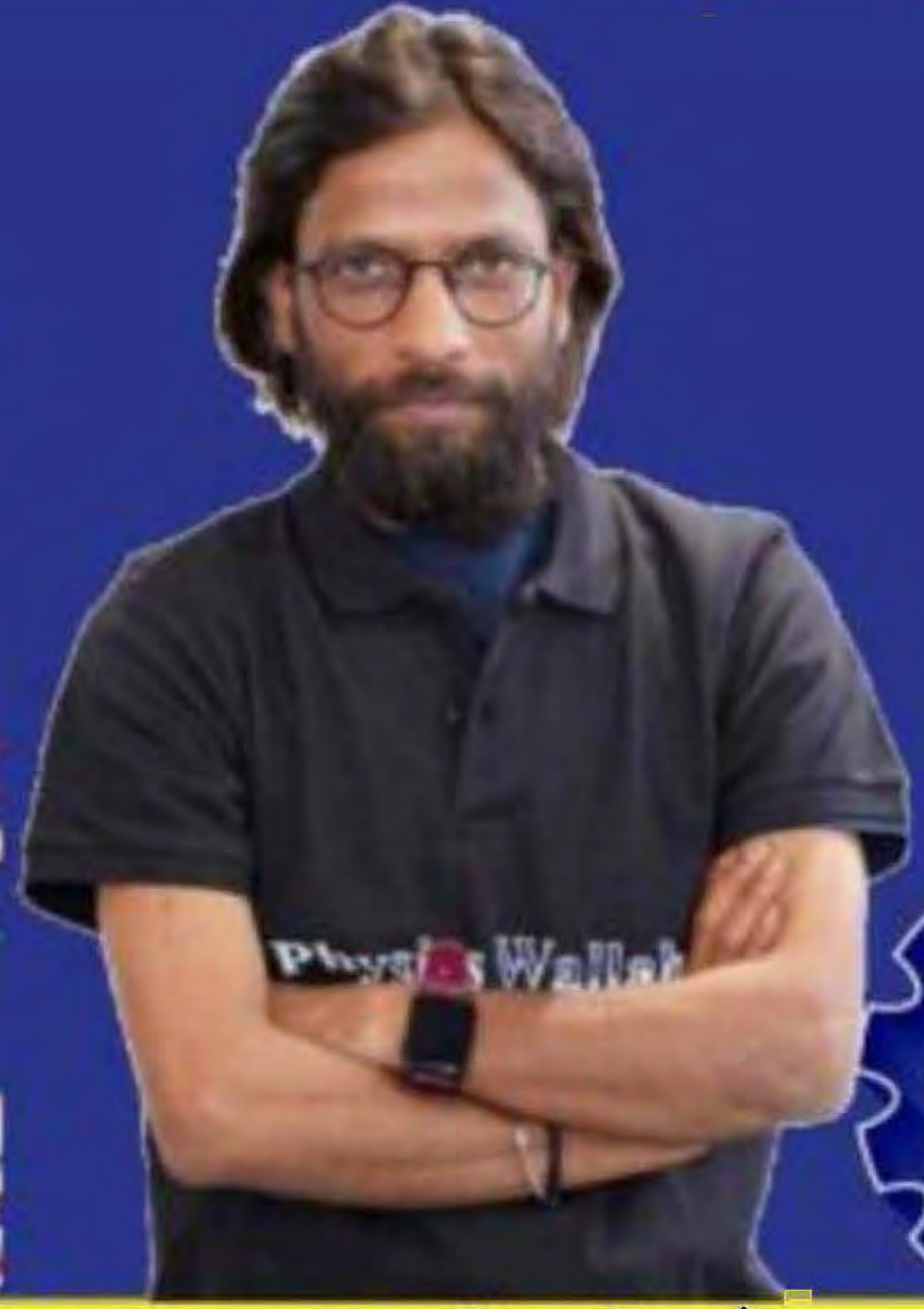
# GATE (ALL BRANCHES)

