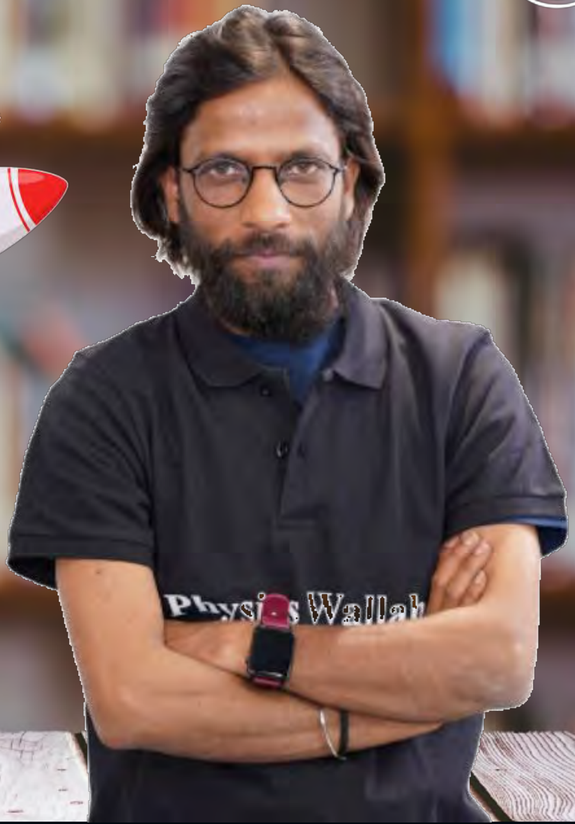
GATE-All BRANCHES

Engineering Mathematics

LAPLACE TRANSFORM

Discussion Notes



By- Rahul sir

Topic : Laplace transform

#Q. If the Laplace transform of a function f(t) is given by $\frac{s+3}{(s+1)(s+2)}$ then f(0) is.

A. 0

B. 1/2

C. 1

D. 3/2

$$L\{f(t)\} = \frac{s+3}{(s+1)(s+2)}$$

$$L\{f(t)\} = \frac{2}{s+1} - \frac{1}{s+2}$$

$$\frac{1}{s-a}$$

$$e^{at}$$

Applying L^{-1} on both sides —

$$L^{-1} L\{f(t)\} = L^{-1}\left\{\frac{2}{s+1}\right\} - L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$f(t) = 2e^{-t} - e^{-2t}$$

$$L(e^{at}) = \frac{1}{s-a}$$

$\therefore f(0) = 2 \cdot e^{-0} - e^{-2\cdot(0)} = 2 - 1 = 1$

$$\boxed{f(0) = 1}$$

Topic : Laplace transform

#Q. The Laplace transform of sin h (at) is

A $\dfrac{s}{s^2+a^2}$

B $\dfrac{s}{s^2-a^2}$

C $\dfrac{a}{s^2-a^2}$

D $\dfrac{a}{s^2+a^2}$

$$L(\sinh at) = ?$$

$$\sinh at = \frac{e^{at} - e^{-at}}{2}$$

$$L\{(\sinh at)\} = \frac{1}{2}\{L(e^{at}) - L(e^{-at})\}$$

$$= \frac{1}{2}\left\{\frac{1}{(s-a)} - \frac{1}{(s+a)}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{(s-a)} - \frac{1}{(s+a)}\right\} \rightarrow \frac{\cancel{s}+a-\cancel{s}+a}{2(s^2-a^2)}$$

$$= \frac{a}{(s^2-a^2)}$$

#Q. The Laplace transform F(s) of the exponential function $f(t) = e^{at}$ when t is greater than equal to 0, where a is a constant and (s - a) > 0, is

A. $\frac{1}{s+a}$

B. $\frac{1}{s-a}$

C. $\frac{1}{a-s}$

D. ∞

$$L(f(t)) = \int_0^\infty e^{-st} \cdot f(t) \cdot dt$$

$$L(e^{at}) = \int_0^\infty e^{at} \cdot e^{-st} \cdot dt = \int_0^\infty e^{-(s-a) \cdot t} \cdot dt$$

$$\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{-1}{s-a}(0-1)$$

$$\boxed{L(e^{at}) = \frac{1}{s-a}}$$

Topic : Laplace transform

#Q. The value of $\int_0^\infty \frac{1}{1+x^2}dx + \int_0^\infty \frac{\sin x}{x}dx$ is

