

Non-regular languages - The pumping lemma

Non-Regular Languages

- However $L = \{a^n b^n : n \geq 0\} = \bigcup_{n \geq 0} L_n$ doesn't seem to be a regular language at all!
- We need an infinite number of states to build this automaton!
- (Observe that you cannot use the fact that regular languages are closed under union because we have an infinite union)

Is this a proof?

NO! In fact consider:

$L' = \{s : s \text{ contains equal number of } a \text{ and } b\}$

$L'' = \{s : s \text{ contains equal number of } ab \text{ and } ba\}$

L' is indeed not regular but L'' is regular!

WE NEED A MATHEMATICAL PROOF!!!

A proof that there is no FA that accepts L or L' .

The pumping lemma

For every infinite *regular* language L
there exists a pumping length $n > 0$ such that
for any string s in L with length $|s| \geq n$
we can write $s = xyz$
with $|xy| \leq n$ and $|y| \geq 1$
such that $xy^iz \in L$ for every $i \geq 0$.

Proof

- If L is regular then there exists a DFA M which accepts L . Set n to be M 's number of states.
- L is infinite, there exists a string s with length greater than n .
- The number of states is n , the accepting path for s is of length at most n .
- The string is of length at least n , there is a part in the path that is repeated.

Proof

- Split s into 3 parts x, y, z with y being the first repeated part.
- Since we have n states the first repetition should take place ($|y| \geq 1$) in at most n transitions ($|xy| \leq n$).
- Since the path under y is a loop we can follow it as many times as we want (maybe none).
- Thus xy^iz for any $i \geq 0$ should lead us to the same accepting state as xyz .

Prove that L is not regular

- Given is an infinite language L .

Prove that L is not regular

- Given is an infinite language L .
(*What if L is finite?*)

Prove that L is not regular

- Given is an infinite language L
- If L is regular

Prove that L is not regular

- Given is an infinite language L
- If L is regular
- Pumping lemma holds:
 - *There exists* a pumping length n such that
 - *for all* proper strings s in L
 - *there is* a splitting of s in x,y,z (with the desired properties) such that
 - *for all* i xy^iz is in L.

Prove that L is not regular

- Given is an infinite language L
- If L is regular
- Pumping lemma holds:
 - *There exists* a pumping length n such that
 - *for all* proper strings s in L
 - *there is* a splitting of s in x,y,z (with the desired properties) such that
 - *for all* i xy^iz is in L.


$$|s| \geq n$$


$$|y| \geq 1 \text{ and } |xy| \leq n$$

Prove that L is not regular

- Given is an infinite language L
- If L is regular the pumping lemma holds.
- The negation of the pumping lemma:
 - *For all* pumping lengths n
 - *there exists* a proper string s in L such that
 - *for every possible* splitting of s in x,y,z (with the desired properties)
 - *there is* an i for which xy^iz is *not* in L.

Prove that L is not regular

- Given is an infinite language L
- Assume L is regular (pumping lemma holds)
- Prove the negation of the pumping lemma:
 - *Fix* an *arbitrary* pumping length n .
 - *Specify* a proper string s in L.
 - Show that *for every possible* splitting of s in x,y,z (with the desired properties),
 - *there is* an i for which xy^iz is *not* in L.

Prove that L is not regular

- Given is an infinite language L
- Assume L is regular (pumping lemma holds)
- Prove the negation of the pumping lemma:
 - *Fix* an *arbitrary* pumping length n .
 - *Specify* a proper string s in L .
 - Show that *for every possible* splitting of s in x,y,z (with the desired properties),
 - *there is* an i for which xy^iz is *not* in L .
- Contradiction!!!

Prove that L is not regular

- Given is an infinite language L
- Assume L is regular (pumping lemma holds)
- Prove the negation of the pumping lemma:
 - *Fix* an *arbitrary* pumping length n .
 - *Specify* a proper string s in L.
 - Show that *for every possible* splitting of s in x,y,z (with the desired properties),
 - *there is* an i for which xy^iz is *not* in L.
- Contradiction!!! **L is not regular**

Example

- $L = \{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Assume that L is regular. The pumping lemma holds!

