GATE-All BRANCHES
Engineering Mathematics

2024

Vector calculus

Lecture No.- 06

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Recap of previous lecture

Topic Surface integral

Topic Question based on surface integral

Topic Divergence of a vector function

Topics to be Covered

Topic Gauss divergence theorem

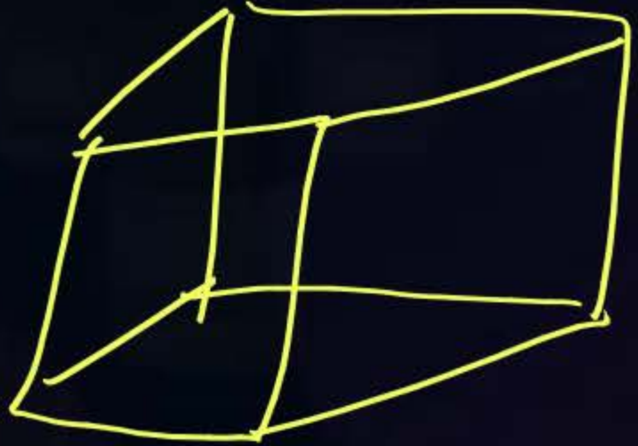
Topic $\boxed{\text{Problems based on gauss divergence theorem}}$

Important for
यांत्रिकी अभियांत्रिकी
(ME)

Topic : Vector calculus

#Q. A vector is defined as $\boxed{f = y\,\hat{\imath} + x\,\hat{\jmath} + z\,\hat{k}}$, where $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are unit vectors in cartesian (x, y, z) coordinate system. The surface integral $\boxed{\oiint f.ds}$ over the closed surface S of a cube with vertices having the following coordinates:

(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 0, 1), (1, 1, 1), (0, 1, 1), (1, 1, 0) is ____.



