Computing the permanental polynomial of $4k$-intercyclic bipartite graphs*

Hitesh Wankhede[1], Ranveer Singh[1], and Ravindra B. Bapat[2,3]

[1] Indian Institute of Technology Indore, Department of Computer Science and Engineering, Indore, India
`hitesh.wankhede@students.iiserpune.ac.in`, `hiteshwankhede9@gmail.com`, `ranveer@iiti.ac.in`

[2] Indian Statistical Institute, Theoretical Statistics and Mathematics Unit, New Delhi, India

[3] Ashoka University, Department of Mathematics, New Delhi, India
`rbb@isid.ac.in`, `ravindra.bapat@ashoka.edu.in`

Abstract. Let G be a bipartite graph with the permanental polynomial $\pi(G, x)$. We define a modified characteristic polynomial $\phi_p(G, x)$ for a bipartite graph by changing the sign of some of the coefficients of its characteristic polynomial. A graph is said to be $4k$-intercyclic if it does not have vertex-disjoint $4k$-cycles. We give an expression to compute $\pi(G, x)$ for $4k$-intercyclic bipartite graphs in terms of $\phi_p(G, x)$ and $\phi_p(G\backslash R, x)$ where R is a $4k$-cycle in G. This class of bipartite graphs is different than the class of bipartite graphs that do not contain an even subdivision of $K_{2,3}$, and for such a class, a method to compute $\pi(G, x)$ using Pfaffian orientation was given by Yan and Zhang in 2004 and Zhang and Li in 2012.

Keywords: Permanental Polynomial · Bipartite Graphs · $4k$-Intercyclic Graphs.

1 Introduction and Preliminaries

We consider simple and undirected graphs. Let G be a graph with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The *adjacency matrix* $A(G) = (a_{ij})$ of a graph G is defined as $a_{ij} = 1$ if v_i and v_j are adjacent and 0 otherwise. The *determinant* and the *permanent* of $A(G)$, are defined as

$$\det(A(G)) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \text{ and } \operatorname{per}(A(G)) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)},$$

respectively, where S_n is the set of all permutation of the set $\{1, 2, \ldots, n\}$ and $\operatorname{sgn}(\sigma)$ is the signature of the permutation σ. While the determinant can be computed in polynomial time using the Gaussian elimination method, and the fastest

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known algorithm runs in $\mathcal{O}(n^{2.3728})$ time [2,3], computing the permanent is notoriously difficult, as it is #P-complete [28]. The "Permanent vs. Determinant Problem" in computational complexity theory is as follows: "Can we express the permanent of matrices as the determinant of a (possibly polynomially larger) matrices?" For an upper bound on the size of the larger matrix, see [14], and for a survey on lower bounds, see [1]. The problem "Given a $(0,1)$-matrix A, under what conditions, changing the sign of some the nonzero entries yields a matrix B such that the $\mathrm{per}(A) = \det(B)$?" is famously known as "Polya Permanent Problem," [25] and it is equivalent to twenty-three other problems listed in [22] including the Even Cycle problem: "Given a directed graph, does it have an even cycle?" More recently, the problem of finding a shortest even cycle was shown to be tractable [8]. Immanants are matrix functions that generalize determinant and permanent, and their complexity dichotomy was also recently studied [11].

The *characteristic polynomial* and the *permanental polynomial* of graph G are defined as

$$\phi(G,x) = \det(xI - A(G)) \text{ and } \pi(G,x) = \mathrm{per}(xI - A(G)),$$

respectively, where I is the identity matrix of order n. The characteristic and the permanental polynomial are graph invariants, and they could be helpful in distinguishing graphs towards Graph Isomorphism Problem [29]. But the permanental polynomial is not studied in great detail as compared to the characteristic polynomial, probably due to its computational difficulty. Some computational evidence suggests that the permanental polynomial is better than the characteristic polynomial while distinguishing graphs [13,19]. We are interested in finding ways to compute the permanental polynomial efficiently; one way to do that is by expressing the permanental polynomial in terms of the characteristic polynomial. For an excellent survey on the permanental polynomial, we refer to [18].

Let

$$\phi(G,x) = \sum_{i=0}^{n} a_i x^{n-i} \text{ and } \pi(G,x) = \sum_{i=0}^{n} b_i x^{n-i}.$$

Then, the interpretation of these coefficients is given using the Sachs formulas as

$$a_i = \sum_{U_i} (-1)^{p(U_i)} 2^{c(U_i)} \text{ [12] and } b_i = (-1)^i \sum_{U_i} 2^{c(U_i)} \text{[24]}, \tag{1}$$

where the summation is taken over all the Sachs subgraphs U_i (subgraphs whose components are either cycles or edges) of G on i vertices, $p(U_i)$ denotes the number of components in U_i, and $c(U_i)$ denotes the number of components in U_i which are cycles.

