ and $T_{\bar{z}\bar{z}}(z,\bar{z})$.*

5.5.2 Other Noether Currents

In this subsection, we compute the Noether current associated to other conformal transformations namely dilatations and Lorentz transformations.

Lorentz Invariance Current

Consider infinitesimal Lorentz transformations with parameters $\omega_{\mu\nu}$. The spacetime and field variations are

$$\frac{\delta x^\rho}{\delta\omega_{\mu\nu}} = \frac{1}{2}\left(\eta^{\rho\mu}x^\nu - \eta^{\rho\nu}x^\mu\right), \quad \frac{\delta\mathcal{F}}{\delta\omega_{\mu\nu}} = \frac{-i}{2}S^{\mu\nu}\Phi.$$

By (B.2.4), the associated conserved current is

$$
\begin{aligned}
j^{\mu\nu\rho} &= \left\{ \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)} \partial_\nu \Phi - \delta^\mu{}_\nu \mathcal{L} \right\} \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)} \frac{\delta \mathcal{F}}{\delta \omega_a} \\
&= T_C^{\mu\nu} x^\rho - T_C^{\mu\rho} x^\nu + i \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)} S^{\nu\rho} \Phi,
\end{aligned}
$$

where we write $T_C^{\mu\nu}$ for the *cannonical* energy momentum tensor

$$
T_C^{\mu\nu} = \frac{1}{2} \left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)} \partial_\nu \Phi - \eta^\mu{}_\nu \mathcal{L} \right),
$$

which is same as the energy momentum tensor we calculated in (5.5.1) upto a factor of half. We also note that the cannonical energy momentum tensor may not be symmetric. Also note that the current corresponding to Lorentz invariance has a nice compact expression modulo the nasty spin generator term. There is a way around to bypass both these problems, that is to make the energy momentum tensor symmetric and obtain a compact expression for the current corresponding to Lorentz invariance. Recall that we are free to add the divergence of an antisymmetric tensor in the current without affecting the conservation law. We will use this freedom. We try looking for a tensor $B^{\rho\mu\nu}$ antisymmetric in first two indices such that with the modified energy momentum tensor

$$
T_B^{\mu\nu} := T_C^{\mu\nu} + \partial_\rho B^{\rho\mu\nu}, \tag{5.5.5}
$$

the Lorentz invariance current is given by

$$
j^{\mu\nu\rho} = T_B^{\mu\nu} x^\rho - T_B^{\mu\rho} x^\nu \tag{5.5.6}
$$

Proposition 5.5.5. *Let*

$$
B^{\mu\rho\nu} = \frac{i}{2} \left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)} S^{\nu\rho} \Phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\rho \Phi\right)} S^{\mu\nu} \Phi + \frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \Phi\right)} S^{\mu\rho} \Phi \right].
$$

Then the modified energy momentum tensor $T_B^{\mu\nu}$ is symmetric and the Lorentz invariance current is given by (5.5.6). The modified energy momentum tensor $T_B^{\mu\nu}$ is called the Belinfante *energy momentum tensor.*

Proof. It is clear that $B^{\mu\rho\nu}$ is antisymmetric in the first twp indices since $S^{\mu\nu} = -S^{\nu\mu}$. Checking the form of Lorentz invariance current is straightforward computation. To see that $T_B^{\mu\nu}$ is symmetric, note that

$$
\partial_\mu j^{\mu\nu\rho} = 0 \implies T_B^{\mu\nu} \delta^\rho{}_\mu - T_B^{\mu\rho} \delta^\nu{}_\mu + x^\rho \partial_\mu T^{\mu\nu} - x^\nu \partial_\mu T^{\mu\rho} = 0.
$$

Now the symmetric property of the energy momentum tensor is immediate from the fact that it is conserved. $\square$

Dilatation Invariance Current

Consider an infinitesimal infinitesimal dilatation with parameter α

$$x'^{\mu} = (1+\alpha)x^{\mu}, \quad \mathcal{F}(\Phi) = (1-\alpha\Delta)\Phi,$$

where Δ is the scaling dimension of Φ. The variations are

$$\frac{\delta x^{\mu}}{\delta\alpha} = x^{\mu}, \quad \frac{\delta\mathcal{F}}{\delta\alpha} = -\Delta\Phi.$$

Thus the conserved current is given by

$$\begin{aligned} j^{\mu}_{D} &= \frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\Phi\right)}x^{\nu}\partial_{\nu}\Phi - \mathcal{L}x^{\mu} + \frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\Phi\right)}\Delta\Phi \\ &= T_{C\ \nu}^{\mu}x^{\nu} + \frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\Phi\right)}\Delta\Phi, \end{aligned}$$

where $T_{C}^{\mu\nu}$ is again the cannonical energy momentum tensor, which now we may assume to be symmetric. Again we have a nasty term in the current and it can again be removed by an appropriate choice of an antisymmetric tensor as we did in the previous case. This time the modified energy momentum tensor becomes traceless which we already concluded based on scale invariance. We will not describe the exact procedure here. The interested reader can look up section 4.2.2 of the Yellow book. Thus we conclude that the dilatation invariance current is given by

$$j^{\mu}_{D} = T^{\mu}_{\ \nu}x^{\nu},$$

where the energy momentum tensor is now symmetric and traceless.

Remark 5.5.6. The form of the current for scale invariance that we have concluded here involves some steps which do not go through for two dimensions. But we will assume it anyway and prove certain results which support our hypothesis.

SCT Invariance Current

An infinitesimal SCT with parameter b^{μ} is given by

$$x'^{\mu} = x^{\mu} + 2(x\cdot b)x^{\mu} - (x\cdot x)b^{\mu}, \quad \mathcal{F}(\Phi) = (1 - ib^{\mu}K_{\mu})\Phi.$$

Following similar methods, we see that the current is given by

$$j^{\mu\nu}_{K} = T_{C\ \rho}^{\mu}(2x^{\rho}x^{\nu} - \eta^{\rho\nu}x^{2}) - \frac{\partial\mathcal{L}}{\partial\left(\partial_{\mu}\Phi\right)}\frac{\delta\mathcal{F}}{\delta b^{\nu}}.$$

We can again do some manipulations and get a cannonical form for the current which we mention without further details:

$$j^{\mu\nu}_{K} = T^{\mu}_{\ \rho}(2x^{\rho}x^{\nu} - \eta^{\rho\nu}x^{2}),$$

where $T^{\mu\nu}$ is the energy momentum tensor.

5.6 Consequences of Conformal Invariance: Quantum Aspects

So far we only discussed the classical aspects of a CFT. We will now discuss the quantum consequences of a CFT. Conformal invariance puts strong restrictions on the quantum theory.

5.6.1 Correlation Functions

In quantum field theory with classical action $S[\Phi]$, we define the correlation function of n number of *fields* $\phi_1, \ldots, \phi_n$ at spacetime points $x_1, \ldots, x_n$ respectively is defined in terms of the path integral

$$\langle \phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)\rangle = \frac{\int [\mathcal{D}\Phi]\phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)\exp(-S[\Phi])}{\int [\mathcal{D}\Phi]\exp(-S[\Phi])}.$$

We can also define the correlation function of *local operators* $\mathcal{O}_1, \ldots, \mathcal{O}_n$ in a similar way:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle = \frac{\int [\mathcal{D}\Phi]\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\exp(-S[\Phi])}{\int [\mathcal{D}\Phi]\exp(-S[\Phi])}.$$

Remark 5.6.1. An important point is the following: in CFT, every local object is called a field as opposed to QFT where we call only the objects Φ appearing in the action as field. Thus Φ, $\partial_\mu\Phi, T^{\mu\nu}$ are all fields and consequently the functional integral measure $[\mathcal{D}\Phi]$ involves all possible fields in the theory.

We will compute several correlation function involving energy momentum tensor and primary fields later.

We can determine the two point correlation function of quasi primary fields exactly upto a normalisation constant using the constraints of conformal invariance. We will assume that the functional integral measure is invariant under conformal transformation[12]. Let us proceed.

Dimension $D \geq 3$

We begin by computing the two point correlation function of two quasi primary spinless fields ϕ_1, ϕ_2. By transformation rule (5.4.4) and invariance of functional integral measure, we obtain the following transformation rule for correlation function:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/D}\left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\Delta_2/D}\langle \phi_1(x_1')\phi_2(x_2')\rangle. \tag{5.6.1}$$

In particular for dilatation $x \to \lambda x$, we get

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \lambda^{\Delta_1+\Delta_2}\langle \phi_1(\lambda x_1)\phi_2(\lambda x_2)\rangle. \tag{5.6.2}$$

[12]this is a heavy assumption and may not hold in general.

Under Lorentz transformation and translation, we can easily check that the Jacobian factor in (5.6.1) is 1 and hence the correlation function remains invariant. This invariance under Lorentz transformation and translation and transformation in(5.6.2) requires that

$$\langle \phi_1 (x_1) \phi_2 (x_2) \rangle = f(|x_1 - x_2|)$$

where $f(x) = \lambda^{\Delta_1+\Delta_2} f(\lambda x)$. Only such function is given by $|x_1 - x_2|^{-\Delta_1-\Delta_2}$. Thus we have

$$\langle \phi_1 (x_1) \phi_2 (x_2) \rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}, \tag{5.6.3}$$

where C_{12} is some function. We are left to impose transformation under SCT. The Jacobian factor for SCT with parameter b^μ can easily be calculated to be

$$\left| \frac{\partial x'}{\partial x} \right| = \frac{1}{(1 - 2b \cdot x + b^2 x^2)^D} \tag{5.6.4}$$

We need to compute the transformation of the term $|x_1 - x_2|$ under SCT to impose the covariance of the correlation function under SCT. Indeed one can easily check that

$$\left| x_i' - x_j' \right| = \frac{|x_i - x_j|}{(1 - 2b \cdot x_i + b^2 x_i^2)^{1/2} \left(1 - 2b \cdot x_j + b^2 x_j^2\right)^{1/2}},$$

for any two spacetime variables x_i, x_j. The correlation function transforms as

$$\begin{aligned} \langle \phi_1 (x_1') \phi_2 (x_2') \rangle &= \frac{C_{12}}{|x_1' - x_2'|^{\Delta_1+\Delta_2}} \\ &= \frac{C_{12}(\gamma_1\gamma_2)^{(\Delta_1+\Delta_2)/2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}, \end{aligned} \tag{5.6.5}$$

where

$$\gamma_i = 1 - 2b \cdot x_i + b^2 x_i^2.$$

Thus covariance of the correlation function and using (5.6.4), (5.6.5) and (5.6.1), we get

$$\begin{aligned} \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} &= \frac{1}{\gamma_1^{D\Delta_1/D} \gamma_2^{D\Delta_2/D}} \frac{C_{12}(\gamma_1\gamma_2)^{(\Delta_1+\Delta_2)/2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \\ &= \frac{C_{12}}{\gamma_1^{\Delta_1} \gamma_2^{\Delta_2}} \frac{(\gamma_1\gamma_2)^{(\Delta_1+\Delta_2)/2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} \end{aligned}$$

This constraint is satisfied only if $\Delta_1 = \Delta_2$. Thus we conclude that

two quasi-primary fields are correlated only if they have the same scaling dimension.

The corresponding correlation function is given by

$$\langle \phi_1 (x_1) \phi_2 (x_2) \rangle = \begin{cases} \frac{C_{12}}{|x_1 - x_2|^{2\Delta_1}} & \text{if } \Delta_1 = \Delta_2 \\ 0 & \text{if } \Delta_1 \neq \Delta_2. \end{cases} \tag{5.6.6}$$

Thus we have determined the correlation function upto a normalisation constant. Similarly, we can determine the three point correlation function. We will not go through the complete details but mention the result:

$$\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{13}^{\Delta_3+\Delta_1-\Delta_2}}, \tag{5.6.7}$$

where

$$x_{ij} = |x_i - x_j|,$$

and C_{123} is again some constant. This impressive feat stops at this stage. For four point correlation function, the result has a lot of freedom and cannot be fixed only using conformal invariance.

Dimension $D = 2$

In two dimensions, we can consider the more general primary fields which transform in a nice way under local conformal transformation. Suppose $\phi_1, \ldots, \phi_n$ are primary fields with conformal dimensions $h_i, \bar{h}_i$. Under a local conformal transformation $z \longrightarrow w(z,\bar{z}),\ \bar{z} \longrightarrow \bar{w}(z,\bar{z})$, the correlation function transforms[13] as

$$\langle \phi_1(w_1,\bar{w}_1)\ldots\phi_n(w_n,\bar{w}_n)\rangle = \prod_{i=1}^{n}\left(\frac{dw}{dz}\right)_{w=w_i}^{-h_i}\left(\frac{d\bar{w}}{d\bar{z}}\right)_{\bar{w}=\bar{w}_i}^{-\bar{h}_i}\langle \phi_1(z_1,\bar{z}_1)\ldots\phi_n(z_n,\bar{z}_n)\rangle\,.$$

We can use (5.6.6) to write the two point function in two dimensions since the steps are the similar. In complex coordinates,

$$z_{ij} = |z_i - z_j| = \sqrt{z_{ij}\bar{z}_{ij}}.$$

So we have

$$\langle \phi_1(z_1,\bar{z}_1)\,\phi_2(z_2,\bar{z}_2)\rangle = \frac{C_{12}}{(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}} \quad \text{if} \quad \begin{cases} h_1 = h_2 = h \\ \bar{h}_1 = \bar{h}_2 = \bar{h} \end{cases}$$

The two point function vanishes if the conformal dimensions of the two fields are different. Similarly, we can write down the three point function. It is given by

$$\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\rangle = C_{123}\frac{1}{z_{12}^{h_1+h_2-h_3}z_{23}^{h_2+h_3-h_1}z_{13}^{h_3+h_1-h_2}}\frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3}\bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1}\bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}.$$

Again the four point function has freedom and cannot be fixed by conformal invariance alone.

