MA 203

Stochastic Process

Example 1: Suppose someone gives you a coin and claims that this coin is biased; that it lands on heads only 48% of the time. You decide to test the coin for yourself. If you want to be 95% confident that this coin is indeed biased, how many times must you toss the coin? ($\in = 0.02$.)

Sol:-

$$P\left\{\left|\frac{S_n}{n}-\mu\right|<\varepsilon\right\} > 1-\frac{\sigma}{n\varepsilon^2}$$

$\mu = p = 0.48$

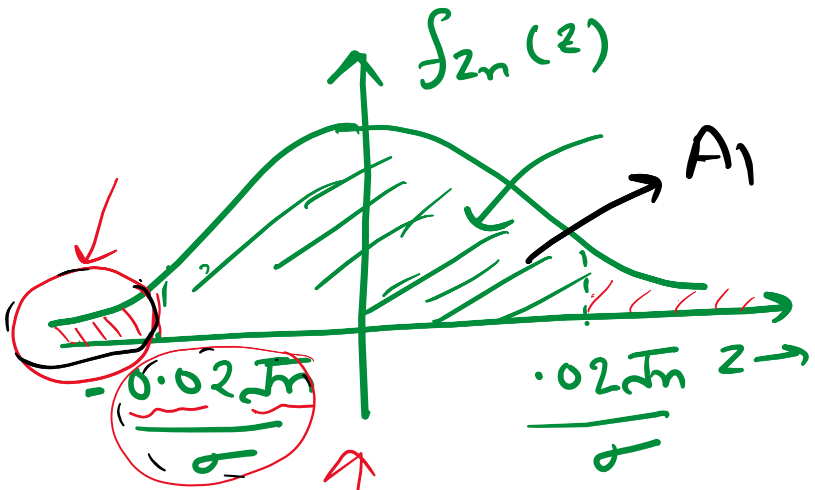
$\sigma^2 = p(1-p) = .48 \times .52$

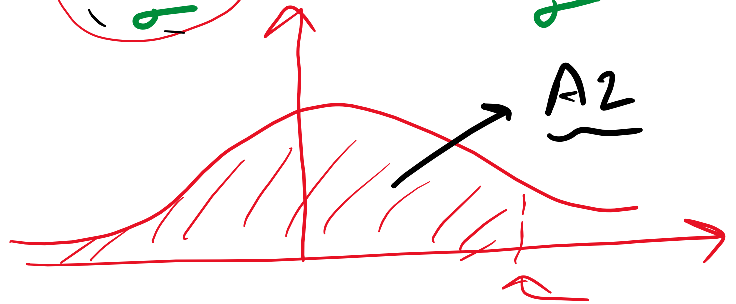
$$P\left\{\left|\frac{S_n - n\mu}{n}\right| < 0.02\right\} > 0.95$$

$$P\left\{\left|\frac{S_n - n\mu}{\sqrt{n}\,\sigma}\right| < \frac{0.02\sqrt{n}}{\sigma}\right\} > 0.95$$

