GATE
DATA SCIENCE AND AI

CALCULUS AND OPTIMIZATION
SINGLE VARIABLE CALCULUS

Lecture No.- 04

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Recape of previous lecture

Topic Evaluation of limits

Topics to be covered

Topic Evaluation of limits

Topic : Single Variable Calculus

#Q. Evaluate: $\lim_{n\to\infty}\frac{1^2+2^2+3^2+.....+n^2}{n^3}$

$$= \lim_{n\to\infty}\frac{1^2+2^2+3^2+\dots+n^2}{n^3}$$

$$= \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}$$

$$= \lim_{n\to\infty}\frac{(n^2+n)(2n+1)}{6n^3}$$

$$= \lim_{n\to\infty}\frac{2n^3+3n^2+n}{6n^3}$$

$$\lim_{n\to\infty}\frac{1^2+2^2+3^2+\dots+n^2}{n^3}$$

$$= \lim_{n\to\infty}\frac{1^2}{n^3}+\frac{2^2}{n^3}+\frac{3^2}{n^3}+\dots+\frac{n^2}{n^3}$$

Wrong method

Infinitesimal

$$1^2+2^2+3^2+\dots+n^2=\Sigma n^2$$

Sum of all square No

$$\Sigma n^2=\frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{2n^3+3n^2+n}{6n^3}$$

$$= \lim_{n\to\infty} \frac{2n^3}{6n^3} + \cancel{\frac{3n^2}{6n^3}} + \cancel{\frac{n}{6n^3}}$$

$$\frac{2}{6} + 0 + 0 = \frac{2}{6} = \frac{1}{3}$$

$\checkmark$ $1+2+3+\dots+n$

$= \sum n = \frac{n(n+1)}{2}$

$\checkmark$ $1^3+2^3+\dots+n^3$

$\sum n^3 = \left[\frac{n(n+1)}{2}\right]^2$

L - Hospital Rule

$$L = \lim_{x \to a} \frac{f(x)}{g(x)}$$

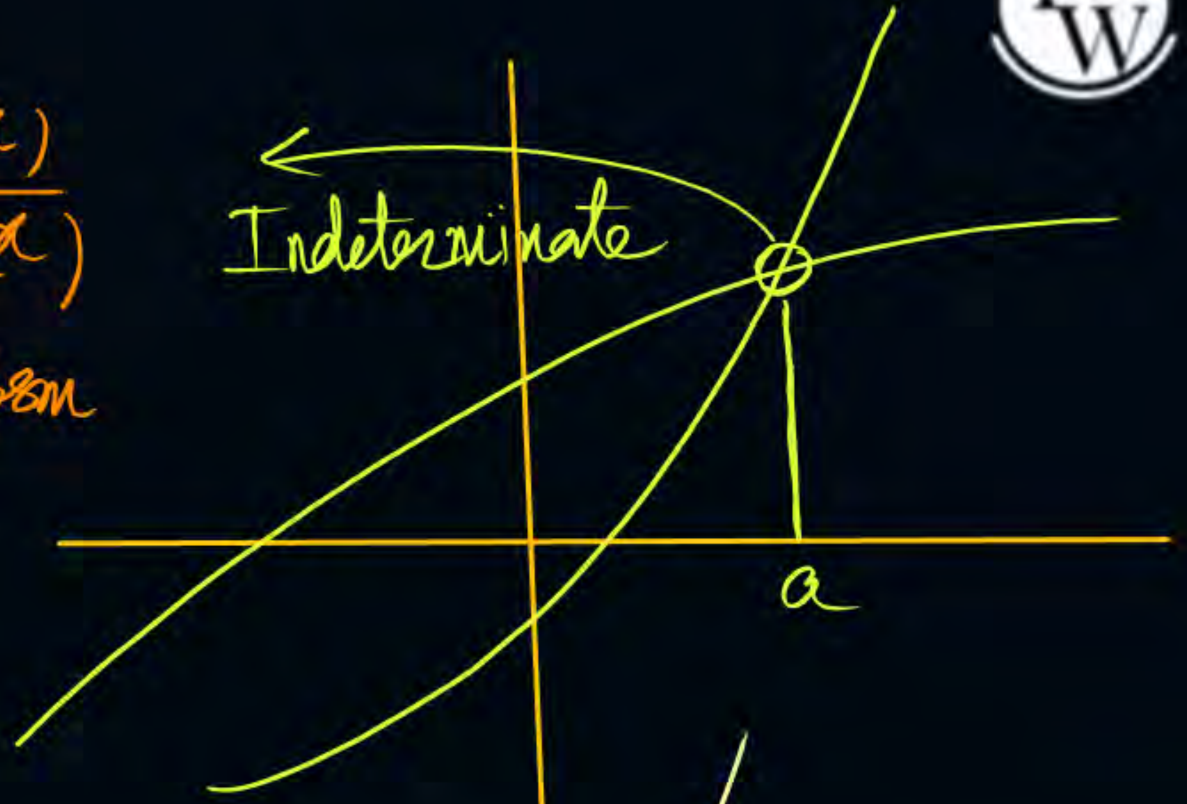
valid $\rightarrow$ $\dfrac{\rightarrow 0}{\rightarrow 0}$ or $\left(\dfrac{\rightarrow \infty}{\rightarrow \infty}\right)$ form

