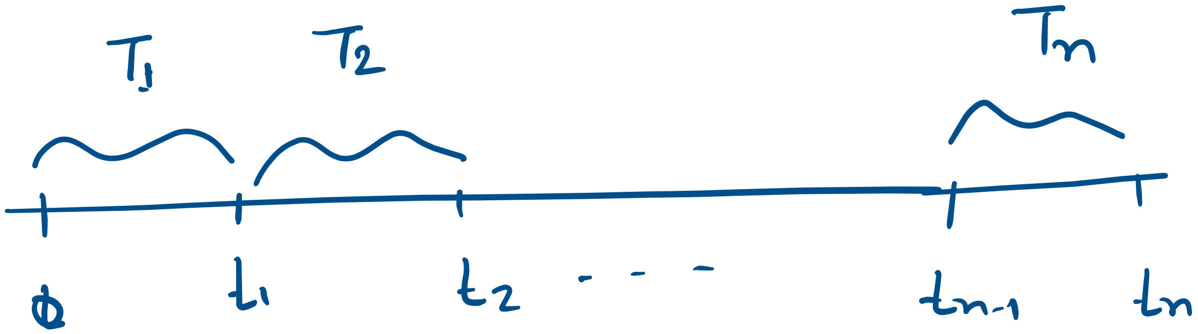
MA 203

Markov Process



$$\boxed{f_{T_n}(t) = \lambda e^{-\lambda t}} \; : \; \text{Exponential distribution}$$

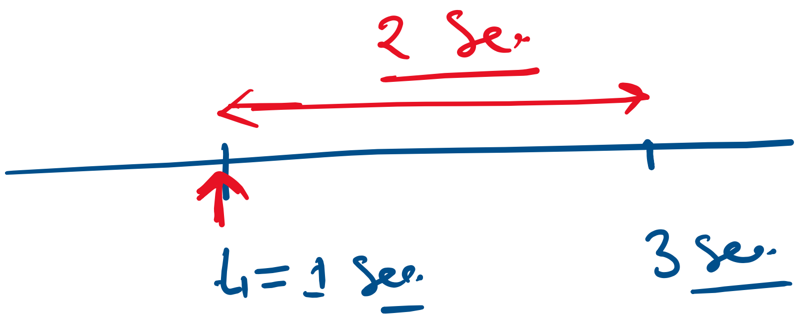
Example 3: Let $N(t)$ be a Poisson process with intensity $\lambda = 2$, and let $T_1, T_2, \ldots$ be the corresponding interarrival times.

(i) Find the probability that first arrival occurs after $t = 0.5$, i.e., $P(T_1 > 0.5)$.

(ii) Given that we have had no arrivals before $t = 1$, find $P(T_1 > 3)$.

(iii) Given that the third arrival occurred at time $t = 2$, find the probability that the fourth arrivals occurs after $t = 4$.

Sol:- (ii)



$P(T_1 > 3)$

$$P\{T_1 > 3 \mid T_1 > 1\} = P\{T_1 > 2\}$$ {Using independent increment Prop.

$$= e^{-\lambda t} = e^{-2\times 2}$$

$$\simeq 0.0183$$

(iii)

$$P\{T_4 > 4 \mid T_3 > 2\}$$

$$= P\{T_4 > 2\}$$

$$= e^{-2\times 2}$$

Markov Process :—

$X(t)$

$t_1 \qquad t_2 \qquad \cdots \qquad t_{n-1} \qquad t_n$

$X(t_1) \qquad X(t_2) \qquad \qquad X(t_{n-1}) \qquad X(t_n)$

$$P\{X(t_n)=J \mid X(t_1)=i_1,\ X(t_2)=i_2, \ldots, X(t_{n-1})=i_{n-1}\}$$

$i_1, i_2, \ldots, i_n$: real number.