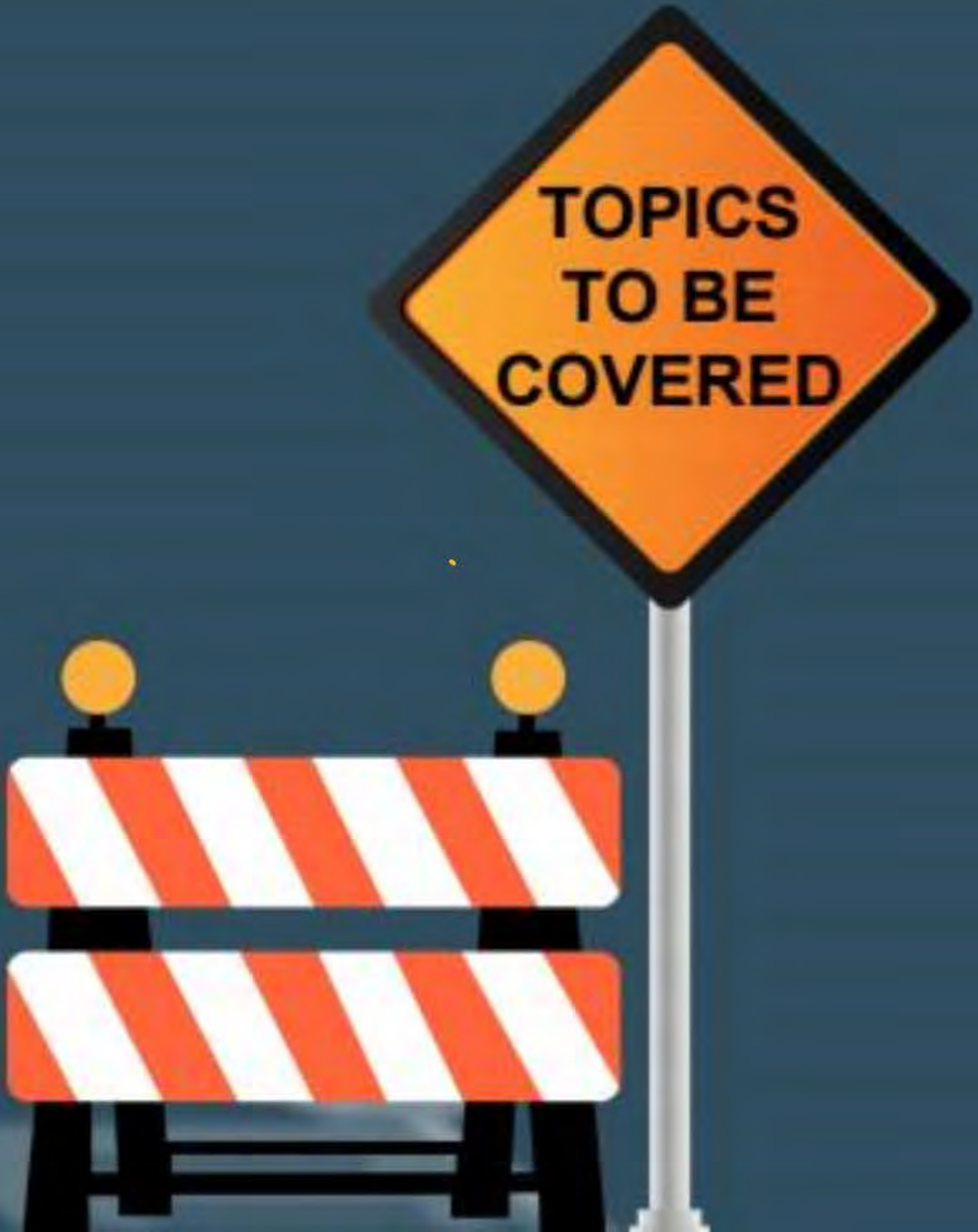
## Engineering Mathematics

**Differential Equation +**
**Partial differential**

Lecture No. 05



By- Rahul Sir




01 Problems based on second order linear differential equation

CASE 02 :- If Roots The Real and Equal.