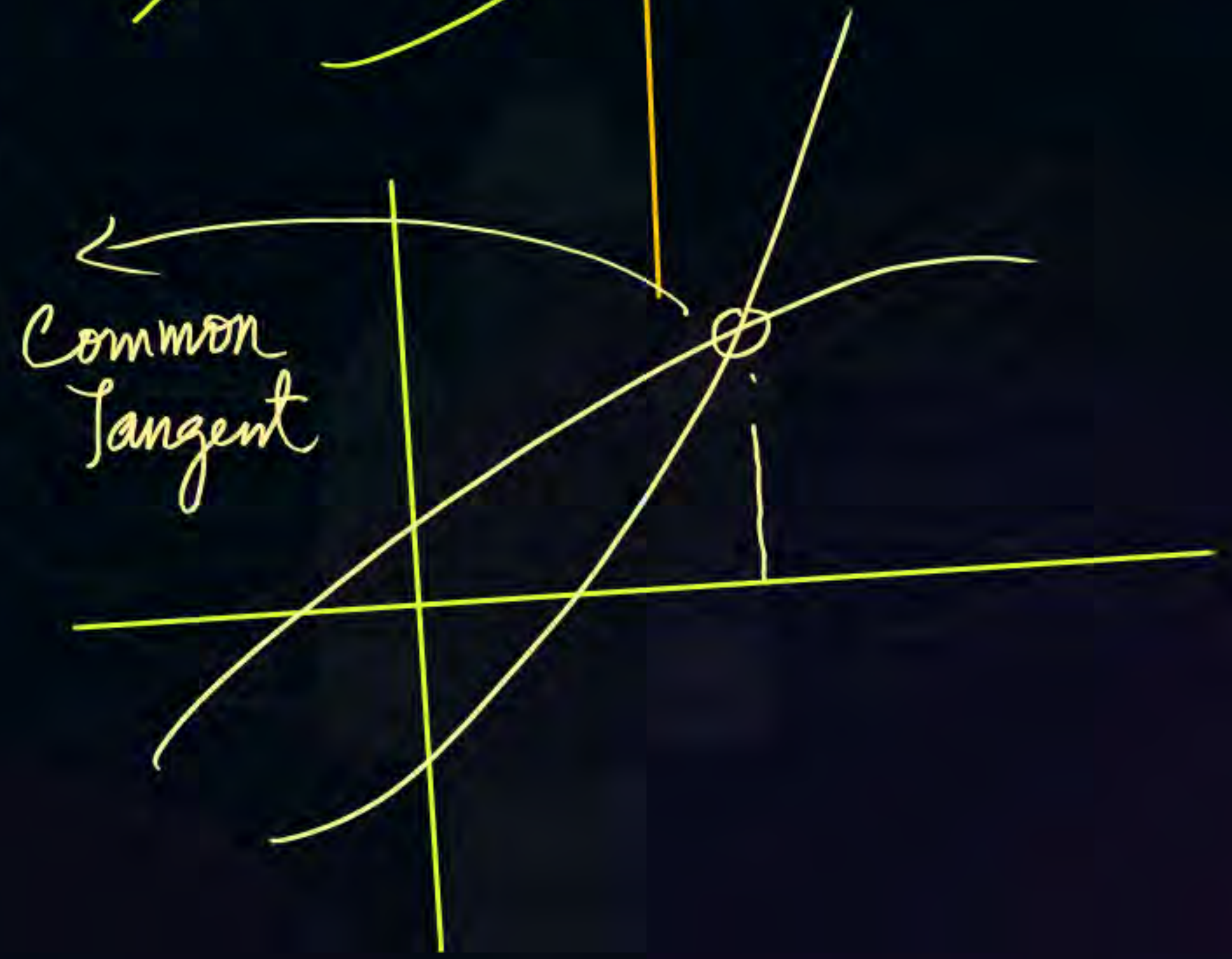
$$L = \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$\left(\dfrac{\rightarrow 0}{\rightarrow 0}\right) \longrightarrow$ Apply L-Hospital Rule $\lim_{x \to a} \dfrac{f'(x)}{g'(x)}$

