MA 203

Transmission of WSS RP Through LTI System

Liner Time Invariant (LTI) System:

LTI

Linearity

Time Invariant

$$x(t) = a_1 x_1(t) + a_2 x_2(t)$$

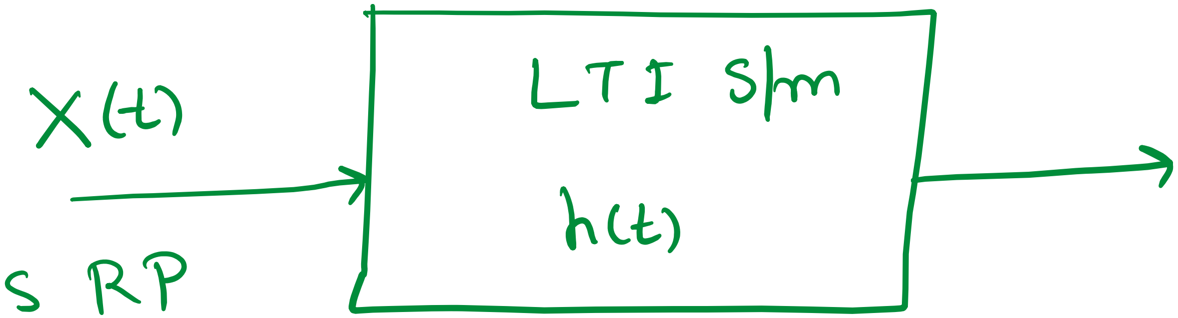
$$\Downarrow$$

$$y(t) = a_1 x_1(t) * h(t) + a_2 x(t) * h(t)$$

$$\begin{tikzcd} x(t),\ \delta(t) \arrow[r] & \boxed{\begin{matrix}\text{LTI} \\ h(t)\end{matrix}} \arrow[r] & \boxed{h(t) * x(t) = \int_{-\infty}^{+\infty} x(t-\alpha)\, h(\alpha)\, d\alpha} \end{tikzcd}$$

$h(t) =$ Impulse Response

Transmission of WSS random process through LTI system:

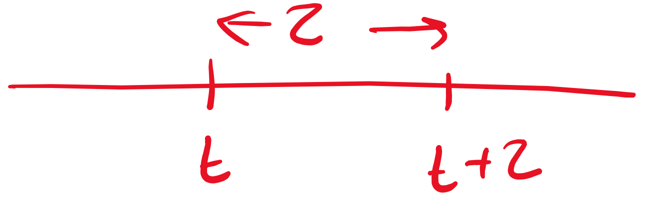


$Y(t) = X(t) * h(t)$

X(t): WSS RP

1. $E[Y(t)] = \mu_Y = $ const.
2. $E[Y(t)\cdot Y(t+\tau)] = R_Y(\tau)$

} WSS



1. whether Y(t) is WSS or not?
2. If Y(t) is WSS, what will be its PSD?

Mean:-

$$\mu_y(t) = E[y(t)] = E\left[x(t) * h(t)\right]$$

$$= E\left[\int_{-\infty}^{+\infty} x(t-\alpha)\, h(\alpha)\, d\alpha\right]$$

$$= \int_{-\infty}^{+\infty} \underbrace{E[x(t-\alpha)]}_{}\, h(\alpha)\, d\alpha$$

$x(t)$ is WSS

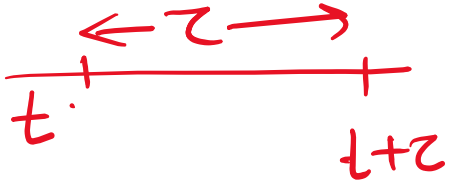
$E[x(t-\alpha)] = \mu_x = \text{const.}$

$$= \underbrace{\int_{-\infty}^{+\infty} \mu_x\, h(\alpha)\, d\alpha}_{}$$

It is NOT the function of time instant t. $\Rightarrow$ const

Autocorrelation:

$$y(t) = \int_{-\infty}^{+\infty} x(t-\alpha)\, h(\alpha)\, d\alpha = x(t) * h(t)$$



