MA203: Function of Two Random Variables

Example 1: Let X and Y are continuous RVs. Find the probability density function of Z where Z = $\frac{X}{Y}$.

Sol:-

$$F_Z(z) = P(Z \le z) = P\left(\frac{X}{Y} \le z\right)$$

$$= P\left(\frac{X}{Y} \le z,\ -\infty < Y \le \infty\right)$$

$$= P\left(\frac{X}{Y} \le z,\ \{Y<0\} \cup \{Y \ge 0\}\right)$$

$$= P\left[\left\{\frac{X}{Y} \le z\right\},\ \left(\{Y<0\} \cup \{Y \ge 0\}\right)\right]$$

$$= P\left[\underbrace{\left\{\frac{X}{Y} \le z\right\}}_{A} \cap \left(\underbrace{\{Y<0\}}_{B} \cup \underbrace{\{Y \ge 0\}}_{\bar{B}}\right)\right]$$

$$= P\left[A \cap (B \cup \bar{B})\right] = P(A, B) + P(A, \bar{B})$$

$$\begin{cases} \underline{Y \ge 0} \\ X \ge YZ \\ \underline{Y < 0} \\ X \le YZ \end{cases}$$

$$= P\left(\frac{X}{Y} \le z,\ Y < 0\right) + P\left(\frac{X}{Y} \le z,\ Y \ge 0\right)$$

$$= P\left(X \ge Yz,\ \underbrace{Y < 0}\right) + P\left(X \le Yz,\ Y \ge 0\right)$$

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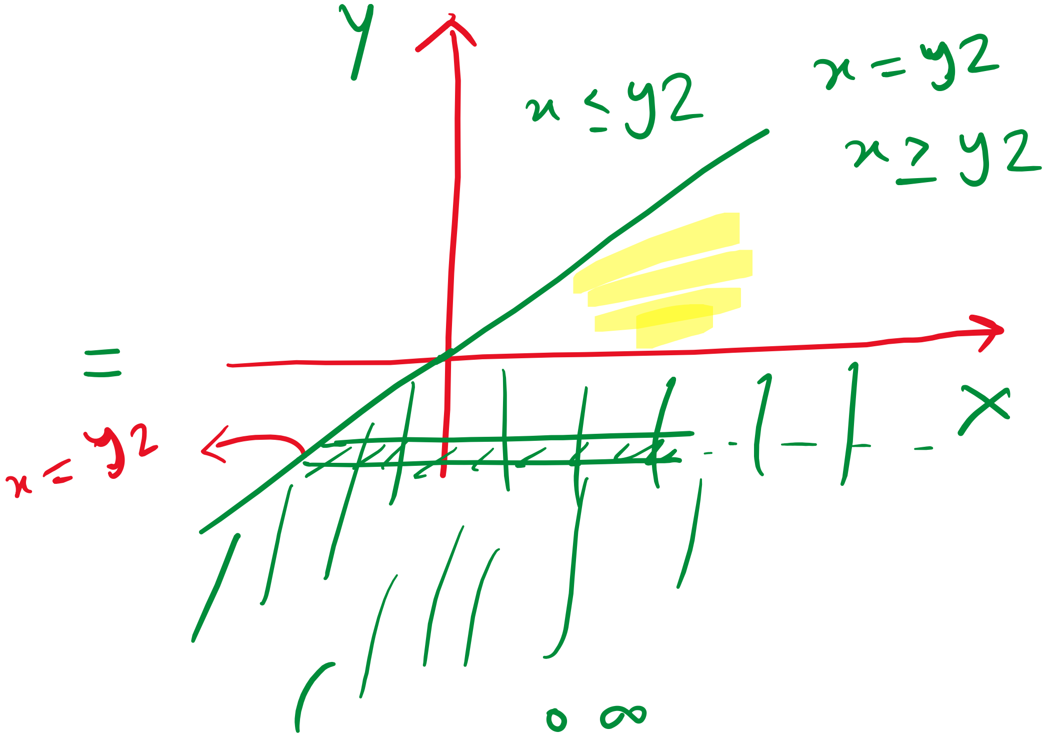

