GATE

DATA SCIENCE AND AI

2024

CALCULUS AND OPTIMIZATION

SINGLE VARIABLE CALCULUS

Lecture No.- 07

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Recap of previous lecture

Topic Evaluation of limits

Topic Evaluation of limits, Mean value theorem

Topics to be covered

Topic Continuity of the function

Topic Differentiability of the functions

Topic Mean value theorem

Differentiability of Function

Differentiability = Smoothness of Function $\longrightarrow$ Derivative (Tangent) or Instaneous Rate

Derivative of $f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Property of Point — GEOMERTICAL Property

at $x = a$

Check the Differentiability

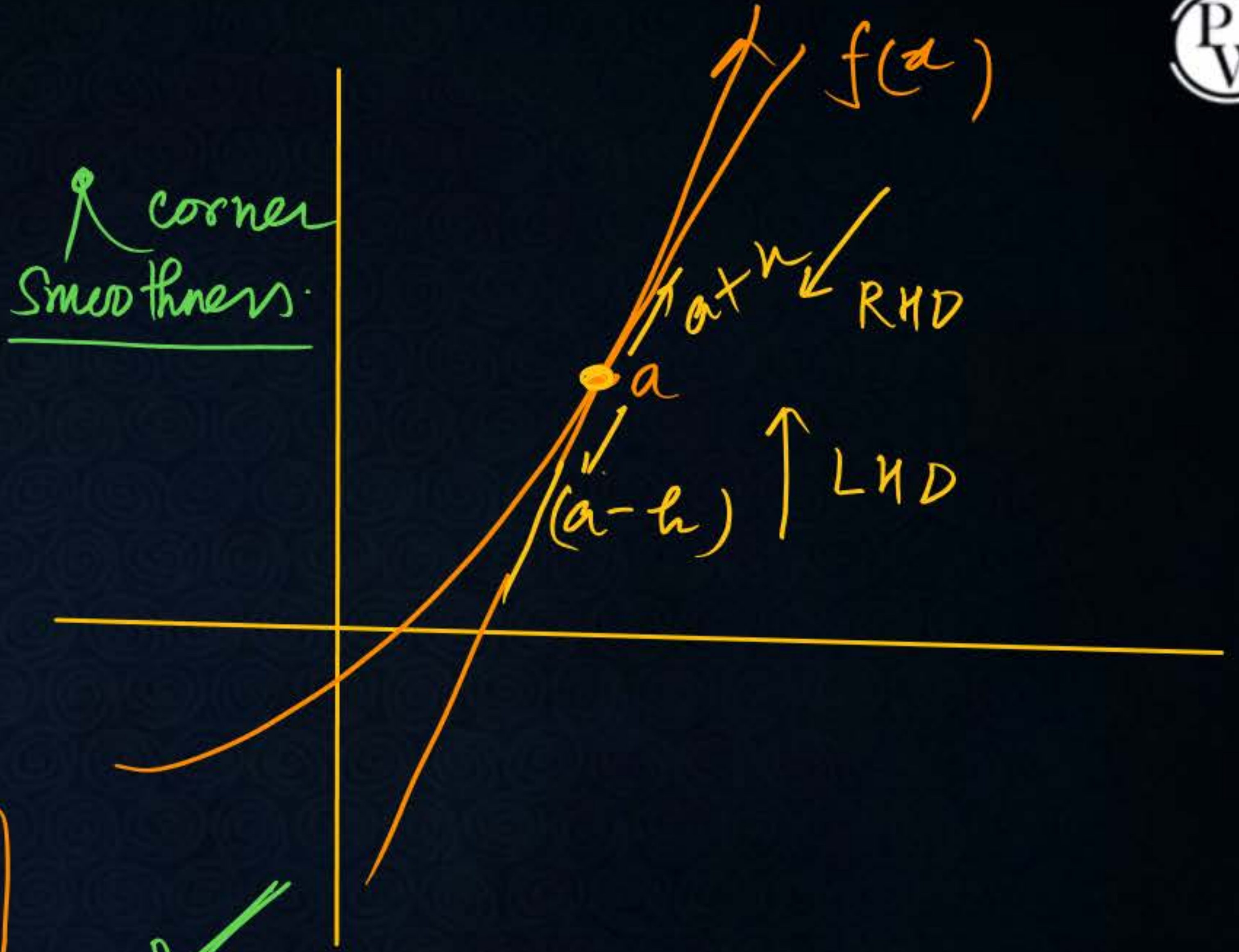
LHD = RHD

$$LHD = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h}$$

$$RHD = \lim_{h\to 0} \frac{f(a+h)-f(a)}{+h}$$

$$\boxed{\lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{+h}}$$

Condition for Differentiability



If function is conti
may or may not be Differentiable

If function is Differentiable Then function is $\boxed{\text{must be continuous.}}$

