GATE (ALL BRANCHES)

Engineering Mathematics

Complex Analysis

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Lecture No. 06

TOPICS TO BE COVERED

01 Calculus of Residues

02 Cauchy Residue theorem

03 Problems based on residue and residue theorem

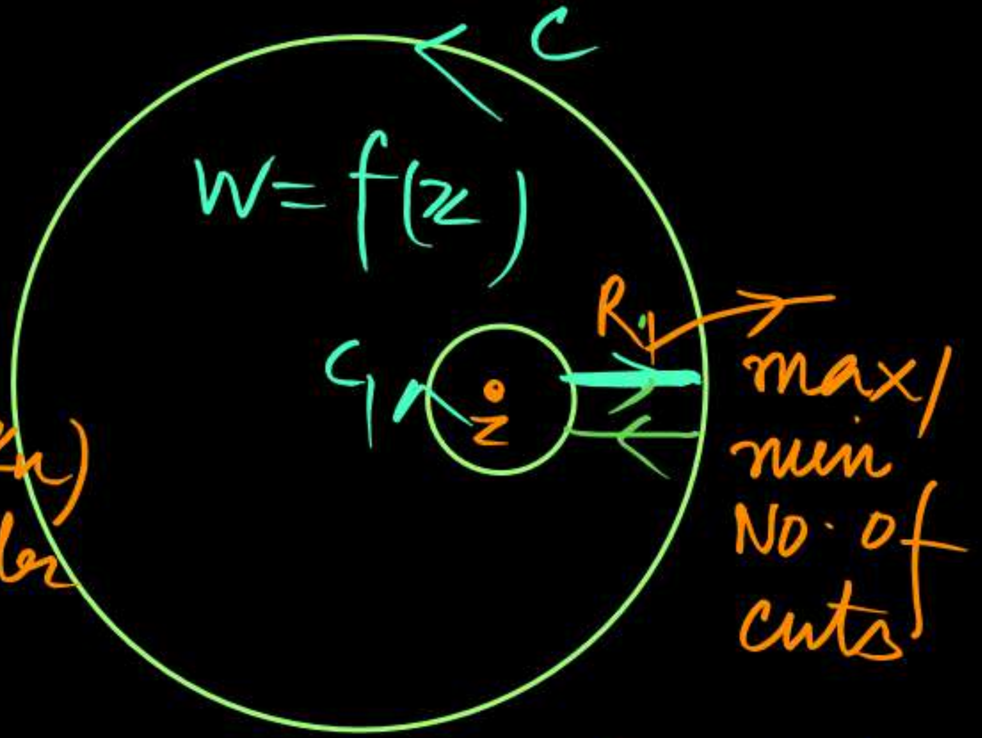
Calculus of Residues:

If $z = z_0$ is a Simple Order Pole

$$\oint f(z)\,dz = \oint \frac{\phi(z)}{(z-z_0)(z-z_1)(z-z_2)\cdots(z-z_n)}$$

$Z = z_0, z_1, z_2 \cdots z_n$ Are Simple order Poles.

= Isolated Singularity



Residues – Remainder.

Residues Simple order Poles.