A $\frac{\pi}{2}$

B π

C $\frac{3\pi}{2}$

D 1

$$I = I_1 + I_2$$

$$I_1 = \int_0^\infty \frac{1}{1+x^2}\,dx \quad \& \quad I_2 = \int_0^\infty \frac{\sin x}{x}\,dx$$

$$I_1 = \tan^{-1}x \Big|_0^\infty \quad \& \quad I_2 = \int_0^\infty \frac{\sin x}{x}\,dx$$

$$= \pi/2 - 0$$

Considing the laplace transform of sin at

$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

By the property of division with 't' :-

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{s^2+a^2}\,ds$$

$$= \int_s^\infty e^{-st}\cdot\frac{\sin at}{t}\cdot dt = \frac{L}{a}\cdot \tan^{-1}\left(\frac{s}{a}\right)\Big|_s^\infty$$

$$\int_s^\infty e^{-st}\cdot\frac{\sin at}{t}\cdot dt = \frac{1}{a}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)\right)$$

Substituting $s=0$ & $a=1$ in both sides of above eqⁿ –

$$\int_s^\infty \frac{\sin at}{t}\,dt = 1\left\{\frac{\pi}{2} - 0\right\} = \frac{\pi}{2}$$

$$\int_s^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

Therefore,

$$I = \left[\tan^{-1} x\right]_0^\infty + \int_0^\infty \frac{\sin x}{x}\,dx$$

$$\left[\because \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}\right]$$

$$I = \tan^{-1}\infty - \tan^{-1} 0 + \pi/2$$

$$I = \frac{\pi}{2} - 0 + \frac{\pi}{2}$$

$$\boxed{I = \pi}$$

Topic : Laplace transform

#Q. The value of the integral $\int_{-\infty}^{\infty} 12\cos(2\pi t)\dfrac{\sin(4\pi t)}{4\pi t}dt$ is ___.

3

$$I = \int_{-\infty}^{\infty} 12\cos(2\pi t)\cdot\frac{\sin(4\pi t)}{4\pi t}\cdot dt$$

$$= 2\times\int_{0}^{\infty} 12\cdot\cos(2\pi t)\cdot\frac{\sin(4\pi t)}{(4\pi t)}\cdot dt$$

$$\int_{-\infty}^{\infty} f(t)\,dt = 2\times\int_{0}^{\infty} f(t)\cdot dt, \quad \text{If } \Big|\ f(-t) = f(t)$$

$$I = 3 \times \int_0^{\infty} \frac{2\sin(4\pi t)\cdot\cos(2\pi t)}{\pi t}\,dt$$

$$I = \frac{3}{\pi}\int_0^{\infty} \frac{\sin 6\pi t + \sin 2\pi t}{(t)}\,dt$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cdot \cos\left(\frac{A-B}{2}\right).$$

$$I = 3 \times \left\{ \int_0^{\infty} \frac{\sin(6\pi t)}{\pi t}\cdot dt + \int_0^{\infty} \frac{\sin 2\pi t}{\pi t}\cdot dt \right\}$$

$$I = I_1 + I_2$$

$$I_1 = 3\int_0^{\infty} \frac{\sin 6\pi t}{\pi t} \cdot dt$$

let $6\pi t = \theta$

$$6\pi \cdot dt = d\theta$$

$$dt = \frac{d\theta}{6\pi}$$

$t = 0,\ \theta = 0$

$t = \infty,\ \theta = \infty$

$$I_1 = 3 \times \int_0^{\infty} \frac{\sin\theta}{\theta/6} \cdot \frac{d\theta}{6\pi}$$

