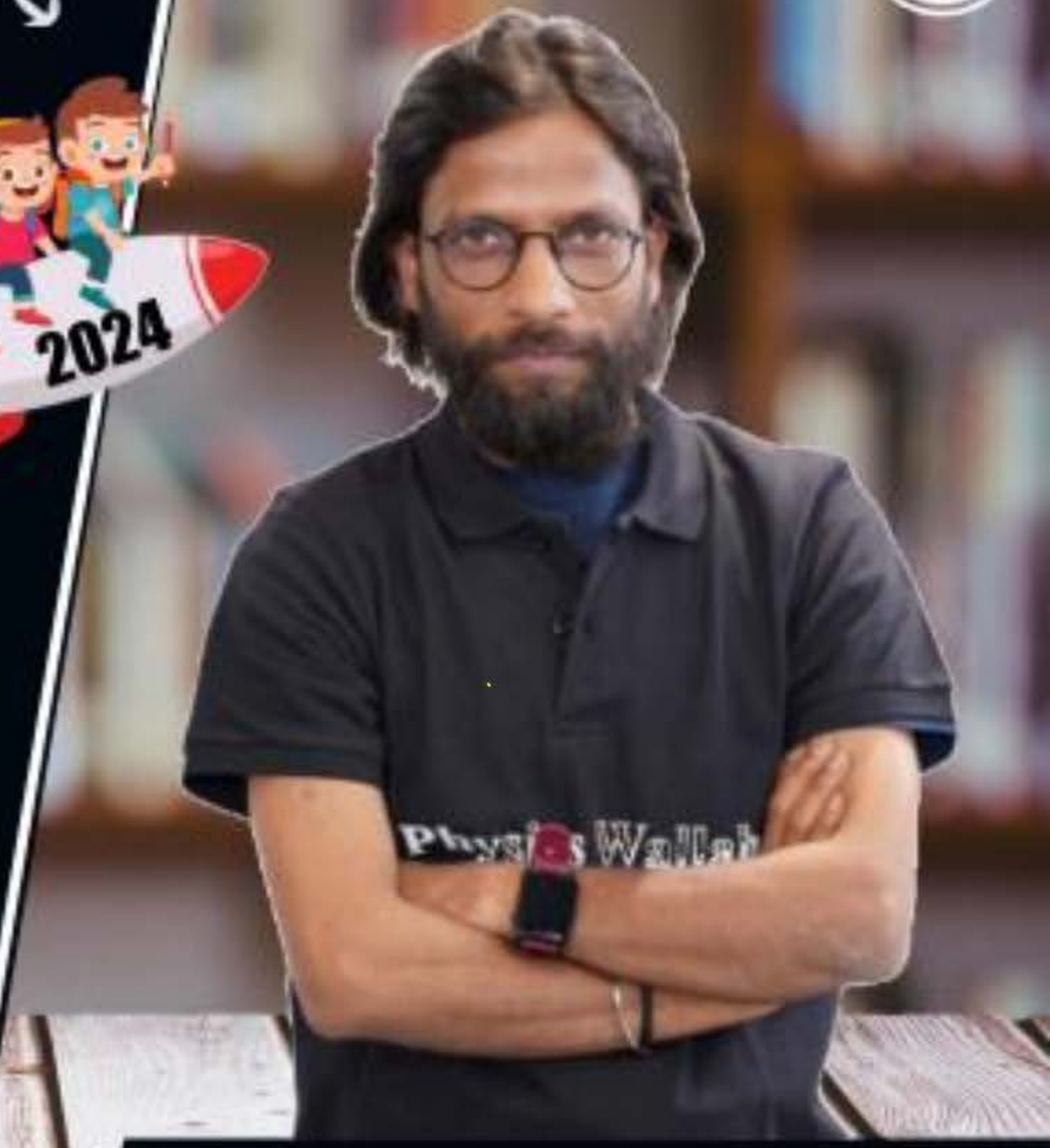
GATE-All BRANCHES
Engineering Mathematics

