Computing the permanental polynomial of $4k$-intercyclic bipartite graphs*

Hitesh Wankhede† Ranveer Singh‡ Ravindra B. Bapat§¶

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Abstract

Let G be a bipartite graph with the permanental polynomial $\pi(G,x)$. We define a modified characteristic polynomial $\phi_p(G,x)$ for a bipartite graph by changing the sign of some of the coefficients of its characteristic polynomial. A graph is said to be $4k$-intercyclic if it does not have vertex-disjoint $4k$-cycles. We give an expression to compute $\pi(G,x)$ for $4k$-intercyclic bipartite graphs in terms of $\phi_p(G,x)$ and $\phi_p(G\backslash R,x)$ where R is a $4k$-cycle in G. This class of bipartite graphs is different than the class of bipartite graphs that do not contain an even subdivision of $K_{2,3}$, and for such a class, a method to compute $\pi(G,x)$ using Pfaffian orientation was given by Yan and Zhang in 2004 and Zhang and Li in 2012.

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1 Introduction and Preliminaries

We consider simple and undirected graphs. Let G be a graph with the vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$. The *adjacency matrix* $A(G)=(a_{ij})$ of a graph G is defined as $a_{ij}=1$ if v_i and v_j are adjacent and 0 otherwise. The *determinant* and the *permanent* of $A(G)$, are defined as

$$\det(A(G))=\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)\prod_{i=1}^{n}a_{i,\sigma(i)} \text{ and } \operatorname{per}(A(G))=\sum_{\sigma\in S_n}\prod_{i=1}^{n}a_{i,\sigma(i)},$$

respectively, where S_n is the set of all permutation of the set $\{1,2,\ldots,n\}$ and $\operatorname{sgn}(\sigma)$ is the signature of the permutation σ. While the determinant can be computed in polynomial time using the Gaussian elimination method, and the fastest known algorithm runs in $\mathcal{O}(n^{2.3728})$ time [2, 3], computing the permanent is notoriously difficult, as it is #P-complete [19]. The "Permanent vs. Determinant Problem" in computational complexity theory is as follows: "Can we express the permanent of matrices as the determinant of a (possibly polynomially larger) matrices?" For an upper bound on the size of the larger matrix, see [10], and for a survey on lower bounds, see [1]. The problem "Given a $(0,1)$-matrix A, under what conditions, changing the sign of some the nonzero entries yields a matrix B such that

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†Indian Institute of Technology Indore, Department of Computer Science and Engineering, Indore, India. hitesh.wankhede@students.iiserpune.ac.in, hiteshwankhede9@gmail.com

‡Indian Institute of Technology Indore, Department of Computer Science and Engineering, Indore, India. ranveer@iiti.ac.in

§Indian Statistical Institute, Theoretical Statistics and Mathematics Unit, New Delhi, India. rbb@isid.ac.in

¶Ashoka University, Department of Mathematics, New Delhi, India. ravindra.bapat@ashoka.edu.in

the per$(A) = \det(B)$?" is famously known as "Polya Permanent Problem," [18] and it is equivalent to twenty-three other problems listed in [15] including the Even Cycle problem: "Given a directed graph, does it have an even cycle?" More recently, the problem of finding a shortest even cycle was shown to be tractable [5]. Immanants are matrix functions that generalize determinant and permanent, and their complexity dichotomy was also recently studied [7].

The *characteristic polynomial* and the *permanental polynomial* of graph G are defined as

$$\phi(G,x) = \det(xI - A(G)) \text{ and } \pi(G,x) = \text{per}(xI - A(G)),$$

respectively, where I is the identity matrix of order n. The characteristic and the permanental polynomial are graph invariants, and they could be helpful in distinguishing graphs towards Graph Isomorphism Problem [20]. But the permanental polynomial is not studied in great detail as compared to the characteristic polynomial, probably due to its computational difficulty. We are interested in finding ways to compute the permanental polynomial efficiently; one way to do that is by expressing the permanental polynomial in terms of the characteristic polynomial. For an excellent survey on the permanental polynomial, we refer to [13]. Let

$$\phi(G,x) = \sum_{i=0}^{n} a_i x^{n-i} \text{ and } \pi(G,x) = \sum_{i=0}^{n} b_i x^{n-i}.$$

Then, the interpretation of these coefficients is given using the Sachs formulas as

$$a_i = \sum_{U_i} (-1)^{p(U_i)} 2^{c(U_i)} \text{ [8] and } b_i = (-1)^i \sum_{U_i} 2^{c(U_i)} \text{[17]}, \tag{1}$$

where the summation is taken over all the Sachs subgraphs U_i (subgraphs whose components are either cycles or edges) of G on i vertices, $p(U_i)$ denotes the number of components in U_i, and $c(U_i)$ denotes the number of components in U_i which are cycles.

