GATE (ALL BRANCHES)

Engineering Mathematics

Complex Analysis

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Lecture No. 03

01 Problems based on Complex functions, C-R equations

Q. Questions

$W = \ln z$ $\log 0 = \text{undefined}$

$= \ln(x+iy)$ $0, 0$ Not analytic

#Q. The function $w = u + iv = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$ is not analytic at the point.

$W = \ln z$

$W = u + iv = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

$W = f(z)$

$\left.\begin{array}{l} z = x+iy \\ W = u+iv \end{array}\right\}$

(A) (0, 0) ✓

(B) (0, 1)

(C) (1, 0)

(D) $(2, \alpha)$

$w = \ln z$ $\quad z = re^{i\theta}$

$= \ln(\underset{\mathrm{I}}{r}\,\underset{\mathrm{II}}{e^{i\theta}})$

$= \ln r + i\theta \ln_e e$

$= \ln r + i\theta$

$= \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

$r = \sqrt{x^2+y^2}$

$\rightarrow$ modulus.

$\theta = \text{Argument} = \tan^{-1}(y/x)$

Questions

#Q. For the function of a complex variable w = lnz (where w = u + jv and z = x + jy) the u = constant lies get mapped int he z-plane as

(A) Set of radial straight lines

(B) ✓ Set of concentric circles

(C) Set of confocal hyperbolas

(D) Set of confocal ellipses

$w = \ln z$

$w = u + jv$

$z = x + jy$

$i = j$

electrical engineering

$u(x, y) = \text{constant}$

$w = \ln z$

$= \underbrace{\frac{1}{2}\ln(x^2+y^2)}_{u(x,y)} + \underbrace{i\tan^{-1}\left(\frac{y}{x}\right)}_{+iv(x,y)}$