\# $y = f(x) = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

find the derivative of $|x|$

$\dfrac{dy}{dx} = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$

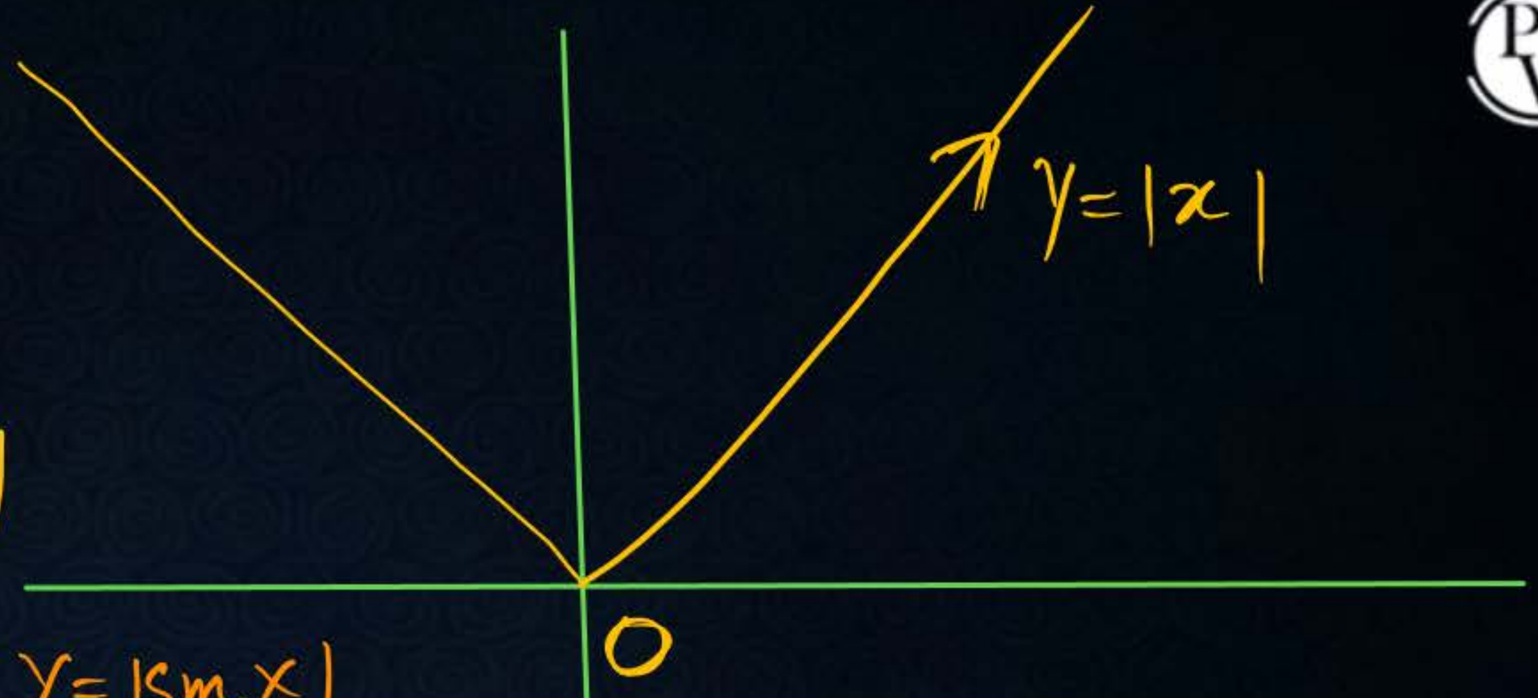
$y = |x|$

both sides squaring

✓ $y^2 = x^2$

Diff. w.r.t to x

$2y\dfrac{dy}{dx} = 2x$ $\quad \dfrac{dy}{dx} = \dfrac{x}{y}$ $\quad \dfrac{dy}{dx} = \dfrac{x}{|x|}$ or $\dfrac{|x|}{x}$



