GATE (ALL BRANCHES)

Engineering Mathematics

Complex Analysis

Lecture No. 01

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TOPICS TO BE COVERED

01 Complex Arithmetic ✓ complex arithmetic

12th fundamentals

02 Problems based on Complex Arithmetic

Complex Analysis → functions of complex

- C-R equations
- Harmonic conjugates
- Complex Integral
- Residue.

Complex Number:

$x^2 - 10x + 40 = 0$ quadratic equation

$$x = \text{Root} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{+10 \pm \sqrt{(10)^2 - 4 \times 1 \times 40}}{2 \times 1}$$

$$x = \frac{+10 \pm \sqrt{100 - 160}}{2}$$

$$x = \frac{10 \pm \sqrt{-60}}{2}$$

$$x = 5 \pm \sqrt{-15}$$

Roots

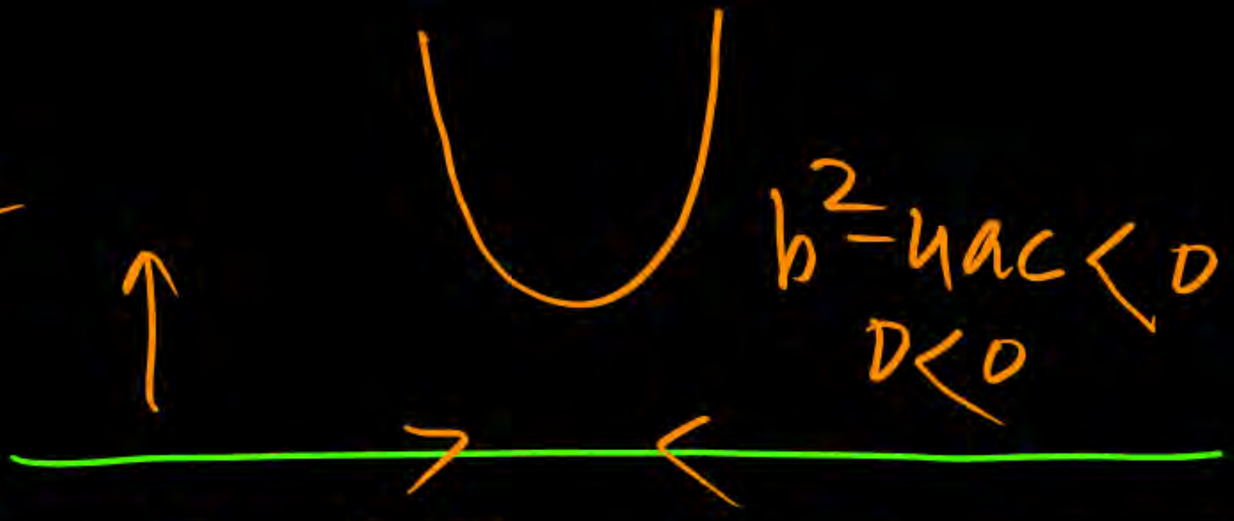
$$\begin{cases} x_1 = \dfrac{5 + \sqrt{-15}}{2} \\ x_2 = \dfrac{5 - \sqrt{-15}}{2} \end{cases}$$

