GATE
DATA SCIENCE AND AI

2024

CALCULUS AND OPTIMIZATION
SINGLE VARIABLE CALCULUS

Lecture No.- 10

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Recap of previous lecture

Topic Taylor series

Topic Maxima and Minima

Topics to be covered

Topic Maxima and minima

Topic : Single Variable Calculus

DA
DS
TN. Imp

#Q. The shortest distance between the line $y - x = 1$ and the curve $x = y^2$ is :

$y - x = 1$
$x = y^2$

A $\frac{3\sqrt{2}}{2}$

B $\frac{3\sqrt{4}}{4}$

C $\frac{3\sqrt{2}}{8}$

D $\frac{3\sqrt{2}}{16}$

$y^2 = 4ax$

$y^2 = x \qquad 4a = 1, \quad a = \frac{1}{4}$

$\longrightarrow$ Parametric co-ordinate

$(at^2, 2at)$ or $\boxed{(t^2, t)}$

$Y - x = 1$

$\frac{ax_1 + by_1 + c_1}{\sqrt{a^2+b^2}}$

$y - x = 1$

$x = y^2$

$\begin{cases} \text{Shortest} - \text{minima} \\ \text{Farthest} - \text{maxi} \end{cases}$

$y = x - 1 \longrightarrow t^2, t$

$= \frac{|-t^2 + t - 1|}{\sqrt{(-1)^2 + (1)^2}} = \frac{|-(t^2 - t + 1)|}{\sqrt{2}}$

$= \frac{t^2 - t + 1}{\sqrt{2}}$

$f(t) = t^2 - t + 1$

$f'(t) = 2t - 1 = 0$

$t = \frac{1}{2}$ min

Distance $= \frac{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right) + 1}{\sqrt{2}} = \frac{3\sqrt{2}}{2 \times 4}$

$= \frac{3\sqrt{2}}{8}$

Topic : Single Variable Calculus

#Q. The function $y = x^x$ has :

$y = x^x$

$y = x^x \longrightarrow$ Local max or Local min

$\frac{dy}{dx} =$

A. No local minimum value

B. A local minimum value at $x = \frac{1}{e}$

C. No local maximum value

D. A local maximum value at $x = \frac{1}{e}$

$$y = x^x = (\text{function})^{\text{function}}$$

both sides Taking log

$$\log y = x \log x$$

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \log x \cdot 1$$

$$\boxed{\frac{dy}{dx} = (1 + \log x) \cdot x^x}$$

$$1 + \log x = 0$$

$$\log x = -1$$

$$\log x = -\log_e e$$

$$\log x = \log_e e^{-1}$$

$$\boxed{x = \frac{1}{e}}$$

$$f''\left(\frac{1}{e}\right) > 0$$

minima.

Topic : Single Variable Calculus

#Q. The area of the greatest isosceles triangle that can be inscribed in a given ellipse having its vertex coincident with one end of the major axis.

A $\frac{\sqrt{3}}{4}ab$

B $\frac{\sqrt{3}}{2}ab$

C $\frac{3\sqrt{3}}{4}ab$

D $\sqrt{3}ab$

Topic : Single Variable Calculus

#Q. The semi-vertical angle of a cone of given total surface and maximum volume is :

H.W

A $\sin^{-1}\left(\frac{1}{3}\right)$

B $30°$

C $\sin^{-1}\left(\frac{1}{4}\right)$

D $60°$

Topic : Single Variable Calculus

#Q. A box of constant volume c is to be twice as long as it is wide. The cost per unit area of the material on the top and four sides is three times the cost for bottom. The are the most economical height of the box is :

