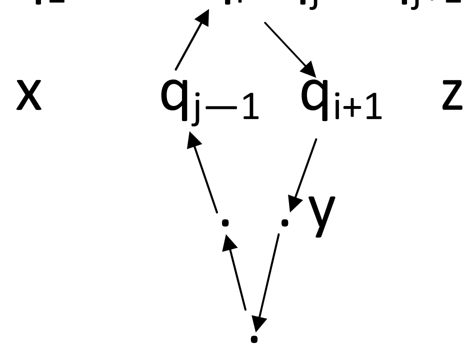
Pumping Lemma for Regular languages

This lemma gives a necessary condition for a language to be regular ie if a language is regular it must satisfy this condition.

Lemma : Let L be a regular language. Then there exists a constant $p > 0$ such that if $w \in L$ with $|w| \geq p$, we must have $w = x\,y\,z$ with $|x\,y| \leq p$, $|y| > 0$ and $x\,y^k\,z \in L$ for $k \geq 0$.

Proof : Since L is regular it is accepted by some DFA M. Let p be the number of states of M. Then for $w \in L$ with $|w|=k \geq p$, M will go thru the states $q_0, q_1, \ldots q_p, \ldots q_k$ where q_0 is the start state and q_k is a final state. Since the sequence $q_0, q_1, \ldots, q_p$ has $p + 1$ elements and there are only p distinct states in M, by PHP there are i,j such that $0 \leq i < j \leq p$ with $q_i = q_j$. Let the string x take M from q_0 to q_i, y take M from q_i to q_j (ie back to q_i) and z take M from q_j to q_k.

$$q_0 \to q_1 \to \ldots\ldots q_i = q_j \to q_{j+1} \to \ldots q_k$$



Since $j \leq p$, $|x\,y| \leq p$. Since $i < j$, $|y| > 0$. Finally for $k \geq 0$, $x\,y^k\,z$ takes M from the start state q_0 to a final state. Hence M accepts $x\,y^k\,z$ ie $x\,y^k\,z \in L$ for $k \geq 0$.

Application of Pumping Lemma to prove that certain languages are not regular.

1) $L=\{0^n1^n \mid n \geq 0\}$. Assume L is regular and let p be the PL constant. Take $w = 0^p1^p$ which is in L and $|w| \geq p$. By PL $w = xyz$, $|xy| \leq p$, $|y| = m > 0$ and $xy^kz \in L$ for $k \geq 0$. Since $|xy| \leq p$, xy and hence y must consist entirely of 0's ie $y = 0^m$. Thus $xy^0z = 0^{p-m}1^p \in L$ which is a contradiction since $m > 0$.

This contradiction proves that L is not regular. Such a proof is always by contradiction. We assume that the given L is not regular. We then choose a suitable $w \in L$ depending on the PL constant p s.t. $|w| \geq p$. Then by PL $w = xyz$ s.t. $|xy| \leq p$, $|y| > 0$ and $xy^kz \in L$ for $k \geq 0$. Then we show that either xz or some xy^kz for $k > 1$ is not in L. This is a contradiction which proves that L is not regular. If we use xz we say that y has been pumped out and if we use xy^kz for $k > 1$, we say tha y^{k-1} has been pumped in. In the above proof we used $w = 0^p1^p$ and pumped out.

2) Palindromes over {0,1} : Take $w =0^p10^p$ and pump out.

3) Language of balanced parantheses : Take $(^p\ )^p$ and pump out

4) $L = \{0^n10^n \mid n \geq 1\}$. Take $w = 0^p10^p$ and pump out

5) $L = \{0^n1^m2^n \mid m,n \geq 0\}$. Take $w = 0^p1^p2^p$ and pump out.

6) $L = \{0^n\ 1^m \mid n \leq m\}$. Take $w = 0^p1^p$ and pump in

7) $L = \{0^n1^{2n} \mid n \geq 1\}$. Take $w =0^p1^{2p}$ and pump out

8) $L = \{0^n \mid n \text{ a perfect square}\}$. Take $w = 0^{p\wedge 2}$. $|xy| \leq p$ and hence $1 < |y| \leq p$. Therefore $|xyz| = p^2 < |xy^2z| \leq p^2+ p < p^2+2p+1= (p+1)^2$. Hence $|xy^2z|$ cannot be a perfect square – a contradiction.

9) $L = \{0^n \mid n \text{ a perfect cube}\}$ HW

10) $L = \{uu \mid u \in \{0,1\}^*\}$. Take $w = 0^p10^p1$. Since w = uu, one 1 should be in the first u and one 1 in the second. After pumping in

in the first u there are more 0's giving the contradiction.