$u(x,y) = c$

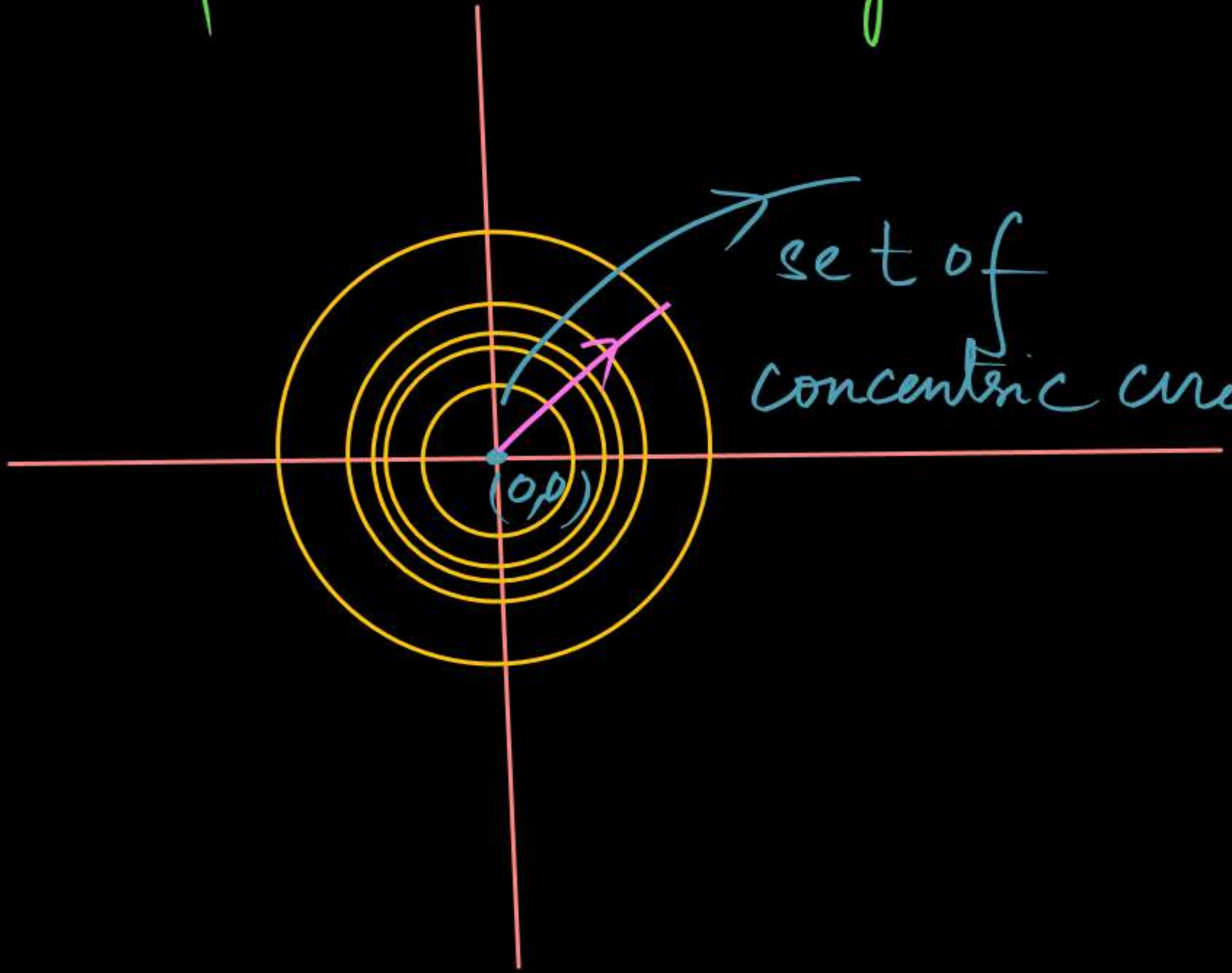
$\frac{1}{2}\ln(x^2+y^2) = c$

$\ln(x^2+y^2) = 2c$

$x^2+y^2 = e^{2c}$

$\underbrace{x^2+y^2 = (e^c)^2 = c^2}_{\text{circle.}}$

If constant is change in Nature $x^2+y^2=c^2$ arbitrary constant



$$\left.\begin{aligned} x^2+y^2&=1 \\ x^2+y^2&=2 \\ x^2+y^2&=3 \\ x^2+y^2&=4 \end{aligned}\right\}$$

Q. Questions

$\phi(x,y) \rightarrow \psi(x,y)$ $\quad u(x,y) \rightarrow v(x,y)$

$\psi(x,y) \rightarrow \phi(x,y)$ $\quad v(x,y) \rightarrow u(x,y)$

#Q. Potential function ϕ is given as $\phi = x^2 - y^2$. What will be the stream function ψ withe the condition $\psi = 0$ at $x = 0, y = 0$?

(A) $2xy$ ✓

(B) $x^2 + y^2$

(C) $x^2 + 2y^2$

(D) $2x^2 y^2$

$$\left.\begin{aligned}\phi(x,y) &= x^2 - y^2\\ \psi(x,y) &= ?\end{aligned}\right] \text{Harmonic Conjugate}$$

If $\psi = 0$ at $x = 0, y = 0$

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

Using C-R equations
$$\left.\begin{aligned}\frac{\partial \phi}{\partial x} &= \frac{\partial \psi}{\partial y}\\ \frac{\partial \phi}{\partial y} &= -\frac{\partial \psi}{\partial x}\end{aligned}\right\}$$

$$\frac{\partial \phi}{\partial x} = 2x$$

$$\frac{\partial \phi}{\partial y} = -2y$$

$$d\psi = -\frac{\partial \phi}{\partial y}dx + \frac{\partial \phi}{\partial x}dy$$

Exact Diff. Equ^n

$$d\psi = \underbrace{2y\,dx + 2x\,dy}_{M dx + N dy \text{ form}}$$

$d\psi = 2y\,dx + 2x\,dy$

both sides Integrate It

$\int d\psi = \underbrace{\int 2y\,dx}_{\text{Treating } y \text{ as a constant}} + \underbrace{\int \cancel{2x}^{\,0}\,dy}_{\text{Independent of } x}$

