GATE

ALL BRANCHES

Engineering Mathematics

Multivariable Calculus and Vector Calculus

Discussion Notes (Part-01)

DPP 01

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Topic : Vector Calculus

#Q. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$,then the value of div $\left(\frac{\vec{r}}{r^3}\right)$ is

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad r = |\vec{r}|$

$$\boxed{\mathrm{div}\left(\frac{\vec{r}}{r^3}\right)}$$

$$\frac{\vec{r}}{r^3} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$$

electrostatics

$|\vec{r}| = r = \sqrt{x^2+y^2+z^2}$

A. 1

B. 2

C. 0

D. -1

$$\frac{\vec{r}}{r^3} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{(x^2+y^2+z^2)^{3/2}}$$

$$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

$\text{div } \vec{F}$

$$\text{Vector} \xrightarrow[\text{Div } \vec{F}]{\nabla \cdot \vec{F}} \text{Scaler}$$

$$\nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$F_1 = \frac{x}{(x^2+y^2+z^2)^{3/2}} \qquad F_2 = \frac{y}{(x^2+y^2+z^2)^{3/2}}$$

$$F_3 = \frac{z}{(x^2+y^2+z^2)^{3/2}}$$

$$\nabla \cdot \vec{F} = \text{div } \vec{F} = \frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{\partial}{\partial y}\left[\frac{y}{(x^2+y^2+z^2)^{3/2}}\right] + \frac{\partial}{\partial z}\left[\frac{z}{(x^2+y^2+z^2)^{3/2}}\right]$$

$$\boxed{\nabla \cdot \vec{F} = 0}$$

$$\nabla \cdot \left(\frac{\vec{r}}{r^3}\right) = 0$$

$$\boxed{\text{Option (c)}}$$

$$\nabla \cdot \boxed{\frac{\vec{r}}{r^3}} = 0$$

Topic : Vector Calculus

#Q. For $a > 0, b > 0$ let $\vec{F} = \frac{x\hat{j} - y\hat{i}}{b^2x^2 + a^2y^2}$ be a planner vector field. Let $C = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = a^2 + b^2|$ be the circle oriented anti-clockwise. Then $\oint \vec{F} \cdot d\vec{r} =$

$\vec{F} = \frac{x\hat{j} - y\hat{i}}{b^2x^2 + a^2y^2}$ $x^2 + y^2 = (a^2 + b^2)$

Line integral $\oint \vec{F} \, d\vec{r}$

A. $2\pi / ab$

B. 2π

C. $2\pi ab$

D. 0

$$\vec{F} = \frac{x\hat{j} - y\hat{i}}{b^2x^2 + a^2y^2}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\oint \vec{F}\cdot d\vec{r} = \oint \frac{x\hat{j} - y\hat{i}}{b^2x^2 + a^2y^2}\left(dx\,\hat{i} + dy\,\hat{j}\right)$$

