GATE-All BRANCHES
Engineering Mathematics

Vector calculus

Lecture No.- 04

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Recap of Previous Lecture

Topic Concept of curl

Topic Greens theorem and Stokes theorem

Topic Problems based on Green's theorem and stokes theorem

Topics to be Covered

Topic Gradient of a scalar function

Topic Directional derivative

Topic Problems based on gradient,directional derivative.

Gradient of a Scaler Point Function:

Gradient → Three dimensional change in x, y, z Direction

Scaler Point Function $\xrightarrow[\text{(del)}]{\nabla}$ vector Point function

$\phi(x,y,z) = c$

$\nabla\phi$ = vector Point function

$\phi(x,y,z) = c$ (Level surface)
(contours)

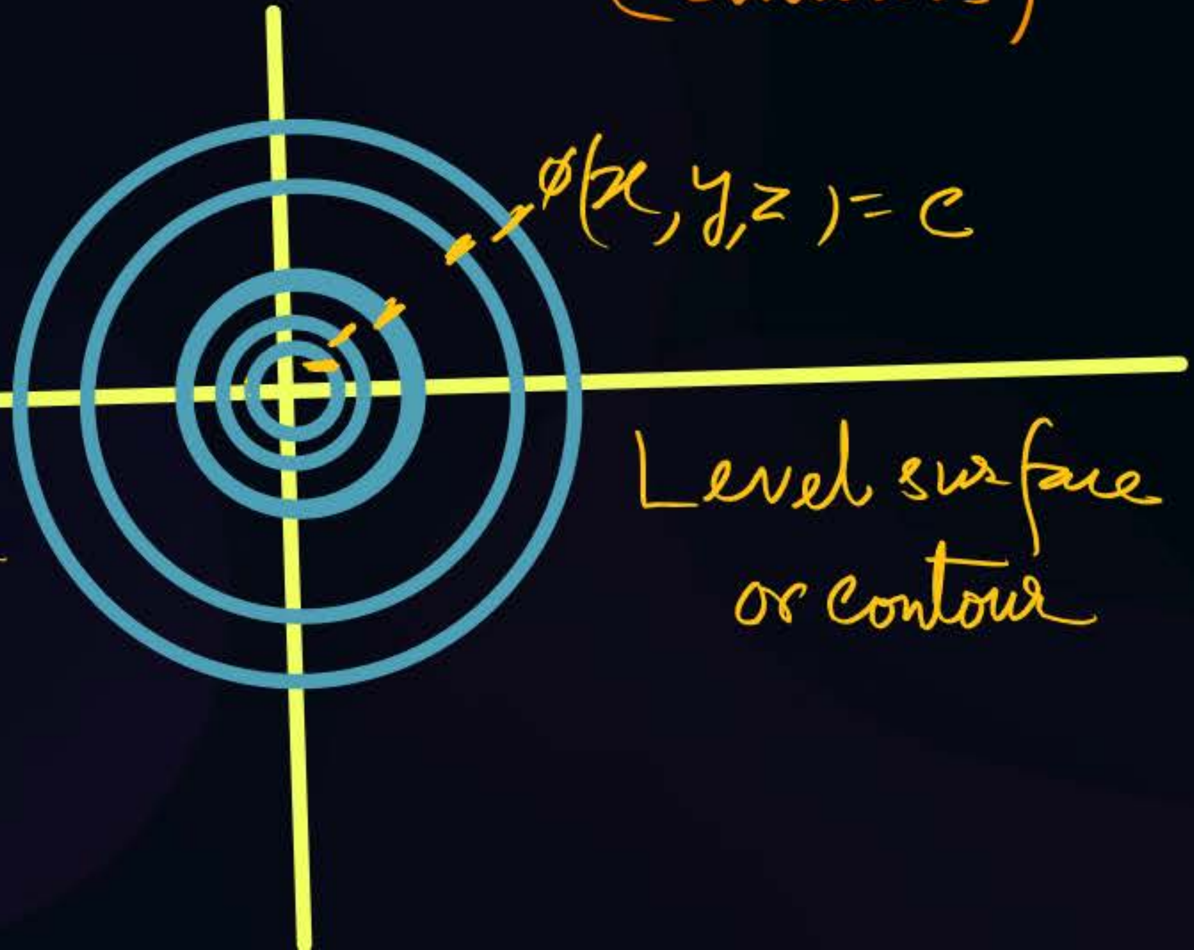
$x^2+y^2=1$

$x^2+y^2=2$

$x^2+y^2=3$

$x^2+y^2=4$

$x^2+y^2=5$



$\phi(x,y,z)=c$

Level surface or contour

$\phi(x,y,z) = c$

$\phi(x, y, z) = c$ Level surface

<u>Temp gradient</u>

Scaler

$\phi(x, y, z) = c$ (level surface)

