Data Science and Artificial Intelligence

Probability and Statistics

Bivariate Random Variable

Lecture No.- 04

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Topics to be Covered

Topic Problem Based on Bivariate Random Variable 02 (Univariate / Bivariate)

Topic : Problem Based on Bivariate Random Variable

Q1. A fair coin is tossed 3 times. Let the random variable X denote the number of heads in 3 tosses of the coin. Find the probability density function of X.

A. $\left(\frac{3}{x}\right)\left(\frac{1}{2}\right)^{2x}\left(\frac{1}{2}\right)^{2-x}$

B. $\left(\frac{3}{2x}\right)\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{1-x}$

C. $\left(\frac{3}{x}\right)\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{3-x}$

D. $\left(\frac{3}{x}\right)\left(\frac{1}{2}\right)^{x}\left(\frac{1}{2}\right)^{4-x}$

$\frac{1}{144}$ ✓

$\int_1^2 + \int_2^3 + \int_3^4 + \int_4^5 + \int_5^6 + \dots$

Q2. If the probability of a random variable X is given by

$f(x) = k(2x-1)$, $x = 1, 2, 3 \dots, 12$. Find k.

$f(x) = K(2x-1)$

$x = 1, 2, 3 \dots 12$

find $K =$ ✓

$\sum x = \frac{x(x+1)}{2}$

$1, \dots 12$

$= \frac{12(12+1)}{2}$

$= \frac{\cancel{12}^{6} \times 13}{\cancel{2}} = 78$

$\boxed{f(x) = K(2x-1)}$ $x = 1, 2, 3 \dots 12$

$\sum$ all Prob. $= 1$

$\sum_{x=1}^{12} 2Kx - K = 1$

$2K \sum_{x=1}^{12} x - K \sum_{x=1}^{12} 1 = 1$

$2K \times \frac{12 \times 13}{2} - K \times 12 = 1$

$1+1+1+1+1+1+1+1+1+1+1+1 = 12$

$\boxed{K = \frac{1}{144}}$

Q3. The density function for the continuous random variable X is

$$f_x(x) = \begin{cases} e^{-X} for\, x > 0 \\ 0\; for\, x \leq 0 \end{cases}$$

Find the Probability P [x ≤ 2| x > 1]

$$f(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

find the Prob. $P\left(X \leq 2 / X > 1\right) = \dfrac{P(X \leq 2 \cap X > 1)}{P(X > 1)}$

$P\left(\frac{A}{B}\right)$

B is already Happend

$\frac{A}{B}$ Happening

$$= \frac{\int_1^2 e^{-x}\,dx}{\int_1^\infty e^{-x}\,dx}$$

✓ Nr $\int_1^2 e^{-x}\,dx = \left[-e^{-x}\right]_1^2 = -e^{-2}+e^{-1}$

$= e^{-1}-e^{-2}$

✓ Dr $\int_1^\infty e^{-x}\,dx = \left[-e^{-x}\right]_1^\infty = -e^{-\infty}+e^{-1}$

$= 0+e^{-1}$

$$P\left(\frac{X\le 2 \wedge X>1}{P(X>1)}\right) = \frac{e^{-1}-e^{-2}}{e^{-1}}$$

✓ $= \boxed{1-e^{-1}} = \underline{0.632}$

Topic : Problem Based on Bivariate Random Variable

Q4. Suppose the random variable X has a probability density function

Conti Random var.

$$f(x) = \begin{cases} kx^3 e^{-x/2}, & x > 0 \\ 0 & otherwise \end{cases}$$

$$f(x) = \begin{cases} Kx^3 e^{-\frac{x}{2}} & x > 0 \\ 0 & otherwise \end{cases}$$

The value of k is

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$= \boxed{\int_{-\infty}^{\infty} Kx^3 e^{-x/2}\,dx = 1}$$

A. 1/96

B. 96

C. 8/3

D. 1/4

$$= \int_0^\infty K e^{-x/2} x^3 \, dx = 1$$

$$\Rightarrow K \int_0^\infty e^{-\frac{x}{2}} x^3 \, dx = 1$$

$t = \frac{0}{2} = 0$ $\qquad \frac{x}{2} = t$ $\qquad x = 2t$

$t = \frac{\infty}{2} = \infty$ $\qquad dx = 2dt$

$$\Rightarrow K \int_0^\infty e^{-t} \cdot (2t)^3 \cdot 2dt = 1$$

$$\Rightarrow 16K \underbrace{\int_0^\infty e^{-t} t^3 \, dt}_{\text{Convert The Gamma function}} = 1$$