[13] assuming that the functional integral measure is invariant.

5.6.2 Ward Identities

In this section, we will write the Ward identities associated to conformal invariance. Ward identities is reviewed in Appendix B.3. We recall the general form of Ward identity. For a classical continuous symmetry with generator G_a and conserved current j_a^μ, the ward identity is given by

$$\frac{\partial}{\partial x^\mu}\left\langle j_a^\mu(x)\Phi_1(x_1)\cdots\Phi_n(x_n)\right\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)\left\langle \Phi_1(x_1)\cdots G_a\Phi_i(x_i)\cdots\Phi_n(x_n)\right\rangle.$$

We will write $X \equiv \Phi_1(x_1)\cdots\Phi_n(x_n)$ to simplify notations.

Translation invariance

The current associated to translation invariance is the energy momentum tensor $T^\mu_{\ \nu}$. So the Ward identity takes the form

$$\partial_\mu\left\langle T^\mu{}_\nu X\right\rangle = -\sum_i \delta(x-x_i)\frac{\partial}{\partial x_i^\nu}\langle X\rangle. \tag{5.6.8}$$

Lorentz invariance

The current associated to Lorentz invariance is

$$j^{\mu\nu\rho} = T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu.$$

Using the generator of Lorentz transformation given in (5.4.1), the Ward identity is given by

$$\partial_\mu\left\langle\left(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu\right)X\right\rangle = \sum_i \delta(x-x_i)\left[\left(x_i^\nu\partial_i^\rho - x_i^\rho\partial_i^\nu\right)\langle X\rangle - iS_i^{\nu\rho}\langle X\rangle\right]$$

where $S_i^{\nu\rho}$ is the spin generator appropriate for the ith field of the set X. We can simplify this Ward identity by using the Ward identity (5.6.8). The left hand side becomes

$$\left\langle(\partial_\mu T^{\mu\nu})x^\rho X - (\partial_\mu T^{\mu\rho})x^\nu X + (T^{\mu\nu}\delta^\rho{}_\mu - T^{\mu\rho}\delta^\nu{}_\mu)X\right\rangle$$

Using the Ward identity (5.6.8), we get

$$\begin{aligned}
\partial_\mu\left\langle\left(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu\right)X\right\rangle &= \left\langle\left(T^{\rho\nu}-T^{\nu\rho}\right)X\right\rangle - \sum_i \delta(x-x_i)\left[\partial_i^\nu\langle x^\rho X\rangle - \partial_i^\rho\langle x^\nu X\rangle\right] \\
&= \left\langle\left(T^{\rho\nu}-T^{\nu\rho}\right)X\right\rangle - \sum_i \delta(x-x_i)\left[x_i^\rho\partial_i^\nu\langle X\rangle - x_i^\nu\partial_i^\rho\langle X\rangle\right. \\
&\qquad\qquad\left. +\delta^{\nu\rho}\langle X\rangle - \delta^{\rho\nu}\langle X\rangle\right] \\
&= \left\langle\left(T^{\rho\nu}-T^{\nu\rho}\right)X\right\rangle - \sum_i \delta(x-x_i)\left[x_i^\rho\partial_i^\nu\langle X\rangle - x_i^\nu\partial_i^\rho\langle X\rangle\right]
\end{aligned}$$

Thus the Ward identity for Lorentz invariance reduces to

$$\langle (T^{\rho\nu} - T^{\nu\rho})\, X\rangle = -i \sum_i \delta\,(x - x_i)\, S_i^{\nu\rho} \langle X\rangle. \tag{5.6.9}$$

It states that the energy momentum tensor is symmetric within correlation functions, except at the position of the other fields of the correlator.

Dilatation invariance

Using the generator $D = -ix^\nu \partial_\nu - i\Delta$ and conserved charge $j^\mu = T^\mu_{\ \nu} x^\nu$ of dilatation invariance, the Ward identity is given by

$$\partial_\mu \langle T^\mu_{\ \nu} x^\nu X\rangle = -\sum_i \delta\,(x - x_i) \left[x_i^\nu \frac{\partial}{\partial x_i^\nu} \langle X\rangle + \Delta_i \langle X\rangle \right]$$

Here again the derivative ∂_μ may act on $T^\mu{}_\nu$ and on the coordinate. By similar manipulations as above, we get the Ward identity corresponding to dilatation invariance

$$\left\langle T^\mu_{\ \mu} X\right\rangle = -\sum_i \delta\,(x - x_i)\, \Delta_i \langle X\rangle, \tag{5.6.10}$$

where Δ_i is the scaling dimension of Φ_i. This Ward identity says that the energy momentum tensor is traceless within the correlator, except at the position of the other fields of the correlator.

5.6.3 Ward Identity in Two Dimensions

We wish to write the Ward identities corresponding to conformal invariance in complex coordinates. We observe that the spin generator $S_{\mu\nu}$ in two dimensions acts on a primary field ϕ as a multiple of the antisymmetric tensor $\varepsilon_{\mu\nu}$

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2}i \\ -\frac{1}{2}i & 0 \end{pmatrix}, \quad \epsilon^{\mu\nu} = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix}.$$

Thus the Ward identities take the form

$$\begin{aligned} \frac{\partial}{\partial x^\mu} \langle T^\mu_{\ \nu}(x) X\rangle &= -\sum_{i=1}^n \delta\,(x - x_i)\, \frac{\partial}{\partial x_i^\nu} \langle X\rangle \\ \epsilon_{\mu\nu} \langle T^{\mu\nu}(x) X\rangle &= -i \sum_{i=1}^n s_i \delta\,(x - x_i)\, \langle X\rangle \\ \left\langle T^\mu_{\ \mu}(x) X\right\rangle &= -\sum_{i=1}^n \delta\,(x - x_i)\, \Delta_i \langle X\rangle, \end{aligned} \tag{5.6.11}$$

where X denotes a collection of n primary fields and s_i is the spin of the ith field. To convert these into complex coordinates, we need the following lemma.

Lemma 5.6.2. *In two dimensions, we have for $x = (z, \bar{z})$*

$$\delta(x) = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z} = \frac{1}{\pi}\partial_z\frac{1}{\bar{z}}. \tag{5.6.12}$$

Proof. We will prove the first relation and the second is similar. We will show that for any holomorphic function $f(z)$ in a neighbourhood M of 0,

$$\frac{1}{\pi}\int_M d^2x f(z)\partial_{\bar{z}}\frac{1}{z} = f(0).$$

To do this, we need a version of Gauss's theorem in complex coordinates. For a vector field F^μ, Gauss theorem gives

$$\int_M d^2x \partial_\mu F^\mu = \int_{\partial M} d\xi_\mu F^\mu,$$

where $d\xi_\mu$ is an outward directed differential of circumference, orthogonal to the boundary ∂M of the domain of integration. We can raise the index of the differential $d\xi_\mu$ using $\epsilon_{\mu\nu}$ which amounts to using a counterclockwise orientation on ∂M. We have $d\xi_\mu = \epsilon_{\mu\rho}ds^\rho$ where ds^ρ is $(dz, d\bar{z})$. Thus Gauss's theorem becomes

$$\begin{aligned}\int_M d^2x\partial_\mu F^\mu &= \int_{\partial M}\{dz\epsilon_{\bar{z}z}F^{\bar{z}} + d\bar{z}\epsilon_{z\bar{z}}F^z\} \\ &= \frac{1}{2}i\int_{\partial M}\{-dzF^{\bar{z}} + d\bar{z}F^z\}\end{aligned} \tag{5.6.13}$$

Here the contour ∂M circles counterclockwise. Taking $F^\mu = (0, f(z)/z)$, we get

$$\begin{aligned}\frac{1}{\pi}\int_M d^2x f(z)\partial_{\bar{z}}\frac{1}{z} &= \frac{1}{\pi}\int_M d^2x\partial_{\bar{z}}\left(\frac{f(z)}{z}\right) \\ &= \frac{1}{2\pi i}\int_{\partial M} dz\frac{f(z)}{z} \\ &= f(0),\end{aligned}$$

where we used the fact that $\partial_{\bar{z}}f(z) = 0$. □

Substituting the delta function in (5.6.11) using (5.6.12), we easily see that the Ward identities take the form

$$
\begin{aligned}
2\pi\partial_z \langle T_{\bar{z}z} X\rangle + 2\pi\partial_{\bar{z}} \langle T_{zz} X\rangle &= -\sum_{i=1}^{n} \partial_{\bar{z}} \frac{1}{z-w_i} \partial_{w_i} \langle X\rangle \\
2\pi\partial_z \langle T_{\bar{z}\bar{z}} X\rangle + 2\pi\partial_{\bar{z}} \langle T_{z\bar{z}} X\rangle &= -\sum_{i=1}^{n} \partial_{z} \frac{1}{\bar{z}-\bar{w}_i} \partial_{\bar{w}_i} \langle X\rangle \\
2\langle T_{z\bar{z}} X\rangle + 2\langle T_{\bar{z}z} X\rangle &= -\sum_{i=1}^{n} \delta\left(x-x_i\right) \Delta_i \langle X\rangle \\
-2\langle T_{z\bar{z}} X\rangle + 2\langle T_{\bar{z}z} X\rangle &= -\sum_{i=1}^{n} \delta\left(x-x_i\right) s_i \langle X\rangle
\end{aligned}
\tag{5.6.14}
$$

Adding and subtracting the last two equations of (5.6.14) and using (5.6.12) gives two new equations

$$
\begin{aligned}
2\pi \langle T_{\bar{z}z} X\rangle &= -\sum_{i=1}^{n} \partial_{\bar{z}} \frac{1}{z-w_i} h_i \langle X\rangle \\
2\pi \langle T_{z\bar{z}} X\rangle &= -\sum_{i=1}^{n} \partial_{z} \frac{1}{\bar{z}-\bar{w}_i} \bar{h}_i \langle X\rangle
\end{aligned}
\tag{5.6.15}
$$

where we used the fact that for primary fields $h_i = (\Delta_i + s_i)/2$ and $\bar{h}_i = (\Delta_i - s_i)/2$. Inserting these relations into the first two equations of (5.6.14), we get

$$
\begin{aligned}
\partial_{\bar{z}} \left[\langle T(z,\bar{z}) X\rangle - \sum_{i=1}^{n} \left(\frac{1}{z-w_i} \partial_{w_i} \langle X\rangle + \frac{h_i}{\left(z-w_i\right)^2} \langle X\rangle \right) \right] &= 0 \\
\partial_{z} \left[\langle \bar{T}(z,\bar{z}) X\rangle - \sum_{i=1}^{n} \left(\frac{1}{\bar{z}-\bar{w}_i} \partial_{\bar{w}_i} \langle X\rangle + \frac{\bar{h}_i}{\left(\bar{z}-\bar{w}_i\right)^2} \langle X\rangle \right) \right] &= 0
\end{aligned}
$$

where we have defined

$$
T(z,\bar{z}) = -2\pi T_{zz}(z,\bar{z}), \quad \bar{T}(z,\bar{z}) = -2\pi T_{\bar{z}\bar{z}}(z,\bar{z}). \tag{5.6.16}
$$

This says that the expression in the square bracket in the above equation is holomorphic and antiholomorphic respectively. In particular T and $\bar{T}$ are functions of z and $\bar{z}$ respectively. Thus we may write

$$
\langle T(z) X\rangle = \sum_{i=1}^{n} \left\{ \frac{1}{z-w_i} \partial_{w_i} \langle X\rangle + \frac{h_i}{\left(z-w_i\right)^2} \langle X\rangle \right\} + \text{reg.}, \tag{5.6.17}
$$

where "reg." stands for a holomorphic function of z, regular at $z = w_i$. A similar expression holds for the antiholomorphic part.

Conformal Ward Identity

We want to write the Ward identity as variation of the path integral due to infinitesimal conformal transformation, so that all the three Ward identity can be combined in a single Ward identity called the conformal Ward identity.

Theorem 5.6.3. *Let $x^\mu \longrightarrow x'^\mu = x^\mu + \varepsilon^\mu(x)$ be an infinitesimal conformal transformation. Then we have*

$$\delta_{\varepsilon,\bar{\varepsilon}}\langle X\rangle = -\frac{1}{2\pi i}\oint_C dz\varepsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_C d\bar{z}\bar{\varepsilon}(\bar{z})\langle \bar{T}(\bar{z})X\rangle,$$

where C is a closed curve in the complex plane containing the positions of all the fields in X.

