MA 203

DTMC

DTMC

Let $\{X_n\}$: Discrete-time RP

- Γ: is finite | countably infinite
- ϕ: finite | countably infinite
- It satisfies Markov prop.

$X_1, X_2, \ldots, X_n, \ldots$

$n = 1, 2, \ldots$

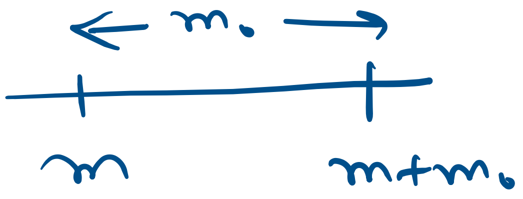
$$p_j = P\{X_n = j\}$$

$j \in \phi$

$$p_{jk}(m,n) = P\{X_n = k \mid X_m = j\}$$

Transition Probability

$j, k \in \phi$



When DTMC is homogeneous,

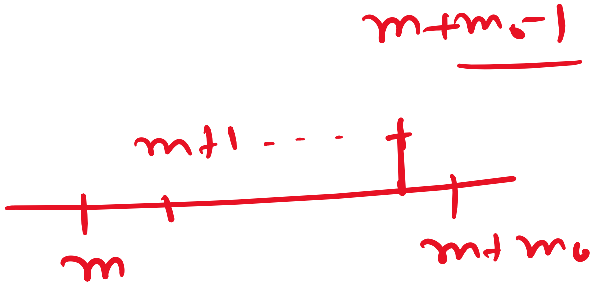
$$p_{jk}(m,n) = P\{X_n = k \mid X_m = j\}$$

$n - m = m_0$

$$= P\{X_{m+m_0} = k \mid X_m = j\}$$

m_0

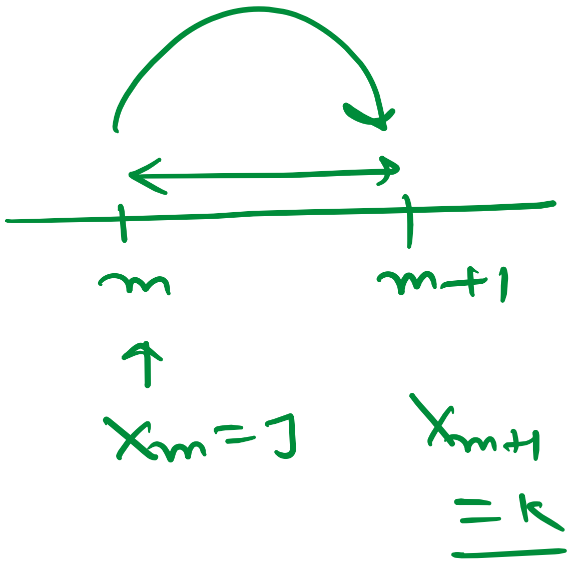
$$= \underbrace{p_{jk}(m_0)}_{m_0\text{-steps transition prob.}}$$



$m_0 = 1$

$$\underbrace{p_{JK}(1)}_{\text{1-step prob.}} = P\{X_{m+1} = k \mid X_m = J\}$$

$= p_{JK}$



Transition Probability Matrix:

$$\boxed{P = [\underbrace{p_{JK}}]} \to 3\times 3$$

$$\Phi = \{0, 1, 2\}$$

$$P = \begin{bmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{bmatrix} 3\times 3$$

$$p_{00} = P\{X_{m+1} = 0 \mid X_m = 0\}$$

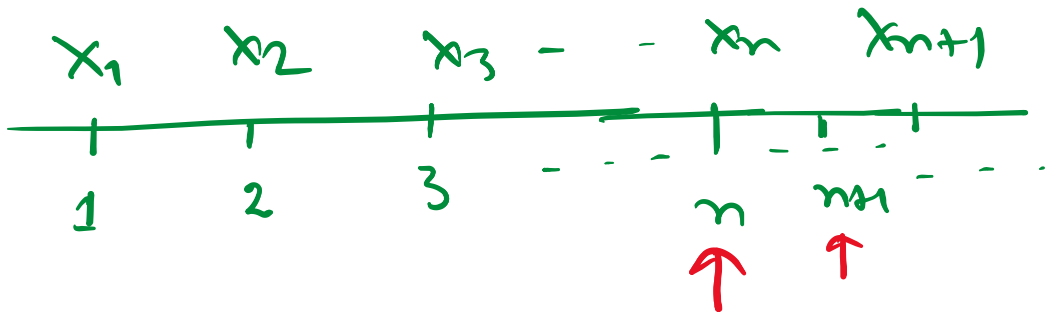
Prop. of Transition Probability Matrix:-

