GATE-All BRANCHES
Engineering Mathematics

Multivariable calculus

Lecture No.- 03

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Recap of Previous Lecture

Topic Change the order of integration

Topic Change the variables

Topic Question based on change of variables

Topics to be Covered

Topic Concept of length of curve

Topic Volume of solid revolution

Topic Maxima and minima with two variables

Topic Problems based on length of curve, Volume of solid revolution, Maxima and minima

Length of curve :

If $y = f(x)$ given

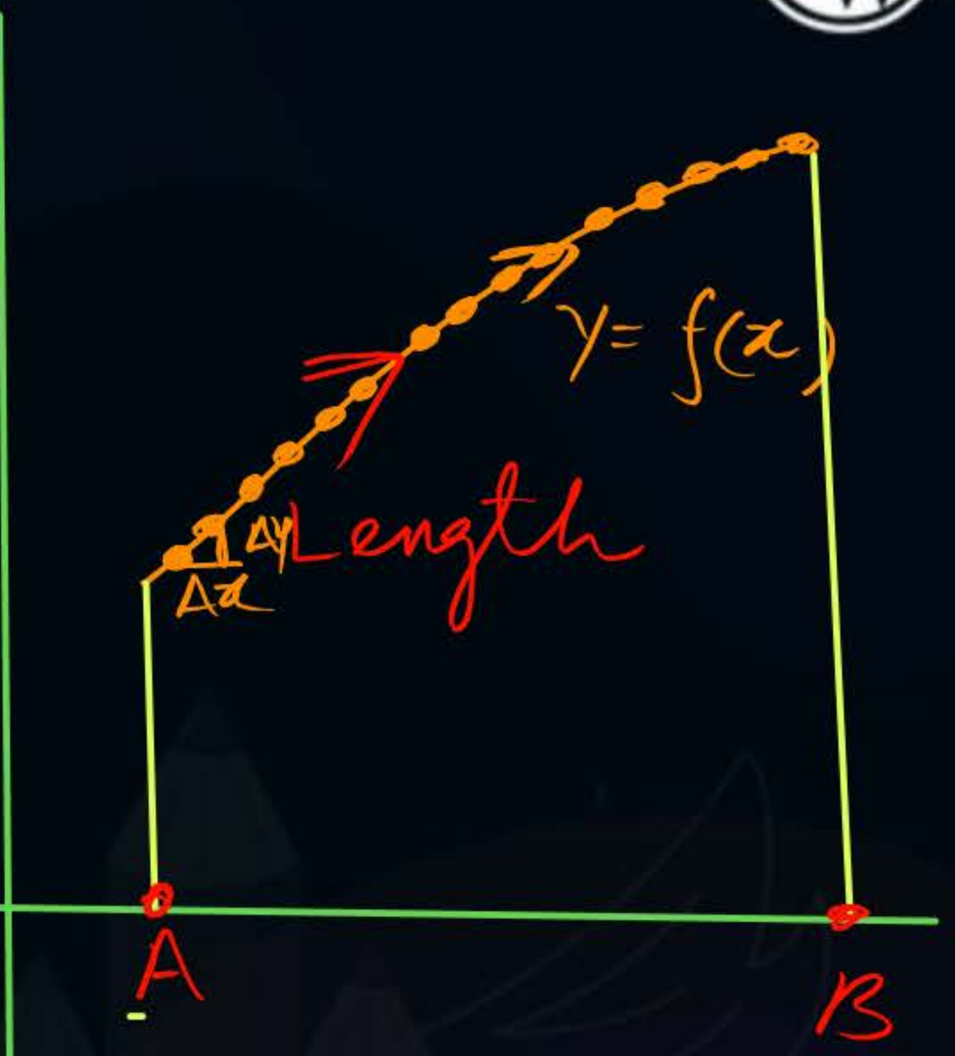
Length of curve

$$L = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$y = x$ 1 to 2

$$L = \int_1^2 \sqrt{1+(1)^2}\, dx$$

$$\boxed{L = \sqrt{2}(2-1) = \sqrt{2}}$$



Case (2) given curve $x = f(y)$ $\quad y = y_1$ to $y = y_2$

Length of curve $L = \int_{y_1}^{y_2} \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$