When the graph G is bipartite, that is, when there are no odd cycles, we have $a_i = b_i = 0$ for each odd i [6, 17]. Hence,

$$\phi(G,x) = \sum_{i=0,2,4,\ldots} a_i x^{n-i} \text{ and } \pi(G,x) = \sum_{i=0,2,4,\ldots} b_i x^{n-i}.$$

For a bipartite graph G, we define $f_i = b_i - (-1)^{i/2} a_i$ for each i, and introduce a modified characteristic polynomial and also a new graph polynomial as

$$\phi_p(G,x) = \sum_{i=0,2,4,\ldots} (-1)^{i/2} a_i x^{n-i} \text{ and } f(G,x) = \sum_{i=0,2,4,\ldots} f_i x^{n-i},$$

respectively, such that we have

$$\pi(G,x) = \phi_p(G,x) + f(G,x). \tag{2}$$

In the graph G, we denote a cycle of length k by C_k. Similarly, we denote a cycle of length $4k$ by C_{4k} and refer to it by $4k$-*cycle*. A graph is called C_{4k}-*free* if it does not contain any $4k$-cycle. A graph is called *intercyclic* if it does not contain two vertex-disjoint cycles. Similarly, a graph is called $4k$-*intercyclic* if it does not contain two vertex-disjoint $4k$-cycles (see Figure 1 for an example).

In 1985, Borowiecki proved the following.

Theorem 1.1. *[6] Let G be a bipartite graph with the spectrum $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then, G is C_{4k}-free if and only if its per-spectrum is $\{i\lambda_1, i\lambda_2, \ldots, i\lambda_n\}$*[1].

[1]The *spectrum* and the *per-spectrum* of a graph G are the set of all roots of its characteristic polynomial and the permanental polynomial, respectively.

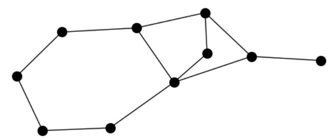

Figure 1: Example of a $4k$-intercyclic bipartite graph.

By inspecting the proof of this theorem, we notice that a bipartite graph G is C_{4k}-free if and only if $\pi(G,x) = \phi_p(G,x)$ (see Corollary 2.3). As a result, the permanental polynomial of C_{4k}-free bipartite graphs can be computed directly through the modified characteristic polynomial. Yan and Zhang, in 2004, found that the permanental polynomial of a larger class of bipartite graphs can be computed using the characteristic polynomial of some oriented graph. They proved the following.

Theorem 1.2. *[22] Let G be a bipartite graph with n vertices containing no subgraph, which is an even subdivision of $K_{2,3}$. Then there exists an orientation G^e of G such that $\pi(G,x) = \det(xI - A(G^e))$, where $A(G^e)$ denotes the skew adjacency matrix of G^e.*

Later Zhang and Li, in 2012, proved the converse of this statement.

Theorem 1.3. *[23] There exists an orientation G^e of a bipartite graph G such that $\pi(G,x) = \det(xI - A(G^e))$ if and only if G contains no even subdivision of $K_{2,3}$.*

Zhang and Li also show that bipartite graphs that do not contain an even subdivision of $K_{2,3}$ are planar and admit Pfaffian orientation. They also give characterization and recognition of such graphs, which leads to a polynomial time algorithm to compute the permanental polynomial of such bipartite graphs.

Borowiecki [6] had posed the open problem, "Characterize graphs with purely imaginary per-spectrum." Since the eigenvalues of a skew adjacency matrix are purely imaginary, Theorem 1.3 is a partial solution to this open problem. Note that the class of C_{4k}-free bipartite graphs is a subset of the class of bipartite graphs that do not contain an even subdivision of $K_{2,3}$. In our earlier work [21], we construct bipartite graphs with purely imaginary per-spectrum containing an even subdivision of $K_{2,3}$.