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

old

$Sol^n = C.F = c_1 e^{r_1 x} + c_2 e^{r_2 x} + c_3 e^{r_3 x} + \dots + c_n e^{r_n x}$

SECOND-order D.E

Roots

$y = c_1 e^{-x} + c_2 e^{-x}$

$= e^{-x}[c_1 + c_2]$

$\underbrace{-1, -1}$

both Dependent

$A e^{-x}$

→ SECOND order

Constant 1 ✗

Constant $c_1$, $c_2$

→ Some modification

$y = e^{rx}$ is a $sol^n$ of D E

$[r^2 e^{rx} + 2r e^{rx} + e^{rx}] = 0$

$\Rightarrow [r^2 + 2r + 1] e^{rx} = 0$

$(r+1)^2 = 0$

$r = -1, -1$

"Roots Are real and Equal"

Independent Solutions

$y_1 = e^{rx} \longrightarrow e^{-x}$

$y_2 = xe^{rx} \longrightarrow xe^{-x}$

$y_3 = x^2 e^{rx}$

$y_4 = x^3 e^{rx}$

$\vdots$

$y_n = x^n e^{rx}$

If Roots Are Real and Equal

$\boxed{r = -1, -1}$

$r_1, r_2 > r$

C.F

$y = c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots$

$y = e^{rx} + xe^{rx} + x^2 e^{rx} + x^3 e^{rx} + \cdots$

$r_1, r_2, r_3 > r = 1, 1, 1$

$e^x, xe^x, x^2 e^x$

$r_1, r_2, r_3, r_4 \to r$ $-2, -2, -2, -2$

$e^{-2x}$

$xe^{-2x}$

$x^2 e^{-2x}$

$x^3 e^{-2x}$

If Roots ARE real and Equal

$W \neq 0$

Independent sol$^n$

$$C.F = c_1 y_1 + c_2 y_2 + c_3 y_3 + \dots + c_n y_n$$

$$y = C.F = c_1 e^{rx} + c_2 x e^{rx} + c_3 x^2 e^{rx} + \dots + c_n x^n e^{r_n x}$$

Ex: Roots $r = -1, -1$

$$\boxed{C.F = y = \text{solution} = (c_1 + c_2 x) e^{-x}}$$

$y_1 = e^{-x}$

$y_2 = x e^{-x}$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

✓ solution

## CASE NO-3

If Roots Are complex / Imaginary

$C.F = c_1 y_1 + c_2 y_2 + c_3 y_3 + --- + c_n y_n$

$Z = a \pm ib$

$a$ = Real Part

$b$ = Imaginary Part

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$

SECOND order

Put $y = e^{rx}$ is a sol$^n$ of D.E

$\Rightarrow [r^2 + r + 1] e^{rx} = 0$

$\Rightarrow r^2 + r + 1 = 0$

$$r = \frac{-1 \pm i\sqrt{3}}{2}$$

Roots ARE Imaginary

complex Roots

$$r = \frac{-1 \pm i\sqrt{3}}{2}$$

$$= \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$$

$$\underbrace{c_1 e^{\left(\frac{-1+i\sqrt{3}}{2}\right)x} + c_2 e^{\left(\frac{-1+i\sqrt{3}}{2}\right)}}_{\text{solve.}}$$

$$\boxed{\cos x + i \sin x = e^{ix}}$$

$$Z = a \pm ib$$

$$\boxed{y = \text{solution } C.F = e^{ax}\left[c_1 \cos bx + c_2 \sin bx\right]}$$

→ Roots Are Complex / Imaginary.

$$Z = r = \frac{-1 \pm i\sqrt{3}}{2}$$

$a$, $b$.

$$C.F = e^{-\frac{x}{2}}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\right)x + c_2 \sin\left(\frac{\sqrt{3}}{2}\right)x\right]$$

$$\left.\begin{aligned} y_1 &= e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}}{2}\right)x \\ y_2 &= e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}}{2}\right)x \end{aligned}\right] \text{Independent solution}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$$

# Q. Questions

#Q. Given the ordinary differential equation $\boxed{\frac{d^2y}{dx^2}+\frac{dy}{dx}-6y=0}$ with $y(0) = 0$ and $\frac{dy}{dx}(0)=1$, the value of $y(1)$ is ______. (Correct to two decimal places).