$$\text{grad}\,\phi = \nabla\phi = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\phi(x, y, z)$$

vector

$$\text{grad}\,\phi = \hat{i}\underbrace{\frac{\partial\phi}{\partial x}}_{\text{change in X Direction}} + \hat{j}\underbrace{\frac{\partial\phi}{\partial y}}_{\text{change in Y Direction}} + \hat{k}\underbrace{\frac{\partial\phi}{\partial z}}_{\text{change in Z direction}}$$

$$\nabla\phi = \text{grad}\,\phi = \hat{i}\frac{(\phi_2 - \phi_1)}{\underbrace{(x_2 - x_1)}_{\text{Potential Diff. (x)}}} + \hat{j}\frac{(\phi_2 - \phi_1)}{\underbrace{(y_2 - y_1)}_{\text{Potential Diff. (y)}}} + \hat{k}\frac{(\phi_2 - \phi_1)}{\underbrace{(z_2 - z_1)}_{\text{Potential Diff. (z)}}}$$

$$\nabla r = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right] r$$

Scaler Point function

$$= \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}$$

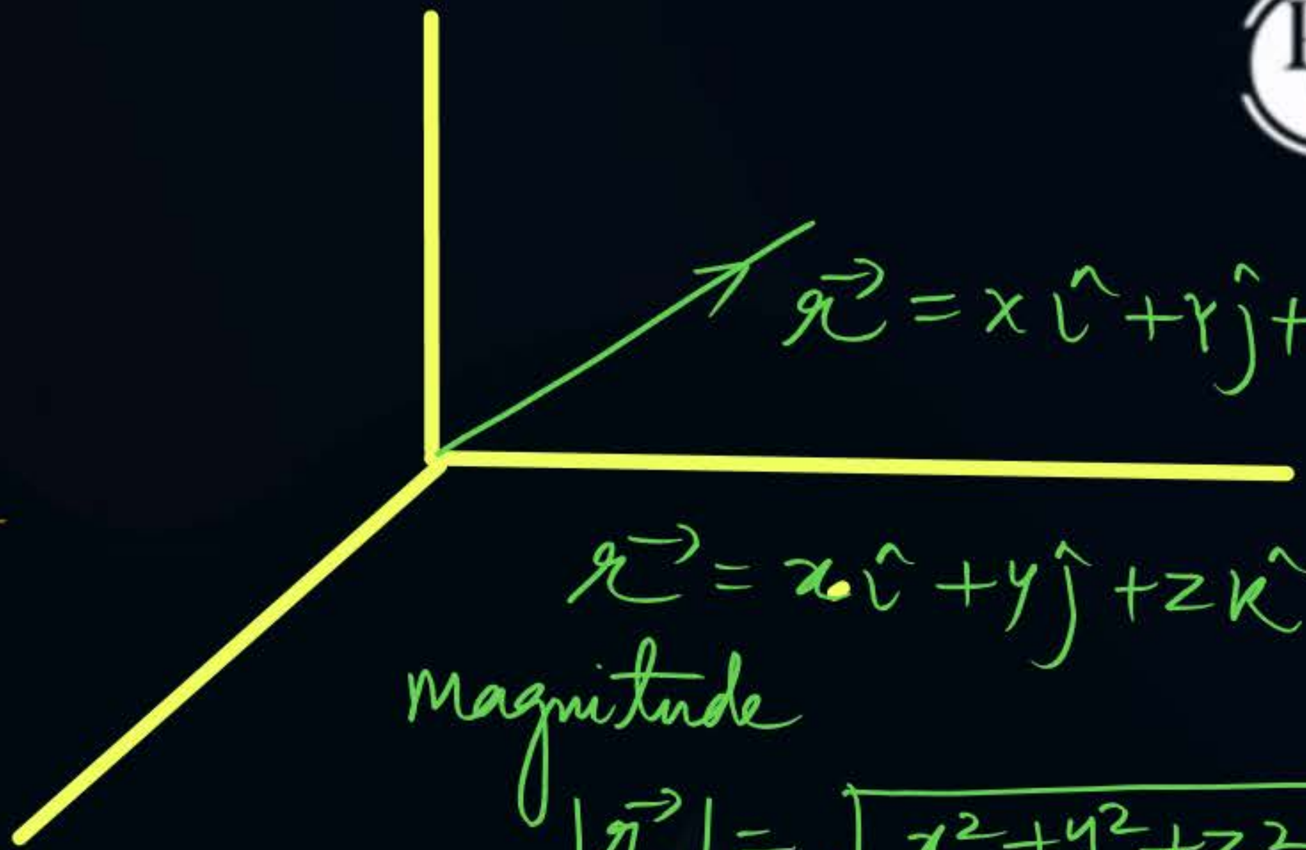
$$\nabla r = \hat{i}\frac{x}{r} + \hat{j}\frac{y}{x} + \hat{k}\frac{z}{r}$$