1. $p_{ij} \geq 0 \quad \forall\, i, j \in \phi$

2. $$\sum_{j \in \phi} p_{ij} = 1 \; ; \; i \in \phi$$

Example 1: Suppose a person can be in one of two states "healthy" or "sick". Let $\{X_n\}, n = 0, 1, 2, \cdots$, refer to the state at time n. Here,

$$X_n = \begin{cases} 1; if\ healthy \\ 0; if\ Sick \end{cases}.$$



$\Phi = \{\text{Sick}, \text{healthy}\}$

$\Phi = \{0, 1\}$

$p_{00} = P\{X_{n+1} = 0 \mid X_n = 0\} = \alpha$

$p_{11} = P\{X_{n+1} = 1 \mid X_n = 1\} = \beta$

Sol:

Step 1:- $\phi = \{0, 1\}$

Step 2:-

$$P = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}$$

Step 3:-

$$p_{00} = P\{X_{n+1} = 0 \mid X_n = 0\} = \alpha$$

$$p_{01} = P\{X_{n+1} = 1 \mid X_n = 0\} = 1-\alpha$$

$$p_{11} = P\{X_{n+1} = 1 \mid X_n = 1\} = \beta$$

$$p_{10} = P\{X_{n+1} = 0 \mid X_n = 1\} = 1-\beta$$

Example 2: A factory has two machines and one repair crew. Assume that probability of any one machine breaking down on a given day is α. Assume that if the repair crew is working on a machine, the probability that will complete the repairs in a day is β. For simplicity, ignore the probability of a repair completion or a breakdown taking place except at the end of a day. Let X_n be the number of machines in operation at the end of the nth day. Assume the behaviour of X_n can be modelled as a Markov chain. Find transition probability matrix?

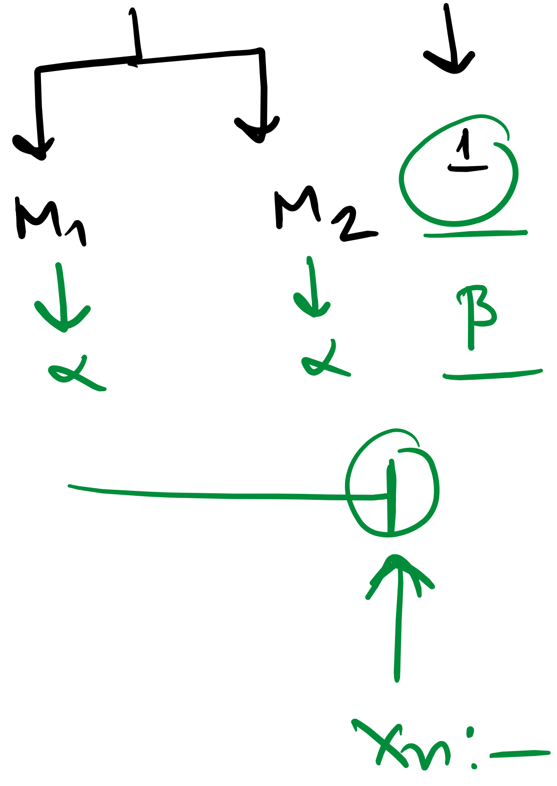
Sol:

Step 1:- X_n: no. of machine in operation

$$\Phi = \{0, 1, 2\}$$

Step 2:-

$$P = \begin{bmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{bmatrix}_{3\times 3}$$



Step3:-

$$p_{00} = P\left\{X_{n+1}=0 \mid X_n=0\right\} = 1-\beta$$

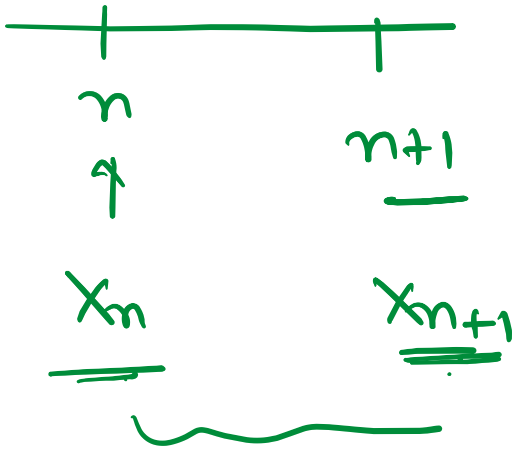
$$p_{01} = P\left\{X_{n+1}=1 \mid X_n=0\right\} = \beta$$

