GATE (ALL BRANCHES)

Engineering Mathematics

**Differential Equation +
Partial differential**

Lecture No. 09

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TOPICS TO BE COVERED

01. Variable Separable Method
02. Problem based on variable separable method
03. Wave Equation, Heat Equation, Laplace Equation
04. Classification of Partial D.E
05. Problems based on classifications of P.D.E

Syllabus

Q. Questions

#Q. Solve the first order Partial Differential equation,

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$$

$\frac{dy}{dx} \rightarrow y = f(x)$

$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \longrightarrow u = u(x,t) = u(X(x), T(t))$

$u(x,0) = 6e^{-3x}$

$\underbrace{\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u}$

First order P.D.E

Two variables.

$u = u(x,t)$

Independent vari

(x,t) dependent var.

Using variable separable method

$u = XT$

$\boxed{u = X(x)T(t)}$

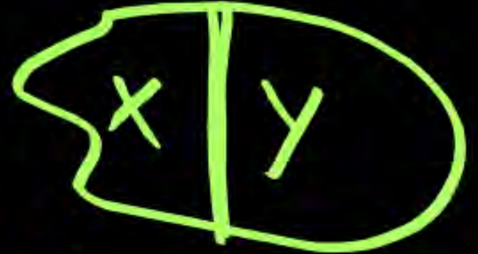
$\boxed{\frac{\partial u}{\partial x\,p} = 2\frac{\partial u}{\partial t\,q} + u}$

$u = X(x)Y(y)$

$u = P(p)Q(q)$

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$$

Put $u = XT$



$$u = X(x)\,T(t)$$

$$\frac{\partial}{\partial x}(XT) = 2\frac{\partial}{\partial t}(XT) + XT \qquad \underbrace{\frac{\partial X}{\partial x}}_{X'} T = 2\underbrace{\frac{\partial T}{\partial z}}_{T'} \cdot X + XT$$

$$\Rightarrow X'T = 2T'X + XT$$

$$\Rightarrow (X' - X)T = 2T'X$$

Variable seprate It

$$\Rightarrow \frac{X' - X}{X} = \frac{2T'}{T} = k$$

$$✓\ \frac{X' - X}{X} = k \qquad ✓\ \frac{2T'}{T} = k$$

$$\boxed{\frac{X'-X}{X} = k}$$

$$X = \frac{X'}{X} - 1 = k$$

$$= \frac{X'}{X} = (k+1)$$

both sides Integrate

$$\Rightarrow \int \frac{X'}{X}\,dx = \int (k+1)\,dx$$

$$\Rightarrow \ln X = (k+1)x + c$$

$$X = e^{(k+1)x} \cdot e^{c}$$

$$e^{c} = A$$

$$\boxed{X = Ae^{(k+1)x}}$$

$$\frac{2T'}{T} = k$$

$$\int \frac{T'}{1} = \int \frac{k}{2}\,dt$$

$$= \ln T = \frac{k}{2}t + c_1$$

$$T = e^{(k/2)t} \cdot e^{c_1}$$

$$\boxed{T = Be^{(k/2)t}}$$

$$\int \frac{f'(x)}{f(x)} = \ln[f(x)]$$

$$u(x,t) = Ae^{(k+1)x} \cdot Be^{\left(\frac{k}{2}\right)t}$$

$$= De^{(k+1)x} e^{\left(\frac{k}{2}\right)t}$$

$$u(x,t) = D e^{(k+1)x} e^{\left(\frac{k}{2}\right)t}$$

$$u(x,0) = 6e^{-3x}$$

$$\Rightarrow u(x,0) = 6e^{-3x} = De^{(k+1)x} e^{0}$$

$$\boxed{D = 6}$$

$$k+1 = -3$$

$$k = -3-1 = -4$$

$$\boxed{u(x,t) = 6e^{-3x-2t}} \checkmark$$

Question

↓

Journey

Q. Questions

#Q. Solve the Partial Differential equation,

$$\boxed{\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0}$$

SECOND order PDE

Put $z = X(x)Y(y)$

$$\frac{\partial^2 (XY)}{\partial x^2} - 2\frac{\partial (XY)}{\partial x} - \frac{\partial (XY)}{\partial y} = 0$$

$$\frac{\partial^2 X}{\partial x^2}Y - 2\frac{\partial X}{\partial x}Y - \frac{\partial Y}{\partial y}X = 0$$

$$X''Y - 2X'Y - Y'X = 0$$

variable seprate It

$$X''Y - 2X'Y - Y'X = 0$$

$$X''Y - 2X'Y = Y'X$$

$$\frac{X'' - 2X'}{X} = \frac{Y'}{Y} = k$$

Slide-4

case 1

$$\frac{X''-2X'}{X}=k$$

$$X''-2X'-kX=0$$

O.D.E with constant cofficients

$$\Rightarrow \boxed{\frac{\partial^2 X}{\partial x^2}-2\frac{\partial X}{\partial x}-kX=0}$$

