# GATE
# DATA SCIENCE AND AI

2024

## CALCULUS AND OPTIMIZATION
## SINGLE VARIABLE CALCULUS


Lecture No.- 08

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# Recape of previous lecture

**Topic** Differentiability of function

**Topic** Mean value theorem

# Topics to be covered

Mrg 12 to 2

max/min one variable

**Topic** Taylor series

**Topic** Maxima and Minima

only one variable

[Optimization)

max/min $R^1$

$R^2$

$R^3$

Optimization $R^n$

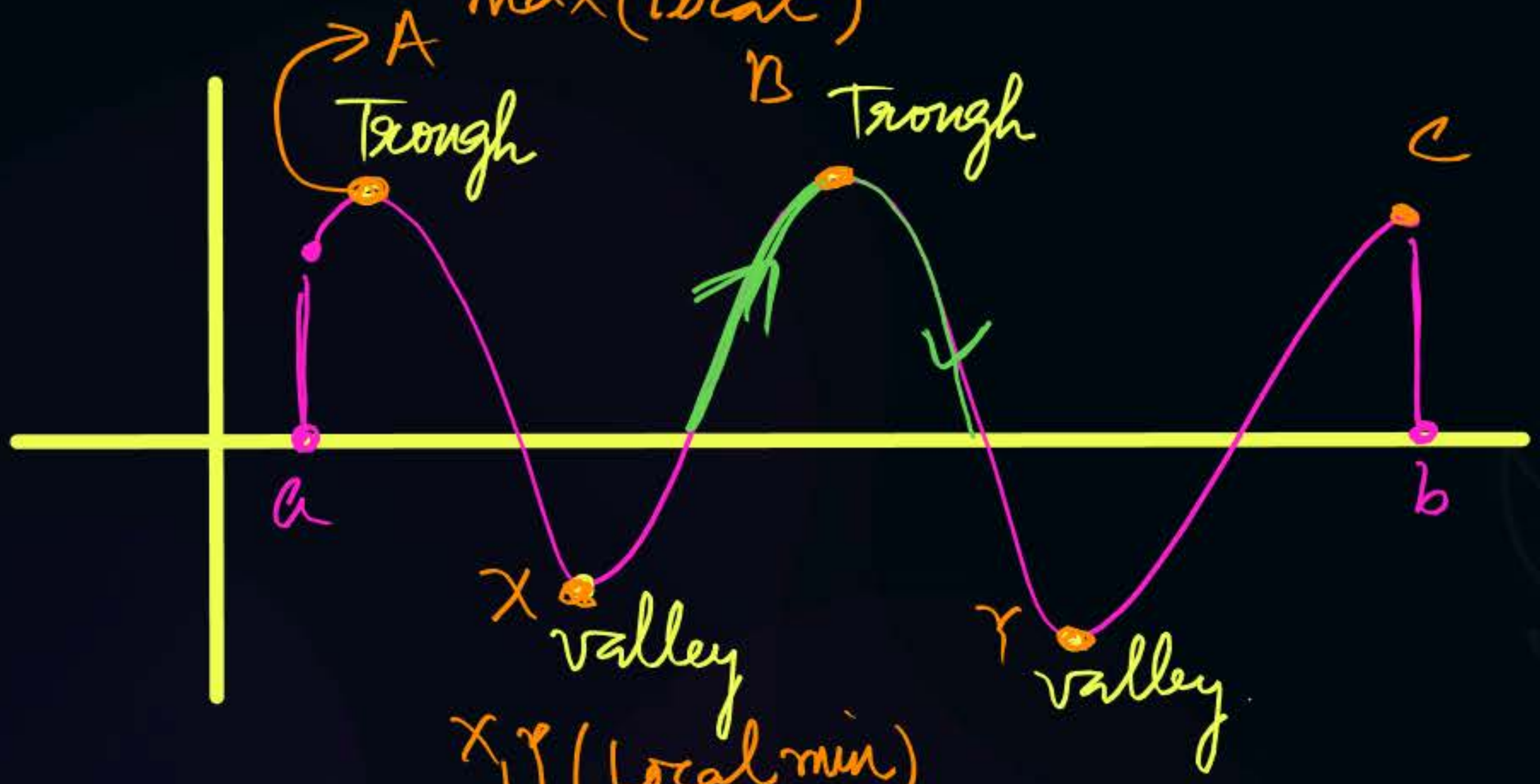
## # # Max/Min of one Variable:

$y = f(x)$ [one variable]

$x$ = Independent variable $y$ = Dependent variable.

