GATE-All BRANCHES
Engineering Mathematics

Vector Calculus

Lecture No.- 02

By- Rahul sir

Recap of Previous Lecture

Topic Concept of line integral

Topic Problem based on line integral

Topic Concept and problems based of curl

Topics to be Covered

Line Integrals

Topic Concept of curl

Topic Greens theorem and Stokes theorem

Topic Problems based on Green's theorem and stokes theorem

Monday - Differential Equⁿ
+
Complex Analysis

Recording monday

16 LECTURES
+
with DPP
+
weekly TEST

START

Laplace transform 03

fourier series 02

Numerical methods 03

Live (Next weekend)

Curl of a vector Function:

A) Vector Function
B) Scaler Function

vector function $\xrightarrow{\nabla(\text{del})}$ Scaler Function

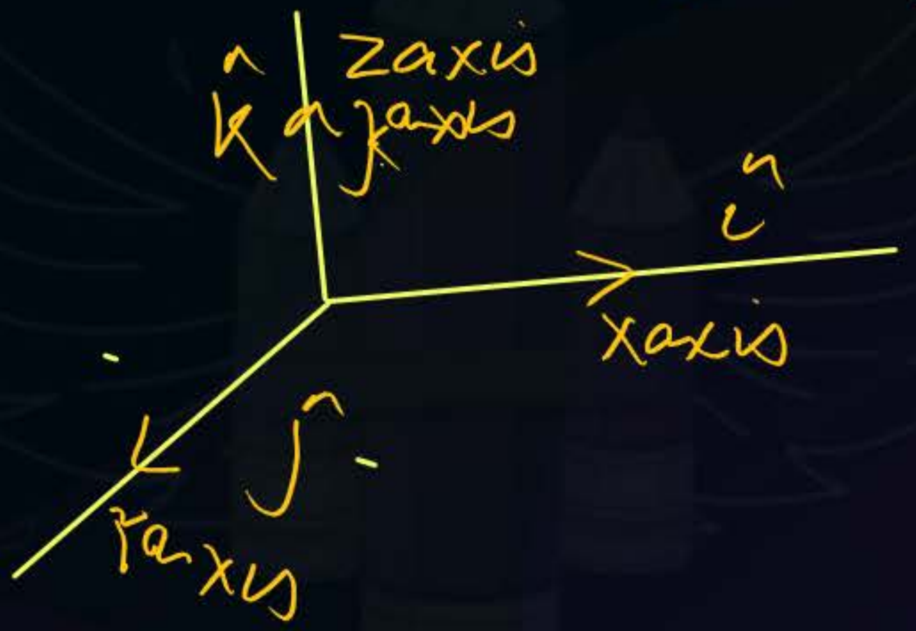
Scaler Functions $\xrightarrow{\nabla(\text{del})}$ Vector functions

Vector functions $\xrightarrow{\nabla(\text{del})}$ Vector function (another form)

∇(del) operator $\longrightarrow$ Three Dimensional change in x, y, z

Operator

$$\boxed{\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}}$$



Curl of a vector Function:

$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

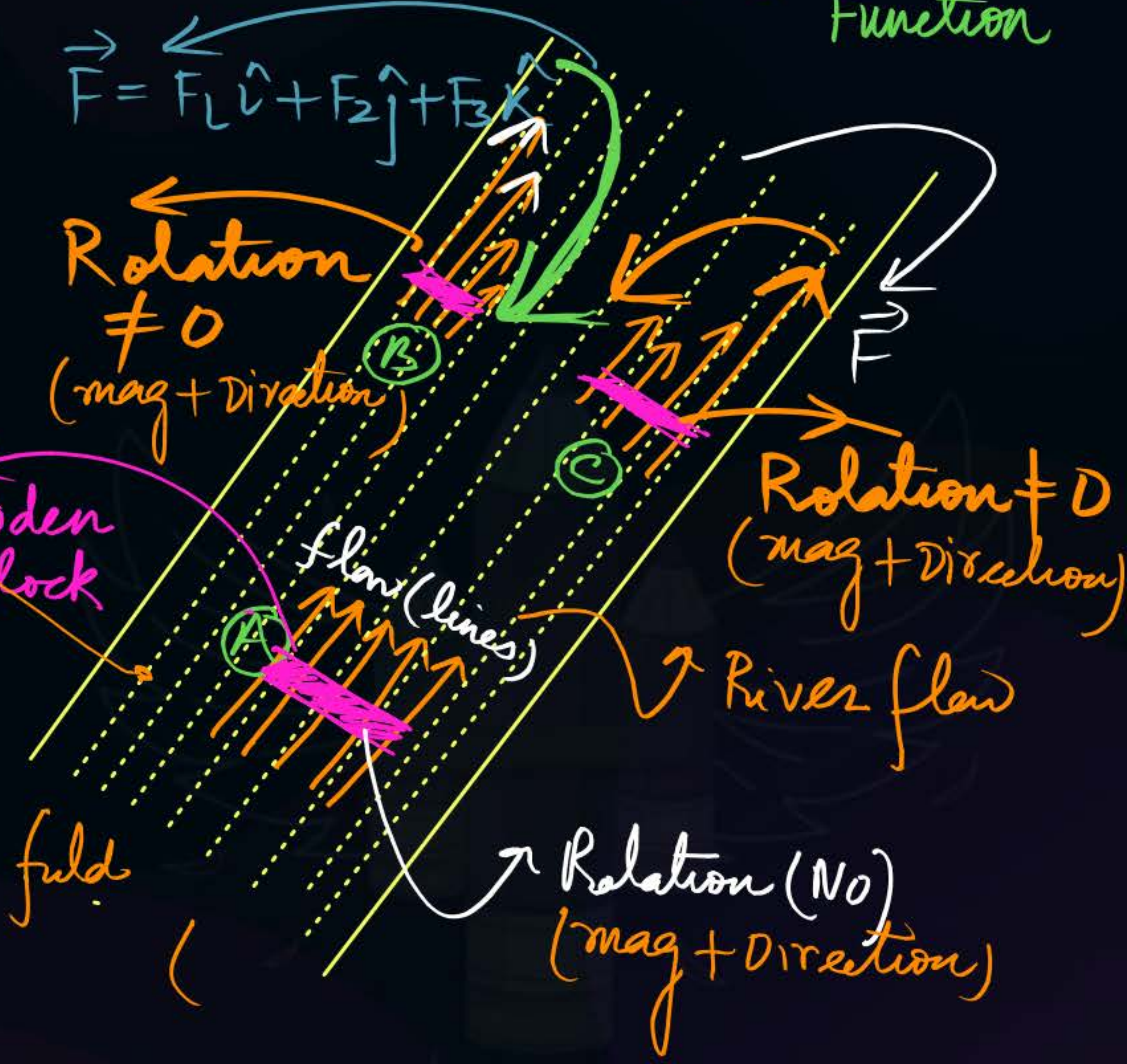
Curl $\vec{F}$ = Vector quantity

= Cross product

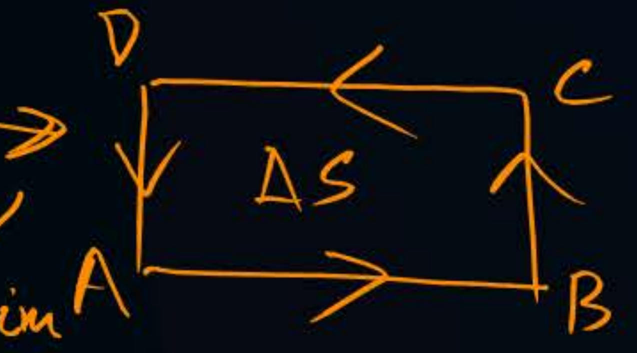
$\nabla \times \vec{F}$ = curl $\vec{F}$

Vector Function $\xrightarrow[\nabla\,(\text{del})]{}$ Another vector Function

enlarged



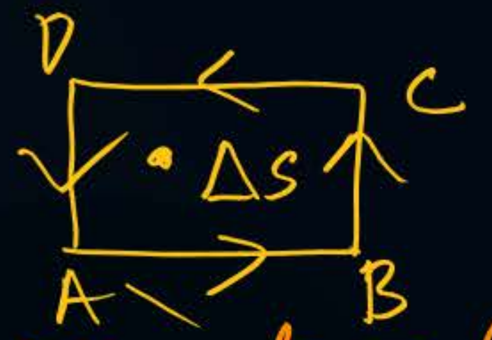
Infinitesimal point — Enlarge — Lim



$$\nabla \times \vec{F} = \underbrace{\lim_{\Delta S \to 0} \frac{\oint \vec{F} \cdot d\vec{r}}{\Delta S}}_{\text{magnitude}} \cdot \hat{n}$$

