Data Science and Artificial Intelligence

Probability and Statistics

Bivariate Random Variable

Lecture No.-03

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Topics to be Covered

One Dimensional

Topic: Problems Based on One Dimensional Random Variable

Countable Discrete

0 1 2

$$\sum_{i=0} P[X = x_i] = 1$$

Probability mass function

Continuous uncountable

$$\int_R f(x)\,dx = 1$$

Density function

cdf $F_X(x) = P_X[X \le x_i]$

Topic : Problems Based on 1D Random Variable

$$\boxed{P(X \geq a) = \int_a^{\infty} f(x)\,dx}$$

Q1. If X is a continuous random variable whose probability density function is given by

$$f(x) = \begin{cases} k\left(5x - 2x^2\right), & 0 \leq x \leq 2 \\ 0, & \text{Otherwise} \end{cases}$$

$$f(x) = \begin{cases} K\left(5x - 2x^2\right) & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Then $P(x \geq 1)$ is

A. 3/14

B. 4/5

C. 14/17

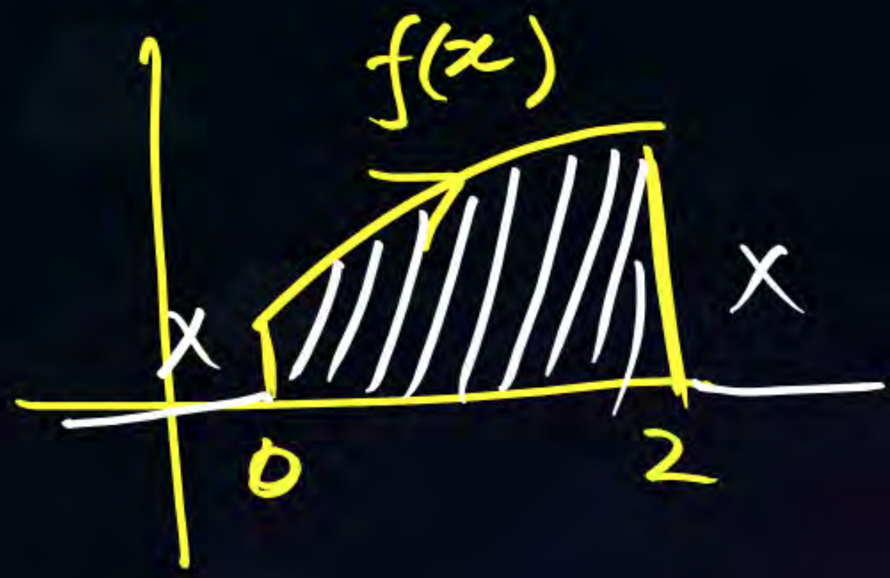
D. 17/28

$$P(X \geq 1) = \int_1^2 K(5x - 2x^2)\,dx$$

$\boxed{0 \text{ to } 1}$

In terms of K.

K remove



$$\int_0^2 K(5x-2x^2)\,dx = 1$$

$$= K\left(\frac{5x^2}{2} - \frac{2x^3}{3}\right]_0^2 = 1$$

$$\Rightarrow K\left[\left(\frac{5(2)^2}{2} - \frac{2(2)^3}{3}\right] - 0\right] = 1$$

$$\Rightarrow \boxed{K = \frac{3}{14}}$$

If finding constant K

using valid pdf $= 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

✓ $P(X \geq 1) = \int_1^2 \frac{3}{14}(5x-2x^2)\,dx$

$$= \frac{3}{14}\int_1^2 (5x-2x^2)\,dx = \frac{3}{14}\left[\frac{5x^2}{2} - \frac{2x^3}{3}\right]_1^2 = \frac{17}{28}$$

Topic : Problems Based on 1D Random Variable

$f_X(x) = P_X(x)$

Q2. $\boxed{P_x(X) = Me^{(-2|x|)} + Ne^{(-3|x|)}}$ is the probability density function for the real random variable X, over the entire x-axis, M and N are both positive real numbers. The equation relating M and N is

A. $M + \frac{2}{3}N = 1$

B. $2M + \frac{1}{3}N = 1$

C. $M + N = 1$

D. $M + N = 3$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \quad \text{or} \quad \int_{-\infty}^{\infty} P_X(x)\,dx = 1$$