Proof. We have

$$\begin{aligned}\partial_\mu\left(\varepsilon_\nu T^{\mu\nu}\right) &= \varepsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}\left(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu\right)T^{\mu\nu} + \frac{1}{2}\left(\partial_\mu\varepsilon_\nu - \partial_\nu\varepsilon_\mu\right)T^{\mu\nu}\\ &= \varepsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}\left(\partial_\rho\varepsilon^\rho\right)\eta_{\mu\nu}T^{\mu\nu} + \frac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha\varepsilon_\beta\epsilon_{\mu\nu}T^{\mu\nu}\end{aligned} \tag{5.6.18}$$

where we used

$$\begin{aligned}\frac{1}{2}\left(\partial_\mu\varepsilon_\nu + \partial_\nu\varepsilon_\mu\right) &= \frac{1}{2}\left(\partial_\rho\varepsilon^\rho\right)\eta_{\mu\nu}\\ \frac{1}{2}\left(\partial_\mu\varepsilon_\nu - \partial_\nu\varepsilon_\mu\right) &= \frac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha\varepsilon_\beta\epsilon_{\mu\nu}\end{aligned} \tag{5.6.19}$$

First equation is same as (5.1.5) for $D = 2$, the second equation can be verified component wise. Now under a general infinitesimal transformation with parameters ω_a and generator $G_a^{(i)}$ in the representation Φ_i,

$$\begin{aligned}x'^\mu &= x^\mu + \omega_a\frac{\delta x^\mu}{\delta\omega_a}\\ \Phi_i'(x) &= \Phi_i(x) - i\omega_a G_a^{(i)}\Phi_i(x).\end{aligned}$$

Thus we have

$$\delta_\omega X = -i\sum_{i=1}^{n}\Big(\Phi_1\left(x_1\right)\cdots G_a^{(i)}\Phi_i\left(x_i\right)\cdots\Phi_n\left(x_n\right)\Big)\omega_a\left(x_i\right).$$

This implies that

$$\delta_\omega\langle X\rangle = -i\omega_a\sum_{i=1}^{n}\Big\langle\Phi_1\left(x_1\right)\cdots G_a^{(i)}\Phi_i\left(x_i\right)\cdots\Phi_n\left(x_n\right)\Big\rangle. \tag{5.6.20}$$

We now apply this formula to infinitesimal conformal transformation. We have seen that a general infinitesimal conformal transformation (without SCT) has the form given by (5.2.1)

$$\varepsilon_\mu = a_\mu + b_{\mu\nu} x^\nu,$$

where the symmetric and antisymmetric part of $b_{\mu\nu}$ parametrises dilatation and Lorentz transformation. The generators of translation, dilatation and Lorentz transformation in representation Φ_i is $-i\partial_\mu, -i\Delta_i$ and $s_i \epsilon_{\mu\nu}$ where Δ_i and s_i is the scaling dimension and spin of the field Φ_i. Using (5.6.20) and noting that the symmetric and antisymmetric part of $b_{\mu\nu}$ is precisely the left hand side of (5.6.19), we have

$$\begin{aligned}\delta_\varepsilon \langle X\rangle = -\sum_{i=1}^{n} & \left[\varepsilon^\nu(x_i) \left\langle \Phi_1(x_1) \cdots \partial_\nu \Phi_i(x_i) \cdots \Phi_n(x_n)\right\rangle\right. \\ & + \alpha \left\langle \Phi_1(x_1) \cdots \Delta_i \Phi_i(x_i) \cdots \Phi_n(x_n)\right\rangle \\ & \left. + i\omega_{\mu\nu} \left\langle \Phi_1(x_1) \cdots s_i \epsilon^{\mu\nu} \Phi_i(x_i) \cdots \Phi_n(x_n)\right\rangle\right],\end{aligned} \tag{5.6.21}$$

where

$$\alpha = \frac{1}{2}\partial \cdot \varepsilon, \quad \omega_{\mu\nu} = \frac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha \varepsilon_\beta \epsilon_{\mu\nu}.$$

Next, using (5.6.18) we have

$$\int_M d^2x \partial_\mu \left\langle T^{\mu\nu}(x)\varepsilon_\nu(x) X\right\rangle = \int_M d^2x \varepsilon_\nu(x)\partial_\mu \langle T^{\mu\nu} X\rangle + \alpha(x)\langle T^\mu{}_\mu X\rangle + \omega_{\mu\nu}(x)\langle T^{\mu\nu} X\rangle,$$

where M is a domain containing the positions of all the fields in the string X. We now use the Ward identities (5.6.11), we get

$$\begin{aligned}\int_M d^2x \partial_\mu \left\langle T^{\mu\nu}(x)\varepsilon_\nu(x) X\right\rangle = -\sum_{i=1}^{n}\int_M d^2x \delta(x-x_i) & \left[\varepsilon_\nu(x) \left\langle \Phi_1(x_1) \cdots \partial^\nu \Phi_i(x_i) \cdots \Phi_n(x_n)\right\rangle\right. \\ & + \alpha \left\langle \Phi_1(x_1) \cdots \Delta_i \Phi_i(x_i) \cdots \Phi_n(x_n)\right\rangle \\ & \left. + i\omega_{\mu\nu} \left\langle \Phi_1(x_1) \cdots s_i \epsilon^{\mu\nu} \Phi_i(x_i) \cdots \Phi_n(x_n)\right\rangle\right].\end{aligned}$$

Using (5.6.21), we conclude that

$$\delta_\varepsilon \langle X\rangle = \int_M d^2x \partial_\mu \left\langle T^{\mu\nu}(x)\epsilon_\nu(x) X\right\rangle.$$

Using (5.6.13) for $F^\mu = \langle T^{\mu\nu}(x)\varepsilon_\nu(x) X\rangle$, we obtain

$$\delta_{\varepsilon,\bar{\varepsilon}}\langle X\rangle = \frac{1}{2} i \int_C \left[-dz \left\langle (T^{z\bar{z}} + T^{\bar{z}\bar{z}})\varepsilon_{\bar{z}} X\right\rangle + d\bar{z} \left\langle (T^{zz} + T^{\bar{z}z})\varepsilon_z X\right\rangle\right],$$

where $\varepsilon = \epsilon^z$ and $\bar{\varepsilon} = \varepsilon^{\bar{z}}$ and the contour C is the boundary curve of M. Now by (5.5.3), we see that

$$\langle T^{z\bar{z}} X\rangle = \frac{1}{4}\left\langle T^\mu{}_\mu(x) X\right\rangle = \langle T^{\bar{z}z} X\rangle = \frac{1}{4}\left\langle T^\mu{}_\mu(x) X\right\rangle,$$

which by the Ward identity (5.6.11) vanished if x is different from all of the positions of the fields in X. Since C goes around those positions, we obtain

$$\delta_{\varepsilon,\bar{\varepsilon}}\langle X\rangle = \frac{1}{2}i\int_C \left[-dz\,\langle T^{\bar{z}\bar{z}}\varepsilon_{\bar{z}}X\rangle + d\bar{z}\,\langle T^{zz}\varepsilon_z X\rangle\right].$$

Substituting the definition (5.6.16), we obtain the conformal Ward identity:

$$\delta_{\varepsilon,\bar{\varepsilon}}\langle X\rangle = -\frac{1}{2\pi i}\oint_C dz\varepsilon(z)\langle T(z)X\rangle + \frac{1}{2\pi i}\oint_C d\bar{z}\bar{\varepsilon}(\bar{z})\langle \bar{T}(\bar{z})X\rangle.$$

□

5.6.4 Operator Product Expansion and Primary Operators in 2d CFT

When the position of two local operators in a correlator approaches each other, the correlation function diverges. This divergence is typical in quantum field theories and reflects the infinite fluctuations of quantum fields when "measured" at a precise position. An operator product expansion (OPE) exactly captures this feature. We define OPE precisely now.

Definition 5.6.4. Suppose $\mathcal{O}_k$ be the local operators in a CFT. For any two local operators $\mathcal{O}_i(z,\bar{z})$ and $\mathcal{O}_j(w,\bar{w})$, an OPE of $\mathcal{O}$ and $\mathcal{O}'$ is a relation of the form:

$$\mathcal{O}(z,\bar{z})\mathcal{O}'(w,\bar{w}) = \sum_k C_{ij}^k(z-w,\bar{z}-\bar{w})\mathcal{O}_k(w,\bar{w}),$$

where $C_{ij}^k(z-w,\bar{z}-\bar{w})$ are functions of $z-w,\bar{z}-\bar{w}$ which diverge as $z\to w$.

Some remarks are in order.

Remark 5.6.5. (i) OPEs are always understood to be operator to be substituted in a time ordered correlation function:

$$\langle \mathcal{O}_i(z,\bar{z})\mathcal{O}_j(w,\bar{w})X\rangle = \sum_k C_{ij}^k(z-w,\bar{z}-\bar{w})\,\langle \mathcal{O}_k(w,\bar{w})X\rangle\,,$$

where X is an string of local operators.

(ii) The string of operators X above is arbitrary their position must be distinct from the positions of $\mathcal{O}_i,\mathcal{O}_j$.

(iii) OPEs have singular behaviour as $z\to w$, which is all we care about. So in many case we write an OPE of operators $A(z)$ and $B(w)$ as

$$A(z)B(w) \sim \sum_{n=1}^{N}\frac{(AB)(w)}{(z-w)^n},$$

where (AB), called the composite field of A and B, are non singular at $z=w$ and $\sim$ indicates that the above relation is true modulo nonsingular terms. Thus every OPE has infinite number of nonsingular extra terms which we don't bother writing.

As an example, observe that in (5.6.17), we proved that for a primary field $\phi(w,\bar{w})$ with conformal dimensions $(h,\bar{h})$ we have

$$\begin{aligned} T(z)\phi(w,\bar{w}) &\sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}) \\ \bar{T}(\bar{z})\phi(w,\bar{w}) &\sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w}). \end{aligned}$$

This OPE is characteristic of primary fields. Thus we may define primary operators alternatively by their OPE with energy momentum tensor.

Definition 5.6.6. A field $\phi(z,\bar{z})$ is called primary with conformal dimensions $(h,\bar{h})$, if the operator product expansion between the energy momentum tensors and $\phi(z,\bar{z})$ takes the following form:

$$\begin{aligned} T(z)\phi(w,\bar{w}) &\sim \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{z-w}\partial_w\phi(w,\bar{w}), \\ \bar{T}(\bar{z})\phi(w,\bar{w}) &\sim \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\phi(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w,\bar{w}). \end{aligned}$$

We will now discuss OPEs and primary operators with the help of an example.

Example: Free Scalar Field

Consider a massless scalar field $X(\sigma)$ where σ covers a 2 dimensional manifold. The action is given by

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\ \partial_\alpha X \partial^\alpha X.$$

The classical equation of motion for this action can easily be computed. It is given by

$$\partial^2 X = 0.$$

To find the quantum consequences, we can use Ehrenfest's theorem which states that the expectation value of operators satisfy the classical equations of motion. We will derive this explicitly. To do this we need the following lemma.

Lemma 5.6.7. *Let $F[\phi]$ be a functional of field $\phi^\alpha(x)$ which vanishes on the boundary. Then the following holds*

$$\int [\mathcal{D}\phi]\frac{\delta F[\phi]}{\delta\phi^\alpha(x)} = 0.$$

Proof. Suppose x varies over a manifold M and $\alpha \in J$ where J is an index set. The variation of $F[\phi]$ is given as

$$\delta F[\phi] := F[\phi+\delta\phi] - F[\phi] = \int_M dx \sum_{\alpha\in J}\frac{\delta F[\phi]}{\delta\phi^\alpha(x)}\delta\phi^\alpha(x).$$

We now have to construct the functional integral measure appropriately. We do this by discretising space and hen taking the continuum limit. We now use DeWitt's notation to discretise the spacetime M. Put

$$i = (\alpha, x) \in I := J \times M, \quad \phi^i := \phi^\alpha(x), \quad i \in I.$$

To discretize spacetime M, we run i over a finite set I so that we now only have finitely many variables ϕ^i, $i \in I$, in the theory. The functional derivative becomes a partial derivative

$$\frac{\partial F[\phi]}{\partial \phi^i}.$$

We thus have

$$\delta F := F[\phi + \delta\phi] - F[\phi] = \sum_{i \in I} \frac{\partial F[\phi]}{\partial \phi^i} \delta\phi^i,$$

and the functional integral measure is simply the product of finitely many measures:

$$\left[\prod_{j \in I} \int d\phi^j\right] \xrightarrow{\text{continuum limit}} \int [\mathcal{D}\phi].$$

This gives

$$\left[\prod_{j \in I} \int d\phi^j\right] \frac{\partial F[\phi]}{\partial \phi^i} \xrightarrow{\text{continuum limit}} \int [\mathcal{D}\phi] \frac{\delta F[\phi]}{\delta \phi^\alpha(x)}.$$

The left hand side in above equation is zero on account of the integral of a total derivative and the boundary condition satisfied by F and the proof is complete. □

Using Lemma 5.6.7, we get

$$0 = \int [\mathcal{D}X] \frac{\delta e^{-S[X]}}{\delta X(\sigma)} = \int [\mathcal{D}X] e^{-S[X]} \left[\frac{1}{2\pi\alpha'} \partial^2 X(\sigma)\right].$$

Thus we get

$$\left\langle \partial^2 X(\sigma) \right\rangle = 0,$$

which is Ehrenfest's theorem.

