# GATE (ALL BRANCHES)

## Engineering Mathematics

# Complex Analysis

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Lecture No. 02

# TOPICS TO BE COVERED

01 Functions of Complex Variable

02 Derivative of complex variable

03 Cauchy Reimann functions, Harmonic functions, Harmonic Conjugates

04 Problems based on Complex functions, C-R equations

# Complex Functions:

**Real Functions**

$y = f(x)$

$x$ = Real No

$y$ = Real No.

**Complex Function**

$W = f(z)$

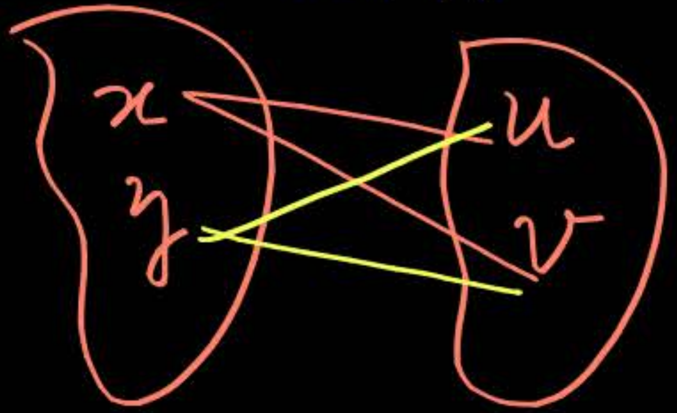
$z = x + iy$

$W = u + iv$

$u + iv = f(x + iy)$

$$\boxed{W = f(z) = u(x,y) + iv(x,y)}$$

$\hookrightarrow$ complex Function




$u(x,y)$ = Real Part of $f(z)$

$v(x,y)$ = Real part of $f(z)$

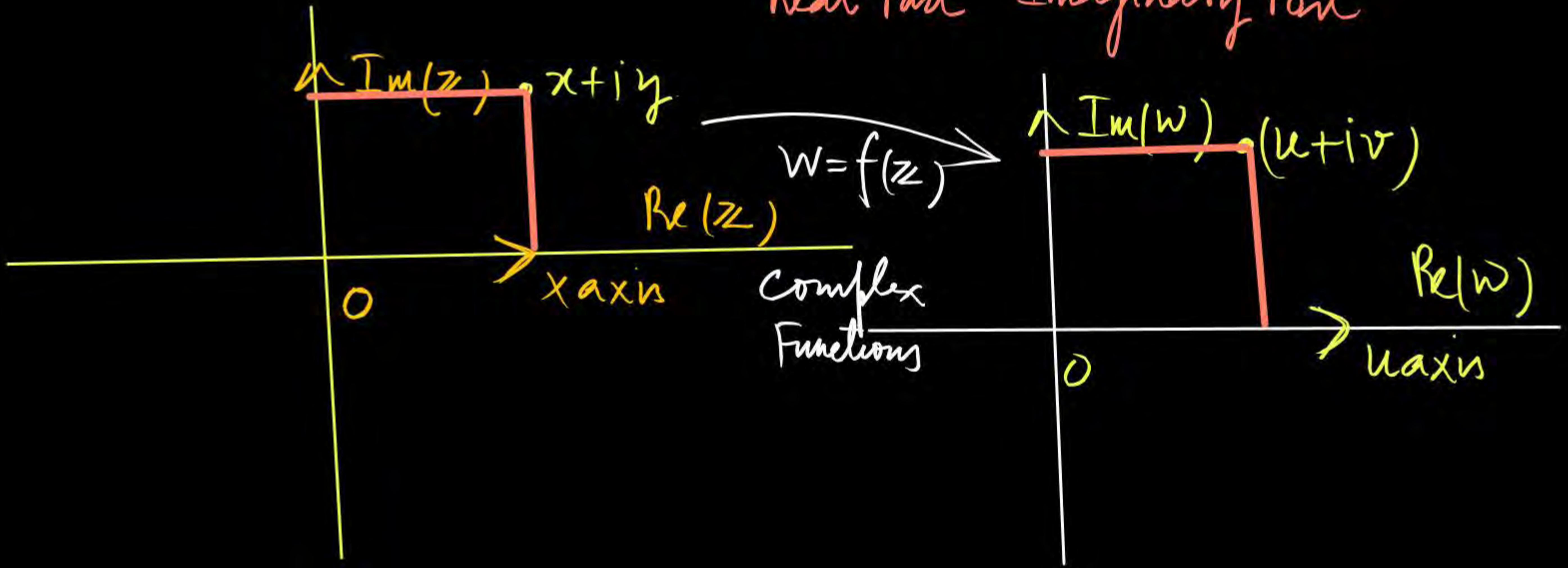
$W = f(z) = z^2$ Complex Function

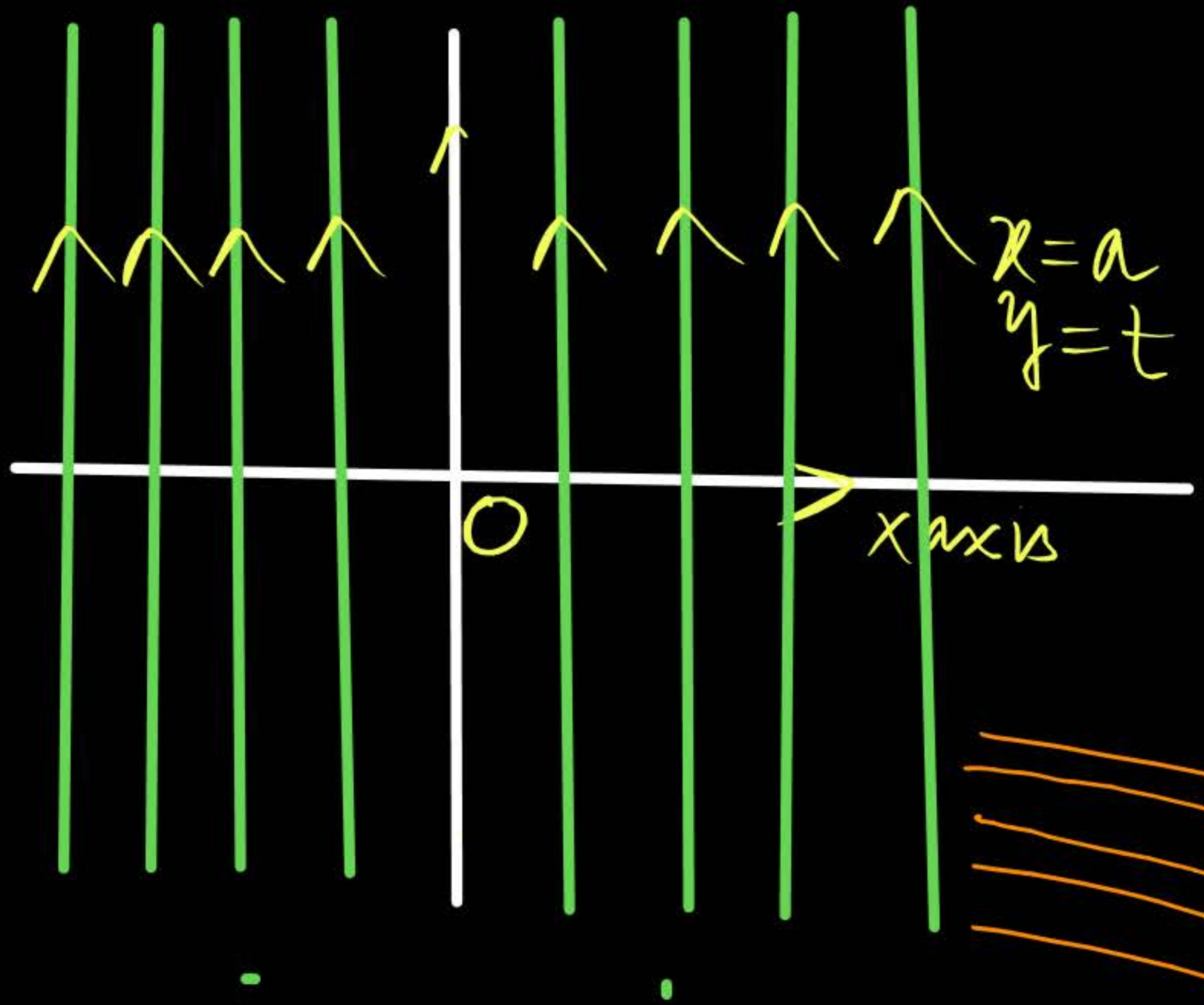
Put $z = (x+iy)$

$$W = f(z) = (x+iy)^2 = \underbrace{(x^2-y^2)}_{\text{Real Part}} + \underbrace{2ixy}_{\text{Imaginary Part}}$$

