GATE (ALL BRANCHES)

Engineering Mathematics

Differential Equation + Partial differential

Lecture No. 08

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TOPICS TO BE COVERED

01 Exact D.E & Orthogonal Trajectory

02 Problems based on Exact D.E

Exact Differential Equation : (First order)

✓ Exact
✓ Reducible to exact

$M\,dx + N\,dy = 0$ Where $M(x,y)$ or $N(x,y)$ Are Functions of x, y only.

If Equation is exact

$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ my Equation is exact

✓ **Solution**

$$\boxed{\int M\,dx + \int N\,dy = c}$$

y Treating as a constant

Independent Term of x

Test $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

Q. Questions

#Q. Solve the exact differential equation $(y^2 + 3)dx + (2xy - 4)dy = 0$

$\frac{\partial M}{\partial y} = 2y$

$\frac{\partial N}{\partial x} = 2y$

$M(x,y) = y^2+3$

$N(x,y) = 2xy-4$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Solution of this exact Diff. Eqn

$\int (y^2+3)\,dx + \int (\cancel{2xy}^{\,0} - 4)\,dy = C$

y as a constant

Independent Term of x

$y^2\int 1\,dx + \int 3\,dx$

$\boxed{y^2x + 3x - 4y = C}$

Q. Questions

#Q. Solve the exact differential equation $\underbrace{(2xy+1)}_{M(x,y)}dx + \underbrace{(x^2+4y)}_{N(x,y)}dy = 0$

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

$$2x = 2x$$

this Equation is exact

$$\int (2xy+1)\,dx + \int (x^2+4y)\,dy = c$$

y Treating as a const

Independent of x.

$$\frac{\cancel{2}x^2}{\cancel{2}}y + x + \frac{4y^2}{2} = c$$

$$\boxed{x^2y + x + 2y^2 = c}$$

Q. Questions

#Q. Solve the exact differential equation $\underbrace{(3x^2 + 2)dx - }(x^2 + y)dy = 0$

$M(x,y) = 3x^2 + 2$

$N(x,y) = -(x^2+y)$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ fail

$0 = -2x$ exactness

Non exact

Reducible to exact

Q. Questions

#Q. Solve the exact differential equation

$$\left(y\sec^2 x + (\sec x)(\tan x)\right)dx + \left(\tan x + 2y\right)dy = 0$$

$M = y\sec^2 x + \sec x \tan x$

$N = \tan x + 2y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Test exactness.

$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

$\frac{\partial M}{\partial y} = \sec^2 x + 0$

$\frac{\partial M}{\partial y} = \sec^2 x$

$\frac{\partial N}{\partial x} = \sec^2 x + 0$

$\int_{y \text{ Treating const.}} (y\sec^2 x + \sec x \tan x)\,dx + \int (\cancel{\tan x} + 2y)\,dy$

Independent of x

$\Rightarrow y\tan x + \sec x + \frac{\cancel{2}y^2}{\cancel{2}} = c_1$

$\boxed{y\tan x + \sec x + y^2 = c_1}$

Q. Questions

#Q. Solve the exact differential equation $(3xy^2 + 2y)dx + (2x^2y + x)dy = 0$

→ Non exact D.E   H.W

→ convert to exact D.E

Q. Questions

#Q. Solve the exact differential equation $(\underbrace{2xy^2 - y}_{})dx + (2x - x^2y)dy = 0$

CASE 03

$M\,dx + N\,dy = 0$

✓ $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = f(y)$ Integrating $e^{\int f(y)dy}$

$\frac{1}{N}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(x)$

$\boxed{I.F = e^{\int f(x)dx}}$

$= \frac{1}{(2xy^2 - y)}\left[2 - 2xy - 4xy - 1\right]$

$= \frac{1}{(2xy^2 - y)}\left[2 - 1 - 6xy\right]$

$= \frac{1}{y(2xy - 1)}\left[3 - 6xy\right] = -\frac{1}{y\cancel{(1-2xy)}}\frac{3\cancel{(1-2xy)}}{} = -\frac{3}{y}$

$f(y) = -\frac{3}{y}$

Integrating factor $= e^{\int f(y)\,dy} = e^{\int -\frac{3}{y}\,dy}$

$(2xy^2 - x)\,dx + (2x - x^2y)\,dy = 0$

$= e^{-3\ln y}$

Non exact D.E

$= e^{-\ln y^3}$

↳ change – convert

$\boxed{I.F = \frac{1}{y^3}}$ ✓

Exact $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\int \frac{(2xy^2 - x)}{y^3}\,dx + \int \frac{(\cancel{2x} - \cancel{x^2 y})}{y^3}\,dy = c$

$\boxed{\frac{x^2}{y} - \frac{x^2}{2y^3} + 0 = c_1}$

y Treating as a const

X Independent

Q. Questions

#Q. Solve the exact differential equation $(x^4 + y^4)dx - \boxed{(x^3y)}dy = 0$

xy^3 Not Integrale

CASE 01 $Mdx + Ndy = 0$

$M(x,y)$ or $N(x,y)$ Are Homogenous Function

$M(x,y) = x^4 + y^4$

$N(x,y) = -xy^3$

Homogenous

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ Non exact

fails — Test for exactness.

