GATE (ALL BRANCHES)

Engineering Mathematics

Differential Equation + Partial differential

Lecture No. 06

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TOPICS TO BE COVERED

01 Non Homogeneous Linear Differential Equation

02 Solution of Non-Homogeneous Linear DE

03 Problems based on Non-Homogeneous DE

04 Cauchy Euler Linear DE

Non-Homogenous Linear Diff. Eqn:

$$\frac{d^2y}{dx^2} = D^2y$$
$$\frac{dy}{dx} = Dy$$
$$\frac{d^3y}{dx^3} = D^3y$$

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X$$

SECOND order Linear D.E with Constant coefficients

If $X \neq 0$ (Non Homogenous)

$X \neq 0$ (Steady state or force apply)

$$[D^2y + PDy + Qy] = X$$

$X = e^x, \sin(x), \log x, x^2, \ldots$ etc

$$\boxed{\text{P.I particular Integral} = \frac{X}{[D^2 + PD + Q]}}$$

Complementary function C.F

$$C.F = C_1e^{r_1x} + C_2e^{r_2x}$$
$$= (C_1 + C_2x)e^{rx}$$
$$= e^{ax}[\cos bx + \sin bx]$$

$X = 0$

$$\Rightarrow [D^2 y + PDy + Qy] = X$$

$$\Rightarrow [D^2 + PD + Q]\, y = X$$

P.I $\Rightarrow$ Particular Integral $\Rightarrow \dfrac{X}{[D^2 + PD + Q]}$

solution of Differential Equation

$$y = C.F + P.I$$

complete solution when $X \neq 0$

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X \quad \text{Where } X = e^{ax+b}$$

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = e^{ax+b}$$

$$= \underbrace{[D^2 + PD + Q]}_{f(D)} y = \underline{e^{ax+b}} \text{ or } e^{ax} \text{ or constant}$$

$$\begin{cases} 5 = e^{0x} \cdot 5 \\ 10 = e^{0x} \cdot 10 \end{cases}$$

$$\text{Particular Integral} = \frac{X(e^{ax+b})}{[D^2 + PD + Q]} \nearrow f(D)$$

Put $D = a$

$$= \frac{e^{ax+b}}{f(a)}$$

If $f(a) \neq 0$

If $f(a) = 0$

$$\text{Particular Integral} = \frac{x \cdot e^{ax+b}}{f'(a)} \quad f'(a) \neq 0$$

Again $f'(a) = 0$ Then

$$\text{Particular Integral} = \frac{x^n e^{ax+b}}{f^n(a)} \quad f^n(a) \neq 0 \quad = \frac{x^2 e^{ax+b}}{f''(a)}$$

n^{th} derivative

Ex:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{2x+1}$$

$$[r^2 - 5r + 6]e^{rx} = 0$$

$$\Rightarrow [r^2 - 3r - 2r + 6]e^{rx} = 0$$

$$= \boxed{r = 2 \quad r = 3}$$

$$C.F = c_1 e^{2x} + c_2 e^{3x}$$

$$P.I = \frac{e^{2x+1}}{(D^2 - 5D + 6)}$$

Put $D = a$

$D = 2$

Differentiate I.t

$$= x\frac{e^{2x+1}}{2D-5}$$

$$= x\frac{e^{2x+1}}{4-5}$$

$$= \frac{e^{2x+1}}{-1} = -e^{2x+1} \cdot x$$

Complete solution

$$= c_1 e^{2x} + c_2 e^{3x} - e^{(2x+1)} \cdot x$$

Type 02 :

