MA203: Random Vectors

1. IID RVs
2. Correlation Matrix
3. Covariance Matrix
4. Multiple Jointly Gaussian RVs

$X_1, X_2, \ldots, X_n$

$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ \vdots \\ X_n \end{bmatrix}$$

$\underline{n\text{:- dimensional}}$ Random Vector

$\cancel{f_{X_1}}$ $\quad f_{X_1, X_2}(x_1, x_2)$

$$\underbrace{f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n)} = \underbrace{f_{\underline{X}}(\underline{x})}$$

$\underline{Ex:-}$

1. $$\underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}}_{n} f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n \underline{= 1}$$

$$f_{X_1}(x_1) = \underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}}_{n-1} f_{X_1}(x_1, x_2, \ldots, x_n)\, dx_2\, dx_3 \ldots dx_n$$

Independent RVs: The RVs are called (mutually) independent if and only if

$$f_{X_1,X_2,\cdots,X_n}(x_1,x_2,\cdots,x_n) = f_{X_1}(x_1)f_{X_2}(x_2)\cdots f_{X_n}(x_n) \Rightarrow$$

$$= \prod_{i=1}^{n} f_{X_i}(x_i)$$

$$\boxed{\prod_{i=1}^{2} x_i = x_1 x_2}$$

For example, if $X_1, X_2, \cdots, X_n$ are independent Gaussian RVs, then

$$f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{1}{2}\frac{(x_i-\mu_i)^2}{\sigma_i^2}}$$

$$f_{X_1,X_2,\cdots,X_n}(x_1,x_2,\cdots,x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}}\, e^{-\frac{(x_i-\mu_i)^2}{2\sigma_i^2}}$$

where $\mu_i = E[X_i]$ and $\sigma_i^2 = E[(X_i-\mu_i)^2]$.

Identically Distributed RVs: The RVs $X_1, X_2, \cdots, X_n$ are called identically distributed if each RV has the same marginal distribution function, that is,

$$F_{X_1}(x_1) = F_{X_2}(x_2) = \cdots = F_{X_n}(x_n).$$

Ex:-

$$\begin{array}{cccccc} & X_1 & & X_2 & \cdots & X_n \\ H & T & H & T & H & T \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & 1 & 0 & 1 & 0 & 1 \end{array}$$

$p_{X_1}(0) = 1/2 \quad p_{X_1}(1) = 1/2$

Independent and Identically Distributed (IID) RVs:

The RVs $X_1, X_2, \cdots, X_n$ are called iid if $X_1, X_2, \cdots, X_n$ are mutually independent and each of $X_1, X_2, \cdots, X_n$ has the same marginal distribution function.

Independent

Identically distributed $\rightarrow$ iid

$$p_{X_1, X_2, \ldots, X_n}(1, 1, \ldots, 1) = \left(\tfrac{1}{2}\right) \times \left(\tfrac{1}{2}\right) \cdots \left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)^n$$

Mean Vector: The mean vector of X, denoted by μ_X, is defined as

$$\mu_X = E[\boldsymbol{X}] = \mathrm{E}[X_1 X_2 \;\ldots\; X_n]^t$$

$$= \left[[E[X_1]\; E[X_2] \;\ldots E[X_n]]\right]^t$$

$$= [\mu_{X_1}\; \mu_{X_2} \;\ldots\; \mu_{X_n}]^t$$

$$X = \begin{bmatrix} \boxed{X_1} \\ \boxed{X_2} \\ \vdots \\ X_n \end{bmatrix}$$

$$X^t = [\boxed{X_1} \; \boxed{X_2} \; \cdots \; X_n]$$

$$EX = [$$

Correlation Matrix: The correlation matrix of a random vector $X = [X_1, X_2, \cdots, X_n]$ is defined as

$$R_{XX} = EXX^t$$

$$= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}_{1\times n} = E\begin{bmatrix} x_1^2 & x_1x_2 & \cdots & x_1x_n \\ x_2x_1 & x_2^2 & \cdots & x_2x_n \\ \vdots & & & \\ x_nx_1 & x_nx_2 & \cdots & x_n^2 \end{bmatrix}$$

$$\boxed{E[x_1x_2] = E[x_2x_1]}$$

$$= \begin{bmatrix} Ex_1^2 & Ex_1x_2 & \cdots & Ex_1x_n \\ Ex_2x_1 & Ex_2^2 & \cdots & Ex_2x_n \\ \vdots & & & \\ Ex_nx_1 & Ex_nx_2 & \cdots & Ex_n^2 \end{bmatrix}$$

Ex:- $X = [x_1 \ x_2]$

$$R_{xx} = \begin{bmatrix} E x_1^2 & E x_1 x_2 \\ E x_2 x_1 & E x_2^2 \end{bmatrix}$$

