GATE (ALL BRANCHES)

Engineering Mathematics

Differential Equation + Partial differential

Lecture No. 03

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TOPICS TO BE COVERED

01 Problems based on Homogeneous Differential Equation

02 Non Homogeneous Differential Equation

03 Questions based on Non Homogeneous Differential Equation

04 Linear Differential Equation Imp

Reducible to Homogenous form :

$$f(x,y) = \frac{ax+by+c}{a'x+b'y+c'}$$

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$$

Non Homogenous.

Check wheather this function Homogenous or Not

$$f(kx,ky) = \frac{a\cdot kx+b\cdot ky+c}{a'kx+b'ky+c'} = \frac{k\left[\boxed{ax+by}+c/k\right]}{k\left[\boxed{a'x+b'y}+c'/k\right]}$$

Homogenous

Homogenous

Non Homogenous

Remove The Non-Homogenous Part

Convert to The Homogenous Eqn

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$$

Put $\begin{cases} x = X+h \\ y = Y+k \end{cases}$ $\quad \frac{dy}{dx} = \frac{dY}{dX}$

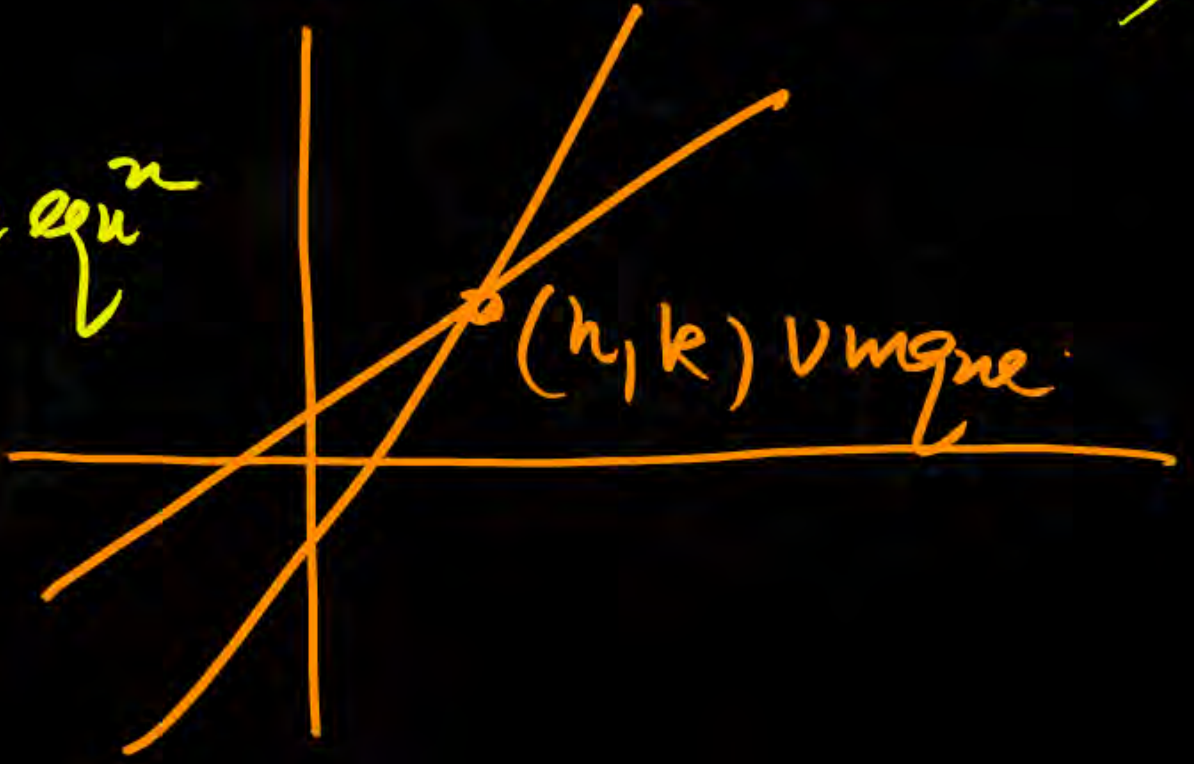
$x = X+h$
$y = Y+k$
$dx = dX$
$dy = dY$

where h, k Are constants and X, Y Are function of x, y

$$\frac{dY}{dX} = \frac{a(X+h)+b[Y+k]+c}{a'(X+h)+b'[Y+k]+c'} = \frac{aX+bY+(\boxed{ah+bk+c})}{a'X+b'Y+(\boxed{a'h+b'k+c'})}$$

