MA203: Function of Two Random Variables

X & Y

$$Z = g(X, Y)$$

$\to$ X^2+Y^2 / $\sqrt{X^2+Y^2}$

$\to$ $\max(X, Y)$ / $\min(X, Y)$

$\to$ $X+Y$ / $X-Y$

$\to$ $\frac{X}{Y}$ / XY

Application:

X - - - - - $\bigcirc$ $\longrightarrow$ $Z = X+Y$

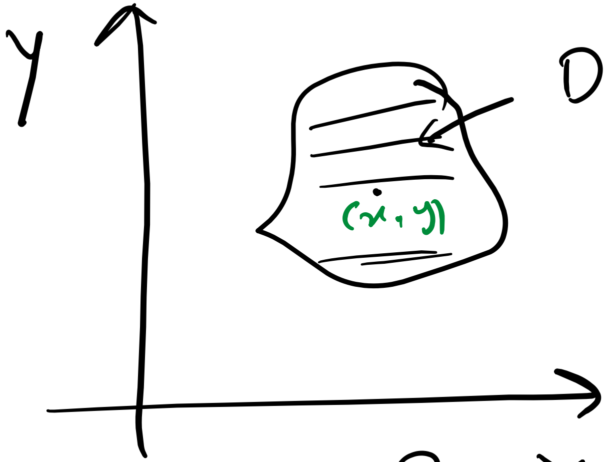
X: Transmitted Signal

Y: noise

Received Signal

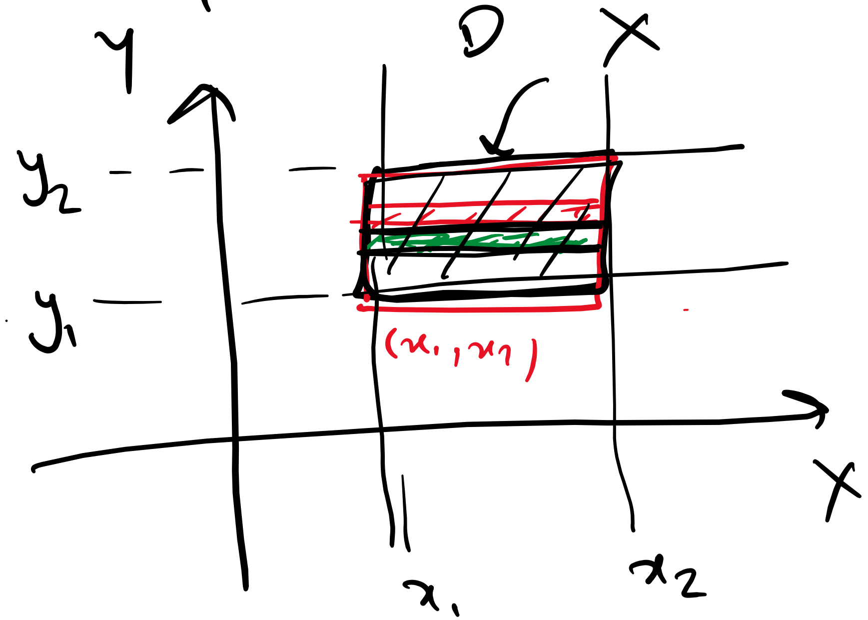
$Z = XY$

<u>Prerequisite:</u> Let X and Y are two continuous RVs. Then the probability that X and Y belongs to regions *D* is given as