$$P\left\{|Z_n| < \frac{0.02\sqrt{n}}{\sigma}\right\} > 0.95$$

$$\simeq P\left\{Z_n < \frac{0.02\sqrt{n}}{\sigma}\right\} > 0.95$$





$1-2Q\left(\frac{0.02\sqrt{n}}{\sigma}\right) \geq 0.95$

$\Rightarrow$

$n =$

$\Phi\left(\frac{0.02\sqrt{n}}{\sigma}\right) \geq 0.95$

$$\boxed{P(Z_n \leq z) = \Phi(z) = \int_{-\infty}^{z} e^{-t^2/2}\,dt}$$

$\frac{0.02\sqrt{n}}{\sigma} \geq 1.645$

$\sqrt{n} \geq \frac{1.645 \times \sigma}{0.02}$

$\sigma^2 = p(1-p)$

$= .48 \times .52$

$=$

$\Phi^{-1}(0.95) = 1.645$

$n \geq \left\{\frac{1.645 \times \sigma}{0.02}\right\}^2$

$\boxed{n = 1692} \Rightarrow$ CLT

WLLN $\Rightarrow \boxed{n = 12480}$

$$\boxed{\Phi(x) = 1 - Q(x)}$$

Stochastic Process

A Review of Last Class

1. A stochastic process is defined as a function of two arguments $X(s,t), s \in S$ and $t \in T$.

2. A stochastic process maps each sample point to a waveform.

3. If we sample a stochastic process $X(s,t)$ at time instants $t_1, t_2, \cdots, t_n$, we get RVs $X(t_1), X(t_2), \cdots, X(t_n)$, respectively.

$\Downarrow$

4. We can define stochastic process as a time varying random variable.

5. Stochastic Process/ Random Process/ Chance process

6. Notation: $X(t) \equiv X(s,t)$

Parameter Space: The set T is called parameter space where $t \in T$ may denote time, length, or any other quantity.

Continuous-Time Random Process: If the index set T is continuous, $\{X(t), t \geq 0\}$ is called a continuous time process.

Example: Suppose $X(t) = Acos(\omega_0 t + \Phi)$, where A and ω_0 are constant and Φ is uniformly distributed between 0 and 2π.

$X(t)$; $t \geq 0$

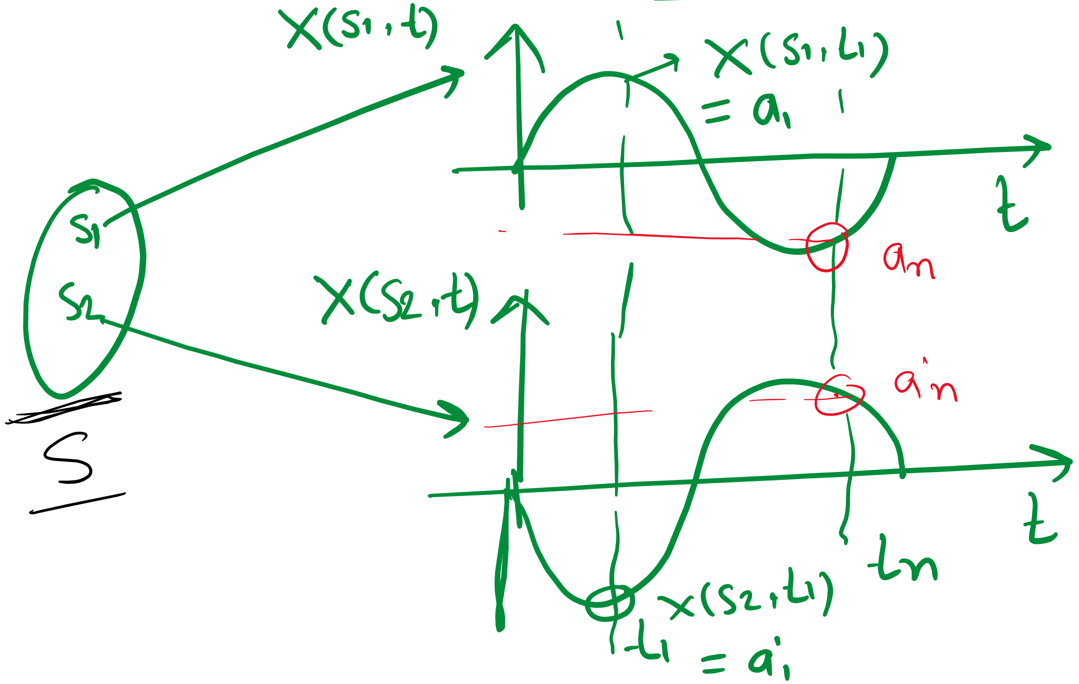
Discrete Time Random Process: If the index set T is a countable set, $\{X(t), t \geq 0\}$ is called a discrete-time process. Such a random process can be represented as $\{X[n], n \in Z\}$ and called a random sequence.