Proposition 11. [9,24] A graph G is bipartite if and only if $a_i = b_i = 0$ for each odd i.

Hence, for a bipartite graph G, we have

$$\phi(G,x) = \sum_{i=0,2,4,\dots} a_i x^{n-i} \text{ and } \pi(G,x) = \sum_{i=0,2,4,\dots} b_i x^{n-i}.$$

For a bipartite graph G, we define $f_i = b_i - (-1)^{i/2} a_i$ for each i, and introduce a modified characteristic polynomial and also a new graph polynomial as

$$\phi_p(G,x) = \sum_{i=0,2,4,\ldots} (-1)^{i/2} a_i x^{n-i} \text{ and } f(G,x) = \sum_{i=0,2,4,\ldots} f_i x^{n-i},$$

respectively, such that we have

$$\pi(G,x) = \phi_p(G,x) + f(G,x). \tag{2}$$

In the graph G, we denote a cycle of length k by C_k. Similarly, we denote a cycle of length $4k$ by C_{4k} and refer to it by $4k$-*cycle*. A graph is called C_{4k}-*free* if it does not contain any $4k$-cycle. A graph is called *intercyclic* if it does not contain two vertex-disjoint cycles. Similarly, a graph is called $4k$-*intercyclic* if it does not contain two vertex-disjoint $4k$-cycles (see Figure 1 for an example).

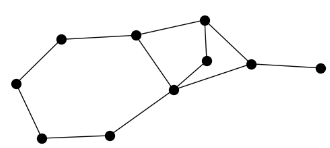

Fig. 1. Example of a $4k$-intercyclic bipartite graph.

In 1985, Borowiecki proved the following.

Theorem 12. *[9] Let G be a bipartite graph with the spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then, G is C_{4k}-free if and only if its per-spectrum is $\{i\lambda_1, i\lambda_2, \ldots, i\lambda_n\}$[4].*

By inspecting the proof of this theorem, we notice that a bipartite graph G is C_{4k}-free if and only if $\pi(G,x) = \phi_p(G,x)$ (see Corollary 23). As a result, the permanental polynomial of C_{4k}-free bipartite graphs can be computed directly through the modified characteristic polynomial. Yan and Zhang, in 2004, found that the permanental polynomial of a larger class of bipartite graphs can be computed using the characteristic polynomial of some oriented graph. They proved the following.

Theorem 13. *[31] Let G be a bipartite graph with n vertices containing no subgraph, which is an even subdivision of $K_{2,3}$. Then there exists an orientation G^e of G such that $\pi(G,x) = \det(xI - A(G^e))$, where $A(G^e)$ denotes the skew adjacency matrix of G^e.*

Later Zhang and Li, in 2012, proved the converse of this statement.

[4] The *spectrum* and the *per-spectrum* of a graph G are the set of all roots of its characteristic polynomial and the permanental polynomial, respectively.

Theorem 14. *[32] There exists an orientation G^e of a bipartite graph G such that $\pi(G,x) = \det(xI - A(G^e))$ if and only if G contains no even subdivision of $K_{2,3}$.*

Zhang and Li also show that bipartite graphs that do not contain an even subdivision of $K_{2,3}$ are planar and admit Pfaffian orientation. They also give characterization and recognition of such graphs, which leads to a polynomial time algorithm to compute the permanental polynomial of such bipartite graphs.

Borowiecki [9] had posed the open problem, "Characterize graphs with purely imaginary per-spectrum." Since the eigenvalues of a skew adjacency matrix are purely imaginary, Theorem 14 is a partial solution to this open problem. Note that the class of C_{4k}-free bipartite graphs is a subset of the class of bipartite graphs that do not contain an even subdivision of $K_{2,3}$. In our earlier work [30], we construct bipartite graphs with purely imaginary per-spectrum containing an even subdivision of $K_{2,3}$.

Let G be a graph with an orientation G^e such that $\pi(G,x) = \det(xI - A(G^e))$. Then, $\pi(G,x) = (x - i\lambda_1)(x - i\lambda_2)\ldots(x - i\lambda_n)$ for some real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$. When k is odd, i is a factor of the coefficient b_k of x^{n-k} . Since the coefficients of $\pi(G,x)$ must be real, it follows that $b_k = 0$ for all odd k, and by Proposition 11, G is bipartite. Theorem 14 can now be reformulated as follows.