location

Picture

Photo

Roots Are complex
$D < 0$



$\sqrt{+\text{Negative No}}$
= Does Not exists

New co-ordinate

$x_1 = 5 + \sqrt{-15}$ $\quad x_2 = 5 - \sqrt{-15}$

This Roots cheat with 2 Dimension

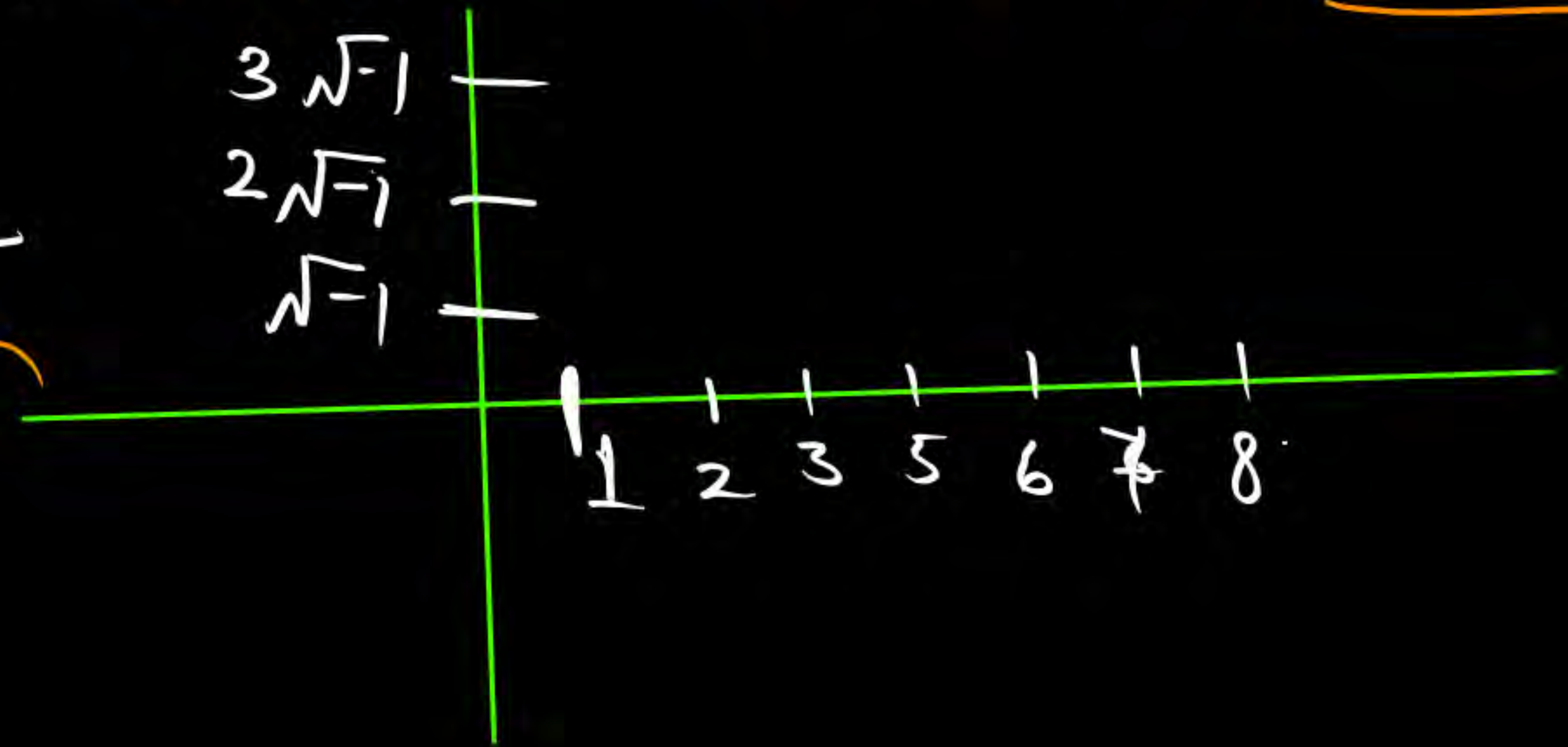
$x = 5 \pm \sqrt{-15}$

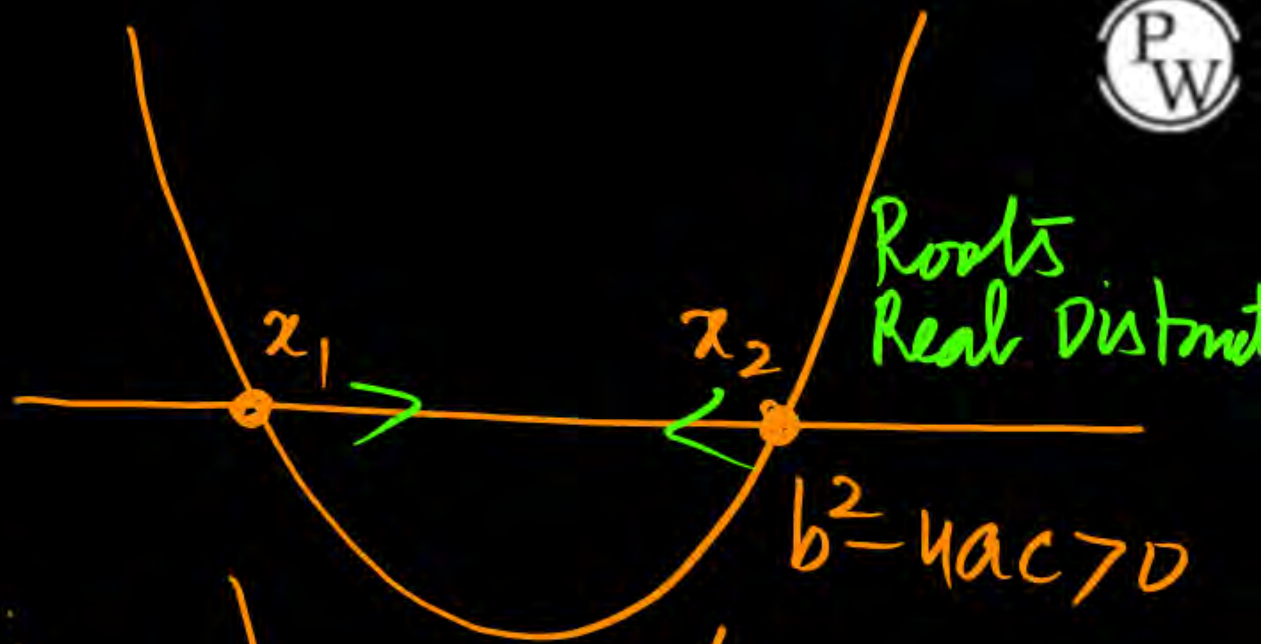
Ist part

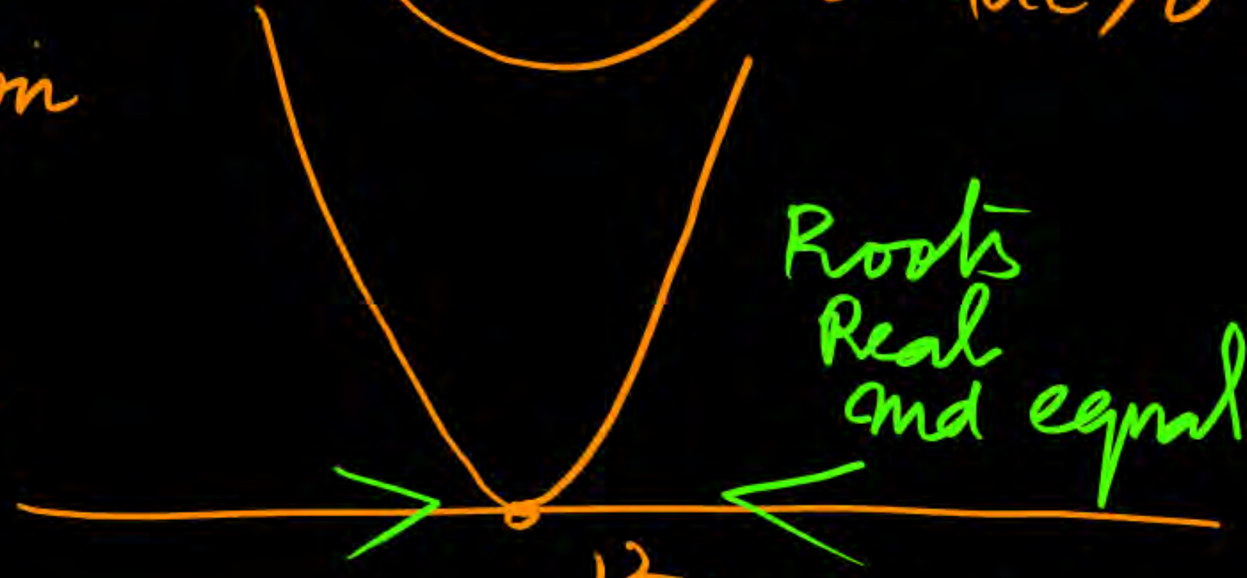
SECOND PART

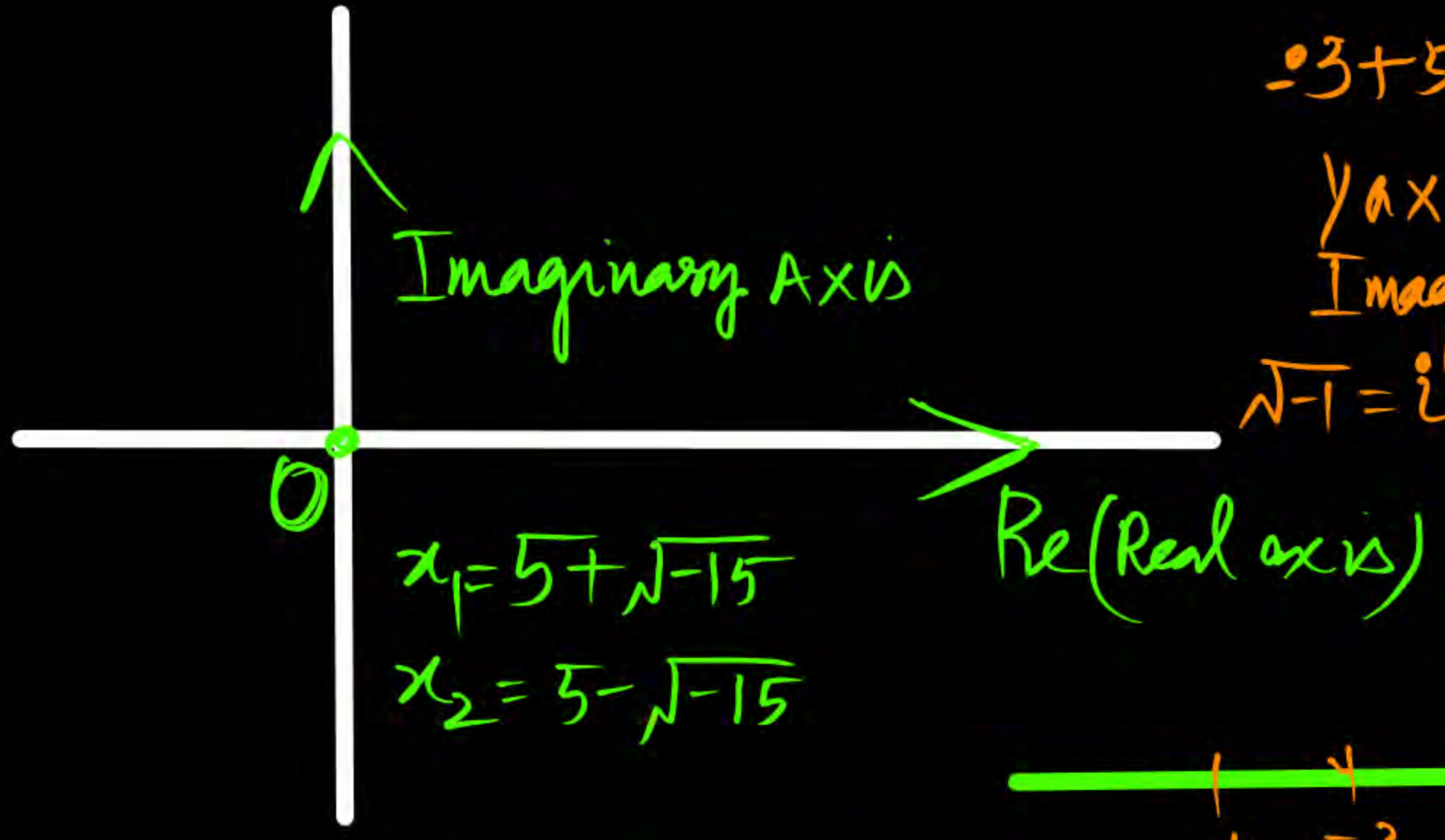
Imaginary

real





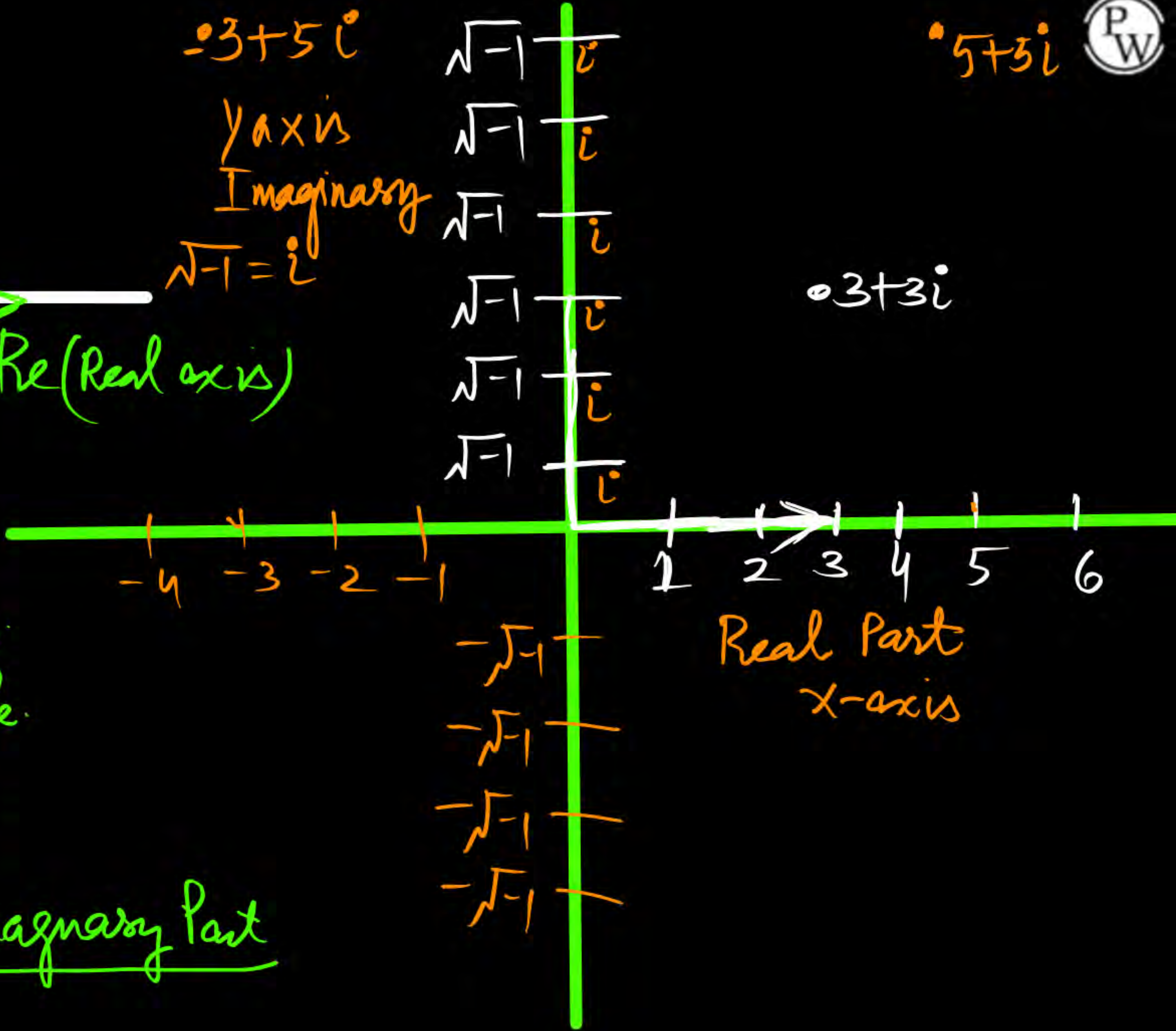




Z = complex No.
One variable.

