GATE-All BRANCHES

Engineering Mathematics

Fourier Series

DPP 01

Discussion Notes

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Topic : FOURIER SERIES

#Q. The following function is defined over the interval [-L, L]:

$f(x) = px^4 + qx^5$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left\{a_n \sin\left(\frac{\pi x}{L}\right) + b_n \cos\left(\frac{\pi x}{L}\right)\right\},$$

fourier series

If it is expressed as a Fourier series,

a_n

Which options amongst the following are true?

A a_n, n = 1,2,..., ∞ depend on p

B a_n, n = 1,2,..., ∞ depend on p

C b_n, n = 1,2,..., ∞ depend on p

D a_n, n = 1,2,..., ∞ depend on q

$f(x) = px^4 + qx^5$

using fourier coefficient formula,

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L}\int_{-L}^{L} (px^4 + qx^5) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L}\int_{-L}^{L} px^4 \sin\left(\frac{n\pi x}{L}\right) \cdot dx + \frac{1}{L}\int_{-L}^{L} qx^5 \sin\left(\frac{n\pi x}{L}\right) \cdot dx$$

$$a_n = 0 + \frac{1}{L}\int_{-L}^{L} q x^4 \sin\left(\frac{n\pi x}{L}\right) dx$$

$\because p x^4 \sin\left(\frac{n\pi x}{L}\right)$ is an odd fn, thus

a_n depends on q

$$f(x) = p x^4 + \underline{\underline{q x^5}}$$

using fourier coeffn formula -

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L}\int_{-L}^{L} (px^4 + qx^5) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{L}\int_{-L}^{L} px^4 \cos\left(\frac{n\pi x}{L}\right) dx + \frac{1}{L}\int_{-L}^{L} qx^5 \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} px^4 \cos\left(\frac{n\pi x}{L}\right) dx + 0$$

$\because qx^5 \cos\left(\frac{n\pi x}{L}\right)$ is an odd function

So b_n depends on p.

#Q. The Fouier cosine series of a function is given by :

$$f(x) = \sum_{n=0}^{\infty} f_n \cos nx$$

For $f(x) = \cos^4 x$, the numerical value of $(f_4 + f_5)$ is __________.(round off to three decimal places).

$$f(x) = \cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2$$

$$\Rightarrow f(x) = \frac{1}{4}\left[1 + \cos^2 2x + 2\cos 2x\right]$$

$$= \frac{1}{4}\left[1 + \frac{1+\cos 4x}{2} + 2\cos 2x\right]$$

$$= \frac{1}{4} + \frac{1}{8} + \frac{\cos 4x}{8} + \frac{\cos 2x}{2}$$

$$= \frac{3}{8} + 0\cdot\cos x + \frac{1}{2}\cdot\cos 2x + 0\cdot\cos 3x + \frac{1}{8}\cdot\cos 4x + 0\cdot\cos 5x + \cdots$$

from cosine fourier series

$F_4 = \frac{1}{8}$ & $f_5 = 0$

$f_4 + f_5 = \frac{1}{8} = 0.125$

given data:-

$f(x) = x^3$ $\qquad$ $\underline{\underline{[-1,1]}}$

The general fourier series expansion is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{c}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{c}\right).$$

Comparing $(\alpha, \alpha+2c)$ with $[-1,1]$

$$\boxed{\alpha = -1 \ \& \ c = 1}$$

$$a_0 = \frac{1}{c} \cdot \int_{-1}^{1} f(x) \cdot dx = \frac{1}{c} \int_{-1}^{1} x^3 dx = 0$$

$\left\{ \because f(x) \text{ is an } \cancel{\text{odd}} \text{ function.} \right\}$

$$a_n = \frac{1}{c} \cdot \int_{-1}^{1} f(x) \cos n\pi x \, dx = \frac{1}{c} \int_{-1}^{1} x^3 \cdot \cos n\pi x \, dx = 0$$

$$b_n = \frac{1}{c} \cdot \int_{-1}^{1} f(x) \cdot \sin n\pi x \cdot dx = \frac{1}{c}$$

$$\int_{-1}^{1} x^3 \sin n\pi x \, dx \neq \underline{\underline{0}}$$

$\therefore$ The fourier series expansion of x^3 in $[-1, 1]$ has only sine terms in it.

