# GATE (ALL BRANCHES)

## Engineering Mathematics

**Differential Equation +**
**Partial differential**

**Lecture No. 02**

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Variable seprable method

- 01 Problems based on Variable Separable method
- 02 Homogeneous Differential Equation
- 03 Reducible to Homogeneous
- 04 Problems based on Homogeneous and non Homogeneous D.E

6 to 7:30

## CASE 02: Reduced to Variable Seprable form:

$$\frac{dy}{dx} = X(x)\,Y(y) \text{ or } \frac{X(x)}{Y(y)}$$

$X(x)$ is a function of $x$ only
$Y(y)$ is a function of $y$ only.

$$\frac{dy}{dx} = X(x) + Y(y) \text{ or } X(x) - Y(y)$$

Using var. seprable method

Using reduced to variable seprable form:

$$\frac{dy}{dx} = f(ax+by+c)$$

$$\frac{dy}{dx} = f(ax+bx+c)$$ where a, b, c Are constants

$ax+by+c=t$

both sides Differentiate It

$$a\cdot 1 + b\frac{dy}{dx} + 0 = \frac{dt}{dx}$$

$$b\frac{dy}{dx} = \frac{dt}{dx} - a$$

$$\boxed{\frac{dy}{dx} = \frac{1}{b}\left[\frac{dt}{dx} - a\right]}$$

$$\Rightarrow \frac{1}{b}\left[\frac{dt}{dx} - a\right] = f(t)$$

$\Rightarrow$ Now seprate the variable and get the solution of differential Equation

## Q. Questions

#Q. The general solution of the differential equation $\frac{dy}{dx} = \cos(x + y)$, with $c$ as a constant, is

(a) $y + \sin(x + y) = x + c$

(b) $\tan\left(\frac{x+y}{2}\right) = y + c$

(c) $\cos\left(\frac{x+y}{2}\right) = x + c$

(d) $\tan\left(\frac{x+y}{2}\right) = x + c$

$\frac{dy}{dx} = \cos(x+y)$ Put $x+y=t$

→ var. separable ✗

→ Reduced to var. separable ✓

$x+y=t$

both sides Diff. It

$\Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$, $\frac{dt}{dx} - 1 = \frac{dy}{dx}$

$\Rightarrow \frac{dt}{dx} - 1 = \cos t$

$\Rightarrow \frac{dt}{dx} = 1 + \cos t$ Now variable Seprate It

$$\int \frac{dt}{1+\cos t} = \int dx$$

both sides Integrate It

$$\Rightarrow \int \frac{dt}{\cancel{1}+2\cos^2\frac{t}{2}\cancel{-1}} = \int dx$$

Solution of D.E

$$\Rightarrow \frac{1}{2}\int \sec^2\frac{t}{2}\,dt = \int dx$$

$$\boxed{\tan\left(\frac{x+y}{2}\right) = x+C}$$

$$\Rightarrow \frac{1}{\cancel{2}} \tan\frac{t}{2}\,\cancel{2} = x+C$$

Ans

$$= \tan\frac{t}{2} = x+C$$

#Q. Which one of the following is the general solution of the first order differential equation $\frac{dy}{dx} = (x + y - 1)^2$ where x, y are real?

(a) $y = 1 + x + \tan^{-1}(x + c)$, where c is a constant

(b) $y = 1 + x + \tan(x + c)$, where c is a constant

(c) $y = 1 - x + \tan^{-1}(x + c)$, where c is a constant

(d) $y = 1 - x + \tan(x + c)$, where c is a constant

$$\frac{dy}{dx} = (x+y-1)^2$$

Put $x+y-1 = t$

= both sides Diff. It

$$= 1 + \frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{dt}{dx} - 1\right)$$

$$\frac{dt}{dx} - 1 = t^2$$

$$\Rightarrow \frac{dt}{dx} = 1 + t^2$$

Now sepnate The variables.

$$\Rightarrow \int \frac{dt}{1+t^2} = \int dx$$

$$\Rightarrow \tan^{-1} t = x + c$$

$$= \quad t = \tan(x+c)$$

$$= \quad x+y-1 = \tan(x+c)$$

## Q. Questions

#Q. Which one of the following curves correctly represents the equation $\frac{df}{dx} + 2f = 3$