Curl $\vec{F}$

(mag + Direction)



per unit surface Total Line Integral

Infinitesimal.

x, y, z Rotations

$$\boxed{\text{Curl } \vec{F} = \lim_{\Delta S \to 0} \frac{\oint \vec{F} \cdot d\vec{r}}{\Delta S} \cdot \hat{n}}$$

Where $\hat{n}$ = Outward drawn Normal

$\hat{i}, \hat{j}, \hat{k}$

All Curl sum

Curl — Line Integral

for small element

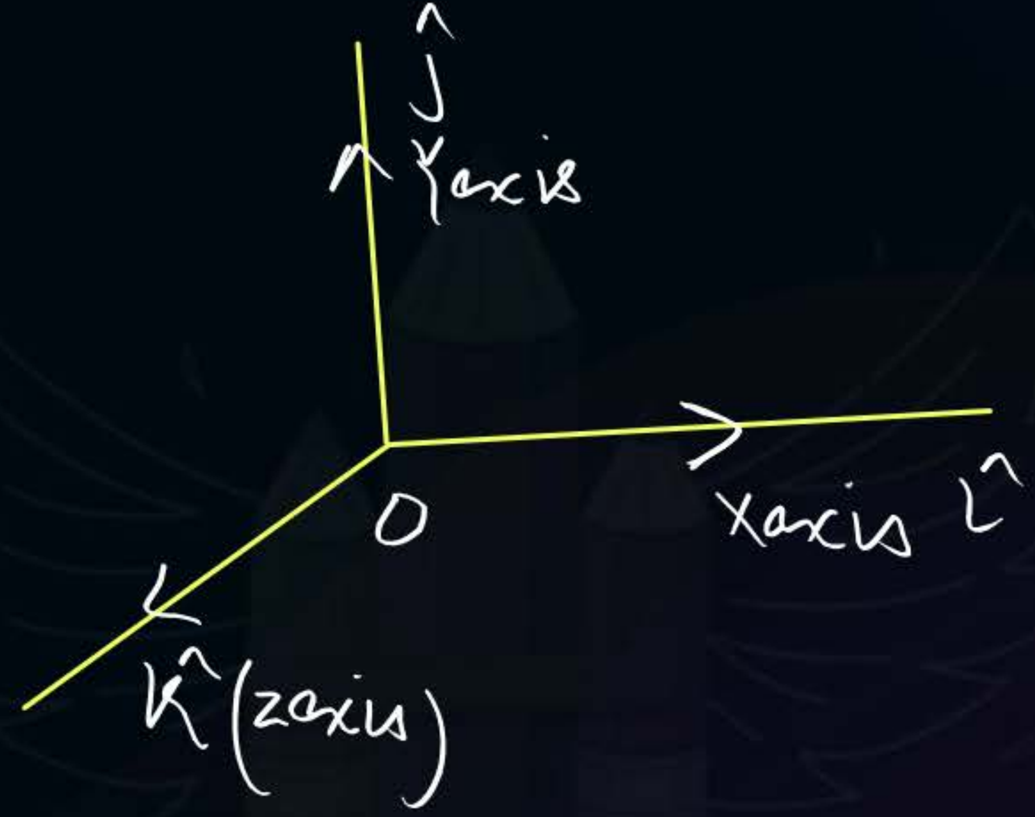
$$\nabla \times \vec{F} = \underbrace{\lim_{ds \to 0} \frac{\oint \vec{F}\, d\vec{r}}{ds}}_{\text{magnitude + Direction}} \hat{n} \longrightarrow \text{vector quantity (mag + Direction)}$$

Proof

How to evaluate the Curl:

If $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$



$\hat{n}$ change in YZ Direction

$$\nabla \times \vec{F} = \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) \text{ XZ Direction}$$

$$+\hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \text{ XY Direction}$$

Cross product Area/volume

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Vector quantity $\longrightarrow$ Another Vector quantity

magnitude + Direction

$$\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i}$$

$$-\hat{j}\underbrace{\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)}$$

XZ Direction

$$\hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

X-Y Direction

Topic : Vector calculus

#Q. If $\vec{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j} + (y^2 - xy)\hat{k}$