$X=e^{rx}$ is a solution of D.E

$$\Rightarrow [r^2-2r-k]e^{rx}=0$$

$$\Rightarrow r^2-2r-k=0$$

$$\Rightarrow r=\frac{+2\pm\sqrt{4+4k}}{2}$$

$$\boxed{r=1\pm\sqrt{1+k}}$$

$$\frac{Y'}{Y}=k$$

both sides Integrate I.r

$$\Rightarrow \int\frac{Y'}{Y}dy=\int k\,dy$$

$$\Rightarrow \ln Y=ky+c$$

$$\Rightarrow Y=e^{ky+c}$$

$$\Rightarrow Y=e^{ky}.e^{c}$$

$$\Rightarrow \boxed{Y=Ae^{ky}}$$

A

$$\lambda = 1 \pm \sqrt{1+K}$$

Roots Are Real and Distinct

$$X = \boxed{C.F = c_1 e^{(1+\sqrt{1+K})x} + c_2 e^{(1-\sqrt{1+K})x}}$$

$$P.I = 0$$

Solution $u = XY$

$$= \left[c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}\right] e^{ky} A$$

Q. Questions

#Q. Solve the Partial Differential equation,

$$\underbrace{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

Wave Equ^n — Periodic
$u(x, t)$

Put $u = X(x)\,T(t)$

X is a function of x only

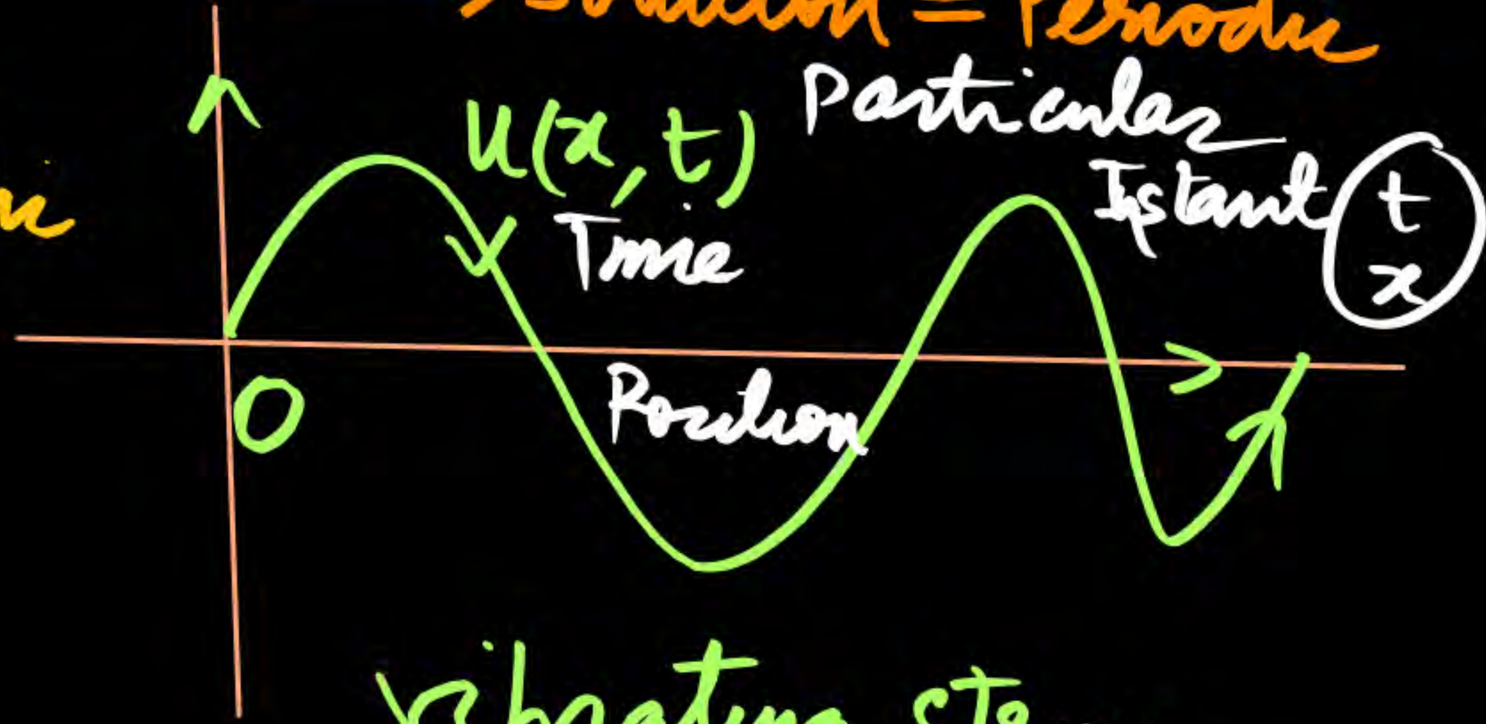
T is a function of t only.

