MA203: Joint PDF of Functions of Two Random Variable

Suppose $\underset{\sim}{X}$ & $\underset{\sim}{Y}$

$\underset{\sim}{Z_1} = \underbrace{g_1(X,Y)}$ $\underset{\sim}{Z_2} = g_2(X,Y)$

$f_{Z_1,Z_2}(z_1, z_2) = ?$

$g_1(x,y)$ & $g_2(x,y)$

$\max(x,y)$

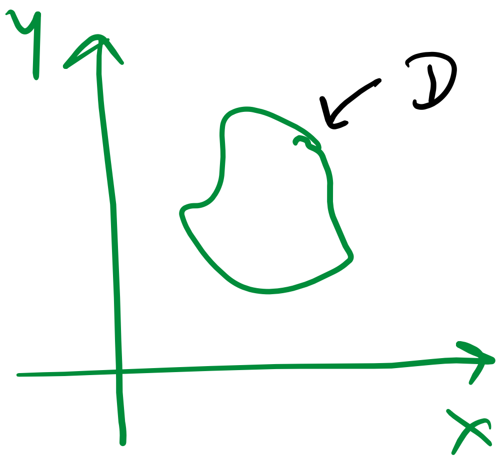
$\frac{x}{y}$

$x+y \,/\, x-y$

Method 1:

Step 1: $F_{Z_1,Z_2}(z_1,z_2) = P(Z_1 \leq z_1, Z_2 \leq z_2) = P[g_1(X,Y) \leq z_1,\ g_2(X,Y) \leq z_2]$

$$= \iint_{(x,y)\in D} f_{X,Y}(x,y)\,dx\,dy$$



Step 2: $f_{Z_1,Z_2}(z_1,z_2) = \dfrac{\partial^2}{\partial z_1 \partial z_2} F_{Z_1,Z_2}(z_1,z_2)$

$$= \frac{\partial^2}{\partial z_1 \partial z_2} \iint_{(x,y)\in D} f_{X,Y}(x,y)\,dx\,dy$$

Solve using Leibnitz Rule

Method 2:

Let X and Y be jointly continuous RVs with joint probability density function (PDF) $f_{X,Y}(x,y)$. The joint PDF of $Z_1 = g_1(X,Y)$ and $Z_2 = g_2(X,Y)$ is defined as

$$f_{Z_1,Z_2}(z_1,z_2) = \sum_{i=1}^{n} \frac{f_{X,Y}(x_i,y_i)}{|J(x_i,y_i)|} = \frac{f_{X,Y}(x_1,y_1)}{|J(x_1,y_1)|} + \frac{f_{X,Y}(x_2,y_2)}{|J(x_2,y_2)|} + \ldots + \frac{f_{X,Y}(x_n,y_n)}{|J(x_n,y_n)|}.$$

Where

for $i = 1, 2, 3, \ldots n$, (x_i, y_i) are n roots of $z_1 = g_1(x,y)$ and $z_1 = g_2(x,y)$,

①   ②

and

$$J(x,y) = \begin{vmatrix} \frac{\partial g_1(x,y)}{\partial x} & \frac{\partial g_1(x,y)}{\partial y} \\ \frac{\partial g_2(x,y)}{\partial x} & \frac{\partial g_2(x,y)}{\partial y} \end{vmatrix}.$$

: Jacobian Matrix

$Y = g(x)$

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{\left| \frac{dg(x)}{dx} \right|_{x=x_i}}$$

where $x_i, i = 1, 2, \ldots, n$, are roots of $y = g(x)$ Eq.

Example 1: Let X and Y be jointly continuous RVs with PDF $f_{X,Y}(x,y)$. Let $Z_1 = X + Y$ and $Z_2 = X - Y$. Find the joint PDF of Z_1 and Z_2?

$z_1, z_2, x, y \in (-\infty, \infty)$

Sol.: Given $Z_1 = g_1(X,Y) = X + Y$ and $Z_2 = g_2(X, Y) = X - Y$

$z_1 = x + y$ — (1)

$z_2 = x - y$ — (2)

Step 1: Find the roots of Equation (1) and Equation (2), i.e., (x_i, y_i) for $i = 1, 2, 3, \ldots, n$.