Theorem 15. *There exists an orientation G^e of a graph G such that $\pi(G,x) = \det(xI - A(G^e))$ if and only if G is a bipartite graph containing no even subdivision of $K_{2,3}$.*

Theorem 15 suggests that the orientation approach in computing the permanental polynomial only works for the class of bipartite graphs that do not contain an even subdivision of $K_{2,3}$. In this article, we give a formula to compute $\pi(G,x)$ for the class of $4k$-intercyclic bipartite graphs (a superset of the class of C_{4k}-free bipartite graphs). This is done by expressing $f(G,x)$ in terms of the modified characteristic polynomial of the graph G and its subgraphs. Our approach is combinatorial rather than based on Pfaffian orientation. Note that the class of $4k$-intercyclic bipartite graphs is different than and not a subset of the class of bipartite graphs that do not contain an even subdivision of $K_{2,3}$.

2 Main result

Theorem 21. *Let G be a $4k$-intercyclic bipartite graph. Then,*

$$\pi(G,x) = \phi_p(G,x) + 4 \sum_{R \in \mathcal{C}_{4k}(G)} \phi_p(G \backslash R, x),$$

where $\mathcal{C}_{4k}(G)$ denotes the set of all $4k$-cycles in G.

To prove this theorem, we need the following lemma.

Lemma 22. *Let G be a bipartite graph. Then, for each $i = 0, 2, 4, \dots,$ we have*

$$f_i = \sum_{j=1,3,5,\dots} 2^{j+1} \sum_{\substack{U_i \text{ containing} \\ \text{exactly } j \text{ } 4k\text{-cycles}}} 2^t,$$

where U_i denotes a Sachs subgraph on i vertices.

Proof. In a bipartite graph, there can be two types of cycles: $4k$-cycles or $(4k+2)$-cycles. Hence, a Sachs subgraph U_i on i vertices can be written as

$$U_i = \{C_{4k_1} \cup \cdots \cup C_{4k_s}\} \cup \{C_{4l_1+2} \cup \cdots \cup C_{4l_t+2}\} \cup \underbrace{\{K_2 \cup \cdots \cup K_2\}}_{r-\text{times}},$$

where s and t are the number of $4k$-cycles and $(4k+2)$-cycles respectively. Then, $p(U_i) = s + t + r$ and $c(U_i) = s + t$ and

$$i = 4(k_1 + \cdots + k_s) + 4(l_1 + \cdots + l_t) + 2(t + r).$$

Check that $s + t + r \equiv i/2 + s \pmod 2$. Using this fact, the coefficients of the characteristic polynomial and the permanental polynomial given in Equation 1 can be written as

$$(-1)^{i/2} a_i = \sum_{U_i} (-1)^s 2^{s+t} \text{ and } b_i = \sum_{U_i} 2^{s+t},$$

respectively ($(-1)^i = 1$ since i is even). Since $f_i = b_i - (-1)^{i/2} a_i$, we get

$$f_i = \sum_{U_i} (1 - (-1)^s)\, 2^{s+t} = \sum_{\substack{U_i \text{ containing an odd} \\ \text{number of } 4k\text{-cycles}}} 2^{s+1} 2^t \tag{3}$$

$$= \sum_{j=1,3,5,\dots} 2^{j+1} \sum_{\substack{U_i \text{ containing} \\ \text{exactly } j \text{ } 4k\text{-cycles}}} 2^t.$$

Note that the contribution in Equation 3 of the Sachs subgraphs in which we have exactly an even number of $4k$-cycles vanishes.

Proof (Theorem 21). Since the graph G is $4k$-intercyclic, the graph $G \backslash R$ is C_{4k}-free for any $R \in \mathcal{C}_{4k}(G)$. Similarly, any Sachs subgraph U_i of G can contain at most one $4k$-cycle, that is, $s \leq 1$. Using Lemma 22, we have

$$f_i = 4 \sum_{\substack{U_i \text{ containing} \\ \text{exactly one } 4k\text{-cycle}}} 2^t.$$

Let $l_R = i - |V(R)|$, and let W_{l_R} denote a Sachs subgraph in $G \backslash R$. Then, there is a one-to-one correspondence between the Sachs subgraphs U_i in G containing R and the Sachs subgraphs W_{l_R} in $G \backslash R$. Hence,

$$f_i = 4 \sum_{R \in \mathcal{C}_{4k}(G)} \sum_{W_{l_R}} 2^t.$$

Observe that

$$\pi(G\backslash R, x) = \sum_{l_R=0,2,4,\ldots} \sum_{W_{l_R}} 2^t x^{(n-|V(R)|)-l_R}.$$