### Relative maxima and Relative Minima (Local max/Local min)

(given in open Interval or Interval is Not given)




X, Y (Local min)

" If Function is Increasing

$\frac{dy}{dx} > 0$ $f'(x) > 0$

" If Function Decreasing

$\frac{dy}{dx} < 0$ $f'(x) < 0$

(A, B, C) — [Highest interval bound] — global max

(X, Y) — [Lowest] — global min

## How to Calculate Local max / Min (Relative Max/Min) → open interval / Interval is Not given

Max | Min

Step (A) $f'(x) = 0$ (Stationary Pt)

derivative exist — stationary Pt

$x_0$ = stationary Point

Step (B) Calculate $f''(x_0) > 0$ (Minima)

$f''(x_0) < 0$ (Maxima)

$f''(x_0) = 0$ (Point of Inflection / Neither max Nor min / Saddle Point)

Taylor SERIES

Step (C) Max value — $x_0$ $y = f(x_0)$

min value — $x_0$ $y = f(x_0)$

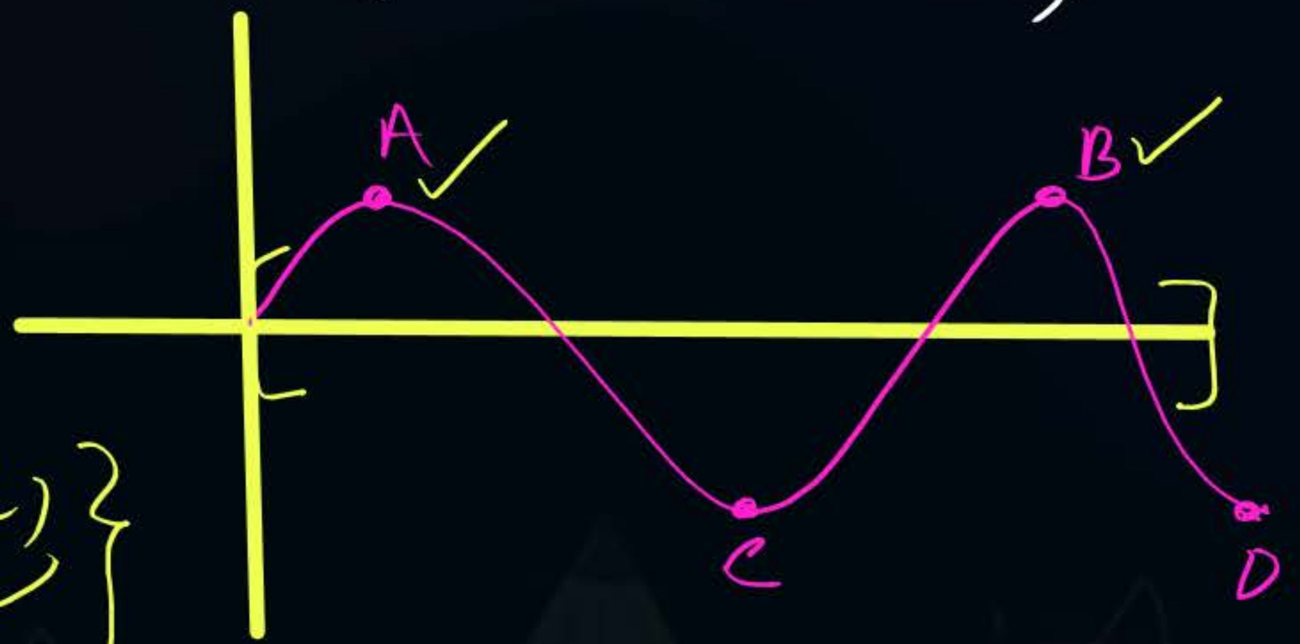
## How to Calculate Global max | global min :

Absolute max | absolute min

(A) step ① $\boxed{f'(x) = 0}$

Evaluate The stationary Pt to $y = f(x)$

(Interval is closed)