$$\nabla r = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{\vec{r}}{r}$$

Scaler function

$$\boxed{\nabla r = \frac{\vec{r}}{r}} \rightarrow$$ vector function

grad



$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

magnitude

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\boxed{r^2 = x^2 + y^2 + z^2}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

w.r.t x

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

w.r.t to y

$$\frac{\partial r}{\partial y} = \frac{y}{r} \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla r = \frac{\vec{r}}{r} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r} = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \hat{k}\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$\underbrace{\sqrt{x^2+y^2+z^2}}_{\text{X Direction}}$ $\underbrace{\sqrt{x^2+y^2+z^2}}_{\text{Y Direction}}$ $\underbrace{\sqrt{x^2+y^2+z^2}}_{\text{Z Direction}}$

Direction cosines x-axis

Direction cosines

$$\begin{cases} \cos\alpha = \dfrac{x}{r} \\ \cos\beta = \dfrac{y}{r} \\ \cos\gamma = \dfrac{z}{r} \end{cases}$$

$$\boxed{\nabla r = \frac{\vec{r}}{r}}$$

$\nabla[\text{Scaler}] = \text{Vector}$

$\xrightarrow{\quad}$ grad

$$\boxed{\nabla[f(r)] = f'(r)\frac{\vec{r}}{r}}$$

$$\nabla[r^n] = n r^{n-1}\frac{\vec{r}}{r} = n r^{n-2}\vec{r}$$

$$\nabla[f(r)] = f'(r)\frac{\vec{r}}{r}$$

$$\nabla[\log r] = \frac{1}{r}\cdot\frac{\vec{r}}{r} = \frac{\vec{r}}{r^2}$$

$$\nabla\left[\frac{1}{r}\right] = -\frac{1}{r^2}\,\frac{\vec{r}}{r} = -\frac{\vec{r}}{r^3}$$

