GATE (ALL BRANCHES)

Engineering Mathematics

Complex Analysis

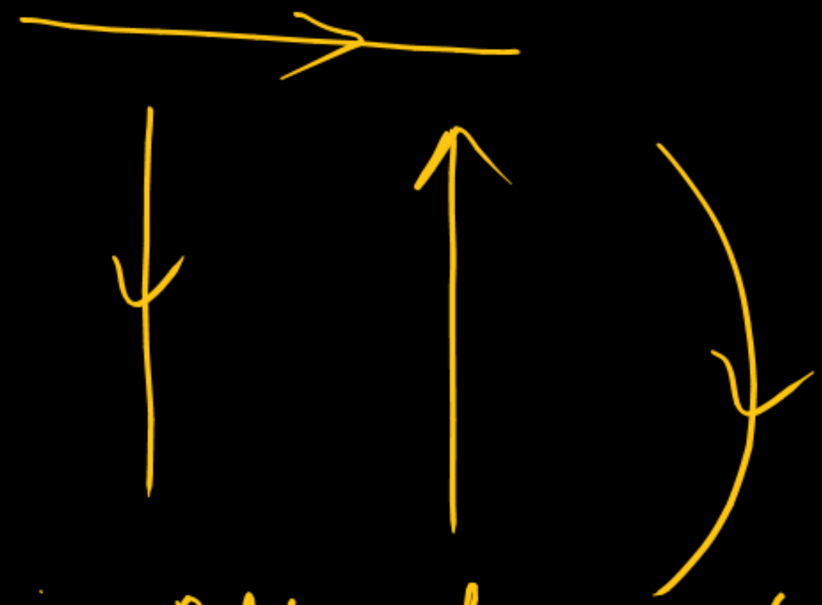
Lecture No. 04

By- Rahul Sir

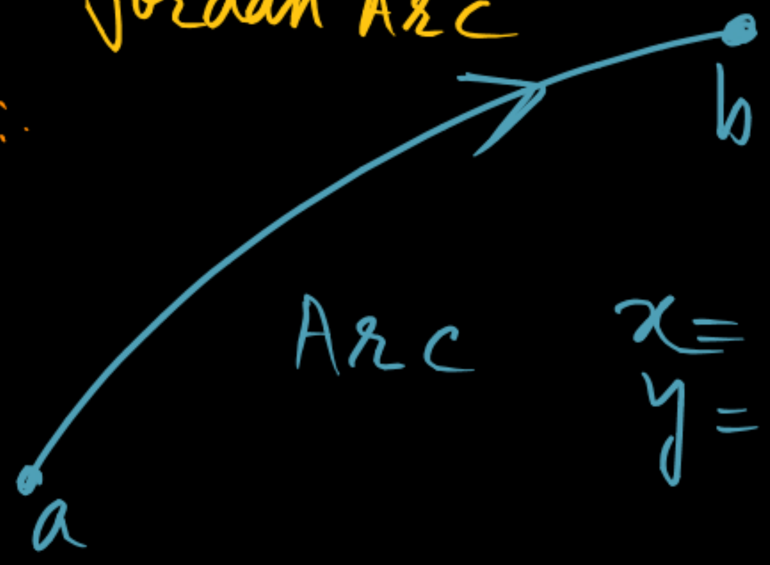
01 Complex Integral , Cauchy Integral Theorem, Cauchy Integral Formula

Complex Integration:

Arc or contours:



Jordan Arc



$\left.\begin{array}{l} x \\ y \end{array}\right]$ t value SAME

Arc $\begin{array}{l} x = \phi(t) \\ y = \psi(t) \end{array}$ Parametric form.

