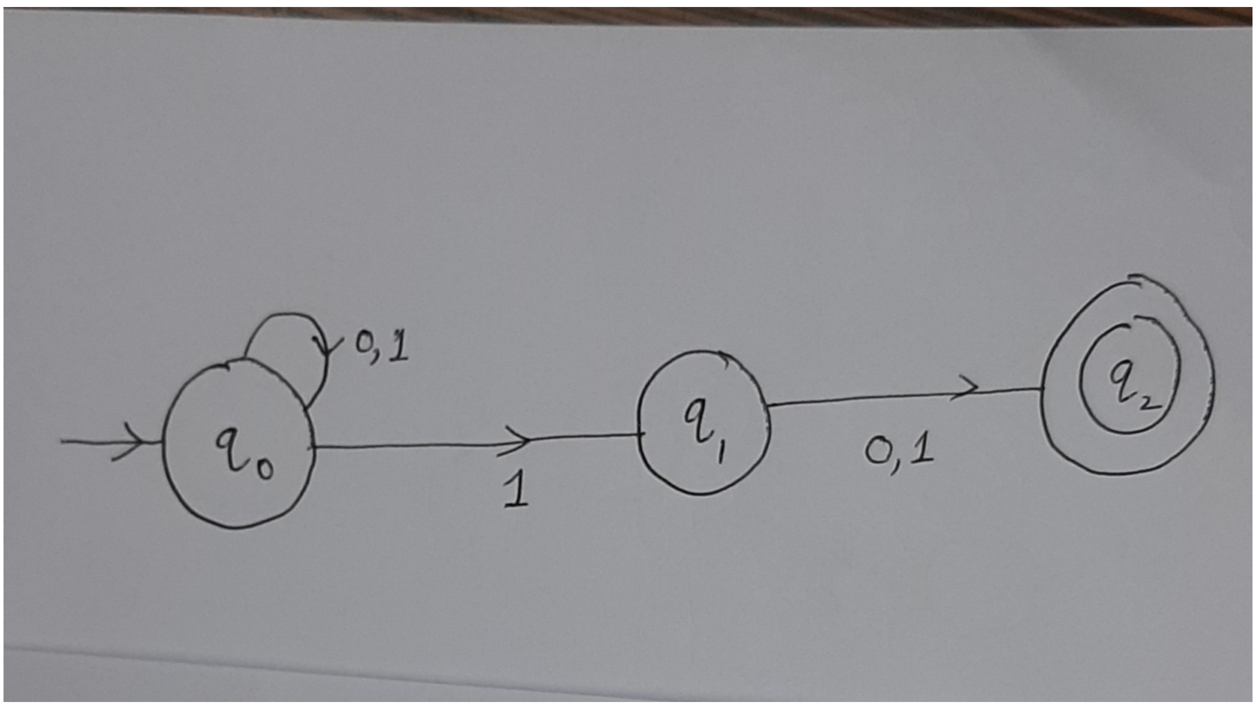
Non-deterministic Finite Automaton (NFA)

Example : second symbol from the end is 1

We can skip the first few symbols till we get a 1 which we guess is the second symbol from the end and if the guess is verified we accept.



δ(Q,a) is a subset of Q ie $\delta : Q \times \Sigma \to P(Q)$
Tansition can be to zero or more states. If transition is to zero states we do not have to mention this in a textual specification. For example we can specify this NFA by ($\{q_0,q_1,q_2\}$,

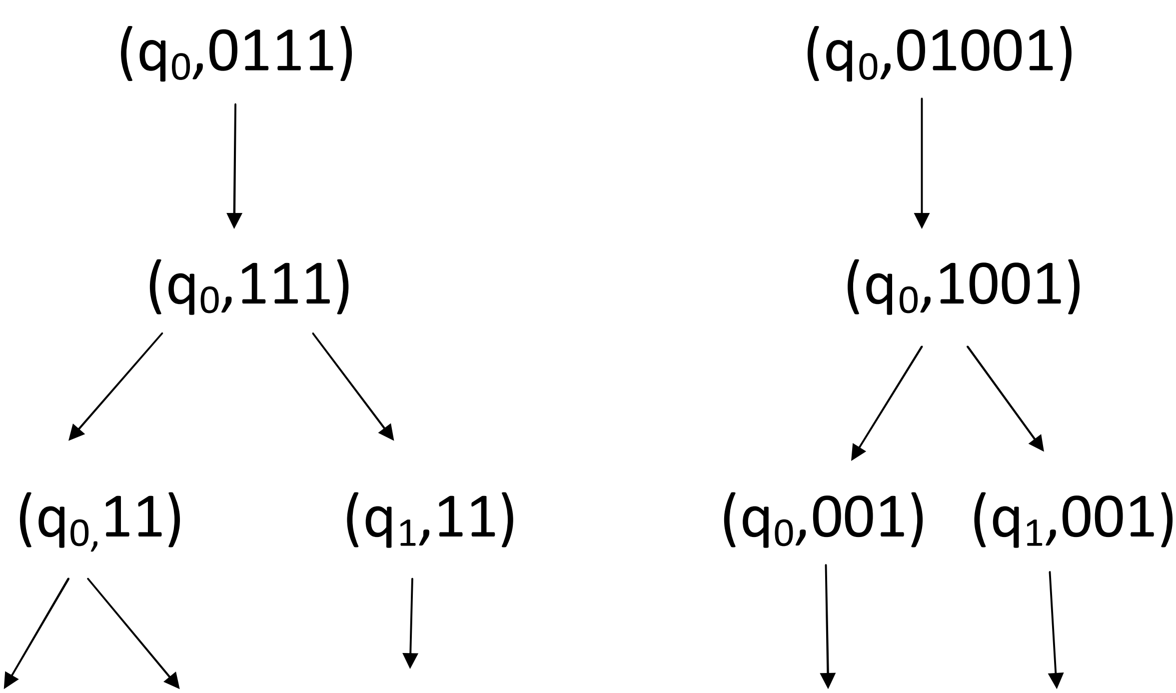
$\{0,1\},\delta,q_0,\{q_2\}$ where $\delta(q_0,0)=\{q_0\},\delta(q_0,1)=\{q_0,q_1\},\delta(q_1,0)=\delta(q_1,1)=\{q_2\}$. $\delta(q_2,0)=\delta(q_2,1)=\varphi$ need not be mentioned.

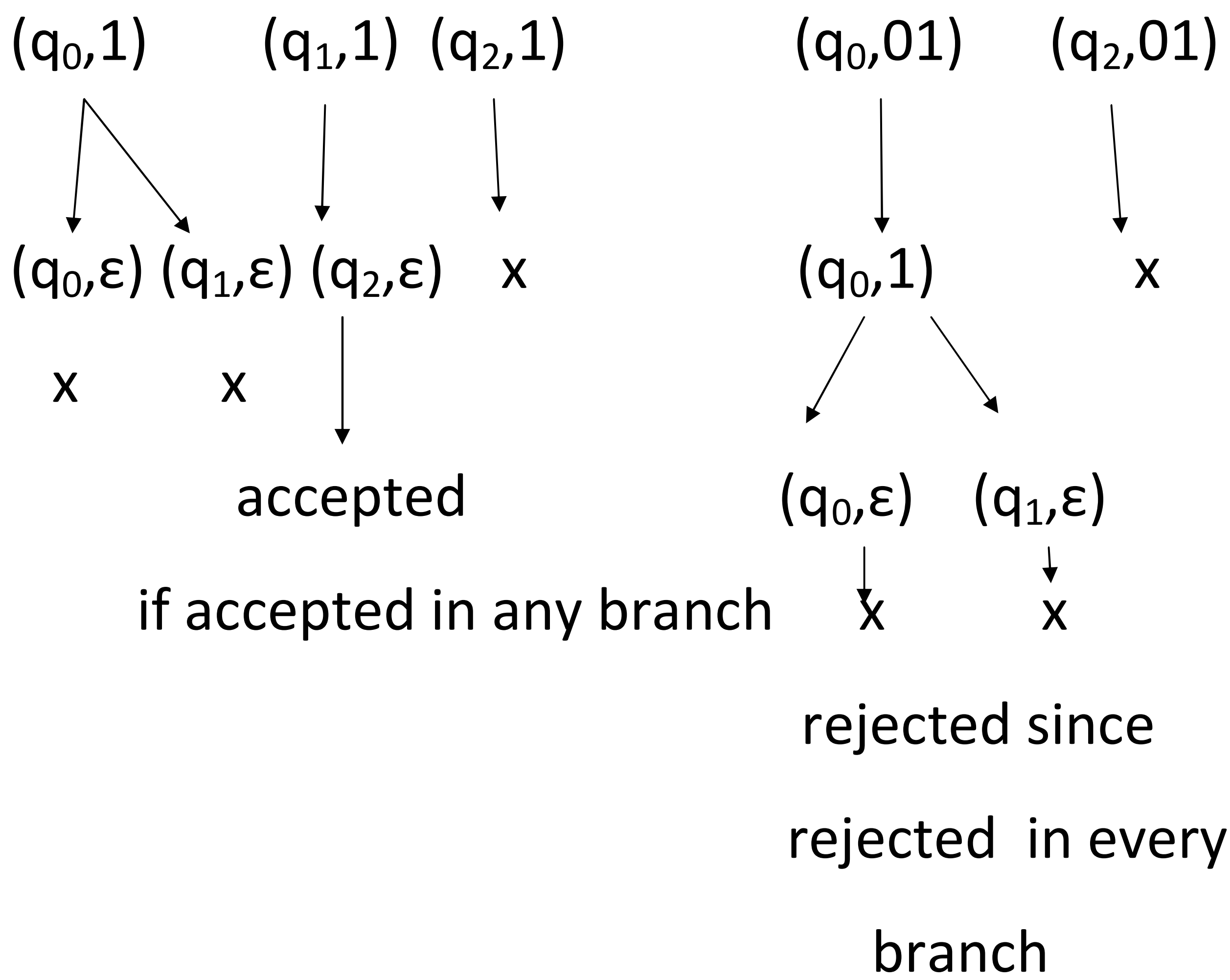
Here

	0	1
$\rightarrow q_0$	$\{q_0\}$	$\{q_0, q_1\}$
q_1	$\{q_2\}$	$\{q_2\}$
q_2	ϕ	ϕ

Computation proceeds in a tree.

Input 0111 Input 01001





Any DFA is a NFA taking any entry in the transition table to be a singleton set. Conversely any NFA has an equivalent DFA because of the following theorem.

Theorem : If $M = (Q, \Sigma, \delta, q_0, F)$ where $\delta : Q \times \Sigma \rightarrow P(Q)$ is a NFA then $L(M) = L(M')$ where M' is the DFA $(Q', \sum, \delta', S_0, F')$ where $Q' = P(Q)$,

$\delta'(S,a) = \cup_{q \in S} \delta(q, a)$ for $S \in P(Q)$, $S_0 = \{q_0\}$, and

$F' = \{S \mid S \cap F \neq \varphi\}$. We omit the proof.

Corollary : A language L is regular iff L = L(M) for a NFA M.