$$\int_0^{\infty} \frac{\sin t}{t} \cdot dt = \frac{\pi}{2}.$$

$$I_1 = \frac{3}{\pi}\int_0^{\infty} \frac{\sin\theta}{\theta} \cdot d\theta$$

$$= \frac{3}{\pi} \times \frac{\pi}{2} = \frac{3}{2}$$

let $I_2 = 3\int_0^\infty \frac{\sin 2\pi t}{\pi t} \cdot dt$

$2\pi t = \theta$

$2\pi dt = d\theta$

$dt = \frac{d\theta}{2\pi}$

when $t = 0, \quad \theta = 0$

$t = \infty, \quad \theta = \infty$

$I_2 = 3 \times \int_0^\infty \frac{\sin\theta}{\theta/2} \cdot \frac{d\theta}{2\pi} = \frac{3}{\pi}\int_0^\infty \frac{\sin\theta}{\theta} \cdot d\theta$

$I_2 = \frac{3}{\pi}\int_0^\infty \frac{\sin\theta}{\theta} \cdot d\theta$

$= \frac{3}{\pi} \times \frac{\pi}{2} = \frac{3}{2}$

$\boxed{I_1 + I_2 = 2 \cdot 3}$

Topic : Laplace transform

#Q. Laplace transform of $\cos(\omega t)$ is $\frac{s}{s^2+\omega^2}$. The Laplace transformation of $e^{-2t}\cos(4t)$ is

A. $\frac{s-2}{(s-2)^2+16}$

B. $\frac{s+2}{(s-2)^2+16}$

C. $\frac{s-2}{(s+2)^2+16}$

D. $\frac{s+2}{(s+2)^2+16}$

$L(\cos \omega t) = \frac{s}{s^2+\omega^2}$

By first shift property -

$L(e^{at} \cdot \cos \omega t) = \frac{(s-a)}{(s-a)^2+\omega^2}$

Substitute $a = -2$ & $\omega = 4$

$L\{e^{-2t} \cdot \cos 4t\} = \frac{s-(-2)}{(s-(-2))^2+4^2}$

$$= \frac{s+2}{(s+2)^2+16}$$

$$\boxed{L\{e^{-2t} \cdot \cos 4t\} = \frac{s+2}{(s+2)^2+16}}$$

Topic : Laplace transform

#Q. The function f(t) satisfies the differential equation $\frac{d^2 f}{dt^2} + f = 0$ and the auxiliary conditions, $f(0) = 0, \frac{df}{dt}(0) = 4$ .The Laplace transform of f(t) is given by

A $\frac{2}{s+1}$

B $\frac{4}{s+1}$

C $\frac{4}{s^2+1}$

D $\frac{2}{s^4+1}$

given $\frac{d^2f}{dt^2} + f = 0$; $f(0) = 0$, $\frac{df}{dt}(0) = 4$.

The auxiliary eqⁿ is $m^2 + 1 = 0$

& $m = \pm i$

The solⁿ is $CF = c_1 \cos t + c_2 \sin t$

given $f(0) = 0$

$c_1 \cdot \cos 0 + c_2 \cdot \sin 0 = 0$

$\boxed{c_1 \cdot 1 = 0}$ $\boxed{c_1 = 0}$

$f(t) = c_2 \sin t$

$\frac{df(t)}{dt} = c_2 \cos t$

$c_2 \times 1 = 4$

$\boxed{c_2 = 4}$

$f(t) = 4\sin t$

$L\{f(t)\} = L\{4\sin t\}$

$= 4L\{\sin t\}$

$= 4 \times \frac{1}{s^2+1}$

$\boxed{= \frac{4}{s^2+1}}$

Topic : Laplace transform

#Q. The inverse Laplace transform of the function $F(s)=\frac{1}{s(s+1)}$ is given by :

A $f(t)=\sin t$

B $f(t)=e^{-t}\sin t$

C $f(t)=e^{-t}$

D $f(t)=1-e^{-t}$

$F(s) = \frac{1}{s(s+1)}$

$F(s) = \frac{1}{s} - \frac{1}{s+1}$ $\longrightarrow$ partial fraction

$L^{-1}(F(s)) = L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$

$f(t) = 1 - e^{-t}$ ✓

$\boxed{L(e^{at}) = \frac{1}{s-a}}$

$L(t^n) = \frac{n!}{s^{n+1}} \rightarrow L\left(\frac{1}{s}\right) = 1$

Topic : Laplace transform

#Q. The Laplace transform of a function f(t) is $\frac{1}{s^2(s+1)}$. The function f(t) is

