Data Science and Artificial Intelligence

Probability and Statistics

Bivariate Random Variable

Lecture No.-02

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Recap of Previous Lecture

Topic Bivariate Random Variable Part-1

Bivariate Random variable.

Joint Prob. $\left\{ P[X=x_i, Y=y_j] = P[X=x_i \wedge Y=y_j] \right.$

Simultaneously

Topics to be Covered

Topic Bivariate Random Variable Continued

Slide 3

Bivariate Random Variables:

$\left.\begin{matrix} X \\ Y \end{matrix}\right]$ Two dimensional Random Variable.

3 Balls.

$\xrightarrow{\text{3cells}}$

cell I	cell 2	cell 3

3cells.

3 balls put 3 cells.

$\left.\begin{matrix} \text{cell 1} \\ \text{cell 2} \\ \text{cell 3} \end{matrix}\right] \rightarrow$ No ball in any cell (Repetation allowed)

b_1	b_2	b_3
$b_1 b_2$	—	b_3
—	b_3	$b_1 b_2$
$b_1 b_2 b_3$	—	—

X = random variable.

Y = random variable

simultaneously jointly work $\left\{\begin{matrix} X = \text{No. of balls in cell 1} \\ Y = \text{No. of cells Are occupied} \end{matrix}\right.$

Y = No. of cells Are occupied

Y = 1 cell
= 2 cell
= 3 cell

Y = 1, 2, 3

X = No of balls in cell 1

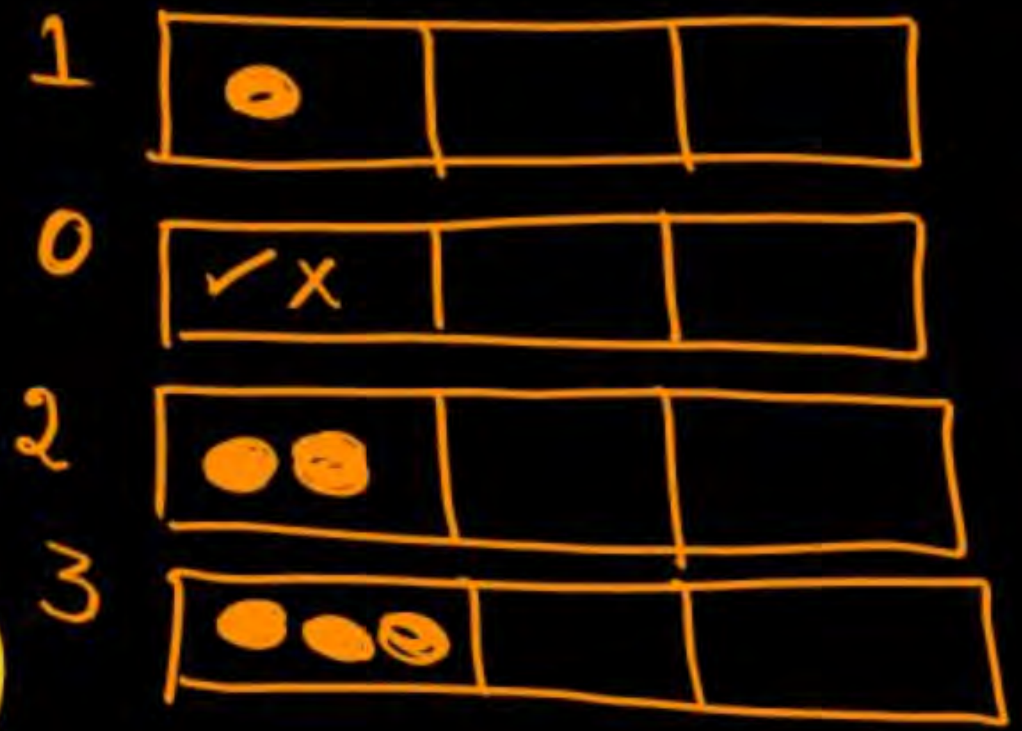
X — 0 ball
— 1 ball
— 2 ball
— 3 ball

X = 0, 1, 2, 3

P[X = No. of balls ∩ Y = No. of cells occupied]