$u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$





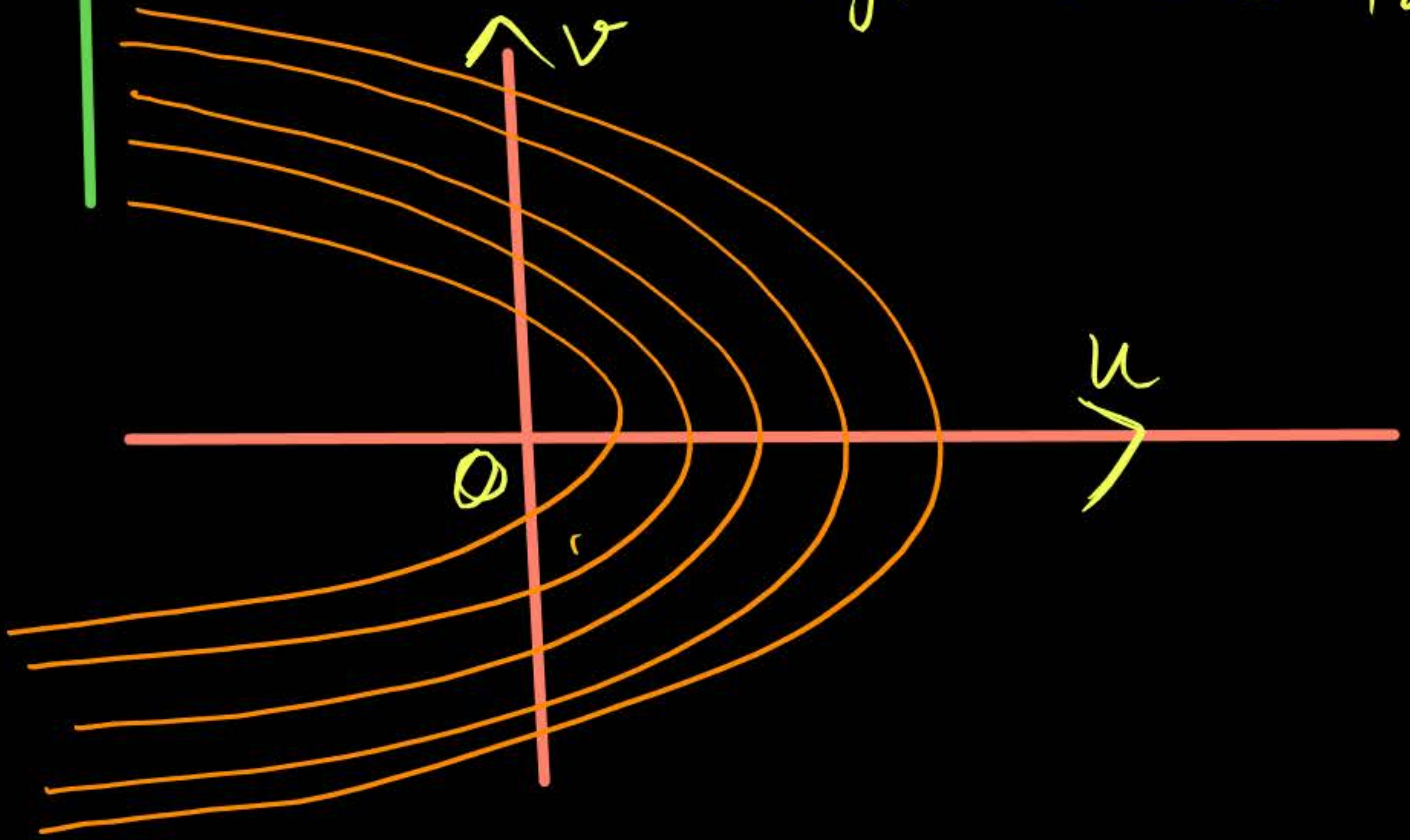

$$f(z) = z^2 = (x+iy)^2$$
$$= (x^2-y^2) + 2ixy$$

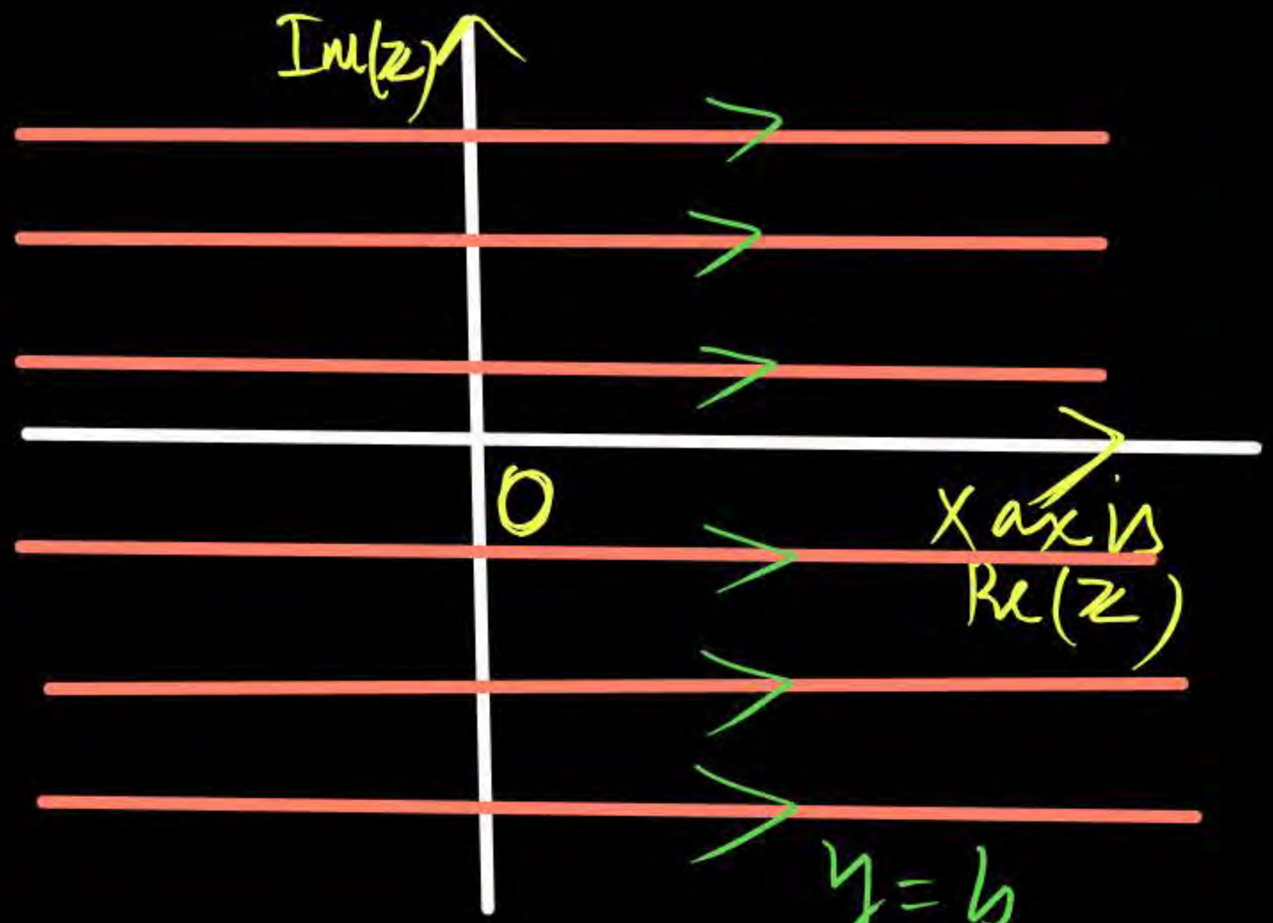
$u(x,y) = x^2-y^2 \quad v(x,y) = 2xy$

$$\left.\begin{aligned} u(x,y) &= a^2-t^2 \\ v(x,y) &= 2at \end{aligned}\right]$$

Parabola Parametric form:

