GATE-All BRANCHES
Engineering Mathematics

NUMERICAL METHODS

Lecture No.- 03

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Recap of previous lecture

Topic Numerical integration

Topic Simpsons 1/3 rule

Topic Simpsons 3/8 rule

Topic Trapezoidal rule

Topic Problems based on numerical integration

Topics to be covered

Topic Eulers method ✓ ✓ Proof

Topic Runge kutta method (2^{nd} order) ✓

Topic Runge kutta method (3rd order) ✓ Not required

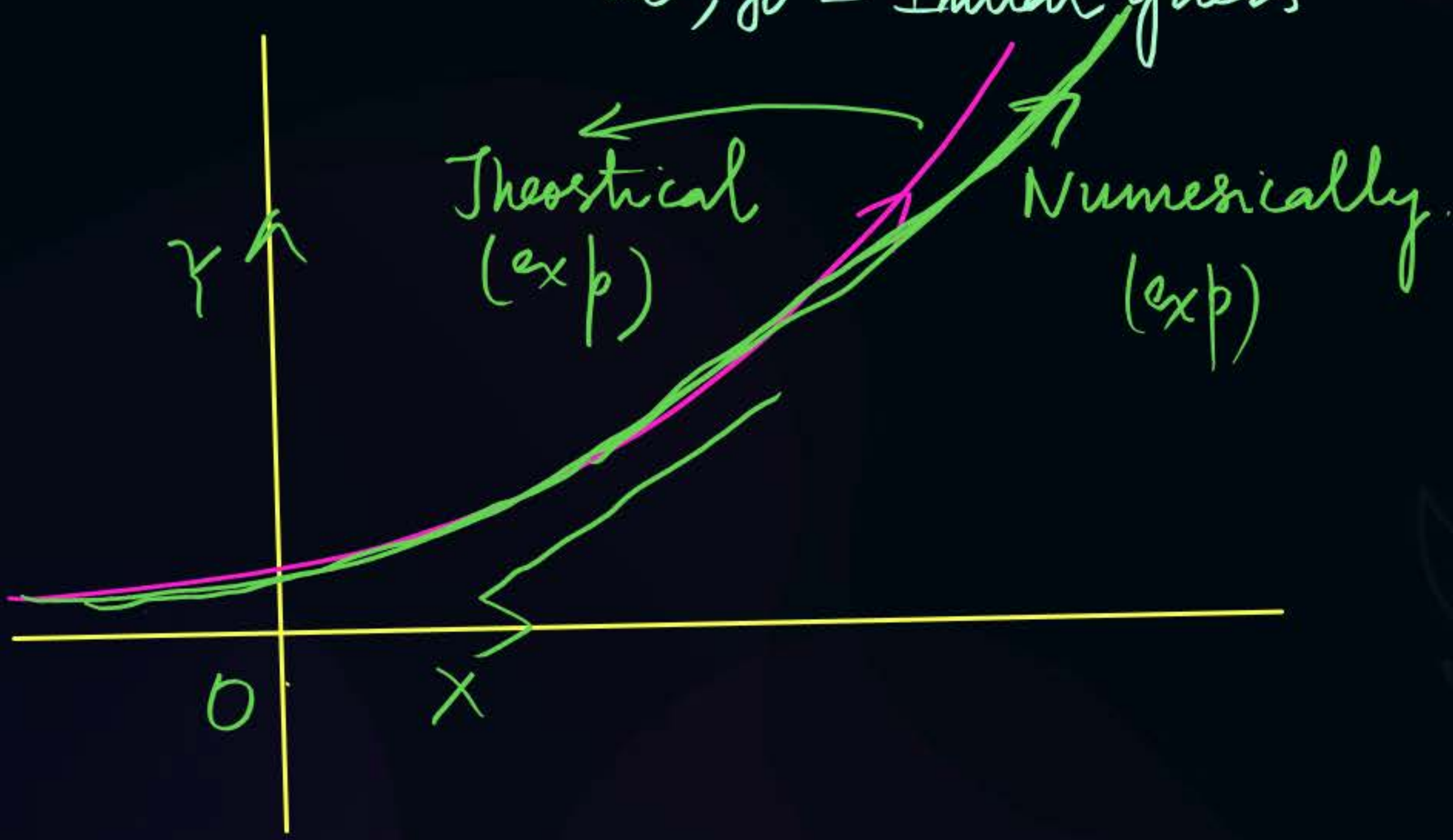
Topic Runge kutta method (4th order) → electrical

