Pumping Lemma for CFL

Lemma : Let L be a CFL. Then there exists a constant $p > 0$ such that for any w in L with $|w| \geq p$, we can write $w = u\, v\, x\, y\, z$ with $|v\, x\, y| \leq p$, $|v\, y| > 0$ and for $k \geq 0$ $u\, v^k\, x\, y^k\, z$ is in L.

We omit the proof. As in the case of Regular Languages, the proof of the Pumping Lemma for CFL uses the Pigeon Hole Principle.

Applications :

1) $L = \{a^n\, b^n\, c^n \mid n \geq 0\}$ is not a CFL.

   Suppose L is a CFL and let p be the Pumping Lemma constant. Then take $w = a^p\, b^p\, c^p$. Obviously w is in L and $|w| \geq p$. Hence $w = u\, v\, x\, y\, z$ with $|vxy| \leq p$, $|vy| > 0$ and for $k \geq 0$ $u\, v^k\, x\, y^k\, z$ is in L. Since $|v\, x\, y| \leq p$, $v\, x\, y$ cannot have a's and c's both.

   a..a..a b..b..b c..c..c

   <…….><…….><……..>

p p p

If v x y does not have a's then uv^0xy^0z has p a's and less than 2 p b's and c's - cannot be in L – a contradiction.

If v x y does not have c's then uv^0xy^0z has p c's and less than 2 p a's and b's – cannot be in L – a contradiction.

This proves that L is not a CFL.

2) L = { w w | w is in {0, 1]* } is not a CFL.

See Book – Example 7.21.

3) $L = \{0^i 1^j \mid j = i^2\}$ is not a CFL.

Let L be a CFL and let p be the Pumping Lemma constant.

Take $w = 0^p1^{p^2}$. We have w = u v x y z, $|v x y| \leq p$ ie $|v y| \leq p$. Suppose v y has α 0's and β 1's . Then $u v^0 x y^0 z = 0^{p-\alpha} 1^{p^2 - \beta}$ is in L. Hence $p^2 - \beta = (p - \alpha)^2$.

Similarly $u v^2 x y^2 z$ is in L. Hence $p^2 + \beta = (p + \alpha)^2$. Thus $2 p^2 = 2 p^2 + 2 \alpha^2$ and hence α = 0 and therefore $\beta > 0$. On the other hand $0^p 1^{p^2 - \beta}$ is in L which is a contradiction. Hence L is not a CFL.

4) HW : $L = \{0^n \mid n \text{ a prime}\}$ is not a CFL.

5) HW : $L = \{0^n \mid n \text{ a perfect square}\}$ is not a CFL.

Hint for 4) and 5) : as for the proof that these languages are not regular.

6) L = Palindromes where number of 0's is equal to number of 1's. Let $w = 0^p1^{2p}0^p$ where p is the PL constant. Then by PL $w = uvxyz$ where $|vxy| \leq p$, $|vy| > 0$ and uv^kxy^kz is in L for $k \geq 0$. If v has some 0 from the left side of w then y cannot have any 0 from the right side and vice versa. In this case then by pumping out we will get a string which is not a palindrome. Otherwise vxy will be entirely 1 and in that case by pumping out

the number of 1’s will be less and again we get a contradiction. Thus L is not a CFL.