The Propagator. We now want to compute the propagator for X. We again use path integral for this. Recall that the propagator in position space is the correlation function $\langle X(\sigma) X(\sigma') \rangle$ which is given by path integral

$$\langle X(\sigma) X(\sigma') \rangle = \frac{1}{Z} \int [\mathcal{D}X] X(\sigma) X(\sigma') e^{-S[X]},$$

where Z is the partition function of the theory given by

$$Z = \int [\mathcal{D}X] e^{-S[X]}.$$

Proposition 5.6.8. *The two point correlation function $\langle X(\sigma)X(\sigma')\rangle$ of the massless scalar field is*

$$\langle X(\sigma)X(\sigma')\rangle = -\frac{\alpha'}{2}\ln(\sigma-\sigma')^2 + const. \tag{5.6.22}$$

Proof. Using Lemma 5.6.7, we see that

$$0 = \int [\mathcal{D}X]\frac{\delta e^{-S[X]}X(\sigma')}{\delta X(\sigma)} = \int [\mathcal{D}X]e^{-S[X]}\left[\frac{1}{2\pi\alpha'}\partial_\sigma^2 X(\sigma)X(\sigma') + \delta(\sigma-\sigma')\right]$$

Dividing throughout by the partition function, we get

$$\left\langle \partial_\sigma^2 X(\sigma)X(\sigma')\right\rangle = -2\pi\alpha'\delta(\sigma-\sigma')$$

So we find that the propagator satisfies the differential equation

$$\partial_\sigma^2\langle X(\sigma)X(\sigma')\rangle = -2\pi\alpha'\delta(\sigma-\sigma').$$

We now solve this differential equation. Since this correlator has to be translation and rotation invariant, thus it should only depend on the norm of separation i.e. $|\sigma-\sigma'|$. Put

$$r = |\sigma-\sigma'| \quad \text{and} \quad K(r) = \langle X(\sigma)X(\sigma')\rangle,$$

then the differential equation (5.6.23) in polar coordinates (r,θ) becomes

$$\begin{aligned}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial K(r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 K(r)}{\partial\theta^2} &= -2\pi\alpha'\delta(\sigma-\sigma')\\ \implies \frac{1}{r}\frac{d}{dr}\left(r\frac{dK(r)}{dr}\right) &= -2\pi\alpha'\delta(\sigma-\sigma').\end{aligned} \tag{5.6.23}$$

Let $\mathbb{D}_r$ be a disc of radius r centred at σ'. We now integrate both sides of (5.6.23) on $\mathbb{D}_r$ with respect to σ, we get

$$\begin{aligned}\int_0^{2\pi} d\theta \int_0^r \rho d\rho \frac{1}{r}\frac{d}{d\rho}\left(\rho\frac{dK(\rho)}{d\rho}\right) &= -2\pi\alpha'\int_{\mathbb{D}_r} d^2\sigma\delta(\sigma-\sigma')\\ \implies 2\pi r\frac{dK(r)}{dr} &= -2\pi\alpha'\\ \implies K(r) &= -\alpha'\ln r + \text{const.}\end{aligned}$$

Thus we conclude that

$$\langle X(\sigma)X(\sigma')\rangle = -\frac{\alpha'}{2}\ln(\sigma-\sigma')^2 + \text{const.}$$

□

Note that the correlator has a divergence as $\sigma \to \sigma'$. This is a common feature of all quantum theories as explained in Subsection 5.6.4. In complex coordinates, the propagator looks as

$$\langle X(z,\bar{z})X(w,\bar{w})\rangle = -\frac{\alpha'}{2}[\ln(z-w)+\ln(\bar{z}-\bar{w})] + \text{ const.}$$

Operator Product Expansions. Taking partial derivatives of (5.6.22), we obtain the correlation function of the derivatives of X. Explicitly, we get

$$\begin{aligned} \langle \partial_z X(z,\bar{z})\partial_w X(w,\bar{w})\rangle &= -\frac{\alpha'}{2}\frac{1}{(z-w)^2} + \text{reg.}, \\ \langle \partial_{\bar{z}} X(z,\bar{z})\partial_{\bar{w}} X(w,\bar{w})\rangle &= -\frac{\alpha'}{2}\frac{1}{(\bar{z}-\bar{w})^2} + \text{reg.} \end{aligned} \tag{5.6.24}$$

Note that the classical equations of motion in complex coordinates is given by

$$\partial_z\partial_{\bar{z}}X(z,\bar{z}) = 0,$$

which enables us to write $X(z,\bar{z})$ as a sum of a holomorphic (left moving mode) and an antiholomorphic (right moving mode) function:

$$X(z,\bar{z}) = X(z) + \bar{X}(\bar{z}).$$

In the following we shall only consider the holomorphic field $X(z)$. We have already proved that the OPE of the field $\partial X \equiv \partial_z X(z)$ with itself is

$$\partial X(z)\partial X(w) \sim -\frac{\alpha'}{2}\frac{1}{(z-w)^2}.$$

Note that exchanging the two factors does not affect the correlator which is a characteristic of Bosonic fields. To know if these fields are primary or not, we need the energy momentum tensor. The energy momentum tensor associated with the free massless scalar field is

$$T_{\mu\nu} = \frac{1}{2\pi\alpha'}\left(\partial_\mu X\partial_\nu X - \frac{1}{2}\eta_{\mu\nu}\partial_\rho X\partial^\rho X\right).$$

In the notation of (5.6.16), in complex coordinates the energy momentum tensor is given by

$$T(z) = -\frac{1}{\alpha'}\partial X\partial X.$$

Similarly we can also calculate $\bar{T}(\bar{z})$.

Care must be taken when interpreting the energy momentum tensor as a quantum operator since it involves product of operators. In cannonical quantisation, we could have normal ordered the above expression by putting annihilation operator to the right of creation operators so that the vacuum expectation value vanishes. Here also we do the same but without referring to creation and annihilation operators. More explicitly, the exact meaning of the above expression is

$$T(z) = -\frac{1}{\alpha'} : \partial X \partial X : \equiv -\frac{1}{\alpha'} \lim_{w \to z} (\partial X(z) \partial X(w) - \langle \partial X(z) \partial X(w) \rangle).$$

We now compute the OPE of ∂X with $T(z)$.

Proposition 5.6.9. *The field ∂X is a primary field with conformal dimension $(h, \bar{h}) = (1, 0)$.*

Proof. We need to compute the correlation function

$$\langle T(z) \partial X(w) \rangle \equiv \langle 0 | \mathcal{T}(T(z) \partial X(w)) | 0 \rangle,$$

where $\mathcal{T}$ is the time ordering operator. We can use Wick's theorem to compute this time ordered product. Recall that by Wick's theorem, the time ordered product $\mathcal{T}(\phi_1 \dots \phi_n)$ of n fields is the normal ordered product $: \phi_1 \dots \phi_n :$ plus all possible contractions where a contraction of a pair of fields means that we replace the pair by the correlation function of the pair. So we obtain

$$\begin{aligned} \mathcal{T}(T(z) \partial X(w)) = -\frac{1}{\alpha'} \Big(&: \partial X(z) \partial X(z) \partial X(w) : + : \partial X(z) \partial \overbracket{X(z) : \partial X}(w) \\ &+ : \partial \overbracket{X(z) \partial X(z) : \partial X}(w) \Big), \end{aligned}$$

where the square bracket indicates contraction. Thus we see that

$$\begin{aligned} \langle T(z) \partial X(w) \rangle &= -\frac{1}{\alpha'} \left(\langle 0 | : \partial X(z) \partial X(z) \partial X(w) : | 0 \rangle + 2 \langle 0 | \partial X(z) | 0 \rangle \langle \partial X(z) \partial X(w) \rangle \right) \\ &= \frac{\langle \partial X(z) \rangle}{(z-w)^2} + \text{reg.} \end{aligned}$$

where we used (5.6.24). Note that the reg. term contains the vacuum expectation value of the normal ordered product $: T(z) \partial X(w) :$. Thus in standard form, the OPE has the form

$$T(z) \partial X(w) \sim \frac{\partial X(z)}{(z-w)^2}.$$

We can expand $\partial X(z)$ around $z = w$:

$$\partial X(z) = \partial X(w) + \partial_w^2 X(w)(z-w) + O((z-w)^2).$$

This gives

$$T(z) \partial X(w) \sim \frac{\partial X(w)}{(z-w)^2} + \frac{\partial_w^2 X(w)}{(z-w)}. \tag{5.6.25}$$

Similarly we can calculate the OPE of ∂X with $\bar{T}(\bar{z})$. Thus according to Definition 5.6.6, ∂X is a primary operator of conformal weight $(h, \bar{h}) = (1, 0)$. □

Corollary 5.6.10. *Higher order derivatives $\partial^n X$, $n > 1$ of the field X are not primary operators.*

Proof. Using (5.6.25), we see that

$$T(z)\partial^2 X(w) \sim \partial_w \left[\frac{\partial X(w)}{(z-w)^2} + \dots\right] \sim \frac{2\partial X(w)}{(z-w)^3} + \frac{2\partial_w^2 X(w)}{(z-w)^2}.$$

□

Corollary 5.6.11. *The field $:e^{ikX}:$ is a primary field with conformal dimensions $h = \bar{h} = \alpha' k^2/4$.*

Proof. We have

$$\begin{aligned}
:\partial X(z)\partial X(z)::e^{ikX(w)}: &= \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} :\partial X(z)\partial X(z)::\underbrace{X(w)X(w)\cdots X(w)}_{n\text{ terms}}: \\
&= \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} :\partial X(z)\partial X(z):X(w)\cdots X(w)\cdots X(w)\cdots X(w): \\
&\qquad + :\partial X(z)\partial X(z)::X(w)\cdots X(w)\cdots X(w): \\
&\qquad + :\partial X(z)\partial X(z)::X(w)\cdots X(w)\cdots X(w): \\
&= \sum_{n=0}^{\infty} \frac{(ik)^n}{n!}\left(-\frac{n(n-1)\alpha'^2}{4(z-w)^2}\right):X^{n-2}(w): + \frac{2n\alpha'}{z-w}:\partial X(z)X^{n-1}(w): + \text{reg.} \\
&= -\frac{k^2\alpha'}{4}\frac{:e^{ikX(w)}:}{(z-w)^2} - \frac{ik\alpha'}{z-w}\sum_{n=1}^{\infty}\frac{(ik)^{n-1}}{(n-1)!}:\partial X(z)X^{n-1}(w): + \text{reg.} \\
&= -\frac{k^2\alpha'^2}{4}\frac{:e^{ikX(w)}:}{(z-w)^2} - \frac{ik\alpha'^2 :\partial X(z)e^{ikX(w)}:}{z-w} + \text{reg.},
\end{aligned}$$

where we performed $n(n-1)$ contractions in first term and $2n$ contractions in the second and third term in the second step. Now observe that in $z \to w$

$$\frac{\partial_z X(z):e^{ikX(w)}:}{z-w} - \frac{\partial_w X(w):e^{ikX(w)}:}{z-w} \sim \text{regular}.$$

Thus we can add and subtract $\frac{\partial_w X(w):e^{ikX(w)}:}{z-w}$ in the last step of the calculation of $T(z):e^{ikX(w)}:$ to get

$$\begin{aligned}
T(z):e^{ikX(w)}: &= -\frac{1}{\alpha'}:\partial X(z)\partial X(z)::e^{ikX(w)}: \\
&= \frac{k^2\alpha'}{4}\frac{:e^{ikX(w)}:}{(z-w)^2} + \frac{ik:\partial_w X(w)e^{ikX(w)}:}{z-w} + \text{reg.} \\
&= \frac{k^2\alpha'}{4}\frac{:e^{ikX(w)}:}{(z-w)^2} + \frac{:\partial_w e^{ikX(w)}:}{z-w} + \text{reg.}
\end{aligned}$$

$\square$

Remark 5.6.12. The fields $V_k(z) =: e^{ikX(z)} :$ are called *vertex operators* in CFT and we will meet them again while discussing the application of CFT in string theory. Corollary 5.6.11 also shows that the spectrum of the free boson CFT is continuous.

Lastly we check whether the energy momentum tensor is a primary operator or not.