$Z = X + iy$

= Real Part + i Imaginary Part



complex No $= z = x+iy$

$z = x+iy$

$\bar{z} = x-iy$ → conjugate of a complex No

i

$\mathrm{Im}(z)$

$-x+iy$ $\qquad$ $z = x+iy$

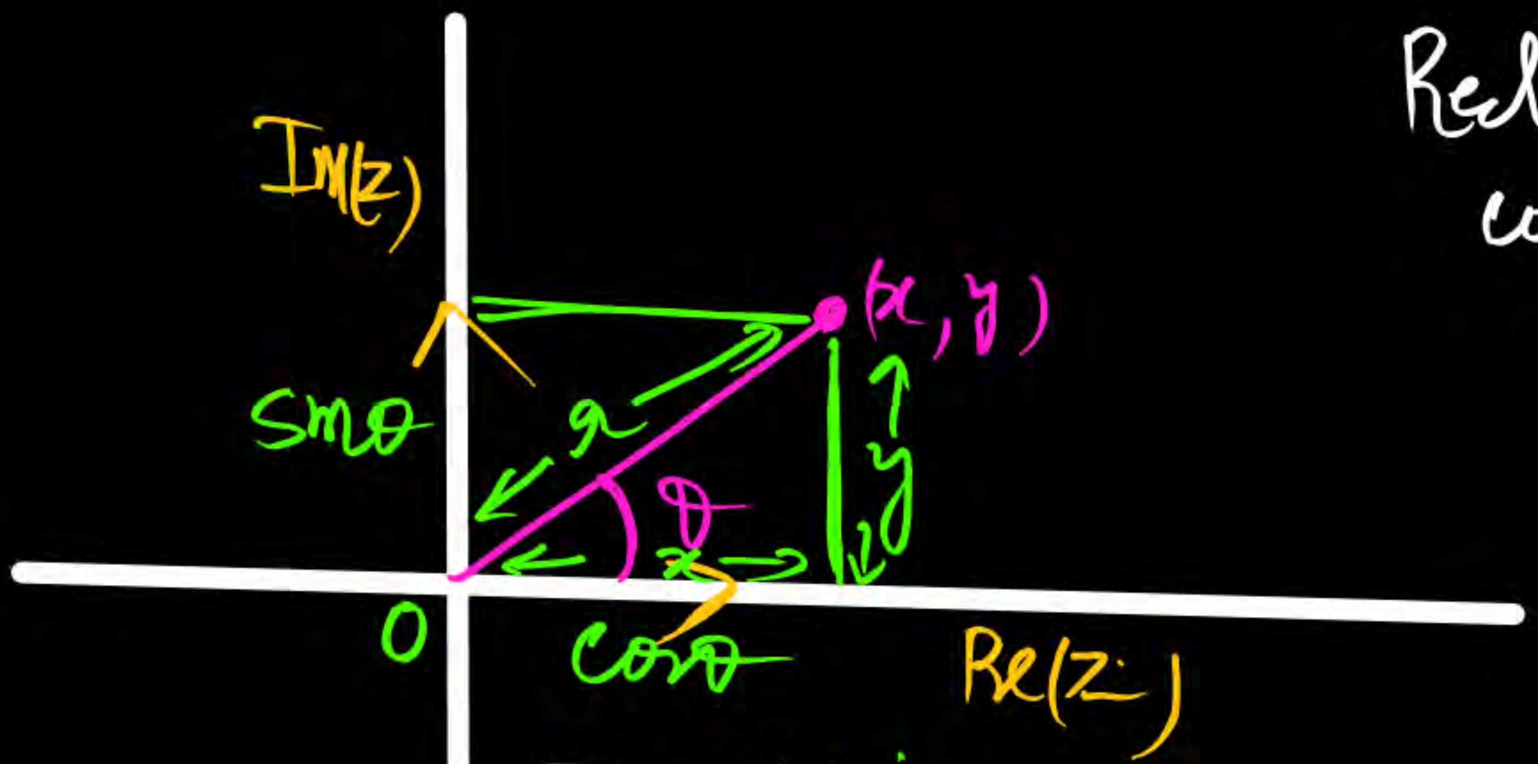
$\mathrm{Re}(z)$

$-x-iy$ $\qquad$ $z = x-iy$

$5+3i$

z^c or z^* or $\bar{z}$ $5-3i$

$z = x+iy$ $\quad$ $\bar{z}$ = complex conjugate $= \underline{x-iy}$



$z = x + iy$

Polar form =

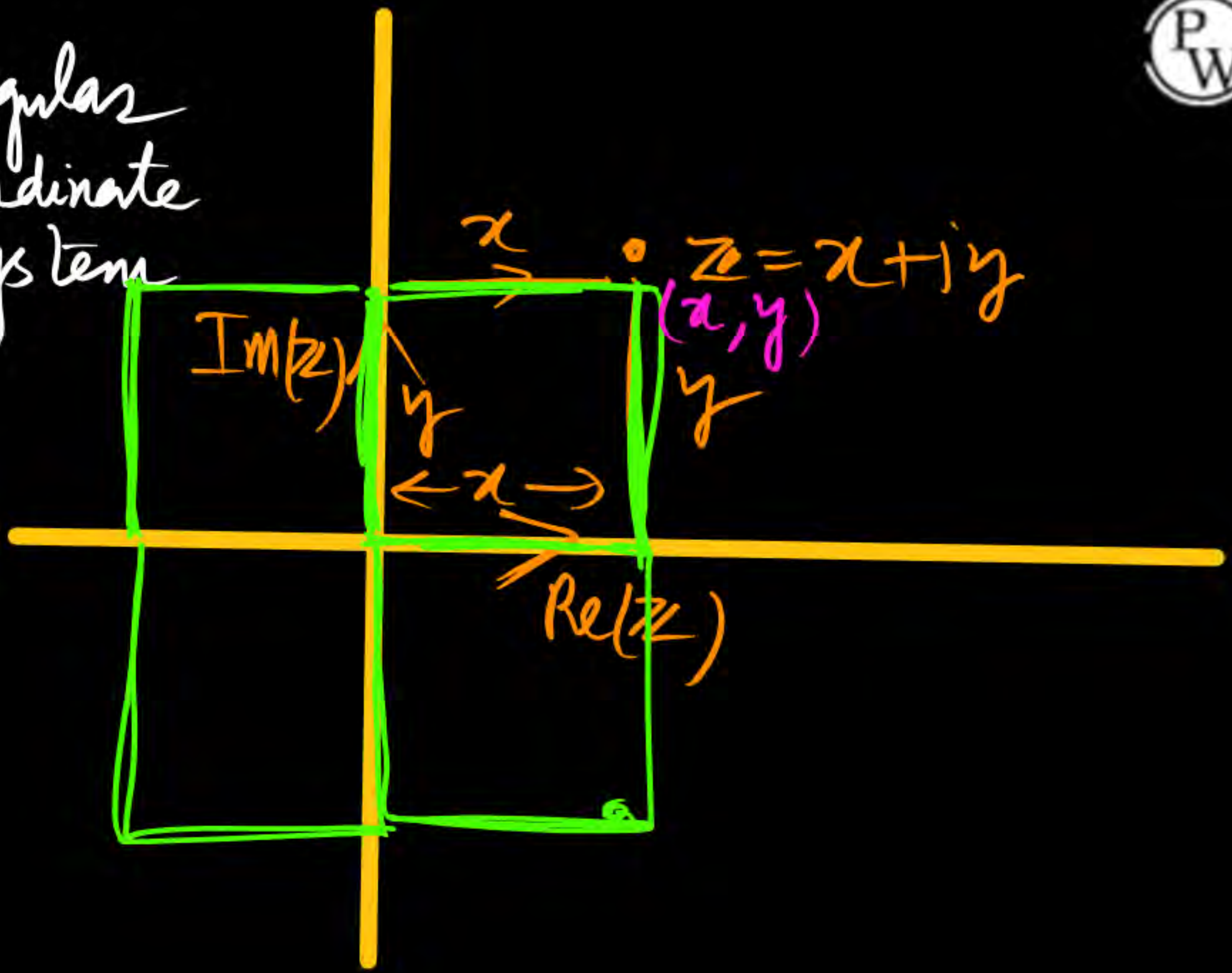
$x = r\cos\theta$

$y = r\sin\theta$

$$\boxed{z = r[\cos\theta + i\sin\theta]}$$

$r = \sqrt{x^2 + y^2}$ $\quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Rectangular co-ordinate system