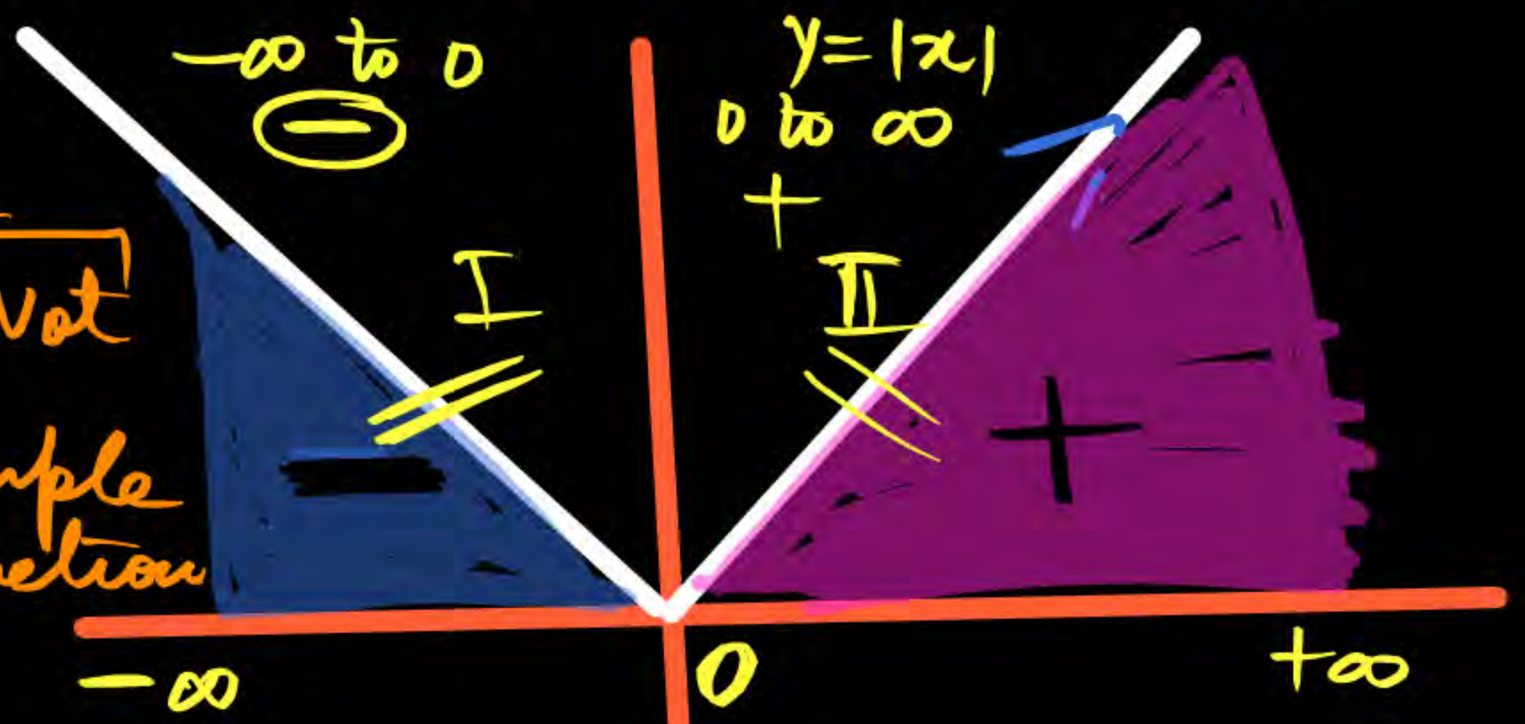
$$f_X(x) = Me^{-2|x|} + Ne^{-3|x|}$$

$$\int_{-\infty}^{\infty} Me^{-2|x|} + Ne^{-3|x|}\,dx = 1$$

Compound Function] # Not a simple function

$$f(x) = \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases} \quad |x|$$

] graph Draw



$$= \int_{-\infty}^{0} Me^{-2|x|} + Ne^{-3|x|}\,dx + \int_{0}^{\infty} Me^{-2|x|} + Ne^{-3|x|}\,dx = 1$$

$$= \int_{-\infty}^{0} Me^{-2(-x)}\,dx + Ne^{-3(-x)}\,dx + \int_{0}^{\infty} Me^{-2x} + Ne^{-3x}\,dx = 1$$

$$= \int_{-\infty}^{0} (Me^{+2x} + Ne^{3x})\, dx + \int_{0}^{\infty} Me^{-2x} + Ne^{-3x}\, dx = 1$$