$$\left.\begin{aligned} x &= at^2 \\ y &= 2at \end{aligned}\right]$$





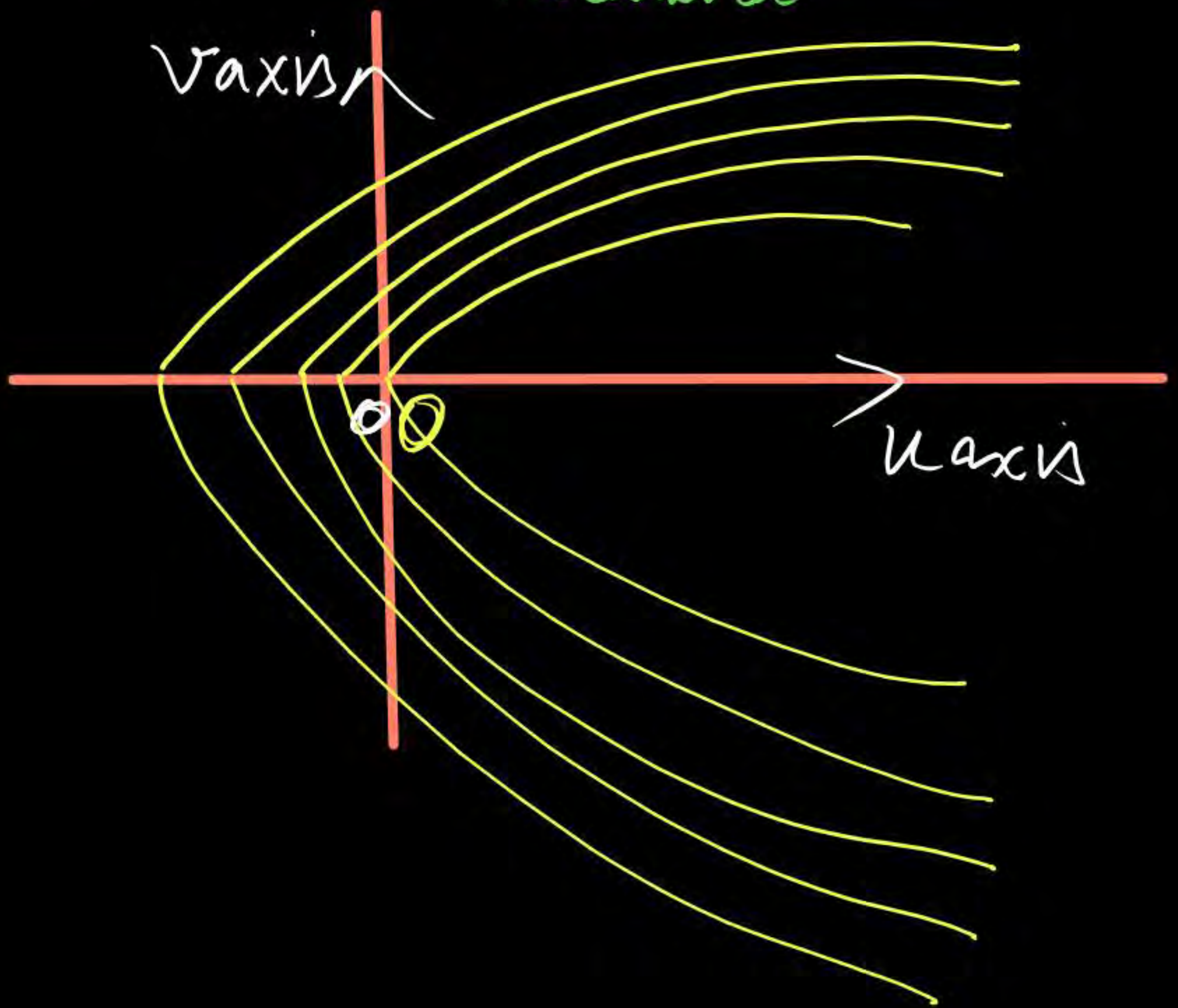

$y = b$

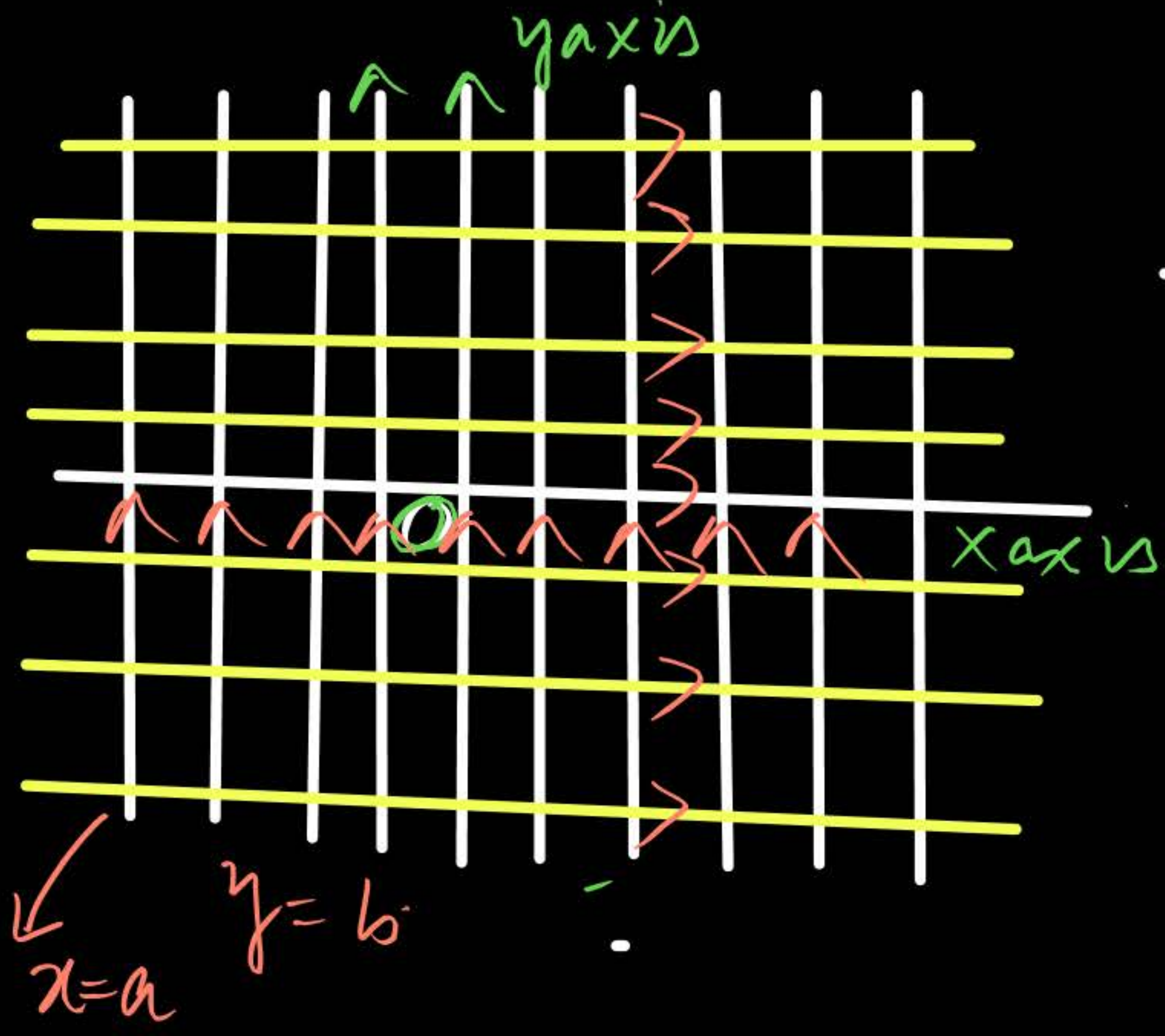
$x = t$

$u(x,y) = x^2 - y^2 \quad v(x,y) = 2xy$

$u(x,y) = t^2 - b^2 \quad v(x,y) = 2bt$

Parabola



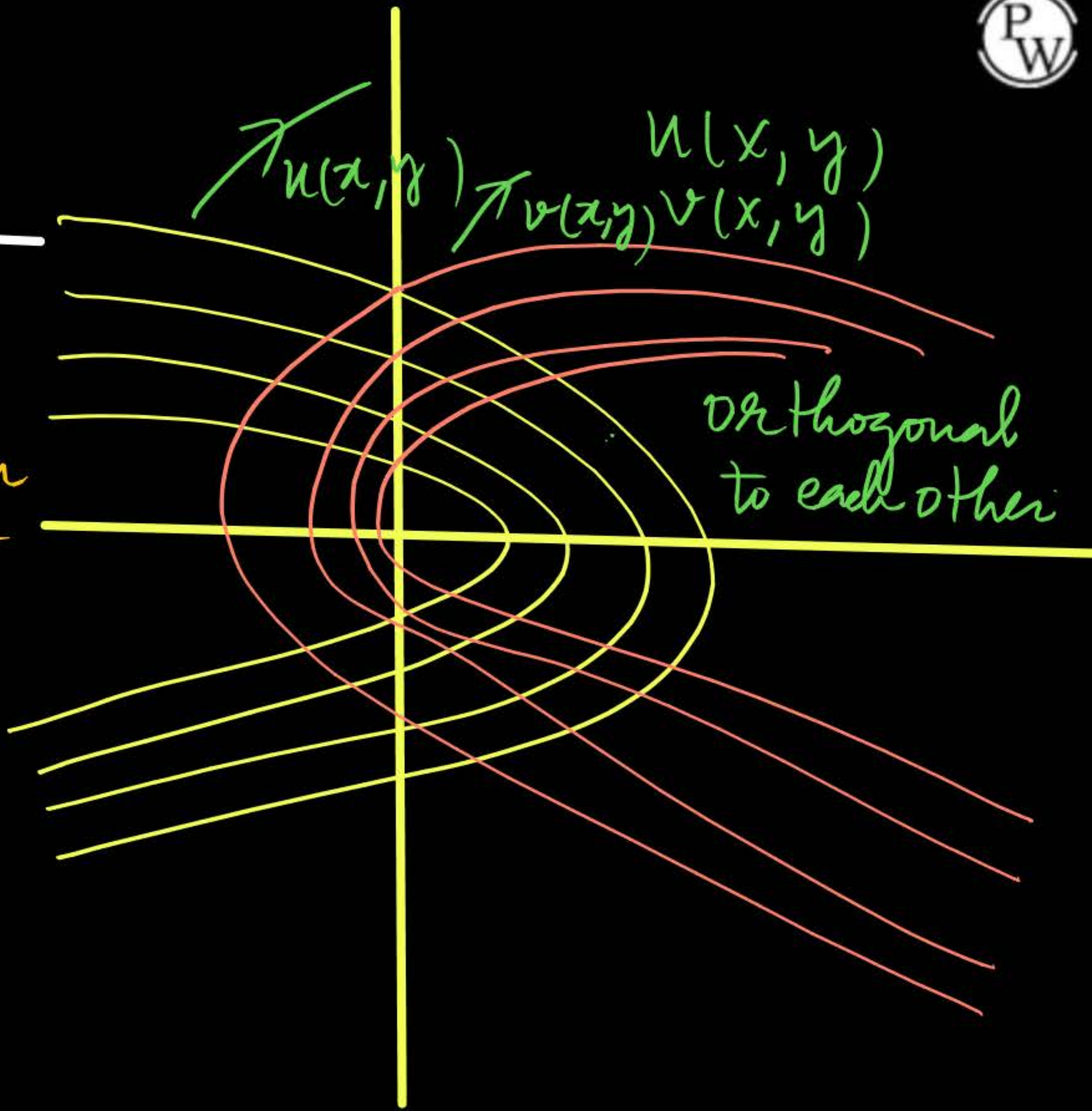