Euler's method :

Numerical Differential equ^n [First order)

$$\frac{dy}{dx} = f(x_0, y_0)$$

x_0, y_0 = Initial guess



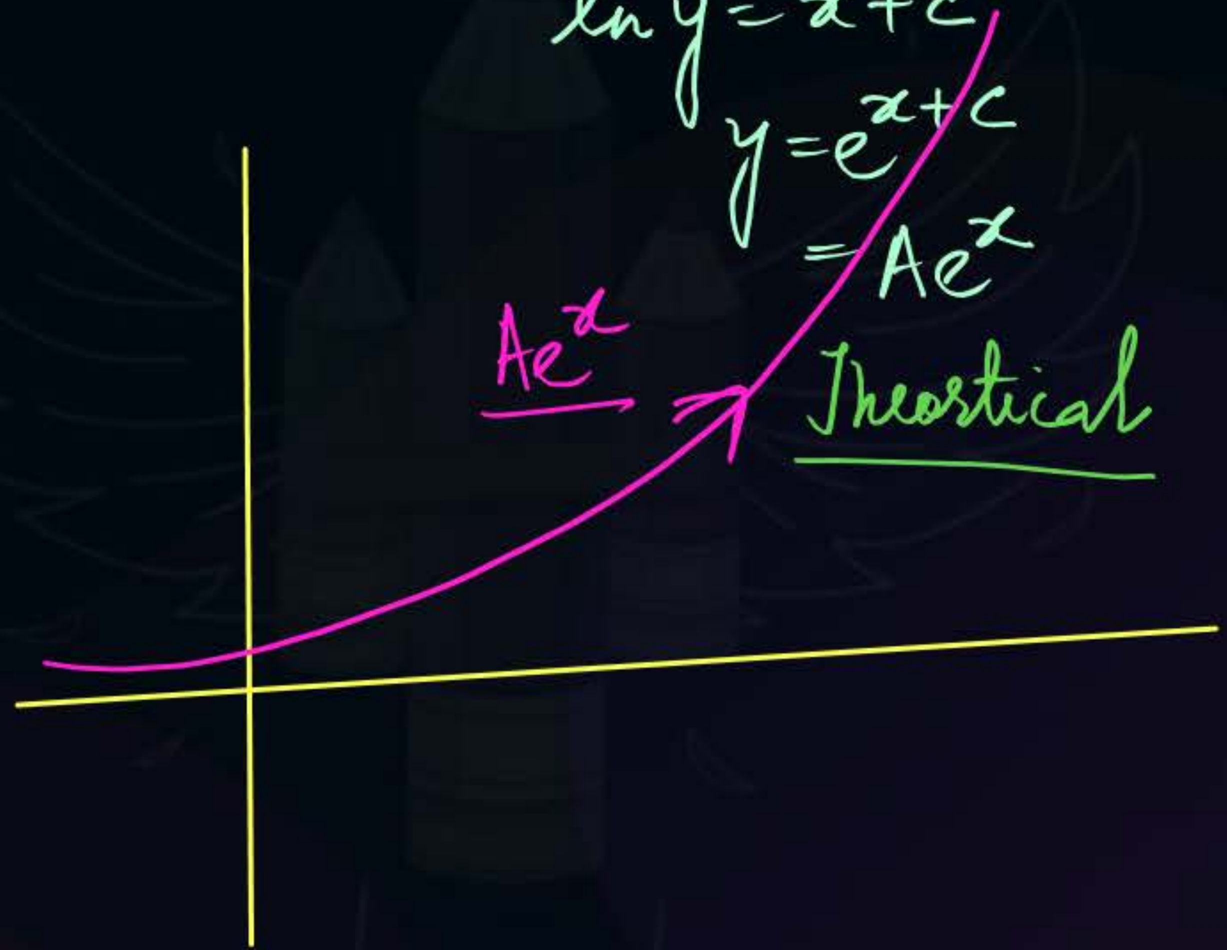
$$\frac{dy}{dx} = f(x_0, y_0) = y$$

$$\frac{dy}{dx} = y \qquad \frac{dy}{y} = dx$$

$$\ln y = x + c$$

$$y = e^{x+c}$$

$$= Ae^x$$



$$\frac{dy}{dx} = f(x_0, y_0)$$

Initial guess $= x_0$

$$x_1 = x_0 + h \qquad x_1 - x_0 = h$$

$$x_2 = x_0 + 2h \qquad x_2 - x_0 = h$$

$$x_3 = x_0 + 3h$$

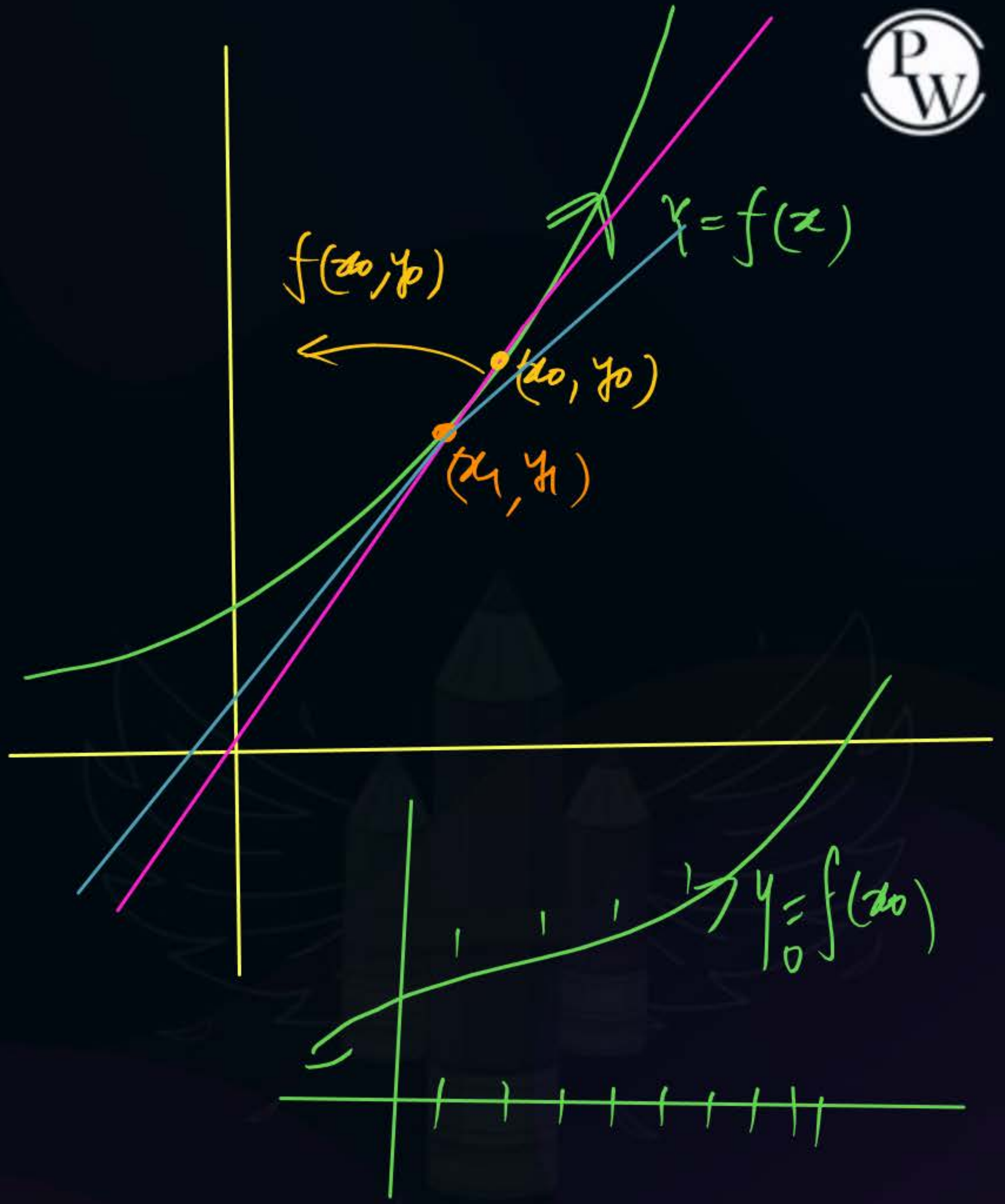
$$\boxed{x_n = x_0 + nh}$$

$A(x_0, y_0) \quad B(x_1, y_1)$

Slope $\dfrac{y_1 - y_0}{x_1 - x_0} = f(x_0, y_0)$

$$y_1 - y_0 = h f(x_0, y_0)$$

$$\boxed{y_1 = y_0 + h f(x_0, y_0)}$$



$A(x_2, y_2) \quad B(x_1, y_1)$

$$\frac{y_2 - y_1}{x_2 - x_1} = f(x_1, y_1)$$

$$= \quad y_2 - y_1 = h\,f(x_1, y_1)$$

$$\boxed{y_2 = y_1 + h\,f(x_1, y_1)}$$

$A(x_3, y_3) \quad B(x_2, y_2)$

$$\frac{y_3 - y_2}{x_3 - x_2} = f(x_2, y_2)$$

$$\boxed{y_3 = y_2 + h\,f(x_2, y_2)}$$

$$\boxed{y_{n+1} = y_n + h\,f(x_n, y_n)}$$

→ Euler's method for Differential eqn

(Forward-Euler)

