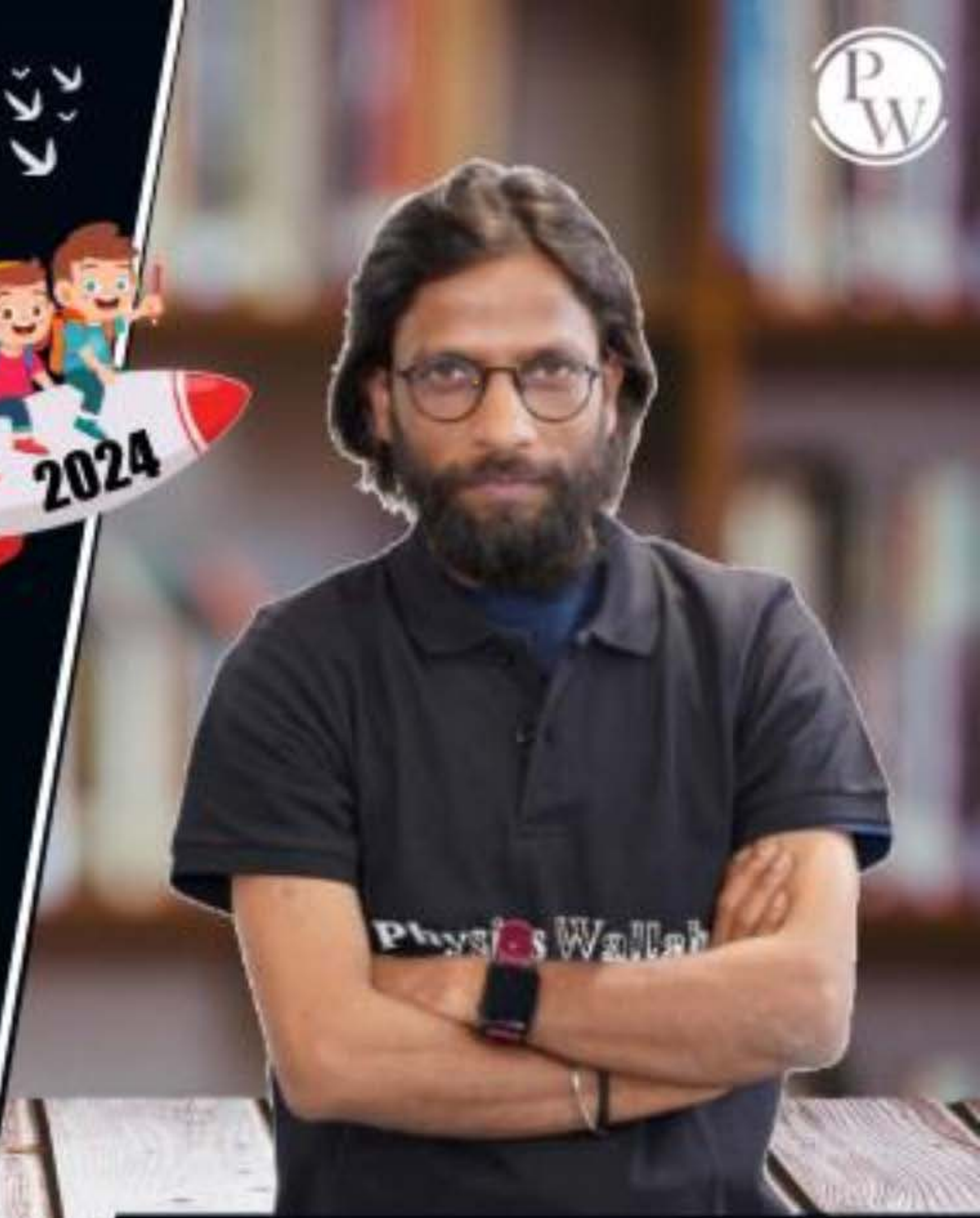
GATE
DATA SCIENCE AND AI

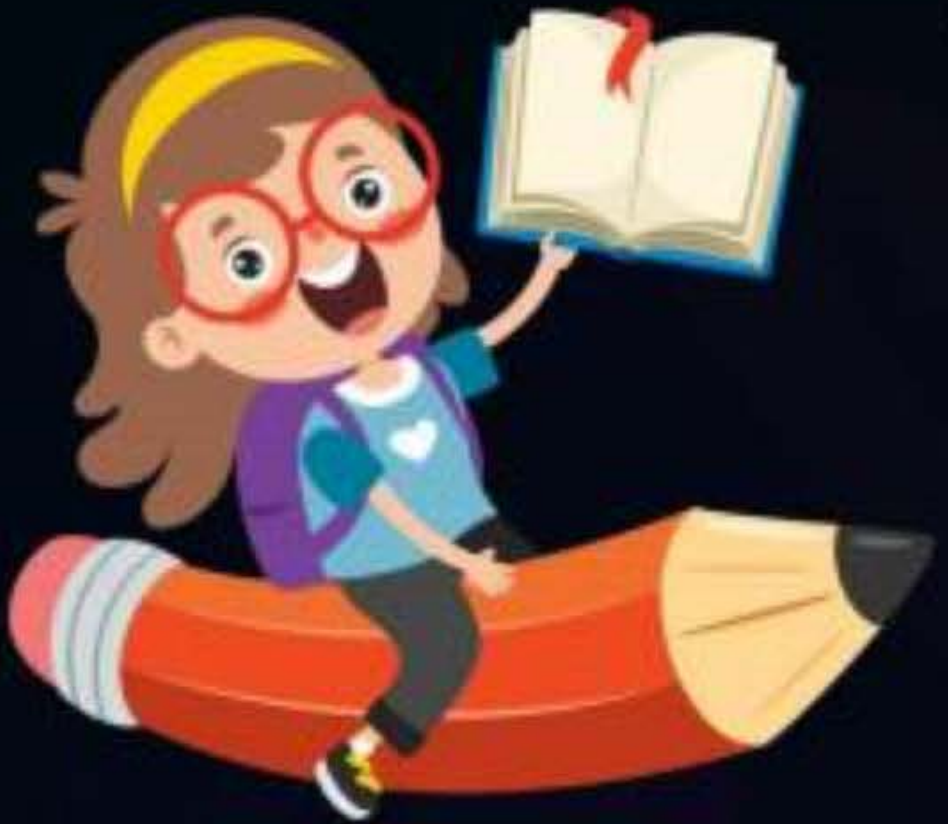
CALCULUS AND OPTIMIZATION
SINGLE VARIABLE CALCULUS

Lecture No.- 05



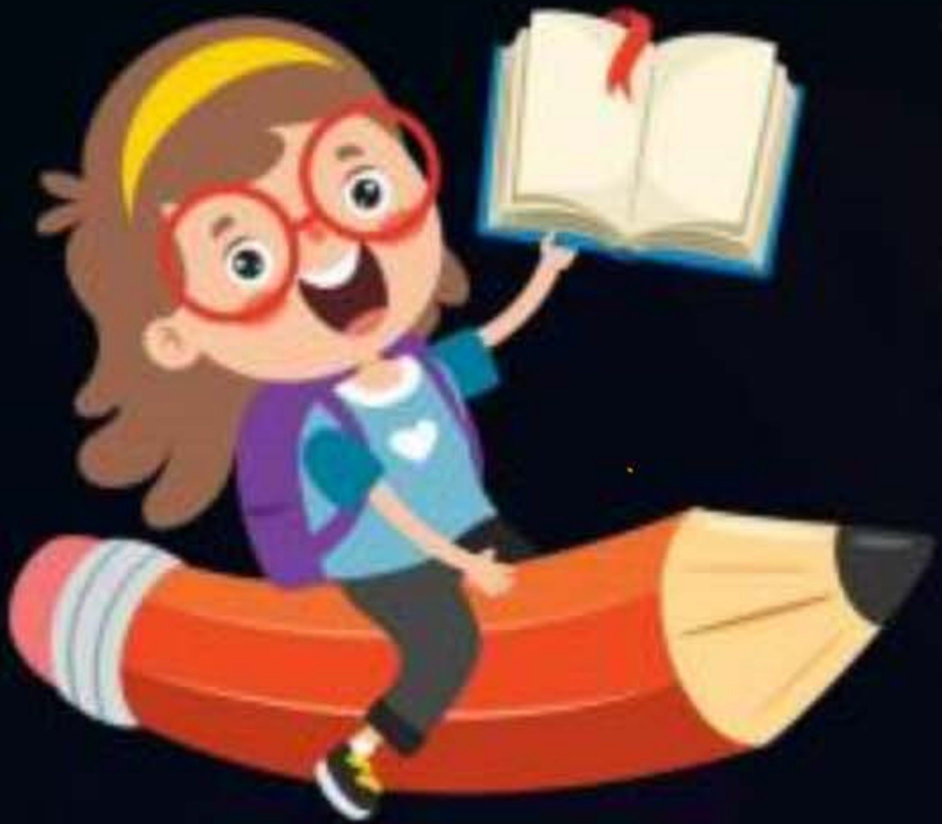
By- Rahul sir

Recape of previous lecture



Topic Evaluation of limits

Topics to be covered



Topic Evaluation of limits

Topic Evaluation of limits, Mean value theorem

CASE 02:

$$L = \lim_{x\to a}[f(x)]^{g(x)} = \lim_{x\to a}[\text{function}]^{\text{function}}$$

"Indeterminate Forms" $(\to\infty)^{\to\infty}$

$\boxed{(\to 1)^{\to\infty}}$

Trick

Taking Log both sides

$$\text{Log } L = \lim_{x\to a} \underbrace{g(x)\, \ln[f(x)]}_{\text{multiply}}$$

$$\boxed{L = e^{\lim_{x\to a} g(x)\ln[f(x)]}}$$

$\underbrace{g(x)\ln f(x)}_{\text{expression}}$ ⟶ change The $\dfrac{f(x)}{g(x)}$ L-Hospital Rule