Jordan Arc → t – SAME

Contour curve

Closed curve

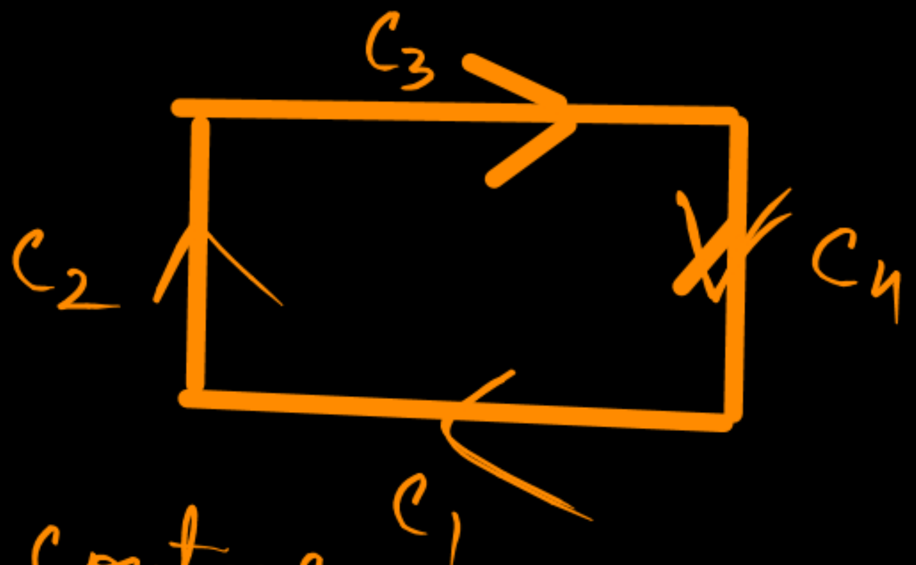
If t is different $z(t) = x(t) + iy(t)$

$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right]$ different t is different

Then It is called a contour.

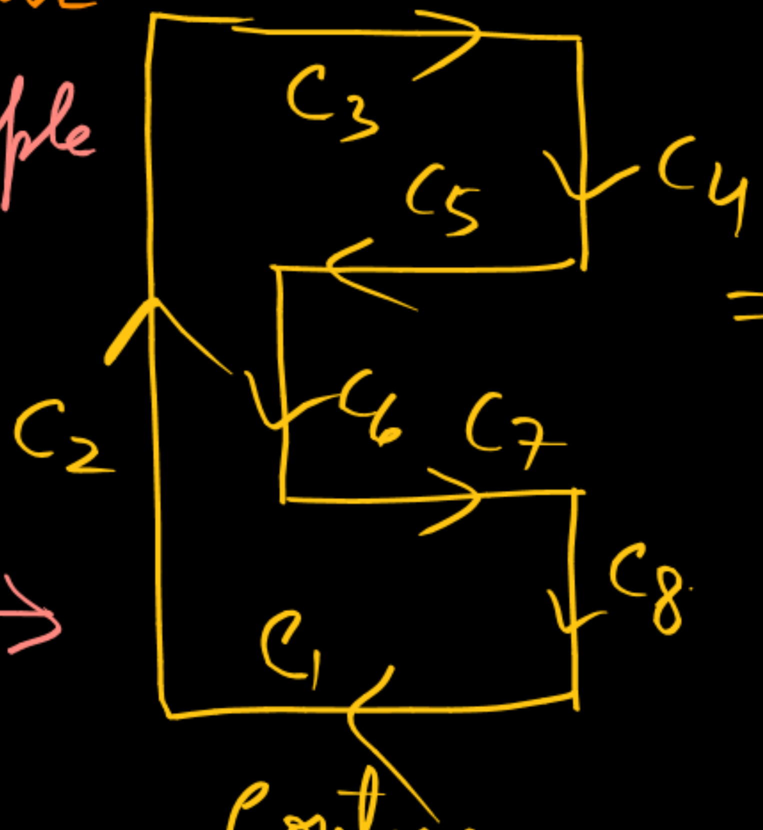
Contour closed

$\begin{array}{l} x = \cos t \\ y = \sin t \end{array}$ t is diff.



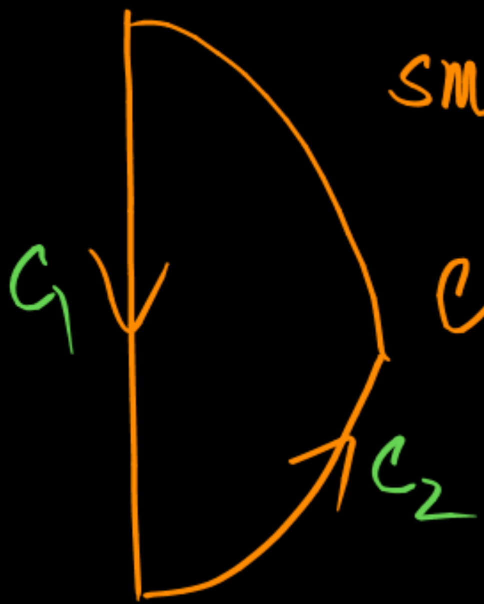
Contour

many simple
Are Add



= contour
closed
curve

Contour



smooth - analytic

Contour

contour

Follow the C-R- equation

Satisfied the $W = f(z)$
Cauchy Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