Topic : Numerical Methods

Euler method

#Q. An explicit forward Euler method is used to numerically intergrate the differential equation $\frac{dx}{dt} = y$

using a time step of 0.1. With the initial condition y (0) = 1, the value of y (1) computed by this method is ______ (*Correct to two decimal places*) :

H.W

Topic : Numerical Methods

Euler

H.W

#Q. Consider the equation $\frac{du}{dt} = 3t^2 + 1$ with $u = 0$ at $t = 0$. This is numerically solved by using the forward Euler method with a step size $\Delta t = 2$. The absolute error in the solution in the end of the first time step is____.

Runge-Kutta (First order)

$$\boxed{y_1 = y_0 + h f(x_0, y_0)}$$

$\frac{dy}{dx} = f(x_0, y_0)$

x_0 = Initial guess.

y_0 = Initial guess.

$\boxed{\text{Euler's method}}$

h = step size.

Runge-Kutta (SECOND order)

$$\boxed{y_1 = y_0 + \frac{1}{2}(K_1 + K_2)}$$

Here

$K_1 = h f(x_0, y_0)$

$K_2 = h f(x_0 + h, y_0 + K_1)$

Diff. Equ^n

Runge-Kutta (Third order)

$$\boxed{y_1 = y_0 + \frac{1}{6}(K_1 + 4K_2 + K_3)}$$

where

$K_1 = h f(x_0, y_0)$

$K_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$

$K_3 = h f(x_0 + h, y_0 + K')$

where $\boxed{K' = h f(x_0 + h, y_0 + K_1)}$

R-K method Fourth order

$$\frac{dy}{dx} = F(x_0, y_0) \qquad x_0 = \; \longrightarrow x_1 \qquad y_0 = \; \longrightarrow y_1$$

$$\boxed{y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)}$$

$$K_1 = hf(x_0, y_0)$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$

$$\boxed{y_2 = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)}$$

$$K_1 = hf(x_1, y_1) \qquad K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right)$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right)$$

$$K_4 = hf(x_1 + h, y_1 + K_3)$$

$$y_3 = y_2 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

#Q. Consider the first order initial value problem $y' = y + 2x - x^2, y(0) = 1, (0, \leq x < \infty)$ with exact solutions $y(x) = x^2 + e^x$. For $x = 0.1$, the percentage difference between the exact solution and the solution obtained using a single iteration of the second – order Runge kutta method with step size $h = 0.1$ is ______.