✓ **Case (3)** given curve

$x = f(t) \quad y = g(t)$

$x = \cos t$

$y = \sin t$

Length of curve

$$L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$L = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$L = 2\pi$$

Total derivative — stream $\phi(x,y)$, Potential $\psi(x,y)$

CE/ME/EMT

(A) Given Function $u = f(x,y)$ $\left.\begin{matrix} x \\ y \end{matrix}\right]$ Independent variable

$\left.\begin{matrix} x = f(t) \\ y = g(t) \end{matrix}\right]$ single var. (ordinary deri)

$u = f(x, y)$ Using Partial derivative

Total derivative

$$\boxed{\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}}$$

Chain Rule.

$\boxed{u = f(x, y)}$ Find the Total derivative

2 variable
del
operator
Partial

du = Total derivative

$$\boxed{du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} dy}$$

$y = f(x)$
Simple

If $u = f(x, y, z)$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

Question: $u = x^2 + y^3$ $\quad x = t^2$, $y = t^3$

$\frac{du}{dt}$ = Total derivative

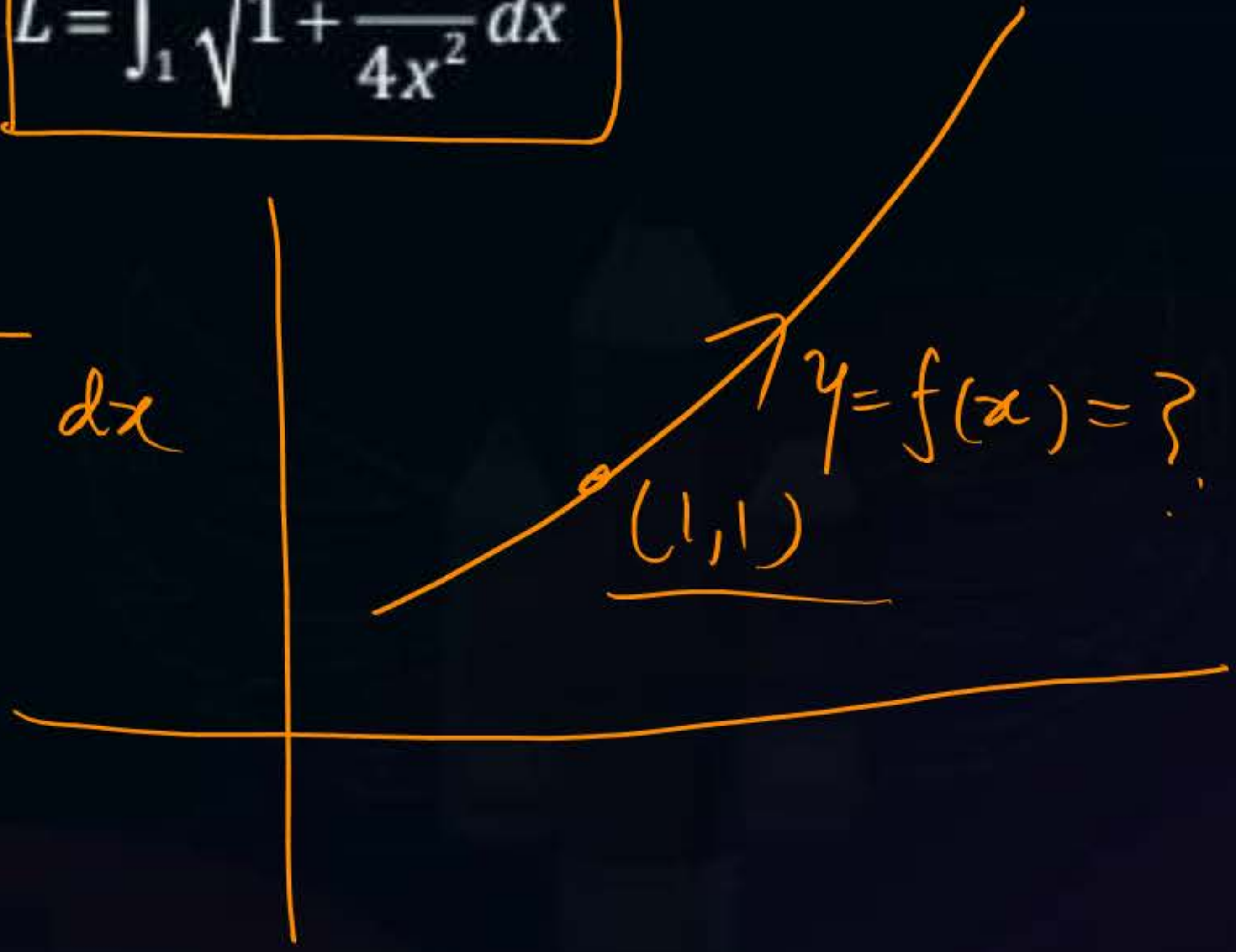
$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= 2x (2t) + 3y^2 \cdot 3t^2 = 2 \times t^2 \times 2t + 3(t^3)^2 \cdot 3t^2$$

