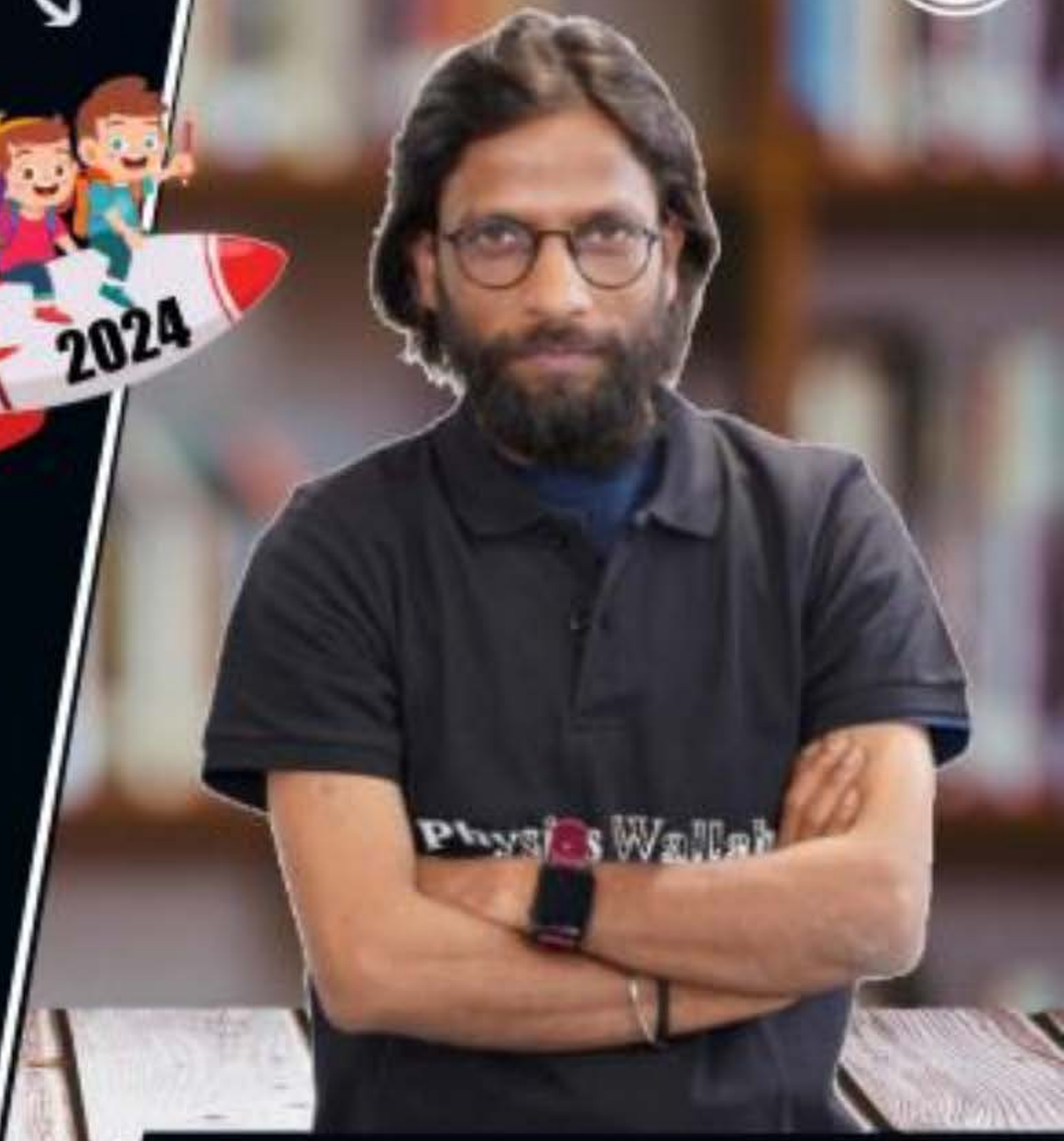
GATE-All BRANCHES
Engineering Mathematics