A. $t-1+e^{-t}$

B. $t+1+e^{-t}$

C. $-1+e^{-t}$

D. $2t+e^{t}$

$$L(f(t)) = F(s) = \frac{1}{s^2(s+1)}$$

using partial fraction over here -

$$F(s) = \frac{1}{s^2(s+1)} = A/s + B/s^2 + C/(s+1)$$

$$F(s) = \frac{As(s+1) + B(s+1) + Cs^2}{s^2(s+1)} = \frac{1}{s^2(s+1)}$$

$A+C=0$; $A+B=0$; $\underline{\underline{B=1}}$

$\boxed{A=-1}$, $\boxed{C=1}$

$$\frac{1}{s^2(s+1)} = \frac{-1}{s} + \frac{1}{s^2} + \frac{1}{(s+1)}.$$

Applying inverse laplace on both sides:-

$$L^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = L^{-1}\left(\frac{-1}{s}\right) + L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left\{\frac{1}{(s+1)}\right\}$$

$$L^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = -1+t+e^{-t}$$

$$\left\{ L\{e^{at}\} = \frac{1}{s-a} \right\}$$

Topic : Laplace transform

#Q. The inverse Laplace transform of $\frac{1}{(S^2+S)}$ is

A. $1+e^{t}$

B. $1-e^{t}$

C. $1+e^{-t}$

D. $1-e^{-t}$

$$f(s) = \frac{1}{s^2+s} = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

$$L^{-1}(f(s)) = L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right)$$

$$= L^{-1}\left\{\frac{1}{s-0}\right\} - L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= e^{0\cdot t} - e^{-1\cdot t}$$

$$= \boxed{1 - e^{-t}}$$

Topic : Laplace transform

#Q. Laplace transform for the functions f (x) = cos h (ax) is

A. $\dfrac{a}{s^2 - a^2}$

B. $\dfrac{s}{s^2 - a^2}$

C. $\dfrac{a}{s^2 + a^2}$

D. $\dfrac{s}{s^2 + a^2}$

$f(x) = \cosh(ax)$

$$\cosh ax = \frac{e^{ax} + e^{-ax}}{2}$$

Taking laplace transform, we get -

$$\frac{1}{2}\left\{ L\left(e^{ax}\right) + L\left\{e^{-ax}\right\} \right\}$$

$$\frac{1}{2}\left\{ \left(\frac{1}{s-a}\right) + \frac{1}{(s+a)} \right\} = \frac{2s}{2(s^2-a^2)} = \frac{s}{s^2-a^2}$$

Topic : Laplace transform

#Q. The solution of $\frac{d^2y}{dt^2} - y = 1,$ which additional satisfies $y\big|t = 0 = \frac{dy}{dt}\Big|_{t=0} = 0$ in the Laplace s-domain is

A $\frac{1}{s(s+1)(s-1)}$

B $\frac{1}{s(s+1)}$

C $\frac{1}{s(s-1)}$

D $\frac{1}{(s-1)}$

$\frac{d^2y}{dt^2} - y = 1 \quad y(0) = 0 \;\&\; \left.\frac{dy}{dt}\right|_{t=0} = 0$

Applying Laplace transformation on both sides of DE.

