# MA203: Covariance, Correlation and Conditional Expectation

Let X and Y are continuous RVs. The joint moment of order $m+n$ is defined as

$$\boxed{E[X^mY^n] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^m y^n f_{X,Y}(x,y)dxdy.}$$

$$\left\{ E[x^m] = \int_{-\infty}^{+\infty} x^m f_X(x)\,dx \right.$$

The joint central moment of order $m+n$ is defined as

$$E[(X-\mu_X)^m(Y-\mu_Y)^n] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(x-\mu_X)^m(y-\mu_Y)^n f_{X,Y}(x,y)dxdy$$

where $\mu_X = E[X]$ and $\mu_Y = E[Y]$.

$$E[(x-\mu_x)^m] = \int_{-\infty}^{+\infty} (x-\mu_x)^m f_x(x)\,dx$$

$m = n = 1$ ; $E[XY]$ = Second Order moment

Let X and Y are discrete RVs. The joint moment of order $m+n$ is defined as

$$\boxed{E[X^m Y^n] = \sum_{(x,y)\in R_X\times R_Y}\sum x^m y^n p_{X,Y}(x,y).}$$

The joint central moment of order $m+n$ is defined as

$$E[(X-\mu_X)^m (Y-\mu_Y)^n] = \sum_{(x,y)\in R_X\times R_Y}\sum (x-\mu_X)^m (y-\mu_Y)^n p_{X,Y}(x,y)$$

where $\mu_X = E[X]$ and $\mu_Y = E[Y]$.

Covariance of Two RVs: The covariance of two RVs X and Y is defined as

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)].$$

$Cov(X,Y)$ is also denoted as $\sigma_{X,Y}$.

$Cov(X,Y)$ $\quad$ $\sigma_{X,Y}$

$$\begin{aligned} Cov(X,Y) &= E[(X-\mu_X)(Y-\mu_Y)] \\ &= E[XY - Y\mu_X - X\mu_Y + \mu_X\mu_Y] \\ &= E[XY] - \mu_X E[Y] - \mu_Y E[X] + \mu_X\mu_Y \\ &= E[XY] - \mu_X\mu_Y \end{aligned}$$

Properties:

1. $Cov(X,Y) = Cov(Y,X)$
2. $Cov(X,X) = var(X)$
3. $Cov(aX,Y) = aCov(X,Y)$
4. $Cov\left(\sum_{i=1}^{m} X_i, \sum_{j=1}^{n} Y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} Cov(X_i, Y_j)$

$Var(X) = E[(X-\mu_X)^2]$

$a$ is constant

Ex:-

$$Cov(X_1+X_2, Y_1+Y_2) = \sum_{i=1}^{2}\sum_{j=1}^{2} Cov(X_i, Y_j)$$

$$= \sum_{i=1}^{2}\{Cov(X_i+Y_1) + Cov(X_i+Y_2)\}$$

$$= Cov(X_1, Y_1) + Cov(X_1, Y_2) + Cov(X_2, Y_1) + Cov(X_2, Y_2)$$

Correlation Co-efficient: The ~~covariance~~ Correlation Co-efficient of two RVs X and Y is defined as

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$\rho_{x,y} = \frac{E[xy] - \mu_x \mu_y}{\sigma_x \sigma_y}$$

$$= \frac{E[xy] - \mu_x \mu_y}{\sqrt{E[(x-\mu_x)^2]} \sqrt{E[(y-\mu_y)^2]}}$$

Positive correl-ation

$X \longleftrightarrow Y$

price of petrol ↑ Inflation ↑

Negative

$X \longleftrightarrow Y$

Temp Temp.