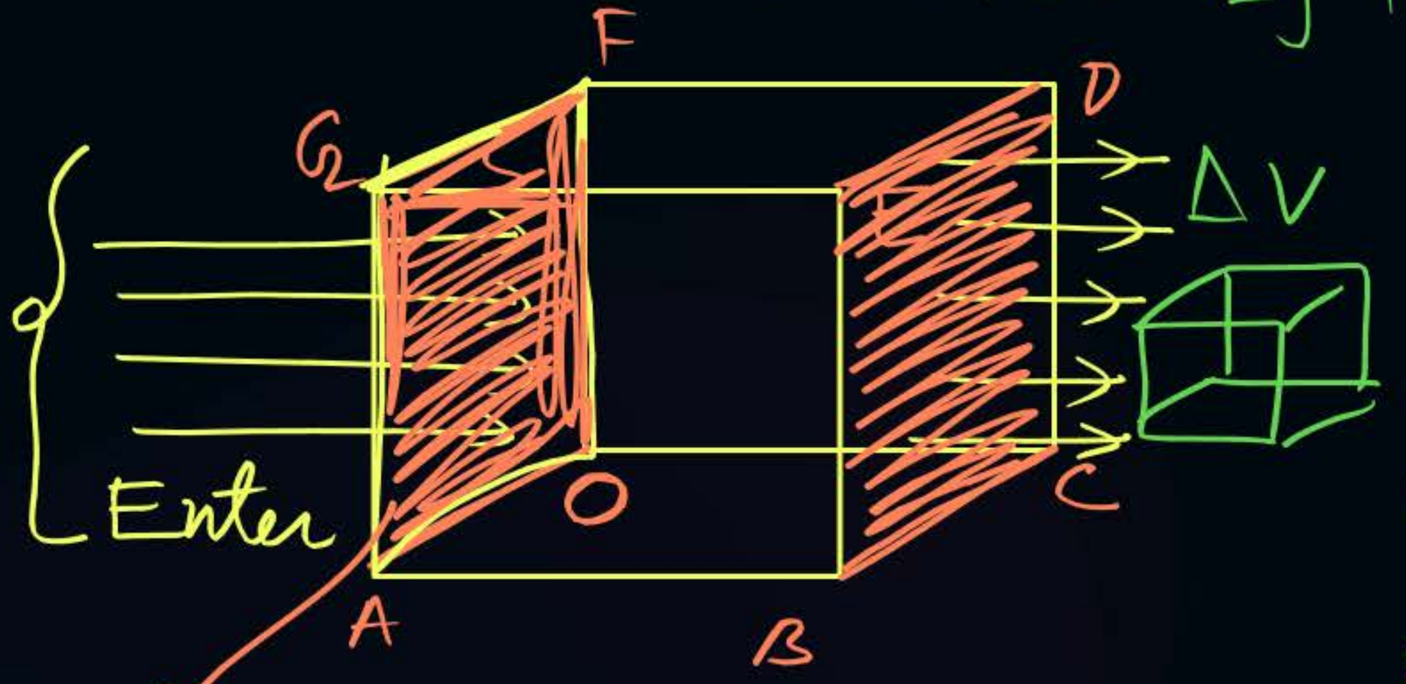
$a = 1$ cube

Divergence of a Vector Function:

Vector function $\xrightarrow{\text{div} \vec{F}}$ Scaler Function

$\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

vector $\xrightarrow{\nabla \cdot \vec{F}}$ scaler. (dot Product)



Surface Integral (How much flued Are enter)