LAPLACE TRANSFORM

Lecture No.- 02



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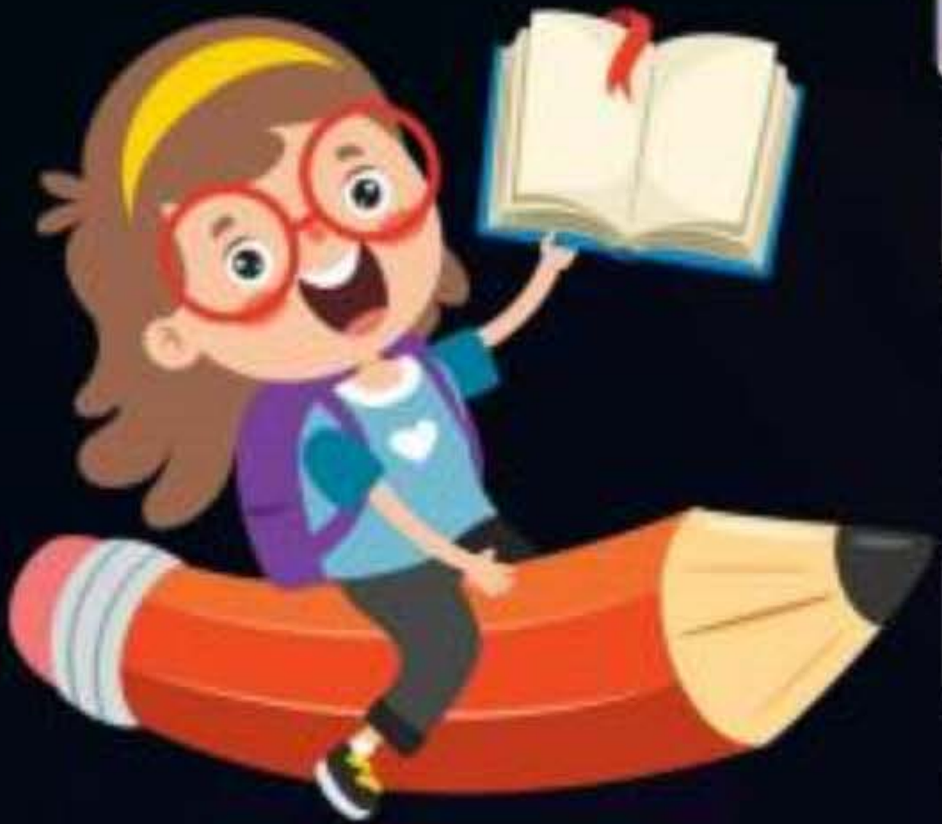
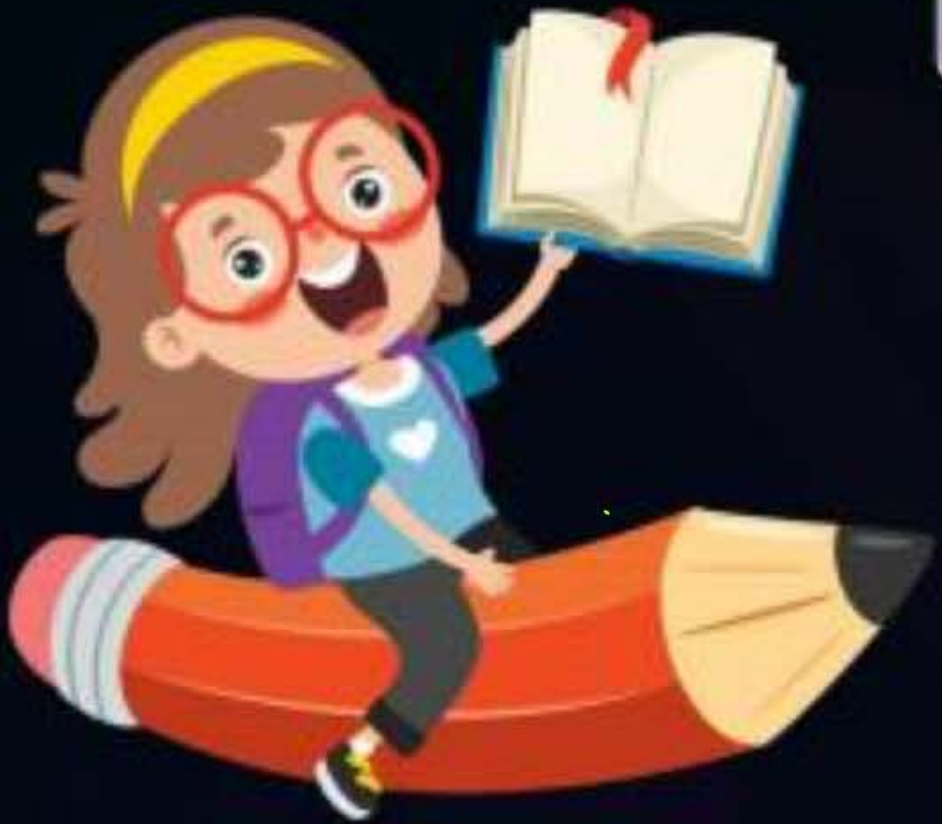
Recape of previous lecture



