GATE-All BRANCHES
Engineering Mathematics

2024

Vector Calculus

Lecture No.- 03

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Recap of Previous Lecture

Topic $\boxed{\text{Concept of curl}}$

Topic Greens theorem and Stokes theorem

(2d)
Line integrals

3d
line

Topics to be Covered

Topic Concept of curl

Topic Greens theorem and Stokes theorem

Topic Problems based on Green's theorem and stokes theorem

Topic: Vector calculus

#Q. If $\vec{A} = xy\,\hat{a}_x + x^2\hat{a}_y$ then $\oint \vec{A}\cdot d\bar{r}$ over the path shown in the figure is

$M = xy \quad \frac{\partial M}{\partial y} = x$

$N = x^2 \quad \frac{\partial N}{\partial x} = 2x$

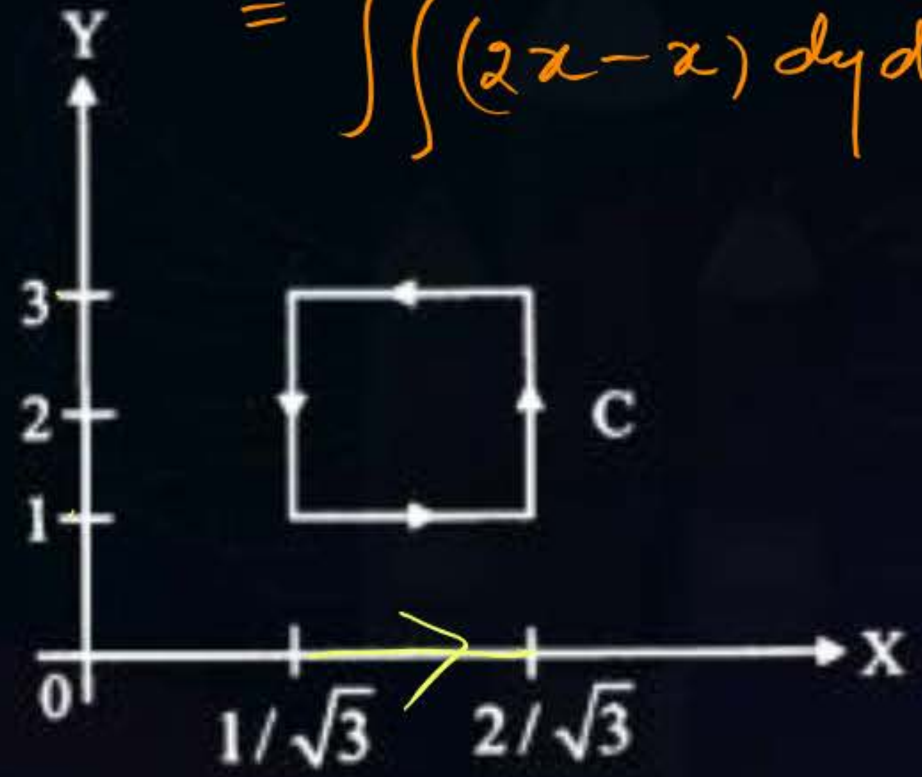
Using green's Theorem

$\oint M\,dx + N\,dy = \iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$

$= \iint (2x - x)\,dy\,dx$

$\Rightarrow \iint x\,dy\,dx$

$= \int_{\frac{1}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}}\int_{1}^{3} x\,dy\,dx = 1$



A) 0

B) $2/\sqrt{3}$

C) 1 ✓

D) $2\sqrt{3}$

$$= \int_{\frac{1}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}} x\,dx \int_{1}^{3} dy$$

$$= 2\int_{\frac{1}{\sqrt{3}}}^{\frac{2}{\sqrt{3}}} x\,dx = 1$$

Topic: Vector calculus

Calculate $\oint \vec{F} \, d\vec{r}$

3 dimensional work done — <u>Stokes Theorem</u>

#Q. Calculate $\oint \vec{F} \cdot d\vec{l}$ where $\boxed{\vec{F} = 2y^2\hat{i} + 3x^2\hat{j} - (2x+z)\hat{k}}$, C is the Boundary of the triangle whose vertices $\underset{A}{\underline{(0,0,0)}}$ $\underset{B}{\underline{(2,0,0)}}$ $\underset{C}{\underline{(2,2,0)}}$

Using stokes Theorem $\oint \vec{F} \, d\vec{r} = \iint \underline{(\nabla \times \vec{F})\hat{n} \, ds}$

$$(\nabla \times \vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y^2 & 3x^2 & -(2x+z) \end{vmatrix} = \hat{i}\left[\cancel{\frac{\partial}{\partial y}(-(2x+z))}^{\,0} - \cancel{\frac{\partial}{\partial z}(3x^2)}^{\,0}\right] - \hat{j}\left[\frac{\partial}{\partial x}(-(2x+z)) - \cancel{\frac{\partial}{\partial z}(2y^2)}^{\,0}\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(3x^2) - \frac{\partial}{\partial y}(2y^2)\right]$$

$$(\nabla \times \vec{F}) = 2\hat{j} + (6x-4y)\hat{k}$$

Using stokes Theorem

$$\oint \vec{F}\cdot d\vec{r} = \iint \left[(2\hat{j} + (6x-4y)\hat{k}\right]\hat{n}\, ds$$

