GATE (ALL BRANCHES)

Engineering Mathematics

Complex Analysis

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Lecture No. 05

TOPICS TO BE COVERED

01 Problems based on Cauchy Integral Theorem

Cauchy Integral formula

$$\boxed{\int_{\Gamma} f(z)\,dz = 0}$$

(any simple connected domain)

Cauchy Integral formula

$$\boxed{\int_{\Gamma} \frac{\phi(z)}{(z-z_0)}\,dz = 2\pi i\,\phi(z_0)} \rightarrow$$ simple order Pole

Cauchy Integral formula for n^{th} order Pole:

✓ $$\boxed{\int \frac{\phi(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!}\phi^{n}(z_0)}$$

$\phi^{n}(z_0)$ → n^{th} derivative

$(z-z_0)^{n+1}$ → n^{th} order Pole.

2^{nd} order:

$$\frac{\phi(z)}{(z-z_0)^2},\ \frac{\phi(z)}{(z-z_0)^3}$$

Q. Questions

#Q. Consider the integral line $I = \int_c \left(x^2 + iy^2\right) dz$, where $z = x + iy$. The line c shown in the figure below.

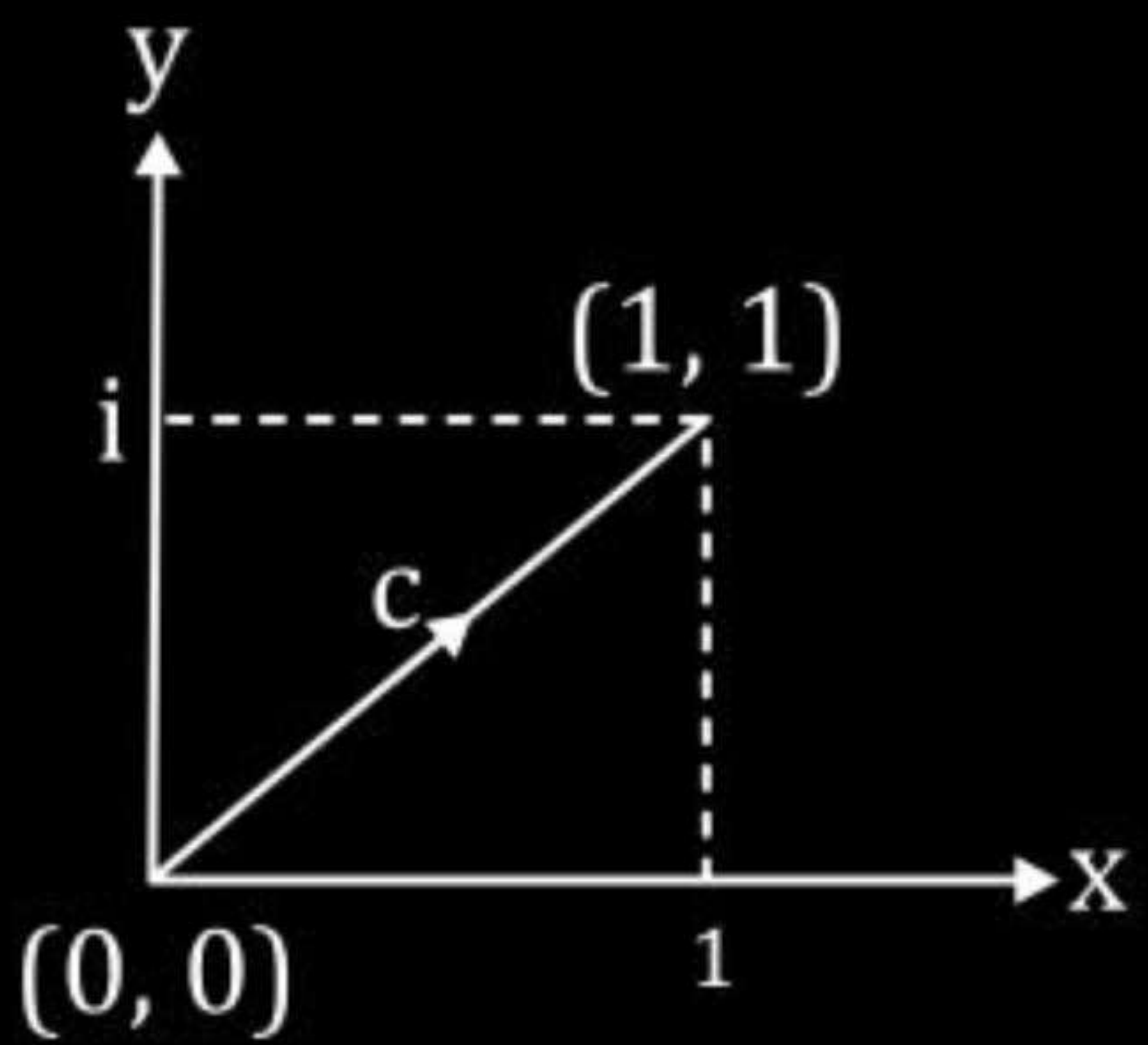
The value of I is

A $\frac{1}{2}i$

B $\frac{2}{3}i$

C $\frac{3}{4}i$

D $\frac{4}{5}i$



Q. Questions

#Q. The value of the contour integral in the complex-plane $\oint \frac{z^3 - 2z + 3}{z-2} dz$ along the contour $\boxed{|z| = 3}$, taken counter-clockwise is

