Introduction to String Theory

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Chapter 1

The Free Relativistic Particle

Let $\mathbb{R}^{1,D-1}$ denote the $D-$dimensional spacetime. We denote a typical vector $X^\mu \in \mathbb{R}^{1,D-1}$ by

$$X^\mu = (X^0, X^i)$$

where $(X^i) \in \mathbb{R}^{D-1}$. We sometimes write $X^\mu = (X^0, \vec{X})$. Our signature for the Minkowski space is

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \ldots, 1).$$

We use natural units $\hbar = c = 1$. Consider a free particle moving in $D-$dimensional spacetime. We want to describe its dynamics. We want our theory to be relativistic, meaning that the theory that we develop must be invariant under Lorentz transformation.

1.1 The Action of a Free Relativistic Particle

We begin by writing down a Lorentz invariant action for the free particle. The most natural choice for a Lorentz invariant action is choosing a Lorentz scalar and the cannonical choice is to chose the length of the world line traced by the particle in spacetime. Put

$$S = \int dt \ L = -m \int dt \sqrt{1 - \dot{\vec{X}}^2}, \quad X^0 = t, \dot{\vec{X}} = \frac{d\vec{X}}{dt}, \tag{1.1.1}$$

where m is a parameter which we will identify with the mass of the particle. We can now compute the conjugate momentum of the system in the usual way. We have

$$P^i = \frac{\delta S}{\delta \dot{X}^i} = \frac{-m(2\dot{X}^i)}{-2\sqrt{1 - \dot{\vec{X}}^2}} = \frac{m\dot{X}^i}{\sqrt{1 - \dot{\vec{X}}^2}}.$$

The Hamiltonian of the system is given by

$$H = \vec{P} \cdot \dot{\vec{X}} - L = \frac{m\dot{\vec{X}}^2}{\sqrt{1 - \dot{\vec{X}}^2}} + m\sqrt{1 - \dot{\vec{X}}^2} = \frac{m}{\sqrt{1 - \dot{\vec{X}}^2}},$$

which we recognise as the usual relativistic energy of a free particle and hence m is identified as the mass of the particle. Note that the action is not manifestly Lorentz invariant as we are treating the first component of the spacetime vector differently from the remaining components. But we want an action which is manifestly Lorentz invariant. One way to obtain such an action is to promote t to be an independent variable and then parametrize the spacetime coordinates by some other parameter say τ. So put $t = X^0$ and parametrize $X^\mu = (X^0, X^i)$ as

$$X^\mu = X^\mu(\tau).$$

By a simple application of chain rule, we have

$$dt = \frac{dX^0}{d\tau} d\tau.$$

The action can then be written as

$$\begin{aligned} S = -m \int d\tau \sqrt{\left(\frac{dX^0}{d\tau}\right)^2 - \left(\frac{d\vec{X}}{dt}\right)^2 \left(\frac{dX^0}{d\tau}\right)^2} &= -m \int d\tau \sqrt{-\left[-\left(\frac{dX^0}{d\tau}\right)^2 + \left(\frac{d\vec{X}}{d\tau}\right)^2\right]} \\ &= -m \int d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}}. \end{aligned}$$

Remark 1.1.1. It seems that we have added a new degree of freedom to our system, namely X^0. Later we will see that this is not the case as our system will have *reparametrization invariance* also called *diffeomorphism invariance* which will make one of the degree of freedom redundant.

It is now clear that S can be interpreted as the length of the worldline it traces in spacetime.

1.2 Symmetries of the Action

Let us now look at the symmetries of our system:

1.2.1 Poincaré invariance

This is a manifest global symmetry of the system.

$$X^\mu \to \widetilde{X}^\mu = \Lambda^\mu_\nu X^\nu + \xi^\mu,$$

where $\xi^\mu \in \mathbb{R}^{1,D-1}$ is a constant vector. We have

$$\begin{aligned} \frac{d\widetilde{X}^\mu}{d\tau}\frac{d\widetilde{X}_\mu}{d\tau} = \eta_{\mu\nu}\frac{d\widetilde{X}^\mu}{d\tau}\frac{d\widetilde{X}^\nu}{d\tau} = \eta_{\mu\nu}\Lambda^\mu_\rho \frac{dX^\rho}{d\tau}\Lambda^\nu_\sigma\frac{dX^\sigma}{d\tau} = \left(\Lambda^\mu_\rho \eta_{\mu\nu}\Lambda^\nu_\sigma\right)\frac{dX^\rho}{d\tau}\frac{dX^\sigma}{d\tau} &= \eta_{\rho\sigma}\frac{dX^\rho}{d\tau}\frac{dX^\sigma}{d\tau} \\ &= \frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}, \end{aligned}$$

where we used the property of Lorentz transformations

$$\Lambda^T \eta \Lambda = \eta.$$

This implies that

$$\widetilde{S} = -m \int d\tau \sqrt{-\frac{d\widetilde{X}^\mu}{d\tau}\frac{d\widetilde{X}_\mu}{d\tau}} = -m \int d\tau \sqrt{-\frac{dX^\mu}{d\tau}\frac{dX_\mu}{d\tau}} = S.$$

We could have directly concluded this by the fact that the action is the length of a curve and hence a Lorentz scalar. So it does not transform under Lorentz transformations.

1.2.2 Diffeomorphism Invariance

We can reparametrize the world line by changing the parameter τ:

$$\tau \to \widetilde{\tau} = \widetilde{\tau}(\tau).$$

where $\widetilde{\tau}(\tau)$ is a monotonic function[1] of τ. The integration measure changes according to the usual change of variable rule. Next under reparametrization we have

$$\widetilde{X}^\mu(\widetilde{\tau}(\tau)) = X^\mu(\tau).$$

Hence we see that the transformed action is

$$S = -m \int d\widetilde{\tau} \left|\frac{d\tau}{d\widetilde{\tau}}\right| \sqrt{-\frac{d\widetilde{X}^\mu}{d\widetilde{\tau}}\frac{d\widetilde{X}_\mu}{d\widetilde{\tau}}\left(\frac{d\widetilde{\tau}}{d\tau}\right)^2} = -m \int d\widetilde{\tau} \left|\frac{d\tau}{d\widetilde{\tau}}\frac{d\widetilde{\tau}}{d\tau}\right| \sqrt{-\frac{d\widetilde{X}^\mu}{d\widetilde{\tau}}\frac{d\widetilde{X}_\mu}{d\widetilde{\tau}}} = \widetilde{S}.$$

This is a local symmetry of the theory - a gauge symmetry as it depends on the the local coordinates of the spacetime. It is also a continuous symmetry of the action. As is well know, gauge symmetries are not really symmetries in the sense that we do not have an associated conserved charge, rather it is a redundancy in the description of the theory which we need to fix when we go to quantum theory by a process called gauge fixing. We now return to the resolution of Remark 1.1.1. Since the time component of spacetime vector is monotonically increasing, we can reparametrize the worldline in such a way that

$$\widetilde{\tau} = X^0(\tau) = t.$$

Fixing the redundancy of the system we get back to our original action. This shows that we have not increased the number of degrees of freedom of the theory by introducing a parameter.

[1] monotonicity is a techinical requirement for reparametrization. Basically what we need is that as we increase τ, we should traverse the worldline in one given direction and not flip between positions. Generally, $\widetilde{\tau}$ is assumed to be increasing so that we travel the worldline in the same direction as in the original parametrization.

1.3 Quantisation

We will now try to quantise the system. We will illustrate four different methods of quantisation, each with its own advantage. This will help us when we go to the string action.

1.3.1 First Method

We quantise our original action in (1.1.1) directly using the Dirac prescription. The conjugate momentum and the Hamiltonian was calculated to be

$$P^i = \frac{m\dot{X}^i}{\sqrt{1-\dot{\vec{X}}^2}}, \quad H = \frac{m}{\sqrt{1-\dot{\vec{X}}^2}}.$$

where the dot represents derivative with respect to t. We promote the fields to operators with the standard substitution $P^i = -i\partial_i$ where $\partial_i = \frac{\partial}{\partial X^i}$ and introduce the wavefunction which satisfies the Schrödinger equation with the above Hamiltonian. Let $\phi(t, X^i)$ be the wavefunction. Then the Schroödinger equation is given by

$$i\frac{\partial \phi}{\partial t} = H\phi.$$

This implies that

$$-\frac{\partial^2 \phi}{\partial t^2} = H^2\phi.$$

Next, one can easily check that

$$H^2 = \vec{P}^2 + m^2.$$

Thus the Schroödinger equation becomes

$$-\frac{\partial^2 \phi}{\partial t^2} = (-\partial_i^2 + m^2)\phi$$

which implies

$$(\partial_\mu \partial^\mu - m^2)\phi = 0. \tag{1.3.1}$$

We can now solve (1.3.1) and get all the quantum dynamics of the system.

Remark 1.3.1. We can recognise (1.3.1) with the Klein-Gordan equation in field theory. There is one crucial difference in our case and the field theory Klein-Gordan equation. In field theory we quantise quantum fields while in our case (relativistic quantum mechanics), we quantise wavefunctions.

1.3.2 Second Method

We will now denote the τ derivative by dot. That is

$$\dot{X}^\mu = \frac{dX^\mu}{d\tau}.$$

Momentum conjugate to X^μ is

$$P^\mu = \frac{\delta S}{\delta \dot{X}^\mu} = \frac{m\dot{X}^\mu}{\sqrt{-\dot{X}^\mu \dot{X}_\mu}}.$$

One easily sees that

$$P^\mu P_\mu + m^2 = 0. \tag{1.3.2}$$

Eq. (1.3.1) is a constraint. Note that we have not yet appealed to the equation of motion of the action to derive Eq. (1.3.1). Such constraints which follow directly from the definition of the conjugate momenta are called *primary constraints*. The number of primary constraints in a system is equal to the number of zero eigenvalues of the Hessian matrix

$$\frac{\partial P^\mu}{\partial \dot{X}^\nu} = \frac{\partial^2 L}{\partial \dot{X}^\mu \dot{X}^\nu}.$$

Note that by the Inverse Function Theorem we need that all eigenvalues of $\frac{\partial P^\mu}{\partial \dot{X}^\nu}$ be nonzero if we want to express P^μ as a function of $\dot{X}^\mu$. Hence in a system with primary constraint, we cannot express P^μ as functions of $\dot{X}^\mu$.

Remark 1.3.2. Any system with "τ"$-$reparametrization invariance has primary constraints.

The Hamiltonian for the system is

$$H = P^\mu \dot{X}_\mu - L = \frac{m\dot{X}^\mu \dot{X}_\mu}{\sqrt{-\dot{X}^\nu \dot{X}_\nu}} + m\sqrt{-\dot{X}^\nu \dot{X}_\nu} = 0.$$

This is not surprising. Vanishing Hamiltonian signals that nothing changes if we pick another parametrization. To quantise the system, we follow Dirac prescription. We promote the fields to operators and the constraint to an operator equation and demand that the wavefunction $|psi(X)$ satisfy the operator equation:

$$(P^\mu P_\mu + m^2)\Psi(X) = 0.$$

The Schrödinger equation is

$$i\frac{\partial \Psi}{\partial \tau} = H\Psi = 0.$$

This simply implies that the wavefunction does not depend on the parametrization - something that we expected. After the standard substitution

$$P^\mu = -i\partial_\mu, \quad \text{where } \partial_\mu = \frac{\partial}{\partial X^\mu},$$

the operator equation (1.3.1) becomes

$$(\partial_\mu \partial^\mu - m^2)\Psi = 0. \tag{1.3.3}$$

This is the same equation that we got in the first method. Hence we again get the same dynamics.

1.3.3 Third Method - Introducing Einbein

Note that both the equivalent actions above have squareroots which makes it difficult to quantise when we go to path-integral quantisation. So, we somehow want to get rid of the squareroot. Moreover the two previous actions cannot be generalised to massless particles due to the m factor in front of the action. Both these problems can be fixed on the expense of introducing another auxiliary field - an einbein in the action which will be fixed by its equation of motion in the classical theory. To be more precise, consider the action

$$S_e = \frac{1}{2}\int d\tau \left(\frac{\dot{X}^\mu \dot{X}_\mu}{e} - em^2\right),$$

where $e = e(\tau)$ is the auxiliary einbein field. Varying the action with respect to e gives

$$\delta S = \frac{1}{2}\int d\tau \left(-\frac{\dot{X}^\mu \dot{X}_\mu}{e^2} - m^2\right)\delta e.$$

Thus $\delta S = 0$ implies

$$e = \frac{\sqrt{-\dot{X}^\mu \dot{X}_\mu}}{m}. \tag{1.3.4}$$

Not that the equation of motion of e is an algebraic equation and hence the field e is not dynamical. If we now plug the expression for e from (1.3.4) in the action S_e we get

$$\begin{aligned} S_e = \frac{1}{2}\int d\tau \left(-m\frac{\dot{X}^\mu \dot{X}_\mu}{\sqrt{-\dot{X}^\mu \dot{X}_\mu}} - m^2\frac{\sqrt{-\dot{X}^\mu \dot{X}_\mu}}{m}\right) &= -\frac{2m}{2}\int d\tau \sqrt{-\dot{X}^\mu \dot{X}_\mu} \\ &= S. \end{aligned}$$

So both the actions are really the same. Thus the two actions are equivalent classically and give the same dynamics. We now want to quantise this action. The conjugate momentum corresponding to e is

$$P_e = \frac{\partial L}{\partial \dot{e}} = 0.$$

The momentum conjugate to X^μ is

$$P^\mu = \frac{\partial L}{\partial \dot{X}_\mu} = \frac{2}{2e}\dot{X}^\mu \implies \dot{X}^\mu = eP^\mu. \tag{1.3.5}$$

The Hamiltonian of the system is given by

$$H = \dot{X}^\mu P_\mu - L = eP^\mu P_\mu - \frac{m}{2e}e^2 P^\mu P_\mu + \frac{m^2}{2}e = \frac{e}{2}(P^\mu P_\mu + m^2),$$

where we used (1.3.5). The Poisson bracket

$$\{P_e, H\}_{P.B.} = \frac{\partial P_e}{\partial P_e}\frac{\partial H}{\partial e} = \frac{1}{2}(P^\mu P_\mu + m^2).$$

But since $P_e = 0$, we get

$$H = \frac{e}{2}(P^\mu P_\mu + m^2) = 0. \tag{1.3.6}$$

The next step in the quantisation process is to promote fields to operators and use the standard operator substitution for P^μ. Suppose the wavefunction of the system is $\psi = \psi(X, e)$. Then the operator eqaution corresponding to $P_e = 0$ implies

$$-i\frac{\partial \psi}{\partial e} = 0.$$

This means that the wavefunction is independent of the einbein - again something that we expected physically. The operator equation corresponding to (1.3.6) gives

$$(P^\mu P_\mu + m^2)\psi = 0 \implies (\partial_\mu \partial^\mu - m^2)\psi = 0.$$

Thus we get the same quantum dynamics as in the previous two methods.

1.3.4 Fourth Method - Gauge Fixing

We begin by observing that S_e has diffeomorphism symmetry. Indeed if we choose another parametrization $\widetilde{\tau} = \widetilde{\tau}(\tau)$, then

$$X^\mu(\tau) \to \widetilde{X}^\mu(\widetilde{\tau}(\tau)) = X^\mu(\tau)$$

and

$$\frac{\partial X^\mu}{\partial \tau} = \frac{\partial \widetilde{X}^\mu}{\partial \widetilde{\tau}}\frac{\partial \widetilde{\tau}}{\partial \tau}$$

and using (1.3.4)

$$e(\tau) = \sqrt{-\frac{\partial \widetilde{X}^\mu}{\partial \tau}\frac{\partial \widetilde{X}_\mu}{\partial \tau}} = \sqrt{-\frac{\partial \widetilde{X}^\mu}{\partial \widetilde{\tau}}\frac{\partial \widetilde{X}_\mu}{\partial \widetilde{\tau}}}\left|\frac{\partial \widetilde{\tau}}{\partial \tau}\right| = \widetilde{e}(\widetilde{\tau})\left|\frac{\partial \widetilde{\tau}}{\partial \tau}\right|.$$

Thus we see that

$$\begin{aligned} S_e &= \frac{1}{2}\int d\widetilde{\tau}\left|\frac{\partial \tau}{\partial \widetilde{\tau}}\right|\left(-\frac{\partial \widetilde{X}^\mu}{\partial \widetilde{\tau}}\frac{\partial \widetilde{X}_\mu}{\partial \widetilde{\tau}}\left(\frac{\partial \widetilde{\tau}}{\partial \tau}\right)^2\frac{1}{\widetilde{e}(\widetilde{\tau})}\left|\frac{\partial \widetilde{\tau}}{\partial \tau}\right|^{-1} - m^2\widetilde{e}(\widetilde{\tau})\left|\frac{\partial \widetilde{\tau}}{\partial \tau}\right|\right) \\ &= \frac{1}{2}\int d\widetilde{\tau}\left(-\frac{\partial \widetilde{X}^\mu}{\partial \widetilde{\tau}}\frac{\partial \widetilde{X}_\mu}{\partial \widetilde{\tau}}\frac{1}{\widetilde{e}(\widetilde{\tau})} - m^2\widetilde{e}(\widetilde{\tau})\right) \\ &= \widetilde{S}_e. \end{aligned}$$

So before we go on quantising the system, we will fix a gauge. We choose a reparametrization $\widetilde{\tau}(\tau)$ such that

$$\widetilde{e}(\widetilde{\tau}) = 1.$$

With this gauge choice, when we go to the quantum theory, we will have to take care of the equation of motion of the einbein and impose it as operator equation with the chosen gauge. We follow the standard procedure of quatization by promoting fields to operators. The equation of motion for e with the above gauge choice becomes:

$$\widetilde{e}(\widetilde{\tau})^2 = -\frac{1}{m^2}\frac{\partial \widetilde{X}^\mu}{\partial \widetilde{\tau}}\frac{\partial \widetilde{X}_\mu}{\partial \widetilde{\tau}} = 1.$$

Using (1.3.5), the equation of motion of einbein becomes

$$P^\mu P_\mu + m^2 = 0. \tag{1.3.7}$$

With the chosen gauge, the action becomes

$$S = \frac{1}{2}\int d\tau \left(\dot{X}^\mu \dot{X}_\mu - m^2\right),$$

where we removed the tildes for brevity. Using (1.3.6) and the gauge choice along with (1.3.7), the Hamiltonian is given by

$$H = \frac{1}{2}(P^\mu P_\mu + m^2) = 0.$$

With the standard substitution for the momentum operator $P_\mu = -i\partial_\mu$, the wavefunction ψ of the system satisfies

$$(\partial_\mu \partial^\mu - m^2)\psi = 0.$$

Hence we again get the same dynamics.

Chapter 2

The Relativistic String

We now want to write an action of the a free relativistic string - the fundamental objects in string theory. As we discussed in the previous chapter, we need to start with a Lorentz invariant action. Since the string is a two dimensional object, it traces a surface called the *worldsheet* in the spacetime. The most natural choice of the action would then be the surface area of the worldsheet traced by the string. We begin by deriving the action of the relativistic string.

2.1 Nambu-Goto Action

The surface traced by the string can be parametrized by two parameters (σ, τ). Let the worldsheet coordinates be $X^\mu(\sigma, \tau)$. To calculate the area of the worldsheet, we will use the worldsheet coordinates. Infinitesimal change in the parameters σ and τ along the worldsheet coordinates is

$$\delta\sigma = \frac{\partial X^\mu}{\partial \sigma} d\sigma, \quad \delta\tau = \frac{\partial X^\mu}{\partial \tau} d\tau.$$

Note that the area of the parallelogram determined by two vectors $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\begin{aligned}
\|\mathbf{A}\|\|\mathbf{B}\| \sin\theta = \|\mathbf{A}\|\|\mathbf{B}\|\sqrt{1-\cos^2\theta} &= \sqrt{\mathbf{A}^2\mathbf{B}^2 - \frac{(\mathbf{A}\cdot\mathbf{B})^2}{\mathbf{A}^2\mathbf{B}^2}\mathbf{A}^2\mathbf{B}^2} \\
&= \sqrt{(\mathbf{A}\cdot\mathbf{A})(\mathbf{B}\cdot\mathbf{B}) - (\mathbf{A}\cdot\mathbf{B})^2} \\
&= \left(\det \begin{bmatrix} \mathbf{A}\cdot\mathbf{A} & \mathbf{A}\cdot\mathbf{B} \\ \mathbf{A}\cdot\mathbf{B} & \mathbf{B}\cdot\mathbf{B} \end{bmatrix}\right)^{\frac{1}{2}},
\end{aligned}$$

where $\|\mathbf{A}\|^2 = \mathbf{A}\cdot\mathbf{A} = \mathbf{A}^2$. So the infinitesimal area of the parallelogram on the worldsheet determined by the vectors $\delta\sigma$ and $\delta\tau$ is

$$d\text{Area} = \left[-\det\left(\partial_\alpha X^\mu \partial_\beta X_\mu\right)\right]^{\frac{1}{2}}, \qquad \boldsymbol{\sigma} = \sigma^\alpha \equiv (\sigma, \tau), \quad \alpha = 1, 2.$$

The minus sign indicates the fact that one of the vectors is timelike ($X^2 < 0$). The Nambu-Goto action is then defined by

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_M d\sigma d\tau \sqrt{-\det\left(\partial_\alpha X^\mu \partial_\beta X_\mu\right)}, \tag{2.1.1}$$

where M is the surface traced by the string, α' is called the *Regge slope*. The reason for this name will be evident in later chapters. We often write

$$h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu.$$

The action can then be written as

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_M d\sigma d\tau \mathcal{L}_{NG}, \quad \mathcal{L}_{NG} = \left[-\det(h_{\alpha\beta})\right]^{\frac{1}{2}}.$$

The worldsheet is in general a curved manifold embedded in spacetine. In the language of differential geometry, $h_{\alpha\beta}$ is called the Pullback metric from the ambient spacetine. The factor of $\frac{1}{2\pi\alpha'}$ can be interpreted as string tension.

2.1.1 Symmetries of the Nambu-Goto action

S_{NG} has global symmetries as well as local symmetries. Let us look at them more closely.