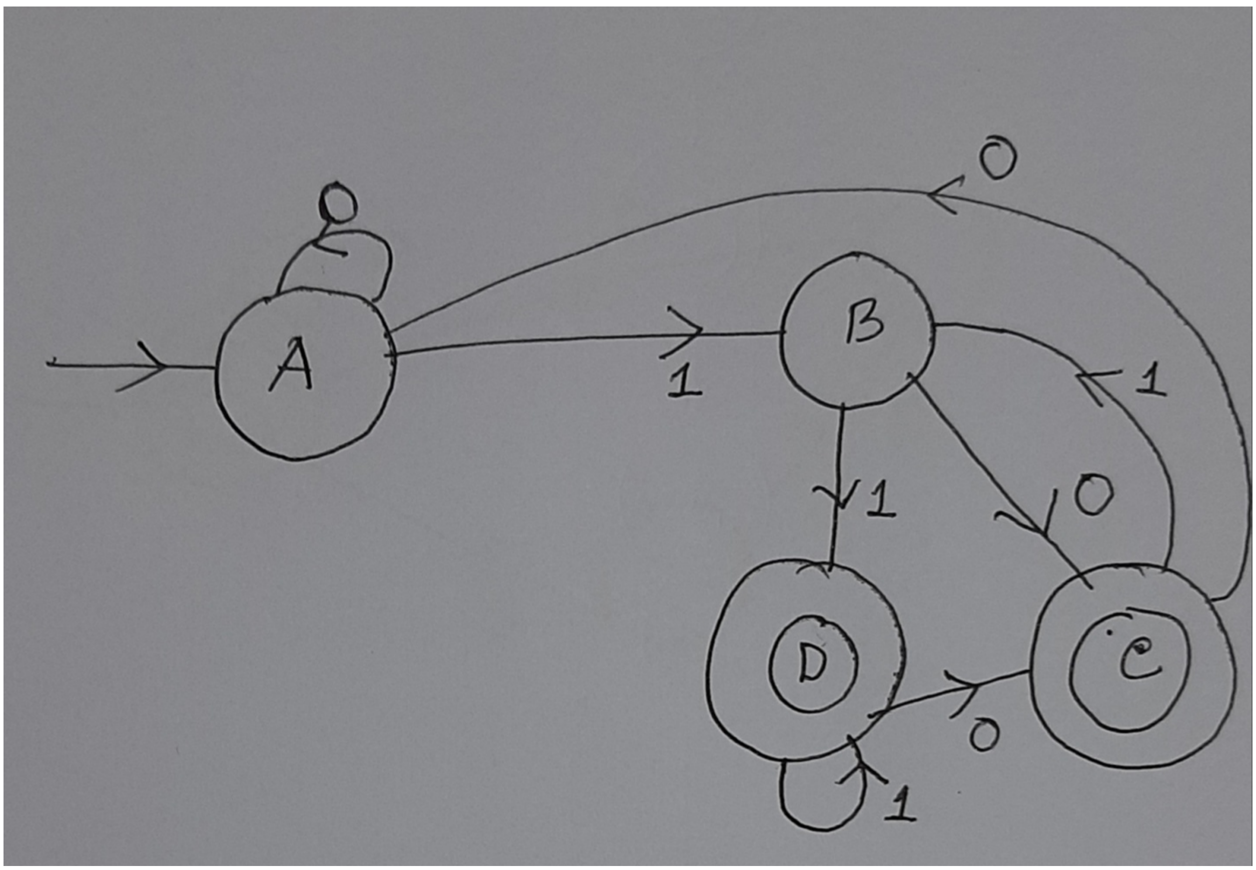
We do not get a bigger class of language using NFA's.

Conversion of a NFA to an equivalent DFA :

For the NFA that we started with :

		0	1
	$\rightarrow q_0$	$\{q_0\}$	$\{q_0, q_1\}$
M	q_1	$\{q_2\}$	$\{q_2\}$
	$* q_2$	φ	φ

		0	1
	→ A $\{q_0\}$	$\{q_0\}$	$\{q_0, q_1\}$
M’	B $\{q_0,q_1\}$	$\{q_0,q_2\}$	$\{q_0,q_1,q_2\}$
	* C $\{q_0,q_2\}$	$\{q_0\}$	$\{q_0,q_1\}$
	* D $\{q_0,q_1,q_2\}$	$\{q_0,q_2\}$	$\{q_0,q_1,q_2\}$



Convert NFA to a DFA

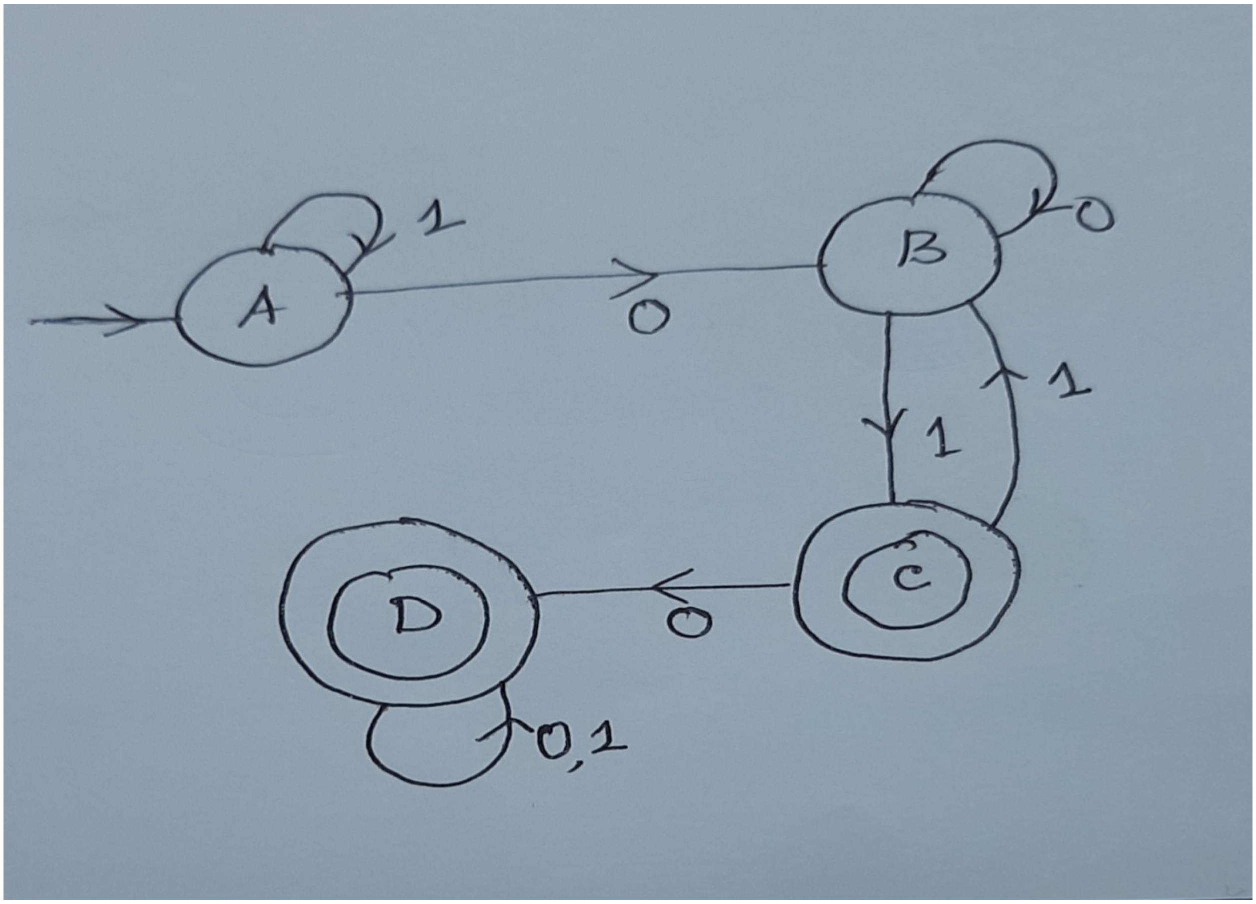
0	1

→ p	{p,q}	{p}
q	φ	{r}
* r	{p,r}	{q}

({p,q,r},{0,1},δ,p,{r}) where δ(p,0)={p,q},δ(p,1)={p} ,δ(q,1)={r}, δ(r,0)={p,r} and δ(r,1)={q}.

Converting

		0	1
→ A	{p}	{p,q}	{p}
B	{p,q}	{p,q}	{p,r}
* C	{p,r}	{p,q,r}	{p,q}
* D	{p,q,r}	{p,q,r}	{p,q,r}



HW 1 Convert NFA to DFA

		0	1
→	p	{p,r}	{q}
	q	{r,s}	{p}
*	r	{p,s}	{r}
*	s	{q,r}	

({p,q,r,s}{0,1},δ,p,{r,s}) where δ(p,0)={p,r},

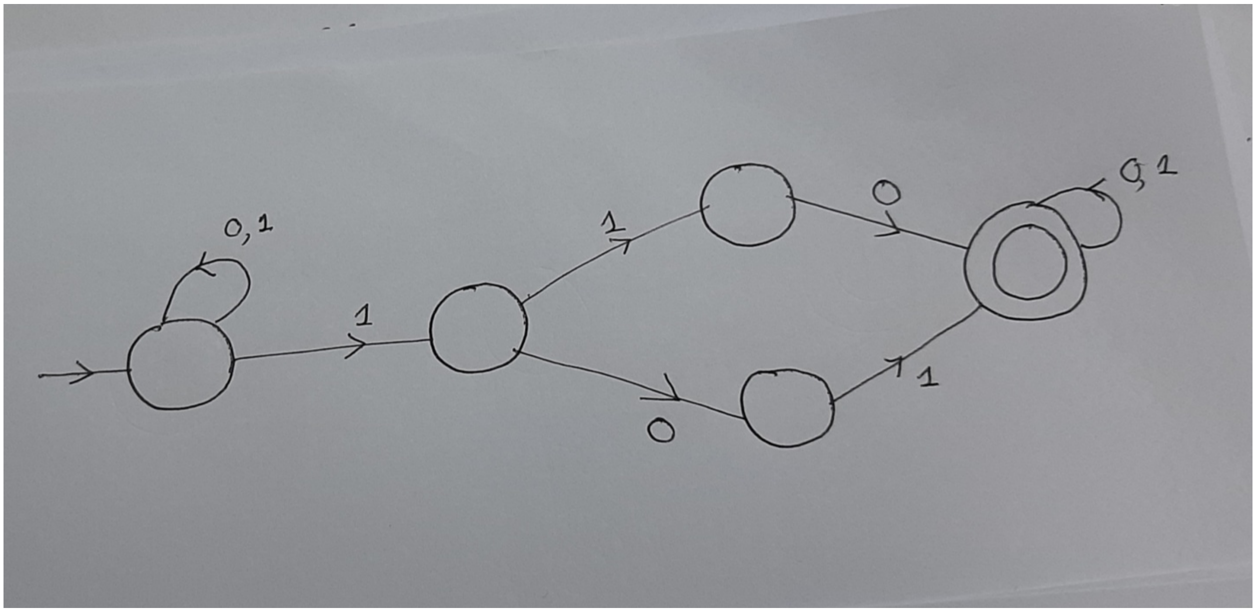
$\delta(p,1)=\{q\},\delta(q,0)=\{r,s\},\delta(q,1)=\{p\},(r,0)=\{p,s\},\delta(r,1)=\{r\},\delta(s,0)=\{q,r\}$.

HW 2 Convert NFA to DFA and find language

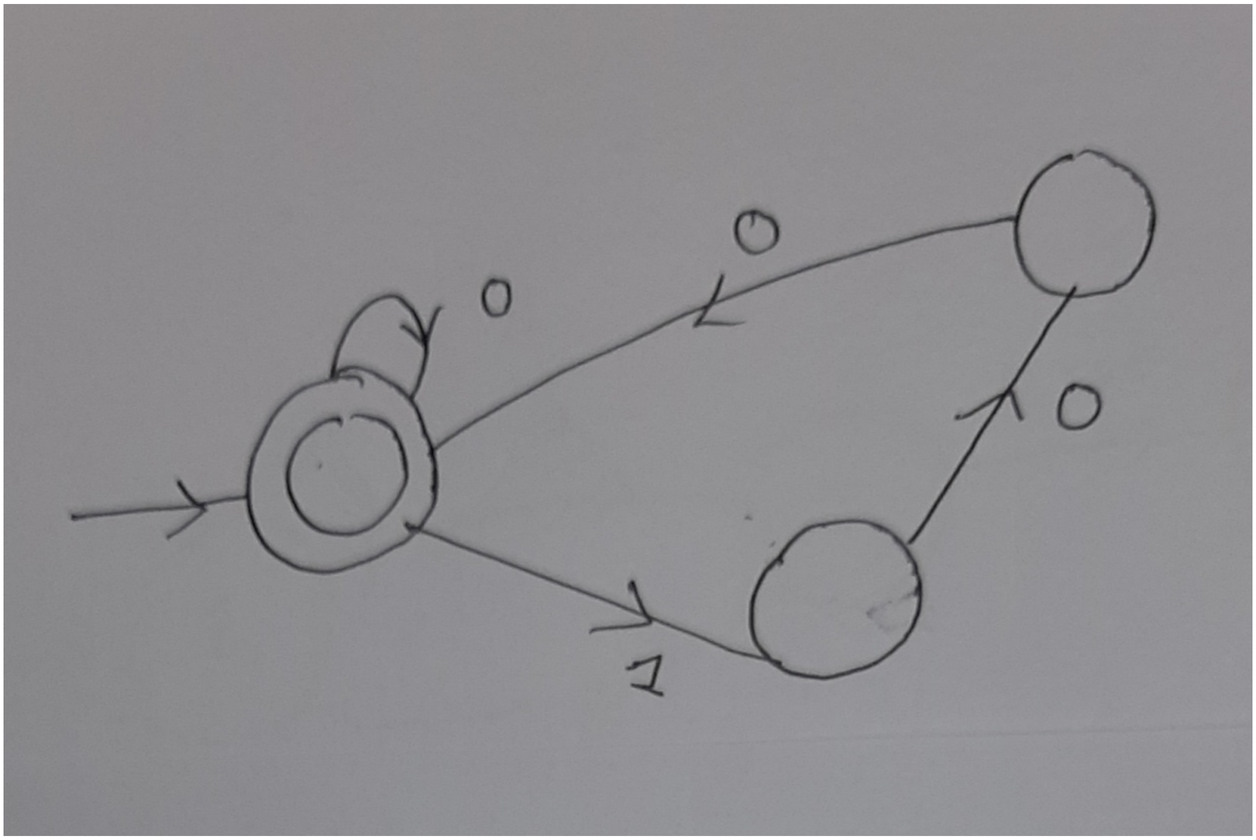
		0	1
→	p	{p,q}	{p}
	q	{r,s}	{t}
	r	{p,r}	{t}
*	s	φ	φ
*	t	φ	φ

$(\{p,q,r,s,t\},\{0,1\},\delta,p,\{s,t\})$ where $\delta(p,0)=\{p,q\}$, $\delta(p,1)=\{r\},\delta(q,0)=\{r,s\},\delta(q,1)=\{t\},\delta(r,0)=\{p,r\}$ and $\delta(r,1)=\{t\}$.