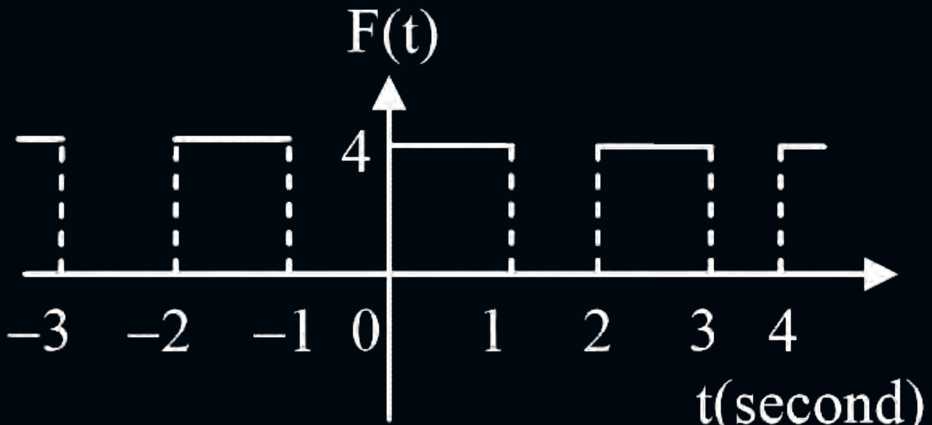
#Q. F(t) is a periodic square wave function as shown. It takes only two values, 4 and 0, and stay at each of these values for 1 second before changing. What is the constant term in the Fourier series expansion of F(t)?

A 1

B 2

C 3

D 4



given data:- $f(t)$ takes two values 4 & 0;

let us consider $[-1, 1]$.

(∵ since the period of function is 2 seconds.)

The general fourier series expansion is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{c}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{c}\right)$$

where;

$$a_0 = \frac{1}{c} \int_{\alpha}^{\alpha+2c} f(x)\, dx = \frac{1}{1} \cdot \int_{-1}^{1} f(x)\, dx$$

$$= \int_{-1}^{0} 0 \cdot dx + \int_{0}^{1} 4 \cdot dx = 4$$

Since $a_0 = 4$

$$\Rightarrow \frac{a_0}{2} = 2$$

Topic : FOURIER SERIES

#Q. Let f(t) be an even function, i.e. f(– t) = f(t) for all t. Let the Fourier transform of f(t) be defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt.$$

Suppose $\frac{dF(\omega)}{d\omega}$ for all ω ,and F(0) = 1
Then.

A f(0) < 1

B f(0) > 1

C f(0) = 1

D f(0) = 0

given function is even

$$\frac{d}{d\omega} F(\omega) = -\omega F(\omega) \quad \text{——} \ (1)$$

from differentiation property -

$$t f(t) = j \frac{d}{d\omega} F(\omega)$$

Applying inverse fourier transfer to the above equation -

$$-jt \ f(t) = j \frac{d}{d\omega} f(t)$$

$$\frac{d}{dt} f(t) = -t f(t) \quad \text{—— (2)}$$

from eq^n (1) & (2) it is clear that f(t) is gaussian function it can be written as;

$$f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$

$$\boxed{f(0) < 1}$$

$$f(0) = \frac{1}{\sqrt{2\pi}} = 0.3989$$

#Q. The Fourier series to represent $x - x^2$ for $-\pi \le x \le \pi$ is given by.

$$x - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

The value of a_0 (round off to two decimal places), is __________.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\,dx$$

$\left\{ \because x \text{ is an odd fn} \right\}$

$$= \frac{1}{\pi}\cdot 2 \cdot \int_{0}^{\pi} -x^2\,dx$$

$$= -2/\pi \left(\frac{x^3}{3}\right)_0^{\pi} = -\frac{2}{\pi}\cdot\frac{\pi^3}{3} = -\frac{2\pi^2}{3} = \underline{\underline{-6.58}}$$

Topic : FOURIER SERIES

#Q. A periodic function f(t), with a period of 2 π , is represented as its Fourier series,

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt \sum_{n=1}^{\infty} b_n \sin nt \div$$

If

$$f(t) = \begin{cases} A\sin t, & 0 \le t \le \pi \\ 0, & \pi < t < 2\pi \end{cases},$$

the Fourier series coefficients a_1 and b_1 of f (t) are.

