GATE (ALL BRANCHES)

Engineering Mathematics

Differential Equation + Partial differential

Lecture No. 04

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TOPICS TO BE COVERED

01 Problems based on Linear Differential equation

02 Reducible to Linear Differential Equation

03 Higher order Linear Differential Equations with constant coefficients

Linear Diff. Equ^n [Reducible Form]

$$\underline{f'(y)}\frac{dy}{dx} + \underbrace{f(y)}_{\text{Middle Term}} P(x) = Q(x)$$

Derivative of Middle term

Where P and Q Are Function of x only.

Linear D.E $\quad \frac{dy}{dx} + Py = Q$

$f'(y)\frac{dy}{dx}$ = Non Linear term = (Dependent var) × derivative of Dependent variable

$f(y) = t \quad$ w.r.t to x

$f'(y)\frac{dy}{dx} = \frac{dt}{dx} \qquad \frac{dt}{dx} + t \cdot P(x) = Q(x)$

This is Linear D.E which is linear in t

$$\frac{dt}{dx} + t\,P(x) = Q(x)$$

Integrating Factor $= e^{\int P(x)\,dx}$

Solution of Differential Eqn.

$$\boxed{t\cdot(I\cdot F) = \int (RHS)\cdot(I\cdot F)\,dx + C}$$

Soln of D.E

Bernoulli Eqn

Reduced to Linear D.E

$$\frac{dy}{dx} + Py = Qy^n$$

Non Linear

Linear

$$\frac{dy}{dx} + Py = Q$$

$$\frac{dy}{dx} + Py = Qy^n \longrightarrow \frac{dy}{dx} + Py = Q$$

Divide via y^n

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} + Py^{1-n} = Q$$

$y^{1-n} = t$

both sides Differentiate It

$$(1-n)\, y^{\cancel{1}-n-\cancel{1}} \frac{dy}{dx} = \frac{dt}{dx}$$

$$(1-n)\frac{1}{y^n}\frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{1}{y^n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dt}{dx}$$

$$\Rightarrow \frac{1}{(1-n)}\frac{dt}{dx} + Pt = Q$$

$$= \frac{dt}{dx} + P(1-n)t = Q(1-n)$$

Where n is a Integer

Which is Linear in t

Integrating factor $= e^{\int P(1-n)dx}$

$$\boxed{t \times (I.F) = \int Q(1-n)\cdot e^{\int(1-n)dx}\, dx + C}$$

Q. Questions

#Q. A curve passes through the point $(x = 1, y = 0)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{x^2+y^2}{2y} + \frac{y}{x}$$

The equation is

$$\frac{dy}{dx} = \frac{(x^2+y^2)}{2y} + \frac{y}{x}$$

→ solution

$y = 0$ at $x = 0$

(a) $ln\left(1+\frac{y^2}{x^2}\right)=x-1$

(b) $ln(1+{}^{y}/_{x})=x-1$

(c) $\frac{1}{2}\,ln\left(1+\frac{y^2}{x^2}\right)=x-1$

(d) $\frac{1}{2}\,ln(1+{}^{y}/_{x})=x-1$

(Do yourself) H.W

Q. Questions

#Q. Consider the differential equation $\left(t^2-81\right)\frac{dy}{dt}+5ty=\sin(t)$ with $y(1)=2\pi$. There exists a unique solution for this differential equation when t belongs to the interval

(a) (-2, 2)

(b) (-10, 10)

(c) (-10, 2)

(d) (0, 10)

$$\frac{dy}{dt}+\frac{5t}{(t^2-81)}=\frac{\sin t}{(t^2-81)}$$

$$P=\frac{5t}{(t^2-81)}$$

$$Q=\frac{\sin t}{(t^2-81)}$$

$$I.F=e^{\int\frac{5t}{(t^2-81)}dt}$$

Solution

$$y\cdot e^{\int\frac{5t}{(t^2-81)}}=\int\frac{\sin t}{(t^2-81)}e^{\int\frac{5t}{(t^2-81)}}dt+c$$