$$= \iint \left[(2\hat{j} + (6x-4y)\hat{k}\right]\hat{k}\, dy\, dx$$

$$= \iint (6x-4y)\, dy\, dx$$

$$\Rightarrow \int_{x=0}^{2}\int_{y=0}^{x} (6x-4y)\, dy\, dx$$

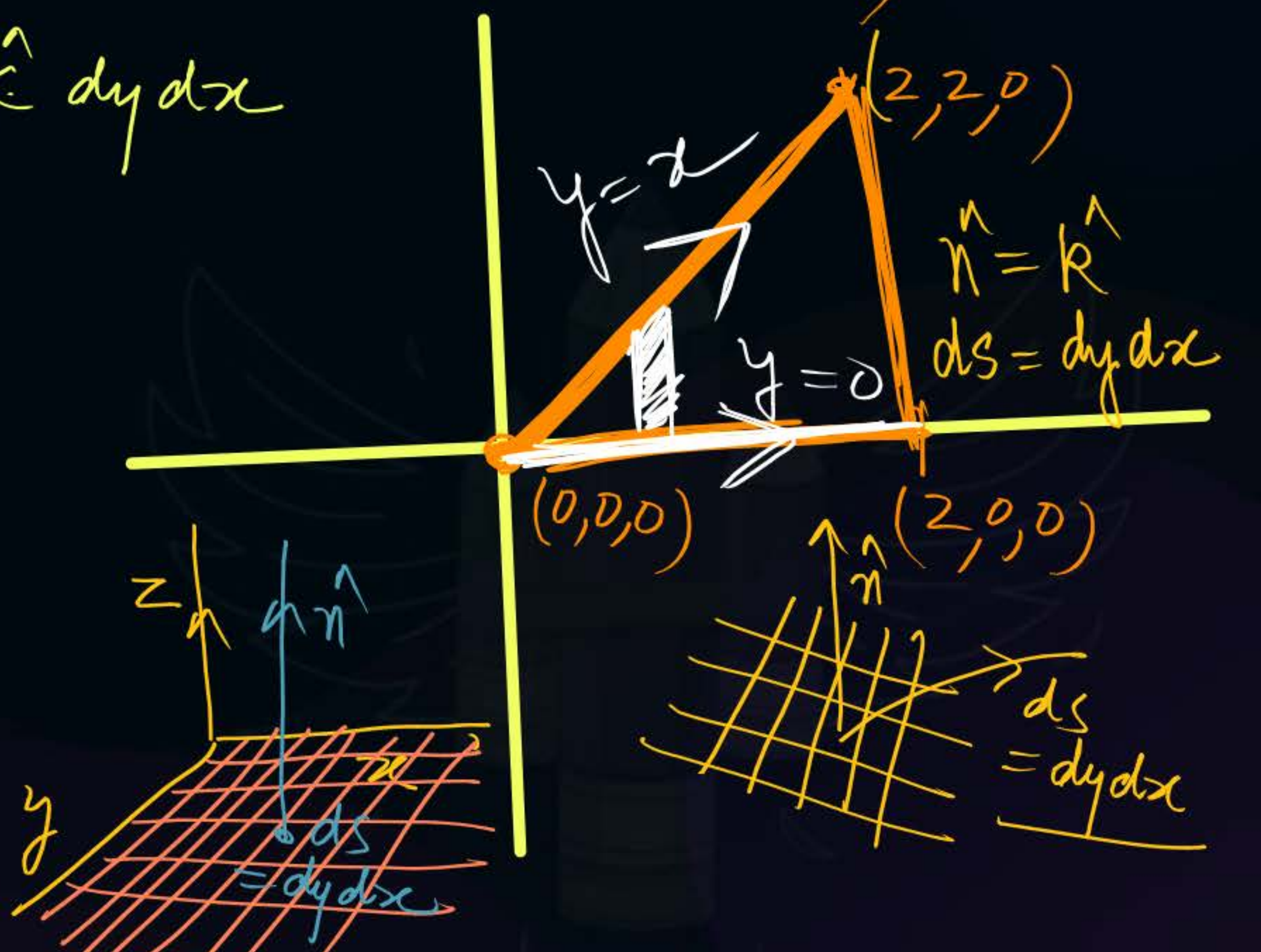
$(x_1\ y_1\ z_1)$

A $(0,0,0)$

B $(2,0,0)$

C $(2,2,0)$

$Z=0$ xy Plane



$$= \int_0^2 dx \left[\int_0^x (6x - 4y)\, dy \right]$$

$$= \int_0^2 dx \left[6xy - \frac{4y^2}{2} \right]_0^x$$

$$= \int_0^2 dx \left[6x^2 - 2x^2 \right]$$

$$= \int_0^2 4x^2\, dx$$

$$= \left[\frac{4x^3}{3} \right]_0^2 = \frac{32}{3}$$

Topic: Vector calculus

$\oint F_1 dx + F_2 dy = \oint \vec{F} \cdot d\vec{r}$ (2d)