Reparametrization Invariance

If we choose another parametrization for the worldsheet $\widetilde{\tau}(\sigma,\tau), \widetilde{\sigma}(\sigma,\tau)$ then the Jacobian of the variable change is

$$J = \det\begin{bmatrix} \frac{\partial\widetilde{\sigma}}{\partial\sigma} & \frac{\partial\widetilde{\sigma}}{\partial\tau} \\ \frac{\partial\widetilde{\tau}}{\partial\sigma} & \frac{\partial\widetilde{\tau}}{\partial\tau} \end{bmatrix}$$

and the worldsheet coordinates changes as

$$\frac{\partial X^\mu}{\partial\sigma^\alpha} = \frac{\partial\widetilde{X}^\mu}{\partial\widetilde{\sigma}^\beta}\frac{\partial\widetilde{\sigma}^\beta}{\partial\sigma^\alpha}.$$

This gives

$$h_{\alpha\beta} = \frac{\partial\widetilde{X}^\mu}{\partial\widetilde{\sigma}^\gamma}\frac{\partial\widetilde{X}_\mu}{\partial\widetilde{\sigma}^\delta}\frac{\partial\widetilde{\sigma}^\gamma}{\partial\sigma^\alpha}\frac{\partial\widetilde{\sigma}^\delta}{\partial\sigma^\beta} = \widetilde{h}_{\alpha\beta}\frac{\partial\widetilde{\sigma}^\gamma}{\partial\sigma^\alpha}\frac{\partial\widetilde{\sigma}^\delta}{\partial\sigma^\beta}.$$

Thus we have

$$\det\left(h_{\alpha\beta}\right) = \det\left(\widetilde{h}_{\alpha\beta}\right) J^2,$$

where we used the fact that $J = \det\left(\frac{\partial\widetilde{\sigma}^\alpha}{\partial\sigma^\beta}\right)$. Plugging everything in the action, we see that

$$S_{NG} = -\frac{1}{2\pi\alpha'}\int_M d\widetilde{\sigma}d\widetilde{\tau} J^{-1}\left(-\det\left(\widetilde{h}_{\alpha\beta}\right)\right)^{\frac{1}{2}} J = \widetilde{S}_{NG}.$$

Reparametrization invariance is also called *diffeomorphism* invariance and is a *gauge symmetry* of the action. We can write the infinitesimal version of the reparametrization as follows:

$$\sigma^\alpha \to \widetilde{\sigma}^\alpha = \sigma^\alpha + \xi^\alpha + O(\xi^2). \tag{2.1.2}$$

Under this change in parameter we have

$$X^\mu(\sigma^\alpha) \to \widetilde{X}^\mu(\widetilde{\sigma}^\alpha) = X^\mu(\sigma^\alpha).$$

We have

$$\widetilde{X}^\mu(\widetilde{\sigma}^\alpha) = X^\mu(\sigma^\alpha) = X^\mu(\widetilde{\sigma}^\alpha - \xi^\alpha) = X^\mu(\widetilde{\sigma}^\alpha) - \xi^\alpha \partial_\alpha X^\mu,$$

where we used Taylor's theorem. This gives

$$\delta X^\mu = \widetilde{X}^\mu(\widetilde{\sigma}^\alpha) - X^\mu(\widetilde{\sigma}^\alpha) = -\xi^\alpha \partial_\alpha X^\mu \tag{2.1.3}$$

Poincaré Invariance

The worldsheet coordinates transform under the Poincaré transformation as follows:

$$X^\mu \to \widetilde{X}^\mu = \Lambda^\mu{}_\nu X^\nu + c^\mu, \tag{2.1.4}$$

where $\Lambda^\mu{}_\nu$ is a Lorentz transformation and c^μ is a constant vector. This is a manifest symmetry of the action. Poincaré invariance is a global symmetry of the action. The infinitesimal version is often calculated in a first course in quantum field theory. We will record it here for later use.

$$\delta X^\mu = a^\mu_\nu X^\nu + b^\mu, \quad (a_{\mu\nu} = -a_{\nu\mu}). \tag{2.1.5}$$

2.1.2 Equations of Motion

We begin by expanding out the determinant in the action. We get

$$\det(h_{\alpha\beta}) = \det(\partial_\alpha X^\mu \partial_\beta X_\mu) = X'^2 \dot{X}^2 - (X' \cdot \dot{X})^2,$$

where

$$X' = \frac{\partial X}{\partial \sigma} \quad ; \quad \dot{X} = \frac{\partial X}{\partial \tau} \quad \& \quad X^2 = X^\mu X_\mu.$$

So we have

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\sigma d\tau \left[-X'^2 \dot{X}^2 + \left(X' \cdot \dot{X} \right)^2 \right]^{\frac{1}{2}},$$

where we have written $X \equiv X^\mu$. The conjugate momenta are given by

$$\Pi^\tau_\mu = \frac{\partial L_{NG}}{\partial \dot{X}^\mu} = -\frac{1}{2\pi\alpha'} \left[\frac{\left(\dot{X} \cdot X' \right) X'_\mu - (X'^2) X_\mu}{\sqrt{\left(X' \cdot \dot{X} \right)^2 - (X'^2 \dot{X}^2)}} \right]$$

$$\Pi^\sigma_\mu = \frac{2\mathcal{L}_{NG}}{\partial X'^\mu} = -\frac{1}{2\pi\alpha'} \left[\frac{\left(\dot{X} \cdot X' \right) \dot{X}_\mu - (X'^2) \dot{X}_\mu}{\sqrt{\left(X' \cdot \dot{X} \right)^2 - \left(X'^2 \dot{X}^2 \right)}} \right].$$

Observe that

$$\frac{\partial^2 \mathcal{L}_{NG}}{\partial \dot{X}^\mu \partial \dot{X}^\nu} \cdot \dot{X}^\nu = \frac{\partial \Pi^\tau_\mu}{\partial \dot{X}^\nu} \dot{X}^\nu = 0,$$
$$\frac{\partial^2 \mathcal{L}_{NG}}{\partial \dot{X}^\mu \partial \dot{X}^\nu} \cdot X'^\nu = \frac{\partial \Pi^\tau_\mu}{\partial \dot{X}^\nu} X'^\nu = 0.$$

So the Hessian $\frac{\partial^2 \mathcal{L}_{NG}}{\partial \dot{x}^\mu \partial \dot{x}^\nu}$ has two zero eigenvalues with eigenvectors $\dot{X}^\mu, X'^\mu$ must have two constraints. We can check that

$$\Pi^\tau_\mu X'^\mu = 0, \quad \Pi^\tau_\mu \Pi^{\tau\mu} + \frac{1}{4\pi^2 \alpha'^2} X'^\mu X'_\mu = 0. \tag{2.1.6}$$

These are one set of constraints. Another set of constraints arise from the fact that

$$\frac{\partial^2 \mathcal{L}_{NG}}{\partial X'^\mu \partial X'^2} \dot{X}^\nu = 0 \quad \text{and} \quad \frac{\partial^2 \mathcal{L}_{NG}}{\partial X'^\mu \partial X'^\nu} X'^\nu = 0.$$

The resulting constraints are

$$\Pi^\sigma_\mu \dot{X}^\mu = 0, \quad \Pi^\sigma_\mu \Pi^{\sigma\mu} + \frac{1}{4\pi^2 \alpha^2} \dot{X}^\mu \dot{X}_\mu = 0. \tag{2.1.7}$$

The Hamiltorian

$$\mathcal{H}^\sigma = \Pi^\sigma_\mu X'^\mu - \mathcal{L}_{NG} = 0; \quad \& \quad \mathcal{H}^\tau = \Pi^\tau_\mu \dot{X}^\mu - \mathcal{L}_{NG} = 0.$$

So the dynamics is determined by constraints. The equation of motion is given by

$$\frac{\partial \Pi^\tau_\mu}{\partial \tau} + \frac{\partial \Pi^\sigma_\mu}{\partial \sigma} = 0. \tag{2.1.8}$$

We can also write the equation of motion in another way. Recall that

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int_M d\sigma d\tau \sqrt{-h}; \quad h = \det h_{\alpha\beta}.$$

From general relativity we have

$$\delta\sqrt{-h} = \frac{1}{2}\sqrt{-h} h^\alpha \beta \delta h_{\alpha\beta}$$

So

$$\frac{\delta \mathcal{L}_{NG}}{\delta (\partial_\alpha X^\mu)} = -\frac{1}{2\pi\alpha'} \left(\frac{1}{2} \sqrt{-h} h^{\alpha\beta} (2\partial_\beta X_\mu) \right)$$

So equation of motion is

$$\partial_\alpha \left(\frac{\partial \mathcal{L}_{NG}}{\partial (\partial_\alpha X^\mu)} \right) = 0$$

which gives

$$\partial_\alpha \left(\sqrt{-h} h^{\alpha\beta} (\partial_\beta X_\mu) \right) = 0.$$

2.2 The Polyakov Action

The final goal of studying string action is to quantise the action and analyse the spectrum that we obtain. The first challenge that we face when we try to quantise the Nambu-Goto action is the squareroot in the action. It is generally tricky to quantise such complicated actions when we go to path integral quantisation. This is why we will use the fourth method of quantisation introduced in Chapter 1. To this end, consider the following action:

$$S_p = -\frac{1}{4\pi\alpha'}\int_M d\sigma d\tau \sqrt{-g} g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu, \tag{2.2.1}$$

where $g = \det(g^{\alpha\beta})$ and $g^{\alpha\beta}$ is an auxiliary background field which plays the role of the einbein in the fourth method of quantisation. This action is called the Polyakov action. The auxillary field $g_{\alpha\beta}$ is a dynamical metric on the world-sheet with Lorentzian signature $(-,+)$. Thus the action S_P can be viewed as a bunch of scalar fields $X^\mu(\sigma,\tau)$ coupled to a $2d$ gravity theory.

2.2.1 Equivalence of S_P and S_{NG}

Let us find the equations of motion of $g_{\alpha\beta}$. Varying S_P with respect to $g_{\alpha\beta}$ gives two terms. We get

$$\delta S_P = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \left[\sqrt{-g}\delta g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X\mu - \frac{1}{2}\sqrt{-g} g_{\alpha\beta}\delta g^{\alpha\beta} g^{ab}\partial_a X^\mu \partial_b X\mu\right].$$

So

$$\delta S_P = 0 \implies \sqrt{-g}\delta g^{\alpha\beta}\left(\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} g_{\alpha\beta} g^{ab}\partial_a X^\mu \partial_b X_\mu\right) = 0.$$

Here we used

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\alpha\beta}\delta g^{\alpha\beta} = \frac{1}{2}\sqrt{-g} g^{\alpha\beta}\delta g_{\alpha\beta}.$$

So the equation of motion of $g_{\alpha\beta}$ is

$$\partial\alpha X^\mu \partial\beta X_\mu = \frac{1}{2} g_{\alpha\beta} g^{ab}\partial_a X^\mu \partial_b X_\mu.$$

Or

$$\partial_\alpha X^\mu \partial_\beta X_\mu = \frac{1}{2} g_{\alpha\beta}\partial^c X^\mu \partial_c X_\mu. \tag{2.2.2}$$

Taking determinant both sides we get

$$\det\left(\partial_\alpha X^\mu \partial_\beta X_\mu\right) = \det\left(\frac{1}{2} g_{\alpha\beta}\partial^c X^\mu \partial_c X_\mu\right).$$

Since $g_{\alpha\beta}$ is 2×2, we get

$$\begin{aligned}\det\left(\partial\alpha X^\mu \partial_\beta X_\mu\right) &= \frac{1}{4}\left(\partial^c X^\mu \partial_c X_\mu\right)^2 g\\ \implies \sqrt{-\det\left(\partial_\alpha X^\mu \partial_\beta X_\mu\right)} &= \frac{1}{2}\sqrt{-g}\left(\partial^c X^\mu \partial_c X_\mu\right).\end{aligned}$$

Substituting this in S_P gives S_{NG}. Thus we see that S_P and S_{NG} are equivalent classically. These two actions presumably gives same quantum dynamics but a rigorous proof is lacking. Indeed path-integral quantisation of S_{NG} is rather difficult to perform due to squareroot and manipulating it to get results involves similar tricks as we have used in the $S_{NG} \longrightarrow S_P$ transition.

2.2.2 Equation of Motion

Let us vary the action with respect to X^μ with $\delta X^\mu(\sigma, \tau_0) = \delta X^\mu(\sigma, \tau_1) = 0$ for some initial and final value τ_0, τ_1 respectively of the parameter τ. Assuming that the string length is ℓ, we have

$$
\begin{aligned}
\delta S_P &= -\frac{1}{4\pi\alpha'} \int_{\tau_0}^{\tau_1} d\tau \int_0^\ell d\sigma \left[2\partial_\alpha X^\mu \left(\partial^\alpha \delta X_\mu\right)\right] \\
&= -\frac{1}{2\pi\alpha'} \int_{\tau_0}^{\tau_1} d\tau \int_0^\ell d\sigma \left[\partial_\alpha \left(\partial^\alpha X^\mu \delta X_\mu\right) - \left(\partial_\alpha \partial^\alpha X^\mu\right) \delta X_\mu\right] \\
&= \frac{1}{2\pi\alpha'} \int_{\tau_0}^{\tau_1} d\tau \int_0^\ell d\sigma \left(\partial_\alpha \partial^\alpha X^\mu\right) \delta X_\mu - \frac{1}{2\pi\alpha'} \int_{\tau_0}^{\tau_1} d\tau \int_0^\ell d\sigma \left(\partial_\sigma \left(\partial^\sigma X^\mu \delta X_\mu\right) - \partial_\tau \left(\partial^\tau X^\mu \delta X_\mu\right)\right) \\
&= \frac{1}{2\pi\alpha} \int_{\tau_0}^{\tau_1} d\tau \int_0^\ell d\sigma \left(\partial_\alpha \partial^\alpha X^\mu\right) \delta X_\mu + \underbrace{\frac{1}{2\pi\alpha'} \int_0^\ell d\sigma \left(\partial^\tau X^\mu\right) \delta X_\mu \Big|_{\tau_0}^{\tau_1}}_{= 0 \text{ as } \delta X^\mu(\sigma, \tau_0) = \delta X^\mu(\sigma, \tau_1) = 0} \\
&\qquad + \frac{1}{2\pi\alpha'} \int_{\tau_0}^{\tau_1} d\tau \underbrace{\left(\partial^\sigma X^\mu \delta X_\mu\right) \Big|_0^\ell}_{\text{surface term}} .
\end{aligned}
$$

To get the equations of motion, we need the surface term to go to zero. Physically we distinguish between two cases - the closed string and the open string. We will deal with the two cases separately.

Closed Strings

We normalise the string length so that $\ell = 2\pi$. Closed string then means that the ends of the string are joined together in a smooth fashion to form a loop. This means that $X^\mu(\sigma, \tau)$ are periodic in σ with period 2π :

$$X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau).$$

This implies that $\delta X_\mu(0,\tau) = \delta X_\mu(2\pi,\tau) = 0$. Thus the equation of motion for closed strings is

$$\partial_\alpha \partial^\alpha X^\mu = 0. \tag{2.2.3}$$

Open Strings

The ends of open string are free and so we need to impose boundary conditions on the ends of the open string. We normalise the length of the string to $\ell = \pi$. We impose the boundary condition such that the surface term vanishes. There are three ways for this to happen - atleast one of the two $\partial^\sigma X^\mu$ and δX_μ or the combination $\partial^\sigma X^\mu \delta X_\mu$ must be zero at $\sigma = 0$ and $\sigma = \pi$. Hence we have three different bondary condition:

1. **Dirichlet boundary condition:** $\delta X_\mu = 0 \quad$ at $\sigma = 0, \pi$.
2. **Neumann boundary condition:** $\partial_\sigma X_\mu = 0$ at $\sigma = 0, \pi$.
3. **Robin boundary condition:** $\partial_\sigma X_\mu \delta X^\mu = 0$ at $\sigma = 0, \pi$.

The first two boundary conditions have been studied in detail in literature and we will also analyse each boundary condition along with mixed boundary condition in detail as we progress in our study.

2.2.3 Symmetries of S_P

As with S_{NG}, we can directly read off two obvious symmetries of S_P:

Reparametrization Invariance

If we transform the parameters as $\sigma^\alpha \longrightarrow \widetilde{\sigma}^\alpha = \widetilde{\sigma}^\alpha(\boldsymbol{\sigma})$ then the scalar fields X^μ transform as

$$X^\mu(\sigma,\tau) \longrightarrow \widetilde{X}^\mu\left(\widetilde{\sigma}^\alpha\right) = X^\mu\left(\sigma^\alpha\right)$$

and the world-sheet metric $g_{\alpha\beta}$ transforms in the usual way

$$g_{\alpha\beta} \longrightarrow \widetilde{g}_{\alpha\beta}\left(\tilde{\sigma}^\alpha\right) = \frac{\partial \sigma^\gamma}{\partial \widetilde{\sigma}^\alpha} \frac{\partial \sigma^\delta}{\partial \widetilde{\sigma} \beta} g_{\gamma\delta}(\sigma).$$

We can find the infinitesimal transformation under $\sigma^\alpha \longrightarrow \sigma^\alpha = \sigma^\alpha - \eta^\alpha$, where η^α is small, using Lie derivative. Indeed under infinitesimal transformation

$$\delta g_{\alpha\beta} = \mathcal{L}_\eta g_{\alpha\beta} = \nabla_\alpha \eta_\beta + \nabla_\beta \eta_\alpha,$$

where ∇_α is the Leve-Civita covariant derivative with the usual Levi-Civita connection

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}\left(\partial_\beta g_{\gamma\delta} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\rho\gamma}\right)$$

Also $\sqrt{-g}$ changes as $\delta(\sqrt{-g}) = \partial_\alpha\left(\eta^\alpha \sqrt{-g}\right)$. The Polyakov action S_P is easily seen to be invariant under reparametrizations. This is a gauge symmetry of the action.

Poincaré Invariance

This is a global symmetry of the action.

$$X^{\mu} \longrightarrow \widetilde{X}^{\mu} = \Lambda^{\mu}_{v} X^{v} + \xi^{\mu}_{\rho}$$

for some constant ξ^{μ}. The infinitesimal version of this transformation is given in 2.1.1.

Weyl Invariance

There is another gauge invariance called Weyl symmetry. Under this $X^{\mu} \longrightarrow X^{\mu}$ and the metric transforms as

$$g_{\alpha\beta} \longrightarrow \widetilde{g}_{\alpha\beta} = \Omega^2(\boldsymbol{\sigma}) g_{\alpha\beta}$$

or infinitesimally if $\Omega^2(\boldsymbol{\sigma}) = e^{2\phi(\boldsymbol{\sigma})}$ then

$$\delta g_{\alpha\beta} = 2\phi(\boldsymbol{\sigma}) g_{\alpha\beta}.$$

To see that this is a symmetry of the action, note that $\sqrt{-g} \longrightarrow \Omega^2(\boldsymbol{\sigma})\sqrt{-g}$ as

$$\det\left(\Omega^2 g_{\alpha\beta}\right) = \Omega^4(\boldsymbol{\sigma}) \det\left(g_{\alpha\beta}\right)$$

and $g^{\alpha\beta} \longrightarrow (\Omega(\boldsymbol{\sigma}))^{-2} g^{\alpha\beta}$. Thus factors from $\sqrt{-g}$ and $g^{\alpha\beta}$ cancel.

Remark 2.2.1. Weyl transformation is not a coordinate transformation. Rather it is a local change of scale under which the theory is invariant. More precisely, this scale change preserves angles between as the metric transforms conformally.

Remark 2.2.2. Weyl transformation is unique to two dimensions since $\sqrt{-g}g^{\alpha\beta}$ remain invariant under $g_{\alpha\beta} \longrightarrow \Omega^2 g_{\alpha\beta}$ only in two dimensions.

Chapter 3

The Closed String

In chapter 2, we found the correct action for relativistic strings namely the Polyakov action. We also found the equations of motion arising from the action and depending on the type of string - open or closed, we imposed boundary conditions. In this chapter, we will solve the classical equations of motion for the closed string and also quantise the theory using two different procedures. We will also analyse the closed string spectrum.

3.1 The Closed Classical String

As we saw in the previous chapter, the Polyakov action has two gauge symmetries. Hence to find the equations of motion, we first need to fix a gauge. This means that we should make an appropriate choice of the background metric using our gauge symmetries.

3.1.1 Fixing a Gauge

We have two diffeomorphism invariance namely for σ, τ and three independent metric components. Write

$$g_{\alpha\beta} = \begin{pmatrix} g_{\sigma\sigma} & g_{\sigma\tau} \\ g_{\tau\sigma} & g_{\tau\tau} \end{pmatrix} \qquad \text{then} \qquad g_{\sigma\tau} = g_{\tau\sigma}.$$

Now since $g_{\alpha\beta}$ has signature $(-,+)$, at least locally one out of $g_{\sigma\sigma}$ and $g_{\tau\tau}$ must be positive. Under diffeomorphism we have

$$g_{\alpha\beta} \longrightarrow \widetilde{g}_{\alpha\beta} = \frac{\partial\sigma^\gamma}{\partial\widetilde{\sigma}^\alpha}\frac{\partial\sigma^\delta}{\partial\widetilde{\sigma}^\beta}g_{\gamma\delta}.$$

This gives

$$\widetilde{g}_{\sigma\sigma} = \left(\frac{\partial\sigma}{\partial\widetilde{\sigma}}\right)^2 g_{\sigma\sigma} + \left(\frac{\partial\tau}{\partial\widetilde{\sigma}}\right)^2 g_{\tau\tau} + 2\frac{\partial\sigma}{\partial\widetilde{\sigma}}\frac{\partial\tau}{\partial\widetilde{\sigma}}g_{\sigma\tau}$$
$$\widetilde{g}_{\tau\tau} = \left(\frac{\partial\sigma}{\partial\widetilde{\tau}}\right)^2 g_{\sigma\sigma} + \left(\frac{\partial\tau}{\partial\widetilde{\tau}}\right)^2 g_{\tau\tau} + 2\frac{\partial\sigma}{\partial\widetilde{\tau}}\frac{\partial\tau}{\partial\widetilde{\tau}}g_{\sigma\tau}$$

and

$$\widetilde{g}_{\sigma\tau} = \widetilde{g}_{\tau\sigma} = \frac{\partial\sigma}{\partial\widetilde{\sigma}}\frac{\partial\sigma}{\partial\widetilde{\tau}}g_{\sigma\sigma} + \frac{\partial\tau}{\partial\widetilde{\sigma}}\frac{\partial\tau}{\partial\widetilde{\tau}}g_{\tau\tau} + \frac{\partial\tau}{\partial\widetilde{\sigma}}\frac{\partial\sigma}{\partial\widetilde{\tau}}g_{\tau\sigma} + \frac{\partial\sigma}{\partial\widetilde{\sigma}}\frac{\partial\tau}{\partial\widetilde{\tau}}g_{\sigma\tau}.$$

Now suppose in a neighbourhood of (σ, τ), $g_{\tau\tau} > 0$ then we put $\widetilde{g}_{\sigma\tau} = \widetilde{g}_{\tau\sigma} = 0$ and $\widetilde{g}_{\sigma\sigma} = -g_{\tau\tau}$. Thus we have a system of two first order partial differential equations to solve for two function $\widetilde{\sigma}(\sigma, \tau)$ and $\widetilde{\tau}(\sigma, \tau)$ that is we need to solve for $\widetilde{\sigma}(\sigma, \tau)$ and $\widetilde{\tau}(\sigma, \tau)$ from

$$\begin{aligned}
&\left(\frac{\partial\sigma}{\partial\widetilde{\sigma}}\right)^2 g_{\gamma\sigma} + \left(\frac{\partial\tau}{\partial\widetilde{\sigma}}\right)^2 g_{\tau\tau} + 2\frac{\partial\sigma}{\partial\widetilde{\sigma}}\frac{\partial\tau}{\partial\widetilde{\sigma}}g_{\sigma\tau} = -g_{\tau\tau} \\
&\frac{\partial\sigma}{\partial\widetilde{\sigma}}\frac{\partial\sigma}{\partial\widetilde{\tau}}g_{\sigma\sigma} + \frac{\partial\tau}{\partial\widetilde{\sigma}}\frac{\partial\tau}{\partial\widetilde{\tau}}g_{\tau\tau} + \frac{\partial\tau}{\partial\widetilde{\sigma}}\frac{\partial\sigma}{\partial\widetilde{\tau}}g_{\tau\sigma} + \frac{\partial\sigma}{\partial\widetilde{\sigma}}\frac{\partial\tau}{\partial\widetilde{\tau}}g_{\sigma\tau} = 0.
\end{aligned}$$

Solution to this exists atleast locally by Cauchy-Kowalevski theorem since the coefficient functions are real analytic. Thus we have transformed $g_{\alpha\beta}$ to $g_{\tau\tau}\eta_{\alpha\beta}$ using the two diffeomorphisms. Since $g_{\tau\tau} = e^{\phi(\boldsymbol{\sigma})}$ thus we now use Weyl rescaling to transform

$$g_{\alpha\beta} \longrightarrow e^{-\phi(\boldsymbol{\sigma})}g_{\alpha\beta} = \eta_{\alpha\beta}.$$

This gauge is called *Conformal gauge.*

Remark 3.1.1. Any $2d$ metric can be made flat using Wely invariance: Suppose $g'_{\alpha\beta} = e^{\phi(\boldsymbol{\sigma})}g_{\alpha\beta}$ then one can easily check that

$$\sqrt{-g'}R' = \sqrt{g}\left(R - \nabla^2\phi\right).$$

If we choose of such that $\nabla^2\phi = R$ then $R' = 0$. But in $2d$ vanishing Ricci scalar implies that Riemann curvature tensor is zero. Since in $2d$ one can show that

$$R_{\alpha\beta\gamma\delta} = \frac{R}{2}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right).$$

Hence the metric is flat.