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=X$$

$X=\sin(ax+b)$ or $\cos(ax+b)$

only sin/cos

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=\sin(ax+b) \text{ or } \cos(ax+b)$$

$$\text{Particular Integral}=\frac{\sin(ax+b)}{[\underbrace{D^2+PD+Q}_{f(D)}]}$$

Put $D^2=-a^2$

$\boxed{D=\sqrt{-a^2}\ \times}$

$$=\frac{\sin(a\cancel{x}+b)}{f(D)}$$

$D^3=D^2D$
$=-a^2D$

$D^4=D^2D^2$
$=-a^2\times-a^2$
$=a^4$

$$=\frac{\sin(a\cancel{x}+b)}{f(-a^2)}$$

$D^2=-a^2$

$f(-a^2)\neq0$

$f(-a^2)=0$

$$P.I=\frac{x\sin(ax+b)}{f'(-a^2)}$$

$$\boxed{P.I=\frac{x^n\sin(ax+b)}{f^n(-a^2)}}$$

$D = \text{Diff.}$

$\frac{1}{D} = \text{Integ}$

$$[D^2+3D+2]\,y = \sin(2x+1)$$

$$= [D^2+3D+2] = 0$$

$y = e^{rx}$ is solution of D.E

$$\Rightarrow [r^2+3r+2]\,e^{rx} = 0$$

$$\Rightarrow r^2+2r+r+2 = 0$$

$$\Rightarrow r(r+2)+1(r+2) = 0$$

$$\left.\begin{aligned} r &= -2 \\ r &= -1 \end{aligned}\right]$$

$$C.F = c_1 e^{-2x} + c_2 e^{-x}$$

Particular Integral

$$= \frac{\sin(2x+1)}{[D^2+3D+2]}$$

Put $D^2 = -a^2$

$D^2 = -4$

$$\Rightarrow \frac{\sin(2x+1)}{[-4+3D+2]} = \frac{\sin(2x+1)}{(3D-2)}\frac{(3D+2)}{(3D+2)}$$

$$= \frac{3D[\sin(2x+1)] + 2\sin(2x+1)}{9D^2-4}$$

$D^2 = -a^2$

$D^2 = -4$

$$= \frac{3\,\boxed{\cos(2x+1)\cdot 2} + 2\sin(2x+1)}{-40} \quad \text{Ans}$$

CASE 03 :

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = x^m$$

$X = x^m$

algebraic function

$$\Rightarrow [D^2 + PD + Q]y = x^m$$

$$P.I = \frac{x^m}{[\underbrace{D^2 + PD + Q}]}$$

Make The Term $[1 + f(D)]$ Type.

$D^2 + PD + Q$

$$\rightarrow Q\left[\frac{D^2 + PD}{\underbrace{Q}_{f(D)}} + 1\right]$$

$$= Q\left[1 + \left[\frac{D^2 + PD}{Q}\right]\right]$$

$$P.I = \frac{x^m}{[1 + f(D)]} = x^m(1 + f(D))^{-1}$$

Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \cdots$$

Now get the particular Integral.

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=1+x^2$$

$$D(1+x^2)=2x$$
$$D^2(1+x^2)=2$$
$$D^3(1+x^2)=0$$

Particular Integral $=\dfrac{(1+x^2)}{[D^2+3D+2]}=\dfrac{(1+x^2)}{2}\left[1+\boxed{\dfrac{D^2+3D}{2}}\right]^{-1}$

$$=\frac{(1+x^2)}{2}\left[1-\left(\frac{D^2+3D}{2}\right)+\frac{-1(-1-1)}{2}\left(\frac{D^2+3D}{2}\right)^2+---\right]$$

$$=\frac{(1+x^2)}{2}\left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2+---\right]$$

$$=\left[\left(\frac{1+x^2}{2}\right)-\left(\frac{D^2+3D}{2}\right)\left(\frac{1+x^2}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2\left(\frac{1+x^2}{2}\right)+--\right]$$

$$\frac{D^2+3D}{2}\left(\frac{1+x^2}{2}\right)$$
$$=\frac{1}{4}\left[D^2(1+x^2)+3D(1+x^2)\right]$$
$$=\frac{1}{4}\left[2+3\cdot 2x\right]$$
$$=\frac{1}{4}\left[2+6x\right]$$

$$=\left[\frac{(1+x^2)}{2}-\frac{1}{4}(2+6x)+\frac{9}{2}\right]$$ Ans

Third Term $\left(\frac{D^2+3D}{2}\right)^2\left(\frac{1+x^2}{2}\right)$

$$= \frac{1}{4}\left[(D^4+9D^2+6D^3)(1+x^2)\right]$$

$$= \frac{1}{4}\left[D^4(1+x^2)+9D^2(1+x^2)+6D^3(1+x^2)\right]$$

$$= \frac{1}{4}\left[0+9\cdot 2+0\right]$$

$$= \frac{1}{4}\times 18 = \frac{9}{2}$$

Q. Questions

#Q. Consider the following second – order differential equation: y" - 4y + 3y = 2t – 3t^2.