$= \underbrace{P[X = x_i, Y = y_j]}$ = Two dimensional random variable.

bivariate random variable [simultaneously together)



$X = 0, 1, 2, 3$ (No. of balls in cell 1)

$Y = 1, 2, 3$ (No. of cells occupied)

$P[X=0, Y=1] = P[X = \text{No ball in cell 1}, Y = 1 \text{ cell occupied}]$

$P[X=0, Y=2] = P[X = \text{No ball in cell 1}, Y = 2 \text{ cell occupied}]$

$P[X=1, Y=1] = P[X = \text{one ball in cell 1}, Y = 1 \text{ cell occupied}]$

X \ Y	1	2	3
0	(0,1)	(0,2)	(0,3)
1	(1,1)	(1,2)	(1,3)
2	(2,1)	(2,2)	(2,3)
3	(3,1)	(3,2)	(3,3)

(0,1) (0,2) (0,3)
(1,1) (1,2) (1,3)
(2,1) (2,2) (2,3)
(3,1) (3,2) (3,3)

Topic : Bivariate Random Variable

$X = (0, 2)$ No ball in cell 1 and 2 cells occupied

Topic : Bivariate Random Variable

X = No. of ball in cell 1
Y = No. of cells occupied

$P[X=0, Y=1] \Rightarrow \frac{2}{27}$

$P[X=0, Y=2] \Rightarrow \frac{6}{27}$

$P[X=0, Y=3] \Rightarrow 0$

$P[X=1\ Y=1] \Rightarrow 0$

$P[X=1, Y=2] \Rightarrow \frac{6}{27}$

$P[X=1\ Y=3] \Rightarrow 6/27$

$P[X=2, Y=1] \Rightarrow 0$

X \ Y	1	2	3
0	(0,1)	(0,2)	(0,3)
1	(1,1)	(1,2)	(1,3)
2	(2,1)	(2,2)	(2,3)
3	(3,1)	(3,2)	(3,3)

$P[X=2, Y=2] = \frac{6}{27}$

$P[X=2, Y=3] = 0$

$P[X=3, Y=1] = \frac{1}{27}$

$P(X=3, Y=2) = 0$

$P[X=3, Y=3) = 0$

Let X, Y be Two Random Variable Then:

(Discrete bivariate Random Variable]

Joint Probability Mass Function = Joint Prob: $P[X=x_i, Y=y_j] = P[X=x_i \wedge Y=y_j]$ = both Simultaneously occur

Joint Prob: Table:

A) $P[X=x_i, Y=y_j] \geq 0$

B) $\sum_{\substack{i=0 \\ j=0}}^{n} P[X=x_i, Y=y_j]$ Column Add $= 1$

X \ Y	1	2	3	Total
0	$\frac{2}{27}$ (0,1)	$\frac{6}{27}$ (0,2)	0 (0,3)	$8/27$
1	0 (1,1)	$6/27$ (1,2)	$6/27$ (1,3)	$12/27$
2	0 (2,1)	$6/27$ (2,2)	0 (2,3)	$6/27$
3	$1/27$ (3,1)	0 (3,2)	0 (3,3)	$1/27$
Total	$\frac{3}{27}$	$\frac{18}{27}$	$\frac{6}{27}$	1

(0,1) (0,2) (0,3)

$P[X=0] = \frac{8}{27} = \frac{2}{27} + \frac{6}{27} + 0$

$P[X=1] = \frac{12}{27}$

$P[X=2] = \frac{6}{27}$

$P[X=3] = \frac{1}{27}$

$P[Y=1] = \frac{3}{27}$

$P[Y=2] = \frac{18}{27}$

$P[Y=3] = \frac{6}{27}$

X \ Y	1 ✓	2 ✓	3 ✓	
0	$\frac{2}{27}$ (0,1)	$\frac{6}{27}$ (0,2)	0 (0,3)	$8/27$
1	0 (1,1)	$6/27$ (1,2)	$6/27$ (1,3)	$\frac{12}{27}$
2	0 (2,1)	$6/27$ (2,2)	0 (2,3)	$6/27$
3	$\frac{1}{27}$ (3,1)	0 (3,2)	0 (3,3)	$1/27$
	$3/27$	$18/27$	$6/27$	1

