GATE-All BRANCHES
Engineering Mathematics

Vector calculus

Lecture No.- 05

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Recap of Previous Lecture

Topic Gradient of a scalar function

Topic Directional derivative

Topic Problems based on gradient,directional derivative.

Topics to be Covered

D.E + complex – Monday

Topic Question based on gradient — directional derivative

Topic Surface integral

Topic Question based on surface integral

Topic : Vector calculus

#Q. Find the directional derivative of $f(x, y, z) = x^2 - y^2 + 2z$ at the point P(1, 2, 3) in the direction of the line $\overrightarrow{PQ}$ where Q (5, 0, 4).

$$f = x^2 - y^2 + 2z$$

$$P\,(1,2,3) \xrightarrow[\text{derivative}]{\text{Directional}} Q\,(5,0,4)$$

Directional derivative $= (\text{grad } f) \cdot \hat{PQ}$

$$P\,(1,2,3) \xrightarrow{\overrightarrow{PQ}} Q\,(5,0,4$$

$$\nabla f = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left[x^2 - y^2 + 2z\right]$$

$$\nabla f = \hat{i}\cdot 2x + \hat{j}(-2y) + \hat{k}(2)$$

$$(\nabla f)_{(1,2,3)} = \hat{i} \times 2 \times 1 + \hat{j}(-2\times 2) + \hat{k}(2)$$

$$= 2\hat{i} - 4\hat{j} + 2\hat{k}$$

Directional derivative

$$= (2\hat{i} - 4\hat{j} + 2\hat{k})\left(\frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{21}}\right)$$

$$= \frac{8 + 8 + 2}{\sqrt{21}}$$

$$= \frac{18}{\sqrt{21}}$$

$\hat{i} + 2\hat{j} + 3\hat{k}$

$P(1,2,3)$ $5\hat{i} + 0\hat{j} + 4\hat{k}$

$Q(5,0,4)$

P $\overrightarrow{PQ}$ Q

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= 5\hat{i} + 0\hat{j} + 4\hat{k} - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$\overrightarrow{PQ} = 4\hat{i} - 2\hat{j} + \hat{k}$$

$$\widehat{PQ} = \frac{4\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{16 + 4 + 1}}$$

Topic : Vector calculus

#Q. Find the directional derivative of $f(x, y, z) = xy^2 + yz^2 + zx^2$, along to tangent to a curve $x = t, y = t^2, z = t^3$ at the point (1, 1, 1)

$$f = xy^2 + yz^2 + zx^2$$

$$\text{grad } f = \nabla f = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\left[xy^2 + yz^2 + zx^2\right]$$

$$\nabla f = \hat{i}(y^2 + 2zx) + \hat{j}[2xy + z^2) + \hat{k}[2zy + x^2]$$

$$(\nabla f)_{(1,1,1)} = \hat{i}(3) + 3\hat{j} + 3\hat{k}$$

along to Tangent to a curve $\quad x = t,\ y = t^2,\ z = t^3$

$$\boxed{\begin{aligned} \vec{r}(t) &= x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} \\ \vec{r}(t) &= t\hat{i} + t^2\hat{j} + t^3\hat{k} \\ \frac{d\vec{r}}{dt} &= \hat{i} + 2t\hat{j} + 3t^2\hat{k} \end{aligned}}$$

$x=1 \quad y=1 \quad z=1$

$t = 1 = 1$

$t^2 = 1 = \pm 1$

$t^3 = 1 = 1, w, w^2$

$t = 1$

$$\frac{d\vec{r}}{dt} = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\frac{d\hat{r}}{dt} = \frac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}}$$

$t^3 = 1, \quad 1, w, w^2$

$w = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

$w^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Directional Derivative $= (3\hat{i} + 3\hat{j} + 3\hat{k})\left(\dfrac{\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{14}}\right)$

$= 18/\sqrt{14}$

Topic : Vector calculus

Directional derivative
$= \nabla f \cdot \hat{a}$

#Q. The magnitude of the directional derivative of the function $f(x, y) = x^2 + 3y^2$ in a direction $\boxed{\text{normal to the circle } x^2 + y^2 = 2,}$ at the point (1,1), is

$N = \frac{\text{grad } f}{|\text{grad } f|} \quad \frac{2x\hat{i} + 2y\hat{j}}{2\sqrt{2}} = \frac{x\hat{i} + y\hat{j}}{\sqrt{2}}$