SECOND Order D.E

$y = e^{rx}$ is a soln.

$\Rightarrow [\underbrace{r^2+r-6}]e^{rx} = 0$

$r^2+r-6=0$

$r^2+3r-2r-6=0$

$r(r+3)-2(r+2)=0$

$r=+2$

$r=-3$

$C.F = y = c_1 e^{-3x} + c_2 e^{+2x}$

Apply The Intial conditions

$y(0) = 0$

$0 = c_1 e^0 + c_2 e^0$

$\boxed{c_1 + c_2 = 0}$

$$\left(\frac{dy}{dx}\right)_{x=0} = 1$$

$$y = c_1e^{-3x} + c_2e^{2x}$$

$$\frac{dy}{dx} = c_1(-3)e^{-3x} + c_2(2)e^{2x}$$

$$1 = c_1(-3)e^{0} + c_2(2)e^{0}$$

$$\boxed{1 = -3c_1 + 2c_2} \qquad \boxed{c_1 + c_2 = 0}$$

$$c_1 = -c_2$$

$$1 = -3[-c_2] + 2c_2$$

$$1 = 3c_2 + 2c_2$$

$$5c_2 = 1$$

$$c_2 = \frac{1}{5} \qquad c_1 = \frac{-1}{5}$$

STEP 01

Solution

STEP 02

apply Intial condition

$c_1$, $c_2$ ?

Solution

$$y = c_1 e^{-3x} + c_2 e^{2x}$$

$$y = -\frac{1}{5} e^{-3x} + \frac{1}{5} e^{2x}$$

$$y(1) = -\frac{1}{5} e^{-3} + \frac{1}{5} e^{2}$$

$y \downarrow x$

Ans

# Q. Questions

#Q. The position of a particle y(t) is described by the differential equation:

$\frac{d^2y}{dt^2} = -\frac{dy}{dt} - \frac{5y}{4}$. The initial conditions are $y(0) = 1$ and $\left.\frac{dy}{dt}\right|_{t=0} = 0$.

The position (accurate to two decimal places) of the particle at $t = \pi$ is ____.

Put $y = e^{rt}$ is a sol$^n$ of D.E

Roots

Ans

$$\frac{d^2y}{dt^2} = -\frac{dy}{dt} - \frac{5y}{4}$$

$$\Rightarrow \frac{d^2y}{dt^2} + \frac{dy}{dt} + \frac{5}{4}y = 0$$

$$= \left[r^2 + r + \frac{5}{4}\right]e^{rt} = 0$$

# Q. Questions

#Q. Consider the differential equation 3y" (x) + 27 y(x) = 0 with initial conditions y(0) = 0 and y' (0) = 2000. The value of y at x = 1 is _____.

$$3y''(x) + 27y(x) = 0$$

$$y''(x) + 9y(x) = 0$$

$y(x) = e^{rx}$ is a solⁿ.

$$[r^2 + 9]e^{rx} = 0$$

$$\boxed{r = \pm 3i}$$

$$y = e^{0x}[c_1 \cos 3x + c_2 \sin 3x]$$

$$y(0) = 0$$

$$y'(0) = 2000$$

$$0 = e^{0x}[c_1 \underline{\cos 0} + c_2 \sin 0]$$

$$0 = c_1 \times \underline{1} + c_2 \times 0$$

$$\boxed{c_1 = 0}$$

$$2000 = -c_1 \sin 3x \times 3 + c_2 \cos 3x \times 3$$

$$\boxed{c_2 = \frac{2000}{3}}$$

Solution of D.E

$$y = c_1 \cos 3x + c_2 \sin 3x$$

$c_1 = 0 \quad c_2 = \frac{2000}{3}$

$$\boxed{y = C.F = \frac{2000}{3} \sin 3x}$$

$$\boxed{y = \frac{2000}{3} \sin 3}$$ Ans.

