GATE-All BRANCHES
Engineering Mathematics

Numerical Methods

DPP 01

Discussion Notes

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#Q. The quadratic equation $2x^2 - 3x + 3 = 0$ is to be solved numerically starting with an initial guess as $x_0 = 2$. The new estimate of x after the first iteration using Newton Raphson method is ___1___.

$f(x) = \underbrace{2x^2 - 3x + 3 = 0}$

$f(2) = 8 - 6 + 3 = 5$

$f'(2) = 4x - 3 = 8 - 3 = 5$

N-R Method

$$\boxed{x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{f(2)}{f'(2)} = 2 - \cancel{5/5} = 1$$

#Q. Starting with x =1 , the solution of the equation $x^3 + x = 1$, after two iterations of Newton-Raphson's method (*up to two decimal places*) is __________.

$x_0 = 1$

$f(x) = x^3 + x - 1 = 0$

$f(1) = 1$

$f'(1) = 3x^2 + 1 = 4$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1}{4} = \frac{3}{4} = 0.75$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$0.75 - \frac{f(0.75)}{f'(0.75)} = 0.686$

#Q. Solve the equation x = 10 cos(x) using the Newton-Raphson method. The initial guess is x = π/4. The value of the predicted root after the first iteration, up to second decimal is ____________.

$x - 10\cos x = f(x)$

$f'(x) = 1 + 10\sin x$

$\boxed{x_0 = \frac{\pi}{4}}$

$= 1.564$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{\pi}{4} - \frac{\left(\pi/4 - 10\cos\pi/4\right)}{\{1 + 10\sin(\pi/4)\}}$

Topic : Numerical Methods

#Q. The root of the function $f(x) = x^3 + x - 1$ obtained after first iteration on application of Newton-Raphson scheme using an initial guess of $x_0 = 1$ is

A) 0.682

B) 0.686

C) 0.750

D) 1.000

x_2

x_1

$x_1 = ?$ N-R

$f(x) = x^3 + x - 1$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$f(1) = 1 + \cancel{1} - \cancel{1}$
$= 1$

$f'(1) = 3x^2 + 1$
$= 3 + 1 = 4$

$1 - \frac{1}{4} = \frac{3}{4}$
$= 0.75$

$f(0.5) = (0.5)^3 + 2(0.5)^2 + 3(0.5) - 1$

#Q. Newton-Raphson method is used to find the roots of the equation, $x^3+2x^2+3x-1=0$. If the initial guess is $x_0 = 1$, then the value of x after 2^{nd} iteration is 0.3043

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 1 - \frac{5}{10}$

$= 0.5$

$f(x) = x^3 + 2x^2 + 3x - 1 = 0$

$f'(x) = 3x^2 + 4x + 3 \rightarrow 3\times(0.5)^2 + 4\times0.5 + 3$

$f'(1) = 10$

$x_2 = 0.5 - \frac{f(0.5)}{f'(0.5)}$

$f(1) = \cancel{1} + 2 + 3 \cancel{- 1} = 5$

#Q. The Newton-Raphson method is used to solve the equation f(x) = $x^3 - 5x^2 + 6x - 8 = 0$. Taking the initial guess as x = 5, the solution obtained at the end of the first iteration is 4.29.

$f(x) = x^3 - 5x^2 + 6x - 8 = 0$

$f'(x) = 3x^2 - 10x + 6$

$x_0 = 5$

$f(5) = 125 - 125 + 30 - 8 = 22$

$f'(5) = 75 - 50 + 6$

$= 31$

$x_1 = 5 - \frac{22}{31}$

$= \frac{155 - 22}{31}$

4.29 ←

Secant Method :- $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$ ⟶ ① generalised

#Q. The quadratic equation $x^2 - 4x + 4 = 0$ is to be solved numerically, starting with the initial guess $x_0 = 3$. The Newton-Raphson method is applied once to get a new estimated and then the Secant method is applied once using the initial guess and this new estmate. The estimated value of the root after the application of the Secant method is ______________.

$f'(x_0) = 2 \times 3 - 4 = 2$

$f(x) = x^2 - 4x + 4 = 0$

$f'(x) = 2x - 4$

$f(x_0) = 9 - 12 + 4 = 1$

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$= 3 - \dfrac{1}{2} = \dfrac{5}{2}$

eqn ①

$x_2 = 2.33$

#Q. In Newton-Raphson iterative method, the initial guess value (X_{ini}) is considered as zero while finding the roots of the equation : $f(x) = -2 + 6x - 4x^2 + 0.5x^3$. The correction Δx, to be added to X_{ini} in the first iteration is 0.33.