$= \frac{\hat{i} + \hat{j}}{\sqrt{2}} \quad (1,1)$

$\nabla f = 2x\hat{i} + 6y\hat{j}$

$\nabla f = 2\hat{i} + 6\hat{j}$

A. $4\sqrt{2}$

B. $5\sqrt{2}$

C. $7\sqrt{2}$

D. $9\sqrt{2}$

$D \cdot D = (2\hat{i} + 6\hat{j})\left(\frac{\hat{i} + \hat{j}}{\sqrt{2}}\right)$

$= \frac{2 + 6}{\sqrt{2}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$

Topic : Vector calculus

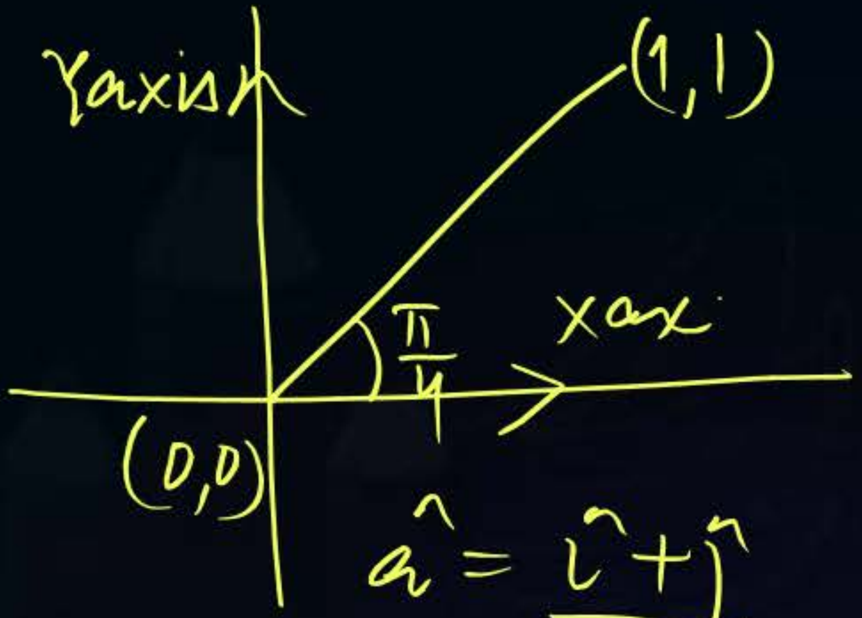
$\nabla f_{(1,1)} =$ d·d $= (2\hat{i}+2\hat{j})\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}\right)$

$= 2\sqrt{2}$ ✓

#Q. The directional derivative of the function $f(x, y) = x^2 + y^2$ along a line directed from (0, 0) to (1, 1), evaluated at the point x = 1, y = 1 is