$$\nabla[r^2] = 2r\cdot\frac{\vec{r}}{r} = \underline{2\cdot\vec{r}}$$

$\frac{\vec{a}}{|a|}$

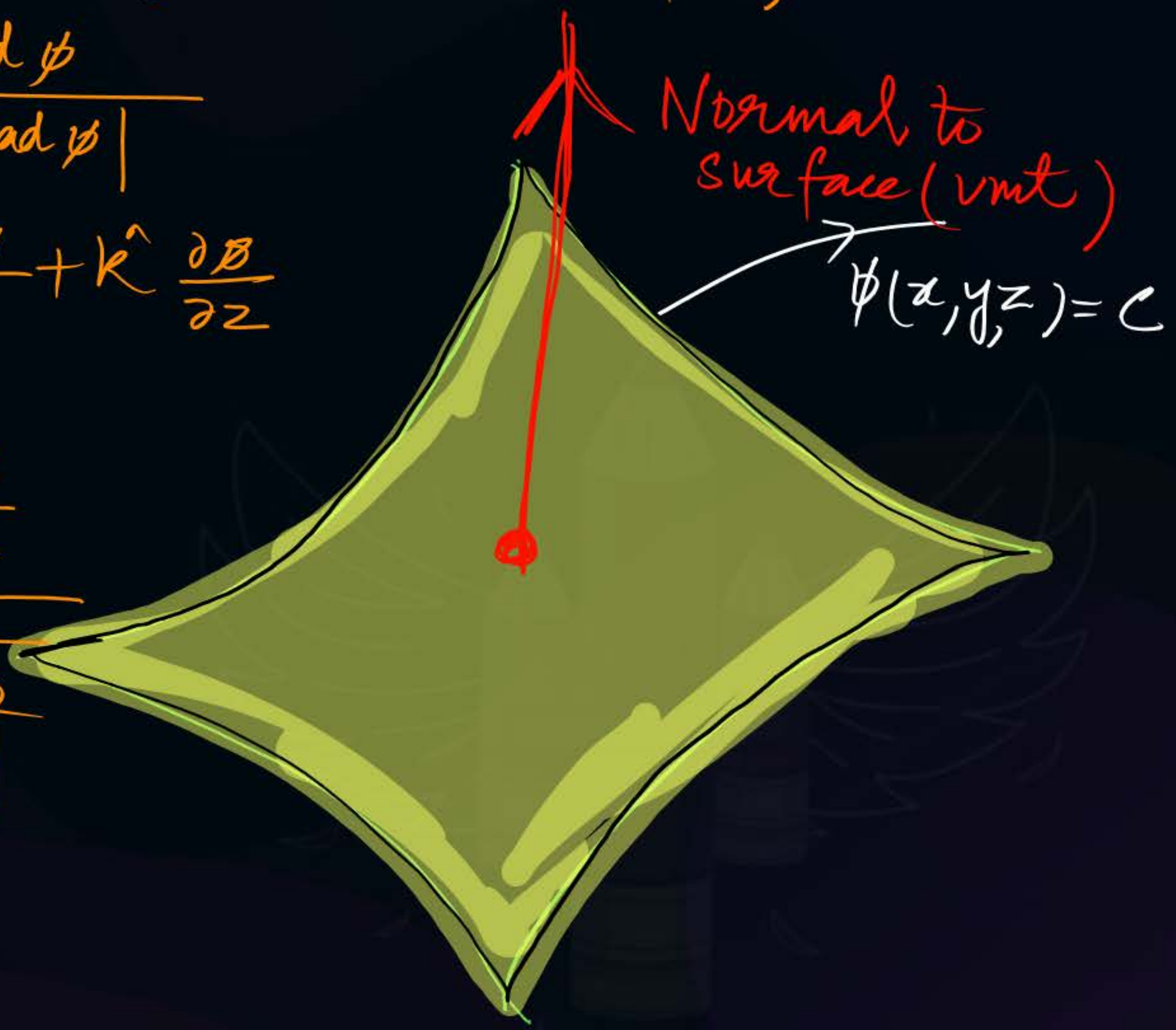
Normal to The Surface:

$\phi(x,y,z) = c$ (level surface)

Normal to The Surface $= \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$

$$\text{grad}\,\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$N = \frac{\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}}{\sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2}}$$