$$\frac{\partial^2}{\partial t^2}(XT) = c^2 \frac{\partial^2}{\partial x^2}(XT)$$

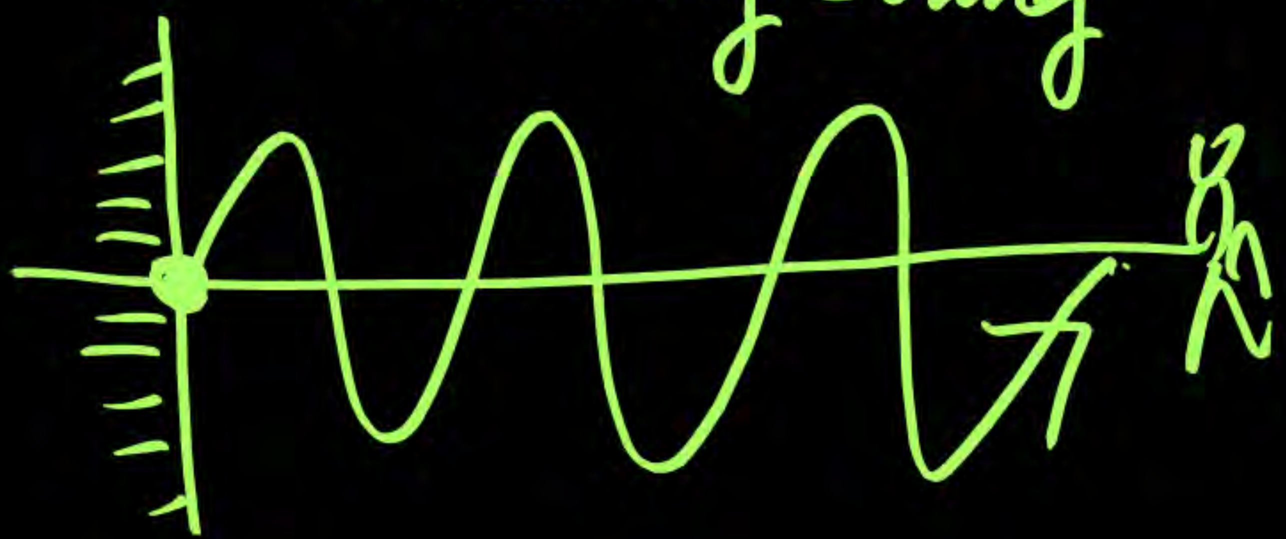
$$\boxed{XT'' = c^2 X'' T}$$

Wave Equation c = medium

→ solution = Periodic



vibrating string



$$XT'' = c^2 X'' T$$

Using variable seprate It

$$\frac{T''}{T} = c^2 \frac{X''}{X}$$

$\to K^2$ K positive

$-K^2$ Negative

0 ZERO

Maths Point of view

Physics/Eng.

$\to$ Periodic solution

CASE 01:

$$\frac{T''}{T} = k^2$$

$$T'' - Tk^2 = 0$$

$$\frac{\partial^2 T}{\partial t^2} - Tk^2 = 0$$

$T = e^{rt}$ is solution of D.E

$$(r^2 - k^2)e^{rt} = 0$$

$$\boxed{r = \pm k}$$

$$\boxed{T = c_1 e^{kt} + c_2 e^{-kt}}$$

$$c^2 \frac{X''}{X} = k^2$$

$$X'' - \frac{k^2}{c^2} X = 0$$

$$\frac{\partial^2 X}{\partial x^2} - \frac{k^2}{c^2} X = 0$$

$$X = e^{rx}$$

$$r^2 e^{rx} - \frac{k^2}{c^2} e^{rx} = 0$$

$$r^2 - \frac{K^2}{c^2} = 0$$

$$\Rightarrow r = \pm \frac{K}{c}$$

Roots Are Real and Distinct

$$X = c_3 e^{\left(\frac{K}{c}\right)x} + c_4 e^{\left(-\frac{K}{c}\right)x}$$

Solution of P.D.E

$$u = u(x,t) = X(x)T(t)$$

$$\boxed{u(x,t) = \left[c_1 e^{Kt} + c_2 e^{-Kt}\right]\left[c_3 e^{\left(\frac{K}{c}\right)x} + c_4 e^{\left(-\frac{K}{c}\right)x}\right]}$$