$\nabla f = 2x\hat{i} + 2y\hat{j}$

$(\nabla f)_{(1,1)} = 2\hat{i} + 2\hat{j}$



$\hat{a} = \frac{\hat{i}+\hat{j}}{\sqrt{2}}$

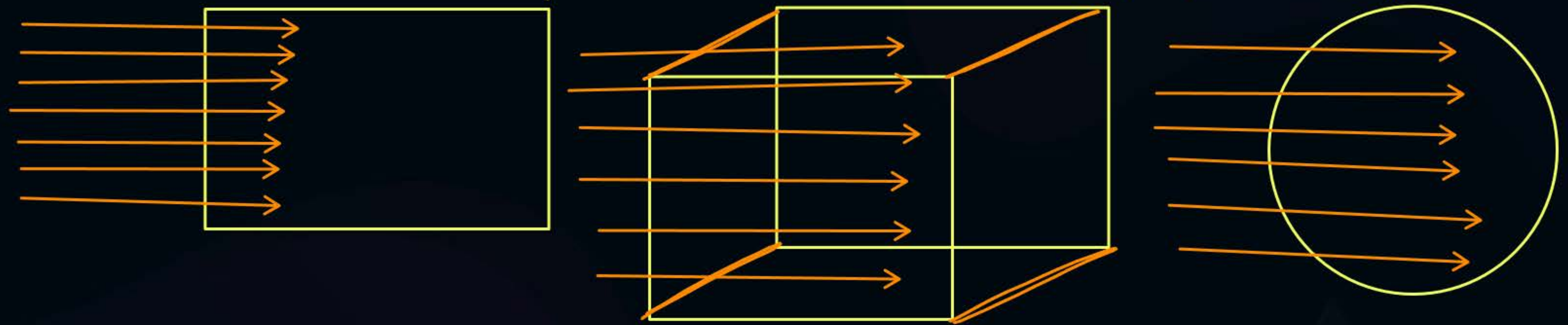
A. $2\sqrt{2}$

B. 2

C. $4\sqrt{2}$

D. $\sqrt{2}$

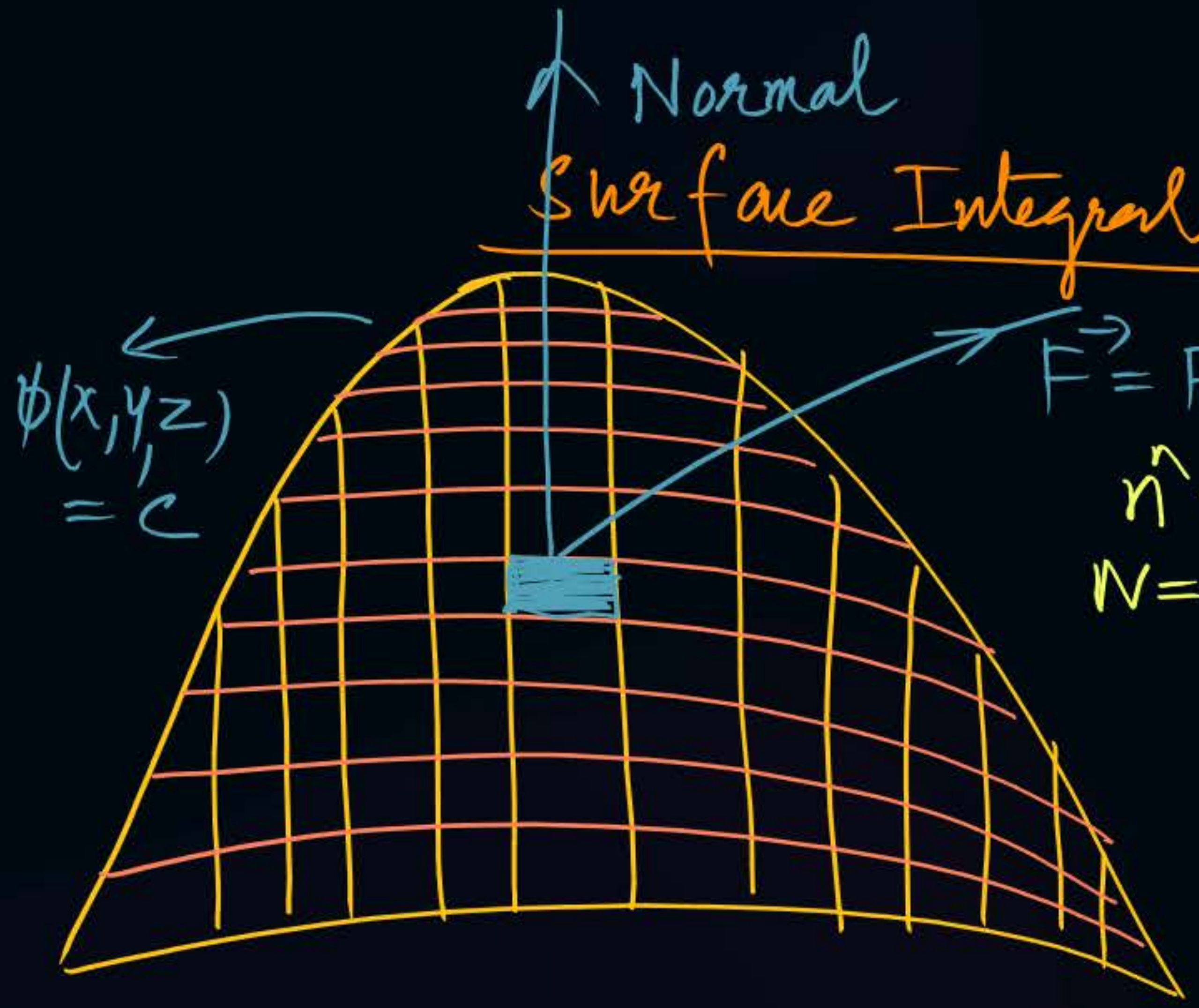
Surface Integral = Flux = Total No. of field Lines/electric field.