$f(0) = 0$

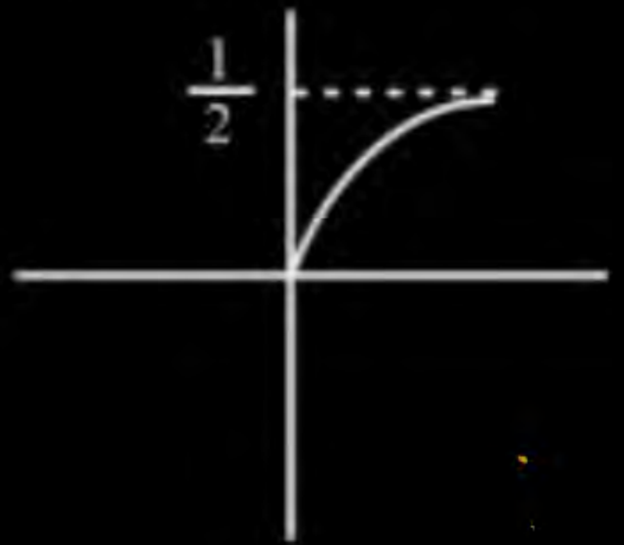
$y = f(x) = \frac{3}{2}\left(1 - e^{-2x}\right)$

At $x = 0$

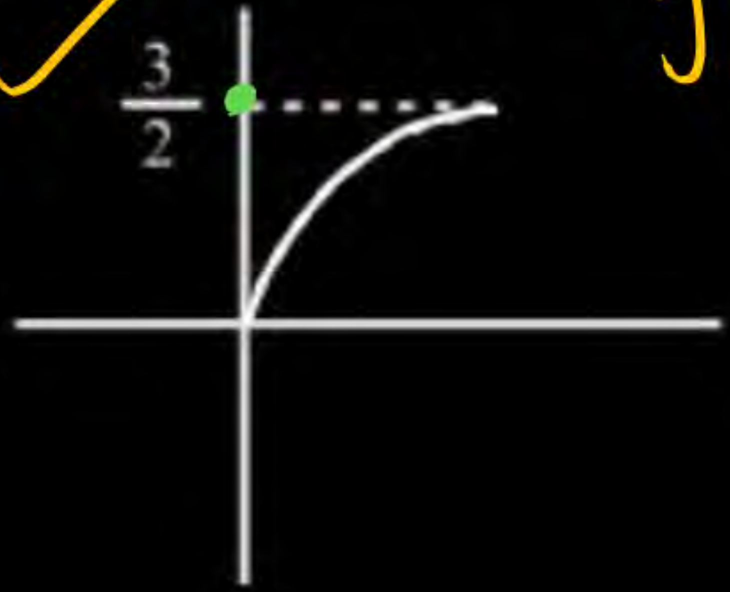
$= \frac{3}{2}\left(1 - e^{0}\right)$

$= \underline{\underline{0}}$

(a)




(b)

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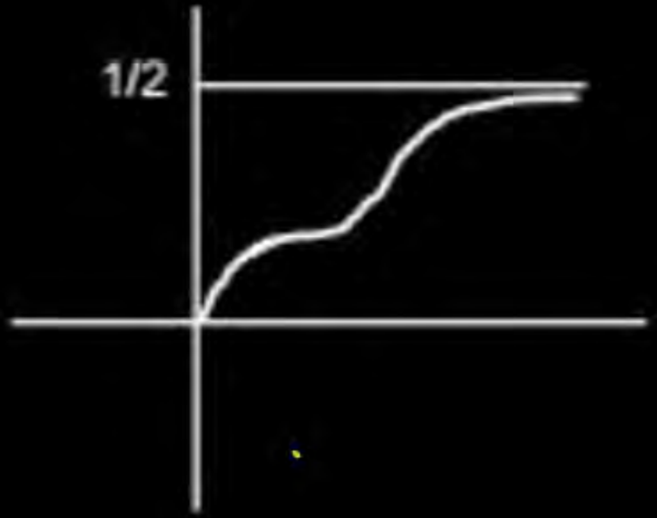
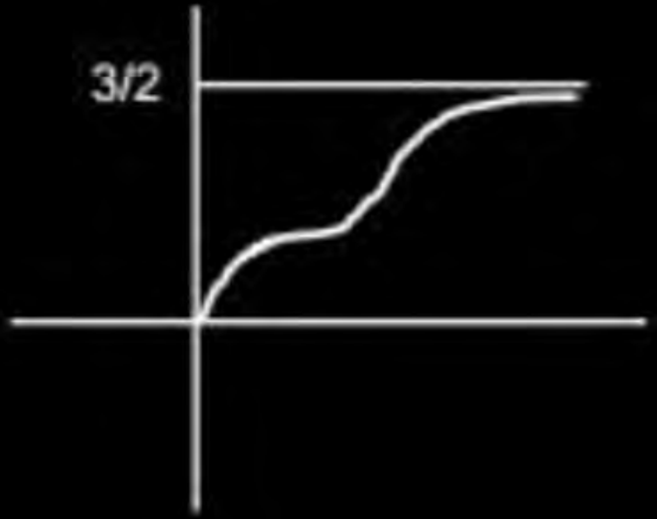