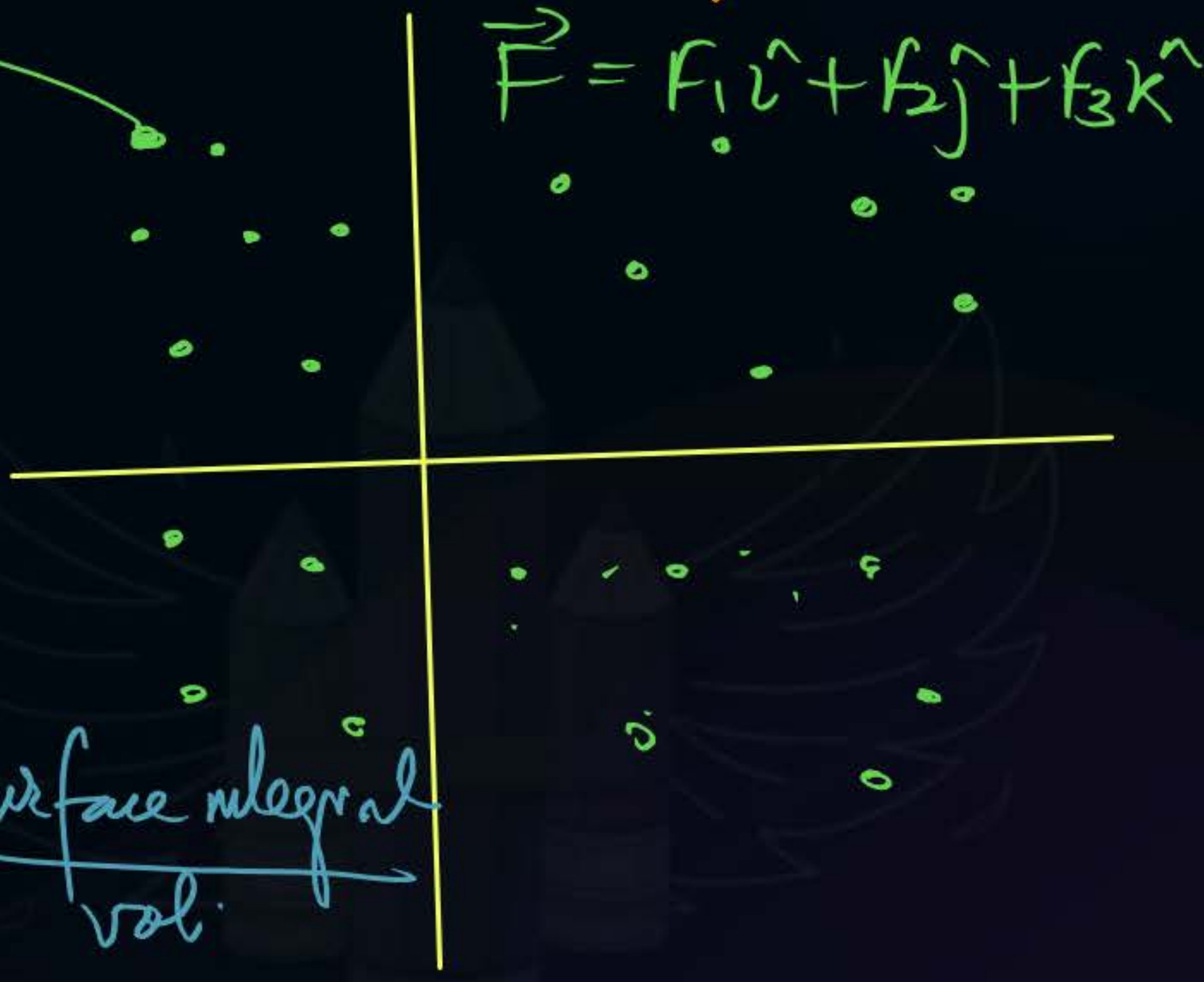
Enlarge

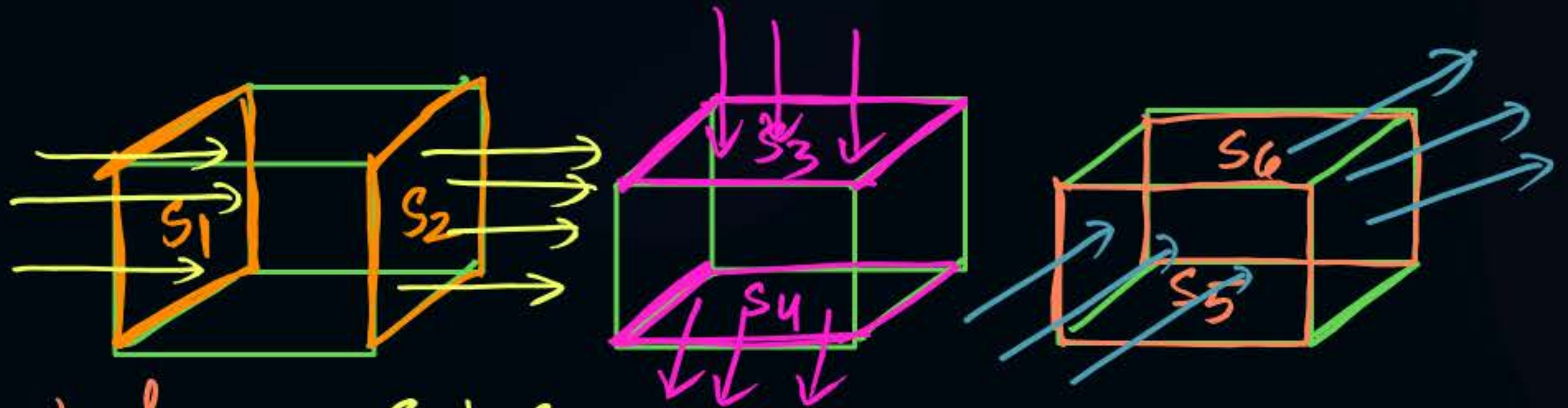
surface

Unit cube

$$= \lim_{\Delta V \to 0} \frac{\oiint \vec{F} \cdot \hat{N} \, ds}{\Delta V}$$

$$\text{Div} \vec{F} = \lim_{\Delta V \to 0} \frac{\oiint \vec{F} \cdot \hat{N} \, ds}{\Delta V} = \frac{\text{surface integral}}{\text{vol.}}$$





volume $= S_1 + S_2 + S_3 + S_4 + S_5 + S_6 =$ Total surface integral

$$\nabla \cdot \vec{F} = \lim_{dV \to 0} \frac{\oiint \vec{F} \cdot \hat{n}\, ds}{dV}$$

where $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

$$\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$$

$$\text{div} \vec{F} = \nabla \cdot \vec{F} = \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] \left[F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k} \right]$$

