MA 203

Poisson Random Process

IID Random Process:-

For any finite choice of time instants $t_1, t_2, t_3, \ldots, t_n$, if the RVs $X(t_1), X(t_2), \ldots, X(t_n)$ are jointly independent with a common CDF, then $X(t)$ is called an IID random process.

$X(t)$: $t_1 \quad t_2 \ldots \quad t_n$

$x(t_1) \quad x(t_2) \ldots x(t_n)$

Thus, for an IID RP $X(t)$,

Independent & Identically distributed

$$F_{X(t_1), X(t_2), \ldots, X(t_n)}(x_1, x_2, \ldots, x_n) = F_{X(t_1)}(x_1) \cdot F_{X(t_2)}(x_2) \cdots F_{X(t_n)}(x_n)$$

$$= \{F_X(x)\}^n$$

where $F_{X(t_1)}(x_1) = F_{X(t_2)}(x_2) = \cdots = F_{X(t_n)}(x_n) = F_X(x)$

Independent Increment RP:- A RP is called an independent increment process if for any $n > 1$ and $t_1 < t_2 < t_3 \cdots < t_n \in \Gamma$, the set of n RVs $X(t_1), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ are jointly independent RVs.

$t_1 < t_2 < t_3 \cdots$ $\quad n = 3 \quad X(t)$

$X(t_1), \quad X(t_2) - X(t_1), \quad X(t_3) - X(t_2)$

Stationary Increment RP:-
If the CDF of $X(t+r) - X(t'+r)$ is same as that of $X(t) - X(t')$, for any choice of t, t' and r, then $X(t)$ is called stationary increment process.

$t \quad t' \quad t+r \quad t'+r$

$X(t) \quad X(t') \qquad X(t+r) \quad X(t'+r)$

r

$X(t) - X(t') = X(t+r) - X(t'+r)$

Counting Process:- A counting process represents the number of occurrences of an event over interval $(0, t]$.

Notation: $\{N(t), t \geq 0\}$

$N(t)$: Number of arrivals $[0, t]$ $(0, t]$ $t \geq 0$

Ex:- Number of customers arriving at a departmental store during $(0, t]$.

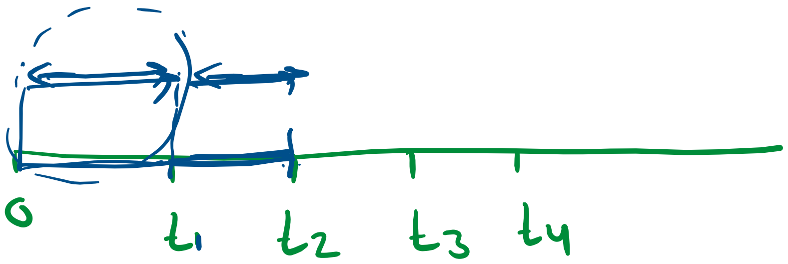
Note that it is a continuous-time discrete-state R.P.

Poisson Process:- The counting process $\{N(t), t \geq 0\}$ is called Poisson Process with the rate parameter λ if

(i) $N(0) = 0$

(ii) $N(t)$ is an independent increment process

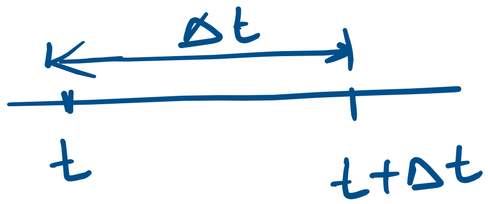
Ex:-



$N(t_3) - N(t_2)$, $N(t_4) - N(t_3)$, . . . are independent.

$(0, t_3]$ $(0, t_2]$

(iii) $P\{N(\Delta t) = 1\} = \lambda \Delta t + O(\Delta t)$



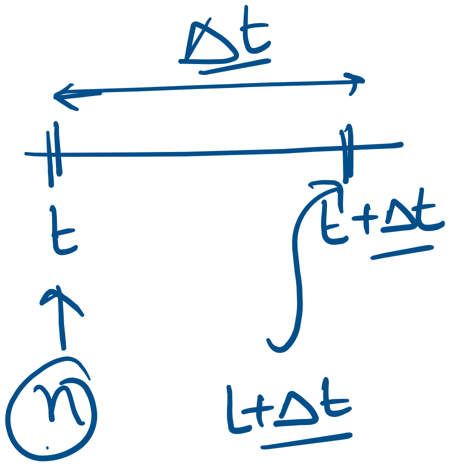
where, $O(\Delta t)$ implies any function such that

$$\lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t} = 0$$

(iv) $P\{N(\Delta t) \geq 2\} = O(\Delta t)$

Distribution :—

$P(N(t+\Delta t) = n)$ = Prob of occurrence of n events up to time $t+\Delta t$



$$= P\{N(t)=n,\ N(\Delta t)=0\} + P\{N(t)=n-1,\ N(\Delta t)=1\}$$

$$+ P\{N(t)<n-1,\ N(\Delta t)\geq 2\}$$

using the independent increment property

$$= P\{N(t)=n\}\left(1-\lambda\Delta t - O(\Delta t)\right) + P(N(t)=n-1)\left(\lambda\Delta t + O(\Delta t)\right)$$

$$+ P(N(t)=n-1)\ O(\Delta t)$$

$$\lim_{\Delta t \to 0} \frac{P\{N(t+\Delta t)=n\} - P\{N(t)=n\}}{\Delta t} = \frac{d}{dt}\left[\frac{P\{N(\Delta t+t)=n\} - P(N(t)=n)}{\Delta t}\right]$$