2024

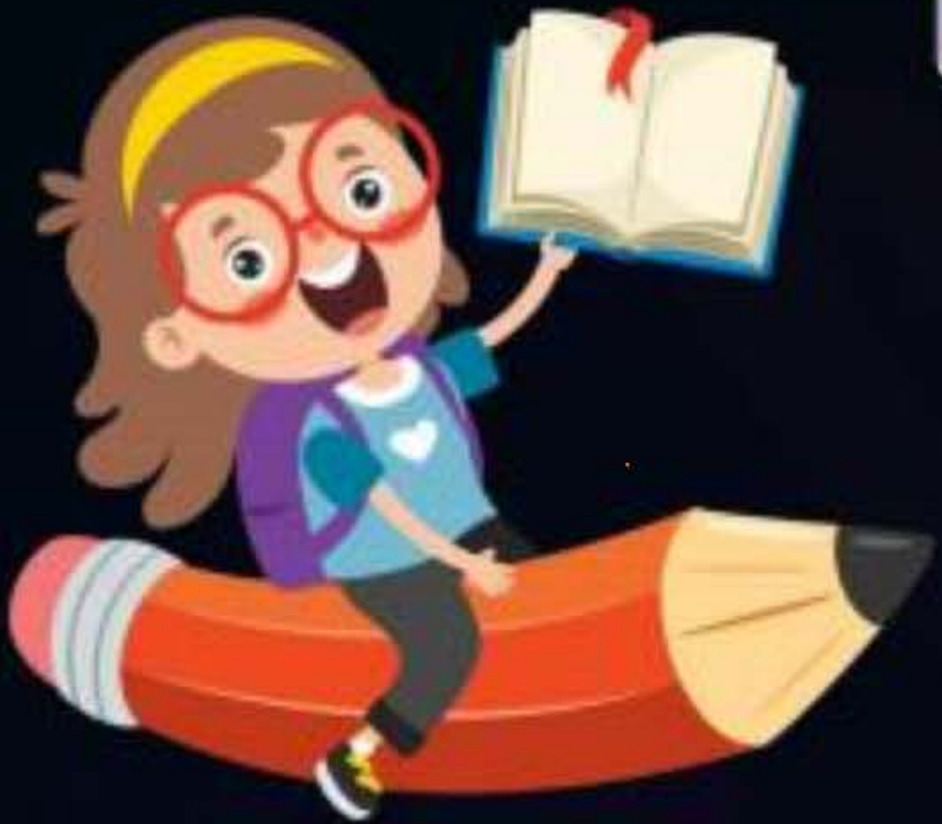
LAPLACE TRANSFORM

Lecture No.- 03



By- Rahul sir

Recape of previous lecture



Topic Problems based on laplace transformation

Topics to be Covered

Topic Solution of differential equation using laplace transforms

Topic Problems based on solution of differential equations

Inverse Laplace Transform: — max chance = 2 marks.

$$L[f(t)] = f(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$L^{-1}[f(s)] = f(t) = (\text{Inverse Laplace Transform})$$

$$L^{-1}\left[\frac{1}{(s^2+1)}\right] = \sin t \qquad L^{-1}\left(\frac{1}{s}\right) = 1 \qquad L^{-1}\left[\frac{1}{(s-a)}\right] = e^{at}$$

$$s \text{ domain} \xrightarrow[\text{Inverse Laplace Transform}]{\text{Process}} t \text{ domain}$$

Topic : Laplace Transform

#Q. The inverse Laplace Transform of $\frac{1}{(s^2+2s)}$ is,

$$L^{-1}\frac{1}{(s^2+2s)} = \frac{1-e^{-2t}}{2}$$

A. $(1-e^{-2t})$

B. $\frac{(1+e^{+2t})}{2}$

C. $\frac{(1-e^{+2t})}{2}$

D. $\frac{(1-e^{-2t})}{2}$

$$L^{-1}\left[\frac{1}{(s^2+2s)}\right] = L^{-1}\frac{1}{[s(s+2)]}$$

$$\frac{1}{\underbrace{s(s+2)}_{\text{Linear Term}}} = \frac{A}{s} + \frac{B}{(s+2)}$$ A and B Are constants

$$A = \frac{1}{(s+2)}\Bigg]_{s=0} = \frac{1}{2}$$

$$B = \frac{1}{s}\Bigg]_{s=-2} = -\frac{1}{2}$$

$$L^{-1}\left[\frac{1}{s(s+2)}\right] = L^{-1}\left[\frac{1}{2}\,\frac{1}{s} - \frac{1}{2}\,\frac{1}{(s+2)}\right]$$

$$= \frac{1}{2}\left[L^{-1}\frac{1}{s} - L^{-1}\left[\frac{1}{(s+2)}\right]\right]$$