H.W

A $\left(\frac{9c}{16}\right)^{1/3}$

B $\left(\frac{9c}{32}\right)^{1/3}$

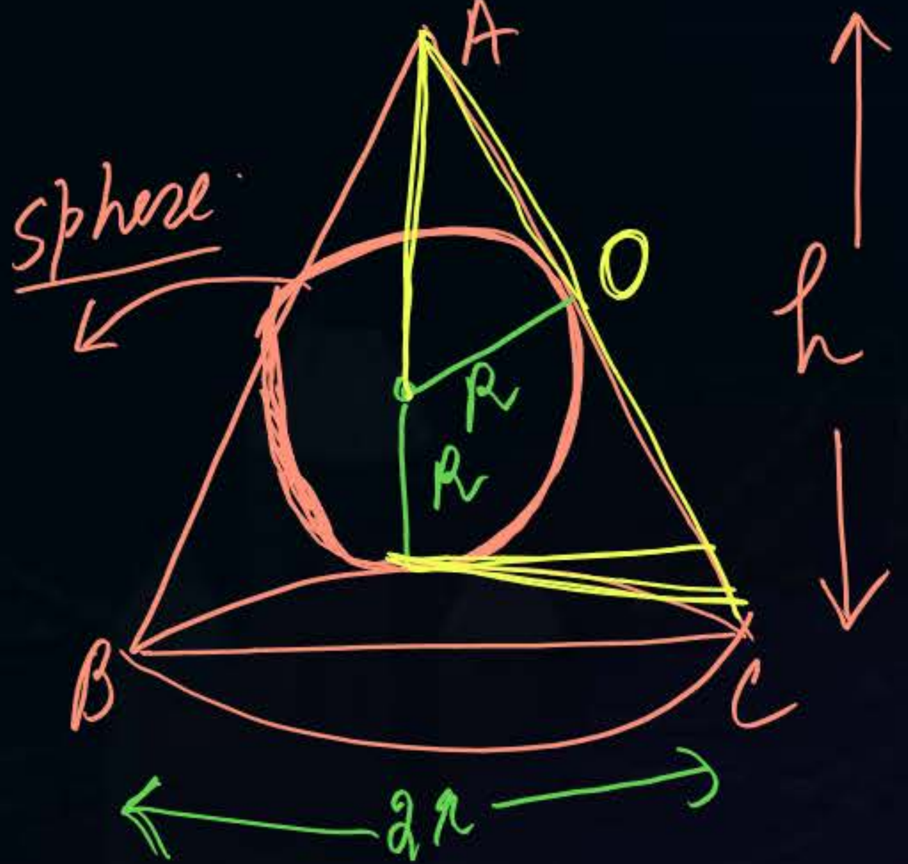
C $\left(\frac{16c}{81}\right)^{1/3}$

D $\left(\frac{32c}{81}\right)^{1/3}$

Do yourself

#Q. A cone is circumscribed about a sphere of radius R. The volume of the cone is minimum if its height is :

Hint – using similar
1) Triangle
2) volume – $V = \frac{1}{3}\pi r^2 h$



A 3 R

B 4 R

C 5 R

D $2\sqrt{2}\,R$

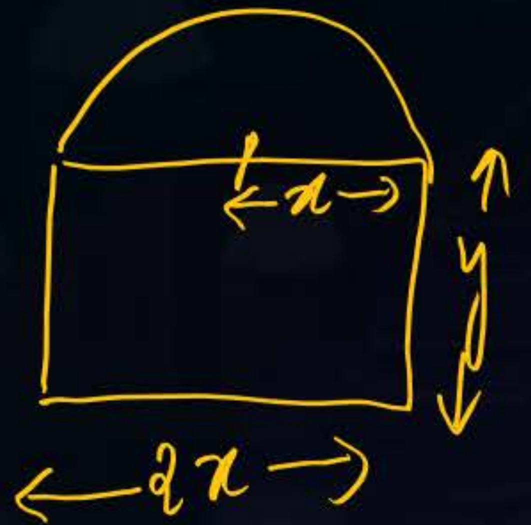
Topic : Single Variable Calculus

Imp Read - 2 times

Any How

#Q. A window is in the form of a rectangle surmounted by a semi-circle. The total area of window is fixed. What should be the ratio of the area of the semi-circular part and the rectangular part so that the total perimeter is minimum ?