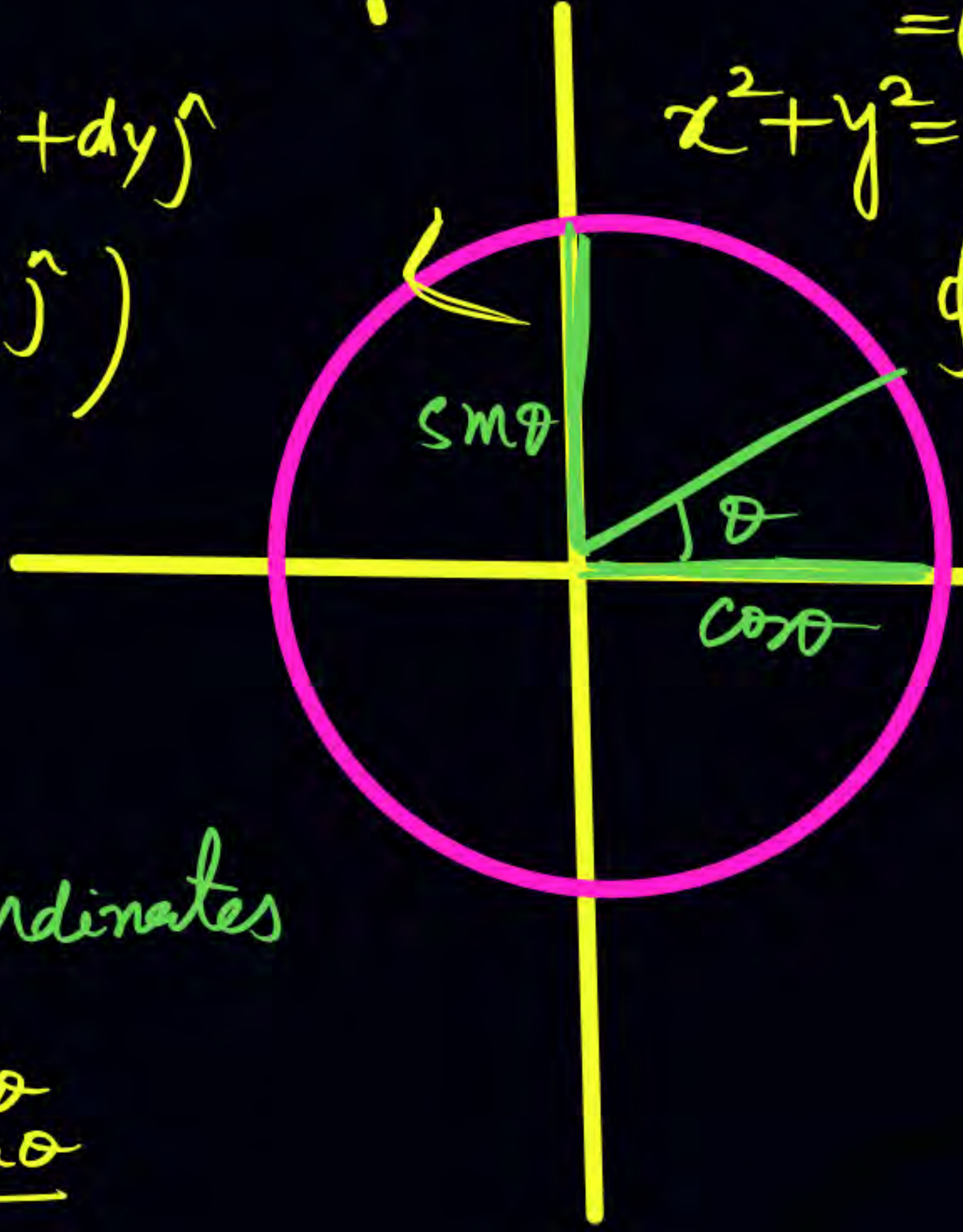
$$\oint \vec{F}\cdot d\vec{r} \Rightarrow \oint \frac{x\,dy - y\,dx}{b^2\underline{x^2} + a^2\underline{y^2}}$$

$$= \left(\sqrt{a^2+b^2}\right)^2$$

$$x^2 + y^2 = (a^2 + b^2)$$

$$\oint \vec{F}\cdot d\vec{r}$$

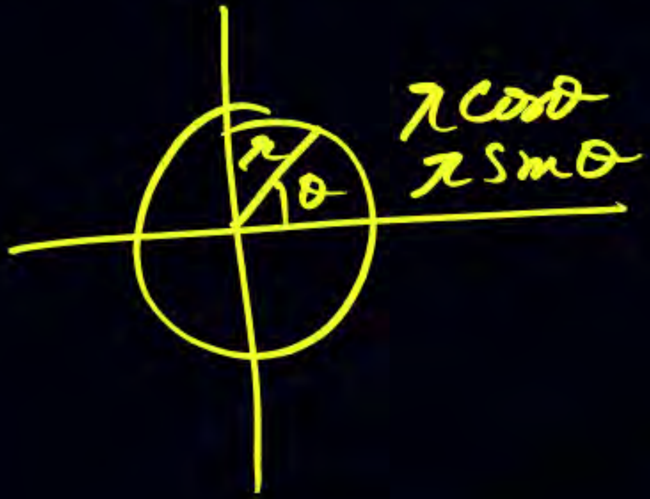
$$\sqrt{a^2+b^2}$$



Techiques / strategy $\longrightarrow$ change The Polar co-ordinates

$x = r\cos\theta$

$y = r\sin\theta$



$x = \sqrt{a^2+b^2}\,\cos\theta$

$y = \sqrt{a^2+b^2}\,\sin\theta$

$$x = \sqrt{a^2+b^2}\cos\theta \qquad y = \sqrt{a^2+b^2}\sin\theta$$

$$dx = -\sqrt{a^2+b^2}\sin\theta\, d\theta$$

$$dy = \sqrt{a^2+b^2}\cos\theta\, d\theta$$

Line Integral $\oint \vec{F}\, d\vec{r} = \oint \dfrac{x\,dy - y\,dx}{b^2x^2 + a^2y^2}$