10 AM to 11:30 AM

10 PM to 11:30 PM

**Properties:**

1. $\rho_{X,Y} = \rho_{Y,X}$ ✓
2. $|\rho_{X,Y}| \leq 1$ ✓ $\Rightarrow$ $-1 \leq \rho_{x,y} \leq +1$
3. $\rho_{X,X} = 1$
4. $\rho_{X,-X} = -1$
5. $\rho_{aX+b,Y} = \rho_{X,Y}$ where $a$ *and* $b$ *are constants and* $a > 0$

Uncorrelated RVs: Two RVs X and Y are called uncorrelated if

$$Cov(X,Y) = 0$$

$$\rho_{X,Y} = 0$$

$$\Rightarrow \quad \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = 0$$

$$\Rightarrow \quad Cov(X,Y) = 0 \Rightarrow E[XY] - \mu_X \mu_Y = 0$$

or $\boxed{E[XY] = \mu_X \mu_Y}$

$\Rightarrow$ X & Y are independent, $E[XY] = \mu_X \mu_Y$, therefore independent RVs are always uncorrelated.

$\Rightarrow$ But if X & Y are uncorrelated, then it is not necessary they are independent.

**Example 2:** The joint PMF of X and Y is given in the Table. Find $\rho_{X,Y}$?

| Y \ X | 0 | 1 | 2 | $p_Y(y)$ |
|---|---|---|---|---|
| 0 | 1/8 | 1/8 | 0 | 1/4 |
| 1 | 1/8 | 1/4 | 1/8 | 1/2 |
| 2 | 0 | 1/8 | 1/8 | 1/4 |
| $p_X(x)$ | 1/4 | 1/2 | 1/4 | |

$R_X = \{0, 1, 2\}$

$R_Y = \{0, 1, 2\}$

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[XY] - \mu_X \mu_Y}{\sqrt{E[X^2] - \mu_X^2}\sqrt{E[Y^2] - \mu_Y^2}}$$

$$E[X] = \sum_{x \in R_X} x\, p_X(x) = 0 \times p_X(0) + 1 \times p_X(1) + 2 \cdot p_X(2)$$

$$= 0 + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1$$

$$E[Y] = \sum_{y \in R_Y} y\, p_Y(y) = 1$$

$$E[XY] = \sum_{(x,y) \in R_X \times R_Y} \sum xy\, p_{X,Y}(x,y) = \sum_{x \in R_X} x \{0 \cdot \cancel{p_{X,Y}(x,0)}^{0} + 1 \cdot p_{X,Y}(x,1) + 2 \cdot p_{X,Y}(x,2)\}$$

$$= 0 + 1 \cdot \{1 \cdot p_{X,Y}(1,1) + 2 \cdot p_{X,Y}(1,2)\} + 2 \cdot \{1 \cdot p_{X,Y}(2,1) + 2 \cdot p_{X,Y}(2,2)\}$$

$$E[XY] = 5/4$$

$$E[X^2] = \sum_{x \in R_X} x^2 \, p_X(x) = 1 \times p_X(1) + 2^2 \times p_X(2) = 1 \times \frac{1}{2} + 4 \times \frac{1}{4} = 3/2$$

$$E[Y^2] = \sum_{y \in R_Y} y^2 \, p_Y(y) = 3/2$$

$$\rho_{X,Y} = \frac{E[XY] - \mu_X \mu_Y}{\sqrt{E[X^2] - \mu_X^2}\,\sqrt{E[Y^2] - \mu_Y^2}} = \frac{5/4 - 1}{\sqrt{\frac{3}{2} - 1}\,\sqrt{\frac{3}{2} - 1}}$$

$$= \frac{1/4}{\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}} = \underset{\sim}{1/2}$$

# Conditional Expectation

**If X and Y are continuous RVs**, then the conditional PDF of Y given $X = x$ given by

$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$; provided that $f_X(x) \neq 0$.

The conditional expectation of Y given $X = x$ is defined as

$$\boxed{\mu_{Y|X=x} = E[Y|X = x] = \int_{-\infty}^{+\infty} y f_{Y|X}(y|x)dy}$$

**If X and Y are discrete RVs**, then the PMF of Y given $X = x$ given by

$p_{Y|X}(y|x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$; provided that $p_X(x) \neq 0$.