$$\% \text{ Error} = \left| \frac{y(0.1)_{exact} - y_{0.1}(RK)}{y_{0.1}(Rk)} \right| \times 100$$

$$\left.\begin{array}{l} \frac{dy}{dx} = y + 2x - x^2 \\ y(0) = 1 \end{array}\right] \text{Normal}$$

$$\left. h = 0.1 \right] \text{R-K } 2^{nd} \text{ order}$$

$$\frac{dy}{dx} = y + 2x - x^2$$

$h = 0.1$

SECOND-order (RK-method)

$$\boxed{y(x) = x^2 + e^x}$$

$$y(0.1) = (0.1)^2 + e^{0.1} = 1.152$$

Using SECOND-order RK method

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

$x_0 = 0 \quad y_0 = 1$

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$hf(x_0 + h, y_0 + k_1)$$

$$k_1 = hf(x_0, y_0)$$

$$= 0.1[y_0 + 2x_0 - x_0^2]$$

$$= 0.1[1 + 2\times 0 - 0^2]$$

$$= \underline{0.1}$$

$x_0 \longrightarrow x_0 + h$

$y_0 \longrightarrow y_0 + k$

$$hf(x_0 + h, y_0 + k_1) = h(y_0 + k_1) + 2(x_0 + h) - (x_0 + h)^2$$

$$\boxed{k_2 = 0.129}$$

$$y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$$

$$= 1 + \frac{1}{2}(0.1 + 0.129) = 1.1145$$

% Error

$$= \left[\frac{y_{1(exact)} - y_{1(RK)}}{y_{1(RK)}}\right] \times 100$$

$$= \left|\frac{1.1152 - 1.1145}{1.1145}\right| \times 100 = \underline{0.061}$$

Topic : Numerical Methods

#Q. Consider a differential equation $\frac{dy(x)}{dx} - y(x) = x$ with the initial condition $y(0) = 0$. Using Euler's first order method with a step size of 0.1 the value of $y(0.3)$ is

A. 0.01

B. 0.031

C. 0.0631

D. 0.1

EC
2010

$$\frac{dy}{dx} - y(x) = x \text{ with Initial condition}$$

$$y(0) = 0 \qquad h = 0.1$$

$$x_0 = 0 \qquad y(0.3) = \checkmark$$

$$y_0 = 0$$

x_0 | x_1 = 0.1 | x_2 = 0.2 | 0.3 x_3

$$\frac{dy}{dx} = (x+y) \qquad h = 0.1$$

$$y_0 = 0 \qquad x_0 = 0$$

$$y_1 = y_0 + h f(x_0, y_0) \qquad f(x_0, y_0) = x_0 + y_0$$

$$y_1 = 0 + 0.1\left[x_0 + y_0\right]$$

$$= 0 + 0.1\left[0 + 0\right]$$

$$\boxed{y_1 = 0}$$

$$y_2 = y_1 + h f(x_1, y_1)$$

$$= 0 + 0.1(x_1 + y_1)$$

$$= 0 + 0.1(0.1 + 0)$$

$$= 0.01$$

$$y_3 = y_2 + h f(x_2, y_2)$$

$$= 0.01 + 0.1(0.2 + 0.01)$$

$$= 0.031$$

$$y_4 = y_3 + h f(x_3, y_3) = 0.031 + 0.1(x_3 + y_3)$$

$$= \underline{0.031}$$

2 mins Summary

Topic One

Topic Two

Topic Three

Topic Four

Topic Five

Euler

RK-method 2nd order + 3rd order + 4th order ✓

THANK - YOU

Topics to be Covered