$$x\,dy - y\,dx = \left(\sqrt{a^2+b^2}\cos\theta\right)\left(\sqrt{a^2+b^2}\cos\theta\right) + \left(\sqrt{a^2+b^2}\sin\theta\right)\sqrt{a^2+b^2}\sin\theta$$

$$= (a^2+b^2)\cos^2\theta + (a^2+b^2)\sin^2\theta$$

$$b^2x^2 + a^2y^2 = b^2\cdot(a^2+b^2)\cos^2\theta + a^2(a^2+b^2)\sin^2\theta$$

$$= (a^2+b^2)\left[b^2\cos^2\theta + a^2\sin^2\theta\right]$$

$$\oint \vec{F}\, d\vec{r} = \int \frac{\cancel{[a^2+b^2]}\,\underbrace{[\cos^2\theta + \sin^2\theta]}_{1}}{\cancel{(a^2+b^2)}\,[b^2\cos^2\theta + a^2\sin^2\theta]}\, d\theta$$

$$\oint F\, d\vec{r} = \oint \frac{1}{b^2\cos^2\theta + a^2\sin^2\theta}\, d\theta$$

$$= \oint_0^{2\pi} \frac{1}{b^2\cos^2\theta + a^2\sin^2\theta}\, d\theta$$

Hint divide via $\cos^2\theta$

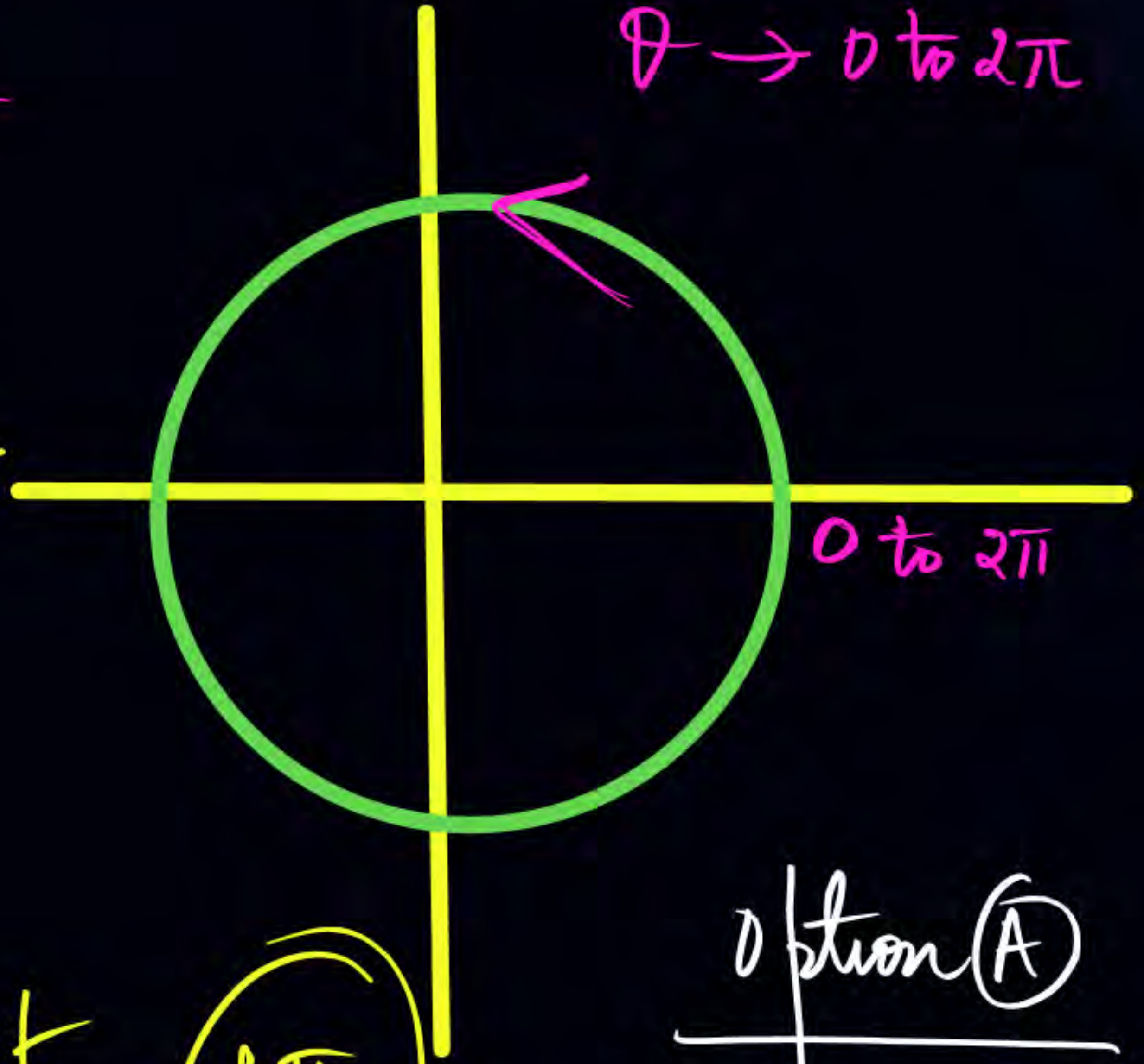
$$= \oint_0^{2\pi} \frac{\sec^2\theta}{\frac{b^2\cos^2\theta}{\cos^2\theta} + \frac{a^2\sin^2\theta}{\cos^2\theta}}\, d\theta$$