(B) Global max $= \left\{ \underline{f(A), f(B), f(C)} \right.$ 

max value

= global max

global min $= \{ \underline{f(X), f(Y)} \}$

= min value
= Least value ] global min

## Topic : Single Variable Calculus

#Q. The function $y = x^2 + \dfrac{250}{x}$ at $x = 5$ attains

A. Maximum

B. Minimum

C. Neither

D. 1

H.W

## Topic : Single Variable Calculus

#Q. For the function $f(x) = x^2e^{-x}$, the maximum occurs when $x$ is equal to

A. 2

B. 1

C. 0

D. –1

$f(x) = x^2e^{-x}$

step (A) $f'(x) = 0$

$\frac{d}{dx}(x^2e^{-x}) = 0$

$-x^2e^{-x} + e^{-x}2x = 0$

$e^{-x}[-x^2+2x] = 0$

$(-x+2)x = 0$

$x=2$, $x=0$ stationary Point

Max value $= x^2e^{-x}$
$= (2)^2e^{-2} = 4e^{-2}$

$x=2$ max "Local max | Local min"

Interval Not given

$f''(x) = -(-x^2e^{-x} + e^{-x}2x) + e^{-x}2 - 2xe^{-x}$

$= x^2e^{-x} - e^{-x}2x + e^{-x}2 - 2xe^{-x}$

$= x^2e^{-x} - 4xe^{-x} + 2e^{-x}$

$x=2$ $= 4e^{-2} - 8e^{-2} + 2e^{-2} = -2e^{-2} < 0$

$x=2$ local max $f''(2) < 0$ | $0 - 0 + 2e^0 = 2 > 0$

#Q. The maximum value of $\boxed{f(x) = x^3 - 9x^2 + 24x + 5}$ in the interval [1, 6] is

$\boxed{\text{Global max}}$

A 21

B 25

C 41

D 46

$$f(x) = x^3 - 9x^2 + 24x + 5 \qquad [1, 6]$$

Step (A) Diff. w.r.t to $x$

$$f'(x) = 3x^2 - 18x + 24 = 0$$

$$3(x^2 - 6x + 8) = 0$$

$$= x^2 - 6x + 8 = 0$$

$$\Rightarrow x^2 - 4x - 2x + 8 = 0$$

$$\Rightarrow x(x-4) - 2(x-4) = 0$$

$$\Rightarrow \boxed{\begin{array}{l} x = 2 \\ x = 4 \end{array}}$$ Stationary Point


$$f(x) = x^3 - 9x^2 + 24x + 5$$

$$\left.\begin{array}{l} f(1) = 1 - 9 + 24 + 5 = 21 \\ f(2) = 8 - 36 + 48 + 5 = 25 \\ f(4) = 64 - 144 + 96 + 5 = 21 \end{array}\right\} \text{local maxima}$$

$$f(6) = 216 - 36 \times 9 + 24 \times 6 + 5 = 41$$

max at $x = 6$ max value $= 41$

#Q. For $0 \leq t < \infty$, the maximum value of the function $f(t) = e^{-t} - 2e^{-2t}$ occurs at

$f'(t) = 0$

$-e^{-t} + 4e^{-2t} = 0$

$-e^{-t}(-1 + 4e^{-t}) = 0$

$-1 + 4e^{-t} = 0$

$4e^{-t} = 1$

$e^{-t} = \frac{1}{4}$

$t = \log_e 4$

$f''(t) < 0$ max

$f''(\log_e 4) < 0$ max at

$x = \log_e 4$

A $t = \log_e 4$

B $t = \log_e 2$

C $t = 0$

D $t = \log_e 8$

#Q. The maximum value of the function $f(x) = \ln(1 + x) - x$ (where x > –1) occurs at x = ____.

$\boxed{x=0}$

$f'(x) = 0$

$\frac{1}{1+x} - 1 = 0$

$\frac{1}{1+x} = 1$

$x = 0$

$f''(x) = -\frac{1}{(1+x)^2}$

$x = 0$

max

## Topic : Single Variable Calculus

{Max value at $x=3$
value = 6

#Q. The maximum value of $f(x) = 2x^3 - 9x^2 + 12x - 3$ in the interval $0 \le x \le 3$ is ____.