Remark 3.1.2. Can the world-sheet metric be made flat globally? Depends on the topology of the space. Locally the metric can be made flat using the three gauge symmetries. Suppose we could extend this locally flat metric to whole worldsheet. This means that the whole worldsheet is covered by a coordinate chart which is flat. This in turn means that the Ricci scalar identically vanishes on the worldsheet. Topologically since in $2d$, the Euler characteristic χ of a manifold satisfies

$$\chi \propto \int_M R.$$

Thus a necessary condition of the extension to be possible is that $\chi = 0$.

We have fixed a gauge. Now we need to find the equation of motion of $g_{\alpha\beta}$ and impose it as a constraint on the classical system after substituting $g_{\alpha\beta} = \eta_{\alpha\beta}$. We have already calculated the equation of motion in subsection 2.2.1 but we can recast it in terms of energy momentum tensor which is often more useful. We begin by writing the gauge fixed action:

$$S_P^{gf} = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \partial_\alpha X^\mu \partial^\alpha X_\mu. \tag{3.1.1}$$

The equation of motion for X^μ is

$$\partial^\alpha \partial_\alpha X^\mu = 0. \tag{3.1.2}$$

Next we have

$$\frac{\delta S}{\delta g^{\alpha\beta}} = -\frac{1}{4\pi\alpha'}\left[-\frac{\sqrt{-g}}{2} g_{\alpha\beta}\partial_c X^\mu \partial^c X_\mu + \sqrt{-g}\partial_\alpha X^\mu \partial_\beta X_\mu\right].$$

Define the energy momentum tensor[1] by

$$T_{\alpha\beta} = -4\pi\alpha' \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g_{\alpha\beta}}.$$

We get

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} g_{\alpha\beta}\partial_c X^\mu \partial^c X_\mu.$$

So that

$$T_{\alpha\beta}\big|_{g_{\alpha\beta}=\eta_{\alpha\beta}} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\partial_c X^\mu \partial^c X_\mu.$$

The equation of motion for $g_{\alpha\beta}$ was

$$\partial_\alpha X^\mu \partial_\beta X_\mu = \frac{1}{2} g_{\alpha\beta}\partial_c X^\mu \partial' X_\mu.$$

So our constraint is $T_{\alpha\beta} = 0$. Written in terms of components:

$$T_{01} = \dot{X}^\mu X'_\mu = 0 \quad T_{11} = T_{00} = \dot{X}^2 - \frac{1}{2}\left(-\left(-\dot{X}^2 + X'^2\right)\right) = \frac{1}{2}\left(\dot{X}^2 + X'^2\right).$$

So we have to impose two constraints

$$\dot{X}^\mu X'_\mu = 0, \quad \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0. \tag{3.1.3}$$

So the equation of motion is a wave equation along the two constraints. We will now solve it.

[1] note that this is not the usual definition of energy momentum tensor. In general relativity (GR) we have different normalisation. In GR the energy moemntum tensor is given by $T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\alpha\beta}}$.

3.1.2 Solving the Equation of Motion: Mode Expansion

We will use the lightcone coordinates to solve the equations of motion (3.1.2) subject to constraints (3.1.3). Introduce the lightcone coordinates

$$\sigma^{\pm} = \tau \pm \sigma,$$

then

$$\partial_+ = \partial_\tau + \partial_\sigma, \quad \partial_- = \partial_\tau - \partial_\sigma.$$

With this, the equation of motion $\partial_\alpha \left(\partial^\alpha X^\mu\right) = 0$ reduces to

$$\partial_+ \partial_- X^\mu = 0. \tag{3.1.4}$$

Indeed we have

$$\partial_+ \partial_- X^\mu = \partial_+ \left(\partial_\tau X^\mu - \partial_\sigma X^\mu\right) = \partial_{\tau\tau} X^\mu - \partial_{\tau\sigma} X^\mu - \partial_{\sigma\tau} X^\mu - \partial_{\sigma\sigma} X^\mu = 0.$$

The most general solution to $\partial_+ \partial_- X^\mu = 0$ is given by

$$X^\mu(\sigma, \tau) = X_L^\mu(\sigma+) + X_R^\mu(\sigma^-) \tag{3.1.5}$$

for arbitrary functions X_L and X_R. For closed strings, we have the periodicity condition

$$X^\mu(\sigma + 2\pi, \tau) = \quad X^\mu(\sigma, \tau).$$

This implies that X^μ can be written as a Fourier series. More precisely, we have

$$\begin{aligned} X_L^\mu\left(\sigma^+\right) &= \frac{x^\mu}{2} + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \widetilde{\alpha}_n^\mu e^{-in\sigma^+} \\ X_R^\mu\left(\sigma^-\right) &= \frac{x^\mu}{2} + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\sigma^-}. \end{aligned} \tag{3.1.6}$$

The functions X_L^μ are called left movers and X_R^μ are called right movers.

Remark 3.1.3. 1. The factors $\alpha', \frac{1}{n}$ have been chosen for convenience when we quantise the system.

2. X_L^μ and X_R^μ are not periodic due to the linear term σ^+, σ^- but the combination $X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right)$ is periodic as σ cancels from the combination $\sigma^+ + \sigma^- = 2\tau$.

3. The quantities x^μ and p^μ are the position and momentum of the center of mass of the string. We will prove this explicitly. Observe that for the Polyakov action,

$$\Pi_\mu^\tau = \frac{\partial \mathcal{L}_P}{\partial \dot{X}^\mu} = -\frac{1}{4\pi\alpha'} \frac{\partial}{\partial \dot{X}^\mu} \left[-\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu\right] = \frac{1}{2\pi\alpha'} \dot{X}_\mu.$$

So

$$P^{\mu} = \int_0^{2\pi} d\sigma \frac{1}{\partial \pi \alpha'} \dot{X}^{\mu} = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \dot{X}_L^{\mu}\left(\sigma^+\right) + \dot{X}_R^{\mu}\left(\sigma^-\right) = \frac{1}{2\pi\alpha'} 2\pi\alpha' p^{\mu} = p^{\mu},$$

and

$$q^{\mu} = \frac{1}{2\pi} \int_0^{2\pi} d\sigma X^{\mu} = \frac{1}{2\pi} \int_0^{2\pi} d\sigma X_L^{\mu}\left(\sigma^+\right) + X_R^{\mu}\left(\sigma^-\right) = \frac{1}{2\pi}\left[2\pi x^{\mu} + 2\pi\alpha' p^{\mu}\tau\right] = x^{\mu} + \alpha' p^{\mu}\tau.$$

So we see that p^{μ} is indeed the momentum and x^{μ} is the position of center of mass of the string at $\tau = 0$.

4. The coordinate functions X^{μ} is real. So $(X_L^{\mu})^{\star} = X_L^{\mu}$ and $(X_R^{\mu})^{\star} = X_R^{\mu}$. This means that the coefficients α_n^{μ} and $\widetilde{\alpha}_n^{\mu}$ satisfy

$$(\alpha_n^{\mu})^{\star} = \alpha_{-n}^{\mu} \quad \text{and} \quad (\widetilde{\alpha}_n^{\mu})^{\star} = \alpha_{-n}^{\mu} \quad \forall n \in \mathbb{Z}\backslash\{0\}.$$

Recall that we had two constraints

$$\dot{X}^{\mu} X_{\mu}' = 0 \quad \text{and} \quad \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0.$$

In Light-cone coordinates, these transform to

$$\begin{aligned}
&\left(\frac{\partial_+ + \partial_-}{2}\right) X^{\mu} \left(\frac{\partial_+ - \partial_-}{2}\right) X_{\mu} = 0 \\
\implies &\left(\partial_+ X^{\mu} + \partial_- X^{\mu}\right)\left(\partial_+ X_{\mu} - \partial_- X_{\mu}\right) = 0 \\
\implies &\left(\partial_+ X^{\mu}\right)^2 - \left(\partial_- X^{\mu}\right)^2 = 0 \\
\implies &\left(\partial_+ X^{\mu}\right)^2 = \left(\partial_- X^{\mu}\right)^2.
\end{aligned}$$

The second constrain becomes

$$\begin{aligned}
&\left(\left(\frac{\partial_+ + \partial_-}{2}\right) X^{\mu}\right)^2 + \left(\left(\frac{\partial_+ - \partial_-}{2}\right) X^{\mu}\right)^2 = 0 \\
\implies &\left(\partial_+ X^{\mu}\right)^2 + \left(\partial_- X^{\mu}\right)^2 + 2\partial_+ X^{\mu} \partial_- X_{\mu} + \left(\partial_+ X^{\mu}\right)^2 + \left(\partial_- X^{\mu}\right)^2 - 2\partial_+ X^{\mu} \partial_- X_{\mu} = 0 \\
\implies &\left(\partial_+ X^{\mu}\right)^2 + \left(\partial_- X^{\mu}\right)^2 = 0.
\end{aligned}$$

Combining these two we get the constraint

$$\left(\partial_+ X^{\mu}\right)^2 = 0 = \left(\partial_- X^{\mu}\right). \tag{3.1.7}$$

We now impose this constraint on the Fourier modes. We have

$$\begin{aligned}
\partial_- X^{\mu} = \partial_- X_R^{\mu} &= \frac{\alpha' p^{\mu}}{2} + \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}\backslash\{0\}} \alpha_n^{\mu} e^{-in\sigma^-} \\
&= \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}} \alpha_n^{\mu} e^{-in\sigma^-},
\end{aligned}$$

where we have defined

$$\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu.$$

The constraint $(\partial_- X^\mu)^2 = 0$ gives

$$\left(\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \alpha_{n\mu} e^{-in\sigma^-}\right)\left(\sqrt{\frac{\alpha'}{2}} \sum_{m\in\mathbb{Z}} \alpha_m^\mu e^{-im\sigma^-}\right) = \frac{\alpha'}{2} \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{n-k}\right) e^{-in\sigma^-} = 0$$

where we used Cauchy product formula and $bm\alpha_k \equiv \alpha_k^\mu$. If we define

$$L_n := \frac{1}{2} \sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{n-k}, \tag{3.1.8}$$

then the constraint becomes $L_n = 0$ for every $n \in \mathbb{Z}$. Similarly the constraint $(\partial_+ X^\mu)^2 = 0$ gives $\widetilde{L}_n = 0$ for every $n \in \mathbb{Z}$ where

$$\widetilde{L}_n := \frac{1}{2} \sum_{k\in\mathbb{Z}} \widetilde{\boldsymbol{\alpha}}_k \cdot \widetilde{\boldsymbol{\alpha}}_{n-k}, \tag{3.1.9}$$

and $\widetilde{\alpha}_0^\mu = \alpha_0^\mu$. The quantities L_n and $\widetilde{L}_n$ are called *Virasoro generators*. The constraints $L_0 = 0 = \widetilde{L}_0$ are particularly interesting as they contain information about the physical degrees of freedom of the string - the string momentum. We have

$$L_0 = \frac{1}{2} \sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{-k}, \quad \widetilde{L}_0 = \frac{1}{2} \sum_{k\in\mathbb{Z}} \widetilde{\boldsymbol{\alpha}}_k \cdot \widetilde{\boldsymbol{\alpha}}_{-k}.$$

In relativistic mechanics, we know that

$$p^\mu p_\mu = -M^2$$

where M is the rest mass of the particle. Since

$$p^\mu p_\mu = \frac{2}{\alpha'} \boldsymbol{\alpha}_0^2 = \frac{2}{\alpha'} \widetilde{\boldsymbol{\alpha}}_0^2,$$

we see that the constraints $L_0 = \widetilde{L}_0 = 0$ implies

$$\frac{1}{2} \sum_{n\neq 0} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n - \frac{\alpha'}{4} M^2 = 0 = \frac{1}{2} \sum_{n\neq 0} \widetilde{\boldsymbol{\alpha}}_{-n} \cdot \widetilde{\boldsymbol{\alpha}}_n - \frac{\alpha'}{4} M^2.$$

This gives

$$M^2 = \frac{4}{\alpha'} \sum_{n>0} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n = \frac{4}{\alpha'} \sum_{n>0} \widetilde{\boldsymbol{\alpha}}_{-n} \cdot \widetilde{\boldsymbol{\alpha}}_n. \tag{3.1.10}$$

This is called the *level matching condition* and will be crucial when we analyse the spectrum of the quantised theory.

3.2 Quantisation of Closed String

There are two ways to quantise the Polyakov action. One is the cannonical quantisation using Dirac's presciption. The other is Feynman's path integral quantisation. The cannonical quantisation procedure involves two ways as we are dealing with a gauge theory:

- **Covariant quantisation:** Change cannonical Poisson brackets to commutators and impose the constraint obtained by fixing a gauge as an operator equation to be satisfied by the states X^μ which are now operators. This method is manifestly Lorentz invariant but gives rise to negative norm states called *ghosts*. These decouple from the theory in the critical dimension $D = 26$.

- **Lightcone quantisation:** In this method we first solve the constraints to classify all classically distinct states and then we quantise the physical states. We break Lorentz invariance in the process and later obtain the same critical dimension $D = 26$ to ensure Lorentz invariance.

We will look at both of these quantisation scheme in detail now.

3.3 Covariant Quantisation

We have D scalar fields $X^\mu, \mu = 0, 1, \ldots, D-1$ and two constraints

$$\dot{X}^\mu X'_\mu = 0 \quad \text{and} \quad \dot{X}^2 + X'^2 = 0.$$

3.3.1 Poisson Brackets

Let us begin by computing the classical Poisson brackets.

(i) Equal τ Poisson bracket $\{X^\mu(\sigma, \tau), X^\nu\,(\sigma', \tau)\}_{P.B.} = 0$.

Proof. For Polyakov action, we have $\Pi^\tau_\mu \sim \dot{X}_\mu$. We bill use the notation $\Pi_\mu := \Pi^\tau_\mu$ everywhere unless stated explicitly. Thus this P.B. is obvious. $\square$

(ii) Equal τ Poisson bracket $\{\Pi^\mu(\sigma, \tau), \Pi^\nu\,(\sigma', \tau)\}_{P.B.} = 0$.

Proof. Obvious from the fact that $\Pi^\tau_\mu \sim \dot{X}_\mu$. $\square$

(iii) Equal τ Poisson bracket $\{X^\mu(\sigma, \tau), \Pi^\nu\,(\sigma', \tau)\}_{P.B.} = \eta^{\mu\nu}\delta(\sigma - \sigma')$.

Proof. By definition

$$
\begin{aligned}
\{X^\mu(\sigma,\tau),\Pi^\nu(\sigma',\tau)\}_{P.B.} &= \eta^{\rho\lambda}\frac{\partial X^\mu(\sigma,\tau)}{\partial X^\rho(\sigma,\tau)}\frac{\partial \Pi^\nu(\sigma',\tau)}{\partial \Pi^\lambda(\sigma,\tau)} \\
&= \eta^{\rho\lambda}\delta^\mu_\rho\delta^\nu_\lambda\delta(\sigma-\sigma') \\
&= \eta^{\mu\nu}\delta(\sigma-\sigma')
\end{aligned}
$$

□

From these Poisson brackets, we can easily calculate the Poisson brackets for $x^\mu, p^\mu, \alpha^\mu_n, \widetilde{\alpha}^\mu_n$ We have

$$
\begin{aligned}
\{x^\mu,p^\nu\}_{P.B.} &= \eta^{\mu\nu}, \quad \{\widetilde{\alpha}^\mu_m,\alpha^\nu_n\}_{P.B.}=0 \\
\{\alpha^\mu_m,\alpha^\nu_n\}_{P.B.} &= \{\widetilde{\alpha}^\mu_m,\widetilde{\alpha}^\nu_n\} = -im\eta^{\mu\nu}\delta_{m+n,0}.
\end{aligned}
\tag{3.3.1}
$$

Using these, Poisson brackets, we can get a algebra satisfied by the Virasoro generators.

Lemma 3.3.1. *The Virasoro generators satisfy the Virasoro algebra also called Witt algebra:*

$$
\{L_n,L_m\}_{P.B.} = 2(m-n)L_{m+n}, \quad \left\{\widetilde{L}_n,\widetilde{L}_m\right\}_{P.B.} = i(m-n)\widetilde{L}_{n+m}, \quad \left\{\widetilde{L}_n,L_m\right\}_{P.B.} = 0. \tag{3.3.2}
$$

Proof. We have

$$
\begin{aligned}
\{L_n,L_m\}_{P.B.} &= \left\{\sum_{l\in\mathbb{Z}}\boldsymbol{\alpha}_{n-l}\cdot\boldsymbol{\alpha}_l,\sum_{k\in\mathbb{Z}}\boldsymbol{\alpha}_{m-k}\cdot\boldsymbol{\alpha}_k\right\}_{P.B.} \\
&= \sum_{l,k\in\mathbb{Z}}\left\{\eta_{\mu\nu}\alpha^\mu_{n-l}\alpha^v_l,\eta_{\rho\sigma}\alpha^\rho_{m-k}\alpha^\sigma_k\right\}_{P.B.}
\end{aligned}
$$

Using

$$
\begin{aligned}
\{AB,CD\}_{P.B.} &= \{A,CD\}_{P.B.}B + A\{B,CD\}_{P.B.} \\
&= C\{A,D\}_{P.B.}B + \{A,C\}_{P.B.}DB + AC\{B,D\}_{P.B.} + A\{B,C\}_{P.B.}D,
\end{aligned}
$$

we get

$$
\begin{aligned}
\{L_n,L_m\}_{P.B.} = \sum_{l,k\in\mathbb{Z}}\eta_{\mu\nu}\eta_{\rho\sigma}\big[&\alpha^\mu_{m-k}\left\{\alpha^\mu_{n-l},\alpha^\sigma_k\right\}_{P.B.}\alpha^\nu_l + \left\{\alpha^\mu_{n-l},a^\rho_{m-k}\right\}_{P.B.}\alpha^\nu_l\alpha^\sigma_k \\
&+ \alpha^\mu_{n-l}\alpha^\rho_{m-k}\left\{\alpha^v_k,\alpha^\sigma_k\right\}_{P.B.} + \alpha^\mu_{n-k}\left\{\alpha^\nu_l,\alpha^\rho_{m-k}\right\}_{P.B.}\alpha^\sigma_k\big] \\
= \sum_{l,k\in\mathbb{Z}}\eta_{\mu\nu}\eta_{\rho\sigma}\big[&-\alpha^\rho_{m-k}\eta^{\mu\sigma}i(n-l)\delta_{n-l+k,0}\alpha^\nu_l - i(n-l)\eta^{\mu\rho}\delta_{n-l+m-k,0}\alpha^\nu_l\alpha^\sigma_k \\
&- il\delta_{l+k,0}\eta^{\nu\sigma}\alpha^\mu_{n-l}\alpha^\rho_{m-k} - il\eta^{\nu\rho}\delta_{l+m-k,0}\alpha^\mu_{n-k}\alpha^\sigma_k\big] \\
= -i\sum_{k\in\mathbb{Z}}\big[&\eta_{\nu\rho}\alpha^\rho_{m-k}\alpha^\nu_{n+k}k + \eta_{\nu\sigma}\alpha^\nu_{n+m-k}\alpha^\sigma_k(k-m)+ \\
&+ \eta_{\mu\rho}\alpha^\mu_{n+k}\alpha^\rho_{m-k}k + \eta_{\mu\sigma}(k-m)\alpha^\mu_{n+m-k}\alpha^\sigma_k\big],
\end{aligned}
$$

where we used (3.3.1). Replacing $m-k$ by k in first and k by $k-n$ in third sum we get

$$\begin{aligned}\{L_n, L_m\}_{P.B.} &= -i\sum_{k\in\mathbb{Z}}\left[\eta_{\nu\rho}\alpha_k^\rho\alpha_{n+m-k}^\nu(m-k)+\eta_{\nu\sigma}\alpha_{n+m-k}^\nu\alpha_k^\sigma(k-m)+\right.\\ &\qquad\qquad \left.+\,\eta_{\mu\rho}\alpha_k^\mu\alpha_{m+n-k}^\rho(n-k)+\eta_{\mu\sigma}(k-m)\alpha_{n+m-k}^\mu\alpha_k^\sigma\right]\\ &= -i\sum_{k\in\mathbb{Z}}\eta_{\mu\nu}\alpha_{n+m-k}^\mu\alpha_k^\nu(n-m)\\ &= i(m-n)L_{m+n}.\end{aligned}$$

Similarly, we get all other Poisson brackets. □

3.3.2 Cannonical Commutation Relations

Following the usual way, promote the scalar fields X^μ to operator valued fields and impose the cannonical commutation relation following the rule:

$$\{\cdot,\cdot\}_{P.B.} = \frac{1}{i}[\cdot,\cdot].$$

Using the Poisson brackets for X^μ, Π^μ, we get the following commutation relations:

$$\begin{gathered}[X^\mu(\sigma,\tau),\Pi^\nu(\sigma',\tau)] = i\eta^{\mu\nu}\delta(\sigma-\sigma')\\ [X^\mu(\sigma,\tau),X^\nu(\sigma',\tau)] = 0 = [X^\mu(\sigma,\tau),\Pi^\nu(\sigma',\tau)].\end{gathered}$$

For the Fourier modes, using (3.3.1), we get

$$\begin{gathered}[x^\mu,p^\nu] = i\eta^{\mu\nu}\\ [\alpha_n^\mu,\alpha_m^\nu] = m\eta^{\mu\nu}\delta_{m+n,0} = [\widetilde{\alpha}_m^\mu,\widetilde{\alpha}_n^\nu]\end{gathered} \tag{3.3.3}$$

and all other combinations are zero. These commutation relations are similar to those of creation and annihilation operators. Indeed if we define

$$a_n^\mu = \frac{1}{\sqrt{n}}\alpha_n^\mu \quad , \quad (a_n^\mu)^\dagger = \frac{1}{\sqrt{n}}\alpha_{-n}^\mu \quad , \quad n>0,$$

then we will get the usual commutation relations:

$$\left[a_n^\mu,(a_m^\mu)^\dagger\right] = i\delta_{nm}.$$

Similarly we can put

$$\widetilde{a}_n^\mu = \frac{1}{\sqrt{n}}\widetilde{\alpha}_n^\mu \quad , \quad (\widetilde{a}_n^\mu)^\dagger = \frac{1}{\sqrt{n}}\widetilde{\alpha}_{-n}^\mu \quad , \quad n>0,$$

then we get the commutation relations:

$$\left[\widetilde{a}_n^\mu,(\widetilde{a}_m^\mu)^\dagger\right] = i\delta_{nm}.$$

So for every scalar field X^μ, $\mu=0,1,\ldots,D-1$ we have two family of creation and annihilation operators corresponding to the Left movers and the right movers.

Remark 3.3.2. We cannot directly get the commutation relations satisfied by the Virasoro generators from the Virasoro algebra. In subsequent sections, we will calculate the quantum algebra of Virasoro generators from the commutation relations of the Fourier modes. As we will discover later, the quantum algebra of Virasoro has an extra *central charge* term and hence the quantum algebra is the central extension of the Witt algebra. We will see that this is related related to the fact the Weyl symmetry which is a symmetry of the classical action does not survive quantisation.