$\left.\begin{array}{l} ah+bk+c=0 \\ a'h+b'k+c'=0 \end{array}\right]$ both are straight line equ^n

Solve The value of h, k



Homogenous

$$\frac{dY}{dX} = \frac{aX+bY+0}{a'X+b'Y+0} = \frac{aX+bY}{a'X+b'Y}$$

$$\frac{dy}{dx} = \frac{ax+by}{a'x+b'y}$$

If Equ^n is Homogenous Then
Put $y = vx$

$$v + x\frac{dv}{dx} = \frac{ax+bvx}{a'x+b'vx} = \frac{(a+bv)}{(a'+b'v)}$$

Now Seprate The variable and get The Solution of D.E

- ✓ Linear D.E
- ✓ Reduced to linear D.E
- ✓ Variable seprable.
- ✓ Reduced to V.S
- ✓ Homogenous
- ✓ Reduced to Homo

Linear Differential Equation :

$y = f(x)$

Dep. variable

Independent variable

$\frac{dy}{dx} \cdot y$ = Non Linear

DX (D) =

$x \frac{dx}{dy}$ = Non-Linear

Dependent

derivative

$x \frac{dy}{dx}$ = Linear

$y \frac{dx}{dy}$ = Linear

$\sin x \frac{dy}{dx}$ = Linear

$\sin y \frac{dy}{dx}$ = Non linear

$t^2 \frac{dt}{dx}$ = Non linear.

(A) degree always 1

$\left(\frac{dy}{dx}\right)$ Degree

(B) (Dependent var) × (Dependent vari derivative)

C) No Non Linear function in dep. variable is Not Present in the eqn.

$t^2 \frac{dx}{dt}$ = Linear

$$\underbrace{\frac{dy}{dx}+y=e^{x}}_{D\cdot E}$$

Using Hit and trial method → Not Integrate → D·E Not solved

A) ✓ Var. separable form.
B) ✓ Reduced var. sep.
(C) ✓ Homogeneous.
(D) ✓ Reduced to Homogeneous.

Using Hit and trial method $\boxed{\text{multiply both Sides } e^{x}}$

Important key Point

multiply करने पर D·E solve

$$\underline{\frac{dy}{dx}e^{x}+ye^{x}}=xe^{x}$$

$$\frac{d}{dx}[ye^{x}]=\underbrace{\frac{dy}{dx}e^{x}+y\cdot e^{x}}=\frac{d}{dx}(ye^{x})$$

I II

$$\Rightarrow \frac{d}{dx}(ye^x) = xe^x$$

both sides Integrate It

$$\Rightarrow \int \frac{d}{dx}(ye^x) = \int xe^x$$

$$\Rightarrow \boxed{ye^x = (x-1)e^x + c}$$ solution of D.E

D

↓

I

↓

D.E

Hit and trial method ⟶ factor multiply = ?

D.E solve

Linear D.E :

$$\frac{dy}{dx} + Py = Q$$ Where P and Q Are Function of x only.

Integrating factor $= e^{\int P\,dx}$

Linear form $ax+by+c=0$

$$\Rightarrow e^{\int P\,dx}\cdot\frac{dy}{dx} + Pye^{\int P\,dx} = Qe^{\int P\,dx}$$

$$\Rightarrow \frac{d}{dx}\left(ye^{\int P\,dx}\right) = Qe^{\int P\,dx}$$

both sides Integrate It

$$\int\frac{d}{dx}\left(ye^{\int P\,dx}\right) = \int Qe^{\int P\,dx}dx$$

Solution of this D.E

$$\boxed{y\cdot(I\cdot F) = \int(RHS)(I\cdot F)\,dx + C}$$

sol^n

$$\frac{dy}{dx} + Py = Q$$

$$\frac{dx}{dy} + Px \Rightarrow Q$$ Where P and Q Are function of y only

Integrating Factor $= e^{\int P\,dy}$

solution

$$\boxed{x(I \cdot F) = \int (RHS)(I \cdot F)\,dy + C}$$ solution of D.E

Q. Questions

#Q. The solution of the equation $\frac{dQ}{dt} + Q = 1$ with Q = 0 at $t = 0$ is

(a) $Q(t) = e^{-t} - 1$

(b) $Q(t) = 1 + e^{-t}$

(c) $Q(t) = 1 - e^{t}$

(d) $Q(t) = 1 - e^{-t}$

$\frac{dQ}{dt} + MQ = N$

where M and N Are function of t.

