GATE (ALL BRANCHES)

Engineering Mathematics

Differential Equation +
Partial differential

Lecture No. 07

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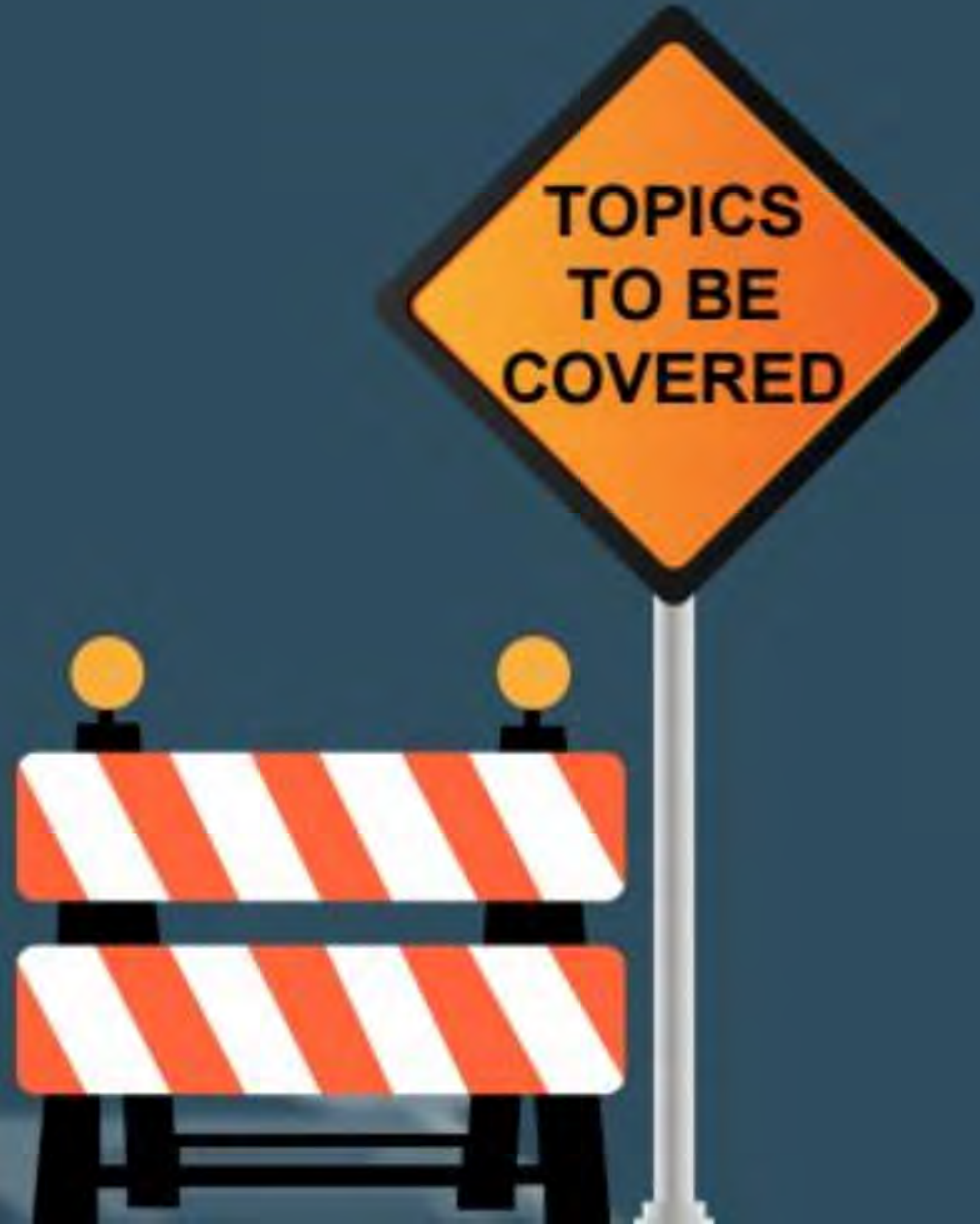

01 Variation of Parameters ✓ generalized P.I

04 Cauchy Euler DE ✓ constant coefficient
✓ variable coefficients

✓ P.I ✓ e^{ax+b}

$\sin(ax+b)$

x^n

Variation of Parameters :-

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X$$

P and Q Are Constants

X is a function of x only.