OR

$\left(\dfrac{\rightarrow \infty}{\rightarrow \infty}\right) \longrightarrow$





$$\underset{x\to a}{\mathrm{Lt}}\frac{f(x)}{g(x)} = \underset{x\to a}{\mathrm{Lt}}\frac{\overset{\text{Diff.}}{\overbrace{f'(x)}}}{\underset{\text{Diff.}}{\underbrace{g'(x)}}} = \underset{x\to a}{\mathrm{lt}}\frac{f''(x)}{g''(x)} = \underset{x\to a}{\mathrm{lt}}\frac{f''(x)}{g''(x)}$$

$\dfrac{\to 0}{\to 0}$ = remove This form

Again $\dfrac{\to 0}{\to 0}$ form

again apply

$\dfrac{\to 0}{\to 0}$

Not removed

Till $\left(\dfrac{\to 0}{\to 0}\right)$ form $\left(\dfrac{\to \infty}{\to \infty}\right)$ से

remove

Plug in Limit $\longrightarrow$ Evaluation The limit ✓

Templates :

(A) $$\boxed{\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{x}{\sin x}=\lim_{x\to 0}\frac{\sin^{-1}x}{x}=\lim_{x\to 0}\frac{x}{\sin^{-1}x}=\lim_{x\to 0}\frac{\tan^{-1}x}{x}=\lim_{x\to 0}\frac{x}{\tan x}=\lim_{x\to 0}\frac{x}{\tan x}=\lim_{x\to 0}\frac{x}{\tan^{-1}x}=1}$$

(A) $\lim_{x\to 0}\frac{\sin x}{x}=\left(\frac{\to 0}{\to 0}\right)$ Form apply L-Hospital Rule

$$=\lim_{x\to 0}\frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x)}=\lim_{x\to 0}\frac{\cos x}{1}=\cos 0^{\circ}=1$$

$$\lim_{x\to 0}\frac{x}{\sin x}=\lim_{x\to 0}\frac{\frac{d}{dx}(x)}{\frac{d}{dx}(\sin x)}=\lim_{x\to 0}\frac{1}{\cos x}=1$$

$$\lim_{x\to 0}\frac{\tan x}{x}\ \left(\frac{\to 0}{\to 0}\right)=\lim_{x\to 0}\frac{\sec^2 x}{1}=1$$

B) $$\lim_{x\to 0}\frac{\sin^{-1}x}{x}=1$$

$$=\lim_{x\to 0}\frac{\frac{1}{\sqrt{1-x^2}}}{1}=1$$

(B) $\boxed{\lim_{x\to 0} \frac{\sin x}{x} = 1}$

$$\lim_{f(x)\to 0} \frac{\sin[f(x)]}{f(x)} = 1$$

c) $\boxed{\lim_{x\to 0} \frac{e^x - 1}{x} = 1}$

$$\lim_{x\to 0} \frac{e^x - 1}{x} = \frac{\to 0}{\to 0} = \lim_{x\to 0} \frac{\frac{d}{dx}(e^x - 1)}{\frac{d}{dx}(x)}$$

$$\Rightarrow \lim_{x\to 0} \frac{e^x}{1} = e^0 = 1$$

D) $\boxed{\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a}$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}(a^x - 1)}{\frac{d}{dx}(x)} = \lim_{x\to 0} \frac{a^x \log_e a}{1}$$

$\left(\frac{\to 0}{\to 0}\right)$ form

$$= a^0 \log_e a$$

$$= \log_e a$$

#Q. The value of limit $\lim_{x\to\frac{\pi}{2}} (\sec x - \tan x)$ is:

$$= \underset{x\to\frac{\pi}{2}}{\mathrm{lt}} \frac{-\cos x}{-\sin x} = \underset{x\to\frac{\pi}{2}}{\mathrm{lt}}(\cot x) = 0$$

$$\boxed{\underset{x\to\frac{\pi}{2}}{\mathrm{Lt}} [\sec x - \tan x] = 0}$$

$$\left.\begin{aligned}\tan 90^\circ &= \infty \\ \cot 90^\circ &= \frac{1}{\infty} = 0\end{aligned}\right\}$$