$\boxed{t=9}$ $\boxed{t=-9}$

$\xrightarrow[t=-9,9]{}$ does Not exists

$t\in[-2,2]$

Slide-5

Q. Questions

#Q. $\frac{dy}{dx}+\frac{xy}{1-x^2}=xy^{1/2}$

$$\frac{dy}{dx}+\frac{xy}{(1-x^2)}=xy^{1/2}$$

$$=\frac{1}{y^{1/2}}\frac{dy}{dx}+\frac{xy}{y^{1/2}(1-x^2)}=x$$

$$=\frac{1}{y^{1/2}}\frac{dy}{dx}+\frac{x}{(1-x^2)}y^{1/2}=x$$

$$P=\frac{x}{2(1-x^2)}\quad Q=\frac{x}{2}$$

$$\Rightarrow y^{1/2}=t$$

$$\Rightarrow 2\frac{dt}{dx}+\frac{x}{(1-x^2)}\cdot t=x$$

$$\frac{1}{2}y^{-1/2}\frac{dy}{dx}=\frac{dt}{dx}$$

$$\frac{dt}{dx}+\frac{x}{2(1-x^2)}\cdot t=\frac{x}{2}$$

$$\frac{1}{y^{1/2}}\frac{dy}{dx}=2\frac{dt}{dx}$$

Integrating factor $= e^{\int P\,dx}$

$= e^{\int \frac{x}{2(1-x^2)}dx}$

$I = \frac{1}{2}\int \frac{x}{(1-x)(1+x)}$ Using Partial Fraction

Solution of D.E

$$\boxed{t\cdot(I.F) = \int (RHS)(I.F)\,dx}$$ solution of D.E

Q. Questions

#Q. $y^2\frac{dy}{dx}=x+y^3$

$y^2\frac{dy}{dx}=x+y^3$

$y^2\frac{dy}{dx}-y^3\cdot 1=x$

$y^3=t$

$3y^2\frac{dy}{dx}=\frac{dt}{dx}$

$\Rightarrow \frac{1}{3}\frac{dt}{dx}-t\cdot 1=x$

$\Rightarrow \frac{dt}{dx}-3t\cdot 1=3x$

$P=-3 \quad Q=3x$

$y^2\frac{dy}{dx}$ = Non Linear

Linear D.E

$\frac{dy}{dx}+Py=Q.$

Integrating factor

$=e^{\int P dx}$

$=e^{\int -3 dx}$

$=\underline{e^{-3x}}$

$$t \cdot e^{-3x} = \int 3x\, e^{-3x}\, dx$$

$$= y^3 e^{-3x} = \frac{1}{3}\int \underset{\mathrm{I}}{t}\, \underset{\mathrm{II}}{e^{-t}}\, dt$$

$3x = t$

$3dx = dt$

$dt = \frac{dt}{3}$

$t \quad e^{-t}$

$1 \quad -e^{-t}$

$0 \quad +e^{-t}$

$= -te^{-t} - e^{-t}$

$$= y^3 e^{-3x} = \frac{1}{3}\left[-y^3 e^{-y^3} - e^{-y^3}\right] + c$$

Ans

2 Marks
100%
GATE

Higher Order Linear Diff. Equation:

$$x^n K_1 \frac{d^n y}{dx^n} + x^{n-1} K_2 \frac{d^{n-1} y}{dx^{n-1}} + x^{n-2} K_3 \frac{d^{n-2} y}{dx^{n-2}} + \dots + K_n y = X$$

Where $K_1, K_2, K_3 \dots K_n$ Are Constants X is a Function of 'x'

Two Parts Break

✓ Constant coefficients

✓ Variable coefficients

(A) $K_1 \frac{d^n y}{dx^n} + K_2 \frac{d^{n-1} y}{dx^{n-1}} + \dots + K_n y = X$

Linear Diff. Equ^n with constant cofficients

(B) $K_1 x^n \frac{d^n y}{dx^n} + K_2 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + K_n y = X$

Linear Diff. with variable cofficients

LINEAR D.E with constant coefficients:

$$K_1\frac{d^n y}{dx^n}+K_2\frac{d^{n-1}y}{dx^{n-1}}+K_3\frac{d^{n-2}y}{dx^{n-2}}+\dots+K_n y=X$$

Where $K_1, K_2, K_3 \dots K_n$ Are constants 'X' is a Function of x only.