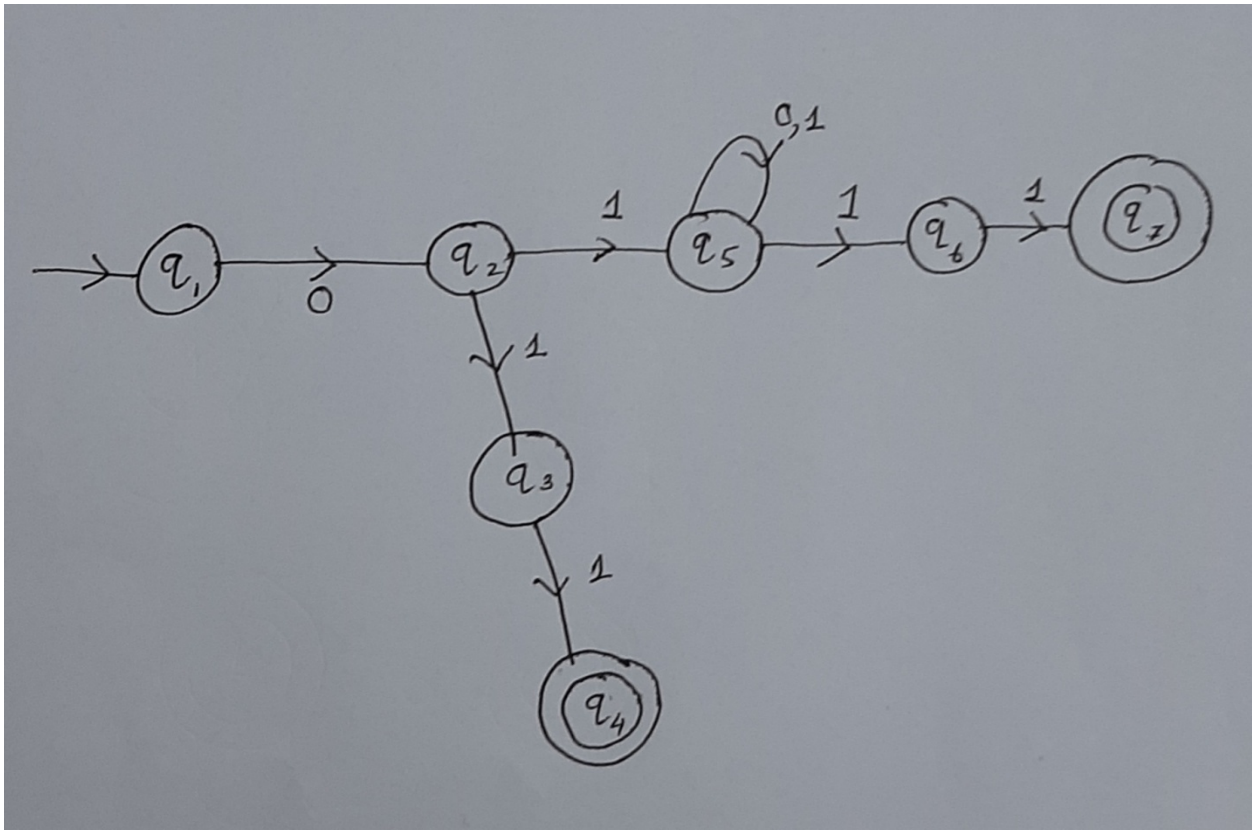
NFA for strings over {0,1] having 101 or 110 as substring



Every 1 followed by 00



NFA for strings that start with 01 and end with 11. Convert this to a DFA.



	0	1
$\rightarrow q_1$	$\{q_2\}$	φ
q_2	φ	$\{q_3,q_5\}$
q_3	φ	$\{q_4\}$
$* \; q_4$	φ	φ
q_5	$\{q_5\}$	$\{q_5,q_6\}$

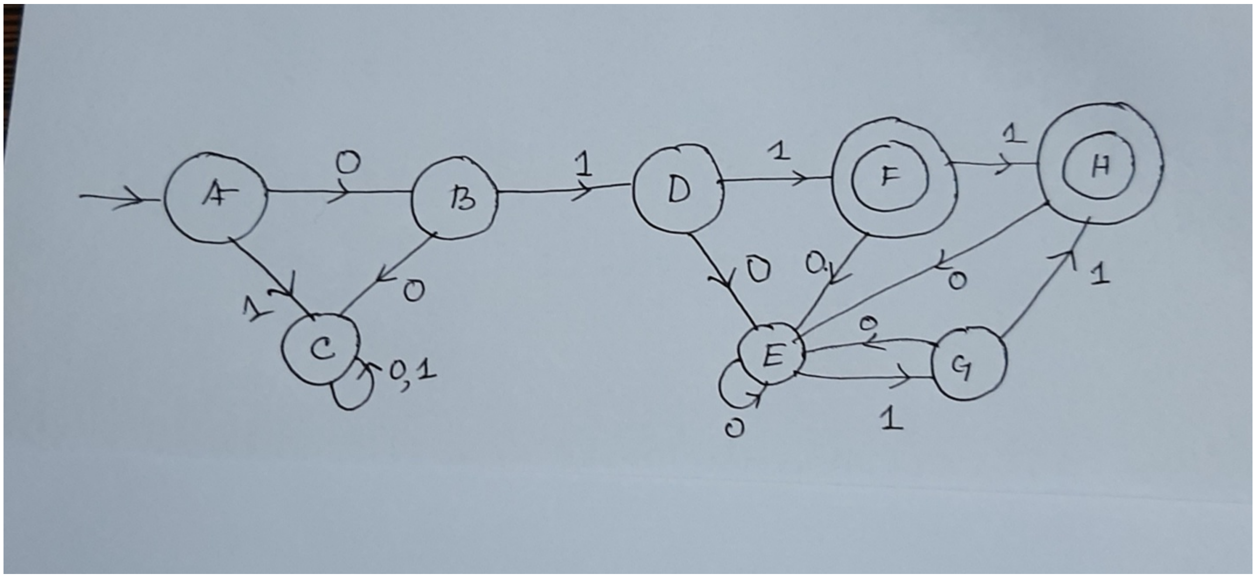
	Converting	0	1
$\rightarrow$ A	$\{q_1\}$	$\{q_2\}$	φ
B	$\{q_2\}$	φ	$\{q_3,q_5\}$
C	φ	φ	φ
D	$\{q_3,q_5\}$	$\{q_5\}$	$\{q_4,q_5,q_6\}$
E	$\{q_5\}$	$\{q_5\}$	$\{q_5,q_6\}$

q_6	φ	$\{q_7\}$
* q_7	φ	φ

* F	$\{q_4,q_5,q_6\}$	$\{q_5\}$	$\{q_5,q_6,q_7\}$
G	$\{q_5,q_6\}$	$\{q_5\}$	$\{q_5,q_6,q_7\}$
* H	$\{q_5,q_6,q_7\}$	$\{q_5\}$	$\{q_5,q_6,q_7\}$

Or

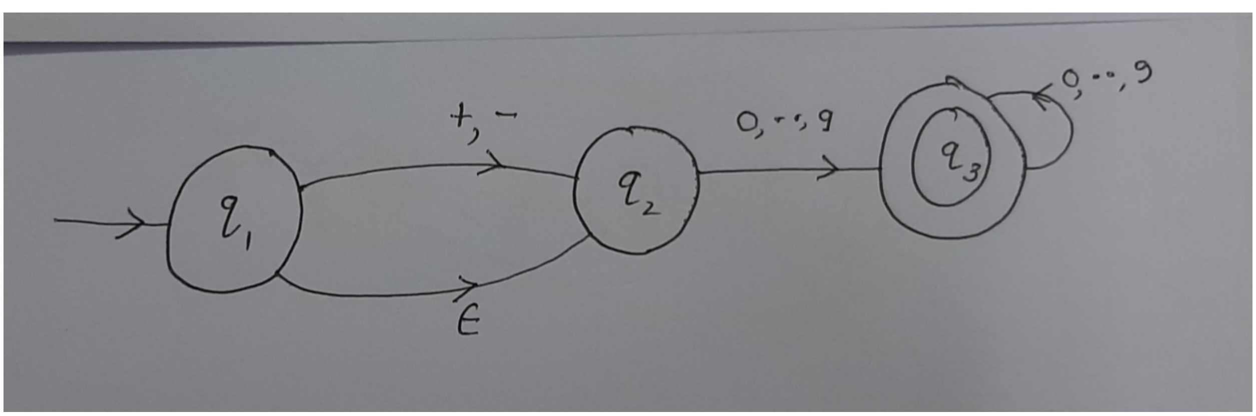
	0	1
→A	B	C
B	C	D
C	C	C
D	E	F
E	E	G
* F	E	H
G	E	H
* H	E	H



ε-NFA or NFA with ε-transitions :

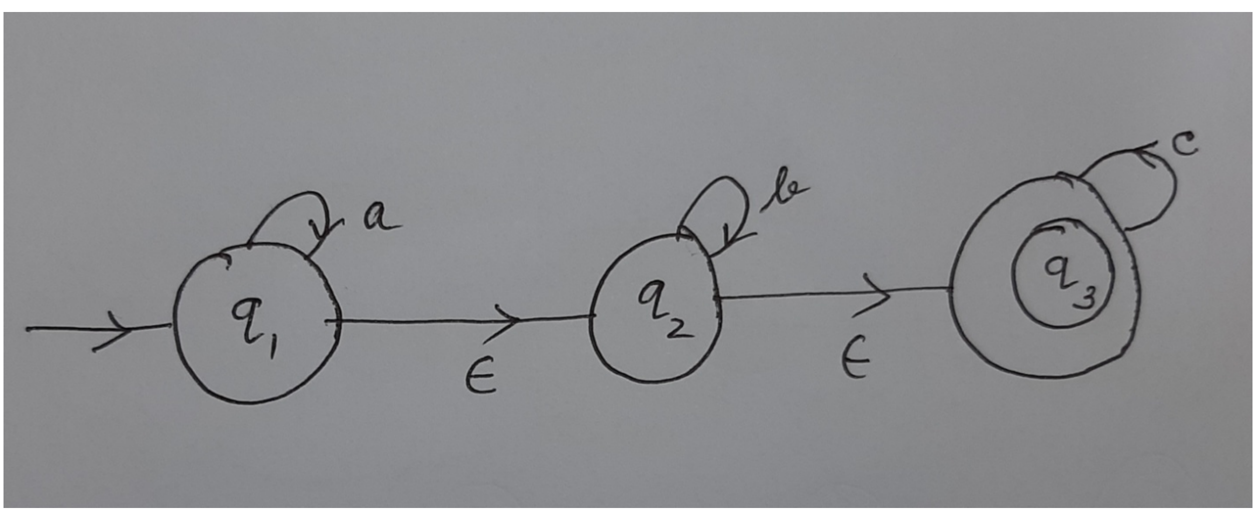
ε-transition : transition which does not consume a symbol, merely changes state