#Q. Find $\oint_C (2xy - x^2)dx - (x^2 + y^2)dy$ where C is the closed curve of the Region Bounded by $y = x^2, y^2 = x$

$\oint (2xy - x^2)dx - (x^2 + y^2)dy$ = (2 dimensional)

Green's Theorem

$F_1 = 2xy - x^2$

$\frac{\partial F_1}{\partial y} = 2x$

$\oint F_1 dx + F_2 dy = \iint \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dy\, dx$

$\frac{\partial F_2}{\partial x} = -2x$

apply green's Theorem

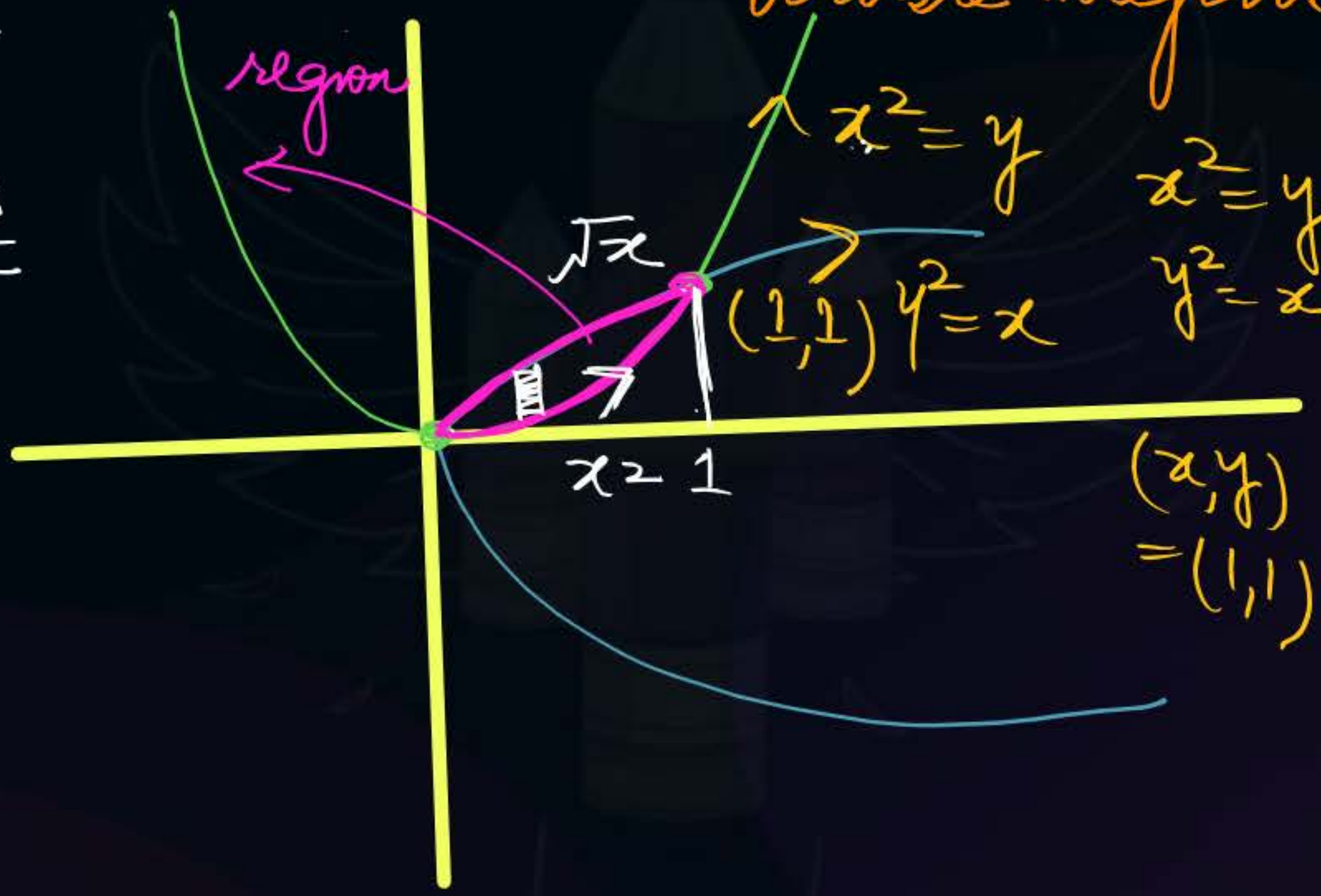
green's Theorem (2 d) (x, y)

$$\oint \vec{F}\,d\vec{r} = \oint F_1\,dx + F_2\,dy = \iint \underbrace{(-2x - 2x)}\,dy\,dx = \underbrace{\iint -4x\,dy\,dx}_{\text{Volume via double integration}}$$

Plot The Curve: $y = x^2$, $y^2 = x$

$$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} -4x\,dy\,dx = -\frac{3}{5}$$

Ans



$x^2 = y$
$y^2 = x$

$(x, y) = (1, 1)$

Topic: Vector calculus

$Mdx + Ndy = \quad M = -y^2 \quad N = xy \quad \left(\frac{\partial N}{\partial x}\right) = y \quad \left(\frac{\partial M}{\partial y}\right) = -2y$