- **Topic** Introduction to laplace transforms
- **Topic** Laplace transform - fundamental function
- **Topic** Shifting property
- **Topic** Multiplication of laplace transforms
- **Topic** Division of laplace transforms
- **Topic** Unit step function and Dirac delta function

Topics to be Covered



Topic ✓ Problems based on laplace transformation

② ✓ Unit step function

③ ✓ Inverse laplace Transform

#Q. *Find the Laplace transform of* $e^{-3t}(\cos 4t + 3\sin 4t)$

$$L[e^{-3t}\cos 4t + 3e^{-3t}\sin 4t]$$

$$L[\cos 4t] = \frac{s}{(s^2+16)}$$

$$L(3e^{-3t}\sin 4t) = \frac{3\times 4}{(s+3)^2+16}$$

$$= \frac{12}{(s+3)^2+16}$$

$$L[e^{-3t}\cos 4t] = \frac{(s+3)}{(s+3)^2+16}$$

$$= \frac{(s+3)}{(s+3)^2+16} + \frac{12}{(s+3)^2+16}$$

Topic : Laplace Transform

$$\left(\sqrt{t}+\frac{1}{\sqrt{t}}\right)^3 = \checkmark\ \underline{H.W}$$

#Q. *Find the Laplace transform of*

A $\left(\sqrt{t}+\dfrac{1}{\sqrt{t}}\right)^3$

B $\cos h\, at \sin bt$

C $\sin h^2\, 2t$

D $e^{-t}\cos t \cos 2\,t$

2) $\cosh at \sin bt = \left(\dfrac{e^{at}+e^{-at}}{2}\right)\sin bt$

$\cosh at = \dfrac{e^{at}+e^{-at}}{2}$

$\sinh at = \dfrac{e^{at}-e^{-at}}{2}$

$= \dfrac{1}{2} L\left[e^{at}\sin bt + e^{-at}\sin bt\right]$

$= \underline{H.W}$

3) H.W

4) $e^{-t}\cos t\cos 2t = e^{-t}\left[\frac{1}{2}\times 2\cos t\cos 2t\right]$

H.W

Function is

Piecewise Defined

#Q. Find the Laplace transform of $f(t)$ defined as

$$f(t) = \begin{cases} \frac{1}{T}, & \text{when } \boxed{0 < t < T} \\ 1, & \text{when } \boxed{t > T} \end{cases}$$

Defination Based

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = f(s)$$

$$L[f(t)] = \int_0^T e^{-st} \cdot \frac{1}{T}\,dt + \int_T^\infty e^{-st}(1)\,dt$$

$$\boxed{L[f(t)] = \frac{1-e^{-st}}{Ts^2}}$$

Topic : Laplace Transform

#Q. If $F(t) = \dfrac{e^{at} - \cos bt}{t}$, Find the laplace transform of $F(t)$.

$$F(t) = \frac{e^{at} - \cos bt}{t} \qquad L[F(t)] \qquad L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} f(s)\,ds$$

$$= L\left[\frac{e^{at} - \cos bt}{t}\right] = \int_s^{\infty} \frac{1}{(s-a)} - \frac{s}{(s^2+b^2)}$$