$\psi = 2yx + 0 + C$

$\psi(x,y) = 2xy + C$

$0 = 0 + C$

$\boxed{C = 0}$

$\boxed{\psi(x,y) = 2xy}$

Harmonic Conjugate

$M\,dx + N\,dy = 0$

Exact Diff. Equ^n

$\underbrace{\int M\,dx}_{\text{Treating } y \text{ as a const}} + \underbrace{\int N\,dy}_{\text{Independent of } x} = \text{constant}$

apply Initial conditions

$\psi(x,y) = 0$

$\psi = 0$

$x = 0$

$y = 0$

Questions

#Q. An analytic function of a complex variable $z = x + iy$ is expressed as $f(z) = u(x, y) + iv(x, y)$ where $i = \sqrt{-1}$. If $u = xy$ then the expression for v should be

(A) $\frac{(x+y)^2}{2}+k$

(B) $\frac{x-y^2}{2}+k$

(C) $\frac{y^2-x^2}{2}+k$

(D) $\frac{(x-y)^2}{2}+k$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Using Cauchy Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$\int dv = \int_{y \text{ as constant}} -x\,dx + \int_{x \text{ Independent}} y\,dy$$

$u(x,y) = xy$

$v(x,y) = ?$

$$\frac{\partial u}{\partial x} = y$$

$$\frac{\partial u}{\partial y} = x$$

$$v(x,y) = -\frac{x^2}{2} + \frac{y^2}{2} + c$$

$$\boxed{v(x,y) = \frac{y^2-x^2}{2} + c}$$

Questions

#Q. If $f(x + iy) = x^3 - 3xy^2 + i\,\phi(x, y)$ where $i = \sqrt{-1}$ and $f(x + i\,y)$ is an analytic function then $\phi(x, y)$ is

(A) $y^3 - 3x^2 y$

(B) $3x^2 y - y^3$

(C) $x^4 - 4x^3 y$

(D) $xy - y^2$

$\psi(x,y) = x^3 - 3xy^2$

$\phi(x,y) = ?$

$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$

$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$

$\frac{\partial \psi}{\partial x} = 3x^2 - 3y^2$

$\frac{\partial \psi}{\partial y} = -6xy$

$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy$

$d\phi = \frac{\partial \psi}{\partial y}dx - \left(\frac{\partial \psi}{\partial x}\right)dy$

$\int d\phi = \int -6xy\,dx - \int (\cancel{3x^2}^{\,0} - 3y^2)\,dy$

$= \phi(x,y) = -6\frac{x^2}{2}y + \frac{y^3}{\cancel{3}}\cancel{3} + c$

$\phi(x,y) = y^3 - 3x^2 y$

Slide-7

Q. Questions

#Q. For an analytic function $f(x + iy) = u(x, y) + i\,v(x, y)$, u is given by $u = 3x^2 - 3y^2$. The expression for v, considering k is to be constant is

(A) $3y^2 - 3x^2 + k$

(B) $6x - 6y + k$

(C) $6y - 6x + k$

(D) $6xy + k$ ✓

$u(x,y) = 3x^2 - 3y^2$

$v(x,y) =$

Using C-R eqn

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

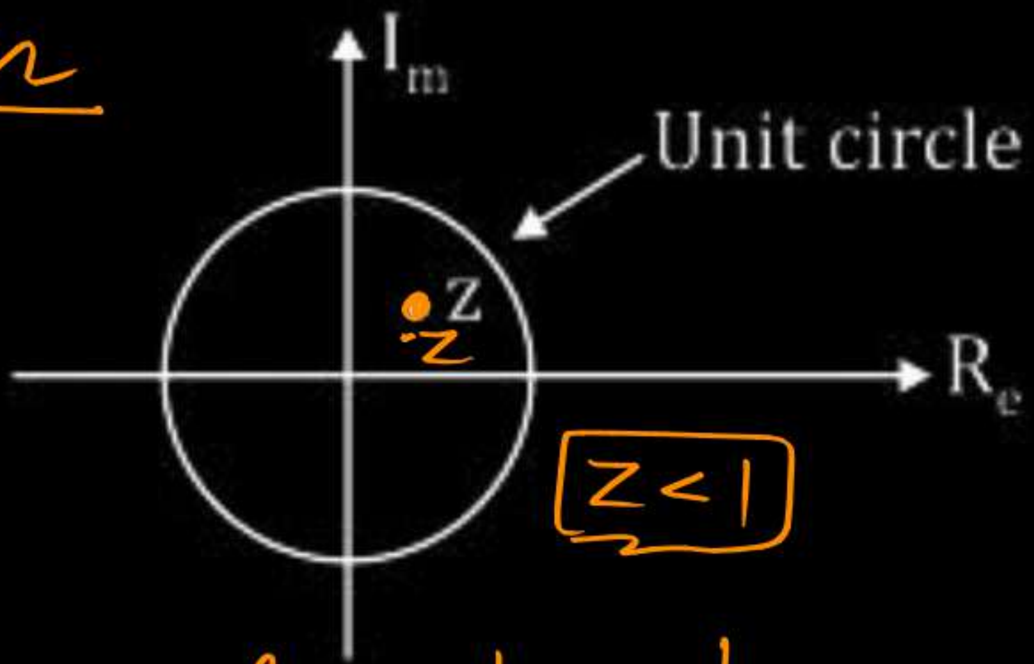
$$\int dv = \int -(-6y)\,dx + \cancel{\int 6x\,dy} \to 0$$

