GATE

ALL BRANCHES

Engineering Mathematics

Multivariable Calculus and Vector Calculus

Discussion Notes (Part-02)

DPP 01

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Topic : Vector Calculus

$\int \vec{F} \cdot d\vec{r}$ $x = e^t \cos t$ $y = e^t \sin t$ $t \to 0$ to π

$M\,dx + N\,dy = c$

$\int M\,dx$ y Treating as a const $+ \int N\,dy$ Independent of y $= c$

#Q. Let $\vec{F} = (3 + 2xy)\hat{\imath} + (x^2 - 3y^2)\hat{\jmath}$ and let L be the curve $\vec{r} = e^t \sin t\,\hat{\imath} + e^t \cos t\,\hat{\jmath}, 0 \le t \le \pi$. Then $\int_L \vec{F} \cdot d\vec{r} =$

A $e^{-3\pi} + 1$

B $e^{-6\pi} + 2$

C $e^{6\pi} + 2$

D $e^{3\pi} + 1$

$\oint \vec{F}\,d\vec{r} = \oint (3 + 2xy)\,dx + (x^2 - 3y^2)\,dy$

$M = 3 + 2xy$, $N = (x^2 - 3y^2)$

$= \int (3 + 2xy)\,dx + \int (\cancel{x^2} - 3y^2)\,dy = c$

y constant Indep of y

$= 3x + x^2y - y^3 = c$

$= \int d(3x + x^2y - y^3)\,dx$

$$= \int \left[(3x + x^2 y - y^3) \right]_{(0,0)}^{(0,e^{-\pi})}$$

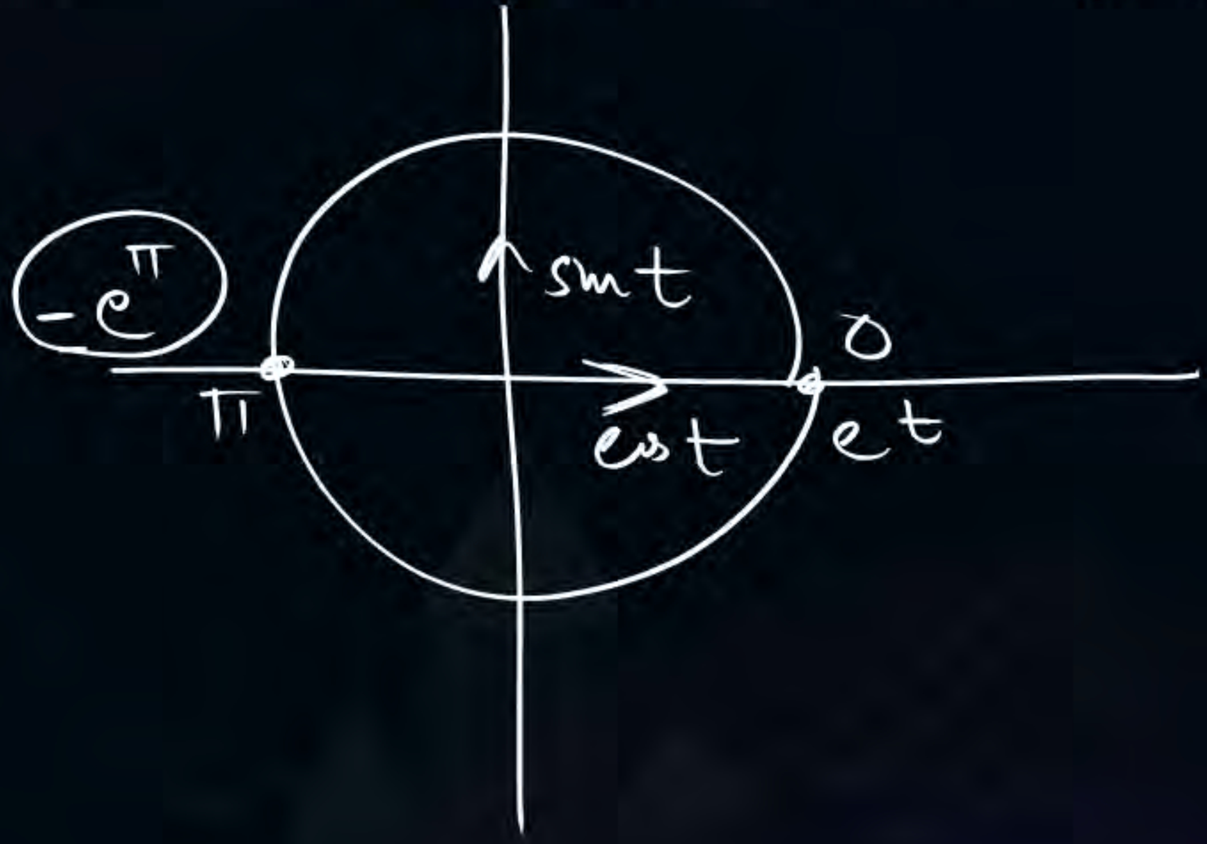
$$= (e^{3\pi} + 1)$$

$\vec{r}(t) = e^t \cos t\, \hat{i} + e^t \sin t\, \hat{j}$

$= e^t [\cos t\, \hat{i} + \sin t\, \hat{j}]$

$$\oint M\,dx + N\,dy = e$$

$$\int d(M\,dx + N\,dy)$$



Hint $\oint \vec{F} \cdot d\vec{r}$

$x = e^t \cos t$

$y = e^t \sin t$

$t \to 0$ to π

$$= e^{3\pi} + 1$$

Topic : Vector Calculus

Telegram

#Q. If the triple integral over the region bounded by the planes $2x + y + z = 4, x = 4, y = 0, z = 0$ is given by $\int_0^2 \int_0^{\lambda(x)} \int_0^{\mu(x,y)} dzdydz$, then the function