r = modulus
= Distance Between origin to Point

θ = arguement or Angle or amplitude $y = r\sin\theta$

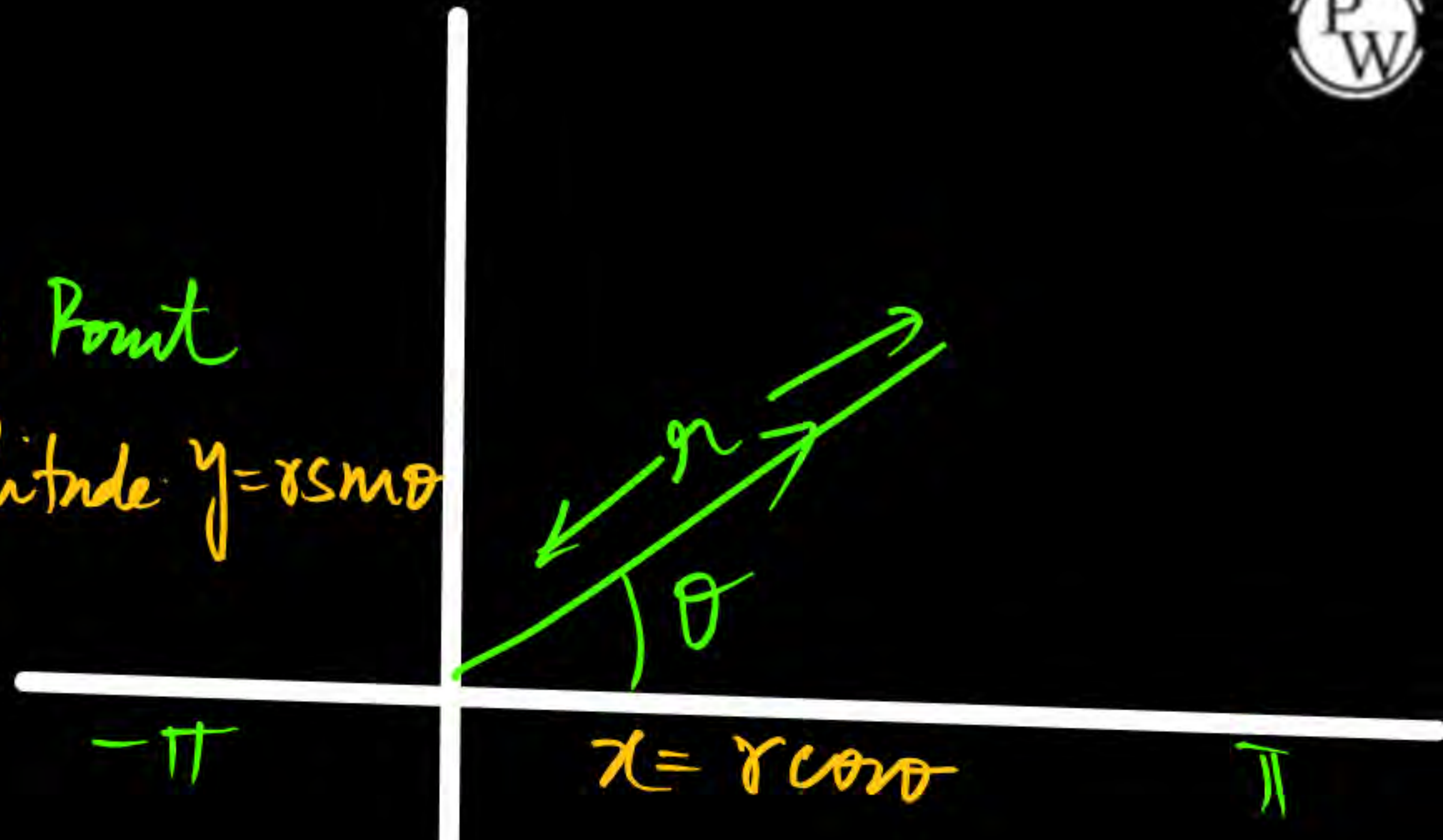
$$Z = x + iy$$

Polar form $x = r\cos\theta$ — ①
$y = r\sin\theta$ — ②

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Polar form $\longrightarrow r[\cos\theta + i\sin\theta]$



$$\boxed{-\pi \le \arg(z) \le \pi}$$

θ = amplitude or arguement
r = modulas

$$z = x + iy \xrightarrow[\text{Polar form}]{} r[\cos\theta + i\sin\theta]$$

$$-\pi \le \arg(z) \le \pi$$

$z_1 = 1 + i$

$= x + iy$

$x = 1$

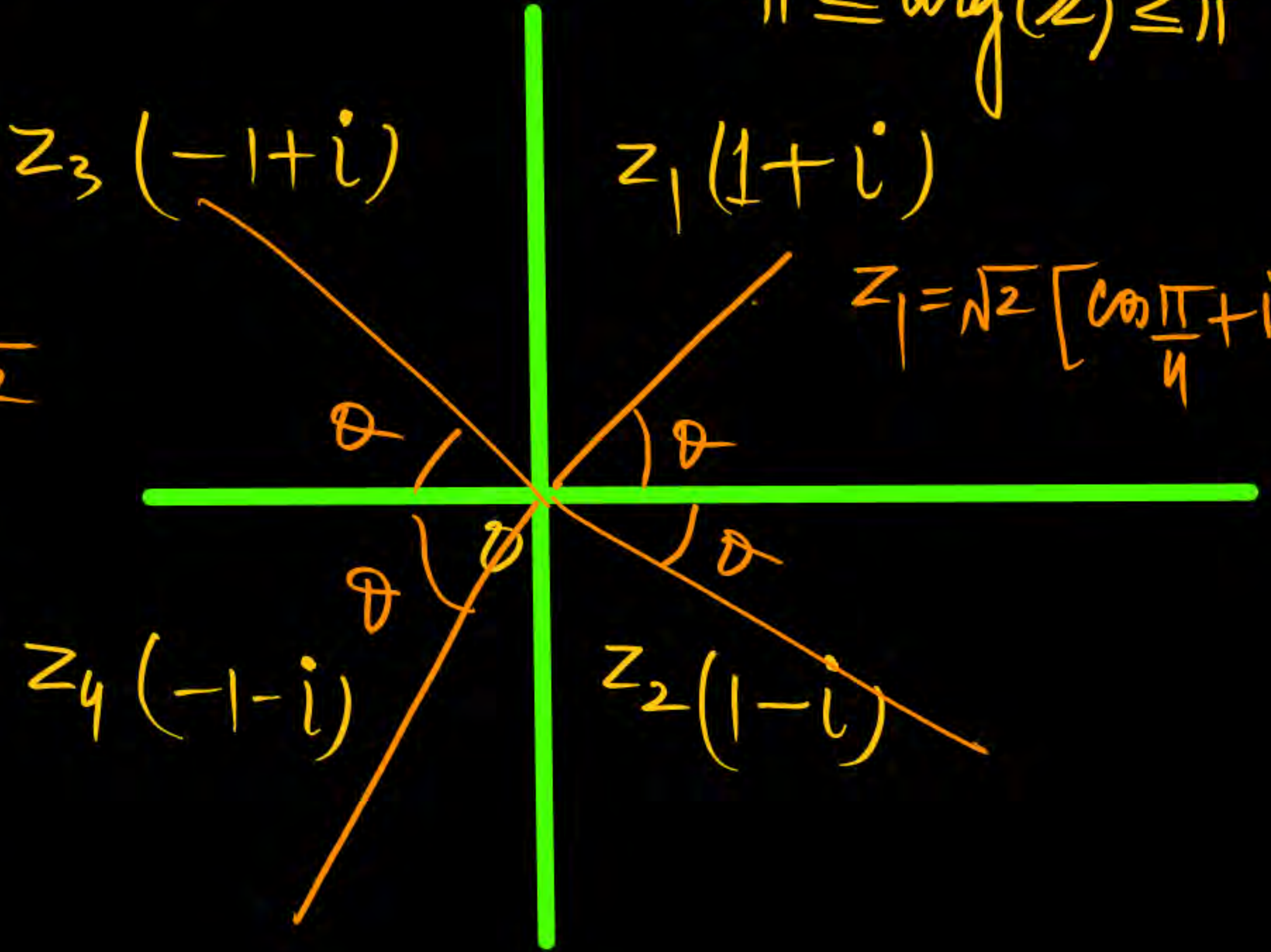
$y = 1$ $\quad r = \text{modulus}$

$= \sqrt{(1)^2 + (1)^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

$\theta = \frac{\pi}{4} \quad r = \sqrt{2}$

$$\boxed{\text{Polar form} = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]}$$



$z_1 = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$

$$z_2 = 1 - i$$

$$r = \sqrt{(1)^2 + (1)^2} = \sqrt{2}$$

$$x = 1 \quad y = -1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(-\frac{1}{1}\right)$$

$$= \tan^{-1}(-1)$$

$$\theta = -\frac{\pi}{4}$$

$$r = \sqrt{2}$$

$$\theta = -\frac{\pi}{4}$$

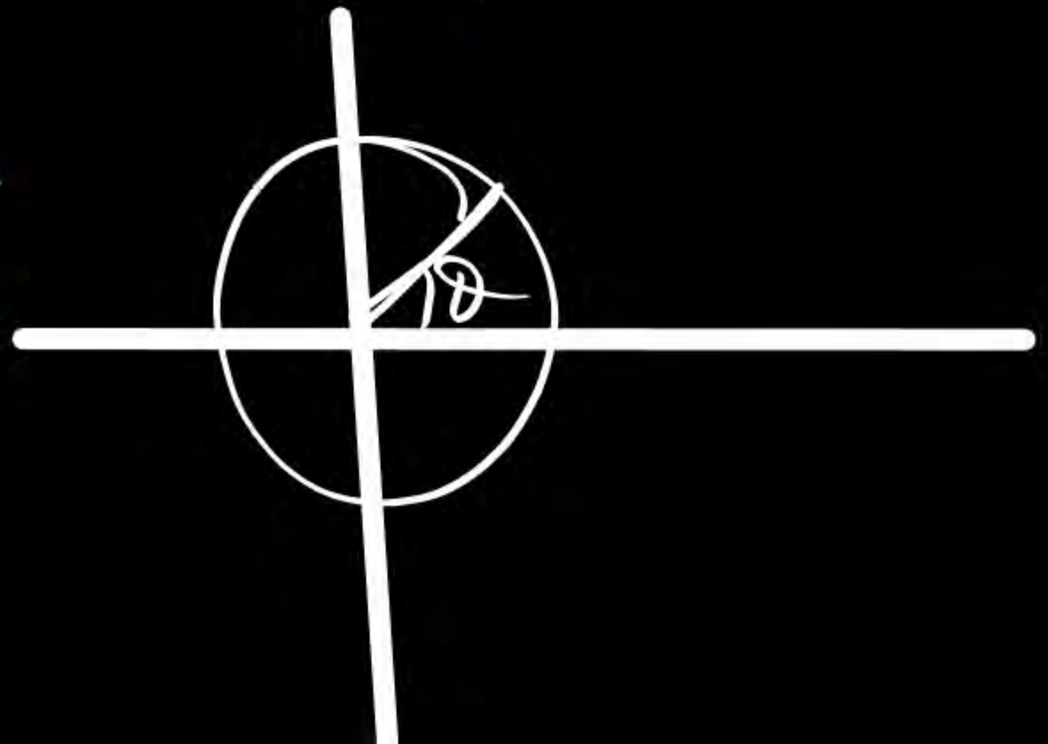
Polar form:

$$= \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]$$

$$= \sqrt{2}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]$$

Euler's form: $z = \cos\theta + i\sin\theta = e^{i\theta}$

$$\frac{1}{z} = \cos\theta - i\sin\theta = e^{-i\theta}$$



$$z = e^{i\theta}$$

Euler form $z = e^{i\theta} = [\cos\theta + i\sin\theta]$

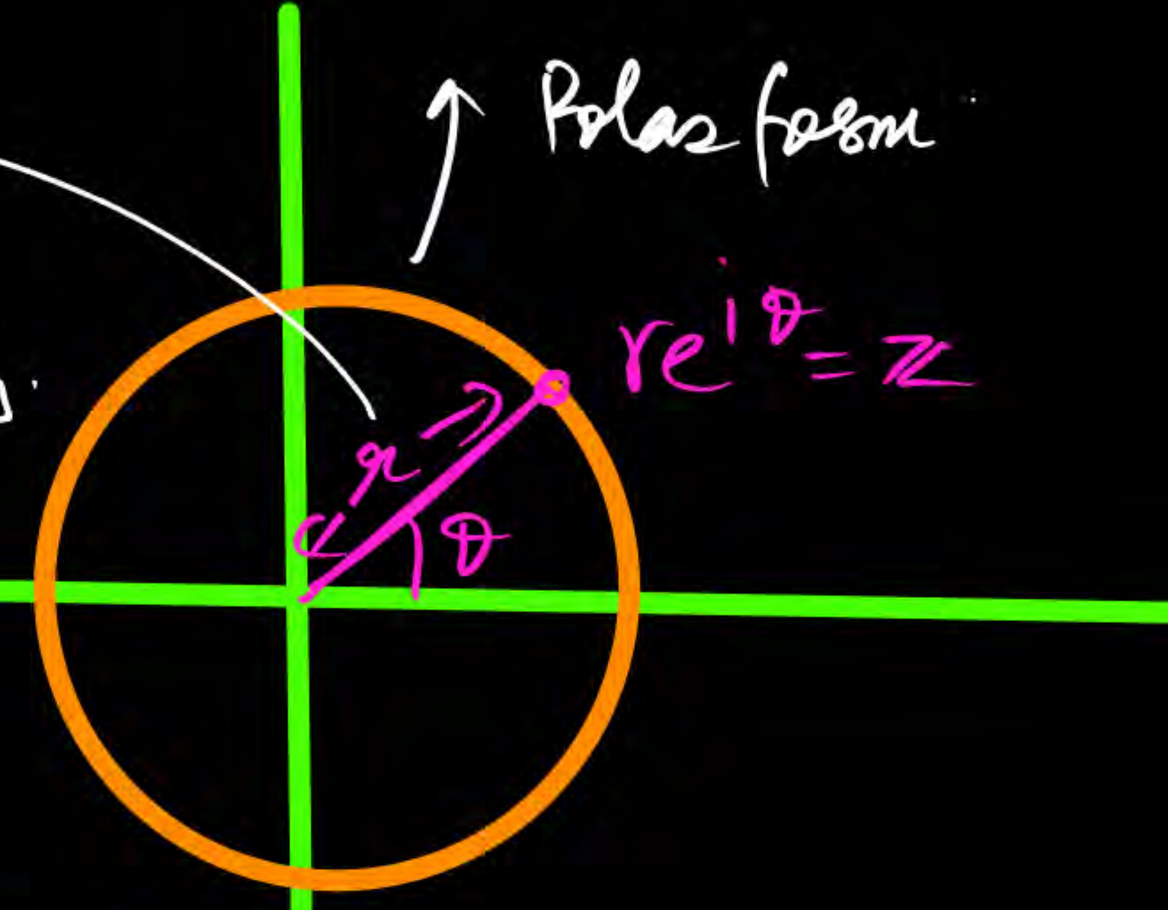
Polar form $z = r[\cos\theta + i\sin\theta] = re^{i\theta}$

$r \to |z|$

$$\boxed{z = |z|e^{i\theta}}$$

→ Polar form

|z| modulus of z



CASE (A) If $z_1 = r_1 e^{i\theta_1}$

$z_2 = r_2 e^{i\theta_2}$

$r_1 = r_2 = 1$ unit circle

$z_1 = e^{i\theta_1}$ $z_2 = e^{i\theta_2}$

$$\boxed{\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2}$$

$$\boxed{\arg(z_1 z_2) = \arg z_1 + \arg z_2}$$

Powers of complex Numbers:

$z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n$

$$\boxed{z^n = \cos n\theta + i\sin n\theta}$$

Demovier Law

$z = \cos\theta - i\sin\theta$

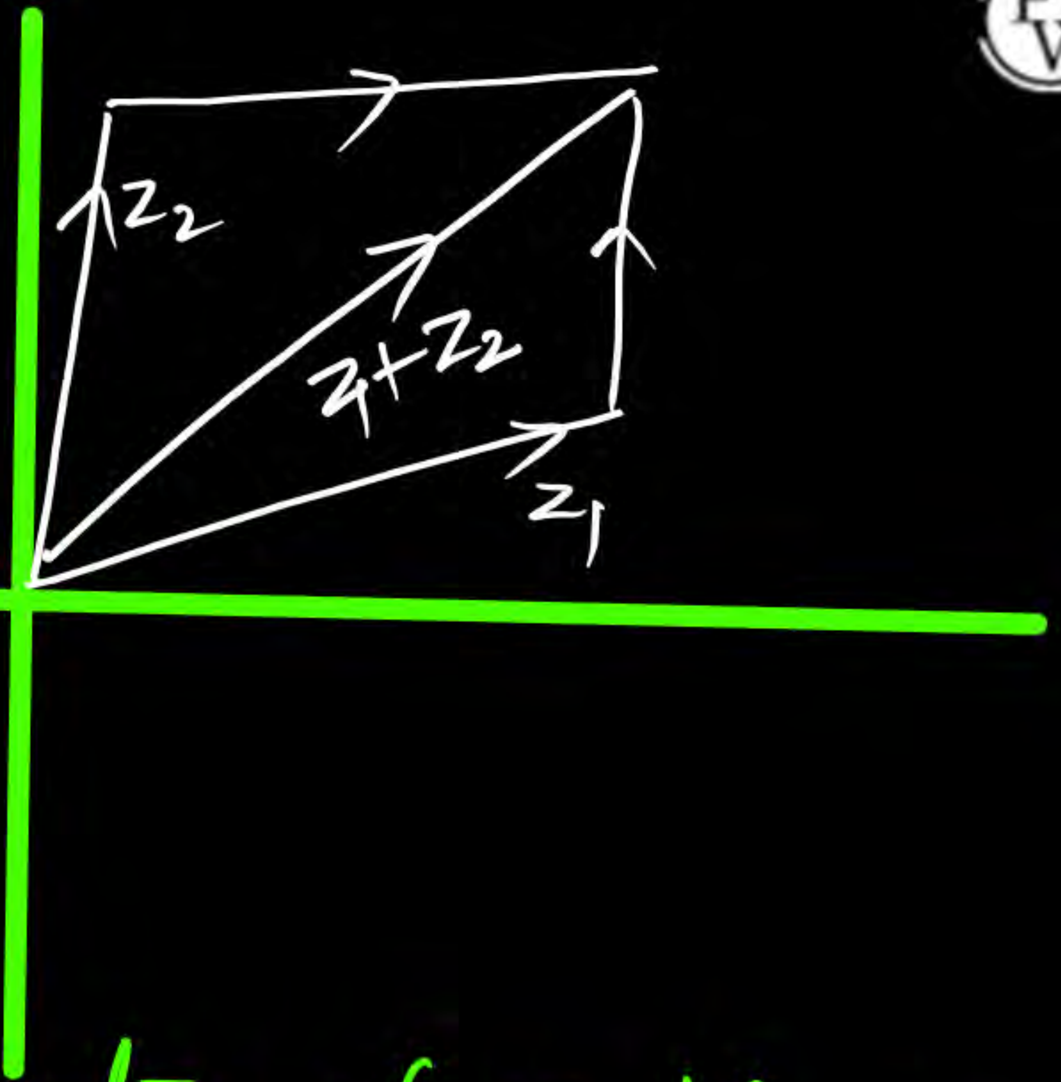
$$\boxed{z^n = (\cos n\theta - i\sin n\theta)}$$

Powers of complex No

Addition of Complex No

$z_1 + z_2$

or $z_1 - z_2$



multiplication of complex No

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$i = \sqrt{-1}$ $\quad i^3 = -i$