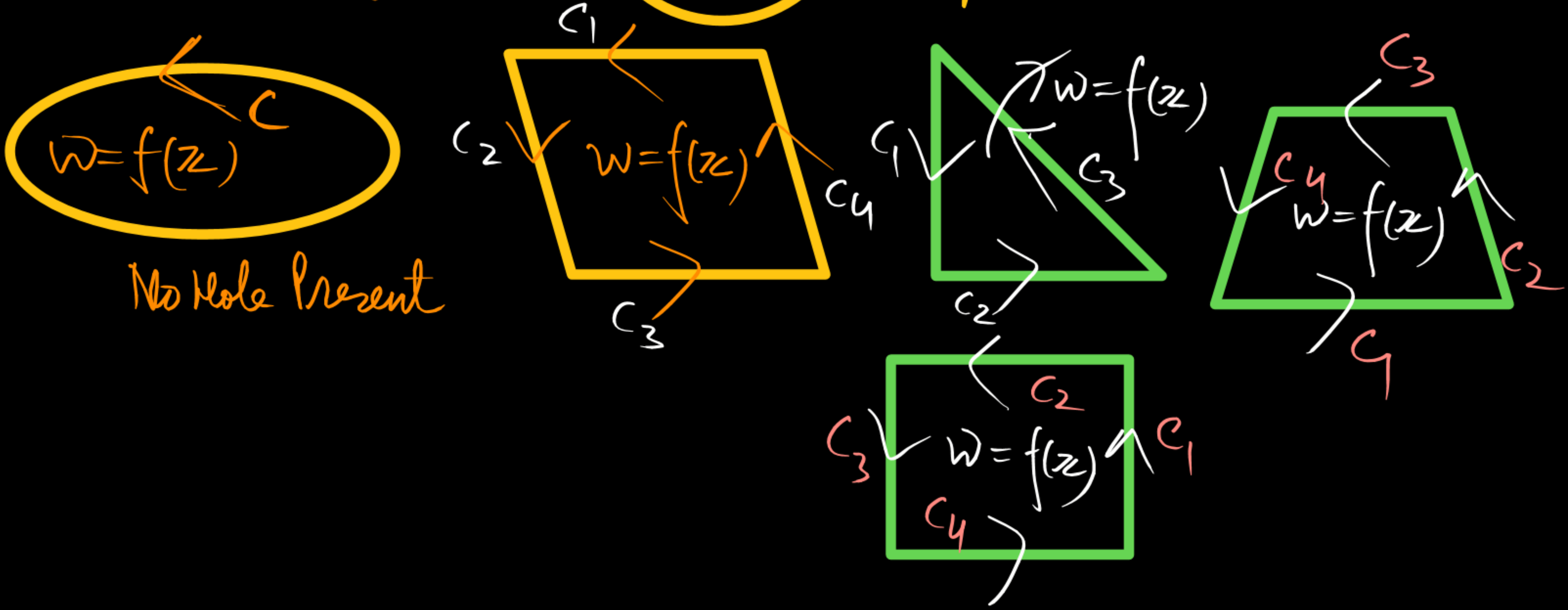
Types of Domain :

1) No Hole Present

"simple connected domain"

$W = f(z) = u(x,y) + i v(x,y)$

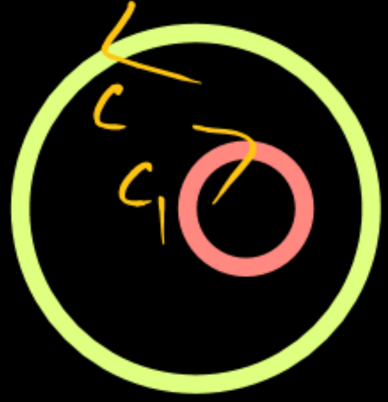
$W = f(z)$ satisfies the C-R equation



No Hole Present

2) Multiple Connected Domain:

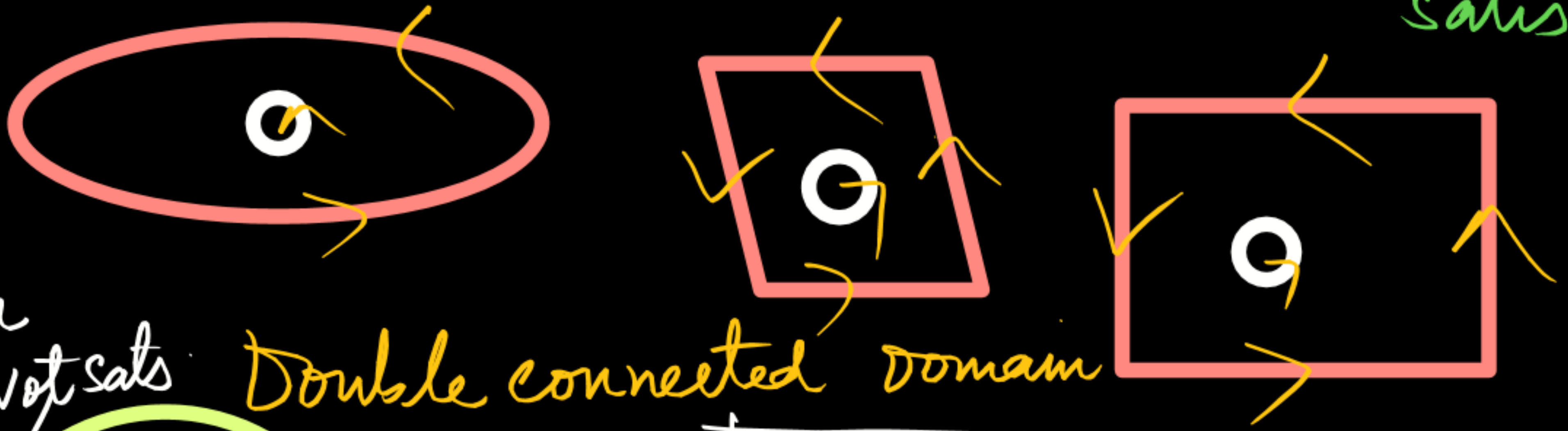
Double connected domain



$W = f(z) = u(x,y) + i\,v(x,y)$

$W = f(z)$ is Not analytic

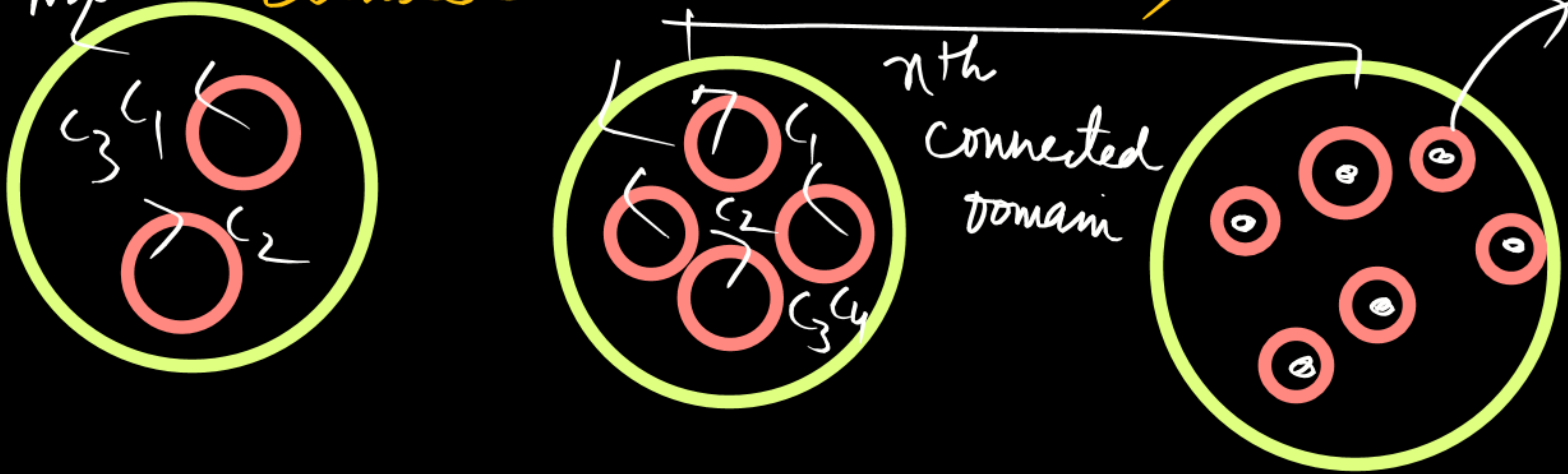
C-R equations is Not satisfied.



Double connected domain

C-R equⁿ Not satis

Triply connected domain



n^{th} connected domain

Derivative does Not

Holes: Not analytic

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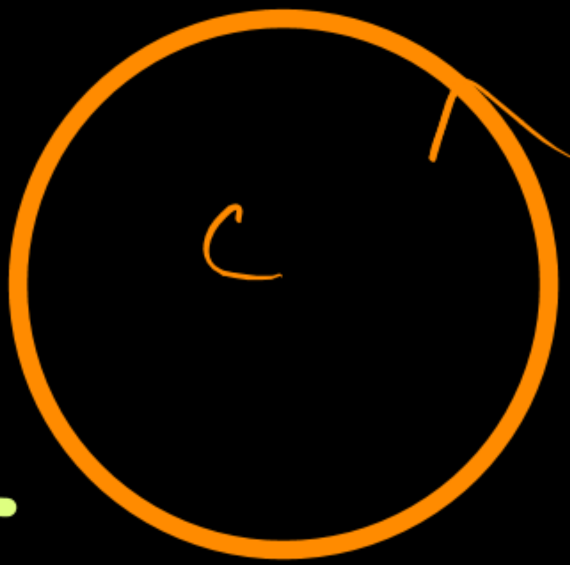
In Simple connected domain
(No Holes Present)

$$w=f(z)=u(x,y)+iv(x,y)$$

C-R equations Are satisfied the $w=f(z)$.