$$P((x,y)\in D) = \iint_{(x,y)\in D} f_{X,Y}(x,y)\,dx\,dy$$

Ex:—



$$P((x,y)\in D) = \int_{y_1}^{y_2}\int_{x_1}^{x_2} f_{X,Y}(x,y)\,dx\,dy$$

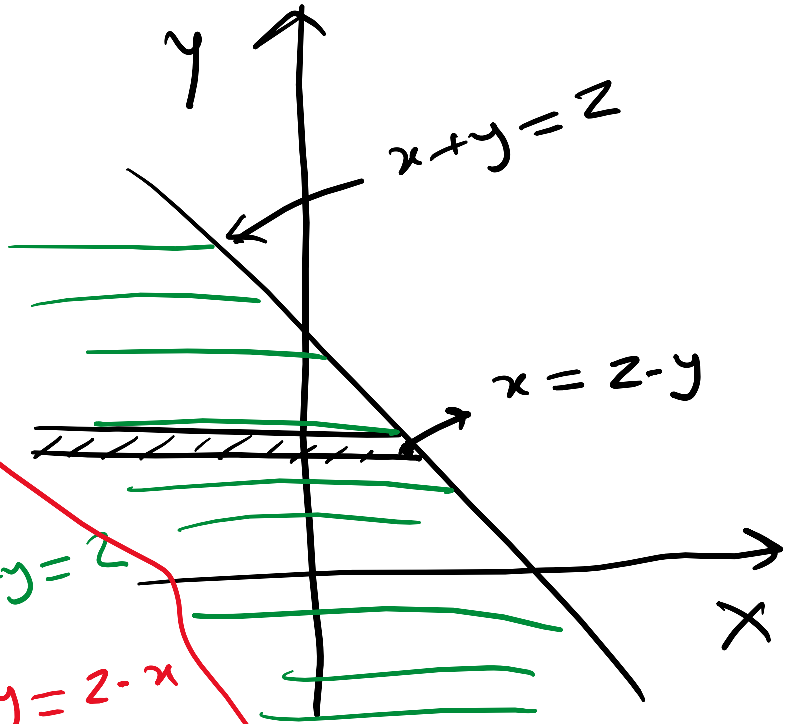
Example 1: Let X and Y are two continuous RVs. Find the probability density function of Z where Z = X+Y.

Sol:-

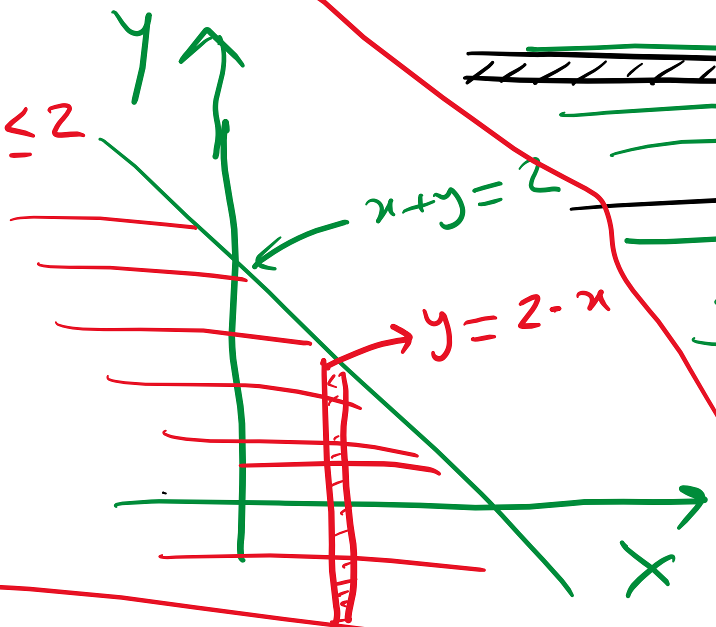
$$F_Z(z) = P(Z \le z) = P(X+Y \le z)$$

Step 1:-

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{z-y} f_{X,Y}(x,y)\,dx\,dy$$



$$F_Z(z) = \int_{-\infty}^{z-x}\int_{-\infty}^{+\infty} f_{X,Y}(x,y)\,dx\,dy$$



Step2:-

$$f_Z(z) = \frac{dF_Z}{dz} = \frac{d}{dz}\int_{-\infty}^{+\infty}\int_{-\infty}^{z-y} f_{X,Y}(x,y)dxdy = \frac{d}{dz}\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{z-y} f_{X,Y}(x,y)dx\right\}dy$$

Using Leibnitz Rule

$$f_Z(z) = 0 - 0 + \int_{-\infty}^{+\infty}\frac{\partial}{\partial z}\left\{\int_{-\infty}^{z-y} f_{X,Y}(x,y)dx\right\}dy$$

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Using Leibnitz Rule

$$f_Z(z) = \int_{-\infty}^{+\infty}\left\{\frac{d(z-y)}{dz}f_{X,Y}(z-y,y) - \frac{d(-\infty)}{dz}f_{X,Y}(-\infty,y) + \int_{-\infty}^{z-y}\frac{\partial f_{X,Y}(x,y)}{\partial z}\right\}dy$$

0

0

$$\boxed{f_Z(z) = \int_{-\infty}^{+\infty} f_{X,Y}(z-y,y)dy}$$

Leibnitz Rule: $\frac{d}{dx}\int_{a(x)}^{b(x)} h(x,y)\,\mathrm{d}y = \mathrm{h}\big(x,b(x)\big)\times\frac{db(x)}{dx} - \mathrm{h}\big(x,a(x)\big)\times\frac{da(x)}{dx} + \int_{a(x)}^{b(x)}\frac{\partial h(x,y)}{\partial x}dy$

If X and Y are independent:

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-y) f_Y(y) dy = f_X(z) * f_Y(y)$$