(d)



$\frac{df}{dx} + 2f = 3 \quad f(0) = 0$

Using variable seprable method

$\Rightarrow \frac{df}{dx} = 3 - 2f$

$\Rightarrow \frac{df}{3-2f} = dx$

both sides integrate It

$\int \frac{df}{3-2f} = \int dx$

$-\frac{1}{2} \ln(3-2f) = x + c$

$I = \int \frac{df}{3-2f}$

$3 - 2f = t$

$-2\,df = dt \quad df = \frac{-dt}{2}$

$\Rightarrow -\int \frac{dt}{2(t)} = -\frac{1}{2} \ln|t|$

$= -\frac{1}{2} \ln|3-2f|$

$-\frac{1}{2} \ln(3-2f) = x + c$

$= \boxed{-\frac{1}{2} \ln(3-2f) = x + c}$

$= -\frac{1}{2} \ln(3 - 2 \times 0) = 0 + c$

$= \boxed{c = -\frac{1}{2} \ln 3}$

$$= -\frac{1}{2}\ln(3-2f) = x - \frac{1}{2}\ln 3$$

$$= \frac{1}{2}\ln 3 - \frac{1}{2}\ln(3-2f) = x$$

$$= \frac{1}{2}\ln\left(\frac{3}{3-2f}\right) = x$$

$$\left(\frac{3}{3-2f}\right) = e^{2x}$$

$$3 = 3e^{2x} - 2fe^{2x}$$

$$3 - 3e^{2x} = -2fe^{2x}$$

$$\frac{3e^{2x}-3}{2e^{2x}} = f$$

$$f = \frac{3}{2} - \frac{3}{2}e^{-2x}$$

$$\boxed{f = \frac{3}{2}\left(1 - e^{-2x}\right)}$$

# Q. Questions

#Q. Let $f(t)$ be a Non-negative function defined on interval [0, 1]

$$\int_0^x \sqrt{1-[f'(t)]^2}\,dt = \int_0^x f(t)\,dt$$

$$f(0)=0 \qquad 0\le x\le 1$$

(a) $f(1/2)<1/2, f(1/3)<1/3$

(b) $f(1/2)<1/2, f(1/3)>1/3$

(c) $f(1/2)>1/2, f(1/3)<1/3$

(d) $f(1/2)>1/2, f(1/3)>1/2$

$$\int_0^x \sqrt{1-[f'(t)]^2}\,dt = \int_0^x f(t)\,dt$$

Remove The Integral

Apply Newton-Leibnitz Rule.

$$\sqrt{1-[f'(x)]^2}\cdot\frac{d}{dx}(x) - \sqrt{1-[f'(0)]^2}\cdot\frac{d}{dx}(0)$$

$$= f(x)\frac{d}{dx}(x)$$

$$\Rightarrow \sqrt{1-[f'(x)]^2}\cdot 1 = f(x)$$

Put $y = f(x)$

$y' = \frac{dy}{dx} = f'(x)$

$$\Rightarrow \sqrt{1-\left(\frac{dy}{dx}\right)^2} = y$$

both sides Square It

$$\Rightarrow 1-\left(\frac{dy}{dx}\right)^2 = y^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = 1-y^2$$