A x + y

B x – y

C x

D y

$$= \int_{x=0}^{4} \int_{y=0}^{y-2x} \int_{z=0}^{4-2x-y} dz\,dy\,dz$$

$$2x+y+z=4$$
$$x=4$$
$$y=0$$
$$z=0$$

$$2x+y+0=4$$
$$y=4-2x$$

$$\lambda(x)+\mu(x) = \cancel{y}-2x+4-2x-\cancel{y}$$
$$= 4-4x$$

Topic : Vector Calculus

PW

The value of the Integral

$$\int_0^1 \int_{x=0}^{1-y^2} y \sin(\pi(1-x)^2)\, dx\, dy$$

#Q. The value of the integral

$\int_{y=0}^{1} \int_{x=0}^{1-y^2} y\, sin(\pi(1-x)^2)\, dxdy$ is

A) $1/2\pi$

B) 2π

C) $\pi/2$

D) $2/\pi$

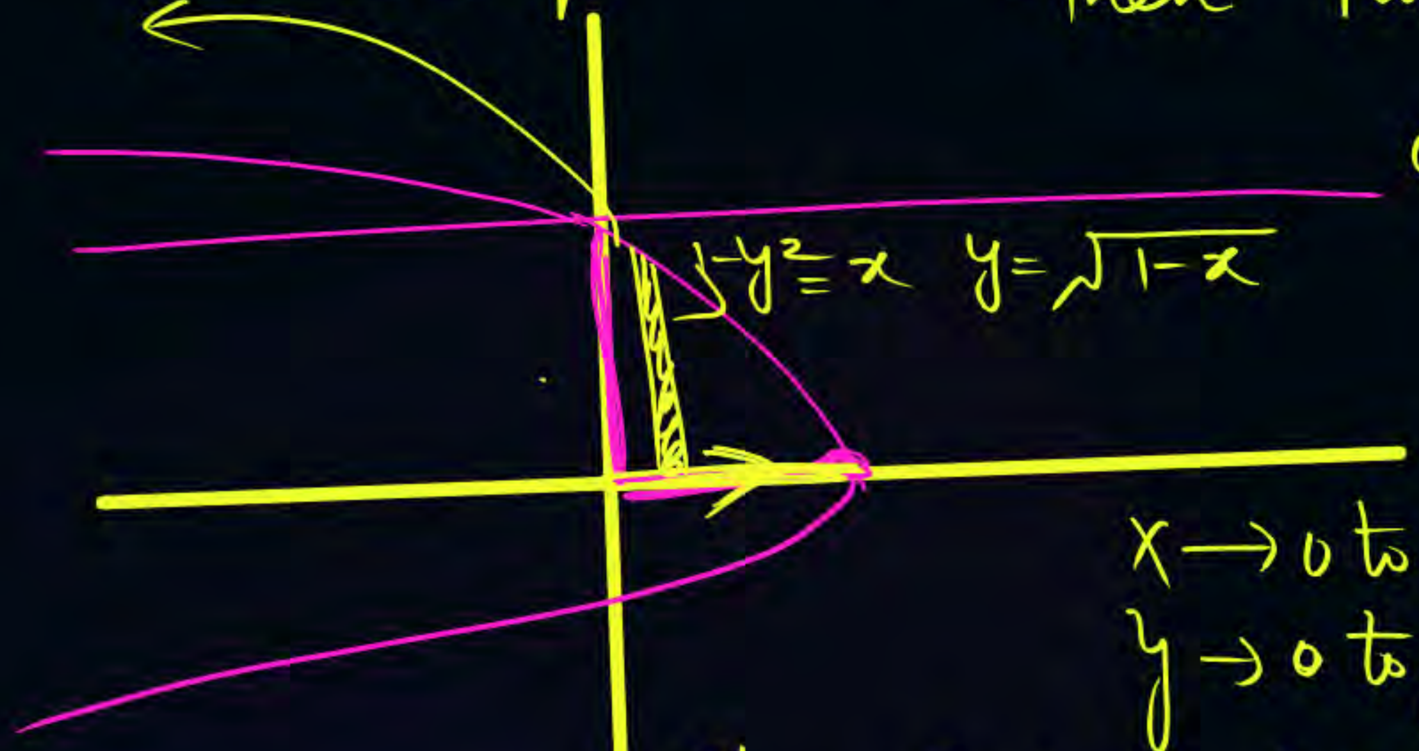
Plot The Limit