Proposition 5.6.13. *The energy momentum tensor $T(z)$ is not a primary operator.*

Proof. Again using Wick's theorem, we have

$$
\begin{aligned}
T(z)T(w) &= \frac{1}{\alpha'^2} : \partial X(z)\partial X(z) :: \partial X(w)\partial X(w) : \\
&= \frac{1}{\alpha'^2}\Big(: \partial X(z)\partial X(z) :: \partial X(w)\partial X(w) : +2 : \partial X(z)\partial X(z) :: \partial X(w)\partial X(w) : \\
&\qquad\qquad + 4 : \partial X(z)\partial X(z) :: \partial X(w)\partial X(w) : \Big) \\
&= \frac{2}{\alpha'^2}\left(-\frac{\alpha'}{2}\frac{1}{(z-w)^2}\right)^2 - \frac{4}{\alpha^2}\frac{\alpha'}{2}\frac{: \partial X(z)\partial X(w) :}{(z-w)^2} + \text{reg.}
\end{aligned}
$$

where we used (5.6.24). Here again the reg. term includes the first normal ordered product. Again substituting $\partial X(z) = \partial X(w) + \partial_w^2 X(w)(z-w) + O((z-w)^2)$, we obtain

$$
\begin{aligned}
T(z)T(w) &= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} - \frac{2}{\alpha'}\frac{\partial_w^2 X(w)\partial X(w)}{z-w} + \text{reg.} \\
&= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}
\end{aligned}
\tag{5.6.26}
$$

Thus the OPE obviates the fact that $T(z)$ is not a primary operator. $\square$

Example: Free Fermionic System

Consider a free Majorana fermion in two dimensions with Euclidean metric. The action is given by

$$
S = \frac{g}{2}\int d^2x \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi, \tag{5.6.27}
$$

where the Dirac matrices satisfy the Clifford algebra

$$
\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}.
$$

With $\eta^{\mu\nu} = \text{diag}(1,1)$, one choice of the Dirac matrices could be

$$
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

With this choice, the action can be simplified and we get

$$S = g\int d^2x(\overline{\psi}\partial\overline{\psi} + \psi\bar{\partial}\psi),$$

where we have written the two component Majorana fermion as $\Psi = (\psi, \overline{\psi})$. The equation of motions are

$$\partial\overline{\psi} = 0, \quad \bar{\partial}\psi = 0,$$

whose solutions are any holomorphic function $\psi(z)$ and any antiholomorphic function $\overline{\psi}(\bar{z})$.

The Propagator. We now calculate the propagator $\langle\Psi_i(\boldsymbol{x})\Psi_j(\boldsymbol{y})\rangle$ for $i, j = 1, 2$. As usual, the first step is to express the action in the form:

$$S = \frac{1}{2}\int d^2\boldsymbol{x}d^2\boldsymbol{y}\Psi_i(\boldsymbol{x})A_{ij}(\boldsymbol{x},\boldsymbol{y})\Psi_j(\boldsymbol{y}).$$

From the action in (5.6.27), we can identify A_{ij} with

$$A_{ij}(\boldsymbol{x},\boldsymbol{y}) = g\delta(\boldsymbol{x}-\boldsymbol{y})(\gamma^0\gamma^\mu)_{ij}\partial_\mu.$$

The propagator is then the inverse of A_{ij}:

$$K_{ij}(\boldsymbol{x},\boldsymbol{y}) \equiv \langle\Psi_i(\boldsymbol{x})\Psi_j(\boldsymbol{y})\rangle = (A^{-1})_{ij}(\boldsymbol{x},\boldsymbol{y}).$$

From the Gaussian integral of Grassmann variables, it is known that K_{ij} satisfies the differential equation:

$$g\delta(\boldsymbol{x}-\boldsymbol{y})(\gamma^0\gamma^\mu)_{i\ell}\partial_\mu K_{\ell j}(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y})\delta_{ij}. \tag{5.6.28}$$

We now return back to the notation $\boldsymbol{x} \to (z,\bar{z}), \boldsymbol{y} \to (w,\bar{w})$. Then (5.6.28) takes the form

$$2g\begin{pmatrix}\partial_{\bar{z}} & 0\\ 0 & \partial_z\end{pmatrix}\begin{pmatrix}\langle\psi(z,\bar{z})\psi(w,\bar{w})\rangle & \langle\psi(z,\bar{z})\bar{\psi}(w,\bar{w})\rangle\\ \langle\bar{\psi}(z,\bar{z})\psi(w,\bar{w})\rangle & \langle\bar{\psi}(z,\bar{z})\bar{\psi}(w,\bar{w})\rangle\end{pmatrix} = \frac{1}{\pi}\begin{pmatrix}\partial_{\bar{z}}\frac{1}{z-w} & 0\\ 0 & \partial_z\frac{1}{\bar{z}-\bar{w}}\end{pmatrix},$$

where we used the following representation of the delta function as in Lemma 5.6.2:

$$\delta((x)) = \frac{1}{\pi}\partial_z\frac{1}{\bar{z}} = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z}. \tag{5.6.29}$$

The propagator is now easily seen to be

$$\begin{aligned}\langle\psi(z,\bar{z})\psi(w,\bar{w})\rangle &= \frac{1}{2\pi g}\frac{1}{z-w}\\ \langle\bar{\psi}(z,\bar{z})\bar{\psi}(w,\bar{w})\rangle &= \frac{1}{2\pi g}\frac{1}{\bar{z}-\bar{w}}\\ \langle\psi(z,\bar{z})\bar{\psi}(w,\bar{w})\rangle &= \langle\bar{\psi}(z,\bar{z})\psi(w,\bar{w})\rangle = 0.\end{aligned} \tag{5.6.30}$$

Operator Product Expansions. After differentiation, we get using (5.6.30)

$$\begin{aligned}\langle \partial_z \psi(z,\bar{z})\psi(w,\bar{w})\rangle &= -\frac{1}{2\pi g}\frac{1}{(z-w)^2}\\ \langle \partial_z \psi(z,\bar{z})\partial_w\psi(w,\bar{w})\rangle &= -\frac{1}{\pi g}\frac{1}{(z-w)^3}.\end{aligned}$$

The OPE of two Fermions then reads

$$\psi(z)\psi(w) = \frac{1}{2\pi g}\frac{1}{z-w} + \text{reg.} \tag{5.6.31}$$

Note that the OPE reflects the anticommutativity of the Fermionic wavefunctions. The energy momentum tensor can be evaluated using the general expression in (5.5.2):

$$\begin{aligned}T^{\bar{z}\bar{z}} &= 2\frac{\partial \mathcal{L}}{\partial \bar{\partial}\Phi}\partial\Phi = 2g\psi\partial\psi\\ T^{zz} &= 2\frac{\partial \mathcal{L}}{\partial \partial\Phi}\bar{\partial}\Phi = 2g\bar{\psi}\bar{\partial}\bar{\psi}\\ T^{z\bar{z}} &= 2\frac{\partial \mathcal{L}}{\partial \partial\Phi}\partial\Phi = -2g\psi\bar{\partial}\psi.\end{aligned}$$

Note that the energy momentum tensor is not symmetric but it becomes symmetric onshell. We need not worry about this because we can always use Belinfante construction described above Proposition 5.5.5. In the notation of (5.6.16), we have

$$T(z) = -\pi g : \psi(z)\partial\psi(z) :,$$

where the normal ordering is again defined as

$$: \psi\partial\psi : (z) = \lim_{z\to w}\left[\psi(z)\partial\psi(w) - \langle \psi(z)\partial\psi(w)\rangle\right].$$

Proposition 5.6.14. *The Fermion wavefunction $\psi(z)$ is a holomorphic primary field of conformal dimension $h = \frac{1}{2}$.*

Proof. We calculate the OPE of $\psi(z)$ with $T(z)$. We have

$$\begin{aligned}T(z)\psi(w) &= -\pi g : \psi(z)\partial\psi(z) : \psi(w)\\ &= \frac{1}{2}\frac{\partial X(z)}{z-w} + \frac{1}{2}\frac{\psi(z)}{(z-w)^2} + \text{reg.}\\ &= \frac{1}{2}\frac{\psi(w)}{(z-w)^2} + \frac{1}{2}\frac{\partial X(w)}{z-w} + \text{reg.},\end{aligned}$$

where we carried over $\psi(z)$ over to $\partial X(z)$ resulting in a minus sign and we used the argument as in Corollary 5.6.11 to replace $\psi(z)$ and $\partial\psi(z)$ by $\psi(w)$ and $\partial\psi(w)$ respectively. ∎

Theorem 5.6.15. *The stress tensor $T(z)$ satisfies the OPE*

$$T(z)T(w) = \frac{1}{4}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg.$$

Proof. The proof is similar to the free Boson case but with larger number of contractions. $\square$

Example with Foresight: The Ghost System

Consider the following action:

$$S = \frac{g}{2}\int d^2x b_{\mu\nu}\partial^\mu c^\nu,$$

where both fields b, c are Fermionic fields and $b_{\mu\nu}$ is symmetric traceless tensor. These fields are called *ghosts* or *reparametrisation ghosts* and will appear in our discussion when we discuss path integral quantisation. The classical equations of motion are

$$\partial^\alpha b_{\alpha\mu} = 0, \quad \partial^\alpha c^\beta + \partial^\beta c^\alpha = 0.$$

In complex notation, we usually write $c = c^z, \bar{c} = c^{\bar{z}}$ and $b = b^{zz}, \bar{b} = b^{\bar{z}\bar{z}}$. The equations of motion then reads

$$\bar{\partial} b = 0, \quad \partial\bar{b} = 0;$$
$$\bar{\partial} c = 0, \quad \partial\bar{c} = 0, \quad \partial c = -\bar{\partial}\bar{c}.$$

Propagator. It is calculated as usual by writing the action as

$$S = \frac{1}{2}\int d^2\boldsymbol{x} d^2\boldsymbol{y} b_{\mu\nu}((x)) A^{\mu\nu}_\alpha(\boldsymbol{x}m\boldsymbol{y})c^\alpha(\boldsymbol{y}),$$

so that

$$A^{\mu\nu}_\alpha(\boldsymbol{x}m\boldsymbol{y}) = \frac{1}{2}g\delta^\nu_\alpha\delta(\boldsymbol{x}-\boldsymbol{y})\partial^\mu.$$

The factor of $1/2$ takes care of the double counting of the terms in the sum since $b_{\mu\nu}$ and hence $A^{\mu\nu}$. As usual the propagator is given by $K = A^{-1}$ where K satisfies the differential equation

$$\frac{1}{2}g\delta^\mu_\alpha\partial^\nu K^\beta_{\mu\nu}(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y})\delta_{\alpha\beta}.$$

Solving this, one gets

$$b(z)c(w) = \frac{1}{\pi g}\frac{1}{z-w} + \text{reg}.$$

This immediately gives us the OPE

$$\langle c(z)b(w)\rangle = \frac{1}{\pi g}\frac{1}{z-w}$$
$$\langle b(z)\partial c(w)\rangle = -\frac{1}{\pi g}\frac{1}{(z-w)^2}$$
$$\langle \partial b(z)c(z)\rangle = \frac{1}{\pi g}\frac{1}{(z-w)^2}.$$

The usual energy momentum tensor is given by

$$T_C^{\mu\nu} = \frac{g}{2}\left(b^{\mu\alpha}\partial^\nu c_\alpha - \eta^{\mu\nu}b^{\alpha\beta}\partial_\alpha c_\beta\right).$$

It turns out that the canonical energy momentum tensor as above is not symmetric even on-shell, so we need the full machinery of Belinfante construction to make the energy momentum tensor symmetric. We add $\partial_\rho B^{\rho\mu\nu}$ as in Proposition 5.5.5 where

$$B^{\rho\mu\nu} = -\frac{1}{2}g(b^{\nu\rho}c^\mu - b^{\nu\mu}c^\rho)$$

It can then be shown that the Belinfane energy momentum tensor given by

$$T_B^{\mu\nu} = \frac{g}{2}\left[b^{\mu\alpha}\partial^\nu c_\alpha + b^{\nu\alpha}\partial^\mu c_\alpha + \partial_\alpha b^{\mu\nu}c^\alpha - \eta^{\mu\nu}b^{\alpha\beta}\partial_\alpha c_\beta\right]$$

is symmetric and traceless onshell. In complex coordinates, we have

$$T(z) = \pi g : (2\partial c\; b + c\partial b) :$$

The OPE of T with c can again be calculated using Wick's theorem:

$$\begin{aligned}
T(z)c(w) &= \pi g : (2\partial c\; b + c\partial b) : c(w) \\
&= -\frac{c(z)}{(z-w)^2} + 2\frac{\partial_z c(z)}{z-w} + \text{reg.} \\
&= -\frac{c(z)}{(z-w)^2} + 2\frac{\partial_w c(w)}{z-w} + \text{reg.}
\end{aligned}$$

Therefore c is a primary field with conformal weight $h = -1$. Similarly, we have

$$\begin{aligned}
T(z)b(w) &= \pi g : (2\partial c\; b + c\partial b) : b(w) \\
& 2\frac{b(w)}{(z-w)^2} + \frac{\partial_w b(w)}{z-w}.
\end{aligned}$$

This implies that b is a primary field with conformal weight $h = 2$. Note that in both the OPEs we used the anticommutativity of b and c. Finally we have

$$\begin{aligned}
T(z)T(w) &= \pi g^2 : (2\partial c(z)b(z) + c(z)\partial b(z)) :: (2\partial c(w)b(w) + c(w)\partial b(w)) : \\
&= \frac{-13}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}
\end{aligned}$$

Thus we see that this OPE has same form as the previous two examples except for the coefficient of the quartic term. One can tacitly modify this coefficient by modifying the action in such a way that the OPE of b and c remains the same but the energy momentum tensor changes. To be precise, we subtract a total derivative $: \partial(cb)$ from the original action. This gives the new energy momentum tensor to be

$$T(z) = \pi g : \partial c\; b : .$$

This new theory is called the *simple ghost system*. New OPEs are

$$T(z)c(w) = \frac{\partial c(w)}{z-w} + \text{reg.}$$
$$T(z)b(w) = \frac{b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w} + \text{reg.}$$

These OPE imply that in simple ghost theory, c is a primary field with conformal dimension $h = 0$ and b is a primary field of dimension $h = 1$. The OPE of T with T is

$$T(z)T(w) = \frac{-1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}$$

Thus we see that the new coefficient is simply -1.