$$= 4\oint_0^{\frac{2\pi}{4}} \frac{\sec^2\theta}{b^2 + a^2\tan^2\theta}$$

$$\oint \vec{F}\, d\vec{r} = 4\oint_0^{\pi/2} \frac{\sec^2\theta}{b^2 + a^2\tan^2\theta}$$

$\tan\theta = t$

$$= \frac{2\pi}{ab}$$



Option (A)

Topic : Vector Calculus

#Q. If $\vec{F}(x, y) = (3x - 8y)\hat{\imath} + (4y - 6xy)\hat{\jmath}$ for $(x, y) \in \mathbb{R}^2$, then $\oint_C \vec{F} \cdot d\vec{r}$, where C is the boundary of the triangular region bounded by the lines $x = 0, y = 0$ and $x + y = 1$ oriented in the anti-clockwise direction is

A 5/2

B 3

C 4

D 5

$$\vec{F}(x,y) = (3x-8y)\hat{i} + (4y-6xy)\hat{j} \text{ for } (x,y) \in \mathbb{R}^2$$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint (3x-8y)\hat{i} + (4y-6xy)\hat{j}\,[dx\,\hat{i} + dy\,\hat{j}]$$

$$= \oint \underbrace{(3x-8y)}_{M}\,dx + \underbrace{(4y-6xy)}_{N}\,dy$$

$M = (3x-8y)$ $\quad \dfrac{\partial M}{\partial y} = -8$

$N = (4y-6xy)$ $\quad \dfrac{\partial N}{\partial x} = -6y$

Using Green's Theorem:

$$\oint M\,dx + N\,dy = \iint_{\text{region}} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx = \iint (-6y + 8)\,dy\,dx$$

$$= \int_x \int_y (8 - 6y)\,dy\,dx$$

X–Limit using vertical strip:

$x \to 0$ to 1

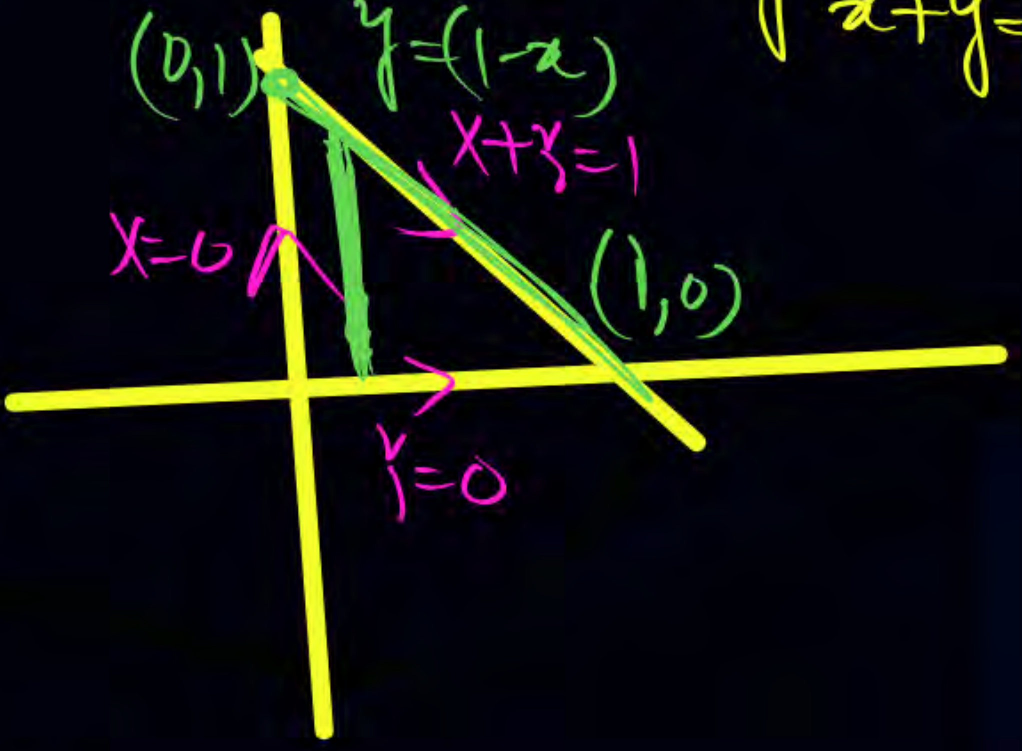
Y–Limit using vertical strip:

$y \to 0$ to $(1-x)$

$x = 0$, $y = 0$, $x + y = 1$

$$= \int_0^1 \int_0^{(1-x)} (8 - 6y)\,dy\,dx$$

$$= \int_0^1 dx \int_0^{1-x} (8 - 6y)\,dy = 5$$



Topic : Vector Calculus

#Q. If $F(x, y, z) = xy^2 + 3x^2 - z^3$,then the value of $\boxed{\overset{\text{grad } f}{\nabla F(x, y, z)}}$ at (2, –1, 4) is equal to

A. $13i - 4j - 48k$

B. $i - 4j - k$

C. $13i + j - 6k$

D. $-13i + 4j - 6k$

$F(x, y, z) = xy^2 + 3x^2 - z^3$

$\nabla F = \hat{i}\frac{\partial F}{\partial x} + \hat{j}\frac{\partial F}{\partial y} + \hat{k}\frac{\partial F}{\partial z}$

$= \hat{i}\frac{\partial}{\partial x}(xy^2 + 3x^2 - z^3) + \hat{j}\frac{\partial}{\partial y}(xy^2 + \cancelto{0}{3x^2} - \cancelto{0}{z^3})$

$+ \hat{k}\frac{\partial}{\partial z}(xy^2 + 3x^2 - z^3)$

$\text{grad } F = \nabla F = \hat{i}\left[y^2 + 6x - 0\right] + \hat{j}\left[2xy\right] + \hat{k}\left[-3z^2\right]$

$(\nabla F)_{(2,-1,4)} = \hat{i}\left[1 + 12\right] + \hat{j}\left[-4\right] + \hat{k}\left[-16 \times 3\right]$

$= 13\hat{i} - 4\hat{j} - 48\hat{k}$

Topic : Vector Calculus

#Q. Let F be a vector field given by $\vec{F}(x,y,z) = -y\hat{\imath} + 2xy\hat{\jmath} + z^3\hat{k}$ for $(x,y,z) \in \mathbb{R}^3$ .If C is the curve of intersection of the surfaces $\boxed{x^2 + y^2 = 1}$ and $y + z = 2$, then which of the following is (are) equal to $\left|\oint_C \vec{F} \cdot d\vec{r}\right|$?

$x^2+y^2=1$
$y+z=2$

A $\int_0^{2\pi}\int_0^1 (1 + 2r\sin\theta) r\,dr\,d\theta$

B $\int_0^{2\pi}\left(\frac{1}{2} + \frac{2}{3}\sin\theta\right) d\theta$

C $\int_0^{2\pi}\int_0^1 (1 + 2r\sin\theta)\,dr\,d\theta$

D $\int_0^{2\pi}(1 + \sin\theta)\,d\theta$

$$\vec{F}(x,y,z) = -y\hat{i} + 2xy\hat{j} + z^3\hat{k}$$

$$\oint \vec{F}\,d\vec{r} = \oint(-y\hat{i} + 2xy\hat{j} + z^3\hat{k})(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$\Rightarrow \oint -y\,dx + 2xy\,dy + z^3\,dz$$

