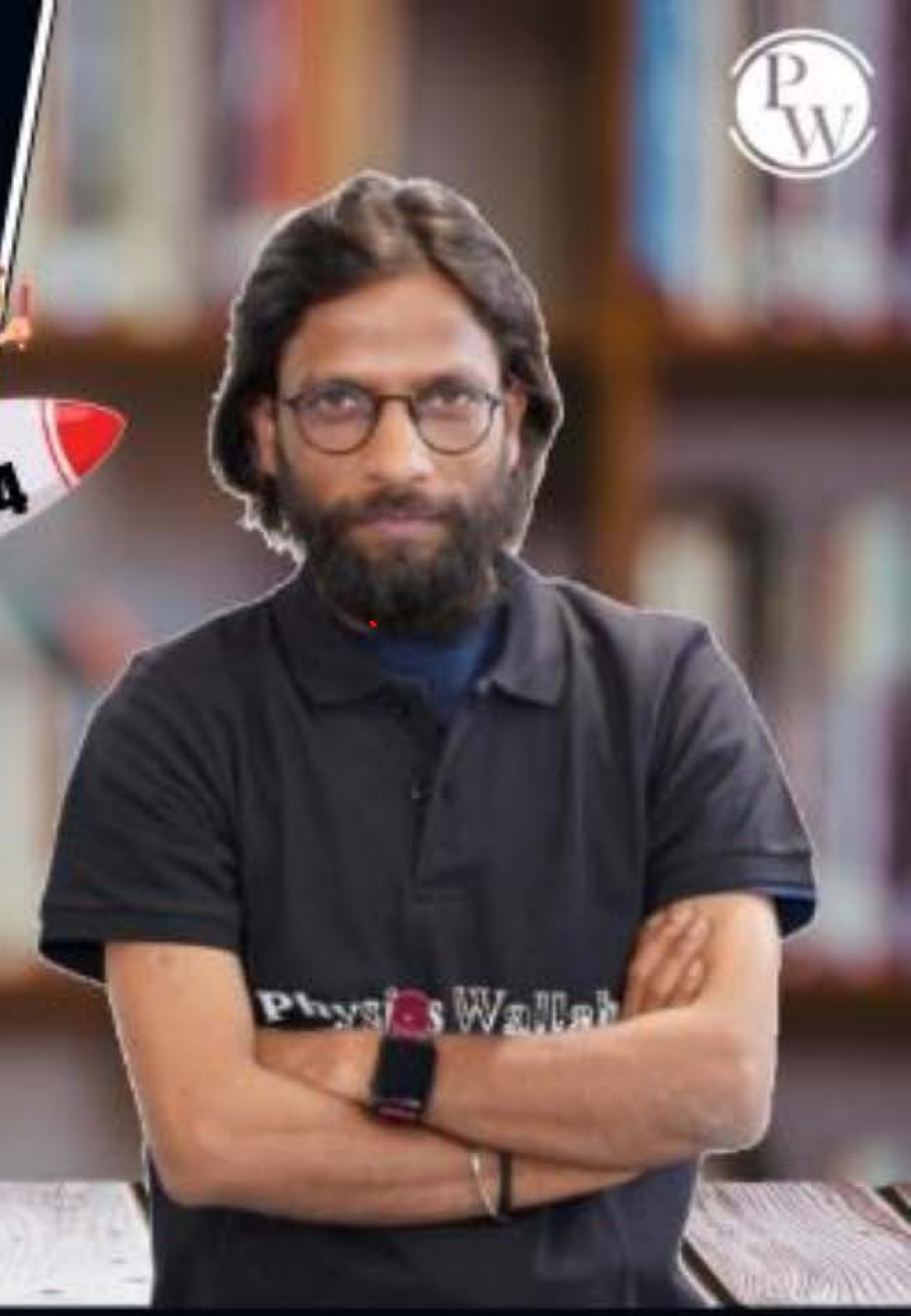
GATE
DATA SCIENCE AND AI

CALCULUS AND OPTIMIZATION
SINGLE VARIABLE CALCULUS

Lecture No.- 03