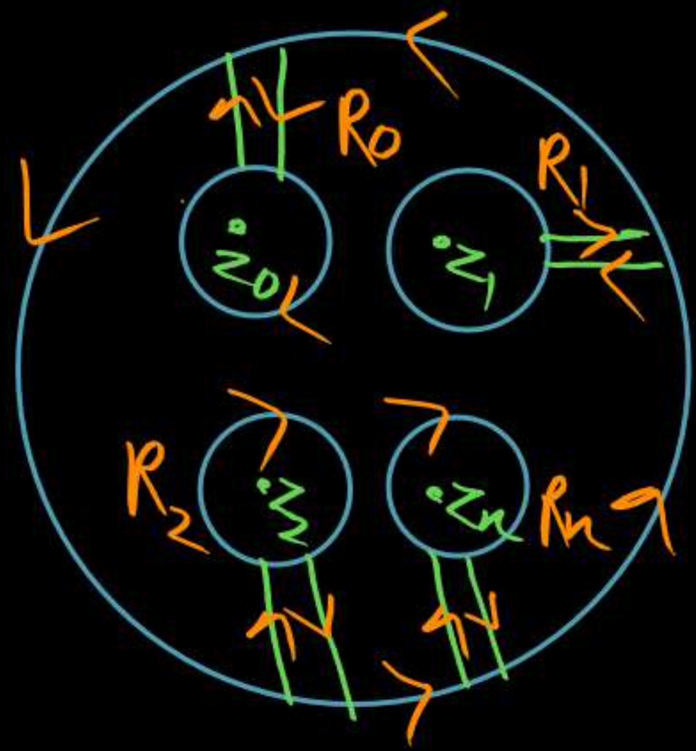
$R_0 = ?$

$R_1 = ?$

$R_2 = ?$

$R_3 = ?$

$R_n = ?$

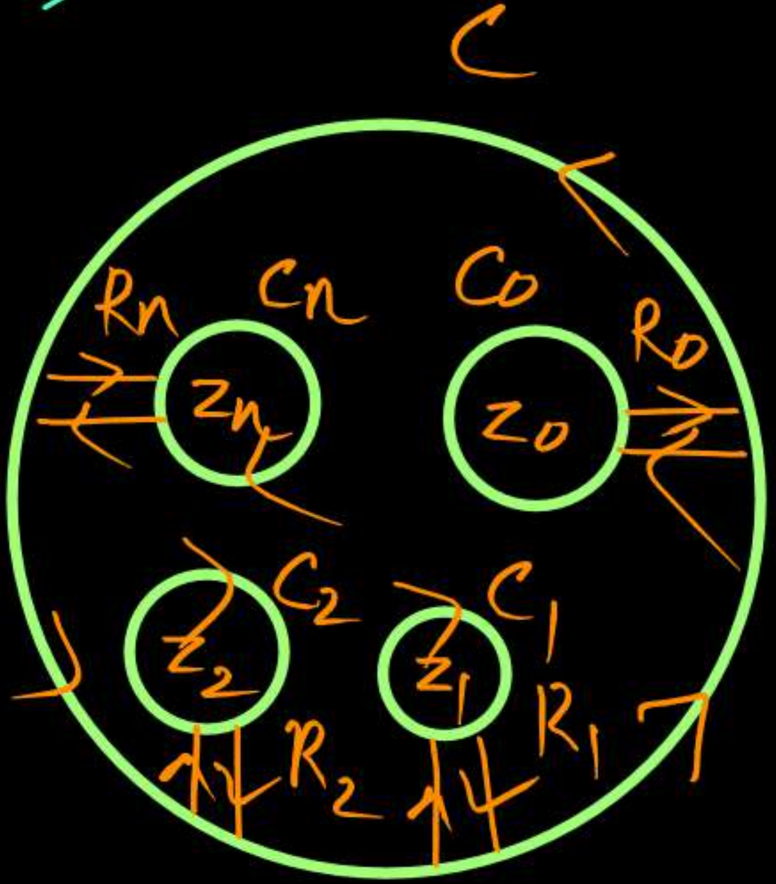


Calculating Residue (for Simple Order Pole:

for $z = z_0$

If simple order are given then

Residues $$\boxed{R_0 = \lim_{z \to z_0} (z - z_0) f(z)}$$



for $z = z_1$ $\quad R_1 = \lim_{z \to z_1} (z - z_1) f(z)$

$z = z_2$ $\quad R_2 = \lim_{z \to z_2} (z - z_2) f(z)$

for any simple order Pole. $$\boxed{R_i = \lim_{z \to z_i} (z - z_i) f(z)}$$

Simple order Pole.

for Double order OR n^{th} order Pole:

for $z = z_0$ is a n^{th} order Pole.

$$\text{Res}[f(z); z = z_0] = \frac{1}{(n-1)!} \underset{z \to z_0}{\text{Lt}} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z)$$

$$\boxed{\text{Res}[f(z); z = z_0] = \frac{1}{(n-1)!} \underset{z \to z_0}{\text{Lt}} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z)}$$