Example: Suppose $X[n] = Acos(\omega_0 n + \Phi)$, where A and ω_0 are constant and Φ is uniformly distributed between 0 and 2π.

$X[n]$; $n = -2, -1, 0, 1, 2, 3, \ldots$

State Space: The set S is the set of all possible values of $X(t)$ for all t and is called the state space where $X(t): S \to A_t$, $A_t \subseteq \mathbb{R}$, $S = \cup A_t$.

The state is the value taken by $X(t)$ at a time t, and set of all such states is called the state space.



$A_{t_1} = \{a_1, a_1'\}$

$\vdots$

$A_{t_n} = \{a_n, a_n'\}$

$\phi = \bigcup_{t_i \in n} A_{t_i}$

Discrete-State Process: A random process is discrete-state if state-space is finite or countable.

if ϕ contains countable | countably infinite elements

Continuous-State Process: A random process is continuous-state if state-space is uncountable.

if ϕ contains uncountable element

Probabilistic Structure of a Random Process:

$X(t)$, t_1 t_2 t_3 . . . . t_n

$X(t_1)$ $X(t_2)$ $X(t_3)$ . . . . $X(t_n)$

Mean of a RP:

t

Mean of the RP $X(t)$ at time t = $\mu_X(t) = E[X(t)]$ t_1, t_2

Auto-correlation Function: Auto-correlation function of the RP $X(t)$ at times t_1 and t_2 is defined as

t_1 t_2 Self

$$R_X(t_1, t_2) = E[X(t_1).X(t_2)]$$

$X(t_1)$ $X(t_2)$

$t_1 = t_2$

$$R_X(t_1, t_1) = E[X^2(t_1)]$$

Auto-Covariance Function: The autocovariance function of RP $X(t)$ at time instants t_1 and t_2 is defined by

$$C_X(t_1, t_2) = E\left[\left(X(t_1) - \mu_X(t_1)\right)\left(X(t_2) - \mu_X(t_2)\right)\right]$$

$t_1 = t_2$

$$C_X(t_1, t_1) = E\left[\left(X(t_1) - \mu_X(t_1)\right)^2\right] = Var(X(t_1))$$

Correlation Coefficient: The correlation-coefficient of RP $X(t)$ at time instants t_1 and t_2 is defined by

$$\rho_X(t_1, t_2) = \frac{\boxed{C_X(t_1, t_2)}}{\sqrt{\underbrace{C_X(t_1, t_1)}_{Var(X(t_1))}\underbrace{C_X(t_1, t_1)}_{Var(X(t_2))}}} =$$

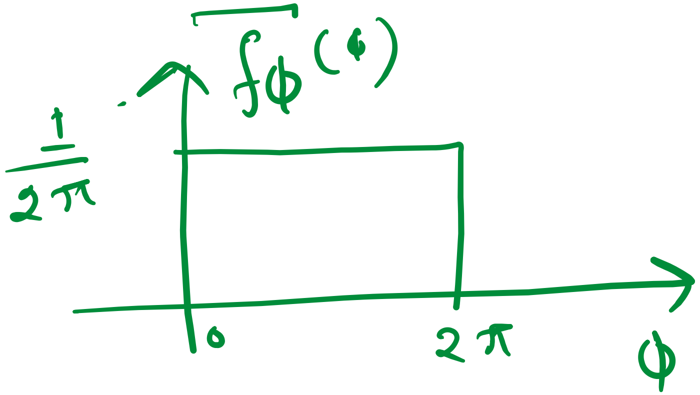
Example 1: $X(t) = A\cos(\omega_0 t + \Phi)$ where A and ω_0 are constants and Φ is uniformly distributed between 0 and 2π. Find the mean and autocorrelation of $X(t)$.

