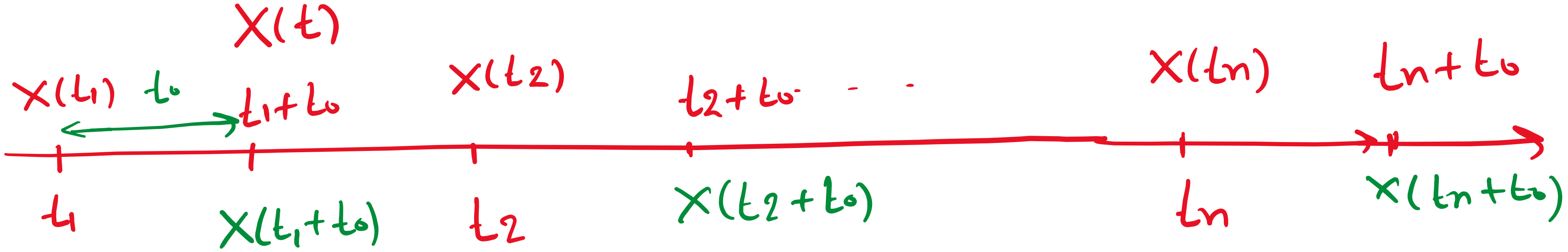
MA 203

1. Strict-Sense Stationary (SSS) RP
2. Wide-Sense Stationary (WSS) RP
3. Power Spectral Density (PSD)

Strict-Sense Stationary (SSS) Random Process: A random process is called strict-sense stationary if its probability structure is invariant with time.



$F_{X(t_1), X(t_2), \ldots, X(t_n)}(x_1, x_2, \ldots, x_n) =$ Joint CDF of $X(t_1), X(t_2), \ldots, X(t_n)$

$F_{X(t_1+t_0), X(t_2+t_0), \ldots, X(t_n+t_0)}(x_1, x_2, \ldots, x_n) =$ Joint CDF of $X(t_1+t_0), X(t_2+t_0), \ldots, X(t_n+t_0)$

If

$$F_{X(t_1), X(t_2), \ldots, X(t_n)}(x_1, x_2, \ldots, x_n) = F_{X(t_1+t_0), X(t_2+t_0), \ldots, X(t_n+t_0)}(x_1, x_2, \ldots, x_n)$$

$\Rightarrow$ We can say the process $X(t)$ is strict-sense stationary.

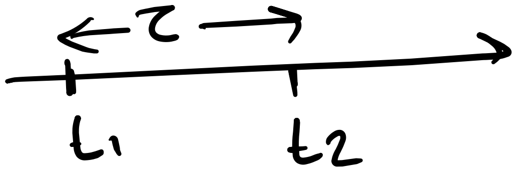
$\Rightarrow$ Joint CDF is invariant w.r.t time axis.

Wide-Sense Stationary (WSS) Random Process: A random process $X(t)$ is called a WSS if

1. $E[X(t)] = \mu_X = \text{const.}$
   → NOT function of time.

2. $R_X(t_1, t_2) = E[X(t_1)\,X(t_2)]$
   $= R_X(t_2 - t_1)$
   $= R_X(\tau)$
   → Auto-correlation function depends only on time shift τ.

$t_1 \qquad t_2 = t_1 + \tau$



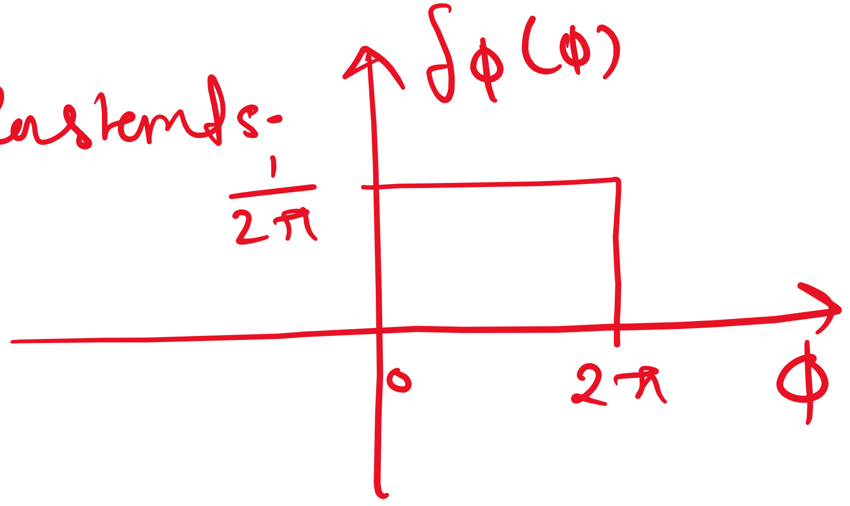
⇒ If these two conditions are satisfied, we can say that the given RP is WSS.