$x_{ini} = 0$

$\Delta x = 0.333...$

$f(x) = -2 + 6x - 4x^2 + 0.5x^3$

$f'(x) = 6 - 8x + 1.5x^2$

$x = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{(-2)}{(6)} = \frac{1}{3} = 0.333$

#Q. When the Newton-Raphson method is applied to solve the equation $f(x) = x^3 + 2x - 1 = 0$, the solution at the end of the first iteration with the initial guess value as $x_0 = 1.2$ is

eqn x_1 ①

$f(x) = x^3 + 2x - 1 = 0$

$f'(x) = 3x^2 + 2$

$f'(1.2) = 3 \times (1.2)^2 + 2$

$f(1.2) = (1.2)^3 + 2 \times 1.2 - 1$

$x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$x_0 = 1.2$

A. -0.82

B. 0.49

C. 0.705

D. 1.69

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

#Q. A numerical solution of the equation $f(x) = x + \sqrt{x} - 3 = 0$ can be obtained using Newton Raphson method. If the starting value is $x = 2$ for the iteration, the value of x that is to be used in the next step is

$f(x) = x + \sqrt{x} - 3$ $\qquad f'(x) = 1 + \frac{1}{2\sqrt{x}}$

$f(2) = 2 + \sqrt{2} - 3$

$= \sqrt{2} - 1$

$x_0 = 2$

A 0.306

B 0.739

C ~~1.694~~

D 2.306

$f'(2) = 1 + \frac{1}{2\sqrt{2}}$

$x_1 = 2 - \frac{\sqrt{2} - 1}{1 + \frac{1}{2\sqrt{2}}}$

Calculator

#Q. Let $x^2 - 117 = 0$. The iterative steps for the solution using Newton-Raphson's Method is given by

$f(x) = x^2 - 117$

$f'(x) = 2x$

$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^2 - 117}{2x_k} = \frac{x_k^2 + 117}{2x_k}$

A. $x_{k+1} = \frac{1}{2}\left(x_k + \frac{117}{x_k}\right)$

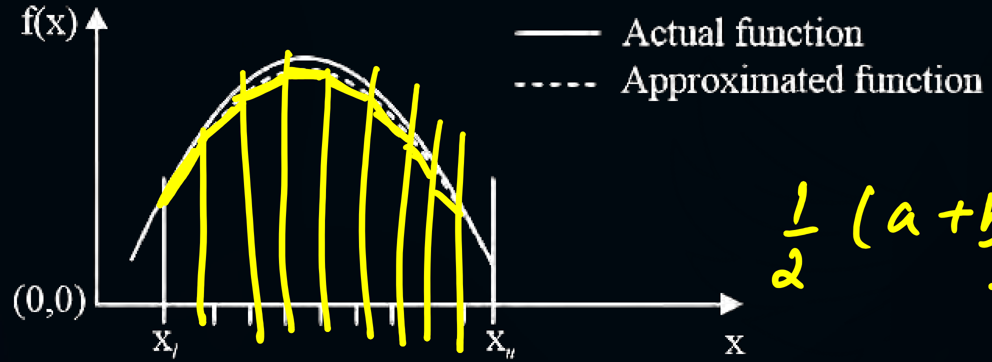
B. $x_{k+1} = x_k - \frac{117}{x_k}$

C. $x_{k+1} = x_k - \frac{x_k}{117}$

D. $x_{k+1} = x_k - \frac{1}{2}\left(x_k + \frac{117}{x_k}\right)$

Numerically computed < Actual area

#Q. A function f(x), that is smooth and convex-shaped between interval (x_1, x_u) is shown in the figure. This function is observed at odd number of regularly spaced points. If the area under the function is computed numerically, then.



$\frac{1}{2}(a+b) \times h$

appr < actual

→ appr curve compared to the actual smooth fn.

Topic : Numerical Methods

I_s I_e

#Q. Consider the definite integral $\int_1^2 (4x^2 + 2x + 6)dx$
Let I_e be the exact value of the integral. If the same integral is estimated using Simpson's rule with 10 equal subintervals, the value is I_s. The percentage error is defined as $e = 100 \times (I_e - I_s)/ I_e$. The value of e is

Same as exact value $\int_a^b f(x)\,dx$

A 2.5

$I_e = I_s$

B 3.5

≤ 2

C 1.2

D 0

By Simpson's $\frac{1}{3}$rd rule gives zero error

#Q. Numerically integrate $f(x) = 10x - 20x^2$ from lower limit a = 0 to upper limit b = 0.5. Using Trapezoidal rule with five equal subdivision. The value (in units round off to two decimal places) obtained is ___________ .