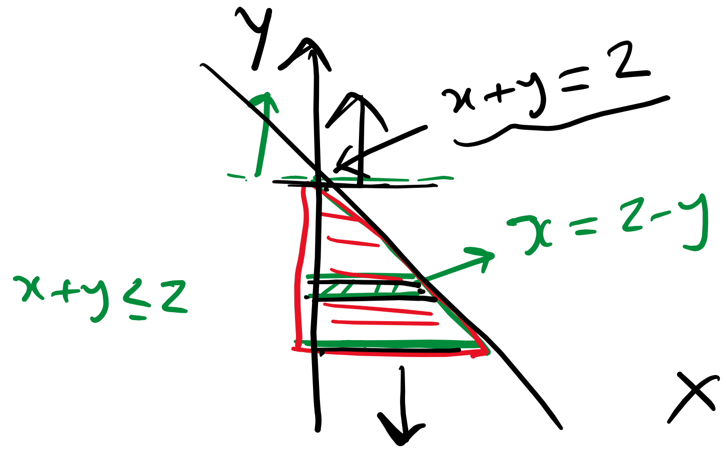
Example 2: Let X and Y are two non-negative continuous RVs. Find the probability density function of Z where Z = X + Y.

$X \geq 0 \; ; \; Y \geq 0$

Sol:-

Step I:-

$$F_Z(z) = P(Z \leq z) = \int_0^z \int_0^{z-y} f_{X,Y}(x,y)\, dx\, dy$$



$$f_Z(z) = \frac{d}{dz}\int_0^z\int_0^{z-y} f_{X,Y}(x,y)\,dx\,dy = \frac{d}{dz}\int_0^z\left\{\int_0^{z-y} f_{X,Y}(x,y)\,dx\right\}dy$$

Using Leibnitz Rule

$$f_Z(z) = \frac{d(z)}{dz}\left\{\int_0^{z-z} f_{X,Y}(x,z)\,dx\right\} - \frac{d(0)}{dz}\left\{\int_0^{z-0} f_{X,Y}(x,0)\,dx\right\} + \int_0^z \frac{\partial}{\partial z}\left\{\int_0^{z-y} f_{X,Y}(x,y)\,dx\right\}dy$$

0 0 0

$$f_Z(z) = \int_0^z \frac{\partial}{\partial z}\left\{\int_0^{z-y} f_{X,Y}(x,y)\,dx\right\}dy$$

Using Leibnitz Rule

$$f_Z(z) = \int_0^z\left\{\frac{d(z-y)}{dz} f_{X,Y}(z-y,y) - \frac{d(0)}{dz} f_{X,Y}(z-y,y) + \int_0^{z-y}\frac{\partial f_{X,Y}(x,y)}{\partial z}\right\}dy$$

0 0

$$\boxed{f_Z(z) = \int_0^z f_{X,Y}(z-y,y)\,dy}$$

Leibnitz Rule: $\frac{d}{dx}\int_{a(x)}^{b(x)} h(x,y)\,\mathrm{dy} = \mathrm{h}\big(x,b(x)\big)\times\frac{db(x)}{dx} - \mathrm{h}\big(x,a(x)\big)\times\frac{da(x)}{dx} + \int_{a(x)}^{b(x)}\frac{\partial h(x,y)}{\partial x}dy$

Example: Suppose X and Y are independent RVs such that

$$f_X(x) = \lambda e^{-\lambda x}; \lambda > 0, x \geq 0$$

and

$f_Y(y) = \lambda e^{-\lambda y}; \lambda > 0, y \geq 0.$

Find the probability density function of Z where $Z = X + Y$.

$$\boxed{f_Z(z) = \int_0^z f_{X,Y}(z-u, u)\, du}$$

Sol:-

$$f_Z(z) = \int_0^z f_X(z-u)\, f_Y(u)\, du$$

$$= \int_0^z \lambda \cdot e^{-(z-u)\lambda} \cdot \lambda e^{-\lambda u} \cdot du$$

$$= \int_0^z \lambda^2 \cdot e^{-\lambda(z-\cancel{u}+\cancel{u})}\, du = \int_0^z \lambda^2 e^{-\lambda z}\, du$$

$$= \lambda^2 z e^{-z\lambda}$$

$\Downarrow$

Erlang Distribution.

Example: Let X and Y are continuous RVs. Find the probability density function of Z where $Z = \frac{X}{Y}$.

Sol:- Step 1:-

$$F_Z(z) = P(Z \le z) = P\left(\frac{X}{Y} \le z\right)$$

$$=$$

$\frac{X}{Y} \le z$

$X \le Yz$

$Y > 0$

$Y < 0$

$X \ge Yz$