Line Integral

$$\frac{\text{Total No. of Line Integrals}}{\text{Total No. of field lines}} = \text{Surface Integral OR flux}$$

Surface Integral:



$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$\hat{n}$ = unit vector (Plane)

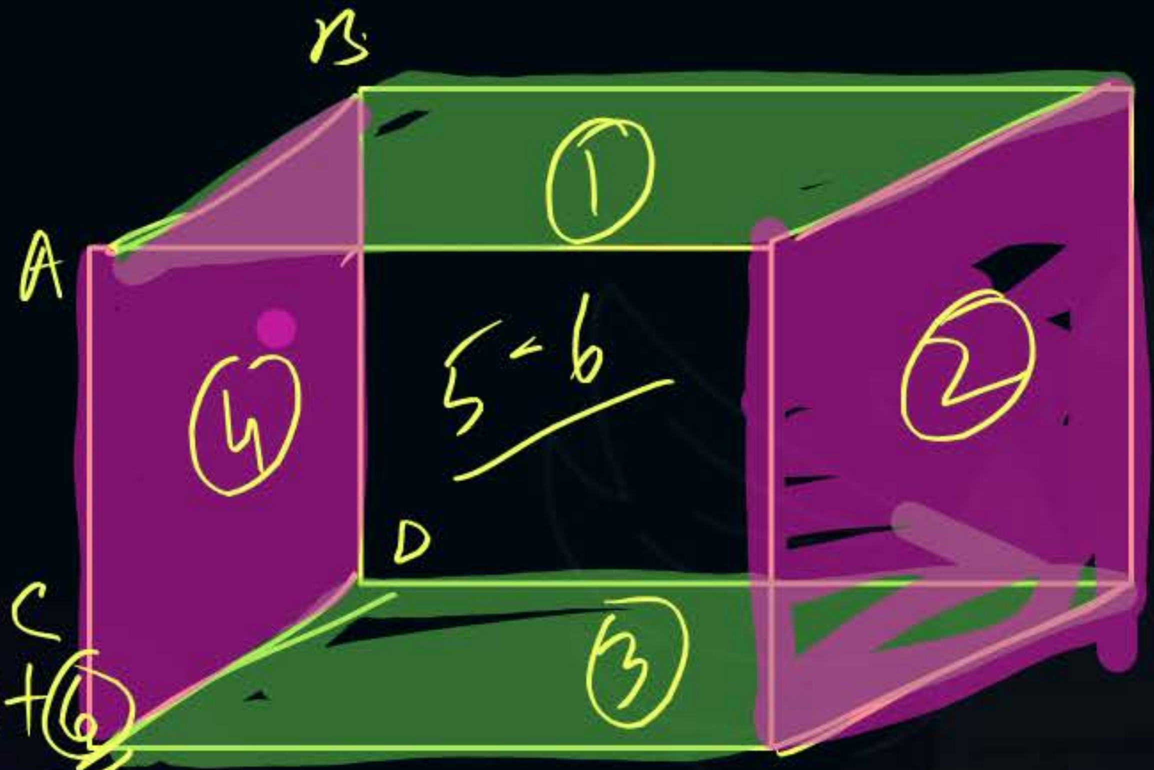
N = surface Normal

surface integral

$= \iint \vec{F}\, \hat{N}\, ds \qquad \vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