Sol:-

$X(t) \longrightarrow$

$$X(t_1) = A\cos(\omega_0 t_1 + \phi)$$

$$X(t_2) = A\cos(\omega_0 t_2 + \phi)$$



$$X(t) = A\cos(\omega_0 t + \phi)$$

$\downarrow$

$t = 1$ sec.

$$X(1) = A\cos(\omega_0 \cdot 1 + \phi) = g(\phi)$$ where ϕ is a random variable

$$\boxed{Y = g(X)}$$

$$\boxed{E[Y] = \int_{-\infty}^{+\infty} g(x) \cdot f_X(x)\, dx}$$

$$E[x(t)] = \int_{-\infty}^{+\infty} \underbrace{A\cos(\omega_0 t + \phi)}_{g(\phi)} \cdot f_\phi(\phi) \cdot d\phi$$

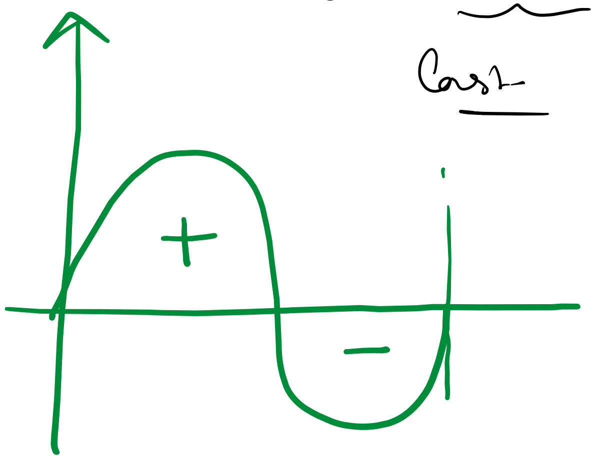
$$= \int_0^{2\pi} A\cos(\omega_0 t + \phi)\,\frac{1}{2\pi}\,d\phi$$

$$= \frac{A}{2\pi}\int_0^{2\pi} \cos(\omega_0 t + \phi)\,d\phi$$

$$\cos(\phi + \theta_0)$$

$$\theta_0 = \omega_0 t$$

Const



$$= \frac{A}{2\pi} \cdot 0 = 0$$

$$R_X(t_1, t_2) = E\left[X(t_1)\, X(t_2)\right]$$

$$= E\left[A \cos(\omega_0 t_1 + \phi) \cdot A \cos(\omega_0 t_2 + \phi)\right]$$

$$= \frac{A^2}{2} E\left[2 \cos(\omega_0 t_1 + \phi) \cos(\omega_0 t_2 + \phi)\right]$$

$$\left\{ 2 \cos A \cos B = \cos(A-B) + \cos(A+B) \right\}$$

$$R_X(t_1, t_2) = \frac{A^2}{2} E\left[\cos(\omega_0 (t_1 - t_2)) + \cos\{\omega_0 (t_1 + t_2) + 2\phi\}\right]$$

$$= \frac{A^2}{2} \cos \omega_0 (t_1 - t_2) + \frac{A^2}{2} E\left[\cos\left(\omega_0 (t_1 + t_2) + 2\phi\right)\right]$$

$$= \frac{A^2}{2} \cos \omega_0 (t_1 - t_2) + \underline{B}$$

$$B = \frac{A^2}{2}\int_0^{2\pi} \cos\left(\omega_0(t_1+t_2)+2\phi\right)\cdot \underbrace{\frac{1}{2\pi}}_{f_\phi(\phi)}\, d\phi$$

$$= \frac{A^2}{4\pi}\underbrace{\int_0^{2\pi} \cos\left(\omega_0(t_1+t_2)+2\phi\right) d\phi}_{0}$$

$$\boxed{R_x(t_1,t_2) = \frac{A^2}{2}\cos\omega_0(t_1-t_2)}$$