$\frac{dQ}{dt} + Q = 1$ with $Q = 0$ at $t = 0$

$M = 1 \quad N = 1$

$I.F = e^{\int 1\,dt} = e^{t}$

$Q \cdot e^{t} = \int e^{t} \cdot 1\,dt$

$Qe^{t} = e^{t} + c$

$0 = e^{0} + c \quad c = -1$

$Qe^{t} = e^{t} - 1$

$Q = 1 - e^{-t}$

Questions

#Q. Let f : R →R be a continuous function which satisfies

$$f(x) = \int_0^x f(t)dt$$, then the value of $f(\ln 5)$

Apply Newton Leibnitz Rule

$f'(x) = f(x)\frac{d}{dx}\cdot 1$

$f'(x) = f(x)$

$\frac{dy}{dx} = y$

$f'(x) = \frac{dy}{dx}$

$y = f(x)$

$\int \frac{dy}{y} = \int dx$

$= \ln y = x + c$

$\boxed{y = Ae^x}$ A = unknown

$f(x) = \int_0^x f(t)dt$

$x=0 \quad f(0) = \int_0^0 f(t)dt = 0$

$$f(x) = y = Ae^{x}$$

$$A = 0$$

Solution $\boxed{y = 0}$

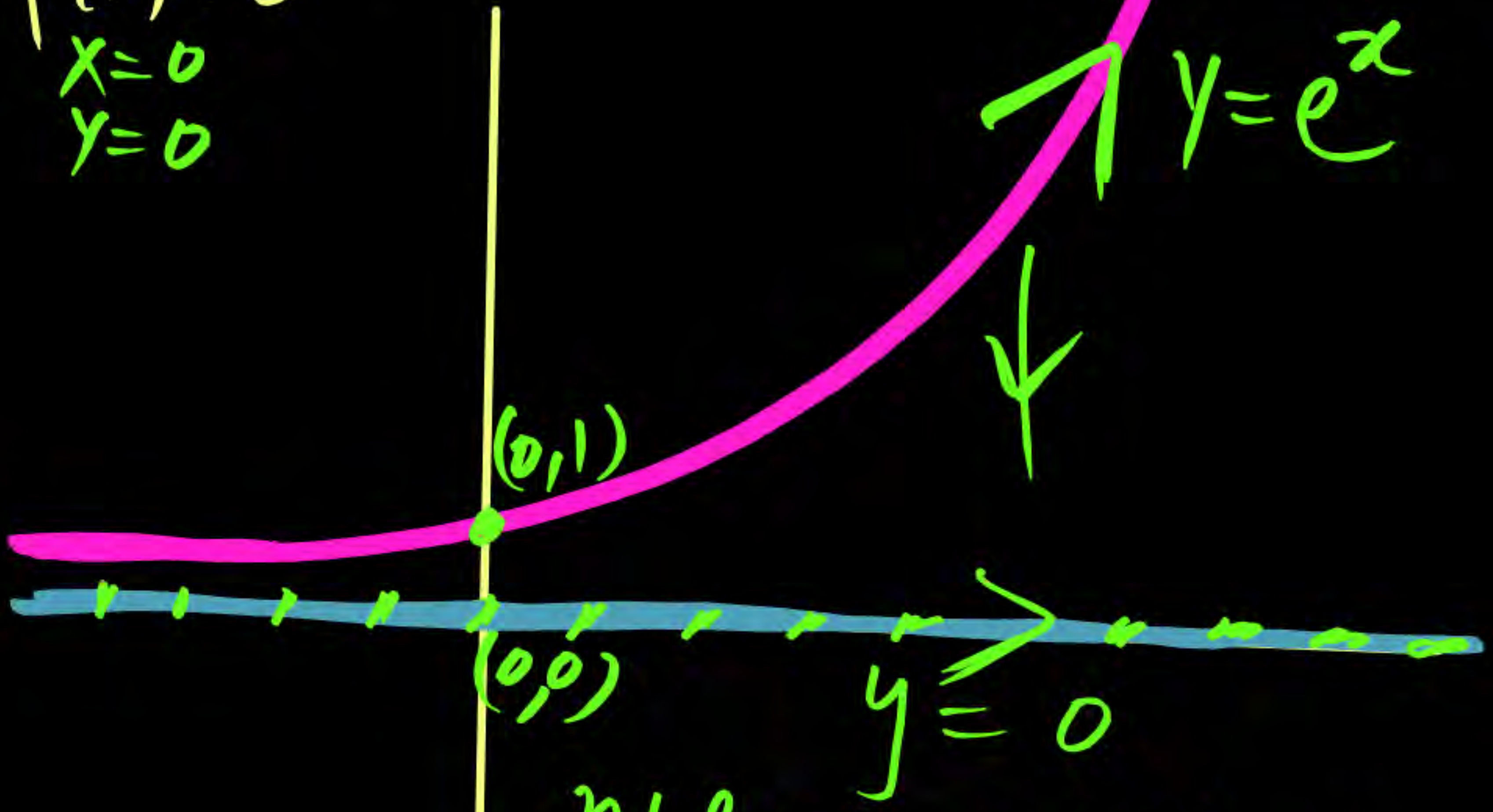
$$\boxed{y(\ln 5) = 0}$$

Ans

$$f(0) = 0$$

$$x = 0$$

$$y = 0$$



$$y(\ln 5) = 0 \qquad y(\ln 2) = 0$$

$$y(81) = 0$$

$$y(9) = 0$$

Questions

#Q. $\frac{dy}{dx}=\frac{2+(x-y)}{3+2x-2y}$

✓ Solve

✓ H.W

Do yourself

$$\frac{dy}{dx}=\frac{2+(x-y)}{3+2(x-y)}$$

Put
$x-y=t$
And solve
The D.E

Q. Questions