$$= \underline{\frac{1}{2}\left[1-e^{-2t}\right]}$$ Ans

s^2+2s = Non linear Function

Laplace Transform — Linear system

$$\frac{1}{(s^2+2s)} = \frac{1}{\underbrace{s(s+2)}}$$

Multiply Non linear

Non linear Term = linear Term

Tool → Partial fraction

$$\frac{f(x)}{(x-a)(x-b)} = \frac{A}{(x-a)} + \frac{B}{(x-b)}$$ Cover-up

$$A = \frac{f(x)}{(x-b)}\Bigg|_{x=a} \qquad B = \frac{f(x)}{(x-a)}\Bigg|_{x=b}$$

#Q. The inverse Laplace transforms of $\dfrac{1}{s^2(s+1)}$ is

A $\frac{1}{2}t^2 e^{-t}$

B $\frac{1}{2}t^2 + 1 - e^{-t}$

C $t - 1 + e^{-t}$

D $\frac{1}{2}t^2\left(1 - e^{-t}\right)$

$$L^{-1}\frac{1}{s^2(s+1)} = \frac{1}{s}\left[\frac{1}{\underbrace{s(s+1)}_{\text{partial fraction}}}\right]$$

$$= \frac{1}{s}\left[\frac{1}{(s)} - \frac{1}{(s+1)}\right]$$

$$= \frac{1}{s^2} - \frac{1}{s(s+1)}$$

$$= \frac{1}{s^2} - \left[\frac{1}{s} - \frac{1}{(s+1)}\right]$$

$$= \frac{1}{s^2} - \frac{1}{s} + \frac{1}{(s+1)}$$

$$= \frac{t^{2-1}}{(2-1)!} - 1 + e^{-t}$$

$$= \underline{t - 1 + e^{-t}}$$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{(s+1)}$$

$s = 0$

$s+1 = 0$

$s = -1$

$$A = \left.\frac{1}{(s+1)}\right|_{s=0} = 1$$

$$B = \left.\frac{1}{s}\right|_{s=-1} = -1$$

$$\boxed{L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}}$$

Topic : Laplace Transform

GATE ME यान्त्रिकी अभियान्त्रिकी

#Q. The Inverse Laplace transform of the function, is

$$\frac{s+5}{(s+1)(s+3)}$$

$$L^{-1}\left[\frac{s+5}{(s+1)(s+3)}\right]$$

$$\frac{s+5}{(s+1)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+3)}$$

$$A = \left.\frac{(s+5)}{(s+3)}\right|_{s=-1} = \frac{\cancel{4}}{\cancel{2}} = 2$$

$$B = \left.\frac{(s+5)}{(s+1)}\right|_{s=-3} = \frac{2}{-2} = -1$$

A $2e^{-t} - e^{-3t}$

B $2e^{-t} + e^{-3t}$

C $e^{-t} - 2e^{-3t}$

D $e^{-t} + 2e^{-3t}$

$$= \frac{2}{(s+1)} - \frac{1}{(s+3)}$$

$$= \boxed{2e^{-t} - e^{-3t}}$$

Topic : Laplace Transform

#Q. The inverse Laplace transform of $\dfrac{s+9}{s^2+6s+13}$ is

A. $\cos 2t + 9\sin 2t$

B. $e^{-3t}\cos 2t - 3e^{-3t}\sin 2t$

C. $e^{-3t}\sin 2t + 3e^{-3t}\cos 2t$

D. $e^{-3t}\cos 2t + 3e^{-3t}\sin 2t$

$(s+3)+6$

2×3

$$L^{-1}\left[\frac{s+9}{s^2+6s+13}\right] = L^{-1}\left[\frac{(s+9)}{(s+3)^2+2^2}\right]$$

$$= L^{-1}\left[\frac{(s+3)}{(s+3)^2+(2)^2} + \frac{2\times 3}{(s+3)^2+2^2}\right]$$

$s^2+6s+13$

$s^2+6s+9+4$

$(s+3)^2+(2)^2$

$$= e^{-3t}\cos 2t + 3e^{-3t}\sin 2t$$

Imp. for syllabus

3 to 5

Transform of derivatives :

$L[f(t)] = f(s)$

$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$

$= s f(s) - f(0)$

$\frac{dy}{dt} =$ $\frac{d^2y}{dt^2} =$

$\frac{d^3y}{dt^3} =$

Imp.

$\boxed{L[f'(t)] = s f(s) - f(0)}$ ⟶ one initial value

$\boxed{L[f''(t)] = s^2 f(s) - s\underline{f(0)} - \underline{f'(0)}}$ ⟶ Two Initial values.