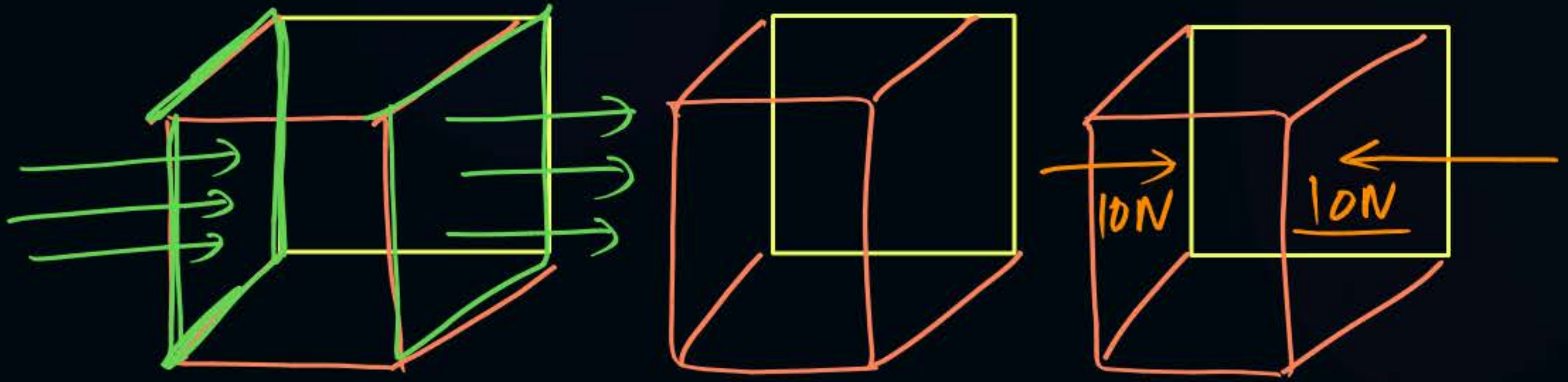
$$\boxed{\text{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

If $\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$

$$\text{div} \vec{r} = \nabla \cdot \vec{r}$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1 = 3$$



Field conservative

Incoming − outgoing = 0
Enter − Exit = 0
Source − sink = 0

① If $\text{div}\,\vec{F} = 0$ (Solenoidal behaviour)

Inductor (circle balance)

$\boxed{\text{Div}\,\vec{F} = 0}$

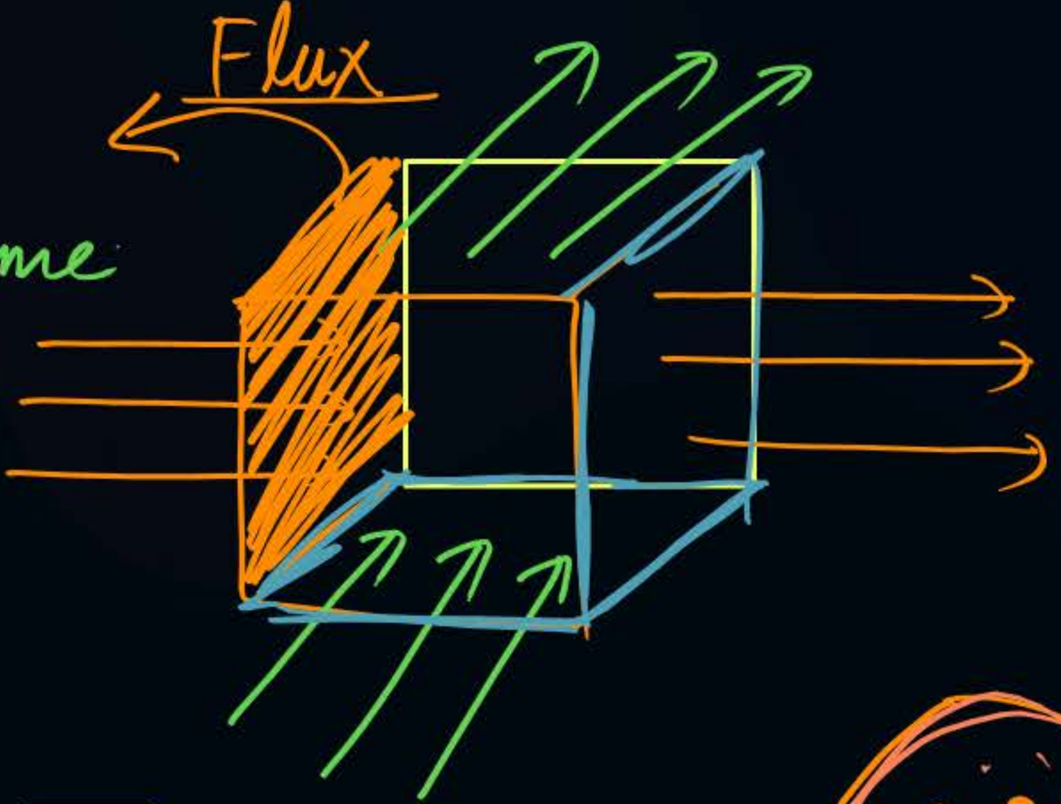
② If $\text{div}\,\vec{F} = 0$ conservative field
= Incompressible flows $\boxed{\text{div}\,\vec{F} = 0}$