3.3.3 Constructing the Fock Space

We will now construct the Fock space of the theory. We begin by constructing the ground state. We now have the creation and annilation operators to define the vacuum of the theory Denote it by $|0\rangle$. Then we demand:

$$\alpha_n^\mu|0\rangle = 0 = \widetilde{\alpha}_n^\mu|0\rangle \qquad \text{for } \mu = 0, 1, \dots, D-1;\ n > 0.$$

Note that this condition alone does not uniquely fix the ground state. This is because, the ground state here is quite different from the one in field theory in the sense that there is a string specified by the center of mass position x^μ and momentum p^μ. So we denote the ground state by $|0; p^\mu\rangle$ which now has the property that

$$\widehat{p}^\mu|0; p^\mu\rangle = p^\mu\,|0; p^\mu\rangle\,, \tag{3.3.4}$$

where p^μ is the momentum of the string. So the ground state of the theory is now defined by

$$\alpha_n^\mu|0\rangle = 0 = \widetilde{\alpha}_n^\mu|0\rangle \qquad \text{for } \mu = 0, 1, \dots, D-1;\ n > 0 \tag{3.3.5}$$

and (3.3.4). A general excitation of the string is

$$\left(\alpha_{-1}^{\mu_1}\right)^{n_{\mu_1}}\left(\alpha_{-2}^{\mu_2}\right)^{n_{\mu_2}}\cdots\left(\widetilde{\alpha}_{-1}^{\nu_1}\right)^{n_{\nu_1}}\left(\widetilde{\alpha}_{-2}^{\nu_2}\right)^{n_{\nu_2}}\cdots|0; p^\mu\rangle.$$

Each excited state has interpretation of a particle. Hence we have infinitely many species of particles in this theory.

3.3.4 Ghosts

We immediately come across a problem. The theory has negative norm states – the so called ghost states[2]. Since $\eta^{00} = -1 < 0$ we have

$$\begin{aligned} \left[\alpha_n^0, \alpha_{-n}^0\right] &= \left[\alpha_n^0, \left(\alpha_{-n}^0\right)^\dagger\right] = -n \qquad \text{and} \\ \left[\widetilde{\alpha}_n, \tilde{\alpha}_n^0\right] &= \left[\widetilde{\alpha}_n^0, \left(\tilde{\alpha}_{-n}^0\right)^\dagger\right] = -n \end{aligned}$$

[2]these are different from Fadeev-Popov ghosts

Consider states of the form $|\psi\rangle = \alpha_{-m}^0 |0; p^\mu\rangle$ for $m > 0$. For these states we have

$$\begin{aligned}\langle \psi \mid \psi\rangle &= \left\langle p^\mu; 0 \left| \left(\alpha_{-m}^0\right)^\dagger \alpha_{-m}^0 \right| 0; p^\mu \right\rangle \\ &= \left\langle p^\mu; 0 \left| \alpha_m^0 \alpha_{-m}^0 \right| 0; p^\mu \right\rangle \\ &= \left\langle p^\mu; 0 \left| -m + \alpha_{-m}^0 \alpha_m^0 \right| 0; p^\mu \right\rangle \\ &= -m \left\langle p^\mu; 0 \mid 0; p^\mu \right\rangle + \left\langle p^\mu; 0 \left| \left(\alpha_m^0\right)^\dagger \alpha_m^0 \right| 0; p^\mu \right\rangle \\ &= -m < 0.\end{aligned}$$

Ghosts are problematic because these are in contradiction to the probabilistic interpretation of norm in Quantum mechanics. Our only hope is to apply the constraints and hope that these ghosts decouple from our theory. That is indeed the case when we fix the dimension of spacetime to be 26.

3.3.5 Normal Ordering and the Quantum Virasoro Algebra

As discussed in the previous section, the constraints in terms of Fourier modes is given by the vanishing of the Virasoro generators. But now, the Fourier modes are no more scaler valued functions but are operators on the Hilbert space. From the commutation relations (3.3.3), we see that the Virasoro generators

$$L_n = \frac{1}{2} \sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{n-k} \quad \text{and} \quad \widetilde{L}_n = \frac{1}{2} \sum_{k\in\mathbb{Z}} \widetilde{\boldsymbol{\alpha}}_k \cdot \widetilde{\boldsymbol{\alpha}}_{n-k}$$

can be defined unambiguously for $n \neq 0$ as $\alpha_k^\mu, \alpha_{n-k}^\nu$ and the respective tildes commute for $n \neq 0$. For $n = 0$, we have

$$L_0 = \frac{1}{2} \sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{-k}, \quad \widetilde{L}_0 = \frac{1}{2} \sum_{k\in\mathbb{Z}} \widetilde{\boldsymbol{\alpha}}_k \cdot \widetilde{\boldsymbol{\alpha}}_{-k},$$

but since $\alpha_k^\mu, \alpha_{-k}^\nu$ and the respective tildes do not commute, the definition of L_0 and $\widetilde{L}_0$ is ambiguous in the quantum theory. We need to pick an ordering convention to define L_0 and $\widetilde{L}_0$. The natural choice is the *normal ordering* – we put annihilation operators $\alpha_n^\mu, \;\; n > 0$ to the right of creation operator $\alpha_n^\mu, \;\; n < 0$. With this choice of normal ordering, we put

$$:L_0: \; = \frac{1}{2} \sum_{k\in\mathbb{Z}} :\boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{-k}: \; = \sum_{k=1}^{\infty} \boldsymbol{\alpha}_{-k} \cdot \boldsymbol{\alpha}_k + \frac{1}{2} \boldsymbol{\alpha}_0^2, \quad :\widetilde{L}_0: \; = \frac{1}{2} \sum_{k\in\mathbb{Z}} :\widetilde{\boldsymbol{\alpha}}_k \cdot \widetilde{\boldsymbol{\alpha}}_{-k}: \; = \sum_{k=1}^{\infty} \widetilde{\boldsymbol{\alpha}}_{-k} \cdot \widetilde{\boldsymbol{\alpha}}_k + \frac{1}{2} \widetilde{\boldsymbol{\alpha}}_0^2.$$

Under the commutation relation on $\alpha_n^\mu, \tilde{\alpha}_n^\mu$ and the choice of normal ordering, we can calculate the quantum Virasoro algebra. We will show that

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} \left(n \left(n^2 - 1\right)\right) \delta_{m+n,0}.$$

where c is called the central charge. Recall that the classical Poisson bracket is

$$\{L_n, L_m\}_{P.B.} = i(m-n)L_{m+n}.$$

This extra term is due to *conformal anomaly* which is due to the breaking of Weyl symmetry in the quantum theory. As we will prove later, the expectation value of the trace of the energy momentum tensor $\langle T^\alpha_{\ \alpha}\rangle \propto R$ where R is the Ricci scaler. The nonvanishing of the trace implies that the Weyl symmetry is broken and hence we have the conformal anomaly. Hence the quantum Virasoro algebra is the central extension of the Witt algebra. We will not define this term precisely here. From now on, we will omit the colons in the Virasoro generators but they are assumed to be normal ordered. We begin by proving a lemma.

Lemma 3.3.3. *For any $m, n \in \mathbb{Z}$, we have*

$$[\alpha^\mu_m, L_n] = m\alpha^\mu_{m+n}, \quad \left[\widetilde{\alpha}^\mu_m, \widetilde{L}_n\right] = m\widetilde{\alpha}^\mu_{m+n}.$$

Proof. With the choice normal ordering we have

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}} :\boldsymbol{\alpha}_{n-k}\cdot\boldsymbol{\alpha}_{\boldsymbol{k}}:.$$

So we have

$$[\alpha^\mu_m, L_n] = \frac{1}{2}\sum_{k\in\mathbb{Z}} [\alpha^\mu_m : \boldsymbol{\alpha}_{n-k}\cdot\boldsymbol{\alpha}_k]$$

Now using $[A, BC] = [A,B]C + B[A,C]$ we get

$$\begin{aligned}
[\alpha^\mu_m, L_n] &= \frac{1}{2}\sum_{k\in\mathbb{Z}} \left\{\eta_{\rho\sigma}:\alpha^\rho_{n-k}\left[\alpha^\mu_m, \alpha^\sigma_k\right]: + \eta_{\rho\sigma}:\left[\alpha^\mu_m, \alpha^\rho_{n-k}\right]\alpha^\rho_k:\right\} \\
&= \frac{1}{2}\sum_{k\in\mathbb{Z}} \left\{\eta_{\rho\sigma}\left(\alpha^\rho_{n-k} m\eta^{\mu\sigma}\delta_{m+k,0} + \alpha^\sigma_k\eta^{\mu\rho}m\delta_{m+n-k,0}\right)\right\} \\
&= \frac{1}{2}\left\{\eta^\mu_\sigma\alpha^p_{n+m}m + n^\mu_\sigma\alpha^\sigma_{m+n} - m\right\} \\
&= m\alpha^\mu_{m+n}.
\end{aligned}$$

The proof for the tildes is identical. □

Theorem 3.3.4. *For any $m, n \in \mathbb{Z}$, we have*

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}\left(n\left(n^2-1\right)\right)\delta_{m+n,0}, \\
\left[\widetilde{L}_n, \widetilde{L}_m\right] &= (n-m)\widetilde{L}_{n+m} + \frac{c}{12}\left(n\left(n^2-1\right)\right)\delta_{m+n,0}.
\end{aligned}$$

Proof. We have

$$
\begin{aligned}
[L_m, L_n] &= \frac{1}{2}\sum_{k\in\mathbb{Z}}[:\boldsymbol{\alpha}_{m-k}\cdot\boldsymbol{\alpha}_k:, L_n] \\
&= \frac{1}{2}\sum_{k\leq 0}[\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m-k}, L_n] + \frac{1}{2}\sum_{k=1}^{\infty}[\boldsymbol{\alpha}_{m-k}\cdot\boldsymbol{\alpha}_k, L_n] \\
&= \frac{1}{2}\sum_{k\leq 0}\boldsymbol{\alpha}_k\cdot[\boldsymbol{\alpha}_{m-k}, L_n] + [\boldsymbol{\alpha}_k, L_n]\cdot\boldsymbol{\alpha}_{m-k} \\
&+ \frac{1}{2}\sum_{k\geq 1}\boldsymbol{\alpha}_{m-k}\cdot[\boldsymbol{\alpha}_k, L_n] + [\boldsymbol{\alpha}_{m-k}, L_n]\cdot\boldsymbol{\alpha}_k \\
&= \frac{1}{2}\sum_{k\leq 0}\{(m-k)\boldsymbol{\alpha}_\kappa\cdot\boldsymbol{\alpha}_{m+n-k} + k\boldsymbol{\alpha}_{n+k}\cdot\boldsymbol{\alpha}_{m-k}\} \\
&+ \frac{1}{2}\sum_{k\geq 1}\{k\boldsymbol{\alpha}_{m-k}\cdot\boldsymbol{\alpha}_{n+k} + (m-k)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k\},
\end{aligned}
$$

where in the second line we broke the normal ordering in the two sums and used Lemma **??** in last line. We now shift the second and third sum by n i,e. substitute $n+k$ by k. We get

$$
\begin{aligned}
[L_m, L_n] =& \frac{1}{2}\sum_{k\leq 0}(m-k)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} + \frac{1}{2}\sum_{k\leq n}(k-n)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} \\
&+ \frac{1}{2}\sum_{k\geq n+1}(k-n)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k + \frac{1}{2}\sum_{k\geq 1}(m-k)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k.
\end{aligned}
$$

Now we need to normal order the second and the third sum. If $n > 0$ then the second sum is not normal ordered and if $n \leq 0$ then the third sum is not normal ordered. We will assume $n > 0$ and proceed. One can get the result for $n \leq 0$ case using the same process. Breaking the second and fourth sum at 0 and $n+1$ respectively, we get

$$
\begin{aligned}
[L_m, L_n] = \frac{1}{2}\Bigg[&\sum_{k\leq 0}(m-k)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} + \sum_{k\leq 0}(k-n)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} + \sum_{k=1}^{n}(k-n)d_k\cdot\boldsymbol{\alpha}_{m+n-k} \\
&+ \sum_{k\geq n+1}(k-n)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k + \sum_{k=1}^{n}(m-k)\delta_{n+n-k}\cdot\boldsymbol{\alpha}_k + \sum_{k\geq n+1}(m-k)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k\Bigg] \\
= \frac{1}{2}\Bigg[&\sum_{k\leq 0}(m-n)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} + \sum_{k\geq n+1}(m-n)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k \\
&+ \sum_{k=1}^{n}(k-n)\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k} + \sum_{k=1}^{n}(m-k)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k\Bigg]
\end{aligned}
$$

We will now use $\left[\boldsymbol{\alpha}_k^\mu, \boldsymbol{\alpha}_{m+n-k}^\nu\right] = \eta^{\mu\nu} k \delta_{m+n,0}$ to normal order the third sum. We get

$$
\begin{aligned}
[L_m, L_n] = \frac{1}{2}\Bigg[& \sum_{k\leq 0}(m-n)\boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{m+n-k} + \sum_{k\geq n+1}(m-n)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k \\
& + \sum_{k=1}^{n}(k-n)\left(\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k + \eta^\mu_\mu k\delta_{m+n,0}\right) + \sum_{k=1}^{n}(m-k)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k\Bigg] \\
= \frac{1}{2}\Bigg[& \sum_{k\leq 0}(m-n)\boldsymbol{\alpha}_k \cdot \boldsymbol{\alpha}_{m+n-k} + \sum_{k\geq 1}(m-n)\boldsymbol{\alpha}_{m+n-k}\cdot\boldsymbol{\alpha}_k \\
& + \sum_{k=1}^{n}(k-n)k\underbrace{n^\mu_\mu}_{D}\delta_{m+n,0}\Bigg] \\
&= (m-n)\frac{1}{2}\sum_{k\in\mathbb{Z}} {:}\boldsymbol{\alpha}_k\cdot\boldsymbol{\alpha}_{m+n-k}{:} + \frac{D}{2}\delta_{m+n,0}\sum_{k=1}^{n}\left(k^2 - nk\right) \\
&= (m-n)L_{m+n} + \frac{D}{2}\delta_{m+n,0}\left(\frac{n(n+1)(2n+1)}{6} - n\frac{n(n+1)}{2}\right) \\
&= (m-n)L_{m+n} + \frac{D}{2}\delta_{m+n,0}\left(\frac{n(n+1)}{2}\left(\frac{2n+1}{3} - n\right)\right) \\
&= (m-n)L_{m+n} + \frac{D}{2}\delta_{m+n,0}\frac{n(n+1)}{2}\left(\frac{1-n}{3}\right) \\
&= (m-n)L_{m+n} + \frac{D}{2}\delta_{m+n,0}\frac{n\left(1-n^2\right)}{6} \\
&= (m-n)L_{m+n} + \frac{D}{12}m\left(m^2-1\right)\delta_{m+n,0},
\end{aligned}
$$

where we replaced n by $-m$ in last step. The proof for the tildes is identical. □

Remark 3.3.5. We can also derive the structure of the central charge term by using Jacobi identity of the Lie bracket. We will rederive this algebra using the tools of conformal field theory.

Remark 3.3.6. In case of only free Bosonic fields, $c = \eta^\mu_\mu = D$ i,e. each scalar field contributes one unit to central charge. When we will rederive this algebra using conformal field theory and quantise the string using path integral, we will calculate the contribution of Fadeev-Popov ghosts to the central charge.

3.3.6 Imposing the Constraints

Recall that the constraints are $L_n = 0 = \widetilde{L}_n$ but this cannot be directly imposed on the Hilbert space of the theory. Indeed if $|\phi\rangle$ is any quantum mechanical state then for any

$n \in \mathbb{Z}$

$$0 = \langle \phi \,|[L_n, L_{-n}]|\, \phi \rangle = 2n \,\langle \phi \,|L_0|\, \phi \rangle + \frac{c}{12} n \left(n^2 - 1\right) \langle \phi \mid \phi \rangle$$

which does not hold if $n \neq 0$. So we cannot impose $L_n|\phi\rangle = 0$ for all n. So the alternative method of imposing the constraint would be to demand that the positive modes annihilate the physical states of the theory:

$$L_n|\text{phys}\rangle = 0, \quad \widetilde{L}_n|\text{phys}\rangle = 0, \quad n > 0, \tag{3.3.6}$$

where $|\text{phys}\rangle$ are the physical states of the theory. This way of imposing the constraints is equivalent to requiring that the matrix elements of all L_n (and the tildes) for $n \neq 0$ vanish. Indeed, we easily see that $L_n^\dagger = L_{-n}$ for $n \neq 0$, thus

$$\langle \text{phys}'|L_n|\text{phys}\rangle = 0, \quad \langle \text{phys}'|L_n|\text{phys}\rangle = 0, \quad \forall n.$$

We are left with imposing the constraint for L_0 and $\widetilde{L}_0$. Recall that we have an ordering ambiguity in defining L_0 and $\widetilde{L}_0$. We now define them using the normal ordering convention we have chosen and impose the constraints $L_0 = 0 = \widetilde{L}_0$ by shifting them by a constant which we will determine later:

$$(L_0 - a)|\text{phys}\rangle = 0 = \left(\widetilde{L}_0 - a\right)|\text{phys}\rangle \quad \text{Mass-shell condition.} \tag{3.3.7}$$

The constant a is called the *normal ordering constant*. In the classical theory, we saw that the constraints $L_0 = 0 = \widetilde{L}_0$ gave us the level matching condition. We want to understand its quantum version. Noting that

$$\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}} p^\mu = \widetilde{\alpha}_0^\mu, \qquad \text{and} \quad p^\mu p_\mu = -M^2,$$

we see that (3.3.7) can be written as

$$\left(N - \frac{\alpha'}{4} M^2 - a\right)|\text{phys}\rangle = 0$$
$$\left(\widetilde{N} - \frac{\alpha'}{4} M^2 - a\right)|\text{phys}\rangle = 0,$$

where

$$N = \sum_{k=1}^{\infty} \boldsymbol{\alpha}_{-k} \cdot \boldsymbol{\alpha}_k \quad \text{and} \quad \widetilde{N} = \sum_{k=1}^{\infty} \widetilde{\boldsymbol{\alpha}}_{-k} \cdot \widetilde{\boldsymbol{\alpha}}_k \tag{3.3.8}$$

are the number operators. Thus the *quantum level matching condition* is

$$M^2 = \frac{4}{\alpha'}(N - a) = \frac{4}{\alpha'}(\widetilde{N} - a). \tag{3.3.9}$$

Since the number operator gives the number of excitations of the string, we see that the number of left-moving and right moving excitations are equal. Thus quantum level matching condition imply equal number of left-moving and right-moving modes.

3.3.7 The No Ghost Theorem

As we discussed in previous sections, our theory has ghosts. We also have a free parameter – the normal ordering constant to fix. It turns out that we can prove a no ghost theorem:

Theorem 3.3.7. (No ghost theorem) *The ghosts decouple in the critical dimension $D = 26$ and with $a = 1$.*

To prove this theorem we follow the following strategy: The normal ordering constant a and the dimension D are not arbitrary in the Quantum theory. For some values of a and D, negative norm states are part of the physical Hilbert space while at other values of a and D, the physical Hilbert space is positive definite. The transition then occurs at the value of a and D where zero norm states become physical. Our strategy will be to find that value of a D where zero norm states become physical – the so called *spurious states.*

Spurious States

Definition 3.3.8. A state $|\psi\rangle$ is called spurious if it satisfies the mass-shell condition and is orthogonal to all physical states. That is

$$(L_0 - a)\,|\psi\rangle = 0 \quad \text{and} \quad \langle \phi \mid \psi\rangle = 0 \quad \forall \quad |\phi\rangle \text{ physical} .$$

Lemma 3.3.9. *A general spurious state is of the form*

$$|\psi\rangle = \sum_{n=1}^{\infty} L_{-n}\,|\chi_n\rangle$$

where $|\chi_n\rangle$ are some states satisfying the modified mass-shell condition

$$(L_0 - a + n)\,|\chi_n\rangle = 0, \quad \forall n \geq 1.$$

Proof. By definition, we have

$$\langle \phi \mid \psi\rangle = 0 \quad \forall \quad |\phi\rangle \quad \text{physical.}$$

We know that

$$L_n|\phi\rangle = 0 \quad \forall n > 0$$

Thus we can write

$$|\psi\rangle = \sum_{n=1}^{\infty} L_{-n}\,|\chi_n\rangle \quad \Big(\text{since} \quad L_{-n}^{\dagger} = L_n\Big)$$

for some state $|\chi_n\rangle$. Mass-shell condition implies

$$(L_0 - a)\,|\psi\rangle = 0$$
$$\Rightarrow \quad \sum_{n=1}^{\infty} (L_0 L_{-n} - a L_n)\,|\chi_n\rangle = 0.$$

By quantum Virasoro algebra $L_0 L_{-n} = L_{-n} L_0 + n L_{-n}$. We get

$$
\begin{aligned}
&\sum_{n=1}^{\infty} (L_{-n} L_0 + n - a L_{-n}) \left|\chi_n\right\rangle = 0 \\
\Rightarrow \quad &\sum_{n=1}^{\infty} L_{-n} (L_0 - a + n) \left|\chi_n\right\rangle = 0 \\
\Rightarrow \quad &(L_0 - a + n) \left|\chi_n\right\rangle = 0 \quad \forall \quad n > 0.
\end{aligned}
$$

$\square$

Definition 3.3.10. The states $\left|\chi_n\right\rangle$ satisfying $(L_0 - a + n)\left|\chi_n\right\rangle = 0$ are called level n states.