$\boxed{\text{H.W}}$

## Topic : Single Variable Calculus

Important
GATE

#Q. For a right angled triangle, if the sum of the lengths of the hypotenuse and a side is kept constant, in order to have maximum area of the triangle, the angle between the hypotenuse and the side is

$h + \text{base} = c$

A $12°$

B $36°$

C $60°$

D $45°$

side + Hyponetuse = constant

$$x + \sqrt{x^2+y^2} = c$$

$$\sqrt{x^2+y^2} = (c-x)$$

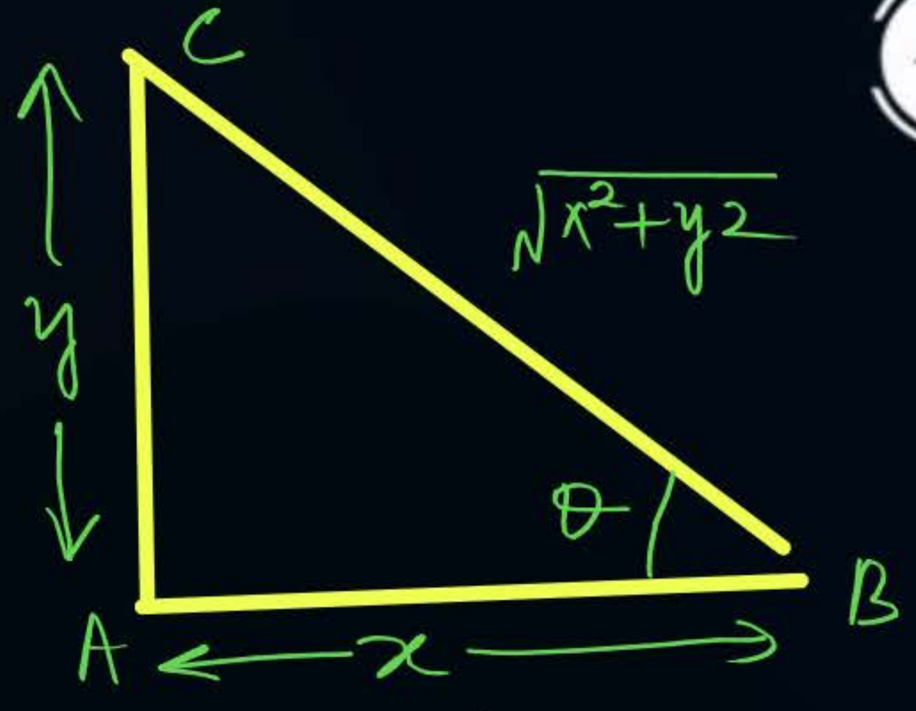
both sides Square It

$$\left(\sqrt{(x^2+y^2}\right)^2 = (c-x)^2$$

$$\cancel{x^2} + y^2 = c^2 + \cancel{x^2} - 2cx$$

$$\boxed{y^2 = c^2 - 2cx}$$

$$\boxed{y = \sqrt{c^2 - 2cx}}$$




$$\text{Area} = \frac{1}{2} x y$$

$$= \frac{1}{2} \times \text{base} \times \text{Height}$$

$$= \frac{1}{2} \times x \times y$$

$$\underbrace{A^2_{max}}_{f(x)} = \frac{1}{4}x^2y^2$$

$$f(x) = \frac{1}{4}\underline{x^2}\,\underline{(c^2-2cx)}$$

$$f'(x) = 0$$

$$\frac{1}{4}\left[x^2(0-2c) + 2x(c^2-2cx)\right] = 0$$

$$\boxed{x = \frac{c}{3}}$$

$$y = \sqrt{c^2-2cx}$$

$$= \sqrt{c^2-2c\times\frac{c}{3}}$$

$$= \sqrt{c^2-\frac{2c^2}{3}} = \sqrt{\frac{3c^2-2c^2}{3}} = \frac{c}{\sqrt{3}}$$