$w=f(z)$
$u(x,y)+iv(x,y)$

Cauchy Integral Theorem Says That $\boxed{\oint_{\text{closed } C} f(z)\,dz = 0}$

<u>Proof</u>:

$$w=f(z)=u(x,y)+iv(x,y)$$

$$z=(x+iy)$$

$$dz=dx+idy$$

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$$\oint f(z)\,dz = \oint (u+iv)(dx+idy)$$ → complex Integral

$$= \oint u\,dx + iv\,dx + iu\,dy - v\,dy$$ ← Real Integrals

$$= \oint \underbrace{(u\,dx - v\,dy)}_{M\,dx + N\,dy} + i\oint \underbrace{(v\,dx + u\,dy)}_{M\,dx + N\,dy}$$

Using greens Theorem

$$\oint M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$$

$$= \iint \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dy\,dx + i\iint_{\text{region}} \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dy\,dx$$

Using C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$= \iint \left(\cancel{\frac{\partial u}{\partial y}} - \cancel{\frac{\partial u}{\partial y}}\right) dy\,dx + i\iint \left(\cancel{\frac{\partial v}{\partial y}} - \cancel{\frac{\partial v}{\partial y}}\right) dy\,dx = 0$$

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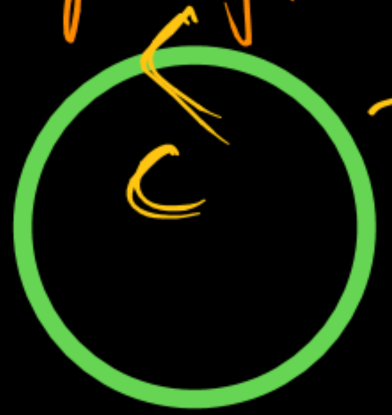
If $w=f(z)$

In simple connected domain, Cauchy Integral said that

$$\boxed{\oint f(z)\,dz = 0}$$

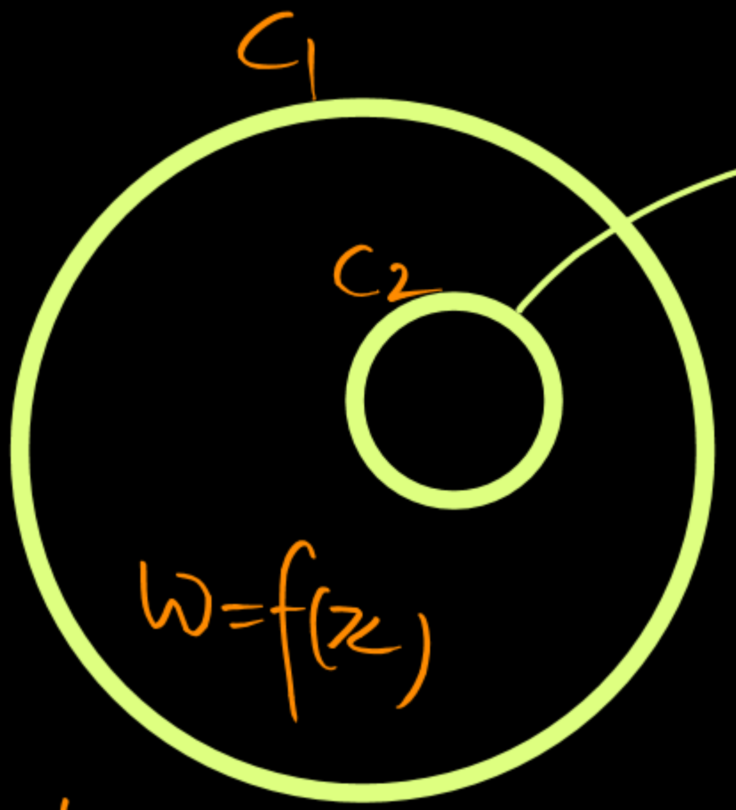
Line Integral $\oint f(z)\,dz = 0$



$\oint f(z)\,dz = 0$

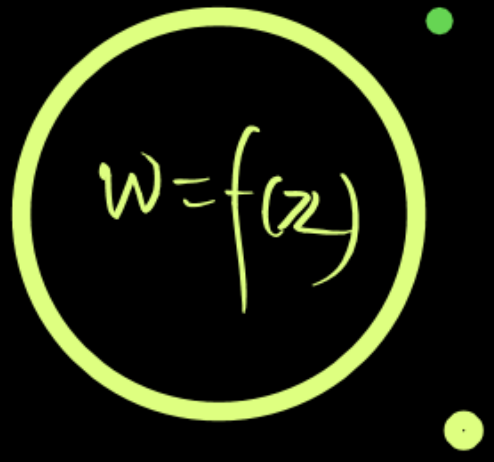
" $w=f(z)$
C-R equ^n
Not satisfied "