Lemma 3.3.11. *Any spurious state $|\psi\rangle$ can be written as*

$$|\psi\rangle = L_{-1}\left|\chi_1\right\rangle + L_{-2}\left|\chi_2\right\rangle$$

where $\left|\chi_1\right\rangle$ and $\left|\chi_2\right\rangle$ are level 1 level 2 states i,e. they satisfy

$$(L_0 - a + 1)\left|\chi_1\right\rangle = 0 \quad \textit{and} \quad (L_0 - a + 2)\left|\chi_2\right\rangle = 0.$$

Proof. By Lemma 3.3.9, we have

$$|\psi\rangle = \sum_{n=1}^{\infty} L_{-n}\left|\chi_n\right\rangle,$$

where $(L_0 - a + n)\left|\chi_n\right\rangle = 0$. We will show that $L_{-n}\chi_n$ can be written as $L_{-1}\left|\chi_1\right\rangle + L_{-2}\left|\chi_2\right\rangle$ for some level 1 and level 2 states $\left|\chi_1\right\rangle$ and $\left|\chi_2\right\rangle$ respectively for all $n \geq 3$. Let us begin with the base case. Note that by quantum Virasoro algebra, we have

$$[L_{-L}, L_{-2}] = (-1+2)L_{-2-1} + 0 = L_{-3}.$$

Thus

$$L_{-3}\left|\chi_3\right\rangle = [L_{-1}, L_2]\left|\chi_3\right\rangle = L_{-1}\left(L - 2\left|\chi_3\right\rangle\right) + L_2\left(-\chi_{-1}\left(\chi_3\right)\right).$$

Take $\left|\chi_1\right\rangle = L_2\left|\chi_3\right\rangle$ and $\left|\chi_2\right\rangle = -\chi_1\left|\chi_3\right\rangle$. It remains to show that $\left|\chi_1\right\rangle$ and $\left|\chi_2\right\rangle$ are Level 1 and level 2 states respectively. Indeed since $(L_0 - a + 3)\left|\chi_3\right\rangle = 0$, we have

$$
\begin{aligned}
(L_0 - a + 1) L_{-2}\left|\chi_3\right\rangle &= (L_0 L - 2 + L - 2(-a+1))\left|\chi_3\right\rangle \\
&= (L_{-2}L_0 + 2L_{-2} + L_{-2}(-a+1))\left(\chi_3\right\rangle \\
&= L_2 (L_0 - a + 3)\left|\chi_3\right\rangle \\
&= 0,
\end{aligned}
$$

where we used the quantum Virasoro algebra: $L_0L_{-2} = L_{-2}L_0 + 2L_{-2}$. Similarly we have

$$\begin{aligned}
(L_0 - a + 2)\left(-L_{-1}\left|\chi_3\right\rangle\right) &= -\left(L_{-1}L_0 + L_{-1}(-a+2)\right)\left|\chi_3\right\rangle \\
&= -\left(L_0L_{-1} + L_{-1} + L_{-1}(-a+2)\right)\left|\chi_3\right\rangle \\
&= -L_{-L}\left(L_0 - a + 3\right)\left|\chi_3\right\rangle \\
&= 0.
\end{aligned}$$

For any n, we assume that $L_{-n+1}\left|\chi_{n-1}\right\rangle$ can be written as $L_{-1}\left|\chi_1\right\rangle + L_{-2}\left|\chi_2\right\rangle$. Then since

$$L_{-n} = \frac{1}{n}\left[L_{-1}, L_{-n+1}\right],$$

so that

$$L_n\left|\chi_n\right\rangle = \frac{1}{n}L_{-1}\left(L_{-n+1}\left|\chi_n\right\rangle\right) + \frac{1}{n}L_{-n+1}\left(-L_{-1}\left|\chi_n\right\rangle\right).$$

Following similar method as in the base case, we can show that $-\frac{1}{n}L_{-1}|\chi_n\rangle$ is a level $n-1$ state. Indeed observe that

$$\begin{aligned}
-\frac{1}{n}\left(L_0 - a + n - 1\right)L_{-1}\left|\chi_n\right\rangle &= -\frac{1}{n}\left(L_{-1}L_0 + L_{-1} + L_1(-a+n-1)\right)\left|\chi_n\right\rangle \\
&= -\frac{1}{n}L_{-1}\left(L_0 - a + n\right)\left|\chi_n\right\rangle \\
&= 0
\end{aligned}$$

So using induction hypothesis, we get

$$L_{-n}\left|\chi_n\right\rangle = L_{-1}\left(\frac{1}{n}L_{n+1}\left|\chi_n\right\rangle\right) + L_{-1}\left|\widetilde{\chi}_1\right\rangle + L_{-2}\left|\widetilde{\chi}_2\right\rangle$$

for some level 1 state $\left|\widetilde{\chi}_1\right\rangle$ a level 2 state $\left|\widetilde{\chi}_2\right\rangle$. It is also clear that $\frac{1}{n}L_{n+1}\left|\chi_n\right\rangle$ is a level 1 state. Thus define

$$\left|\widehat{\chi}_1\right\rangle = \left|\widetilde{\chi}_1\right\rangle + \frac{1}{n}L_{n+1}\left|\chi_n\right\rangle \quad \left|\widehat{\chi}_2\right\rangle = \left|\widetilde{\chi}_2\right\rangle$$

so that

$$L_{-n}\left|\chi_n\right\rangle = L_{-1}\left|\widehat{\chi}_1\right\rangle + L_{-2}\left|\widehat{\chi}_2\right\rangle$$

where $\widehat{\chi}_1$ and $\widehat{\chi}_2$ are level 1 and level 2 states respectively. □

Norm Zero States

Note that the spurious states are orthogonal to all physical states. Thus if we require the spurious states themselves to be physical we must have

$$\langle\psi \mid \psi\rangle = 0.$$

Thus all physical spurious states are norm-zero states. We will now find values of a and D so that all spurious states become physical. Any spurious state decouples from all physical process as they have zero norm.

Physical Spurious States

In view of Lemma 3.3.11, it is sufficient to find the values of a and D such that the spurious states $L_{-1}\left|\chi_1\right\rangle$ and $L_{-2}\left|\chi_2\right\rangle$ become physical where $\left|\chi_1\right\rangle$ and $\left|\chi_2\right\rangle$ are level 1 and level 2 states respectively.

Theorem 3.3.12. *Let $\left|\chi_1\right\rangle$ be a level 1 state satisfying $L_m\left|\chi_1\right\rangle = 0$ for all $m > 0$.Then the spurious state $\left|\psi\right\rangle = L_{-1}\left|\chi_1\right\rangle$ is physical if and only if $a = 1$.*

Proof. ($\implies$) Suppose $L_{-1}\left|\chi_1\right\rangle$ is physical. Then $L_1 L_{-L}\left|\chi_1\right\rangle = 0$ as physical states $\left|\phi\right\rangle$ satisfy $L_m|\phi\rangle = 0$ for all $m > 0$. We get

$$\begin{aligned}
L_1 L_{-1}\left|\chi_1\right\rangle &= \left(L_1 L_1 + 2L_0\right)\left|\chi_1\right\rangle \\
&= 2L_0\left|\chi_1\right\rangle \\
&= 2(a-1)\left|\chi_1\right\rangle,
\end{aligned}$$

since $\left|\chi_1\right\rangle$ is a level 1 state satisfying $\left(\lfloor_0 - a + 1\right)\left|\chi_1\right\rangle = 0$. Thus $L_1 L_{-1}\left|\chi_1\right\rangle = 0 \Rightarrow a = 1$.
($\Longleftarrow$) If $a = 1$ then backtracking above steps we get $L_1 L_{-1}\left|\chi_1\right\rangle = 0$. To check that $L_m L_{-1}\left|\chi_1\right\rangle = 0$, we proceed inductively. We have the base case. Next

$$L_m L_{-l}\left|x_1\right\rangle = L_{-1} L_m\left|\chi_1\right\rangle + (m+1) L_{m-1}\left|\chi_1\right\rangle = 0,$$

since $L_m\left|\chi_1\right\rangle = 0$ by assumption and $L_{m-1}\left|\chi_1\right\rangle = 0$ by induction hypothesis. Next thing to check is

$$\left(L_0 - a\right) L_{-1}\left|\chi_1\right\rangle = 0, \quad (a = 1).$$

Indeed

$$\begin{aligned}
L_0 L_{-1}\left|\chi_1\right\rangle &= L_{-1} L_0\left|\chi_1\right\rangle + L_{-1}\left|\chi_1\right\rangle \\
&= 0 + L_{-1}\left|\chi_1\right\rangle,
\end{aligned}$$

since $\left(L_0 - a + 1\right)\left|\chi_1\right\rangle = L_0\left|\chi_1\right\rangle = 0$. □

Next we look at level 2 spurious states. A general level 2 spurious state is

$$\left|\psi\right\rangle = \left(L_2 + \gamma L_1 L_1\right)\left|\chi_2\right\rangle.$$

We will show that $\left|\psi\right\rangle$ is physical if and only if $\gamma = \frac{3}{2}$ and $D = 26$.

Theorem 3.3.13. *Let $\left|\chi_2\right\rangle$ be a level 2 state satisfying $L_m\left|\chi_2\right\rangle = 0$ for all $m > 0$. Then the spurious state $\left|\psi\right\rangle = \left(L_{-2} + \gamma L_{-1}\left[_{-1}\right)\left|\chi_2\right\rangle\right.$ is physical if and only if $\gamma = \frac{3}{2}$ and $D = 26$.*

Proof. ($\implies$) Suppose $\left|\psi\right\rangle$ is physical, then we must have $\langle\psi|\psi\rangle = 0$ since $\left|\psi\right\rangle$ is spurious. Next we demand $L_m|\psi\rangle = 0$ for all $m > 0$. In particular $L_1|\psi\rangle = 0$. We have

$$\begin{aligned}
&\left(L_1 L_{-2} + \gamma L_{\perp} L_{-1} L_{-1}\right)\left|\chi_2\right\rangle = 0 \\
&\Rightarrow \left(L_{-2} L_1 + 3L_{-1} + \gamma L_{-1} L_1 L_{-1} + 2\gamma L_0 L_{-1}\right)\left|\chi_2\right\rangle = 0 \\
&\Rightarrow \left(L_1 + 3L_{-1} + \gamma L_{-1} L_{-1} L_1 + 2\gamma L_{-1} L_0 + 2\gamma L_{-1} L_0 + 2\gamma L_{-1}\right)\left|\chi_2\right\rangle = 0 \\
&\Rightarrow L_{-1}\left(3 + 4\gamma L_0 + 2\gamma\right)\left|\chi_2\right\rangle = 0,
\end{aligned}$$

where we used $L_1|\chi_2\rangle = 0$. Now since $L_0|\chi_2\rangle = -|\chi_2\rangle$, we get

$$\begin{aligned}
&L_{-1}(3 + y(-1)\gamma + 2\gamma)\,|\chi_2\rangle = 0 \\
\Rightarrow &(3 - 2\gamma)L_{-1}\,|\chi_2\rangle = 0 \\
\Rightarrow &\gamma = \frac{3}{2}
\end{aligned}$$

So

$$|\psi\rangle = \left(L_{-2} + \frac{3}{2}L_{-1}L_{-1}\right)|x_2\rangle\,.$$

Next we impose $L_2|\psi\rangle = 0$. We have

$$\begin{aligned}
&L_2\left(L_{-2} + \frac{3}{2}L_{-1}L_1\right)|\chi_2\rangle = 0 \\
\Rightarrow\ &\left[L_2, L_{-2} + \frac{3}{2}L_{-1}L_{-1}\right]|\chi_2\rangle + \left(L_{-2} + \frac{3}{2}L_{-1}L_{-L}\right)L_2\,|\chi_2\rangle = 0 \\
\Rightarrow\ &\left(4L_0 + \frac{c}{12}2(3)\delta_{0,0} + \frac{3}{2}\left[L_2, L_1L_{-1}\right]\right)|\chi_2\rangle = 0,
\end{aligned}$$

where we used our assumption that $L_2\,|\chi_2\rangle = 0$. Now since

$$\begin{aligned}
[L_2,\ L_{-1}L_{-1}] &= [L_2, L_1]\,L_1 + L_{-1}\,[L_2, L_1] \\
&= 3L_1L_{-1} + 3L_{-1}L_1 \\
&= 3\,(L_{-1}L_1 + 2L_0) + 3L_{-1}L_1 \\
&= 6L_{-1}L_1 + 6L_0.
\end{aligned}$$

So we have

$$\begin{aligned}
&\left(4L_0 + \frac{c}{2} + \frac{3}{2}\,(6L_{-1}L_1 + 6L_0)\right)|\chi_2\rangle = 0 \\
\Rightarrow\ &\left(13L_0 + 9L_{-1}L_1 + \frac{c}{2}\right)|\chi_2\rangle = 0 \\
\Rightarrow\ &c = 26,
\end{aligned}$$

where we used $L_1|\chi_2\rangle = 0$ and $L_0|\chi_2\rangle = -|\chi_2\rangle$. In free Bosonic string theory, we know that $c = \eta^\mu_\mu = D$, so $D = 26$.
($\Longleftarrow$) Assuming $D = 26$, $\gamma = \frac{3}{2}$, we can show that $L_1|\psi\rangle = 0$ and $L_2|\psi\rangle = 0$ back tracking the steps. For $m \geq 3$, it is easily proved using induction as in the proof of Theorem 3.3.12.

Finally we need to show that $(L_0 - 1)|\psi\rangle = 0$. To see that this is true, observe that

$$\begin{aligned}
(L_0 - 1)\left(L_{-2} + \frac{3}{2}L_{-1}L_1\right)|\chi_2\rangle &= \left(L_0L_{-2} + \frac{3}{2}L_0L_{-1}L_{-1}\right)|\chi_2\rangle - |\psi\rangle \\
&= \left(L_{-2}L_0 + 2L_2 + \frac{3}{2}L_{-1}L_0L_1 + \frac{3}{2}L_{-1}L_{-1}\right)|\chi_2\rangle - |\psi\rangle \\
&= \left(L_2(-1) + 2L_{-2} + \frac{3}{2}L_{-1}L_{-1}L_0 + \frac{3}{2}L_{-1}L_{-1} + \frac{3}{2}L_{-1}L_1\right)|\chi_2\rangle - |\psi\rangle \\
&= \left(L_{-2} - \frac{3}{2}L_{-1}L_{-1} + \frac{3}{2}L_{-1}L_{-1} + \frac{3}{2}L_{-1}L_{-1}\right)|\chi_2\rangle - |\psi\rangle \\
&= 0,
\end{aligned}$$

where we used $L_0|\chi_2\rangle = -|\chi_2\rangle$. □

Thus we have shown that infinite classes of spurious states of zero norm appear in our theory when $D = 26$ and $a = 1$. Thus we have determined the boundary where positive norm states turn into negative norm states. Thus for these values of a and D, the ghosts decouple from the theory as infinitely many zero norm states appear in our theory. There are *non-critical string theories* free of ghosts for $a \leq 1$ and $D \leq 25$ but we will not pursue it here. We conclude the covariant quantisation of closed strings with the result that the spectrum is well defined and ghost free in the critical dimension. We will arrive at the same result in the next section using another quantisation scheme.

3.4 Lightcone Quantisation

In lightcone quantisation, we begin by solving the constraints first and separating the physical degrees of freedom. Before we begin, let us discuss about reparametrizations, conformal transformations and Weyl rescaling.

Given any reparametrization of the worldsheet, it corresponds to choosing a different coordinate chart for the manifold. This has no physical consequence as all points, curves remain same on the manifold (worldsheet). Thus any diffeomorphism automatically preserves circular and hyperbolic angles. On the other hand coordinate transformations which transform the metric as

$$g_{\mu\nu} \longrightarrow \Omega^2(\sigma)g_{\mu\nu}(\sigma)$$

are called conformal transformations. These transformations preserve angles (circular as well as hyperbolic) Another version of Conformal transformations are maps between manifolds. Let (M, g) and $(N, \tilde{g})$ be Riemannian manifolds and $\varphi : M \longrightarrow N$ be a smooth map. Then φ is said to be a conformal map if the pullback $\varphi^*\widetilde{g} = \Omega^2 g$ for some smooth function Ω. Writing $x' = \varphi(x)$ we see that

$$\widetilde{g}_{\mu\nu}(x')\frac{\partial x^\mu}{\partial x^\rho}\frac{\partial x' v}{\partial x^\sigma} = \Omega^2(x)g_{\rho\sigma}.$$

Thus angles are preserved. In particular if $M = N$ and $\widetilde{g} = g$ then

$$g_{\mu\nu}\left(x'\right) \frac{\partial x'}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} = \Omega^2(x) g_{\rho\sigma}$$

which are usual conformal transformations. Weyl rescalings on the other hand, are completely different. They are not coordinate transformations. These do not act on the parametrizations but act on the metric. Since the metric is only scaled thus angles are preserved.

3.4.1 Residual Gauge Freedom: Lightcone Gauge

We have already fixed a gauge i,e. chosen two reparametrizations and used Weyl rescaling to fix the metric to $\eta_{\mu\nu}$. But we have some residual gauge symmetry. Indeed consider a reparametrization $\sigma^{\alpha} \longrightarrow \widetilde{\sigma}^{\alpha} = \widetilde{\sigma}^{\alpha}(\boldsymbol{\sigma})$ such that the metric changes by $\eta_{\alpha\beta} \longrightarrow \widetilde{\eta}_{\mu\beta} = \Omega^2(\boldsymbol{\sigma})\eta_{\mu\nu}$. This can then be undone by a weyl rescaling. ($d\sigma d\tau$ introduces a Jacobian in the action which again ensures that the resulting action is Weyl invariant and we can again use Weyl rescaling). These are exactly the conformal transformations. Thus we see that

"diffeomorphisms = conformal $\times$ Weyl".

Next we need to find all such transformation. We will do this in the so called *lightcone coordinates*. Introduce

$$\sigma^{\pm} = \tau \pm \sigma \implies \tau = \frac{\sigma^+ + \sigma^-}{2}, \quad \sigma = \frac{\sigma^+ - \sigma^-}{2}.$$

The metric is given by

$$\begin{aligned} ds^2 = -d\tau^2 + d\sigma^2 &= -\frac{1}{4}\left(d\sigma^+ + d\sigma^-\right)^2 + \frac{1}{4}\left(d\sigma^+ - d\sigma^-\right)^2 \\ &= -\frac{1}{4}d\sigma^{+2} - \frac{1}{4}d\sigma^{-2} - \frac{1}{2}d\sigma^+ d\sigma^- + \frac{1}{4}d\sigma^{+2} + \frac{1}{4}d\sigma^{-2} - \frac{1}{2}d\sigma^+ d\sigma^- \\ &= -d\sigma^+ d\sigma^-. \end{aligned}$$

So a reparametrization $\sigma^+ \longrightarrow \widetilde{\sigma} + (\sigma^+)$ and $\sigma^- \longrightarrow \widetilde{\sigma}^-(\sigma^-)$, ds^2 simply changes by scaling. Indeed

$$ds^2 = -\frac{\partial\sigma^+}{\partial\widetilde{\sigma}^+} d\widetilde{\sigma}^+ \frac{\partial\sigma^-}{\partial\widetilde{\sigma}^-} d\sigma^- = -\frac{\partial\sigma^+}{\partial\widetilde{\sigma}^+}\frac{\partial\sigma^-}{\partial\widetilde{\sigma}} d\widetilde{\sigma}^+ d\widetilde{\sigma}.$$

Note that the reparametrizations are single variable. We would like to fix the remnant gauge. The choice that we will make here is called *lightcone gauge*. Introduce

$$X = \frac{1}{\sqrt{2}}\left(X^0 \pm X^{D-1}\right).$$

Such a choice breaks Lorentz invariance in classical as well as quantum theory as we have picked a special time and space part while Lorentz transformations mixes space and time coordinates. So when we quantise our system, we will look for conditions that restores the Lorentz invariance. It is now easy to see that

$$ds^2 = -2dX^+dX^- + \sum_{i=1}^{D-2} \left(dX^i\right)^2.$$

So the metric $n_{++} = 0 = n_{--}$ and $n_{+-} = n_{-+} = -1$ and $\quad \eta_{ii} = 1 \quad \forall \quad \dot{i} = 1, 2, \ldots, D-2$ and all other elements vanish. So any vector $A^\mu = (A^+, A^-, A^i)$ is lowered as

$$A_\mu = (-A_-, -A_+, \quad A_i)$$

and the dot product is

$$A^\mu B_\mu = -A^+B_- - A^-B_+ + A^iB^i.$$

Solution of the equation of motion is

$$X^+ = X_L^+\left(\sigma^+\right) + X_R^+\left(\sigma^-\right).$$

To see this, note that

$$X^\mu = X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right)$$

so that

$$\begin{aligned} X^+ &= \frac{1}{\sqrt{2}}\left(X^0 + X^{D-1}\right) = \frac{1}{\sqrt{2}}\left[X_L^0\left(\sigma^+\right) + X_L^{D-1}\left(\sigma^+\right) + X_R^0\left(\sigma^-\right) + X_R^{D-L}\left(\sigma^-\right)\right] \\ &= X_L^+\left(\sigma^+\right) + X_R^+\left(\sigma^-\right). \end{aligned}$$

We now fix our gauge. Note that X^+ satisfies the wave equation $\partial_+\partial_-X^+ = 0$. Now note that a reparametrization $\widetilde{\sigma}^+ = \widetilde{\sigma}^+\left(\sigma^+\right)$ and $\widetilde{\sigma}^- = \widetilde{\sigma}^-\left(\sigma^-\right)$ corresponds to

$$\widetilde{\tau} = \frac{\sigma^+ + \sigma^-}{2}, \quad \widetilde{\sigma} = \frac{\widetilde{\sigma}^+ - \widetilde{\sigma}^-}{2}.$$

But $\widetilde{\tau}$ has to satisfy $\partial_+\partial_-\widetilde{\tau} = 0$. So we can choose

$$\widetilde{\tau} = \frac{X^+}{\alpha' p^+} - x^+.$$

This is called *lightcone gauge*. The coordinate X^- still satisfies the wave equation

$$\partial + \partial - X^- = 0$$

The usual solution is

$$X^- = X_L^-\left(\sigma^+\right) + X_R\left(\sigma^-\right).$$

Let us look at the constraints in lightcone gauge. We had the constraint $(\partial_+ X)^2 = 0 = (\partial_- X)^2$ with $X = (X^+, X^-, X^0)$. So we get

$$(\partial_+ X)^2 = -2\partial_+ X^- \partial_+ X^+ + \sum_{i=1}^{D-2} \left(\partial_+ X^i\right)^2$$
$$(\partial_- X)^2 = -2\partial_- X^- \partial_- X^+ + \sum_{i=1}^{D-2} \left(\partial_- X^i\right)^2 .$$

Since

$$\partial_+ X^+ = \frac{\alpha' p^+}{2} = \partial_- X^+ \quad \left(\text{as} \quad \tau = \frac{\sigma^+ + \sigma^-}{2}\right),$$

the constraints $(\partial_+ X)^2 = 0 = (\partial_- X)^2$ gives

$$\begin{aligned} \partial_+ X^- &= \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \left(\partial_+ X^i\right)^2 \\ \partial_- X^- &= \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \left(\partial_- X^2\right)^2 . \end{aligned} \tag{3.4.1}$$

Thus we see that in lightcone gauge the $D-2$ scalar fields determine X^- upto an additive constant coming from integration. Indeed we see that if we write the mode expansion of $X^-_{L/R}$

$$X_L^- \left(\sigma^+\right) = \frac{1}{2} x^- + \frac{\alpha'}{2} p^- \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \widetilde{\alpha}_n^- e^{-in\sigma^+}$$
$$X_R^- \left(\sigma^-\right) = \frac{1}{2} x^- + \frac{\alpha'}{2} p^- \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^- e^{-in\sigma^-}$$

then x^- is coming as the integration constant while all other terms p^- and $\widetilde{\alpha}_n^-, \alpha_n^-$ is determined in terms of $\widetilde{\alpha}_n^i, \alpha_n^i$ and p^+. Indeed if we write

$$\partial_+ X_L^- = \sqrt{\frac{\alpha}{2}} \sum_{n\in\mathbb{Z}} \widetilde{\alpha}_n^- e^{in\sigma^+} \quad \text{with} \quad \widetilde{\alpha}_0^- = \sqrt{\frac{\alpha'}{2}} p^-$$
$$\partial_- X_R^- = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}}^{n} \alpha_n e^{-in\sigma^-} \quad \text{with} \quad \alpha_0^- = \sqrt{\frac{\alpha'}{2}} p^- .$$

Then substituting $\left(\partial_+ X^2\right)^2$ using Fourier modes of X^i in (3.4.1), we get by comparing coefficients of $e^{-in\sigma^\pm}$ that

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}} \frac{1}{p^+} \sum_{m\in\mathbb{Z}} \sum_{i=1}^{D-1} \alpha_{n-m}^i \alpha_m^i$$
$$\widetilde{\alpha}_n^- = \frac{1}{\sqrt{2\alpha'}} \frac{1}{p^+} \sum_{m\in\mathbb{Z}} \sum_{i=1}^{D-2} \widetilde{\alpha}_{n-m}^i \widetilde{\alpha}_m^i .$$

For $n = 0$, we get two expressions for p^-:

$$\frac{\alpha' p^-}{2} = \frac{1}{2p^+} \sum_{i=1}^{D-2} \left(\frac{\alpha'}{2} p^i p^i + \sum_{n \neq 0} \alpha^i_{-n} \alpha^i_n \right) = \frac{1}{2p^+} \sum_{i=1}^{D-2} \left(\frac{\alpha'}{2} p^i p^i + \sum_{n \neq 0} \widetilde{\alpha}^i_{-n} \widetilde{\alpha}^i_n \right).$$

Using $p^\mu = (p^+, p^-, p^i)$ we see that

$$M^2 = -p^\mu p_\mu = 2p^+ p^- - \sum_{i=1}^{D-2} p^i p^i.$$

Using the above equality for $\frac{\alpha' p^-}{2}$ above we get

$$M^2 = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \alpha^i_{-n} \alpha^i_n = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \widetilde{\alpha}^i_{-n} \widetilde{\alpha}^i_n,$$

where we used

$$\sideset{}{'}\sum_{n \neq 0} \alpha^i_{-n} \alpha^i_n = 2 \sideset{}{'}\sum_{n>0} \alpha^i_{-n} \alpha^i_n.$$

The oscillators $\alpha^i_n, \widetilde{\alpha}^i_n$ are called *transverse oscillators*. These are physical excitations in the sense that knowing α^i_n and $\widetilde{\alpha}'_n$ determines all other modes. Thus the most general classical solution can be determined in terms of $2(D-2)$ oscillator modes $\alpha^i_n, \widetilde{\alpha}^i_n$ and a bunch of zero modes $p^\pm, p^i, x^\pm$.