A $a_1 = \frac{A}{\pi}; b_1 = 0$

B $a_1 = \frac{A}{2}; b_1 = 0$

C $a_1 = 0 \; ; b_1 = \frac{A}{\pi}$

D $a_1 = 0 \; ; b_1 = \frac{A}{2}$

given data:-

$$f(t) = \begin{cases} A\sin t & 0 \le t \le \pi \\ 0 & \pi < t < 2\pi \end{cases}$$

$$a_1 = \frac{1}{\pi} \cdot \int_0^{2\pi} f(t) \cdot \cos t \cdot dt$$

$$a_1 = \frac{1}{\pi}\left\{\int_0^{\pi} A \sin t \cdot \cos t \cdot dt\right\}$$

$$a_1 = \frac{1}{\pi} \times A/2 \times \int_0^{\pi} \sin 2t \cdot dt$$

$$a_1 = \frac{A}{2\pi} \times \left\{ -\frac{\cos 2t}{2} \Big|_0^{\pi} \right\} = \frac{-A}{4\pi} \{1-1\}$$

$$= 0$$

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} f(t) \cdot \sin t \cdot dt$$

$$= \frac{1}{\pi} \int_0^{\pi} (A \sin t \cdot \sin t \cdot dt)$$

$$b_1 = \frac{A}{\pi}\int_0^{\pi} \sin\tau \cdot d\tau$$

$$b_1 = \frac{2A}{\pi}\int_0^{\pi/2} \sin\tau \cdot d\tau = \frac{\cancel{2}A}{\cancel{\pi}} \cdot \frac{1}{\cancel{2}} \times \frac{\cancel{\pi}}{2}$$

$$\boxed{b_1 = \frac{A}{2} \;\; \& \;\; a_1 = 0}$$

Topic : FOURIER SERIES

#Q. The fourier cosine series for an even functions f (x) is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx).$$

The value of the coefficient a_2 for the function f (x) = cos2 (x) in [0, π]

A. – 0.5

B. 0.00

C. 0.5

D. 1.0

given data!- $f(x) = \cos^2 x$ in $[0, \pi]$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cdot \cos nx$$

we know that -

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$$\cos^2 x = \frac{1}{2} + \frac{1}{2} \cdot \cos 2x = a_0 + a_2 \cdot \cos 2x$$

$a_2 = \frac{1}{2} = 0.5 \longrightarrow$ on comparing

Topic : FOURIER SERIES

#Q. For the function $f(x) = \begin{cases} -2 & -\pi < x < 0 \\ 2 & 0 < x < \pi \end{cases}$ The value of a_n in the Fourier Series expansion of f(x) is

A. 2

B. 4

C. 0

D. -2

given data -

$$f(x) = \begin{cases} -2; & -\pi < x < 0 \\ 2; & 0 < x < \pi \end{cases}$$

The general fourier series expansion of $f(x)$ is given by -

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where;

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot \cos nx \cdot dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\times\left\{\int_{-\pi}^{0} -2\cos nx\,dx + \int_{0}^{\pi} 2\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-2\frac{\sin nx}{n}\Big|_{-\pi}^{0} + 2\frac{\sin nx}{n}\Big|_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{2}{n}\times 0 + 2\cdot 0\right\} = 0 \quad \boxed{a_n = 0}$$

Topic : FOURIER SERIES

#Q. The Fourier series of the function $f(x)=\begin{cases} 0 & -\pi < x \le 0 \\ \pi - x & 0 < x < \pi \end{cases}$

In the interval $[-\pi, \pi]$ is

$$f(x)=\frac{\pi}{4}+\frac{2}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\ldots\right]+\left[\frac{\sin x}{1}+\frac{\sin 2x}{2}+\frac{\sin 3x}{2}+\ldots\right]$$

The convergence of the above Fourier series at x = 0 gives.