$$\Rightarrow \frac{dy}{dx} = \pm\sqrt{1-y^2}$$

This form is D.E

Using Var. separable method

$$\int \frac{dy}{\sqrt{1-y^2}} = \int \pm\, dx$$

$$\sin^{-1} y = \pm x + c$$

$$\boxed{y = \pm \sin(x+c)}$$

$$y = \sin(x+c)$$

$$\Rightarrow f(x) = \sin(x+c)$$

$$\Rightarrow 0 = \sin(0+c)$$

$$\Rightarrow \boxed{c = 0}$$

$$f(x) = \sin x$$

$$f(0) = 0$$

$$x = 0 \quad f(0) = 0$$

$$\sin x < x$$

$$f(x) < x$$

$$\boxed{f\left(\frac{1}{2}\right) < \frac{1}{2},\ f\left(\frac{1}{3}\right) < \frac{1}{3}}$$

Ans.

A

## Homogenous Differential Equ$^n$ :

$f(tx, ty) = t^n f(x, y)$

# Homogenous Function :

$f(x,y) = x^2 + y^2 + xy$ This Function is Homogenous OR Not

Rule for Homogenous function

$f(kx, ky) = k^n f(x,y)$ [Scaling]

Check the condition

$f(kx, ky) = k^2x^2 + k^2y^2 + kx\,ky$

$f(kx, ky) = k^2[x^2 + y^2 + xy]$

$$\boxed{f(kx, ky) = k^2 f(x,y)}$$

function is Homogenous.

(ii) $f(x,y) = x^3 + y^2$

$f(kx, ky) = k^3x^3 + k^2y^2$

$\Rightarrow k^2[kx^3 + y^2]$

$f(kx, ky) \neq k^n f(x,y)$

This function is Not Homogenous.

1) If $\frac{f(x)}{g(x)}$ Where $f(x), g(x)$ is Homogenous function

$\frac{f(x)}{g(x)}$ is also Homogenous function

$f(x)\cdot g(x)$ is also Homogenous function

# Type 03 Homogenous Eqn

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} \text{ or } f(x,y)\,g(x,y)$$

Where $f(x,y)$ or $g(x,y)$ both are Homogenous function

Put $\boxed{y = Kx}$ OR $\boxed{x = Ky}$ $\quad \frac{dx}{dy} = K + y \cdot \frac{dK}{dy}$

both sides Diff. w.r.t to $x$

$$\frac{dy}{dx} = K + x\frac{dK}{dx}$$

$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ Put $y = Kx$

Put The value of $\frac{dy}{dx}$

$$K + x\frac{dK}{dx} = \frac{f(K)}{g(K)}$$

Now variable Seprate It and get The solution of D.E

## Q. Questions

#Q. A curve passing through the point $\underset{}{\overset{(x,\,y)}{\underline{\left(1, \pi/6\right)}}}$. Let the slope of the curve at each point $(x, y)$ be $\frac{y}{x} + sec\left(\frac{y}{x}\right)$. Then the equation of the curve $x > 0$

(a) $\sin\left(\frac{y}{x}\right) = \ln(x) + 1/2$

(b) $\cos(y/x) = \log x + \frac{1}{2}$

(c) $\sin\left(\frac{2y}{x}\right) = \log x + 2$

(d) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

$$\frac{dy}{dx} = \frac{y}{x} + \sec\left(\frac{y}{x}\right)$$

This form

Homogeneous form

$$f(x,y) = f(kx, ky)$$

$$= \frac{ky}{kx} + \sec\left(\frac{ky}{kx}\right)$$

$$= \frac{y}{x} + \sec\left(\frac{y}{x}\right) \checkmark$$

Put $y = kx$

$$\frac{dy}{dx} = \frac{y}{x} + \sec\left(\frac{y}{x}\right)$$

Put $y = kx$, $\boxed{K = \frac{y}{x}}$

$$K + x\frac{dK}{dx} = \frac{K\cancel{x}}{\cancel{x}} + \sec\left(\frac{K\cancel{x}}{\cancel{x}}\right) = K + \sec K$$