Division by t

$$L[e^{at}] = \frac{1}{(s-a)}$$

$$L[\cos bt] = \frac{s}{(s^2+b^2)}$$

$$= \int_s^\infty \frac{1}{(s-a)} - \int_s^\infty \frac{s}{(s^2+b^2)} ds$$

$$= \int_s^\infty \frac{1}{(s-a)} - \frac{1}{2}\int \frac{2s}{(s^2+b^2)} ds$$

$$= \int \frac{1}{(s-a)} ds = \ln(s-a)$$

$$= \left[\ln(s-a) - \frac{1}{2}\ln(s^2+b^2)\right]_s^\infty$$

$$\frac{1}{2}\int \frac{2s}{(s^2+b^2)} ds = \frac{1}{2}\ln(s^2+b^2)$$

$$= \left[\frac{2\ln(s-a) - \ln(s^2+b^2)}{2}\right]_s^\infty$$

$$= \frac{1}{2}\left[\ln(s-a)^2 - \ln(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\ln\frac{(s-a)^2}{(s^2+b^2)}\right]_s^\infty$$

$$= \frac{1}{2}\ln\left[\frac{s^2+b^2}{(s-a)^2}\right] \checkmark$$

H.W

#Q. Find the Laplace transform of

A $t^3 e^{-3t}$ — shifting

B $t \sin^2 3t$ — multiplication

C $\dfrac{1-\cos t}{t^2}$ — Division via t

Topic : Laplace Transform

H·W

#Q. Find the Laplace transforms of

A $\dfrac{e^{-at}-e^{-bt}}{t} = \int_s^{\infty} \dfrac{1}{(s+a)} - \dfrac{1}{(s+b)}$

$= \ln\left[\dfrac{s+b}{(s+a)}\right]$

Using Division rat

B $\dfrac{\cos at - \cos bt}{t} = \dfrac{1}{2}\ln\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$

#Q. *Find the Laplace transform of the following functions :*

A $\dfrac{e^{-t}\sin t}{t}$

$$= \int_s^\infty \frac{1}{(s+1)^2+1}\, ds$$

B $\dfrac{1-\cos 2t}{t}$

$$= \int_s^\infty \frac{1}{s} - \frac{s}{(s^2+4)}\, ds$$

Topic : Laplace Transform

#Q. Evaluate : $I = \int_0^{\infty} \underbrace{t^3 e^{-t} \sin t}\, dt$

$L[\sin t] = \dfrac{1}{(s^2+1)}$

$$\boxed{L[t^3 \sin t] = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{(s^2+1)}\right) \Rightarrow \frac{24s(s-1)}{(s^2+1)^4}}$$

A $\underbrace{\int_0^{\infty} t^3 e^{-t} \sin t \, dt}$ ✓

B $\int_0^{\infty} \dfrac{e^{-t} \sin^2 t}{t} dt$ ✓

$$\int_0^\infty e^{-t}\frac{\sin t}{t}\,dt$$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{(s^2+1)} = \frac{\pi}{2} - \tan^{-1}s$$

$$\int_0^\infty e^{-st}f(t)\,dt = f(s)$$

$$\int_0^\infty e^{-st}\cdot\left(\frac{\sin t}{t}\right)dt = \frac{\pi}{2} - \tan^{-1}s$$

$s = 1$

$$= \int_0^\infty e^{-t}\left(\frac{\sin t}{t}\right)dt = \frac{\pi}{2} - \tan^{-1}1$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

$\log_e \quad \ln e$

$$I = \frac{1}{4}\log 5$$

$$\int_0^\infty e^{-t}\frac{\sin^2 t}{t}\,dt \Rightarrow$$

$$L[\sin^2 t] = L\left[\frac{1-\cos 2t}{2}\right] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{(s^2+4)}\right] = \frac{1}{2}\left[\int_s^\infty \frac{1}{s} - \frac{1}{2}\int_s^\infty \frac{s}{(s^2+4)}\right]$$

$\cos 2t = \cos^2 t - \sin^2 t$

$= 1 - 2\sin^2 t$

$$= \frac{1}{2}\left[\ln s - \frac{1}{2}\ln(s^2+4)\right]$$

$$= \frac{1}{2}\times\frac{1}{2}\ln\left(\frac{s^2+4}{s^2}\right)$$

$$= \frac{1}{4}\ln\left(\frac{s^2+4}{s^2}\right)$$

$$\int_0^\infty e^{-st} f(t)\,dt = f(s)$$

$$\int_0^\infty e^{-st}\cdot\frac{\sin^2 t}{t}\,dt = \frac{1}{4}\ln\left(\frac{s^2+4}{s^2}\right)$$