$X = \frac{e^{3x}}{x^2}, e^{3x}\sin x, \frac{1}{1+\sin x}, \frac{1}{1+\cos x}, \log x, \tan x \ldots$ etc

C.F = Complementary Function

$$\boxed{C.F = C_1 y_1 + C_2 y_2 + C_3 y_3 + \cdots + C_n y_n}$$

y_1, y_2, y_3 Are function of x only

$C_1, C_2, C_3 \ldots C_n$ Are constants

Particular Integral $\boxed{P.I = -y_1 \int \frac{y_2 X}{W} dx + y_2 \int \frac{y_1 X}{W} dx}$

Where W = Wronskian $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$
(n×n) matrix

Variation of Parameters

Complete Solution $\boxed{C.F + P.I = y}$

Illustration 01

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$$

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

Put $y = e^{rx}$ is a solution of D.E

$$\underbrace{(r^2 - 6r + 9)}\, e^{rx} = 0$$

$$\Rightarrow (r-3)^2 = 0 \quad \boxed{r = 3, 3}$$

$$\boxed{C.F = (c_1 + c_2 x)e^{3x}}$$

$$\underbrace{C.F = c_1 e^{3x} + c_2 x e^{3x}}$$

$$C.F = c_1 y_1 + c_2 y_2 + \ldots c_n y_n$$

$$\boxed{y_1 = e^{3x}} \quad \boxed{y_2 = e^{3x} \cdot x}$$

Wronskian

$$W = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3xe^{3x} + e^{3x} \end{vmatrix}$$

$$\boxed{W = e^{6x}}$$

Particular Integral $= -y_1\int \frac{y_2 X}{W}\,dx + y_2\int \frac{y_1 X}{W}\,dx.$

$$= -e^{3x}\int \frac{xe^{3x}\cdot\frac{e^{3x}}{x^2}}{e^{6x}} + xe^{3x}\int \frac{e^{3x}\cdot\frac{e^{3x}}{x^2}}{e^{6x}}$$

$$= -e^{3x}\int \frac{1}{x}\,dx + xe^{3x}\int \frac{1}{x^2}\,dx$$

$$= -e^{3x}\ln x + \cancel{x}e^{3x}\left(-\frac{1}{\cancel{x}}\right)$$

✓ P.I $\boxed{= -e^{3x}\ln x - e^{3x}}$

$y_1(x) = e^{3x}$

$y_2(x) = xe^{3x}.$

$X = \frac{e^{3x}}{x^2}$

$W = e^{6x}$

$y = C.F + P.I$

$= (C_1 + C_2x)e^{3x} - e^{3x}\ln x - e^{3x}$

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X \begin{cases} e^{ax+b} \\ \sin(ax+b) \\ x^m \end{cases}$$

$$\left\{ \begin{array}{l} \dfrac{e^{3x}}{x^2} \\ \tan x \\ \sec^2 x \\ \sec 2x \\ \ln x \\ \dfrac{1}{1+\sin x} \end{array} \right.$$

millions. combination of Functions

2) variation of Parameters $\dfrac{d^2y}{dx^2}+y=\dfrac{1}{1+\sin x}$

$$P.I = -y_1\int \frac{y_2 X}{W}\,dx + y_2\int\frac{y_1 X}{W}\,dx$$

$$= -\cos x\int \frac{\sin x\cdot \frac{1}{1+\sin x}\,dx}{1}$$

I_1

$$+\sin x\int\frac{\cos x}{1+\sin x}\,dx$$

I_2

$$(r^2+1)=0$$

$$r=\pm i$$

Roots Are complex

$$C.F = c_1\cos x + c_2\sin x$$

$$y_1(x)=\cos x$$

$$y_2(x)=\sin x$$

$$W=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}$$

$$=\cos^2 x+\sin^2 x$$

$$\boxed{W=1}$$

$$X=\frac{1}{1+\sin x}$$

$$I_1 = \int \frac{\sin x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}$$

$$= \int \frac{\sin x - \sin^2 x}{1-\sin^2 x}\, dx$$

$$= \int \frac{\sin x - \sin^2 x}{\cos^2 x}\, dx$$

$$= \int \tan x \sec x\, dx - \int \tan^2 x\, dx$$

$$= \sec x - \int (\sec^2 x - 1)\, dx$$

$$\boxed{I_1 = \sec x - \tan x + x + c}$$

$$I_2 = \int \frac{\cos x}{1+\sin x}\, dx$$

$$= \int \frac{f'(x)}{f(x)}\, dx$$

$$= \ln[f(x)] + c$$

$$\boxed{I_2 = \ln[1+\sin x] + c}$$

for Third order:

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \neq 0$$

Cauchy-Euler Linear Differential Equ^n: (variable coefficients)

