Non-regular languages - The pumping lemma

Non-Regular Languages

- However $L = \{a^n b^n : n \geq 0\} = \bigcup_{n \geq 0} L_n$ doesn't seem to be a regular language at all!
- We need an infinite number of states to build this automaton!
- (Observe that you cannot use the fact that regular languages are closed under union because we have an infinite union)

Is this a proof?

NO! In fact consider:

L’ = {s : s contains equal number of a and b}

L’’ = {s : s contains equal number of ab and ba}

L’ is indeed not regular but L’’ is regular!

<u>WE NEED A MATHEMATICAL PROOF!!!</u>

A proof that there is no FA that accepts L or L’.

The pumping lemma

For every infinite *regular* language L

there exists a pumping length $n > 0$ such that

for any string s in L with length $|s| \geq n$

we can write $s = xyz$

with $|xy| \leq n$ and $|y| \geq 1$

such that xy^iz *in L* for every $i \geq 0$.

Proof

- If L is regular then there exists a DFA M which accepts L. Set n to be M’s number of states.
- L is infinite, there exists a string s with length greater than n.
- The number of states is n, the accepting path for s is of length at most n.
- The string is of length at least n, there is a part in the path that is repeated.

Proof

- Split s into 3 parts x, y, z with y being the first repeated part.
- Since we have n states the first repetition should take place ($|y| \geq 1$) in at most n transitions ($|xy| \leq n$).
- Since the path under y is a loop we can follow it as many times as we want (maybe none).
- Thus xy^iz for any $i \geq 0$ should lead us to the same accepting state as xyz.

Prove that L is not regular

- Given is an infinite language L.

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  (What if L is finite?)

Prove that L is not regular

- Given is an infinite language L
- If L is regular

Prove that L is not regular

- Given is an infinite language L
- If L is regular
- Pumping lemma holds:
  - *There exists* a pumping length n such that
  - *for all* proper strings s in L
  - *there is* a splitting of s in x,y,z (with the desired properties) such that
  - *for all* i xy^iz is in L.

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$|s| \geq n$

$|y| \geq 1$ and $|xy| \leq n$

Prove that L is not regular

- Given is an infinite language L
- If L is regular the pumping lemma holds.
- The negation of the pumping lemma:
  - *For all* pumping lengths n
  - *there exists* a proper string s in L such that
  - *for every possible* splitting of s in x,y,z (with the desired properties)
  - *there is* an i for which xy^iz is *not* in L.

Prove that L is not regular

- Given is an infinite language L
- Assume L is regular (pumping lemma holds)
- Prove the negation of the pumping lemma:
  - *Fix* an *arbitrary* pumping length n.
  - *Specify* a proper string s in L.
  - Show that *for every possible* splitting of s in x,y,z (with the desired properties),
  - *there is* an i for which xy^iz is *not* in L.

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- Assume L is regular (pumping lemma holds)
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- Contradiction!!!

Prove that L is not regular

- Given is an infinite language L
- Assume L is regular (pumping lemma holds)
- Prove the negation of the pumping lemma:
  - *Fix* an *arbitrary* pumping length n.
  - *Specify* a proper string s in L.
  - Show that *for every possible* splitting of s in x,y,z (with the desired properties),
  - *there is* an i for which xy^iz is *not* in L.
- Contradiction!!! **L is not regular**

Example

- $L = \{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Assume that L is regular. The pumping lemma holds!

Example

- L = $\{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Fix an arbitrary pumping length *k* for L.

Example

- $L = \{a^n b^n : n \geq 0\}$ is not regular.

Proof:

The string $s = a^k b^k$ should be in the language.

Example

- L = $\{a^n b^n : n \geq 0\}$ is not regular.

Proof:

s is proper: $|s| = 2k$ is greater than k

Example

- $L = \{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Consider all possible splittings of $a^k b^k$ in the form xyz with the desired properties:

- $|xy| \leq k$ and
- $|y| \geq 1$

Example

- L = $\{a^n b^n : n \geq 0\}$ is not regular.

<u>Proof:</u>

Consider all possible splittings of $a^k b^k$ in the form xyz with the <u>desired properties</u>:

- $|xy| \leq k$ and
- $|y| \geq 1$

$$\overbrace{a\ a\ a\ a\ a\ a\ a\ a\ a}^{k}\ \overbrace{b\ b\ b\ b\ b\ b\ b\ b\ b}^{k}$$

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Example

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Proof:

Consider all possible splittings of $a^k b^k$ in the form xyz with the desired properties

- $y = a^m$, for $1 \leq m \leq k$

$$\overbrace{a\,a\,a\,a\,a\,a\,a\,a\,a}^{k}\;\overbrace{b\,b\,b\,b\,b\,b\,b\,b\,b}^{k}$$

$$\underbrace{a\,a\,a\,a\,a}_{x}\;\underbrace{a\,a\,a}_{y}\;\underbrace{a\,b\,b\,b\,b\,b\,b\,b\,b\,b}_{z}$$

Example

- $L = \{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Consider all possible splittings of $a^k b^k$ in the form xyz with the desired properties

- $y = a^m$, for $1 \leq m \leq k$
- for $i = 2$, $xy^2z = a^{k+m}b^k$ is not in L!

$$\underbrace{a\ a\ a\ a\ a}_{x}\ \underbrace{a\ a\ a}_{y}\ \underbrace{a\ a\ a}_{y}\ \underbrace{a\ b\ b\ b\ b\ b\ b\ b\ b\ b}_{z}$$

Example

- $L = \{a^n b^n : n \geq 0\}$ is not regular.

<u>*Proof:*</u>

Consider all possible splittings of $a^k b^k$ in the form xyz with the <u>desired properties</u>

- xy^2z is not in L!

CONTRADICTION!

$$\underbrace{a\ a\ a\ a\ a}_{x}\ \underbrace{a\ a\ a}_{y}\ \underbrace{a\ a\ a}_{y}\ \underbrace{a\ b\ b\ b\ b\ b\ b\ b\ b\ b}_{z}$$

Try it yourself

- Show that the following language is not a regular language:

$$L_p' = \{ab^jc^j \mid j \geq 0\}$$

- Members: a, abc, abbcc, abbbccc, …

Try it yourself

- Show that the following language is not a regular language:

$$L_p' = \{ab^jc^j \mid j \geq 0\}$$

- Negation of the pumping lemma (reminder):
  - *Fix* an *arbitrary* pumping length n.
  - *Specify* a proper string s in L.
  - Show that *for every possible* splitting of s in x,y,z (with the desired properties),
  - *there is* an i for which xy^iz is *not* in L.