0 to ∞ OR −∞ to +∞

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} \, dx$$

exponetial + alegebraic

$$16K \boxed{\int_0^\infty e^{-t} t^3 \, dt} = 1$$

$n - 1 = 3$

$\boxed{n = 4}$

$16K \times \Gamma 4 = 1$

$16K \times 3! = 1$

$$\boxed{K = \frac{1}{96}}$$

Q5. Let X be a continuous random variable with pdf

$$f_X(x) = \begin{cases} cx^2, \text{ for } 0 < x \leq 1, \\ 0 \quad \text{otherwise} \end{cases}$$

$\int_0^1 cx^2\,dx = 1$

$\boxed{c = 3}$

$f(x) = cx^2 \quad 0 < x \leq 1$

$= 0 \quad$ otherwise

For some positive constant c. The value of $P\left(X \leq \frac{2}{3} \middle| X > \frac{1}{3}\right)$

$P\left(X \leq \frac{2}{3} / X > \frac{1}{3}\right)$

$$P\left(X \leq \frac{2}{3} / X > \frac{1}{3}\right) = \frac{P\left(X \leq \frac{2}{3} \cap X > \frac{1}{3}\right)}{P\left(X > \frac{1}{3}\right)} = \frac{\int_{1/3}^{2/3} 3x^2\,dx}{\int_{1/3}^{1} 3x^2\,dx}$$

$= \frac{7}{26}$

A. 3/26

B. 5/26

C. 7/26 ✓

D. 11/26

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Q6. Suppose the random variable X has the probability density function

$$f(x) = \begin{cases} ce^{x/3}, & x \leq 0, \\ ce^{-x/3}, & x > 0, \end{cases}$$

Do yourself

For some positive constant c. The value of P [x > 6/x > 0] is

A. e^{-2}

B. ce^{-2}

C. 0

D. $1-e^{-2}$

Q7. Let X be a discrete random variable with probability function $P(X = x) = \frac{2}{3^x}$ for $x = 1, 2, 3, \ldots$. What is the probability that X is even?

A. 1/4

B. 2/7

C. 1/3

D. 2/3

$P(X=x) = \frac{2}{3^x}$ $x = 1, 2, 3.$

What is the Probability X is even.

X	2	4	6	8	10
$P(X=x)$	$\frac{2}{3^2}$	$\frac{2}{3^4}$	$\frac{2}{3^6}$	$\frac{2}{3^8}$	$\frac{2}{3^{10}}$

$= P(X=2) + P(X=4) + P(X=6) + P(X=8) + \text{------}$

Discrete Random [Countable SET)

$= \frac{2}{3^2} + \frac{2}{3^4} + \frac{2}{3^6} + \frac{2}{3^8} + \frac{2}{3^{10}} + \text{-----}$

$= \frac{2}{9}\left[1 + \frac{1}{9} + \left(\frac{1}{9}\right)^2 + \left(\frac{1}{9}\right)^3 + \left(\frac{1}{9}\right)^4 + \text{--}\right]$ SERIES

G.P.

$a + ar + ar^2 + ar^3 + \text{---}$

$\Rightarrow \frac{2}{9}\left[1 + \frac{\frac{1}{9}}{1 - \frac{1}{9}}\right]$

$= \frac{2}{9}\left[1 + \frac{\frac{1}{\cancel{9}}}{\frac{8}{\cancel{9}}}\right] = \frac{1}{4}$

$S_\infty = \frac{a}{1-r} = \frac{\text{1st term}}{1 - \text{common ratio}}$

$r = \frac{T_2}{T_1} =$

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Q8. Let $f(x) = \dfrac{k\,|\,x\,|}{(1+|\,x\,|)^4}, -\infty < x < \infty$