Example – signed decimal integer : optional +, - followed by nonempty string of decimals :

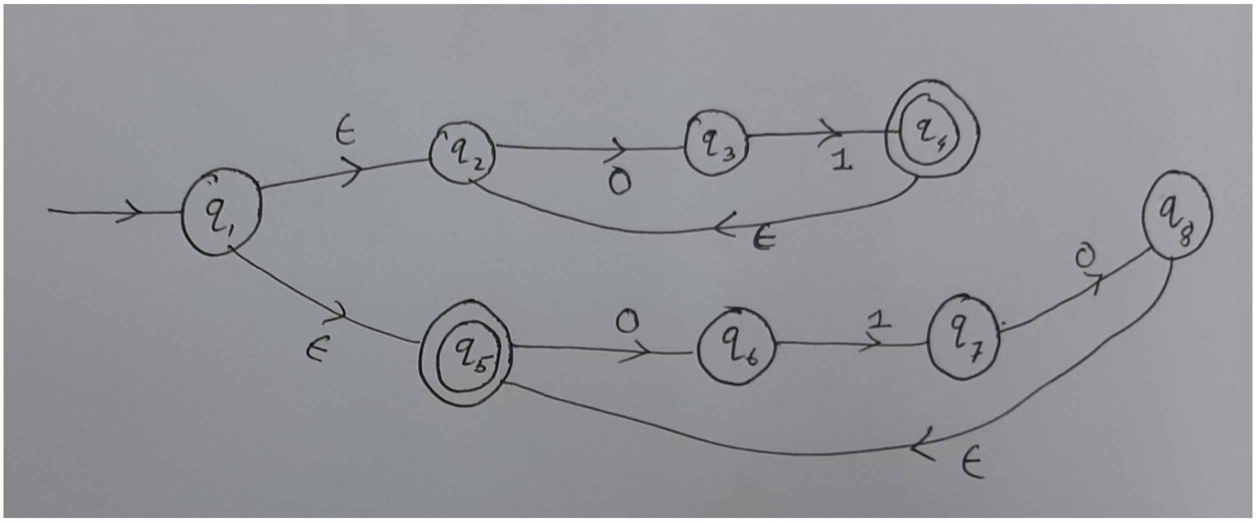


	ε	+, -	0,…,9
$\rightarrow q_1$	$\{q_2\}$	$\{q_2\}$	φ
q_2	φ	φ	$\{q_3\}$
$* \ q_3$	φ	φ	$\{q_3\}$

Formally an ε-NFA is a 5-tuple $M = (Q, \Sigma, \delta, q_0, F)$ where Q, Σ, q_0 and F are as in NFA and $\delta : Q \times \Sigma_e \rightarrow P(Q)$ where $\Sigma_e = \Sigma \cup \{\varepsilon\}$ with ε not in Σ. If $q' \in \delta(q, \varepsilon)$ then $(q, w) \rightarrow (q', w)$. For L = 0 or more a's followed by 0 or more b's followed by 0 or more c's :



Either 01 repeated one or more times or 010 repeated zero or more times.



		ε	0	1
→	q_1	$\{q_2, q_5\}$	φ	φ
	q_2	φ	$\{q_3\}$	φ
	q_3	φ	φ	$\{q_4\}$
*	q_4	$\{q_2\}$	φ	φ
*	q_5	φ	$\{q_6\}$	φ
	q_6	φ	φ	$\{q_7\}$

q_7	φ	$\{q_8\}$	φ
q_8	$\{q_5\}$	φ	φ

A DFA is automatically an ε-NFA with no ε-transitions and rest like DFA-NFA conversion ie other transition-table entries are singleton sets.

For the conversion of an ε-NFA to a DFA we need the notion of ε-closures.

ε-closure of a state :

ec(q) = set of all states that can be reached by a sequence of zero or more ε-transitions from q

Thus $q \in ec(q)$. Also if $p \in ec(q)$ and $p' \in \delta(p,\varepsilon)$ then $p' \in ec(q)$.

ε-closure of a set of states :

$ec(E) = \cup_{q \in E}\, ec(q)$.

A set E is defined to be ε-closed if $ec(E) = E$.

Easy to prove that ec(ec(E)) = ec(E), ie ec(E) is ε-closed for any E. also union of ε-closed states is ε-closed.

Theorem : Let $M = (Q, \sum, \delta, q_0, F)$ be an ε-NFA.

Then M is equivalent to the DFA $M' = (Q', \sum, \delta',$ $q_0', F')$ where Q' = set of ε-closed subsets of Q, $\delta'(E,a) = \mathrm{U}_{q\in E}\, ec(\delta(q,a))$, $q_0' = ec(\{q_0\})$ and $F' = \{E \mid$ E is ε-closed and $E \cap F \neq \varphi\}$. We omit the proof.

Thus a language is regular iff it is accepted by some ε-NFA.

Conversion of ε-NFA to DFA :

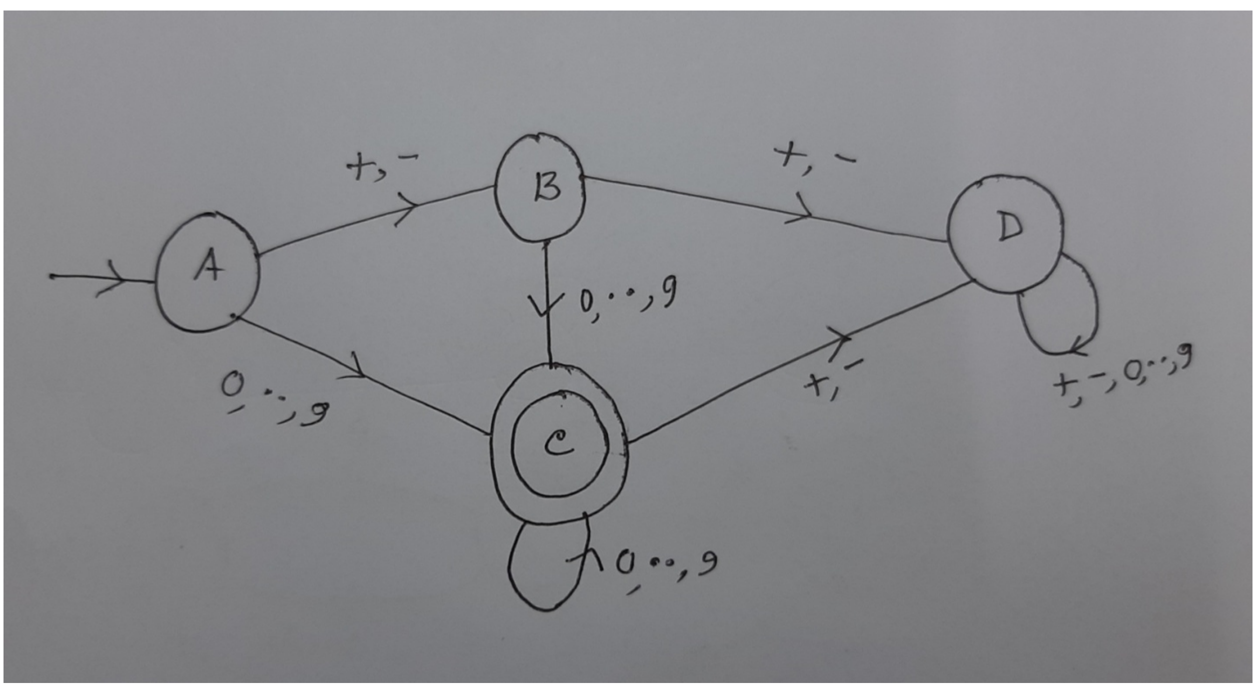
Signed decimal integers

	ε	+,-	0,…,9
$\rightarrow q_1$	$\{q_2\}$	$\{q_2\}$	φ
q_2	φ	φ	$\{q_3\}$

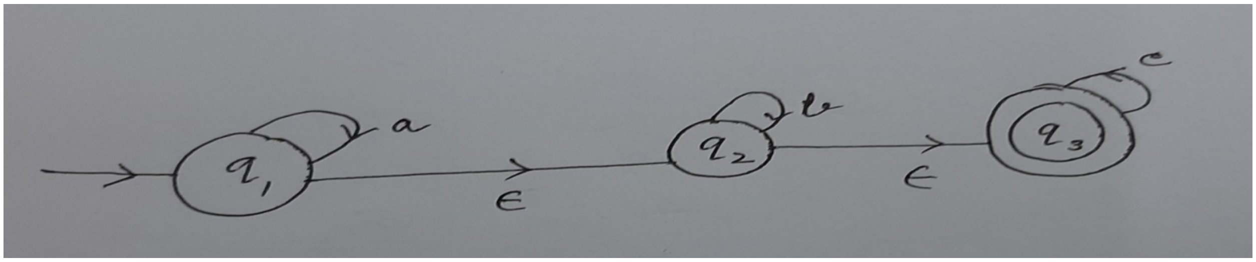
*	q_3	φ	φ	$\{q_3\}$

$ec(q_1) = \{q_2, q_1\}$ $ec(q_2) = \{q_2\}$ $ec(q_3) = \{q_3\}$

		+,-	0,...,9
$\rightarrow$A	$\{q_1,q_2\}$	$\{q_2\}$	$\{q_3\}$
B	$\{q_2\}$	φ	$\{q_3\}$
* C	$\{q_3\}$	φ	$\{q_3\}$
D	φ	φ	φ

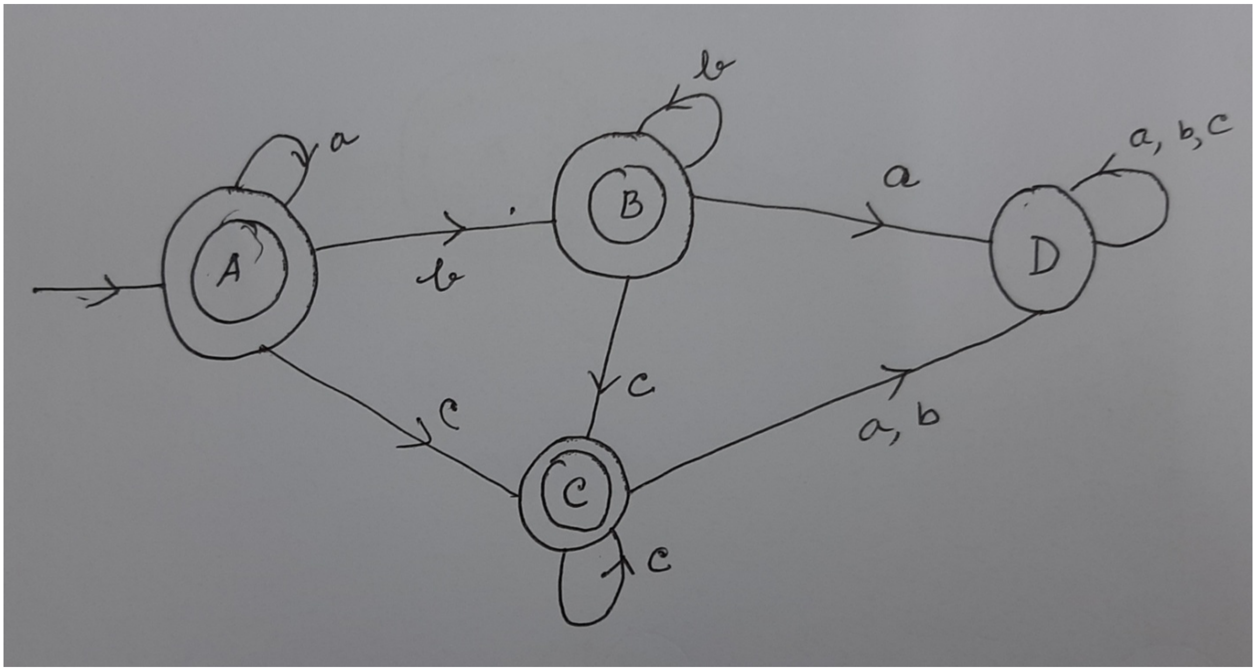


0 or more a’s followed by 0 or more b’s followed by 0 or more c’s. Convert to DFA.