$w = f(z)$

Complex Function

$u = x^2 - y^2$

$v = 2xy$



$$W = f(z) = \underbrace{u(x,y)}_{\text{Real Part}} + i\,\underbrace{v(x,y)}_{\text{Imaginary Part}}$$

Fluid Dynamics

$$W = f(z) = \underbrace{\phi(x,y)}_{\substack{\text{Potential Function} \\ \text{(Real Part)}}} + i\,\underbrace{\psi(x,y)}_{\substack{\text{Stream Function} \\ \text{Imaginary}}}$$

Complex Functions

$$\left\{\begin{array}{l} W = f(z) = e^{z} \\ W = \sin z \\ W = \cos z \\ W = z^2 \\ W = z^3 \\ W = 1 + z^2 \end{array}\right.$$

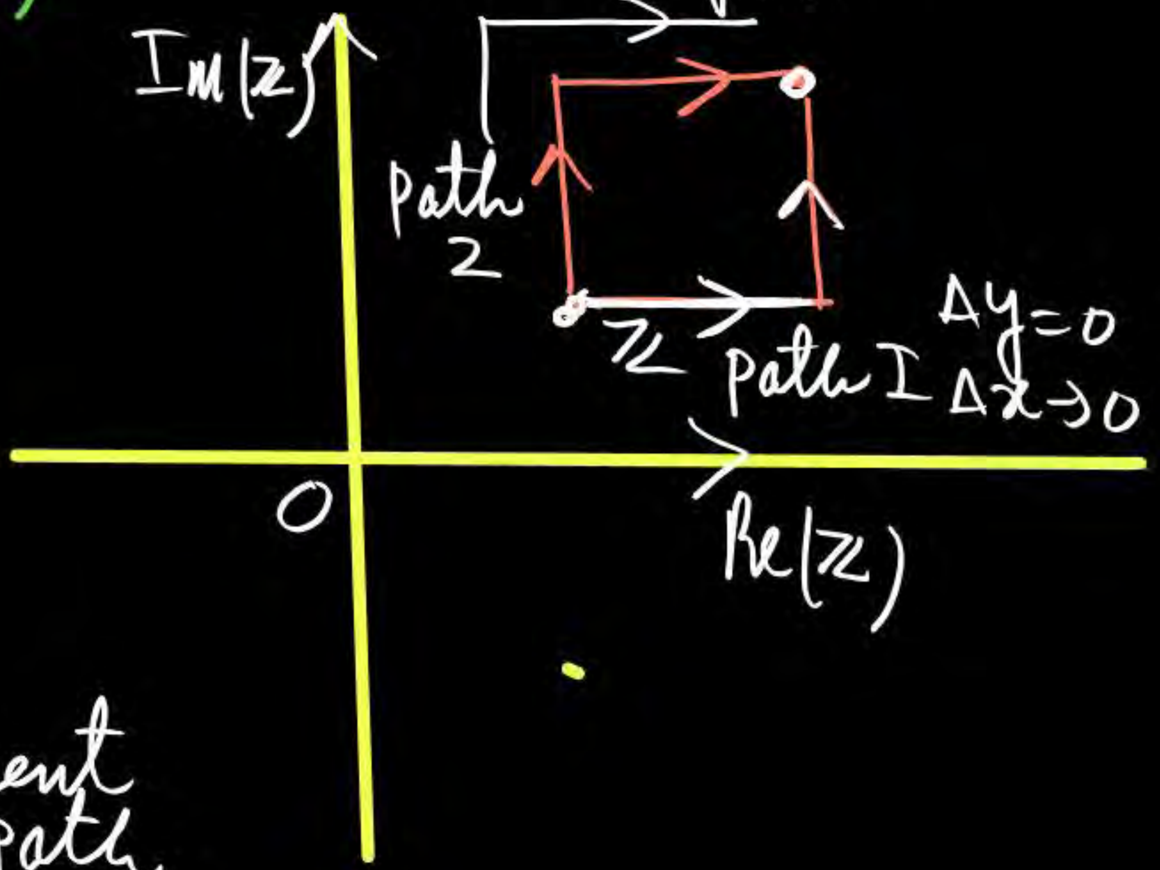
# derivative of complex Functions :