3.4.2 Quantisation

The usual way of quantisation is to compute the classical Poisson brackets and use Dirac prescription. As we did in covariant quantisation, using the Poisson brackets, the following commutation relations are obvious:

$$\begin{aligned} \left[x^i, p^j\right] = i\delta^{ij}, \quad \left[x^-, p^+\right] = -i, \quad \left[x^+, p^-\right] = -i \\ \left[\alpha^i_n, \alpha^j_m\right] = n\delta^{ij}\delta_{m+n,0} = \left[\widetilde{\alpha}^i_n, \widetilde{\alpha}^j_m\right]. \end{aligned} \tag{3.4.2}$$

The ground state is again $|0; p^\mu\rangle$ with $|0\rangle$ being the string. To build the Fock space, we impose

$$\widehat{p}^\mu |0; p^\mu\rangle = p^\mu |0; p^\mu\rangle, \quad \widetilde{\alpha}^i_n |0; p^\mu\rangle = 0 = \alpha^i_n |0; p^\mu\rangle, \quad \forall \quad n > 0 \quad \mu = 1, 2, \cdots, D-1.$$

We act with $\alpha^i_{-n}, \widetilde{\alpha}^i_{-n}, n > 0$ to build the Fock space. Notice that i runs only over spatial index $i = 1, 2 \cdots, D-1$, so the theory does not have ghosts by construction. Its time to impose the constraints. As we had in covariant quantisation, level matching with normal ordering implies

$$M^2 = \frac{4}{\alpha'}(N - a) = \frac{4}{\alpha'}(\widetilde{N} - a),$$

where now the number operators are

$$N = \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n\neq 0}\alpha_n^i\alpha_n^i \quad \text{and} \quad \widetilde{N} = \frac{1}{2}\sum_{i=1}^{D-1}\sum_{n\neq 0}\widetilde{\alpha}_n^i\widetilde{\alpha}_n^i$$

and a is again the normal ordering constant which we again fix by requiring that the spectrum be Lorentz invariant. Note that

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{D-1}\sum_{n\neq 0}\alpha_{-n}^i\alpha_n^i &= \frac{1}{2}\sum_i\left[\sum_{n>0}\alpha_{-n}^i\alpha_n^i + \sum_{n<0}\alpha_{-n}^i\alpha_n^i\right] \\
&= \frac{1}{2}\sum_i\sum_{n>0}\alpha_{-n}^i\alpha_n^i + \frac{1}{2}\sum_i\left[\sum_{n<0}\alpha_n^i\alpha_{-n}^t - n\right] \\
&= \sum_i\sum_{n>0}\alpha_n^i\alpha_n^i + \frac{D-2}{2}\sum_{n>0}n,
\end{aligned}$$

where we used the commutator $\left[\alpha_n^i, \alpha_{-n}^i\right] = n$. The last sum is divergent but we need to extract physics out of this divergence. The result is the appearance of *Casimir force*. We will do this in two ways.

UV Cut-off $\varepsilon \ll 1$

Write

$$\sum_{n>0}n \longrightarrow \sum_{n>0}ne^{-\varepsilon n} = -\frac{\partial}{\partial\varepsilon}\sum_{n>0}e^{-\varepsilon n} = -\frac{\partial}{\partial\varepsilon}\left[\left(L - e^{-\varepsilon}\right)^{-1}\right].$$

Now

$$\begin{aligned}
-\frac{\partial}{\partial\varepsilon}\left[\frac{1}{1-e^{-\varepsilon}}\right] &= \frac{e^{-\varepsilon}}{\left(1-e^{-\varepsilon}\right)^2} = \frac{\left(1-\varepsilon+\frac{\varepsilon^2}{2}+O\left(\varepsilon^3\right)\right)}{\left(1-1+\varepsilon-\frac{\varepsilon^2}{2}+\cdots\right)^2} \\
&= \frac{\left(1-\varepsilon+\frac{\varepsilon^2}{2}+O\left(\varepsilon^3\right)\right)}{\varepsilon^2\left(1-\frac{\varepsilon}{2}+\dots\right)^2} \\
&= \frac{1}{\varepsilon^2}\left(t-\varepsilon+\frac{\varepsilon^2}{2}+O\left(\varepsilon^3\right)\right)\left(1+2\frac{\varepsilon}{2}-2\frac{\varepsilon^2}{3!}+\frac{3}{4}\varepsilon^2+O\left(\varepsilon^3\right)\right) \\
&= \frac{1}{\varepsilon^2}\left(1+\frac{\varepsilon^2}{2}-\varepsilon^2-\frac{2}{6}\varepsilon^2+\frac{3}{4}\varepsilon^2+O\left(\varepsilon^3\right)\right) \\
&= \frac{1}{\varepsilon^2}-\frac{1}{12}+O(\varepsilon).
\end{aligned}$$

The $\frac{1}{\varepsilon^2}$ must be renormalised away. After renormalising and taking $\varepsilon \to 0$, we get the odd result

$$\sum_{n=1}^{\infty}n = -\frac{1}{12}. \tag{3.4.3}$$

Zeta Function Regularisation

The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C}.$$

The series defining $\zeta(s)$ converges absolutely and uniformly on compact subsets of the half plane $\{s \in \mathbb{C} : \mathrm{Re}(s) > 1\}$ and hence $\zeta(s)$ is holomorphic on this half plane. Moreover, the Riemann zeta function admits a unique analytic continuation to the whole $s-$plane. To be precise, Riemann in 1859 proved the following integral representation of the Riemann zeta function:

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\Big\}(s) = \frac{1}{s(s-1)} + \int_1^{\infty} W(x)\left(x^{s/2} + x^{\frac{1-s}{2}}\right)\frac{dx}{x}, \tag{3.4.4}$$

where

$$W(x) = \sum_{n=1}^{\infty} e^{n^2\pi x}$$

and $\Gamma(s)$ is the gamma function. The integral on the right hand side of (3.4.5) converges for all $\mathbb{C}$. So this integral gives an analytic continuation of $\zeta(s)$. Indeed putting

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

we see that $\xi(s)$ is an entire function and satisfies

$$\xi(1-s) = \xi(s).$$

From the fact that $\xi(s)$ is entire, we see that $\zeta(s)$ (analytically continued) has simple zeros at $s = -2n,\ n \in \mathbb{N}$ corresponding to poles of $\Gamma\left(\frac{s}{2}\right)$.[3] Now at $s = -1$, we have that $\xi(-1) = \xi(2)$. This implies

$$\begin{aligned} &2\pi^{1/2}\Gamma\left(-\frac{1}{2}\right)\zeta(-1) = 2\pi^{-1}\Gamma(1)\zeta(2) \\ \Rightarrow \quad &\pi^{1/2}\left(-\frac{1}{2}\right)\sqrt{\pi}\zeta(-1) = \pi^{-1}\frac{\pi^2}{6} \\ &\Rightarrow \zeta(-1) = -\frac{1}{12}. \end{aligned}$$

So we see that both of the computation gives same result.

[3]These are called the *trivial zeros* of $\zeta(s)$. The *Riemann hypothesis* says that all other *non trivial zeros* of $\zeta(s)$ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$. This is still an open problem and a million dollar problem announced by the Clay Mathematical Institute.

3.4.3 String Spectrum

With the above regularisation, the level matching condition becomes

$$M^2 = \frac{4}{\alpha'}\left[\sum_{i=1}^{D-2}\sum_{n>0}\alpha^2_{-n}\alpha^i_n - \frac{D-2}{24}\right] = \frac{4}{\alpha'}\left[:N: - \frac{D-2}{24}\right]$$
$$= \frac{4}{\alpha'}\left[\sum_{i=1}^{D-2}\sum_{n>0}\widetilde{\alpha}^i_{-n}\widetilde{\alpha}^i_n - \frac{D-2}{24}\right] = \frac{4}{\alpha'}\left[:\widetilde{N}: - \frac{D-2}{24}\right].$$

We also identify the normal ordering constant as

$$a = \frac{D-2}{24}.$$

Let us look at the ground state $|0;p^\mu\rangle$. By our definition of vacuum $|0\rangle$, we have

$$:N:|0;p^\mu\rangle = 0 = :\widetilde{N}:|0;p^\mu\rangle\,.$$

So Level matching gives

$$M^2 = -\frac{D-2}{6\alpha'} < 0.$$

These are particles weth negative mass-squared. These are called *Tachyons*. These are a problem in Bosonic string theory. But when we study superstring theory where we include Fermionic fields on the worldsheet, then these states automatically vanish. Now let us look at excited states. First excited state is obtained by acting α^i_{-1} and $\widetilde{\alpha}^i_{-1}$. To see this observe that for $n > 0$,

$$\begin{aligned} N\alpha^j_{-n}|0;p^\mu\rangle &= \sum_{i=1}^{D-1}\sum_{k=1}^{\infty}\alpha^i_{-k}\alpha^i_k\alpha^j_{-n}|0;p^\mu\rangle \\ &= \left[\sum_i\sum_{k=1}^{\infty}\alpha^i_{-k}\alpha^j_{-n}\alpha^i_k + k\delta^{ij}\delta_{k-n,0}\alpha^i_{-k}\right]|0;p^\mu\rangle \\ &= n\alpha_{-n}|0;p^\mu\rangle\,. \end{aligned}$$

So α^i_{-1} and $\widetilde{\alpha}^i_{-1}$ give first excited states. Thus level matching requires us to act α^i_{-1} and $\widetilde{\alpha}_{-1}$ together. So the first excited states are $\alpha^i_{-1}\widetilde{\alpha}^j_{-1}|0;p^\mu\rangle$. Mass of each of these states is

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{D-2}{24}\right). \tag{3.4.5}$$

3.4.4 Fixing Lorentz Invariance

Our states are labelled by indices $i, j = 1, 2, \ldots, D-1$ and hence these transform as vectors with respect to the group $SO(D-2) \hookrightarrow SO(1, D-1)$ where $SO(1, D-1)$ is the full Lorentz group. But finally we want our states to fit into some representation of the Lorentz group $S0(1, D-1)$. Here we invoke Wigner's classification of representations of Poincaré group. From the discussion in Appendix A[4], we see that if we want our states $\alpha^i_{-1}\widetilde{\alpha}^j_{-1}|0;p^\mu\rangle$ to transform as some representation of the Lorentz group, then these states must be massless as these states fit into the representation of the little group $SO(D-2)$ which is the little group corresponding to massless representation. Thus (3.4.5) implies that $D = 26$ which also gives $a = 1$. Thus we have recovered the critical dimension by requiring that the first excited state be representations of the Lorentz group. We still need to make sure that the higher excited states also transform as some representations of the Lorentz invariant and we now have no choice other than to hope that with the values of a and D that we have chosen, we somehow manage to embed the higher excited states into the representation of Lorentz group. This is indeed the case. We will show this for the second excited state but one can check that the allhigher excited states fit into some massive representation of the Lorentz group. We first note from (3.4.5) that all higher excited states are massive with the values of D that we have chosen. So by Wigner's classification, all these states must fit into some representation of $SO(D-1)$ as the little group for massive representations is $SO(D-1)$. For $N = \widetilde{N} = 2$, the states are

$$\begin{array}{ll} \alpha^i_{-1}\alpha^j_{-1}|0;p^\mu\rangle\,,\alpha^i_{-2}|0;p^\mu\rangle & \text{– Right moving} \\ \widetilde{\alpha}^i_{-1}\widetilde{\alpha}^j_{-1}|0;p^\mu\rangle\,,\widetilde{\alpha}^i_{-2}|0;p^\mu\rangle & \text{– Left moving.} \end{array}$$

Since $\alpha^i_{-1}, \alpha^j_{-1}$ commute, in the right moving sector there are a total of

$$\begin{aligned} \frac{1}{2}(D-2)(D-1) + (D-2) &= (D-2)\left(\frac{D-1+2}{2}\right) \\ &= \frac{(D-2)(D+1)}{2} \\ &= \frac{1}{2}D(D-1) - 1 \end{aligned}$$

states. These easily fit into the symmetric traceless representation of $SO(D-1)$. Infact one can prove that all higher excited states fit into some representation of $SO(D-1)$. Hence we have recovered Lorentz invariance by fixing the dimension of spacetime.

There is one other way to explicitly check that we have recovered Lorentz invariance: We compute the conserved charges and currents corresponding to the global Poincaré symmetry $X^\mu \longrightarrow \Lambda^\mu_\nu X^\nu + C^\mu$ of the action and require that they satisfy Poincaré algebra. Let us begin with translations $X^\mu \longrightarrow X^\mu + C^\mu$. One can compute the Noether current. It turns out to be:

$$P^\alpha_\mu = \frac{1}{2\pi\alpha'}\partial^\alpha X_\mu.$$

[4] I recommend going through Appendix A to understand the uses of Wigner's classification in string theory context.

It is easy to see that $\partial_\alpha P^\alpha_\mu = 0$ as $\partial_\alpha \partial^\alpha X_\mu = 0$ on-shell. Next the Noether charge corresponding to Lorentz transformation $X^\mu \longrightarrow \Lambda^\mu_\nu X^\nu$ is

$$J^\alpha_{\mu\nu} = P^\alpha_\mu X_\nu - P^\alpha_\nu X_\mu.$$

We can again check that $\partial_\alpha J^\alpha_{\mu\nu} = 0$. Indeed we have

$$\begin{aligned}
\partial_\alpha J^\alpha_{\mu\nu} &= \left(\partial_\alpha P^\alpha_\mu\right) X_\nu + P^\alpha_\mu \partial_\alpha X_\nu - \left(\partial_\alpha P^\alpha_\nu\right) X_\mu - P^\alpha_\nu \partial_\alpha X_\mu \\
&= P^\alpha_\mu \partial_\alpha X_\nu - P^\alpha_\nu \partial_\alpha X_\mu \\
&= \frac{1}{2\pi\alpha'} \left(\partial^\alpha X_\mu \partial_\alpha X_\nu - \partial^\alpha X_\nu \partial_\alpha X_\mu\right) \\
&= 0.
\end{aligned}$$

The conserved charges corresponding to J^τ_μ is

$$M_{\mu\nu} = \int_0^\pi d\sigma J^\tau_{\mu\nu}.$$

Now using the mode expansion for X^μ we get

$$\begin{aligned}
M^{\mu\nu} = \int_0^\pi d\sigma \left(X^\mu \Pi^\nu - X^\nu \Pi^\mu\right) &= \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu\right) \\
&= l^{\mu\nu} + E^{\mu\nu} + \widetilde{E}^{\mu\nu},
\end{aligned}$$

where

$$\begin{aligned}
l^{\mu\nu} &= x^\mu p^\nu - x^\nu p^\mu, \\
E^{\mu\nu} &= -i \sum_{n=1}^\infty \frac{1}{n} \left(\alpha^\mu_{-n} \alpha^\nu_n - \alpha^\nu_n \alpha^\mu_n\right), \\
\widetilde{E}^{\mu\nu} &= -i \sum_{n=1}^\infty \frac{1}{n} \left(\widetilde{\alpha}^\mu_{-n} \widetilde{\alpha}^\nu_n - \widetilde{\alpha}^\nu_{-n} \widetilde{\alpha}^\mu_n\right).
\end{aligned}$$

The first piece $l^{\mu\nu}$ is the orbital angular momentum of the string while the other two pieces arise from excited states. Classically, one can check that the Poisson bracket for $M_{\mu\nu}$ satisfies Lorentz algebra. In covariant quantisation, it is easy to check that $M_{\mu\nu}$ satisfies the Lorentz algebra but in lightcone quantisation, things are not so clear. In lightcone gauge, we must be able to produce the Lorentz algebra i,e.

$$[M^{\rho\sigma}, M^{\tau\nu}] = \eta^{\sigma\tau} M^{\rho\nu} - \eta^{\rho\tau} M^{\sigma\nu} + \eta^{\rho\nu} M^{\sigma\tau} - \eta^{\sigma\nu} M^{\rho\tau}.$$

The only bracket which is non trivial is $[M^{i-}, M^{j-}] = 0$. This commutator involves p^- and α^-_n which has been fixed in lightcone gauge in terms of other transverse oscillators. A messy calculation gives

$$\begin{aligned}
\left[M^{i-}, M^{j-}\right] = &\frac{2}{(p^+)^2} \sum_{n>0} \left(\left[\frac{D-2}{24} - 1\right] n + \frac{1}{n} \left[a - \frac{D-2}{24}\right] \right) \left(\alpha^i_{-n} \alpha^j_n - \alpha^j_{-n} \alpha^i_n\right) \\
&+ + \frac{2}{(p^+)^2} \sum_{n>0} \left(\left[\frac{D-2}{24} - 1\right] n + \frac{1}{n} \left[a - \frac{D-2}{24}\right] \right) \left(\widetilde{\alpha}^i_{-n} \widetilde{\alpha}^j_n - \widetilde{\alpha}^j_{-n} \widetilde{\alpha}^i_n\right),
\end{aligned}$$

which is 0 if and only if $a = 1$ and $D = 26$. This is consistent with our earlier derivation of the critical dimension.

3.4.5 First String Excitation

The first excited states are massless representations of the little group $SO(D-2)$. There are a total $(D-2)^2$ particles in the first excitation. So we want to get the irreducible representations of $SO(D-2)$ of dimension $(D-2)^2$ so that each irreducible factor would correspond to an elementary particle by Wigner's proposal. Using the method of Young Tableau, we can prove that the tensorial representation of $SO(D-2)$ of dimension $(D-2)^2$ consists of three irreducible parts:

	Traceless symmetric $\oplus$	**Antisymmetric** $\oplus$	**Trace (Scalar)**
Dim:	$\dfrac{(D-2)(D-1)}{2} - 1$	$\dfrac{(D-2)(D-3)}{2}$	1

Following the usual method of constructing field theory from representations, we attach a tensor field to each of these representation. We get three particles.

1. $G_{\mu\nu}(X)$: the traceless symmetric tensor field which we will identify with *graviton*.
2. $B_{\mu\nu}(X)$: the antisymmetric tensor field. This is sometimes called the *Kalb-Ramond field.*
3. $\Phi(X)$: the trace part of the tensor representations. This scaler field is called the *dilaton.*

To see that these fields arise in our theory, we decompose the first excited state as follows:

$$\alpha_1^i \widetilde{\alpha}_{-1}^j \left|0;p^\mu\right\rangle = \underbrace{\left(\alpha_{-1}^{(i}\widetilde{\alpha}_{-1}^{j)} - \frac{1}{D-2}\delta^{ij}\alpha_{-1}^k\widetilde{\alpha}_{-1}^k\right)\left|0;p^\mu\right\rangle}_{\text{symmetric traceless}} + \underbrace{\alpha_{-1}^{[i}\widetilde{\alpha}_{-1}^{j]}\left|0;p^\mu\right\rangle}_{\text{antisymmetric}} + \frac{1}{D-2}\underbrace{\delta^{ij}\alpha_{-1}^k\widetilde{\alpha}_{-1}^k\left|0;p^\mu\right\rangle}_{\text{trace}},$$

where $(,)$ and $[,]$ are the symmetrized and antisymmetrized indices. The traceless symmetric field $G_{\mu\nu}$ is particularly interesting as it represents massless spin 2 particle. We will identify this field with the metric of spacetime, the graviton because Weinberg in [DOI: 10.1103/PhysRev.138.B988] showed that any interacting theory of massless spin 2 particle is Einstein's gravity. Later we will explicitly derive Einstein's field equations from this field.

Chapter 4

Open Strings and D-Branes

In the previous Chapter, we quantised the closed string and found that the spectrum contains three particles including the graviton. In this Chapter, we will quantise the open strings with different boundary conditions.

4.1 Solving the Equations of Motion

We have already found the equations of motion of the open string subject to three different boundary conditions in Subsection 2.2.2. As already mentioned in Subsection 2.2.2, we will normalise the length of the string so that $\sigma \in [0, \pi)$. We will now solve the equations of motion for the first two boundary conditions.

4.1.1 Neumann Boundary Condition at Both Ends (NN)

This means that

$$\partial_\sigma X^\mu = 0 \quad \text{for} \quad \sigma = 0, \pi.$$

Since the equation of motion is

$$\partial_\alpha \partial^\alpha X^\mu = 0,$$

we again have

$$X^\mu(\sigma, \tau) = X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right)$$

with

$$X_L^\mu\left(\sigma^+\right) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}_n^\mu e^{-in\sigma^+}$$

and

$$X_R^\mu\left(\sigma^-\right) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}.$$

Now $\sigma = \frac{\sigma^+ - \sigma^-}{2}$, so $\sigma = 0 \Rightarrow \sigma^+ = \widetilde{\sigma} = \tau/2$. Since

$$\partial_\sigma X^\mu = \frac{1}{2}\left(\partial_+ X^\mu - \partial_- X^\mu\right),$$

the condition $\partial_\sigma X^\mu = 0$ implies $\partial_+ X^\mu = \partial_- X^\mu$. Using the Fourier expansion above, we get

$$\alpha' p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \widetilde{\alpha}_n^\mu e^{-in\sigma^+}\Bigg|_{\sigma=0,\pi} = \alpha\prime p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \alpha_n^\mu e^{-in\sigma^-}\Bigg|_{\sigma=0,\pi}.$$

At $\sigma = 0$, we get

$$\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \left(\widetilde{\alpha}_n^\mu - \alpha_n^k\right) e^{-n\pi\tau_2} = 0 \quad \Rightarrow \quad \widetilde{\alpha}_n^\mu = \alpha_n^\mu \quad \forall\; n \neq 0.$$

So we have

$$X^\mu = x^\mu + 2p^\mu \alpha' \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau}\left(e^{-in\sigma} + e^{in\sigma}\right).$$

This gives

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma). \tag{4.1.1}$$

We can check that the boundary condition at $\sigma = \pi$ is automatically satisfied. Again we can check that x^μ le p^μ are center of mass position and momentum of the string. Constraints are

$$(\partial_+ X)^2 = 0 = (\partial_- X)^2.$$

With the given Fourier expansion, we still have the same classical constraints

$$L_n = 0 \text{ where } L_n = \frac{1}{2} \sum_{k\in\mathbb{Z}} \boldsymbol{\alpha}_{n-k} \cdot \boldsymbol{\alpha}_k \quad \forall n \in \mathbb{Z}$$

where now $\alpha_0^\mu = \sqrt{2\alpha'} p^\mu$. The Poisson bracket for α_n^μ are still the same.

$$\{\alpha_m^\mu, \alpha_n^\nu\}_{P.B.} = -im\eta^{\mu\nu}\delta_{m+n,0}, \quad \{x^\mu, p^\nu\}_{P.B.} = \eta^{\mu\nu}. \tag{4.1.2}$$

Virasoro algebra is also the same

$$\{L_m, L_n\}_{P.B.} = -i(m-n)L_{m+n}.$$

The Poisson bracket for the Fourier modes and the Virasoro generators remain the same.