$$\boxed{v(x,y) = 6xy + k}$$

Questions

#Q. A point z has been plotted in the complex plane as shown in the figure below

$W = \frac{1}{z}$ outer

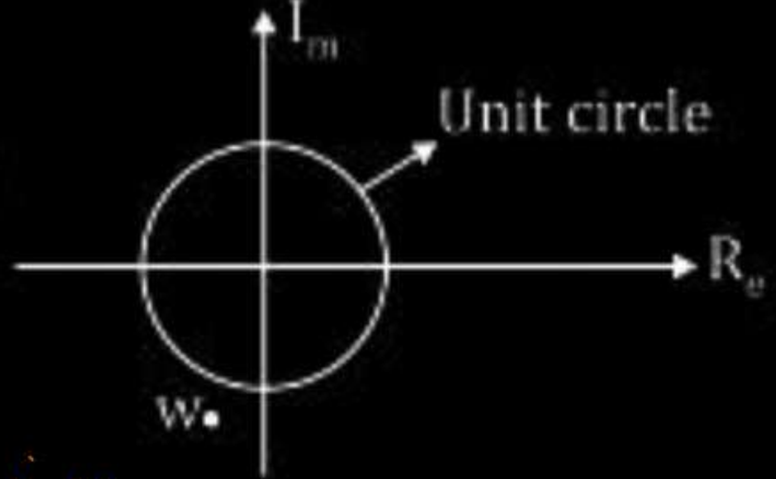


$\boxed{|z| < 1}$

The plot of the complex number w = 1/z

(A)

(B)



$(+\ -)$

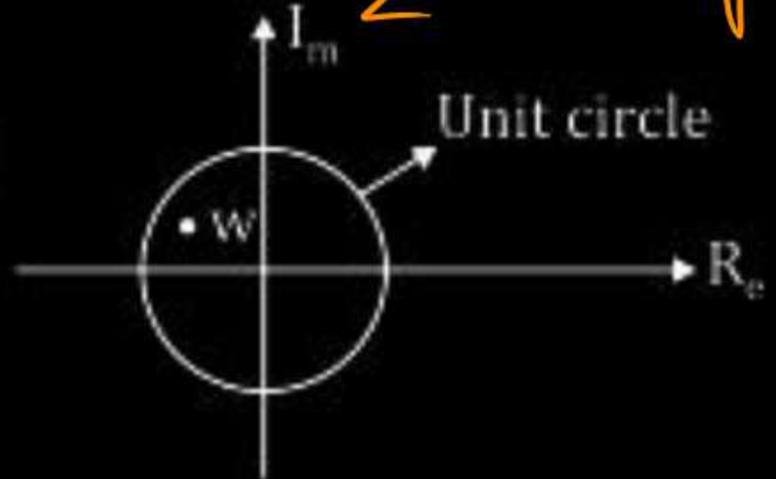
$W = f(z) = \frac{1}{z} = \frac{x - iy}{x^2+y^2} = \frac{x}{(x^2+y^2)} - i\frac{y}{(x^2+y^2)}$