A $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

B $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$

C $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

D $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=\frac{\pi}{4}$

The $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \pi - x & 0 \le x < \pi \end{cases}$

and fourier series is –

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \cdots\right] + \left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]$$

at $x = 0$, it is a point of discontinuity, the fourier series converges to $\frac{1}{2}\left[f(0^-) + f(0^+)\right]$

Where $f(0^-) = \lim_{x \to 0} (\pi - x) = \pi$ —

$f(0^+) = 0$

Put $x = 0$ in fourier series —

$$f(0) = \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} - - \cdot\right]$$

$$\frac{\pi}{2} = \frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} - - - -\right]$$

$$\frac{\pi}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} - - -\right]$$

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}}$$

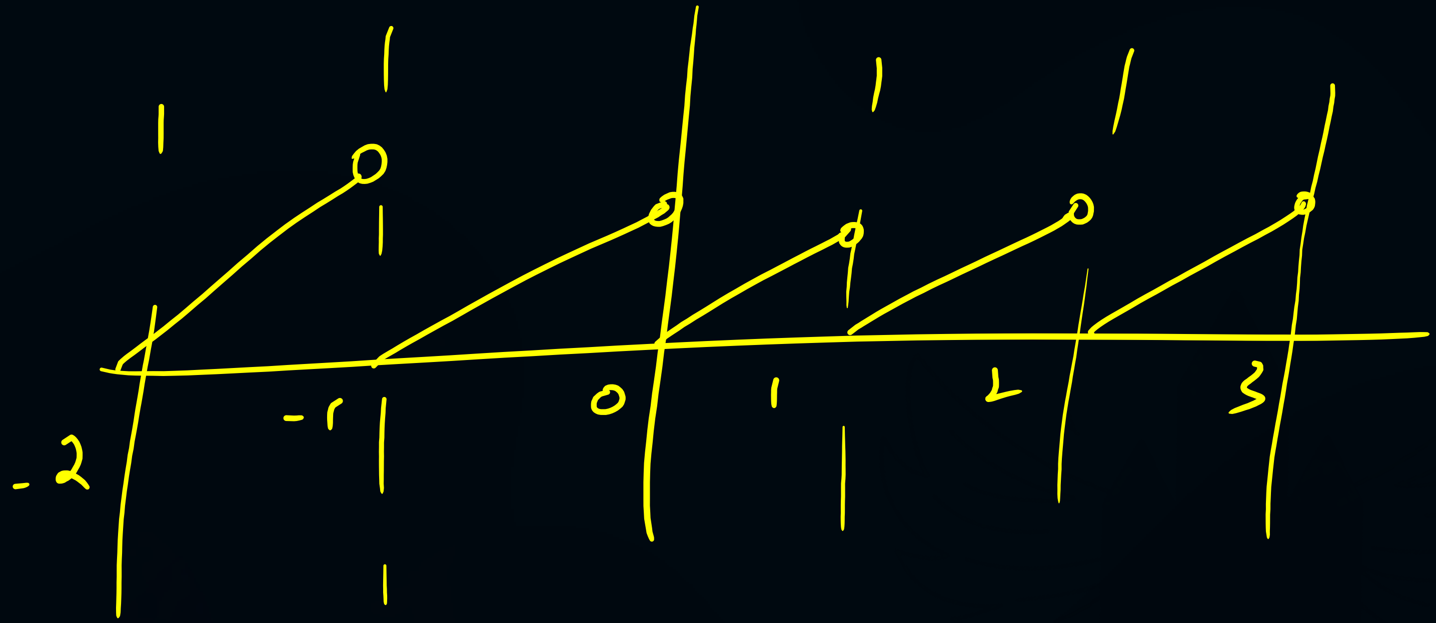
Topic : FOURIER SERIES

#Q. Let g: [0, ∞] [0, ∞) be a function defined by g(x) = x− [x], where [x] represents the interger part of x. (That is the largest interger which is less than or equal to x).The value of the constant term in the Fouier Series expansion of g(x) is ________.