$y = |\sin x|$

$\dfrac{dy}{dx} = ?$

$\dfrac{dy}{dx} = \dfrac{x}{|x|}$ or $\dfrac{|x|}{x}$

If $x > 0$ 1

$x < 0$ -1

$x = 0$ undefined

$f'(x) = \lim\limits_{h \to 0} \dfrac{|x+h| - |x|}{h} = \dfrac{x}{|x|}$ or $\dfrac{|x|}{x}$

Topic : Single Variable Calculus

#Q. Show that the functions f (x) = $|x^2 - 4|$ is not differentiable at x =2.

$$f(x) = \begin{cases} x^2 - 4 & ; \quad x \le -2 \\ 4 - x^2 & ; \quad -2 < x < 2 \\ x^2 - 4 & ; \quad x \ge 2 \end{cases}$$

Topic : Single Variable Calculus

$\rightarrow$ Property Pt Diff.

Interval =Cont.

#Q. For the given functions: $f(x)=\begin{cases}\dfrac{x^2}{2} & ; \ 0\le x<1\\ 2x^2-3x+\dfrac{3}{2} & ; \ 1\le x\le 2\end{cases}$

$\boxed{A, B, C}$

$\boxed{MSQ}$

$f'(x)=\begin{cases}x & 0\le x\le 1\\ 4x-3 & 1\le x\le 2\end{cases}$

which of the following is (are) correct :

(A) at $x=1\rightarrow$ LHL RHL

LHL RHL (B)

A f(x) is continuous $\forall x\in[0,2]$

B f′(x) is continuous $\forall x\in[0,2]$

(C) X

C f″(x) is discontinuous at x =1

D f″(x) is continuous $\forall x\in[0,2]$

#Q. Discuss the differentiability f (x) at $\boxed{x = -1}$ if $f(x) = \begin{cases} 1 - x^2 ; & x \leq -1 \\ 2x + 2 ; & x > -1 \end{cases}$

$x = -1$

$f(-1) = 1 - x^2 = 1 - (-1)^2$
$= 1 - 1 = 0$

$f(x) = \begin{cases} 1 - x^2 & x \leq -1 \\ 2x + 2 & x > -1 \end{cases}$

Left Hand derivative

$LHD = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$

$= \lim_{h \to 0} \frac{f(-1-h) - 0}{-h} = \lim_{h \to 0} \frac{-h^2 - 2h}{-h}$

$f(x) = 1 - x^2$

$= \lim_{h \to 0} \frac{(h^2 + 2h)}{h} = \lim_{h \to 0} \frac{\cancel{h}(h+2)}{\cancel{h}} = 2$

LHD

$f(-1-h) = 1 - (-1-h)^2$
$= 1 - (1+h)^2 = \cancel{1} - (\cancel{1} + h^2 + 2h)$

$$f(x) = 2x + 2$$

$$f(-1+h) = 2(-1+h) + 2$$

$$RHD = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{2(-1+h) + 2 - 0}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{-2} \cancel{+2} + 2\cancel{h}}{\cancel{h}}$$

$$= 2$$

$$\left.\begin{aligned} LHD &= 2 \\ RHD &= 2 \end{aligned}\right] \text{at } x = -1$$

Diff.

Topic : Single Variable Calculus

#Q. At x = 0 the given functions. $f(x)=\begin{cases} x^2\sin\frac{1}{x} & ; \quad x\neq 0 \\ 0 & ; \quad x=0 \end{cases}$ is :

A. Discontinuous

B. Differentiable

C. Non-differentiable

D. None of these

$$f(x)=\begin{cases} x^2\sin\left(\frac{1}{x}\right) & x\neq 0 \\ 0 & x=0 \end{cases}$$

$$LHD = \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}$$

$$= \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{f(-h)-f(0)}{-h}$$

$$= \lim_{h\to 0}\frac{h^2\sin\left(\frac{1}{-h}\right)}{-h}$$

$$= \lim_{h\to 0}\frac{-h^2\sin\frac{1}{h}}{-h} = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0\times[-1 \text{ to } 1]$$

$$= 0$$

—— continuous

THIS function is diff

LHD = RHD

$$f(0)=0$$

$$f(-h) = h^2\sin\left(\frac{1}{-h}\right) = -h^2\sin\frac{1}{h}$$

$$f(0)=0$$

$$RHD = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = 0$$

LHD = RHD

function diff.