#Q. If $x^2\left(\frac{dy}{dx}\right)+2xy=\frac{2\ln x}{x}$ and $y(1)=0$ then what is $y(e)$?

(a) e

(b) 1

(c) $\frac{1}{e}$

(d) $\frac{1}{e^2}$

$$x^2\frac{dy}{dx}+2xy=\frac{2\ln x}{x}$$

$$=\frac{dy}{dx}+\frac{2x}{x^2}y=\frac{2\ln x}{x^3}$$

$$\Rightarrow \frac{dy}{dx}+\frac{2y}{x}=\frac{2\ln x}{x^3}$$ form

$P=\frac{2}{x}$ $Q=\frac{2\ln x}{x^3}$

$I.F=x^2$

$\frac{dy}{dx}+Py=Q$

P and Q function of x

$$I.F=e^{\int P dx}$$

$$=e^{\int\frac{2}{x}dx}$$

$$=e^{2\ln x}$$

$$=e^{\ln x^2}=x^2$$

$$y\cdot x^2=\int\frac{2\ln x}{\cancel{x^3}x}\cdot\cancel{x^2}=2\int\frac{\ln x}{x}\,dx$$

Put $\ln x=t$

$$\frac{dx}{x}=dt$$

$$\int\frac{\ln x}{x}=\int t\,dt$$

$$=\frac{t^2}{2}+c$$

$$=\frac{(\ln x)^2}{2}+c$$

$$y\,x^2=2\frac{(\ln_e x)^2}{2}+c$$

Apply Intial condition $y(1)=0$

$$0=\frac{\boxed{\ln_e(1)}^2}{2}+c \qquad \boxed{c=0}$$

$$y\,x^2=\frac{(\ln_e x)^2}{\cancel{2}}\cancel{2}+0$$

$$y=\frac{(\ln_e x)^2}{x^2}$$

$$y(e)=\frac{(\ln_e e)^2}{e^2}=\frac{1}{e^2}$$

Q. Questions

#Q. The solution of the differential equation $\frac{dy}{dx}+2xy=e^{-x^2}$ with $y(0) = 1$ is