$a = 0$ to $b = 0.5$

$h = \frac{b-a}{n} = \frac{0.5 - 0}{5} = 0.1$

x	0	0.1	0.2	0.3	0.4	0.5
$f(x)$	0	0.8	1.2	1.2	0.8	0
	y_0	y_1	y_2	y_3	y_4	y_n

$$I = \int_0^{0.5} (10x - 20x^2)\,dx$$

$$\frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + y_3 + y_4)\right] = 0.40$$

Topic : Numerical Methods

(1)

#Q. The integral $\int_0^1 (5x^3 + 4x^2 + 3x + 2)\,dx$ is estimated numerically using three alternative methods namely the rectangular, trapezoidal and Simpson's rules with a common step size . In the context, which one of the following statements is TRUE ?

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A Simpson's rule as well as rectangular rule of estimation will give non-zero error.

B Only Simpson's rule of estimation will give zero error.

C Simpson's rule, rectangular rule as well as trapezoidal rule of estimation will give non-zero error

D Only the rectangular rule of estimation will give zero error.

$\left|\frac{A-B}{A}\right| \times 100 = 5.182\%$

$\frac{\pi/2}{2}\left[(8+4\cos x)_{x=0} + (8+4\cos x)\Big|_{x=\pi/2}\right] = 5\pi = B$

#Q. For the intergral $\int_0^{\pi/2}(8+4\cos x)dx$, the absolute percentage error in numerical evaluation with the Trapezoidal rule, using only the end points, is ___5.2%___. (*round off to one decimal place*)

(i) Analytic

$\int_0^{\pi/2}(8+4\cos x)\,dx = 8x + 4\sin x\Big|_0^{\pi/2}$

$= 8\times\pi/2 + 4\times 1$

$= 4\pi + 4 = A$

(ii) Trapezoidal rule:-

$h/2(y_0+y_n)$

collinear

#Q. P(0,3), Q(0.5, 4), and R(1,5) are three points on the curve defined by f(x). Numerical integration is carried out using both Trapezoidal rule and Simpson's rule with in limits x = 0 and x = 1 for the curve. The difference between the two results will be.

P(0,3); Q(0.5,4); R(1,5) are collinear.

$\int_0^1 f(x).dx$ will be 0 in b/w the trapezoidal, simpson's & the analytical intr

A. 0

B. 0.25

C. 0.5

D. 1

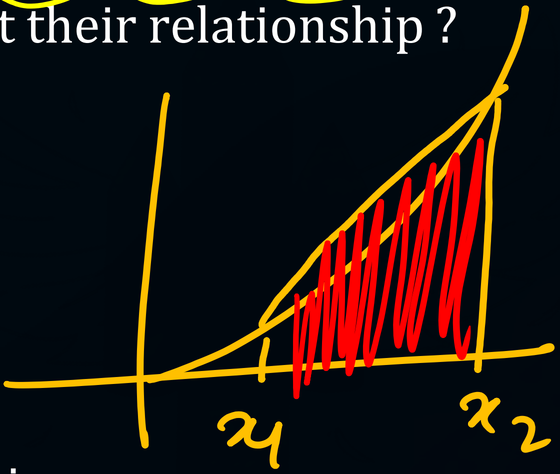
Topic : Numerical Methods

#Q. The integral $\int_{x_1}^{x_2} x^2 dx$ with $x_2 > x_1 > 0$ is evaluated analytically as well as numerically using a single application of the trapezoidal rule. If I is the exact value of the integral obtained analytically and J is the approximate value obtained using the trapezoidal rule, which of the following statements is correct their relationship ?

A $J > I$

B $J < I$

C $J = I$



D Insufficient data to determine the relationship.

$x_0 = 0$

$x_1 = 0.8$

$h = \underline{0.4}$

$x_n = \underline{0.4}$

#Q. For step-size, $\Delta x = 0.4$, the value of following integral using Simpson's 1/3 rule is ________________.

$$\int_0^{0.8}(0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5)dx$$

$f(x_0) = 0.2$

$f(x_1) = 2.456$

$f(x_n) = 0.232$

$$I = \int_0^{0.8} f(x)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + y_n\right]$$

$$= \frac{0.4}{3}\left[0.2 + 4 \times 2.456 + 0.232\right]$$

$$= \underline{1.367}$$

Topic : Numerical Methods

$\int_0^4 (x^4+10)\,dx = \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right] =$

#Q. Find the magnitude of error (Correct to two decimal places) in the estimation of following intergral using Simpson 1/3 rule. Take the step length as 1. $\int_0^4 \left(x^4+10\right)dx$