$$\cancel{K} + x\frac{dK}{dx} = \cancel{K} + \sec K$$

$$x\frac{dK}{dx} = \sec K$$

Now variable Seprate It

$$\frac{dK}{\sec K} = \frac{dx}{x}$$

both sides Integrate It

$$\int \frac{dK}{\sec K} = \int \frac{dx}{x}$$

$$\Rightarrow \int \cos K \, dK = \int \frac{dx}{x}$$

$$\Rightarrow \sin K = \ln x + C$$

$$\Rightarrow \sin\left(\frac{y}{x}\right) = \ln x + C$$

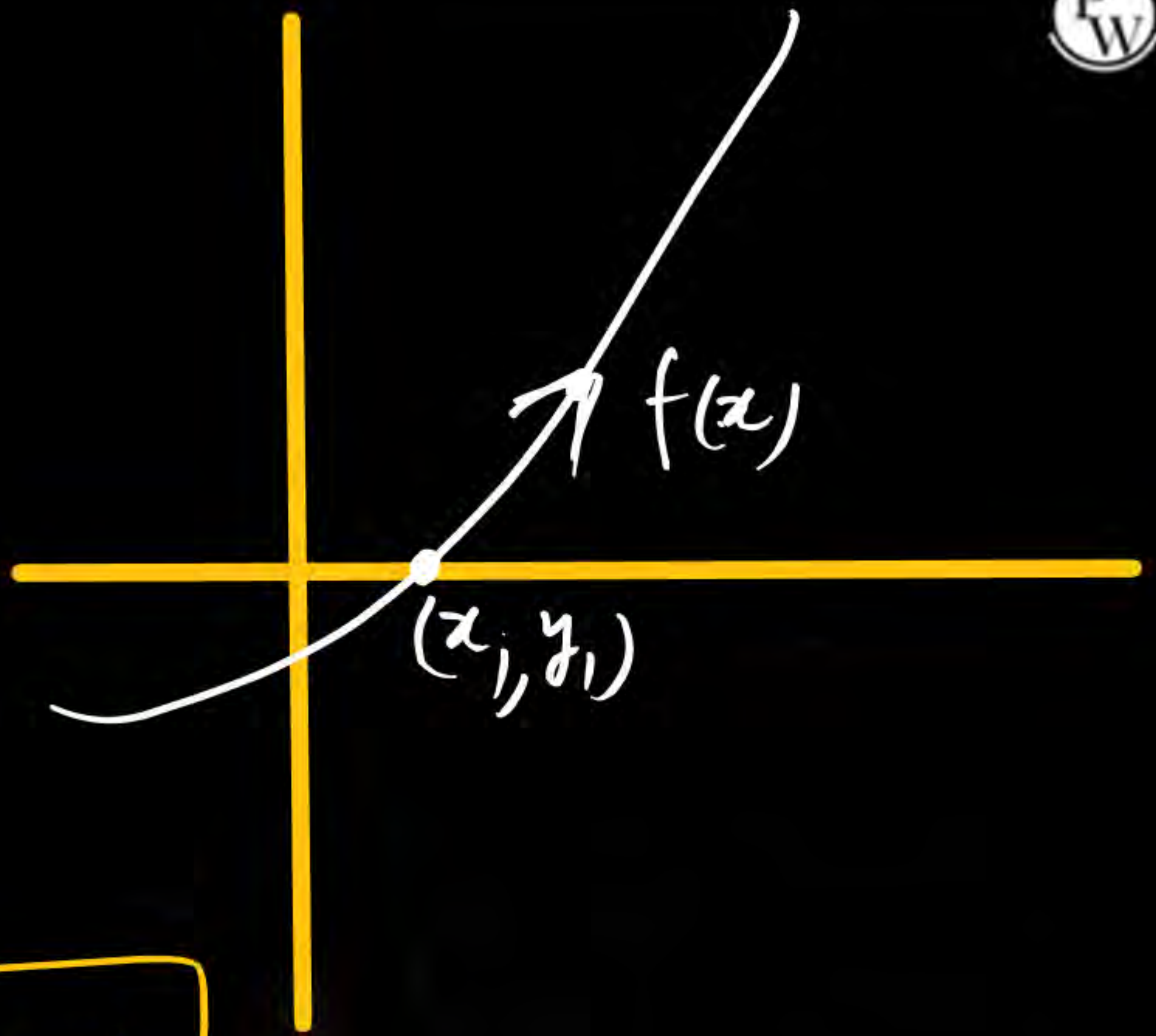
$$\Rightarrow \sin\left(\frac{y}{x}\right) = \ln x + c$$

at $\left(1, \frac{\pi}{6}\right)$ $x = 1$ $y = \frac{\pi}{6}$

$$\Rightarrow \sin\left(\frac{\frac{\pi}{6}}{1}\right) = \ln 1 + c$$

$$\boxed{c = \frac{1}{2}}$$

Solution of the D.E $= \boxed{\sin\left(\frac{y}{x}\right) = \ln x + \frac{1}{2}}$