$W = f(z) = \frac{1}{z} = \frac{1}{(x+iy)}$

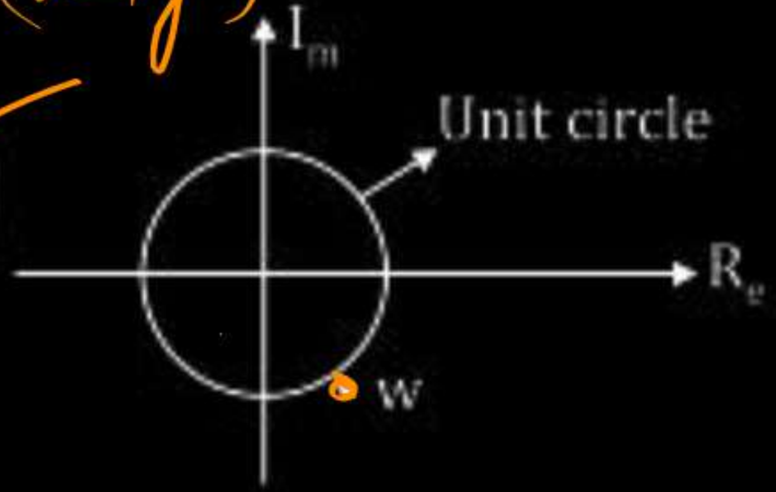
$= \frac{x - iy}{(x-iy)(x+iy)}$

$= \frac{x-iy}{x^2+y^2}$

(C)



(D) ✓



Q. Questions

#Q. The real part of an analytic function f(z) where z = x + jy is given by $e^{-y}\cos(x)$. The imaginary part of f(z) is

$u(x,y)$

(A) $e^{y}\cos(x)$

(B) $e^{-y}\sin(x)$

(C) ✓ $-e^{y}\sin(x)$

(D) $-e^{-y}\sin(x)$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$\int dv = \int -(-e^{-y}\cos x)\,dx + \int \cancel{(-e^{-y}\sin x)\,dy}^{\,0}$$

$$v = -e^{-y}\sin x + 0 + k$$

$$\boxed{v(x,y) = -e^{-y}\sin x}$$

$u(x,y) = e^{-y}.\cos x$

$v(x,y) = ?$

Using C-R equⁿ

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial x} = -e^{-y}\sin x$$

$$\frac{\partial u}{\partial y} = -e^{-y}\cos x$$

Q. Questions

#Q. Let S be the set of points in the complex plane corresponding to the unit circle. (i.e., $S = \{z : |Z| = 1\}$). Consider the function $f(z) = zz^*$ where z^* denotes the complex conjugate of z. The f(z) maps S to which one of the following in the complex plane?

$f(z) = zz^*$

z^* denotes complex conjugate

(A) Unit circle

(B) $\boxed{\text{Horizontal axis line segment from origin to (1, 0)}}$ ✓

$z = x+iy \quad z^* = x-iy$

(C) The point (1, 0)