In this article, we give the formula to compute $\pi(G,x)$ for the class of $4k$-intercyclic bipartite graphs (a superset of the class of C_{4k}-free bipartite graphs). This is done by expressing $f(G,x)$ in terms of the modified characteristic polynomial of the subgraphs of G. Our approach is combinatorial rather than based on Pfaffian orientation. Note that the class of $4k$-intercyclic bipartite graphs is different than and not a subset of the class of bipartite graphs that do not contain an even subdivision of $K_{2,3}$.

2 Main result

Theorem 2.1. *Let G be a $4k$-intercyclic bipartite graph. Then,*

$$\pi(G,x) = \phi_p(G,x) + 4 \sum_{R \in \mathcal{C}_{4k}(G)} \phi_p(G \backslash R, x),$$

where $\mathcal{C}_{4k}(G)$ denotes the set of all $4k$-cycles in G.

To prove this theorem, we need the following lemma.

Lemma 2.2. *Let G be a bipartite graph. Then, for each $i = 0, 2, 4, \ldots,$ we have*

$$f_i = \sum_{j=1,3,5,\ldots} 2^{j+1} \sum_{\substack{U_i \text{ containing} \\ \text{exactly } j \text{ } 4k\text{-cycles}}} 2^t,$$

where U_i denotes a Sachs subgraph on i vertices.

Proof. In a bipartite graph, there can be two types of cycles: $4k$-cycles or $(4k+2)$-cycles. Hence, a Sachs subgraph U_i on i vertices can be written as

$$U_i = \{C_{4k_1} \cup \cdots \cup C_{4k_s}\} \cup \{C_{4l_1+2} \cup \cdots \cup C_{4l_t+2}\} \cup \underbrace{\{K_2 \cup \cdots \cup K_2\}}_{r-\text{times}},$$

where s and t are the number of $4k$-cycles and $(4k+2)$-cycles respectively. Then, $p(U_i) = s+t+r$ and $c(U_i) = s+t$ and

$$i = 4(k_1 + \cdots + k_s) + 4(l_1 + \cdots + l_t) + 2(t+r).$$

Check that $s+t+r \equiv i/2+s \pmod 2$. Using this fact, the coefficients of the characteristic polynomial and the permanental polynomial given in Equation 1 can be written as

$$(-1)^{i/2}a_i = \sum_{U_i}(-1)^s 2^{s+t} \text{ and } b_i = \sum_{U_i} 2^{s+t},$$

respectively ($(-1)^i = 1$ since i is even). Since $f_i = b_i - (-1)^{i/2}a_i$, we get

$$\begin{aligned} f_i = \sum_{U_i} \left(1-(-1)^s\right) 2^{s+t} &= \sum_{\substack{U_i \text{ containing an odd} \\ \text{number of } 4k\text{-cycles}}} 2^{s+1}2^t \qquad (3) \\ &= \sum_{j=1,3,5,\ldots} 2^{j+1} \sum_{\substack{U_i \text{ containing} \\ \text{exactly } j \text{ } 4k\text{-cycles}}} 2^t. \end{aligned}$$

Note that the contribution in Equation 3 of the Sachs subgraphs in which we have exactly an even number of $4k$-cycles vanishes.

□

Proof of Theorem 2.1. Since the graph G is $4k$-intercyclic, the graph $G\backslash R$ is C_{4k}-free for any $R \in \mathcal{C}_{4k}(G)$. Similarly, any Sachs subgraph U_i of G can contain at most one $4k$-cycle, that is, $s \leq 1$. Let $l_R = i - |V(R)|$, and let W_{l_R} denote a Sachs subgraph in $G\backslash R$. Then, there is a one-to-one correspondence between the Sachs subgraphs U_i in G containing R and the Sachs subgraphs W_{l_R} in $G\backslash R$. Using Lemma 2.2, we have

$$f_i = 4 \sum_{R\in\mathcal{C}_{4k}(G)} \sum_{W_{l_R}} 2^t.$$

Observe that

$$\pi(G\backslash R, x) = \sum_{l_R=0,2,4,\ldots} \sum_{W_{l_R}} 2^t x^{(n-|V(R)|)-l_R}.$$