$y=0,\ y=1,\ x=0,\ \boxed{x=1-y^2}$

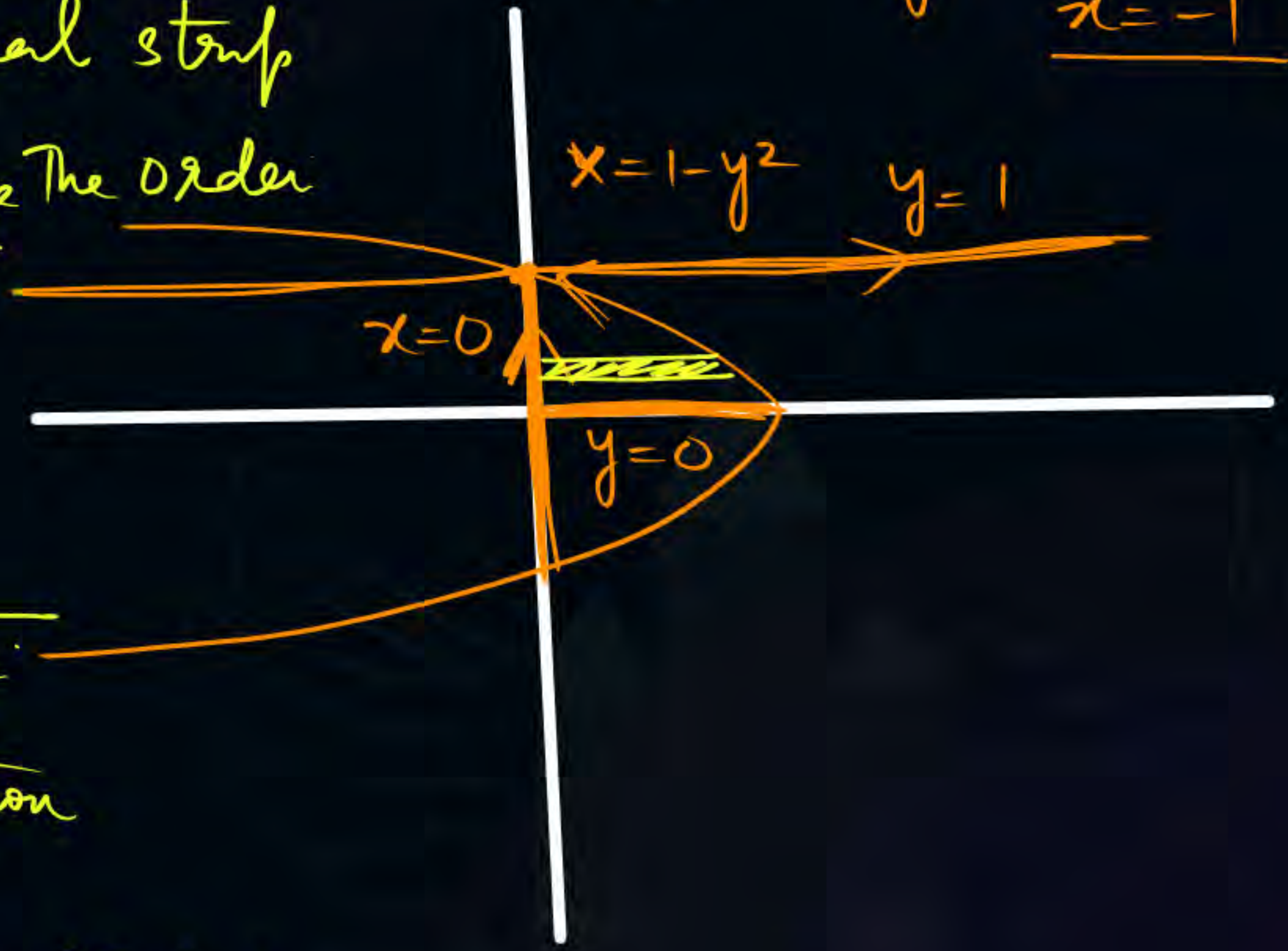
$(x+1)=y^2 \qquad x+1=0,\ x=-1$

If y 0 to 1 Then Horizonal strip

vertical strip



Change The order $\Longleftarrow$



$x \to 0$ to 1

$y \to 0$ to $\sqrt{1-x}$

Change The order of Integration

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x}} y \sin\left[\pi(1-x)^2\right] dy\, dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x}} y \sin[\pi(1-x)^2]\, dy\, dx$$

$$= \int_0^1 \sin(\pi(1-x)^2)\, dx \int_0^{\sqrt{1-x}} y\, dy$$

$$= \int_0^1 \sin[\pi(1-x)^2]\, dx \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x}}$$

$$= \int_0^1 \frac{(1-x)}{2} \sin[\pi(1-x)^2]\, dx$$

If $\pi(1-x)^2 = t$

$$2\pi(1-x)(-1)\,dx = dt$$

$$(1-x)\,dx = -\frac{1}{2\pi}\,dt$$

$$= \frac{1}{2\pi}$$ Ans

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Extra XE/EE/ME/CE/EC

Rotating about y-axis is

$$= \int_{y=a}^{b} 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

#Q. The area of the surface generated by rotating the curve $x = y^3, 0 \leq y \leq 1$ .about the y-axis is