Ex:-

$$X(t) = A\cos(\omega_0 t + \phi)$$

where A & ω_0 are constants.

ϕ is a RV

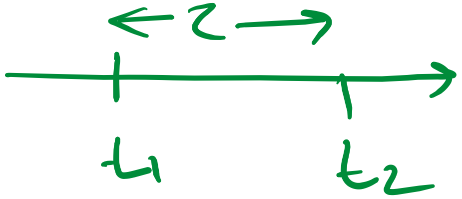


1. $E[X(t)] = 0$;

2. $R_X(t_1, t_2) = \frac{A^2}{2}\cos(\omega_0(t_1 - t_2))$ $\quad t_2 = t_1 + \tau$

$$= \frac{A^2}{2}\cos(\omega_0(t_2 - t_1))$$

$$= \frac{A^2}{2}\cos\omega_0(\tau) = \underline{R_X(\tau)}$$



$R_X(0) = \frac{A^2}{2}\cos(0) = \frac{A^2}{2}$ $\Rightarrow$ $R_X(0)$ represents power of RP $X(t)$.

Power Spectral Density (PSD) of WSS RP:

$X(t)$ is a WSS RP

$X(t) \rightarrow R_X(\tau)$

$$\underbrace{S_X(f)}_{\text{Power spectral density (PSD) of a R.P } X(t)} = \underbrace{\int_{-\infty}^{+\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau}_{\text{Fourier transform of } R_X(\tau)}$$

$$\underbrace{R_X(\tau)}_{\text{Autocorrelation function}} = \underbrace{\int_{-\infty}^{+\infty} S_X(f) \cdot e^{j2\pi f\tau}\, df}_{\text{Inverse fourier Transform of } S_X(f)}$$