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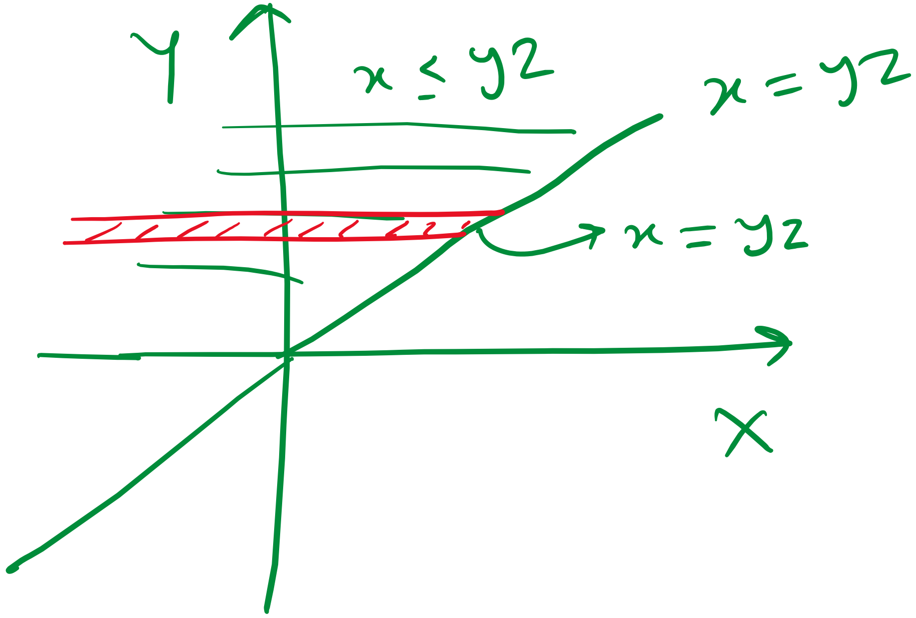


$$F_Z(z) = \int_{-\infty}^{0}\int_{yz}^{\infty} f_{X,Y}(x,y)\,dx\,dy + \int_{0}^{\infty}\int_{-\infty}^{yz} f_{X,Y}(x,y)\,dx\,dy$$

$$f_Z(z) = \frac{d}{dz}\int_{-\infty}^{0}\left\{\int_{yz}^{\infty} f_{X,Y}(x,y)dx\right\}dy + \frac{d}{dz}\int_{0}^{\infty}\left\{\int_{-\infty}^{yz} f_{X,Y}(x,y)dx\right\}dy$$

$\Downarrow$ $\Downarrow$

Using Leibnitz Rule

$$f_Z(z) = \int_{-\infty}^{0}\frac{\partial}{\partial z}\left\{\int_{yz}^{\infty} f_{X,Y}(x,y)dx\right\}dy + \int_{0}^{\infty}\frac{\partial}{\partial z}\left\{\int_{-\infty}^{yz} f_{X,Y}(x,y)dx\right\}dy$$

Using Leibnitz Rule

$$f_Z(z) = \int_{-\infty}^{0}\left\{-\frac{\partial(yz)}{\partial z}f_{X,Y}(yz,y)\right\}dy + \int_{0}^{\infty}\left\{\frac{\partial(yz)}{\partial z}f_{X,Y}(yz,y)\right\}dy$$

$\Downarrow$ $\Downarrow$

$$f_Z(z) = \int_{-\infty}^{0}\{-y\}f_{X,Y}(yz,y)dy + \int_{0}^{\infty}\{y\}f_{X,Y}(yz,y)dy$$

$$= \int_{-\infty}^{0}|y|\,f_{X,Y}(yz,y)dy + \int_{0}^{\infty}|y|\,f_{X,Y}(yz,y)dy = \int_{-\infty}^{+\infty}|y|\,f_{X,Y}(yz,y)dy$$

Leibnitz Rule: $\frac{d}{dx}\int_{a(x)}^{b(x)} h(x,y)\,\mathrm{dy} = \mathrm{h}\big(x,b(x)\big)\times\frac{db(x)}{dx} - \mathrm{h}\big(x,a(x)\big)\times\frac{da(x)}{dx} + \int_{a(x)}^{b(x)}\frac{\partial h(x,y)}{\partial x}dy$

$$f_Z(z) = \int_{-\infty}^{0} |y| f_{X,Y}(yz, y)dy + \int_{0}^{\infty} |y| f_{X,Y}(yz, y)dy = \int_{-\infty}^{+\infty} |y| f_{X,Y}(yz, y)\, dy$$

If X and Y are independent:

$$\boxed{f_Z(z) = \int_{-\infty}^{+\infty} |y| f_X(yz) f_Y(y)\, dy}$$

Example 2: X and Y are independent zero mean Gaussian RVs with unity standard deviation. Find the probability density function of RV Z where Z $= \frac{X}{Y}$..