$i^2 = -1$ $\quad i^4 = 1$

Cube Root of Unity :

$$x^3 = 1$$

$$x = (1)^{\frac{1}{3}} = \text{Polar co-ordinate}$$

$$1 = 1 + i \cdot 0 = x + iy$$

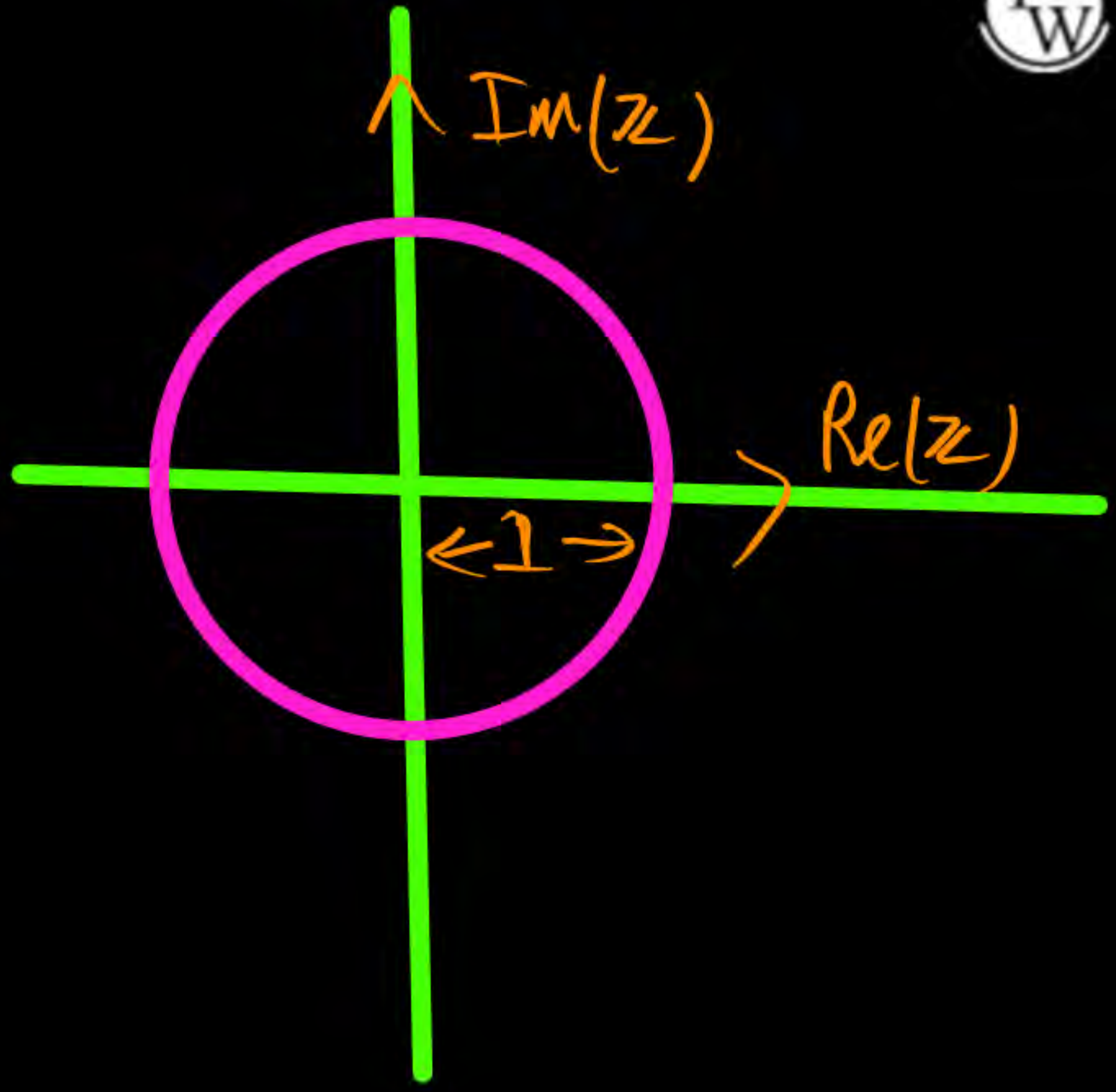
$$\text{mod} = r = \sqrt{x^2 + y^2}$$

$$r = \sqrt{1+0} = \sqrt{1} = 1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{0}{1}\right) = 0$$

$$\boxed{\theta = 0^\circ}$$

$$\text{Polar form} = r[\cos\theta + i\sin\theta] = 1[\cos 0 + i\sin 0]$$



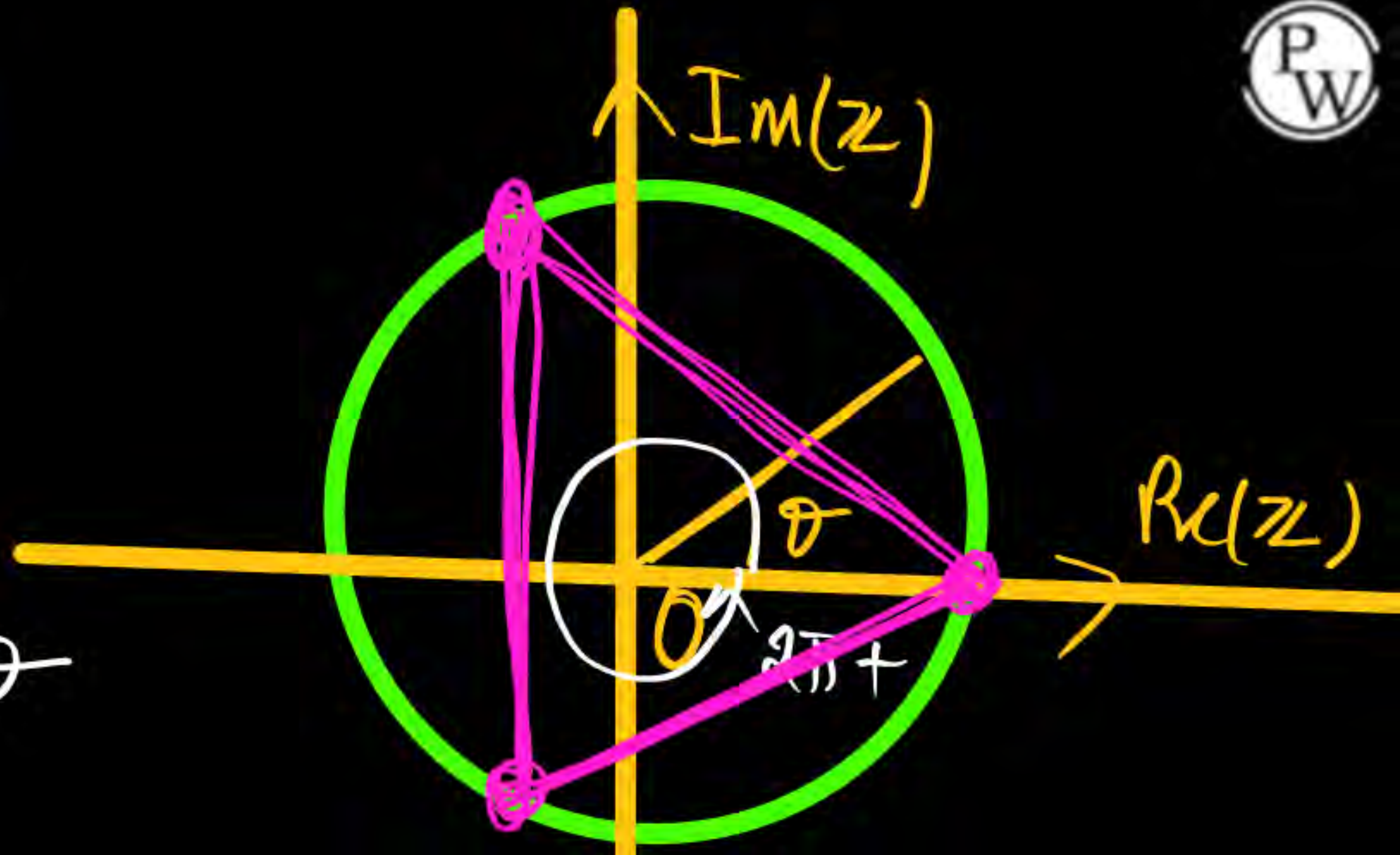
$$x = [\cos 0 + i\sin 0]^{\frac{1}{3}}$$

$$x_n = [\cos 2n\pi + i\sin 2n\pi]^{\frac{1}{3}}$$

Using Demovier law

$$[\cos\theta + i\sin\theta]^n = \cos n\theta + i\sin n\theta$$

$$x_n = \left[\cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}\right]$$

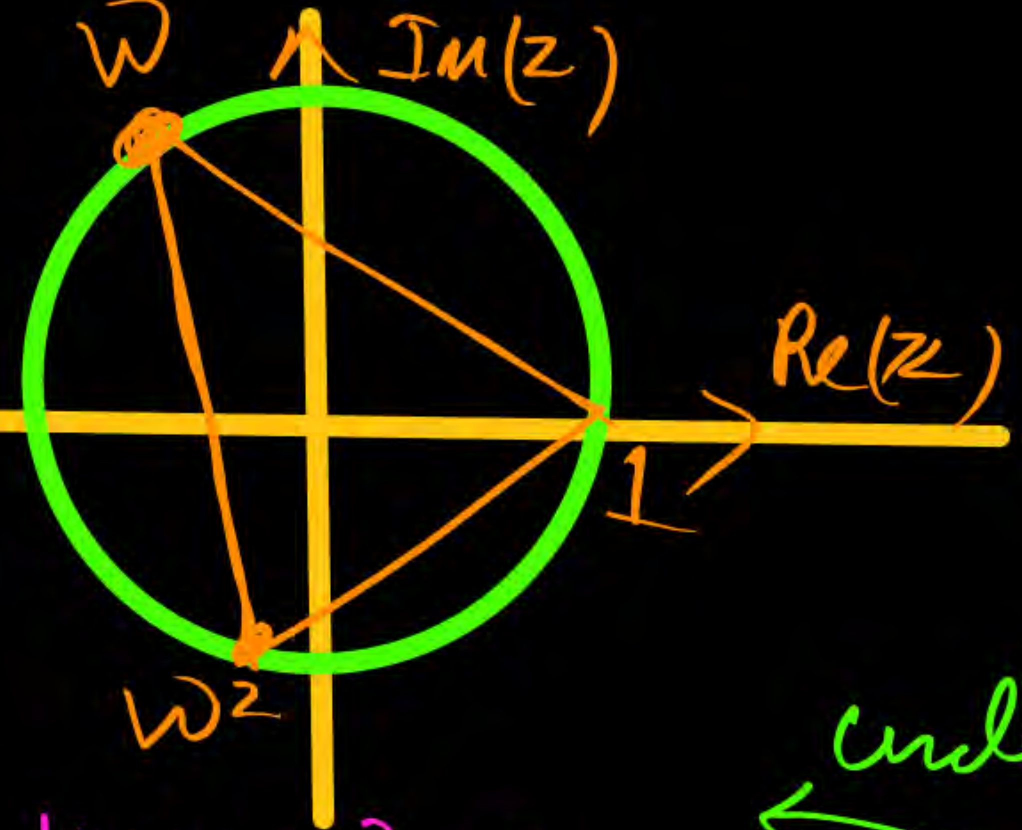


$n=0,1,3$

$$\begin{cases} x_1 = \cos 0 + i\sin 0 = 1 & = 1 \\ x_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} & = \omega \\ x_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} & = \omega^2 \end{cases}$$