$$\frac{\overset{\to N}{\ln f(x)}}{\underset{\to 0}{\dfrac{1}{g(x)}}} \quad \frac{\to 0}{\to 0} \ \& \ \frac{\to\infty}{\to\infty}$$

$$L = \lim_{x \to a} \left(f(x)\right]^{g(x)}$$

$$\text{Ans} = e^{A}$$

$$\boxed{\text{Where } A = \lim_{x \to a} [f(x) - 1]\, g(x)}$$

special form $(\to 1)^{\to \infty}$

$$\boxed{\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e}$$

$$L = \lim_{x \to 0} (\underbrace{1+x}_{f(x)})^{\frac{1}{x}}_{g(x)}$$

$1^{\to \infty}$ form

Orginal method

both sides taking log

$$\log L = \lim_{x \to 0} \frac{1}{x} \ln(1+x) = \lim_{x \to 0} \frac{\log(1+x)}{x} = \frac{\to 0}{\to 0}$$

$$= \lim_{x \to 0} \frac{\frac{1}{(1+x)}}{1} = 1$$

$$\log L = 1$$

$$\boxed{L = e^{1}} \text{ Ans}$$

II[nd] method only for $\to 1^{\to \infty}$ $= e^{A}$

$$= e^{1} = 1$$

$$A = \lim_{x \to 0} [\cancel{1} + x \cancel{- 1}] \frac{1}{x}$$

L-Hospital

Plug in Template

$$L = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}$$

log

Ist method Taking log both sides

$$\log L = \lim_{x \to \infty} x \log\left(\frac{x+1}{x}\right) = x \log\left(1 + \frac{1}{x}\right) = \text{convert } \frac{f(x)}{g(x)}$$

$$\lim_{x \to \infty} \frac{\log\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} = \frac{\to 0}{\to 0}$$

$$= \lim_{x \to \infty} \frac{\frac{1}{\left(1+\frac{1}{x}\right)}\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{-\frac{1}{x^2}}}$$

$$= \frac{1}{\frac{x+1}{x}} = \frac{x}{x+1} = \frac{1}{1+\frac{1}{x}}$$

$$= 1$$

$$\log L = 1$$

$$L = e$$

IInd method $\to 1^{\to \infty}$

$$= \lim_{x \to \infty} \left(\cancel{1} + \frac{1}{x} - \cancel{1}\right) x = 1$$

$$= e$$

Topic : Single Variable Calculus

#Q. Evaluate : $\lim_{x\to 0}(1-2x)^{1/x}$

Using Log.