Now consider the polynomial

$$\begin{aligned} f(G,x) = \sum_{i=0,2,4,\ldots} f_i x^{n-i} &= 4 \sum_{R\in\mathcal{C}_{4k}(G)} \sum_{l_R=0,2,4,\ldots} \sum_{W_{l_R}} 2^t x^{(n-|V(R)|)-l_R} \\ &= 4 \sum_{R\in\mathcal{C}_{4k}(G)} \pi(G\backslash R, x). \end{aligned}$$

Hence, using Equation 2, we get

$$\pi(G,x) = \phi_p(G,x) + 4 \sum_{R\in\mathcal{C}_{4k}(G)} \pi(G\backslash R, x).$$

Since $G\backslash R$ is C_{4k}-free, the application of this expression leads to $\pi(G\backslash R, x) = \phi_p(G\backslash R, x)$. This proves the theorem.

□

Example. Consider the $4k$-intercyclic graph G shown in Figure 1. It contains three 4-cycles and two 8-cycles, and removal of each of them from the graph yields the following subgraphs: $P_5 \cup K_1$, $P_4 \cup K_2$, $P_4 \cup K_1 \cup K_1$, K_2 and $K_1 \cup K_1$, respectively. Then, using Theorem 2.1,

$$\pi(G,x) = \phi_p(G,x) + 4\left(\phi_p(P_5 \cup K_1, x) + \phi_p(P_4 \cup K_2, x) + \phi_p(P_4 \cup K_1 \cup K_1, x)\right. \\ \left. + \phi_p(K_2, x) + \phi_p(K_1 \cup K_1, x)\right).$$

We need to do the following computations: $\phi_p(G,x) = x^{10} + 12x^8 + 40x^6 + 47x^4 + 18x^2 + 1$, $\phi_p(P_5 \cup K_1, x) = x^6 + 4x^4 + 3x^2$, $\phi_p(P_4 \cup K_2, x) = x^6 + 4x^4 + 4x^2 + 1$, $\phi_p(P_4 \cup K_1 \cup K_1, x) = x^6 + 3x^4 + x^2$, $\phi_p(K_2) = x^2 + 1$, $\phi_p(K_1 \cup K_1, x) = x^2$. Hence, we get $\pi(G,x) = x^{10} + 12x^8 + 52x^6 + 91x^4 + 58x^2 + 9$. Note that Theorem 1.2 and 1.3 are not applicable for this graph G as it contains $K_{2,3}$.

The following corollary shows that Theorem 2.1 is a generalization of Borowiecki's proof idea for computational purposes at least.

Corollary 2.3. *[6] A bipartite graph G is C_{4k}-free if and only if $\pi(G,x) = \phi_p(G,x)$.*

Proof. The forward implication easily follows from Theorem 2.1. Suppose $\pi(G,x) = \phi_p(G,x)$ holds, then from Equation 3, we have

$$f_i = \sum_{\substack{U_i \text{ with an odd} \\ \text{number of } 4k\text{-cycles}}} 2^{s+1} 2^t = 0,$$

for each i. Suppose, on the contrary, that G contains a $4k$-cycle for some k. Then, there exists a Sachs subgraph $U_{4k} = C_{4k}$, and it contains an odd number of $4k$-cycles. Hence, $f_{4k} \neq 0$, and we get a contradiction which concludes that G is C_{4k}-free. □

Computation of $\pi(G,x)$ for a $4k$-intercyclic bipartite G depends on the number of $4k$-cycles in it by Theorem 2.1. Let d be the number of $4k$-cycles in G, and m be the number of edges in G. We recall that a $4k$-cycle, if it exists, can be found in $\mathcal{O}(m^{2-(\frac{1}{k}-\frac{1}{(2k+1)})})$ [4], and all cycles of a given length in general graphs can be found in $\mathcal{O}(f * occurrences)$ where f is the time required to find one occurrence [16]. It follows that checking whether a given graph G is $4k$-intercyclic, that is, whether it contains two disjoint $4k$-cycles, can be done in $\mathcal{O}(m^2 d)$. It would be interesting to see what would be an upper bound on the number of $4k$-cycles in a bipartite graph. We also mention that intercyclic graphs [14], odd-intercyclic graphs [9], graphs without two disjoint S-cycles [11], intercyclic digraphs [12] have been studied. It would also be an interesting problem to find a structural characterization of $4k$-intercyclic bipartite graphs.

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