GATE
DATA SCIENCE AND AI

CALCULUS AND OPTIMIZATION
SINGLE VARIABLE CALCULUS

Lecture No.- 09

By- Rahul sir

Recap of previous lecture

Topic — Taylor series

Topic — Maxima and Minima

or Optimization

Topics to be covered

Topic Maxima and minima

Taylor SERIES :

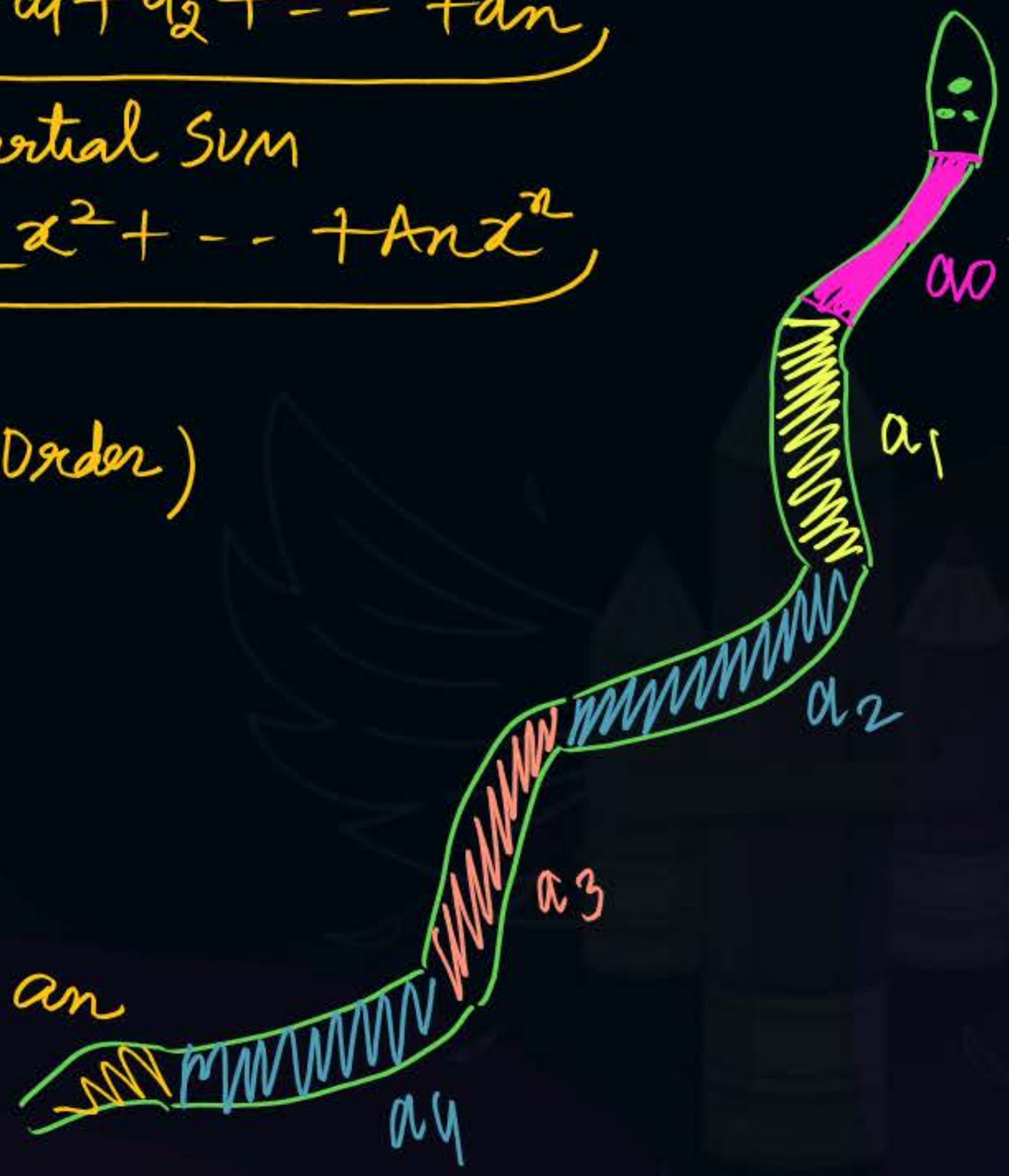
Calculus optimization $\boxed{09 \text{ to } 11}$ ✓

(A) Maclaurins Series :

SUM of n^{th} Term $= \underbrace{a_0 + a_1 + a_2 + \cdots + a_n}_{\text{Partial Sum}}$

SERIES $f(x) = \underbrace{A_0 + A_1 x + A_2 x^2 + \cdots + A_n x^n}_{n^{th} \text{ Sum}}$

(Increasing Order)