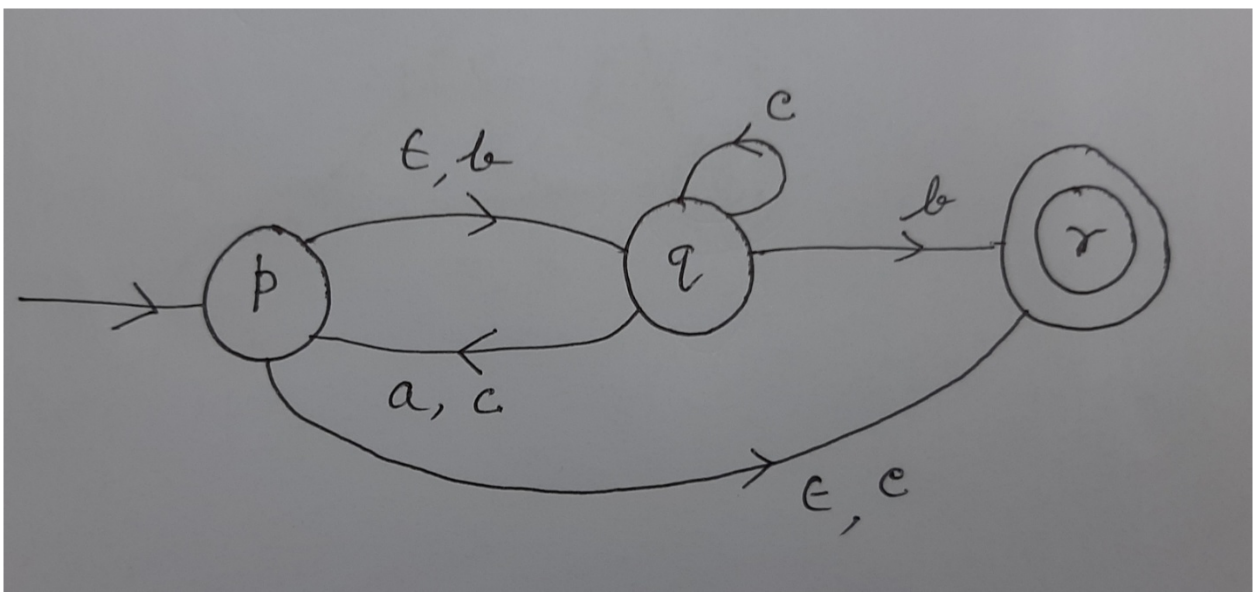


	ε	a	b	c	
→q_1	$\{q_2\}$	$\{q_1\}$	φ	φ	$ec(q_1) = \{q_1,q_2,q_3\}$
q_2	$\{q_3\}$	φ	$\{q_2\}$	φ	$ec(q_2) = \{q_2,q_3\}$
*q_3	φ	φ	φ	q_3	$ec(q_3) = \{q_3\}$

		a	b	c
→ * A	$\{q_1,q_2,q_3\}$	$\{q_1,q_2,q_3\}$	$\{q_2,q_3\}$	$\{q_3\}$
* B	$\{q_2,q_3\}$	φ	$\{q_2,q_3\}$	$\{q_3\}$
*C	$\{q_3\}$	φ	φ	$\{q_3\}$
D	φ	φ	φ	φ

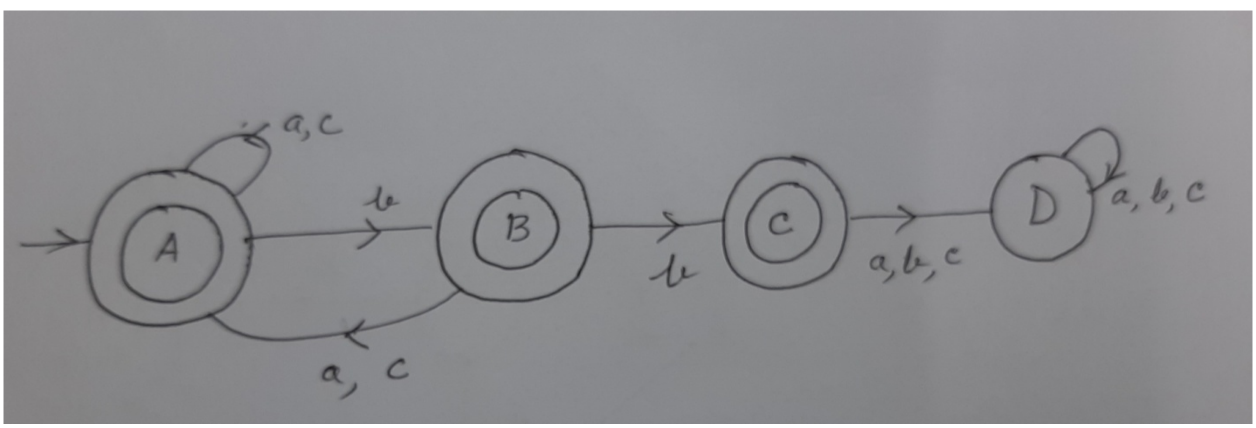


Convert to DFA and describe the language



	ε	a	b	c	
→p	{q,r}	φ	{q}	φ	ec(p) = {p,q,r}
q	φ	{p}	{r}	{p,q}	ec(q) = {q}
* r	φ	φ	φ	φ	ec(r) = {r}

		a	b	c
→*A	{p,q,r}	{p,q,r}	{q,r}	{p,q,r}
*B	{q,r}	{p,q,r}	{r}	{p,q,r}
*C	{r}	φ	φ	φ
D	φ	φ	φ	φ

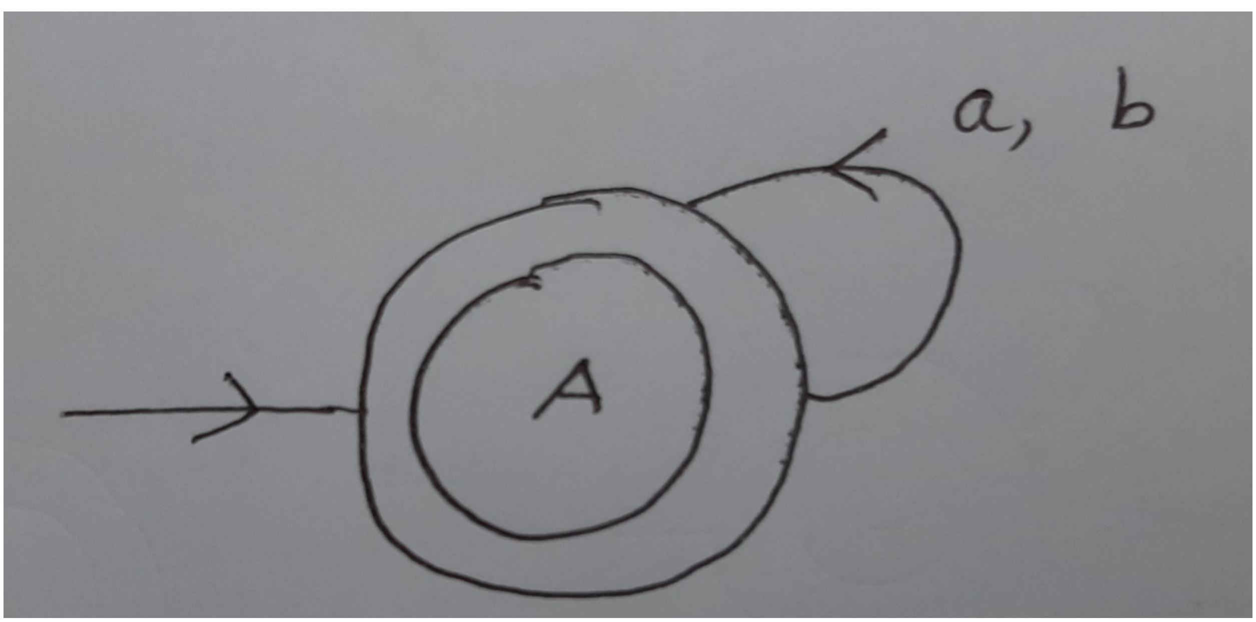


All strings over {a,b,c} such that either there is no bb or bb comes only at the end.

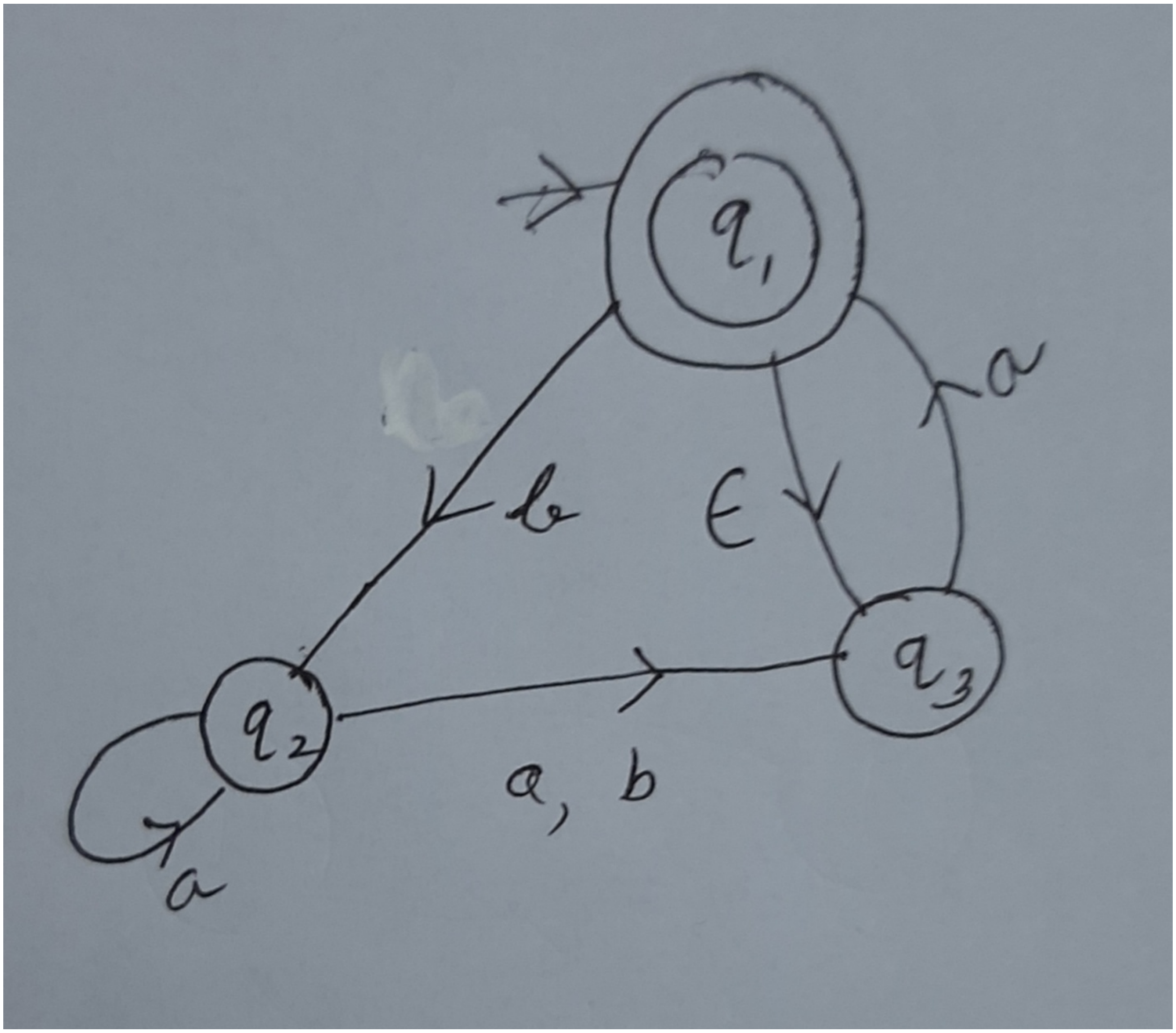
Convert to equivalent DFA

	ε	a	b	
→ p	{r}	{q}	{p,r}	ec(p) = {p,q,r}
q	φ	{p}	φ	ec(q) = {q}
*r	{p,q}	{r}	{p}	ec(r) = {p,q,r}