$\rightarrow$ 245.33

0, 4

$h = 1$

0, 1, 2, 3, 4

$x_0 \quad x_1 \quad x_2 \quad x_3 \quad x_4$

$y_0 = f(x_0) = 10$

$y_1 = f(x_1) = 11$

$y_2 = f(x_2) = 26$

$y_3 = f(x_3) = 91$

$y_4 = f(x_4) + 10 = 266$

Topic : Numerical Methods

Flywheel energy $= \int_0^{360} T \cdot d\theta = \int_{x_1}^{x_2} f \cdot dx$

$\rightarrow \frac{1}{3} \cdot \frac{\pi}{3} [$

#Q. Torque exerted on a flywheel over a cycle is listed in the table. Flywheel energy (in J per unit cycle) using Simpson's rule is

Angle (degree)	0	60	120	180	240	300	360
Torque (N m)	0 y_0	1066 y_1	–323 y_2	0 y_3	323 y_4	–355 y_5	0 y_6

$= n = 6$

A 542

B 993

C 1444

D 1986

$h = 60^\circ = \frac{\pi}{3}$

$\Rightarrow \frac{h}{3} [(y_0 + y_n) + 4(y_1 + y_3 \ldots) + 2(y_2 + y_4 + \ldots)]$

$= 992.74]$

$\int_0 f(x).dx$

#Q. The Table below gives values of a function F (x) obtained for values of x intervalues of 0.25.
The value of the intergral of the function between the limits 0 to 1 using Simpson's rule is

X	0	0.25	0.5	0.75	1.0
F(x)	1	0.9412	0.8	0.64	0.50
	y_0	y_1	y_2	y_3	y_n

$n = 4$

$h = \frac{b-a}{n}$

$h = 0.25$

$\frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3)+2y_2\right]$

A. 0.7854

B. 2.3562

C. 3.1416

D. 7.5000

Topic : Numerical Methods

#Q. Consider the differential equation $\frac{dy}{dx} = 4(x+2) - y$ For the initial condition y = 3 at x = 1, the value of y at x = 1.4 obtained using Euler's method with a step- size of 0.2 is 6.9.
(*round off to one decimal place*)

$\frac{dy}{dx} = 4(x+2) - y$

$f(x,y) = 4(x+2) - y$

$h = 0.2$

(y_2) has to followed

	x_0	x_1	x_2
x	1	1.2	1.4
y	3	?	?

Topic : Numerical Methods

$\frac{dx}{dt} = 4t + t = f(t, x)$

at $t = 0$; $x = x_0$; $h = \Delta t = 0.2$

from RK- 4th order Method

$x_{i+1} = x_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

#Q. Consider an ordinary differential equation $\frac{dx}{dt} = 4t + t$. If $x = x_0$ at $t = 0$, the increment in x calculated using Runge-kutta fourth order multi-step method with a step size of $\Delta t = 0.2$ is :

where $k_1 = f(t_0, x_0)$

$k_2 = f\left(t_0 + \frac{h}{2}, x_0 + \frac{k_1}{2}\right)$

$k_3 = f\left(t_0 + \frac{h}{2}, x_0 + \frac{k_2}{2}\right)$

$k_4 = f(t_0 + h, x + h)$

A. 0.22

B. 0.44

C. 0.66

D. 0.88

$$\boxed{\begin{aligned} k_1 &= f(0, x_0) \\ &= 4 \times 0 + 4 = 4 \end{aligned}}$$

2 mins Summary

Topic One

Topic Two

Topic Three

Topic Four

Topic Five

Rectangular	0 degree (x^0)
Trapezoidal	1 degree (x^1)
Simpson's $\frac{1}{3}$	2 degree (x^2
Simpson's $\frac{3}{8}$	3 degree (x^3)

(23) By euler's method :-

$y_1 = y_0 + h\, f(x_0, y_0)$

$= 3 + 0.2\, f(1, 3)$, $x_0 = 1$; $y_0 = 3$

$= 4.8 = y_1$

$x_1 = 1.2$, $y_1 = 4.8$

$y_2 = y_1 + h\,(f(x_1, y_1))$

$= 4.8 + 0.2 f(1.2, 4.8)$

$= 4.8 + 0.2\,(4(1.2+2) - 4.8)$

$= 4.8 + 1.6 = 6.4$

(24.)

$k_2 = f\left(0 + \frac{0.2}{2}, x_0 + \frac{k_1}{2}\right)$

$= 4(0.1) + 4$

$= 4.4$

$k_3 = f\left(0 + \frac{0.2}{2}, x_0 + \frac{k_2}{2}\right)$

$= 4(0.1) + 4 = 4.4$

$k_4 = f(0 + 0.2, x_0 + k_3)$

$= 4(0.2) + 4 = 4.8$

$x_1 - x_0 = \frac{0.2}{6} = 0.88$

THANK - YOU

Topics to be Covered