$\boxed{\text{Marginal Probability}}$

only Target with one value

$P[X_i, Y_j] \rightarrow P[X_i]$

$P[X_i, Y_j] \rightarrow P[Y_j]$

X static

Y Dynamic

Marginal Prob:

$$P[X=x_i] = P[X=x_i \wedge Y=y_j] \Rightarrow P[X=x_1 \wedge Y=y_1] + P[X=x_1 \wedge Y=y_2) + P[X=x_1 \wedge Y=y_3) + - - -$$

$$\begin{cases} P[X=x_i] = \sum_j P[X=x_i, Y=y_j] \\ P[Y=y_j] = \sum_i P[X=x_i, Y=y_j] \end{cases}$$

Marginal Probability Distribution

for X

X	0	1	2	3
$P[X=x_i]$	$\frac{8}{27}$	$\frac{12}{27}$	$\frac{6}{27}$	$\frac{1}{27}$

marginal prob. Dis for Y.

Y	1	2	3	
$P[Y=y_j]$	$\frac{3}{27}$	$\frac{18}{27}$	$\frac{6}{27}$	

Throwing A Two Die

$\boxed{\text{Die A, Die B}}$

Die A → 1, 2, 3, 4, 5, 6

Die B → 1, 2, 3, 4, 5, 6

✓ X = No. of dots Die A

✓ Y = No. of dots Die B.

] Simultaneously occur

$P[X = x_i, Y = y_j]$

$= P[X = \text{No. of dots Die A} \cap Y = \text{No. of dots Die B}]$

$P[X=2, Y=3] = \frac{1}{36}$

X \ Y	1	2	3	4	5	6
1	(1,1) 1/36	(1,2) 1/36	(1,3) 1/36	(1,4) 1/36	(1,5) 1/36	(1,6) 1/36
2	(2,1) 1/36	(2,2) "	(2,3) "	(2,4) "	(2,5) "	(2,6) "
3	(3,1) 1/36	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1) 1/36	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1) 1/36	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6) 1/36

$P[X=1, Y=1] = \frac{1}{36}$ $P[X=1, Y=2] = \frac{1}{36}$

$P[X=2, Y=2] = \frac{1}{36}$

Marginal Probability:

$$P[X=x_i, Y=y_j]$$

Marginal Prob $\Rightarrow$ I am Intersted only Single Random variable

marginal Prob

$$\left\{ P[X=1] = P[X=1, Y=1] + P[X=1, Y=2] + P[X=1, Y=3] + P[X=1, Y=4] + P[X=1, Y=5] + P[X=1, Y=6] = \frac{1}{6} \right.$$

$$\boxed{P[X=x_i] = \sum_j P[X=x_i, Y=y_j]}$$

X \ Y	1	2	3	4	5	6	X
1	1/36	1/36	1/36	1/36	1/36	1/36	1/6
2	1/36	1/36	1/36	1/36	1/36	1/36	1/6
3	1/36	1/36	1/36	1/36	1/36	1/36	1/6
4	1/36	1/36	1/36	1/36	1/36	1/36	1/6
5	1/36	1/36	1/36	1/36	1/36	1/36	1/6
6	1/36	1/36	1/36	1/36	1/36	1/36	1/6
Y	1/6	1/6	1/6	1/6	1/6	1/6	1

$$\sum_i \sum_j P[X=x_i, Y=y_j] = 1$$

Marginal Prob. Distribution of X

X	1	2	3	4	5	6
$P[X=x_i, Y=y_j]$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

Marginal Probability

$$\left\{\begin{aligned} P[X=1] &= \frac{1}{6} \\ P[X=2] &= \frac{1}{6} \\ P[X=3] &= \frac{1}{6} \\ P[X=4] &= \frac{1}{6} \\ P[X=5] &= \frac{1}{6} \\ P[X=6] &= \frac{1}{6} \end{aligned}\right.$$

Marginal Prob. of Y:

$$P[Y=y_j] = P[X=x_1, Y=y_1] + P[X=x_2, Y=y_1] + P[X=x_3, Y=y_1] + P[X=x_4, Y=y_1] + P[X=x_5, Y=y_1] + P[X=x_6, Y=y_1]$$

Imp:

$$\boxed{P[Y=y_j] = \sum_i P[X=x_i, Y=y_j]}$$ —— marginal Prob. of y.

