MA 203: Convergence of Sequence of RVs

1. Convergence in Mean Square

2. Convergence in Distribution

3. Convergence in Almost Sure

4. Convergence in Probability

$A_1, A_2, \ldots, A_n, \ldots$

$$\left\{ A_n = \frac{1}{n^2} \right\}$$

$$\underbrace{\lim_{n \to \infty} A_n} < \infty$$

$$\boxed{\lim_{n \to \infty} \frac{1}{n^2} = 0}$$

$X_1, X_2, \ldots, X_n, \ldots$

Four Methods

1. Mean Square
2. Distribution
3. Probability
4. Almost Sure

law

Convergence in Mean Square Sense: A random sequence $\{X_i\}_{i=1}^{\infty}$ is said to converge in the mean-square sense (m.s) to a random variable X if

$E[|(X_n - X)|^2] \to 0$ as n $\to \infty$.

We also write

$\{X_n\} \to X.$ $$\boxed{\{X_n\} \xrightarrow{m.s} X}$$

$X_1, X_2, \ldots, X_n, \ldots$

$$\boxed{\lim_{n \to \infty} E[|(X_n - X)|^2] = 0}$$

Example 1: Suppose $\{X_n\}_{n=1}^{\infty}$ be a sequence of RVs with

$$P(\{X_n = n\}) = \frac{1}{n^2}$$

and

$$P(\{X_n = 0\}) = 1 - \frac{1}{n^2}.$$

Examine if $\{X_n\}_{n=1}^{\infty}$ converges to $\{X = 0\}$ in the m.s sense.

$\{X_n\}_{n=1}^{\infty} = X_1, X_2, \ldots, X_n, \ldots$

$(S, \mathcal{F}, P)$

Sol:-

$$\boxed{\lim_{n\to\infty} E[(X_n - X)^2] = 0}$$

$$\Rightarrow \lim_{n\to\infty} E[(X_n - 0)^2] = \lim_{n\to\infty} E[X_n^2]$$

$$= \lim_{n\to\infty} \left\{ 0^2 \times p_{X_n}(X_n = 0) + n^2 \times p_{X_n}(X_n = n) \right\}$$

$$= \lim_{n\to\infty} \left\{ 0^2 \times \left(1 - \frac{1}{n^2}\right) + \cancel{n^2} \times \frac{1}{\cancel{n^2}} \right\}$$

$$= \lim_{n\to\infty} \{ 0 + 1 \}$$

$$= \lim_{n\to\infty} 1 = 1$$

Given sequence $\{X_n\}_{n=1}^{\infty}$ is not converging in mean square sense.

X_n: Discrete RV $\quad R_{X_n} = \{0, n\}$

$$E[X^2] = 0^2 \times \underbrace{P(X_n = 0)} + n^2 \times \underbrace{P(X_n = n)}$$

$P_{X_n}(0) = P(X_n = 0)$

$P_{X_n}(n) = P(X_n = n)$

Example 2: Suppose $\{X_n\}_{n=1}^{\infty}$ be a sequence of RVs with

$$P(\{X_n = 1\}) = \frac{1}{n}$$

and

$$P(\{X_n = 0\}) = 1 - \frac{1}{n}.$$

Examine if $\{X_n\}_{n=1}^{\infty}$converges to $\{X = 0\}$ in the m.s sense.

Sol:-

$$E[(X_n - X)^2] = E[X_n^2]$$

$$= 0^2 \times P(X_n = 0) + 1^2 \times P(X_n = 1)$$

$$= 0 \times \left(1 - \frac{1}{n}\right) + 1 \times \frac{1}{n}$$

$$= 1/n$$

$$\lim_{n \to \infty} E[(X_n - X)^2] = \lim_{n \to \infty} 1/n = 0 \Rightarrow$$

Convergence in Distribution: Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of RVs with CDF $F_{X_1}(x_1), F_{X_2}(x_2), \cdots, F_{X_n}(x_n), \cdots$, respectively.

We say that $\{X_i\}_{i=1}^{\infty}$ converge in distribution to X, $X_n \xrightarrow{d} X$, if

$$\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$$ for all x at which $F_X(x)$ is continuous.

$$\boxed{\lim_{n\to\infty} F_{X_n}(x) = F_X(x)}$$ $\forall\, x$ at which $F_X(x)$ is continuous

1. $\lim_{n\to\infty} F_{X_n}(x) =$ ____

1. $\lim_{n\to\infty} F_{X_n}(x)$ must be a CDF

$$F_{X_n}(x) = \begin{cases} 1 ; & x \geq n \\ 0 ; & \text{o.w.} \end{cases} \Rightarrow \lim_{n\to\infty} F_{X_n}(x) = \underbrace{\begin{cases} 1 ; & x \geq \infty \\ 0 ; & \text{o.w.} \end{cases}}$$

