MA 203

Poisson Random Process

Poisson Random Process:-

Notation: $\{N(t), t \geq 0\}$

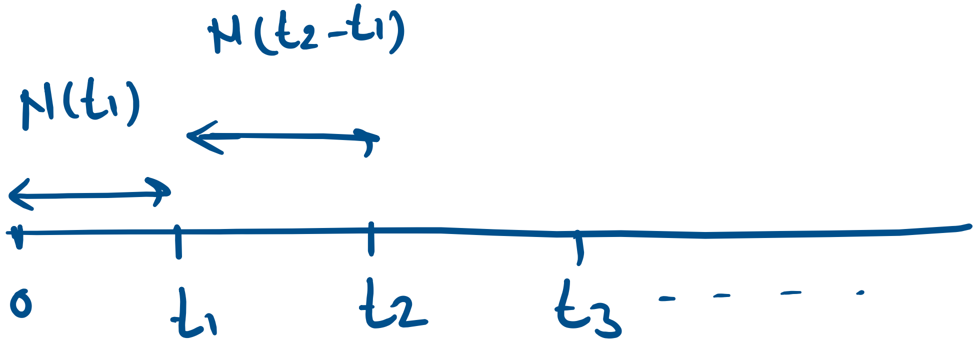
$N(t)$: Number of arrivals in the time interval $(0, t]$.

$$\underbrace{P\{N(t)=k\}}_{\text{Probability of } k \text{ arrival in } (0,t]} = \frac{(\lambda t)^k e^{-\lambda t}}{\lfloor k}$$

where λ is arrival rate.

It is a type of Counting process.

2. It is an independent increment process.



$N(t_1)$: Number of arrivals in $(0, t_1]$

$N(t_2 - t_1)$: # arrivals in $(t_1, t_2]$

$N(t_1)$ & $N(t_2 - t_1)$ are independent RVs.

$N(t_2)$: $(0, t_2]$

$N(t_1)$: $(0, t_1)$

$N(t_2 - t_1) = N(t_2) - N(t_1)$

Example: A petrol pump serves on the average 30 cars per hour. Find the probability that during a period of 5 minutes (i) no car comes to the station, (ii) exactly 3 cars come to the station and (iii) more than 3 cars come to the station.

Sol:-

$\lambda = 30$ cars / hour

$\lambda t_0 = 30$

$t_0 = 1$ hr.

$\lambda = 30/60$ cars / hr.

$= 1/2$ cars / minute

(i) $t = 5$ minutes, $k = 0$

$P\{N(5) = 0\} = e^{-\lambda t} = e^{-1/2 \times 5} = e^{-5/2}$

(ii) $t = 5$ minutes; $k = 3$.

$$P\{M(5) = 3\} = \frac{(\lambda t)^3 e^{-\lambda t}}{\lfloor \underline{k}}$$

$$= \frac{(1/2 \times 5)^3 \, e^{-\frac{1}{3}\times 5}}{\lfloor \underline{3}}$$

(iii) $$P\{N(5) > 3\} = 1 - P\{N(5) = 0\} - P\{N(5) = 1\}$$

$$- P\{N(5) = 2\} - P\{M(5) = 3\}$$

Example 2: Number of customers arriving at a grocery store can be modelled by a Poisson process with intensity $\lambda = 10$ customers per hour.

(i) Find the probability that there are two customers between 10:00 AM and 10:20AM.

(ii) Find the probability that there are 3 customers between 10:00AM and 10:20AM and 7 customers between 10:20AM and 11:00 AM.

Sol:-

$t_0 = 1$ hr.

$\lambda t_0 = 10$ customers

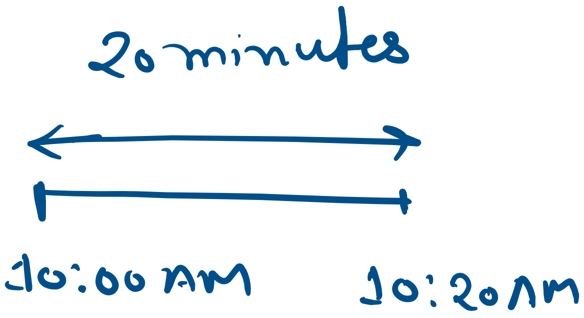
$\lambda = 10/t_0$ customers/hr.