Linear D.E with variable coefficients

$$x^n\frac{d^ny}{dx^n}+K_1\frac{d^{n-1}y}{dx^{n-1}}x^{n-1}+K_2\frac{d^{n-2}y}{dx^{n-2}}x^{n-2}+\cdots+K_ny=X$$

variable coefficients

Transform OR change

$$\frac{d^ny}{dx^n}+P\frac{d^{n-1}y}{dx^{n-1}}+Q\frac{d^{n-2}y}{dx^{n-2}}+\cdots+Ny=X$$

Constant coefficients

$y=C.F+P.I$

for $n=2$ (SECOND order)

$$x^2\frac{d^2y}{dx^2}+K_1x\frac{dy}{dx}+K_2y=X$$ (Second order Cauchy-Euler) (Variable coefficients)

X is a Function of x only.

Using Hit and trial method

Change: $$\frac{d^2y}{dx^2}+K_1\frac{dy}{dx}+K_2y=X$$

$$x=e^t \quad t=\ln x \quad \frac{dt}{dx}=\frac{1}{x}$$

✓ $$\underbrace{\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}}_{\text{Chain Rule}}=\frac{1}{x}\frac{dy}{dt}$$

$$\frac{d}{dt}=D$$

$$\frac{dy}{dt}=Dy$$ ✓

$$\boxed{x\frac{dy}{dx}=Dy}$$

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$$\frac{dy}{dx}=\frac{1}{x}\cdot\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2}=\left(-\frac{1}{x^2}\right)\frac{dy}{dt}+\underbrace{\frac{d^2y}{dt^2}\cdot\frac{dt}{dx}}_{\text{Chain Rule}}\cdot\frac{1}{x}$$

$$\frac{d^2y}{dx^2}=-\frac{dy}{dt}\cdot\frac{1}{x^2}+\frac{d^2y}{dt^2}\frac{1}{x}\cdot\frac{1}{x}$$

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$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{d^2y}{dt^2}\cdot\frac{1}{x^2}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = -\frac{dy}{dt} + \frac{d^2y}{dt^2} = -Dy + D^2y = \underbrace{D(D-1)y}$$

$$\Rightarrow x^3\frac{d^3y}{dx^3} = \underbrace{D(D-1)(D-2)y}$$

$$\Rightarrow x^4\frac{d^4y}{dx^4} = \underbrace{D(D-1)(D-2)(D-3)y}$$

$\boxed{x = e^t}$

Remove The variables

$$x^2\frac{d^2y}{dx^2} + K_1 x\frac{dy}{dx} + K_2 y = X \Rightarrow \underbrace{\left[D(D-1)y + K_1 Dy + K_2 y\right]}_{\text{Constant cofficients}} = \underline{f(t)}$$

Q. Questions

#Q. Consider the ordinary differential equation $\boxed{x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0}$. Given the values of $\boxed{y(1) = 0}$ and $\boxed{y(2) = 2}$, the value of $y(3)$ (round off to 1 decimal place), is ___.

2nd order
Cauchy-Euler

$D(D-1)y$ $\quad Dy$

$$\boxed{x = e^t}$$

$$= [D(D-1)y - 2Dy + 2y] = 0$$

$$\Rightarrow [D^2 - D - 2D + 2]y = 0$$

$$\Rightarrow [D^2 - 3D + 2]y = 0$$

x x

✓ t

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 0$$

C.F ✓ P.I = 0

Put $y = e^{rt}$ is a sol^n of D.E

$$[r^2 - 3r + 2]e^{rt} = 0$$

$$= [r^2 - 2r - r + 2]e^{rt} = 0$$

$$= r(r-2) - 1(r-2) = 0$$

$$\boxed{r = 1 \quad r = 2}$$

If Roots Are real and distinct

$C.F = c_1 e^{t} + c_2 e^{2t}$

convert x $\boxed{x = e^{t}}$

$C.F = c_1 x + c_2 x^2$

$P.I = 0$

Complete solution

$y = c_1 x + c_2 x^2$

apply The Initial Conditions

$y(1) = 0$

$0 = c_1 \times 1 + c_2 (1)^2$

$\boxed{c_1 + c_2 = 0}$

$c_1 = -c_2$

$\left\{ \begin{array}{l} y(1) = 0 \\ y(2) = 2 \end{array} \right]$

In Terms of x

$2 = c_1 \cdot 2 + 4c_2$

$2 = 2c_1 + 4c_2$

$2 = 2 \times -c_2 + 4c_2$

$-2c_2 + 4c_2 = 2$

$2c_2 = 2$

$c_2 = 1$

$c_1 = -1$

$C.F = y = -x + x^2$

$$y = c_1 x + c_2 x^2$$

$$y = -x + x^2$$

$$y(3) = -3 + (3)^2$$

$$= 9 - 3$$

$$= 6 \text{ Ans}$$

Q. Questions

#Q. A differential equation is given as $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 4$. The solution of the differential equation in terms of arbitrary constants C_1 and C_2 is

