GATE-All BRANCHES
Engineering Mathematics

LAPLACE TRANSFORM

Lecture No.- 01

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Recap of previous lecture

Topic Gauss divergence theorem

Topic Problems based on gauss divergence theorem

Important for याँत्रिकी अभियांत्रिकी (ME)

Topics to be Covered

- **Topic** Introduction to laplace transforms
- **Topic** Laplace transform - fundamental function
- **Topic** Shifting property
- **Topic** Multiplication of laplace transforms
- **Topic** Division of laplace transforms
- **Topic** Unit step function and Dirac delta function

Control-EE ME-control system

Laplace Transform → Differential eqn → Using Laplace Transform

1 mark - question
2 mark - 1 question

IES/XE/CE/ME ✓

Previous YEARS — ME/CE → only CE/ME previous YEARS

EE/EC — As a extra

Signal
Communication
Network
Control system

Laplace Transform → Target → Integration

ROC/bilateral

Introduction to Laplace Transform:

Domain $\xrightarrow[\text{Transform}]{\text{Laplace}}$ Domain

Algebraic Domain $\xrightarrow[\text{Transform}]{\text{Laplace}}$ Integral Domain

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$s = x + iy$

$= \sigma + j\omega$

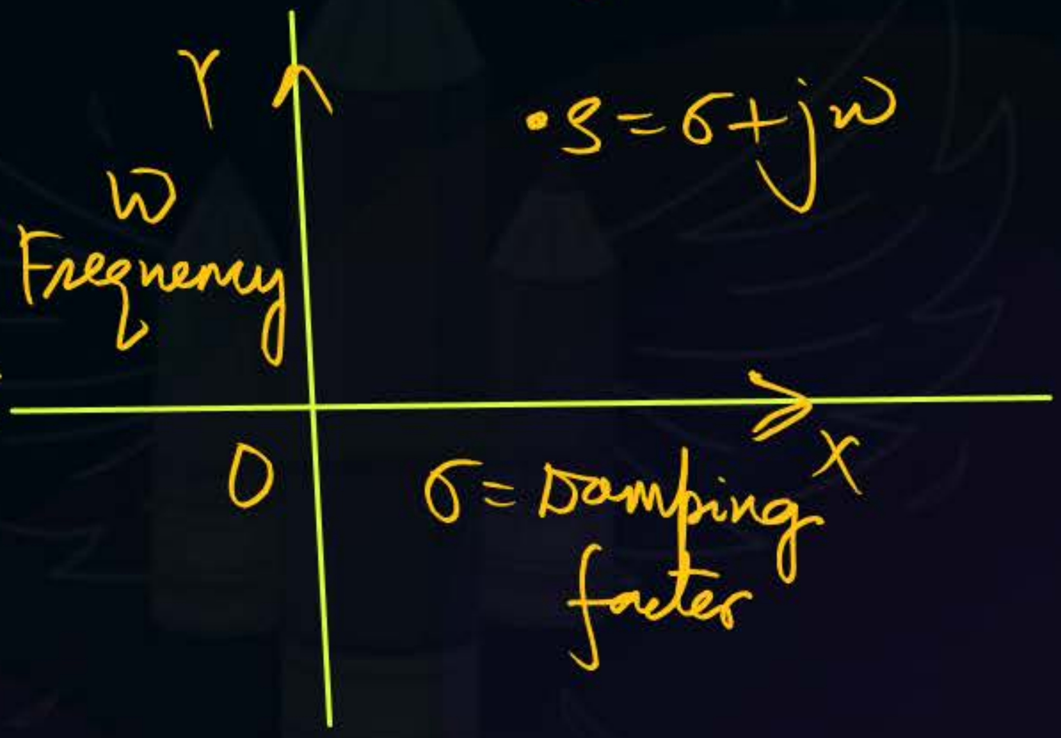
✓ only

t form $\longrightarrow$ s form

$$\boxed{L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt}$$

= Unilateral Laplace Transform

$$L[f(t)] = \int_{-\infty}^{\infty} e^{-st} f(t)\,dt \text{ (bilateral Laplace)}$$