$$\phi(x,y,z) = x^2+y^2+z^2 = 1 \quad \text{at } (1,1,1)$$

Sphere

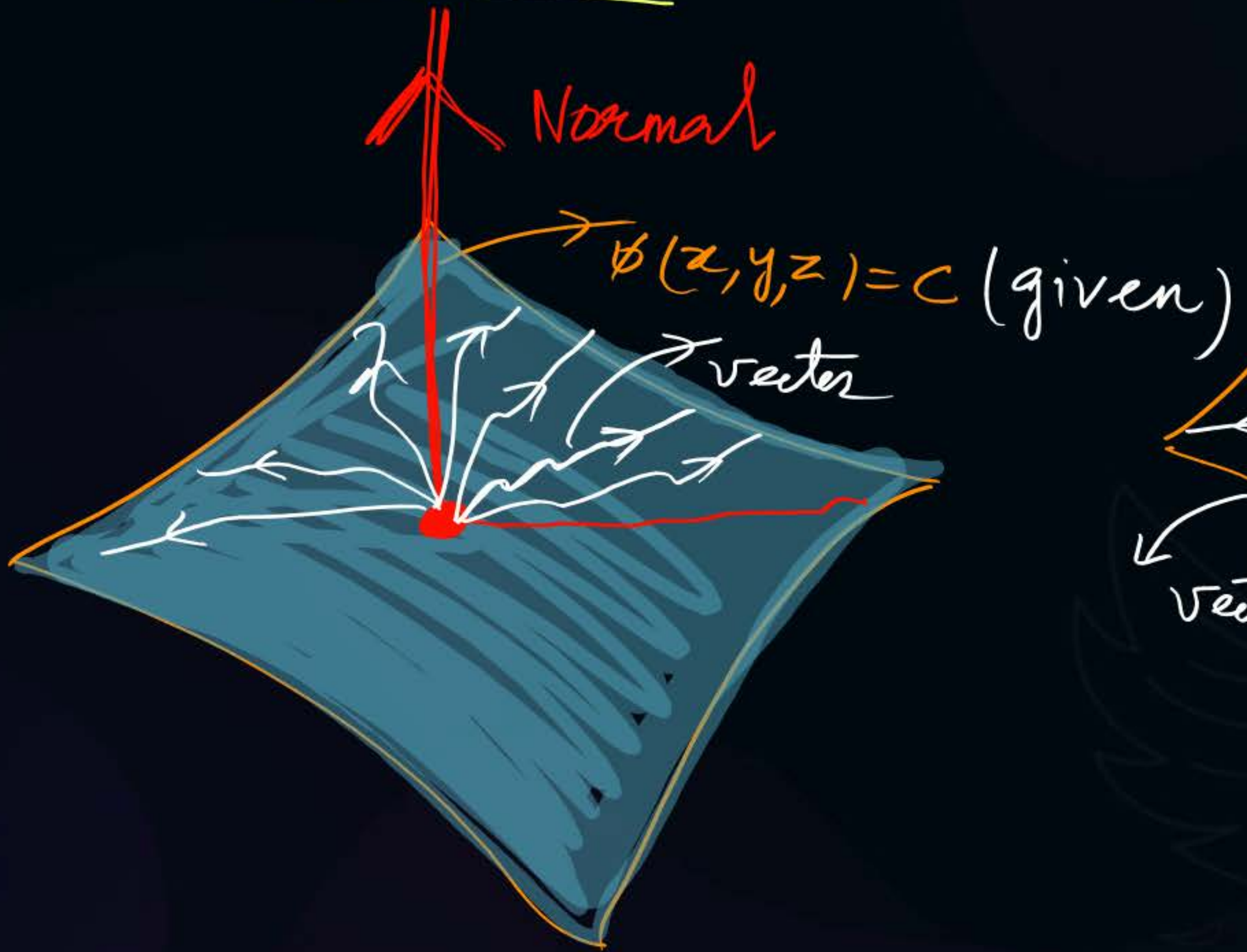
$$x^2+y^2+z^2=1$$

$$N = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{(2x)^2+(2y)^2+(2z)^2}}$$

$$= \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{2\sqrt{x^2+y^2+z^2}}$$

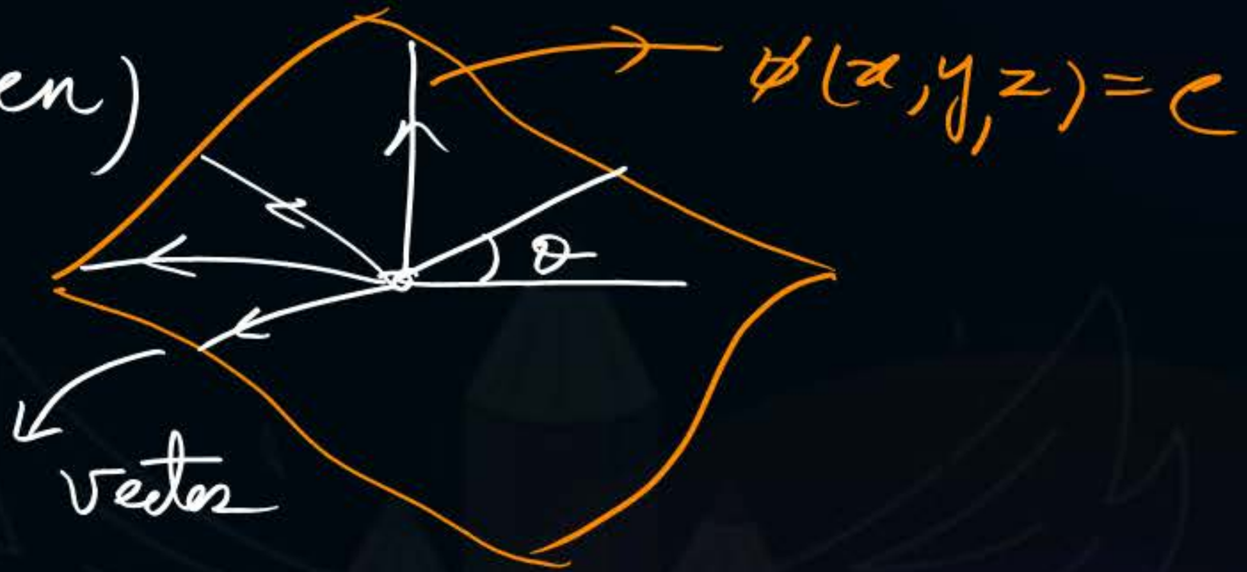
$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{1+1+1}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{3}} = \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$$