$$f(x) = \frac{k|x|}{(1+|x|)^4} \quad -\infty < x < \infty$$

Then the value of k for which $f(x)$ is a probability density function is

A. 1/6

B. 1/2

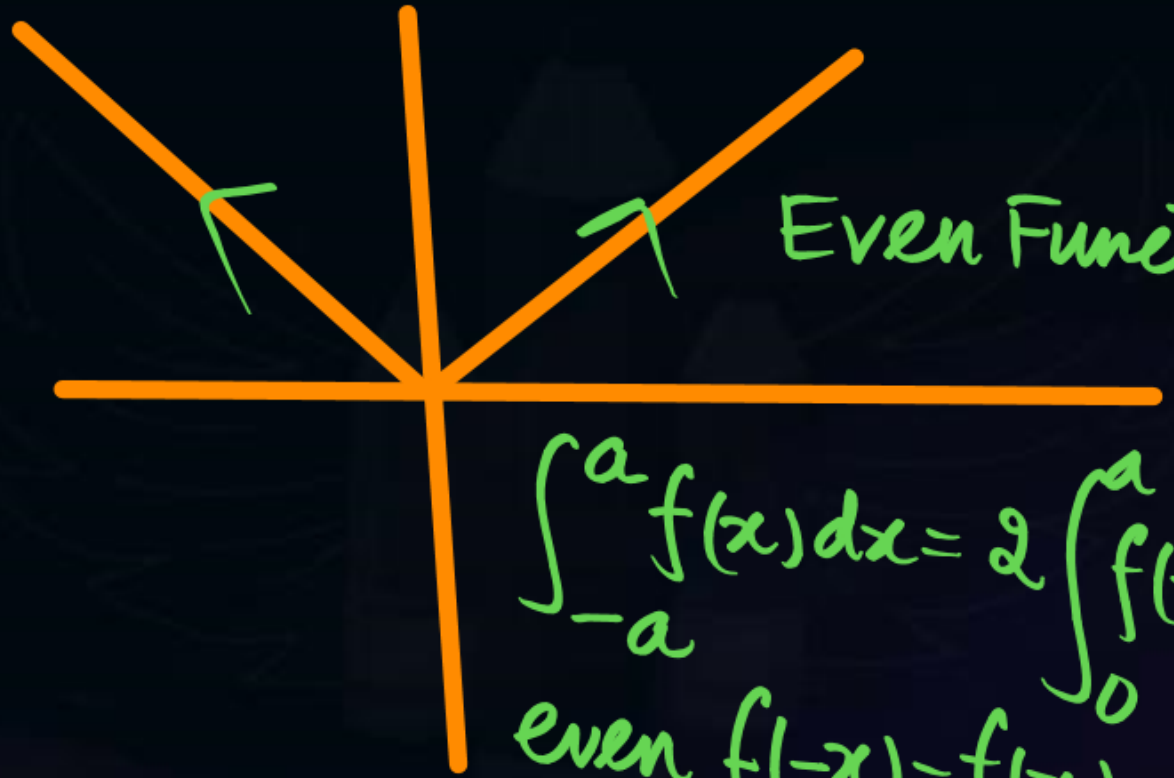
C. 3

D. 6

$$\int_{-\infty}^{\infty} \frac{k|x|}{1+|x|^4}\,dx = 1$$

$$= 2\int_0^{\infty} \frac{kx}{1+x^4}\,dx = 1$$

$$= 2\int_0^{\infty} \frac{kx}{1+x^4}$$



Even Function

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$$

even $f(-x) = f(x)$

$$I = \int_{-\infty}^{\infty} \frac{K|x|}{(1+|x|)^4}$$ 100%

|x| even Function

Beta Function

$$\int_0^{\infty} \frac{K \cdot x}{(1+x)^4} dx$$

Beta Function

$$= 2\int_0^{\infty} \frac{Kx}{(1+x)^4} = 2\int_0^{\infty} \frac{K \cdot x^{l-1}}{(1+x)^{l+m}}$$

bracket

$$= 2\int_0^{\infty} \frac{K \cdot x}{(1+x)^4} = 2\int_0^{\infty} \frac{K \cdot x^{2-1}}{(1+x)^{2+2}}$$

$$\frac{x^{l-1}}{(1+x)^{l+m}} = \frac{\Gamma l \, \Gamma m}{\Gamma l+m}$$

$$= 2 \times k \frac{\Gamma 2 \, \Gamma 2}{\Gamma 2+2}$$

$$= \frac{2k \cdot 1}{3!} = 1$$

$k=3$

Q10. A continuous random variable X has density function

$$f(x)=\begin{cases} 2x & 0<x<\frac{1}{2} \\ \frac{4-2x}{3} & \frac{1}{2}\le x<2 \\ 0, & \text{elsewhere} \end{cases}$$

