<u>**MA 203**</u>

1. Central Limit Theorem
2. Stochastic Process

Central Limit Theorem (CLT)

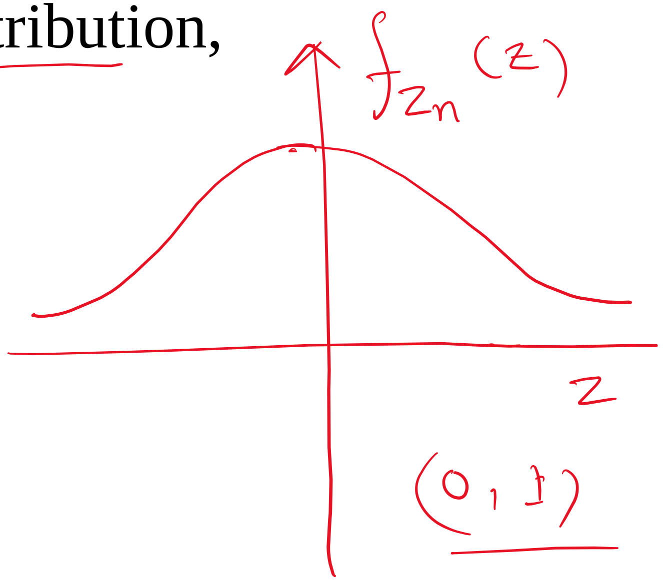
Theorem: Let $X_1, X_2, \cdots, X_n, \cdots$ be a sequence of IID RVs defined on the probability space (S, F, P). Assume that $E[X_i] = \mu$ and $var(X_i) = \sigma^2$.

Define,

$$Z_n = \frac{\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} E[X_i]}{\sqrt{var\left(\sum_{i=1}^{n} X_i\right)}}; \; n = 1, 2, \cdots$$

Then, for larger n, Z_n (approximately) follow the standard normal distribution,

$$P(Z_n \le z) \approx \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$



Proof: Assume that MGF of $X_i's$ exist

$$M_{Z_n}(t) = E\left[e^{\left(\frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma}\right)t}\right]$$

$$= e^{-\frac{\sqrt{n}\mu t}{\cancel{\sqrt{n}}\sigma}} E\left[e^{\left(\frac{\sum_{i=1}^{n} X_i}{\sqrt{n}\sigma}\right)t}\right]$$

$$= e^{-\frac{\sqrt{n}\mu t}{\cancel{\sqrt{n}}\sigma}} \left\{E\left[e^{\left(\frac{X_1}{\sqrt{n}\sigma}\right)t}\right]\right\}^n$$

We know that

$$M_X(t) = 1 + E[X]t + \frac{E[X^2]}{2} + \cdots$$

$$\ln M_X(t) = \ln\left\{1 + \left(E[X]t + \frac{E[X^2]}{2} + \cdots\right)\right\}$$

$$M_{Z_n}(t) = E\left[e^{Z_n t}\right]$$

$$Y_1$$

$$= e^{-\frac{\sqrt{n}\mu t}{\sigma}} \left\{E\left[e^{\frac{X_1 t}{\sqrt{n}\sigma}}\right] \cdots E\left[e^{\frac{X_n t}{\sqrt{n}\sigma}}\right]\right\}$$

$$= \boxed{M_{Y_1}(t) \cdot M_{Y_2}(t) \cdots M_{Y_n}(t)}$$

$$= \{M_{Y_1}(t)\}^n$$

$$= \left\{E\left[e^{\frac{X_1}{\sqrt{n}\sigma}t}\right]\right\}^n$$

$$\left\{e^x = 1 + x + \frac{x^2}{2!} + \cdots\right\}$$

$$\ln M_{Z_n}(t) = \frac{-\sqrt{n}\,\mu t}{\sigma} + n \ln\left(1 + \frac{\mu t}{\sqrt{n}\,\sigma} + \frac{(\sigma^2+\mu^2)t^2}{2\,n\sigma^2} + \cdots\right)$$

$\left\{ E(x^2) = \sigma^2 + \mu^2 \right\}$

$$= \frac{-\sqrt{n}\,\mu t}{\sigma} + n\left\{\left(\frac{\mu t}{\sqrt{n}\,\sigma} + \frac{(\sigma^2+\mu^2)t^2}{2\,n\sigma^2} + \cdots\right) - \frac{1}{2}\left(\frac{\mu^2 t^2}{n\sigma^2} + \cdots\right) + \frac{1}{3}(\cdots) - \cdots\right\}$$

$x \quad x^2 \quad x^3$

$$= \frac{-\sqrt{n}\,\cancel{\mu t}}{\cancel{\sigma}} + \left\{\left(\frac{\sqrt{n}\,\cancel{\mu t}}{\cancel{\sigma}} + \frac{\cancel{\sigma^2} t^2}{2\cancel{\sigma^2}} + \cancel{\frac{\mu^2 t^2}{2\sigma^2}} + \cdots\right) + \left(-\cancel{\frac{1}{2}\frac{\mu^2 t^2}{\sigma^2}} + \cdots\right)\right\}$$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

as $n \to \infty$

$$\lim_{n\to\infty} \ln M_{Z_n}(t) = \frac{t^2}{2}$$

$$M_{Z_n}(t) = e^{\frac{t^2}{2}} \Rightarrow \text{Moment Generating function of } N(0,1)$$

$$\boxed{Z_n \sim N(0,1)}$$

$\{Z_n\} \xrightarrow{d} N(0,1)$ as $n \to \infty$

Example 1: Suppose someone gives you a coin and claims that this coin is biased; that it lands on heads only 48% of the time. You decide to test the coin for yourself. If you want to be 95% confident that this coin is indeed biased, how many times must you toss the coin? ($\in = 0.02$.)

Stochastic Process

Definition: A stochastic process is defined as a function of two arguments $X(s,t), s \in S$ and $t \in T$.

Sampling of Random Process:

Parameter Space: The set T is called parameter space where $t \in T$ may denote time, length, or any other quantity.

Continuous-Time Random Process: If the index set T is continuous, $\{X(t), t \geq 0\}$ is called a continuous time process.

Example: Suppose $X(t) = Acos(\omega_0 t + \Phi)$, where A and ω_0 are constant and Φ is uniformly distributed between 0 and 2π.

Discrete Time Random Process: If the index set T is a countable set, $\{X(t), t \geq 0\}$ is called a discrete-time process. Such a random process can be represented as $\{X[n], n \in Z\}$ and called a random sequence.

Example: Suppose $X[n] = Acos(\omega_0 n + \Phi)$, where A and ω_0 are constant and Φ is uniformly distributed between 0 and 2π.

State Space: The set S is the set of all possible values of $X(t)$ for all t and is called the state space where $X(t): S \to A_t$, $A_t \subseteq \mathbb{R}$, $S = \cup A_t$.

Discrete-State Process: A random process is discrete-state if state-space is finite or countable.

Continuous-State Process: A random process is continuous-state if state-space is uncountable.

Probabilistic Structure of a Random Process:

Mean of a RP: $E[X(t)]$