Rotating about x-axis is

$$= \int_{x=a}^{b} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

A $\frac{\pi}{27}10^{3/2}$

B $\frac{4\pi}{3}(10^{3/2}-1)$

C $\frac{\pi}{27}(10^{3/2}-1)$

D $\frac{4\pi}{3}10^{3/2}$

Extra $x = y^3$

$$\frac{dx}{dy} = 3y^2$$

Surface Rotated X-axis

$$S = \int_{y=a}^{b} 2\pi y^3 \sqrt{1+(3y^2)^2}\, dy$$

$$= \int_{0}^{1} 2\pi y^3 \sqrt{1+9y^4}\, dy$$

If $y^4 = t$

$$= \frac{\pi}{27}\left(10^{3/2}-1\right)$$

Surface generated (Rotated) y-axis

$$S = \int_{y=a}^{b} 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\cdot dy$$

surface generated Rotated X-axis

$$S = \int_{x=a}^{b} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

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$$\text{Surface Area} = \int_{x=a}^{b}\int_{y=c}^{d}\sqrt{\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2+1}\;dy\,dx$$

#Q. The area of the part of surface of the paraboloid $x^2 + y^2 + z = 8$ lying inside the cylinder $x^2 + y^2 = 4$ is

A. $\frac{\pi}{2}(17^{\frac{3}{2}} - 1)$

B. $\pi(17^{\frac{3}{2}} - 1)$

C. $\frac{\pi}{6}(17^{\frac{3}{2}} - 1)$

D. $\frac{\pi}{3}(17^{\frac{3}{2}} - 1)$

$x^2+y^2+z=8$ Lying Inside The clyinder $x^2+y^2=4$

$\underbrace{x^2+y^2+z=8}_{\text{paraboloid}}$

$$\boxed{\text{Surface Area} = \int_x \int_y \sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\; dy\, dx}$$

$x^2+y^2+z=8$

$Z=(8-x^2-y^2)=f(x,y)$

$\frac{\partial z}{\partial x}=-2x$

$\frac{\partial z}{\partial y}=-2y$

$$=\int_x \int_y \sqrt{(-2x)^2+(-2y)^2+1}\; dy\, dx$$

$$=\int_x \int_y \sqrt{4x^2+4y^2+1}\; dy\, dx$$

$$=\int_x \int_y \sqrt{4(x^2+y^2)+1}\; dy\, dx$$

$$= \int_x \int_y \sqrt{4(x^2+y^2)+1}\, dy\, dx$$

$$= \int_r \int_\theta \sqrt{4r^2+1}\; r\, dr\, d\theta$$

$$= \int_0^2 dr \int_{\theta=0}^{2\pi} \sqrt{4r^2+1}\, d\theta$$

$$= \int_0^2 \sqrt{(4r^2+1)}\; \underline{r\, dr} \cdot \underline{\int_0^{2\pi} d\theta}$$

$$= \frac{\cancel{2}\pi}{\cancel{2}} \int \sqrt{4t+1}\; dt$$

$$= \pi \int_0^4 \sqrt{4t+1}\; dt = \frac{\pi}{6}\left(17^{3/2}-1\right)$$

$r^2 = t$

$2r\, dr = dt$

$r\, dr = \frac{dt}{2}$

$0 = t$

$2^2 = t$

$t = 4$

$r = 0$ to 2

$\theta = 0$ to 2π

$x^2+y^2 = 4$

$x^2+y^2 = (2)^2$

$r = 2$

$\sin\theta$

$\cos\theta$

CHANGE the Polar co-ordinates

$dy\, dx = r\, dr\, d\theta$

$x^2+y^2 = r^2$

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#Q. Length of the arc of the curve

$y = \log \sec x$ from $x = 0$ to $x = \frac{\pi}{3}$ is equal to

A $\log(2-\sqrt{3})$

B $\log(1-\sqrt{3})$

C $\log(1+\sqrt{3})$

D $\log(2+\sqrt{3})$

$y = \log \sec x.$

$\frac{dy}{dx} = \frac{1}{\sec x}\frac{d}{dx}(\sec x)$

$= \frac{1}{\cancel{\sec x}} \cdot \cancel{\sec x} \tan x$

$= \underline{\tan x}$

$Y = \log \sec x$

from $x = 0$ to $\frac{\pi}{3}$

Length of the curve

$L \Rightarrow \int_{x=a}^{b} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$