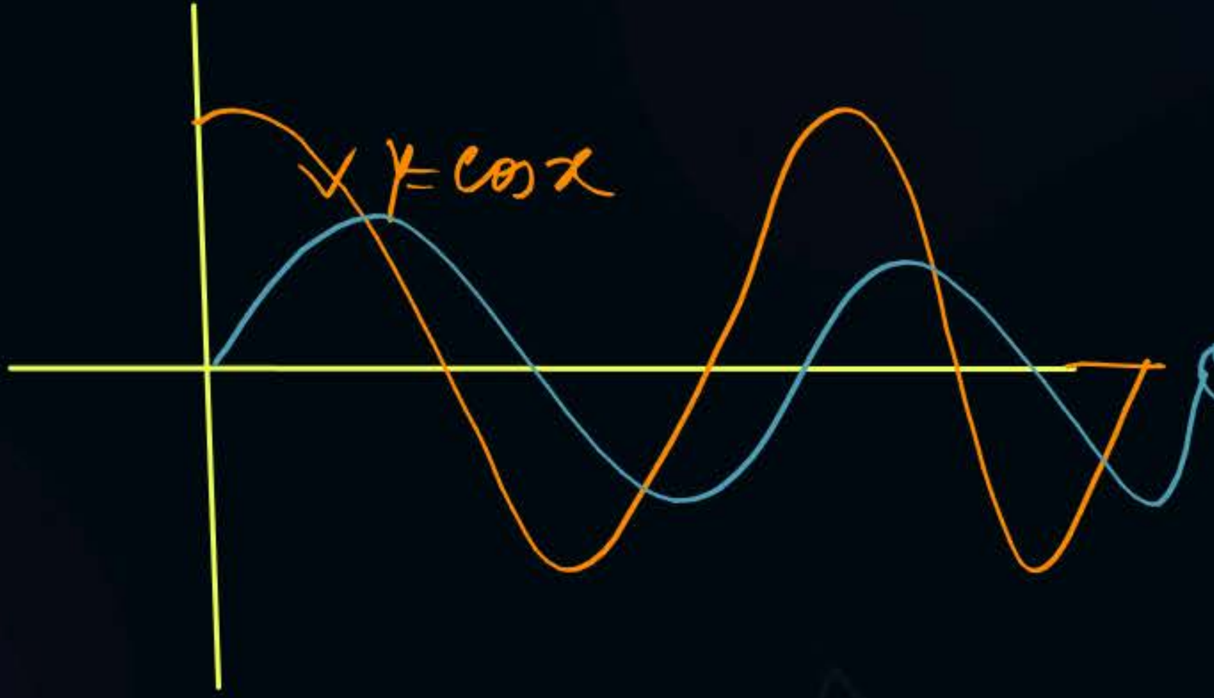
Snake growth in terms of Sequence.

$$f(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_n x^n \text{ — SUM}$$

[Number line]

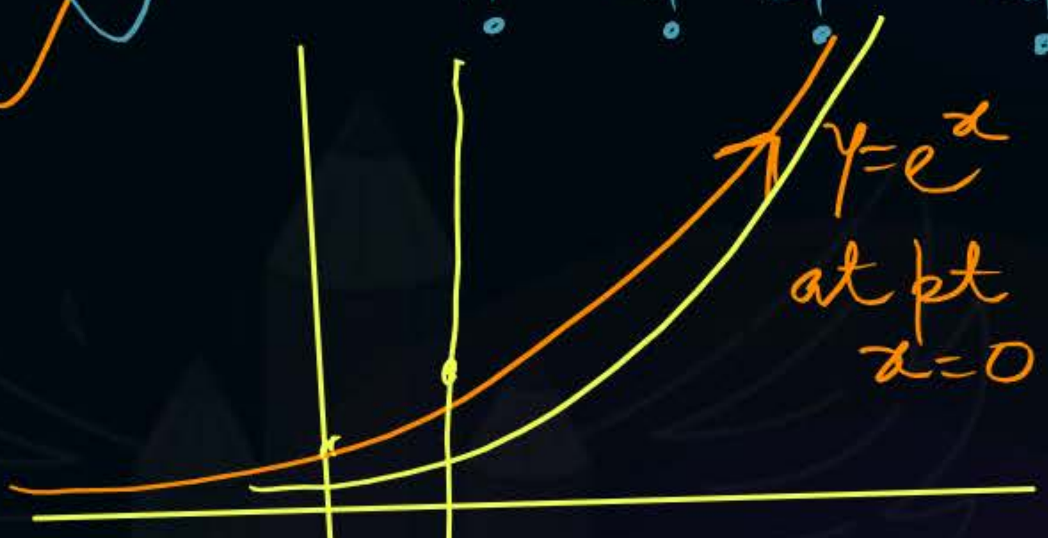
SERIES – $f(x)$
at $x = 0$ = problem (A)

SERIES – $f(x)$
at $x = a$ = problem No (2)