$\oint f(z)\,dz \neq 0$



Remove It

Doubly connected domain



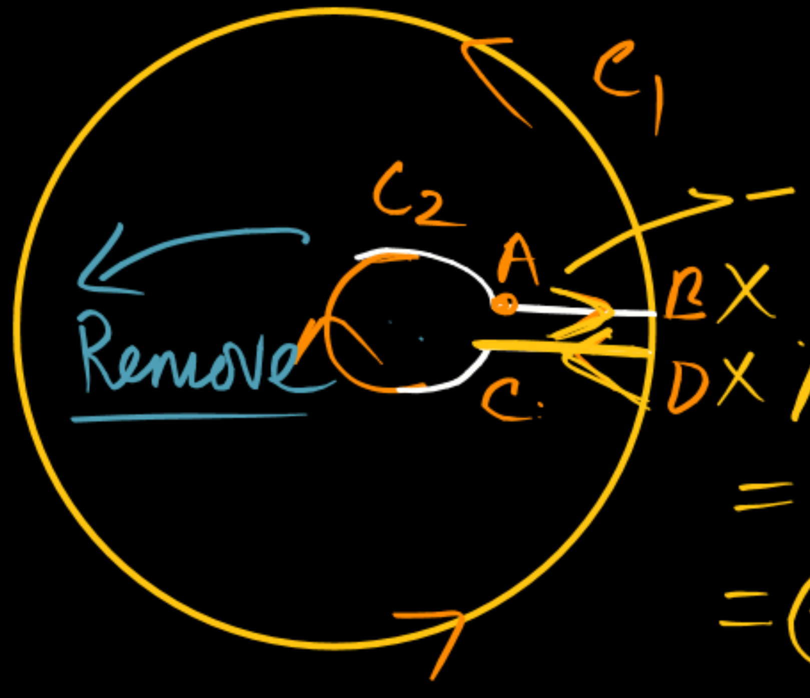
$\oint f(z)\,dz = 0$

Simple connected domain

Conversion of double connected domain to Simple Connected domain:

Using Cauchy Integral Theorem:

STRATEGY: Max/min No cuts

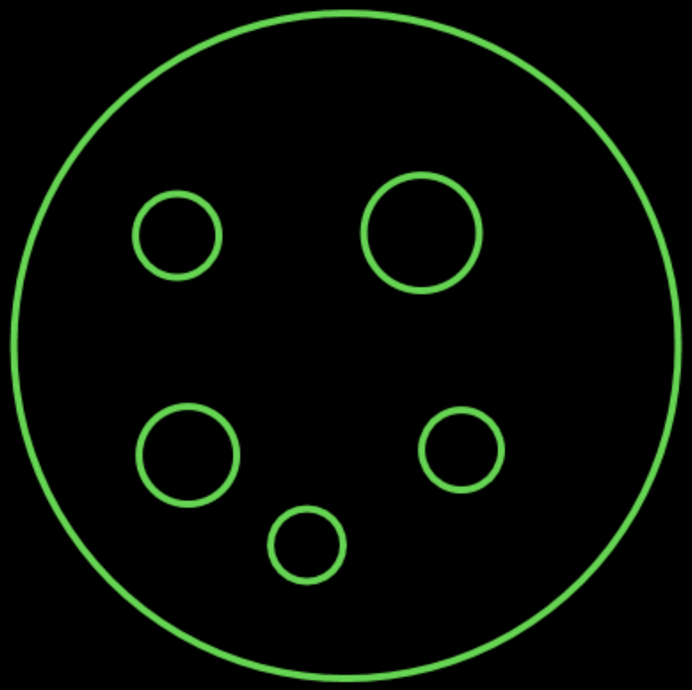


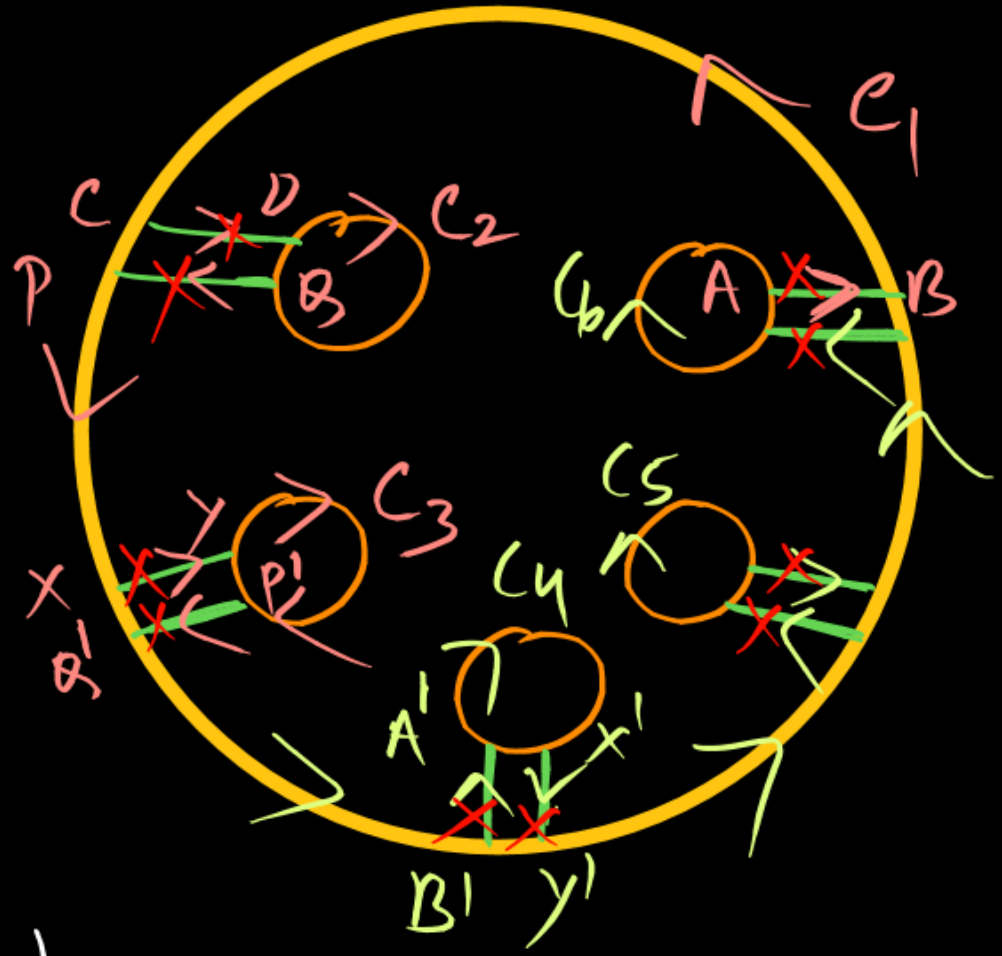
$AB = CD = -AB$

$$\oint_{AB} f(z)\,dz + \oint_{C_1} f(z)\,dz + \oint_{CD} f(z)\,dz + \oint_{C_2} f(z)\,dz = 0$$

$$\cancel{\oint_{AB} f(z)\,dz} + \oint_{C} f(z)\,dz - \cancel{\oint_{AB} f(z)\,dz} - \oint_{C_2} f(z)\,dz = 0$$

$$\oint_{C} f(z)\,dz = \oint_{C_1} f(z)\,dz$$





$w = f(z)$

$\oint f(z)\,dz = 0$

If any multiple connected domain $\xrightarrow{\text{convert}}$ Simple connected domain

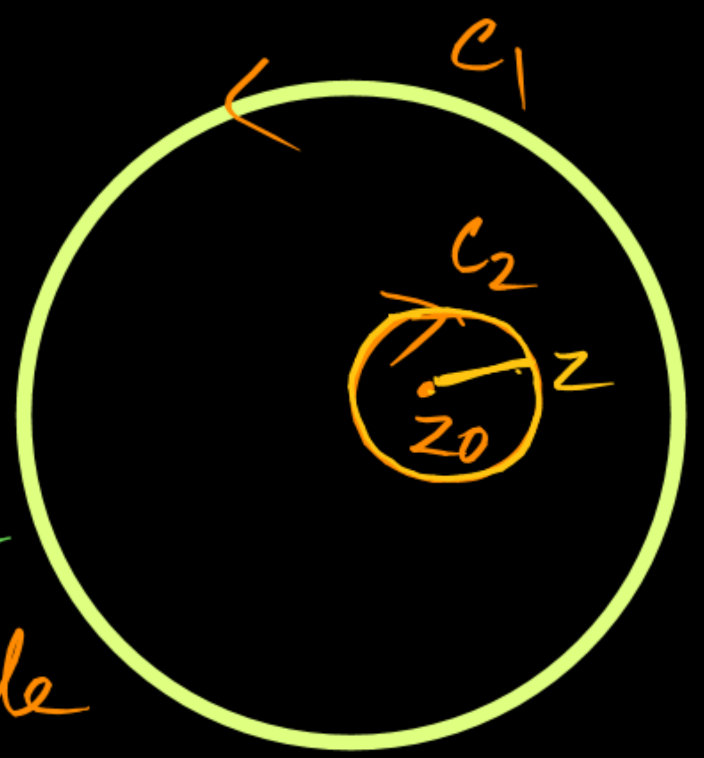
$\rightarrow \oint f(z)\,dz = 0$

Cauchy Integral formula:

$$W = f(z) = u(x,y) + i\,v(x,y)$$

$W = f(z)$ is NOt analytic.