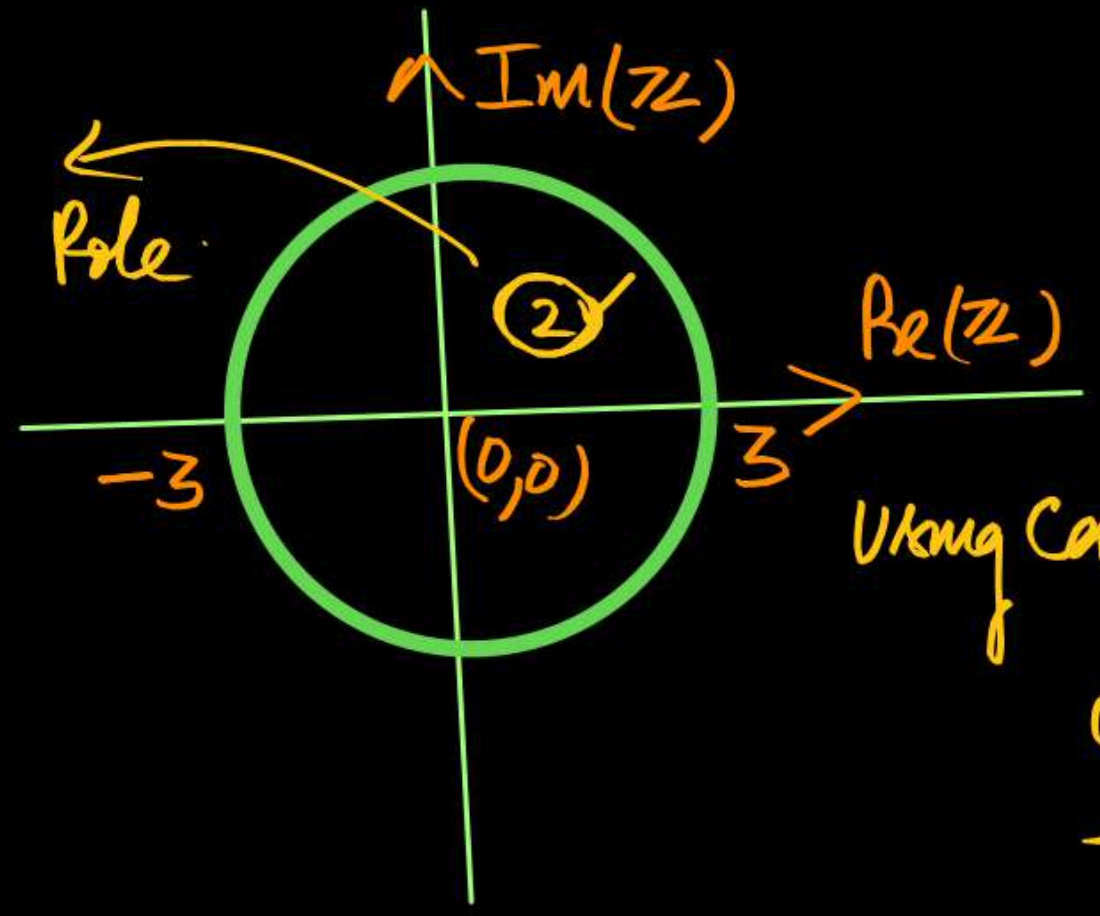
A) $-18\pi i$

B) 0

C) $14\pi i$

D) $48\pi i$

STEP 01 Plot the contour



$$\oint \frac{z^3-2z+3}{\boxed{(z-2)}} dz$$ contour $|z|=3$ circle center $(0,0)$ radius 3

$$\oint \frac{\phi(z)}{(z-z_0)} = 2\pi i\, \phi(z_0)$$

If z_0 Pole lie in the domain

$z-2=0$ $z=2$ is a simple order Pole.

Using Cauchy Integral formula

$$\oint \frac{z^3-2z+3}{(z-2)} dz = 2\pi i \left[z^3-2z+3\right]_{z=2}$$

$$= 2\pi i[7] = \underline{14\pi i}$$

Q. Questions

#Q. The contour C given below is on the complex place $z = x + jy$, where $j = \sqrt{-1}$.

$j = i$

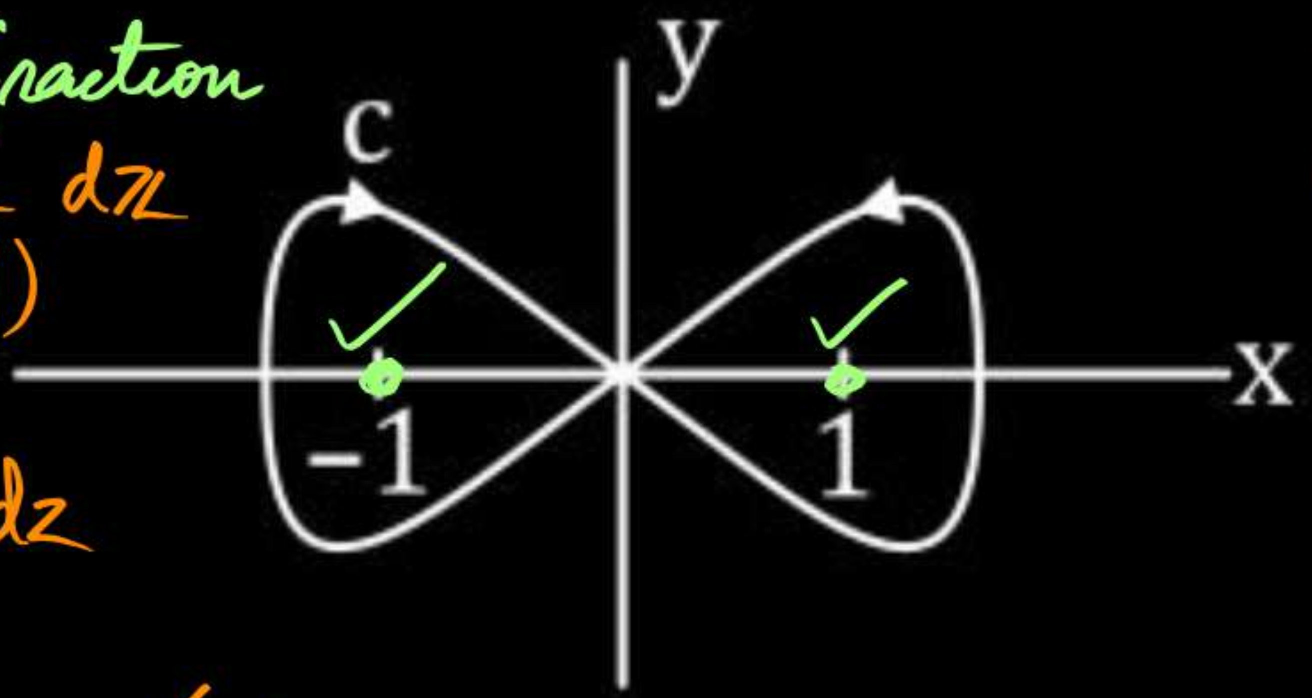
The value of the integral $\frac{1}{\pi j}\int_C \frac{dz}{z^2-1}$ is $\underline{0}$.

$$\frac{1}{\pi j}\oint_C \frac{dz}{\underbrace{(z-1)(z+1)}_{\text{Partial fraction}}}$$

$$\frac{1}{\pi j}\oint_C \frac{1}{2(z-1)} - \frac{1}{2}\cdot\frac{1}{(z+1)}\,dz$$

$$= \frac{1}{\pi j}\frac{1}{2}\oint_C \frac{dz}{(z-1)} - \oint \frac{dz}{(z+1)}\,dz$$

$$= \frac{1}{2\pi j}\left[\cancel{2\pi j \times 1} - \cancel{2\pi j \times 1}\right] = 0$$



D^r Poles $D^r = 0$

$(z-1)(z+1) = 0$

$$\left.\begin{array}{l} z = 1 \\ z = -1 \end{array}\right]$$

$$\frac{1}{(z-1)(z+1)} = \frac{A}{(z-1)} + \frac{B}{(z+1)}$$

$$A = \left.\frac{1}{(z+1)}\right|_{z=1} = \frac{1}{2}$$

$$B = \left.\frac{1}{(z-1)}\right|_{z=-1} = \frac{-1}{2}$$

SECOND method :-

$$\frac{1}{\pi j}\oint \frac{dz}{(z^2-1)} = \frac{1}{\pi j}\oint \frac{dz}{(z-1)(z+1)}$$

If both Poles Are lie in The domain

$$= \frac{1}{\pi j}\left[\oint \frac{\frac{1}{(z-1)}}{(z+1)}\,dz + \oint \frac{\frac{1}{(z+1)}}{(z-1)}\,dz\right]$$