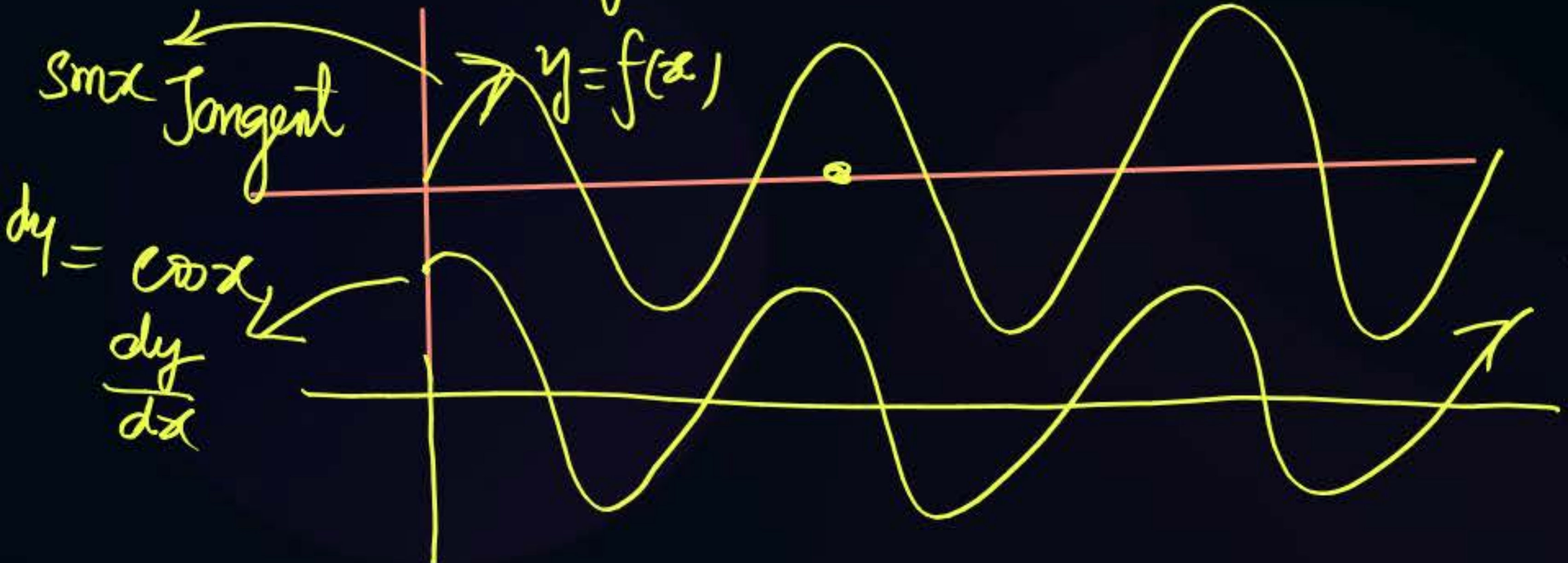


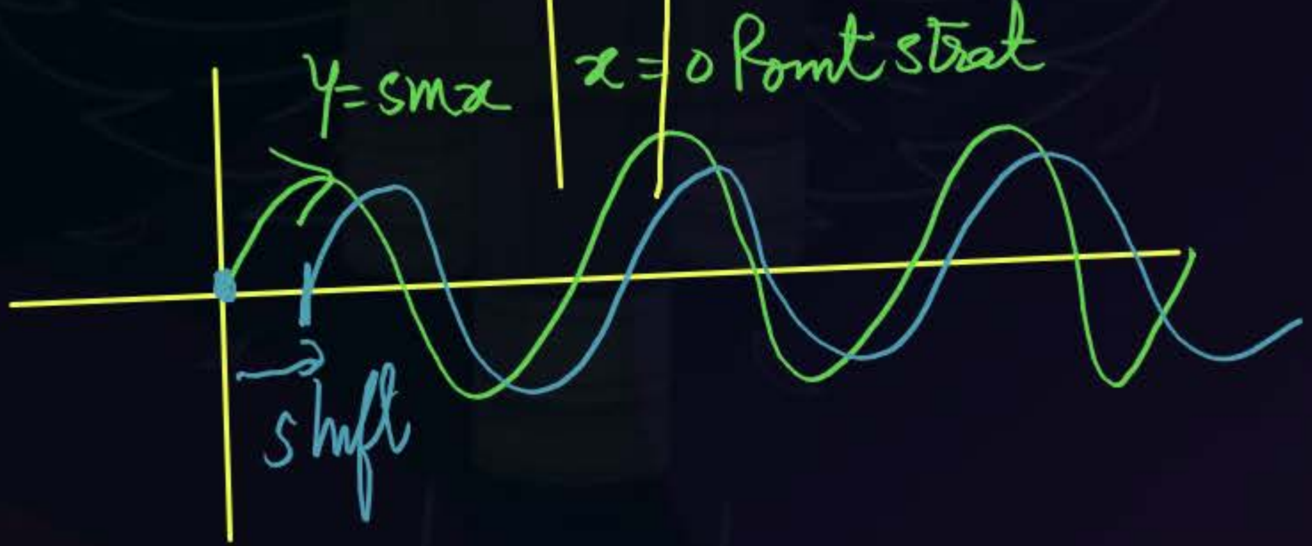
$f(x) = e^x$ → SERIES