Example

- $L = \{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Fix an arbitrary pumping length k for L .

Example

- $L = \{a^n b^n : n \geq 0\}$ is not regular.

Proof:

The string $s = a^k b^k$ should be in the language.

Example

- $L = \{a^n b^n : n \geq 0\}$ is not regular.

Proof:

s is proper: $|s| = 2k$ is greater than k

Example

- $L = \{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Consider all possible splittings of $a^k b^k$ in the form xyz with the desired properties:

- $|xy| \leq k$ and
- $|y| \geq 1$

Example

- $L = \{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Consider all possible splittings of $a^k b^k$ in the form xyz with the desired properties:

- $|xy| \leq k$ and
- $|y| \geq 1$

The diagram shows the string $a^k b^k$ with 10 'a's and 10 'b's. A blue bracket above the first 10 characters is labeled with a yellow k . Another blue bracket above the next 10 characters is labeled with a yellow k . This illustrates a split of the string into xy and z where $|xy| = 10 \leq k$ and $|y| \geq 1$.

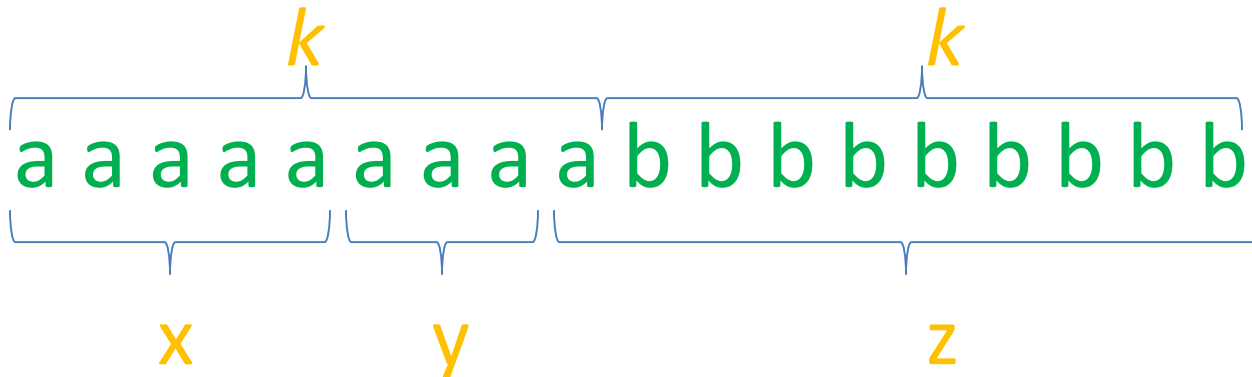
Example

- $L = \{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Consider all possible splittings of $a^k b^k$ in the form xyz with the desired properties:

- $|xy| \leq k$ and
- $|y| \geq 1$



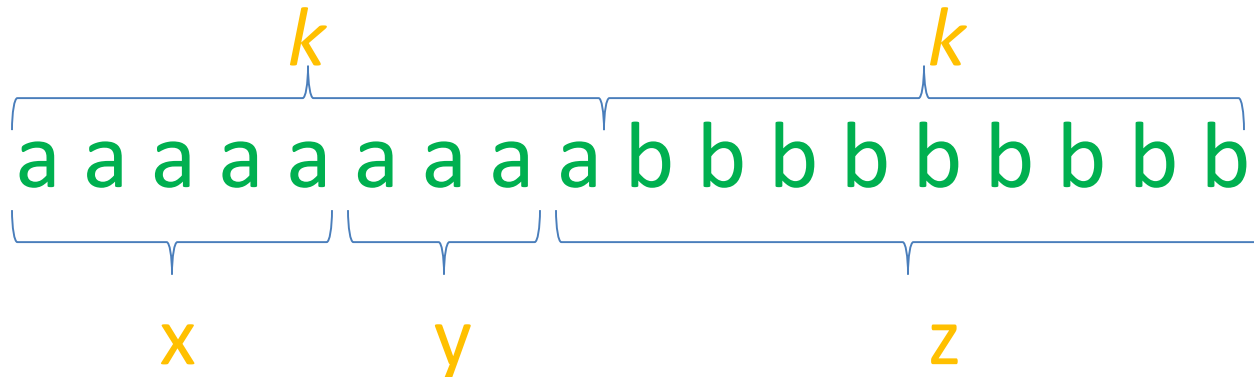
Example

- $L = \{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Consider all possible splittings of $a^k b^k$ in the form xyz with the desired properties