$\frac{\pi}{4}$



A $\frac{\pi}{4}$

B $\frac{1}{4}$

C $\frac{3\pi}{4}$

D $\frac{1}{2}$

Area of whole geometrical figure

$$A = \underset{\text{Area}}{\text{Rectangle}} + \text{Area of semicircle}$$

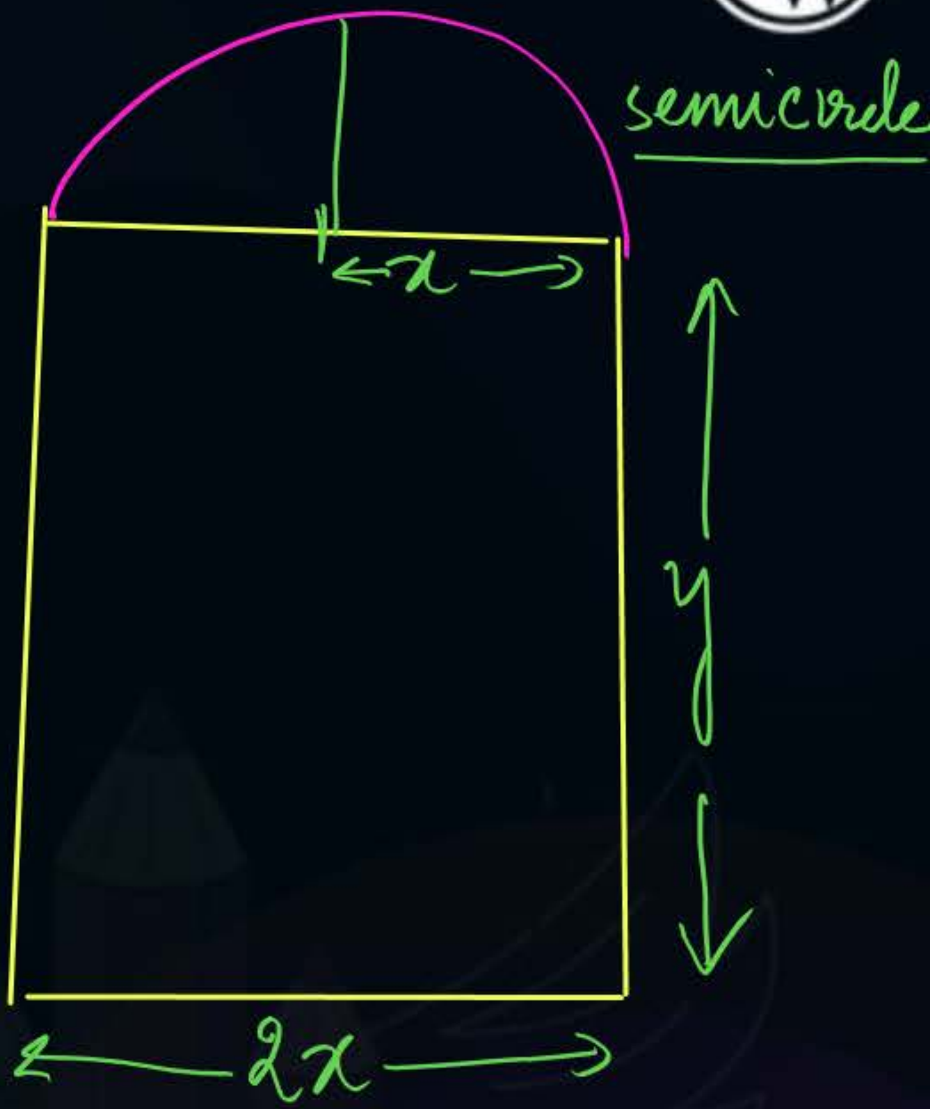
$$\boxed{A = 2x \cdot y + \frac{\pi}{2} \cdot x^2} \quad \text{—} \quad (1)$$

$$\text{Perimeter} = \underbrace{2x + 2y}_{\substack{\text{Rectangle} \\ \text{Perimeter}}} + \underbrace{\pi x}_{\substack{\text{circle} \\ \text{perimeter}}}$$

Whole function convert in x

$$\boxed{P(x) = 2x + 2y + \pi x} \quad \text{—} \quad (2)$$

$$P(x) = 2x + \pi x + \bigcirc$$



$$A = 2xy + \frac{\pi}{2}x^2$$

$$\boxed{\frac{1}{x}\left(A - \frac{\pi}{2}x^2\right) = 2y}$$

$$P(x) = 2x + 2y + \pi x$$

$$\boxed{P(x) = 2x + \pi x + \frac{1}{x}\left(A - \frac{\pi}{2}x^2\right)}$$

Perimeter

→ minimize (optimization)