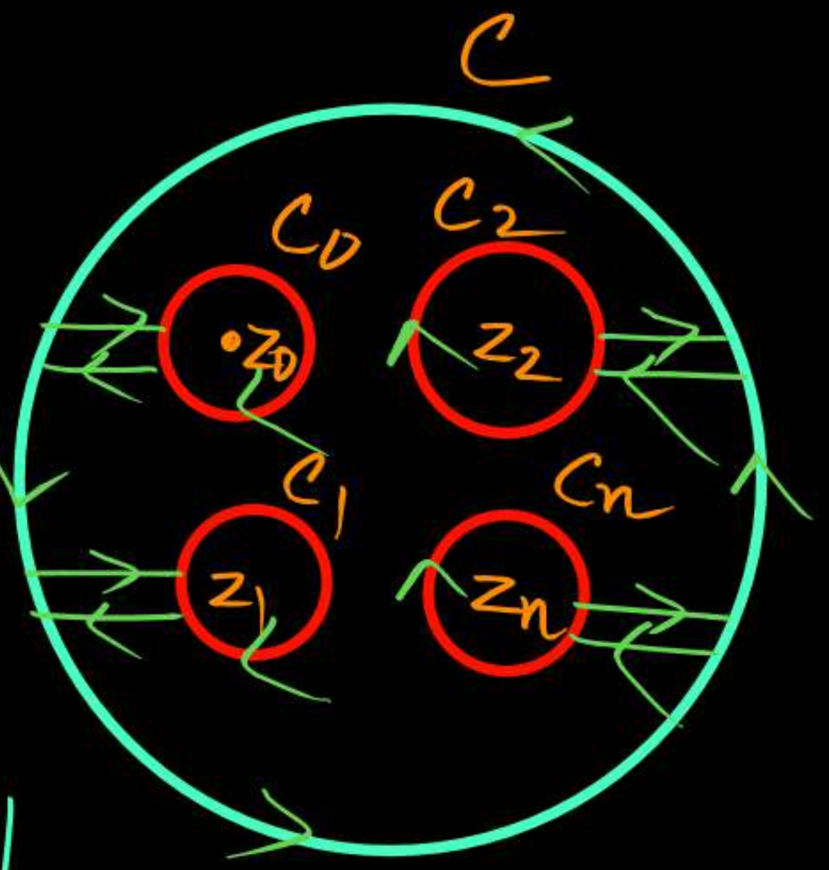
n^{th} order Pole

n^{th} order Pole.

Cauchy Residues Theorem :-

$$\oint f(z)\,dz = 2\pi i\,[\text{sum of residues}]$$

$$= 2\pi i\,[R_0 + R_1 + R_2 + \cdots + R_n]$$



✓ Cauchy Integral formula

✓ Cauchy residues Theorem

$$\boxed{\oint f(z)\,dz = 2\pi i \sum_{i=1}^{n} R_i}$$

Q. Questions

$$\oint \frac{1-2z}{z(z-1)(z-2)}\,dz \rightarrow 2\pi i \sum_{i=1} R_i$$

$f(z)$

#Q. The residues of a complex function $X(z) = \frac{1-2z}{z(z-1)(z-2)}$ at its poles are

(a) $\frac{1}{2}, -\frac{1}{2}$ and 1

(b) $\frac{1}{2}, \frac{1}{2}$ and -1

(c) ✓ $\frac{1}{2}, 1$ and $-\frac{3}{2}$

(d) $\frac{1}{2}, -1$ and $-\frac{3}{2}$

$$X(z) = \frac{1-2z}{z(z-1)(z-2)}$$

$z(z-1)(z-2) = 0$ $\quad D^r = 0 \rightarrow$ Poles.

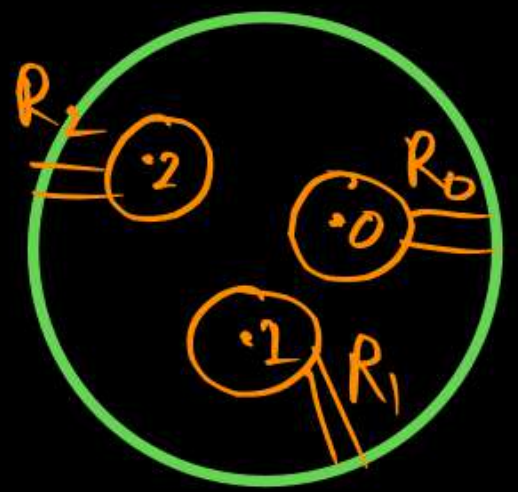
$z = 0 \quad z = 2$

$z = 1$

$$\text{Res at } [f(z); z = z_0] = \lim_{z \to z_0} (z - z_0) f(z)$$

$$\text{Res at } [f(z); z = 0] = \lim_{z \to 0} \cancel{(z-0)} \cdot \frac{1-2z}{\cancel{z}(z-1)(z-2)}$$

$$\boxed{R_0 = \frac{1}{2}}$$



$$\text{Res}[f(z); z = 1] = \lim_{z \to 1} \cancel{(z-1)} \cdot \frac{(1-2z)}{z\cancel{(z-1)}(z-2)} = \frac{-1}{-1} = 1$$