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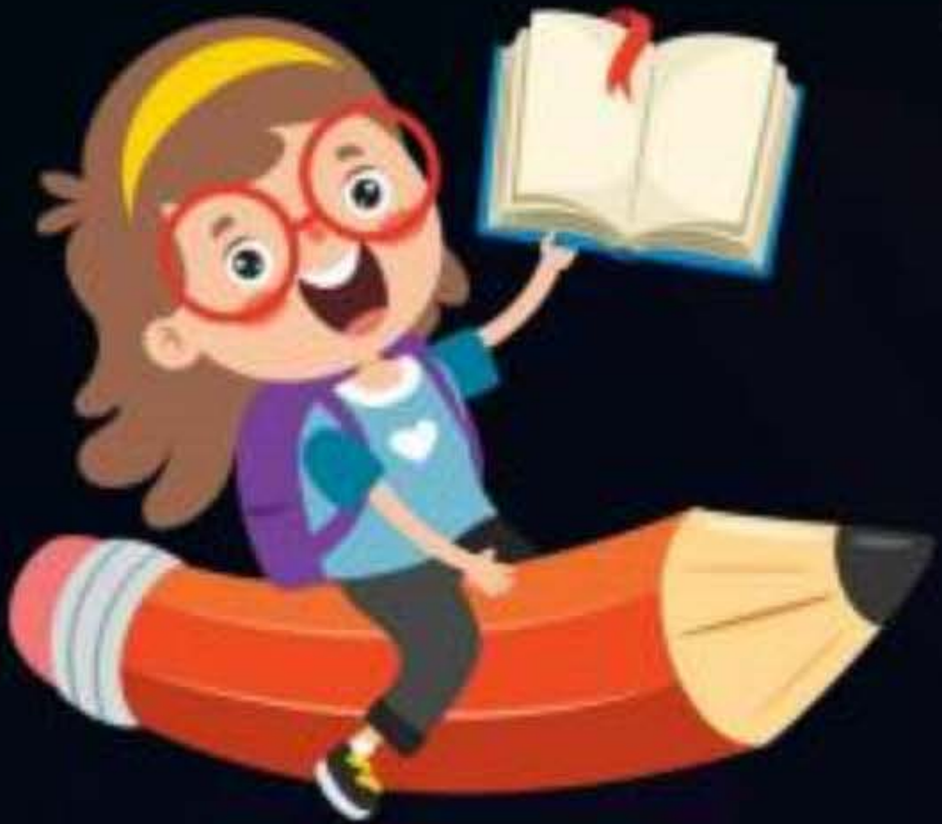
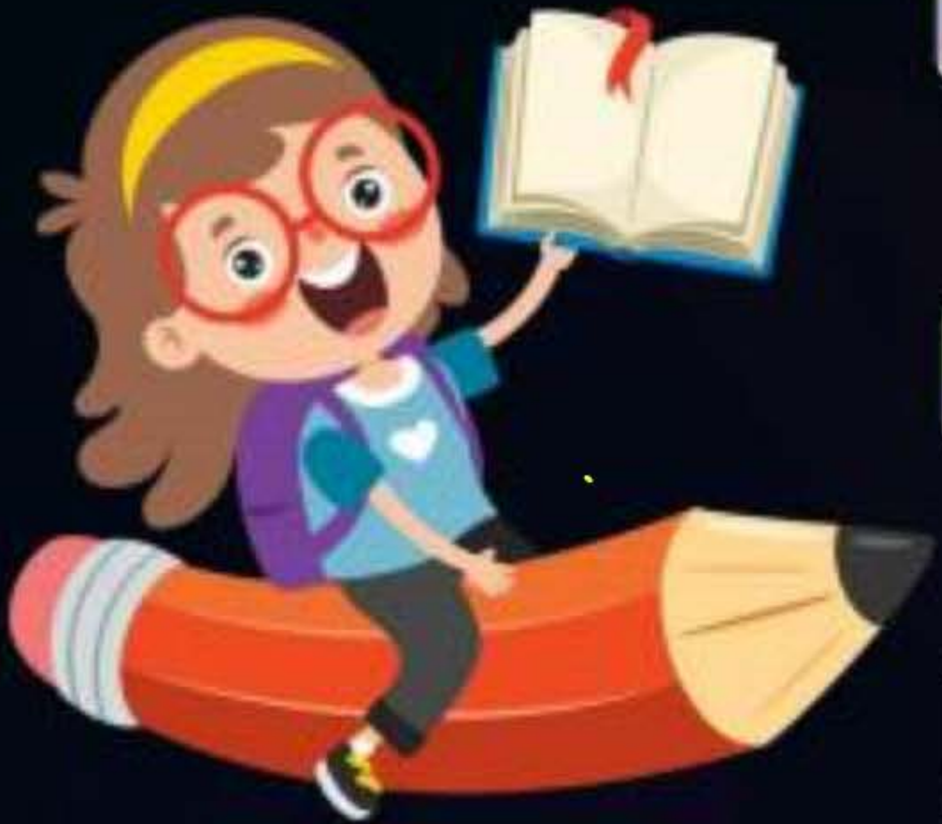
Recape of previous lecture