✓ Do yourself

Find P $[0.25 < x \le 1.25]$

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Q11. Let X be a continuous random variable with probability density function

$$f(x) = \frac{1}{2} e^{-|x-1|}, \quad -\infty < x < \infty$$

Find the value of $P(1 < |X| < 2)$

10th $|x|<a$, $|x|>a$

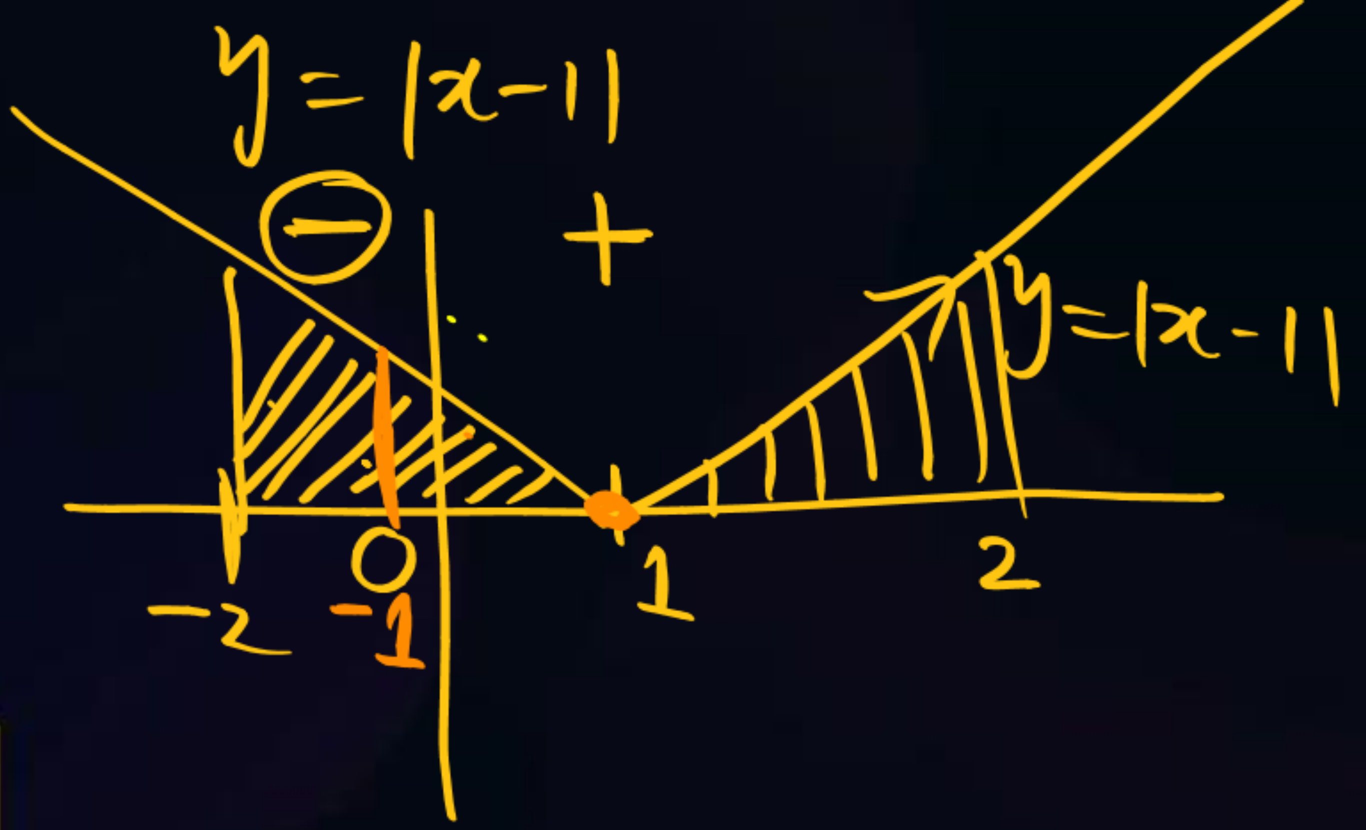
$f(x) = \frac{1}{2} e^{-|x-1|}$

$P[1<|x|<2] = |x|>1$, OR $|x|<2$

$\left\{ \begin{array}{l} (-\infty,-1) \cup (1,\infty) \\ -2<x<2 \end{array} \right.$

$\Rightarrow \int_{-2}^{-1} \frac{1}{2} e^{-(-(x-1))} dx + \int_{1}^{2} \frac{1}{2} e^{-(x-1)} dx$