$$A_{max} = \frac{1}{2}xy \text{ — } A\,max\ proof$$

$$A^2_{max} = \frac{1}{4}x^2y^2 \rightarrow A^2\,max$$

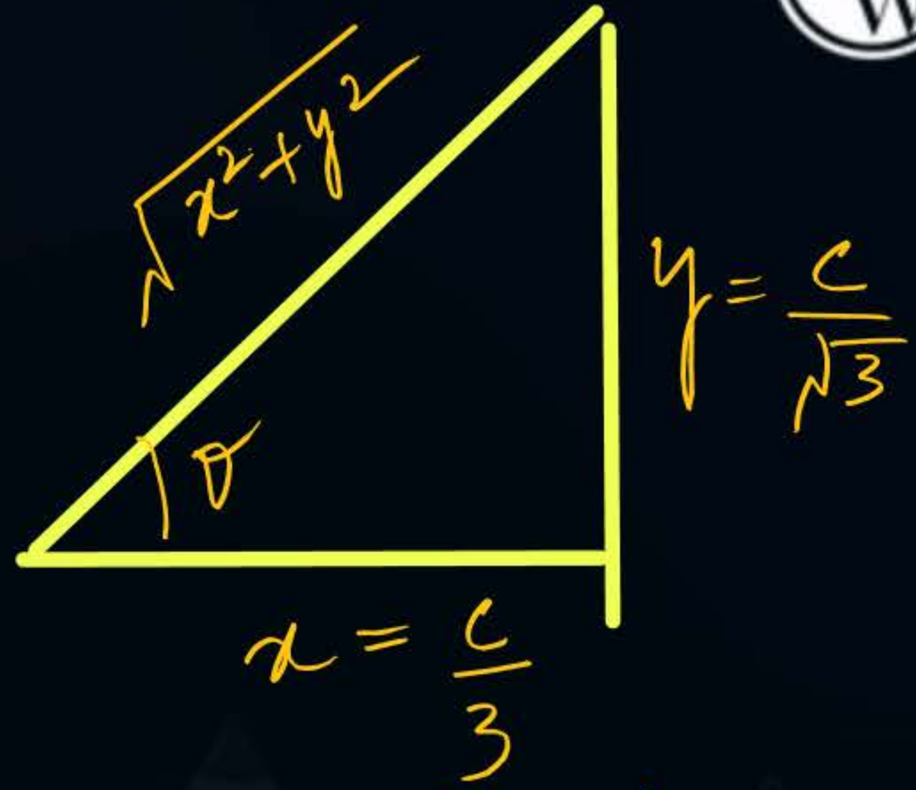
$$A^3\,max$$

$$\frac{1}{4}\left[-2cx^2+2xc^2-\cancel{4c}x^2\right) = 0$$

$$-6cx^2+2xc^2 = 0$$

$$\left.\begin{array}{l} x=0 \\ x=c/3 \end{array}\right]$$

$$\boxed{\theta = 60^\circ}$$




$$\tan\theta = \frac{y}{x} = \frac{\frac{c}{\sqrt{3}}}{\frac{c}{3}}$$

$$= \sqrt{3} \quad \frac{3\cancel{c}}{\cancel{c}\sqrt{3}}$$

## Topic : Single Variable Calculus

#Q. Let $f(x) = xe^{-x}$. The maximum value of the function in the interval $(0, \infty)$ is

Local max/min

H.W

A $e^{-1}$

B $e$

C $1 - e^{-1}$

D $1 + e^{-1}$

MVT = LMVT

Rolles Theorem apply

#Q. A function $f(x) = 1 - x^2 + x^3$ is defined in the closed interval [–1, 1]. The value of $x$, in the open interval (–1,1) for which the mean value theorem is satisfied, is