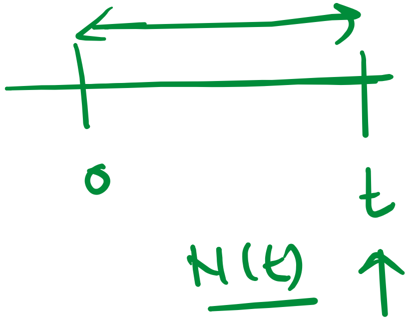
$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= -\lambda \left[P(N(t)=n) - P(N(t)=n-1) \right]$$

or $\frac{d}{dt} P(N(t)=n) = -\lambda \left[P(N(t)=n) - P(N(t)=n-1) \right]$ —— (A)

The above is a first-order linear differential equation with initial condition $P\{N(0)=n\}=0$.

$\underline{P(N(t)=0)}$



$$\frac{d}{dt} P(N(t)=0) = -\lambda P(N(t)=0) - \underbrace{P(N(t)=-1)}_{0}$$

$$\frac{d}{dt} P(N(t)=0) = -\lambda P(N(t)=0) \Rightarrow \boxed{\frac{dy}{dx} = -\lambda y}$$

$$\boxed{y = e^{-\lambda x}}$$

$$\Rightarrow \boxed{P(N(t)=0) = e^{-\lambda t}} \quad -(1)$$

put $\underline{n=1}$ in Eq. (A)

$$\frac{d}{dt} P\{N(t)=1\} = -\lambda P\{N(t)=1\} - \lambda P\{N(t)=1-1\}$$

or $$\frac{d}{dt} P\{N(t)=1\} = -\lambda \{P(N(t)=1)\} - \lambda P\underbrace{(N(t)=0)}$$

$$= -\lambda P(N(t)=1) - \lambda e^{-\lambda t}$$

or $$\frac{d}{dt} P\{N(t)=1\} = -\lambda P\{N(t)=1\} - \lambda e^{-\lambda t}$$

$$\boxed{P(N(0)=1) = 0}$$

$$P(N(t)=1) = \lambda t\, e^{-\lambda t} = \frac{(\lambda t)\, e^{-\lambda t}}{\lfloor 1}$$

$n=0 \qquad P(N(t)=0) = e^{-\lambda t}$

$n=1 \qquad P(N(t)=1) = \dfrac{(\lambda t)\, e^{-\lambda t}}{\lfloor 1}$

$n=2 \qquad P(N(t)=2) = \dfrac{(\lambda t)^2 \left(e^{-\lambda t}\right)}{\lfloor 2}$

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

$n=k \qquad \boxed{P(N(t)=k) = \dfrac{(\lambda t)^k\, e^{-\lambda t}}{\lfloor k}} \Rightarrow$ k arrivals in the interval of $(0,t]$

At a particular time instant, $t = t_1$

$$P\{N(t_1) = k\} = \frac{(\lambda t_1)^k e^{-\lambda t_1}}{\lfloor k}$$

$\lambda t_1 = \lambda_0$

$(0, t_1]$

$$\frac{\lambda_0^k e^{-\lambda_0}}{\lfloor k}$$

$$\boxed{\begin{array}{l} \text{Mean of Poisson Process,} = \lambda t_1 = \lambda_0 \\ \text{Variance} = \lambda t_1 = \lambda_0 \end{array}}$$

Example:- A petrol pump serves on the average of 30 cars per hour. Find the probability that during a period of 5 minutes (i) No car comes to the station (ii) exactly 3 cars comes to the station and (iii) more than 3 cars comes to the station.

Sol:- Average arrival $= 30$ cars/Hour

$= 30$ cars/60 minutes

$= 1/2$ car/minutes

$(0, 5]$;

(i) $P(N(5)=0) = ?$

(i) $P(N(5)=0) =$

$$\boxed{P(N(t)=k) = \frac{(\lambda t)^k e^{-\lambda t}}{\lfloor k}}$$

$\underline{k=0}$

$$P(N(5)=0) = \frac{e^{-\lambda t}}{\lfloor 0} = e^{-\lambda t} = e^{-\frac{1}{2}\times 5}$$

$$= e^{-5/2}$$

(ii) $$P(N(5)=3) = \frac{e^{-1/2\times 5}}{\lfloor 3}\left(\frac{1}{2}\times 5\right)^3 = 0.2138$$

(iii) $$P(N(5)>3) = 1 - \underbrace{P(N(5)=0)} - \underbrace{P(N(5)=1)} - P(\underline{N(5)=2}) - P(\underline{N(5)=3})$$