N = Normal to the surface

ds = surface element



$① + ② + ③ + ④ + ⑤ + ⑥$

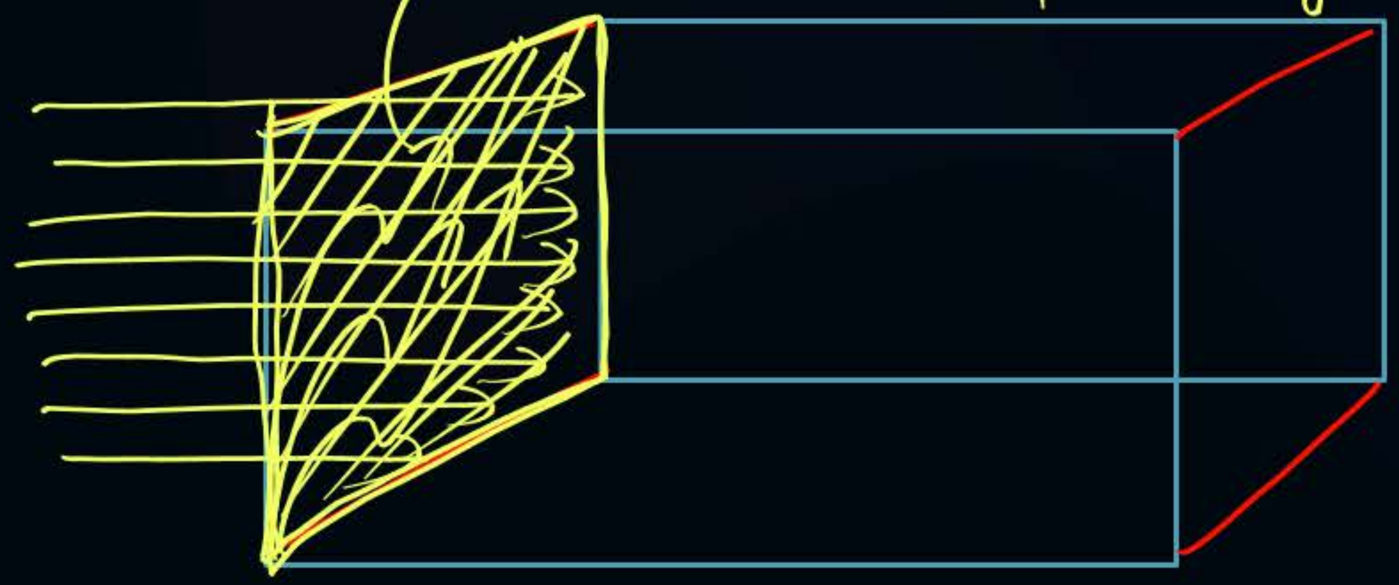
surface Integral = volume

Surface Integral

$$= \iint \vec{F} \, \hat{N} \, ds$$

$$\hat{N} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$$

Plane – surface integral



$$|\hat{n}| = 1$$
$$|\hat{k}| = 1$$
$$|\hat{j}| = 1$$

$$\cos\theta = \frac{dA}{ds}, \quad ds = \frac{dA}{\cos\theta}$$

$$ds = \frac{dy\,dx}{\cos\theta}$$

$$\hat{n}\cdot\hat{k} = |\hat{n}|\,|\hat{k}|\cos\theta$$

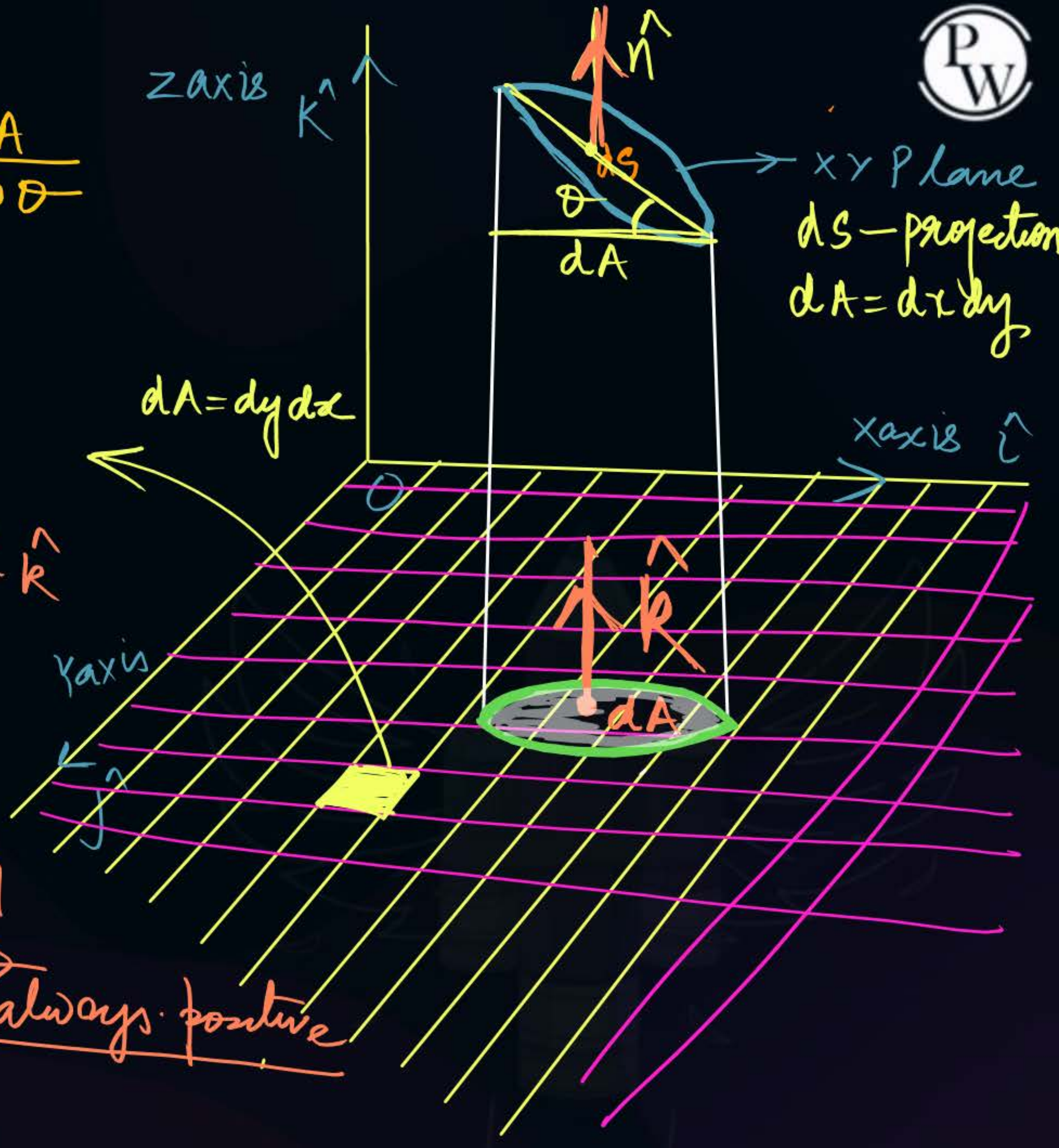
Using Dot Product