$\left(s^2 F(s) - s \cdot y(0) - y'(0)\right) - F(s)$

$= \frac{1}{s}$

$= (s^2 - 1) F(s) = \frac{1}{s}$

$F(s) = \frac{1}{s(s^2-1)} = \frac{1}{s(s+1)(s-1)}$

Topic : Laplace transform

#Q. The value of the integral is $2\int_{-\infty}^{\infty}\left(\frac{\sin 2\pi t}{\pi t}\right)dt$ equal to

A. 0

B. 0.5

C. 1

D. 2

$$2\int_{-\infty}^{\infty} \frac{\sin 2\pi t}{\pi t} \cdot dt$$

$$f(t) = f(-t) = \frac{\sin(2\pi t)}{\pi t}$$

$$I = 2\int_{-\infty}^{\infty} \frac{\sin 2\pi t}{\pi t} \cdot dt = 4\int_{0}^{\infty} \frac{\sin 2\pi t}{\pi t} \cdot dt$$

$$= \frac{4}{\pi}\int_{0}^{\infty} \frac{\sin 2\pi t}{t} \cdot dt$$

from Laplace transform $\qquad L\left[\frac{f(t)}{t}\right] = \int_{0}^{\infty} F(s) \cdot ds$

we know $\int_0^\infty \frac{\sin t}{t} = \frac{\pi}{2}$

$$\int_0^\infty \frac{\sin 2\pi t}{t} = \frac{\pi}{2}$$

$$\boxed{\text{Hence } I = \frac{4}{\pi} \cdot \frac{\pi}{2} = 2}$$

Topic : Laplace transform

#Q. Let $X(s) = \dfrac{3s+5}{s^2+10s+21}$ be the Laplace Transform of a signal x (t). Then, x (0^+) is

A. 0

B. 3

C. 5

D. None of these

using the theorem -

$$x(0^+) = \lim_{s \to \infty} s \cdot X(s)$$

$$\lim_{s \to \infty} \left(\frac{s(3s+5)}{s^2+10s+21} \right)$$

$$\lim_{s \to \infty} \frac{3 + 5/s}{1 + 10/s + 21/s^2} = 3$$

Topic : Laplace transform

#Q. Consider the differential equation :

$$\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+y(t)=\delta(t) \text{ with } y(t)\big|t=0=-2 and \frac{dy}{dt}\Big|_{t=0^+}=0$$

The numerical value of $\left.\frac{dy}{dt}\right|_{t=0}$ is

A. -2

B. -1

C. 0

D. 1

given data -

$$\frac{d^2y(t)}{dt} + 2\frac{dy(t)}{dt} + y(t) = \delta(t)$$

$$y(t)\Big|_{t=0} = -2 \Rightarrow y(0) = -2$$

$$\frac{dy}{dt}\Big|_{t=0} = 0 \Rightarrow y'(0) = 0$$

differential eq^n can be written as -

$$\boxed{y'' + 2y' + y = \delta(t)}$$

Taking laplace transform on both sides:—

$$\left[s^2 Y(s) - s\,y(0) - y'(0)\right] + 2\left[s\,Y(s) - y(0)\right] + Y(s) = 1$$

$$\left(s^2+2s+1\right) Y(s) + 2s + 4 = 1$$

$$\left(s^2+2s+1\right) Y(s) + 2s + 4 = 1$$

$$\boxed{Y(s) = \frac{-3-2s}{(s+1)^2}}$$

$$Y(s) = \frac{-3}{(s+1)^2} - 2\left[\frac{(s+1)-1}{(s+1)^2}\right]$$

$$Y(s) = \frac{-3}{(s+1)^2} - \frac{2}{s+1} + \frac{2}{(s+1)^2} = \frac{-1}{(s+1)^2}$$

$$\frac{-2}{(s+1)}$$

$$y(t) = \mathcal{L}^{-1}\left(Y(s)\right)$$

$$= \mathcal{L}^{-1}\left[-\frac{1}{(s+1)^2} - \frac{2}{(s+1)}\right]$$

2 mins Summary

Topic One

Topic Two

Topic Three

Topic Four

Topic Five

$$y = -te^{-t} - 2e^{-t}$$

$$L^{-1}\left\{\frac{1}{s-a}\right\} \Leftarrow e^{at}$$

$$\frac{dy}{dt} = te^{-t} - e^{-t} + 2e^{-t}$$

$$= e^{-t}(t-1+2)$$

$$= e^{-t}(t+1)$$

$$\left.\frac{dy}{dt}\right|_{t=0} = 1$$

THANK - YOU

Topics to be Covered