$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2}$; $f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$

$\mu_X = 0$; $\sigma_X = 1$ $\qquad$ $\mu_Y = 0$; $\sigma_Y = 1$

$$f_Z(z) = \int_{-\infty}^{+\infty} |y| f_{X,Y}(yz, y)\, dy = \int_{-\infty}^{+\infty} |y| f_X(yz)\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{+\infty} |y| \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2 z^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

$$= \int_{-\infty}^{+\infty} \underbrace{|y| \frac{1}{2\pi} \cdot e^{-(1+z^2)\frac{y^2}{2}}\, dy}_{\text{Even function of } y} = \frac{1}{\pi} \int_0^{\infty} y\, e^{-(1+z^2)y^2/2}\, dy$$

put $(1+z^2)\frac{y^2}{2} = u$

$(1+z^2) \cdot \frac{2y}{2}\, dy = du$

$\Rightarrow (1+z^2)\, y\, dy = du$

$$= \frac{1}{\pi} \int_0^{\infty} e^{-u} \frac{du}{1+z^2}$$

$$= \frac{1}{\pi(1+z^2)} \left\{ -e^{-u} \right\} \Big|_0^{\infty} = \underbrace{\frac{1}{\pi(1+z^2)}}$$

Example 3: Let X and Y are continuous RVs. Find the probability density function of Z where Z = $max(X,Y)$.

Sol:-

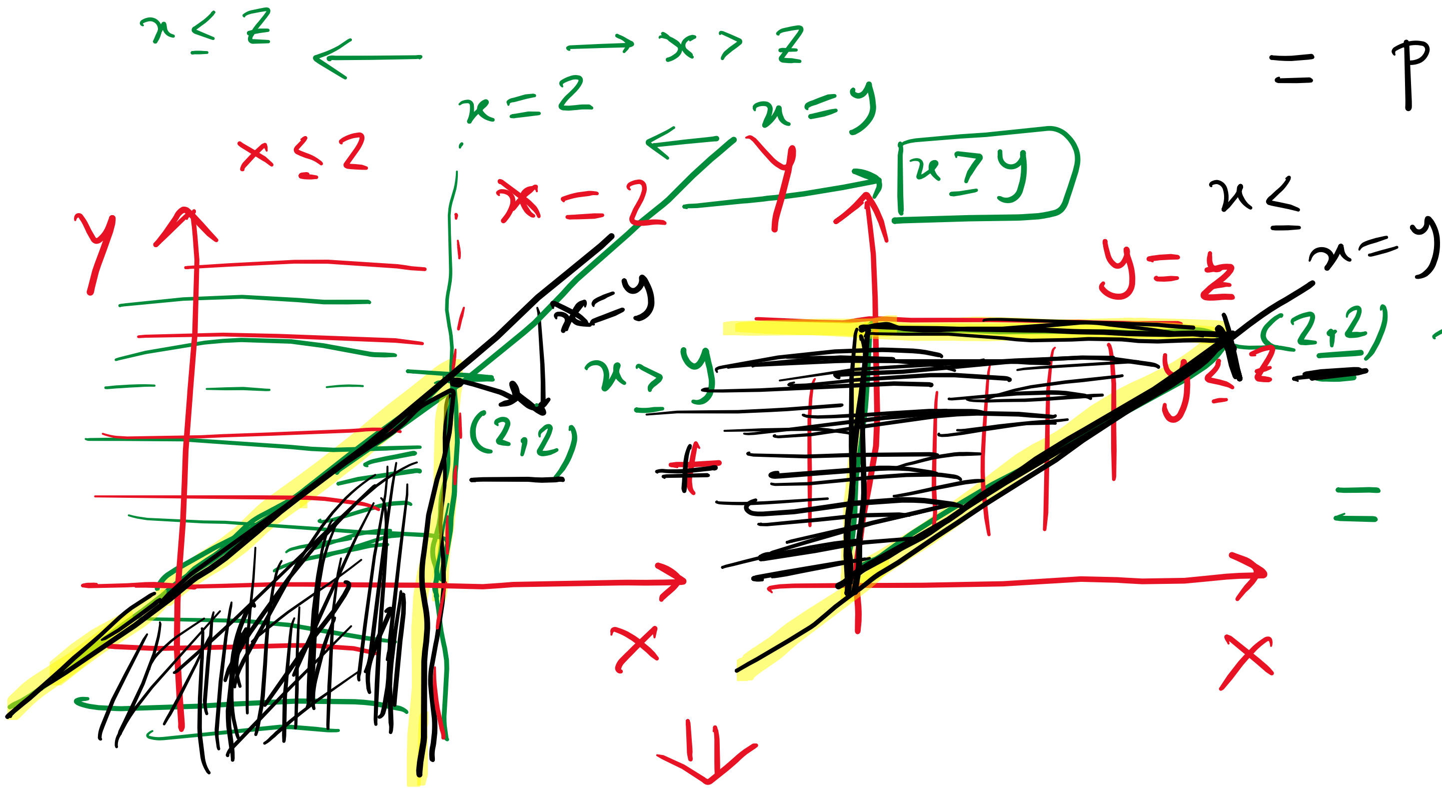
$$Z = \max(X,Y) = \begin{cases} X \; ; \; X \geq Y \\ Y \; ; \; X < Y \end{cases}$$