only $\to 1^{\to\infty}$

$= \to 1^{\to\infty}$ form $= \underline{e^{-2}}$

$L = \lim_{x\to 0}\left[\cancel{1}-2x-\cancel{1}\right]\frac{1}{x}$

$= -2$

$\boxed{L = e^{-2}}$

$\lim_{x\to 0}\underbrace{(1-2x)}_{f(x)}^{\frac{1}{x}g(x)}$

$\lim_{x\to 0}(1-2x)^{\frac{1}{x}} = \lim_{x\to 0}\frac{1}{x}\log(1-2x)$

$= \underline{e^{-2}}$

Topic : Single Variable Calculus

#Q. **Evaluate :** $\lim_{x\to 1} x^{\cot \pi x}$ $\longrightarrow (\to 1)^{\to \infty}$

$= \lim_{x\to 1} x^{\cot \pi x} \longrightarrow e^{A}$

Where $A = \lim_{x\to 1} [f(x)-1]\, g(x)$

$= \lim_{x\to 1} [x-1] \cot \pi x$

$= \lim_{x\to 1} (x-1) \cot \pi x$

$= \lim_{x\to 1} \frac{(x-1)}{\tan \pi x} = \frac{1}{\pi}$

$= \underline{e^{1/\pi}}$

SC
0

$\lim_{x\to 1} \frac{(x-1)\pi}{\tan \pi x \; \pi}$

$= \frac{1}{\pi} \lim_{x\to 1} \frac{(\pi x - \pi)}{\tan(\pi - \pi x)}$

$= +\frac{1}{\pi} \lim_{x\to 1} \frac{\pi - \pi x}{\tan(\pi - \pi x)}$

Template

$= +\frac{1}{\pi} \times 1 = +\frac{1}{\pi}$

$\boxed{\text{Ans} = e^{+1/\pi}}$ $= e^{\frac{1}{\pi}}$

Topic : Single Variable Calculus

#Q. The value of $\lim_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$ is:

[1^∞ Type of indeterminate form]

$\lim_{x\to a}\left(2-\frac{a}{x}\right)^{\tan\frac{\pi x}{2a}} \Rightarrow 1^{\to\infty}$ form

$= e^{A}$

$A = \lim_{x\to a}\left(2-\frac{a}{x}-1\right)\tan\frac{\pi x}{2a}$

$= \lim_{x\to a}\frac{\left(1-\frac{a}{x}\right)}{\cot\frac{\pi x}{2a}} = \frac{+\frac{a}{x^2}}{-\csc^2\frac{\pi x}{2a}\,\frac{\pi}{2a}}$

$= -\frac{2}{\pi}$

A) e

B) e^{π}

C) $e^{-2/\pi}$ ✓

D) $e^{2/\pi}$

$A = -\frac{2}{\pi}$

Ans $= e^{-2/\pi}$

Not $\left(\rightarrow 1^{\rightarrow\infty}\right)$

#Q. The value of $L = \lim_{x\to 0}(1/x)^{\sin x}$ is:

[∞^0 Type of indeterminate form]

$\boxed{\left(\rightarrow 1^{\rightarrow\infty}\right) \text{ Not valid}}$

A 0

B 1/2

C 1

D None of these

$$L = \lim_{x\to 0}\left(\frac{1}{x}\right)^{\sin x} \rightarrow \infty^{\rightarrow 0}$$