$L = \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2 x}\, dx$

$$L = \int_0^{\frac{\pi}{3}} \sqrt{1+\tan^2 x}\, dx$$

$$\Rightarrow \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x}\, dx$$

$$\Rightarrow \boxed{\int_0^{\frac{\pi}{3}} \sec x\, dx}$$

$$\Rightarrow \underline{\underline{\text{Ans}}}$$

$$\sec^2 x - \tan^2 x = 1$$

$$\sec^2 x = 1 + \tan^2 x$$

$$\int \sec x\, dx = \underline{\log(\sec x + \tan x)}$$

$$\int \underline{\sec x \cdot dx}$$

Topic : Vector Calculus

$[x]$ = greatest Integer of x

#Q. For a real number x, define $[x]$ to be the smallest integer greater than or equal to x. Then

3

$$\int_0^1 \int_0^1 \int_0^1 ([x] + [y] + [z])\, dx\, dy\, dz =$$

$$I = \int_0^1 \int_0^1 \int_0^1 ([x]+[y]+[z])\, dx\, dy\, dz$$

$$\Rightarrow \int_0^1 \int_0^1 \int_0^1 (1+1+1)\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^1 \int_0^1 3\, dx\, dy\, dz$$

$z=$ $y=$ $x=$

$$= 3\int_0^1 dz \int_0^1 dy \int_0^1 dx$$

$$= 3\int_0^1 dz \int_0^1 dy\, (1)$$

$$= 3 \times 1 \times 1 \times 1 = 3$$

greatest Integer of x

$0 \leq x < 1$

$[x] = 1$

$[y] = 1 \quad 0 \leq y < 1$

$[z] = 1 \quad 0 \leq z < 1$

Topic : Vector Calculus

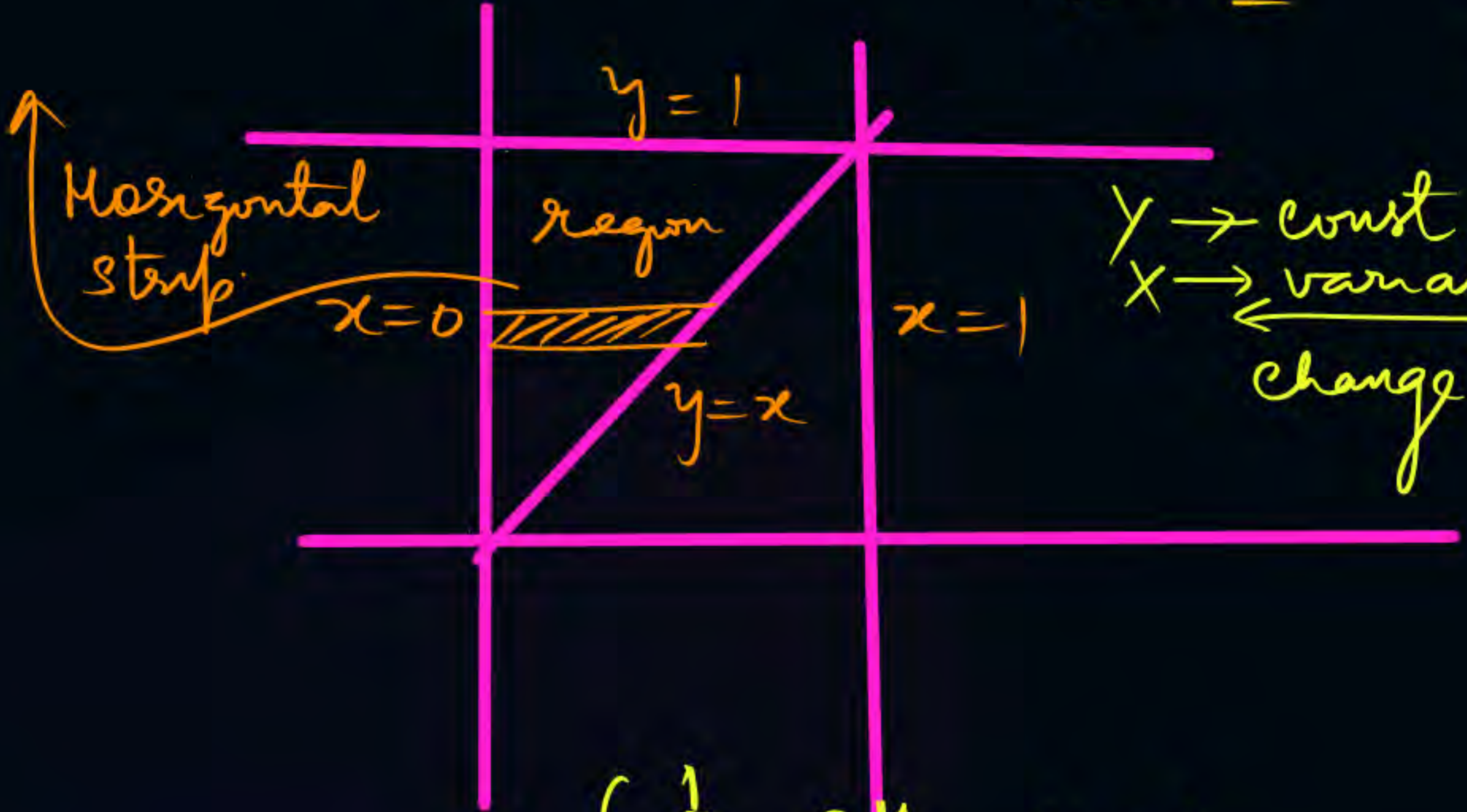
#Q. The value of the integral $\int_0^1 \int_x^1 y^4 e^{xy^2}\, dy dx$ is ___

(correct up to three decimal places)