Topic :Double integration

#Q. Let f be an increasing differential function, if the curve y = f(x) passes through (1, 1) and has length $\boxed{L = \int_1^2 \sqrt{1 + \frac{1}{4x^2}}\, dx}$ Then the curve

$$L = \int_1^2 \sqrt{1 + \frac{1}{4x^2}}\; dx$$



A. $y = \ln(\sqrt{x}) - 1$

B. $y = 1 - \ln(\sqrt{x})$ (B)

C. $y = \ln(1 + \sqrt{x})$

D. $y = 1 + \ln(\sqrt{x})$ (D)

$$L = \int_1^2 \sqrt{1+\left(\frac{1}{4x^2}\right)}\,dx = \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\underbrace{\int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx}_{\text{Formula } L}$$

$$\cancel{1} + \frac{1}{4x^2} = \cancel{1} + \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4x^2}$$

$$\Rightarrow \frac{dy}{dx} = \pm\frac{1}{2x}$$

Using variable seprable

$$\int dy = \pm\frac{1}{2}\int \frac{dx}{x}$$

$$y = \pm\frac{1}{2}\ln x + C$$

$$y = \pm\ln(x)^{1/2} + c$$

$$\boxed{y = \pm\ln\sqrt{x} + c}$$

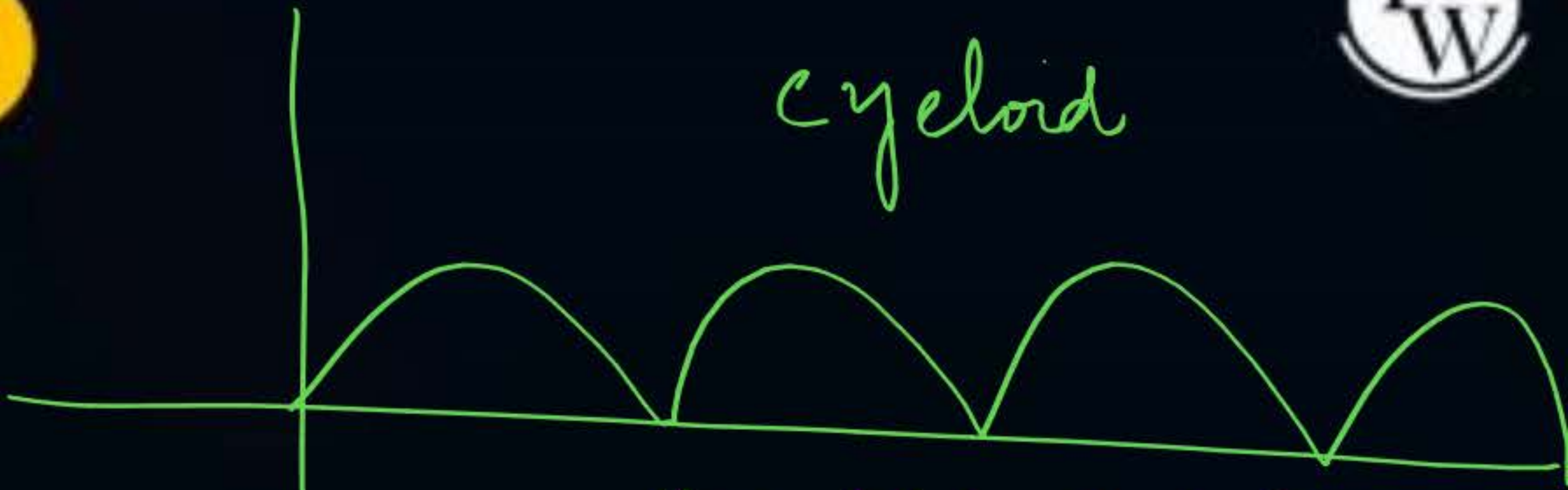
$y = f(x)$ Passes Through $(1,1)$

$$1 = \pm\ln\sqrt{1} + c$$

$$c = 1$$

$$y = \pm\ln\sqrt{x} + 1$$

Topic :Double integration



#Q. Length of the arc of the cycloid

$x = a\,(t - \sin t)$

$y = a(1 - \cos t)$

$x = a(1 - \sin t)$

$y = a(1 - \cos t)$

$t \to 0$ to 2π

$$L = \int_{t=0}^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

A. 8a

B. 4a

C. $4\sqrt{2}a$

D. $2\sqrt{2}a$

2 Marks CE/ME/EE/EC

Maxima and minima (Two variables)

$$\boxed{Z = f(x, y)}$$

Multivariable

Some standard Notation

(A) $\frac{\partial z}{\partial x}$ = Diff. w.r.t to x = P

(B) $\frac{\partial z}{\partial y}$ = Diff. w.r. to to y = q

(C) $\frac{\partial^2 z}{\partial x^2} = r$

D) $\frac{\partial^2 z}{\partial y \partial x}, \frac{\partial^2 z}{\partial x \partial y} = s$