Covariance Matrix: The covariance matrix of a random vector $X = [X_1, X_2, \cdots, X_n]$ is defined as

$$C_X = E\underline{(X - \mu_X)}\underline{(X - \mu_X)}^t$$

μ_X : mean vector of Random Vector X

$$= E \begin{bmatrix} X_1 - \mu_{X_1} \\ X_2 - \mu_{X_2} \\ \vdots \\ X_n - \mu_{X_n} \end{bmatrix} \begin{bmatrix} X_1 - \mu_{X_1} & X_1 - \mu_{X_2} & \cdots & X_n - \mu_{X_n} \end{bmatrix}$$

$$= E \begin{bmatrix} (X_1 - \mu_{X_1})^2 & (X_1 - \mu_{X_1})(X_2 - \mu_{X_2}) & \cdots & (X_1 - \mu_{X_1})(X_n - \mu_{X_n}) \\ (X_2 - \mu_{X_2})(X_1 - \mu_{X_1}) & (X_2 - \mu_{X_2})^2 & \cdots & (X_2 - \mu_{X_2})(X_n - \mu_{X_n}) \\ \vdots & \vdots & \ddots & \vdots \\ (X_n - \mu_{X_n})(X_1 - \mu_{X_1}) & (X_n - \mu_{X_n})(X_2 - \mu_{X_2}) & \cdots & (X_n - \mu_{X_n})^2 \end{bmatrix}$$

$$
= \begin{bmatrix} E(x_1-\mu_{x_1})^2 & E(x_1-\mu_{x_1})(x_2-\mu_{x_2}) & \cdots & E(x_1-\mu_{x_1})(x_n-\mu_{x_n}) \\ E(x_2-\mu_{x_2})(x_1-\mu_{x_1}) & E(x_2-\mu_{x_2})^2 & \cdots & E[x_2-\mu_{x_2})(x_n-\mu_{x_n}) \\ \vdots & & & \\ E[x_n-\mu_{x_n})(x_1-\mu_{x_1}) & E(x_n-\mu_{x_n})(x_2-\mu_{x_2}) & & E[x_n-\mu_{x_n})^2 \end{bmatrix}
$$

$$
= \begin{bmatrix} Var(x_1) & Cov(x_1, x_2) & \cdots & Cov(x_1, x_n) \\ Cov(x_2, x_1) & Var(x_2) & \cdots & Cov(x_2, x_n) \\ \vdots & & & \\ Cov(x_n, x_1) & Cov(x_n, x_2) & \cdots & Var(x_n) \end{bmatrix}
$$

Ex: $X = [x_1 \quad x_2]$

$$C_x = \begin{bmatrix} Var(x_1) & Cov(x_1, x_2) \\ Cov(x_2, x_1) & Var(x_2) \end{bmatrix}$$

$=$

Properties of Covariance Matrix:

1. C_X is a symmetric matrix because $Cov(X_i, X_j) = Cov(X_j, X_i)$.

Uncorrelated RVs: n RVs $X_1, X_2, \cdots, X_n$ are called uncorrelated if for each (i, j), $i = 1, 2, \cdots, n$ and j $= 1, 2, \cdots, n$, $i \neq j$,

$$Cov(X_i, X_j) = 0.$$

$Cov(X_1, X_2) = 0$

$Cov(X_3, X_4) \neq 0$

If $X_1, X_2, \cdots, X_n$ are uncorrelated, C_X will be a diagonal matrix.

$X_1, X_2, \cdots, X_n$

$$C_X = \begin{bmatrix} Var(X_1) & 0 & \cdots & 0 \\ 0 & Var(X_2) & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & Var(X_n) \end{bmatrix}$$

Example 1: Let Z = $[X\ Y]^t$ be a random vector with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{x}; 0 < x < 1; 0 < y < x \\ 0; O.w. \end{cases}$$

Find Correlation matrix R_{ZZ} and Covariance Matrix C_Z.

Note that X and Y are random Variables

Sol:- (i) $E[X]$ (ii) $E[Y]$ (iii) $E[XY]$

(iv) $E[X^2]$ (v) $E[Y^2]$

$$R_{ZZ} = \begin{bmatrix} EX^2 & EXY \\ EYX & EY^2 \end{bmatrix}$$

$E[X] = ?$

$$E[X^m Y^n] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^m y^n f_{X,Y}(x,y)\,dx\,dy$$