$W = f(z)$ is a complex Function Then derivative of $f(z)$

derivative of complex Function $= f'(z) = \frac{d}{dz}[f(z)] = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

$W = f(z) = u(x,y) + iv(x,y)$

$z$ is Diff. or Not

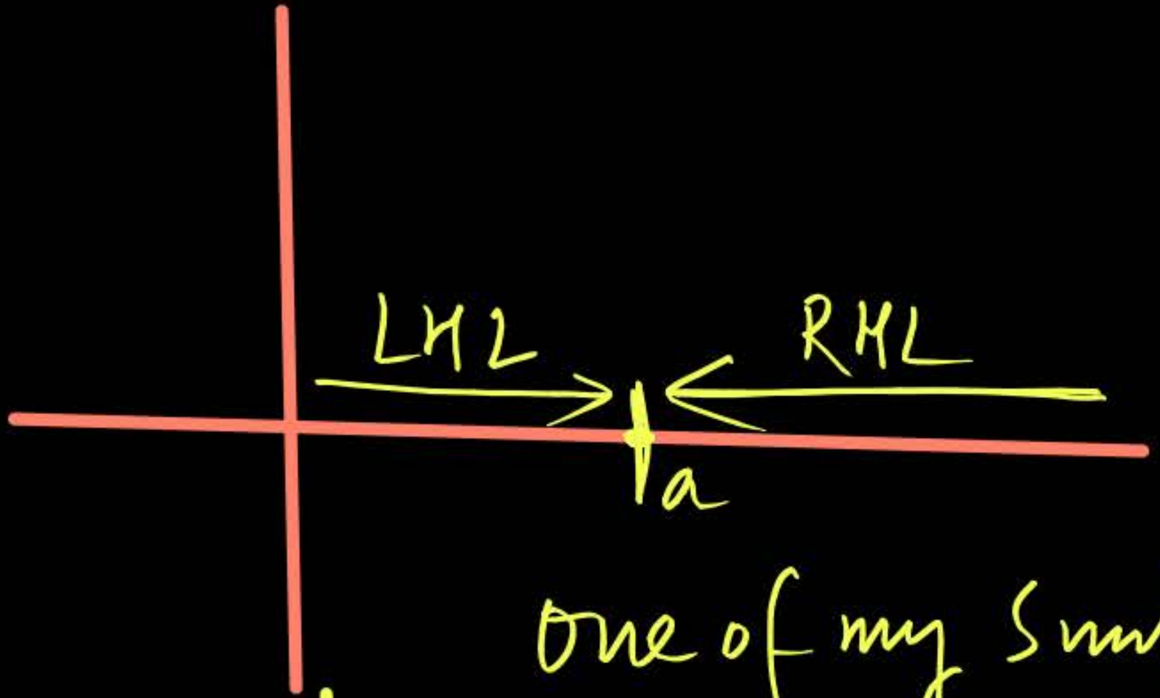



• Real No — Real Line