For $n=2$ (SECOND order)

$$\Rightarrow \boxed{K_1\frac{d^2y}{dx^2}+K_2\frac{dy}{dx}+K_3 y=X}$$

Where K_1, K_2, K_3 are constants 'X' is a function of 'x' only

CHECK The Linearity:

Using The superposition Principle

$$\begin{cases} L[X_1 \pm X_2] = L[X_1] \pm L[X_2] \\ L[cX_1] = cL[X_1] \end{cases}$$

Check The Linear Property

$$L[x_1+x_2] = L[x_1] + L[x_2] \qquad y \longrightarrow x_1+x_2$$

$$K_1\frac{d^2(x_1+x_2)}{dx^2} + K_2\frac{d}{dx}(x_1+x_2) + K_3(x_1+x_2)$$

$$\Rightarrow \underbrace{\left[K_1\frac{d^2x_1}{dx^2} + K_2\frac{dx_1}{dx} + K_3x_1\right]}_{L[x_1]} + \underbrace{\left[K_1\frac{d^2x_2}{dx^2} + K_2\frac{dx_2}{dx^2} + K_3x_2\right]}_{L[x_2]}$$

$$L[x_1] + L[x_2]$$

This is a Linear Differential Equ^n

Holds The Linear Property $\boxed{L[x_1+x_2] = L[x_1] + L[x_2]}$

c_1, c_2

$$K_1 \frac{d^2y}{dx^2} + K_2 \frac{dy}{dx} + K_3 y = X$$

This is Linear D.E

Where K_1, K_2, K_3 Are const.

X is a Function of x only

If $X = 0$ (Homogenous system)

= Transient state

= Natural osullations

= No force apply.

Solution

y = Natural osullations

OR

Transients or Homogenous Eqn

$\boxed{y = \text{Complementary function}}$

✓ Non Homogenous. If $X \neq 0$, $X = e^x$, $\log x$, $\sin x$, $\cos x$, x^2, etc