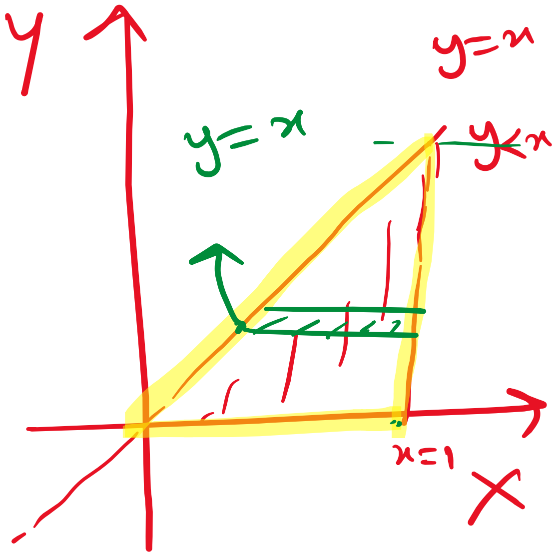
$$C_Z = \begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(Y,X) & Var(Y) \end{bmatrix}$$

$E[X] \Rightarrow m=1, n=0$; $E[Y] \Rightarrow m=0, n=1$

$E[X^2] \Rightarrow m=2, n=0$; $E[Y^2] \Rightarrow m=0, n=2$

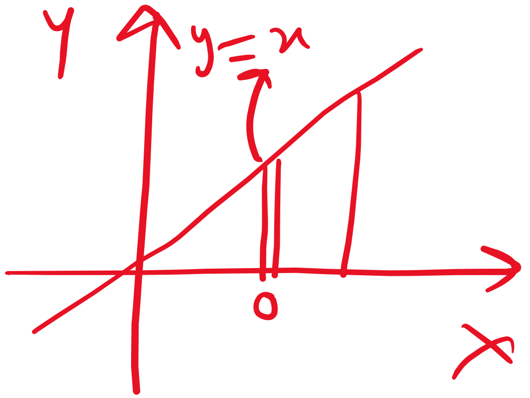
$$E[X] = \int_0^1 \int_y^1 x \, f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_0^1 \int_y^1 x \times \frac{1}{x}\, dx\, dy = \int_0^1 (1-y)\, dy = 1/2$$



$$E[Y] = \int_0^1 \int_0^x y \frac{1}{x}\, dy\, dx$$

$$= \int_0^1 \frac{1}{x} \cdot \left\{ \frac{y^2}{2} \right\}\Big|_0^x dx = \int_0^1 \frac{x}{2}\, dx = 1/4$$



$$E[XY] = \int_0^1 \int_0^x xy \frac{1}{x} \, dy \, dx = 1/6$$

$$E[X^2] = \int_0^1 \int_y^1 x^2 \frac{1}{x} \, dx \, dy = 1/3$$

$$E[Y^2] = \int_0^1 \int_0^x y^2 \frac{1}{x} \, dy \, dx = 1/9$$

$$R_2 = \begin{bmatrix} 1/3 & 1/6 \\ 1/9 & 1/6 \end{bmatrix}$$

$$Var(x) = E[x^2] - \mu_x \mu_x$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{3} - \frac{1}{4} = 1/12$$

$$C_z = \begin{bmatrix} Var(x) & Cov(x,y) \\ Cov(y,x) & Var(y) \end{bmatrix}$$

$$Var(y) = E[y^2] - \mu_y^2 = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}$$

$$Cov(x,y) = E[xy] - \mu_x \mu_y$$

$$= \frac{1}{6} - \frac{1}{2} \times \frac{1}{4} = \frac{1}{6} - \frac{1}{8} = 1/24$$

$$C_z = \begin{bmatrix} 1/12 & 1/24 \\ 1/24 & 7/144 \end{bmatrix}$$

Multiple Jointly Gaussian RVs: For any positive integer n, $X_1, X_2, \cdots, X_n$ represent n jointly RVs. These n RVs define a random vector $X = [X_1, X_2, \cdots, X_n]^t$.

These RVs are called jointly Gaussian if the RVs $X_1, X_2, \cdots, X_n$ have joint PDF function given by

$$\boxed{f_{X_1, X_2, \cdots, X_n}(x_1, x_2, \cdots, x_n) = \frac{e^{-\frac{1}{2}(X-\mu_X)^t C_X^{-1}(X-\mu_X)}}{(\sqrt{2\pi})^n \sqrt{det(C_X)}}}$$

where, $C_X = E(X-\mu_X)(X-\mu_X)^t$ is the covariance matrix and $\mu_X = \left[\mu_{X_1}\ \mu_{X_2}\ \ldots\ \mu_{X_n}\right]^t$ is the mean vector of X.

Property-1: If $X_1, X_2, \cdots, X_n$ are jointly Gaussian, then the marginal PDF of each of $X_1, X_2, \cdots, X_n$ is Gaussian.

$(X-\mu_X)^t \quad C_X^{-1} \quad (X-\mu_X) \quad det(C_X)$

1 ✓ 2 ✓ 3 ✓ 4 ✓

Property-2: If the jointly Gaussian RVs $X_1, X_2, \cdots, X_n$ are uncorrelated, then $X_1, X_2, \cdots, X_n$ are independent.

Jointly Gaussian

Uncorrelated

$\rightarrow$ Independent