Remark 5.6.16. Note that the OPE of $T(z)$ with itself in all the three examples above fails to be that of a primary operator only on account of the $(z-w)^{-4}$ term, without which $T(z)$ would be a primary operator of conformal weight $(h, \bar{h}) = (2, 0)$. The coefficient of the $(z-w)^{-4}$ term thus decides this fact. This is a general feature of CFTs. The coefficient of this inverse quartic term is called the *central charge* which we now explore in the next subsection.

5.6.5 Central Charge

We begin by proving the OPE structure of the energy momentum tensor in general CFTs.

Theorem 5.6.17. *The energy momentum tensor in a 2d unitary CFT*[14] *satisfies the OPE:*

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \textit{reg.}$$
$$\bar{T}(\bar{z})\bar{T}(\bar{w}) = \frac{\bar{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{w})}{\bar{z}-\bar{w}} + \textit{reg.}$$

The constant $c, \bar{c}$ are called the central charges of the CFT.

Proof. We begin by observing that since the stress tensor is a symmetric tensor of rank 2, thus it must represent a spin $s = 2$ representation of the Lorentz group. Next, the stress tensor has scaling dimension 2. Thus the general form of the OPE of $T(z)$ with itself will be of the form:

$$T(z)T(w) = \cdots + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}$$

where the dots in front indicate higher order singularity. Other singular terms in the OPE have the form:

$$\frac{\mathcal{O}_n}{(z-w)^n},$$

[14] we will define and discuss this notion in next section.

which implies that the scaling dimension of $\mathcal{O}_n$ will be

$$\Delta[\mathcal{O}_n] = 4 - n$$

since the left hand side has scaling dimension 4 and $(z-w)^{-n}$ has scaling dimension $-n$. We will shortly prove that any unitary CFT cannot have conformal weights $h, \bar{h} < 0$. Thus the scaling dimension of $\mathcal{O}_n$ cannot be negative which implies that the most singular term that can appear in the OPE is $(z-w)^{-4}$. So the OPE reduces to

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{\mathcal{O}(w)}{(z-w)^3} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}$$

where $\mathcal{O}(w)$ is some field of scaling dimension 1 and c is a complex number. We now only need to rule out the $(z-w)^{-3}$ term. Note that this term violates the $T(z)T(w) = T(w)T(z)$ which is required since we interpret OPEs as operators inside a correlation function. So now it suffices to show that $(z-w)^{-1}$ term does indeed satisfy this upto regular terms. To see this, note that we can expand Taylor $T(z)$ around w:

$$T(z) = T(w) + \partial_w T(w)(z-w) + O((z-w)^2) \implies \partial_z T(z) = 0 + \partial_w T(w) + O(z-w).$$

Thus

$$\begin{aligned}
T(w)T(z) &= \frac{c/2}{(w-z)^4} + \frac{2T(z)}{(w-z)^2} + \frac{\partial_z T(z)}{w-z} + \text{reg.} \\
&= \frac{c/2}{(w-z)^4} + \frac{2(T(w) + \partial_w T(w)(z-w) + O((z-w)^2))}{(w-z)^2} + \frac{\partial_w T(w) + O(z-w)}{w-z} + \text{reg.} \\
&= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.} \\
&= T(z)T(w),
\end{aligned}$$

where the higher order terms in the expansion have been included in the regular terms. This trick does not work for $(z-w)^{-3}$ term. Similar analysis gives the OPE of $\bar{T}(\bar{z})$. $\square$

Appendices

Appendix A

Wigner's Classification of Representations of Poincaré Group

In this appendix, we will briefly review Wigner's little group method of classifying the irreducible representations of the Poincaré group. The idea is mathematically enlightening and and motivated a lot of research in representation theory. But here we will not delve into the mathematically rigorous treatment, the interested reader can look up here for a summary of the mathematical theory. Rather we will take a more physical approach on the lines of Weinberg's quantum theory of fields. We begin by discussing Wigner's proposal of interpreting elementary particles as irreducible representations of the Poincaré group. We assume familiarity with basic terminology of topology.

A.1 Projective Representations

Let $|\Psi\rangle$ be a state in Hilbert space $\mathcal{H}$. Note that any two states $|\Psi\rangle$ and $|\Phi\rangle$ which are nonzero and related by

$$|\Psi\rangle = \lambda|\Phi\rangle \quad \lambda \in \mathbb{C}^* := \mathbb{C}\backslash\{0\} \tag{A.1.1}$$

are the same quantum mechanical states. So it is pertinent to consider the quotient space of $\mathcal{H}^* = \mathcal{H}\backslash\{0\}$ as $\mathbb{P}(\mathcal{H}) := \mathcal{H}^*/\sim$ where $|\Psi\rangle \sim |\Phi\rangle$ if and only if (A.1.1) is true. The quotient space $\mathbb{P}(\mathcal{H})$ is called the *projectivised Hilbert space*. Recall that the probability amplitude of transition from $|\Psi\rangle$ to Φ is given by

$$p(|\Psi\rangle, |\Phi\rangle) = \frac{\langle\Psi \mid \Phi\rangle}{\langle\Psi \mid \Psi\rangle\langle\Phi \mid \Phi\rangle}.$$

In the quotient topology on $\mathbb{P}(\mathcal{H})$, p induces a continuous map[1] on $\mathbb{P}(\mathcal{H})$ which we denote by $\widetilde{p}$. A homeomorphism $T : \mathbb{P}(\mathcal{H}) \longrightarrow \mathbb{P}(\mathcal{H})$ satisfying

$$\widetilde{p}(T[\Psi], T[\Phi]) = \widetilde{p}(|\Psi\rangle, |\Phi\rangle)$$

[1] it is a standard result in quotient topology. See for example Topology by Munkres.

where $[\Phi], [\Psi]$ are equivalence classes in $\mathbb{P}(\mathcal{H})$, is called a *projective automorphism.* The set of all such maps, denoted by $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$, is a group called *projective automorphism group.* The action of this group on $\mathbb{P}(\mathcal{H})$ leaves transition probabilities invariant. Now consider a particle in the Minkowski space $\mathbb{R}^{1,D-1}$. The symmetry group of this space is precisely the Poincaré group[2] which we denote by $\mathcal{P}$. Let two observers $\mathcal{O}$ and $\mathcal{O}'$, related by $\Lambda \in \mathcal{P}$, measure the quantum mechanical particle. In general, there measurement result will reveal different states, say $[\Psi]$ and $[\Psi']$ respectively. Thus physically one expects that transition probabilities in $\mathcal{O}$ and $\mathcal{O}'$ be same. This means that the two states must be related by some projective automorphism:

$$[\Psi] = T_\Lambda[\Psi'], \quad \text{for some} \quad T_\Lambda \in \mathrm{Aut}(\mathbb{P}(\mathcal{H})).$$

If $\mathcal{O} = \mathcal{O}'$ then $T_\Lambda = Id$ and we should have $T_\Lambda = T_{Id} = Id \in \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$. Lastly if a third observer $\mathcal{O}''$, related to $\mathcal{O}'$ by Γ, measures the state then we must impose $T_\Lambda \circ T_\Gamma = T_{\Lambda\circ\Gamma}$. Thus the change of frame induces a representation $\Pi : \mathcal{P} \longrightarrow \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$. This is called the *projective representation.*

A.2 Elementary Particles

The representation $(\Pi, \mathcal{H})$ of the Poincaré group is called irreducible if the only nontrivial closed invariant subspace of $\mathcal{H}$ is $\mathcal{H}$. That is $\Pi(\mathcal{P})(V) \subseteq V$ if and only if $V = \mathcal{H}$. The closed condition is technical: we want the invariant subspace to be a Hilbert space in its own right which is not automatically true in infinite dimensional Hilbert space unless the subspace is closed.

Wigner suggested that the irreducible projective representations of the Poincaré group correspond to elementary particles within the quantum system under consideration. Wigner's argument was as follows: an elementary particle in a quantum mechanical system is a vector in $\mathbb{P}(\mathcal{H})$. As discussed, different observers will see different vectors in $\mathbb{P}(\mathcal{H})$ corresponding to the elementary particle. All these vectors must be related by some projective automorphism. The set of all these vectors constitutes $\mathcal{P}-$invariant subspace of $\mathbb{P}(\mathcal{H})$ and hence we obtain a subrepresention of $(\Pi, \mathcal{H})$. This subrepresentation can be thought of as a subsystem which is elementary if it is irreducible (otherwise it will have more smaller subsystems). This reduces the problem of determining all relativistic free particles in Minkowski spacetime to the mathematical task of finding all irreducible projective representations of the Poincaré group.

[2]mathematically speaking, the symmetry group of a Riemannian manifold $(\mathcal{M}, g)$ is the group of all diffeomorphisms from $\mathcal{M}$ to itself whose pullback preserves the metric.

A.3 Projective Representations of the Poincaré Group

Let us now take a look at the Poincaré group more closely. We begin by defining semidirect product.

Definition A.3.1. Let H and N be groups and suppose there is a a group homomorphism $\phi : H \to \mathrm{Aut}(N)$. Then the semidirect product of H by N, denoted $H \ltimes N$ which has $H \times N$ as underlying set, and multiplication defined by $(h, n) \cdot (h', n') = (hh', n\phi(h)\,(n'))$.

Each element of the Lorentz group $SO(1, D-1)$ defines an automorphism of $\mathbb{R}^{1,D-1}$ defined by matrix multiplication. Thus we can form the semidirect product $SO(1, D-1) \ltimes \mathbb{R}^{1,D-1}$. The physically relevant Poincaré group is the semidirect product of the proper orthochronous Lorentz group and the abelian translation group. That is

$$\mathcal{P} = ISO(1, D-1) = SO(1, D-1)_I \ltimes \mathbb{R}^{1,D-1}$$

where $SO(1, D-1)_I$ is the connected component of identity in the Lorentz group. The Poincaré algebra is generated by the generators of translations and Lorentz transformations denoted by P^μ and $M^{\mu\nu}$ respectively. They satisfy the Poincaré algebra:

$$\begin{aligned}
&i\,[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\sigma\mu}M_{\rho\nu} + \eta_{\sigma\nu}M_{\rho\mu}\\
&i\,[P_\mu, M_{\rho\sigma}] = \eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho\\
&i\,[P_\mu, P_\rho] = 0.
\end{aligned}$$

The third commutator says that P_μ commutes among themselves. So we start with states in $\mathbb{P}(\mathcal{H})$ which are simultaneous eigenvectors of P^μ. We label all other degrees of freedom by σ. We have

$$P^\mu \psi_{q,\sigma} = q^\mu \psi_{q,\sigma}.$$

Note that infinitesimal translations are represented by $U = \mathbb{1} - iP^\mu\varepsilon_\mu$ and repeating this, we obtain finite translations

$$U(\mathbb{1}, a) = e^{-iP^\mu a_\mu}.$$

so that

$$U(\mathbb{1}, a)\psi_{q,\sigma} = e^{-iq\cdot a}\psi_{q,\sigma}.$$

These $U(\mathbb{I}, a)$ are the projective representations of the translation part of the Poincaré group. Usually the physical requirement restricts U to be unitary which restricts P^μ to be Hermitian. Recall that

$$\begin{aligned}
&(\Lambda, a) \cdot (\Lambda', a') = (\Lambda\Lambda', a' + \Lambda a) \quad \text{in} \quad \mathcal{P}\\
&(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda a).
\end{aligned}$$

An infinitesimal Poincaré transformation with parameters ω, ε is unitarily represented as

$$U(\mathbb{1} + \omega, \varepsilon) = \mathbb{1} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} - \varepsilon_\mu P^\mu + \ldots .$$

So For a general $\Lambda \in SO(1, D-1)$ we have

$$U(\Lambda, a)U(\mathbb{1}+\omega, \in)U(\Lambda, a)^{-1} = U\left(\Lambda(\mathbb{1}+\omega)\Lambda^{-1}, \Lambda\varepsilon - \Lambda\omega\Lambda^{-1}a\right).$$

Using infinitesimal version upto linear order in ω, we get

$$U(\Lambda, a)\left[\mathbb{1} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} - \varepsilon_\mu P^\mu\right]U(\Lambda, a)^{-1} = \mathbb{1} + \frac{i}{2}\left(\Lambda\omega\Lambda^{-1}\right)_{\mu\nu}M^{\mu\nu} - \left(\Lambda\varepsilon - \Lambda\omega\Lambda^{-1}a\right)_\mu P^\mu.$$