$$R_X(0) = \int_{-\infty}^{+\infty} S_X(f) \cdot df$$

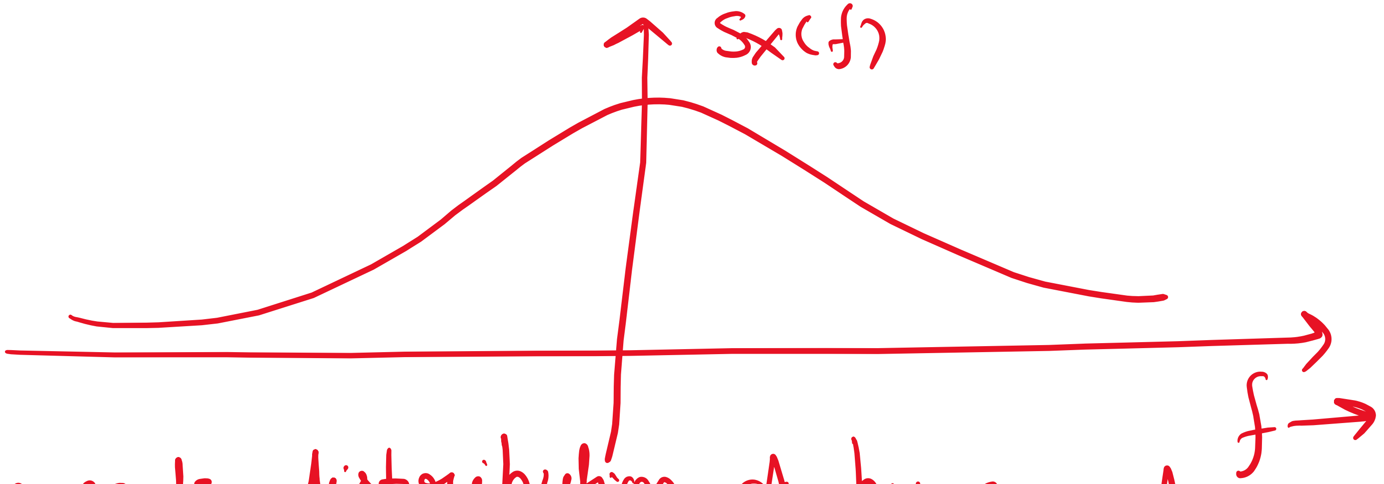
1. $S_x(f) \geq 0$

2. $\underbrace{S_x(f) = S_x(-f)}$

$\hookrightarrow$ $S_x(f)$ is symmetric w.r.t frequency f

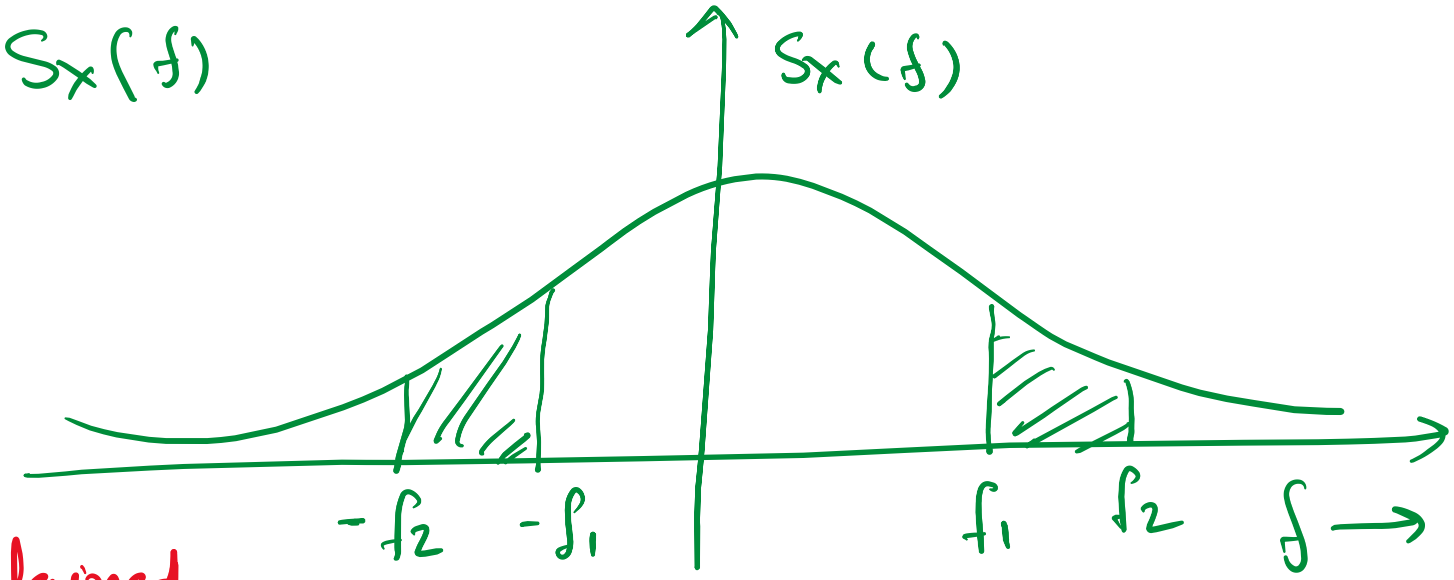
or

$S_x(f)$ is an even function of $\underline{f}$.



PSD represents distribution of power of $X(t)$ with frequency.

$X(t) \longrightarrow S_X(f)$



Frequency Band

$[f_1, f_2]$

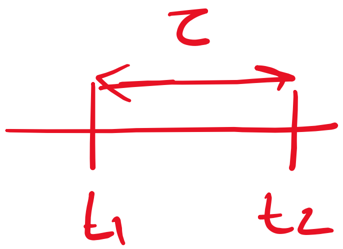
$\Rightarrow$ B.W.

$= \underline{\underline{f_2 - f_1}}$

Total power contained in the frequency band $[f_1, f_2]$ $= \displaystyle\int_{-f_2}^{-f_1} S_X(f)\cdot df + \int_{f_1}^{f_2} S_X(f)\, df$

Example 1: Consider a WSS RP $X(t)$ with autocorrelation

$$R_X(\tau) = \frac{1}{2\pi} e^{-a|\tau|}$$



where $a = 5kHz$.