# Q. Questions

#Q. $\log\left(\frac{dy}{dX}\right)=ax+by$

$$\log\left(\frac{dy}{dx}\right) = ax+by$$

$$\frac{dy}{dx} = e^{ax+by} = e^{ax}e^{by}$$

Now seprate The variables.

$$\Rightarrow \int \frac{dy}{e^{by}} = \int e^{ax}\,dx$$

$$\Rightarrow \int e^{-by}\,dy = \int e^{ax}\,dx$$

$$\Rightarrow \boxed{\frac{e^{-by}}{-b} - \frac{e^{ax}}{a} = c}$$

## Q. Questions

#Q. $\sqrt{1+x^2+y^2+x^2y^2}+xy\frac{dy}{dX}=0$

$$xy\frac{dy}{dx} = -\sqrt{(1+x^2)+y^2(1+x^2)}$$

$$xy\frac{dy}{dx} = -\sqrt{(1+x^2)(1+y^2)}$$

Now seprate The variable and You get The solution

Do yourself

## Q. Questions

#Q. $xy\frac{dy}{dx}=\sqrt{1+x^2}\sqrt{1+y^2}$

H.W

Do yourself

## Q. Questions

#Q. $x\dfrac{dy}{dx}=y(\log y-\log x+1)$

$$\frac{dy}{dx}=\frac{y}{x}\left[\ln\left(\frac{y}{x}\right)+1\right]$$

Put $y=vx$

$$v+x\frac{dv}{dx}=\frac{v\cancel{x}}{\cancel{x}}\left(\ln\left(\frac{vx}{x}\right)+1\right)$$

$$\cancel{v}+x\frac{dv}{dx}=\ln v\cdot v+\cancel{v}$$

$$\int\frac{dv}{v\ln v}=\int\frac{dx}{x}$$

$\ln v=t$

$$\int\frac{dt}{t}=\int\frac{dx}{x}$$

$$\boxed{\ln\ln v=\ln x+\ln c}$$

$$\ln\ln v=\ln(cx)$$

$$\ln v=cx$$

$$\boxed{v=e^{cx}}$$

$$\frac{y}{x}=e^{cx}$$

$$\boxed{y=xe^{cx}}$$

# Q. Questions

#Q. $(x+y+1)\dfrac{dy}{dx}=1$

$$\frac{dy}{dx}=\frac{1}{(x+y+1)}$$

$$=\frac{dt}{dx}-1=\frac{1}{t}$$

$$\frac{dt}{dx}=\frac{1}{t}+1=\frac{1+t}{t}$$

$$=\int\frac{t\,dt}{1+t}=\int dx$$

$$(x+y+1)\frac{dy}{dx}=1$$

$$x+y+1=t$$

Do yourself

$$1+\frac{dy}{dx}=\frac{dt}{dx}$$

$$=\int\frac{t+1-1}{t+1}dt=\int dx$$

$$=\int dt-\int\frac{1}{1+t}dt=\int dx$$

$$=t-\ln|1+t|=x+C$$

$$=x+y+1-\ln(1+x+y+1)=x+C$$

# Q. Questions

#Q. Consider the following differential equation

$$x(y\,dx + x\,dy)\cos\left(\frac{y}{x}\right) = y\,(x\,dy - y\,dx)\sin\left(\frac{y}{x}\right)$$

Which of the following is the solution of the above equation (C is an arbitrary constant)

$\frac{dy}{dx}$ = form.

$y = vx$ Put and
Now get the sol$^n$

Do yourself

(a) $\frac{x}{y}\cos\frac{y}{x} = C$

(b) $\frac{x}{y}\sin\frac{y}{x} = C$

(c) $xy\cos\frac{y}{x} = C$

(d) $xy\sin\frac{y}{x} = C$

# Thank You!

PW Soldiers