$$= \left[\frac{Me^{2x}}{2} + \frac{Ne^{3x}}{3}\right]_{-\infty}^{0} + \left[\frac{\cancel{Me^{-2x}}}{-2} + \frac{\cancel{Ne^{-3x}}}{-3}\right]_{0}^{\infty} = 1$$

$$= \left\{\left[\frac{M}{2} + \frac{N}{3}\right] - \left[\frac{Me^{-\infty}}{2} + \frac{Ne^{-\infty}}{3}\right]\right\} + \left(+\frac{M}{2} + \frac{N}{3}\right) = 1$$

$$\frac{M}{2} + \frac{N}{3} + \frac{M}{2} + \frac{N}{3} = 1$$

$$M + \frac{N}{3} + \frac{N}{3} = 1$$

$$M + \frac{2N}{3} = 1$$

Rohit formula

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$\int e^x dx = e^x + c$

$\int_a^b F(x)\,dx = F(b) - F(a)$

Q3. A continuous random variable X has a probability density function $f(x) = e^{-x}$, $0 < x < \infty$. Then $P\{X > 1\}$ is

A. 0.368

B. 0.5

C. 0.632

D. 1.0

$f(x) = e^{-x} \quad 0<x<\infty$

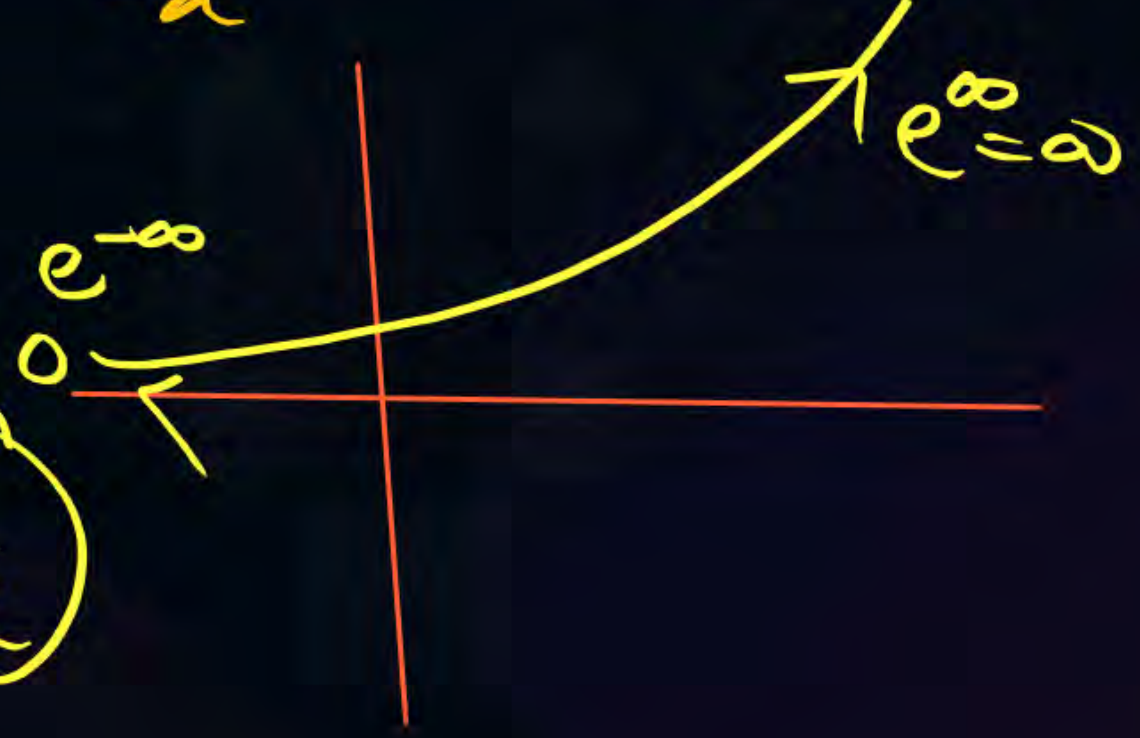
$P(X>1) = \int_1^\infty f(x)\,dx$

$= \int_1^\infty e^{-x}\,dx$

$= \left[-e^{-x}\right]_1^\infty = -e^{-\infty} + e^{-1}$

$= 0 + e^{-1} = e^{-1} = \frac{1}{e}$

$= \frac{1}{2.732}$



Q4. Find the value of λ such that the function f(x) is a valid probability density function ____.

$$f(x) = \lambda(x-1)(2-x) \quad \text{for } 1 \le x \le 2$$
$$= 0 \quad \text{otherwise}$$

$f(x) = \lambda(x-1)(2-x) \quad 1 \le x \le 2$

$= 0$

$$\int_1^2 \lambda(x-1)(2-x)\,dx = 1$$

This is a valid pdf.