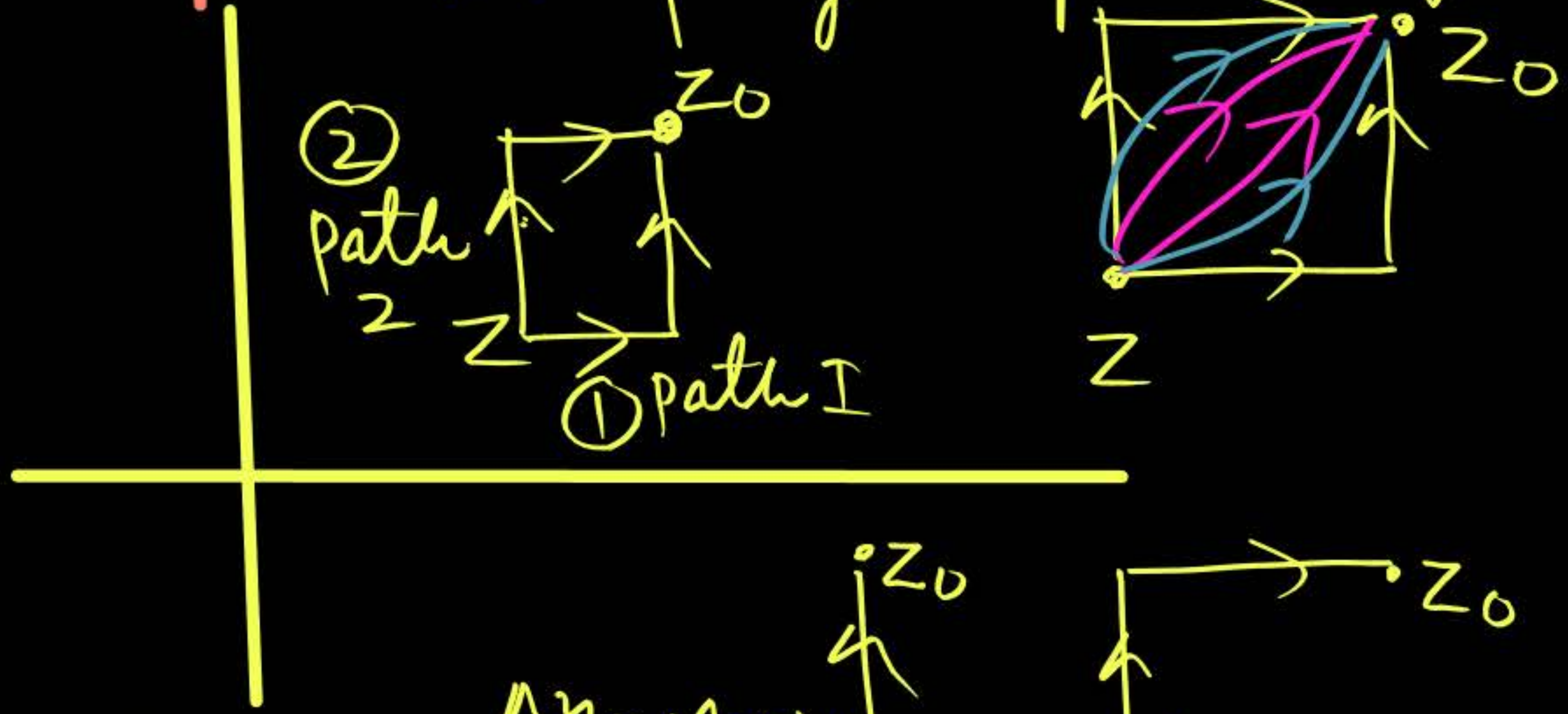
• complex Pt → Disc

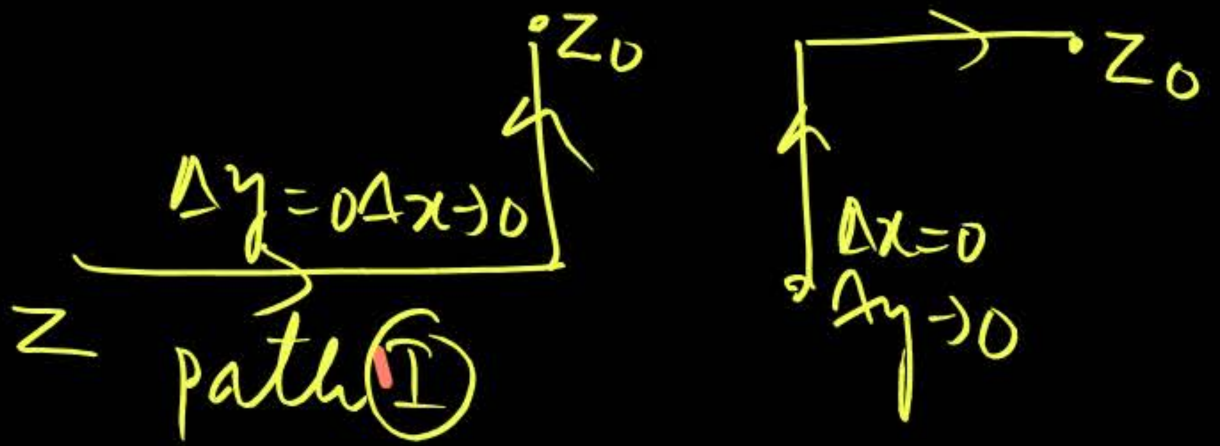

Neighbourhood

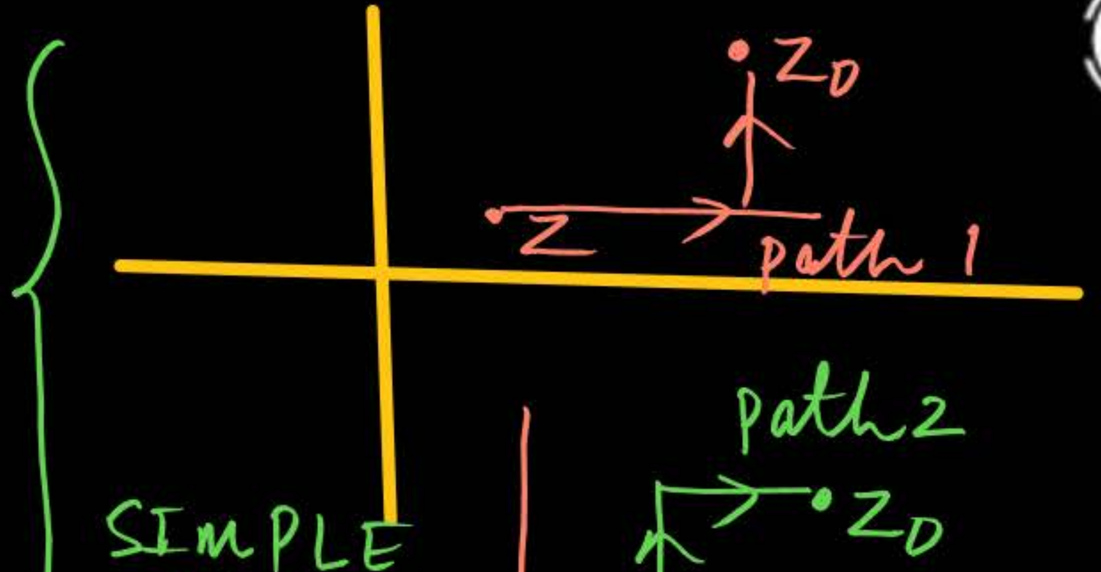

Different path

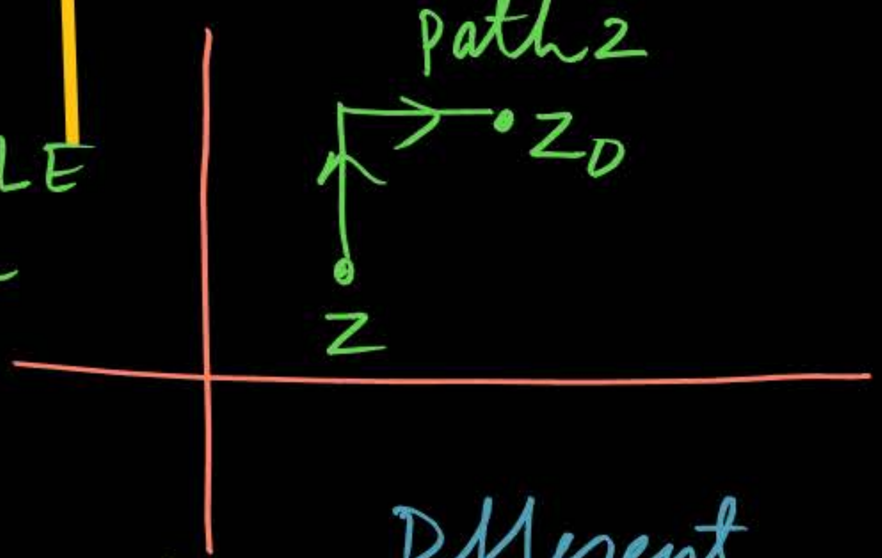


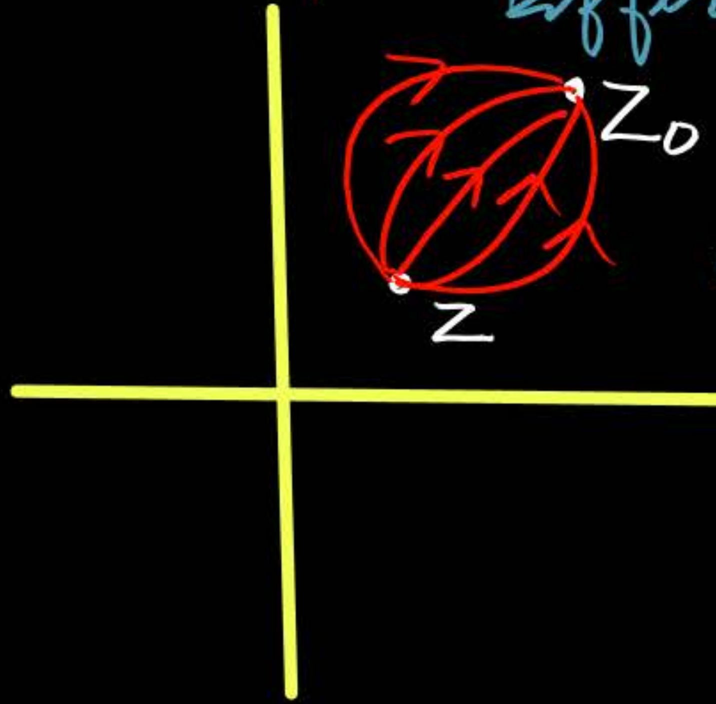

One of my Simplest way.