	a	b
→*A{p,q,r}	{p,q,r}	{p,q,r}



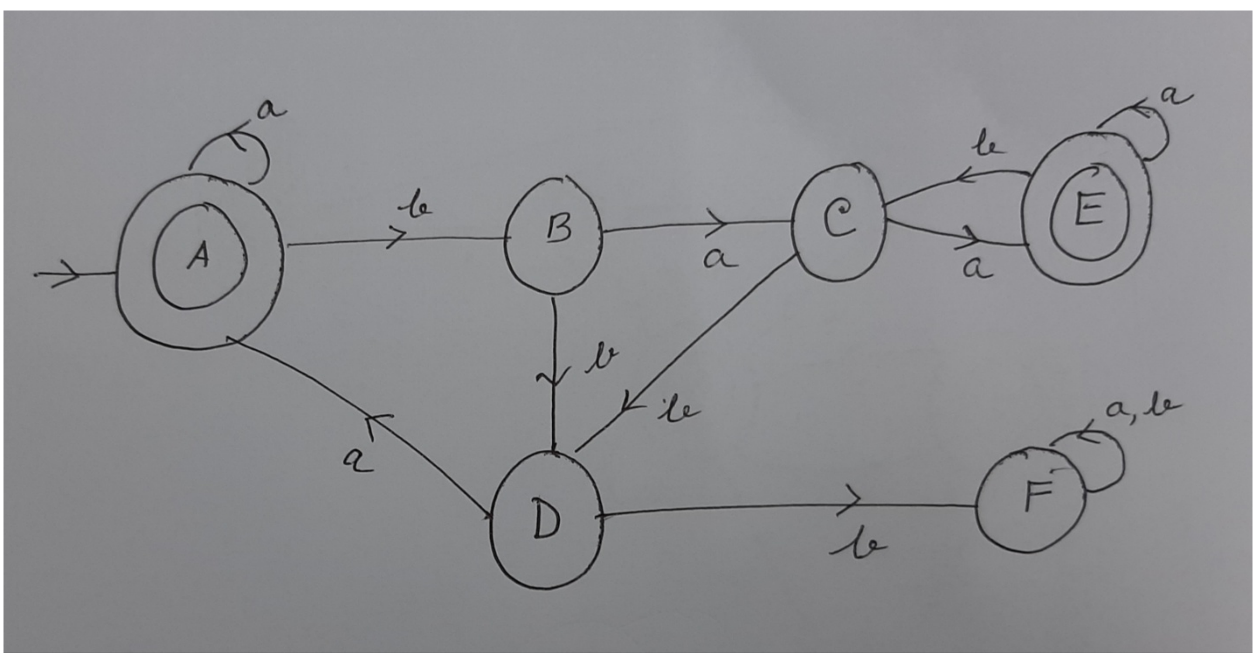
Convert to equivalent DFA



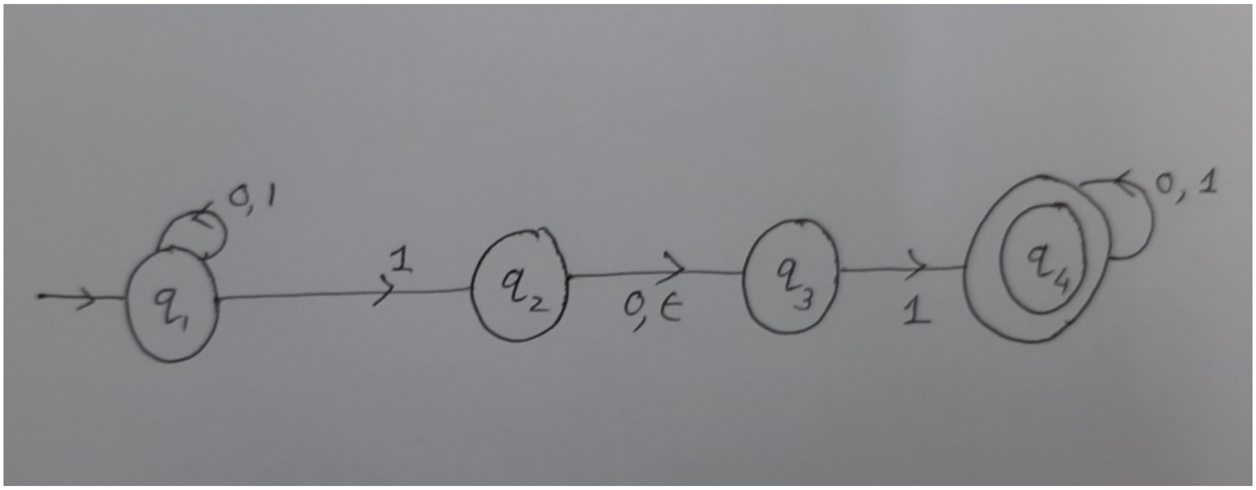
	ε	a	b	
$\rightarrow * q_1$	$\{q_3\}$	φ	$\{q_3\}$	$ec(q_1) = \{q_1, q_3\}$
q_2	φ	$\{q_2, q_3\}$	$\{q_3\}$	$ec(q_2) = \{q_2\}$ $ec(q_3) = \{q_3\}$

q_3	φ	$\{q_1\}$	φ

		a	b
$\rightarrow$*A	$\{q_1,q_3\}$	$\{q_1,q_3\}$	$\{q_2\}$
B	$\{q_2\}$	$\{q_2,q_3\}$	$\{q_3\}$
C	$\{q_2,q_3\}$	$\{q_1,q_2,q_3\}$	$\{q_3\}$
D	$\{q_3\}$	$\{q_1,q_3\}$	φ
*E	$\{q_1,q_2,q_3\}$	$\{q_1,q_2,q_3\}$	$\{q_2,q_3\}$
F	φ	φ	φ

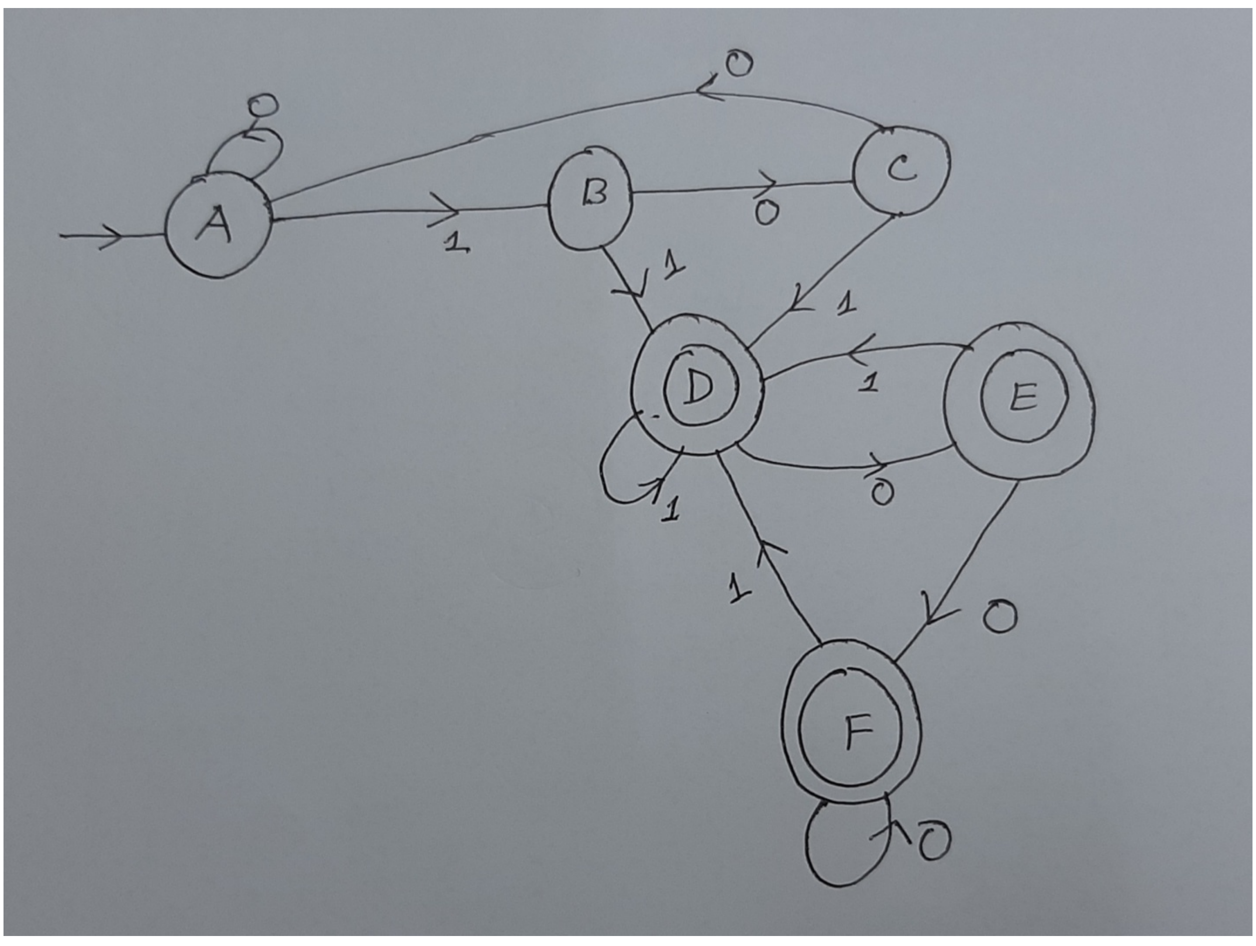


Strings having 101 or 11 as substring

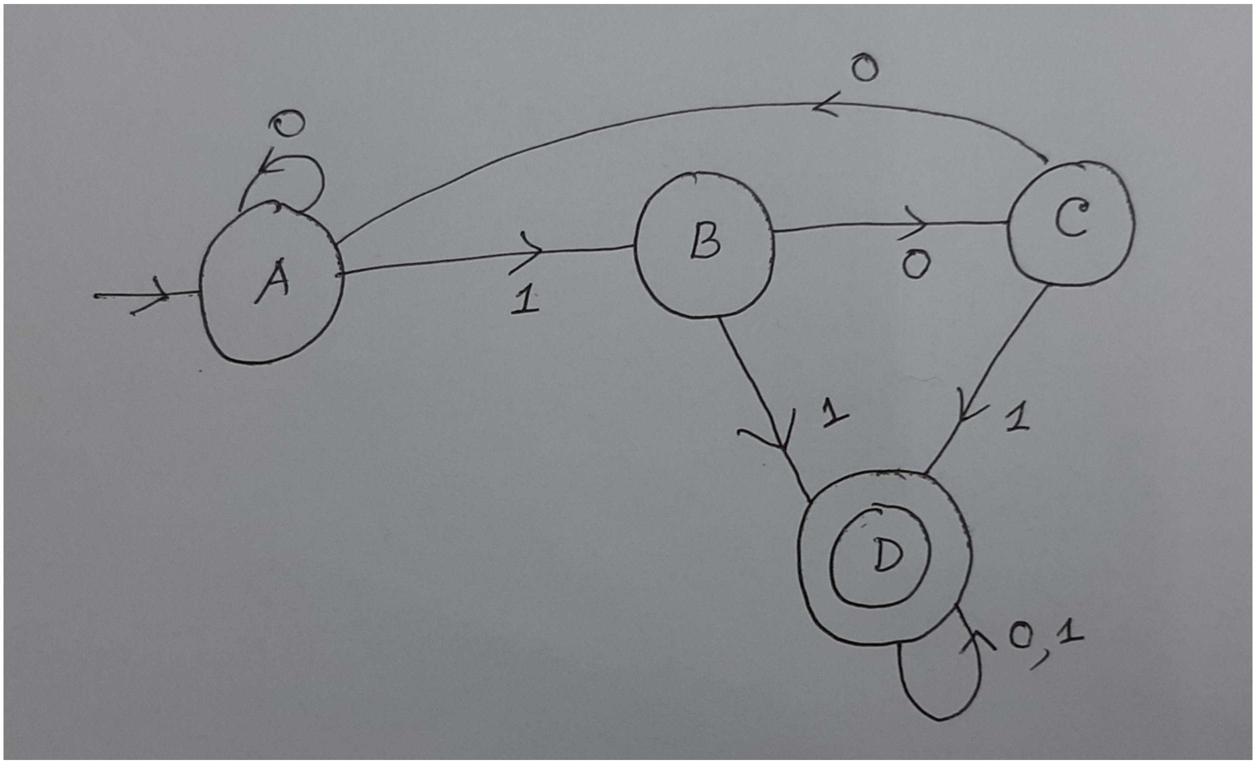


	ε	0	1	
$\rightarrow q_1$	φ	$\{q_1\}$	$\{q_1,q_2\}$	$ec(q_1) = \{q_1\}$
q_2	$\{q_3\}$	$\{q_3\}$	φ	$ec(q_2) = \{q_2,q_3\}$
q_3	φ	φ	$\{q_4\}$	$ec(q_3) = \{q_3\}$
$*q_4$	φ	$\{q_4\}$	$\{q_4\}$	$ec(q_4) = \{q_4\}$

		0	1
→ A	$\{q_1\}$	$\{q_1\}$	$\{q_1\ q_2,q_3\}$
B	$\{q_1,q_2,q_3\}$	$\{q_1,q_3\}$	$\{q_1,q_2,q_3,q_4\}$
C	$\{q_1,q_3\}$	$\{q_1\}$	$\{q_1,q_2,q_3,q_4\}$
*D	$\{q_1,q_2,q_3,q_4\}$	$\{q_1,q_3,q_4\}$	$\{q_1q_2,q_3,q_4\}$
*E	$\{q_1,q_3,q_4\}$	$\{q_1,q_4\}$	$\{q_1,q_2,q_3,q_4\}$
*F	$\{q_1,q_4\}$	$\{q_1,q_4\}$	$\{q_1,q_2,q_3,q_4\}$



A group of final states from which other states cannot be reached can be merged into one.