The conditional expectation of Y given $X = x$ is defined as

$$\boxed{\mu_{Y|X=x} = E[Y|X = x] = \sum_{y \in R_Y} y p_{Y|X}(y|x)}$$

Example 1: The joint PMF of RVs X and Y are given in the Table. Find $E[Y|X=2]$?

| Y \ X | 0 | 1 | 2 | $P_Y(y)$ |
|---|---|---|---|---|
| 0 | 0.25 | 0.10 | 0.15 | 0.50 |
| 1 | 0.14 | 0.35 | $\boxed{0.01}$ | 0.50 |
| $P_X(x)$ | 0.39 | 0.45 | $\boxed{0.16}$ | |

$$\mu_{Y|X} = E[Y|X=2] = \sum_{y \in R_Y} y \, P_{Y|X}(y|2)$$

$$R_Y = \{0, 1\}$$

$$E[Y|X=2] = \sum_{y \in R_Y} y \, P_{Y|X}(y|2)$$

$$= 0 \cdot \cancel{P_{Y|X}(0|2)}^{0} + 1 \cdot P_{Y|X}(1|2)$$

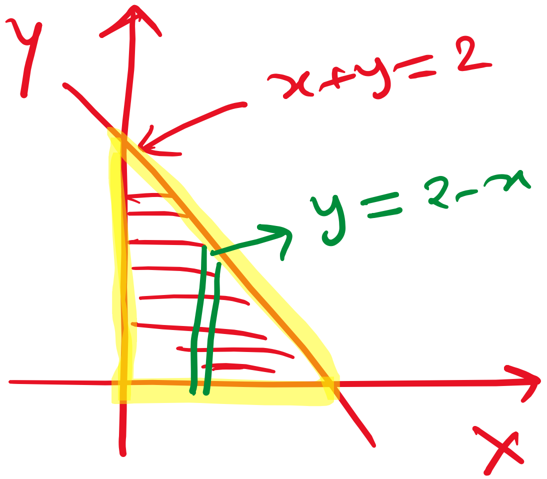
$$= P_{Y|X}(1|2)$$

$$= \frac{P_{X,Y}(2,1)}{P_X(2)} = \frac{0.01}{0.16}$$

$$= 1/16$$

Example 4: Suppose X and Y are jointly uniform RVs with the joint PDF given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2}; & 0 \le x, 0 \le y, x+y \le 2 \\ 0; & o.w. \end{cases}$$




Find $E[Y|X=x]$?

$$E[Y|X=x] = \int_{-\infty}^{+\infty} y \, f_{Y|X}(y|x) \, dy$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x,y) \, dy = \int_{0}^{2-x} \frac{1}{2} \, dy = \frac{1}{2}(2-x)$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{1/2}{\frac{1}{2}(2-x)} = \frac{1}{2-x}$$

$$E[Y|X] = \int_{-\infty}^{+\infty} y \cdot \frac{1}{2-x}\, dy = \int_0^{2-x} \frac{y}{2-x}\, dy$$

$$= \frac{1}{2-x} \left\{ \frac{y^2}{2} \right\}_0^{2-x} = \frac{2-x}{2}$$

$$\boxed{E[Y|X] = g(x) = \frac{2-x}{2}}$$

**Total Expectation Theorem:** $\mathrm{EE}[Y|X] = E[Y]$ and $EE[X|Y] = \mathrm{E}[X]$

$$f_{X,Y}(x,y) = f_X(x) \cdot f_{Y|X}(y|x)$$

$$E[Y|X] = g(X)$$

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x)\, f_X(x)\, dx = \int_{-\infty}^{+\infty} E[Y|X=x]\, f_X(x)\, dx$$

$$= \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} y\, f_{Y|X}(y|x)\, dy \right\} f_X(x)\, dx$$

$$= \int_{-\infty}^{+\infty} y \int_{-\infty}^{+\infty} f_X(x) \cdot f_{Y|X}(y|x)\, dy\, dx$$

$$= \int_{-\infty}^{+\infty} y \int_{-\infty}^{+\infty} f_{X,Y}(x,y)\, dx\, dy = \int_{-\infty}^{+\infty} y\, f_Y(y)\, dy = E[Y]$$