- $y = a^m$, for $1 \leq m \leq k$



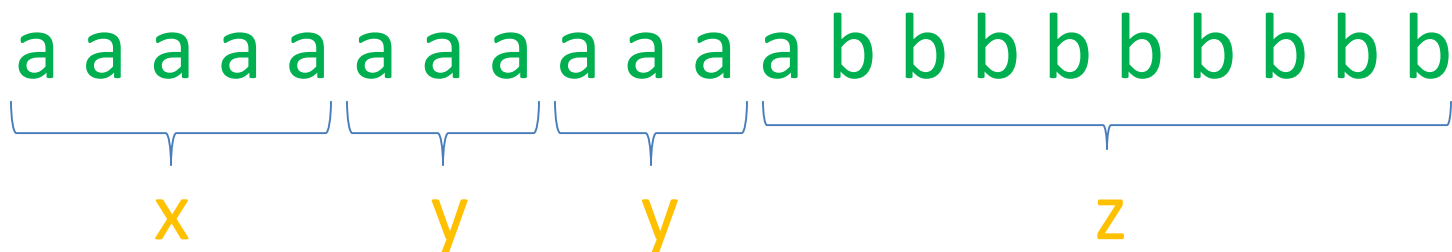
Example

- $L = \{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Consider all possible splittings of $a^k b^k$ in the form xyz with the desired properties

- $y = a^m$, for $1 \leq m \leq k$
- for $i = 2$, $xy^2z = a^{k+m}b^k$ is not in L !



Example

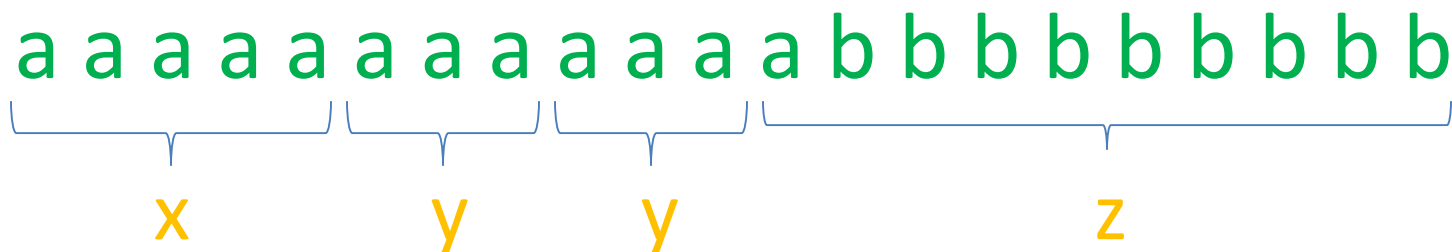
- $L = \{a^n b^n : n \geq 0\}$ is not regular.

Proof:

Consider all possible splittings of $a^k b^k$ in the form xyz with the desired properties

- xy^2z is not in L !

CONTRADICTION!



Try it yourself

- Show that the following language is not a regular language:

$$L_p' = \{ab^j c^j \mid j \geq 0\}$$

- Members: a, abc, abbcc, abbbccc, ...

Try it yourself

- Show that the following language is not a regular language:

$$L_p' = \{ab^j c^j \mid j \geq 0\}$$

- Negation of the pumping lemma (reminder):
 - *Fix* an *arbitrary* pumping length n .
 - *Specify* a proper string s in L .
 - Show that *for every possible* splitting of s in x, y, z (with the desired properties),
 - *there is* an i for which $xy^i z$ is *not* in L .