$\Rightarrow \frac{1}{2}(e^{-1}+e)(e^{-1}+e^{-2})$



$$\int_0^\infty \frac{x^{l-1}}{(1+x)^{m+l}} = \frac{\Gamma l\,\Gamma m}{\Gamma l+m} \qquad = \Gamma l \Rightarrow (l-1)!$$

$$\Gamma m \Rightarrow (m-1)!$$

Q12. The following table represents the joint probability distribution of the discrete random variable (X, Y):

X \ Y	1	2
1	0.1	0.2
2	0.1	0.3
3	0.2	0.1

Find :

(i) F (2, 2), F (3, 2)
(ii) $F_X(3)$
(iii) $F_Y(1)$

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5-10 min

Q13. Consider an experiment of drawing randomly three balls from an urn containing two red, three white, and four blue balls. Let (X, Y) be a bivariate r.v. where X and Y denote, respectively, the number of red and white bails chosen.

$X =$ No. of red ball

$Y =$ No. of white ball

A. Find the range of (X, Y).

B. Find the joint pmf's of (X, Y).

C. Find the marginal pmf's of X and Y.

D. Are X and Y independent?

$P[X = x_i]$

$P[Y = y_j]$

marginal Prob.

$$P[X = x_i \cap Y = y_j] = P[X = x_i]\, P[Y = y_j]$$

X = No. of red ball
Y = No. of white ball

X = No of red ball 0, 1, 2
Y = No. of white ball 0, 1, 2, 3

Joint Prob. Mass Function
$P[X = x_i, Y = y_j]$ where
$i = 0, 1, 2$
$j = 0, 1, 2, 3$

X \ Y	0	1	2	3
0	(0,0)	(0,1)	(0,2)	(0,3)
1	(1,0)	(1,1)	(1,2)	(1,3)
2	(2,0)	(2,1)	(2,2)	(2,3)

Range = (Ω_{xy})

3 balls

$\frac{4C_3}{9C_3}$

$P[X=0, Y=0] = P[X = \text{No red ball} \wedge Y = \text{No white ball}]$

$\Rightarrow \frac{4C_3}{9C_3} = \frac{4}{84}$

$P[X=0, Y=1] \Rightarrow \frac{18}{84}$

$P[X=0, Y=2] = \frac{12}{84}$

$P[X=1, Y=0] = \frac{12}{84}$

$P[X=1, Y=1] = \frac{24}{84}$

$P[X=1, Y=2] = \frac{6}{84}$

$P[X=2, Y=0] = \frac{4}{84}$

$P[X=2, Y=1] = \frac{3}{84}$

$P[X=2, Y=2] =$

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Q16. Consider the binary communication channel shown in Fig. 3-4. Let (X, Y) be a bivariate r.v., where X is the input to the channel and Y is the output of the channel. Let $P(X = 0) = 0.5$, $P(Y = 1|X = 0) = 0.1$ and $P(Y = 0|X = 1) = 0.2$.

(a) Find the joint pmf's of (X, Y)

(b) Find the marginal pmf's of X and Y.

(c) Are X and Y independent?