A) 1

B) 0

C) -1

D) 2

$$\underset{x\to\frac{\pi}{2}}{\mathrm{Lt}} [\sec x - \tan x]$$

$$= \underset{x\to\frac{\pi}{2}}{\mathrm{lt}} \left[\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right]$$

$$= \underset{x\to\frac{\pi}{2}}{\mathrm{lt}} \left[\frac{1-\sin x}{\cos x}\right] \left(\frac{\to 0}{\to 0}\right) \text{Form}$$

$$= \underset{x\to\frac{\pi}{2}}{\mathrm{lt}} \left[\frac{\frac{d}{dx}(1-\sin x)}{\frac{d}{dx}(\cos x)}\right]$$

#Q. The value $\lim_{x\to 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ is:

Using L-Hospital Rule –
– Do yourself

$$\lim_{x\to 0} \frac{\sin(\pi \cos^2 x)}{x^2}$$

Using Template

$$= \lim_{x\to 0} \frac{\sin[\pi(1-\sin^2 x)]}{x^2}$$

$$= \lim_{x\to 0} \frac{\sin(\pi - \pi \sin^2 x)}{x^2}$$

$$= \lim_{x\to 0} \frac{\sin(\pi \sin^2 x)}{x^2}$$

A. π

B. $-\pi$

C. 2π

D. -2π

$$= \lim_{x\to 0} \boxed{\frac{\sin(\pi \sin^2 x)}{x^2}}$$

$$= \lim_{x\to 0} \frac{\sin(\pi \sin^2 x)}{\pi \sin^2 x} \cdot \frac{\pi \sin^2 x}{x^2}$$

$$= \lim_{x\to 0} \frac{\sin(\pi \sin^2 x)}{\pi \sin^2 x} \cdot \pi \left(\frac{\sin x}{x}\right)^2$$

$$= 1 \cdot \pi \cdot 1$$

$$= \pi$$

$$\lim_{f(x)\to 0} \frac{\sin[f(x)]}{f(x)} = 1$$

$$\pi \sin^2 x \longrightarrow 0$$

Topic : Single Variable Calculus

#Q. The limiting value of $x \sin \frac{1}{x}$ as $x \to 0$ is:

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right)$$

$$= \left[\lim_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = 1 \right.$$

wrong approach ✗

A. 1

B. 0 ✓

C. -1

D. Limit does not exist

$\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) = \lim\limits_{x\to 0} \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)} = \underbrace{\lim\limits_{f(x)\to 0} \frac{\sin[f(x)]}{f(x)}}_{f(x)\to 0} = 1$

$\underbrace{\frac{1}{x} \longrightarrow \infty}_{\neq 1}$ (Template Not work)

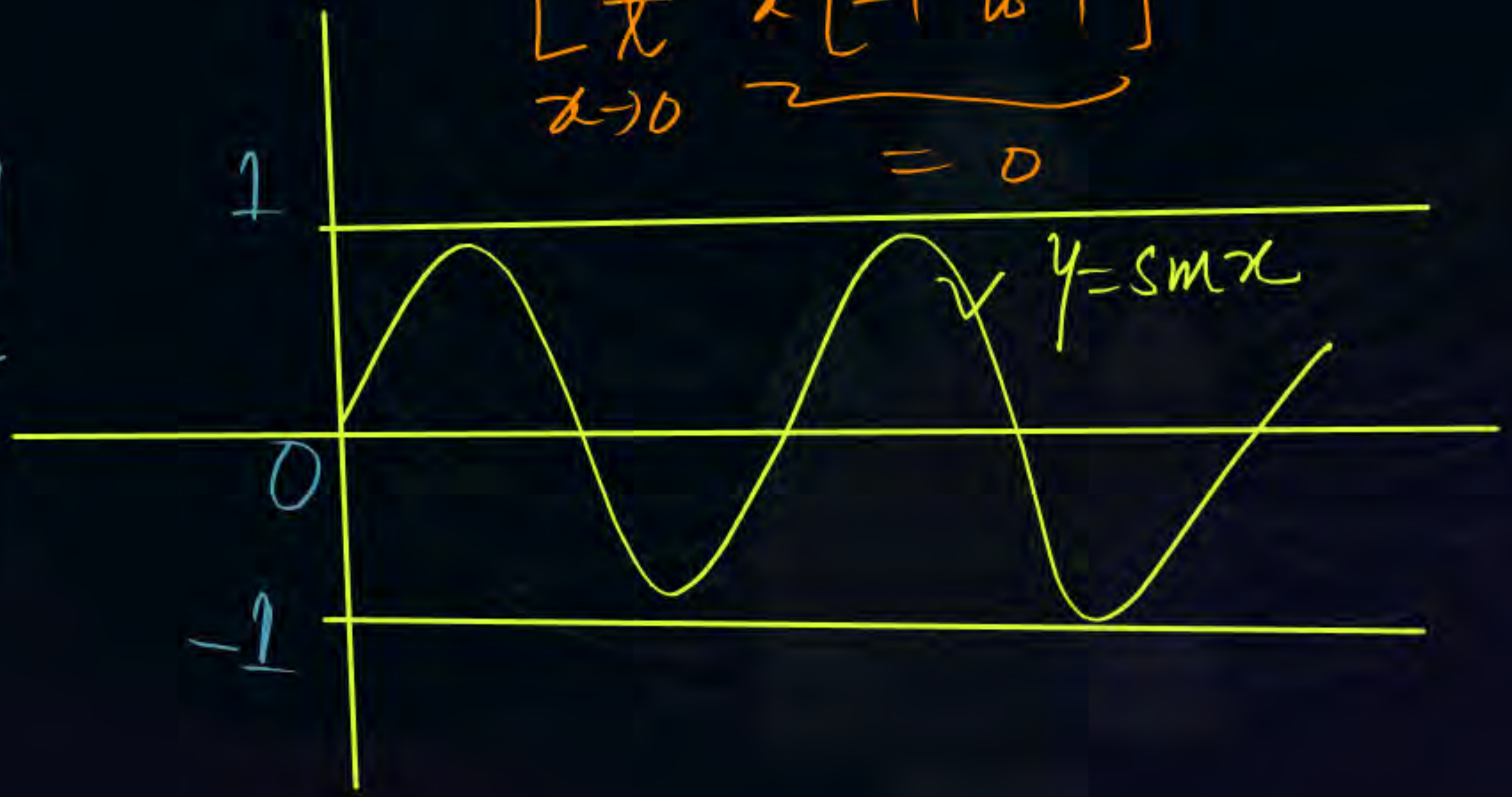
$L = \lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right)$