and C-R equation are Not satisfied



$$f(z) = \frac{\phi(z)}{(z-z_0)^1}$$

$z = z_0$ is Simple Order Pole.

Equation of circle in argand Plane or complex Plane

$$\boxed{z - z_0 = re^{i\theta}}$$

Using Cauchy Integral Theorem

$$\int_{C_1} \frac{\phi(z)\,dz}{(z-z_0)} - \int_{C_2} \frac{\phi(z)}{(z-z_0)}\,dz = 0$$

$$\oint_{C_1} \frac{\phi(z)}{(z-z_0)}\,dz - \oint_{C_2} \frac{\phi(z)}{(z-z_0)}\,dz = 0$$

$$\oint_{C_1} f(z)\,dz - \oint_{C_2} f(z)\,dz = 0$$

$$f(z) = \frac{\phi(z)}{(z-z_0)}$$

Pole

$$\oint_{C_1} \frac{\phi(z)}{(z-z_0)}\,dz = \oint_{C_2} \frac{\phi(z)}{(z-z_0)}\,dz$$

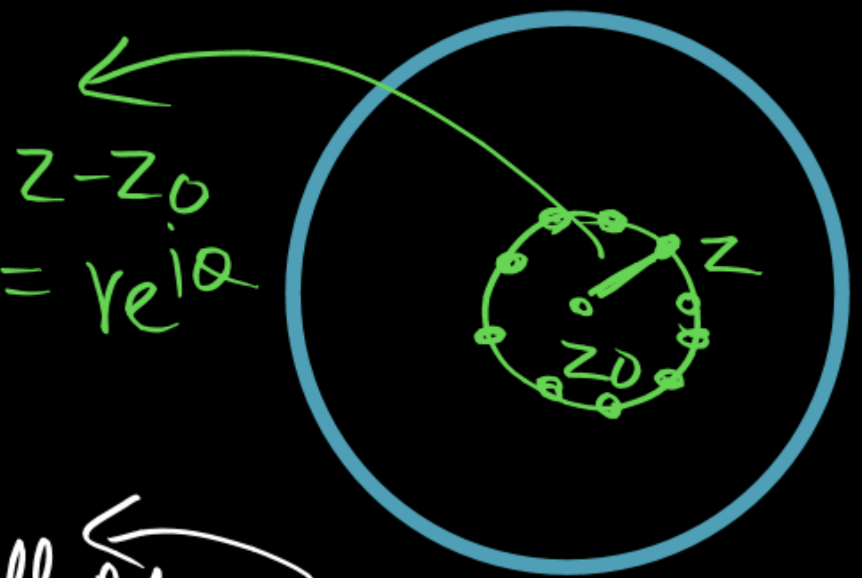
$$\boxed{z - z_0 = re^{i\theta}}$$

$$dz = rie^{i\theta}\,d\theta$$

$$= \oint_{C_2} \frac{\phi(z_0 + re^{i\theta})}{\cancel{re^{i\theta}}} \cdot \cancel{re^{i\theta}}\, i\,d\theta$$

$$= i\oint_{C_2} \phi(z_0 + re^{i\theta})\,d\theta$$

$$= \lim_{r\to 0} i\int_0^{2\pi} \phi(z_0 + \underbrace{\cancel{re^{i\theta}}}_{\to 0})\,d\theta$$



all Pts Are analytic

• $r \to 0$ 0 ~~to~~
$\theta \to 2\pi$

$$= i\int_0^{2\pi} \phi(z_0)\, d\theta$$

$$= 2\pi i\, \phi(z_0)$$

In $f(z) = \displaystyle\int \frac{\phi(z)}{(z-z_0)}\, dz = 2\pi i\, \phi(z_0)$

Cauchy Integral formula for Simple order Pole.

Imp.

$$\boxed{\oint f(z)\, dz = \oint \frac{\phi(z)}{(z-z_0)}\, dz = 2\pi i\, \phi(z_0)}$$

$z = z_0$ simple order Poles.

Cauchy Integral formula

Thank You!

PW Soldiers