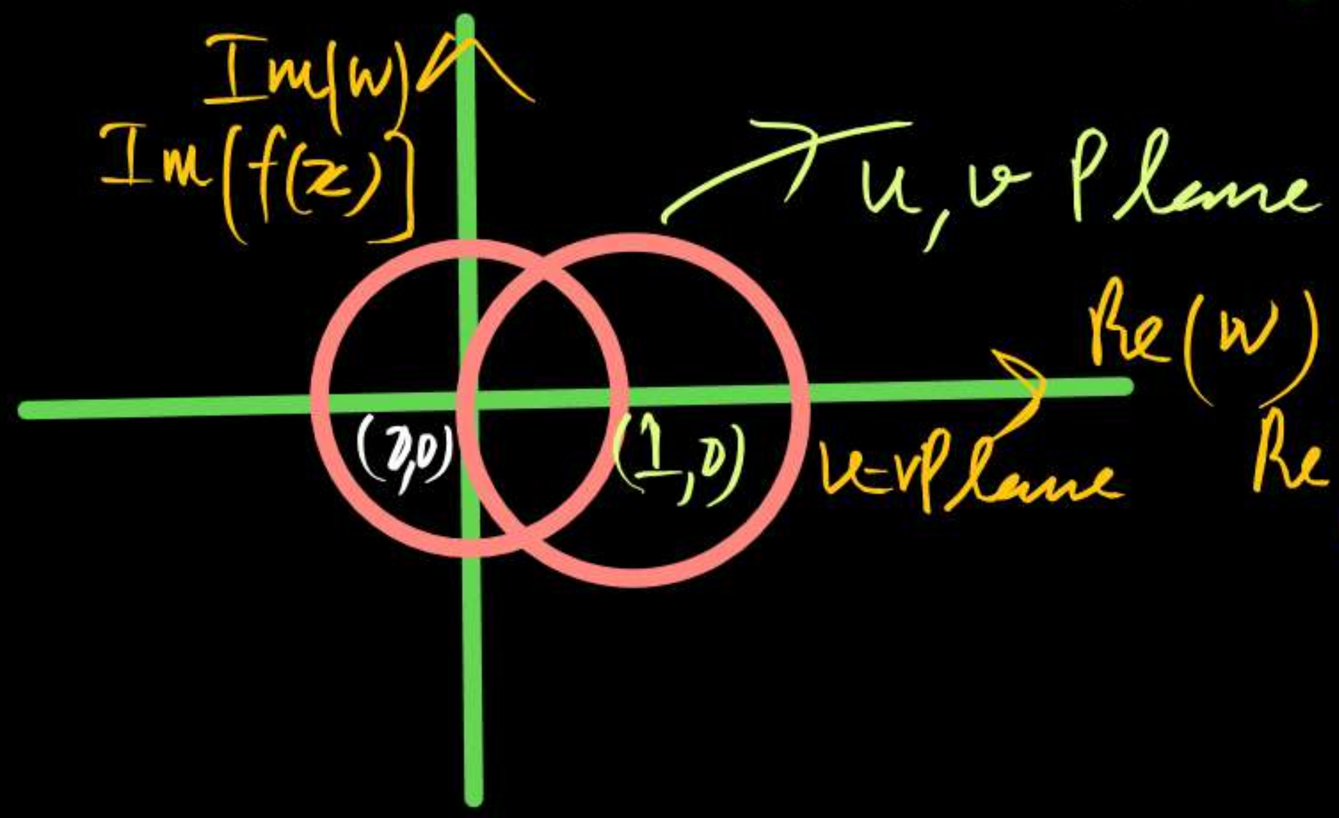
$zz^* = (x+iy)(x-iy) = x^2+y^2 = 1$

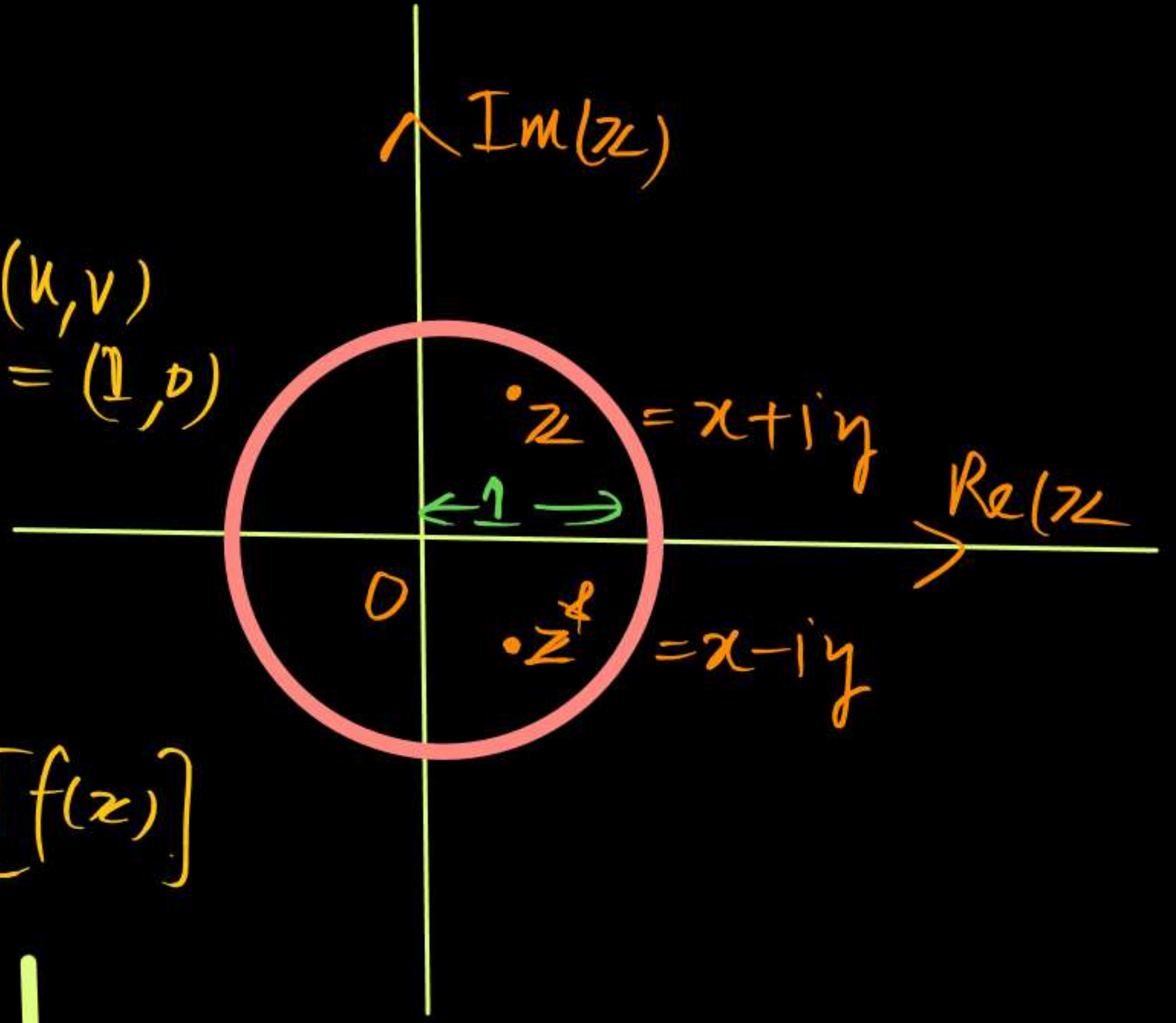
(D) The entire horizontal axis

$$f(z) = z \cdot z^*$$
$$= (x+iy)(x-iy)$$
$$= (x^2+y^2) = 1$$

(u, v)

$$f(z) = 1 = u+iv = 1+i\cdot 0 = (1,0)$$





Q. Questions

#Q. All the values of the multi valued complex function 1^i, where $i = \sqrt{-1}$

(A) Purely imaginary

(B) Real and non negative

(C) On the unit circle

(D) Equal in real and imaginary parts.

Q. Questions

#Q. An analytic function of a complex variable $z = x + i\,y$, where $i = \sqrt{-1}$ is expressed as $f(z) = u(x, y) + i\,v(x, y)$. If $u(x, y) = 2xy$, then $v(x, y)$ must be

(A) $x^2 + y^2 + \text{constant}$

(B) $x^2 - y^2 + \text{constant}$

(C) $-x^2 + y^2 + \text{constant}$ ✓

(D) $-x^2 - y^2 + \text{constant}$

$z = x + iy$

$i = \sqrt{-1}$

$u(x,y) = 2xy$

$v(x,y)$

$$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$\int dv = \int -2x\,dx + \int 2y\,dy$$

$$\boxed{v(x,y) = -x^2 + y^2 + c}$$

Q. Questions

#Q. An analytic function of a complex variable $z = x + iy$ is expressed as $f(z) = u(x, y) + iv(x, y)$, where $i = \sqrt{-1}$. If $u(x, y) = x^2 - y^2$, then expression for $v(x, y)$ in terms of x, y and a general constant c would be

(A) $xy + c$

(B) $\frac{x^2+y^2}{2} + c$

(C) $2xy + c$

(D) $\frac{(x-y)^2}{2} + c$

$u(x,y) = x^2 - y^2$

$\frac{\partial u}{\partial y} = -2y$

$\frac{\partial u}{\partial x} = 2x$

$dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$

$dv = 2y\,dx + 2x\,dy$

both sides Integrate It

$\int dv = \int_{y\ \text{constant}} 2y\,dx + \int 2x\,dx$ Independent of x

$\boxed{v(x,y) = 2xy + c}$

$\boxed{v(x,y) = 2xy + c}$

Q. Questions

Constant $\longrightarrow$ C-R equations

$w = f(z)$

#Q. If $f(z) = (x^2 + ay^2) + ibxy$ is a complex analytic function of $z = x + iy$, where $i = \sqrt{-1}$, then