$$\cos\theta = \frac{\hat{n}\cdot\hat{k}}{|\hat{n}|\,|\hat{k}|} = \frac{\hat{n}\cdot\hat{k}}{1\times 1} = \hat{n}\cdot\hat{k}$$

$$ds = \frac{dy\,dx}{|\hat{n}\cdot\hat{k}|}$$

$$\boxed{\iint_S \vec{F}\,\hat{N}\,ds = \iint_{xy\ \text{Plane}} \vec{F}\cdot\hat{N}\,\frac{dy\,dx}{|\hat{n}\cdot\hat{k}|}}$$

→ always positive



In XY Plane.

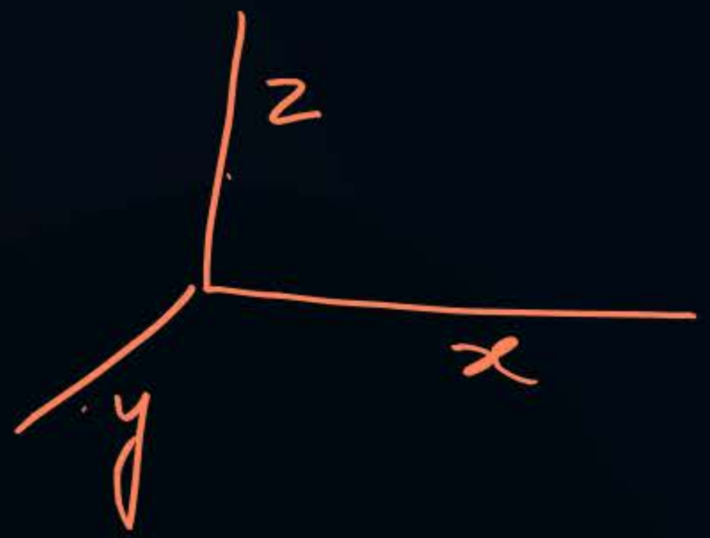
$$\iint \vec{F}\,\hat{N}\,ds = \iint_{xy\ Plane} \vec{F}\,\hat{N}\,\frac{dy\,dx}{|\hat{n}\cdot\hat{k}|}$$

yz Plane

$$\iint_{yz\ Plane} \vec{F}\,\hat{N}\,ds = \iint_{yz\ Plane} \vec{F}\,\hat{N}\,\frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$

zx Plane.