$$\log L = \lim_{x\to 0} \sin x \log\left(\frac{1}{x}\right)$$

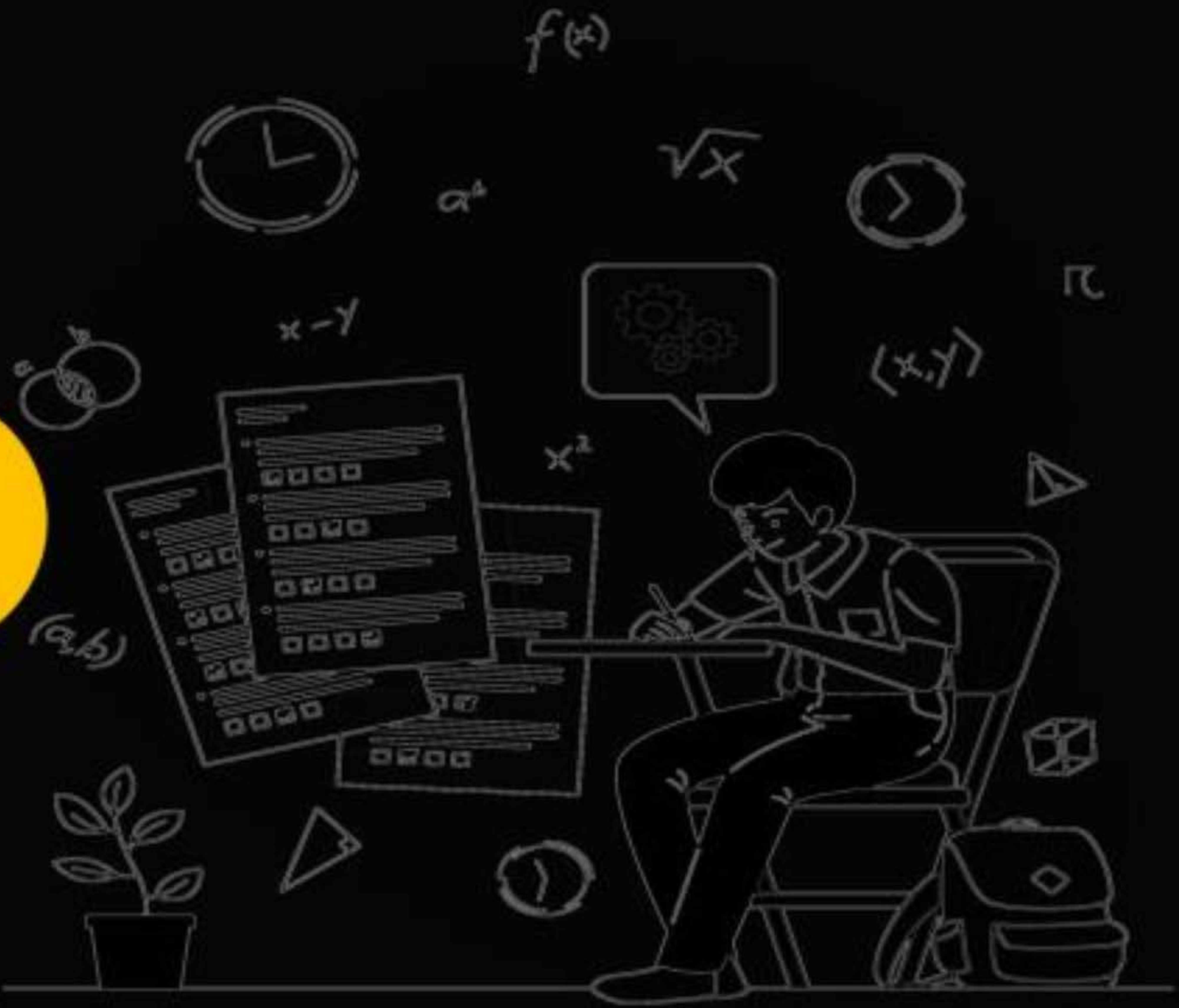
$$\log L = 0$$

$$\boxed{L = e^0 = 1}$$

$$= \lim_{x\to 0} \frac{\log\left(\frac{1}{x}\right)}{1/\sin x}$$

$$\Rightarrow \frac{\frac{1}{\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{-\frac{\cos x}{\sin^2 x}} = 0$$