(a) $y = C_1x^2 + C_2x + 2$

(b) $y = \frac{C_1}{x^2} + C_2x + 4$

(c) $y = C_1x^2 + C_2x + 4$

(d) $y = \frac{C_1}{x^2} + C_2x + 2$

$[D(D-1)y - 2Dy + 2y] = 0$

$= [D^2 - D - 2D + 2]y = 0$

$= [D^2 - 3D + 2]y = 0$

$\boxed{C.F = C_1x + C_2x^2}$

$D = a$

$P.I = \frac{4e^{0t}}{[D^2 - 3D + 2]}$

$= \frac{4}{2} = 2$

$P.I = 2$

$\boxed{y = C_1x + C_2x^2 + 2}$

Q. Questions

$D = \frac{d}{dt}$ $[D(D-1)y - 3Dy + 3y] = 0$

$\Rightarrow [D^2 - D - 3D + 3]y = 0$

$\Rightarrow [D^2 - 4D + 3]y = 0$

$y = e^{rt}$

$[r^2 - 4r + 3] = 0$

$r^2 - 3r - r + 3 = 0$

$r(r-3) - 1(r-3) = 0$

$\boxed{r = 1 \quad r = 3}$

#Q. Consider the homogeneous ordinary differential equation

$$x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = 0, x > 0$$ $\boxed{x = e^t}$

with y(x) as a general solution. Given that y(1) = 1 and y(2) = 14 the value of y(1.5), rounded off to two decimal places, is _____.

$C.F = c_1 e^t + c_2 e^{3t}$

$\boxed{y = c_1 x + c_2 x^3}$

$y = c_1 x + c_2 x^3$

✓ $y(1) = 1$ $\quad \boxed{1 = c_1 + c_2}$

$y(2) = 14$

$14 = 2c_1 + 8c_2$

$y(1) = 1$

$y(2) = 14$

$\boxed{c_1 = -1 \quad c_2 = 2}$

$2c_1 + 8c_2 = 14 \times 1$

$c_1 + c_2 = 1 \times 2$

$\Rightarrow$ $\cancel{2c_1} + 8c_2 = 14$

$\cancel{2c_1} + 2c_2 = 2$

$6c_2 = 12$

$\boxed{c_2 = 2}$

$c_1 + c_2 = 1$

$c_1 + 2 = 1$

$c_1 = -1$

$y = -x + 2x^3$

$\boxed{y(1.5) = -1.5 + 2(1.5)^3}$

$=$ Ans

$= -1.5 + 2(1.5)^3$

$=$

Q. Questions

$[D(D-1)y + Dy - 4y] = 0$

#Q. Consider the differential equation $\boxed{x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0}$ with the boundary conditions of y(0) = 0 and y(1) = 1. The complete solution of the differential equation is

$= [D^2 - \cancel{D} + \cancel{D} - 4]\,y = 0$

$= [D^2 - 4]\,y = 0$ Put $y = e^{rt}$ is a soln

$\Rightarrow [r^2 - 4]\,e^{rt} = 0$

$r = \pm 2$

$C.F = c_1 e^{2t} + c_2 e^{-2t}$

$y = c_1 x^2 + \frac{c_2}{x^2}$

(a) x^2

(b) $\sin\left(\frac{\pi x}{2}\right)$

(c) $e^x \sin\left(\frac{\pi x}{2}\right)$

(d) $e^{-x} \sin\left(\frac{\pi x}{2}\right)$

$$y = c_1 x^2 + \frac{c_2}{x^2}$$

$y(1) = 1 \qquad \boxed{1 = c_1 + c_2}$

$y(0) = 0 \qquad 0 = \underline{c_1 \times 0} + \frac{c_2}{0}$

$$0 = c_2(\vec{\infty})$$

$$\boxed{c_2 = 0}$$

$\underline{1} = c_1 + c_2$

$1 = c_1 + 0$

$c_1 = 1$

Initial condition

$\left. \begin{array}{l} y(0) = 0 \\ y(1) = 1 \end{array} \right]$

$$y = 1 \cdot x^2 + \frac{0}{x^2}$$

$$y = x^2$$

Thank You!

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