#Q. Show that f (x) = x |x| is differentiable x = 0.

$$f(x) = \begin{cases} -x^2 & ; \quad x \le 0 \\ x^2 & ; \quad x > 0 \end{cases}$$

$\boxed{\text{Differentiable at } x = 0}$

Topic : Single Variable Calculus

#Q. At x = 0 the given functions. $f(x)=\begin{cases} x\sin\left(\log x^2\right) & ; \quad x\neq 0 \\ 0 & ; \quad x=0 \end{cases}$ is :

$\boxed{f(a)=0}$

$LHD = \lim_{h\to 0} \frac{f(a-h)-f(a)}{-h} = \lim_{h\to 0} \frac{\cancel{-h}\sin\log h^2 - 0}{\cancel{-h}}$

$f(0-h) = f(-h) = -h\sin(\log(-h)^2)$

$= -h\sin\log h^2$

$= \lim_{h\to 0} \sin(\log h^2) =$ Not defined

does Not exist

Not differentiable

Function Not diff.

A Discontinuous

B Differentiable

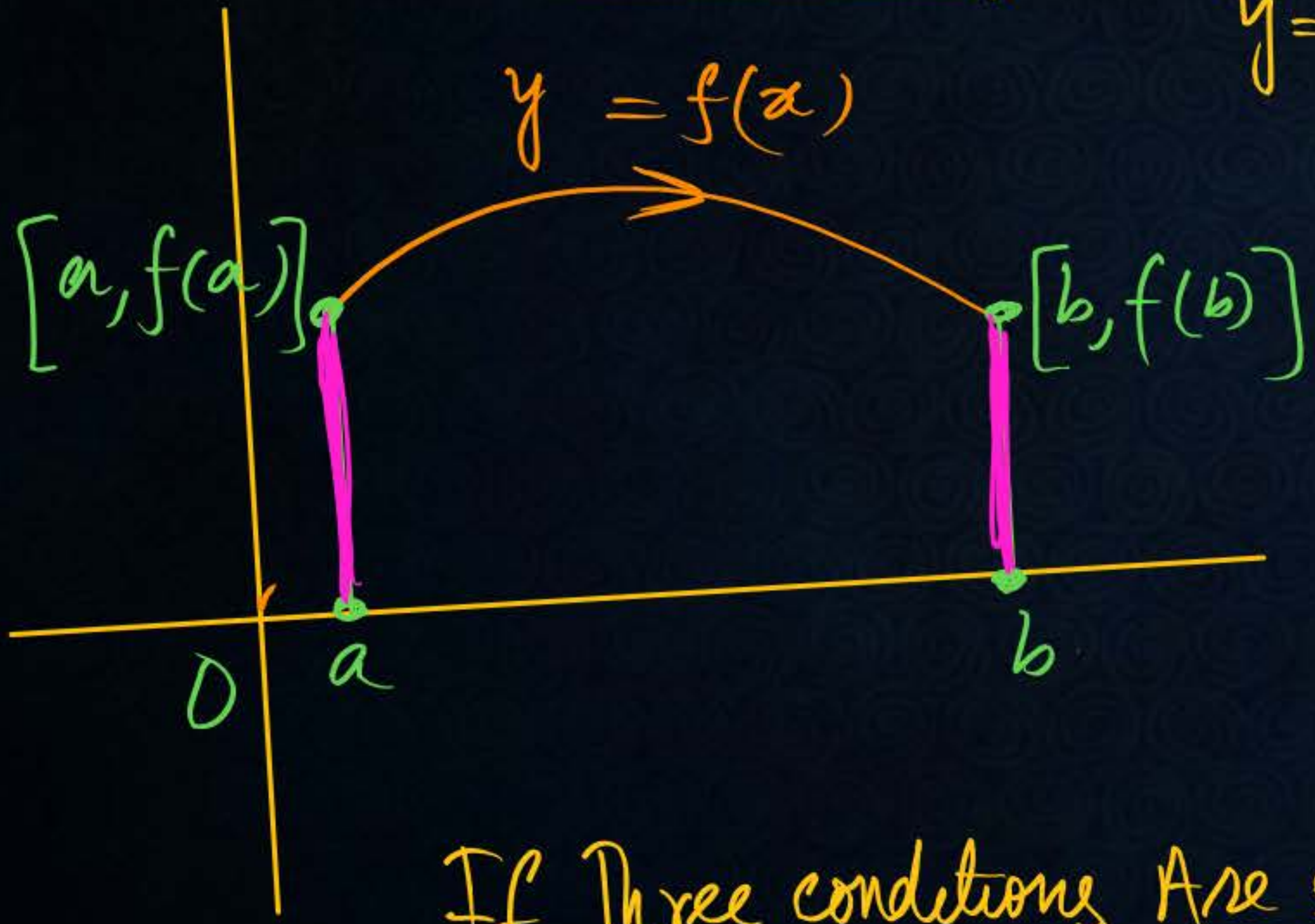
C Non-differentiable

D None of these

MEAN value THEOREM :

- Rolles Theorem
- Lagrange mean value Theorem

Rolles Theorem :