$$\boxed{x-[x]=\{x\}}$$

$\{x\}$

$f(x) = x - [x] = \{x\}$



from the graph, period of $f(x)=l$

The general fourier series expansion of $f(x)$ in the interval $(\alpha, \alpha+2l)$ is given by;

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \cdot \sin\left(\frac{n\pi x}{l}\right)$$

Where $a_0 = \frac{1}{l}\int_{\alpha}^{\alpha+x} f(x)\,dx$

Considering the interval [0, 1]

$\Rightarrow \alpha = 0$ & $c = \frac{1}{2}$

$$\frac{a_0}{2} = \frac{1}{2}\left\{\frac{1}{c}\int_{\alpha}^{\alpha+2c} f(x)\,dx\right\}$$

$$\frac{a_0}{2} = \frac{1}{2} = 0.5$$

$$= \frac{1}{2} \times \frac{1}{\left(\frac{1}{2}\right)} \times \int_0^1 \left(x - [x]\right) dx$$

$$\frac{a_0}{2} = \int_0^1 (x - 0)\cdot dx = \frac{1}{2}$$

Topic : FOURIER SERIES

#Q. The period of the signal $x(t) = 8 \sin\left(0.8\pi t + \frac{\pi}{4}\right)$

A $0.4\ \pi$ s

B $0.8\ \pi$ s

C 1.25 s

D 2.5 s

$x(t) = 8\sin\left(0.8\pi t + \frac{\pi}{4}\right)$

Comparing with the standard form of the signal

$x(t) = A\sin(\omega t + \phi)$

$\omega = 0.8\pi$

$\frac{2\cancel{\pi}}{T} = 0.8\cancel{\pi}$

$T = 2/0.8 = 2.5\text{ s}$

$T = 2.5\text{ sec}$

Topic : FOURIER SERIES

#Q. Choose the function f(t); $-\infty < t < \infty$, for which a Fourier series cannot be defined.

$\exp(-|t|) \cdot \sin(25t)$

A. 3 sin (25 t)

B. 4 cos (20 t + 3) + 2 sin (710 t)

C. exp (– |t|) sin (25 t)

D. 1

(1) $3\sin(2\pi x)$ → fourier series

(2)

(c) → for c, the fourier series can't be defined.

#Q. The Fourier series of a real periodic function has only

P. cosine terms if it is even
Q. sine terms if it is even
R. cosine terms if it is odd
S. sine terms if it is odd.

odd sin terms

even cosine fn

Which of the above statements are correct?

A P and S

P & S

B P and R

A

C Q and S

D Q and R

Topic : FOURIER SERIES

#Q. For the function e^{-x}, the linear approximation around $x = 2$ is

e^{-x} $x=2$

A $(3 - x)\, e^{-2}$

B $1 - x$

C $\left[3 + 2\sqrt{2} - \left(1 + \sqrt{2}\right)x\right]e^{-2}$

D e^{-2}

$$f(x) = f(x_0) + \frac{(x-x_0) \cdot f'(x_0)}{1!} + \frac{(x-x_0)^2 f''(x_0)}{2!} + - - -$$

$$= e^{-2} + (x-2)e^{-2} + \frac{(x-2)^2}{2!}(e^{-2}) - - -$$

$$= e^{-2} + \left(2-x+\frac{(x-2)^2}{2!}\right)e^{-2} + \underbrace{\cdot - - - \cdot}_{\text{higher power to neglect}}$$

$$= e^{-2}(3-x)$$

Topic : FOURIER SERIES

#Q. Which of the following functions would have only odd powers of x in its Taylor series expansion about the point x = 0 ?