Topic Sketching graphs

Topics to be Covered



Topic Some transformation of graphs

Topic Existence of limits

Topic How to evaluate limits

H.W Plot The curve

$$\begin{cases} y = e^x + 1 \\ y = e^x - 1 \\ y = \log_e x + 1 \\ y = \log_e x - 1 \end{cases}$$

Existence of Limit

Existence of Limit $\begin{cases} LHL = RHL = \text{finite} \longrightarrow \text{Limit exists} \\ LHL \neq RHL = \text{Limit does Not exists} \end{cases}$

How to evaluate The Limit:

Indeterminate forms

, $\dfrac{\to 0}{\to 0}$ form

, $\dfrac{\to \infty}{\to \infty}$ form

, $(\to \infty) - (\vec{\infty})$

$(\to \infty)^{\to \infty}$

$(\to 1)^{\to \infty}$

$\vec{\infty} (\to 0)$

$(\to 0)^{\to 0}$

Indeterminate forms

$(\to 1)^{\to \infty}$ approaches

(approaches)

(ϵ, δ)

$(\epsilon)(\delta)$

$\dfrac{\to 1}{\to 0} = \to \infty$

$\dfrac{1}{0} = \vec{\infty}$

Topic : Single Variable Calculus

Approaches –
ϵ , δ

Indeterminate forms

Examples :

$$\left.\frac{\to 0}{\to 0}\right] \text{①}, \quad \left.\frac{\to \infty}{\to \infty}\right] \text{②}, \quad \underbrace{(\to \infty) - (\to \infty)}_{③}$$

$$\underbrace{\to \infty \times \to 0}_{④}, \quad \underbrace{(\to 1)^{\to \infty}}_{⑤}, \quad \underbrace{(\to 0)^{\to 0}}_{⑥}, \quad \underbrace{(\to 1)^{\to \infty}}_{⑦}$$

Indeterminate

1 - exact

Topic : Single Variable Calculus

#Q. Evaluate: $\lim_{x\to 1}\left(4x^3-3x^2+6\right)$.

$\underset{x\to 1}{Lt}\left(4x^3-3x^2+6\right) \to$ No (Indeterminate form)

USE the Plug in Rule

$= 4(1)^3-3\times(1)^2+6$

$= 7$

$\frac{\to 0}{\to 0}, \frac{\to\infty}{\to\infty}, \frac{\to\infty-\to\infty}{}$

Remove this Indeterminate form

Plug-In Limit Rule

$\to$ Last step

Topic : Single Variable Calculus

#Q. Evaluate: $\lim_{x\to 0}\frac{\cos^3 x-3\cos x+7}{3x^2+5x-14}$

$= \lim_{x\to 0}\frac{\cos^3 x-3\cos x+7}{3x^2+5x-14}$

$\Rightarrow$ This is Not a Indeterminate form.

$\rightarrow$ Plug in Limit Rule

$= \frac{\cos^3 0-3\cos 0+7}{3\times 0+5\times 0-14}$

$= \frac{1-3+7}{-14}$

$= \frac{+5}{-14} = \frac{-5}{14}$

Topic : Single Variable Calculus

Indeterminate forms $\frac{\to 0}{\to 0}, \frac{\to \infty}{\to \infty},$

(i) Evaluate: $\lim\limits_{x \to 2} \dfrac{x^3 - 2x - 4}{x^2 - 3x + 2}$

$$\lim_{x \to 2} \frac{(x^3 - 2x - 4)}{(x^2 - 3x + 2)} = \frac{(2)^3 - 2 \times 2 - 4}{(2)^2 - 3 \times 2 + 2}$$

$$= \frac{\to 0}{\to 0} \text{ form}$$

A) Ist strategy

$\boxed{\text{Using Factorization method}}$

$\underbrace{\hphantom{xxxxxxxxxx}}_{\text{Ist strategy}}$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x^2 + 2x + 2)}{(x-1)\cancel{(x-2)}}$$

$$= \lim_{x \to 2} \frac{(x^2 + 2x + 2)}{(x-1)} = \text{Remove The } \frac{\to 0}{\to 0} \text{ form}$$