X = input channel

Y = output channel

$P[X=1] = 1 - P[X=0] = 1 - 0.5 = 0.5$

$P\left(\frac{Y=1}{X=0}\right) = 0.1$

$P\left(\frac{Y=0}{X=0}\right) = 1 - 0.1 = 0.9$

$P\left(\frac{Y=0}{X=1}\right) = 0.2$

$P\left(\frac{Y=1}{X=1}\right) = 1 - 0.2 = 0.8$

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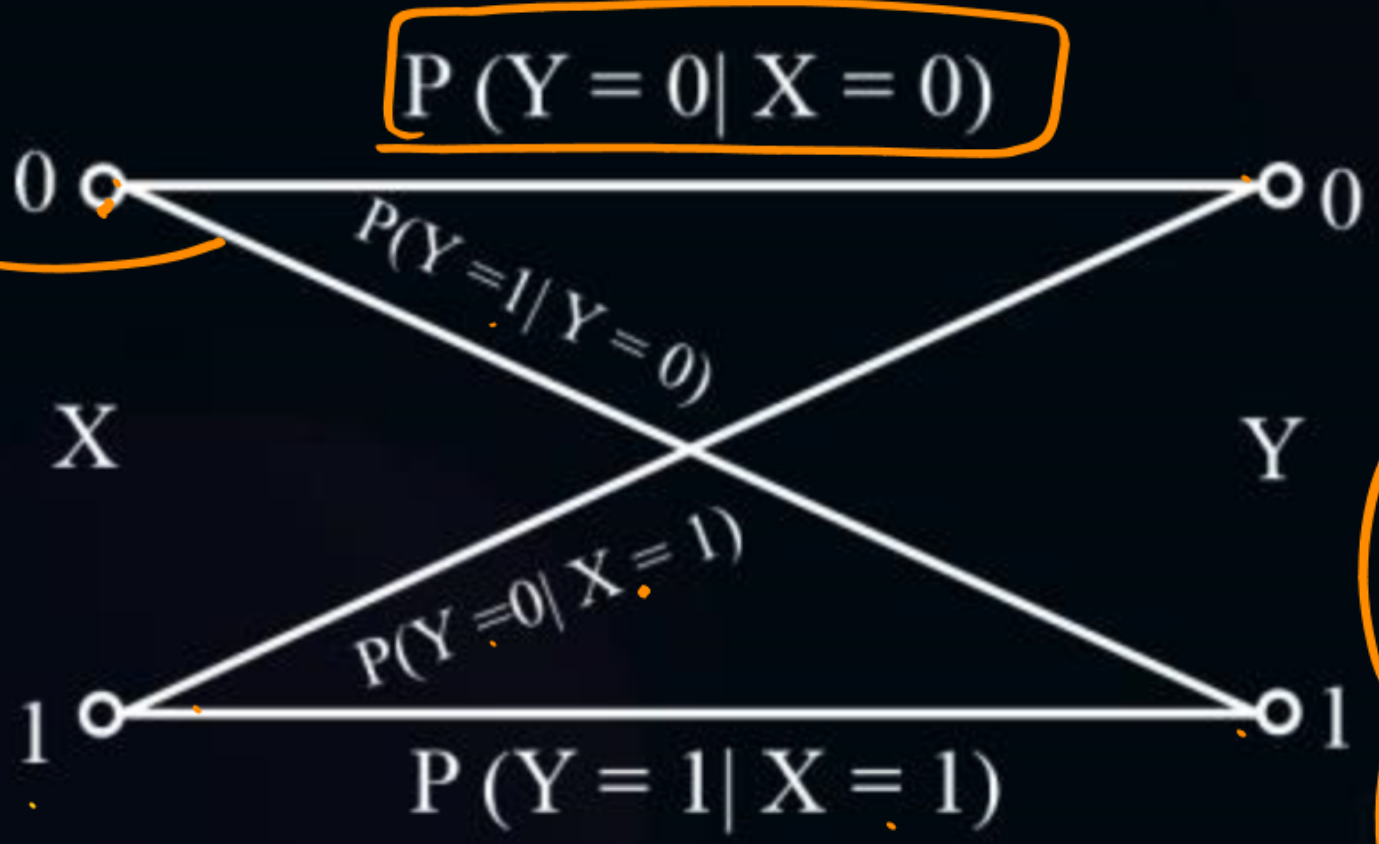
$P\left(\frac{Y=1}{Y=0}\right)$



Binary communication channel

$P(X=0)=0.5$

$P(X=1) = 1-P(X=0)$
$= 1-0.5 = 0.5$

$P\left(\frac{Y=1}{X=0}\right) = 0.1$

$P\left(\frac{Y=0}{X=1}\right) = 0.2$

$P\left(\frac{Y=0}{X=0}\right) = 1-0.1 = 0.9$

$P\left(\frac{Y=1}{X=1}\right) = 1-0.2 = 0.8$

$$P[X=0, Y=0] = P[X=0 \cap Y=0]$$

$$\checkmark \; P\left[\frac{Y=0}{X=0}\right] = \frac{P[Y=0 \cap X=0]}{P[X=0]} = 0.9 \times 0.5 = 0.45$$

$$\checkmark \; P\left[\frac{Y=0}{X=1}\right] = \frac{P[Y=0 \cap X=1]}{P[X=1]} = 0.2 \times 0.5$$

X \ Y	0	1
0	(0,0)	(0,1)
1	(1,0)	(1,1)

$$\boxed{0.45,\ 0.05,\ 0.1,\ 0.4}$$

$$\checkmark \; P\left[\frac{Y=1}{X=1}\right] = \frac{P[Y=1 \cap X=1]}{P[X=1]} = \underline{0.8 \times 0.5}$$

$$P\left[\frac{Y=1}{X=0}\right] = \frac{P[Y=1 \cap X=0]}{P[X=0]} = 0.1 \times 0.5 = \underline{0.05}$$

✓ X and Y Are Not Independent

THANK - YOU