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$

$$= \lambda \int_1^2 (2x - x^2 - 2 + x)\,dx = 1$$

$$\lambda = 6$$

5 min

Prob. of Three dots $= 3K = 3 \times \frac{1}{21} = \frac{1}{7}$

Q5. Consider a die with the property that the probability of a face with 'n' dots showing up is proportional to 'n'. The probability of the face with three dots showing up is $\frac{1}{7}$.

dot $\propto$ prob.

Die

X = No. of dots

$P[X=x_i]$ = prob.

1 dot $\propto 1$, $1 = k \cdot 1$

2 dots $\propto 2$, $2 = k \cdot 2$

3 dots $\propto 3$, $3 = 3 \cdot k$

4 dots $\propto 4$, $4 = 4 \cdot k$

5 dots $\propto 5$, $5 = 5 \cdot k$

6 dots $\propto 6$, $6 = 6 \cdot k$

Discrete Random Variable.

X	1	2	3	4	5	6
$P[X=x_i]$	K	$2K$	$3K$	$4K$	$5K$	$6K$

Prob. of 3 dots $= (3K)$

using Prob. mass function $\sum_{i=1}^{6} P[X=x_i] = 1$

$K + 2K + 3K + 4K + 5K + 6K = 1$

$K = \frac{1}{21}$

Topic : Problems Based on 1D Random Variable

Q6. Let X be a random variable with probability density function

Do yourself

$$f(x)=\begin{cases}0.2 & \text{for} |x| \le 1 \\ 0.1 & \text{for } 1<|x|\le 4 \\ 0 & \text{otherwise}\end{cases}$$

The probability $P(0.5 < x < 5)$ is ____.

continuous random variable

Q7. Lifetime of an electric bulb is a random variable with density $\boxed{f(x) = kx^2}$, where x is measured in years. If the minimum and maximum lifetimes of bulb are 1 and 2 years respectively, then the value of k is ____.

$f(x) = Kx^2$

x = measured in YEARS

If this is a valid prob. density function $\int = 1$

$$\int_{x=1}^{2} Kx^2\,dx = 1$$

$$K\left[\frac{x^3}{3}\right]_1^2 = 1$$

$$= K\left[\frac{8}{3} - \frac{1}{3}\right] = 1$$

$$\boxed{K = \frac{3}{7}}$$ Ans.

Q8. Given that x is a random variable in the range $[0, \infty]$ with a probability density function $\frac{e^{-\frac{x}{2}}}{K}$, the value of the constant K is

$$f(x) = \begin{cases} \frac{e^{-\frac{x}{2}}}{K} & 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

finding constant = valid pdf

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^{\infty} \frac{e^{-\frac{x}{2}}}{K}\,dx = 1$$

$$= \frac{1}{K}\int_0^{\infty} e^{-\frac{x}{2}}\,dx = 1$$

$$I = \int_0^{\infty} e^{-\frac{x}{2}}\,dx$$

$\frac{x}{2} = t$ $\quad \frac{dx}{2} = dt$ $\quad dx = 2dt$

$$= \int_0^{\infty} e^{-t}\,2dt$$

$$2\int_0^{\infty} e^{-t}\,dt$$

$$\boxed{K = 2}$$

↑ solve

Mathematica

Q15. A normal random variable X has the following probability density function

$$f_X(x) = \frac{1}{\sqrt{8\pi}} e^{-\left\{\frac{(x-1)^2}{8}\right\}}, -\infty < x < \infty$$

$$f_X(x) = \frac{1}{\sqrt{8\pi}} e^{-\left(\frac{(x-1)^2}{8}\right)}$$

$-\infty < x < \infty$

Then $\int_1^\infty f_X(x)dx =$

$$\boxed{\int_1^\infty f_X(x)\,dx = \int_1^\infty \frac{1}{\sqrt{8\pi}} e^{-\left(\frac{(x-1)^2}{8}\right)} dx}$$