The curl of $\overrightarrow{F}$ at (1, 1, 1) is equal to

$$\vec{F} = (x^2 - y^2)\hat{i} + 2xy\hat{j} + (y^2 - xy)\hat{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 - y^2) & 2xy & (y^2 - xy) \end{vmatrix}$$

$x = 1$, $y = 1$, $z = 1$

$$\Rightarrow \hat{i}\left[\frac{\partial}{\partial y}(y^2 - xy) - \frac{\partial}{\partial z}(2xy)\right] - \hat{j}\left[\frac{\partial}{\partial x}(y^2 - xy) - \frac{\partial}{\partial z}(x^2 - y^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2 - y^2)\right]$$

$$= \hat{i}\left[(2y - x) - 0\right] - \hat{j}\left[-y - 0\right] + \hat{k}\left[2y - (-2y)\right]$$

$x = 1$, $y = 1$, $z = 1$

$$= \hat{i}\left[(2y - x)\right] + \hat{j}y + \hat{k}\left[4y\right]$$

$$\nabla \times \vec{F} = \underline{\hat{i} + \hat{j} + 4\hat{k}}$$

If $\nabla \times \vec{F} = \text{curl}\, \vec{F} = \vec{0}$ = Irrotational

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0 = \text{Irrotational Vector}$$

Stokes Theorem: $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ → old (Line Integral)

(Three Dimensional vector function/field)

$$\oint \vec{F}\cdot d\vec{r} = \oint F_1\,dx + F_2\,dy + F_3\,dz$$

$$\boxed{= \oint \vec{F}\,d\vec{r} = \iint (\text{curl}\,\vec{F})\,\hat{n}\,ds}$$

3 Dimensional space

Where $\hat{n}$ = outward Drawn Normal

ds = surface element They depend on surface xy, yz, zx

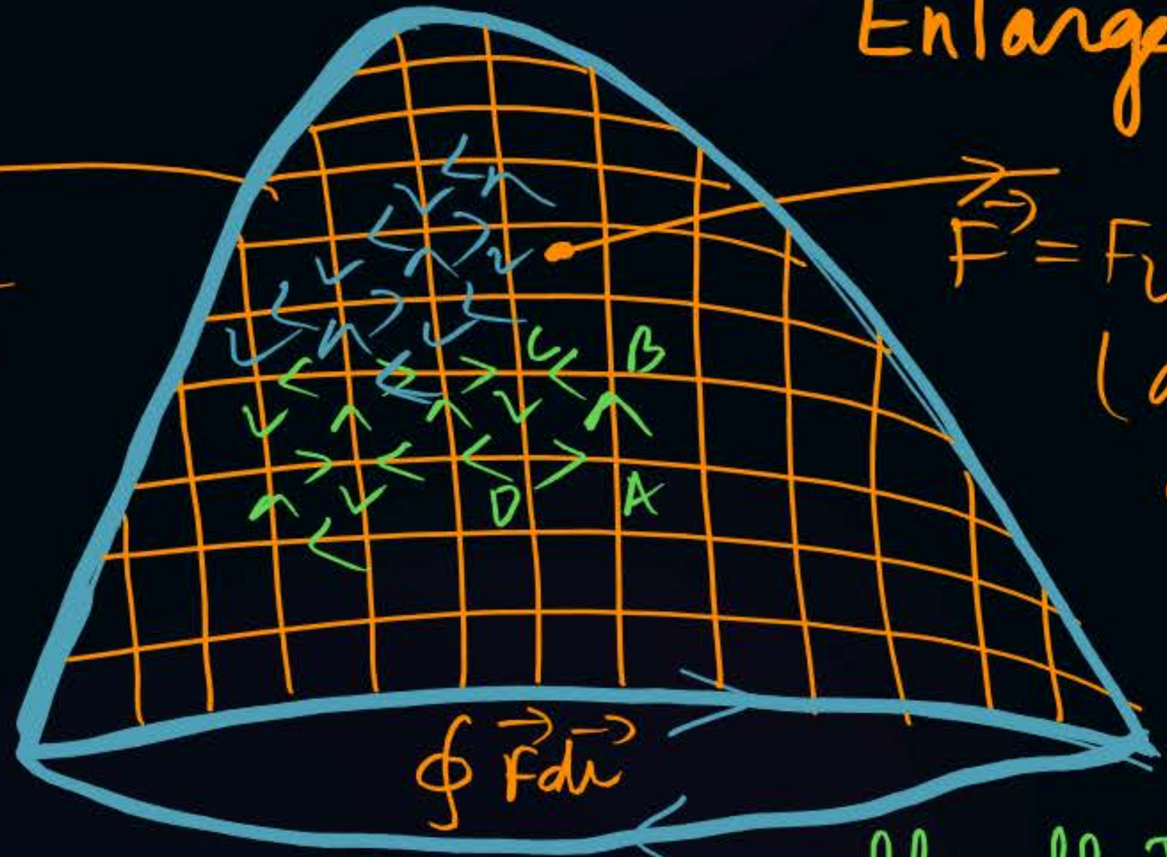
Statement: $$\oint \vec{F}\,d\vec{r} = \iint \underbrace{(\nabla\times\vec{F})\,\hat{n}\,ds}$$