at $x = 0$

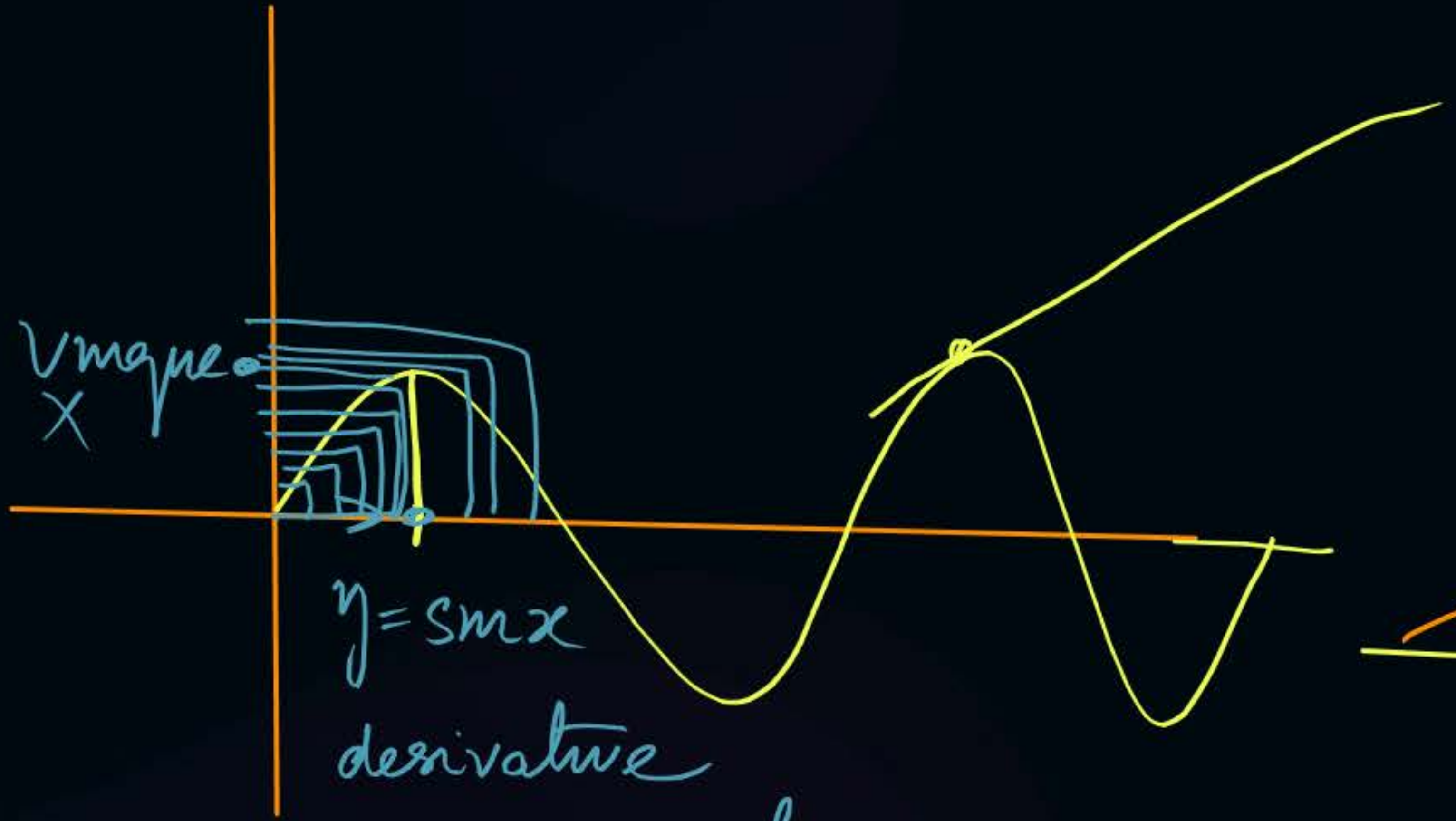
$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$$