2marks: Directional derivative:



derivative $\frac{dy}{dx}, \frac{dx}{dy}$

along straight



Directional derivative $= (\text{grad}\,\phi)\cdot\hat{a} = (\nabla\phi)\hat{a}$

Where $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

$\text{grad}\,\phi = \hat{i}\dfrac{\partial\phi}{\partial x} + \hat{j}\dfrac{\partial\phi}{\partial y} + \hat{k}\dfrac{\partial\phi}{\partial z}$

Directional derivative along

The x-axis is

$\Rightarrow (\nabla\phi)\cdot\hat{i}$

Y axis $= (\nabla\phi)\cdot\hat{j}$

Z axis $= (\nabla\phi)\cdot\hat{k}$

Tangent to The curve

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

x, y, z Are Function in t

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

$\frac{d\vec{r}}{dt}$ = Tangent to a curve

$$\boxed{\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}}$$

Topic : Vector calculus

grad ϕ (8,8)

#Q. A scalar field is given by $\boxed{f = x^{2/3} + y^{2/3},}$ where x and y are the Cartesian coordinates. The derivative of 'f' along the line y = x directed away from the origin at the point (8, 8) is

$$\nabla f = \frac{2}{3} x^{-1/3} \hat{i} + \frac{2}{3} y^{-1/3} \hat{j}$$

$$= \frac{2}{3} \times \frac{1}{(8)^{1/3}} \hat{i} + \frac{2}{3} \frac{1}{(8)^{1/3}} \hat{j}$$

$$= \frac{\cancel{2}}{3} \times \frac{1}{\cancel{2}} \hat{i} + \frac{\cancel{2}}{3} \times \frac{1}{\cancel{2}} \hat{j}$$