change → exact Diff. Equation

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

SEARCH or find The Integrating factor

If $M(x,y)$ and $N(x,y)$ both are Homogenous function

$$\boxed{\text{Integrating factor} = \frac{1}{Mx+Ny}}$$

$$(x^4+y^4)\,dx - (y^3x)\,dy = 0$$

$$Mx+Ny \neq 0$$

$$Mx+Ny = (x^4+y^4)x - (y^3x)y$$

$$= x^5 + \cancel{xy^4} - \cancel{y^4x} = x^5$$

$$\text{Integrating factor} = \frac{1}{Mx+Ny} = \frac{1}{x^5}$$

$$(x^4+y^4)\,dx-(y^3x)\,dy=0$$

Multiply I.F factor $\dfrac{1}{Mx+Ny}$

$$=\frac{(\cancel{x^4}+y^4)}{\cancel{x^5}x}dx-\left(\frac{y^3x}{x^5}\right)dy=0$$

This is exact Diff. Equation

$$=\int\frac{1}{x}dx+\int\frac{y^4}{x^5}dx-\int\cancel{\frac{y^3x}{x^5x^4}}^{\to 0}dy=c$$

y Treating as a constant

Independent of x

$$=\ln x+y^4\left(-\frac{1}{4x^4}\right)-0=c_1$$

$$\int\frac{1}{x^5}=\frac{x^{-5+1}}{-5+1}$$

$$=\frac{x^{-4}}{-4}$$

$$=\frac{-1}{4x^4}$$

$$\boxed{\ln x-\frac{y^4}{4x^4}=c_1}$$

Q. Questions

#Q. Solve the exact differential equation $\underbrace{y(x^2y^2+2)}_{M(x,y)}dx + \underbrace{x(2-2x^2y^2)}_{N(x,y)}dy = 0$

$y f_1(x,y) + x f_2(x,y) = 0$

$$\left\{\begin{aligned} M(x,y) &= y(x^2y^2+2) \\ M(x,y) &= y^3x^2+2y \\ N(x,y) &= 2x-2x^3y^2 \end{aligned}\right.$$

Test for Exactness

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Non exact Differential equation $3y^2x^2+2 \neq 2-6x^2y^2$

$\longrightarrow$ Reduced to Exactness

Integrating factor

$$= \frac{1}{Mx - Ny} \text{ where } Mx - Ny \neq 0$$

$$Mx - Ny = (y^3x^2 + 2y)x - (2x - 2x^3y^2)y$$

$$= y^3x^3 + 2xy - (2xy - 2x^3y^3)$$

$$= y^3x^3 + \cancel{2xy} - \cancel{2xy} + 2x^3y^3$$

$$Mx - Ny = \underline{3x^3y^3}$$

$$\text{I. factor} = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$$

CASE 02

If $y f_1(x,y)\,dx + x f_2(x,y)\,dy = 0$

$$y(x^2y^2+2)\,dx + x(2-2x^2y^2)\,dy = 0$$

$$= (x^2y^3+2y)\,dx + (2x-2x^3y^2)\,dy = 0$$

$\frac{1}{Mx-Ny}$ is a I.F

$\frac{1}{Mx-Ny}$ I.Factor $\rightarrow$ Multiply $\longrightarrow$ exact

$$= \underbrace{\int \frac{(x^2y^3+2y)}{3x^3y^3}dx}_{\text{Treating } y \text{ as constant}} + \underbrace{\int \frac{(2x-2x^3y^2)}{3x^3y^3}dy}_{x \text{ Indep.}} = 0$$

$\longrightarrow$ this equation Pass $\rightarrow$ condition for Exact

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$= \int \frac{\cancel{x^2}\cancel{y^3}}{\cancel{3x^3}_{x}\cancel{y^3}}dx + \int \frac{2\cancel{y}}{3x^3\cancel{y^3}_{y^2}}dx + \cancel{\int \frac{2x}{3x^3y^3}dy}^{0} - \int \frac{2\cancel{x^3}\cancel{y^2}}{3\cancel{x^3}\cancel{y^3}_{y}}dy = c$$

$$= \frac{1}{3}\ln x + \frac{1}{y^2}\frac{2}{3}\int \frac{1}{x^3}dx + \left(-\frac{2}{3}\ln y\right) = c$$

Q. Questions

#Q. A solution of differential equation

$\frac{dy}{dx} = \frac{x^2+y^2+1}{2xy}$ Satisfying y(1) =1 is given by

(a) A system of hyperbolas

(b) A system of Circles

(c) $y^2 = x(1 + x) - 1$

(d) All the above

$\frac{dy}{dx} = \frac{x^2+y^2+1}{2xy}$

$y = vx$ Put

$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2+1}{2x\,vx}$

Method 2

$\boxed{v + x\frac{dv}{dx} = \frac{x^2+v^2x^2+1}{2x^2v}}$

$2xy\,dy = x^2dx + y^2dy + dx$

$\boxed{\frac{2xy\,dy - y^2dx}{x}} = \frac{(x^2+1)dx}{x}$

$$\frac{2xy\,dy - y^2 dx}{x^2} = \frac{(x^2+1)}{x^2} dx$$

$$d\left(\frac{y^2}{x}\right) \begin{matrix} \mathrm{I} \\ \mathrm{II} \end{matrix}$$

$$= \frac{2xy\,dy - y^2 dx}{x^2} = \left(1 + \frac{1}{x^2}\right) dx$$

$$= \frac{\mathrm{I} \cdot \frac{d}{dx}\mathrm{II} - \mathrm{II}\frac{d}{dx}\mathrm{I}}{(\mathrm{II})^2}$$

$$= \int d\left(\frac{y^2}{x}\right) = \int\left(1 + \frac{1}{x^2}\right) dx$$

$$y(1) = 1$$

$$= \frac{y^2}{x} = x - \frac{1}{x} + c$$

$$\boxed{y^2 = (x^2-1) + cx}$$

$$= \frac{y^2}{x} = \frac{(x^2-1)}{x} + c$$

$$c = y(1) = 1$$

$$y^2 = (x^2-1) + cx$$

Thank You!

PW Soldiers