$\phi(z)$ $\phi(z)$

Using cauchy Integral formula

$$= \frac{1}{\pi j}\left[2\pi j\cancel{\left[\frac{1}{(z-1)}\right]_{z=-1}} + 2\pi j\cancel{\left[\frac{1}{z+1}\right]_{z=1}}\right]$$

$$= \underline{\underline{0}}$$

Q. Questions

$$\int \frac{\frac{z+1}{z-2}\,\phi(z)}{(z+2)(z-z_0)}\,dz$$

#Q. The value of integral $\int_C \frac{z+1}{z^2-4}\,dz$ in counter clockwise direction around a circle C of radius 1 with centre at the point $z = -2$ is

$$\int \frac{(z+1)}{(z-2)(z+2)}\,dz$$

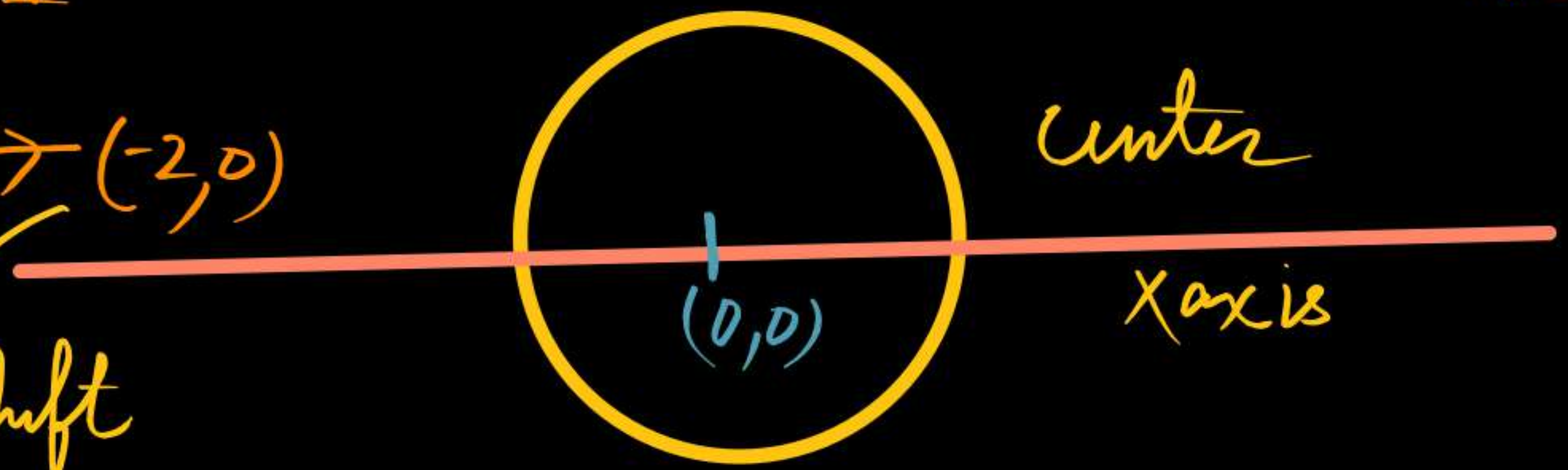
Poles $\left.\begin{array}{l} z=2 \\ z=-2 \end{array}\right]$