Linear Property:

$$L[af(t)+bg(t)+ch(t)]$$

We know that

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = f(s)$$

$$L[af(t)+bg(t)+ch(t)] = aL[f(t)] + bL[g(t)] + cL[h(t)]$$

$$= af(s)+bg(s)+ch(s)$$

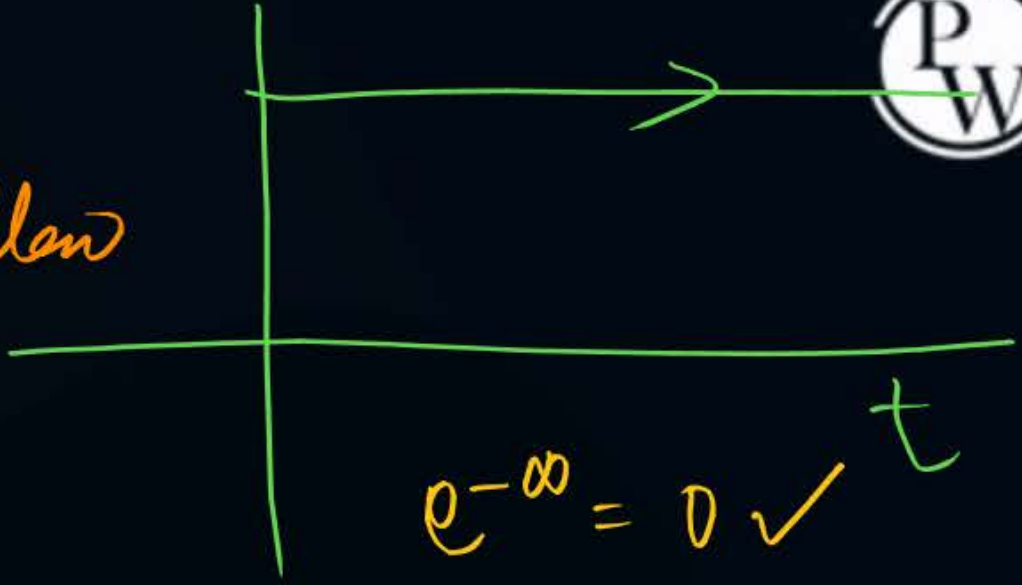
$\boxed{\text{Linear Property}}$

Laplace Transform Depend on Linear System

SOME basic Function : = exponetial follow order

(A) $f(t) = 1$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt =$$

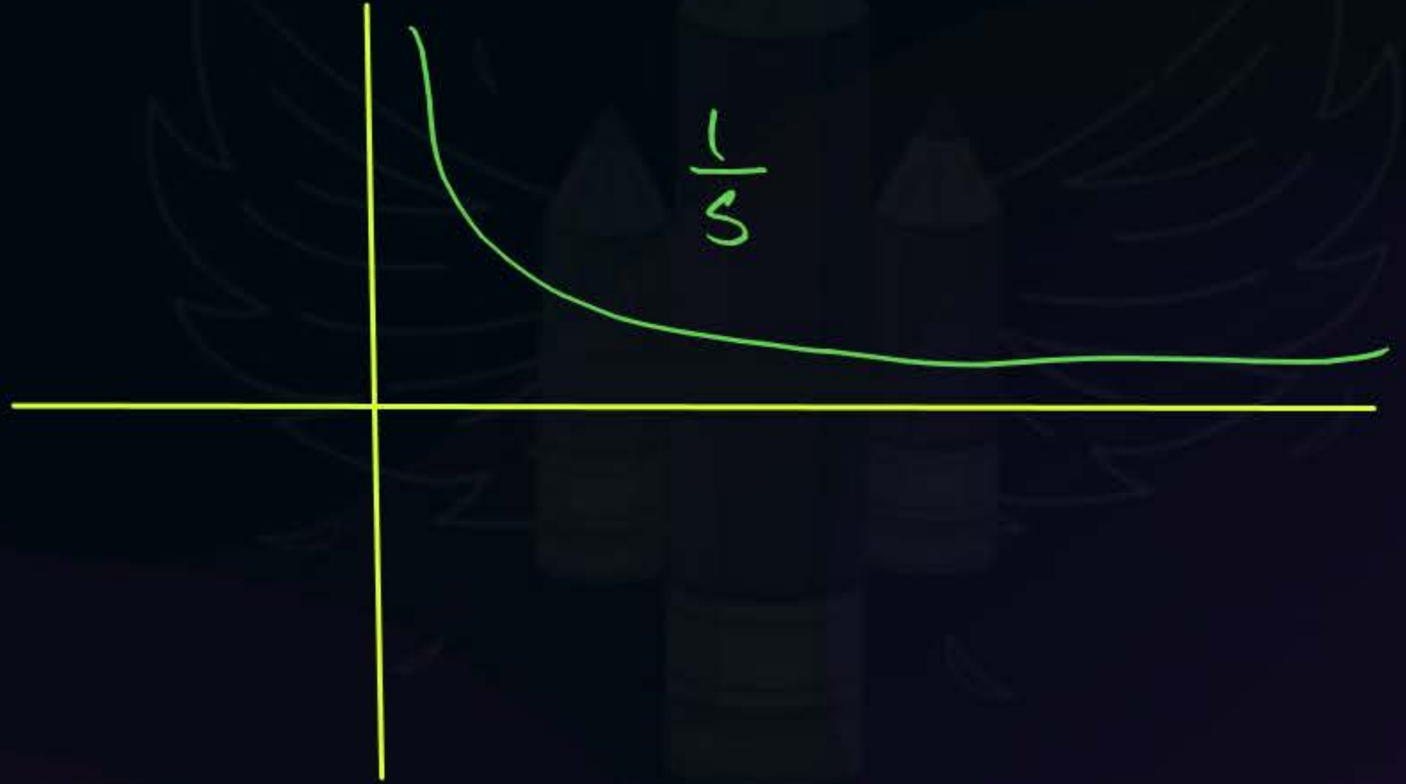


$e^{-\infty} = 0$ ✓

$$L[1] = \int_0^\infty e^{-st}\,1\,dt = \left[\frac{+e^{-st}}{-s}\right]_0^\infty = \left[0 + \frac{1}{s}\right] = \frac{1}{s}$$

$$\boxed{L[1] = \frac{1}{s}}$$

$$\begin{cases} L[5] = \dfrac{5}{s} \\ L[10] = \dfrac{10}{s} \\ L[2] = \dfrac{2}{s} \end{cases}$$



$L[e^{at}] -$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = f(s)$$

$\boxed{t \longrightarrow s}$

$$= \int_0^\infty e^{-st} \cdot e^{at}\,dt$$

$$= \int_0^\infty e^{(a-s)t}\,dt = \frac{1}{(s-a)}$$

$$\boxed{L[e^{at}] = \frac{1}{(s-a)}}$$

$$\boxed{L[e^{-at}] = \frac{1}{(s+a)}}$$

$$L[e^{5t}] = \frac{1}{(s-5)}$$

$a = 5$

$$L[e^{-5t}] = \frac{1}{(s+5)}$$

$$L[e^{t}] = \frac{1}{(s-1)}$$

$$L[e^{-t}] = \frac{1}{(s+1)}$$

Illustration $= 5 + e^t + e^{-5t} + 6e^{-2t}$

$$= L[5] + L[e^t] + L[e^{-5t}] + L[6e^{-2t}]$$