$\hat{n}\cdot\hat{n} = 1$

$$(\nabla\times\vec{F})\,\hat{n}\,ds = \boxed{\lim_{ds\to 0} \frac{\oint \vec{F}\,d\vec{r}}{ds}\,\hat{n}}\,ds\,\hat{n} = \oint \vec{F}\,d\vec{r}$$

Enlarge

House
Honey
bee

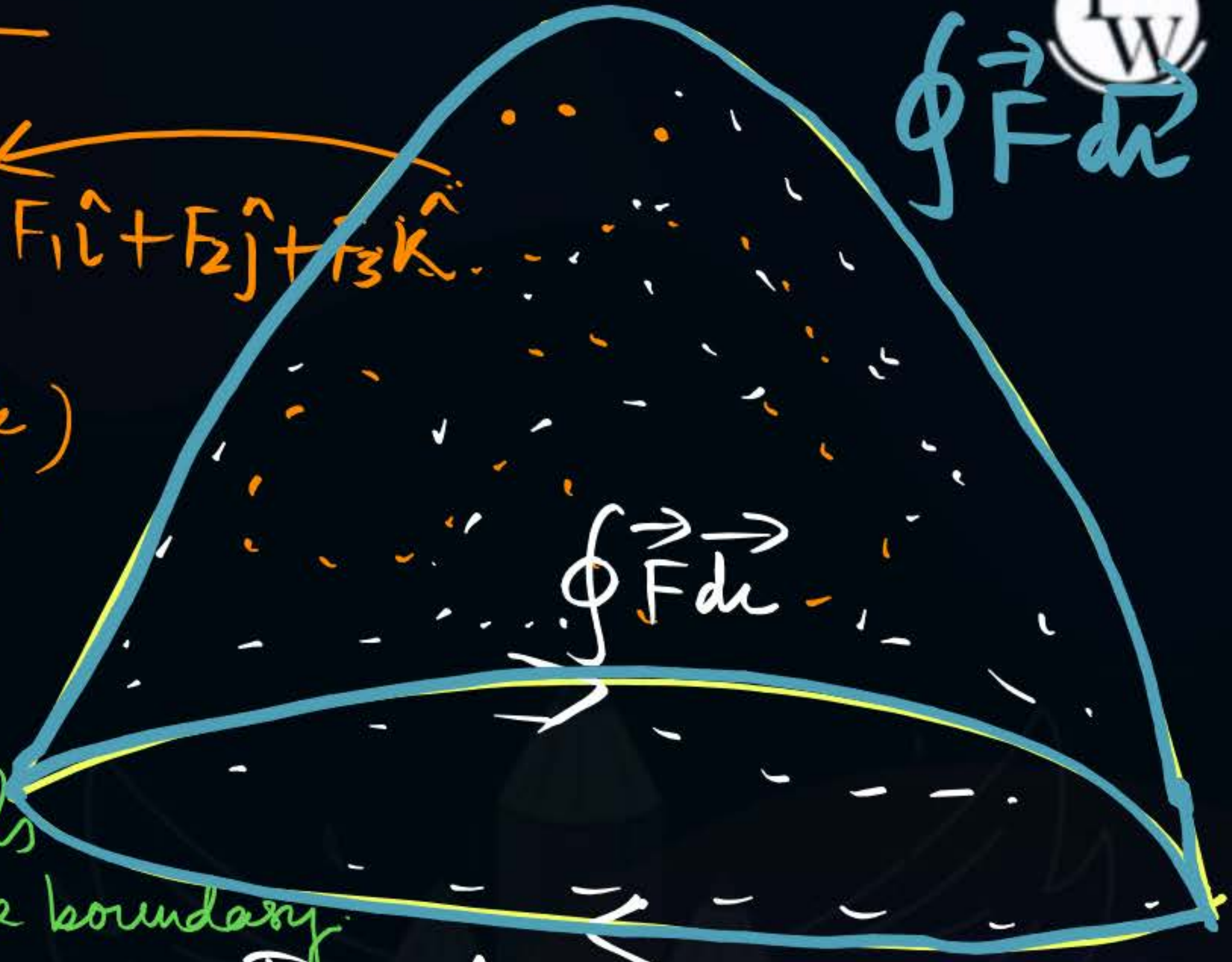


$\oint \vec{F}\,d\vec{r}$