#Q. Value of the integral $\oint_c \boxed{xydy - y^2dx}$ where, c is the square cut from the first quadrant by the line x = 1 and y = 1 will be (Use Green's theorem to change the line integral into double integral)

Using green's Theorem

$\Rightarrow \iint y-(2y)\,dy\,dx$

$= \iint 3y\,dy\,dx$

A. 1/2

B. 1

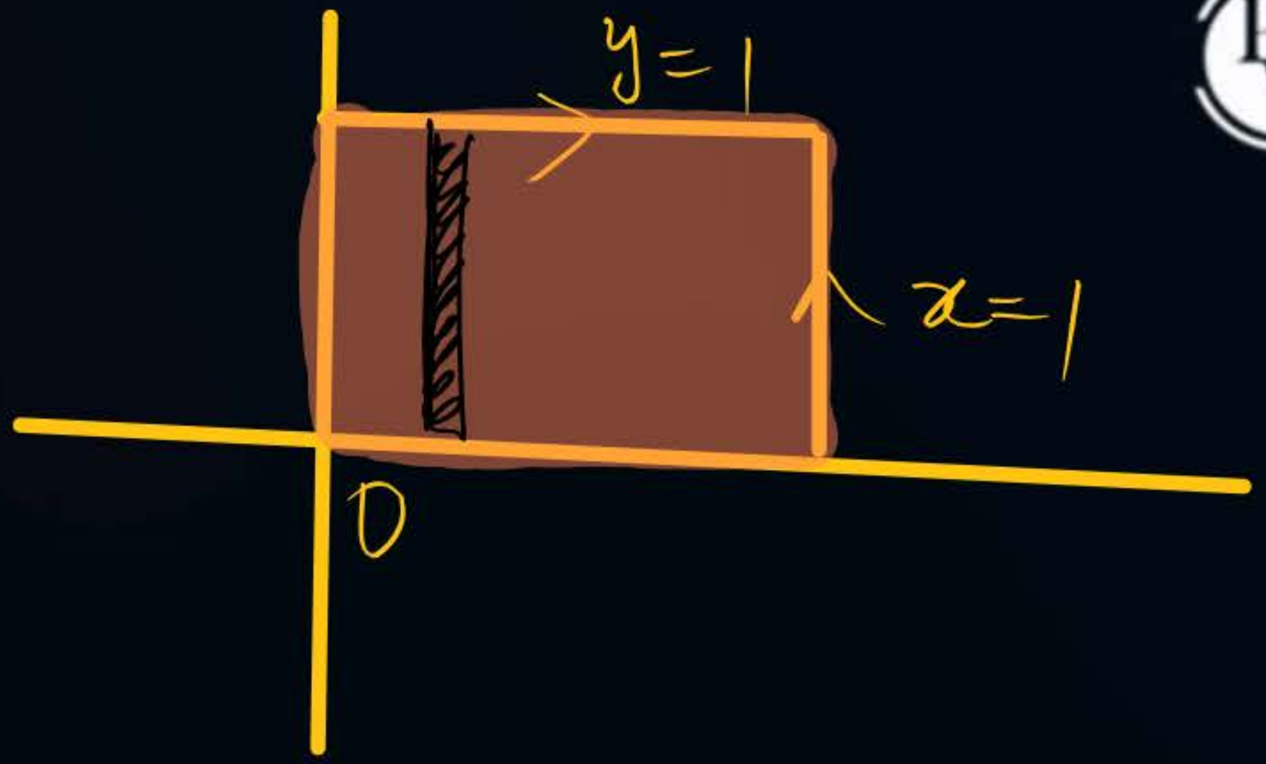
C. 3/2

D. 5/3

$$= \int\int (3y)\, dy\, dx$$

$$= 3\left[\int_0^1 dx \int_0^1 y\, dy\right]$$

$$= \frac{3}{2}$$ ✓



Topic: Vector calculus

$\boxed{\vec{ds} = \hat{n}\, ds}$ $\quad \oint F\, d\vec{r} = \iint (\nabla \times \vec{F})\, \vec{ds}$

#Q. Given $\vec{F} = z\hat{a}_x + x\hat{a}_y + y\hat{a}_z$. If S represents the portion of the $x^2 + y^2 + z^2 = 1$ for $z \geq 0$, then $\int_s (\nabla \times \vec{F}).\overrightarrow{ds}$ is ______.

$= \iint (\nabla \times \vec{F})\, \hat{n}\, ds$

$x^2 + y^2 + z^2 = 1$

Sphere

$\hat{n} = \hat{k}$

$ds = dy\,dx$

$Z = 0$

$Z > 0$

$Z = 0$

$x^2 + y^2 + z^2 = 1$

$x^2 + y^2 = 1$ circle

Rotate

Axis

Sphere

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = (\hat{i} + \hat{j} + \hat{k})$$

Using stokes Theorem

$$\oint \vec{F} \cdot d\vec{r} = \iint (\nabla \times \vec{F}) \cdot d\vec{S}$$

$$= \iint (\hat{i} + \hat{j} + \hat{k}) \cdot \hat{n}\, dS$$

$$= \iint (\hat{i} + \hat{j} + \hat{k}) \cdot \hat{k}\, dy\, dx = \iint dy\, dx = \text{Area of circle}$$

$$= \pi r^2 = \pi (1)^2 = \pi$$

$\oint \vec{F} \cdot d\vec{r} = \int (F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}) \cdot (dx\,\hat{i} + dy\,\hat{j}) = \oint F_1\,dx + F_2\,dy$