A $-\frac{1}{2}$

B $-\frac{1}{3}$

C $\frac{1}{3}$

D $\frac{1}{2}$

LMVT

$$f'(c) = \frac{f(b)-f(a)}{(b-a)}$$

$$-2c+3c^2 = \frac{1-(-1)}{1-(-1)} = \frac{\cancel{2}}{\cancel{2}} = 1$$

$$3c^2-2c-1=0$$

$$c = -\frac{1}{3},\ c = 1$$

$(-1, 1)$

$a = -1$

$b = 1$

$f(x) = 1-x^2+x^3$

$f(-1) = 1-(-1)^2+(-1)^3$

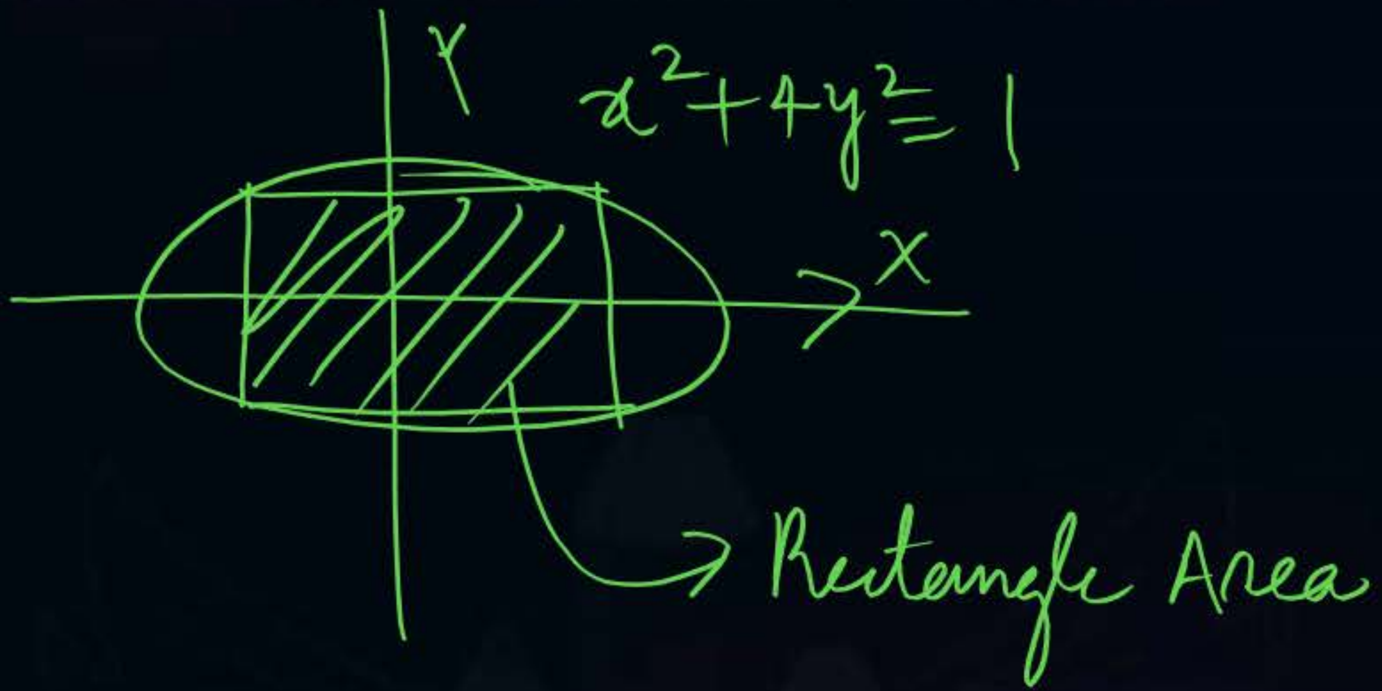
$f(-1) = 1-1-1 = -1$

$f(1) = 1-1+1 = 1$

# Topic : Single Variable Calculus

H.W

#Q. The maximum area (in square units) of a rectangle whose vertices lie on the ellipse $x^2 + 4y^2 = 1$ is ____.



## Topic : Single Variable Calculus

#Q. The maximum value attained by the function $f(x) = x(x-1)(x-2)$ in the interval [1,2] is.

Max value at $x=0$

H.W

## Topic : Single Variable Calculus

M.W

Homework

#Q. The minimum value of the function $f(x)=\frac{1}{3}x\left(x^2-3\right)$ in the interval $\underbrace{-1000 \le x \le 1000}$ occurs at x = ____.

Closed interval — Global max/min

## Topic : Single Variable Calculus

Closed interval [a, b]
global max/min

#Q. Let $f(x) = 3x^3 - 7x^2 + 5x + 6$. The maximum value of $f(x)$ over the interval [0,2] is ________. (upto 1 decimal place)

$f(2) = 12$

max point    max value

H.W

# 2 mins Summary

12 to 2

Topic One

Topic Two

Topic Three

Topic Four

Topic Five

Max | min

global
local

Points of Extremum

12th max/min

# THANK - YOU