$$= P\{X(t_n)=J \mid X(t_{n-1})=i_n\}$$

$X(t_1) \qquad X(t_2) \qquad \cdots \qquad X(t_{n-1})$

$R_1 \qquad R_2 \qquad \qquad \downarrow R_{n-1}$

$i_1 \in R_1 \qquad i_2 \in R_2 \qquad \cdots \qquad i_n \in R_{n-1}$

$$\boxed{P\{X(t_n)=j \mid X(t_1)=i_1, X(t_2)=i_2, \ldots, X(t_{n-1})=i_n\} = P\{X(t_n)=j \mid X(t_{n-1})=i_n\}}$$

$X(t_1), X(t_2), \ldots, X(t_{n-2})$

Markov Property

$X(t)$: Markov RP.

$\Rightarrow$

Γ: Parameter space

ϕ: State space.

$X(t)$: Markov chain $\Rightarrow$ State space ϕ is finite/Countably infinite.

Ex:-

$$1 \quad 2 \quad \cdots \quad n-1 \quad n \quad \cdots$$

$$X_1 \quad X_2 \quad \cdots \quad X_{n-1} \quad X_n \quad \cdots$$

X_1: H (p) $\to 1$; T ($1-p$) $\to 0$

$$S_n = X_1 + X_2 + \cdots + X_n$$

S_n: # of heads in first n toss

$$S_{n+1} ; \quad S_n + X_{n+1}$$

Case-1 $\Rightarrow P\{S_{n+1} = k \mid S_n = k-1\} = p = P\{S_{n+1} = k \mid S_n = k-1, S_{n-1} = i_1, \cdots\}$

Case-2 $\Rightarrow P\{S_{n+1} = k-1 \mid S_n = k-1\} = 1-p$

$$P\{S_{n+1} = k \mid S_n = k-1,\ S_{n-2} = i_2,\ S_{n-3} = i_3, \ldots\}$$

$$S_n = i_1$$

$$= P\{\underbrace{S_{n+1} = k}_{} \mid \underbrace{S_n = k-1}_{}\}$$

Prob. of head in $(n+1)^{th}$ toss

or

k heads in $(n+1)$ train toss

of heads up to n^{th} toss $= k-1$

$$P\{S_{n+1} = k \mid S_n = k-2\} = 0$$

$S_{n-1} = k-2$ or $k-1$

DTMC (Discrete Time Markov Chain)

\# $X(t)$: → Markov Prop.
→ State space finite / Countably infinite
→ Markov chain

\# T : Time space — finite / Countably infinite.

Discrete-Time Markov-Chain

$X[n]$: Discrete - Time RP.

: Markov Prop.

: State space is finite $\{\underline{\Phi}\}$

$\underline{X_n}$

$\{X_n, \ n = 0, 1, 2, \cdots\}$

$X_0, X_1, X_2, \cdots ;$

$i_0, i_1, i_2, \cdots \in \underline{\Phi}$

$\checkmark P\{X_{n+1} = j \mid X_n = i, X_{n-1} = i_2, \cdots, X_0 = i_1\}$

$= P\{X_{n+1} = j \mid X_n = i\}$

$X(t)$: RP.

$X(t_1) \quad X(t_2) \quad \cdots \quad X(t_n)$

$x_1 \in R_1 \qquad x_{n-1} \in R_n$

x_i

$$P\{X(t_n) = J \mid X(t_1) = x_1,\ X(t_2) = x_2, \ldots, X(t_n) = x_{n-1}\}$$

$$= P\{\underbrace{X(t_n) = J}_{\text{nth time instant or nth state}} \mid \underbrace{X(t_{n-1}) = x_{n-1}}_{\text{outcome of } (n-1)^{th} \text{ time instant} \Rightarrow (n-1)^{th} \text{ state}}\}$$

State space is finite / countably infinite

$\Downarrow$

Markov Process $\Longrightarrow$ Markov chain

Time space $\rightarrow$ finite / countably finite : DTMC

$\rightarrow$ uncountable : CTMC

Continuous time Markov chain