$$\iint_{zx\ Plane} \vec{F}\,\hat{N}\,ds = \iint_{zx\ Plane} \vec{F}\,\hat{N}\,\frac{dz\,dx}{|\hat{n}\cdot\hat{j}|}$$



Topic : Vector calculus

$\frac{3a^4}{8} = \text{Ans}$

#Q. Find surface integral $\oiint \vec{F}.\hat{N}\, ds$

$N = \frac{\nabla f}{|\nabla f|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{2\sqrt{x^2+y^2+z^2}}$

$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{a^2}}$

Where $\vec{F}$(x, y, z) = yz$\hat{\imath}$ + zx$\hat{\jmath}$ + $xy\hat{k}$ and S is the part of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ which lies in the first octant. (xy Plane)

$\vec{F}\cdot\hat{N} = (yz\hat{i}+zx\hat{j}+xy\hat{k})\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}\right)$

$= \frac{3xyz}{a}$

$\iint \vec{F}\hat{n}\, ds = \iint \frac{3xyz}{a} \frac{dy\,dx}{(z/a)} = \boxed{3\iint xy\, dy\, dx}$

$x^2+y^2+z^2=a^2 \Rightarrow x^2+y^2=a^2$

In xy Plane $\boxed{z=0}$

$$\Rightarrow \underbrace{\int_{x=0}^{a}\int_{0}^{\sqrt{a^2-x^2}} xy\,dy\,dx}=\frac{3a^4}{8}$$

Volume via Double integrals

Topic : Vector calculus

H.W

#Q. Find surface integral $\oiint \vec{F}.\hat{N}\, ds$

Where $\vec{F}(x, y, z) = (x + y^2)\hat{\imath} - 2x\hat{\jmath} + 2yz\hat{k}$ and S is the part of the plane x+2y+3z=6 in the first octant.

Topic : Vector calculus

M.W

#Q. Find surface integral $\oiint \vec{F}.\hat{N}\, ds$

Where $\vec{F}(x, y, z) = y\hat{\imath} + 2x\hat{\jmath} - z\hat{k}$ and S is the surface of the plane $2x+y=4$ in the first octant cut off by the plane $z=4$.

Topic : Vector calculus

#Q. Find surface integral $\oiint \vec{F}.\hat{N}\, ds$

Where $\vec{F}(x, y, z) = 18z\hat{\imath} - 12\hat{\jmath} + 3y\hat{k}$ and S is the part of the plane 2x + 3y + 6z = 12 located in x – y plane.

$$\vec{F} = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$$

$$\phi(x,y,z) = 2x + 3y + 6z = 12$$

Step ①

find The Normal to The Surface

$$N = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$$

$$\vec{F}\cdot\hat{N} = (18z\hat{i} - 12\hat{j} + 3y\hat{k})\left(\frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}\right)$$

$$\boxed{\vec{F}\cdot\hat{N} = \frac{6}{7}(6z-6+3y)}$$

$$N = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$$

$$N = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{(2)^2+(3)^2+(6)^2}}$$

$$\text{Normal} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}$$

(II) In XY Plane

$$\iint \vec{F}\hat{N}\,ds = \iint \frac{6}{7}(6z-6+3y)\frac{dy\,dx}{|\hat{n}\cdot\hat{k}|}$$

$$\Rightarrow \iint \cancel{\frac{6}{7}}(6z-6+3y)\frac{dy\,dx}{\cancel{\left|\frac{6}{7}\right|}}$$

$$= \iint (6z-6+3y)\,dy\,dx$$

in xy Plane

$$\underline{z=0}$$

$$n_1 = \hat{n}\cdot\hat{i} = \frac{2}{7}$$

$$n_2 = \hat{n}\cdot\hat{j} = \frac{3}{7}$$

$$\boxed{n_3 = \hat{n}\cdot\hat{k} = \frac{6}{7}}$$

$$n_1^2+n_2^2+n_3^2 = 1$$

$$\iint \vec{F}\cdot\hat{N}\,ds = \iint (6z-6+3y)\,\underline{dy}\,dx \longrightarrow \iint \underbrace{(3y-6)\,dy\,dx}$$