$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$
(all magnitude) are equal

$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

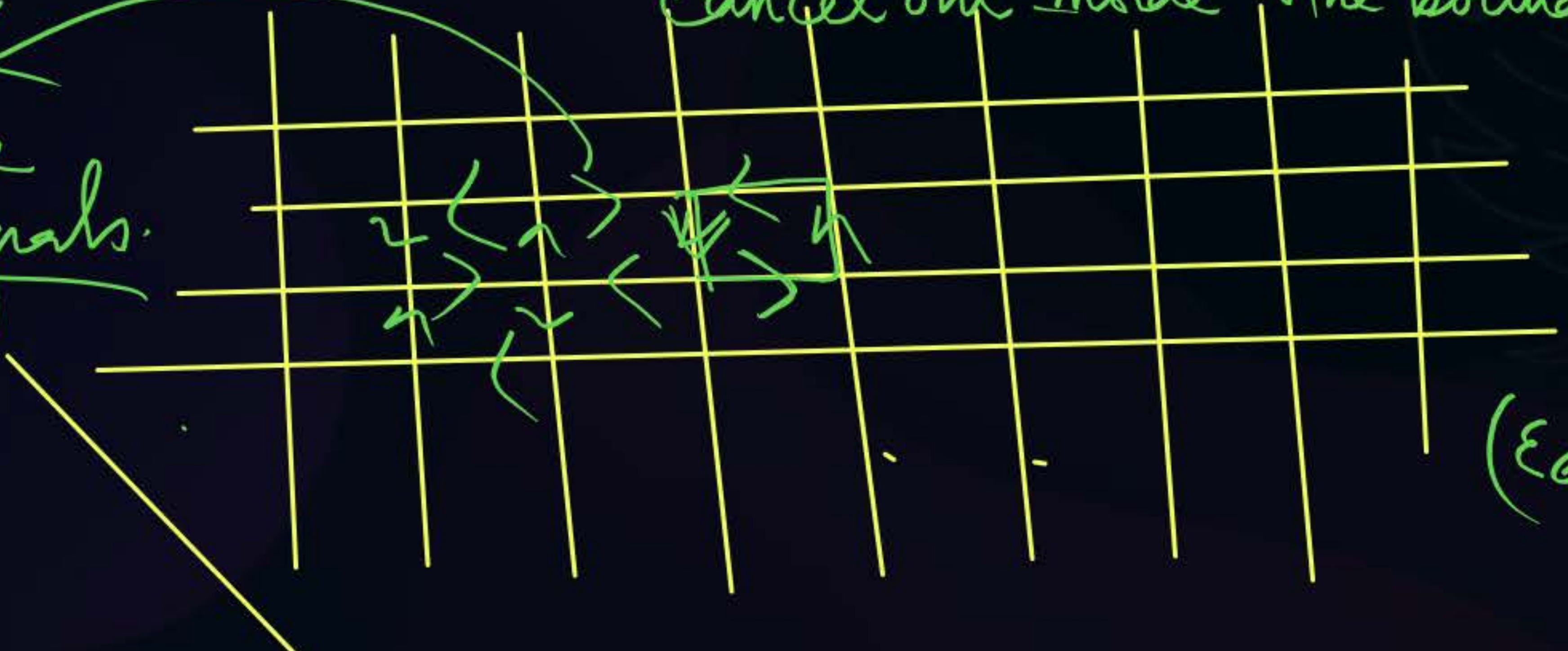
$\oint \vec{F}\,d\vec{r}$



Inverted Pot

all all line Integrals
cancel out inside the boundary.