#Q. The integral $\underbrace{\int_C (y\,dx - x\,dy)}_{\text{2 dimensional } (x, y)}$ is evaluated along the circle $x^2 + y^2 = \frac{1}{4}$ traversed in counter clockwise direction. The integral is equal to

Green's Theorem

$\oint y\,dx - x\,dy$

A. 0

B. $-\frac{\pi}{4}$

C. $-\frac{\pi}{2}$ ✓

D. $\frac{\pi}{4}$

$\oint M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx = \iint (-1-1)\,dy\,dx = \iint -2\,dy\,dx$

$M = y$

$N = -x$

$$= \iint -2\,dy\,dx$$

$$\iint (\text{constant})\,dy\,dx = \text{AREA}$$

Area

$$\iint f(x,y)\,dy\,dx = \text{volume via double Integrals}$$

$x^2+y^2=\frac{1}{4}$

circle.

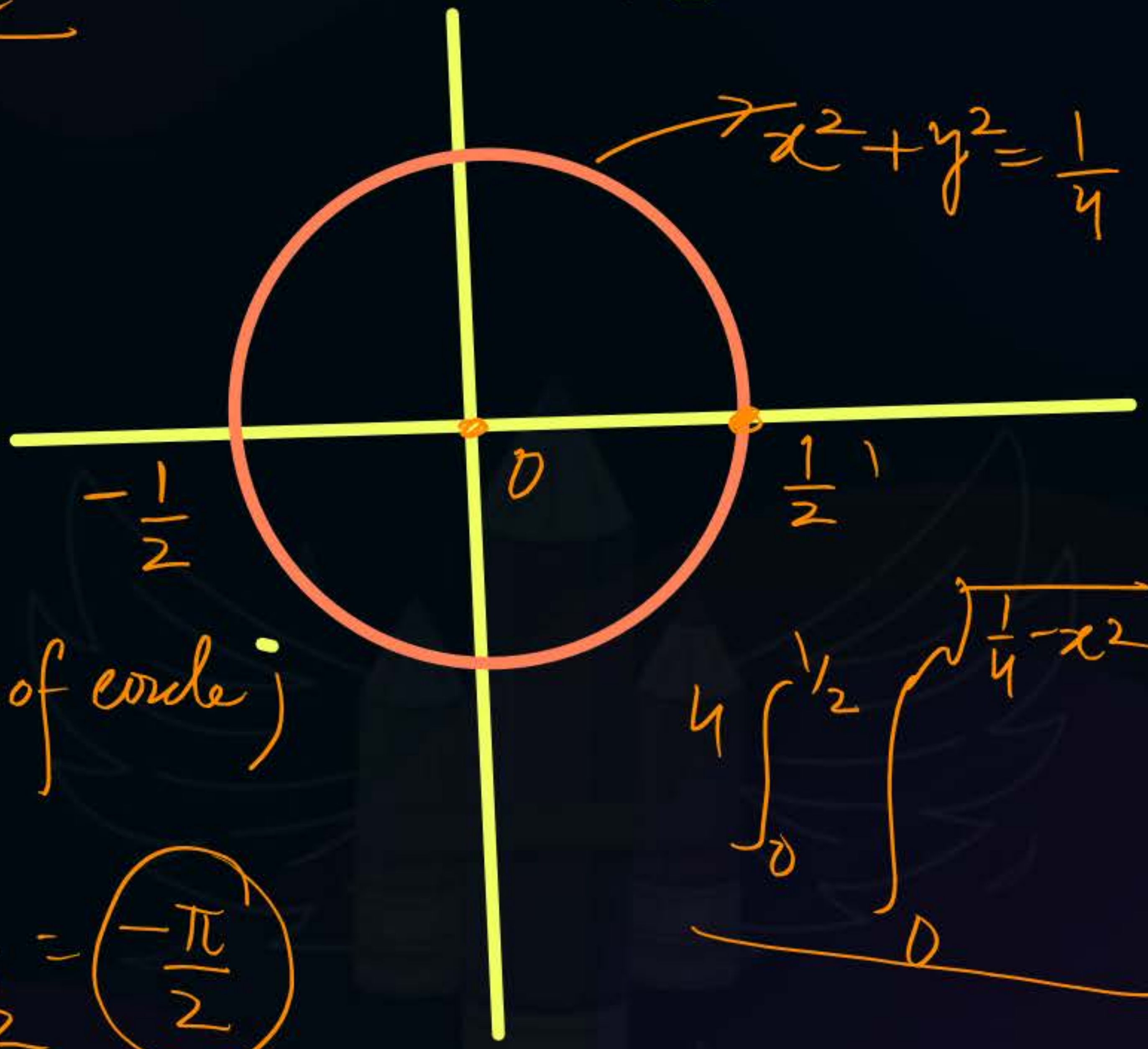
$$= \iint -2\,dy\,dx$$

$$= -2\iint dy\,dx = -2\times(\text{Area of circle})$$

Area of ○

$$= -2\times\pi r^2$$

$$= -\cancel{2}\times\pi\times\frac{1}{\cancel{4}_2} = -\frac{\pi}{2}$$



$$4\int_0^{1/2}\int_0^{\sqrt{\frac{1}{4}-x^2}}$$

Topic: Vector calculus

$\boxed{Ans = \frac{5}{3}}$

$\boxed{ME\ 2010}$

$\boxed{5-7\ min}$

#Q. The value of $\int_C [\overbrace{(3x-8y^2)}^{M}dx + \overbrace{(4y-6xy)}^{N}dy]$, (where C is the