4.1.2 Dirichlet Boundary Condition at Both Ends (DD)

We impose $\delta X^\mu = 0$ at $\sigma = 0, \pi$. This means that $\dot{X}^\mu = 0$ at $\sigma = 0, \pi \quad \forall\ \tau$. Suppose $X^\mu(0,\tau) = x_0^\mu$ and $X^\mu(\pi,\tau) = x_1^\mu$. The constraint is the same. We can still write

$$X^\mu = X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right).$$

where

$$\begin{aligned}
X_L^\mu\left(\sigma^+\right) &= \frac{1}{2}x^\mu + p^\mu\alpha'\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}_n^\mu e^{-in\sigma^+} \\
X_R^\mu\left(\sigma^-\right) &= \frac{1}{2}x^\mu + p^\mu\alpha'\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}.
\end{aligned}$$

But the boundary condition implies

$$\begin{aligned}
X^\mu(0,\tau) = x_0^\mu \Rightarrow \quad & x^\mu + 2p^\mu\alpha'\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\widetilde{\alpha}_n^\mu + \alpha_n^\mu\right)e^{-in\tau} \\
& \Rightarrow p^\mu = 0, \quad x^\mu = x_0^\mu \quad \text{and} \quad \widetilde{\alpha}_n = -\alpha_n^\mu
\end{aligned}$$

But the second condition $X^\mu(\pi,\tau) = x_1^\mu$ is not satisfied. Thus the general solution must have the form

$$X^\mu = x_0^\mu + \frac{x_1^\mu - x_0^\mu}{\pi}\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\sin(n\sigma). \tag{4.1.3}$$

This is gotten by assuming the forms of X_L and X_R as

$$\begin{aligned}
X_L^\mu\left(\sigma^+\right) &= \frac{1}{2}x^\mu + p^\mu\alpha'\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}_n^\mu e^{-in\sigma^+} \\
X_R^\mu\left(\sigma^-\right) &= \frac{1}{2}x^\mu - p^\mu\alpha'\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-},
\end{aligned}$$

so that

$$X^\mu(\sigma,\tau) = x^\mu + 2\alpha' p^\mu\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\widetilde{\alpha}_n^\mu e^{in\sigma^+} + \alpha_n^\mu e^{-in\sigma^-}\right).$$

Then $X^\mu(0,\tau) = x_0^\mu \Rightarrow x^\mu = x_0^\mu \quad$ and $\quad \widetilde{\alpha}_n^\mu = -\alpha_n^\mu$ and $X^\mu(\pi,\tau) = x_1^\mu$ implies

$$\begin{aligned}
& x_0^\mu + 2\alpha' p^\mu\pi + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\left(\widetilde{\alpha}_n^\mu e^{-in\pi} - \widetilde{\alpha}_n^\mu e^{in\pi}\right)e^{-in\tau} = x_1^\mu \\
& \Rightarrow x_0^\mu + 2\alpha' p^\mu\pi = x_1^\mu \\
& \Rightarrow 2\alpha' p^\mu = \frac{x_1^\mu - x_0^\mu}{\pi}.
\end{aligned}$$

There is no center of mass momentum and the center of mass position is $\frac{x_0^\mu + x_1^\mu}{2}$ as is easily computed:

$$q^\mu = \frac{L}{\pi}\int_0^\pi d\sigma x^\mu(\sigma,\tau) = x_0^\mu + \frac{1}{\pi}\frac{x_1^\mu - x_0^\mu}{\pi}\frac{1}{2}\pi^2 + 0 = \frac{x_0^\mu + x_1^\mu}{2}.$$

Next, we find the classical constraints in terms of Fourier modes. The constraints are

$$\left(\partial_+ X^\mu\right)^2 = 0 = \left(\partial_- X^\mu\right)^2.$$

We have

$$\begin{aligned}
\partial_+ X^\mu &= \frac{x_1^\mu - x_0^\mu}{\pi}\frac{1}{2} + \sqrt{2\alpha'}\sum_{n\neq 0}\alpha_n^\mu \frac{1}{2in}\partial_+\left(e^{-in(\tau-\sigma)} - e^{-in(\tau+\sigma)}\right) \\
&= \frac{x_1^\mu - x_0^\mu}{\pi}\frac{1}{2} + \sqrt{\frac{\alpha}{2}}\sum_{n\neq 0}\alpha_n^\mu e^{-in\sigma^+} \\
&= \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}}\alpha_n^\mu e^{-in\sigma^+},
\end{aligned}$$

where

$$\alpha_0^\mu = \frac{1}{\sqrt{2\alpha'}}\frac{x_1^\mu - x_0^\mu}{\pi}.$$

Similarly

$$\partial_- X^\mu = -\sqrt{\frac{\alpha\prime}{2}}\sum_{n\neq 0}\alpha_n^\mu e^{-in\sigma^-},$$

with same $\boldsymbol{\alpha}_0$. Thus constraints are

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}}\boldsymbol{\alpha}_{n-k}\cdot\boldsymbol{\alpha}_k = 0 \quad \forall n\in\mathbb{Z}.$$

All Poisson brackets remain the same.

4.1.3 Neumann at $\sigma = 0$ and Dirichlet at $\sigma = \pi$ (ND)

This means

$$\partial_\sigma X^\mu = 0 \text{ at } \sigma = 0, \tau \text{ and } X^\mu = x^\mu \text{ at } \sigma = \pi, \tau.$$

As usual

$$X^\mu = X_L^\mu\left(\sigma^+\right) + X_R^\mu\left(\sigma^-\right).$$

where

$$\begin{aligned}
X_L^\mu\left(\sigma^+\right) &= \frac{1}{2}x^\mu + p^\mu\alpha'\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\widetilde{\alpha}_n^\mu e^{-in\sigma^+} \\
X_R^\mu\left(\sigma^-\right) &= \frac{1}{2}x^\mu + p^\mu\alpha'\sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}.
\end{aligned}$$

The condition $\partial_\sigma X^\mu = 0$ at $\sigma = 0 \Rightarrow \alpha_n^\mu = \widetilde{\alpha}_n^\mu$ as in previous case. Next

$$X^\mu = x^\mu \text{ at } \sigma = \pi \quad \Rightarrow p^\mu = 0$$
$$i\sqrt{\frac{\alpha'}{2}}2\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\pi) = 0 \quad \forall \quad \tau.$$

This is possible only if $\cos(n\pi) = 0 \quad \forall n \Rightarrow n \in \mathbb{Z} + \frac{1}{2}$. So the sum must actually run over half integers. So we get

$$X^\mu(\sigma,\tau) = x^\mu + i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma). \tag{4.1.4}$$

One can again show that the oscillators, which are now half integral, satisfy the same Poisson bracket. It is easy to check that

$$\partial_\pm X^\mu = \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\alpha_n^\mu e^{-in\sigma^\pm} \Rightarrow (\partial_\pm X^\mu)^2 = \alpha'\frac{1}{2}\sum_{n\in\frac{1}{2}\mathbb{Z}}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\boldsymbol{\alpha}_{n-r}\cdot\boldsymbol{\alpha}_r e^{-in\sigma^\pm},$$

so that the classical constraints are again the same with the same expression for the Virasoro generators:

$$L_n = \frac{1}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\boldsymbol{\alpha}_{n-r}\cdot\boldsymbol{\alpha}_r = 0 \quad \forall\ n \in \frac{1}{2}\mathbb{Z}.$$

4.1.4 Dirichlet at $\sigma = 0$ and Neumann at $\sigma = \pi$ (DN)

Following similar process as in Subsection 4.1.3, we get that

$$X^\mu(\sigma,\tau) = x^\mu + \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}+\frac{1}{2}}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\sin(n\sigma), \tag{4.1.5}$$

and

$$\partial_\pm X^\mu = \pm\sqrt{\frac{\alpha'}{2}}\sum_{n\in 2+\frac{1}{2}}\alpha_n^\mu e^{-in\sigma^\pm}.$$

This gives us the same classical constraints. The Poisson bracket also remains the same.

4.1.5 NN for $0 \le \mu \le p$ and DD for $p+1 \le \mu \le D-1$: D-Branes

This means that

$$\partial_\sigma X^a = 0 \text{ for } a = 0,\cdots,p \text{ at } \sigma = 0,\pi$$
$$X^I(0,\tau) = c^I, \quad X^I(\pi,\tau) = d^I \quad \text{ for } I = p+1,\cdots,D-1.$$

This fixes the endpoints of the string in the $D-p-1$ directions and hence is constrained to move in the $(p+1)-$ dimensional hypersurface. This bypersurface is usually called a *Dp-Brane*. So a $D0$-brane is a particle, a $D1$-brane is itself a string, a $D2$-brane is a membrane and so on. In particular if $p = D-1$ then we get to NN case which means all space is a D-brane, that is we get space filling D-brane. Combining Fourier modes of NN and DD conditions, we get

$$\begin{aligned} X^\mu(\sigma,\tau) &= x^\mu + 2p^\mu\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(n\sigma), \mu = 0,1,\cdots,p \\ X^\mu(\sigma,\tau) &= c^\mu + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu \sin(n\sigma)\cdot \quad \mu = p+1,\cdots,D-1. \end{aligned} \tag{4.1.6}$$

One can also work out the Poisson bracket and show that they remain the same.

4.2 Quantisation

We can again quantise the open string in the cannonical way or using path integral. Here we will discuss the cannonical quantisation. As usual, it can be done in two ways. We will quickly discuss covariant quantisation but the lighcone quantisation will be discussed in some detail.

4.2.1 Covariant Quantisation

Using the classical Poisson brackets (4.1.2), we impose the commutation relations

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}, \quad [\alpha_n^\mu, \alpha_m^\nu] = n\delta_{m+n,0}\eta^{\mu\nu}, \tag{4.2.1}$$

with all others being zero. Construct the Fock space as usual from ground state $|0;p^\mu\rangle$. We will again encounter ghosts which we can again get rid of by choosing a and D as in closed string case by the same spurious state analysis. Infact in open string case we only have one set of Virasoro generators

$$L_n = \sum_r \boldsymbol{\alpha}_{n-r}\cdot\boldsymbol{\alpha}_r,$$

where the summation index and mode index run over integers or half-integers depending on boundary conditions whether NN, DD or DN, ND. The quantum Virasoro algebra is again the same i,e. the central extension of the Witt algebra. Thus the constraints are again imposed as

$$\begin{aligned} L_n|\text{phys}\rangle &= 0 \\ (L_0 - a)\,|\text{phys}\rangle &= 0 \end{aligned}$$

where a is the normal ordering constant. The number operator is

$$N = \sum_{n=1}^{\infty}\left(\alpha_{-n}^\mu\alpha_{\mu,n} + \alpha_n^i\alpha_{i,n}\right) + \sum_{r\in\mathbb{N}_0+\frac{1}{2}}\alpha_{-r}^a\alpha_{a,r},$$

where μ denotes NN direction, i denotes DD direction and a denotes the DN and ND directions and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Again using the spurious state discussion we have $a = 1, D = 26$ for our spectrum to be ghost free. Lorentz invariance is manifest and the the normal ordering constant drops out of any expressions involving angular momentum.

4.2.2 Lightcone Quantisation

As usual, we go to lightcone gauge by introducing

$$X^{\pm} = \frac{1}{\sqrt{2}} \left(X^0 \pm X^p \right)$$

and choosing $X^+ = 2\alpha' p^+ \tau$. It is easy to see that $X^{\pm}$ has to satisfy Neumann boundary condition (due to τ in X^+). The X^+ oscillators are all zero except the zero mode

$$\alpha_0^+ = \sqrt{2\alpha'} p^+.$$

As in closed string case, the oscillators of X^- is determined by the transverse oscillators upto a constant x^-. Let us now impose the commutation relations

$$\begin{aligned} \left[q^-, p^+ \right] &= -i, \left[q^i, pj \right] = i\delta^{ij} \\ \left[\alpha_n^i, \alpha_m^j \right] &= n\delta^{ij}\delta_{n+m,0}. \end{aligned} \tag{4.2.2}$$

We can now construct the Fock space from vacuum $|0; p^\mu\rangle$ by acting α_n^i $m < 0$ on $|0; p^\mu\rangle$. The spectrum is manifestly ghost free. Let us look at the ordering ambiguity. We have

$$\begin{aligned} L_0 - \alpha_0^2 = \sum_{n \neq 0} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n &= \sum_{n>0} \boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_n + \sum_{n<0}^{0} \boldsymbol{\alpha}_{-n} \cdot \boldsymbol{\alpha}_n \\ &= \sum_{n>0} \boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_n + \sum_{n<0} \left(\boldsymbol{\alpha}_n \cdot \boldsymbol{\alpha}_n - n(D-2) \right) \\ &= 2 \left(\sum_{n>0} \boldsymbol{\alpha}_n \boldsymbol{\alpha}_n + \frac{D-2}{2} \sum_{n>0} n \right). \end{aligned}$$

Now the sum above can go over integer or half-integer depending on NN, DD or ND, DN boundary conditions. In integral case, the last term is regularised using zeta function:

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}.$$

If the sum goes over half integers then the last term is regularised using Hurwitz zeta function. The last sum can be written as

$$\sum_{n \in \mathbb{N}_0 + \frac{1}{2}} n = \sum_{n=0}^{\infty} \left(n + \frac{1}{2} \right).$$

The Hurwitz zeta function is defined as

$$\zeta(s,q)=\sum_{n=0}^{\infty}\frac{1}{(n+q)^s}$$

and is holomorphic for $\mathrm{Re}(s)>1$ and $\mathrm{Re}(q)>0$. It can be analytically continued to the whole s-plane for a given value of q in the domain of definition but we do not need the complete sophisticated machinery here rather a simple trick here will do the job. We note that

$$\begin{aligned}\zeta\left(s,\frac{1}{2}\right)&=2^s\sum_{n=0}^{\infty}\frac{1}{(2n+1)^s}\\&=2^s\left[\zeta(s)-\sum_{n=1}^{\infty}\frac{1}{(2n)^s}\right]\\&=2^s\left(\zeta(s)-2^{-s}\zeta(s)\right)\\&=(2^s-1)\zeta(s).\end{aligned}$$

Thus, using the analytic continuation of the Riemann zeta function, we get

$$\zeta\left(-1,\frac{1}{2}\right)=-\frac{1}{2}\zeta(-1)=\frac{1}{24}.$$

We have five boundary conditions. In general we can have a mix of all those boundary conditions. Let i_1 be NN and DD directions and i_2 be ND and DN directions. Then we have

$$L_0-\alpha_0^2=2\sum_{n=1}^{\infty}\alpha_{-n}^{i_1}\alpha_{i_1,n}+2\sum_{n\in\mathbb{N}_0+\frac{1}{2}}\alpha_{-n}^{i_2}\alpha_{i_2,n}+D_1\left(-\frac{1}{12}\right)+D_2\left(\frac{1}{24}\right)$$

where $D_1+D_2=D-2$. Next the number operator is also given as

$$N-a=\sum_{n=1}^{\infty}\alpha_{-n}^{i_1}\alpha_{i,n}+\sum_{n\in\mathbb{N}_0+\frac{1}{2}}\alpha_{-n}^{i_2}\alpha_{i_2,n}-\frac{D_1}{12}+\frac{D_2}{24}.$$

If we consider the first string excitation $\alpha_{-n}^{i}|0;p^\mu\rangle$ where $n=1$ if $i=i_1$ and $n=\frac{1}{2}$ if $i=i_2$ then

$$N\alpha_{-n}^{i}\left|0;p^\mu\right\rangle=n\alpha_{-n}^{i}\left|0;p^\mu\right\rangle.$$

Next the mass-spectrum is calculated using the constraint

$$(L_0-a)\left|\mathrm{phys}\right\rangle=0.$$

Using the expression for (L_0-a) from previous calculation, we get

$$\begin{aligned}&2N-\frac{D_1}{12}+\frac{D_2}{24}-2\alpha'M^2+2\alpha'\left(\frac{\Delta X}{2\pi\alpha'}\right)^2\\ \Rightarrow\quad&\alpha'M^2=N-\frac{D-2}{24}+\frac{D_2}{16}+2\alpha'\left(\frac{\Delta X}{2\pi\alpha'}\right)^2,\end{aligned}$$

where ΔX is the string length in the DD direction. Let us explore the origin of the extra term. Note that in the DD directions,

$$\alpha_0^\mu = \frac{1}{\sqrt{2\alpha'}} \frac{x_1^\mu - x_0^\mu}{\pi} = \frac{\Delta X}{\sqrt{2\alpha'}\pi}.$$

This implies that

$$\alpha_0^2 = 2\alpha' p^2 + \left(\frac{\Delta X}{\sqrt{2\alpha'}\pi}\right)^2 = -2\alpha' M^2 + 2\alpha' \left(\frac{\Delta X}{2\alpha'\pi}\right)^2.$$

The extra term has natural physical interpretation: It is the mass of the string stretched between two branes.

4.2.3 String Spectrum

Let us start with NN boundary conditions. The ground state is $|0; p^\mu\rangle$ and the mass spectrum gives

$$M^2 = -\frac{D-2}{24\alpha'} < 0.$$

So the ground state is Tachyonic. The first excited state is

$$\alpha_{-1}^i |0; p^\mu\rangle$$

which transforms as a vector representation of $SO(D-2)$. Again Wigner's theorem implies that this state is a massless representation. Thus we get

$$D = 1 - \frac{p-2}{24} \Rightarrow D = 26.$$

We can go on constructing the higher excited states and show that $D = 26$ forces all of them to be massive representations of the Lorentz group. At level n, the mass spectrum is

$$\alpha' M^2 = n - 1$$

and at level n, the representation includes a symmetric tensor of rank n (this comes from Young Tableau method which we shall not describe here), thus the maximum spin at each level is (see Appendix B)

$$J_{\max} = n.$$

Hence we have

$$J_{\max} = \alpha' M^2 + 1.$$

If we plot $J_{\max}$ verses M^2 at each level, we get a straight line with slope α'. This is why α' is called the *Regge slope*. All states at a given level satisfy

$$J_{\max} \leq \alpha' M^2 + 1,$$

and since J and M^2 are quantised, all states lie on straight lines with the Tachyon lying on the leading trajectory. These lines are called *Regge trajectories*. Regge trajectories are observed in nature both for Mesons and baryons.

We now consider Dp branes i,e. NN boundary conditions in p+1 directions and DD boundary conditions in $D-p-1$ direction. There are two cases to distinguish.

One Dp Brane

In this case, we have

$$X^{\mu}(0,\tau)=c^I=X^{\mu}(\pi,\tau)\quad \mu=p+1,\ldots,D-1.$$

Thus the ends of the string are constrained to lie on one Dp brane. The ground state is now defined by

$$\alpha^i_n\,|0;p^{\mu}\rangle=0,\quad n>0,\quad i=1,2,\cdots p-1,p+1,\cdots,D-1.$$

Note that the string momentum p^{μ} is actually only in $p+1$ directions. The $SO(1,D-1)$ Lorentz group is broken into $SO(1,p)\times SO(D-P-1)$. Again Lorentz invariance requires $D=26$ and $a=1$ as we can readily see by looking at the mass spectrum of first excited state. To be explicit, the first excited states are $\alpha^i_{-1}\,|0;p^{\mu}\rangle$ for $i=1,2,\ldots,p-1$. transforms as a vector representation of $SO(1,p)$. As is known from quantum field theory, these are gauge fields. We introduce a gauge field $A_i,\ i=0,\ldots,p$ and its quanta represents spin 1 photons. The other oscillators are

$$\alpha^I_{-1}\,|0;p^{\mu}\rangle\,,\quad I=p+1,\ldots,D-1.$$

These transform as scalar representations of $SO(1,p)$ and hence we introduce $D-p-1$ scalar fields ϕ^I. These ϕ^I have physical interpretation of fluctuations of the Dp brane. This suggests that Dp branes are themselves dynamical as we will see later. Although ϕ^I transform as scalars under the $SO(1,p)$ Lorentz group of the Dp brane they transform as vectors as representations of the $SO(D-p-1)$ rotation group. This appears as a global symmetry of the brane world volume. One can also consider ϕ^I as the Goldstone Bosons associated to the spontaneously broken translational symmetry.

Two Dp Branes: String Stretched Between Two Branes

In this case $X^{\mu}(0,\sigma)=X^{\mu}(\pi,\sigma),\quad \mu=p+1,\cdots,D-1$. We get a shift in mass spectrum:

$$\alpha' M^2=N-\frac{D-2}{24}+\alpha'\left(\frac{x_1^{\mu}-x_0^{\mu}}{2\alpha'\pi}\right)^2.$$

Thus the states $\alpha^i_{-1}\,|\Delta x^{\pm},p^i\rangle$ are no longer massless. In general we can stack N such Dp branes on top of each other and denote the massless vector excitation as

$$\alpha^i_{-1}|k,\ell,p^i\rangle$$

where k, ℓ are labels which encode the Dp branes on which the endpoints of the string end. These are called *Chan-Paton labels.* The resulting N^2 states can be embedded in an $N \times N$ matrix and expanded in a complete set of $N \times N$ matrices

$$\left|k, \ell; p^i\right\rangle = \lambda^a_{k\ell} |a; p^i\rangle, \quad a \in \left\{1, \cdots, N^2\right\},$$

where $\lambda^a_{k\ell}$ are called *Chan-Paton factors.* The resulting fields T^k_ℓ, $\left(\phi^I\right)^k_\ell$ and $\left(A^a\right)^k_\ell$ can be fit into Hermitian matrices. The diagonal fields arise from strings ending on same brane. We will later see that $\left(A^a\right)^k_\ell$ are identified with $U(N)$ Yang-Mills gauge Bosons and $\left(\phi^I\right)^k_\ell$ transform in the adjoint representation of $U(N)$.

4.3 Discrete Diffeomorphisms: Oriented verses Nonoriented Strings

Until now, we dealt with oriented string theories that is we have not considered reparametrizations of the form

$$\begin{aligned} \sigma &\to \sigma' = \pi - \sigma \\ \tau &\to \tau' = \tau. \end{aligned}$$

Such a reparametrization respects the periodicity of closed strings and maps the two ends of an open string to each other and reverses the orientation $d\sigma \wedge d\tau$ of the worldsheet. The above discrete diffeomorphism can be implemented by a unitary operator Ω:

$$\Omega X^\mu(\sigma, \tau)\Omega^{-1} = X^\mu(\pi - \sigma, \tau).$$

Since the same operation twice is trivial we demand $\Omega^2 = 1$. So that the only eigenvalues of Ω can be ± 1. These actions can be expressed in terms of the oscillators. For closed string we substitute

$$X^\mu(\sigma, \tau) = x^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \left(\alpha^\mu_n e^{-in\sigma^-} + \widetilde{\alpha}^\mu_n e^{-in\sigma^+}\right).$$

Also the length of the string was normalised to 2π, so that the condition now becomes

$$\Omega X^\mu(\sigma, \tau)\Omega^{-1} = X^\mu(2\pi - \sigma, \tau).$$

This gives

$$\Omega \alpha^\mu_n \Omega^{-1} = \tilde{\alpha}^\mu_n, \quad \Omega \widetilde{\alpha}^\mu_n \Omega^{-1} = \alpha^\mu_n.$$

For open string 6 , we need to differentiate between different boundary conditions. Using similar calculation as for closed strings, we get the following:

- NN boundary condition: $\Omega \alpha^\mu_n \Omega^{-1} = (-1)^n \alpha^\mu_n$.
- DD boundary condition: $\Omega \alpha^\mu_n \Omega^{-1} = (-1)^{n+1} \alpha^\mu_n \quad \Omega x^\mu_{0,1} \Omega^{-1} = x^\mu_{1,0}$.

- ND-DN boundary condition: $\Omega \alpha^{\mu,ND}_{n+\frac{1}{2}} \Omega^{-1} = i(-1)^n \alpha^{\mu,DN}_{n+\frac{1}{2}}$.