Closure properties of Regular languages :

We had earlier discussed some simple closure properties using Cartesian products. Using ε-NFA's closure properties of regular languages more completely.

1) Union : Let L1 = L(M1) and L2 = L(M2) where M1=(Q1,∑1,δ1,q_1,F1) and M2= (Q2,∑2, δ2,p_1,F2) are ε-NFA's. As before we can assume ∑1 =∑2 = ∑ = ∑1 U ∑2. Let Q' be the disjoint union of Q1 and Q2, and Q = {q_0} U Q'. Define ε-NFA M = (Q, ∑,δ,{q_0},F1UF2) where δ(q_0,ε)= {q_1,p_1}, δ(q,a)=δ1(q,a) if q Є Q1 and δ(q,a) = δ2(q,a) if q Є Q2. Then it is easy to prove that L1 U L2 = L(M).

2) As before it is easy to prove that L^c is regular if L is regular.

3) Hence if L1,L2 are regular then L1∩L2 = $(L1^c U L2^c)^c$ is regular.

4) Let $L=L(M)$ where $M=(Q,\sum,\delta,q_0,F)$ is an ε-NFA. Let $M^*=(Q,\sum,\delta^*,q_0,F^*)$ and $M^+ = (Q, \sum, \delta^+,q_0,F)$ be ε-NFA's where δ^+ values same as δ values and additionally $\delta^+(q) = \{q_0\}$ if $q \in F$. Also for M^*, δ^* values are same as δ^+ values of M^+ and additionally $F^* = F \cup \{q_0\}$ then it is easy to see that $L^* = L(M^*)$ and $L^+ = L(M^+)$. Thus if L is regular then L^* and L^+ are also regular.
5) Let $L1 = L(M1)$ and $L2 = L(M2)$ where $M1 = (Q1,\sum1,\delta1,q_0,F1)$ and $M2=(Q2,\sum2,\delta2,p_0,F2)$ then let $M = (Q',\sum1 \cup \sum2,\delta,q_0,F2)$ where Q' is the disjoint union of Q1 and Q2, $\delta(q,a) = \delta1(q,a)$ if $q \in Q1$ and $a \in \sum1$ $\delta(q,a) =$

$\delta 2(q,a)$ if $q \in Q2$ and $a \in \sum 2$ $\delta(q,\varepsilon) = \{p_0\}$.

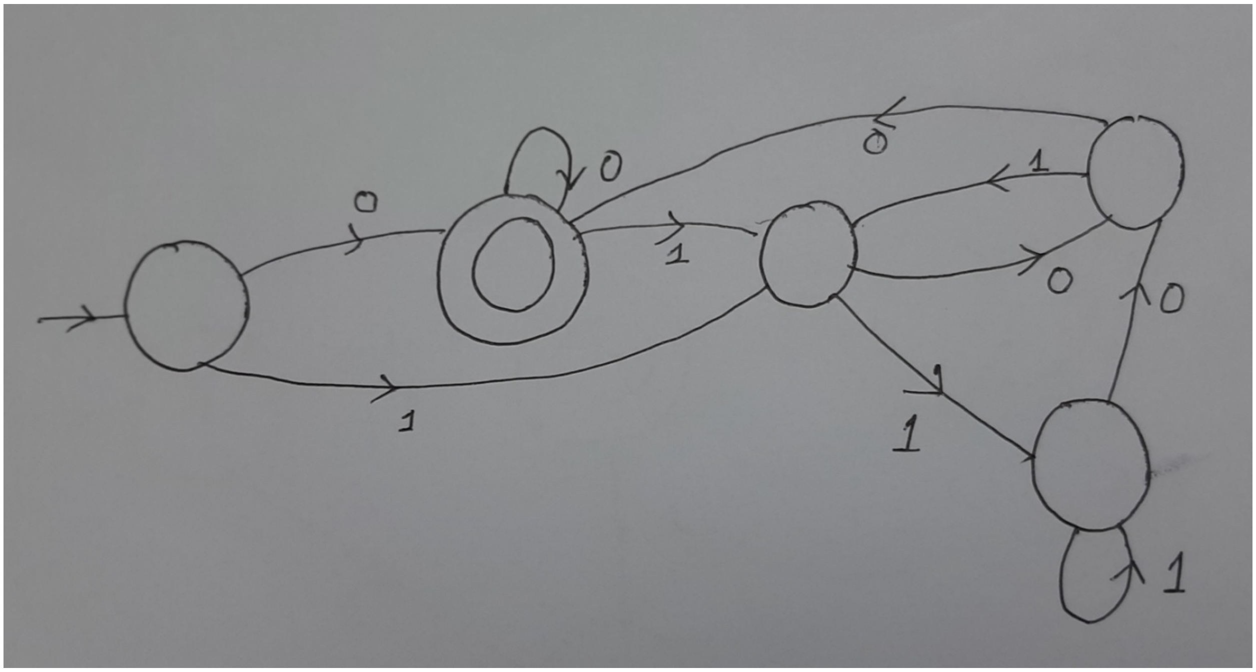
Then it is easy to prove that L(M) = L1 L2. Thus if L1, L2 are regular then L1 L2 is regular.

6) Let L = L(M) where M is an ε-NFA. Construct ε-NFA M^R by :
   a) Choose the start state of M to be the only final state of M^R.
   b) Choose a new start state in M^R and take ε-transitions from this to the final states of M. If there is only one final state of M then it can be taken to be the start state of M^R.
   c) Reverse all arrows.

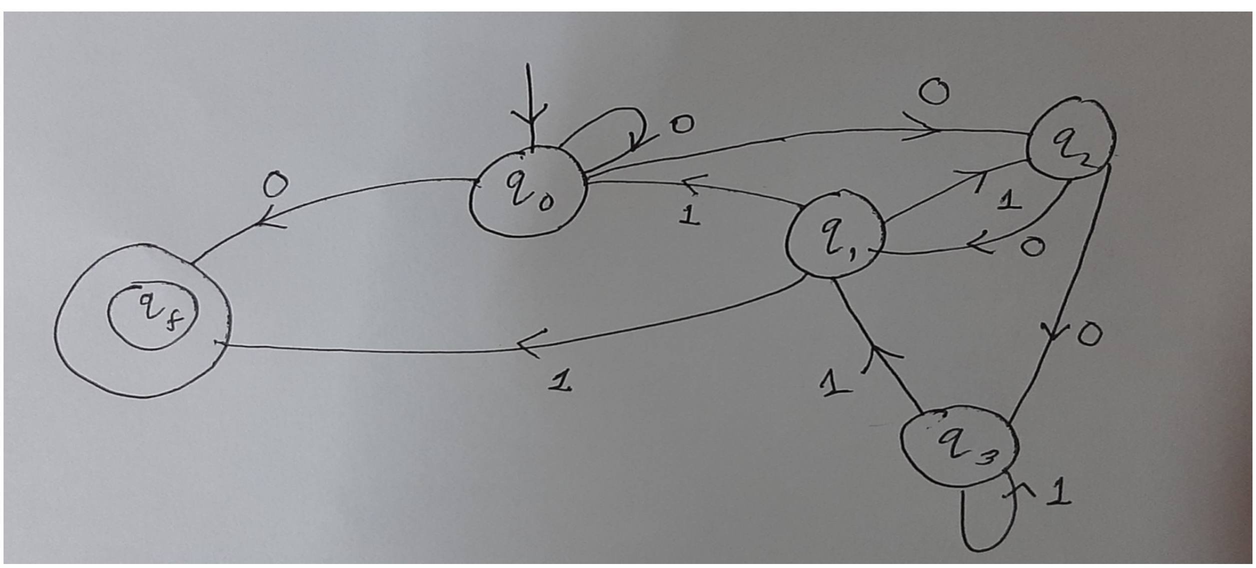
   Then $L(M^R) = L^R$.

Find ε-NFA for reversals of nonempty binary strings which represent multiples of 4 and find an equivalent DFA.

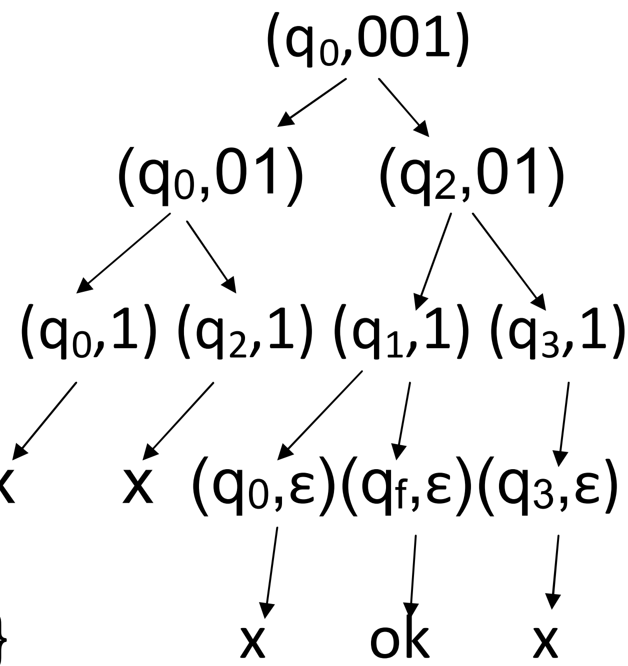
Multiples of 4



Since there is only one final state and there are no ε-transitions in the original automaton, we get a NFA for the reversals.