Comparing coefficients of $\omega_{\mu\nu}$ and ε_μ, we get

$$\begin{aligned} U(\Lambda, a)M^{\mu\nu}U(\Lambda, a)^{-1} &= \left(\Lambda^{-1}\right)^\mu_\lambda \left(\Lambda^{-1}\right)^\nu_\rho \left(M^{\lambda\rho} - a^\lambda P^\rho + a^\rho P^\lambda\right) \\ U(\Lambda, a)P^\rho U(\Lambda, a)^{-1} &= \left(\Lambda^{-1}\right)^\rho_\mu P^\mu. \end{aligned} \tag{A.3.1}$$

Our aim now is to find the projective representation of the Lorentz part of the Poincaré group. Indeed if $U(\Lambda, 0) \equiv U(\Lambda)$ is such a representation then

$$\begin{aligned} P^\mu U(\Lambda)\psi_{p,\sigma} &= U(\Lambda)U(\Lambda)^{-1}P^\mu U(\Lambda)\psi_{p,\sigma} \\ &= U(\Lambda)\Lambda^\mu_\nu P^\nu \psi_{p,\sigma} \\ &= (\Lambda p)U(\Lambda)\psi_{p,\sigma}. \end{aligned}$$

So we must have

$$U(\Lambda)\psi_{p,\sigma} = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda, p)\psi_{\Lambda p,\sigma'}. \tag{A.3.2}$$

In general, this representation is reducible. since this is a unitary representation, a theorem in representation theory says that it is completely reducible, that is it can be written as a direct sum of irreducible representations of invariant subspaces of eigenvectors of P^μ with eigenvalue Λp. Our goal is to classify all such irreducible representations. To do so, we first calculate the orbit of action of Lorentz group on $\mathbb{R}^{1,D-I}$. It is clear that $SO(1, D-1)_I$ fixes p^2 for all $p \in \mathbb{R}^{1,D-1}$ but when $p^2 \leq 0$ then it also fixes the sign of p^0. Accordingly we get the following orbits:

1. $p^2 = m^2 > 0$: one sheeted hyperboloid.
2. $p^2 = -m^2 < 0$: two sheeted hyperboloid corresponding to $p^0 > 0$ or $p^0 < 0$.
3. $p^2 = 0$: cone with vertex at the origin.

Now given any p^μ, one can choose (depending on the orbit of p^μ) a standard q^μ such that

$$p^\mu = L^\mu_\nu(p)q^\nu,$$

where $\quad L_\nu \in SO(1, D-1)_I$. By above discussion

$$\psi_{p,\sigma} = N(p)U\left(L^\mu_\nu(p)\right)\psi_{q,\sigma},$$

where $N(p)$ is some normalesation factor. Now for any $\Lambda \in SO(1, D-1)_I$ we have

$$
\begin{aligned}
U(\Lambda)\psi_{p,\sigma} &= N(p)U(\Lambda)U(L(p))\psi_{q,\sigma} \\
&= N(p)U(L(\Lambda p))U\left(L^{-1}(\Lambda p)\Lambda L(p)\right)\psi_{q,\sigma},
\end{aligned}
$$

where we used property of group representations. Note that

$$
L^{-1}(\Lambda p)\Lambda L(p)q = L^{-1}(\Lambda p)\Lambda p = q.
$$

The set of all such elements of Λ is called the *stability group* of q also called the *little group*. For any two elements $W, \overline{W}$ in the little group of q, we have

$$
U(W)\psi_{q,\sigma} = \sum_{\sigma\prime} D^q_{\sigma,\sigma'}(W)\psi_{q,\sigma'}
$$

and

$$
\begin{aligned}
U(\overline{W}W)\psi_{q,\sigma} &= \sum_{\sigma'} D^q_{\sigma,\sigma'}(W)\sum_{\sigma''} D^q_{\sigma',\sigma''}(\overline{W})\psi_{q,\sigma''} \\
&= \sum_{\sigma',\sigma''} D^q_{\sigma,\sigma'}(W)D^q_{\sigma',\sigma''}(\overline{W})\psi_{q,\sigma''} \\
&= \sum_{\sigma''} D^q_{\sigma,\sigma''}(\overline{W}W)\psi_{q,\sigma''},
\end{aligned}
$$

where

$$
D^q_{\sigma,\sigma''}(\overline{W}W) = \sum_{\sigma'} D^q_{\sigma,\sigma'}(W)D^q_{\sigma',\sigma''}(\overline{W}).
$$

Thus we see that $D^q(W)$ is a representation of the little group. So putting $W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p)$ we have

$$
U(W(\Lambda,p))\psi_{q,\sigma} = \sum_{\sigma'} D_{\sigma,\sigma'}(W(\Lambda,p))\psi_{q,\sigma'}.
$$

So that

$$
\begin{aligned}
U(\Lambda)\psi_{p,\sigma} &= N(p)\sum_{\sigma'} D_{\sigma,\sigma'}(w(\Lambda,p))U(L(\Lambda p))\psi_{q,\sigma'} \\
&= \frac{N(p)}{N(\Lambda p)}\sum_{\sigma'} D_{\sigma,\sigma'}(W(\Lambda,p))\psi_{\Lambda p,\sigma'}.
\end{aligned}
$$

Hence apart from the normalisation factor, the problem of finding unitary irreducible representations of Poincaré group has been reduced to finding unitary irreducible representations of the little group corresponding to each orbit. So we first find the little group corresponding to each orbit.

1. $q^2 = m^2 > 0$: by going to rest frame, we can set q^μ to $q^\mu = (0, 0, \cdots, 0, m)$. Looking at the form of this vector, we can see that the little group is $SO(1, D-2)_I \hookrightarrow SO(1, D-1)_I$.

2. $q^2 = -m^2 < 0$: , by going to rest frame, we can take q^μ to be $q^\mu = (m, \vec{0})$. Clearly the little group is $SO(D-1)$.

3. $q^2 = 0$: the little group computation is not so obvious. Although it turns out to be the Euclidean group $E(D-2) = SO(D-2) \ltimes \mathbb{R}^{D-2}$. This is the isometry group of $\mathbb{R}^{D-2}$ with the Euclidean metric.

In $q^2 = 0$, one case is $q^\mu = 0$ whose stabiliser is the whole Poincaré group $\mathcal{P}$.

Gender	Orbit	Little Group	Unitary Representation
$q^2 = -m^2$	Mass shell	$SO(D-1)$	Massive
$q^2 = -m^2$	Hyperboloid	$SO(1, D-1)_I$	Tachyonic
$q^2 = 0$	Lightcone	$E(D-2)$	Massless
$q^\mu = 0$	Origin	$\mathcal{P}$	Zero Momentum

Physically, Tachyonic representations are not accepted. So we will only deal with the other two. One can use the little group method to find all irreducible representations of the Euclidean group. The idea is to go to the Lie algebra of $E(D-2)$ and identify the "translations" generators and repeat the procedure above. The upshot of this computation is that we get two orbits and the corresponding little groups are called *short little groups*. The corresponding unitary irreducible representations are labelled as *helicity* and *infinite spin*. The analogue of the Lorentz group here is obviously $SO(D-2)$. The short Little group corresponding to infinite spin is $S0(D-3)$ and that for infinite spin is $SO(D-2)$.

Next one can use Young Tableau to embed the irreducible representations of the Little groups in all cases into tensorial representations. For the particular case that we will be dealing with, we would like to find the massless irreducible representations of dimension $(D-2)^2$ of the Poincaré group. It turns out that it is the direct sum of three irreducible parts:

$$\begin{array}{cccc} & \textbf{Traceless symmetric} \oplus & \textbf{Antisymmetric} \oplus & \textbf{Trace (Scalar)} \\ \text{Dim:} & \dfrac{(D-2)(D-1)}{2} - 1 & \dfrac{(D-2)(D-3)}{2} & 1 \end{array}$$

Appendix B

Symmetry Generators: Generators of the Conformal Algebra

In this appendix, we describe the general principle of symmetry transformations at classical and quantum level and the general method of finding the generators of symmetry transformation. We also describe the consequences of symmetries in classical and quantum theories namely, Noether's theorem and Ward identities respectively. As an application, we compute the generators of conformal symmetry and the corresponding charges and describe the Ward identities.

B.1 Continuous Symmetry Transformations

Let $\Phi : \mathbb{R}^{1,D-1} \longrightarrow \mathcal{M}$ be a field from the spacetime to some target manifold. Its dynamics is governed by an action by virtue of the Euler-Lagrange equations. The action generally is a functional of Φ and its first derivatives:

$$S[\Phi] = \int d^D x \mathcal{L}(\Phi, \partial_\mu \Phi),$$

where $\mathcal{L}(\Phi, \partial_\mu \Phi)$ is the Lagrange density. Suppose we transform the field as $\Phi \longrightarrow \Phi' = \mathcal{F}(\Phi)$. Then the action also transforms as $S[\Phi] \longrightarrow S[\Phi'] =: S'[\Phi]$.

Definition B.1.1. (i) A transformation of field $\Phi \longrightarrow \Phi' = \mathcal{F}(\Phi)$ is called a *symmetry* of the action $S[\Phi]$ is under the transformation of field the action remains invariant in the sense that $S'[\Phi] = S[\Phi]$.

(ii) A symmetry $\Phi \longrightarrow \Phi' = \mathcal{F}(\Phi)$ of the action is called a *continuous symmetry* is the transformation of the field is parametrised by a continuous parameter α. A symmetry which is not continuous is called *discrete*.

(iii) A symmetry $\Phi \longrightarrow \Phi' = \mathcal{F}(\Phi)$ is called a *spacetime symmetry* if the field transformation results from a spacetime transformation. Otherwise it is called *internal symmetry.*

(iv) A symmetry $\Phi \longrightarrow \Phi' = \mathcal{F}(\Phi)$ is called *local symmetry* if the field transforms differently at different spacetime points. If the field transforms in exactly the same way at every spacetime point, then it is called *global symmetry*.

Since we will mostly be considering symmetries with respect to conformal transformations which is a spacetime transformation, we will restrict our attention to spacetime symmetries.

B.1.1 Spacetime Symmetries

Consider a spacetime transformation

$$\begin{aligned} &x \longrightarrow x'(x) \\ &\Phi(x) \longrightarrow \Phi'\left(x'\right) \end{aligned} \tag{B.1.1}$$

Under such a transformation, the field Φ changes in two ways: first by the functional change $\Phi' = \mathcal{F}(\Phi)$ where we have expressed the new field Φ' as a function of the old field Φ, and second by the change of argument $x \longrightarrow x'$. Expressing the new field at x', we see that

$$\Phi'\left(x'\right) = \mathcal{F}(\Phi(x)). \tag{B.1.2}$$

This way of looking at symmetry transformations is called *active transformation*. Under such a transformation, the action transforms as

$$\begin{aligned} S' &= \int d^D x \mathcal{L}\left(\Phi'(x), \partial_\mu \Phi'(x)\right) \\ &= \int d^D x' \mathcal{L}\left(\Phi'\left(x'\right), \partial'_\mu \Phi'\left(x'\right)\right) \\ &= \int d^D x' \mathcal{L}\left(\mathcal{F}(\Phi(x)), \partial'_\mu \mathcal{F}(\Phi(x))\right) \\ &= \int d^D x \left|\frac{\partial x'}{\partial x}\right| \mathcal{L}\left(\mathcal{F}(\Phi(x)), \left(\partial x^\nu / 2x'^\mu\right) \partial_\nu \mathcal{F}(\Phi(x))\right), \end{aligned}$$

where $\left|\frac{\partial x'}{\partial x}\right|$ is the Jacobian of variable change. We have changed variables $x \longrightarrow x'$ according to the transformation (B.1.1) and used (B.1.2) and in first two steps finally again made a change of variables in last step.

Example B.1.2. (i) Translation: it is defined as

$$\begin{aligned} &x^\mu \longrightarrow x'^\mu = x^\mu + a^\mu \\ &\Phi'(x+a) = \Phi(x) \end{aligned}$$

It is clear that $S' = S$. The action is invariant under translations, unless it depends explicitly on position.

(ii) Lorentz Transformation: under Lorentz transformation $\Lambda \in \mathrm{SO}(1, D-1)$,

$$
\begin{aligned}
x'^{\mu} &= \Lambda^{\mu}{}_{\nu} x^{\nu} \\
\Phi'(\Lambda x) &= L_{\Lambda} \Phi(x),
\end{aligned}
$$

where we have assumed that the fields transform *linearly* with respect to Lorentz transformation, so that the operators L_Λ furnish a representation of the Lorentz group. Depending on the action and the representation Φ of the Lorentz group, the action may or may not be invariant under Lorentz transformation.