✓ Forced oscillations

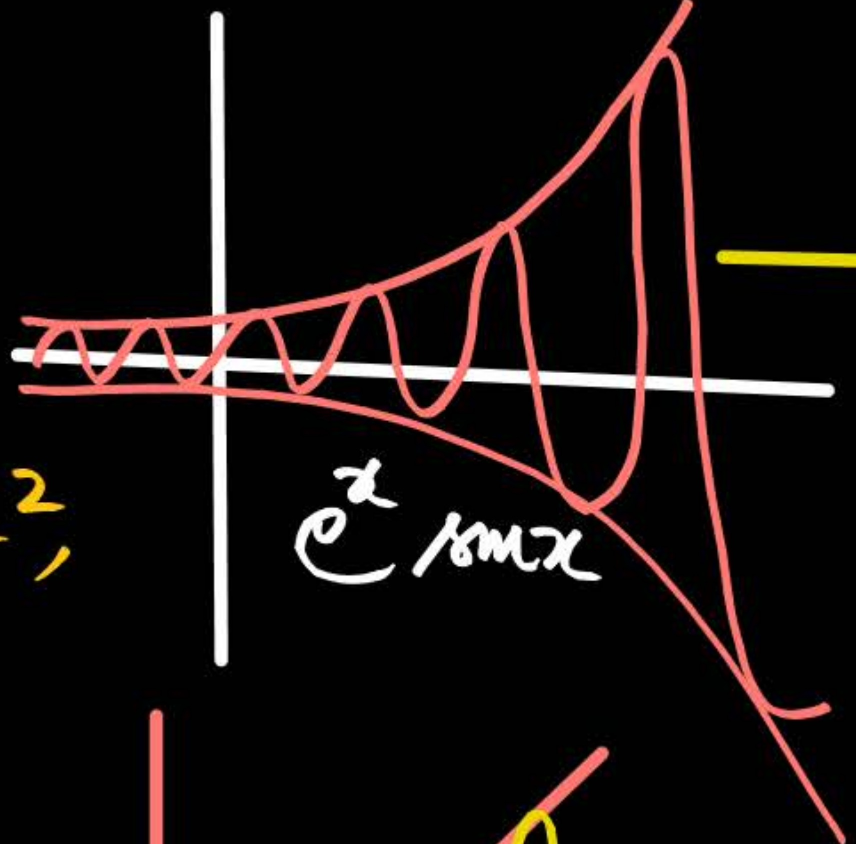
✓ Steady state

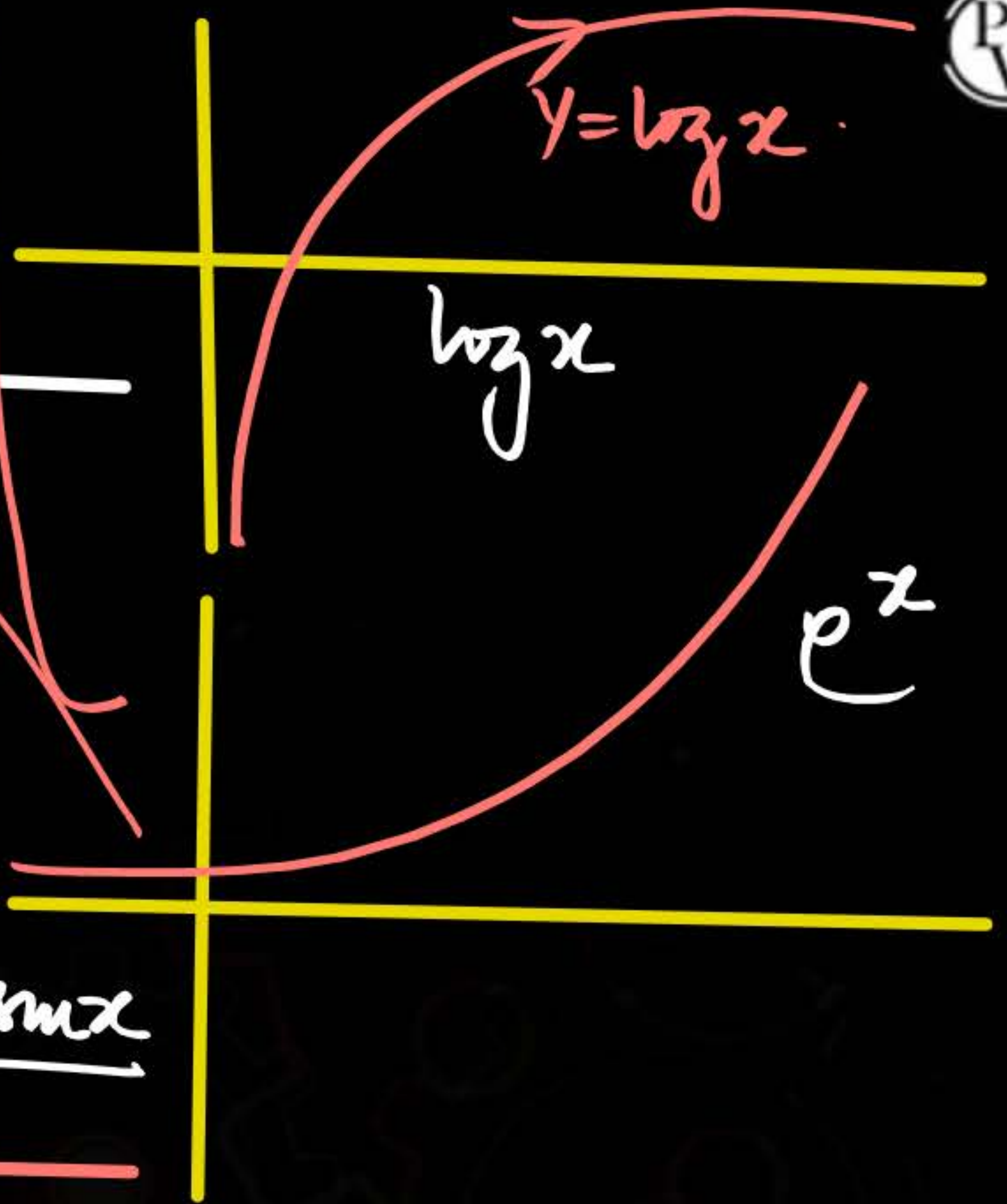
✓ force apply.