x, y — Green's Theorem

region bounded by x = 0, y = 0 and x + y = 1) is ____.

$M = 3x - 8y^2$

$N = 4y - 6xy$

$\frac{\partial N}{\partial x} = -6y$

$\frac{\partial M}{\partial y} = -16y$

$\oint \vec{F}\,d\vec{r} = \int M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$

$= \iint (-6y + 16y)\,dy\,dx$

$= \iint 10y\,dy\,dx$

$x = 0$

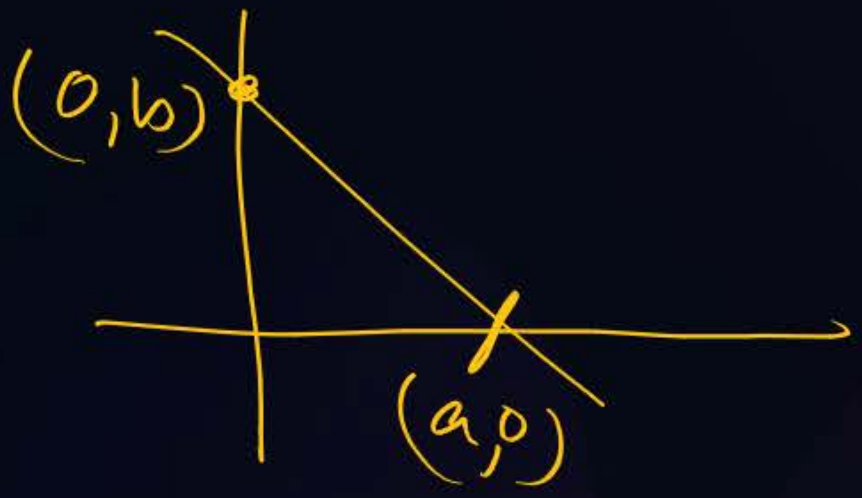
$y = 0$

$x + y = 1$

$$\frac{x}{1} + \frac{y}{1} = 1$$

$(1,0) \qquad (0,1)$

$$\frac{x}{a} + \frac{y}{b} = 1$$



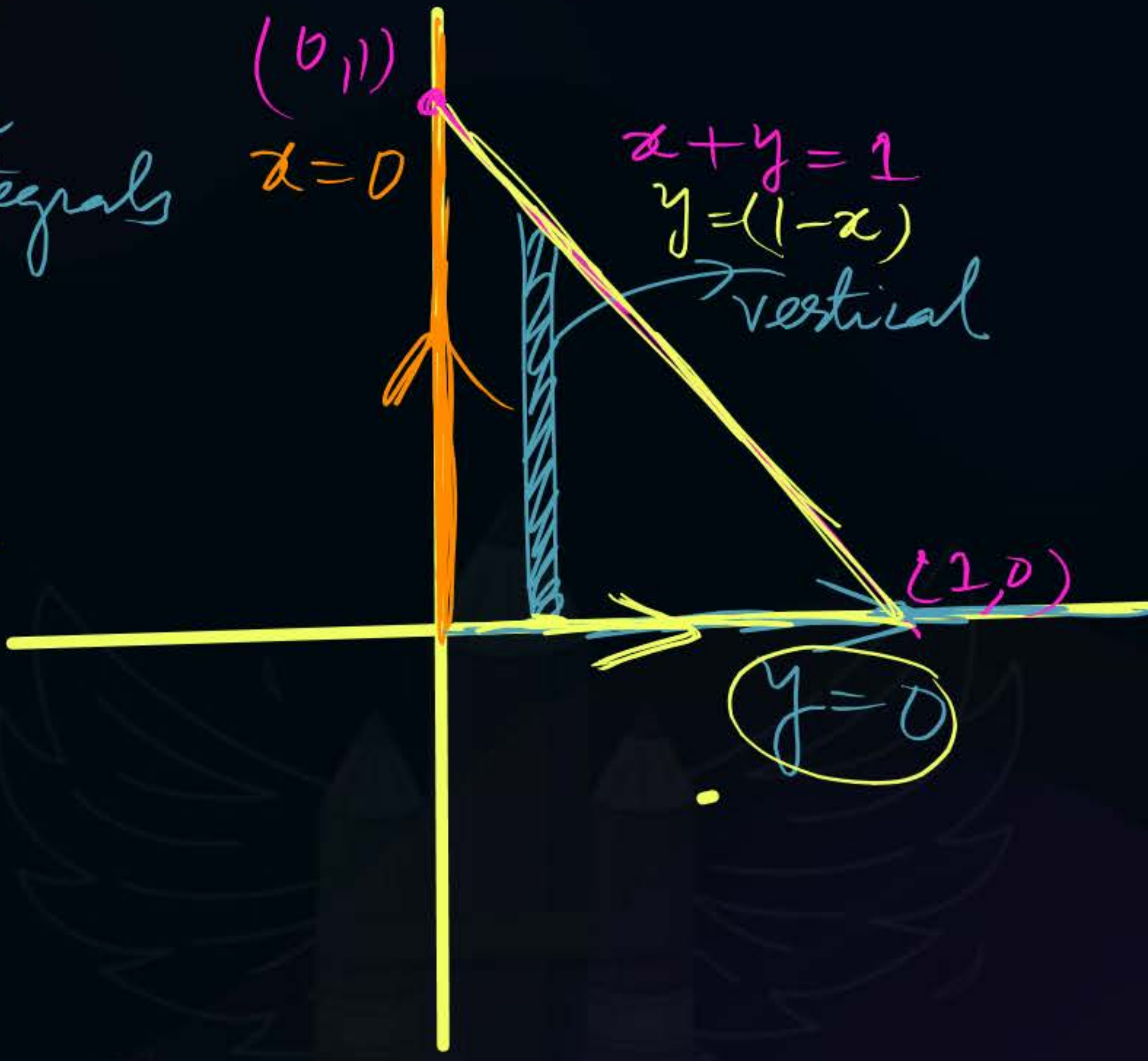
$$\Rightarrow \iint \underbrace{10y\,dy\,dx}_{\text{Volume via double integrals}}$$