$s = 1$

$$\int_0^\infty e^{-t}\frac{\sin^2 t}{t}\,dt = \frac{1}{4}\ln 5$$

Topic : Laplace Transform

ME

#Q. The Laplace transform of sin h (at) is

$L(\sin hat) = \frac{a}{(s^2-a^2)}$

A $\frac{s}{s^2 - a^2}$

B $\frac{s}{s^2 + a^2}$

C $\frac{a}{s^2 - a^2}$

D $\frac{a}{s^2 + a^2}$

$L[\sin h(at)]$

$\sin hat = \frac{e^{at} - e^{-at}}{2}$

$= \frac{1}{2}\left[L(e^{at}) - L(e^{-at})\right]$

$= \frac{1}{2}\left[\frac{1}{(s-a)} - \frac{1}{(s+a)}\right]$

$= \frac{a}{(s^2-a^2)}$

#Q. The Laplace Transform of the following function is

$$f(t) = \begin{cases} \sin t & \text{for } 0 \le t \le \pi \\ 0 & \text{for } t > \pi \end{cases} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\pi \underbrace{e^{-st} \sin t}\,dt + \int_\pi^\infty e^{-st} \cdot 0\,dt = \frac{1 + e^{-\pi s}}{(1+s^2)} \text{ for all } s > 0$$

A. $\frac{1}{1+s^2}$ for all $s > 0$

B. $\frac{1}{1+s^2}$ for all $s < \pi$

C. $\frac{1+e^{-ns}}{1+s^2}$ for all $s > 0$

D. $\frac{e^{-ns}}{1+s^2}$ for all $s > 0$

Topic : Laplace Transform

#Q. The Laplace transform of $6t^3 + 3\sin 4t$ is

$L[(6t^3 + 3\sin 4t)]$

$= 6\dfrac{3!}{s^4} + 3\cdot\dfrac{4}{(s^2+16)}$

$= \dfrac{36}{s^4} + \dfrac{12}{(s^2+16)}$

A. $\dfrac{36}{s^4} + \dfrac{12}{s^2+16}$

B. $\dfrac{36}{s^4} + \dfrac{12}{s^2-16}$

C. $\dfrac{18}{s^4} + \dfrac{12}{s^2-16}$

D. $\dfrac{36}{s^3} + \dfrac{12}{s^2+16}$

Topic : Laplace Transform

#Q. The Laplace transform of $f(t)=\begin{cases}\dfrac{t}{T}, & 0<t<T\\ 1 & t>T\end{cases}$ is

Ans

$\int_0^\infty e^{-st} f(t)\,dt = L[f(t)]$

$= \dfrac{1-e^{-sT}}{s^2T}$

A $\dfrac{-(1-e^{-sT})}{s^2T}$

B $\dfrac{(1-e^{-sT})}{s^2T}$ ✓

C $\dfrac{(1+e^{-sT})}{s^2T}$

D $\dfrac{(1-e^{sT})}{s^2T}$

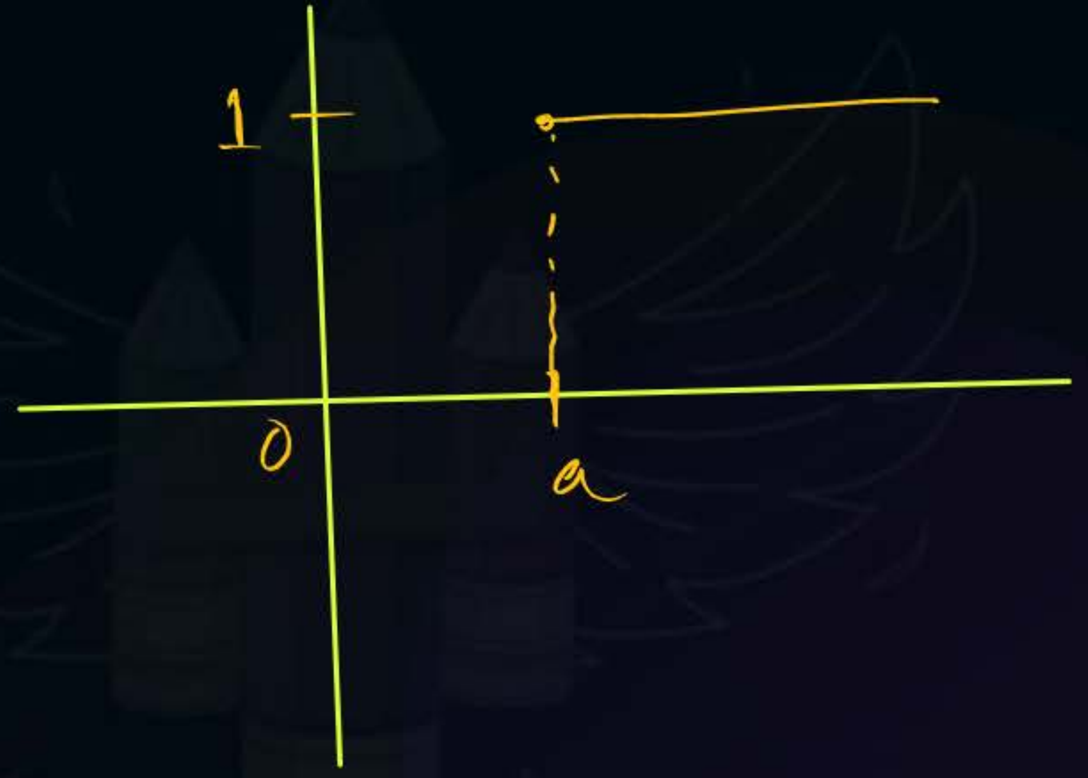
Unit step Function:
OR
Heaviside function