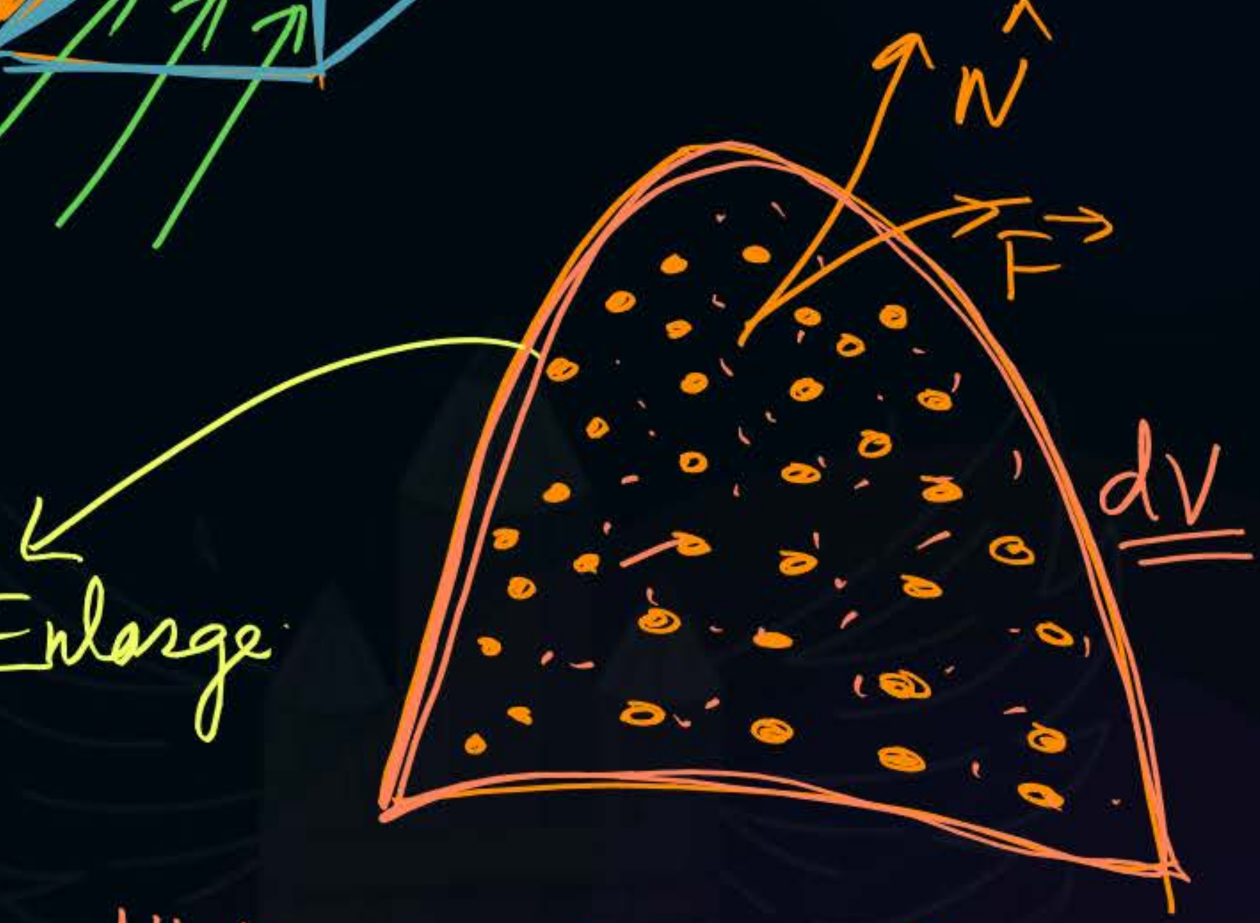
$$S_1 + S_2 + S_3 + \cdots + S_6 = \text{volume}$$