$$\Rightarrow 10\int_0^1 dx \int_0^{1-x} y\,dy$$

$$= 10\int_0^1 dx \left[\frac{y^2}{2}\right]_0^{1-x}$$

$$= 10\int_0^1 \frac{(1-x)^2}{2}\,dx$$

$$= \frac{5}{3}$$



Topic: Vector calculus

$\oint \vec{F} \cdot \vec{dr} =$ green's Theorem $\quad r = \frac{4}{\sqrt{\pi}}$

#Q. The value of the line integral $\oint_C \bar{F}.\bar{r}'ds,$ where C is a circle of radius $\frac{4}{\sqrt{\pi}}$ unit is ____.

$= \iint (2-1)\,dy\,dx = \iint dy\,dx = \pi r^2$
$= \pi \times \left(\frac{4}{\sqrt{\pi}}\right)^2 = 16$

Here, $\bar{F}(x,y) = y\hat{\imath} + 2x\hat{\jmath}$ and $\bar{r}$ is the UNIT tangent vector on the curve C at an arc length s from a reference point on the curve, $\hat{\imath}$ and $\hat{\jmath}$ are the basis vectors in the

x-y Cartesian reference. In evaluating the line integral, the curve has to be traversed in the counter-clockwise direction.

Topic: Vector calculus

#Q. Suppose C is the closed curve defined as the circle $x^2 + y^2 = 1$ with C oriented anticlockwise. The value of $\oint \underbrace{(xy^2dx + x^2ydy)}_{Mdx+Ndy}$ over the curve C equals _____.

$M = xy^2$

$N = x^2y$

$= 0$

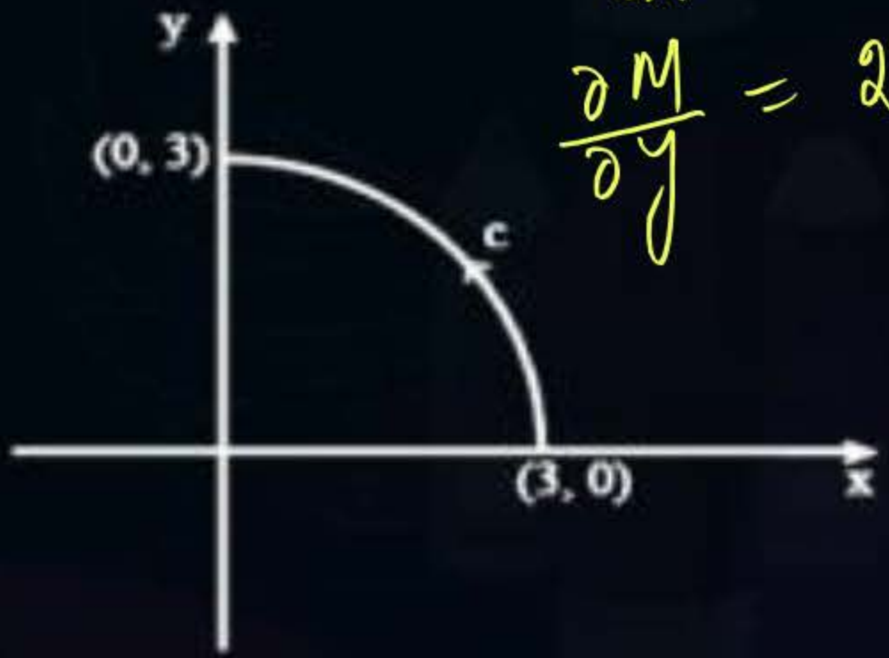
$$\oint Mdx + Ndy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx = \iint (2xy - 2xy)\, dy\,dx = 0$$