Conditional Probability Mass function :- $P\left[\frac{A}{B}\right] = \frac{P(A \cap B)}{P(B)}$

If X and Y Are given Two Dimensional random variables.

$$P\left[\frac{x}{y}\right] = P\left[X \text{ given } Y\right] = P\left[\frac{X=x}{Y=y}\right] = \frac{P[X=x, Y=y]}{P[Y=y]}$$

✓ Y = y already Happened.

$\left(\frac{x}{y}\right)$ is Happeinng.

→ Happeing

$\frac{A}{B} \Rightarrow$ B - done

A →

$$= \text{joint Prob} = P[X=x_i, Y=y_j]$$

$P[Y=y]$ = marginal Prob. of y.

$$P\left[\frac{X=x_i}{Y=y_j}\right] = P\left[\frac{X=x_i, Y=y_j}{P[Y=y_i]}\right] = \underline{\text{conditional prob.}}$$

$P[X \cap Y]$
$P[X = x_i \cap Y = y_j]$

$P\left(\frac{y}{x}\right) = P\left[\frac{Y=y}{X=x}\right] = \frac{P[Y=y, X=x]}{P[X=x]}$

Joint Prob.

Marginal Prob. of X

$P[X=x] \neq 0$

$$\boxed{P\left[\frac{Y=y_j}{X=x_i}\right] = \frac{P[Y=y_j, X=x_i]}{P[X=x_i]}}$$

Conditional Prob.

Independence of Random variables:

If A and B Are Indep.
$P(A \cap B) = P(A)P(B)$

Two Discrete Random variable X, Y.

Simultaneously working.

$$P[\underbrace{X=x_i, Y=y_j}] = P[X=x_i]\,P[Y=y_i].$$

$P[X=x_i \cap Y=y_j]$

Topic : Bivariate Random Variable

5-10 Min

Q1. The following table represents the joint probability distribution of the discrete random variable (X, Y):

X \ Y	1	2	X
1	0.1 (1,1)	(1,2) 0.2	0.3
2	0.1 (2,1)	(2,2) 0.3	0.4
3	0.2 (3,1)	(3,2) 0.1	0.3
Y	0.4	0.6	1

Marginal X

X	1	2	3
$P[X=x]$	0.3	0.4	0.3

marginal Y

Y	1	2
$P[Y=y]$	0.4	0.6

Find :

(i) The marginal distributions.

(ii) The conditional distribution of X given Y = 1.

(iii) $P[(X+Y) < 4]. = P[X=1, Y=1] + P[X=2, Y=1] + P[X=1, Y=2] = 0.4$

Conditional Prob. Distribution

$$P\left[X \text{ given } Y=1\right] = P\left[\frac{X=1}{Y=1}\right] = \frac{P[X=1 \cap Y=1]}{P[Y=1]} = \frac{0.1}{0.4} = \frac{1}{4}$$

$$= P\left[\frac{X=2}{Y=1}\right] = \frac{P[X=2 \cap Y=1]}{P[Y=1]} = \frac{0.1}{0.4} = \frac{1}{4}$$

$$\Rightarrow P\left[\frac{X=3}{Y=1}\right] = P\left[\frac{X=3 \cap Y=1}{P[Y=1]}\right] = \frac{0.2}{0.4} = \frac{1}{2}$$

✓ Conditional Prob. Distribution

X given Y=1	1	2	3
	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

Continue

Q2. Two discrete random variables X and Y have

$\boxed{P[X=0, Y=0] = \frac{2}{9},}$ Joint Prob.

$P[X = 0, Y = 1] = \frac{1}{9},$ Joint Prob. working Simultaneously

$P[X=1, Y = 0] = \frac{1}{9},$ and

$P[X=1, Y = 1] = \frac{5}{9}$

Examine whether X and Y are independent?

If X, Y Are Independent marginal prob.