$= 10/60$ customers/minute

$= 1/6$ customers/minute.

(i) $t = 20$ minutes, $\lambda = 1/6$, $k = 2$.

$$P\{N(20) = 2\} = \frac{(\lambda t)^2 e^{-\lambda t}}{2!}$$

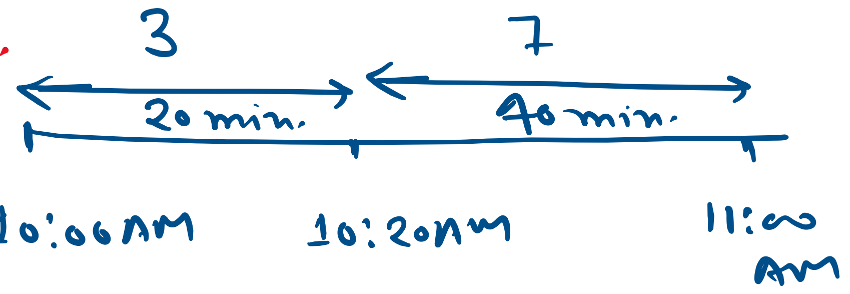


$$= \frac{\left\{\frac{1}{6}\times 20\right\}^2 \left\{e^{-\frac{1}{6}\times 20}\right\}}{2!}$$

$$= \frac{\left\{\frac{10}{3}\right\}^2 e^{-10/3}}{2!} = \frac{\left(\frac{100}{9}\right) e^{-10/3}}{2} \approx \underline{0.2}$$

(ii) $P\{N(20) = 3,\ N(40) = 7\}$

Using independent increment Prop.