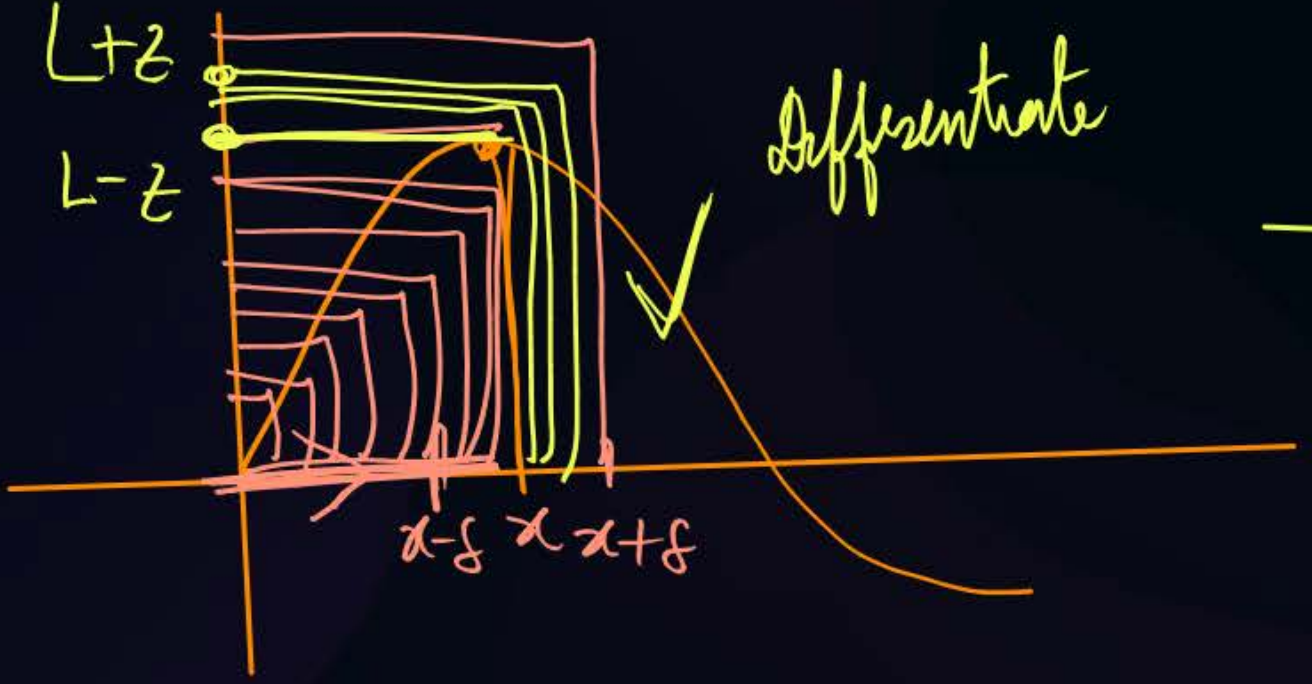
problem Identified ⟶ $f(x)$ = Differentiate

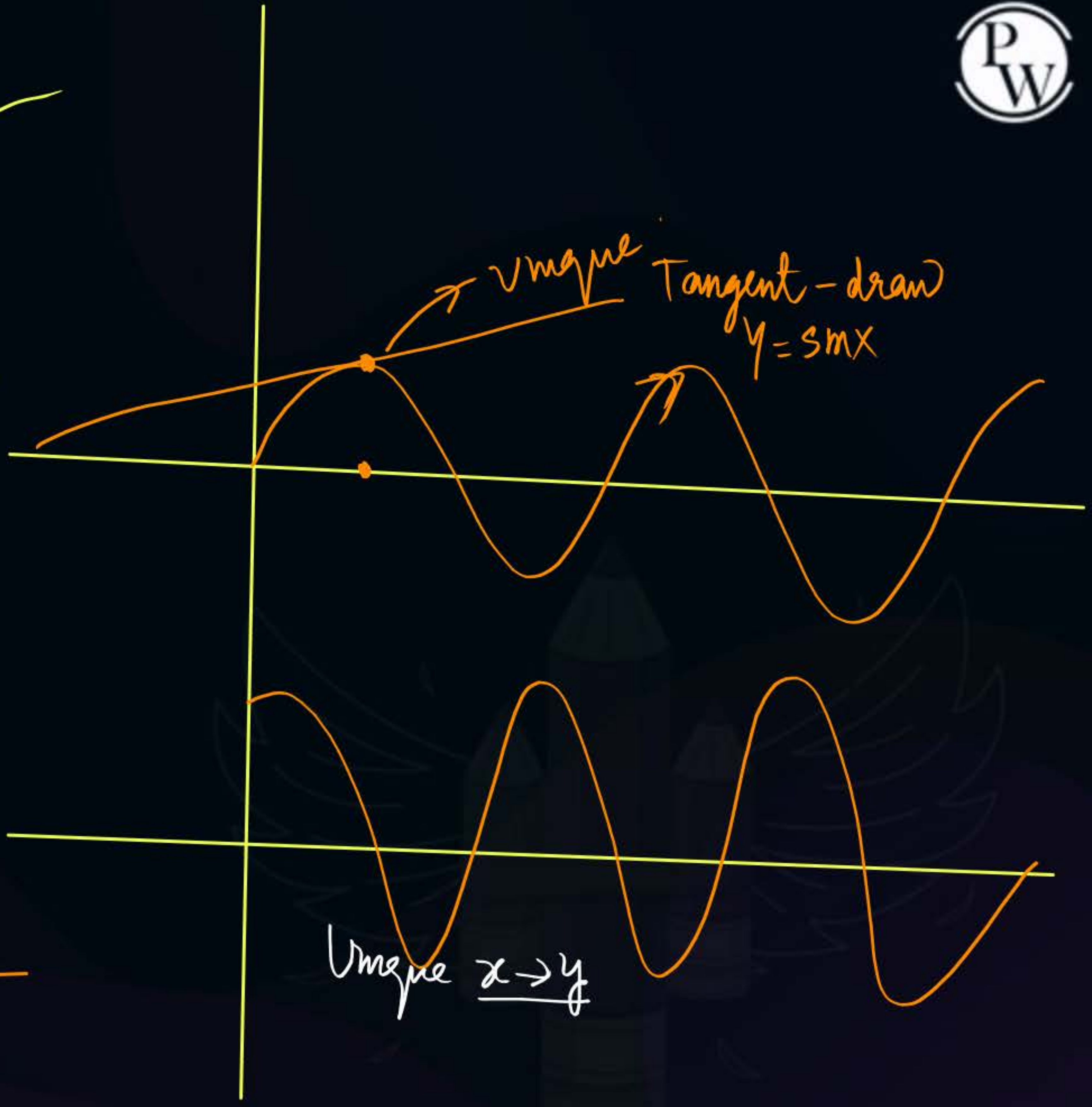






derivative
= unique value





$$f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots + A_n x^n$$

Differentiate w.r.t to x

$$f'(x) = A_1 + 2A_2 x + 3A_3 x^2 + \cdots$$

Again Diff.