$\boxed{P[X=x_i, Y=y_j]} = P[X=x_i]P[Y=y_j]$

$P[X=0, Y=0] = P[X=0]P[Y=0]$

$\frac{2}{9} = \frac{3}{9} \times \frac{3}{9} = \frac{9}{81} \neq \frac{1}{9}$

X \ Y	0	1	X
0	(0,0) $\frac{2}{9}$	(0,1) $\frac{1}{9}$	$\frac{3}{9}$
1	(1,0) $\frac{1}{9}$	(1,1) $\frac{5}{9}$	$\frac{6}{9}$
Y.	$\frac{3}{9}$	$\frac{6}{9}$	1

$P[X=0] = \frac{3}{9}$ $P[X=1] = \frac{6}{9}$

$P[Y=0] = \frac{3}{9}$ $P[X=1] = \frac{6}{9}$

X and Y Are Not Independent

Topic : Bivariate Random Variable

$\sqrt{5-10}$ Min

Q3. The join probability distribution of a pair of random variables is given by the following :

Y / X	1	2	3
1	1/12	0	1/18
2	1/6	1/9	1/4
3	0	1/5	2/15

(i) Evaluate marginal distribution of X.

(ii) Evaluate conditional distribution of Y given X =2

(iii) Obtain P[X + Y < 5]. $= \frac{15}{36}$ Ans

marginal

X	$P[X=x_i]$
1	5/36
2	19/36
3	1/3

(2) conditional Dist of Y given X=2

1	6/19
2	4/19
3	9/19

✓ Conditional Prob. Distribution given $X=2$

$$P\left(\frac{y}{x}\right) = P\left[\frac{Y=y_j}{X=x_i}\right]$$

$P\left(\frac{Y=1}{X=2}\right)$	1	2	3
	$\frac{6}{19}$	$\frac{4}{19}$	$\frac{9}{19}$

$$P\left[\frac{Y=1}{X=2}\right] = \frac{P[Y=1 \wedge X=2]}{P[X=2]} = \frac{\frac{1}{6}}{\frac{19}{36}} = \frac{\cancel{36}}{\cancel{6}\times 19} = \frac{6}{19}$$

$$P\left[\frac{Y=2}{X=2}\right] = \frac{P[Y=2 \wedge X=2]}{P[X=2]} = \frac{\frac{1}{9}}{\frac{19}{36}} = \frac{36}{9\times 19} = \frac{4}{19}$$

$$P\left[\frac{Y=3}{X=2}\right] = \frac{P[Y=3 \wedge X=2]}{P[X=2]} = \frac{\frac{1}{4}}{\frac{19}{36}} = \frac{\cancel{36}\,9}{\cancel{4}\times 19} = \frac{9}{19}$$

Topic : Bivariate Random Variable

5-10 min

$P[X=4] \times P[Y=2] = P[X=4 \cap Y=2]$ YES

Q4. For the following joint probability distribution of (X, Y)

X \ Y	1	2	3
1	1/20	1/10	1/10
2	1/20	1/10	1/10
3	1/10	1/10	1/20
4	1/10	1/10	1/20

$\frac{1}{20}+\frac{1}{10}+\frac{1}{10}$

$P\left[\frac{Y=2}{X=4}\right] = P\left[\frac{Y=2 \cap X=4}{X=4}\right] = \frac{P[X=4 \cap Y=2]}{P[X=4]}$

(i) find the probability that Y = 2 given that X = 4, $\frac{2}{5}$

(ii) find the probability that Y = 2, and $P[Y=2] = \frac{2}{5}$

$P[Y=2]$ = marginal pro. of Y.

(iii) examine if the two events X = 4 and Y = 2 are independent.

YES $P[X=4 \cap Y=2] = P[X=4]\,P[Y=2]$

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Topic : Bivariate Random Variable

Q5. The following table represents the joint probability distribution of the discrete random variable (X, Y):

X \ Y	1	2
1	0.1	0.2
2	0.1	0.3
3	0.2	0.1

Find :

(i) F (2, 2), F (3, 2)

(ii) $F_X(3)$

(iii) $F_Y(1)$

THANK - YOU