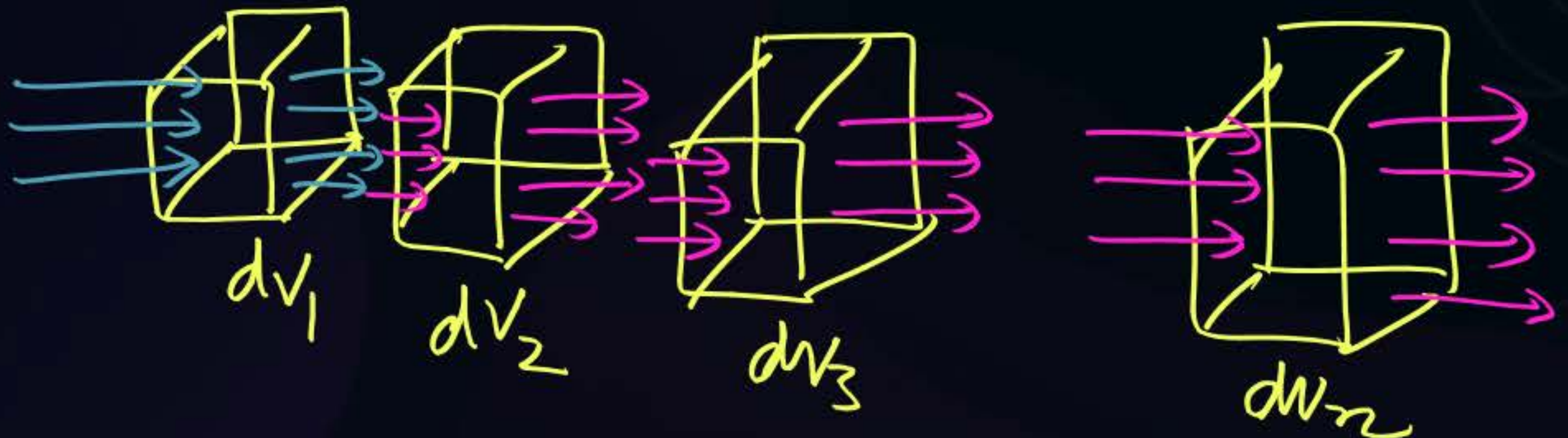
Guass divergence Theorem :

$$\oiint \vec{F}\,\hat{N}\, ds = \iiint \underbrace{(\nabla \cdot \vec{F})}_{\text{div } \vec{F}}\, dV$$

$$\boxed{\oiint \vec{F}\,\hat{N}\, ds = \iiint (\nabla \cdot \vec{F})\, dx\, dy\, dz}$$