(a) $(1+x)e^{x^2}$

(b) $(1+x)e^{-x^2}$

(c) $(1-x)e^{x^2}$

(d) $(1-x)\,e^{-x^2}$

$P = 2x \quad Q = e^{-x^2} \quad I.F = e^{\int 2x\,dx} = e^{x^2}$

$y e^{x^2} = \int e^{x^2} \cdot e^{-x^2}\,dx$

$y e^{x^2} = \int dx$

$y e^{x^2} = x + c$

$1 e^{0} = 0 + c$

$c = 1$

$y e^{x^2} = x + 1$

$y = \frac{(x+1)}{e^{x^2}}$

$y = x e^{-x^2} + e^{-x^2}$

$= e^{-x^2}(x+1)$

Q. Questions

#Q. The solution of $x\frac{dy}{dx}+y=x^4$ with condition $y(1)=\frac{6}{5}$

(a) $y=\frac{x^4}{5}+\frac{1}{x}$

(b) $y=\frac{4x^4}{5}+\frac{4}{5x}$

(c) $y=\frac{x^4}{5}+1$

(d) $y=\frac{x^5}{5}+1$

$$x\frac{dy}{dx}+y=x^4$$

$$\frac{dy}{dx}+\frac{y}{x}=\frac{x^4}{x}$$

$$\frac{dy}{dx}+\frac{y}{x}=x^3$$

$$P=\frac{1}{x} \quad Q=x^3$$

Do yourself

Slide-9

Q. Questions

$y = f(x)$
$y' = dy/dx = f'(x)$

#Q2. Let f : [1, ∞] → (2, ∞) be a differentiable function such that f(1) = 2

$6\int_1^x f(t)dt = 3x\,f(x) - x^3$ $= 6$

Then the value of $f(2)$, $\forall x \geq 1$ Ans

using Leibnitz Rule.

$6f(x)\cdot\frac{d}{dx}(x) = 3[x\cdot f'(x) + f(x)\cdot 1] - 3x^2$

$6f(x) = 3xf'(x) + 3f(x) - 3x^2$

$2f(x) = xf'(x) + f(x) - x^2$

$xf'(x) + f(x) - 2f(x) = x^2$

$xf'(x) - f(x) = x^2$

$f'(x) - \frac{f(x)}{x} = \frac{\cancel{x^2}}{\cancel{x}} = x$

$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = x$

$P = -\frac{1}{x}$ $Q = x$

$I.F = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

Slide-12

Q. Questions

#Q3. Let f(x) be a differentiable function on the interval $(0, \infty)$ such that $f(1) = 1$

$$\lim_{t\to x} \frac{t^2 f(x) - x^2 f(t)}{t-x} = 1 \text{ for } x > 0$$

(a) $\frac{1}{3x} + \frac{2x^2}{3}$

(b) $-\frac{1}{3x} + \frac{4x^2}{3}$

(c) $-\frac{1}{x} + \frac{2}{x^2}$

(d) $\frac{1}{x}$

t → variable
x → const

$\to 0$

$\to 0$

$$\lim_{t\to x} \frac{t^2 f(x) - x^2 f(t)}{t-x} = 1$$

Using L-Hospital Rule.

$$\Rightarrow \lim_{t\to x} \frac{2t f(x) - x^2 f'(t)}{1-0} = 1$$

$$= \lim_{t\to x} \frac{2t f(x) - x^2 f'(t)}{1} = 1$$

$$= 2x f(x) - x^2 f'(x) = 1$$

$$2xf(x) - x^2 f'(x) = 1$$

$$\Rightarrow 2xy - x^2 \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow -x^2 \frac{dy}{dx} + 2xy = 1$$

$$\Rightarrow x^2 \frac{dy}{dx} - 2xy = -1$$

$$\Rightarrow \frac{dy}{dx} - \frac{2xy}{x^2} = -\frac{1}{x^2}$$

$$\Rightarrow \boxed{\frac{dy}{dx} - \frac{2y}{x} = -\frac{1}{x^2}}$$

Do yousef

$$y = f(x)$$

$$\frac{dy}{dx} = f'(x)$$

$$\frac{dy}{dx} - \frac{2y}{x} = -\frac{1}{x^2}$$

This is linear D.E form

$$P = -\frac{2}{x} \quad Q = -\frac{1}{x^2}$$

$$I.F = e^{\int P\,dx} = e^{\int -\frac{2}{x}\,dx}$$

$$= e^{-2\ln x}$$

$$= e^{-\ln x^2} = \frac{1}{x^2}$$

Q. Questions

#Q4. If y(t) is a solution of $(1+t)\frac{dy}{dt} - ty = 1$ and $y(0) = -1$. Then y(1) is equal to