$\neq F_X(x)$

$\boxed{F_X(\infty) = 1}$

2. Continuity

$$F_X(x) = \begin{cases} 0 ; & x < 0 \\ 1 ; & x \geq 1/n \end{cases}$$

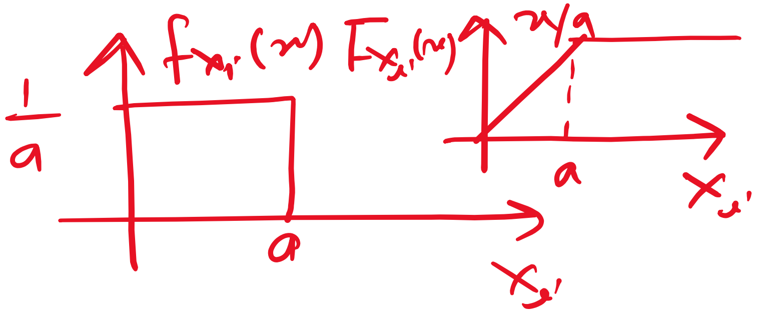
$$F_{X_n}(x) = \begin{cases} 0 ; & x \leq 1/n \\ 1 ; & x > 1/n \end{cases}$$

$$\boxed{F_X(0) \neq \lim_{n\to\infty} F_{X_n}(0)}$$

$$\lim_{n\to\infty} F_{X_n}(x) = \begin{cases} 0 ; & x \leq 0 \\ 1 ; & x > 0 \end{cases}$$

Example: Suppose $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of independent RVs with each RV X_i having the uniform density

$$f_{X_i}(x) = \begin{cases} \frac{1}{a} ; & 0 \le x \le a \\ 0 ; & o.w. \end{cases}$$



Define $Z_n = max(X_1, X_2, \cdots, X_n)$. Examine that $Z_1, Z_2, \cdots, Z_n, \cdots$ converges to RV Z in distribution where

$Z_2 = \max(X_1, X_2)$

$Z_1 = \max(X_1)$

$$F_Z(z) = \begin{cases} 0 ; & z < a \\ 1 ; & z \ge a \end{cases}$$

Sol:

$$\boxed{\lim_{n \to \infty} F_{Z_n}(z) = F_Z(z)}$$

Sol:-

$$F_{Z_n}(z) = P(Z_n \le z)$$

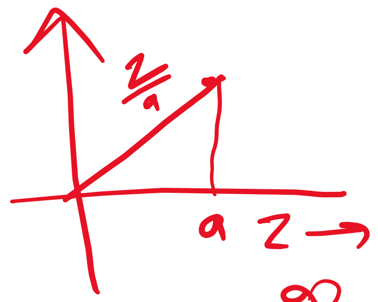
$$= P\left(\max(X_1, X_2, \ldots, X_n) \le z\right)$$

$$= P(X_1 \le z, X_2 \le z, \ldots, X_n \le z)$$

independence

$$= \underline{P(X_1 \le z)}\ \underline{P(X_2 \le z)} \cdots \underline{P(X_n \le z)}$$

Identically distributed



$$\int_a^{\infty} = 0$$

$$\underline{F_{Z_n}(z)} = \begin{cases} 0 \ ; & \text{for } z \le 0 \\ \underline{\left(\frac{z}{a}\right)^n} ; & \text{for } 0 < z < a \\ 1 \ ; & z \ge a \end{cases}$$

$$\underline{\lim_{n \to \infty} F_{Z_n}(z)} = \left\{ \begin{array}{ll} 0 \ ; & z < a \\ 1 \ ; & \underline{z \ge a} \end{array} \right\}$$

$$\boxed{F_Z(\infty) = 1}$$

$$\boxed{F_Z(-\infty) = 0}$$

Almost Sure (a. s.) Convergence or Convergence with Probability 1: Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of RVs defined on the probability space (S, F, P).

The sequence $\{X_n\}_{n=1}^{\infty}$ is said to converge to X almost sure or with probability 1 if

$$P\left(\left\{s \mid \lim_{n\to\infty} X_n(s) = X(s)\right\}\right) = 1 .$$

$s_1, s_2, \cdots, s_m$

$s \in S$

$$A = \left\{ s \mid \lim_{n\to\infty} X_n(s) = X(s) \right\}$$

$$A = \left\{ s_i \mid \lim_{n\to\infty} X_n(s_i) = X(s) \right\} = \{ s_1, s_2, \cdots, s_t \}$$

$t < m$

$$A = \{ s_1, \cdots, s_t \}$$

$t < m$ $\quad P(A) = P\{ s_1, s_2, \cdots, s_m \} = 1$

S: $s_1, s_2, \vdots, s_m$

$$P(s_1, s_2, \cdots s_t) = 1$$

Example:- Suppose $S = \{s_1, s_2, s_3\}$ and $\{X_n\}_{n=1}^{\infty}$ be a sequence of RVs with $X_n(s_1) = 1$, $X_n(s_2) = -1$, $X_n(s_3) = n$

Define a RV X such that

$X(s_1) = 1$, $X(s_2) = -1$, $X(s_3) = 1$

$s_3 = 0$

$\boxed{S}$

H
T
S

$$A = \{ s_i \mid \lim_{n\to\infty} X_n(s_i) = X(s_i) \} = \{s_1, s_2)$$

$$\Rightarrow P(A) = P(s_i \mid \lim_{n\to\infty} X_n(s_i) = X(s_i) \}) = P(s_1, s_2)$$

$P(s_3) = 0$

$\downarrow$

$P\{s_1, s_2) = 1$

Therefore, if $P(s_1, s_2) = 1$