$$F_Z(z) = P(Z \leq z) = P(Z \leq z, \{X \geq Y\} \cup \{X < Y\})$$

$$= P[\underbrace{\{Z \leq z\}}_{A} \cap (\underbrace{\{X \geq Y\}}_{B} \cup \underbrace{\{X<Y\}}_{\bar{B}})]$$

$$= P(Z \leq z, X \geq Y) + P(Z \leq z, X<Y)$$

$$= P(X \leq z, X \geq Y) + P(Y \leq z, X<Y)$$



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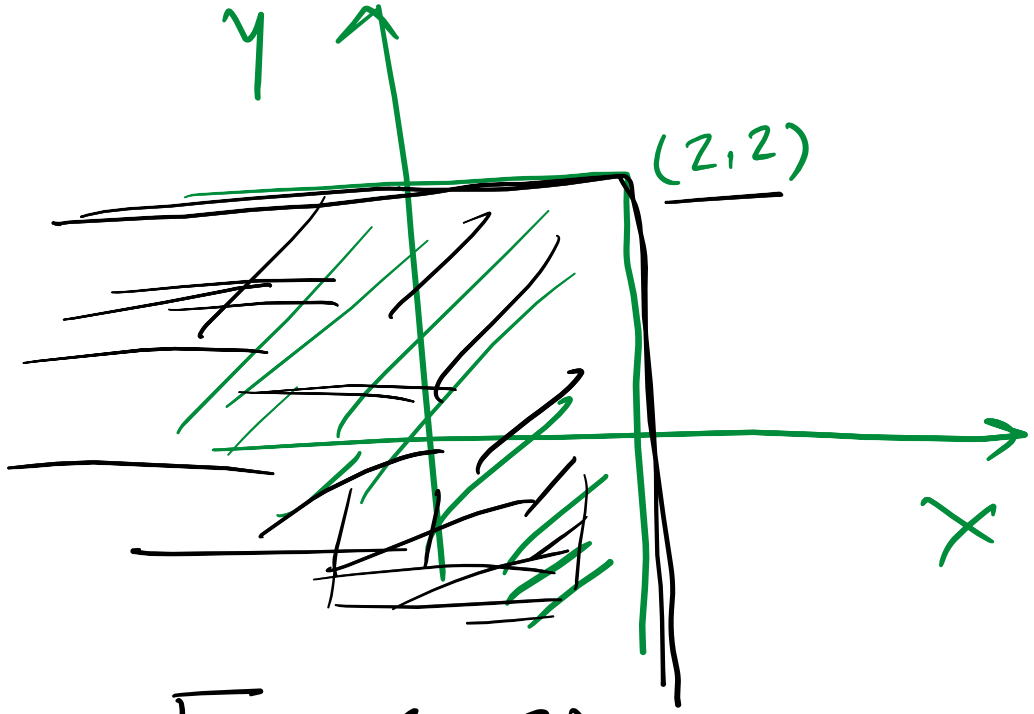


$$F_2(z) = \underline{F_{x,y}(z,z)}$$

$$\boxed{f_2(z) = \frac{d}{dz} F_{x,y}(z,z)}$$

Example 4: Let X and Y are continuous RVs. Find the probability density function of R where R = $\sqrt{X^2+Y^2}$.