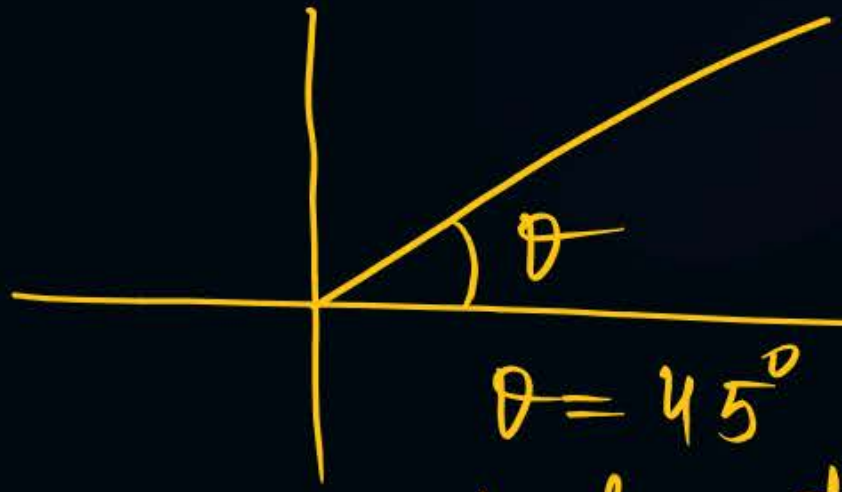
$$= \frac{\hat{i} + \hat{j}}{3}$$

A. $\frac{\sqrt{2}}{3}$

B. $\frac{\sqrt{3}}{2}$

C. $\frac{2}{\sqrt{3}}$

D. $\frac{3}{\sqrt{2}}$

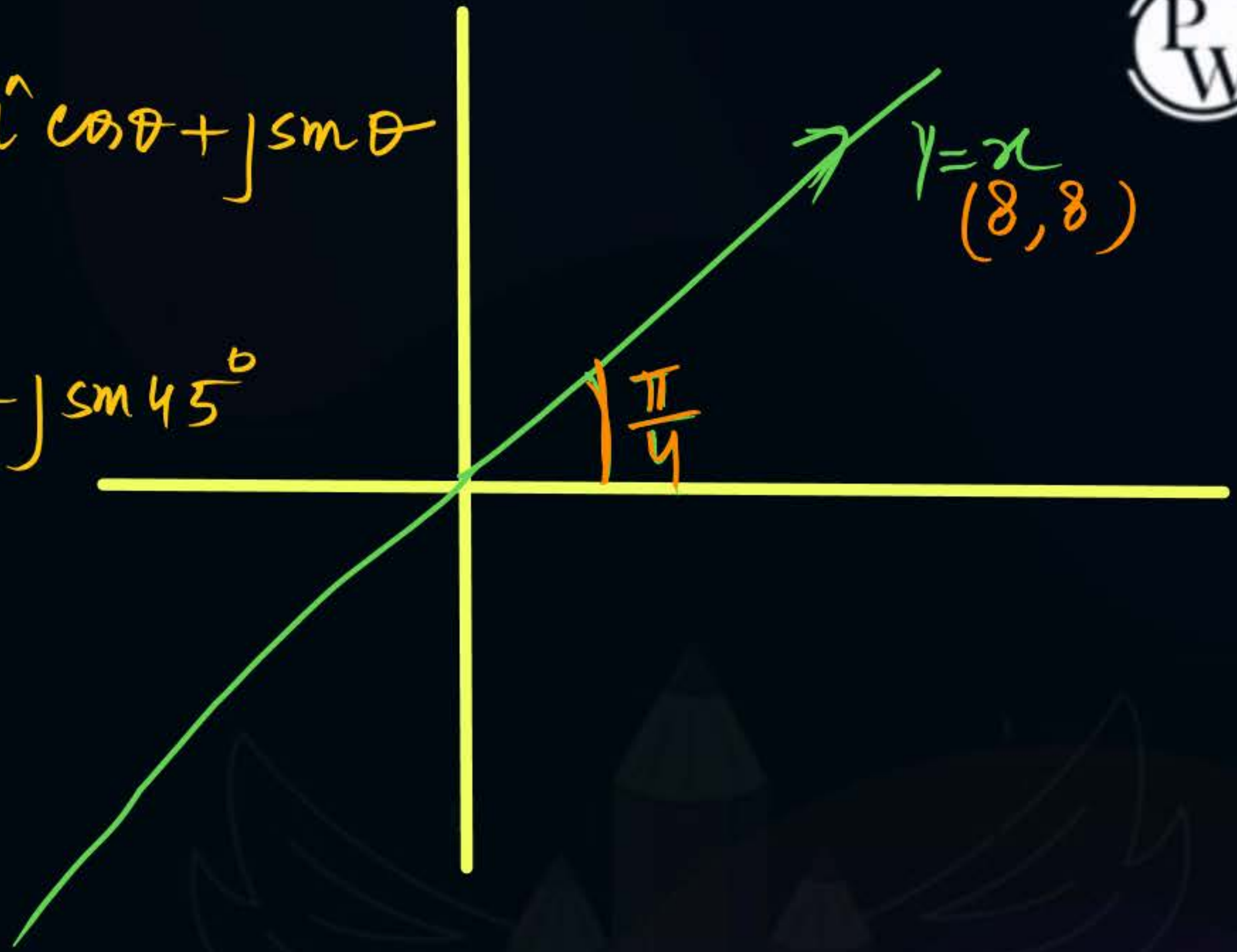


Unit vector = $\hat{i}\cos\theta + \hat{j}\sin\theta$

$\theta = 45^\circ$

Unit vector = $\hat{i}\cos 45^\circ + \hat{j}\sin 45^\circ$

$$\hat{a} = \frac{\hat{i}+\hat{j}}{\sqrt{2}}$$



Directional derivative

$$= (\text{grad} f)\cdot\hat{a}$$

$$= \frac{\hat{i}+\hat{j}}{3} \times \frac{\hat{i}+\hat{j}}{\sqrt{2}}$$

$$= \frac{1+0+0+1}{3\sqrt{2}} = \frac{2}{3\sqrt{2}} = \frac{\sqrt{2}}{3}$$

Topic : Vector calculus

$\rightarrow P(2,1,3)$

#Q. The directional derivative of $\boxed{f(x, y, z) = 2x^2 + 3y^2 + z^2}$ at the point $\boxed{p(2, 1, 3)}$ in direction of the vector $\vec{a} = \bar{\imath} - 2\bar{k}$ is ______. $\vec{a} = \hat{i} - 2\hat{k}$