(iii) Scale Transformations: it is defined as

$$
\begin{aligned}
x' &= \lambda x \\
\Phi'(\lambda x) &= \lambda^{-\Delta} \Phi(x)
\end{aligned}
$$

where λ is the dilation factor and Δ is called the *scaling dimension* of the field Φ. Note that the Jacobian of this transformation is $|\partial x'/\partial x| = \lambda^D$. Thus we have

$$
S' = \lambda^D \int d^D x \mathcal{L}\left(\lambda^{-\Delta}\Phi, \lambda^{-1-\Delta}\partial_\mu \Phi\right)
$$

As an example, consider the action of a massless scalar field φ in spacetime dimension D:

$$
S[\varphi] = \int d^D x \partial_\mu \varphi \partial^\mu \varphi.
$$

It is easily checked that this action is scale invariant if we make the choice

$$
\Delta = \frac{1}{2} D - 1.
$$

B.2 Infinitesimal Transformation and Noether's Theorem

We now consider continuous transformations and study their effect when the parameter is very small. We will keep the parameter only upto linear order. Such a general transformation may be written as

$$
\begin{aligned}
x'^{\mu} &= x^{\mu} + \omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \\
\Phi'\left(x'\right) &= \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x),
\end{aligned}
\tag{B.2.1}
$$

where $\{\omega_a\}$ is a set of infinitesimal parameters.

Definition B.2.1. The *generator* G_a of a symmetry transformation is defined by the following expression

$$
\delta_\omega \Phi(x) \equiv \Phi'(x) - \Phi(x) \equiv -i \omega_a G_a \Phi(x).
$$

Observe that to first order in ω_a,

$$\begin{aligned}\Phi'(x') &= \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) \\ &= \Phi(x') - \omega_a \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \Phi(x') + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x') .\end{aligned}$$

This gives an explicit expression for the generators:

$$iG_a \Phi = \frac{\delta x^\mu}{\delta \omega_a} \partial_\mu \Phi - \frac{\delta \mathcal{F}}{\delta \omega_a} \tag{B.2.2}$$

Example B.2.2. (i) Infinitesimal translation: for an infinitesimal translation by a vector ε^μ (the index a becomes here a spacetime index), we have $\delta x^\mu / \delta \varepsilon^\nu = \delta^\mu_\nu$ and $\mathcal{F}$ is trivial. Thus using (B.2.2), we see that the generator is simply

$$P_\mu = -i\partial_\mu .$$

(ii) Infinitesimal Lorentz transformation: an infinitesimal Lorentz transformation has the form

$$\begin{aligned}x'^\mu &= x^\mu + \omega^\mu{}_\nu x^\nu \\ &= x^\mu + \omega_{\rho\nu} \eta^{\rho\mu} x^\nu .\end{aligned}$$

Using (1.2.1), we can easily see that $\omega_{\mu\nu} = -\omega_{\nu\mu}$. This antisymmetry gives the following variation of coordinates:

$$\frac{\delta x^\mu}{\delta \omega_{\rho\nu}} = \frac{1}{2}\left(\eta^{\rho\mu} x^\nu - \eta^{\nu\mu} x^\rho\right) .$$

The field Φ transforms as

$$\mathcal{F}(\Phi) = L_\Lambda \Phi, \qquad L_\Lambda \approx 1 - \frac{1}{2} i\omega_{\rho\nu} S^{\rho\nu}$$

where $S^{\rho\nu}$ is some Hermitian matrix obeying the Lorentz algebra (generator of Lie algebra in the particular representation in which Φ belongs). Using (B.2.2), we get

$$\frac{1}{2} i\omega_{\rho\nu} L^{\rho\nu} \Phi = \frac{1}{2}\omega_{\rho\nu}\left(x^\nu \partial^\rho - x^\rho \partial^\nu\right)\Phi + \frac{1}{2} i\omega_{\rho\nu} S^{\rho\nu}\Phi$$

where $L^{\rho\nu}$ is the generator. The factor of $\frac{1}{2}$ preceding $\omega_{\rho\nu}$ in the definitions of $L^{\rho\nu}$ and $S^{\rho\nu}$ cancels the double counting of transformation parameters. The generators of Lorentz transformations are thus

$$L^{\rho\nu} = i\left(x^\rho \partial^\nu - x^\nu \partial^\rho\right) + S^{\rho\nu} .$$

In particular, if we take $\Phi(x) = x$, then $\mathcal{F}$ is trivial and the generator is simply $L_{\mu\nu} = i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right)$ which is just the angular momentum operator. Thus on spacetime, Lorentz transformation is generated by the angular momentum operator.

(iii) Infinitesimal dilatation: let $x \longrightarrow x' = (1+\alpha)x$ be an infinitesimal scaling then it is clear that $\delta x/\delta\alpha = x$. Next, the field transforms as $\Phi'((1+\alpha)x) = (1+\alpha)^{-\Delta}\Phi(x)$ where Δ is the scaling dimension of the field Φ. Thus we have

$$\frac{\delta\mathcal{F}}{\delta\alpha} = \frac{\delta}{\delta\alpha}((1-\Delta\alpha)\Phi(x)) = -\Delta\Phi(x).$$

Thus using (B.2.2), we find that the generator of infinitesimal dilation in the representation Φ is

$$D := -ix^\mu\partial_\mu - i\Delta. \tag{B.2.3}$$

B.2.1 Generators of Conformal Transformations

As described in Subsection 5.2.1, conformal transformation in dimensions $D \geq 3$ include four different kinds of transformations. We will now find the *spacetime* generators of those transformations which we directly indicated in Subsection 5.2.1.

(i) Translation: we already computed the generator for any field Φ. In particular, for the field $\Phi(x) = x$, the generator is

$$P_\mu = -i\partial_\mu.$$

(ii) Lorentz transformation: the spacetime generator for Lorentz transformation was computed to be

$$L_{\mu\nu} = i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right).$$

(iii) Dilatation: let α be an infinitesimal dilatation parameter, then $x \longrightarrow x' = (1+\alpha)x$. Thus we have

$$\frac{\delta x^\mu}{\delta\alpha} = x^\mu.$$

Hence the generator D of dilatation can be directly read off from (B.2.2),

$$D = -ix^\mu\partial_\mu.$$

(iv) Special conformal transformation: let b^μ be an infinitesimal SCT parameter. Then spacetime transforms as

$$x'^\mu = x^\mu + 2(x\cdot b)x^\mu - (x\cdot x)b^\mu.$$

Thus we quickly find that

$$\frac{\delta x^\mu}{\delta b^\nu} = 2x_\rho\delta^\rho{}_\nu x^\mu - (x\cdot x)\delta^\mu{}_\nu.$$

Hence the generator is

$$K_\mu = -i(2x_\mu x^\nu\partial_\nu - (x\cdot x)\partial_\mu).$$

(v) Two dimensional infinitesimal conformal transformations: in Subsection 5.3.1, we concluded that infinitesimal conformal transformations in complex coordinates are given by

$$
\begin{aligned}
z' &= z + \varepsilon(z) = z + \sum_{n\in\mathbb{Z}} \varepsilon_n \left(-z^{n+1}\right) \\
\bar{z}' &= \bar{z} + \bar{\varepsilon}(\bar{z}) = \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\varepsilon}_n \left(-\bar{z}^{n+1}\right)
\end{aligned}
$$

So the generator corresponding to $z' = z - \varepsilon_n z^{n+1}$ and $\bar{z}' = \bar{z} - \bar{\varepsilon}_n \bar{z}^{n+1}$ can be computed similar to above cases. Indeed for the transformation of z, we have

$$\frac{\delta z}{\delta \varepsilon_n} = -z^{n+1}.$$

Using (B.2.2)[1] we get,

$$l_n = -z^{n+1}\partial_z.$$

Similarly we get the conjugated generator.

B.2.2 Noether's Theorem

In classical field theory, the dynamics is governed by the action of the classical field. A *classical symmetry* is a symmetry of the action under some transformation of the field. Noether's theorem is a statement about a particularly fruitful consequence of continuous symmetries.

Theorem B.2.3. *Let Φ be a classical field and $S[\Phi]$ be its action. Given a continuous symmetry parametrised by ω_a of the action, there exists a conserved current j^μ_a in the sense that when the classical equations of motion are satisfied then*

$$\partial_\mu j^\mu_a = 0.$$

We will not prove this theorem here as it is a standard result covered in any quantum field theory course. If we explicitly write the transformation of field as

$$
\begin{aligned}
x'^\mu &= x^\mu + \omega_a \frac{\delta x^\mu}{\delta \omega_a} \\
\Phi'\left(x'\right) &= \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x),
\end{aligned}
$$

then the conserved current j^μ_a is given by

$$j^\mu_a = \left\{ \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)} \partial_\nu \Phi - \delta^\mu{}_\nu \mathcal{L} \right\} \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \Phi\right)} \frac{\delta \mathcal{F}}{\delta \omega_a}, \tag{B.2.4}$$

[1] we have to remove i since we are already in complex coordinates.

where $\mathcal{L}$ is the Lagrangian density. Under the transformation, the action changes as follows:

$$\delta S = -\int dx j_a^\mu \partial_\mu \omega_a, \tag{B.2.5}$$

which after integration by parts yields the conservation of current under the assumption that the classical equations of motions are satisfied.

Observe that we can add any antisymmetric tensor tothe current without affecting its conservation. Indeed for

$$j_a^\mu \longrightarrow j_a^\mu + \partial_\nu B_a^{\nu\mu} \quad , \quad B_a^{\nu\mu} = -B_a^{\mu\nu}$$

$\partial_\mu \partial_\nu B_a^{\nu\mu} = 0$ by antisymmetry. Thus although the expression (B.2.4) for j_a^μ is cannonical, it is somewhat ambiguous.

B.3 Quantum Symmetries: Ward Identity

In previous sections, we discussed classical symmetries which has nothing to do with quantum field theory. We now discuss what it means for a classical symmetry to also be a symmetry of the corresponding quantum theory.

In quantum theory, the most important objects are correlation functions. Consider a theory with field Φ with action $S[\Phi]$. The n-point correlation function is given by

$$\langle \Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)\rangle = \frac{\int [\mathcal{D}\Phi]\Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)\exp(-S[\Phi])}{\int [\mathcal{D}\Phi]\exp(-S[\Phi])}.$$

We can say something about the classical symmetry in quantum theory by looking at these correlation function. Indeed, suppose $\Phi \longrightarrow \Phi'$ be a symmetry of the action $S[\Phi]$, that is a classical symmetry. Thus we see that in quantum theory, we need the exponential $\exp(-S[\Phi])$ to be invariant under the transformation. But this is not it. Under this transformation, the integral measure $[\mathcal{D}\Phi]$ may change non trivially and may not remain invariant. Then even if the field transformation is a classical symmetry, it may not be a *quantum symmetry* in the sense that the correlation functions change under the transformation and hence the quantum theory may change entirely.

We have the following theorem if we assume that the integral measure is also invariant under a continuous classical symmetry transformation.

Theorem B.3.1. *Suppose* (B.1.1) *and* (B.1.2) *be a classical symmetry of the action* $S[\Phi]$*. Suppose also that the functional integration measure* $[\mathcal{D}\Phi]$ *is also invariant under* (B.1.2)*. Then we have*

$$\langle \Phi(x_1')\cdots\Phi(x_n')\rangle = \langle \mathcal{F}(\Phi(x_1))\cdots\mathcal{F}(\Phi(x_1))\rangle\,.$$

Proof. The proof is straightforward using change of variables. ∎

Example B.3.2. (i) Under translation $x \longrightarrow x + a$, we have

$$\langle \Phi(x_1 + a) \cdots \Phi(x_n + a) \rangle = \langle \Phi(x_1) \cdots \Phi(x_n) \rangle .$$

(ii) Lorentz invariance of scalar fields results in the following identity

$$\langle \Phi(\Lambda^\mu{}_\nu x_1^\nu) \cdots \Phi(\Lambda^\mu{}_\nu x_n^\nu) \rangle = \langle \Phi(x_1^\mu) \cdots \Phi(x_n^\mu) \rangle .$$

(iii) Scale invariance of scalar fields ϕ_i with scaling dimensions Δ_i gives

$$\langle \phi_1(\lambda x_1) \cdots \phi_n(\lambda x_n) \rangle = \lambda^{-\Delta_1} \ldots \lambda^{-\Delta_n} \langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle .$$

Another consequence of a classical symmetry and the invariance of functional integral measure is the following theorem:

Theorem B.3.3. (Ward Identities) *Let* (B.2.1) *be a classical infinitesimal symmetry of the action* $S[\Phi]$ *with generators* G_a *and corresponding classical conserved current* j_a^μ. *Suppose also that functional integral measure is invariant under the symmetry transformation of fields. Then we have*

$$\frac{\partial}{\partial x^\mu} \langle j_a^\mu(x) \Phi(x_1) \cdots \Phi(x_n) \rangle = -i \sum_{i=1}^{n} \delta(x - x_i) \langle \Phi(x_1) \cdots G_a \Phi(x_i) \cdots \Phi(x_n) \rangle .$$

Ward identity is the quantum version of Noether's theorem. Given a classical symmetry, it survives quantisation if the corresponding Ward identity holds and we say that the classical symmetry is also a quantum symmetry.

Definition B.3.4. A classical symmetry $\Phi \longrightarrow \Phi'$ is called an anomaly if the corresponding Ward identity does not hold. In this case we say that we have a quantum symmetry breaking.

There are other aspects of symmetry braking a quantum level, *spontaneous symmetry breaking* for example in which case the projective unitary representation of the classical symmetry group does not keep the ground state of the quantum theory invariant. We will not delve further into these topics.

We will use Ward identities in conformal field theory very often. Ward identity of fields characterises them as primary or non primary fields.