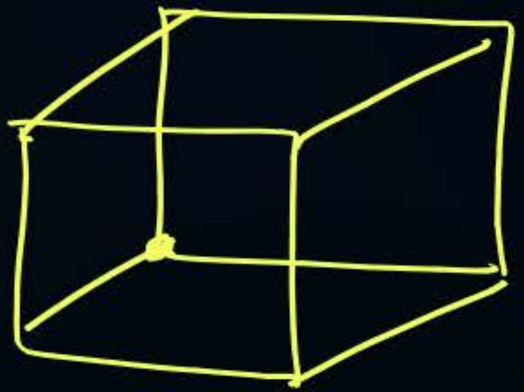


All divergence Are cancel out

$$\vec{f} = y\hat{i} + x\hat{j} + z\hat{k}$$

$$\text{div}\,\vec{f} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z)$$

$$= 0 + 0 + 1 = 1$$



Unit cube

Length = 1

Using Gauss divergence Theorem:

$$\oiint \vec{f}\, ds = \oiiint (\nabla \cdot \vec{F})\, dv$$

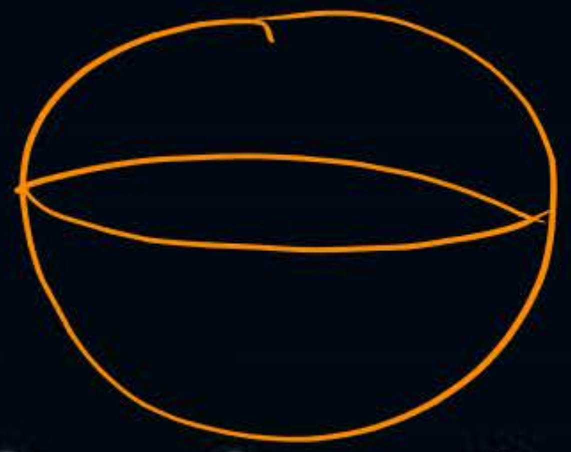
$$= \oiiint 1\, dv$$

$$= \text{volume of cube}$$

$$= a^3 = (1)^3 = \underline{1}$$

Topic : Vector calculus

$x^2+y^2+z^2=9$



sphere

#Q. The surface integral $\iint_S \frac{1}{\pi}(9xi - 3yj)\cdot n\, dS$ over the sphere given by $x^2 + y^2 + z^2 = 9$ is ______.

Divergence → surface Integral

Using Guass Divergence THEOREM

$$\oiint \vec{F}\,\hat{N}\, ds = \iiint (\nabla\cdot\vec{F})\, dv$$

4 December - NO PYQ
NO DPP

$$\vec{F} = \frac{1}{\pi}(9x\hat{i} - 3y\hat{j}) = \frac{9x}{\pi}\hat{i} - \frac{3y}{\pi}\hat{j} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$$F_1 = \frac{9x}{\pi} \qquad F_2 = \frac{-3y}{\pi}$$

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = \text{Dot Product} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}\left(\frac{9x}{\pi}\right) + \frac{\partial}{\partial y}\left(\frac{-3y}{\pi}\right)$$

$$= \frac{9}{\pi} - \frac{3}{\pi} = \frac{6}{\pi}$$

Using Gauss Divergence Theorem

$$\oiint \vec{F}\cdot\hat{N}\,ds = \oiiint\left(\frac{6}{\pi}\right)\cdot dv$$

$$= \frac{6}{\pi}\boxed{\oiiint dy\,dx\,dz} = \frac{6}{\pi}\times\text{volume of sphere}$$

$$= \frac{\cancel{6}}{\cancel{\pi}}\times\frac{4}{\cancel{3}}\cancel{\pi}\times\cancel{(3)^3}\,3^2$$

$$= 216$$

$$\text{Area} = \iint dy\,dx$$

$$\text{Vol} = \iiint dx\,dy\,dz$$

72π

#Q. The surface integral $\iint_S F.ndS$ over the surface S of the sphere $x^2 + y^2 + z^2 = 9$, where $F = (x + y)i + (x + z)j + (y + z)k$ and n is the unit outward surface normal, yields ______.