(a) $\frac{1}{2}$

(b) $e + \frac{1}{2}$

(c) $e - \frac{1}{2}$

(d) $-\frac{1}{2}$

$\frac{dy}{dt} - \frac{t}{(1+t)} y = \frac{1}{(1+t)}$ $P = \frac{-t}{(1+t)}$

$Q = \frac{1}{(1+t)}$

$I.F = e^{\int \frac{-t}{(1+t)} dt}$

$-\int \frac{t+1-1}{1+t} = -\left[\int \frac{t+1}{t+1} - \int \frac{1}{1+t}\right]$

$= -[t - \ln(1+t)] = \ln(1+t) - t$

$= e^{\ln(1+t) - t}$

$= e^{-t} e^{\ln(1+t)} = e^{-t}(1+t)$

$$I \cdot F = e^{-t}(1+t)$$

Solution of D.E

$$= \quad y \cdot [e^{-t}(1+t)] = \int e^{-t}\cancel{(1+t)} \cdot \frac{1}{\cancel{(1+t)}}\, dt$$

$$= \quad y e^{-t}(1+t) = \int e^{-t}\, dt$$

$$= \quad y e^{-t}(1+t) = -e^{-t} + c$$

$$y(0) = -1$$

$$t = 0$$

$$y = -1$$

$$= -1 e^{0}(1+0) = -e^{0} + c$$

$$= -1 = -1 + c$$

$$= -1 + 1 = c$$

$$c = 0$$

$$ye^{-t}(1+t) = -e^{-t} + 0$$

$$y = \frac{-e^{-t}}{e^{-t}(1+t)}$$

$$= -\frac{e^{-t}\cdot e^{t}}{(1+t)}$$

$$y = -\frac{1}{(1+t)}$$

$$y = \frac{-1}{1+1}$$

$$\boxed{y = -\frac{1}{2}}$$ Ans

Q. **Questions**

#Q5. If $y=y(x)$, $\frac{2+\sin x}{(y+1)}\frac{dy}{dx}=-\cos x$

$y(0)=1$, then $y(\pi/2)$ equals to

(a) 1/3

(b) 2/3

(c) -1/3

(d) 1

$$\frac{2+\sin x}{(y+1)}\frac{dy}{dx}=-\cos x$$

$$\frac{dy}{dx}+Py=Q$$

$$\frac{dy}{dx}=\frac{-\cos x(y+1)}{(2+\sin x)}$$

$$\frac{dy}{dx}=\frac{-\cos x\cdot y-\cos x}{2+\sin x}$$

$$\frac{dy}{dx}+y\frac{\cos x}{2+\sin x}=\frac{-\cos x}{2+\sin x}$$

$$P=\frac{\cos x}{2+\sin x}\quad Q=\frac{-\cos x}{(2+\sin x)}$$

Do yourself

$y(0)=1$ and get the solution

Q. Questions

MSQ

#Q7. If $y(x)$ satisfies the D.E.

$y' - y\tan x = 2\sec x$, $y(0) = 0$ then

Do yourself

$\frac{dy}{dx} - y\tan x = 2\sec x$

$\left.\begin{array}{l} P = -\tan x \\ Q = 2\sec x \end{array}\right]$ I.F ✓

✓ Initial value

✓ Ans

(a) $y(\pi/4) = \frac{\pi^2}{8\sqrt{2}}$

(b) $y'(\pi/4) = \frac{\pi^2}{18}$

(c) $y(\pi/4) = \frac{\pi^2}{9}$

(d) $y'(\pi/3) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

Q. Questions

#Q. $\dfrac{dy}{dx}+y=\dfrac{1+\sin x}{1+\cos x}$

$P = 1$

$Q = \dfrac{1+\sin x}{1+\cos x}$

Solution

Do youself

$\dfrac{dy}{dx}+Py=Q$

Questions

#Q. $\left(1+y^2\right)dx=(\tan^{-1}y-x)dy$

to yourself

Thank You!

PW Soldiers