$$\Rightarrow \oint -y\,dx + 2xy\,dy + (2-y)^3(-dy)$$

$$\Rightarrow \oint -y\,dx + [2xy - (2-y)^3)\,dy]$$

$y + z = 2$

$z = 2 - y$

If whole expression convert to the x, y Then apply The green's Theorem

$$\boxed{dz = -dy}$$

apply green's Theorem

$M = -y$ $\qquad \frac{\partial M}{\partial y} = -1$

$N = 2xy - (2-y)^3$ $\qquad \frac{\partial N}{\partial x} = 2y$

$$\oint M\,dx + N\,dy = \iint_{\text{region}} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$$

$$= \boxed{\iint (2y+1)\,dy\,dx}$$

Polar-coordinates

$x^2+y^2=1$ $x = r\cos\theta$

Circle $y = r\sin\theta$

$x = \cos\theta$

$y = \sin\theta$

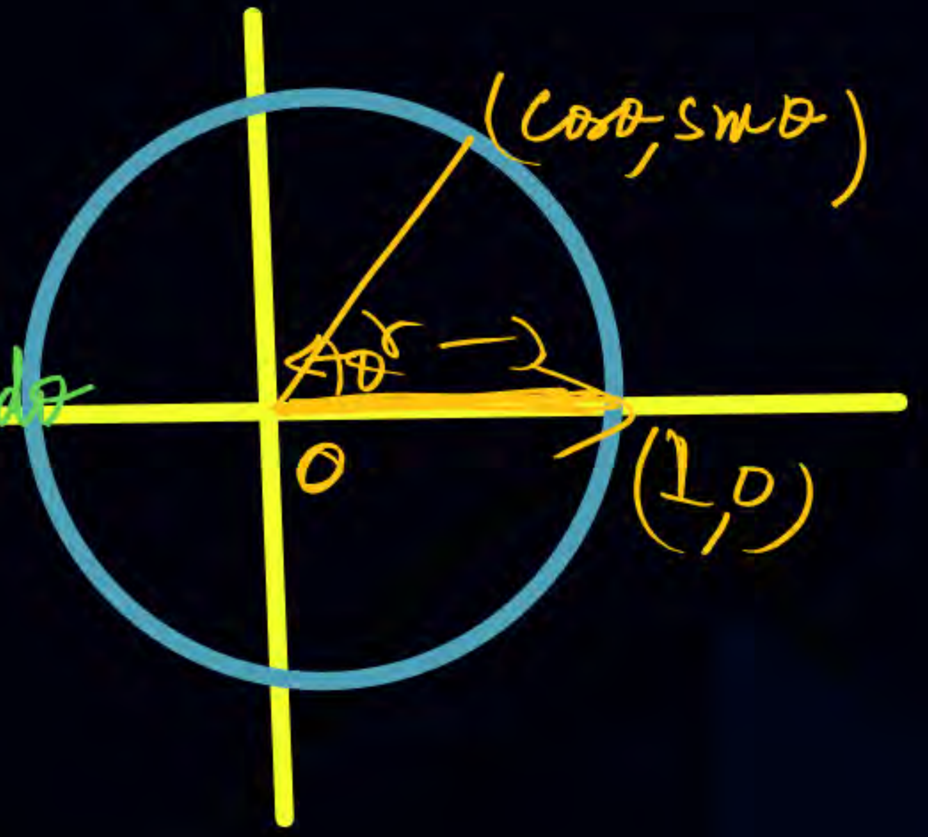
Using Polar co-ordinates

$x = r\cos\theta$ $y = r\sin\theta$

$$\iint f(x,y)\,dy\,dx$$

$$\boxed{dy\,dx = r\,dr\,d\theta}$$

$$= \iint f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$



$x = \cos\theta$ $y = \sin\theta$

$dy\,dx = (r)\,dr\,d\theta$

$= r\,dr\,d\theta$

$$\boxed{dy\,dx = r\,dr\,d\theta}$$

$$\iint (2y+1)\,dy\,dx = \iint (2r\sin\theta + 1)\, r\,dr\,d\theta$$

$r \to 0$ to 1

$$= \int_0^1 \int_0^{2\pi} (1 + 2r\sin\theta)\, r\,dr\,d\theta$$

$$\int_x \int_y \longrightarrow \int_r \int_\theta$$

$$= \int_0^{2\pi} d\theta \left[\frac{r^2}{2} + \frac{2r^3}{3}\sin\theta \right]_0^1$$