$y = f(x)$

(A) $f(x)$ is Differentiable in $\boxed{(a, b)}$

LHD = RHD

(B) $f(x)$ is continuous in $\boxed{[a, b]}$

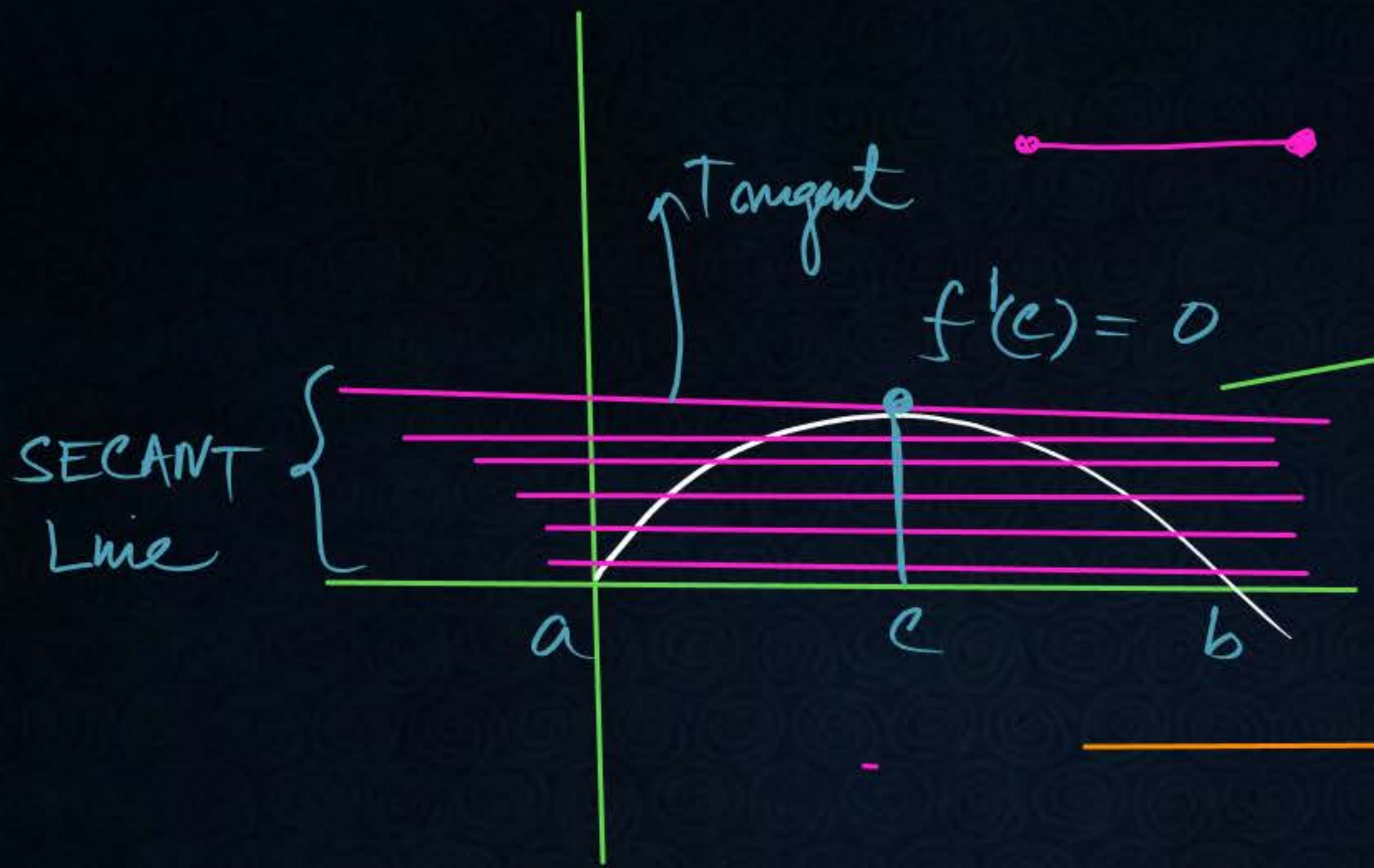
LHL = RHL

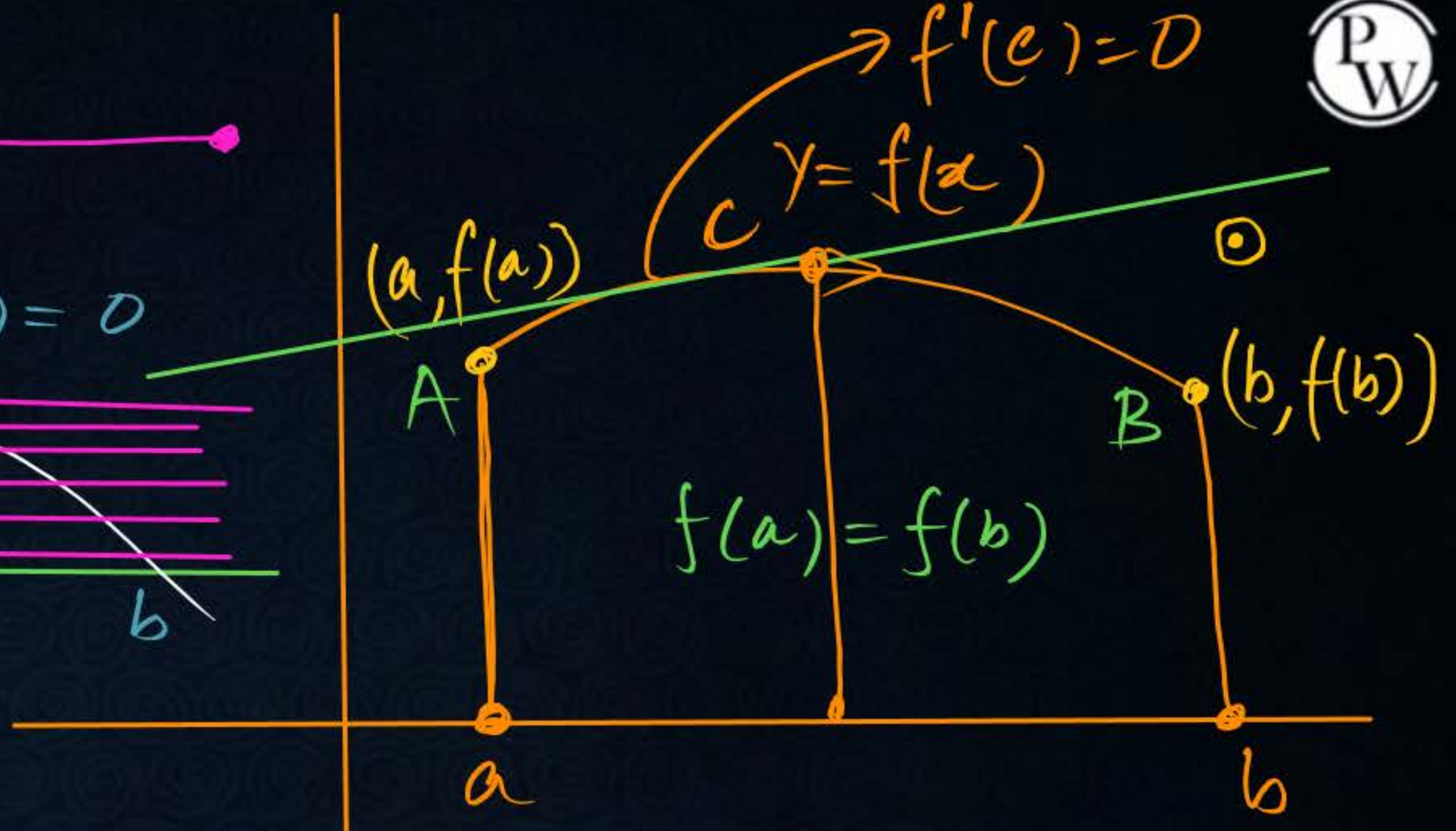
(C) $f(a) = f(b)$