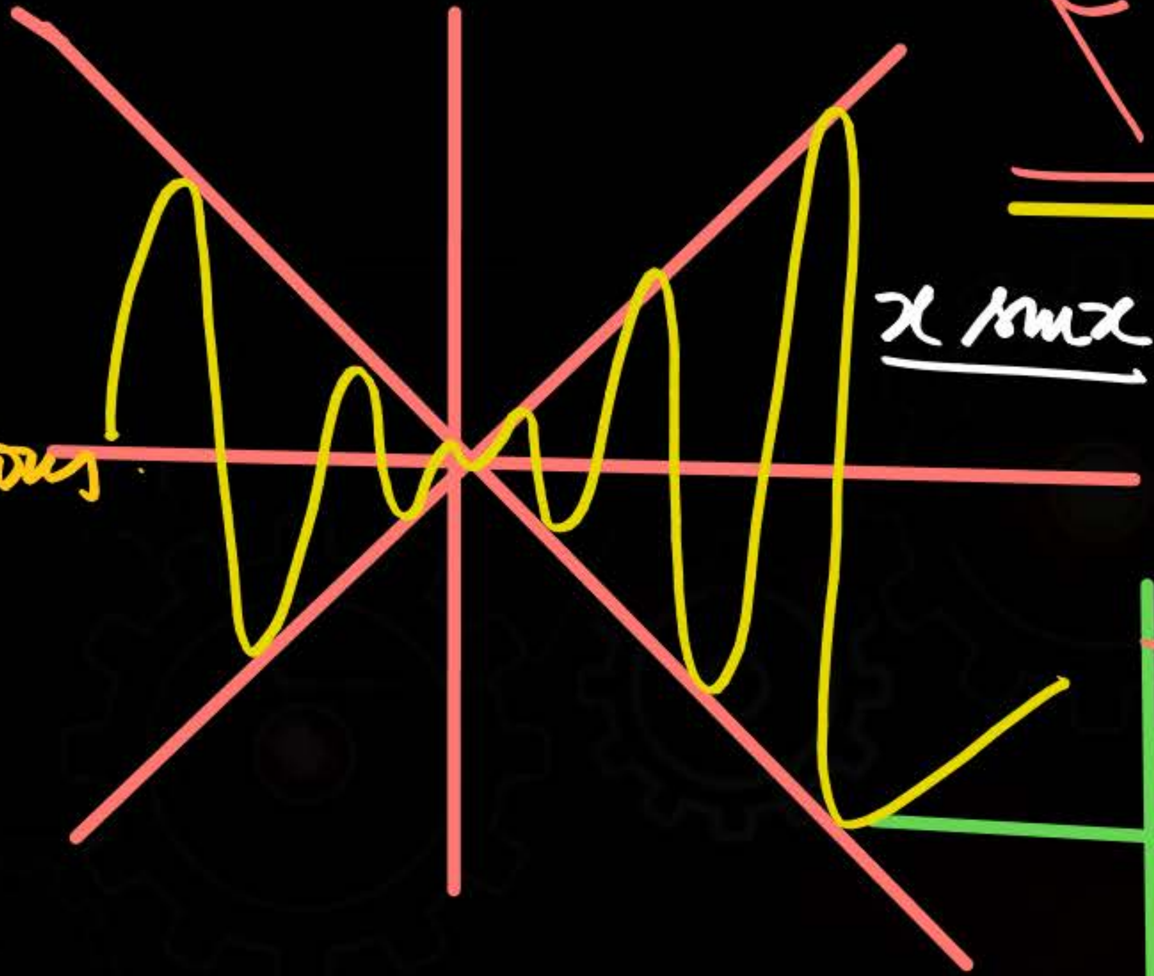
$y =$ Homogenous + Non Homogenous

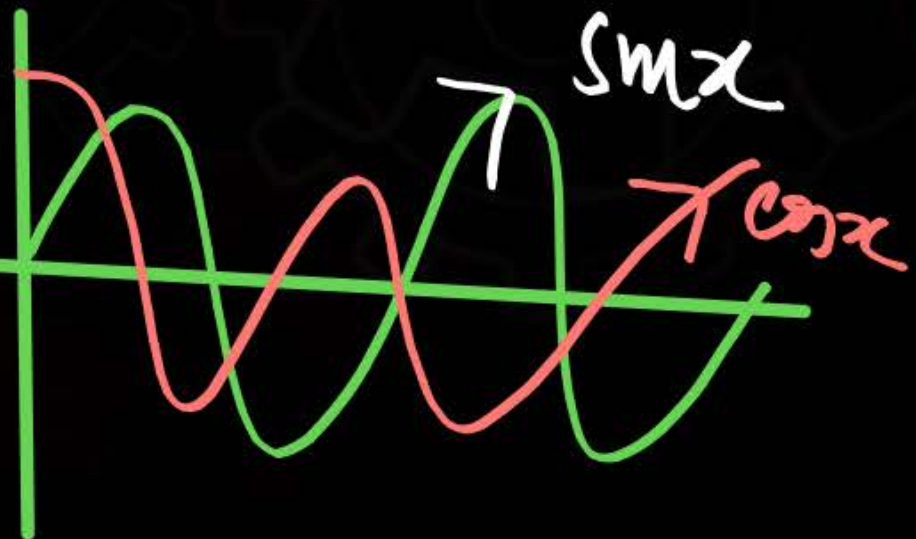
$=$ ZERO + NON ZERO solution

$=$ C.F + Particular Integral









CASE 01 If $X = 0$ $\quad K_1 \frac{d^2y}{dx^2} + K_2 \frac{dy}{dx} + K_3 y = X$

$$K_1 \frac{d^2y}{dx^2} + K_2 \frac{dy}{dx} + K_3 y = 0, \quad \frac{d^2y}{dx^2} + \underbrace{\frac{K_2}{K_1}}_{P} \frac{dy}{dx} + \underbrace{\frac{K_3}{K_2}}_{Q} y = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

Solution of this D.E

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

Using Hit and trial method

Roots

Put $y = e^{rx}$ is a solution of This D.E

$$\frac{d^2}{dx^2}(e^{rx}) + P\frac{d}{dx}(e^{rx}) + Qe^{rx} = 0$$

$$\Rightarrow \boxed{[r^2 + Pr + Q]e^{rx} = 0}$$

$$\frac{d}{dx}(e^{rx}) = re^{rx}$$

$$\frac{d^2}{dx^2}(e^{rx}) = r \cdot re^{rx} = r^2e^{rx}$$

$$\Rightarrow \underbrace{[r^2 + pr + q]}_{\text{Two Roots}} e^{rx} = 0$$

$$\Rightarrow r = \frac{\pm p \pm \sqrt{p^2 - 4q}}{2}$$

✓ $p^2 - 4q = 0$ Roots Real + Equal

✓ $p^2 - 4q > 0$ Roots real + diff.

✓ $p^2 - 4q < 0$ Roots Are Imaginary.

Linear Diff. Equ^n

Solution

$e^{r_1 x}$ $e^{r_2 x}$ $e^{r_3 x}$ $e^{r_4 x}$

$$\boxed{y = c_1 y_1 + c_2 y_2 + c_3 y_3 + c_4 y_4 + \cdots c_n y_n}$$

Linear form

CASE 01:

If Roots Are Real and Distinct

Solution of D.E