$$p_{02} = P\left\{X_{n+1}=2 \mid X_n=0\right\} = 0$$

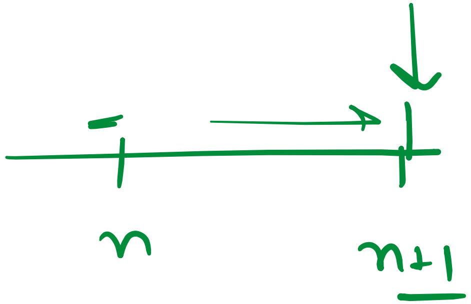
$$p_{10} = P\left\{X_{n+1}=0 \mid X_n=1\right\} = \alpha\cdot(1-\beta)$$

$$p_{11} = P\left\{X_{n+1}=1 \mid X_n=1\right\} = (1-\alpha)(1-\beta) + \beta\cdot\alpha$$

$$p_{12} = P\left\{X_{n+1}=2 \mid X_n=1\right\} = \beta(1-\alpha)$$



$$p_{20} = P\{X_{n+1}=0 \mid X_n = 2\} = \underbrace{\alpha\cdot\alpha}$$



$$p_{21} = P\{X_{n+1}=1 \mid X_n=2\} = \alpha(1-\beta) + \alpha(1-\beta)$$

$$p_{22} = P\{X_{n+1}=2 \mid X_n=2\} = (1-\alpha)\cdot(1-\alpha)$$

Step 4:-

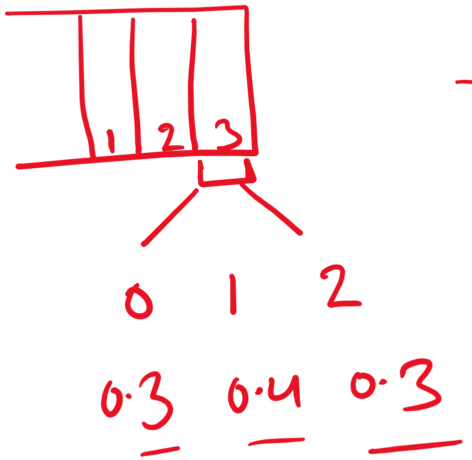
$$P = \begin{bmatrix} 1-\beta & \beta & 0 \\ \alpha(1-\beta) & (1-\alpha)(1-\beta)+\alpha\beta & \beta(1-\alpha) \\ \alpha^2 & 2\alpha(1-\beta) & (1-\alpha)^2 \end{bmatrix}$$

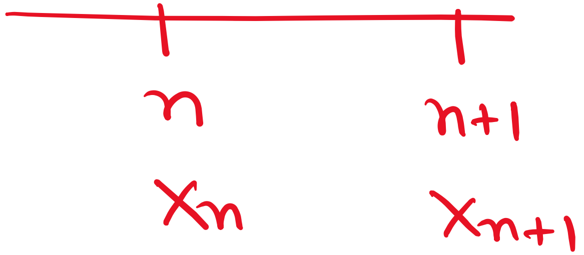
Example 3: The owner of a one-chair barber shop is thinking of expanding the shop capacity because there seem to be too many people are turned away. Observations indicate that in the time required to cut one person's hair there may be 0, 1, and 2 arrivals with probability 0.3, 0.4, and 0.3, respectively. The shop has a fixed capacity of 3 people including the one whose hair is being cut. Any new arrivals who finds 3 people in the barber shop is denied entry. Let X_n be the number of people in the shop at the completion of the nth person's hair cut. $\{X_n\}$ is a Markov chain assuming IID arrival.

Sol:-

$$\Phi = \{0, 1, 2\}$$

Step1:-





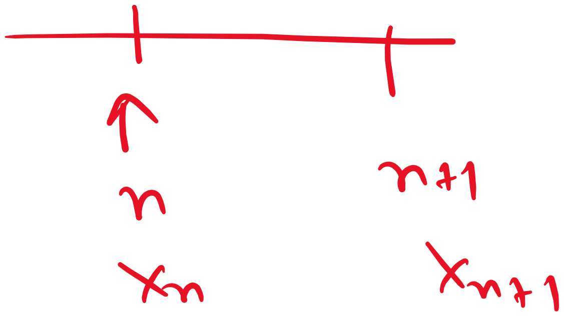
Step2:-

$$P = \begin{bmatrix} p_{00} & p_{01} & p_{02} \\ p_{10} & p_{11} & p_{12} \\ p_{20} & p_{21} & p_{22} \end{bmatrix}$$

3x3

Step 3: –

$$p_{00} = P\{X_{n+1}=0 \mid X_n=0\} = 0.3$$



$$p_{01} = P\{X_{n+1}=1 \mid X_n=0\} = 0.4$$

$$p_{02} = P\{$$