$$\boxed{\begin{array}{c} 1+\omega+\omega^2 = 0 \\ \omega^3 = 1 \end{array}}$$

$x^3 = 1$



$0 \quad 1$

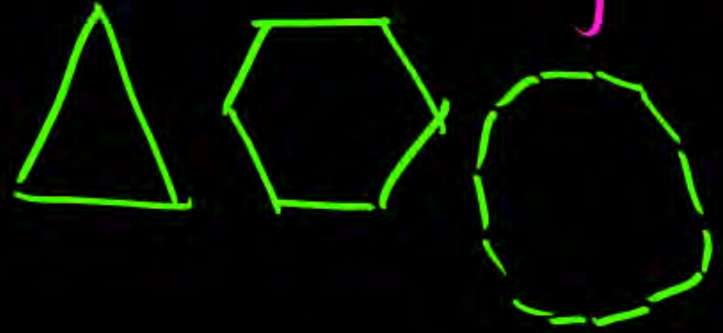
$\frac{2\pi}{3} \quad \omega$

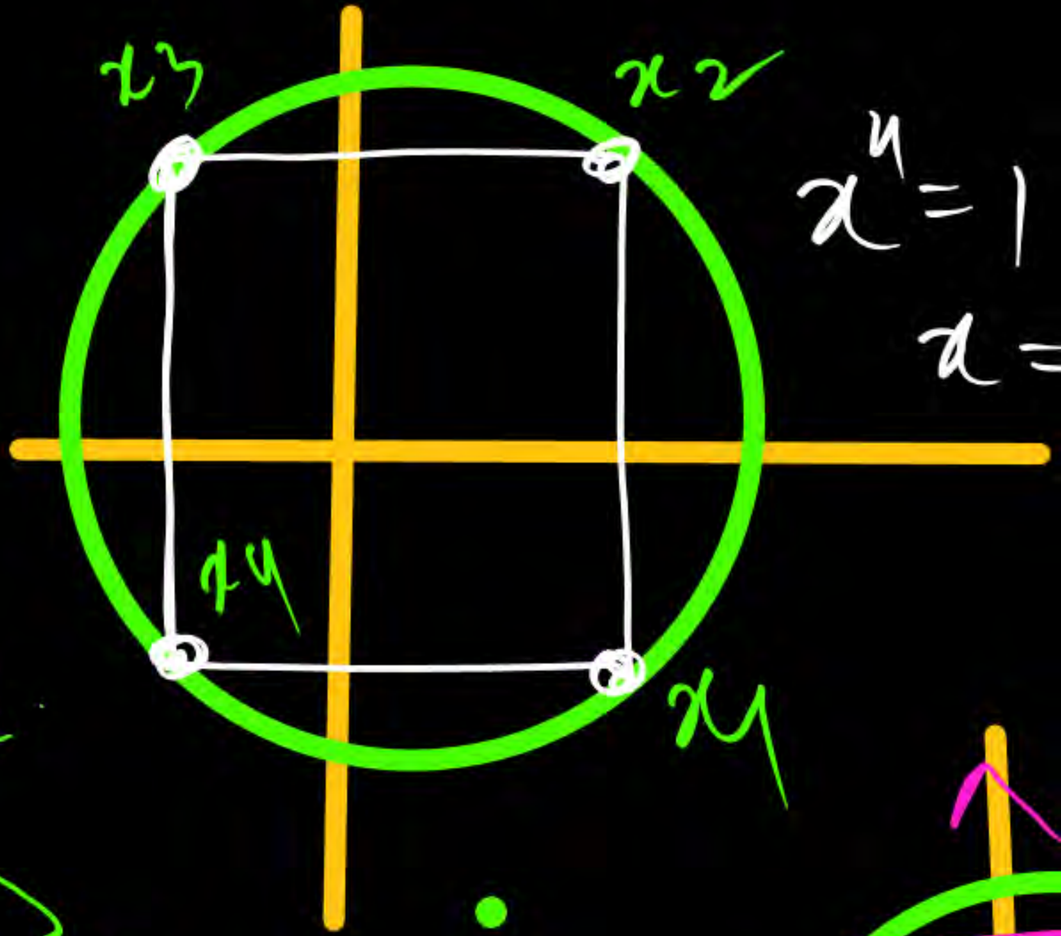
$\frac{4\pi}{3} \quad \omega^2$

$1+\omega+\omega^2=0$

$\omega^3 = 1$

Cube Root of Unity





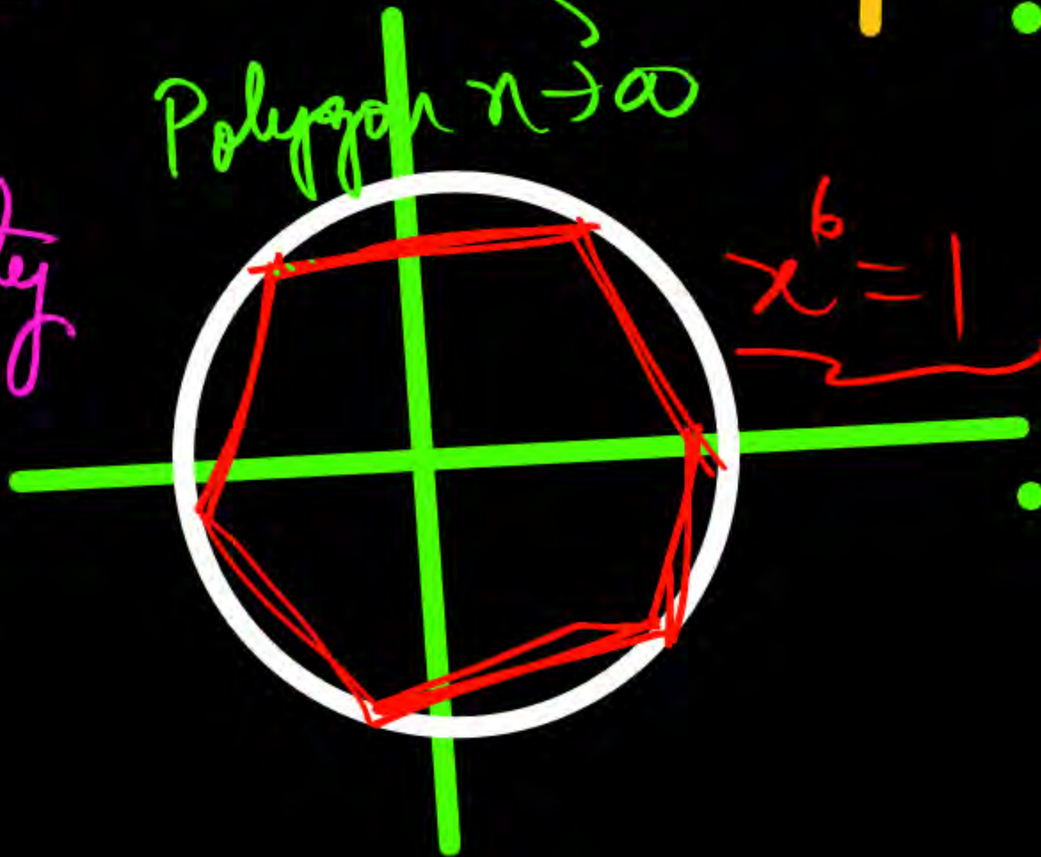
$x^4 = 1$

$x = (1)^{\frac{1}{4}}$

$x_n = 0, 1, 2, 3$

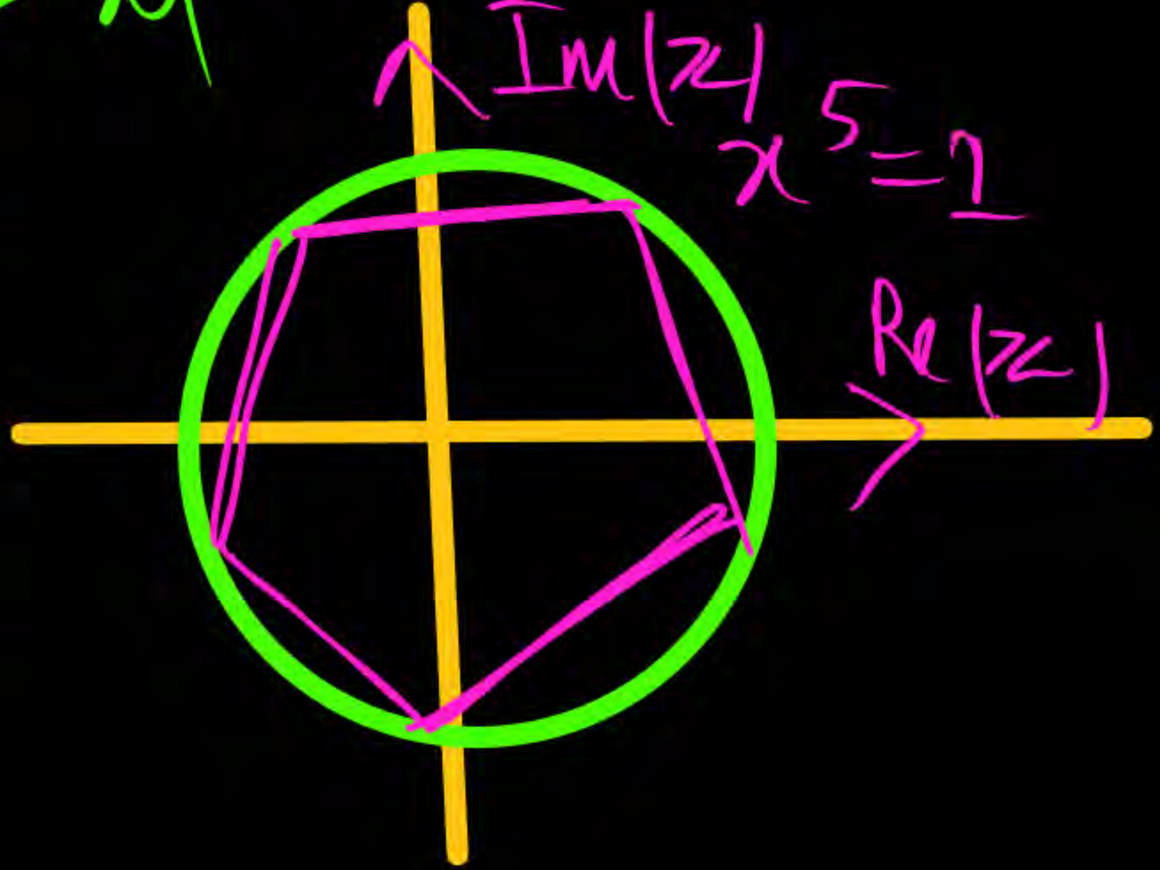
circle

Polygon $n \to \infty$



$x^6 = 1$

$x^5 = 1$



Q. Questions

#Q. The real part of the complex number $z = x + iy$ is given by

(A) $\text{Re}(z) = z - z^*$

(B) $\text{Re}(z) = \frac{z-z^*}{2}$

(C) $\text{Re}(z) = \frac{z+z^*}{2}$

(D) $\text{Re}(z) = z + z^*$

$\text{Re}(z) = \frac{z+z^*}{2}$

$= \frac{x+\cancel{iy}+x-\cancel{iy}}{2}$

$= \frac{\cancel{2}x}{\cancel{2}} = x$

$z^* = x - iy$ Conjugate of a complex No

$z = x + iy$

Real part

Imaginary Part

1) $\text{Re}(z) = x = x+iy-(x-iy)$

$= x+iy-x+iy$

$= 2iy$

2) $\frac{\cancel{x}+iy-\cancel{x}+iy}{2} = iy$

c) $\frac{x+\cancel{iy}+x-\cancel{iy}}{2} = x$

Q. Questions

#Q. $\cos\phi$ can be represented as

(A) $\dfrac{e^{i\phi}-e^{-i\phi}}{2}$

(B) $\dfrac{e^{i\phi}-e^{-i\phi}}{2i}$

(C) $\dfrac{e^{i\phi}+e^{-i\phi}}{i}$

(D) $\dfrac{e^{i\phi}+e^{-i\phi}}{2}$

$e^{i\phi} = \cos\phi + \cancel{i\sin\phi}$ — (1)

$e^{-i\phi} = \cos\phi - \cancel{i\sin\phi}$ — (2)

Add

$e^{i\phi}+e^{-i\phi} = 2\cos\phi$

$$\boxed{\cos\phi = \frac{e^{i\phi}+e^{-i\phi}}{2}}$$

Q. Questions

#Q. i^i, where $i = \sqrt{-1}$ is given by

(A) 0

(B) $e^{-\pi/2}$

(C) $\frac{\pi}{2}$

(D) 1

i^i

i = Polar form = $i = 0 + i \cdot 1$

$r = \sqrt{1} = 1$

$\theta = \tan^{-1}\left(\frac{1}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$

$i^2 = -1$

$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{\frac{i\pi}{2}}$

$= \left[e^{i\pi/2}\right]^i$

$= e^{i^2\frac{\pi}{2}} = e^{-\frac{\pi}{2}}$

Slide-5

Q. Questions

#Q. The complex number $z = x + jy$ which satisfy the equation $|z + 1| = 1$ lie on

(A) A circle with $(1, 0)$ as the centre and radius 1

(B) A circle with $(-1, 0)$ as the centre and radius 1

(C) y - axis

(D) x - axis

Q. Questions

#Q. e^z is a periodic with a period of

(A) 2π

(B) $2\pi i$

(C) π

(D) $i\pi$

$$e^z = e^{x+iy}$$
$$= e^x e^{iy}$$
$$= \underbrace{e^x}_{\text{Real Part}}[\underbrace{\cos y + i\sin y}_{\text{Periodicity } 2n\pi}]$$
$$= e^x[\cos(y+2n\pi) + i\sin(y+2n\pi)]$$
$$= e^x e^{i(y+2n\pi)}$$
$$= e^{(x+iy+2n\pi i)} = e^{z+2n\pi i}$$

repeat Period $= 2\pi i$

Q. Questions

#Q. Consider the circle $|z - 5 - 5i| = 2$ in the complex number plane (x, y) with $z = x + iy$. The minimum distance from the origin to the circle is

(A) $5\sqrt{2} - 2$

(B) $\sqrt{54}$

(C) $\sqrt{34}$

(D) $5\sqrt{2}$

Q. Questions

#Q. If a complex number $\underbrace{z=\frac{\sqrt{3}}{2}+i\frac{1}{2}}$ then z^4 is

(A) $2\sqrt{2}+2i$

(B) ✓ $-\frac{1}{2}+i\frac{\sqrt{3}}{2}$

(C) $\frac{\sqrt{3}}{2}-i\frac{1}{2}$

(D) $\frac{\sqrt{3}}{8}-i\frac{1}{8}$

Power of complex No

$Z=\cos\theta+i\sin\theta$