A. 0

B. $\frac{1}{2}$

C. $1-\frac{1}{e}$

D. 1

$$\frac{(x-1)^2}{8} = t \qquad \boxed{t=0 \quad t=\infty}$$

both sides Differentiate It

$$2(x-1)\,dx = 8\,dt$$

$$(x-1)dx = 4dt$$

$$dx = \frac{4}{(x-1)}dt$$

$$\Rightarrow \int_0^\infty e^{-t} \cdot \frac{4\,dt}{(x-1)}$$

$$\frac{(x-1)^2}{8} = t$$

$$(x-1) = \sqrt{8t}$$

$$\Rightarrow \int_0^\infty e^{-t} \frac{4 \cdot}{\sqrt{8t}} dt$$

Gamma Function

$$= \frac{4}{\sqrt{8}} \int_0^\infty e^{-t} t^{-1/2} dt$$

$$\Gamma n = \int_0^\infty e^{-t} t^{n-1} dt$$

$$= \frac{4}{\sqrt{8}} \boxed{\int_0^\infty e^{-t} t^{-1/2} dt}$$

gamma function

$$\Gamma n = (n-1)!$$

$$\Gamma 4 = (4-1)! = 3! = 6$$

$$\Gamma \frac{1}{2} = \sqrt{\pi}$$

$$\Gamma \frac{7}{2} = \frac{5}{2} \frac{3}{2} \frac{1}{2} \times \sqrt{\pi}$$

$$\Gamma \frac{3}{2} = \frac{1}{2} \Gamma \frac{1}{2} = \sqrt{\pi}/2$$

lock

$$= \frac{4}{\sqrt{8}} \boxed{\int_0^\infty e^{-t}\, t^{-1/2}\, dt}$$

compare It $\int_0^\infty e^{-t}\, t^{n-1}\, dt$

$n-1 = -\frac{1}{2}$

$n = -\frac{1}{2} + 1 = \frac{1}{2}$

$$= \frac{1}{\sqrt{8\pi}} \times \frac{4}{\sqrt{8}} \times \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{1}{\sqrt{8\cancel{\pi}}} \times \frac{4}{\sqrt{8}} \times \cancel{\sqrt{\pi}}$$

$$= \frac{1}{2} \ \underline{\text{Ans}}$$

Q17. The graph of function $f(x)$ is shown in figure for $f(x)$ to be a valid probability density function, the value of h is

valid pdf

Total Area = 1

A. 1/3

B. 2/3

C. 1

D. 3

$$\int_0^1 + \int_1^2 + \int_2^3$$

$$\frac{h}{2} + h + \frac{3h}{2} = 1$$

$$\Rightarrow \boxed{h = \frac{1}{3}}$$

$$\underbrace{\frac{1}{2} \times b_1 \times h_1}_{} + \frac{1}{2} \times b_2 \times h_2 + \frac{1}{2} b_3 \times h_3 = 1$$

$$\frac{1}{2} \times h \times 1 + \frac{1}{2} \times 1 \times 2h + \frac{1}{2} \times 1 \times 3h = 1$$

Topic : Problems Based on 1D Random Variable

Q18. The random variable X takes on the values 1, 2 (or) 3 with probabilities $\frac{2+5P}{5}$, $\frac{1+3P}{5}$ and $\frac{1.5+2P}{5}$ respectively the values of P [redacted]

A. ✓ 0.05, 1.87

B. ✓ 1.90, 5.87

C. ✓ 0.05, 1.10

D. ✓ 0.25, 1.40

X	1	2	3
$P(X=x)$	$\frac{2+5p}{5}$	$\frac{1+3p}{5}$	$\frac{1.5+2p}{5}$

Using Prob. mass function

$$\sum_{i=1}^{3} P[X=x_i] = 1$$

$$\frac{2+5p}{5} + \frac{1+3p}{5} + \frac{1.5+2p}{5} = 1$$

$$\boxed{p = 0.05}$$

Topic : Problems Based on 1D Random Variable

NCERT

Do yourself

Q20. The function p(x) is given by $p(x) = A/x^{\mu}$ where A and μ are constants with $\mu > 1$ and $1 \leq x < \infty$ and $p(x) = 0$ for $-\infty < x < 1$. For p(x) to be a probability density function, the value of A should be equal to

H.W

A. $\mu - 1$

B. $\mu + 1$

C. $1/(\mu - 1)$

D. $1/(\mu + 1)$

THANK - YOU