Now consider the polynomial

$$\begin{aligned} f(G,x) = \sum_{i=0,2,4,\ldots} f_i x^{n-i} &= 4 \sum_{R\in\mathcal{C}_{4k}(G)} \sum_{l_R=0,2,4,\ldots} \sum_{W_{l_R}} 2^t x^{(n-|V(R)|)-l_R} \\ &= 4 \sum_{R\in\mathcal{C}_{4k}(G)} \pi(G\backslash R, x). \end{aligned}$$

Hence, using Equation 2, we get

$$\pi(G,x) = \phi_p(G,x) + 4 \sum_{R\in\mathcal{C}_{4k}(G)} \pi(G\backslash R, x).$$

Since $G\backslash R$ is C_{4k}-free, the application of this expression leads to $\pi(G\backslash R, x) = \phi_p(G\backslash R, x)$. This proves the theorem.

Example 1. Consider the $4k$-intercyclic graph G shown in Figure 1. It contains three 4-cycles and two 8-cycles, and removal of each of them from the graph yields the following subgraphs: $P_5 \cup K_1$, $P_4 \cup K_2$, $P_4 \cup K_1 \cup K_1$, K_2 and $K_1 \cup K_1$, respectively. Then, using Theorem 21,

$$\begin{aligned} \pi(G,x) = \phi_p(G,x) + 4(&\phi_p(P_5\cup K_1,x) + \phi_p(P_4\cup K_2,x) + \phi_p(P_4\cup K_1\cup K_1,x) \\ &+\phi_p(K_2,x) + \phi_p(K_1\cup K_1,x)). \end{aligned}$$

We need to do the following computations: $\phi_p(G,x) = x^{10}+12x^8+40x^6+47x^4+18x^2+1$, $\phi_p(P_5\cup K_1,x) = x^6+4x^4+3x^2$, $\phi_p(P_4\cup K_2,x) = x^6+4x^4+4x^2+1$, $\phi_p(P_4\cup K_1\cup K_1,x) = x^6+3x^4+x^2$, $\phi_p(K_2) = x^2+1$, $\phi_p(K_1\cup K_1,x) = x^2$. Hence, we get $\pi(G,x) = x^{10}+12x^8+52x^6+91x^4+58x^2+9$. Note that Theorem 13 and 14 are not applicable for this graph G as it contains $K_{2,3}$.

The following corollary shows that Theorem 21 is a generalization of Borowiecki's proof idea for computational purposes at least.

Corollary 23. *[9] A bipartite graph G is C_{4k}-free if and only if $\pi(G,x) = \phi_p(G,x)$.*

Proof. The forward implication easily follows from Theorem 21. Suppose $\pi(G,x) = \phi_p(G,x)$ holds, then from Equation 3, we have

$$f_i = \sum_{\substack{U_i \text{ with an odd} \\ \text{number of } 4k\text{-cycles}}} 2^{s+1}2^t = 0,$$

for each i. Suppose, on the contrary, that G contains a $4k$-cycle for some k. Then, there exists a Sachs subgraph $U_{4k} = C_{4k}$, and it contains an odd number of $4k$-cycles. Hence, $f_{4k} \neq 0$, and we get a contradiction which concludes that G is C_{4k}-free.

3 Discussion and further problems

Cyclic structure

Let us recall the idea of coalescence of graphs. Consider two graphs G_1 and G_2 rooted at the vertices r_1 and r_2 respectively. By identifying r_1 in G_1 with r_2 in G_2, we construct a new graph $G_1 \cdot G_2$ called the *coalescence* of G_1 and G_2. Schwenk (1974) proved the following relation for the characteristic polynomial which holds for the permanental polynomial as well.

Lemma 31. *[26] Let* (G_1, r_1) *and* (G_2, r_2) *be two rooted graphs, then*

$$\begin{aligned}\pi(G_1 \cdot G_2, x) = \pi(G_1, x)\pi(G_2 - r_2, x) + \pi(G_1 - r_1, x)\pi(G_2, x)\\ -x\pi(G_1 - r_1, x)\pi(G_2 - r_2, x).\end{aligned}$$

A *block* is a 2-connected component of a graph, and a *cut-vertex* is a vertex whose removal makes the resultant graph disconnected. Let G be a $4k$-intercyclic bipartite graph, and consider its block-decomposition. Using Lemma 31, we can recursively compute $\pi(G, x)$. If H is a C_{4k}-free block of G, then we can substitute $\pi(H, x) = \phi_p(H, x)$. If G is a $4k$-intercyclic bipartite graph, then exactly one of the following holds:

- All the $4k$-cycles are contained in exactly one block. All other blocks are C_{4k}-free.
- All the $4k$-cycles pass through exactly one cut-vertex. All other blocks which do not share this cut-vertex are C_{4k}-free.