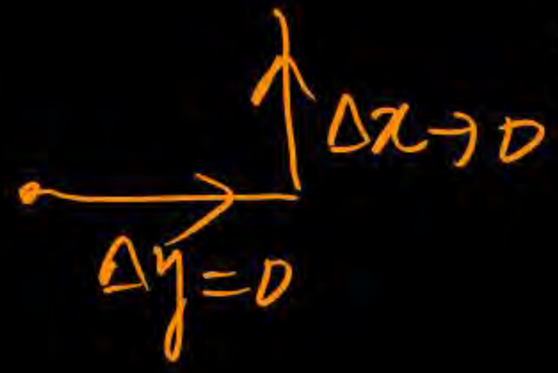




Different angles or Arguement

Path (I)




$$W = f(z) = u(x,y) + i\,v(x,y)$$

$$\Delta z = \Delta x + i\Delta y.$$

derivative of complex Function

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

path(I)

$$f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y)$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y = 0}} \frac{u(x+\Delta x, y+\Delta y) + i\,v(x+\Delta x, y+\Delta y) - [u(x,y) + i\,v(x,y))}{\Delta x + i\Delta y}$$

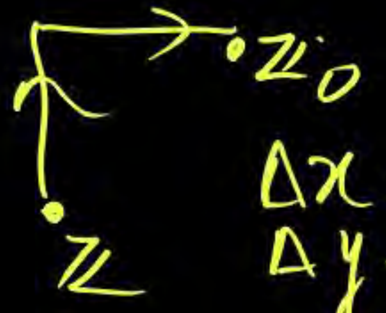
Put $\Delta y = 0$ $\Delta x \to 0$

$$\left[f'(z)\right]_{\text{Path 1}} = \lim_{\Delta x \to 0} \left[\frac{u(\overbrace{x+\Delta x}^{\text{change in } x}, y) - u(x,y)}{\Delta x}\right] + i\left[\frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}\right]$$

$$\left[f'(z)\right]_{\text{path 1}} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

for Path 2:

$$\frac{d}{dz}[f(z)]_{\text{path 2}} = f'(z) = \lim_{\Delta z \to 0} \left.\frac{f(z+\Delta z)-f(z)}{\Delta z}\right|_{\text{path 2}}$$




$\Delta x = 0$

$\Delta y \to 0$

$$= \lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

$$\boxed{[f'(z)]_{\text{path 2}} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}}$$

Condition for existence of derivative

If derivative of $[f(z) \text{ path } 1] = [f'(z)]_{\text{path 2}}$

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Compare with real and Imaginary Parts

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \,;\, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

This condition satisfied the Differentiability

$W = f(z) = u(x,y) + iv(x,y)$ Complex Functions

satisfied the $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \,,\, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$ Cauchy Riemann Equations (C-R equations)

# Differentiable + continuous $\longrightarrow$ Analytic condition

Any complex functions satisfied the C-R equation Then complex function said That Analytic.

$\boxed{\text{C-R equations} \rightarrow \text{Analytic}}$

1) Analytic Function OR Entire Function $\longrightarrow$ C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

2) $W = f(z) = u(x,y) + i\,v(x,y)$

In fluid Dynamics

$u \longrightarrow \phi$

$v \longrightarrow \psi$

$$W = f(z) = \overbrace{\underbrace{\phi(x,y)}_{\text{Potential function}}}^{\text{Real}} + i\,\overbrace{\underbrace{\psi(x,y)}_{\text{Stream function}}}^{\text{Imaginary}}$$

$$\boxed{\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}}$$

C-R equation

# Complex derivative:

$$w = f(z) = u(x,y) + i v(x,y)$$

✓ CASE 01 If $\boxed{u(x,y) \text{ is given}}$

derivative of $f(z) = f'(z) = \boxed{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}$

Using C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$f'(z) = \frac{\partial u}{\partial x} + i\left(-\frac{\partial u}{\partial y}\right) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

If $u(x,y)$ is given

$$\boxed{f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}}$$

CASE 02 If $v(x,y)$ is given

$$f(z) = u(x,y) + i\,v(x,y)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\Rightarrow \boxed{f'(z) = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}}$$

$\rightarrow$ $v(x,y)$ is given

Using C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

2 Marks / Imp # Harmonic Function : $w = f(z) = u(x, y) + i\,v(x, y)$

If any complex Function Satisfies The C-R equation → Analytic Function

If any analytic Function Satisfied the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ For $u(x, y)$

OR $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ for $v(x, y)$

Then function is Harmonic.

2 marks

# Harmonic conjugate :

$u(x, y) \xrightarrow[\text{conjugate}]{\text{Harmonic}} v(x, y)$

$v(x, y) \longrightarrow u(x, y)$

| Functions | Harmonic conjugate |
|---|---|
| $u(x,y)$ | $v(x,y)$ |
| $v(x,y)$ | $u(x,y)$ |

CASE 01 If $u(x,y)$ is given and find the Harmonic conjugate $v(x,y)$

$dv$ = Exact Differential

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$ equation in $v$ terms

$\downarrow$ Using C-R equations

Change in $u$ terms

Using C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$$\boxed{dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy}$$ Equation in $u$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

both sides Integrate It

$$\int dv = \int -\frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dy$$

$v(x,y)$ is a required Harmonic Conjugate.

If you Integrate It Then follow the

$$Mdx + Ndy = 0$$

$$\int M dx + \int N dy = C$$

Treating $y$ as a constant

Independent of $x$

# Case-02

If $v(x,y)$ is given and find The Harmonic Conjugate $u(x,y)$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

Using C-R equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$du = \frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy$$

both sides Integrate It

$$\int du = \int \frac{\partial v}{\partial y}dx - \int \frac{\partial v}{\partial x}dy$$

$u(x, y)$ is required Harmonic Conjugate

Follow The Rule

$$Mdx + Ndy = 0$$

$$\int M dx + \int N dy = C$$

Treating $y$ as a const

Independent of $x$

# Thank You!

**PW Soldiers**