Directional derivative $= (\text{grad}\,\phi)\,\hat{a}$

A -2.785

B -2.145

C -1.789

D 1.000

$$\text{grad}\,\phi = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\left[2x^2 + 3y^2 + z^2\right]$$

$$\text{grad}\,\phi = \hat{i}(2\times 2x) + \hat{j}(6y) + \hat{k}(2z)$$

$$(\nabla\phi)_{(2,1,3)} \Rightarrow 2\times 4\hat{i} + 6\hat{j} + 6\hat{k} = 8\hat{i} + 6\hat{j} + 6\hat{k}$$

$$\text{grad}\,\phi = (8\hat{i} + 6\hat{j} + 6\hat{k})$$

$$\text{Directional derivative} = (8\hat{i} + 6\hat{j} + 6\hat{k}) \times \frac{(\hat{i} - 2\hat{k})}{\sqrt{1 + (-2)^2}}$$

$$= \frac{8 - 12}{\sqrt{5}} = \frac{-4}{\sqrt{5}} \text{ Ans}$$

$\hat{i} \cdot \hat{i} = 1$ $\quad \hat{j} \cdot \hat{j} = 1$

$\hat{i} \cdot \hat{j} = 0$ $\quad k \cdot k = 1$

$\hat{j} \cdot \hat{k} = 0$

Topic : Vector calculus

#Q. The directional derivative of $\boxed{f(x, y)= \frac{xy}{\sqrt{2}}(x + y)}$ at (1, 1) in the direction of the unit vector at an angle of $\frac{\pi}{4}$ with X- axis, is given by ________.

∇f

$$f(x,y) = \frac{xy}{\sqrt{2}}(x+y)$$

$$\nabla f = \text{grad } f = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\left[\frac{xy}{\sqrt{2}}(x+y)\right]$$

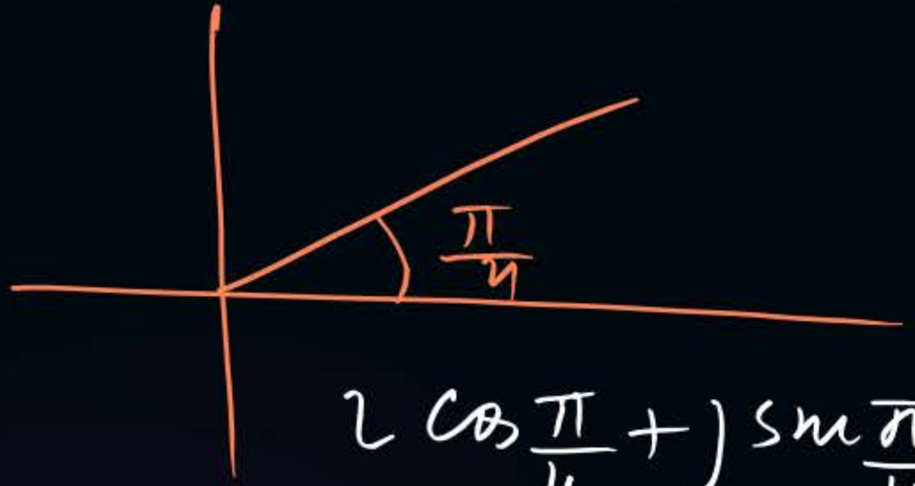
$$= \hat{i}\frac{\partial}{\partial x}\left[\frac{x^2 y}{\sqrt{2}} + \frac{xy^2}{\sqrt{2}}\right] + \hat{j}\frac{\partial}{\partial y}\left[\frac{x^2 y}{\sqrt{2}} + \frac{xy^2}{\sqrt{2}}\right]$$

$$(\text{grad } f)_{(1,1)} = \hat{i}\left[\frac{2xy}{\sqrt{2}} + \frac{y^2}{\sqrt{2}}\right] + \hat{j}\left[\frac{x^2}{\sqrt{2}} + \frac{2xy}{\sqrt{2}}\right] + 0$$

$$= \hat{i}\left[\frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] + \hat{j}\left[\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}}\right]$$

$$= \frac{3\hat{i} + 3\hat{j}}{\sqrt{2}}$$

Directional derivative $= (\text{grad } f)\,\hat{a}$



$2\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}$

$= \frac{\hat{i}+\hat{j}}{\sqrt{2}}$

$= \frac{(3\hat{i}+3\hat{j})}{\sqrt{2}} \times \left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right)$

$= \frac{3+3}{2}$

$= 3$

Topic : Vector calculus

grad v

H.W

#Q. The directional derivative of field $u(x,y,z) = x^2 - 3yz$ in the direction of the vector $(\hat{i} + \hat{j} - 2\hat{k})$ at point (2,-1,4) is ______________

2 mins Summary

Topic One $\longrightarrow$ grad ϕ

Topic Two $\longrightarrow$ Normal

Topic Three $\longrightarrow$ D.D

Topic Four $\longrightarrow$ Tangent

Topic Five $\longrightarrow$ questions

THANK - YOU