(1) + (2)   (1) − (2)

$x_1 = \frac{z_1+z_2}{2}$ and $y_1 = \frac{z_1-z_2}{2}$

Step 2: Find Jacobian matrix $J(x,y)$,

$$J(x,y) = \begin{vmatrix} \frac{\partial g_1(x,y)}{\partial x} & \frac{\partial g_1(x,y)}{\partial y} \\ \frac{\partial g_2(x,y)}{\partial x} & \frac{\partial g_2(x,y)}{\partial y} \end{vmatrix}$$

$\frac{\partial g_1(x,y)}{\partial x} = \frac{\partial}{\partial x}\{x+y\} = 1$; $\frac{\partial g_1(x,y)}{\partial y} = \frac{\partial}{\partial y}\{x+y\} = 1$; $\frac{\partial g_2(x,y)}{\partial x} = \frac{\partial}{\partial x}\{x-y\} = 1$; $\frac{\partial g_2(x,y)}{\partial y} = \frac{\partial}{\partial y}\{x-y\} = -1$

$J(x,y) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$

Step 3: $f_{Z_1,Z_2}(z_1,z_2) = \frac{f_{X,Y}(x_1,y_1)}{|J(x_1,y_1)|} = \frac{1}{2} f_{X,Y}\left(\frac{z_1+z_2}{2}, \frac{z_1-z_2}{2}\right)$

Example 2: Suppose X and Y are two independent Gaussian RVs each with mean 0 and variance σ^2. Given $R = \sqrt{X^2 + Y^2}$ and $\theta = tan^{-1}\left(\frac{Y}{X}\right)$. Find $f_{R,\theta}(r,\theta)$?

Sol.:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}};\ f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}};\ f_{X,Y}(x,y) = f_X(x) f_Y(y);\ f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

$$R = g_1(X,Y) = \sqrt{X^2 + Y^2} \text{ and } \theta = g_2(X,Y) = tan^{-1}\left(\frac{Y}{X}\right)$$

$$r = \sqrt{x^2 + y^2} \quad \text{— } 1$$

$$\theta = tan^{-1}\left(\frac{y}{x}\right) \quad \text{— } 2$$

Step 1: $x = r\cos\theta\ ;\ y = r\sin\theta$

Step 2:

$$J(x,y) = \begin{vmatrix} \frac{\partial g_1(x,y)}{\partial x} & \frac{\partial g_1(x,y)}{\partial y} \\ \frac{\partial g_2(x,y)}{\partial x} & \frac{\partial g_2(x,y)}{\partial y} \end{vmatrix}$$

Where

$$\frac{\partial g_1(x,y)}{\partial x} = \frac{\partial}{\partial x}\left\{\sqrt{x^2+y^2}\right\} = \frac{1}{2\sqrt{x^2+y^2}} \times 2x = \frac{x}{\sqrt{x^2+y^2}}; \quad \frac{\partial g_1(x,y)}{\partial y} = \frac{\partial}{\partial y}\left\{\sqrt{x^2+y^2}\right\} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial g_2(x,y)}{\partial x} = \frac{\partial}{\partial x}\left\{tan^{-1}\frac{y}{x}\right\} = \frac{-y}{x^2+y^2}; \quad \frac{\partial g_2(x,y)}{\partial y} = \frac{\partial}{\partial x}\left\{tan^{-1}\frac{y}{x}\right\} = \frac{x}{x^2+y^2}$$

y

$$J(x,y) = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix} \text{ or } J(rcos\theta, rsin\theta) = \begin{vmatrix} cos\theta & sin\theta \\ \frac{-sin\theta}{r} & \frac{cos\theta}{r} \end{vmatrix} = \frac{cos^2\theta+sin^2\theta}{r} = \frac{1}{r}$$

$J(x_1, y_1)$

Step 3:

$n=1$

$$f_{R,\theta}(r,\theta) = \sum_{i=1}^{n} \frac{f_{X,Y}(x_i,y_i)}{|J(x_i,y_i)|} = \frac{f_{X,Y}(x_1,y_1)}{|J(x_1,y_1)|} = \frac{f_{X,Y}(r\cos\theta, r\sin\theta)}{|J(r\cos\theta, r\sin\theta)|}$$

A

$\frac{1}{r}$

$$f_{x,y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$$

$$f_{X,Y}(r\cos\theta, r\sin\theta) = \frac{1}{2\pi\sigma^2} e^{-\frac{(r^2\cos^2\theta + r^2\sin^2\theta)}{2\sigma^2}} = \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$$