$$\boxed{y = c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots + c_n y_n}$$

y_1, y_2, y_3 Are function of x only

$y = e^{rx}$

$r = 2$ $\quad y_1 = e^{2x}$

$r = 3$ $\quad y_2 = e^{3x}$

Solution $\boxed{y = c_1 e^{2x} + c_2 e^{3x}}$

$$\underbrace{\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0}$$

✓ SECOND order ✓

✓ $X = 0$ Homogenous

✓ LINEAR

Put $y = e^{rx}$ is a solution D.E

$$= \underbrace{[r^2 - 5r + 6]}_{\text{factor}} e^{rx} = 0$$

$r^2 - 3r - 2r + 6 = 0$

$(r-3)(r-2) = 0$

$r = 3$

$r = 2$

$y = c_1 e^{2x} + c_2 e^{3x}$

where c_1 and c_2 Are constants

e^{2x}, e^{3x} Are Independent solution $\left\langle \begin{matrix} 2 \\ 3 \end{matrix} \right]$ Indep.

Check the solutions Dependent / Independent

Wronskian $=$ $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$ Independent Solution

W

SECOND order.

$W = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} \neq 0$

Wronskean $=$ $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$ <u>Dependent solutions</u>

W

If $r \to r_1, r_2, r_3$

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + c_3 e^{r_3 x}$$

Independent condition

Wronskian

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \neq 0$$

Independent

for n Roots

$r_1 \quad r_2 \quad r_3 \quad r_4$

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + c_3 e^{r_3 x} + \cdots + \cdots + c_n e^{r_n x}$$

Thank You!

PW Soldiers