The particular solution of the differential equation is

Particular Integral = $\frac{2t-3t^2}{[D^2-4D+3]}$ algebraic

(a) -2 -2t - t^2

(b) -2t – t^2

(c) 2t – 3t^2

(d) -2 – 2t – 3t^2

$\boxed{P.I = -2-2t-t^2}$

Q. Questions

$$\rightarrow C.F = (C_1 + C_2 x)e^{-3x}$$

#Q. The solution of the differential equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 9x + 6$ with C_1 and C_2 as constants is

$$P.I = \frac{9x+6}{[D^2+6D+9]}$$

$$= \frac{(9x+6)}{9\left[\frac{D^2+6D}{9}+1\right]}$$

$$= \frac{(9x+6)}{9}\left[1 - \left(\frac{D^2+6D}{9}\right) + \ldots\right]$$

$$= x$$

(a) $y = (C_1 x + C_2)e^{-3x}$

(b) $y = C_1 e^{3x} + C_2\, e^{-3x}$

(c) $y = (C_1 x + C_2)\, e^{-3x} + x$

(d) $y = (C_1 x + C_2)\, e^{3x} + x$

Q. Questions

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#Q. The solution of the differential equation $k^2 \frac{d^2y}{dx^2} = y - y_2$ under the boundary conditions