Self Assessment Test



Topic : Single Variable Calculus

#Q. **Evaluate :** $\lim\limits_{x\to 0}\dfrac{x(e^x-1)+2(\cos x-1)}{x(1-\cos x)}$ $= 1$

$$\underset{x\to 0}{\mathrm{Lt}}\ \frac{x(e^x-1)+2(\cos x-1)}{x(1-\cos x)} = \frac{xe^x - x + 2\cos x - 2}{x - x\cos x}$$

$$= \underset{x\to 0}{\mathrm{lt}}\ \frac{(x+1)e^x - 1 + 2(-\sin x)}{x - [-x\sin x + \cos x]} = \frac{xe^x + e^x - 1 - 2\sin x}{x + x\sin x - \cos x}$$

$$= \underset{x\to 0}{\mathrm{lt}}\ \frac{(x+1)e^x + e^x - 2\cos x}{1 + x\cos x + \sin x \cdot 1 + \sin x}$$ Again.

$= 1$

Topic : Single Variable Calculus

$$\underset{x\to\infty}{Lt}\ \frac{x^3-\cos x}{x^2+(\sin x)^2}$$
$= \text{Infinity}$

#Q. $$\lim_{x\to\infty}\frac{x^3-\cos x}{x^2+(\sin x)^2}$$

A ∞

B 0

C 2

D Does not exist

Topic : Single Variable Calculus

#Q. **Evaluate :** $\underset{x \to 0}{\mathrm{Lt}}\ x \sin\left(\frac{1}{x}\right) =$

H.W

A. ∞

B. 0

C. 1

D. Does not exist

Topic : Single Variable Calculus

#Q. $\lim\limits_{\theta \to 0} \dfrac{\sin m\theta}{\theta}$, where m is an integer, is one of the following:

H.W

A m

B $m\pi$

C $m\theta$

D 1

Topic : Single Variable Calculus

#Q. **Evaluate :** $\lim\limits_{x\to 0} \frac{1}{10}\frac{1-e^{-j5x}}{1-e^{-jx}} =$

H.W

Topic : Single Variable Calculus

#Q. **Evaluate :** $\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)$ is $=\frac{1}{2}$

#Q. Evaluate : $\underset{x\to 0}{\text{Lt}} \dfrac{x-\sin x}{1-\cos x}$ is

1)

$$\underset{x\to 0}{Lt} \frac{1-\cos x}{0+\sin x} = \underset{x\to 0}{lt} \frac{1-\cos x}{\sin x} = \frac{\to 0}{\to 0}$$

Again $= \underset{x\to 0}{lt} + \frac{\sin x}{\cos x}$ Again

$= \underset{x\to 0}{lt} \tan x$

$= \underline{\underline{0}}$

$$\frac{1}{\sin x} - \frac{\cos x}{\sin x}$$

$= \dfrac{1-\cos x}{\sin x}$] Totally confused]

$= \operatorname{cosec} x - \cot x$

$= \underline{\to \infty}$

Topic : Single Variable Calculus

#Q. **Evaluate :** $\displaystyle \operatorname*{Lt}_{x\to 0}\left(\frac{e^{2x}-1}{\sin(4x)}\right) =$