$$y(t+\tau) = \int_{-\infty}^{+\infty} x(t+\tau-\beta)\, h(\beta)\, d\beta$$

$$R_y(t,t+\tau) = E[y(t)\cdot y(t+\tau)] = E\left[\left\{\int_{-\infty}^{+\infty} x(t-\alpha) h(\alpha)\, d\alpha\right\}\left\{\int_{-\infty}^{+\infty} x(t+\tau-\beta)\, h(\beta)\, d\beta\right\}\right]$$

$$= E\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(t-\alpha)\, x(t+\tau-\beta)\, h(\alpha)\, h(\beta)\, d\alpha\, d\beta\right]$$

$$E[Y(t)Y(t+\tau)] = E\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \{X(t-\alpha)\,X(t+\tau-\beta)\}\, h(\alpha)\cdot h(\beta)\, d\alpha\, d\beta\right]$$

X(t) is WSS

$$E[X(t_1)X(t_2)] = R_X(t_2-t_1)$$

$$t_1 = t-\alpha$$

$$t_2 = t+\tau-\beta.$$

$$t_2-t_1 = t+\tau-\beta-(t-\alpha)$$

$$= \tau+\alpha-\beta$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underbrace{E\{X(\underbrace{t-\alpha}_{t_1})\,X(\underbrace{t+\tau-\beta}_{t_2})\}}\, h(\alpha)\, h(\beta)\, d\alpha\, d\beta.$$

$$\hookrightarrow R_X(t_2-t_1) = R_X(\tau+\alpha-\beta)$$

$$E[Y(t)\,Y(t+\tau)] = \underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_X(\tau-\beta+\alpha)\, h(\alpha)\, h(\beta)\, d\alpha\, d\beta}$$

This part depends only on time shift τ

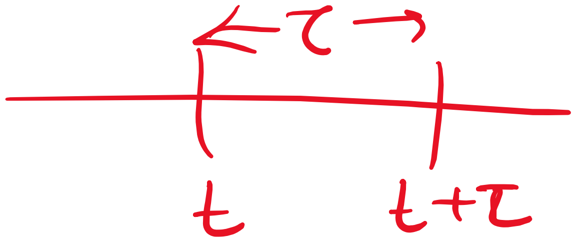
$\Rightarrow$ It does not depend on time t.

$\Rightarrow$ Therefor, $Y(t)$ is a WSS RP.

$\Rightarrow$ 1. $E[Y(t)] = \mu_Y =$ const.

2. $E[Y(t)\,Y(t+\tau)] = R_Y(\tau)$

$\Rightarrow$ $Y(t)$ is a WSS R.P.



$$R_y(\tau) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \underline{R_x(\tau-\beta+\alpha)}\, h(\beta)\cdot h(\alpha)\, d\alpha\, d\beta$$

Put $\bar{\alpha} = -\alpha$

$$d\bar{\alpha} = -\underline{d\alpha}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_x(\tau-\beta-\underline{\bar{\alpha}})\, h(\beta)\, \underline{h(-\bar{\alpha})}\, d\beta\, d(\underline{+\bar{\alpha}})$$

Put $\underline{\bar{h}(\bar{\alpha}) = h(-\bar{\alpha})}$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_x(\underline{\tau}-\underline{\beta}-\underline{\bar{\alpha}})\, \underline{h(\beta)}\, \underline{\bar{h}(\bar{\alpha})}\, d\beta\, \underline{d\bar{\alpha}} \quad \checkmark$$

$$R_y(\tau) = R_x(\tau) * h(\tau) * \bar{h}(\tau)$$

$$\because h(-\alpha) = \bar{h}(\alpha)$$

$$\boxed{R_y(\tau) = R_x(\tau) * h(\tau) * h(-\tau)} \quad -(1)$$

Convolution

Fourier transform of (1)

$$S_y(f) = \int_{-\infty}^{+\infty} R_y(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$= F.T.\left(R_x(\tau)\right)$$

$$F.T.\{R_y(\tau)\} = F.T.\{R_x(\tau) * h(\tau) * h(-\tau)\}$$

using Property of F.T.