Line
integrals.

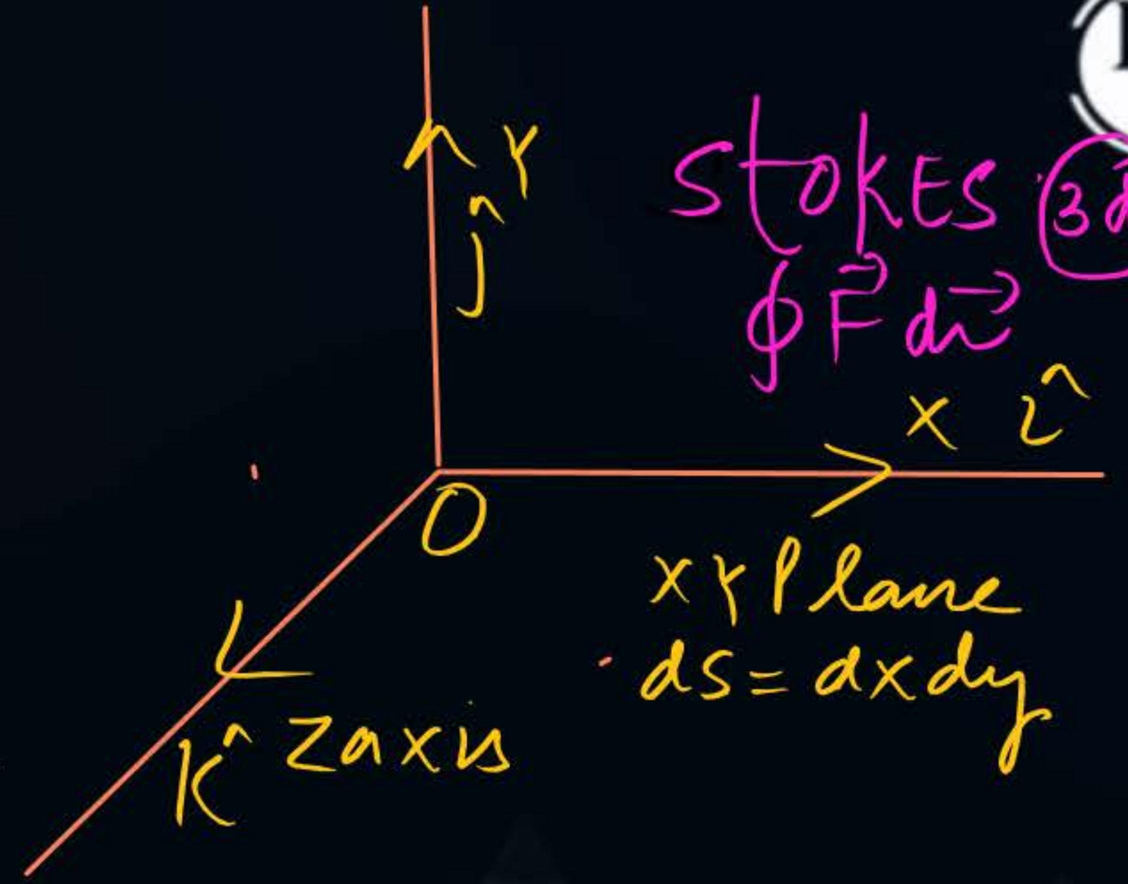


(Equal mag + opposite Direction)

$$\oint F\,d\vec{r} \underset{\text{3 Dimensional space}}{=} \iint (\nabla \times \vec{F})\,\underline{\hat{n}}\,ds$$

Stokes (3d)

$\oint \vec{F}\,d\vec{r}$



In xY Plane.