$$f''(x) = 2A_2 + 6A_3 x + \cdots$$

$$f'''(x) = 6A_3 + \cdots$$

$x = 0$ Point

$$f(0) = A_0$$

$$\frac{f'(0)}{1!} = A_1$$

$$\frac{f''(0)}{2!} = A_2$$

$$A_3 = \frac{f'''(0)}{3!}$$

$$A_4 = \frac{f^{iv}(0)}{4!}$$

Put the value of $A_0, A_1, A_2, A_3, A_4 \cdots$

$$\boxed{f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots}$$

Maclaurins Series

at $x = 0$

$e^x = e^0 + \frac{x}{1!}e^0 + \frac{x^2}{2!}e^0 + \frac{x^3}{3!}e^0 + \frac{x^4}{4!}e^0 + \cdots$

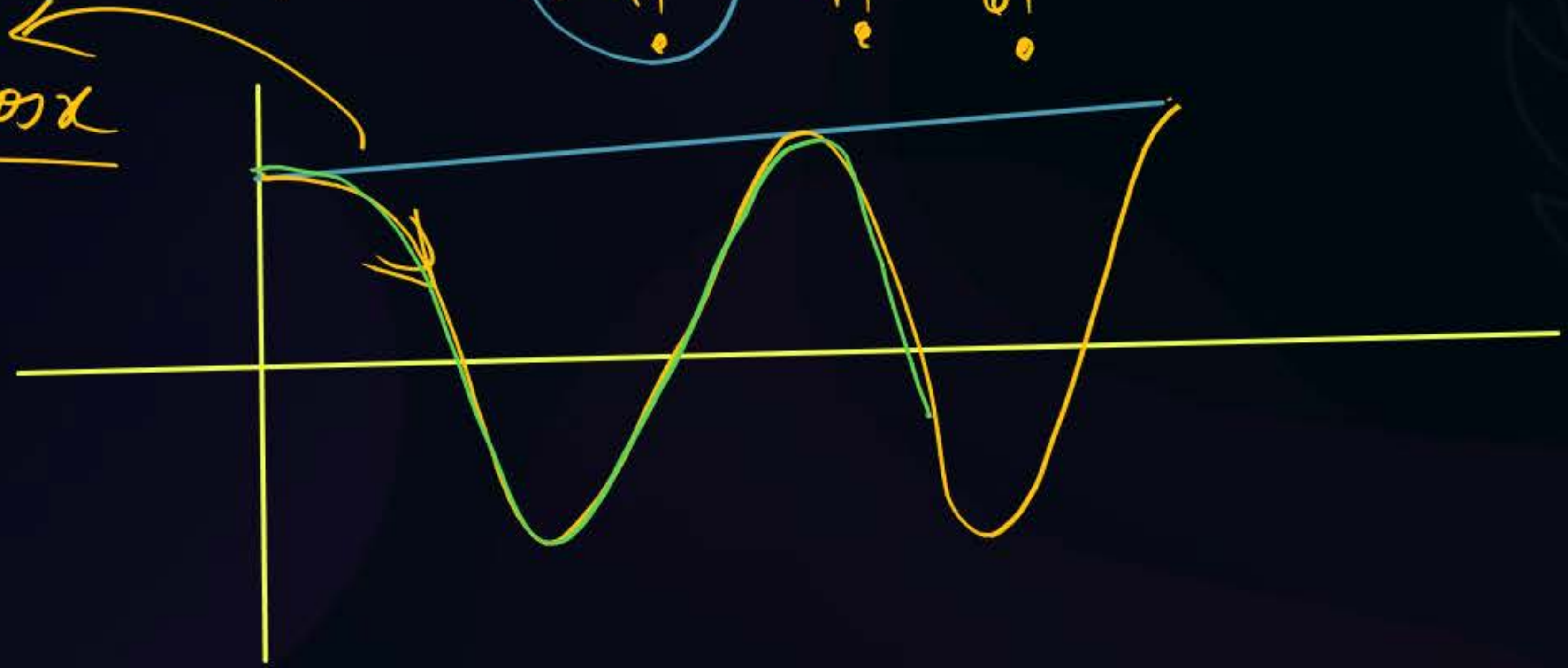
$$\boxed{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}}$$

☆ $f(x)$ Differentiable (must be)