radius = 1

center

$(0,0) \rightarrow (-2,0)$

shift

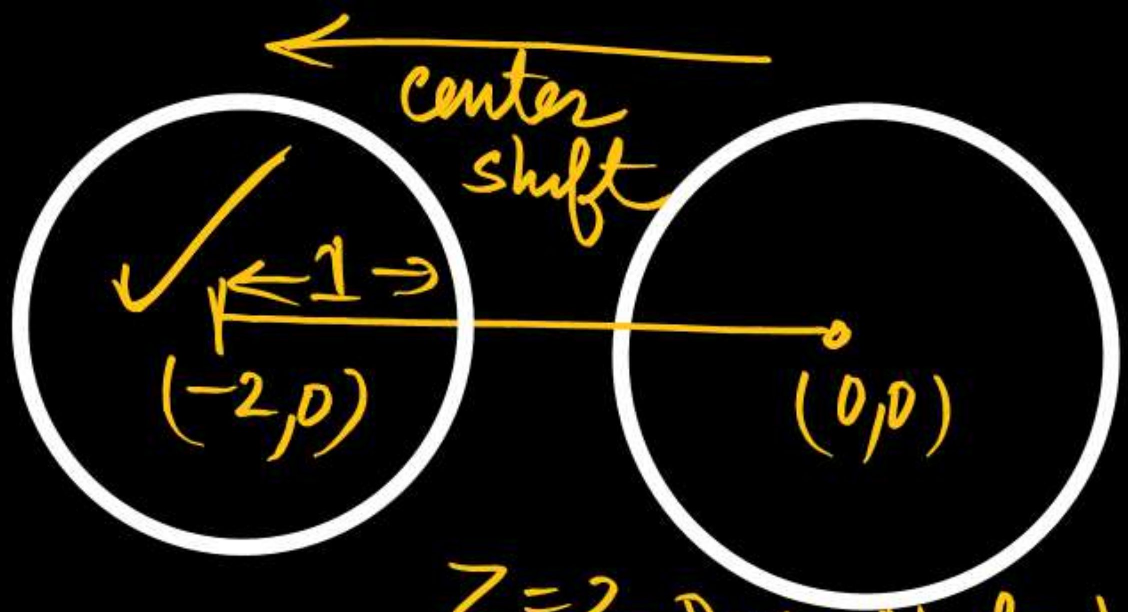


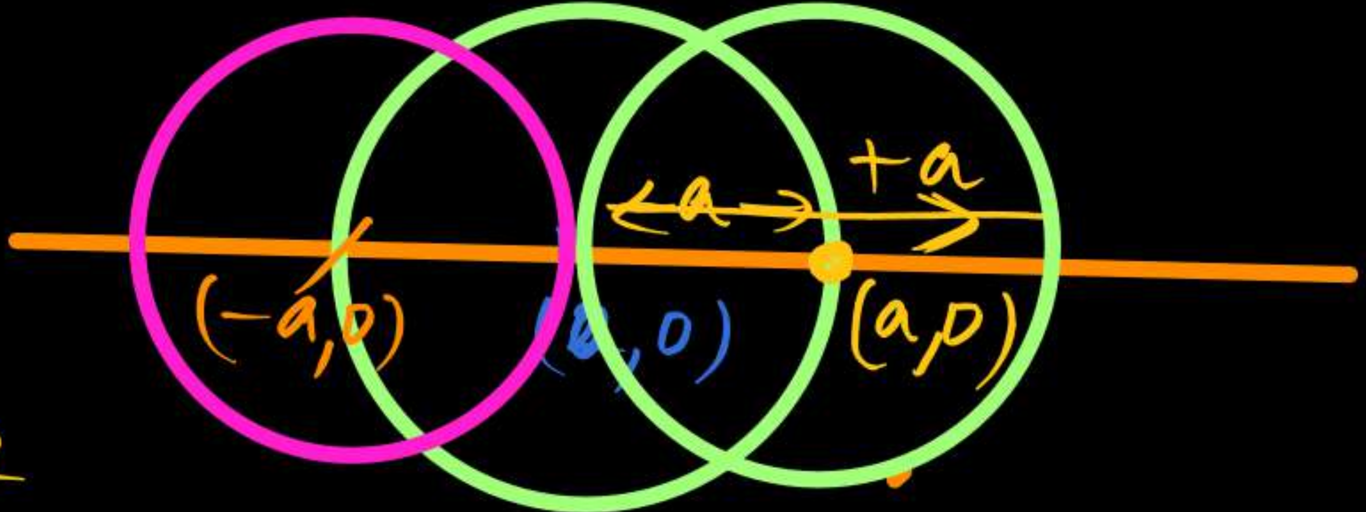
A) $\pi i/2$

B) $2\pi i$

C) $-\pi i/2$

D) $-2\pi i$





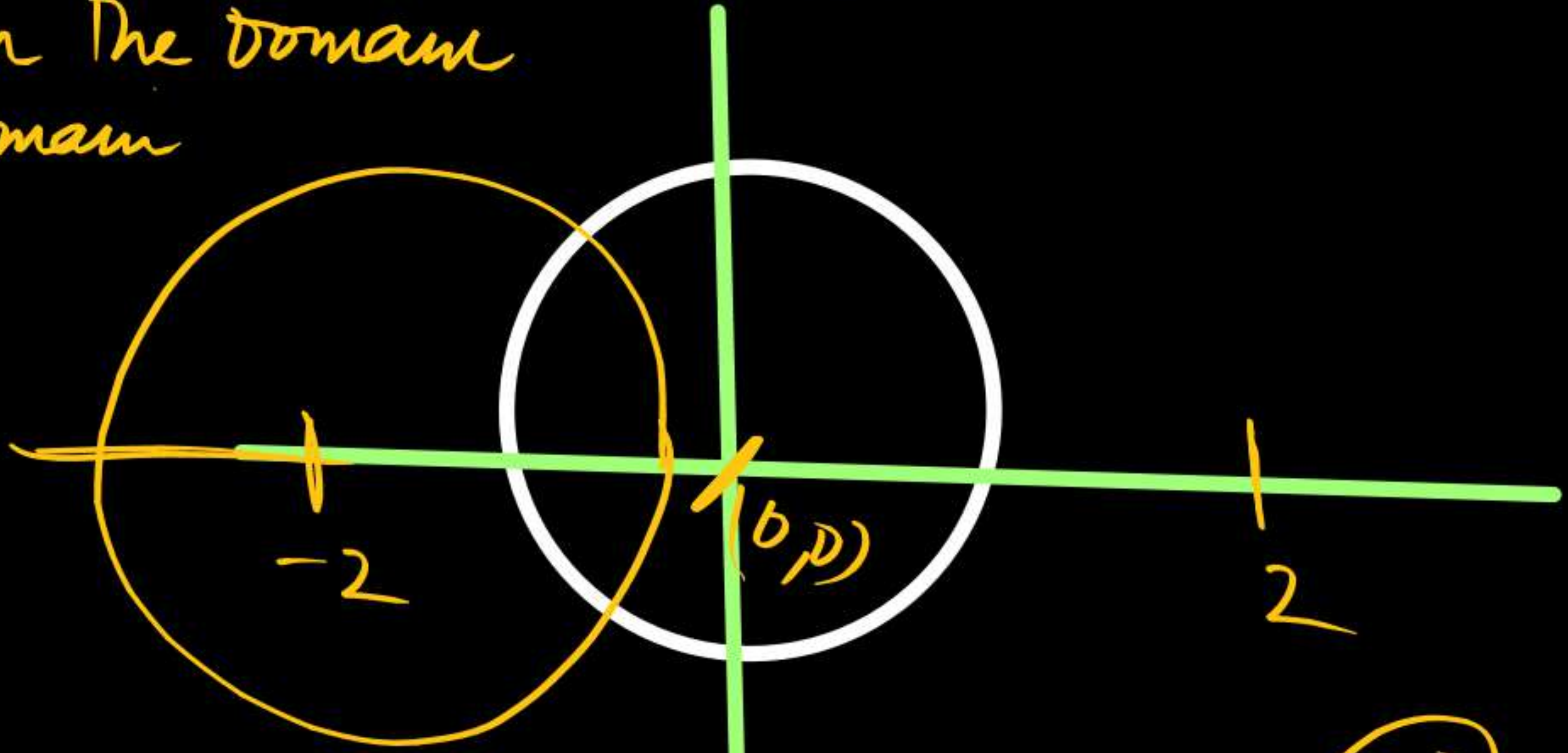
Poles $z = 2$

$z = -2$

$z=2$ Does Not Lie in The Domain

$z=-2$ Lie in The Domain

$$\int \frac{(z+1)}{(z-2)(z+2)}\,dz$$



Using Cauchy Integral formula

$\phi(z)$

$$\int \frac{\phi(z)}{(z-z_0)}\,dz$$

$$\oint \frac{\frac{(z+1)}{(z-2)}}{(z+2)}\,dz = 2\pi i \left[\frac{(z+1)}{z-2}\right]_{z=-2} = 2\pi i \times \frac{-1}{-4} = \frac{\pi i}{2}$$

Q. Questions

#Q. Let z be a complex variable. For a counter-clockwise integration around a unit circle C, centered at origin,