$$\Rightarrow \quad S_y(f) = F.T.\{R_x(\tau)\} \cdot F.T.\{h(\tau)\} \, F.\{h(-\tau)\}$$

$$\Rightarrow \quad S_y(f) = S_x(f) \cdot \underset{\sim}{H(f)} \cdot \underset{\sim}{H^*(f)}$$

$$S_y(f) = S_x(f) \cdot |H(f)|^2$$

Let $\underset{\sim}{h(t)} \longleftrightarrow \underset{\sim}{H(f)}$

$$H(f) = \int_{-\infty}^{+\infty} h(t)\, e^{-j2\pi f t}\, dt$$

$h(-\tau) \longleftrightarrow \underline{H^*(f)}$

$$\boxed{S_y(f) = S_x(f) \cdot |H(f)|^2}$$

PSD of R.P. $Y(t)$ — O/P PSD

PSD of R.P. $X(t)$ — I/P PSD

F.T. of $h(t)$

Gaussian Random Process:

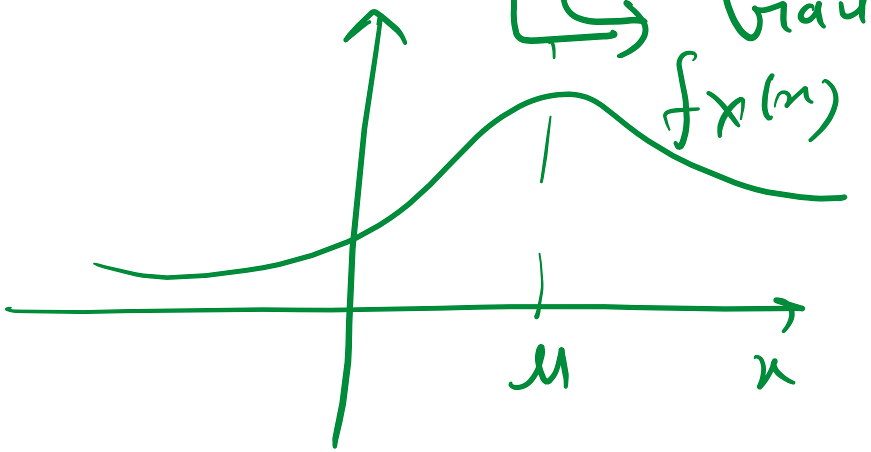
Suppose $X(t)$

$t_1 \quad t_2 \quad \cdots \quad t_n$

$X(t_1) \quad X(t_2) \quad \cdots \quad X(t_n)$

$f_{X(t_1), X(t_2), \ldots, X(t_n)}(x_1, x_2, \ldots, x_n)$

$\hookrightarrow$ Gaussian R.V. are following Gaussian distribution



$\Rightarrow$ $X(t)$ is a Gaussian R.P., then each R.V. $X(t_1), X(t_2), \cdots X(t_n)$ must be Gaussian.

White Noise:- A R.P. is white if it is WSS and

(white R.P.)

$$R_x(\tau) = \frac{\eta}{2}\delta(\tau).$$

White Random Process

1. It is WSS

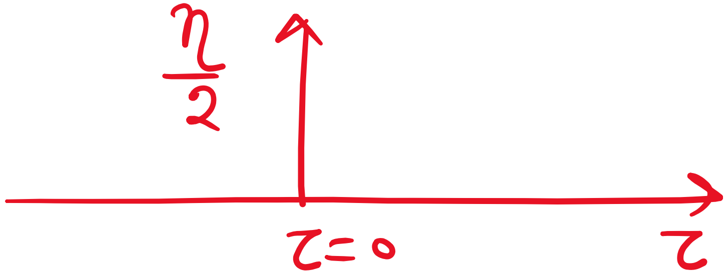
$\rightarrow E[x] = \mu_x = \text{const.}$