By Theorem 21, the computation of permanental polynomial of a $4k$-intercyclic bipartite graphs requires listing all the $4k$-cycles in it. For a general graph on n vertices, all the cycles of length k up to $log\ n$ can be found in polynomial time using the color coding method [6, 4]. For larger k, the problem is intractible. Let d be the number of $4k$-cycles in G, and m be the number of edges in G. In a general graph, a $4k$-cycle, if it exists, can be found in $\mathcal{O}(m^{2-(\frac{1}{k}-\frac{1}{(2k+1)})})$ [5], and there is a randomized algorithm to find all the cycles of a given length in $\mathcal{O}(f * occurrences)$ where f is the time required to find one occurrence [23]. Hence, if we know all the possible lengths of $4k$-cycles to begin with, then checking whether a given graph G is $4k$-intercyclic, that is, whether it contains two disjoint $4k$-cycles, can be done in $\mathcal{O}(m^2 d)$. Otherwise, the problem remains difficult as it may include deciding whether the graph contains a Hamiltonian cycle.

Let G be a general graph with n vertices and m edges. Let $\mathcal{C}(G)$ denote the set of all cycles, and $|\mathcal{C}(G)| = \eta$. Tarjan [27] gave an algorithm to list all the cycles in $\mathcal{O}((mn)(\eta + 1))$ time. The complexity was later improved to $\mathcal{O}((m + n)(\eta + 1))$ by Johnson [15]. For listing cycles in degenerate graphs, see [21]. For a cycle $c \in \mathcal{C}(G)$, let $|c|$ denote the number of edges in c. Birmelé et al. [7] gave an algorithm to list all the cycles in $\mathcal{O}(m + \sum_{c \in \mathcal{C}(G)} |c|)$ time, and this algorithm can be used to address our problem of listing all the $4k$-cycles. This suggests the following.

Problem 32. Find conditions on a 2-connected C_{4k}-free bipartite graph and a $4k$-intercyclic bipartite graph to ensure that the total number of cycles is polynomially bounded.

Recall that the computation of $\pi(G, x)$ for a $4k$-intercyclic bipartite G depends on the number of $4k$-cycles in it by Theorem 21. We also mention that intercyclic graphs [20], odd-intercyclic graphs [10], graphs without two disjoint S-cycles [16], intercyclic digraphs [17] have been studied. Hence, it is also an interesting problem to find a structural characterization and explicit constructions of $4k$-intercyclic bipartite graphs.

Cospectrality

Since $\phi_p(G, x)$ can be recovered from $\phi(G, x)$, it follows from Corollary 23 that two C_{4k}-free bipartite graphs G_1 and G_2 are cospectral ($\phi(G_1, x) = \phi(G_2, x)$) if and only if they are per-cospectral ($\pi(G_1, x) = \pi(G_2, x)$) [9]. Next, we give general result for cospectrality and per-cospectrality that follows from Theorem 21. Let p be some polynomial, and define a class of $4k$-intercyclic bipartite graphs $\mathcal{G}_p = \{G \mid f(G, x) = p\}$ corresponding to p.

Theorem 33. *Let $G_1, G_2 \in \mathcal{G}_p$ for some polynomial p. Then, $\phi(G_1, x) = \phi(G_2, x)$ if and only if $\pi(G_1, x) = \pi(G_2, x)$.*

Next, we give three such classes.

- Let $p = 0$, then $\mathcal{G}_0$ is the class of all C_{4k}-free bipartite graphs.
- Let $p = 4lx^{n-4}$ for a given n and l, then $\mathcal{G}_{4lx^{n-4}}$ is the class of all bipartite graphs with exactly l C_4's and no other cycle such that $G \backslash C$ is an independent set for any cycle C.
- The class of all unicyclic bipartite graphs with a C_4 such that any two graphs in this class are cospectral after the removal of C_4.

For any such class $\mathcal{G}_p$ of bipartite graphs, the permanental polynomial is not any more useful than the characteristic polynomial in distinguishing them.

Problem 34. Find classes of $4k$-intercyclic bipartite graphs for which the expression $f(G, x)$ can be computed quickly and characterize the class of graphs $\mathcal{G}_p$ for various polynomials p.

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