$\underbrace{\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)}_{s}, \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)}_{s}$ $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$

E) $\frac{\partial^2 z}{\partial y^2} = t$

How to Evaluate Max|min (Two variables) $z = f(x, y)$

step ① $\frac{\partial z}{\partial x} = 0$, $\frac{\partial z}{\partial y} = 0$ to get The stationary Point (x_0, y_0)

step ② $rt - s^2$ where $r = \frac{\partial^2 z}{\partial x^2}$

$s = \frac{\partial^2 z}{\partial y \partial x}$, $\frac{\partial^2 z}{\partial x \partial y}$

$t = \frac{\partial^2 z}{\partial y^2}$

Taylor SERIES (Two vari)

(A) $rt - s^2 > 0$, $r > 0$ (min)
And The min value $f(x_0, y_0)$

(B) $rt - s^2 > 0$, $r < 0$ (max)
and max value $f(x_0, y_0)$

(C) $rt - s^2 < 0$ This case is Neither max Nor minima → Saddle Point

D) $rt - s^2 = 0$ (This case is Undecided)

Topic :Double integration

#Q. The function $f(x, y) = x^2y - 3xy + 2y + x$ has

$2xy - 3y + 1 = 0$

$2 \times 1\, y - 3y + 1 = 0$

$-y = -1 \quad y = 1$

A. No local extremum

B. One local maximum but no local minimum

C. One local minimum but not local maximum

D. One local minimum and one local maximum

$\frac{\partial f}{\partial x} = 2xy - 3y + 1$

$\frac{\partial f}{\partial y} = x^2 - 3x + 2$

$\frac{\partial f}{\partial x} = 0 \quad \frac{\partial f}{\partial y} = 0$

$x^2 - 3x + 2 = 0$

$x^2 - 2x - x + 2 = 0$

$x(x-2) - 1(x-2) = 0$

$x = 1$
$x = 2$

$x = 1 \quad x = 2$
$y = 1 \quad y = -1$

$$\frac{\partial^2 f}{\partial x^2} = 2y = r$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2x - 3 = s$$

$$\frac{\partial^2 f}{\partial y^2} = 0 = t$$

at Pt $(1,1)$

$r = 2\times 1 = 2$

$s = 2\times 1 - 3 = -1$

$t = 0$

$rt - s^2$, $r > 0$

$2\times 0 - (-1)^2 < 0$, $-1 < 0$

Saddle Pt

(Neither Max/min)