$r \to 0$ to 1

$\theta \to 0$ to 2π

$$= \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3}\sin\theta \right) d\theta$$

Change of variables

$x = r\cos\theta$

$y = r\sin\theta$

$$\boxed{\iint f(x,y)\,dy\,dx = \iint f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta}$$

$$\iint f(x,y)\,dy\,dx = \iint f(r\cos\theta, r\sin\theta)\,|J|\,dr\,d\theta$$

$$\Rightarrow \iint f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

$$\text{Jacobian } J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}$$

determinat

$$= r\cos^2\theta + r\sin^2\theta = r(\sin^2\theta + \cos^2\theta)$$

$$|J| = r$$

For Any variables:

$$\iint f(x,y)\,dy\,dx = \iint f(u,v)\,|J|\,du\,dv$$ (Change of variables)

Topic : Vector Calculus

(A) (B) (C)
MSQ

MSQ

#Q. If $\vec{F}(x,y,z) = (2x+3yz)\hat{\imath} + (3xz+2y)\hat{\jmath} + (3xy+2z)\hat{k}$ for $(x,y,z) \in \mathbb{R}^3$, then which among the following is (are) true ?

A $\nabla \times \vec{F} = \vec{0}$ ✓

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x+3yz & 3xz+2y & 3xy+2z \end{vmatrix} = \vec{0} \qquad \nabla \times \vec{F} = \vec{0} \ ✓$$

B $\oint_C \vec{F} \cdot d\vec{r} = 0$ along any simple closed curve C

$$\oint \vec{F} \cdot d\vec{r} = \iint (\nabla \times \vec{F}) \cdot \hat{n}\, dS = \iint \vec{0}$$

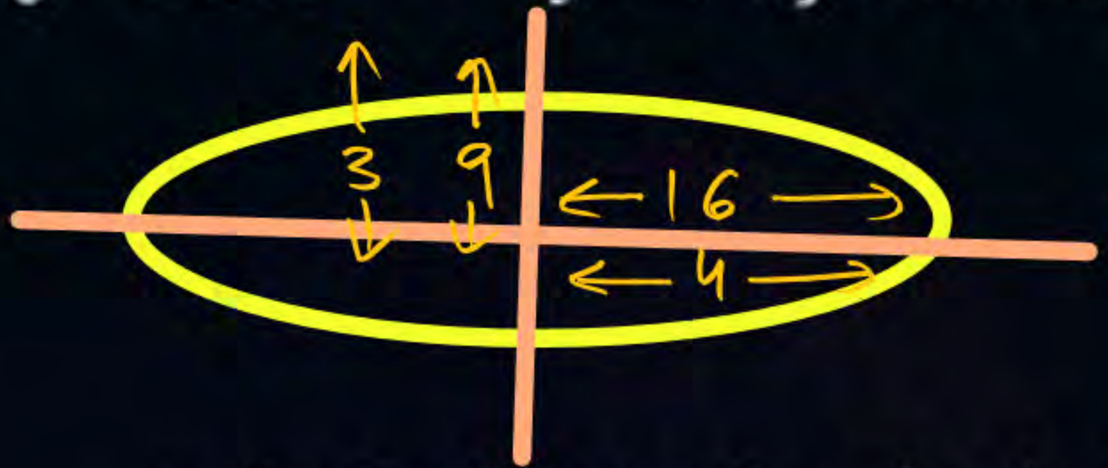
C There exist a scalar function $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla \cdot \vec{F} = \phi_{xx} + \phi_{yy} + \phi_{zz}$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2x+3yz) + \frac{\partial}{\partial y}(3xz+2y) + \frac{\partial}{\partial z}(3xy+2z)$$
$$= 2+2+2 = 6$$

D $\nabla \cdot \vec{F} = 0$ ✗ $\boxed{\operatorname{div} \vec{F} = 6}$

Topic : Vector Calculus

#Q. Let $\vec{F} = -y\hat{\imath} + x\hat{\jmath}$ and let C be the ellipse



$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

oriented counter clockwise. Then the value of $\oint_C \vec{F} \cdot d\vec{r}$) round off to 2 decimal place) is ____

$\vec{F} = -y\hat{i} + x\hat{j}$

$\oint \vec{F} \cdot d\vec{r} = \int -y\,dx + x\,dy$

using green's Theorem.