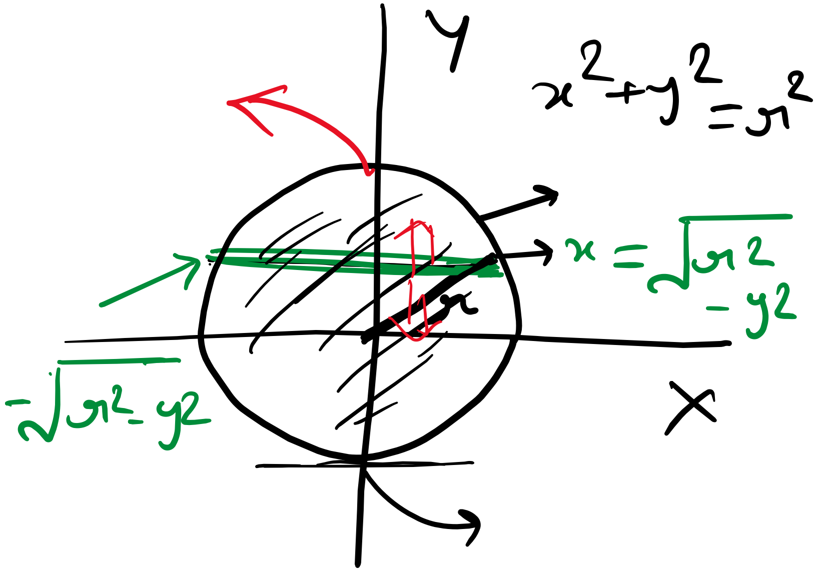
Sol:-

$$F_R(r) = P(R \le r) = P\left(\sqrt{x^2+y^2} \le r\right)$$

Step.1:-

$$= P(x^2+y^2 \le r^2)$$

$$F_R(r) = \int_{-r}^{+r} \int_{-\sqrt{r^2-y^2}}^{+\sqrt{r^2-y^2}} f_{x,y}(x,y)\, dx\, dy$$



$x^2+y^2 \le r^2$ represents the area covered by the Circle $x^2+y^2=r^2$

Step2:-

$$f_R(r) = \frac{dF_R(r)}{dr} = \frac{d}{dr}\int_{-r}^{+r}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} f_{X,Y}(x,y)\,dx\,dy$$

$$f_R(r) = \frac{d}{dr}\int_{-r}^{+r}\left\{\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} f_{X,Y}(x,y)\,dx\right\}dy$$

Using Leibnitz Rule

$$f_R(r) = \frac{d(r)}{dr}\left\{\int_{-\sqrt{r^2-r^2}}^{\sqrt{r^2-r^2}} f_{X,Y}(x,r)\,dx\right\} - \frac{d(-r)}{dr}\left\{\int_{-\sqrt{r^2-r^2}}^{\sqrt{r^2-r^2}} f_{X,Y}(x,-r)\,dx\right\} + \int_{-r}^{+r}\frac{\partial}{\partial r}\left\{\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} f_{X,Y}(x,y)\,dx\right\}dy$$

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$$f_R(r) = \int_{-r}^{+r}\frac{\partial}{\partial r}\left\{\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} f_{X,Y}(x,y)\,dx\right\}dy$$

Leibnitz Rule: $\frac{d}{dx}\int_{a(x)}^{b(x)} h(x,y)\,\mathrm{dy} = \mathrm{h}\big(x,b(x)\big)\times\frac{db(x)}{dx} - \mathrm{h}\big(x,a(x)\big)\times\frac{da(x)}{dx} + \int_{a(x)}^{b(x)}\frac{\partial h(x,y)}{\partial x}dy$

$$f_R(r) = \int_{-r}^{+r} \frac{\partial}{\partial r}\left\{\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} f_{X,Y}(x,y)\,dx\right\} dy$$