What is the power of $X(t)$? What is the power spectral density (PSD)? Further, from PSD, calculate the bandwidth required which contains 90% of the total power.

Sol:-

(i) Power of $X(t) = R_X(0)$

$$= \frac{1}{2a} \cdot e^{-a \cdot 0} = \frac{1}{2a}$$

(ii) PSD $\quad S_X(f) = \int_{-\infty}^{+\infty} R_X(\tau) \cdot e^{-j2\pi f\tau} \, d\tau$

$$S_x(f) = \int_{-\infty}^{+\infty} \frac{1}{2a} e^{-a|\tau|} e^{-j2\pi f\tau} d\tau$$

$$= \frac{1}{2a} \int_{0}^{\infty} e^{-a\tau} e^{-j2\pi f\tau} d\tau + \int_{-\infty}^{0} \frac{1}{2a} \cdot e^{-a(-\tau)} e^{-j2\pi f\tau} d\tau$$

$$= \frac{1}{2a} \cdot \int_{0}^{\infty} e^{-(a+j2\pi f)\tau} d\tau + \frac{1}{2a} \int_{-\infty}^{0} e^{a\tau} e^{-j2\pi \tau} d\tau$$

$$= \frac{1}{2a} \int_{0}^{\infty} e^{-(a+j2\pi f)\tau} d\tau + \frac{1}{2a} \int_{-\infty}^{0} e^{(a-j2\pi f)\tau} d\tau$$

$$\Downarrow$$

$$\tau = -u$$

$$d\tau = -du$$

$$= \frac{1}{2a} \int_{0}^{\infty} e^{-(a+j2\pi f)\tau} d\tau + \frac{1}{2a} \int_{0}^{\infty} e^{-(a-j2\pi f)u} du$$

$$S_x(f) = \frac{1}{2a}\left\{\frac{e^{-(a-j2\pi f)\tau}}{-(a-j2\pi f)}\right\}\Bigg|_0^{\infty} + \frac{1}{2a}\left\{\frac{e^{-(a+j2\pi f)}}{-(a+j2\pi f)}\right\}\Bigg|_0^{\infty}$$

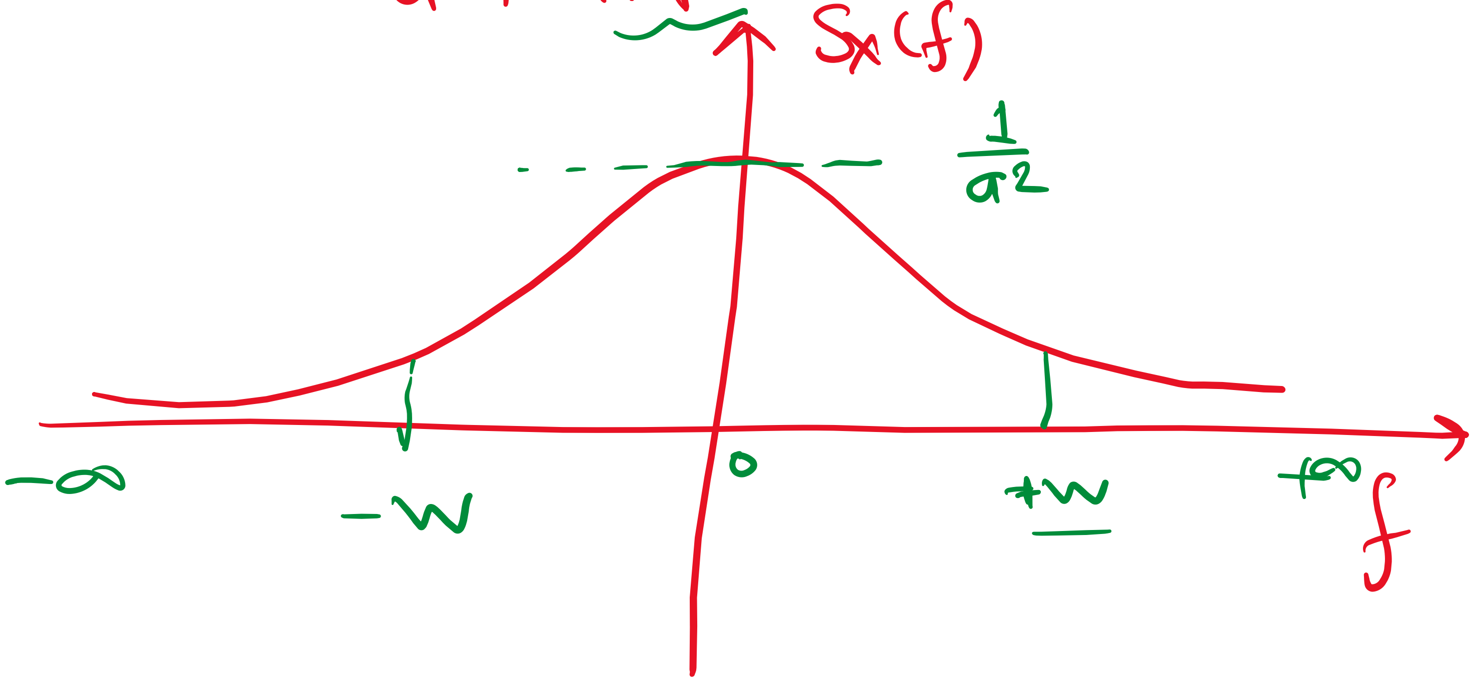
$$= \frac{1}{2a}\left\{\frac{1}{a-j2\pi f}\right\} + \frac{1}{2a}\left\{\frac{1}{a+j2\pi f}\right\}$$