$$\underbrace{\boxed{H(t) \text{ or } u(t)}}_{\text{Unit step function}} = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

$u(t-0)$ → 0 shift

$u(t-a)$ a unit

$$H(t-a) \text{ or } u(t-a) = \begin{cases} 1 & t \ge a \\ 0 & t < a \end{cases}$$



Laplace Transform of unit step Function

✓ $L[u(t)] = \dfrac{1}{s}$ Unit step function

✓ $L[u(t-a)] = \dfrac{e^{-as}}{s}$ unit step function

$L[u(t-5)] = \dfrac{e^{-5s}}{s}$

$L[u(t-5) - u(t-3)]$

$= \dfrac{e^{-5s}}{s} - \dfrac{e^{-3s}}{s}$

$L[u(t) + u(t-3) + \sinh t]$

$= L[u(t)] + L[u(t-3)] + L[\sinh t]$

$= \frac{1}{s} + \frac{e^{-3s}}{s} + \frac{a}{(s^2-a^2)}$

$L(u(t-a) - u(t-b))$

$= \frac{e^{-as}}{s} - \frac{e^{-bs}}{s}$

Second shifting Property:

$L[f(t)] = f(s)$

$$\boxed{L[\underbrace{u(t-a)}_{}\, f(t-a)] = e^{-as} f(s)}$$

SAME → SAME

shift shift

$$L[u(t-a)f(t-a)] = e^{-as}f(s)$$

1) $L[\underbrace{(t-1)^2}_{\text{SAME } (t-1)}\underbrace{u(t-1)}_{\text{SAME shift}}] = e^{-s}\cdot\frac{2}{s^3}$

$= \frac{2e^{-s}}{s^3}$

t^2

$(t-1) \quad u(t-1)$

$f(t) = t^2$

$f(s) = \frac{2!}{s^3}$

2) $L[\sin\underbrace{(t-2)}\,u\underbrace{(t-2)}]$

SAME shift

$= e^{-2s}\cdot\frac{1}{(s^2+1)}$

3) $L[\cos(t-5)u(t-5)] = e^{-5s}\cdot\frac{s}{(s^2+1)}$

3) $L\left[(t^2-2t)\,u(t-1)\right]$

$$= L\left[(\underbrace{t^2-2t+1}_{(t-1)^2}-1)\,u(t-1)\right]$$

$$= L\left[\left[(t-1)^2-1\right]u(t-1)\right]$$

$$= L\left[(t-1)^2 u(t-1) - u(t-1)\right]$$

$$= e^{-s}\cdot\frac{2!}{s^3} - \frac{e^{-s}}{s}$$

Strategy

1) SAME shift

$$L\left[u(t-a)\,f(t-a)\right] = e^{-as}F(s)$$

$$L\left[u(t-a)\right] = \frac{e^{-as}}{s}$$

Topic : Laplace Transform

#Q. If $f(x)=\begin{cases} 0 & \text{for } x<3, \\ x-3 & \text{for } x\geq 3 \end{cases}$ then the Laplace transform of $f(x)$ is