$$= \frac{5}{s} + \frac{1}{(s-1)} + \frac{1}{(s+5)} + \frac{6}{(s+2)}$$

(c) $L[t^n] = \int_0^\infty e^{-st} t^n \, dt$

$\boxed{n \text{ is positive}}$

$st = x$

$s\,dt = dx$

$dt = \frac{dx}{s}$

$s \times 0 = x$

$s \times \infty = x$

$= \int_0^\infty e^{-x} \cdot \left(\frac{x}{s}\right)^n \cdot \frac{dx}{s}$

$= \int_0^\infty e^{-x} \frac{x^n}{s^{n+1}} dx = \frac{1}{s^{n+1}} \underbrace{\int_0^\infty e^{-x} x^n dx}_{\text{gamma function}} = \frac{\Gamma{n+1}}{s^{n+1}}$

$L[t^n] = \frac{\Gamma{n+1}}{s^{n+1}}$

$$\boxed{L[t^n] = \frac{n!}{s^{n+1}}}$$

$L[t^2] = \frac{2!}{s^{2+1}} = \frac{2!}{s^3}$

$L[t^3] = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$

$\Gamma N = \int_0^\infty e^{-x} x^{N-1} dx$

$N-1 = n$

$N = (n+1)$

$\Gamma N = (N-1)!$

(c) $L[\sin at] = \int_0^\infty \underbrace{e^{-st} \sin at}\, dt$

Integration Using Parts

Periodic function

Diff.	Integration
e^{-st}	$\sin at$
$-se^{-st}$	$-\frac{\cos at}{a}$
$s^2 e^{-st}$	$-\frac{\sin at}{a^2}$

$$I = -e^{-st}\frac{\cos at}{a} - \frac{s}{a^2}e^{-st}\sin at - \int \frac{s^2}{a^2}\underbrace{e^{-st}\sin at\, dt}$$

$$I + \frac{s^2}{a^2}I = \left[-e^{-st}\frac{\cos at}{a} - \frac{s}{a^2}e^{-st}\sin at\right]_0^\infty$$

$$I = \frac{a^2}{(s^2+a^2)}\frac{1}{a}\left[-e^{-st}\frac{\cos at}{1} - \frac{s}{a}e^{-st}\sin at\right]_0^\infty$$