(i) $y = y_1$ at $x = 0$ and

(ii) $y = y_2$ at $x = \infty$

Where k, y_1 and y_2 are constant is

(a) $y=(y_1-y_2)e^{-\frac{x}{k^2}}+y^2$

(b) $y=(y_1-y_2)\sinh\left(\frac{x}{k}\right)+y_1$

(c) $y=(y_2-y_1)e^{\frac{-x}{k}}+y_1$

(d) $y=(y_1-y_2)e^{\frac{-x}{k}}+y_2$

$$K^2\frac{d^2y}{dx^2} = y - y_2$$

$$\Rightarrow \underbrace{K^2\frac{d^2y}{dx^2} - y}_{} = -y_2$$

Put $y = e^{rx}$ is a solⁿ of DE

$$K^2 r^2 e^{rx} - e^{rx} = 0$$

$$\left[K^2 r^2 - 1\right] e^{rx} = 0$$

$$K^2 r^2 - 1 = 0$$

Ans $\boxed{r = \pm\frac{1}{K}}$

$$C.F = c_1 e^{\frac{1}{K}x} + c_2 e^{-\frac{1}{K}x}$$

Particular Integral $= \dfrac{-y_2 e^{0x}}{(K^2D^2 - 1)}$

Put $D = 0$

$$\Rightarrow \frac{-y_2 e^{0x}}{K^2 \times 0 - 1}$$

$$P.I = y_2$$

Complete sol^n $= c_1 e^{\frac{1}{K}x} + c_2 e^{-\frac{1}{K}x} + y_2$

$$y = c_1 e^{\frac{1}{K}x} + c_2 e^{-\frac{1}{K}x} + y_2$$

Apply Initial conditions

$y = y_1$ at $x = 0$

$y = y_2$ at $x = \infty$

first condition

$$y_1 = c_1 e^0 + c_2 e^0 + y_2$$

$$\boxed{y_1 - y_2 = c_1 + c_2}$$

$$y_2 = c_1 e^{\infty} + c_2 e^{-\infty} + y_2$$

$$y_2 - y_2 = c_1 e^{\infty} + c_2 e^{-\infty}$$

$$0 = c_1(\infty) + c_2 \times 0$$

$$\boxed{c_1 = 0}$$

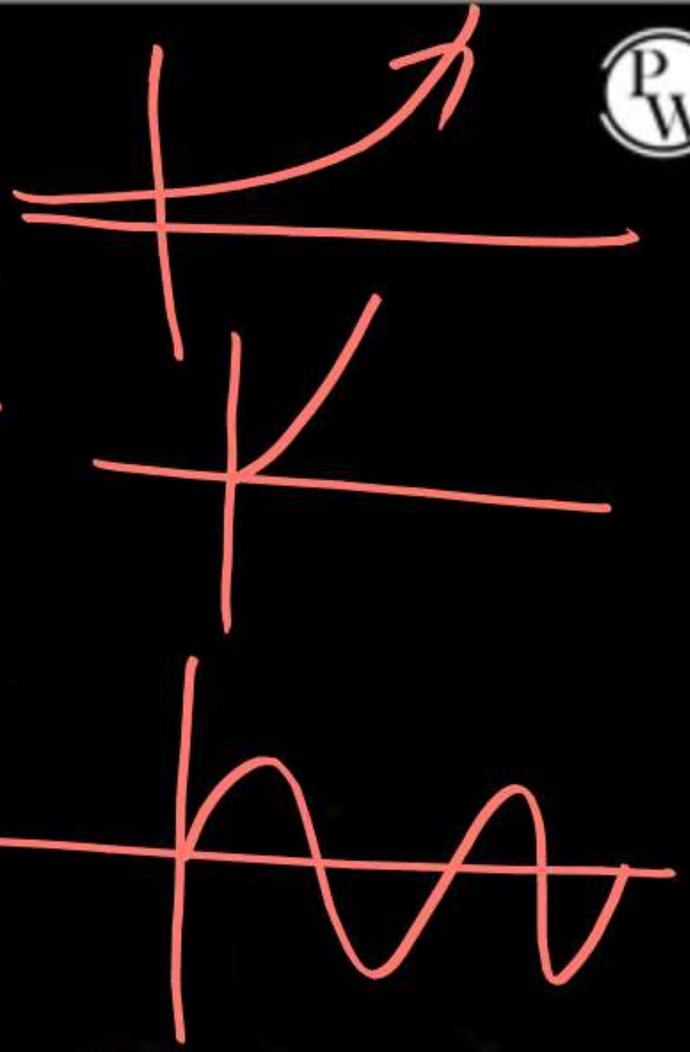
$$c_2 = (y_1 - y_2)$$

solution

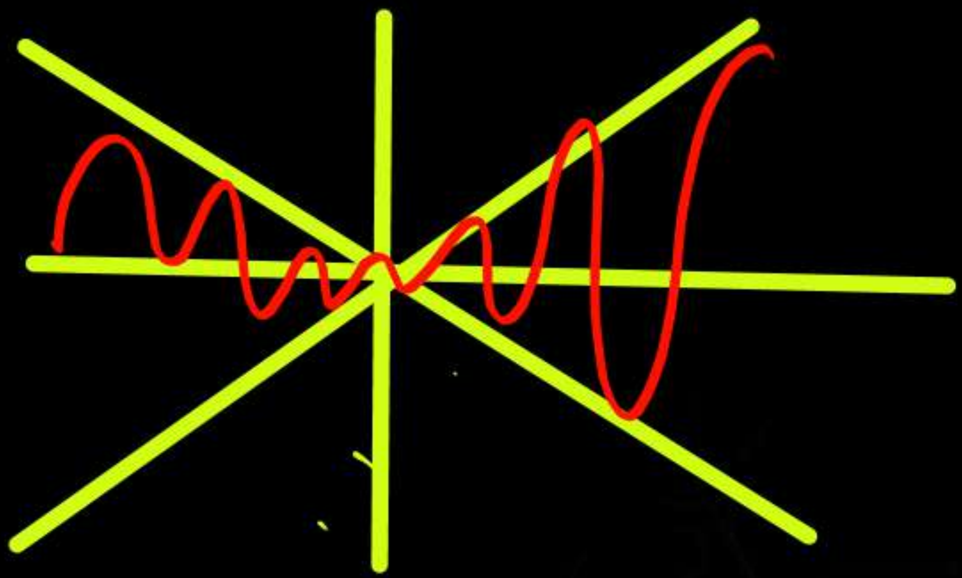
$$\boxed{y = (y_1 - y_2) e^{-\frac{x}{K}}}$$