$$\frac{dP}{dx} = 2 + \pi + \left(\frac{-A}{x^2} - \frac{\pi}{2}\right)$$

$$\frac{dP}{dx} = 0$$

$$(2+\pi) = \frac{A}{x^2} + \frac{\pi}{2}$$

$$x \Rightarrow \sqrt{\frac{2A}{4+\pi}}$$

$$\frac{A}{x} - \frac{\pi}{2}\frac{\cancel{x^2}}{\cancel{x}}$$

$$\frac{A}{x} - \frac{\pi}{2}x$$

$$= -\frac{A}{x^2} - \frac{\pi}{2}$$

$$(2+\pi) - \frac{\pi}{2} = \frac{A}{x^2}$$

$$= \frac{4+2\pi-\pi}{2} = \frac{A}{x^2}$$

$$= \frac{4+\pi}{2} = \frac{A}{x^2}$$

$$= \frac{2}{4+\pi} = \frac{x^2}{A}$$

$$\boxed{x = \sqrt{\frac{2A}{4+\pi}}}$$

$x = \sqrt{\frac{2A}{4+\pi}}$ $\quad f''\left(\frac{2A}{4+\pi}\right) > 0$ min at

$x = \sqrt{\frac{2A}{4+\pi}}$

Area of semicircle

$= \frac{\pi}{2} x^2$

$= \frac{\pi}{2} \sqrt{\frac{2A}{4+\pi}}$

$= \frac{\pi}{\cancel{2}} \times \frac{\cancel{2}A}{4+\pi}$

Area of $\bigcirc$ $= \frac{\pi A}{4+\pi}$

Rectangle Area $A = A - \frac{\pi A}{4+\pi} = \frac{(4+\pi)A - \pi A}{4+\pi} = \frac{4A + \cancel{\pi A} - \cancel{\pi A}}{4+\pi}$

$= \frac{4A}{4+\pi}$

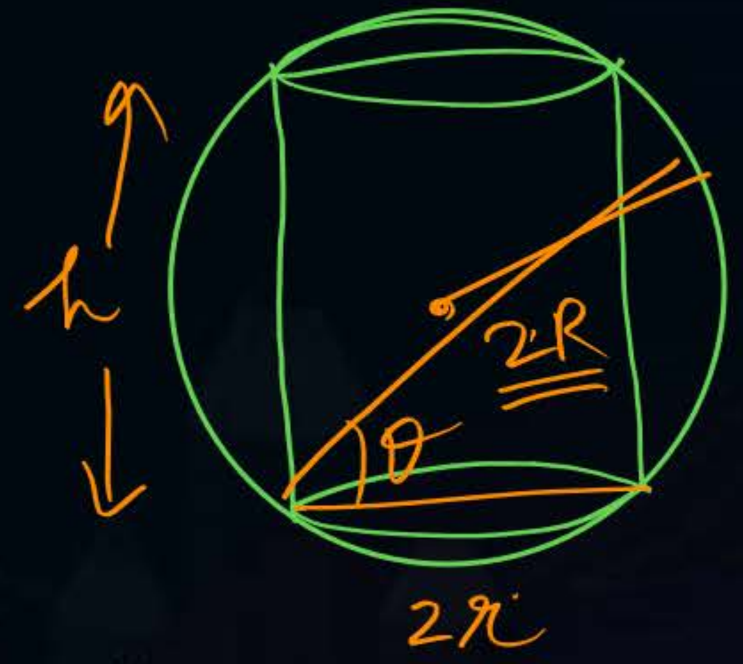
ratio $= \frac{\pi}{4}$

Topic : Single Variable Calculus

#Q. The maximum surface area of a cylinder that can be inscribed in a given sphere of radius R is