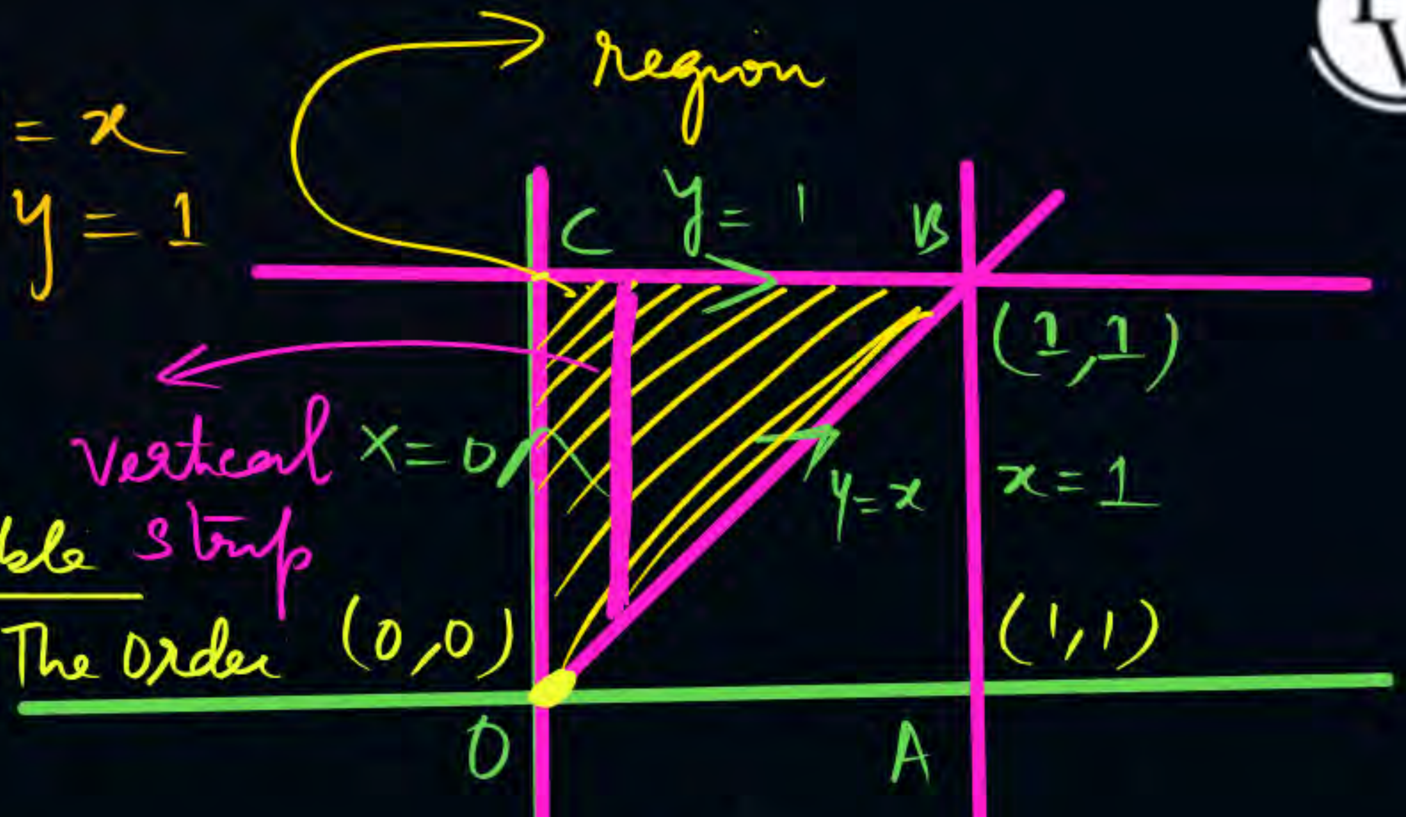
The value of Integral $\int_{x=0}^{1}\int_{y=x}^{1} y^4 e^{xy^2}\, dy\, dx$

e^{x^2} — Not Integrable form

Plot the Curves $x = 0$, $x = 1$, $y = x$, $y = 1$



Y → const
X → variable
change the order



x - Horizontal 0 to 1
$y = x$ to 1

vertical strip

$$\int_{y=0}^{1}\int_{x=0}^{y} y^4 e^{xy^2}\,dx\,dy = \text{change the order}$$

$$e^{xy^2} = \frac{e^{xy^2}}{y^2}$$

$$I = \int_{y=0}^{1} \int_{x=0}^{y} y^4 e^{xy^2}\, dx\, dy$$

$dy\,dx \Longrightarrow dx\,dy$

$$\Rightarrow \int_0^1 y^4\, dy \int_{x=0}^{y} e^{xy^2}\, dx$$

$$\Rightarrow \int_0^1 y^4\, dy \left[\frac{e^{xy^2}}{y^2}\right]_{x=0}^{y}$$

$$\Rightarrow \int_0^1 \cancel{y^4}\, y^2 \left[\frac{e^{y^3}}{\cancel{y^2}}\right] dy$$

$$\Rightarrow \int_0^1 y^2 e^{y^3}\, dy$$

$$I = \int_0^1 y^2 e^{y^3}\, dy$$

Put $y^3 = t$

$3y^2\, dy = dt$

$y^2\, dy = \frac{dt}{3}$

$$I = \frac{1}{3}\int_0^1 e^t\, dt$$

$$I = \frac{e^1 - e^0}{3} = \frac{e-1}{3}$$ Ans

Ans $$\boxed{\int_0^1 \int_x^1 y^4 e^{xy^2}\, dy\, dx = \frac{(e-1)}{3}}$$

THANK - YOU