Plug in Limit $= \dfrac{(2)^2 + 2 \times 2 + 2}{(2-1)} = 10$

$$x^2 - 3x + 2$$
$$= x^2 - 2x - x + 2$$
$$= x(x-2) - 1(x-2)$$
$$= (x-1)(x-2)$$

Topic : Single Variable Calculus

\# 1) $\boxed{\text{Plug-in}}$

(ii) Evaluate: $\lim_{x \to a} \dfrac{x^3 - a^3}{x^2 - ax}$ $= \lim_{x\to a} \dfrac{x^3 - a^3}{(x^2 - ax)} = \dfrac{a^3 - a^3}{a^2 - a^2} = \dfrac{\to 0}{\to 0}$ $\quad \dfrac{\to 0}{\to 0}, \dfrac{\to \infty}{\to \infty}$

Remove The Indeterminate forms.

Factorization

$$= \lim_{x \to a} \frac{\cancel{(x-a)}(x^2+a^2+ax)}{x\cancel{(x-a)}}$$

$$= \lim_{x\to a} \frac{x^2+a^2+ax}{x}$$ Plug-in Limit

$$= \frac{a^2+a^2+a^2}{a}$$

$$= \frac{3a^2}{a} = 3a$$

Topic : Single Variable Calculus

✓ Division method

extra

(iii) Evaluate: $\lim_{x\to 2}\dfrac{x^3-6x^2+11x-6}{x^2-6x+8}$ $= \underset{x\to 2}{\mathrm{lt}}\dfrac{x^3-6x^2+11x-6}{(x^2-6x+8)} = \dfrac{\to 0}{\to 0}$

Remove It

$= \underset{x\to 2}{\mathrm{lt}}\dfrac{(x-1)\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x-4)}$

$= \underset{x\to 2}{\mathrm{lt}}\dfrac{(x-1)(x-3)}{(x-4)}$

Remove It

$\dfrac{\to 0}{\to 0}$ $\dfrac{\to \infty}{\to \infty}$

Plug in Limit $\dfrac{(2-1)(2-3)}{(2-4)}$

$= \dfrac{1\times(-1)}{-2} = \dfrac{1}{2}$

Topic : Single Variable Calculus

#Q. Evaluate: $\lim_{x \to 3} \dfrac{x^3 - 7x^2 + 15x - 9}{x^4 - 5x^3 + 27x - 27}$

A. 2/9

B. 1/9

C. -2/9

D. None of these

Rationilization

#Q. Evaluate: $\lim_{x\to 3} \frac{(\sqrt{3x+7}-4)}{(\sqrt{x+1}-2)} = \frac{\to 0}{\to 0}$ form

Strategy No-2 Square Root

USE — Rationilization

$= \lim_{x\to 3} \frac{\sqrt{3x+7}-4}{\sqrt{x+1}-2} \times \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2} \times \frac{\sqrt{3x+7}+4}{\sqrt{3x+7}+4}$

$= \lim_{x\to 3} \frac{(3x+7-16)}{(x+1-4)} \cdot \frac{\sqrt{x+1}+2}{\sqrt{3x+7}+4}$

$= \lim_{x\to 3} \frac{3\cancel{(x-3)}}{\cancel{(x-3)}} \frac{\sqrt{x+1}+2}{\sqrt{3x+7}+4}$

$= 3 \lim_{x\to 3} \frac{\sqrt{x+1}+2}{\sqrt{3x+7}+4}$

→ Remove Indeterminate form

↓ Plug in Limits

$= 3 \times \frac{\cancel{4}^{1}}{\cancel{8}_{2}} = \frac{3}{2}$

Topic : Single Variable Calculus

#Q. Evaluate: $\lim_{x\to a} \dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}$ $= \dfrac{\to 0}{\to 0}$ remove It