L-Hospital Rule $= \frac{1}{2}$

OR Using Template method

$\displaystyle \operatorname*{Lt}_{x\to 0} \frac{\sin x}{x} = 1$

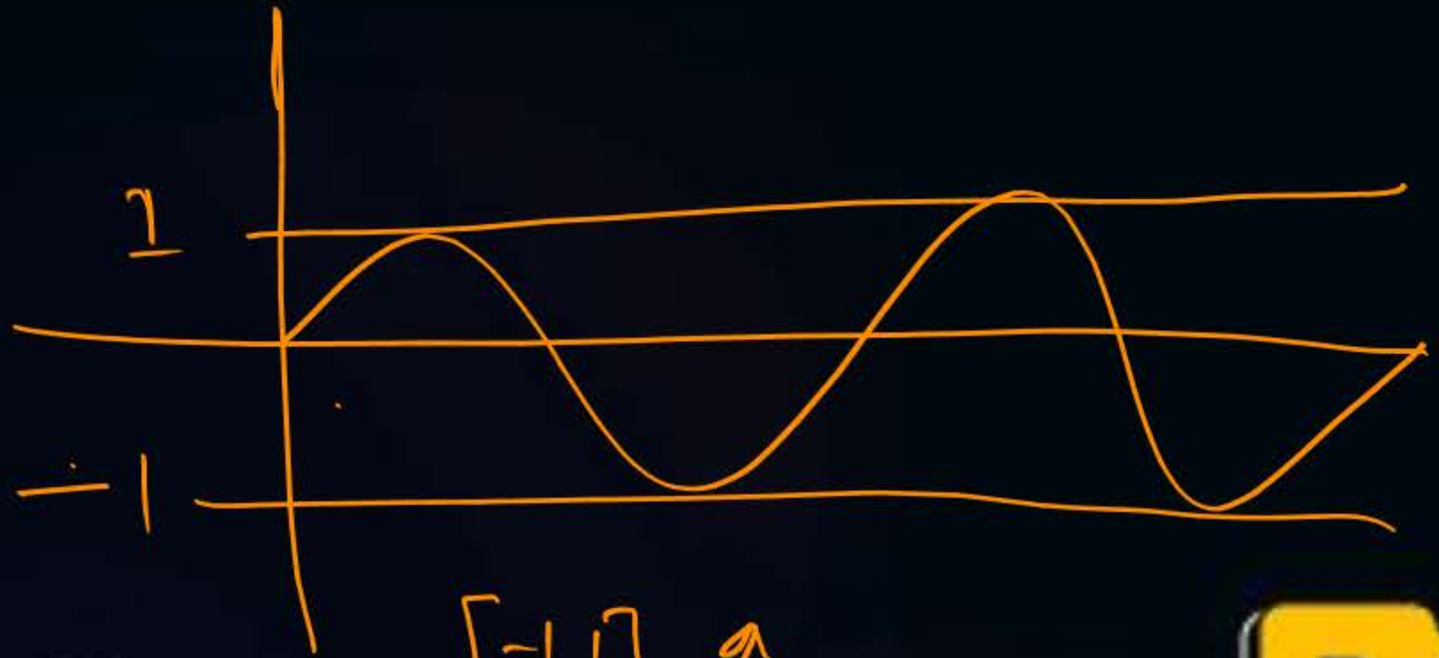
$\displaystyle \operatorname*{Lt}_{x\to 0} \frac{e^x-1}{x} = 1$

$\displaystyle \Rightarrow \operatorname*{Lt}_{x\to 0} \frac{e^{2x}-1}{2x} \cdot \frac{2x}{\sin 4x} \cdot \frac{4x}{4x}$

$= \frac{1}{2}$ <u>Ans</u>

#Q. **Evaluate :** $\underset{x\to\infty}{\mathrm{Lt}}\left(\dfrac{x+\sin x}{x}\right)$ is equal to

$$\underset{x\to\infty}{\mathrm{Lt}}\left(\frac{x+\sin x}{x}\right)$$
$$= \underset{x\to\infty}{\mathrm{lt}}\left(1+\frac{\sin x}{x}\right]$$
$$= 1+\underset{x\to\infty}{\mathrm{lt}}\frac{\sin x}{x}$$
$$= 1+\underset{x\to\infty}{\mathrm{lt}}\frac{[-1,1]}{x} \to 0$$
$$= 1+0$$
$$= 1 \text{ Ans.}$$



$$= \frac{[-1,1]}{\infty} \to 0$$

A $-\infty$

B 0

✓ C 1

D ∞

Topic : Single Variable Calculus

#Q. The expression $\lim_{a \to 0} \frac{x^a - 1}{a}$ is equal to

$\boxed{\underset{x \to 0}{Lt} \frac{a^x - 1}{x} = \log a}$ → Template

A. $\log x$

B. 0

C. $x \log x$

D. ∞

$\underset{a \to 0}{Lt} \frac{x^a - 1}{a}$ ← vari

a — variable
x → constant

$= \frac{\to 0}{\to 0}$ form

$= \underset{a \to 0}{Lt} \frac{x^a - 1}{a}$

$= \underset{a \to 0}{Lt} \frac{\frac{d}{da}(x^a - 1)}{\frac{d}{da}(a)}$

$= \underset{a \to 0}{Lt} \frac{x^a \log x - 0}{1}$

$= x^0 \log x$

$= \underline{\log x}$

Topic : Single Variable Calculus

#Q. **Evaluate :** $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x}$ is equal to