$$= \frac{a}{s^2+a^2}\left[\qquad\right]_0^\infty$$

$$\boxed{L[\sin at] = \frac{a}{(s^2+a^2)}}$$

$$\boxed{L[\cos at] = \frac{s}{(s^2+a^2)}}$$

$$L[\cos \overset{a}{5} t] = \frac{s}{s^2+25}$$

$$L[\sin 3t] = \frac{3}{(s^2+9)}$$

$\sin\alpha, \cos\alpha$

constant

$$L[\sin(t+\alpha)] = L[\sin t \boxed{\cos\alpha} + \cos t \boxed{\sin\alpha}]$$

$$= \cos\alpha\, L[\sin t] + \sin\alpha\, L[\cos t]$$

$$= \cos\alpha \cdot \frac{1}{(s^2+1)} + \sin\alpha \frac{s}{(s^2+1)}$$

✓ $$L[\cos(t+\alpha)] = L[\underline{\cos t}\cos\alpha - \sin\alpha \sin t]$$

$$= \cos\alpha \cdot \frac{s}{(s^2+1)} - \sin\alpha \cdot \frac{1}{(s^2+1)}$$

$$L[\cos at] = \frac{s}{(s^2+a^2)}$$

$$L[\sin at] = \frac{a}{(s^2+a^2)}$$

$L[\underbrace{\cos t \cos 2t}]$

Non Linear $\longrightarrow$ Linear

$L[\cos t \cos 2t] = L\left[\frac{1}{2} \times 2 \underbrace{\cos t \cos 2t}\right]$

$= \frac{1}{2} L[\underbrace{2\cos t \cos 2t}]$ ×

$\left\{ \begin{array}{l} \cos(2t+t) \\ +\cos(2t-t) \end{array} \right.$

$= \frac{1}{2} L[\cos 3t + \cos t]$ $+,-$

$\cos 3t + \cos(-t)$

$\cos 3t + \cos t$

$= \frac{1}{2}\left[L[\cos 3t] + L[\cos t]\right]$

$= \frac{1}{2}\left[\frac{s}{(s^2+9)} + \frac{s}{(s^2+1)}\right]$

Laplace Transform

$\downarrow$ only Linear $(+,-)$

$\cos A \cos B$

$= \cos(A+B) + \cos(A-B)$

✓ Try It

$L[\cos t \sin 2t]$

$L[\sin 2t \sin 3t]$

$L[\sin 2t \cos t]$

Try (NEST)

$$L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right)$$
$$= \frac{1}{2}\left[L(1) - L[\cos 2t]\right]$$
$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{(s^2+4)}\right]$$

$$L[(\cos^2 t)] = L\left[\frac{1+\cos 2t}{2}\right]$$
$$= \frac{1}{2} L\left[1+\cos 2t\right]$$
$$= \frac{1}{2}\left[L[1] + L[\cos 2t]\right]$$

$L[\cos^2 t]$
$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{(s^2+4)}\right]$$

strategy

→ convert to List of Formula

$$\cos 2t = \cos^2 t - \sin^2 t$$
$$= 1 - 2\sin^2 t$$
$$= 2\cos^2 t - 1$$

$$\cos 2t = 1 - 2\sin^2 t$$
$$2\sin^2 t = 1 - \cos 2t$$
$$\sin^2 t = \frac{1-\cos 2t}{2}$$

$L[\sin^3 t]$ = Try It

$L[\cos^3 t]$ =

$\sin 3t$

$\cos 3t$

$$L\left[e^{5t}+e^{-\alpha t}+e^{t}+\sin 3t+\cos 4t+5+t^{5}+t^{6}\right]$$

$$\Rightarrow L\left[e^{5t}\right]+L\left[e^{-\alpha t}\right]+L\left[e^{t}\right]+L\left[\underline{\sin 3t}\right]+L\left[\cos 4t\right]+L[5]$$
$$+L\left[t^{5}\right]+L\left[t^{6}\right]$$

$$\Rightarrow \boxed{\frac{1}{(s-5)}+\frac{1}{(s+\alpha)}+\frac{1}{(s-1)}+\frac{3}{(s^{2}+9)}+\frac{s}{(s^{2}+16)}+\frac{5}{s}+\frac{5!}{s^{6}}+\frac{6!}{s^{7}}}$$