$$\oint_C \frac{1}{5z-4}\,dz = A\pi i,$$

the value of A is

A) 2/5

B) 1/2

C) 2

D) 4/5

$\oint_C \frac{1}{(5z-4)}\,dz = A\pi i$ The value of A

$\int \frac{\phi(z)}{(z-z_0)}\,dz = 2\pi i\,\phi(z_0)$

$|z| = 1$

$z - \frac{4}{5} = 0$

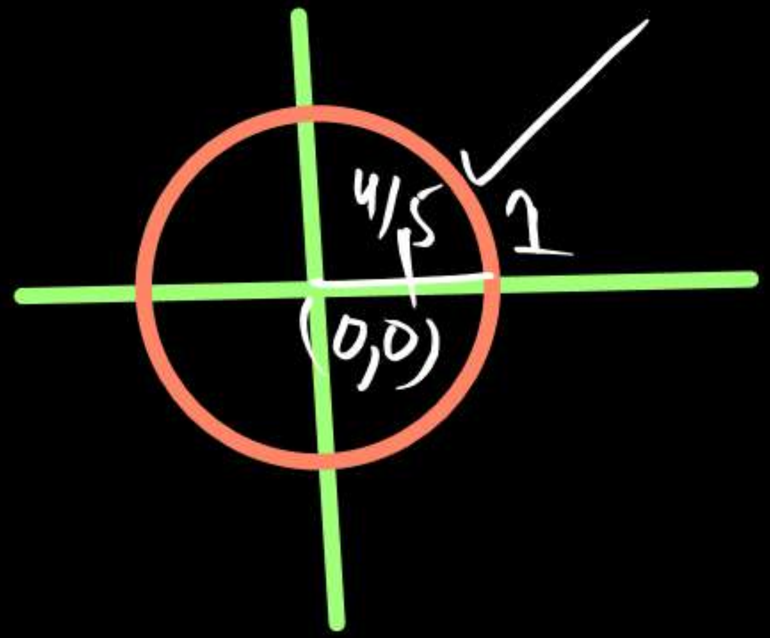
$z = \frac{4}{5}$

$\Rightarrow \frac{1}{5}\oint \frac{1}{\left(z-\frac{4}{5}\right)}\,dz$

$\Rightarrow \frac{1}{5}\,2\pi i\,[1]$

$\Rightarrow \frac{2\pi i}{5} = A\pi i$

$A = \frac{2}{5}$



Q. Questions

#Q. The closed loop line integral

$$\oint_{|z|=5} \frac{z^3+z^2+8}{z+2}\,dz$$

evaluated counter clockwise, is

A. $+8\,j\pi$

B. $+4\,j\pi$

C. $-8\,j\pi$

D. $-4\,j\pi$

Using cauchy Integral THEOREM

$$\oint \frac{\phi(z)}{(z-z_0)}\,dz = 2\pi i\,\phi(z_0)$$

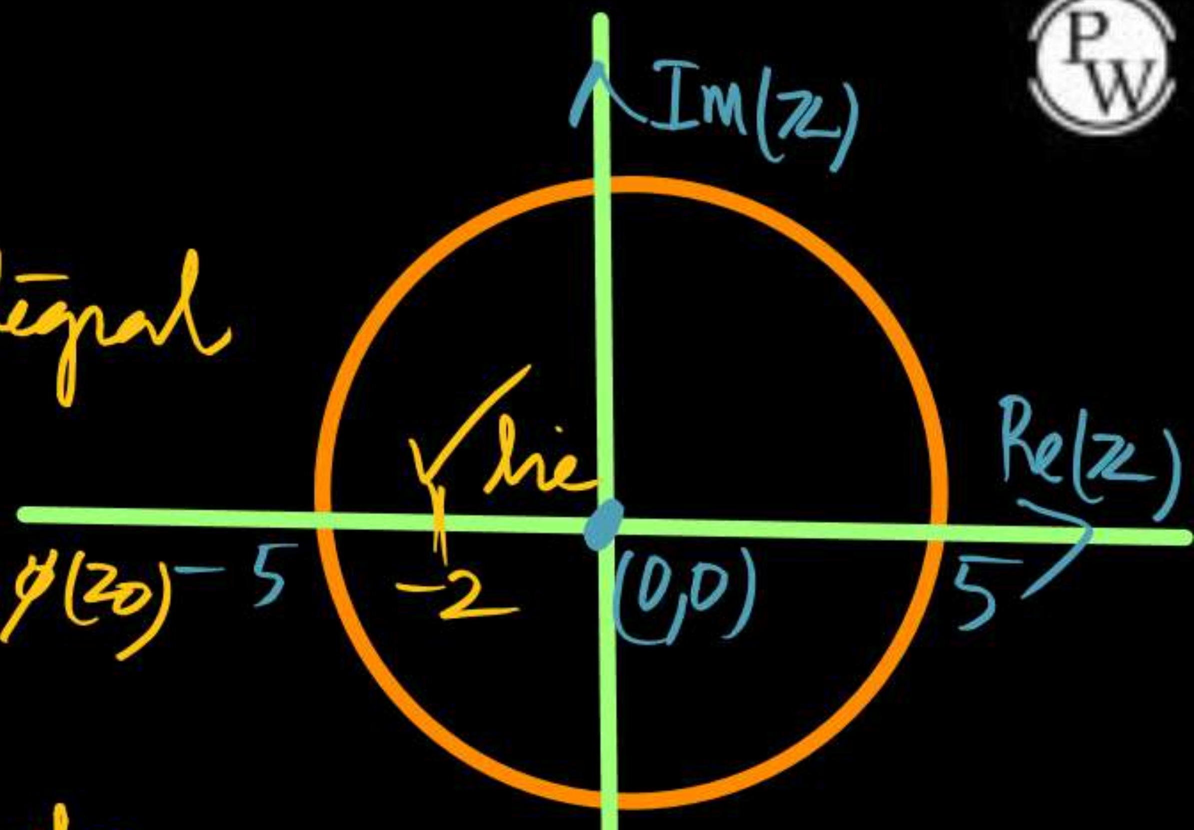
$$\oint \frac{z^3+z^2+8}{z+2}\,dz$$

$$= 2\pi i\,\left[z^3+z^2+8\right]_{z=-2}$$

$$= 2\pi i\,[\cancel{-8}+4\cancel{+8}]$$

$$= 8\pi i$$

$$= 8\pi j$$



$z+2=0$

$z=-2$

This Pole Lie on The Domain

Q. Questions

#Q. The value of the integral $\oint_c \frac{2z+5}{\left(z-\frac{1}{2}\right)\left(z^2-4z+5\right)} dz$ over the contour $|z| = 1$, taken in the anti-clockwise direction, would be

$$\oint \frac{2z+5}{\left(z-\frac{1}{2}\right)\left(z^2-4z+5\right)} dz$$

$\left(z-\frac{1}{2}\right)=0, z=\frac{1}{2}$

$\left(z^2-4z+5\right)=0$

$z = \frac{+4 \pm \sqrt{16-20}}{2}$

$z = \frac{+4 \pm 2i}{2}$

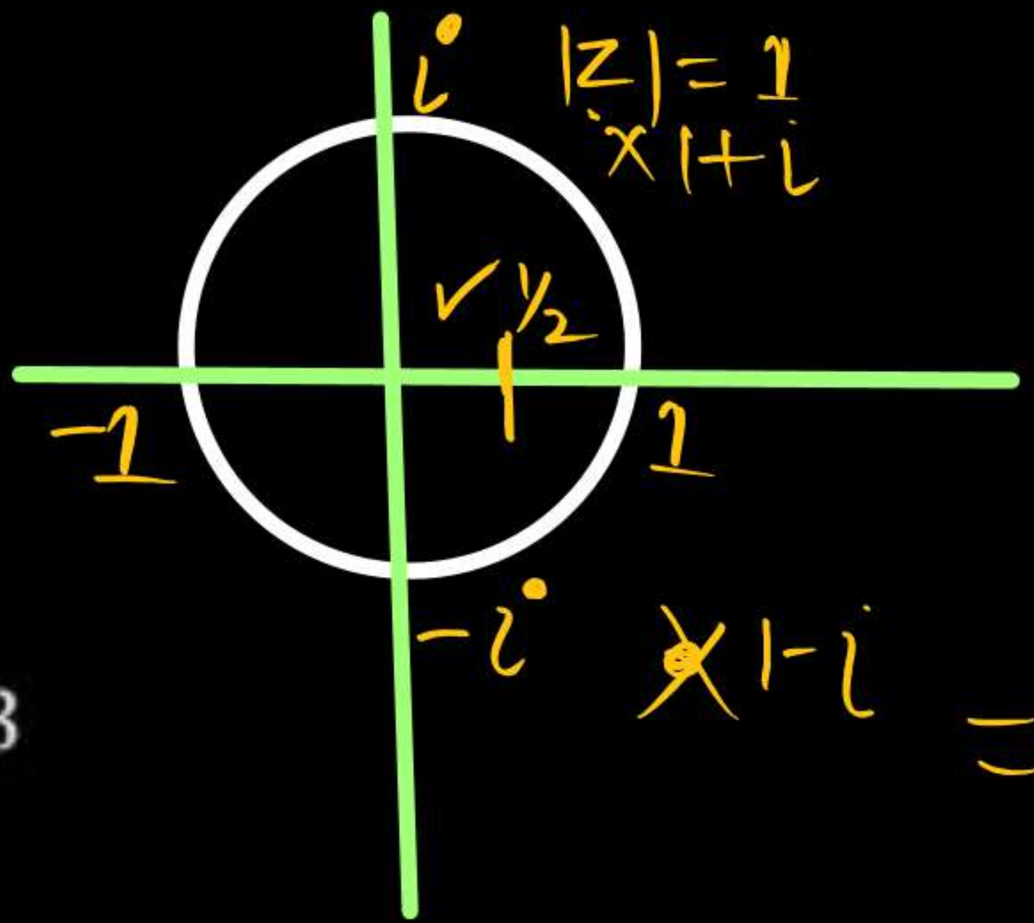
$z = 1 \pm i$

(A) $\frac{24\pi i}{13}$

(B) $\frac{48\pi i}{13}$

(C) 24/13

(D) 12/13



$\Rightarrow$

$\Rightarrow 2\pi i \left[\frac{2z+5}{z^2-4z+5}\right]_{z=\frac{1}{2}}$

$=$

$$= 2\pi i\left[\frac{2z+5}{z^2-4z+5}\right]_{z=\frac{1}{2}}$$

$$= 2\pi i\left[\frac{2\times\frac{1}{2}+5}{\frac{1}{4}-\frac{1}{2}+5}\right] = 2\pi i\left[\frac{6}{\frac{1-2+20}{4}}\right]$$

$$= 2\pi i\left[\frac{6}{\frac{+19}{4}}\right]$$

$$= \frac{12\pi i}{\frac{19}{4}} = \frac{24\pi i}{19}$$

Q. Questions

#Q. The value of the integral $\frac{1}{2\pi j}\int_c \frac{z^2+1}{z^2-1}dz$ where z is a complex number and C is a unit circle with centre at 1 + 0j in the complex plane is $\underline{1}$.