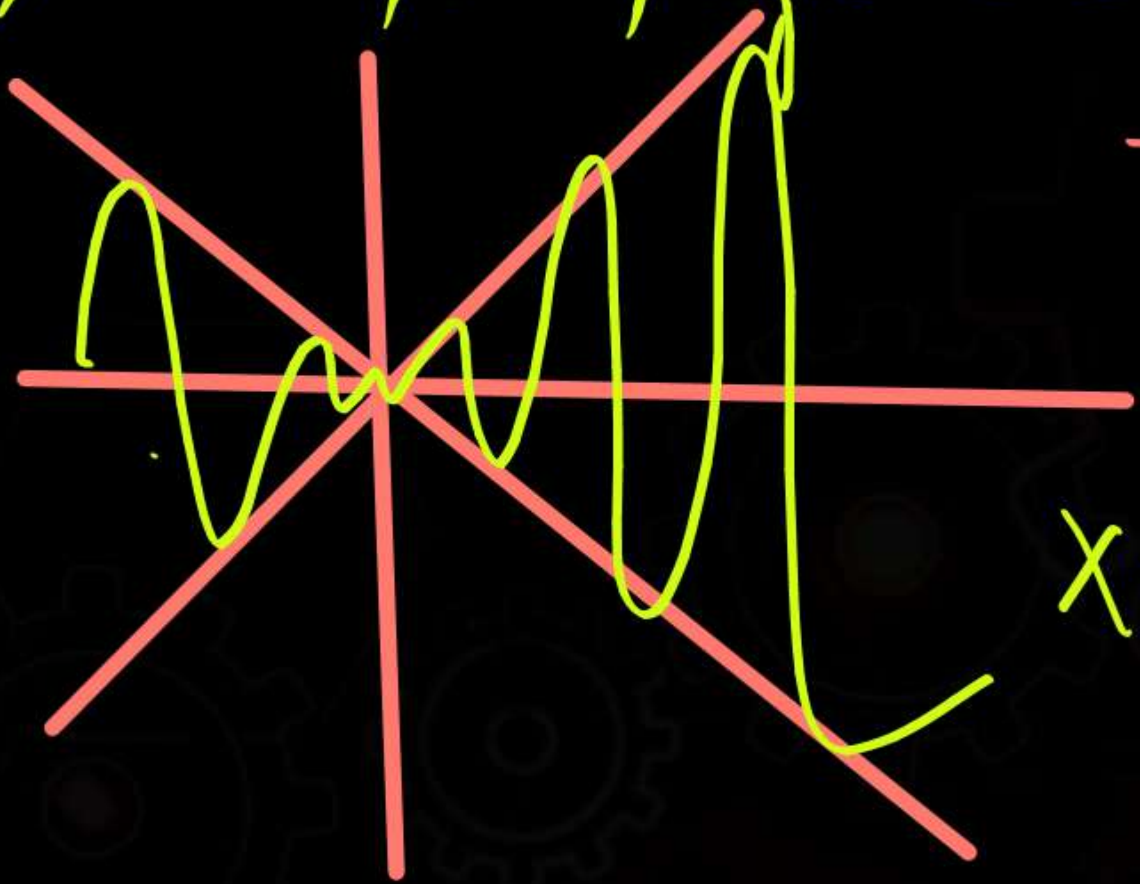
Variation of Parameters :-

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X \quad \begin{cases} e^x \\ x^m \\ \sin x \end{cases}$$



If $X = e^x \sin x,\ x \sin x,\ \tan x,\ \log x$ - - -





$X =$ any kind of function

Thank You!

PW Soldiers