$M = -y \quad N = x$

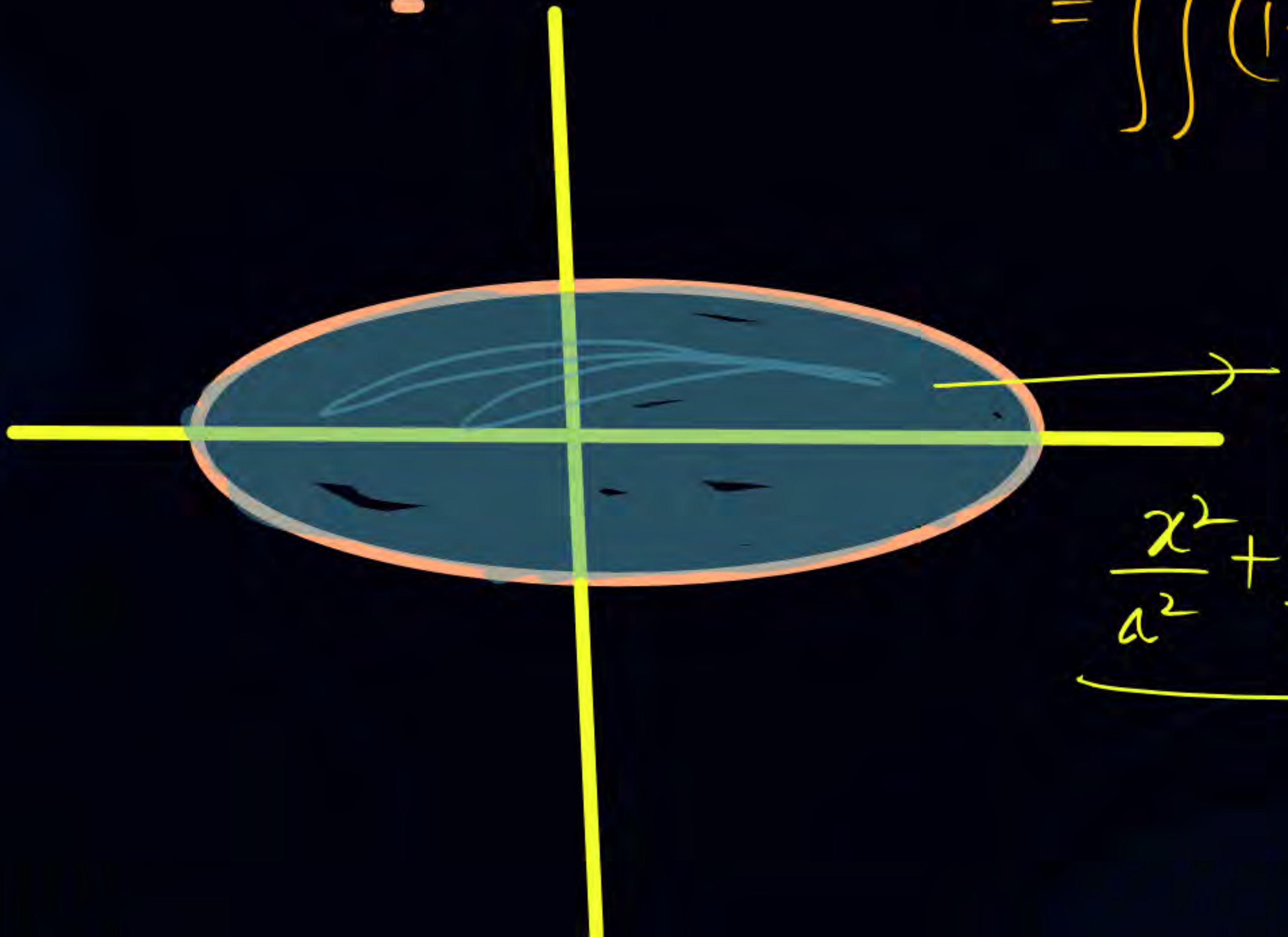
$\frac{\partial M}{\partial y} = -1 \quad \frac{\partial N}{\partial x} = 1$

Using green's Theorem

$$\oint M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$$

$$= \iint (1+1)\,dy\,dx = 2\iint dy\,dx$$

$$= 2 \times \text{Area of ellipse}$$



Area of ellipse $= \pi ab$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$= 2 \times \pi ab = 2 \times 3.14 \times 4 \times 3$$

$$= 24\pi = 24 \times 3.14$$

THANK - YOU