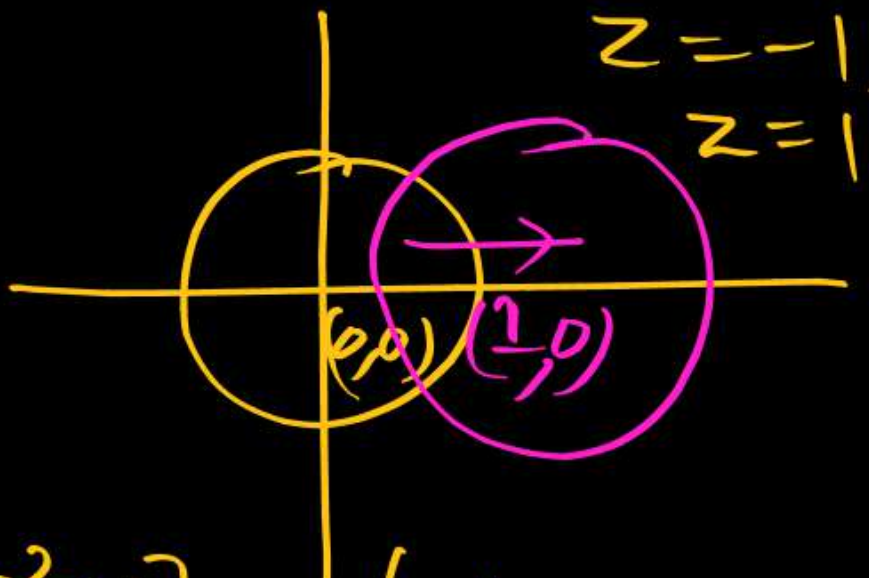
1, 0

$$\frac{1}{2\pi j}\oint \frac{z^2+1}{(z^2-1)}dz = \frac{1}{2\pi j}\oint \frac{z^2+1}{(z-1)(z+1)}dz$$

Poles $\boxed{z=1 \quad z=-1}$

$z=-1$ ✗

$z=1$ ✓



$$= \frac{1}{2\pi j}\int \frac{\frac{(z^2+1)}{(z+1)}}{(z-1)}dz = \cancel{\frac{1}{2\pi j}} \times \cancel{2\pi j}\left[\frac{z^2+1}{z+1}\right]_{z=1}$$

$(-1,0)$ ✗

$= 1$

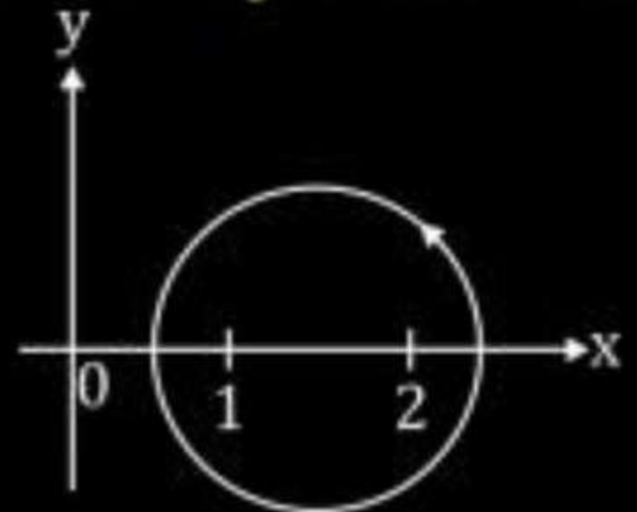
Q. Questions

Do yourself

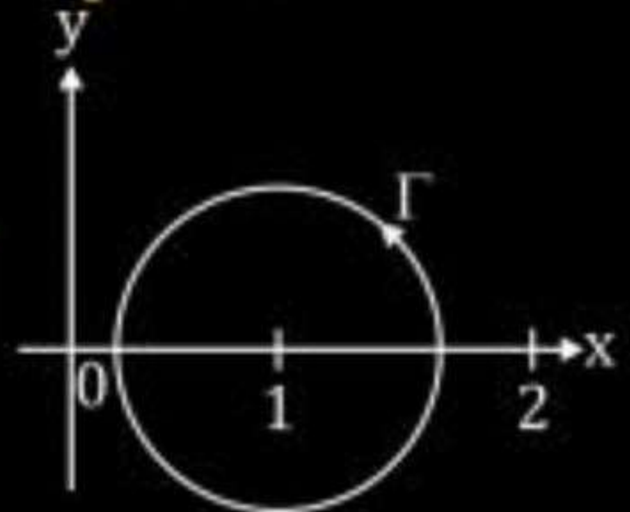
H.W

#Q. The value of $\oint_{\Gamma} \frac{3z-5}{(z-1)(z-2)} dz$ along a closed path Γ is equal to $(4\pi i)$, where $z = x + iy$ and $i =$ is $\sqrt{-1}$. The correct path Γ is

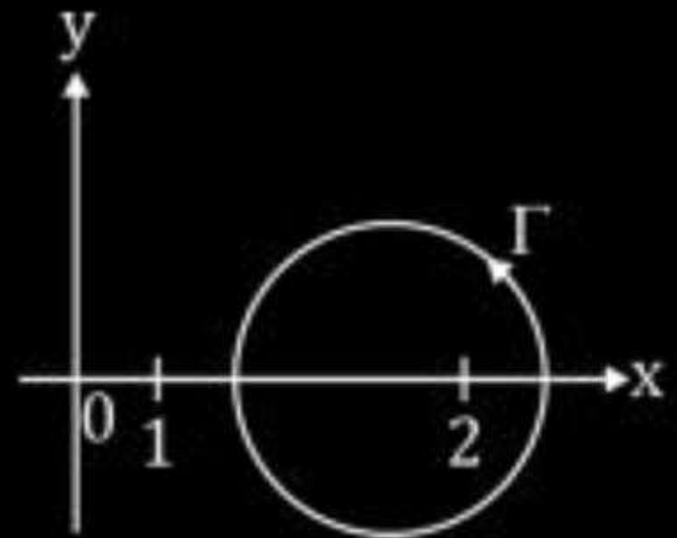
A



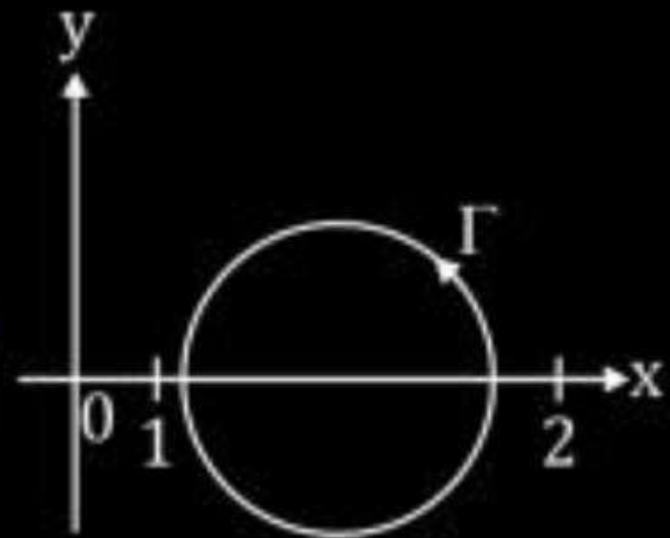
B



C



D



Q. Questions

#Q. An integral I over a counter clockwise circle C is given by $I = \oint_C \frac{z^2-1}{z^2+1} e^z \, dz$. If C is defined as $|z| = 3$, then the value of I is