Differentiable + continuous + $f(a) = f(b)$

If Three conditions Are satisfied

Then [If at least one point $f'(c) = 0$ $\boxed{c \in (a, b)}$





Slope at Pt $x = c$ $\quad (a, f(a)) \quad (b, f(b))$

$\qquad\qquad\qquad\qquad\qquad (x_1, y_1) \quad (x_2, y_2)$

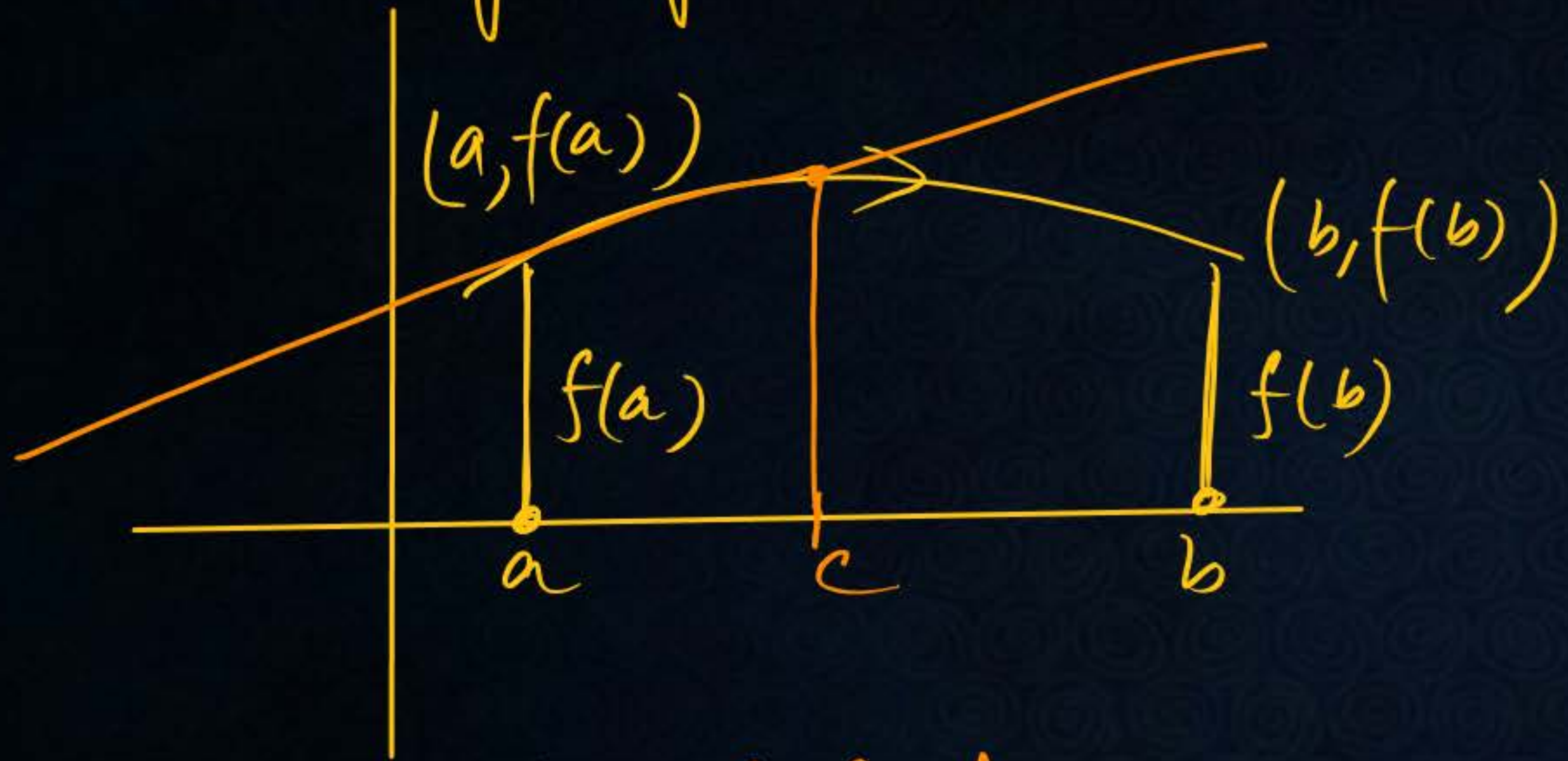
$$f'(c) = \frac{f(b) - f(a)}{(b-a)}$$

Using conditions Rolles Theorem $f(a) = f(b)$

$$f'(c) = \frac{f(b) - f(b)}{(b-a)} = 0$$

$$\boxed{f'(c) = 0 \quad c \in (a, b)}$$

Lagrange MEAN value Theorem :