(A) $a = -1, b = -1$

(B) $a = -1, b = 2$

(C) $a = 1, b = 2$ ✓

(D) $a = 2, b = 2$

$f(z) = (x^2 + ay^2) + ibxy$

Using C-R equ^n

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

\# $u(x,y) = x^2 + ay^2$

$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = 2ay$

\# $v(x,y) = bxy$

$\frac{\partial v}{\partial x} = by$

$\frac{\partial v}{\partial y} = bx$

Using C-R equ^n

$2x = bx \qquad (2-b)x = 0$

✓ $\boxed{b = 2}$ $\qquad 2 - b = 0$

$\boxed{b = 2}$

$2ay = -by$

$(2a - b)y = 0 \qquad 2a = b$

$2a = 2$

$a = 1$

$a = 1 \quad b = 2$

Q. Questions

#Q. F(z) is a function of the complex variable z = x + iy given by $z = x + iy$

F(z)= iz + k Re(z) + i Im(z) $\quad Re(z) = x \quad Im(z) = y$

For what value of *k* will F(z) satisfy the Cauchy-Riemann equations?

$$F(z) = iz + K\,Re(z) + i\,Im(z)$$ — C-R equations Are satisfied

A 0

B 1

C −1

D y

$$F(z) = i(x+iy) + k(x) + i \cdot y$$

$$= ix + i^2 y + kx + iy$$

$$= ix - y + kx + iy$$

$$F(z) = (kx - y) + i(x + y)$$

$$F(z) = (Kx - y) + i(x + y)$$

$$u(x,y) = Kx - y, \quad v(x,y) = (x + y)$$

$$\frac{\partial u}{\partial x} = K \quad \frac{\partial u}{\partial y} = -1$$

$$\frac{\partial v}{\partial x} = 1 \quad \frac{\partial v}{\partial y} = 1$$

Using C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Rightarrow$ apply C-R equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$K = 1 \qquad -1 = -1$$

$$K = 1$$

Q. Questions

#Q. Consider the analytic function $f(z) = x^2 - y^2 + i\,2xy$ of the complex variable $z = x + iy$, where $i = \sqrt{-1}$. The derivative $f'(z)$ is

$$f(z) = (x^2 - y^2) + 2ixy$$

A. $2x + i2y$

B. $x^2 + iy^2$

C. $x + iy$

D. $2x - i2y$

$$f(z) = z^2$$
$$= (x+iy)^2$$
$$= \underbrace{(x^2 - y^2)}_{u(x,y)} + \underbrace{2ixy}_{v(x,y)}$$

$$f'(z) = 2z$$
$$= 2(x+iy)$$
$$= 2x + 2iy$$

Q. Questions

#Q. A harmonic function is analytic if it satisfies he Laplace equation. If $u(x, y) = 2x^2 - 2y^2 + 4xy$ is a harmonic function, then its conjugate harmonic function $v(x, y)$ is

A $-4xy + 2y^2 - 2x^2 +$ constant

B $4xy - 2x^2 + 2y^2 +$ constant

C $4y^2 - 4xy +$ constant

D $2x^2 - 2y^2 + xy +$ constant

$u(x,y) = 2x^2 - 2y^2 + 4xy$

$\partial v = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$

$\partial v = -(-4y + 4x)dx + [4x + 4y) dy$

$\int dv = \underbrace{\int (4y - 4x) dx}_{Y\ \text{constant}} + \underbrace{\int (4x + 4y) dy}_{\text{Independent of } x}$

$\boxed{v(x,y) = 4yx - 2x^2 + 2y^2 + c}$

Q. Questions

#Q. Consider the complex valued function $f(z) = 2z^3 + b\,|z|^3$ where z is a complex variable. The value of b for which the function f(z) is analytic is ______.

$f(z) = 2z^3 + b|z|^3$

$b = 0$

$f(z) = 2z^3$

every polynomial is a differentiable function

$b = 0$

Consider the complex valued function

In Real function

$|x| \longrightarrow$ Not differenti.

$|z|$ is not diff. $(0,0)$

Thank You!

PW Soldiers