$\rightarrow E[x(t)\,x(t+\tau)] = R_x(\tau)$

2. $R_x(\tau) = \frac{\eta}{2}\delta(t)$; $\eta = \text{const.}$

$\delta(t) = \text{Impulse}$

$$\underline{R_x(\tau) = \frac{\eta}{2}\,\delta(\tau)}$$



$R_x(\tau) = 0;\ \tau \neq 0$

$\Rightarrow$ <u>PSD</u>

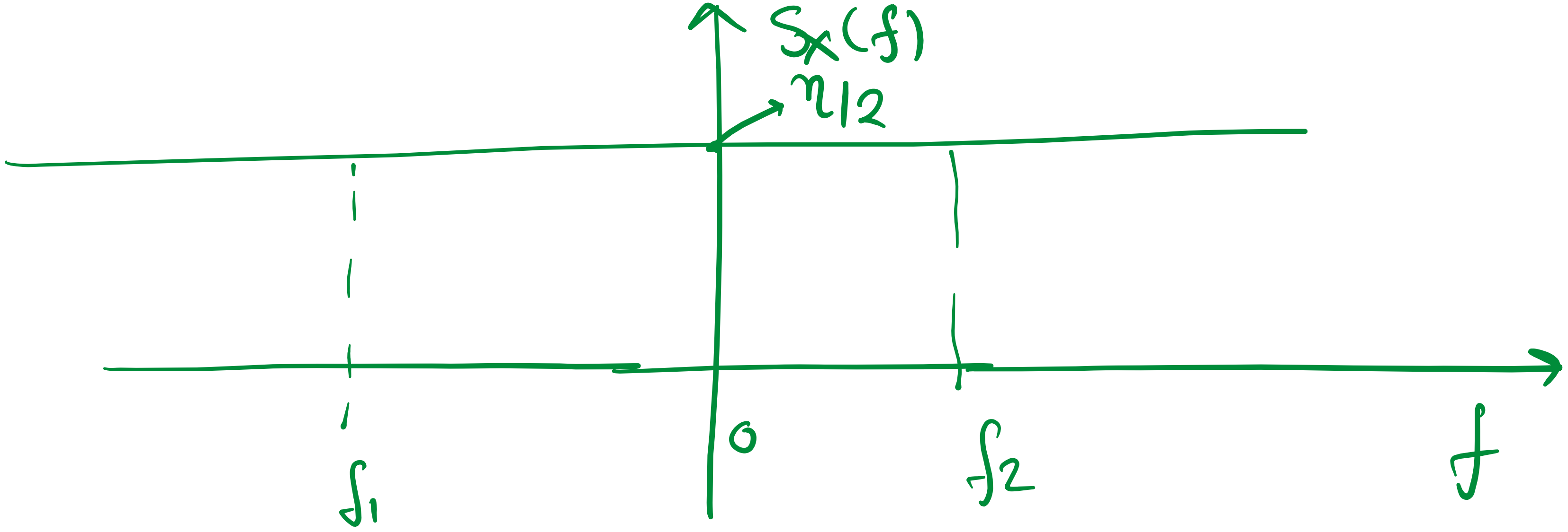
$$S_x(f) = \int_{-\infty}^{+\infty} R_x(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$= \int_{-\infty}^{+\infty} \frac{\eta}{2}\,\delta(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$= \int_{-\infty}^{+\infty} \frac{\eta}{2}\,\delta(\tau)\cdot e^{0}\, d\tau = \frac{\eta}{2}\int_{-\infty}^{+\infty} \delta(\tau)\, d\tau$$

$$= \frac{\eta}{2}$$

$$\boxed{\int_{-\infty}^{+\infty} \delta(\tau) = 1}$$



⇒ Distribution of Power corresponding to each frequency component is same

⇓

⇒ Therefore, It is known as white Random Process.

White Gaussian R.P.

- $X(t)$

  → White & Gaussian → $X(t)$ is white Gaussian R.P.

→ 1. W.S.S.

– 2. $R_X(\tau) = \frac{\eta}{2}\delta(\tau)$

$$\underbrace{f_{X(t_1), X(t_2), \dots, X(t_n)}(x_1, x_2, \dots, x_n)}_{\text{Must be Gaussian}}$$