(A) $f(x)$ is Differentiable (a, b)

(B) $f(x)$ is conti $[a, b]$

(C) $f(a) \neq f(b)$

Average velocity

$$f'(t) = \frac{f(t_0) - f(t)}{t_0 - t}$$

Slope at Point c

LMVT $\boxed{f'(c) = \dfrac{f(b) - f(a)}{(b - a)}}$

Cauchy MEAN value Theorem:

$y_1 = f(x)$

$y_2 = g(x)$

Using LMVT

$$\left\{ \begin{aligned} f'(c) &= \frac{f(b)-f(a)}{(b-a)} \\ g'(c) &= \frac{g(b)-g(a)}{(b-a)} \end{aligned} \right.$$

$$\frac{f'(c)}{g'(c)} = \frac{\frac{f(b)-f(a)}{(b-a)}}{\frac{g(b)-g(a)}{(b-a)}} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Cauchy mean value Theorem:

$$\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$$

$c \in (a, b)$

#Q. It is given that for the function $f(x) = x^3 - 6x^2 + ax + b$ on $[1, 3]$ Rolle's theorem holds with $\boxed{c = 2 + \frac{1}{\sqrt{3}}}$. Find the values a and b if $f(1) = f(3) = 0$.

A $\boxed{a = 11, b = -6}$

B $a = -11, b = 6$

C $a = 11, b = 6$

D $a = -11, b = -6$

$$f(x) = \underbrace{x^3 - 6x^2 + ax + b}, \quad [1,3]$$

$f(1) = f(3) = 0$

Using Rolles Theorem.

$$f(1) = 1^3 - 6\times1^2 + a\times1 + b$$

$$= 1^3 - 6 + a + b$$

$$0 = -5 + a + b$$

$$\boxed{a+b=5} \quad \text{— (1)}$$

$$f(3) = 3^3 - 6\times9 + a\times3 + b$$

$$f(3) = 27 - 54 + 3a + b$$

$$0 = -27 + 3a + b$$

$$3a+b=27$$

$$a+\cancel{b}=5$$

$$\underline{\underset{-}{3a}\cancel{+b}=\underset{-}{27}}$$

$$-2a = -22$$

$$a = 11$$

$$a+b=5$$

$$11+b=5$$

$$b=-6$$

$$c = 2 + \frac{1}{\sqrt{3}}$$ verified

$$f(x) = x^3 - 6x^2 + 11x - 6$$

Using Rolles Theorem.

$$f'(c) = 0$$

$$3c^2 - 12c + 11 = 0$$

$$= \quad a = 3 \quad b = -12 \quad c = 11$$

$$c = \frac{12 \pm \sqrt{144 - 4 \times 11 \times 3}}{2 \times 3}$$

$$c = \frac{12 \pm \sqrt{144 - 132}}{6} = c = 2 \pm \frac{1}{\sqrt{3}}$$

$$c \in [a, b]$$

$$[1, 3]$$

$$\left(2 + \frac{1}{\sqrt{3}}\right) \in [1, 3)$$

$$c = 2 + \frac{1}{\sqrt{3}} \checkmark$$

$$= 2 - \frac{1}{\sqrt{3}} \times$$

reject

Topic : Single Variable Calculus

#Q. If $2a + 3b + 6c = 0$, then the equations $ax^2 + bx + c = 0$ has :

A. At least one real root between 0 and 1

B. No real root between 0 and 1

C. At least one real root between 1 and 2

D. None of these

2 mins Summary

Topic One

Topic Two

Topic Three

Topic Four

Topic Five

MEAN value
LMVT
Rolles

Differentiability

✓ LMVT + Rolles +
Taylor SERIES
✓ Optimization
max/mini

THANK - YOU