Case-2 $x = 2$, $y = -1$

$r = 2\times -1 = -2$

$s = 2\times 2 - 3 = 1$

$t = 0$

$rt - s^2 = (-2)\times 0 - (1)^2$

$= -1 < 0$ Saddle Pt

Neither max Nor min

Topic :Double integration

H.W

#Q. The continuous function f(x, y) is said to have saddle point at (a, b) if

$f_x = \frac{\partial f}{\partial x}$ $f_{xx} = \frac{\partial^2 f}{\partial x^2}$ $\frac{\partial^2 f}{\partial y^2} = f_{yy}$ $f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$

$f_y = \frac{\partial f}{\partial y}$

A $f_x(a,b) = f_y(a,b) = 0$
$f_{xy}^2 - f_{xx}f_{yy} < 0 \text{ at } (a,b)$

B $f_x(a,b) = 0, f_y(a,b) = 0,$
$f_{xy}^2 - f_{xx}f_{yy} > 0 \text{ at } (a,b)$

C $f_x(a,b) = 0, f_y(a,b) = 0,$
$f_{xx} \text{ and } f_{yy} < 0 \text{ at } (a,b)$

D $f_x(a,b) = 0, f_y(a,b) = 0,$
$f_{xy}^2 - f_{xx}f_{yy} = 0 \text{ at } (a,b)$

Topic :Double integration

#Q. The function $f(x, y) = 2x^2 + 2xy - y^3$ has

A. Only one stationary point at (0, 0)

B. Two stationary points at (0, 0) and $\left(\frac{1}{6}, \frac{-1}{3}\right)$

C. Two stationary points at (0, 0) and (1, –1)

D. No stationary point.

$f(x,y) = \underbrace{2x^2 + 2xy - y^3}_{\text{surface}}$

$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$

$$\begin{cases} \frac{\partial f}{\partial x} = 4x + 2y \\ \frac{\partial f}{\partial y} = 2x - 3y^2 \end{cases}$$

$\frac{\partial f}{\partial x} = 0,\quad 4x + 2y = 0$

$2y = -4x$

$\boxed{y = -2x}$

$2x - 3y^2 = 0$

$2x = 3y^2$

$-y = 3y^2$

$\Rightarrow 3y^2 + y = 0$

$\Rightarrow y(3y+1) = 0$

$y = 0$

$y = -\frac{1}{3}$

$\boxed{y = 0 \quad x = 0}$

$\boxed{y = -\frac{1}{3} \quad x = \frac{1}{6}}$

Stationary Pts

Topic :Double integration

$\boxed{200}$ $\boxed{\text{revision}}$

1.30 Hrs. $\rightarrow$ 10–12 marks

#Q. For the function $f(x, y) = x^2 - y^2$ defined on R^2, the point (0, 0) is

$\frac{\partial^2 f}{\partial x^2} = 2 = r$

$\frac{\partial^2 f}{\partial y \partial x} = 0 = s$

$\frac{\partial^2 f}{\partial y^2} = -2 = t$

$\frac{\partial f}{\partial x} = 2x$ $\frac{\partial f}{\partial y} = -2y$

$2x = 0$ $2y = 0$

$x = 0$, $y = 0$

$rt - s^2 = 2 \times -2 - 0 = -4 < 0$

$rt - s^2 < 0$

Neither max Nor minima (Saddle Point)

