Unit 5: Graph Theory

Topic 1: Introduction to Graphs

Outline

Topics to be covered

Graph theory is one of the most important mathematical tools in computer science. Graphs are discrete structures consisting of vertices and edges that connect these vertices.

Topics to be covered in this unit are

1. Graphs (Directed and undirected)
2. Degree of graph
3. Representation of graphs
4. Isomorphism of graphs
5. Paths (Euler and Hamilton)
6. Shortest path problem
7. Planar graph
8. Graph coloring
9. Tree, Spanning tree, Minimum spanning tree

Applications and Objectives

We often model a computer network as a graph, and use the knowledge and techniques in dealing with graphs to solve problems in networks. Thus there are many areas of applications of Graph theory in computer science.

1. Algorithm
2. Computer networks
3. Circuit design
4. Data structures.

The main objective of this unit is to enable you to model problems and learn basic concepts and knowledge.

Outline

Graph

Definition

A **graph** $G = (V, E)$ consists of V, a nonempty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its **endpoints**. An **edge** is said to connect its endpoints.

Depending on the assignment of directions to edges, graphs have three categories.

- **Undirected Graphs:** Edges are undirected. Each undirected edge is associated with a set of two vertices $\{u, v\}$ connecting vertices u and v. If $\{u, v\}$ is an edge in an undirected graph G, then u and v are called **adjacent (or neighbors)** in G. In this case, the edge $e = \{u, v\}$ is called **incident with** the vertices u and v. If u is incident with itself, then there is a **loop** at u in G. The **degree of a vertex** u in G is the number of edges incident with it in G, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex u is denoted by $\deg(u)$.

Graph

- **Directed Graph(Digraph):** Edges are directed. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at vertex u and end at vertex v. If (u, v) is an edge in a directed graph G, then u is said to be **adjacent to** v and v is said to be **adjacent from** u. The vertex u is called the **initial vertex** of (u, v), and v is called the **terminal or end vertex** of (u, v). In a loop, the initial and terminal vertices are the same. In a directed graph the **in-degree** of a vertex v, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. The **out-degree** of v, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)
- **Mixed Graph:** Some edges are directed and some edges are undirected.

Graph

A vertex with degree is zero is called **isolated**, and a vertex of degree 1 is called **pendant**.

If there are m edges connecting the same pair of vertices u and v, then we say that the edge $\{u, v\}$ or (u, v) has multiplicity m.

We use the term graph only to undirected graphs unless it is mentioned, even if many properties are identical for directed and undirected graphs.

Graph

Moreover, depending on multiplicity of an edge and presence of loop, graphs are divided in three categories.

1. **Simple Graphs:** A graph in which each edge connects two different vertices and no two edges connect the same pair of vertices.
2. **Multigraphs:** More than one edge connect the same pair of vertices.
3. **Pseudograph:** Graphs those include loops, and possibly multiple edges connecting the same pair of vertices.

Graph

For example,

- Niche overlap graphs in ecology are simple undirected graphs.

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Problems:

1. How can a graph that models e-mail messages sent in a network be used to find people who have recently changed their primary e-mail address?

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Problems:

1. How can a graph that models e-mail messages sent in a network be used to find people who have recently changed their primary e-mail address?
2. Describe a graph model that represents whether each person at a party knows the name of each other person at the party. Should the edges be directed or undirected? Should multiple edges be allowed? Should loops be allowed?

Properties of Graphs

Theorem (The Handshaking Theorem)

Let $G = (V, E)$ be an undirected graph, then

$$2|E| = \sum_{u \in V} deg(u).$$

Again if $G = (V, E)$ is a directed graph with e edges, then

$$|E| = \sum_{u \in V} deg^{-}(v) = \sum_{u \in V} deg^{+}(v).$$

Problem

Problem: Find degree of each vertex and total number of edges present in the following graphs:

1. Complete graphs K_n
2. Cycles C_n
3. Wheels W_n
4. Complete bipartite graphs $K_{m,n}$

Subgraph

Sometimes we need only part of a graph to solve a problem. When edges and vertices are removed from a graph, without removing endpoints of any remaining edges, a smaller graph is obtained. Such a graph is called a subgraph of the original graph.

Definition

Let $G = (V, E)$ be a graph. Then $H = (W, F)$ is said to be **subgraph** of G if $W \subseteq V$ and $F \subseteq E$. Moreover, the subgraph H is a **proper subgraph** G if $G \neq H$.

For example, C_n are proper subgraph of W_n.

Construction of graph

We can obtain new graphs from a given graph in the following ways:

- **Removing edges of a graph:** Given a graph $G = (V, E)$ and an edge $e \in E$, we can produce a subgraph of G by removing the edge e. The resulting subgraph, denoted by $G - e = (V, E - \{e\})$.
- **Adding edges to a graph:** Given a graph $G = (V, E)$, we can also add an edge e (not present in G) to a graph to produce a new larger graph when this edge connects two vertices already in G. The resultant larger graph is denoted by $G + e = (V, E \cup \{e\})$.
- **Removing vertices of a graph:** Given a graph $G = (V, E)$ and a vertex $u \in V$, we can produce a subgraph of G by removing the vertex u and the edges associated with u.
- **Adding vertices to a graph:** Given a graph $G = (V, E)$, we can also add a vertex u (not present in G) to a graph to produce a new larger graph when this vertex is connected with atleast one vertex already in G.
- **Union of graphs:** The **union** of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$.

Outline

Bipartite graph

A simple graph $G = (V, E)$ is said to be **bipartite** if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a **bipartition** of the vertex set V of G.

For example, K_n, W_n are never bipartite, C_n are bipartite for even n, $K_{m,n}$ are always bipartite.

Bipartite graph

Theorem

A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

- A simple graph is called **regular** if every vertex of the graph has the same degree.
- A regular graph is called n**-regular** if every vertex of the graph has degree n.

Problem:

1. How many vertices does a 4-regular graph with 10 edges have?
2. Show that if G is a bipartite simple graph with v vertices and e edges, then $e \leq \frac{v^2}{4}$.

Outline

Representation of Graphs

There are many useful ways to represent graphs. Sometimes, two graphs have exactly the same form (may not be visible diretcly), in the sense that there is a one-to-one correspondence between their vertex sets that preserves edges. With the help of an appropriate graph representation, its becomes easy to check isomorphism of graphs.

Following are certain ways to represent graphs:

1. **Adjacency list**
2. **Adjacency matrix**
3. **Incidence matrix**

Isomorphism of graphs

Definition

The two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a bijection f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an isomorphism. Two simple graphs that are not isomorphic are called non-isomorphic.

Two isomorphic graphs preserve all properties of the graphs. These properties are called **graph invariants**.

Isomorphism of Graphs

If there are two graphs with the same number of vertices and edges, then it is not so easy to find an isomorphism between the two graphs as there are $n!$ possible bijections between the two graphs with n vertices. It is very difficult to check which function will work for the given graphs. Therefore, the matrix representation is a better way to find isomorphism between two graphs. If the two matrices representing the graphs become identical after making suitable row and column shifting, then the two graphs will be isomorphic.

However, it is always easy to show that two graphs are not isomorphic rather than finding isomorphism between them. To show that two graphs are not identical, it is enough to find a property of one graph which is not there in the second graph. This is because two isomorphic graphs have identical properties.

Thank You

Any Question!!!