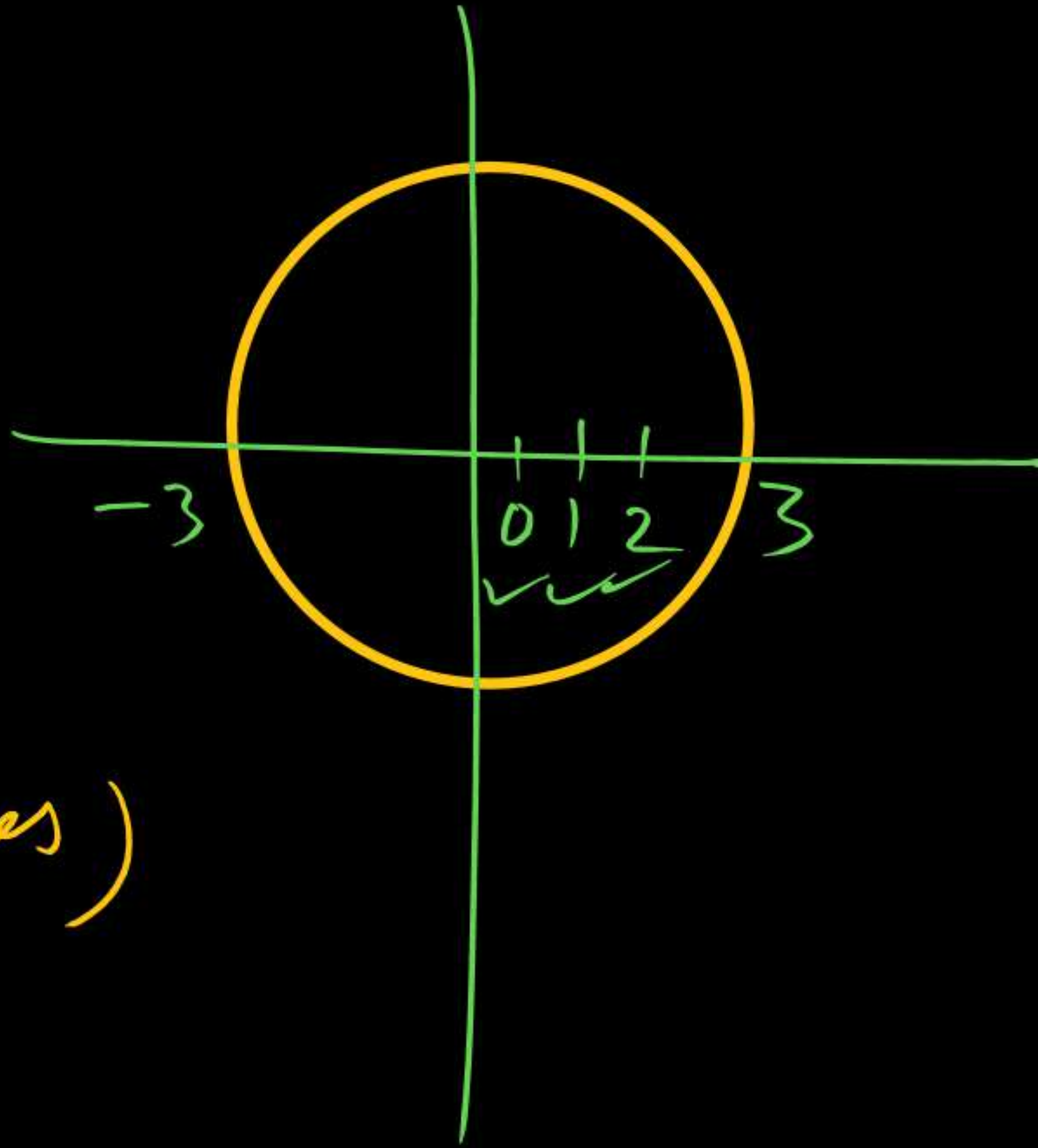
$R_1 = 1$

Slide-3

$$R_2 = \lim_{z \to 2} \cancel{(z-2)} \frac{(1-2z)}{z(z-1)\cancel{(z-2)}} = \frac{-3}{2} = \frac{-3}{2}$$

$R_1 = +\frac{1}{2}$ $\quad R_2 = 1$ $\quad R_3 = -\frac{3}{2}$

$$\boxed{\oint \frac{1-2z}{z(z-1)(z-2)} \qquad |z| = 3}$$



Sum of residue Theorem:

$$\oint f(z)\,dz = 2\pi i \,(\text{sum of residues})$$

$$= 2\pi i \left[R_0 + R_1 + R_2\right]$$

$$= 2\pi i \left[\frac{1}{2} + 1 - \frac{3}{2}\right] = 0$$

Q. Questions

#Q. The integral $\oint f(z)dz$ evaluated around the unit circle on the complex plane for $f(z) = \frac{\cos z}{z}$ is