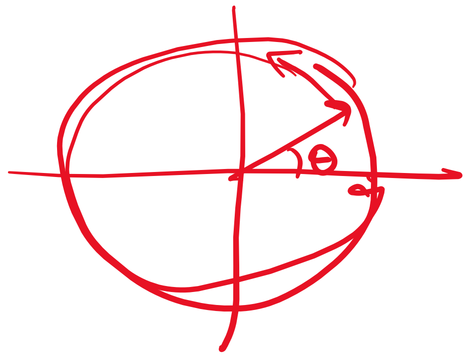
$$f_{R,\theta}(r,\theta) = \frac{f_{X,Y}(r\cos\theta, r\sin\theta)}{|J(r\cos\theta, r\sin\theta)|} = \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \; ; 0 \le \theta \le 2\pi, 0 \le r \le \infty$$

$f_R(r) = ?$

$f_\theta(\theta) = ?$

$$f_R(r) = \int_0^{2\pi} f_{R,\theta}(r,\theta)\, d\theta$$

$$f_\theta(\theta) = \int_0^{\infty} f_{R,\theta}(r,\theta)\, dr$$



$R^2 = x^2 + y^2$

$x^2 + y^2 = r^2$

$$f_R(r) = \int_0^{2\pi} \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} d\theta$$

$$= \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} \int_0^{2\pi} d\theta = \frac{r}{\cancel{2\pi}\sigma^2} e^{-r^2/2\sigma^2} \times \cancel{2\pi}$$

$$= \frac{r}{\sigma^2} e^{-r^2/2\sigma^2} \quad ; 0 \le r < \infty$$

$$f_\Theta(\theta) = \int_0^{\infty} \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2} dr \qquad \text{put } \frac{r^2}{2\sigma^2} = u$$

$$\frac{2r\,dr}{2\sigma^2} = du$$

$$= \int_0^{\infty} \frac{e^{-u} \times \cancel{\sigma^2}}{2\pi\cancel{\sigma^2}} du = \int_0^{\infty} \frac{e^{-u}}{2\pi} du$$

$$= \frac{1}{2\pi}\left\{-e^{-u}\right\}\Big|_0^{\infty} = \frac{1}{2\pi}\left\{-0+1\right\} = \frac{1}{2\pi}$$

Mean / Expectation of function of two RVs

$$Z = g(X, Y) \Rightarrow z = g(x, y)$$

$$\boxed{E[Z] = \int_{-\infty}^{+\infty} z \, f_Z(z) \, dz}$$

$$\boxed{E[Z] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(x, y) \cdot f_{X,Y}(x, y) \, dx \, dy}$$

X

$$E[X] = \int_{-\infty}^{+\infty} x \, f_X(x) \, dx$$

$$Y = g(X)$$

$$E[Y] = \int_{-\infty}^{+\infty} g(x) \cdot f_X(x) \, dx$$

Ex:- The joint Pdf of two RVs X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}xy ; & 0 \leq x \leq 2,\ 0 \leq y \leq 2 \\ 0 ; & o.w. \end{cases}$$

Find the joint expectation of $g(X,Y) = X^2Y$.

Find the $E[Z]$ where $Z = g(X,Y) = X^2Y$.

Sol:-

$$E[Z] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(x,y) \cdot f_{X,Y}(x,y)\, dx\, dy$$

$$E[Z] = \int_0^2 \int_0^2 g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_0^2 \int_0^2 x^2 y \, \frac{xy}{4} \, dx\, dy = \frac{1}{4} \int_0^2 x^3 dx \cdot \int_0^2 y^2 dy$$

$$= \frac{1}{4} \left\{ \frac{x^4}{4} \right\} \Big|_0^2 \left\{ \frac{y^3}{3} \right\} \Big|_0^2$$

$$= \frac{1}{4} \left\{ \frac{16}{4} \right\} \times \left\{ \frac{8}{3} \right\}$$

$$= \underline{8/3}$$

Joint Moment of RVs:- For two continuous RVs X and Y, the joint moment of order $m+n$ is defined as

$$E[X^m Y^n] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^m y^n \, f_{X,Y}(x,y)\, dx\, dy$$

If $m = n = 1$

$E[XY]$: Second order moment.