A e^{-2}

B e

C 1

D e

Topic : Single Variable Calculus

#Q. The value of $\lim_{x \to 0} \frac{1-\cos(x^2)}{2x^4}$ is

$\underset{x\to 0}{Lt} \frac{1-\cos x}{x^2} = \frac{1}{2}$ <u>Ans</u>

A 0

B 1/2

C 1/4

D undefined

$\underset{\boxed{x\to 0}}{Lt} \frac{1-\cos(x^2)}{2x^4}$ $\quad x^2 = t$, $x^4 = t^2$

$= \underset{t\to 0}{lt} \frac{1-\cos t}{2t^2} = \frac{1}{4}$

L-Hospital Rule $\frac{\to 0}{\to 0}$

$= \frac{1}{4}$

Derivative

Dummy var:

$\underset{x\to 0}{Lt} \frac{1-\cos x^2}{2x^4}$

$x^2 = t \quad t \to 0$
$x^4 = t^2$

$\underset{t\to 0}{Lt} \frac{1-\cos t}{2t^2} = \frac{\to 0}{\to 0}$ L-Hospital Rule $\underset{t\to 0}{Lt} \frac{0+\sin t}{4t} = \frac{1}{4}$

Topic : Single Variable Calculus

#Q. **Evaluate :** $\lim_{x\to 0}\left(\dfrac{-\sin x}{2\sin x + x\cos x}\right)$ is ________.

$= -\frac{1}{3}$

$$\lim_{x\to 0} \frac{-\sin x}{2\sin x + x\cos x} = \frac{0}{0} \text{ form}$$

Using L-Hospital

$$= \lim_{x\to 0} \frac{-\cos x}{2\cos x - x\sin x + \cos x}$$

$$= \frac{-1}{2+1}$$

$$= -\frac{1}{3} \text{ Ans}$$

Topic : Single Variable Calculus

#Q. **Evaluate :** $\lim_{x\to 0}\left(\frac{\tan x}{x^2 - x}\right)$ is ______.

$= \lim_{x\to 0}\left(\frac{\tan x}{x^2 - x}\right)$

$= \lim_{x\to 0} \frac{\sec^2 x}{2x-1}$

$= -1$

Topic : Single Variable Calculus

#Q. The value of $\lim\limits_{x\to 1}\dfrac{x^7-2x^5+1}{x^3-3x^2+2}$

A. is 0

B. is – 1

C. is 1

D. Does not exit

$$\mathop{Lt}_{x\to 1}\frac{x^7-2x^5+1}{x^3-3x^2+2}$$

Using L-Hospital Rule

$$=\mathop{lt}_{x\to 1}\frac{7x^6-10x^4}{3x^2-6x}$$

$$=-1 \quad \underline{\text{Ans}}$$

Topic : Single Variable Calculus

#Q. Compute $\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$

A 1

B limit does not exit

C 53/12

D ✓ 108/7

$$\underset{x \to 3}{Lt} \frac{x^4 - 81}{2x^2 - 5x - 3} = \frac{\to 0}{\to 0} \text{ form}$$

L-Hospital Rule

$$\underset{x \to 3}{lt} \frac{4x^3}{4x - 5}$$

$$= \frac{4 \times 27}{7}$$

$$= \frac{108}{7}$$

2 mins Summary

Topic One

Topic Two

Topic Three

Topic Four

Topic Five

$$\lim_{x \to a} [f(x)]^{g(x)} =$$

TIFR-CAM

ISI
TIFR
CMI
IMSc
IIMS - data analytics

$\boxed{27 \text{ lakh}}$

forms

FMS

THANK - YOU

Topics to be Covered