Using stokes Theorem.

$$\oint_{xY\,Plane} F\,d\vec{r} = \iint (\nabla \times \vec{F})\,\hat{k}\,dy\,dx$$

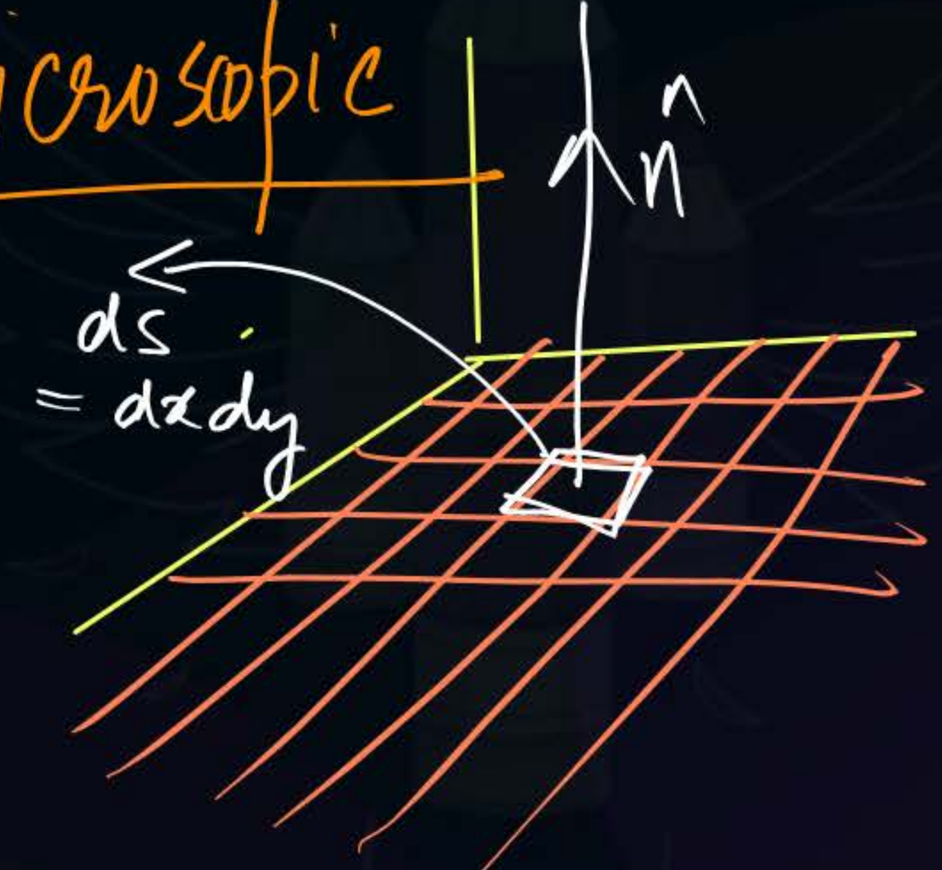
In yz Plane.

$$\oint_{yz\,Plane} \vec{F}\,d\vec{r} = \iint (\nabla \times \vec{F})\,\hat{i}\,dy\,dz$$

zx Plane.

$$\oint_{zx\,Plane} \vec{F}\,d\vec{r} = \iint (\nabla \times \vec{F})\,\hat{j}\,dz\,dx$$

Microsopic



CASE 02 If Field is given (2d)

$$\vec{F} = F_1(x,y)\,\hat{i} + F_2(x,y)\,\hat{j}$$

$\hat{n} = \hat{k}$
(xy Plane)
$ds = dy\,dx$

Using stokes Theorem: $\oint \vec{F}\,d\vec{r} = \iint (\nabla \times \vec{F})\,\hat{n}\,ds$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix} = \underbrace{\hat{i}[0] - \hat{j}[0]}\, + \underbrace{\hat{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]}$$

$$\oint \vec{F}\,d\vec{r} = \iint \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}\,dy\,dx$$

$$\oint \vec{F}\, d\vec{r} = \iint \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dy\, dx$$

$$\boxed{\oint \underbrace{F_1\, dx + F_2\, dy}_{\text{work done}} = \iint_{x\, y} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dy\, dx}$$

Green's Theorem
(2d space)

$$\oint M\, dx + N\, dy = \iint_{x\, y} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy\, dx$$

(region)

Green's Theorem

(Macroscopic)
(2d)
Line integrals

THANK - YOU