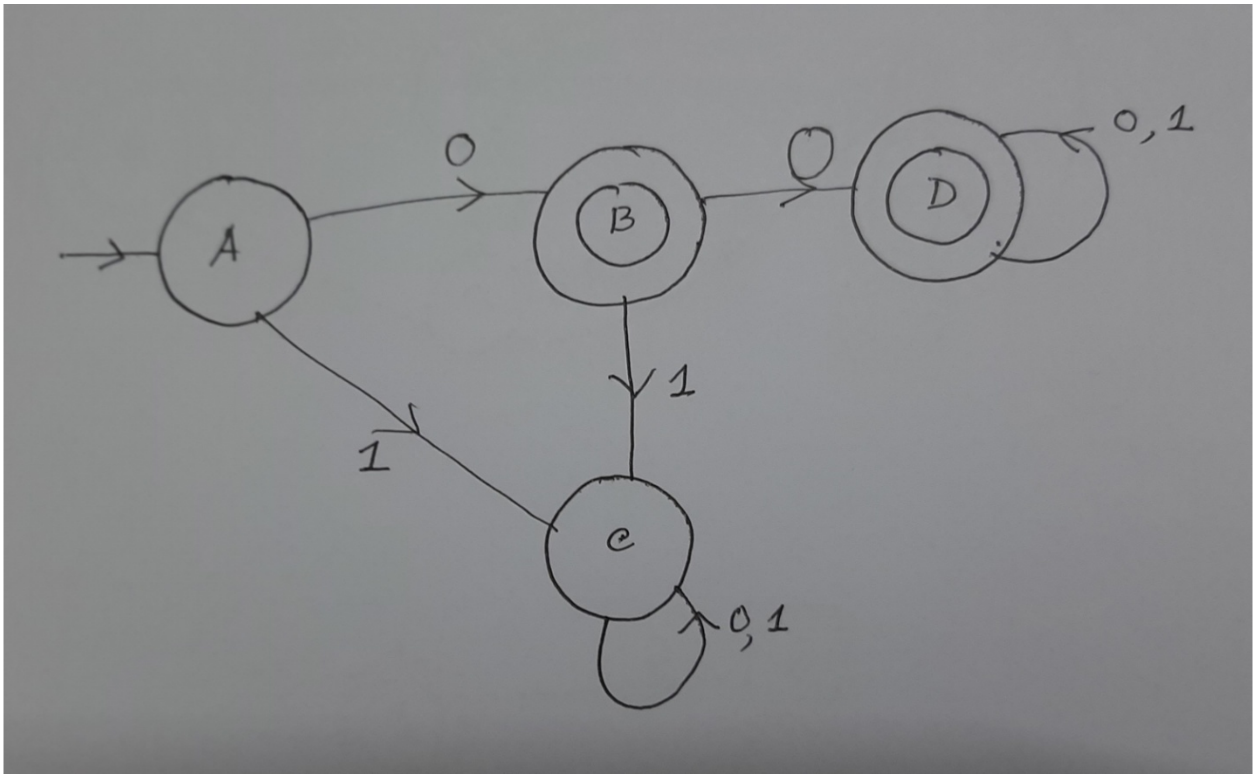


	0	1
$\rightarrow q_0$	$\{q_0,q_2,q_f\}$	φ
q_1	φ	$\{q_0,q_2,q_f\}$
q_2	$\{q_1,q_3\}$	φ
q_3	φ	$\{q_1,q_3\}$
$*\ q_f$	φ	φ



Converting to DFA

		0	1
$\rightarrow$ A	$\{q_0\}$	$\{q_0,q_2,q_f\}$	φ
* B	$\{q_0,q_2,q_f\}$	$\{q_0,q_1,q_2,q_3,q_f\}$	φ
C	φ	φ	φ
* D	$\{q_0,q_1,q_2,q_3,q_f\}$	$\{q_0,q_1,q_2,q_{3,}q_f\}$	$\{q_0,q_1q_2,q_3,q_f\}$



Checking	w	w^R	
	0	0	ok
	1	1	x
	00	00	ok
	01	10	x
	10	01	x
	11	11	x
	"000"	000	ok

“001”   100 x

“010” 010 x

“011” 110 x

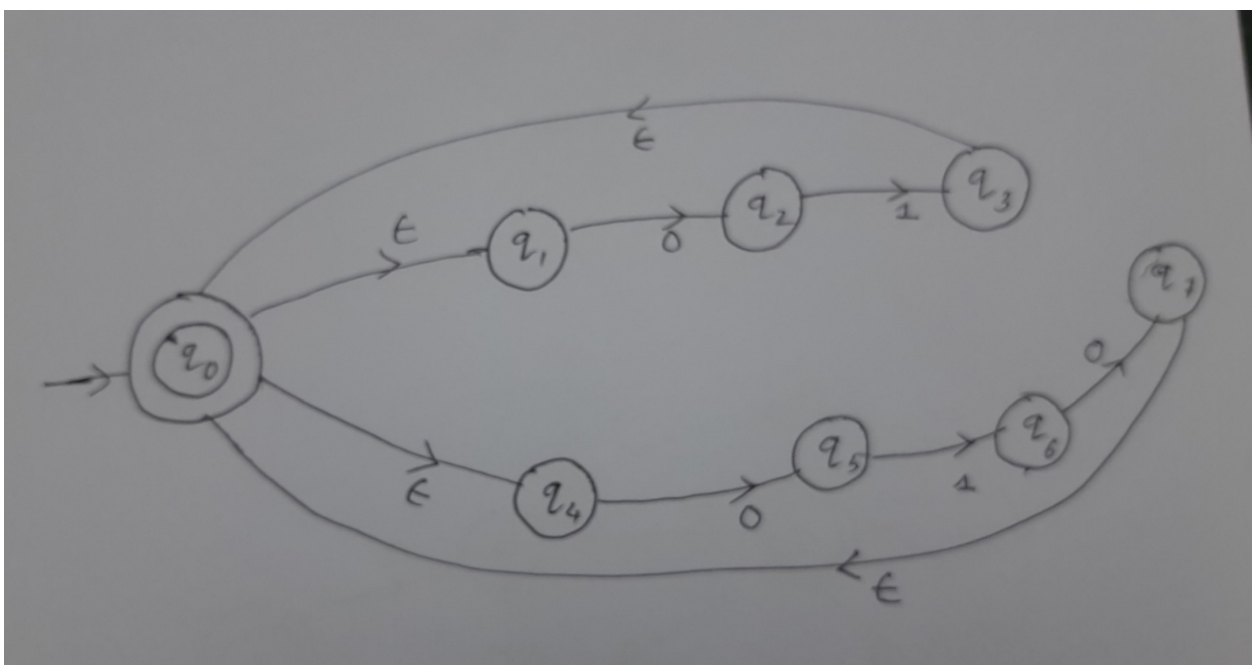
100   001 ok

101   101 x

110   011 x

111   111   x

ε-NFA for (01 U 010)* . Convert to DFA.



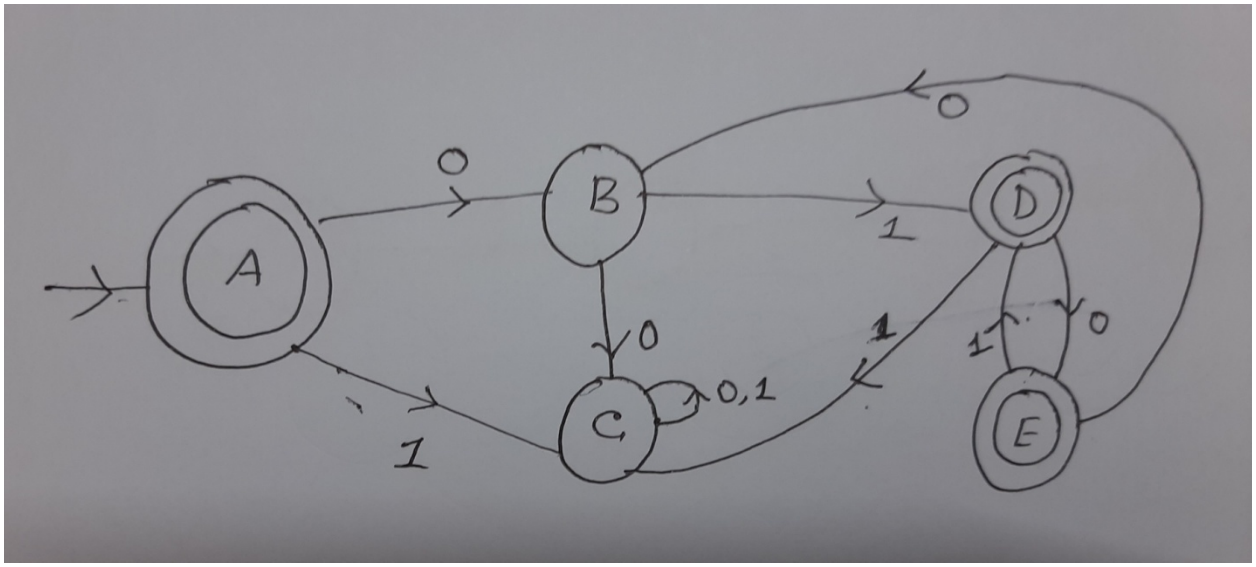
$ec(q_0) = \{q_0,q_1q_4\}$, $ec(q_1) = \{q_1\}$, $ec(q_2) = \{q_2\}$,
$ec(q_3) = \{q_3,q_0,q_1,q_4\}$, $ec(q_4) = \{q_4\}$, $ec(q_5) = \{q_5\}$,
$ec(q_6) = \{q_6\}$, $ec(q_7) = \{q_7,q_0,q_1,q_4\}$

DFA	0	1
→*A $\{q_0,q_1,q_4\}$	$\{q_2,q_5\}$	φ
B $\{q_2,q_5\}$	φ	$\{q_3,q_6,q_0,q_1q_4\}$
C φ	φ	φ
* D $\{q_3,q_6,q_0,q_1,q_4\}$	$\{q_7,q_2,q_5,q_0,q_1,q_4\}$	φ
* E $\{q_7,q_2,q_5,q_0,q_1,q_4\}$	$\{q_2,q_5\}$	$\{q_7,q_2,q_5,q_0,q_1,q_4\}$

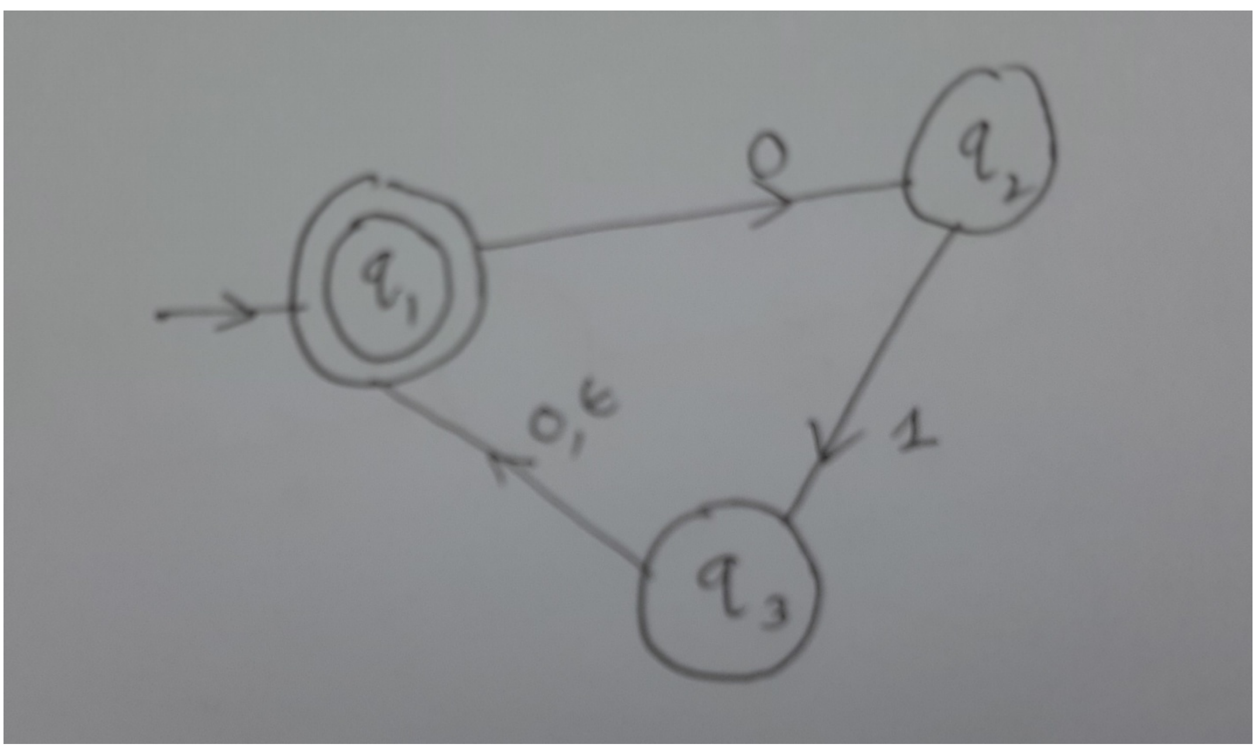
Or	0	1
→*A	B	C
B	C	D
C	C	C

* D   E   C

* E   B   D



Simplified ε-NFA



	ε	0	1	
$\rightarrow *\ q_1$	φ	$\{q_2\}$	φ	$ec(q_1) = \{q_1\}$
q_2	φ	φ	$\{q_3\}$	$ec(q_2) = \{q_2\}$
q_3	$\{q_1\}$	$\{q_1\}$	φ	$ec(q_3) = \{q_1, q_3\}$

Converting		0	1
$\rightarrow *A$	$\{q_1\}$	$\{q_2\}$	φ
B	$\{q_2\}$	φ	$\{q_3, q_1\}$
C	φ	φ	φ
* D	$\{q_3, q_1\}$	$\{q_1, q_2\}$	φ
* E	$\{q_1, q_2\}$	$\{q_2\}$	$\{q_3, q_1\}$

Same DFA