Leibnitz Rule

$$f_R(r) = \int_{-r}^{r}\left\{\frac{d\left(\sqrt{r^2-y^2}\right)}{dr} f_R\left(\sqrt{r^2-y^2}, y\right) - \frac{d\left(-\sqrt{r^2-y^2}\right)}{dr} f_R\left(-\sqrt{r^2-y^2}, y\right) + \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \cancelto{0}{\frac{\partial f_{X,Y}(x,y)}{\partial z}}\, dx\right\} \mathrm{dy}$$

$$f_R(r) = \int_{-r}^{+r}\left\{\frac{r}{\sqrt{r^2-y^2}} f_{X,Y}\left(\sqrt{r^2-y^2}, y\right) + \frac{r}{\sqrt{r^2-y^2}} f_{X,Y}\left(-\sqrt{r^2-y^2}, y\right)\right\} dy$$

$$\boxed{f_R(r) = \int_{-r}^{+r} \frac{r}{\sqrt{r^2-y^2}}\left\{f_{X,Y}\left(\sqrt{r^2-y^2}, y\right) + f_{X,Y}\left(-\sqrt{r^2-y^2}, y\right)\right\} dy}$$

Leibnitz Rule: $\displaystyle \frac{d}{dx}\int_{a(x)}^{b(x)} h(x,y)\,\mathrm{dy} = \mathrm{h}\left(x, b(x)\right) \times \frac{db(x)}{dx} - \mathrm{h}\left(x, a(x)\right) \times \frac{da(x)}{dx} + \int_{a(x)}^{b(x)} \frac{\partial h(x,y)}{\partial x}\,dy$

Example 5: Let X and Y are independent zero mean Gaussian RVs with equal variance σ^2. Find the probability density function of R where $R = \sqrt{X^2 + Y^2}$.

Sol:- $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2}$; $f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-y^2/2}$; $f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$

Given that X and Y are independent

Using Result of Example 4

$\Downarrow$

$$f_R(r) = \int_{-r}^{+r} \frac{r}{\sqrt{r^2-y^2}} \left\{ f_{X,Y}\left(\sqrt{r^2-y^2}, y\right) + f_{X,Y}\left(-\sqrt{r^2-y^2}, y\right) \right\} dy$$

use $f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}$

$$= \int_{-r}^{+r} \frac{r}{\sqrt{r^2-y^2}} \left\{ \frac{1}{2\pi\sigma^2} e^{-\frac{(r^2 \cancel{-y^2} \cancel{+y^2})}{2\sigma^2}} + \frac{1}{2\pi\sigma^2} e^{-\frac{(r^2 \cancel{-y^2} \cancel{+y^2})}{2\sigma^2}} \right\} dy$$

$$= \frac{1}{2\pi\sigma^2} \int_{-r}^{+r} \left\{ \frac{2r}{\sqrt{r^2-y^2}} e^{-\frac{r^2}{2\sigma^2}} \right\} dy = \frac{2}{2\pi\sigma^2} \int_{-r}^{+r} \frac{r}{\sqrt{r^2-y^2}} e^{-r^2/2\sigma^2} dy$$

$$f_R(r) = \frac{2}{2\pi\sigma^2} \int_{-r}^{+r} \frac{r}{\sqrt{r^2 - y^2}} e^{-r^2/_{2\sigma^2}} \, dy$$

$$f_R(r) = \frac{re^{-r^2/_{2\sigma^2}}}{\pi\sigma^2} \int_{-r}^{+r} \frac{1}{\sqrt{r^2 - y^2}} dy$$

$$f_R(r) = \frac{2re^{-r^2/_{2\sigma^2}}}{\pi\sigma^2} \int_{0}^{+r} \frac{1}{\sqrt{r^2 - y^2}} dy$$

Put $y = rsin\theta, dy = rcos\theta d\theta$, and $\frac{dy}{\sqrt{r^2-y^2}} = \frac{rcos\theta d\theta}{rcos\theta} = d\theta$

$$f_R(r) = \frac{2re^{-r^2/_{2\sigma^2}}}{\pi\sigma^2} \int_{0}^{\pi/_2} d\theta$$

$$f_R(r) = \frac{2re^{-r^2/_{2\sigma^2}}}{\pi\sigma^2} \left(\frac{\pi}{2} - 0\right) = \frac{re^{-r^2/_{2\sigma^2}}}{\sigma^2}$$

$$f_R(r) = \frac{re^{-r^2/_{2\sigma^2}}}{\sigma^2} ; r \geq 0$$ $\Rightarrow$ Rayleigh Distribution.