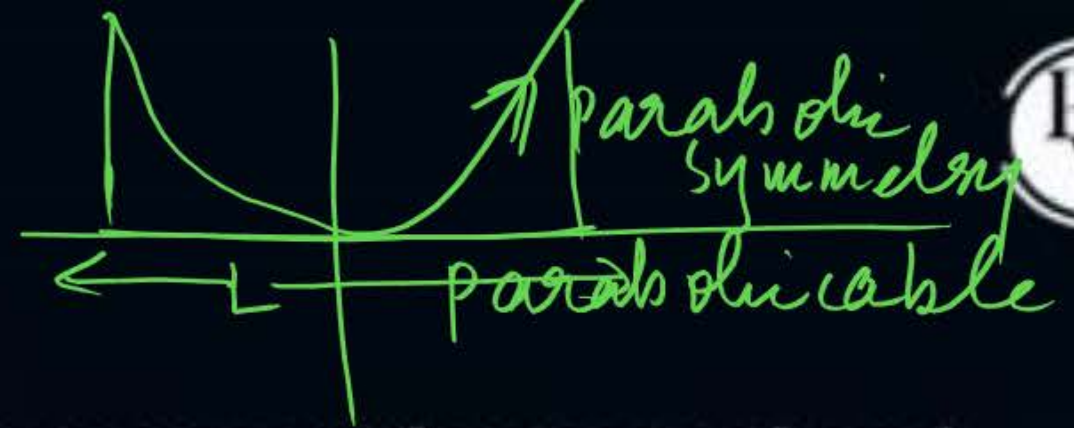
A. A local minimum

B. Neither a local minimum (nor) a local maximum

C. A local maximum

D. Both a local minimum and a local maximum

Topic :Double integration



#Q. A parabolic cable is held between two supports at the same level. The horizontal span between the supports is L. The sag at the mid-span is h. The equation of the parabola is $y = 4h\frac{x^2}{L^2}$, where x is the horizontal coordinate and y is the vertical coordinate with the origin at the centre of the cable. The expression for the total length of the cable is

$$\int_{x_1}^{x_2}\sqrt{\left(\frac{dy}{dx}\right)^2+1} = L \quad dx$$

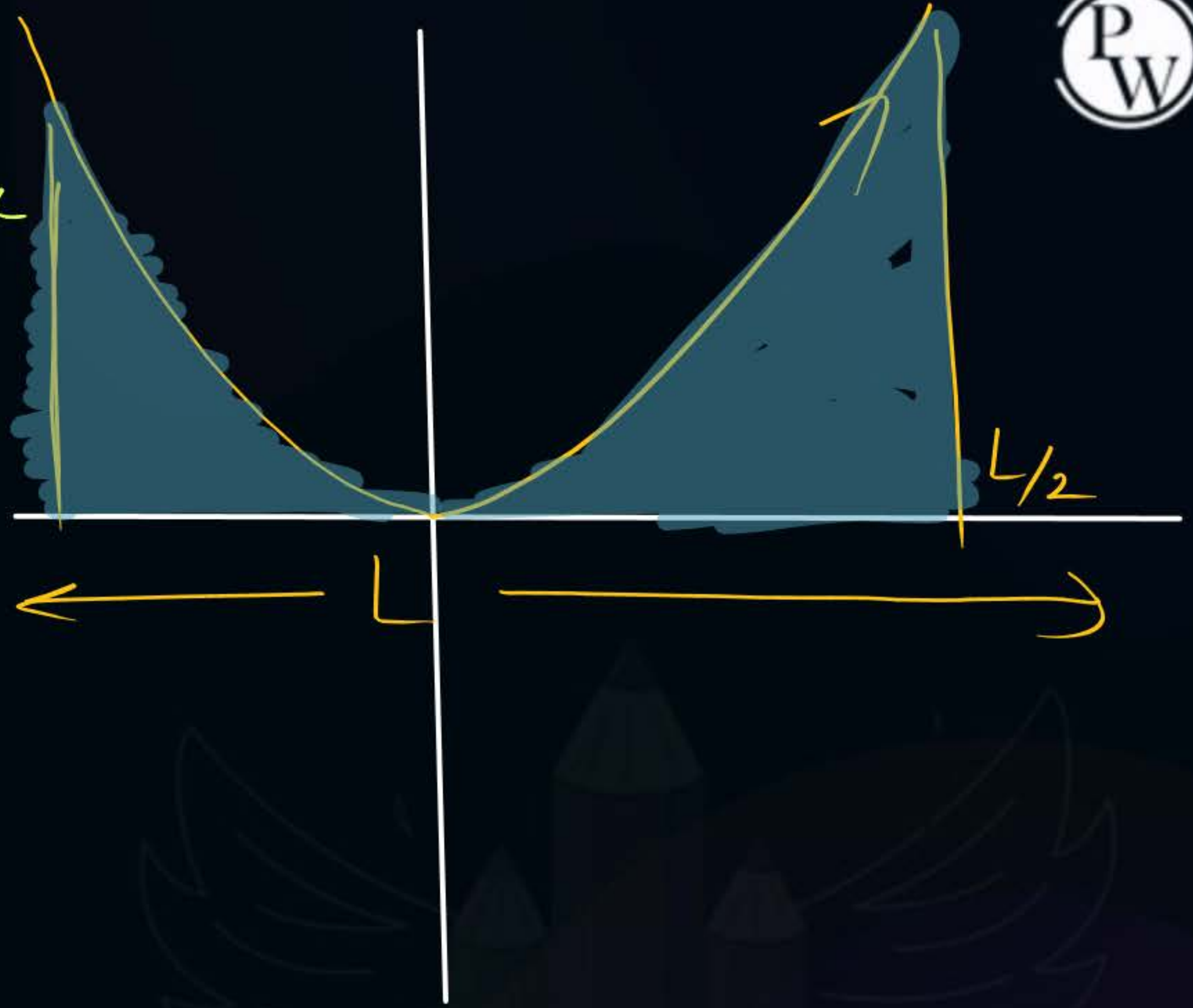
A $\int_0^L \sqrt{1+64\frac{h^2x^2}{L^4}}\,dx$