We need to fix the action of Ω on the ground state. It turns out that $\Omega|0;p^\mu\rangle$ is determined upto a sign which is fixed by the so called *Tadpole cancellation* (will be investigated later). For closed strings, the unoriented string spectrum must be invariant under left moving - right moving sector exchange. This means that of the three massless fields, only graviton and dilaton survives. This is called the *restricted Shapiro-Virasoro model* and the oriented one is called the *extended Shapiro-Virasoro model*.

Let us now turn to the open strings. If Ω acts on the ground state with plus sign, then the unoriented open string spectrum with NN (respectively DD) boundary condition must consist of even (respectively odd) level number. For $2N$ branes stacked on top of each other, one must also consider the action of Ω on the Chan-Paton factors. Since Ω changes orientations $\left(\Omega x^\mu_{1,0} \Omega^{-1} = x^\mu_{0,1}\right)$ we have

$$\Omega \, |k, \ell; p^\mu\rangle = |\ell, k, p^\mu\rangle$$

at massless vector level. This means we only have $N(2N-1)$ (symmetric) surviving Chan-Paton factors. Thus we get a massless vector of a $SO(2N) \subset U(2N)$ gauge theory. If Ω acts with negative sign, the Chan-Paton labels are antisymmetrized and we get a massless vector of a $Sp(2N) \subset U(2N)$ gauge theory where $Sp(2N)$ is the symplectic group of rank N defined as follows:

$$Sp(2N) = \left\{ M \in GL(2N, \mathbb{R}) : MJM^T = J \right\}$$

where

$$J = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

where $\mathbb{I}$ is $N \times N$ identity matrix. This is because the dimension of the antisymmetric representation is

$$(2N)^2 - \frac{2N(2N-1)}{2} = (2N)\left[\frac{4N-(2N-1)}{2}\right] = N(2N+1)$$

which is equal to the real dimension of $Sp(2N)$.

Appendices

Appendix A

Wigner's Classification of Representations of Poincaré Group

In this appendix, we will briefly review Wigner's little group method of classifying the irreducible representations of the Poincaré group. The idea is mathematically enlightening and and motivated a lot of research in representation theory. But here we will not delve into the mathematically rigorous treatment. Rather we will take a more physical approach on the lines of Weinberg's quantum theory of fields. We begin by discussing Wigner's proposal of interpreting elementary particles as irreducible representations of the Poincaré group. We assume familiarity with basic terminology of topology.

A.1 Projective Representations

Let $|\Psi\rangle$ be a state in Hilbert space $\mathcal{H}$. Note that any two states $|\Psi\rangle$ and $|\Phi\rangle$ which are nonzero and related by

$$|\Psi\rangle = \lambda|\Phi\rangle \quad \lambda \in \mathbb{C}^* := \mathbb{C}\backslash\{0\} \tag{A.1.1}$$

are the same quantum mechanical states. So it is pertinent to consider the quotient space of $\mathcal{H}^* = \mathcal{H}\backslash\{0\}$ as $\mathbb{P}(\mathcal{H}) := \mathcal{H}^*/\sim$ where $|\Psi\rangle \sim |\Phi\rangle$ if and only if (A.1.1) is true. The quotient space $\mathbb{P}(\mathcal{H})$ is called the *projectivised Hilbert space.* Recall that the probability amplitude of transition from $|\Psi\rangle$ to Φ is given by

$$p(|\Psi\rangle, |\Phi\rangle) = \frac{\langle\Psi \mid \Phi\rangle}{\langle\Psi \mid \Psi\rangle\langle\Phi \mid \Phi\rangle}.$$

In the quotient topology on $\mathbb{P}(\mathcal{H})$, p induces a continuous map[1] on $\mathbb{P}(\mathcal{H})$ which we denote by $\widetilde{p}$. A homeomorphism $T : \mathbb{P}(\mathcal{H}) \longrightarrow \mathbb{P}(\mathcal{H})$ satisfying

$$\widetilde{p}(T[\Psi], T[\Phi]) = \widetilde{p}(|\Psi\rangle, |\Phi\rangle)$$

[1] it is a standard result in quotient topology. See for example Topology by Munkres.

where $[\Phi], [\Psi]$ are equivalence classes in $\mathbb{P}(\mathcal{H})$, is called a *projective automorphism.* The set of all such maps, denoted by $\mathrm{Aut}(\mathbb{P}(\mathcal{H}))$, is a group called *projective automorphism group.* The action of this group on $\mathbb{P}(\mathcal{H})$ leaves transition probabilities invariant. Now consider a particle in the Minkowski space $\mathbb{R}^{1,D-1}$. The symmetry group of this space is precisely the Poincaré group[2] which we denote by $\mathcal{P}$. Let two observers $\mathcal{O}$ and $\mathcal{O}'$, related by $\Lambda \in \mathcal{P}$, measure the quantum mechanical particle. In general, there measurement result will reveal different states, say $[\Psi]$ and $[\Psi']$ respectively. Thus physically one expects that transition probabilities in $\mathcal{O}$ and $\mathcal{O}'$ be same. This means that the two states must be related by some projective automorphism:

$$[\Psi] = T_\Lambda[\Psi'], \quad \text{for some} \quad T_\Lambda \in \mathrm{Aut}(\mathbb{P}(\mathcal{H})).$$

If $\mathcal{O} = \mathcal{O}'$ then $T_\Lambda = Id$ and we should have $T_\Lambda = T_{Id} = Id \in$Aut$(\mathbb{P}(\mathcal{H}))$. Lastly if a third observer $\mathcal{O}''$, related to $\mathcal{O}'$ by Γ, measures the state then we must impose $T_\Lambda \circ T_\Gamma = T_{\Lambda \circ \Gamma}$. Thus the change of frame induces a representation $\Pi : \mathcal{P} \longrightarrow \mathrm{Aut}(\mathbb{P}(\mathcal{H}))$. This is called the *projective representation.*

A.2 Elementary Particles

The representation $(\Pi, \mathcal{H})$ of the Poincaré group is called irreducible if the only nontrivial closed invariant subspace of $\mathcal{H}$ is $\mathcal{H}$. That is $\Pi(\mathcal{P})(V) \subseteq V$ if and only if $V = \mathcal{H}$. The closed condition is technical: we want the invariant subspace to be a Hilbert space in its own right which is not automatically true in infinite dimensional Hilbert space unless the subspace is closed.

Wigner suggested that the irreducible projective representations of the Poincaré group correspond to elementary particles within the quantum system under consideration. Wigner's argument was as follows: an elementary particle in a quantum mechanical system is a vector in $\mathbb{P}(\mathcal{H})$. As discussed, different observers will see different vectors in $\mathbb{P}(\mathcal{H})$ corresponding to the elementary particle. All these vectors must be related by some projective automorphism. The set of all these vectors constitutes $\mathcal{P}-$invariant subspace of $\mathbb{P}(\mathcal{H})$ and hence we obtain a subrepresention of $(\Pi, \mathcal{H})$. This subrepresentation can be thought of as a subsystem which is elementary if it is irreducible (otherwise it will have more smaller subsystems). This reduces the problem of determining all relativistic free particles in Minkowski spacetime to the mathematical task of finding all irreducible projective representations of the Poincaré group.

[2] mathematically speaking, the symmetry group of a Riemannian manifold $(\mathcal{M}, g)$ is the group of all diffeomorphisms from $\mathcal{M}$ to itself whose pullback preserves the metric.

A.3 Projective Representations of the Poincaré Group

Let us now take a look at the Poincaré group more closely. We begin by defining semidirect product.

Definition A.3.1. Let H and N be groups and suppose there is a a group homomorphism $\phi : H \to \mathrm{Aut}(N)$. Then the semidirect product of H by N, denoted $H \ltimes N$ which has $H \times N$ as underlying set, and multiplication defined by $(h, n) \cdot (h', n') = (hh', n\phi(h)\,(n'))$.

Each element of the Lorentz group $SO(1, D-1)$ defines an automorphism of $\mathbb{R}^{1,D-1}$ defined by matrix multiplication. Thus we can form the semidirect product $SO(1, D-1) \ltimes \mathbb{R}^{1,D-1}$. The physically relevant Poincaré group is the semidirect product of the proper orthochronous Lorentz group and the abelian translation group. That is

$$\mathcal{P} = ISO(1, D-1) = SO(1, D-1)_I \ltimes \mathbb{R}^{1,D-1}$$

where $SO(1, D-1)_I$ is the connected component of identity in the Lorentz group. The Poincaré algebra is generated by the generators of translations and Lorentz transformations denoted by P^μ and $M^{\mu\nu}$ respectively. They satisfy the Poincaré algebra:

$$\begin{aligned}
&i\,[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\sigma\mu} M_{\rho\nu} + \eta_{\sigma\nu} M_{\rho\mu} \\
&i\,[P_\mu, M_{\rho\sigma}] = \eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho \\
&i\,[P_\mu, P_\rho] = 0.
\end{aligned}$$

The third commutator says that P_μ commutes among themselves. So we start with states in $\mathbb{P}(\mathcal{H})$ which are simultaneous eigenvectors of P^μ. We label all other degrees of freedom by σ. We have

$$P^\mu \psi_{q,\sigma} = q^\mu \psi_{q,\sigma}.$$

Note that infinitesimal translations are represented by $U = \mathbb{I} - iP^\mu \varepsilon_\mu$ and repeating this, we obtain finite translations

$$U(\mathbb{I}, a) = e^{-iP^\mu a_\mu}.$$

so that

$$U(\mathbb{I}, a)\psi_{q,\sigma} = e^{-iq\cdot a}\psi_{q,\sigma}.$$

These $U(\mathbb{I}, a)$ are the projective representations of the translation part of the Poincaré group. Usually the physical requirement restricts U to be unitary which restricts P^μ to be Hermitian. Recall that

$$\begin{aligned}
&(\Lambda, a) \cdot (\Lambda', a') = (\Lambda\Lambda', a' + \Lambda a) \quad \text{in} \quad \mathcal{P} \\
&(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda a)\,.
\end{aligned}$$

An infinitesimal Poincaré transformation with parameters ω, ε is unitarily represented as

$$U(\mathbb{I} + \omega, \varepsilon) = \mathbb{I} + \frac{i}{2}\omega_{\mu\nu} M^{\mu\nu} - \varepsilon_\mu P^\mu + \ldots .$$

So For a general $\Lambda \in SO(1, D-1)$ we have

$$U(\Lambda, a)U(\mathbb{I}+\omega, \in)U(\Lambda, a)^{-1} = U\left(\Lambda(\mathbb{I}+\omega)\Lambda^{-1}, \Lambda\varepsilon - \Lambda\omega\Lambda^{-1}a\right).$$

Using infinitesimal version upto linear order in ω, we get

$$U(\Lambda, a)\left[\mathbb{I} + \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu} - \varepsilon_\mu P^\mu\right]U(\Lambda, a)^{-1} = \mathbb{I} + \frac{i}{2}\left(\Lambda\omega\Lambda^{-1}\right)_{\mu\nu}M^{\mu\nu} - \left(\Lambda\varepsilon - \Lambda\omega\Lambda^{-1}a\right)_\mu P^\mu.$$

Comparing coefficients of $\omega_{\mu\nu}$ and ε_μ, we get

$$\begin{aligned} U(\Lambda, a)M^{\mu\nu}U(\Lambda, a)^{-1} &= \left(\Lambda^{-1}\right)^{\mu}_{\lambda}\left(\Lambda^{-1}\right)^{\nu}_{\rho}\left(M^{\lambda\rho} - a^\lambda P^\rho + a^\rho P^\lambda\right) \\ U(\Lambda, a)P^\rho U(\Lambda, a)^{-1} &= \left(\Lambda^{-1}\right)^{\rho}_{\mu}P^\mu. \end{aligned} \tag{A.3.1}$$

Our aim now is to find the projective representation of the Lorentz part of the Poincaré group. Indeed if $U(\Lambda, 0) \equiv U(\Lambda)$ is such a representation then

$$\begin{aligned} P^\mu U(\Lambda)\psi_{p,\sigma} &= U(\Lambda)U(\Lambda)^{-1}P^\mu U(\Lambda)\psi_{p,\sigma} \\ &= U(\Lambda)\Lambda^\mu_\nu P^\nu\psi_{p,\sigma} \\ &= (\Lambda p)U(\Lambda)\psi_{p,\sigma}. \end{aligned}$$

So we must have

$$U(\Lambda)\psi_{p,\sigma} = \sum_{\sigma'} C_{\sigma'\sigma}(\Lambda, p)\psi_{\Lambda p,\sigma'}. \tag{A.3.2}$$

In general, this representation is reducible. since this is a unitary representation, a theorem in representation theory says that it is completely reducible, that is it can be written as a direct sum of irreducible representations of invariant subspaces of eigenvectors of P^μ with eigenvalue Λp. Our goal is to classify all such irreducible representations. To do so, we first calculate the orbit of action of Lorentz group on $\mathbb{R}^{1,D-I}$. It is clear that $SO(1, D-1)_I$ fixes p^2 for all $p \in \mathbb{R}^{1,D-1}$ but when $p^2 \leq 0$ then it also fixes the sign of p^0. Accordingly we get the following orbits:

1. $p^2 = m^2 > 0$: one sheeted hyperboloid.
2. $p^2 = -m^2 < 0$: two sheeted hyperboloid corresponding to $p^0 > 0$ or $p^0 < 0$.
3. $p^2 = 0$: cone with vertex at the origin.

Now given any p^μ, one can choose (depending on the orbit of p^μ) a standard q^μ such that

$$p^\mu = L^\mu_\nu(p)q^\nu,$$

where $L_\nu \in SO(1, D-1)_I$. By above discussion

$$\psi_{p,\sigma} = N(p)U\left(L^\mu_\nu(p)\right)\psi_{q,\sigma},$$

where $N(p)$ is some normalesation factor. Now for any $\Lambda \in SO(1, D-1)_I$ we have

$$\begin{aligned} U(\Lambda)\psi_{p,\sigma} &= N(p)U(\Lambda)U(L(p))\psi_{q,\sigma} \\ &= N(p)U(L(\Lambda p))U\left(L^{-1}(\Lambda p)\Lambda L(p)\right)\psi_{q,\sigma}, \end{aligned}$$

where we used property of group representations. Note that

$$L^{-1}(\Lambda p)\Lambda L(p)q = L^{-1}(\Lambda p)\Lambda p = q.$$

The set of all such elements of Λ is called the *stability group* of q also called the *little group*. For any two elements $W, \overline{W}$ in the little group of q, we have

$$U(W)\psi_{q,\sigma} = \sum_{\sigma\prime} D^q_{\sigma,\sigma'}(W)\psi_{q,\sigma'}$$

and

$$\begin{aligned} U(\overline{W}W)\psi_{q,\sigma} &= \sum_{\sigma'} D^q_{\sigma,\sigma'}(W)\sum_{\sigma''} D^q_{\sigma',\sigma''}(\overline{W})\psi_{q,\sigma''} \\ &= \sum_{\sigma',\sigma''} D^q_{\sigma,\sigma'}(W)D^q_{\sigma',\sigma''}(\overline{W})\psi_{q,\sigma''} \\ &= \sum_{\sigma''} D^q_{\sigma,\sigma''}(\overline{W}W)\psi_{q,\sigma''}, \end{aligned}$$

where

$$D^q_{\sigma,\sigma''}(\overline{W}W) = \sum_{\sigma'} D^q_{\sigma,\sigma'}(W)D^q_{\sigma',\sigma''}(\overline{W}).$$

Thus we see that $D^q(W)$ is a representation of the little group. So putting $W(\Lambda, p) = L^{-1}(\Lambda p)\Lambda L(p)$ we have

$$U(W(\Lambda, p))\psi_{q,\sigma} = \sum_{\sigma'} D_{\sigma,\sigma'}(W(\Lambda, p))\psi_{q,\sigma'}.$$

So that

$$\begin{aligned} U(\Lambda)\psi_{p,\sigma} &= N(p)\sum_{\sigma'} D_{\sigma,\sigma'}(w(\Lambda, p))U(L(\Lambda p))\psi_{q,\sigma'} \\ &= \frac{N(p)}{N(\Lambda p)}\sum_{\sigma'} D_{\sigma,\sigma'}(W(\Lambda, p))\psi_{\Lambda p,\sigma'}. \end{aligned}$$

Hence apart from the normalisation factor, the problem of finding unitary irreducible representations of Poincaré group has been reduced to finding unitary irreducible representations of the little group corresponding to each orbit. So we first find the little group corresponding to each orbit.

1. $q^2 = m^2 > 0$: by going to rest frame, we can set q^μ to $q^\mu = (0, 0, \cdots, 0, m)$. Looking at the form of this vector, we can see that the little group is $SO(1, D-2)_I \hookrightarrow SO(1, D-1)_I$.

2. $q^2 = -m^2 < 0$: , by going to rest frame, we can take q^μ to be $q^\mu = (m, \vec{0})$. Clearly the little group is $SO(D-1)$.

3. $q^2 = 0$: the little group computation is not so obvious. Although it turns out to be the Euclidean group $E(D-2) = SO(D-2) \ltimes \mathbb{R}^{D-2}$. This is the isometry group of $\mathbb{R}^{D-2}$ with the Euclidean metric.

In $q^2 = 0$, one case is $q^\mu = 0$ whose stabiliser is the whole Poincaré group $\mathcal{P}$.

Gender	Orbit	Little Group	Unitary Representation
$q^2 = -m^2$	Mass shell	$SO(D-1)$	Massive
$q^2 = -m^2$	Hyperboloid	$SO(1, D-1)_I$	Tachyonic
$q^2 = 0$	Lightcone	$E(D-2)$	Massless
$q^\mu = 0$	Origin	$\mathcal{P}$	Zero Momentum

Physically, Tachyonic representations are not accepted. So we will only deal with the other two. One can use the little group method to find all irreducible representations of the Euclidean group. The idea is to go to the Lie algebra of $E(D-2)$ and identify the "translations" generators and repeat the procedure above. The upshot of this computation is that we get two orbits and the corresponding little groups are called *short little groups*. The corresponding unitary irreducible representations are labelled as *helicity* and *infinite spin*. The analogue of the Lorentz group here is obviously $SO(D-2)$. The short Little group corresponding to infinite spin is $S0(D-3)$ and that for infinite spin is $SO(D-2)$.

Next one can use Young Tableau to embed the irreducible representations of the Little groups in all cases into tensorial representations. For the particular case that we will be dealing with, we would like to find the massless irreducible representations of dimension $(D-2)^2$ of the Poincaré group. It turns out that it is the direct sum of three irreducible parts:

$$\begin{array}{lccc} & \textbf{Traceless symmetric} \oplus & \textbf{Antisymmetric} \oplus & \textbf{Trace (Scalar)} \\ \text{Dim:} & \dfrac{(D-2)(D-1)}{2} - 1 & \dfrac{(D-2)(D-3)}{2} & 1 \end{array}$$

Appendix B

A Note on Maximum Spin of Level N States

In Chapter 4, we used the fact that the maximum spin in level N representation of the Lorentz group is N. In this appendix, for each spin component, we will produce a level N state and show that its spin eigenvalue corresponding to the particular component is N To make this explicit, we first recall the spin generators

$$E^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha^{\mu}_{-n}\alpha^{\nu}_{n} - \alpha^{\nu}_{n}\alpha^{\mu}_{n}\right), \quad \widetilde{E}^{\mu\nu} = -i\sum_{n=1}^{\infty}\frac{1}{n}\left(\widetilde{\alpha}^{\mu}_{-n}\widetilde{\alpha}^{\nu}_{n} - \widetilde{\alpha}^{\nu}_{n}\widetilde{\alpha}^{\mu}_{n}\right).$$

We will distinguish between open and closed strings. In lightcone quantisation, the relevant spin generators are E^{ij} and $\widetilde{E}_{ij}$ for $1 \leq i, j \leq D-2$. In closed string case, the state corresponding to the spin component E^{ij} and $\widetilde{E}^{ij}$ is given by

$$\Omega^{ij} = \left(\alpha^{i}_{-1} + i\alpha^{j}_{-1}\right)^{N}\left(\widetilde{\alpha}^{i}_{-1} + i\widetilde{\alpha}^{j}_{-1}\right)^{N}|0;p^{\mu}\rangle\,.$$

Now observe that

$$\begin{aligned}
E^{ij}\Omega^{ij} &= \left(\widetilde{\alpha}^{i}_{-1} + i\widetilde{\alpha}^{j}_{-1}\right)^{N}(-i)\sum_{n=1}\frac{1}{n}\left(\alpha^{i}_{-n}\alpha^{j}_{n} - \alpha^{j}_{-n}\alpha^{i}_{n}\right)\left(\alpha^{i}_{-1} + i\alpha^{i}_{-1}\right)^{N}|0;p^{\mu}\rangle \\
&= \left(\widetilde{\alpha}^{i}_{-1} + i\widetilde{\alpha}^{j}_{-1}\right)^{N}(-i)\left(\alpha^{i}_{-1}\alpha^{j}_{1} - \alpha^{j}_{-1}\alpha^{i}_{1}\right)\left(\alpha^{i}_{-1} + i\alpha^{j}_{-1}\right)^{N}|0;p^{\mu}\rangle\,,
\end{aligned}$$

where we used the fact that $\left(\alpha^{i}_{-n}\alpha^{j}_{n} - \alpha^{j}_{-n}\alpha^{i}_{n}\right)$ commutes with $\alpha^{i}_{-1} + i\alpha^{j}_{-1} \quad \forall n > 1$. Now we have

$$\begin{aligned}
\left[\alpha^{i}_{-1}\alpha^{j}_{1} - \alpha^{j}_{-1}\alpha^{i}_{1},\ \alpha^{i}_{-1} + i\alpha^{j}_{-1}\right] &= \alpha^{i}_{-1}\left[\alpha^{j}_{1}, \alpha^{i}_{-1}\right] + i\alpha^{i}_{-1}\left[\alpha^{j}_{1}, \alpha^{j}_{-1}\right] - \alpha^{j}_{-1}\left[\alpha^{i}_{1},\ \alpha^{i}_{-1}\right] - i\alpha^{j}_{-1}\left[\alpha^{i}_{1}, \alpha^{j}_{-1}\right] \\
&= \alpha^{i}_{-1}\delta^{ji} + i\alpha^{i}_{-1} - \alpha^{j}_{-1} - i\alpha^{j}_{-1}\delta^{ij} \\
&= \begin{cases} i\left(\alpha^{i}_{-1} + i\alpha^{j}_{-1}\right) & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}
\end{aligned}$$

So assuming $i \neq j$, we get

$$\begin{aligned}
E^{ij}\Omega^{ij} &= \left(\widetilde{\alpha}^i_{-1} + i\widetilde{\alpha}^j_{-1}\right)^N (-i)\left(\alpha^i_{-1}\alpha^j_1 - \alpha^j_{-1}\alpha^i_1\right)\left(\alpha^i_{-1} + i\alpha^j_{-1}\right)^N |0; p^\mu\rangle \\
&= \left(\widetilde{\alpha}^i_{-1} + i\widetilde{\alpha}^j_{-1}\right)^N (-i)(iN)\left(\alpha^i_{-1} + i\alpha^j_{-1}\right)^N |0; p^\mu\rangle \\
&= N\Omega^{ij}.
\end{aligned}$$

Similarly $\widetilde{E}^{ij}\Omega^{ij} = N\Omega^{ij}$. In case of open strings with NN boundary conditions:

$$E^{ii}\left(\alpha^i_{-1} + i\alpha^j_{-1}\right)^N |0; p^\mu\rangle = N\left(\alpha^i_{-1} + 2\alpha^j_{-1}\right)|0; p^\mu\rangle .$$