(a) $2\pi i$

(b) $4\pi i$

(c) $-2\pi i$

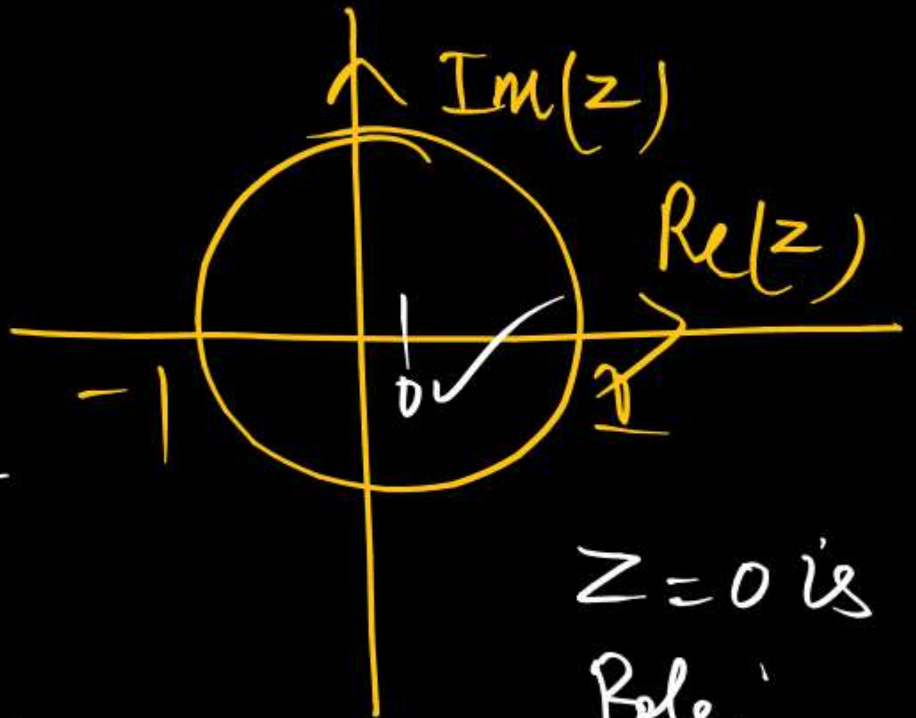
(d) 0

$f(z) = \oint \frac{\cos z}{z} dz$

Res at a simple Pole.

$Res[f(z); z=0] = \lim_{z \to 0} (z-0) \cdot \frac{\cos z}{z}$

$= 1$ ✓



z = 0 is Pole.

Using cauchy residues THEOREM $\oint f(z) dz = 2\pi i \left(\sum R_i\right]$

$= 2\pi i [1] = \underline{2\pi i}$

$$\oint \frac{\cos z}{(z-0)} = \int \frac{\phi(z)}{(z-z_0)} = 2\pi i\, \phi(z_0)$$

$$= 2\pi i \left[\cos z\right]_{z=0}$$

$$= \underline{2\pi i}$$

Cauchy Integral formula

Q. Questions

#Q. Given $X(z) = \frac{z}{(z-a)^2}$ with $|z| > a$, $\underbrace{\text{the residue of } X(z)\, z^{n-1} \text{ at } z = a \text{ for } n \geq 0 \text{ will be}}$