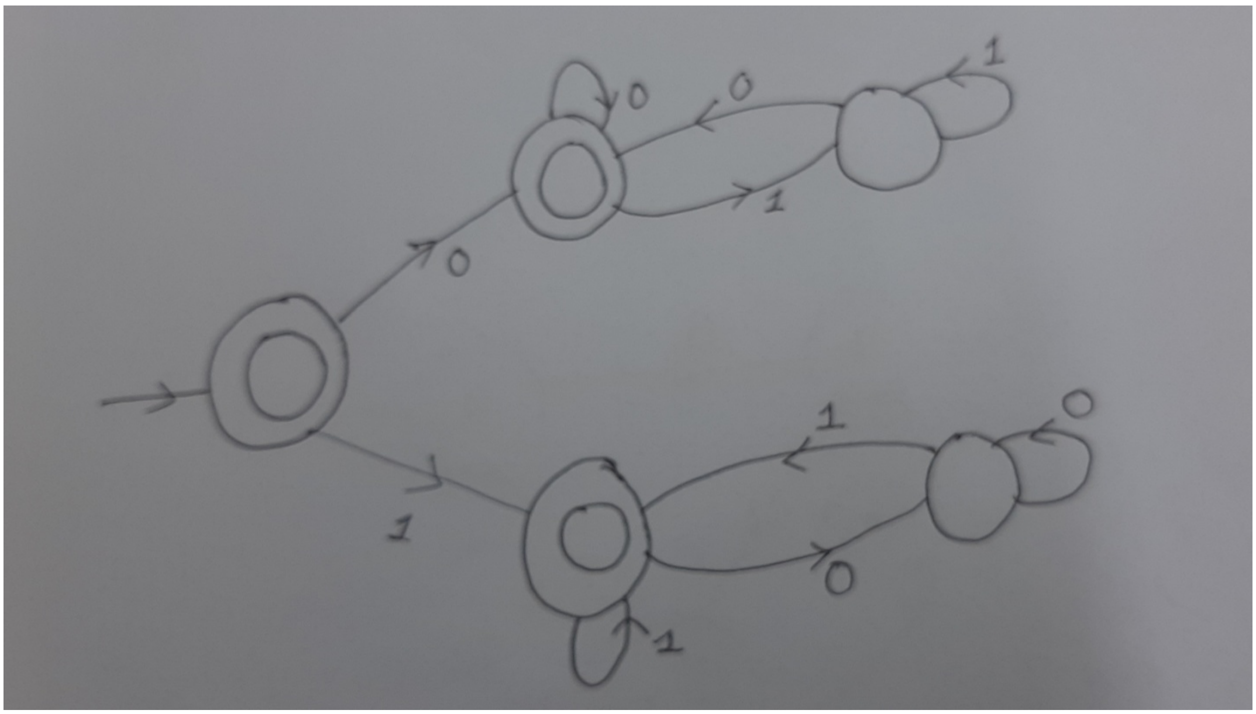
11) L = {u u^R | u Є {0,1}*}. Take w = 0^p110^p and proceed as in 10).

12) L = {u u^- | u Є {0,1}*, u^- complement of u}. Take w = 0^p1^p and pump out

13) Let $n_0(w)$ = number of 0's in w and similarly $n_1(w)$. Let L = {w Є {0,1}* | $n_0(w)$ = $n_1(w)$ }. Take w = 0^p1^p and pump out

14) L = {0^n | n a prime}. Take w = 0^q with q a prime ≥ p. this is possible since there are infinite number of primes. Now w = xyz with |y| = m ≥ 1. So $|xy^{q+1}z| = |xyz| + |y|^q$ = q + qm = q(m +1). Now q ≥ 2 and m+1 ≥ 2, which is a contradiction since a prime cannot have 2 factors ≥ 2. Proof in Book (Example 4.3) more complicated. This proof supplied by Sunand Sharma of 2017 Batch.

15) Prove or disprove L = {w Є {0,1}* | w has equal number of 01 and 10 as substrings} is regular. This is regular since it is given by the DFA



16) Prove or disprove L = {$0^n u 1^n$ |n≥1, u Є {0,1}*} is regular. Let L1 = 0(0+1)*1. Clearly L1 is a subset of L. Take w Є L. Then w = $0^n u 1^n$ n ≥ 1. Let u' = $0^{n-1} u 1^{n-1}$. Then u' Є {0,1}* and w = 0 u' 1 Є L1. Thus L is a subset of L1. Thus L = L1 = 0(0+1)*1 is regular.

17) Prove or disprove $\{0^n 1 u 1^n \mid n \geq 1, u \in \{0,1\}^*\}$ is regular. Assume that L is regular and take $w = 0^p 1 \varepsilon 1^p$. Pumping in we get a contradiction. Hence L is not regular.