$\cos x = \cos 0 + \frac{x}{1!}(-\sin 0) + \frac{x^2}{2!}(-\cos 0) + \frac{x^3}{3!}(\sin 0) + \frac{x^4}{4!}(+\cos 0) + \cdots$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$\cos x$



$$\boxed{e^{-x},\ \log x,\ \sin x,\ (1+x)^n}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}$$

$$e^{0.1} = 1 + \frac{(0.1)}{1!} + \frac{(0.1)^2}{2!} + \frac{(0.1)^3}{3!} + \cdots$$

at Point $x=a$ $(x-a)$ term

SERIES

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + A_3(x-a)^3 + - + A_n(x-a)^n$$

at point a

Diff. w.r.t to x

$$f'(x) = A_1 + 2A_2(x-a) + 3A_3(x-a)^2 + - - -$$

$$f''(x) = 2A_2 + 6A_3(x-a) + - - -$$

at point $\underline{x=a}$

$f(a) = A_0$ $\quad A_2 = \dfrac{f''(a)}{2!}$

$\dfrac{f'(a)}{1!} = A_1$ $\quad A_3 = \dfrac{f'''(a)}{3!}$

$$f(x) = f(a) + (x-a)f'(a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Problem No 2

at $x=a$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Taylor's SERIES at $x=a$

If $\underline{a=0}$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \cdots$$

<u>Maclaurin Series</u>

<u>Condition</u> $\rightarrow$ only allowed for differentiable function

Topic : Single Variable Calculus

10 AM to 12 AM

#Q. Suppose that the amount of money in a bank account after t years is given by

$$\boxed{A(t) = 2000 - 10te^{5-\frac{t^2}{8}}}$$

The minimum and maximum amount of money in the account during the first 10 years that it open occur respectively at:

[0, 10] Interval

A. t = 2, t = 0

B. t = 2, t = 10

C. t = 0, t = 2

D. t = 10, t = 0

$$A(t) = \underline{2000} - 10\underbrace{t\,e^{5-\frac{t^2}{8}}}_{\mathrm{I}\ \mathrm{II}}$$

[0, 10] global max

$$A'(t) = 0$$

$$A'(t) = 0 - 10\left[t \cdot \frac{d}{dt}\left(e^{5-\frac{t^2}{8}}\right) + e^{\left(5-\frac{t^2}{8}\right)} \cdot \frac{d}{dt}(t)\right]$$

$$A'(t) = -10\left[t \cdot e^{5-\frac{t^2}{8}}\left(0 - \frac{\cancel{2}t}{\cancel{8}}\right) + e^{\left(5-\frac{t^2}{8}\right)} \cdot 1\right]$$

$$= -10e^{5-\frac{t^2}{8}}\left[t\left(-\frac{t}{4}\right) + 1\right]$$

$$A'(t) = \underline{-10e^{5-\frac{t^2}{8}}}\left[\frac{-t^2}{4} + 1\right] = \underline{0}$$

$$\boxed{A'(t) = 0}$$

$$-\frac{t^2}{4} = -1$$

$$\frac{t^2}{4} = 1$$

$$t = \pm 2$$

$$t = 2$$

Critical pt

$$f(2) = \checkmark$$

$$f(0) = \checkmark$$

$$f(10) = \checkmark$$

Topic : Single Variable Calculus

H.W

MSQ

multiple select Question

#Q. The function $Q(y) = 3y\,(y+4)^{2/3}$ on $y \in [-5, -1]$ has :

A 0 as $\underbrace{\text{absolute}}_{\text{global}}$ maximum value

B $\underbrace{\text{absolute}}_{\text{global}}$ maximum value at $y = -1$

C -15 as $\underbrace{\text{absolute}}_{\text{global}}$ minimum value

D $\underbrace{\text{absolute}}_{\text{global}}$ minimum value at $y = -\frac{12}{5}$

#Q. The third term in the Taylor's series expansion of e^x about 'a' would be ____.

$e^x = f(x)$

$f'(x) = e^x$

$f''(x) = e^x$

$f''(a) = e^a$

Third term $= f''(a)\dfrac{(x-a)^2}{2!} = \dfrac{e^a}{2!}(x-a)^2 = \dfrac{e^a}{2}(x-a)^2$

A $e^a(x-a)$

B $\dfrac{e^a}{2}(x-a)^2$

C $\dfrac{e^a}{2}$

D $\dfrac{e^a}{6}(x-a)^3$

Topic : Single Variable Calculus

#Q. The Taylor series expansion of sin x about $x = \frac{\pi}{6}$ is given by

A $\frac{1}{2}+\frac{\sqrt{3}}{2}\left(X-\frac{\Pi}{6}\right)-\frac{1}{4}\left(X-\frac{\Pi}{6}\right)^2-\frac{\sqrt{3}}{12}\left(X-\frac{\Pi}{6}\right)^3+\ldots$

B $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$

C $\frac{x-\frac{\Pi}{6}}{1!}-\frac{\left(x-\frac{\Pi}{6}\right)^3}{3!}+\frac{\left(x-\frac{\Pi}{6}\right)^5}{5!}-\frac{\left(x-\frac{\Pi}{6}\right)^7}{7!}$