B $2\int_0^{L/2} \sqrt{1+64\frac{h^3x^2}{L^4}}\,dx$

C $\int_0^{L/2} \sqrt{1+64\frac{h^2x^2}{L^4}}\,dx$

D $2\int_0^{L/2} \sqrt{1+64\frac{h^2x^2}{L^4}}\,dx$

$$= 2\int_0^{\frac{L}{2}} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$



Topic :Double integration

ME 2010

200 - 15

15

15

15

15

#Q. The distance between the origin and the point nearest to it on the surface $\boxed{z^2 = 1 + xy}$ is

Minimum — NEAREST

Farthest — max

A 1

B $\sqrt{3}/2$

C $\sqrt{3}$

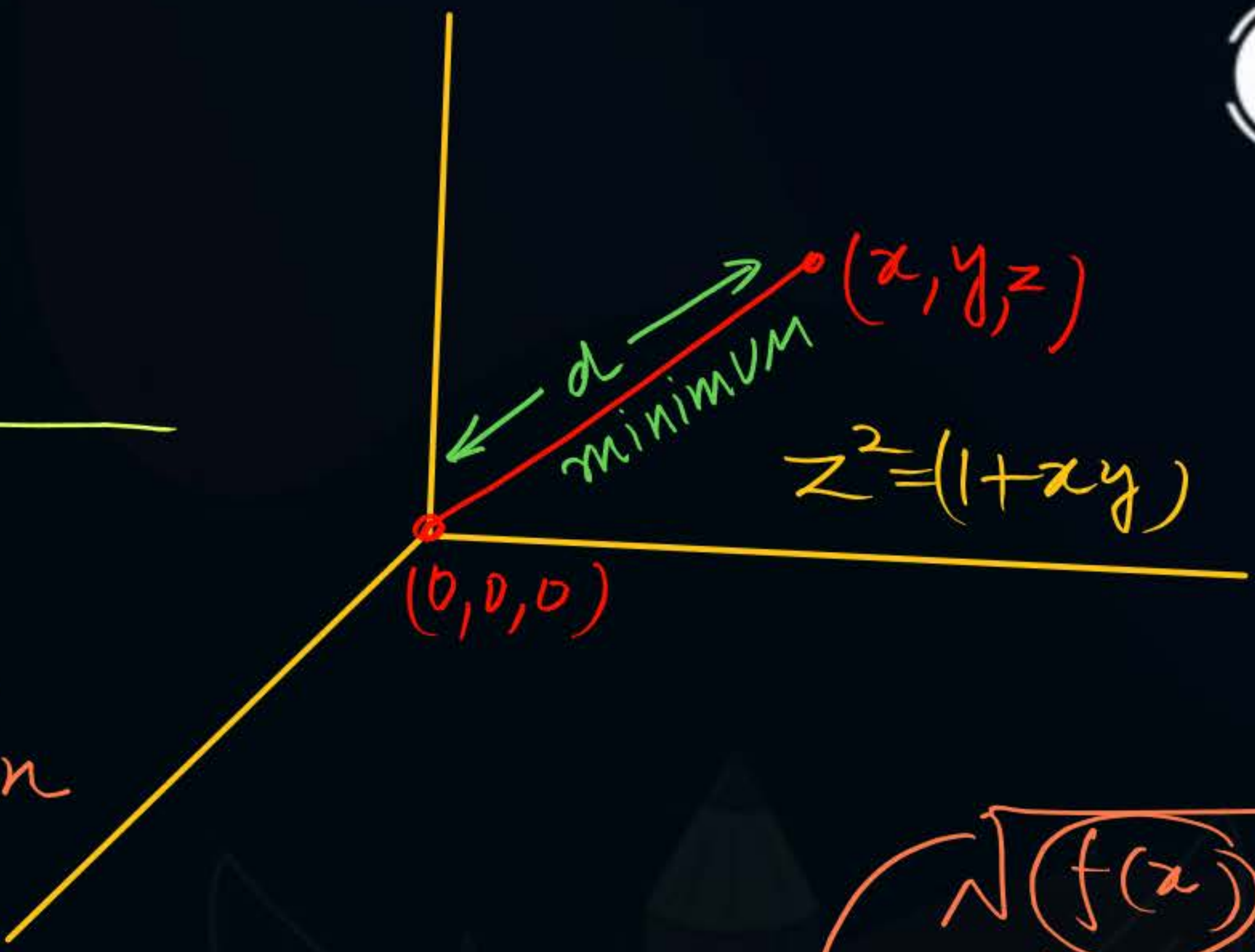
D 2

distance between $(0,0,0)$ to (x,y,z)

$$= d(x,y,z) = \sqrt{\underbrace{x^2+y^2+z^2}_{\text{3 Dimensional}}}$$

$$d(x,y) = \sqrt{(x^2+y^2+1+xy)}$$

$$f(x,y) = \boxed{x^2+y^2+1+xy} \text{ min}$$



$$f(x,y) = x^2+y^2+1+xy$$

$$\frac{\partial f}{\partial x} = 2x+y = 0 \qquad x=0$$

$$\frac{\partial f}{\partial y} = 2y+x = 0 \qquad y=0$$

Stationary Pt

$\sqrt{f(x)}$

min

Square Root min

$$\frac{\partial f}{\partial x} = 2x + y$$

$$\frac{\partial f}{\partial y} = 2y + x$$

Stationary Pt $(0,0)$

Min distance

$$= \sqrt{x^2 + y^2 + 1 + xy}$$

$$= \sqrt{(0)^2 + (0)^2 + 1 + 0 \cdot 0}$$