$$I = \oint \frac{(z^2-1)e^z}{(z^2+1)} \quad |z| = 3$$

A. $-\pi i \sin(1)$

B. $-2\pi i \sin(1)$

C. $-3\pi i \sin(1)$

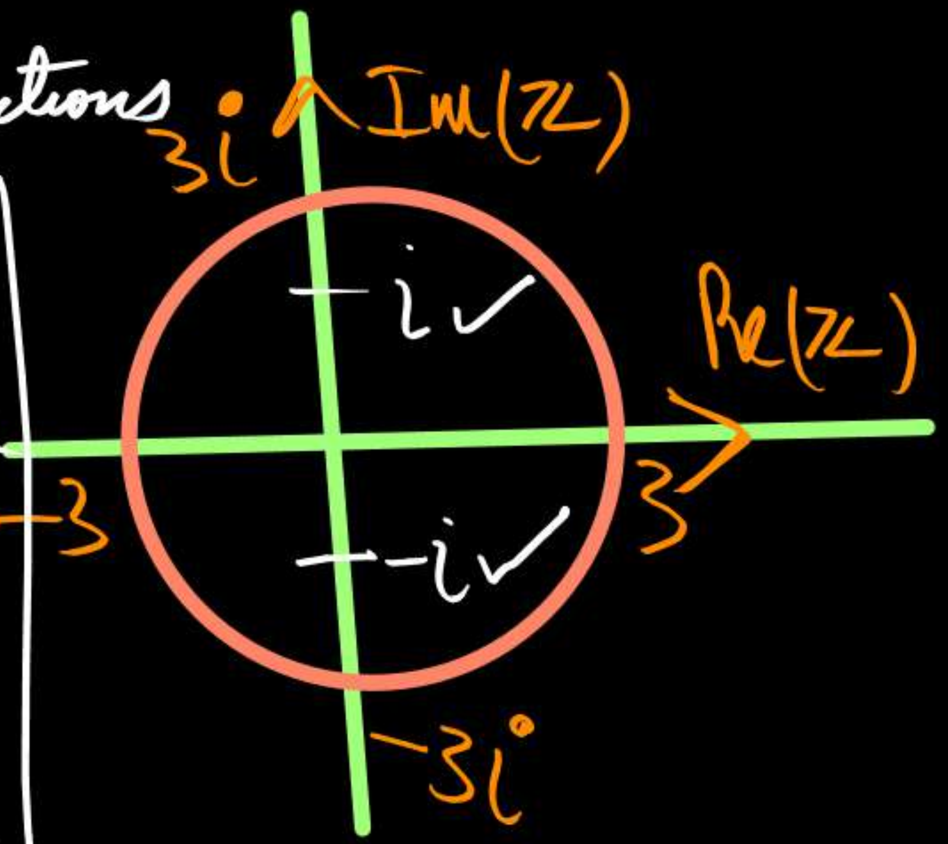
D. $-4\pi i \sin(1)$

$$I = \oint \frac{(z^2-1)e^z}{(z+i)(z-i)}$$

Using Partial fractions

$$= \boxed{\oint \frac{\frac{(z^2-1)e^z}{(z+i)}}{(z-i)} dz} + \boxed{\oint \frac{\frac{(z^2-1)e^z}{(z-i)}}{(z+i)} dz}$$

=



$$= \oint \frac{\frac{e^z(z^2-1)}{(z+i)}}{(z-i)}\,dz + \oint \frac{\frac{e^z(z^2-1)}{(z-i)}}{(z+i)}\,dz$$

Using Cauchy Integral THEOREM

$$= 2\pi i\left[\frac{e^z(z^2-1)}{(z+i)}\right]_{z=i} + 2\pi i\left[\frac{e^z(z^2-1)}{(z-i)}\right]_{z=-i}$$

$$= 2\pi i\left[e^i\frac{(i^2-1)}{2i} + e^{-i}\frac{(i^2-1)}{-2i}\right]$$

$$= \frac{\cancel{2\pi i}}{\cancel{2i}}\left[e^i(-2) + e^{-i}\frac{(\cancel{-}2)}{\cancel{-}1}\right]$$