(a) a^{n-1}

(b) a^n

(c) n an

(d) $n\, a^{n-1}$

$X(z) = \frac{z}{(z-a)^2}$

$|z| > a$

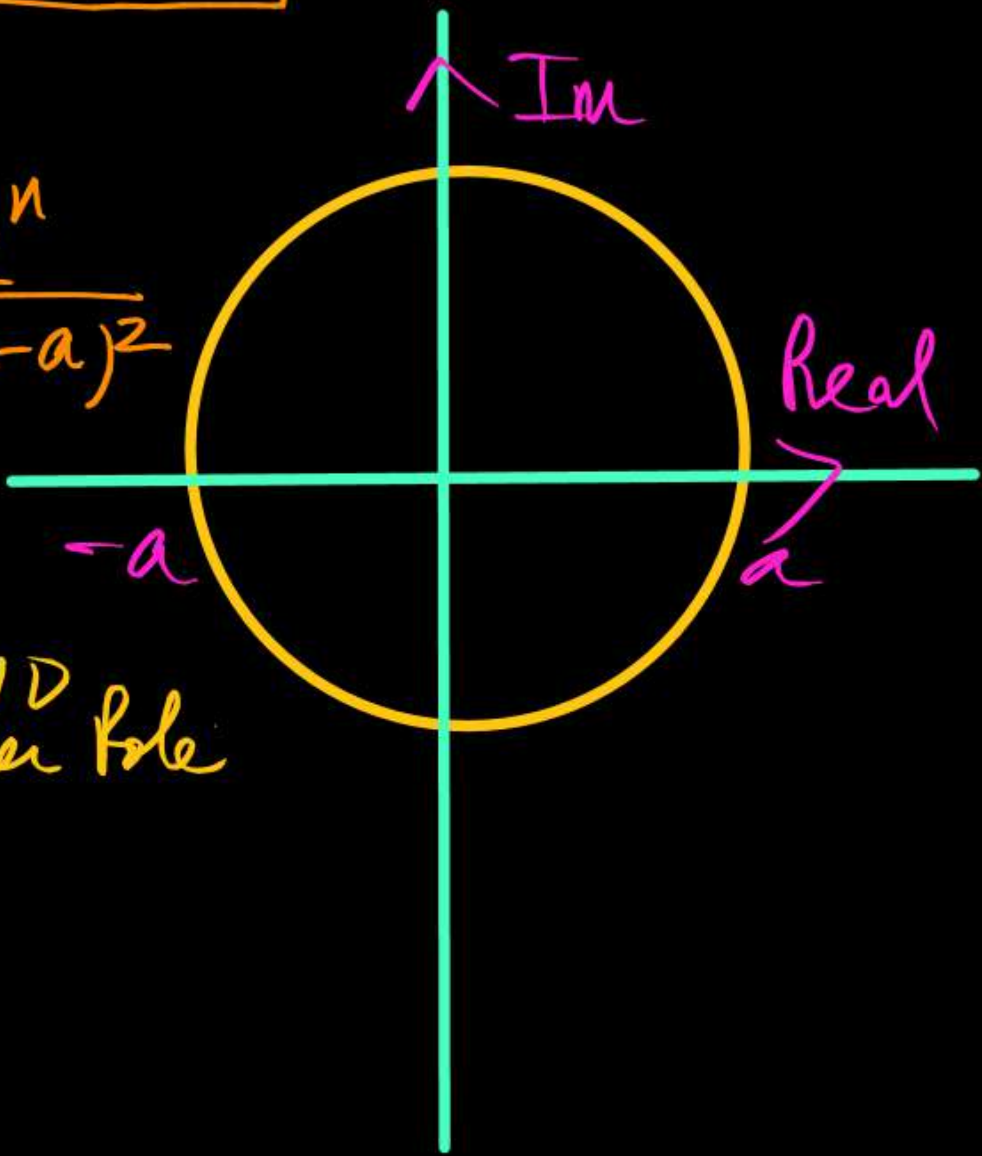
residue of $X(z) z^{n-1}$

$= \frac{z}{(z-a)^2} z^{n-1} = \frac{z^n}{(z-a)^2}$

$=$

$Res[f(z), z = z_0] = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z)$

SECOND order Pole



$$f(z) = \frac{z^n}{(z-a)^2} \qquad n=2$$

$$\text{Res}[f(z); z=a] = \frac{1}{(2-1)!} \lim_{z\to a} \frac{d^{2-1}}{dz^{2-1}} \cancel{(z-a)^2} \cdot \frac{z^n}{\cancel{(z-a)^2}}$$

$$= \frac{1}{1!} \lim_{z\to a} \frac{d}{dz} z^n$$

$$= \lim_{z\to a} \frac{d}{dz} z^n$$

$$= \lim_{z\to a} n z^{n-1}$$

$$\text{Res}[f(z); z=a] = \underline{n a^{n-1}} \quad \underline{\text{Ans}}$$

Q. Questions

#Q. The residue of the function $f(z) = \frac{1}{(z+2)^2(z-2)^2}$ at $z = 2$ is

$\longrightarrow 2^{nd}$ order

(a) $-\frac{1}{32}$

(b) $-\frac{1}{16}$

(c) $\frac{1}{16}$

(d) $\frac{1}{32}$

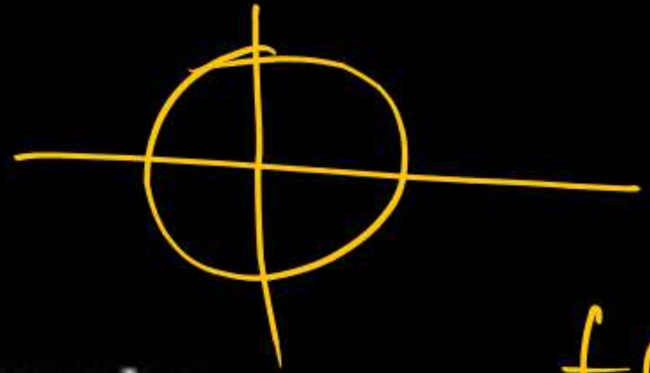
$$\text{Res}[f(z); z=2]$$

$$= \frac{1}{(2-1)!} \lim_{z \to 2} \frac{d^{2-1}}{dz^{2-1}} \cancel{(z-2)^2} \cdot \frac{1}{(z+2)^2 \cancel{(z-2)^2}}$$

$$= \frac{1}{1!} \lim_{z \to 2} \frac{d}{dz} \frac{1}{(z+2)^2}$$

$$= \lim_{z \to 2} \frac{1}{(z+2)^2} = -\frac{1}{32}$$

Q. Questions



#Q. If $f(z) = c_0 + c_1 z^{-1}$, then $\oint_{\substack{unit\\circle}} \frac{1+f(z)}{z} dz$ is given by