$$L[f(x)] = \int_0^{\infty} e^{-sx} f(x)\,dx$$

$$= \int_{-\infty}^{3} e^{-sx}(0)\,dx + \int_3^{\infty} e^{-sx}(3-x)\,dx$$

$$\Rightarrow \frac{1}{s^2} e^{-sx}$$

A. $s^{-2} e^{sx}$

B. $s^2 e^{sx}$

C. s^{-2}

D. $s^{-2} e^{-sx}$

Topic : Laplace Transform

gate

#Q. The Laplace transformation of $e^{-2t} \sin 4t$ is

$L(\sin 4t) = \frac{4}{(s^2+16)}$

$L(e^{-2t} \sin 4t) \Rightarrow \frac{4}{(s+2)^2+16}$

$= \frac{4}{(s^2+4+4s+16)}$

$= \frac{4}{s^2+4s+20}$

A. $\frac{4}{s^2+4s+25}$

B. $\frac{4}{s^2+4s+20}$

C. $\frac{4s}{s^2+4s+20}$

D. $\frac{4}{2s^2+4s+20}$

Topic : Laplace Transform

$e^{\alpha t} \cos \alpha t$

$L(\cos \alpha t) = \frac{s}{(s^2+\alpha^2)}$

$L(e^{\alpha t} \cos \alpha t) = \frac{(s-\alpha)}{(s-\alpha)^2+\alpha^2}$

#Q. The Laplace transform of $e^{at} \cos(at)$ is equal to

A $\frac{(s-\alpha)}{(s-\alpha)^2+\alpha^2}$

B $\frac{(s+\alpha)}{(s+\alpha)^2+\alpha^2}$

C $\frac{1}{(s-\alpha)^2}$

D None of the above

Topic : Laplace Transform

#Q. The Laplace transform of the function $f(t) = e^{at}$ when $t > 0$ and where a is a constant is

$f(t) = e^{at}$

$f(s) = \frac{1}{(s-a)}$

A $\frac{1}{(s-a)}$

B $\frac{1}{(s+a)}$

C $\frac{1}{(s-a)^{-1}}$

D $\frac{1}{(s+a)^{-1}}$

Topic : Laplace Transform

#Q. The Laplace transform of the F(s) on the exponential function f(t) $= e^{at}$ when $t \geq 0$, where a is a constant and $(s - a) > 0$, is

A $\frac{1}{s+a}$

B $\frac{1}{s-a}$

C $\frac{1}{a-s}$

D ∞

Topic : Laplace Transform

#Q. Laplace transform of the function f(x) = cos h (ax) is

$f(x) = \cosh(ax)$

$= \dfrac{e^{ax} - e^{-ax}}{2}$

$= \dfrac{1}{2}\left[\dfrac{1}{(s-a)} - \dfrac{1}{(s+a)}\right]$

$= \dfrac{s}{(s^2 - a^2)}$

A $\dfrac{a}{(s^2 - a^2)}$

B $\dfrac{s}{(s^2 - a^2)}$

C $\dfrac{a}{(s^2 + a^2)}$

D $\dfrac{s}{s^2 + a^2}$

Topic : Laplace Transform

#Q. Evaluate $\int_0^\infty \frac{\sin t}{t} dt$

A π

B $\frac{\pi}{2}$

C $\frac{\pi}{4}$

D $\frac{\pi}{8}$

$$\int_0^\infty \frac{\sin t}{t} dt$$

$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{1}{(s^2+1)} ds$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$\int_0^\infty e^{-st} \frac{\sin t}{t} dt = \frac{\pi}{2} - \tan^{-1} s$$

$s=0$ $\quad$ $s=0$

$$\boxed{\int_0^\infty 1 \times \frac{\sin t}{t} dt = \frac{\pi}{2}}$$

$$\int_0^\infty \frac{\sin t}{t} dt = \frac{\pi}{2}$$

#Q. If L defines the Laplace transform of a function L [Sin (at)] will be equal to

$L[\sin at] = \frac{a}{(s^2+a^2)}$

A $\frac{a}{s^2 - a^2}$

B $\frac{a}{s^2 + a^2}$

C $\frac{s}{s^2 + a^2}$

D $\frac{s}{s^2 - a^2}$

2 mins Summary

Topic One

Topic Two

Topic Three

Topic Four

Topic Five

Problems —

Unit step

SECOND shifting Property

THANK - YOU

Topics to be Covered