$$= \frac{1}{2a}\left\{\frac{1}{a-j2\pi f} + \frac{1}{a+j2\pi f}\right\}$$

$$= \frac{1}{\cancel{2a}} \times \left\{\frac{\cancel{a}+j2\cancel{\pi f} + \cancel{a}-j\cancel{2}\pi f}{(a-j2\pi f)(a+j2\pi f)}\right\}$$

$$\boxed{S_x = \frac{1}{a^2+4\pi^2 f^2}}$$

$$S_x(f) = \frac{1}{a^2 + 4\pi f^2}$$



(iii)

$\frac{1}{2a}$ $[-W, W]$

$$\frac{90}{100} \times \frac{1}{2a} = \int_{-W}^{W} S_x(f) \cdot df$$

$$\frac{0.9}{2a} = \int_{-W}^{+W} \frac{1}{a^2 + 4\pi^2 f^2}\, df$$

$$\Rightarrow \frac{0.9}{2a} = \frac{1}{4\pi^2}\int_{-W}^{+W} \frac{1}{f^2 + \frac{a^2}{4\pi^2}}\, df$$

$$= \frac{1}{4\pi^2}\left\{\frac{2\pi}{a} \tan^{-1}\left\{\frac{f}{a/2\pi}\right\}\right\}_{-W}^{+W}$$

$$\Rightarrow \frac{0.9}{2a} = \frac{1}{4\pi^2}\cdot\frac{2\pi}{a}\left\{\tan^{-1}\left(\frac{2\pi f}{a}\right)\right\}\Bigg|_{-W}^{+W}$$

$$\Rightarrow \frac{0.9}{2\cancel{a}} = \frac{1}{\cancel{4\pi^2}}\cdot\frac{2\cancel{\pi}}{\cancel{a}}\left\{\tan^{-1}\left(\frac{2\pi W}{a}\right) - \tan^{-1}\left(-\frac{2\pi W}{a}\right)\right\}$$

$$\Rightarrow 0.9\pi = 2\tan^{-1}\left(\frac{2\pi W}{a}\right)$$

$$\text{or} \quad \frac{2\pi W}{a} = \tan\left(\frac{0.9\pi}{2}\right)$$

$$W = \frac{a}{2\pi}\underbrace{\tan\left(\frac{0.9\pi}{2}\right)}_{} \longrightarrow 6.3138$$

$$a = 5\,KHz$$

$$\simeq 1.0059$$

$$W \simeq a = 5\,KHz$$

$$[-W, W] = [-5\,\text{kHz}, +5\,\text{kHz}]$$

$$B.W. = 2 \times 5\,\text{kHz}$$

$$= 10\,\text{kHz}$$