(a) $2\pi c_1$

(b) $2\pi(1 + c_0)$

(c) $2\pi j c_1$

(d) $2\pi j(1 + c_0)$

$f(z) = c_0 + \frac{c_1}{z}$

$\oint_{\text{unit circle}} \frac{1 + c_0 + \frac{c_1}{z}}{z} dz = \oint_{|z|=1} \frac{z + c_0 z + c_1}{z^2} dz$

$= \oint_{|z|=1} \frac{z + c_0 z + c_1}{(z-0)^2}$

$\text{Res}[f(z); z = 0]$

$= \frac{1}{(2-1)!} \lim_{z\to 0} \frac{d^{2-1}}{dz^{2-1}} \cancel{(z-0)^2} \cdot \left[\frac{z + c_0 z + c_1}{\cancel{z^2}}\right]$

$= \lim_{z\to 0} \frac{d}{dz}(z + c_0 z + c_1)$

$\boxed{R = 1 + c_0}$

$\text{sum of residues} = 2\pi i[1 + c_0]$

Q. Questions

#Q. If C denotes the counter clockwise unit circle, the value of the contour integral

$\frac{1}{2\pi j}\oint_C Re\{z\}dz$ is ____.

Q. Questions

#Q. The residues of the function $f(z) = \frac{1}{(z-4)(z+1)^3}$, are

$(z-4) \to$ simple order

$(z+1)^3 \to 3^{rd}$ order Pole.

(a) -1/27 and -1/125

(b) 1/125 and -1/125

(c) –1/27 and 1/5

(d) 1/125 and -1/5

$$\mathrm{Res}[f(z); z=4] = \lim_{z\to 4} \cancel{(z-4)} \cdot \frac{1}{\cancel{(z-4)}(z+1)^3} = \frac{1}{125}$$

$$\mathrm{Res}[f(z); z=-1] = \frac{1}{(3-1)!} \lim_{z\to -1} \frac{d^{3-1}}{dz^{3-1}} \cancel{(z+1)^3} \cdot \frac{1}{(z-4)\cancel{(z+1)^3}}$$

$$= \frac{1}{2!} \lim_{z\to -1} \frac{d^2}{dz^2} \frac{1}{(z-4)}$$

$$= -\frac{1}{125}$$

Q. Questions

$$z^2-3z+2 = z^2-2z-z+2$$
$$= z(z-2)-1(z-2)$$

#Q. If C is a circle $|Z| = 4$ and $f(z) = \frac{z^2}{(z^2-3z+2)^2}$, then $\oint_C f(z)dz$ is