## Q. Questions

#Q. Find the solution of $\frac{d^2y}{dx^2} = y$ which passes through origin and the point $\left(\ln 2, \frac{3}{4}\right)$

(a) $y = \frac{1}{2} e^x - e^{-x}$

(b) $\frac{1}{2}(e^x + e^{-x})$

(c) $y = \frac{1}{2}(e^x - e^{-x})$

(d) $\frac{1}{2} e^x + e^{-x}$

$y = e^{rx}$ is a solution of DE

$r^2 e^{rx} = e^{rx}$

$(r^2 - 1) = 0 \quad r = \pm 1$

$y = c_1 e^x + c_2 e^{-x}$

$$\boxed{y = \frac{e^x - e^{-x}}{2}}$$

Ans

$(0, 0)$: $x = 0$, $y = 0$

Ist condition

$0 = c_1 + c_2$

$c_1 = \frac{1}{2}$

$c_2 = -\frac{1}{2}$

$\left(\ln 2, \frac{3}{4}\right)$: $x = \ln 2$, $y = \frac{3}{4}$

IInd condition

$\frac{3}{4} = c_1 e^{\ln 2} + c_2 e^{-\ln 2}$

$\frac{3}{4} = c_1 \times 2 + \frac{c_2}{2}$

## Q. Questions

#Q. The solution of the differential equation $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0$ with $y(0) = y'(0) = 1$ is

$y = e^{rt}$ is a solution $\quad \frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0 \quad y(0) = y'(0) = 1$

Do yourself

(a) $(2 - t)\,e^{t}$

(b) $(1 + 2t)\,e^{-t}$

(c) $(2 + t)\,e^{-t}$

(d) $(1 - 2t)\,e^{t}$

## Q. Questions

#Q. The solution to the differential equation $\frac{d^2u}{dx^2} - k\frac{du}{dx} = 0$ where 'k' is a constant, subjected to the boundary conditions $u(0) = 0$ and $u(L) = U$, is

(a) $u=U\frac{x}{L}$

(b) $u=U\left(\frac{1-e^{kx}}{1-e^{kL}}\right)$

(c) $u=U\left(\frac{1-e^{-kx}}{1-e^{-kL}}\right)$

(d) $u=U\left(\frac{1+e^{kx}}{1+e^{kL}}\right)$

Do yourself

## Q. Questions

$\ddot{y}(t) = \frac{d^2y}{dt^2}$

#Q. The maximum value of the solution y (t) of the differential equation $y(t) + \ddot{y}(t) =$ 0 with initial conditions $\dot{y}(0) = 1$ and y(0) = 1, for $t \geq 0$ is

$\frac{d^2y}{dt^2} + y(t) = 0$

$\left(\frac{dx}{dt}\right)_{t=0} = 1 \quad y(t=0) = 1$

$y(t) = e^{rt}$ is a soln of D.E

$\Rightarrow (r^2+1)e^{rt} = 0$

$\Rightarrow r^2 = -1 \quad \boxed{r = \pm i}$

$\boxed{C.F = y = c_1 \cos t + c_2 \sin t}$

Apply Initial conditions $c_1 = 1, \; c_2 = 1$

$y = C.F = \cos t + \sin t$ — max | min

(a) 1

(b) 2

(c) $\pi$

(d) $\sqrt{2}$

$a\cos t + b\sin t$

Max value

$\sqrt{a^2+b^2}$

Min value

$-\sqrt{a^2+b^2}$

$$-\sqrt{a^2+b^2} \le a\cos t + b\sin t \le \sqrt{a^2+b^2}$$

$$\text{Max value} = y = \cos t + \sin t$$

$$\text{max value} = \sqrt{(1)^2+(1)^2}$$

$$= \sqrt{1+1}$$

$$= \sqrt{2}$$

## Q. Questions

#Q. A function $n(x)$ satisfies the differential equation $\frac{d^2n(x)}{dx^2} - \frac{n(x)}{L^2} = 0$ where L is a constant. The boundary conditions are $\boxed{n(0) = k}$ and $\boxed{n(\infty) = 0}$. The solution to this equation is