$z=0$ in xy Plane

volume via Double integrals

$$2x+3y+6z=12$$

Put $z=0$

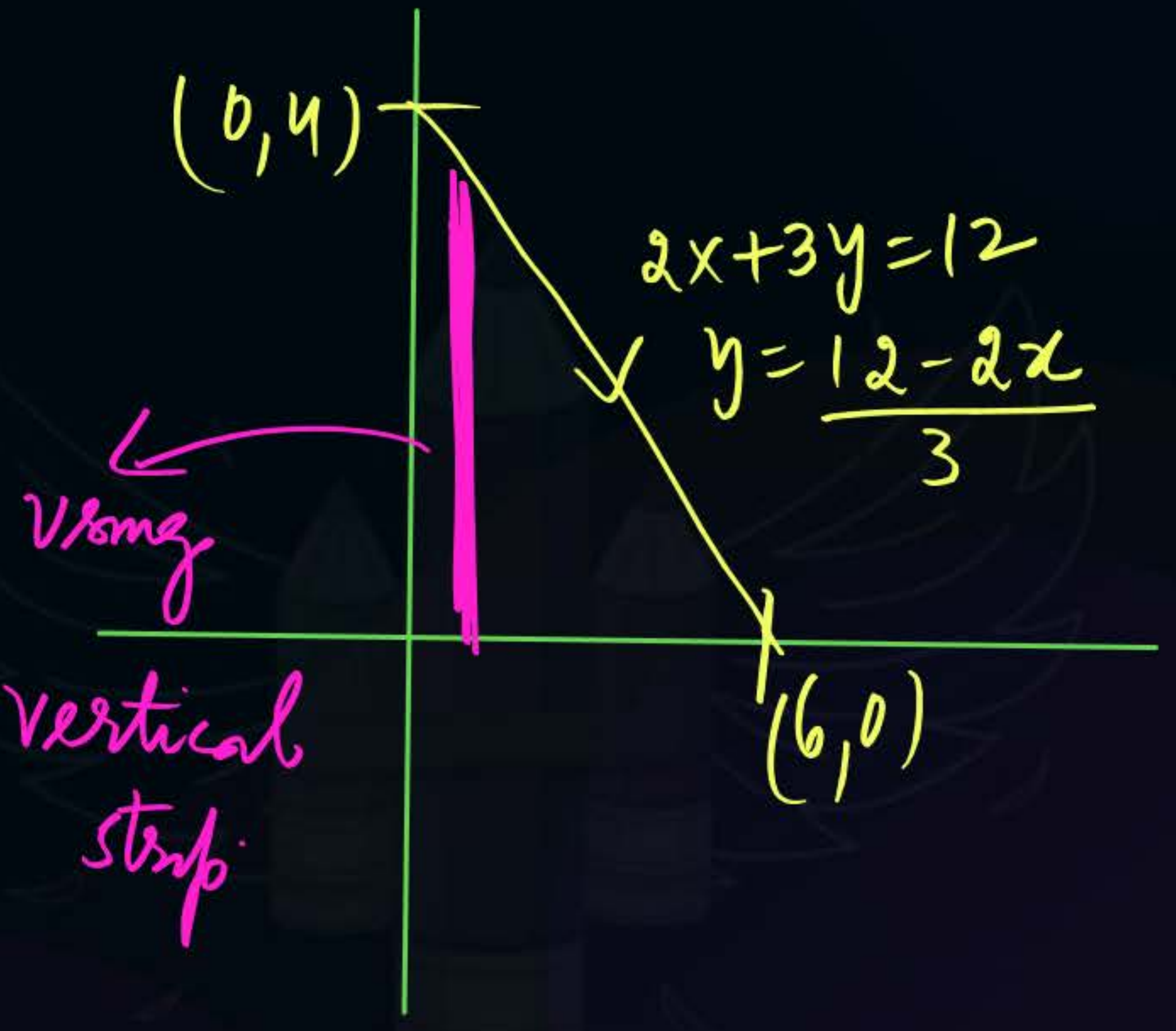
$$2x+3y=12$$

$$\frac{2x}{12}+\frac{3y}{12}=1$$

$$\frac{x}{6}+\frac{y}{4}=1$$

$$\iint \vec{F}\cdot\hat{N}\,ds = \int_{x=0}^{6}\int_{0}^{\frac{12-2x}{3}} (3y-6)\,dy\,dx = \boxed{\cancel{+}24}$$

$$= -24$$



2 mins Summary

Topic One $\longrightarrow$ Directional derivative ✓

Topic Two $\longrightarrow$ surface integral $\iint \vec{F} \cdot \hat{n}\, ds =$ surface

Topic Three Questions

Topic Four

Topic Five

THANK - YOU