$\boxed{L[f'''(t)] = s^3 f(s) - s^2 f(0) - s f'(0) - f''(0)}$

$\boxed{L[f^n(t)] = s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{n-1}(0)}$

given conditions $\left[\begin{array}{l} y(0)=1 \\ y'(0)=2 \\ y''(0)=2 \end{array}\right.$

#Q. Solve the equation :

$$\frac{d^3y}{dt^3}+2\frac{d^2y}{dt^2}-\frac{dy}{dt}-2y=0, \text{ where } y=\cancel{1}\,0, \frac{dy}{dt}=2, \frac{d^2y}{dt^2}=2 \text{ at } t=0$$

$$\frac{d^3y}{dt^3}+2\frac{d^2y}{dt^2}-\frac{dy}{dt}-2y=0$$

$y=0 \quad \frac{dy}{dt}=2$ — ① $\quad y'(0)=2$

$\frac{d^2y}{dt^2}=2$ at $t=0$ — ② $\quad y''(0)=2$

Taking laplace both sides.

$$\boxed{L\left[\frac{d^3y}{dt^3}\right]+2L\left[\frac{d^2y}{dt^2}\right]-L\left[\frac{dy}{dt}\right]-2L[y]=L[0]}$$

$L[f]=f(s)$

$$L^{-1}(y(s)) = y(t)$$

$$= \left[s^3 y(s) - s^2 y(0) - s y'(0) - y''(0)\right] + 2\left[s^2 y(s) - s y(0) - y'(0)\right] - \left[s y(s) - y(0)\right] - 2y(s) = 0$$

$$\begin{cases} y(0) = 1 \quad y'(0) = 2 \\ \qquad\qquad y''(0) = 2 \end{cases}$$

$$= \left[s^3 y(s) - s^2 \times 1 - s\cdot 2 - 2\right] + 2\left[s^2 y(s) - s \times 1 - 2\right] - \left[s y(s) - 1\right] - 2y(s) = 0$$

$$= \left[s^3 y(s) - s^2 - 2s - 2\right] + 2\left[s^2 y(s) - s - 2\right] - \left[s y(s) - 1\right] - 2y(s) = 0$$

$$= s^3 y(s) - s^2 - 2s - 2 + 2s^2 y(s) - 2s - 4 - s y(s) + 1 - 2y(s) = 0$$

$$= \left[s^3 + 2s^2 - s - 2\right] y(s) = s^2 + 2s + 2s + 6 - 1$$

$$= \frac{(s^2 + 4s + 5)}{(s^3 + 2s^2 - s - 2)}$$

$$Y(s) = \frac{(s^2+4s+5)}{(s^3+2s^2-s-2)} = \frac{(s^2+4s+5)}{(s-1)(s+1)(s+2)}$$

Using Partial Fraction

$$\boxed{L^{-1}[Y(s)] = L^{-1}\left[\frac{s^2+4s+5}{(s-1)(s+1)(s+2)}\right]}$$

$y(t) = ✓$

$(s^2-1)(s+2)$

$\boxed{s^3+2s^2-s-2}$ ✓

Topic : Laplace Transform

#Q. Solve the Equation :

$$y'' - 3y' + 2y = 4t + e^{3t}, \text{ where } y(0) = 1 \text{ and } y'(0) = -1$$

$$Y(s) \Rightarrow \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{s^2(s-1)(s-2)(s-3)}$$

$$\boxed{y(t) = 3 + 2t - \frac{e^t}{2} - 2e^{2t} + \frac{1}{2}e^{3t}}$$

Topic : Laplace Transform

$\frac{e^{-t}}{3}[\sin t + \sin 2t]$

#Q. Using Laplace transform, solve the following differntial equation :

$$\boxed{\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=e^{-t}\sin t}, \text{ where } x(0) \text{ and } x'(0)=1.$$

$x(0)=1$

$x'(0)=1$

$$s^2X(s)-sX(0)-X'(0)+2sX(s)-X(0)+5X(s)=\frac{1}{(s+1)^2+1}$$

$$[s^2+2s+5]X(s)-s-1-1=\frac{1}{(s+1)^2+1}$$

$$(s^2+2s+5)X(s)=(s+2)+\frac{1}{(s+1)^2+1}$$

$$X(s)=\frac{(s+2)}{(s^2+2s+5)}+\frac{1}{((s+1)^2+1)(s^2+2s+5)}$$

$$L^{-1}[X(s)]=X(t)$$

Topic : Laplace Transform

#Q. Using Laplace transformation, solve the differential equation

$$\frac{d^2x}{dt^2}+9x=\cos 2t, \text{if } x(0)=1, x\left(\frac{\pi}{2}\right)=-1.$$

H.W

$$=\frac{1}{5}\left[\cos 2t+4\cos 3t+4\sin 3t\right]$$

2 mins Summary

Topic One

Topic Two

Topic Three

Topic Four

Topic Five

✓ Transform of derivative

✓ Problems

Rahul Sir PW

Laplace Transform
GATE PYQ

THANK - YOU

Topics to be Covered