D $\frac{1}{2}$

$\sin x =$ expansion $a = \frac{\pi}{6}$

$= f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \ldots$

$= \sin\left(\frac{\pi}{6}\right) + \left(x-\frac{\pi}{6}\right)\left[\cos\left(\frac{\pi}{6}\right)\right] + \frac{(x-\pi/6)^2}{2!}\left(-\sin\frac{\pi}{6}\right)$

#Q. For the function e^{-x}, the $\boxed{\text{linear approximation}}$ around x = 2 is

" Linear approximation "

using Taylor Series

$$f(x) = \underbrace{\boxed{f(a) + (x-a)f'(a)}}_{\text{"Linear approximation"}} + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

Linear approximation

$= f(a) + (x-a) f'(a)$

$= e^{-2} + (x-2)e^{-2}(-1)$

$= e^{-2}(3-x)$

A $(3 - x)e^{-2}$

B $1 - x$

C $\left[3+2\sqrt{2}-\left(1+\sqrt{2}\right)x\right]e^{-2}$

D e^{-2}

Topic : Single Variable Calculus

#Q. In the Taylor series expansion of $(e^x + \sin x)$ about the point $x = \pi$, the coefficient of $(x - \pi)^2$ is

$$= \frac{f''(a)}{2!}(x-\pi)^2$$

$$= \frac{e^{\pi}}{2!}(x-\pi)^2$$

$$= \frac{e^{\pi}}{2!}$$

$$f(x) = e^x + \sin x$$

$$f''(x) = e^x - \sin x$$

$$f''(\pi) = e^{\pi} - \sin \pi$$

$$= e^{\pi}$$

A e^{π}

B $0.5e^{\pi}$

C $e^{\pi} + 1$

D $e^{\pi} - 1$

$f(x) = \frac{\sin x}{(x-\pi)}$ at $x=\pi$

If $x=\pi$ — Differentiation $\longrightarrow$ Not defined

$f(x) = \frac{\sin x}{(x-\pi)} \longrightarrow$ Not apply Taylor SERIES

Remove The Denominator

$x-\pi = t$

$x = t+\pi$

$\to t = 0$

$$f(t) = \frac{\sin(\pi+t)}{t} = -\frac{\sin t}{t} = -\frac{1}{t}\left[\sin t\right] = -\frac{1}{t}\left[t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\cdots\right]$$

$$= -1+\frac{(x-\pi)^2}{3!}-\frac{(x-\pi)^4}{5!}+\cdots \qquad = -1+\frac{t^2}{3!}-\frac{t^4}{5!}+\cdots$$

Topic : Single Variable Calculus

#Q. The infinite series $1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots$ corresponds to

H.W

A. $\sec x$

B. e^x

C. $\cos x$

D. $1+\sin^2 x$

Topic : Single Variable Calculus

$x=0$

#Q. The Taylor series expansion of $(3 \sin x + 2 \cos x)$ is

expand

A $2+3x-x^2-\frac{x^3}{2}+\ldots$

B $2-3x+x^2-\frac{x^3}{2}+\ldots$

C $2+3x+x^2+\frac{x^3}{2}+\ldots$

D $2-3x-x^2+\frac{x^3}{2}+\ldots$

Topic : Single Variable Calculus

H.W

#Q. Let $f(x) = e^{x+x^2}$ for real x. From among the following, choose the Taylor series approximation of f(x) around x = 0, which includes all powers of x less than or equal to 3.

A $1 + x + x^2 + x^3$

B $1 + x + \frac{3}{2}x^2 + x^3$

C $1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3$

D $1 + x + 3x^2 + 7x^3$

#Q. Taylor series expansion of $f(x)=\int_0^x e^{-\left(\frac{t^2}{2}\right)} dt$ around $x = 0$ has the form $f(x) = a_0 + a_1x + a_2x^2 + \ldots..$

The coefficient a_2 (correct to two decimal places) is equal to ____.

$f(x)=\int_0^x e^{-t^2/2} dt$ around $x=0$ $\quad f(0)=0$

$f(x)=a_0+a_1x+a_2x^2+ - - -$

a_2

$$f(x) = \int_{\phi(x)}^{\psi(x)} f(t)\,dt$$

Newton leibnitz Theorem.

$$f'(x) = f[\psi(x)]\frac{d}{dx}\psi(x) - f[\phi(x)]\frac{d}{dx}[\phi(x)]$$

$$f(x) = \int_0^x e^{-t^2/2}\,dt$$

$$f'(x) = e^{-x^2/2}\cdot 1 - 0$$

$$f'(x) = e^{-x^2/2}$$

$$f''(x) = -x e^{-\frac{x^2}{2}}$$

$$f(0) = \int_0^0 e^{-t^2/2} = 0$$

$$f'(0) = e^0 = 1$$

$$f''(0) = 0$$

$$\text{Coefficient} = \frac{f''(0)}{2!} = \frac{0}{2!} = 0 \quad \underline{\text{Ans}}$$

2 mins Summary

Topic One — Taylor SERIES

Topic Two — Maclaurins Series

Topic Three ✓ Max/min

Topic Four

Topic Five

THANK - YOU