Topic: Vector calculus

$$= \iint (2y+2x) - (2y+2x)\, dy\, dx$$
$$= 0 \quad \underline{\text{Ans}}$$

#Q. As shown in the figure, C is the arc from the point $\boxed{(3, 0)}$ to the point (0, 3) on the circle $x^2 + y^2 = 9$. The value of the integral $\int_C (y^2 + 2yx)dx + (2xy + x^2)dy$ is ____ (up to 2 decimal places).

$$\frac{\partial N}{\partial x} = 2y + 2x$$
$$\frac{\partial M}{\partial y} = 2y + 2x$$



$$\underbrace{\oint (y^2+2xy)\,dx + (2xy+x^2)\,dy}_{2d}$$

$M = y^2 + 2xy \quad N = 2xy + x^2$

Topic: Vector calculus

$M = -y$
$N = x$

$= \iint 1-(-1)\, dy\, dx$

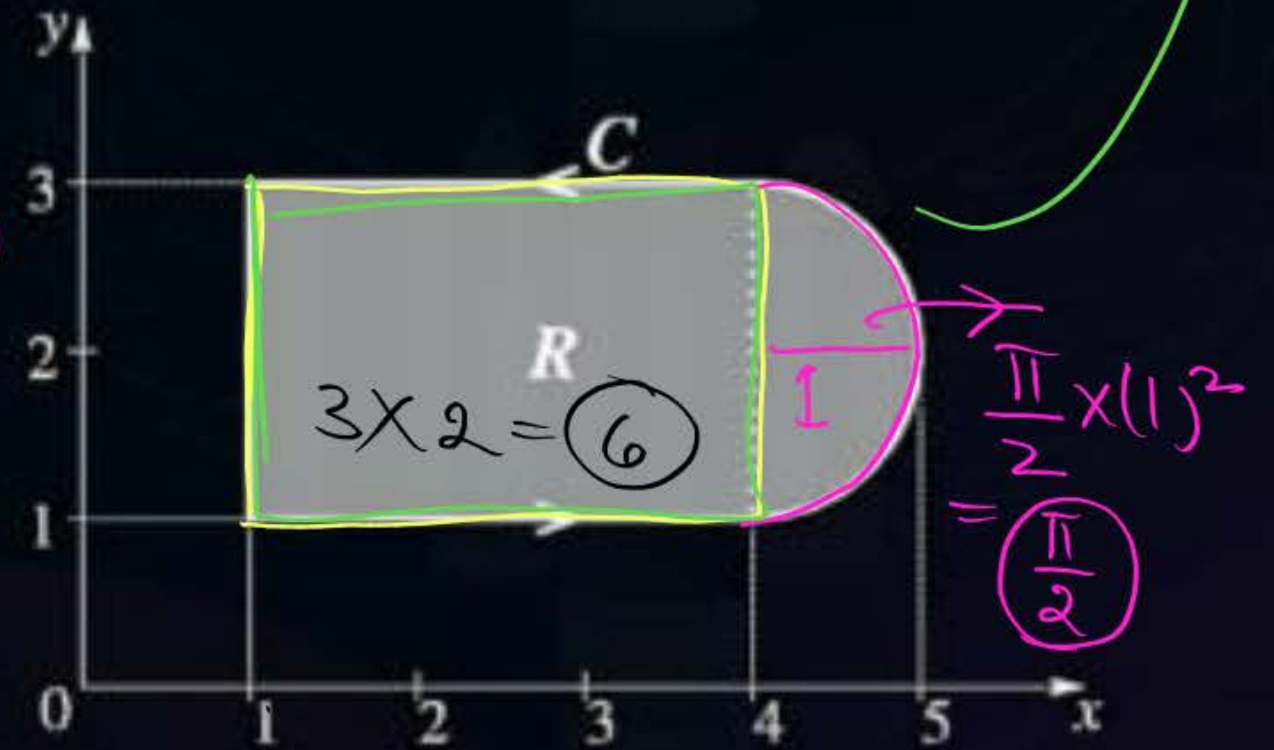
$= \iint +2\, dy\, dx$

$= +2 \iint dy\, dx$

#Q. Consider the line integral $\int_C (xdy - ydx)$the integral being taken in a counterclockwise direction over the closed curve C that forms the boundary of the region R shown in the figure below. The region R is the area enclosed by the union of a 2 × 3 rectangle and a semi-circle of radius 1. The line integral evaluates to

$= +2[\text{Area}] = 2[\square + D]$

$= 2\left[6 + \frac{\pi}{2}\right] = 12 + \pi$



A. $6 + \pi/2$

B. $8 + \pi$

C. $12 + \pi$

D. $16 + 2\pi$

2 mins Summary

Topic One ✓ curl

Topic Two ✓ stokes

Topic Three ✓ green

Topic Four ✓ questions

Topic Five

$x^2+y^2+z^2=1$

$z>0$
$z=0$

$\to$ xy Plane

xy Plane

$z>0$

$\hookrightarrow$ Normal

$\hat{k}$

$= \dfrac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$

surface Integral

x, y, z

THANK - YOU