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$ ——

$\sin x^3$

$x \to x^3$

A. $\sin(x^3)$

B. $\sin(x^2)$

C. $\cos(x^3)$

D. $\cos(x^2)$

$$\sin x^3 = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \cdots \infty$$

Topic : FOURIER SERIES

#Q. In the Taylor series expansion of exp(x) + sin(x) about the point x = π, the coefficient of $(x - \pi)^2$ is

$e^{x} + \sin x$

A exp (π)

B 0.5 exp (π)

C exp (□) + 1

D exp (π) – 1

$f(x) = e^x + \sin x$

Taylor's series is

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

$$\boxed{a = \pi}$$

$$f(x) = f(\pi) + (x-\pi) f'(\pi) + \frac{(x-\pi)^2}{2!} f''(\pi) \cdots$$

coeff^n of $(x-\pi)^2$ is $\frac{f''(\pi)}{2}$ ✗

$$f''(x) = e^{x} - \sin x \Big|_{x=\pi} = \underline{\underline{e^{\pi}}}$$

$$\boxed{\frac{f''(\pi)}{2} = \frac{e^{\pi}}{2} = 0.5\,e^{\pi}}$$

Topic : FOURIER SERIES

#Q. The Taylor series expansion of $\dfrac{\sin x}{x-\pi}$ at $x = \pi$ is given by

A $1+\dfrac{(x-\pi)^2}{3!}+\ldots$

B $-1-\dfrac{(x-\pi)^2}{3!}+\ldots$

C $1-\dfrac{(x-\pi)^2}{3!}+\ldots$

D $-1+\dfrac{(x-\pi)^2}{3!}+\ldots$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$3 \rightarrow 2$

$\therefore \sin(x-\pi) = (x-\pi) - \frac{(x-\pi)^3}{3!} + \frac{(x-\pi)^5}{5!} - \frac{(x-\pi)^7}{7!} + \cdots$

$\Rightarrow -\frac{\sin x}{(x-\pi)} = 1 - \frac{(x-\pi)^2}{3!} + \frac{(x-\pi)^4}{5!} - \frac{(x-\pi)^6}{7!} + \cdots$

$\frac{\sin x}{x-\pi} = -1 + \frac{(x-\pi)^2}{3!} - \frac{(x-\pi)^4}{5!} + \frac{(x-\pi)^6}{7!} \cdots$

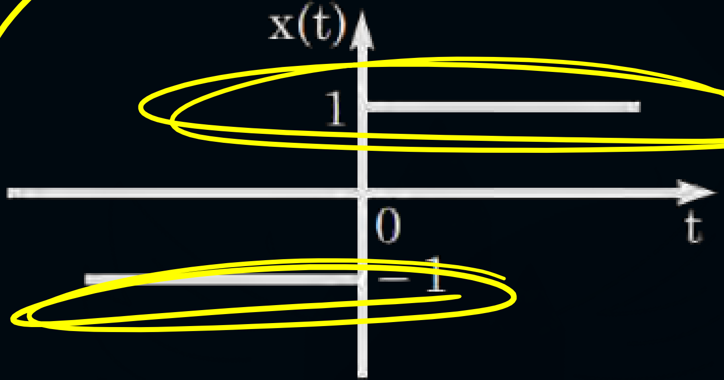
Topic : FOURIER SERIES

-ve

+ve

#Q. The function x(t) is shown in the figure. Even and odd parts of a unit-step function u(t) are respectively,

$-\frac{1}{2}$ $\frac{1}{2}$



A. $\frac{1}{2}, \frac{1}{2}x(t)$

B. $-\frac{1}{2}, \frac{1}{2}x(t)$

C. $\frac{1}{2}, -\frac{1}{2}x(t)$

D. $-\frac{1}{2}, -\frac{1}{2}x(t)$

Even part $\rightarrow$ $\dfrac{u(t)+u(-t)}{2}$

Now, $u(t)=0\ ;\ t<0$

$=1\ ;\ t\geq 0$

$u(-t)=0,\ -t<0$

$=1,\ -t\geq 0$

$u(-t)=1\ ;\ t\leq 0$

$0\ ;\ t>0$

$\dfrac{u(t)+u(-t)}{2}=\dfrac{1}{2}\ ;\ t\leq 0$

$=\dfrac{1}{2}\ ;\ t>0$

$$\underline{\underline{\text{Even}\left[u(t)\right]=\frac{1}{2}}}$$

Odd $u(t)=\dfrac{u(t)+u(-t)}{2}$

$=\begin{cases}-\dfrac{1}{2}, & t\leq 0\end{cases}$

$\frac{1}{2}, t > 0 \Big]$

$\frac{x(t)}{2}$ from given figure

THANK - YOU

Topics to be Covered