$$= \sqrt{1} = 1$$

$$\frac{\partial^2 f}{\partial x^2} = r = 2$$

$$\frac{\partial^2 f}{\partial y \partial x} = 1 = s$$

$$\frac{\partial^2 f}{\partial y^2} = 2 = t$$

$r > 0$

$$rt - s^2 = 2 \times 2 - 1$$

$$= 3 > 0$$

(minima)

Topic :Double integration

#Q. Given a function

$f(x, y) = 4x^2 + 6y^2 - 8x - 4y + 8$,

the optimal values of f(x, y) is

A. A minimum equal to 10/3

B. A maximum equal to 10/3

C. A minimum equal to 8/3

D. A maximum equal to 8/3

$f(x,y) = 4x^2 + 6y^2 - 8x - 4y + 8$

Optimal value: max / min

$\frac{\partial f}{\partial x} = 8x - 8$ $\quad 8x - 8 = 0 \quad x = 1$

$\frac{\partial f}{\partial y} = 12y - 4$ $\quad 12y - 4 = 0 \quad y = \frac{1}{3}$

$\frac{\partial^2 f}{\partial x^2} = 8 = r$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = 0 = s$

$\frac{\partial^2 f}{\partial y^2} = 12 = t$

$rt - s^2 = 8 \times 12 - 0 = 96 > 0$

$r > 0$ minima at Pt $\left(1, \frac{1}{3}\right)$

$f\left(1, \frac{1}{3}\right) = 4 \times (1)^2 + 6\left(\frac{1}{3}\right)^2 - 8 \times 1 - 4 \times \frac{1}{3} + 8 = \frac{10}{3}$

THANK - YOU