Solution but Not Periodic

CASE 02

$$\frac{T''}{T} = -K^2$$

$$T'' + K^2 T = 0$$

$= T = e^{rt}$ is a sol^n of D.E

$$\Rightarrow (r^2 + K^2)e^{rt} = 0$$

$$\Rightarrow r = \pm Ki$$

$$\boxed{T = C_5 \cos Kt + C_6 \sin Kt}$$

$$c^2 \frac{X''}{X} = -K^2 \qquad X = e^{rx}$$

$$X'' + \frac{K^2}{c^2} X = 0$$

$$r^2 + \frac{K^2}{c^2} = 0$$

$$r = \pm \frac{K}{c} i$$

$$\boxed{X = C_7 \cos\left(\frac{K}{c}\right)x + C_8 \sin\left(\frac{K}{c}\right)x}$$

Solution of PDE

$u = XT$

IInd solution

$$\boxed{u = \left[C_7 \cos\left(\frac{K}{c}\right)x + C_8 \sin\left(\frac{K}{c}\right)x\right]\left[C_5 \cos Kt + C_6 \sin Kt\right]}$$

(Periodic)

CASE 03

$$\frac{T''}{T} = 0$$

$$\frac{X''}{X}c^2 = 0$$

$$T'' = 0$$

$$X'' = 0$$

$$\frac{\partial^2 T}{\partial t^2} = 0$$

$$\frac{\partial^2 X}{\partial x^2} = 0$$

$T = e^{rt}$ is a sol^n

$X = e^{rx}$ is a sol^n of D.E

$$\boxed{r = 0, 0}$$

$$r = 0, 0$$

$$T = (C_9 + C_{10}t)e^{0t}$$

$$X = (C_{11} + C_{12}x)e^{0x}$$

$$\boxed{T = C_9 + C_{10}t}$$

$$\boxed{X = C_{11} + C_{12}x}$$

$$\boxed{\text{Solution of D.E} = u(x,t) = (C_9 + C_{10}t)(C_{11} + C_{12}x)}$$

Q. Questions

#Q. Solve the Partial Differential equation,

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$ (HEAT Equation)

$u = XT$

$= K^2$ ✓

$= -K^2$ ✓

$= 0$ ✓

for

HEAT Propagate ⇒ Periodic

Do yourself

Q. Questions

#Q. Solve the Laplace equation,

✓ $$\boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0}$$ Laplace equ^n

✓ Do yourself

$u = XY$

$u = X(x)Y(y)$

Solution: K^2, $-K^2$, 0

Classification of SECOND order P.D.E with constant coefficients :-

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\boxed{\partial y \partial x}} + C\frac{\partial^2 u}{\boxed{\partial y^2}} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G(x, y)$$

Where A, B, C, D, E, F Are constant and G(x, y) is a function of x, y only.

Behaviour Depend on A, B, C

- $B^2 - 4AC = 0$ Parabolic
- $B^2 - 4AC < 0$ Elliptical
- $B^2 - 4AC > 0$ Hyperbolic

Q. Questions

#Q. The partial differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} = 0$$

DEGREE = 1
order = 2

(A) Degree 1, Order 2 ✓

(B) Degree 1, Order 1

(C) Degree 2, Order 1

(D) Degree 2, Order 2

Questions

#Q. Consider the following P.D.E for u(x,y) with constant c>1

$$\boxed{\frac{\partial u}{\partial y} + c\frac{\partial u}{\partial x} = 0}$$

Do yourself

(A) f(x+cy) ✓

(B) f(x-cy) ✓

(C) f(cx+y) ✓

(D) f(cx-y) ✓

Q. Questions

#Q. The number of boundary conditions required to solve the differential equation

$$\underbrace{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}}_{} = 0$$

4 Constants

(A) 1

(B) 2

(C) 0

(D) 4

Under condition 4

Q. Questions

#Q. The type of partial differential equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial x \partial y} + 2\frac{\partial p}{\partial x} - \frac{\partial p}{\partial y} = 0$$

(A) Elliptic

(B) Parabolic

(C) Hyperbolic

(D) None of these

$A = 1$
$B = 1$
$C = 1$

$B^2 - 4AC$
$= (1)^2 - 4 \times 1 \times 1$
$= 1 - 4$
$= -3$

$B^2 - 4AC < 0$

Q. Questions

#Q. Consider the following P.D.E

$$\frac{3\partial^2 p}{\partial x^2} + \frac{3\partial^2 p}{\partial y^2} + \frac{B\partial^2 p}{\partial x \partial y} + 4p = 0$$

For this equation is classified as patrabolic, the value of B^2 is 36

$A = 3$
$B = B$
$C = 3$

$B^2 - 4AC = 0$ (parabolic)

$B^2 = 4AC$

$= 4 \times 3 \times 3 = 36$

$\boxed{B^2 = 36}$

Thank You!

PW Soldiers