Surface Area of cylinder

$= 2\pi r \underline{h} + 2\pi r^2$



A $\pi R^2(1+\sqrt{5})$

B $\pi R^2(\sqrt{5}-1)$

C $\pi R^2(1+\sqrt{3})$

D $\pi R^2(\sqrt{3}-1)$

Surface Area of cylinder

$$S.A = 2\pi r h + 2\pi r^2$$

$$S.A = \pi \cdot (2R\cos\theta)(2R\sin\theta) + 2\pi \cdot (R^2\cos^2\theta)$$

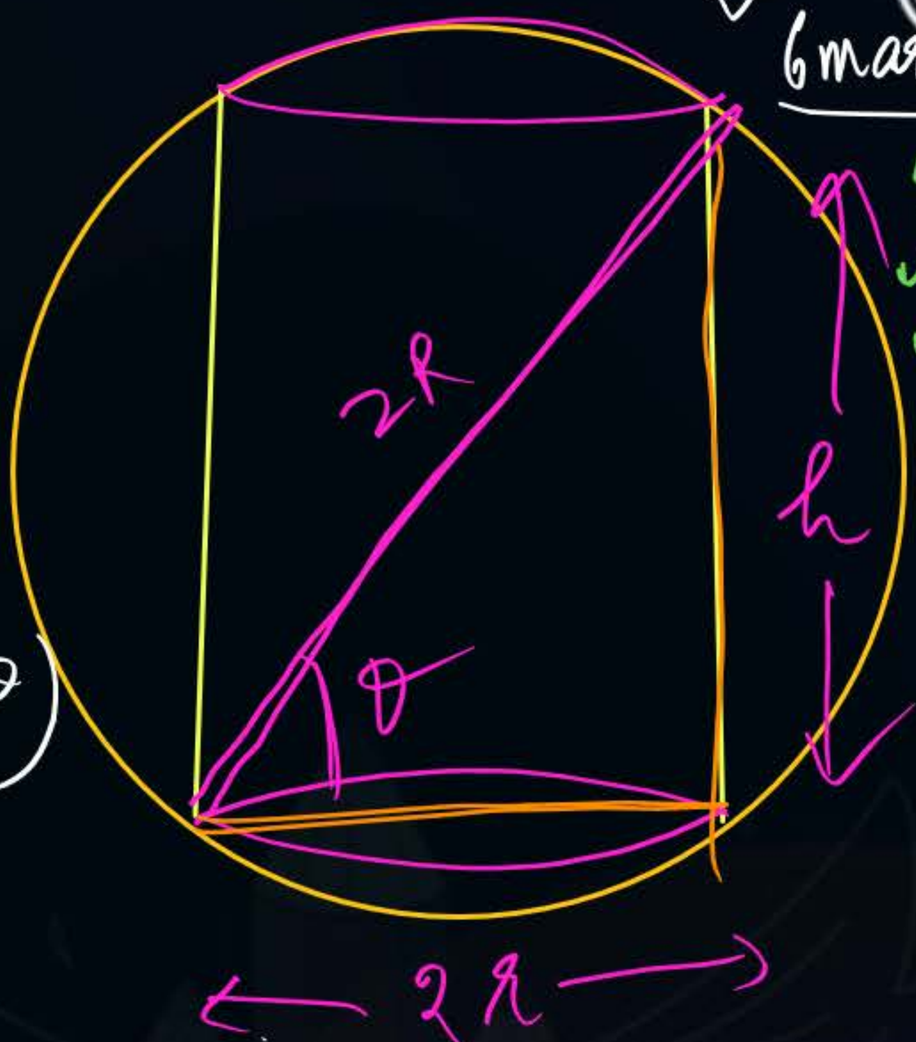
$$\boxed{S.A = \pi \cdot 4R^2 \cos\theta \sin\theta + 2\pi R^2 \cos^2\theta}$$

$$\frac{ds}{d\theta} = \pi \cdot 4R^2 \left[\frac{\cos 2\theta}{4}\right) + 2\pi R^2 (-2\cos\theta \sin\theta)$$

$$\frac{ds}{d\theta} = 0 \qquad f'(s,\theta) = 0$$

$$\pi(4R^2)\frac{\cos 2\theta}{4} + 2\pi R^2\left(-\frac{\sin 2\theta}{4}\right) = 0$$

$$\boxed{\theta = \frac{1}{2}\tan^{-1} 2}$$



6 marks
5-7
Lim
Max
Conti

$2r = 2R\cos\theta$

$h = 2R\sin\theta$

$$\boxed{S_{max} = \pi R^2(1+\sqrt{5})}$$

2 mins Summary

Topic One

Topic Two

Topic Three

Topic Four

Topic Five

Optimization ✓

Theory

DPP sat ✓ $\boxed{\text{DPP02}}$

SUN $\boxed{\text{DPP03}}$

Major 2

9 to 11

11:30 to 1:30

SAT Likelihood Estimation

✗ ✓ ✓ ✓ ✓

2 3 4 5

1) Today to till sat Task
DPP02

DPP02

$\boxed{\text{random variables}}$

Code.org ✓

THANK - YOU

[redacted]