$$= \pi\left[-2e^i + e^{-i}\,2\right]$$

$$= 2i\left[e^{-i} - e^i\right]$$

$$= 2\pi i\left(e^{-i} - e^{i}\right)$$

$$= 2\pi i\left[-2i\sin 1\right]$$

$$= \boxed{-4\pi i \sin 1}$$

$$\cos x + i\sin x = e^{ix}$$

$$x = 1$$

$$\cancel{\cos 1} + i\sin 1 = e^{i}$$

$$\cancel{\cos 1} \mp i\sin 1 = \pm e^{-i}$$

$$2i\sin 1 = e^{i} - e^{-i}$$

$$-2i\sin 1 = e^{-i} - e^{i}$$

Q. Questions

#Q. A simple closed path C in the complex plane is shown in the figure. If

$$\oint_C \frac{e^z}{z^2-1}\,dz = -i\pi A,$$

Do yourself

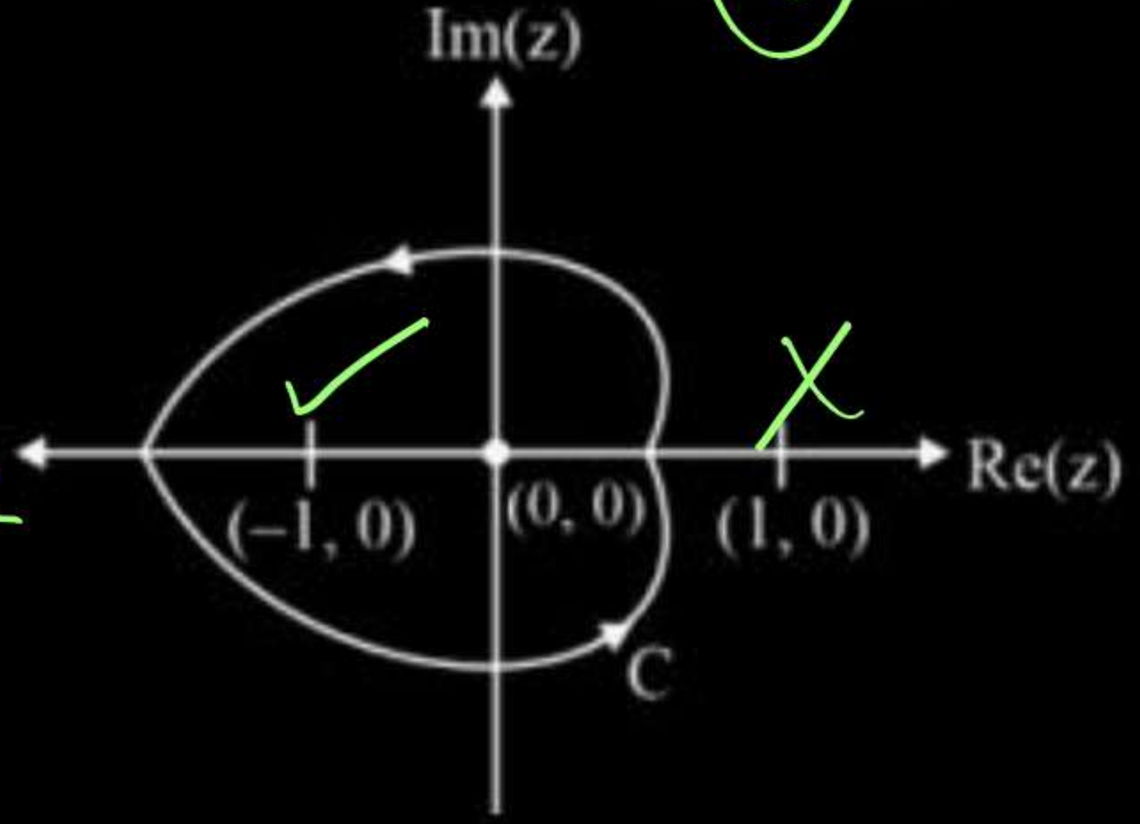
where $i=\sqrt{-1}$, then the value of A is $\frac{1}{e}$ (rounded off to two decimal places).

H.W

$$\oint \frac{e^z}{(z-1)(z+1)}$$

$$= \oint \frac{\frac{e^z}{(z-1)}}{(z+1)}\,dz$$

$$= \left[\frac{e^z}{(z-1)}\right]_{z=-1} 2\pi i =$$



$$= \cancel{2}\pi i\left[\frac{e^{-1}}{-\cancel{2}}\right]$$

$$= -\pi i e^{-1} = -i\pi A$$

$$\boxed{A = \frac{1}{e}}$$

Q. Questions

$\frac{1}{5}$

#Q. Given $z = x + iy$, $i = \sqrt{-1}$. C is a circle of radius 2 with the center at the origin. If the contour C is traversed anticlockwise, then the value of the integral $\frac{1}{2\pi}\int_C \frac{1}{(z-i)(z+4i)}dz$ is ___. (Round off to one decimal place).

$$\frac{1}{2\pi}\oint \frac{1}{(z-i)(z+4i)} = \frac{1}{2\pi}\oint \frac{\frac{1}{(z+4i)}}{(z-i)}dz$$

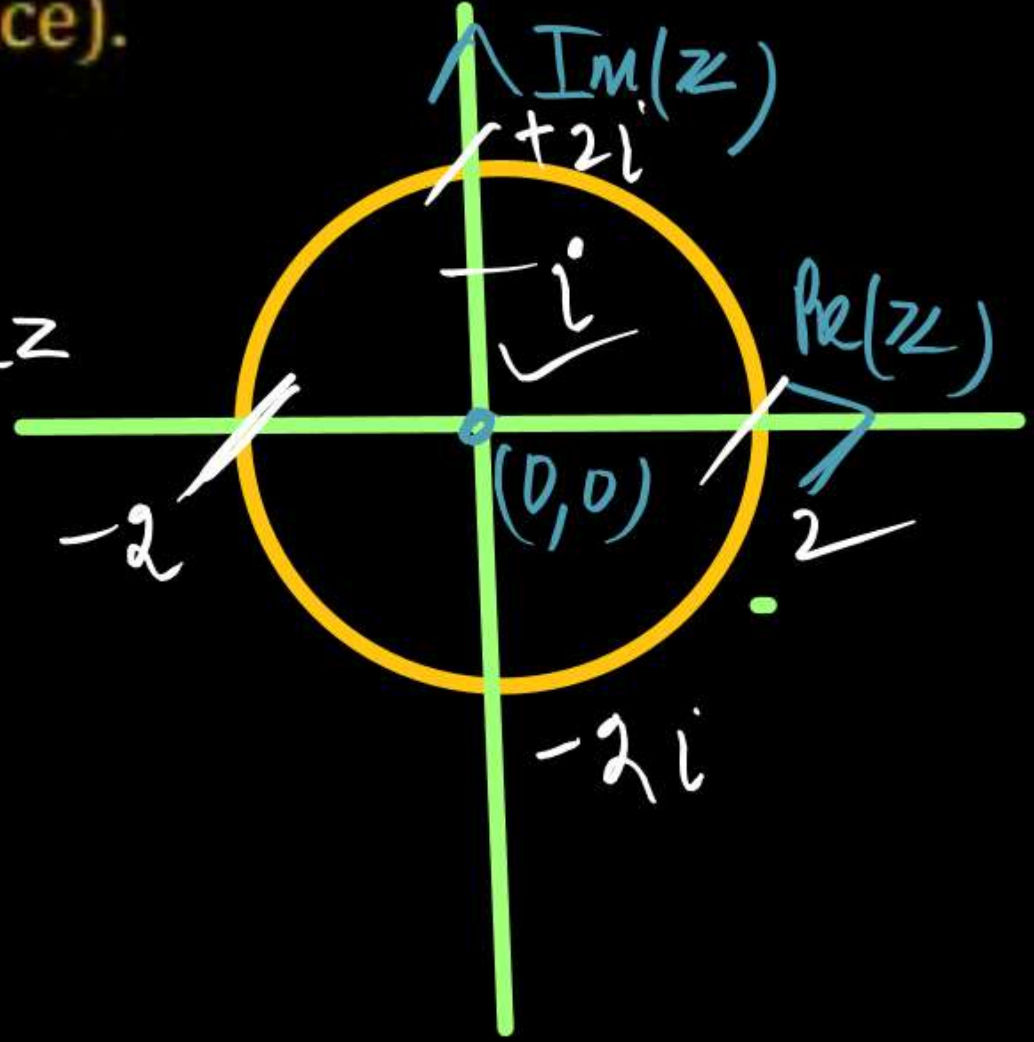
$z = i$

$z = -4i$

Using cauchy Integral formula.

$$= \frac{1}{\cancel{2\pi}}\left[\frac{1}{5i}\right]_{z=i} \times \cancel{2\pi} i$$

$$= \frac{1}{5\cancel{i}} \times \cancel{i} = \frac{1}{5}$$



Q. Questions

#Q. The value of the following complex integral, with C representing the unit circle centered at origin in the counter clock wise sense, is:

$|z|=1$ $\quad \int_C \frac{z^2+1}{z^2-2z}\,dz = -\pi i$

$$\oint \frac{\phi(z)}{(z-z_0)}$$

(a) $8\pi i$

(b) $-8\pi i$

(c) $-\pi i$

(d) πi

$$\oint_C \frac{z^2+1}{z(z-2)}\,dz$$

$$= \oint \frac{\frac{(z^2+1)}{(z-2)}}{z}\,dz = 2\pi i\left[\frac{z^2+1}{z-2}\right]_{z=0} = \cancel{2}\pi i\left[\frac{1}{-\cancel{2}}\right] = -\pi i$$

Thank You!

PW Soldiers