Shifting Property :

$$L[f(t)] = f(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[e^{at} f(t)] = f(s-a)$$

$$L[e^{-at} f(t)] = f(s+a)$$

$$s \xrightarrow{e^{at}} (s-a)$$

$$s \xrightarrow{e^{-at}} (s+a)$$

$L[e^{t} \sin t]$

$$s \longrightarrow (s-1)$$

$$L[\sin t] = \frac{1}{(s^2+1)}$$

$$L[e^{t} \sin t] = \frac{1}{(s-1)^2+1}$$

2) $L[e^{2t} \sin 3t] =$

$$L(\sin 3t) = \frac{3}{(s^2+9)}$$

$$L[e^{2t} \sin 3t] = \frac{3}{(s-2)^2+9}$$

$L(e^{-2t}\cos 3t) \Rightarrow \quad L(\cos 3t) = \frac{s}{(s^2+9)} \qquad s \longrightarrow (s+2)$

$L(e^{-2t}\cos 3t) = \frac{(s+2)}{(s+2)^2+9}$

$L(e^{-2t}\, t^3) \Rightarrow L(t^3) = \frac{3!}{s^4}$

$$\boxed{L(e^{-2t}\, t^3) = \frac{3!}{(s+2)^4}}$$

✓ Try It

Solve It

$L[e^t \sin^2 t] = \qquad \frac{s \longrightarrow (s-1)}{\quad}$

$L[e^{-t}\cos^2 t] = \qquad \frac{s \longrightarrow (s+1)}{\quad}$

Multiplication via t :

$$L[f(t)] = f(s)$$

$$L[t^1 f(t)] = (-1)^1 \frac{d^1}{ds^1}[f(s)]$$

$$L[t^2 f(t)] = (-1)^2 \frac{d^2}{ds^2}[f(s)]$$

Imp:

$$\boxed{L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}[f(s)]}$$

2) $L[\underbrace{t}\, e^{-5t}] = L[e^{-5t}] = \frac{1}{(s+5)}$

$$L[te^{-5t}] = (-1)^1 \frac{d}{ds}\left[\frac{1}{(s+5)}\right] = -1 \times \frac{-1}{(s+5)^2} = \frac{1}{(s+5)^2}$$

1) $L[t^1 \overbrace{\sin t}^{f(t)}]$

$$L[\sin t] = \frac{1}{(s^2+1)}$$

$$L[t \sin t] = (-1)^1 \frac{d}{ds} \frac{1}{(s^2+1)}$$

$$= -\left[\frac{-2s}{(s^2+1)^2}\right]$$

$$L(t \sin t) \Rightarrow \frac{2s}{(s^2+1)^2}$$

$$L(t) = \frac{1}{s^2}$$

$$L(te^{-5t}) = \frac{1}{(s+5)^2}$$

$$L(t\cos t) = (-1)^1 \frac{d}{ds}\frac{s}{(s^2+1)} \checkmark = (-1)\frac{(-\cancel{2}\cdot s^2-)}{\cancel{2}(s^2+1)^2} = \frac{s^2}{(s^2+1)^2}$$

$$L(te^{-t}) = (-1)^1\frac{d}{ds}\frac{1}{(s+1)} = \frac{1}{(s+1)^2}$$

Division via t :

$$L[f(t)] = f(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\boxed{L\left[\frac{f(t)}{t}\right] = \int_s^\infty f(s)\,ds}$$

$$L\left[\frac{f(t)}{t^2}\right] = \int_s^\infty \left[\int_s^\infty f(s)\,ds\right] ds$$

$$\boxed{L\left[\frac{f(t)}{t^n}\right] = \int_s^\infty \left[\int_s^\infty \left[\int_s^\infty \left[\int_s^\infty f(s)\,ds\right] ds\,ds \ldots n \text{ times}\right.\right.\right.}$$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{(s^2+1)}\,ds = \left[\tan^{-1}s\right]_s^\infty \qquad L(\sin t) = \frac{1}{(s^2+1)}$$

$$= \tan^{-1}\infty - \tan^{-1}s = \frac{\pi}{2} - \tan^{-1}s \ \checkmark$$

$$L\left[\frac{e^{-t}-e^{-2t}}{t}\right] = L(e^{-t}) - L(e^{-2t})$$

$$= \int_s^\infty \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

$$= \left[\ln\frac{(s+1)}{(s+2)}\right]_s^\infty = \left[\ln\frac{\left(1+\frac{1}{s}\right)}{\left(1+\frac{2}{s}\right)}\right]_s^\infty = \ln 1 - \ln\frac{(s+1)}{(s+2)}$$

$$= 0 - \ln\left(\frac{s+1}{s+2}\right)$$

$$= \underline{\ln\left(\frac{s+2}{s+1}\right)}$$

2 mins Summary

Topic One

Topic Two

Topic Three

Topic Four

Topic Five

Laplace Transform ✓

functions

Shifting Property

Multiplication t

Division by t

THANK - YOU

Topics to be Covered