$$= P\{N(20) = 3\} \cdot P\{N(40) = 7\}$$

$k = 3,\ t = 20$ $\qquad$ $k = 7,\ t = 40$

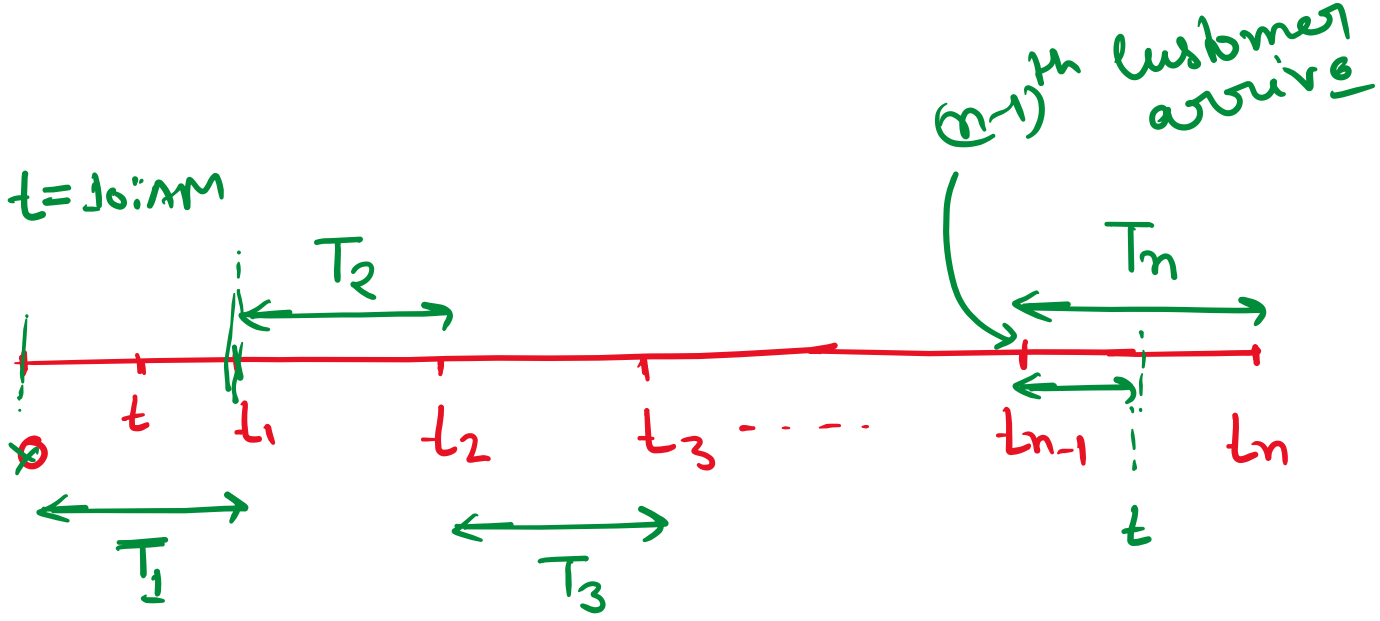
$I_1 = (10:00AM,\ 10:20AM]$

$I_2 = (10:20AM,\ 11:00AM]$

$$= \frac{\left(\frac{1}{6}\times 20\right)^3 e^{-\left(\frac{1}{6}\times 20\right)}}{3!} \times \frac{\left(\frac{1}{6}\times 40\right)^7 \cdot e^{-\frac{1}{6}\times 40}}{7!}$$

$$\approx 0.0325$$

Interarrival Time:



Objective: 1. To determine the pdf of T_n

2. To determine the probability of n^{th} customer at a given time instant.

Let, T_n = time elapsed between the $(n-1)^{th}$ arrival and nth arrival.

T_n is a continuous RV.

Let

$$P\{T_1 > t\} = P\{0 \text{ customer arrive in } (0,t]\}$$
$$= P\{N(t)=0\}$$
$$= e^{-\lambda t}$$

CDF $\quad F_{T_1}(t) = 1 - P\{T_1 > t\} = P\{T_1 \le t\}$
$$= 1 - e^{-\lambda t}$$

PDF

$$f_{T_1}(t) = \frac{d}{dt} F_{T_1}(t)$$

$$= \frac{d}{dt}\left\{1 - e^{-\lambda t}\right\}$$

$$= \lambda e^{-\lambda t}$$

Similarly,

$P(T_n > t) = P(\{0$ customers arrive in the time interval $(t_{n-1}, t_{n-1}+t]\} \mid \{n-1$ customers arrive in the time interval $(0, t_{n-1}]\})$

Using independent increment Prop.

$$P\{T_n > t\} = P\{0 \text{ customer arrive in the time interval } (t_{n-1}, t_{n-1}+t]\}$$

$$= P\{N(t)=0\}$$

$$= e^{-\lambda t}$$

CDF of T_n, $F_{T_n}(t) = 1-P\{N(t)=0\} = 1-P\{T_n > t\}$

$$= P\{T_n \leq t\}$$

$$= 1-e^{-\lambda t}$$

PDF of T_n, $\boxed{f_{T_n}(t) = \lambda e^{-\lambda t}}$

$$E[T_n] = \frac{1}{\lambda} = \int_{-\infty}^{+\infty} t\, f_{T_n}(t)\, dt$$

Thus, the inter-arrival times of a Poisson process with parameter λ are exponentially distributed with pdf

$$f_{T_n}(t) = \lambda e^{-\lambda t} \quad ; \quad \underline{n > 0}, \; t > 0$$

and the mean inter-arrival time of $1/\lambda$.

Example 3: Let $N(t)$ be a Poisson process with intensity $\lambda = 2$, and let $T_1, T_2, \ldots$ be the corresponding interarrival times.

(i) Find the probability that first arrival occurs after $t = 0.5$, i.e., $P(T_1 > 0.5)$.

(ii) Given that we have had no arrivals before $t = 1$, find $P(T_1 > 3)$.

(iii) Given that the third arrival occurred at time $t = 2$, find the probability that the fourth arrivals occurs after $t = 4$.

Sol:- (i) $P\{T_1 > 0.5\}$ = P{no arrival up to time t}

= P{0 arrival in the (0, 0.5]}

$= e^{-2t} = e^{-2\times 0.5} = e^{-1} = 1/e$