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(x+y) + \frac{\partial}{\partial y}(x+z) + \frac{\partial}{\partial z}(y+z)$$
$$= 1 + 0 + 1 = 2$$

Using Gauss Divergence Theorem.

$$\iint \vec{F}\cdot\hat{n}\,ds = \iiint (\nabla\cdot\vec{F})\,dv$$

$$= \iiint 2\,dv = 2 \times \text{volume of sphere}$$

$$= 2 \times \frac{4}{3}\pi r^3$$

$$= 2 \times \frac{4}{3}\pi \times (3)^3$$

$$= 9 \times 8\pi$$

$$= \underline{72\pi}$$

Topic : Vector calculus

7 times

$\nabla \cdot \vec{F} = \text{div}$

$\nabla \times \vec{F} = \text{curl}$

$\nabla \phi = \text{gradient}$

#Q. The value of the integral $\oiint_S \vec{r}.\vec{n}dS$ over the closed surface S bounding a volume V, where $\bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is the position

$$\oiint \vec{r}\cdot\hat{n}\,ds = \iiint (\nabla\cdot\vec{r})\,dv = \iiint 3\,dv$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla\cdot\vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

$$= 3\boxed{\iiint dv} = \underline{3V}$$

A. V

B. 2V

C. 3V ✓

D. 4V

Topic : Vector calculus

$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

div $\vec{u}$ =

#Q. The divergence of the vector field $\vec{u} = e^x(\cos y\,\hat{\imath} + \sin y\,\hat{\jmath})$ is

$\vec{u} = e^x \cos y\,\hat{i} + e^x \sin y\,\hat{j}$

$\text{div}\,\vec{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$

$= \frac{\partial}{\partial x}(e^x \cos y) + \frac{\partial}{\partial y}(e^x \sin y) + \frac{\partial}{\partial z}(0)$

$\Rightarrow e^x \cos y + e^x \cos y$

$= \underline{2e^x \cos y}$

A. 0

B. $e^x \cos y + e^x \sin y$

C. $2e^x \cos y$

D. $2e^x \sin y$

Topic : Vector calculus

Guass Divergence

$\vec{F}$

#Q. The value of the surface integral $\iint_S (9xi - 2yj - zk) \cdot ndS$ over the surface S of the sphere $x^2 + y^2 + z^2 = 9$, where n is the unit outward normal to the surface element dS, is $\boxed{216\pi}$

$$\iint_S \vec{F}\,\hat{N}\,ds = \iiint (\nabla \cdot \vec{F})\,dv$$

$$\vec{F} = 9x\hat{i} - 2y\hat{j} - z\hat{k}$$

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(9x) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(-z)$$

$$\nabla \cdot \vec{F} = 9 - 2 - 1 = 6$$

sphere vol

$$= \iiint 6\,dv = 6\boxed{\iiint dy\,dx\,dz}$$

$$= 6 \times \frac{4}{3}\pi(3)^2 = 216\pi$$

$\hat{n} = \hat{k}$ XY Plane

#Q. Given a vector $\bar{u} = \frac{1}{3}\left(-y^3\hat{\imath} + x^3\hat{\jmath} + z^3\hat{k}\right)$ and $\hat{n}$ as the unit normal vector to the surface of the hemisphere ($x^2 + y^2 + z^2 = 1; z \geq 0$), the value of integral $\int \underbrace{(\nabla \times u)}_{\text{curl } \vec{u}}.\bar{n}\, dS$ evaluated on the curved surface of the hemisphere S is

$= \frac{\pi}{2}$ ✓

A $-\frac{\pi}{2}$

B $\frac{\pi}{3}$

C $\frac{\pi}{2}$

D π

2 mins Summary

Topic One $\longrightarrow$ Divergence $\longrightarrow$ $\lim_{dv \to 0} \frac{\oiint \vec{F} \cdot \hat{n}\, ds}{dv}$

Topic Two $\longrightarrow$ Gauss Divergence $\oiint \vec{F}\, ds = \iiint (\nabla \cdot F) \cdot dv$

Topic Three

Topic Four

Topic Five

THANK - YOU