$n(x) = e^{rx}$ is a solution of D.E

$\Rightarrow \frac{d^2}{dx^2}(e^{rx}) - \frac{e^{rx}}{L^2} = 0$

$\Rightarrow \left[r^2 - \frac{1}{L^2}\right]e^{rx} = 0$

$r = \pm\frac{1}{L}$ constant

$r = \pm\frac{1}{L}$

Roots ARE Real and Different

(a) $n(x) = k\exp\left(\frac{-x}{L}\right)$

(b) $n(x) = k\exp\left(\frac{-x}{\sqrt{L}}\right)$

(c) $n(x) = k^2\exp\left(\frac{-x}{L}\right)$

(d) $n(x) = k^2\exp\left(\frac{-x}{\sqrt{L}}\right)$

Solution of Diff. Eqn$^n$

$$n(x) = C.F = y = c_1 e^{\frac{1}{L}x} + c_2 e^{-\frac{1}{L}x}$$

Apply Initial Conditions

$$n(x) = c_1 e^{\frac{1}{L}x} + c_2 e^{-\frac{1}{L}x}$$

$n(0) = K$

$n(\infty) = 0$

$n(\infty) = 0$

$$0 = c_1 e^{\infty} + c_2^{-\infty}$$

$$= c_1(\infty) + c_2(0)$$

$$\boxed{c_1 = 0}$$

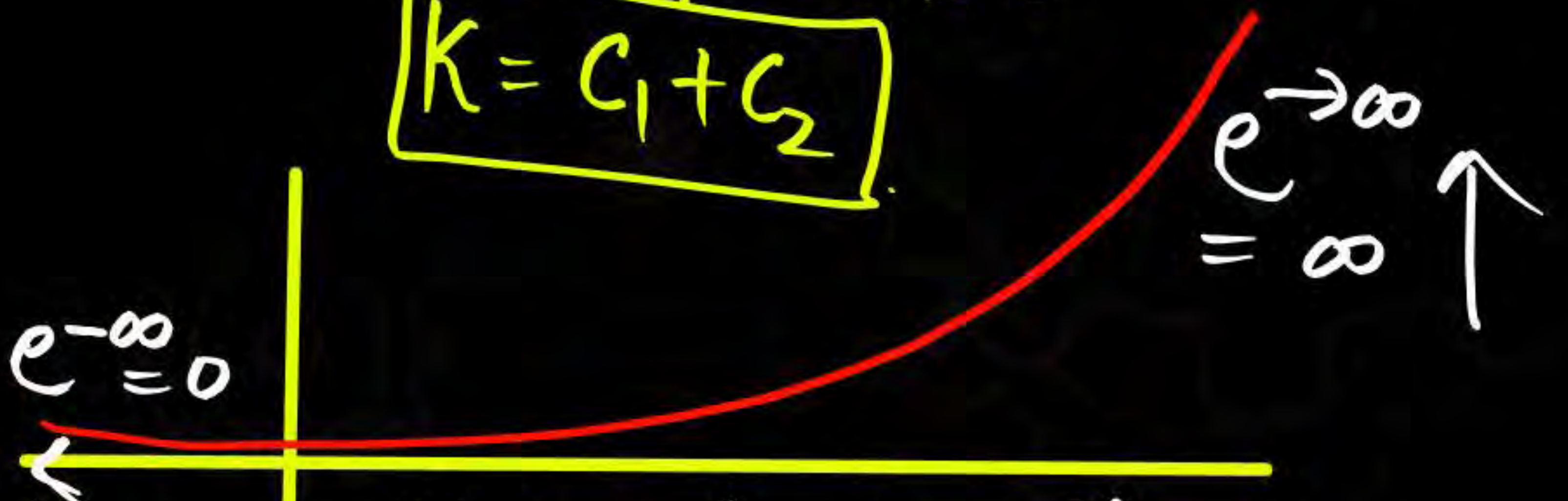
$$c_2 = K$$

1)

$$K = c_1 e^{\frac{1}{L}\times 0} + c_2 e^{-\frac{1}{L}\times 0}$$

$$K = c_1 e^0 + c_2 e^0$$

$$\boxed{K = c_1 + c_2}$$




$y = 0$

$$n(x) = \text{sol}^n = Ke^{-\frac{x}{L}} \quad \underline{\text{Ans}}$$

# Thank You!

**PW Soldiers**