$Z=\frac{\sqrt{3}}{2}+i\frac{1}{2}$

$Z=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$

$Z^4=\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^4$

$=\left(\cos\frac{4\pi}{6}+i\sin\frac{4\pi}{6}\right)$

$Z^4=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$

$$\boxed{Z^4=-\frac{1}{2}+i\frac{\sqrt{3}}{2}}$$

Q. Questions

$i^2 = -1$

#Q. The value of the expression $\frac{-5+i10}{3+4i}$ is

(A) 1 – 2i

(B) 1 + 2i

(C) 2 – i

(D) 2 + i

$\frac{-5+10i}{3+4i}$ = division of complex Number

$\frac{-5+10i}{3+4i} \times \frac{3-4i}{3-4i}$

$= \frac{(-5+10i)(3-4i)}{9-(4i)^2}$

$= \frac{-15+20i+30i-40i^2}{9-16i^2}$

$\frac{-15+40+50i}{9+16}$

$= \frac{25+50i}{25}$

$= 1+2i$

Q. Questions

#Q. Let $w^4 = 16j$. Which of the following cannot be a value of w?

$w^4 = 16i$

$W = (16i)^{1/4}$

Do yourself

Polar form

Demovers law apply

Now get The solution

(A) $2e^{(j2\pi/8)}$

(B) $2e^{(j\pi/8)}$

(C) $2e^{(j5\pi/8)}$

(D) $2e^{(j9\pi/8)}$

Questions

value of

$(1+i)^8$

#Q. Value of $(1+i)^8$, where $i = \sqrt{-1}$, is equal to

(A) 4

(B) 16

(C) 4i

(D) 16i

$$(1+i)^2 = 1+i^2+2i$$
$$= 1-1+2i$$
$$= 2i$$

$$\left[(1+i)^2\right]^4 = (1+i)^8$$
$$= [2i]^4$$
$$= 16i^4 = 1\times 16 = 16$$

Q. Questions

#Q Let $f(z) = (az+b)/(cz+d)$, If $f(z_1) = f(z_2)$ for all $z_1 \neq z_2$, $a = 2$, $b = 4$ and $c = 5$, then d should be equal to_______.

$$f(z) = \frac{az+b}{cz+d} \quad z_1 \neq z_2$$

$$f(z_1) = f(z_2)$$

$$\frac{az_1+b}{cz_1+d} = \frac{az_2+b}{cz_2+d}$$

$$\cancel{acz_1z_2} + adz_1 + bcz_2 + \cancel{bd} = \cancel{acz_1z_2} + bcz_1 + adz_2 + \cancel{bd}$$

$$ad(z_1 - z_2) = bc(z_1 - z_2)$$

$$ad = bc$$

$$2 \times d = 5 \times 4$$

$$d = \frac{5 \times 4}{2} = 10$$

$$d = 10$$

Q. Questions

#Q. Given two complex numbers $z_1 = 5 + (5\sqrt{3}i)$ and $z_2 = (2/\sqrt{3}) + 2i$, the argument of z_1/z_2 in degrees is ________.

$$z_1 = \underset{x_1}{5} + \underset{y_1}{5\sqrt{3}}\,i = \arg z_1 = \tan^{-1}\left(\frac{y_1}{x_1}\right) = \tan^{-1}\left(\frac{\cancel{5}\sqrt{3}}{\cancel{5}}\right) = \frac{\pi}{3}$$

$$z_2 = \underset{x_2}{\frac{2}{\sqrt{3}}} + \underset{y_2}{2}\,i = \arg z_2 = \tan^{-1}\left(\frac{y_2}{x_2}\right) = \tan^{-1}\left(\frac{2}{\frac{2}{\sqrt{3}}}\right) = \frac{\pi}{6}$$

$$\left(\frac{z_1}{z_2}\right) = \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

$$= \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}\ \text{Ans}$$

Q. Questions

#Q. The argument of the complex number (1+i)/(1-i) , where i = $\sqrt{-1}$, is ________.

$$\left(\frac{z_1}{z_2}\right) = \frac{1+i}{1-i}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

Do yourself

Q. Questions

#Q. The product of two complex numbers $1 + i$ and $2 - 5i$ is

(A) $7 - 3i$

(B) $3 - 4i$

(C) $-3 - 4i$

(D) $7+3i$

Product of complex No $= z_1 z_2$

$z_1 = 1+i$

$z_2 = 2-5i$

Q. Questions

Do yourself

#Q. The modulus of the complex number $\left(\frac{3+4i}{1-2i}\right)$ is

(A) 5

(B) $\sqrt{5}$

(C) $1/\sqrt{5}$

(D) $1/5$

$$\frac{3+4i}{1-2i} \times \frac{1+2i}{1+2i}$$

mod $r = |z| = \sqrt{x^2+y^2}$

Ans

Thank You!

PW Soldiers