(a) 1

(b) 0

(c) –1

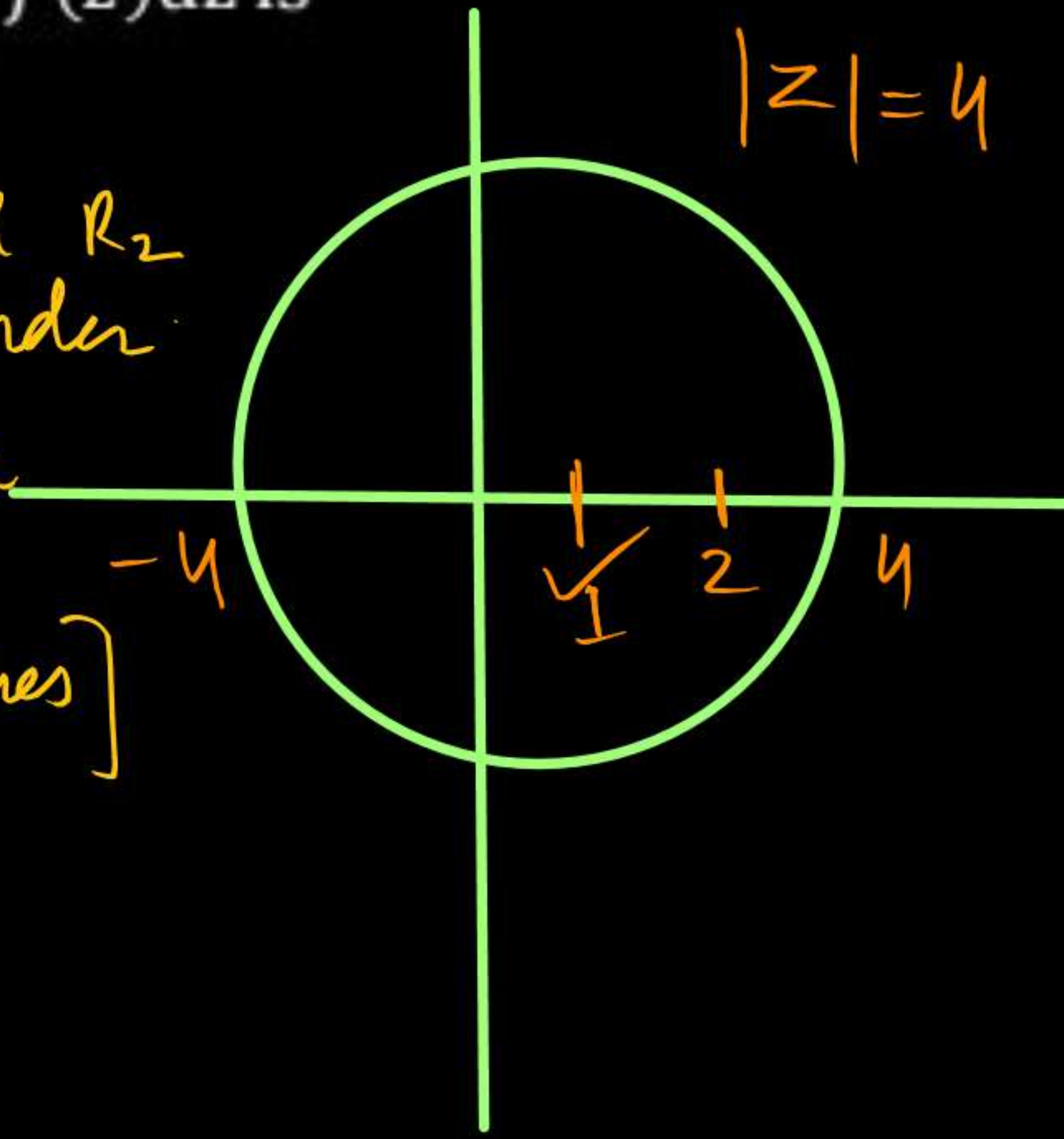
(d) –2

$$f(z) = \frac{z^2}{\underbrace{(z-1)^2}_{R_1}(z-2)^2}$$

R_2 2nd order

R_1 2nd order

$$\oint \frac{z^2}{(z-1)^2(z-2)^2} = 2\pi i\,[\text{sum of residues}]$$

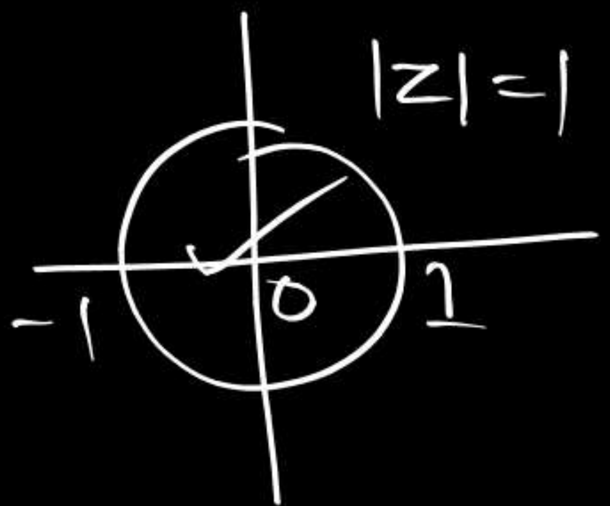


Q. Questions

#Q. The value of the contour integral

$$\frac{1}{2\pi j}\oint_c \left(z+\frac{1}{z}\right)^2 dz$$

Evaluated over the unit circle $|z| = 1$ is 1.



$\text{Res}[f(z); z=0]$

$= \lim_{z \to 0} \cancel{(z-0)} \cdot \frac{z^2+1}{\cancel{z}}$

$= 1$

$\oint \frac{z^2+1}{z} dz = 2\pi j[1] = 2\pi j$

$\frac{1}{2\pi j} \oint \left(z+\frac{1}{z}\right)^2 dz$

$= \frac{1}{2\pi j} \oint \frac{z^2+1}{(z-0)} dz$

simple order Pole

$= \frac{1}{2\pi j} \times 2\pi j$

$= 1$

Thank You!

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