$= \lim\limits_{x\to 0} x\,[\underbrace{-1 \text{ to } 1}_{\text{oscillation}}]$

oscillation Between [−1 to 1]

$= 0$

$\lim\limits_{x\to 0} x\,\underbrace{[-1 \text{ to } 1]}_{=0}$



Topic : Single Variable Calculus

H.W

#Q. The value of $\lim_{x\to\infty} e^x \tan \frac{a}{e^x}$ is:

A a

B 0

C 1

D None of these

#Q. **Evaluate :** $\lim_{x\to1}\frac{2^x-2}{x-1}$

$\lim_{x\to0}\frac{a^x-1}{x}=\log_e a$

USE

$x-1=t$

$x\to1 \quad t\to0$

Using L-Hospital Rule

$\lim_{x\to1}\frac{2^x-2}{(x-1)}=\frac{\to0}{\to0}$

$=\lim_{x\to1}\frac{2^x\ln2-0}{1}=2\ln2$

$=\lim_{x\to1}\boxed{\frac{2^x-2}{(x-1)}}$ = Using Template

$=\lim_{x\to1}\frac{2(2^{x-1}-1)}{(x-1)}$

$=\lim_{x\to1}2\left[\frac{2^{x-1}-1}{(x-1)}\right]$

$=\lim_{t\to0}2\left[\frac{2^t-1}{t}\right]$

$=\lim_{t\to0}\left[\frac{2^t-1}{t}\right]\cdot2$

$=2\ln2=\ln2^2$

$=\ln4$

Topic : Single Variable Calculus

#Q. **Evaluate :** $\lim_{x\to a} \dfrac{e^{\sqrt{x}} - e^{\sqrt{a}}}{x-a}$

$$= \underset{x\to a}{\mathrm{Lt}}\ \frac{e^{\sqrt{a}}\left[e^{\sqrt{x}-\sqrt{a}} - 1\right]}{(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a})}$$

$$= \underset{x\to a}{\mathrm{Lt}}\ \frac{e^{\sqrt{a}}\left[\dfrac{e^{\sqrt{x}-\sqrt{a}} - 1}{\sqrt{x}-\sqrt{a}}\right]}{(\sqrt{x}+\sqrt{a})}$$

$$= \boxed{\frac{e^{\sqrt{a}}}{2\sqrt{a}}}$$

IInd method

= L-Hosp

Topic : Single Variable Calculus

#Q. **Evaluate :** $\lim_{x \to 0} \dfrac{6^x - 2^x - 3^x + 1}{\sin^2 x}$

H.W

2 mins Summary

Topic : Single Variable Calculus

THANK - YOU