1) VSE - rationalization
2) Plug - in Limits

$$= \lim_{x\to a} \frac{(\sqrt{a+2x}-\sqrt{3x})(\sqrt{a+2x}+\sqrt{3x})(\sqrt{3a+x}+2\sqrt{x})}{(\sqrt{3a+x}-2\sqrt{x})(\sqrt{3a+x}+2\sqrt{x})(\sqrt{a+2a}+\sqrt{3a})}$$

$$= \frac{2}{3\sqrt{3}}$$

Topic : Single Variable Calculus

#Q. Evaluate: $\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)$

$$\underset{x\to\infty}{Lt}\left(x-\sqrt{x^2+x}\right)$$

Using Rationalization

$$=\underset{x\to\infty}{lt}\ x-\sqrt{x^2+x}\times\frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}}$$

$$=\underset{x\to\infty}{lt}\frac{\cancel{x^2}-(\cancel{x^2}+x)}{x+\sqrt{x^2+x}}$$

$$=\underset{x\to\infty}{lt}\left\{\frac{-x}{x+\sqrt{x^2+x}}\right\}$$

$$\underset{x\to\infty}{Lt}\frac{\frac{-x}{x}}{\frac{x}{x}+\sqrt{\frac{x^2}{x^2}+\frac{x}{x^2}}}$$

$$=\underset{x\to\infty}{lt}\frac{-1}{1+\sqrt{1+\cancel{\frac{1}{x}}\to 0}}$$

remove lt

$x\to\infty$

$\frac{1}{x}\to 0$

A -1/2 $= -\frac{1}{2}$ Ans

B 1/2

C 1

D -1

$$\lim_{x\to\infty} \frac{-x}{x+\sqrt{x^2+x}}$$

$$\lim_{x\to\infty} \frac{\cancel{x}}{\cancel{x}}\left[\frac{-1}{1+\sqrt{\frac{1}{x}+1}}\right]$$

$$= -\frac{1}{2}$$

$x \to \infty$

$\frac{1}{x} \to 0$

$x \to \frac{1}{y}$

$y \to 0$

#Q. Evaluate: $\lim_{x\to\infty} \frac{x^3-2x^2+3x+1}{5x^3+7x+2}$ $\Rightarrow \frac{\to\infty}{\to\infty}$ form $\longrightarrow$ Remove It $\downarrow$ Plug-in

follow This strategy $\frac{1}{x}\to 0$

$$= \lim_{x\to\infty} \frac{1-\frac{2x^2}{x^3}+\frac{3x}{x^3}+\frac{1}{x^3}}{\frac{5x^3}{x^3}+\frac{7x}{x^3}+\frac{2}{x^3}}$$

$$= \lim_{x\to\infty} \frac{1-\cancel{\frac{2}{x}}^{\to 0}+\cancel{\frac{3}{x^2}}^{\to 0}+\cancel{\frac{1}{x^3}}^{\to 0}}{5+\cancel{\frac{7}{x^2}}^{\to 0}+\cancel{\frac{2}{x^3}}^{\to 0}}$$

$$= \frac{1}{5}$$

\#

$$\lim_{x\to\infty} \frac{px^n+qx^{n-1}+---}{ax^n+bx^{n-1}+---}$$

Poly Poly

If Num (Power) = Den (Power)

= coefficient of Highest Term

$$= \frac{p}{a}$$

Topic : Single Variable Calculus

$$\underset{x\to a}{Lt}\ \frac{x^n-a^n}{x^m-a^m}=\frac{n}{m}a^{n-m}$$
$$=\frac{4}{3}(5)^{4-3}=\frac{20}{3}$$

#Q. Evaluate : $\lim\limits_{x\to 5}\dfrac{x^4-625}{x^3-125}$ $= \dfrac{\to 0}{\to 0}$ form $\longrightarrow$ Remove It

Strategy

Template/Wallpaper

$$\underset{x\to a}{Lt}\ \frac{x^n-a^n}{(x-a)}=na^{n-1}$$

Template

$$\underset{x\to 5}{Lt}\ \frac{\dfrac{x^4-625}{(x-5)}}{\dfrac{x^3-125}{(x-5)}}=\underset{x\to 5}{Lt}\ \frac{\dfrac{x^4-(5)^4}{(x-5)}}{\dfrac{x^3-(5)^3}{(x-5)}}$$

$$=\underset{x\to 5}{Lt}\ \frac{\dfrac{x^4-5^4}{(x-5)}}{\dfrac{(x^3-5^3)}{(x-5)}}=\frac{4\times 5^{4-1}}{3\times 5^{3-1}}=\frac{20}{3}\ \text{Ans}$$

2 mins Summary

Limits – How to evaluate

Topic One

Topic Two

Topic Three

Topic Four

Topic Five

THANK - YOU