Data Science and Artificial Intelligence

Probability and Statistics

Bivariate Random Variable

Lecture No.- 08

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Topics to be Covered

Topic Question Based on Bivariate Continuous Random Variable questions → formula

computer + Data + ↑

Probability

Concept

Learn

pdf

Bayes

fundamentals (undergraduate)

Learning → Habit

70%

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Integ + quadratic

$\boxed{+\ \text{P and C}}$

+ fundamental concepts

+ Limits +

70%

← questions

Topic : Question Based on Bivariate Continuous Random Variables

Marginal PMF = discrete Random

Q1. The joint probability distribution of two random variables X and Y is given by:

$P[X = x_i, Y = y_j] = $ Joint PMF $= P[X = x_i \wedge Y = y_j]$

$P(X = 0, Y = 1) = \frac{1}{3}$, $P(X = 1, Y = -1) = \frac{1}{3}$, and $P(X = 1, Y = 1) = \frac{1}{3}$,

Find

marginal PMF for X

$P_X(x) = \sum_j P[X = x_i, Y = y_j]$

(i) Marginal distributions of X and Y, and

(ii) The conditional probability distribution of X given Y = 1.

$Y = 1$ given $X = x_i$ $\quad P\left[\frac{X = x_i}{Y = 1}\right]$

PMF for Y $\quad P_Y(y) = \sum_i P[X = x_i, Y = y_j]$

$X = 0, 1, -1 \quad Y = 0, -1, +1$

$P[X=0, Y=1] = \frac{1}{3}$

$P[X=1, Y=-1] = \frac{1}{3}$

$P[X=1, Y=1] = \frac{1}{3}$

X and Y bivariate Discrete

$X =$ Integer

$Y =$ Integer

$X = 0$
$Y = 0$

$X \backslash Y$	0	−1	1	
0	(0,0) 0	(0,−1) 0	(0,1) $\cancel{0}$ $\frac{1}{3}$	$\frac{1}{3}$ $P(X=0)$
−1	(−1,0) 0	(−1,−1) 0	(−1,1) 0	0 $P(X=-1)$
1	(1,0) 0	(1,−1) $\frac{1}{3}$	(1,1) $\frac{1}{3}$	$\frac{2}{3}$ $P(X=1)$
	0 $P(Y=0)$	$\frac{1}{3}$ $P(Y=-1)$	$\frac{2}{3}$ $P(Y=1)$	1

Discrete

X	0	−1	1
X	$\frac{1}{3}$	0	$\frac{2}{3}$

Y	0	−1	1
$P(Y=y_i)$	0	$\frac{1}{3}$	$\frac{2}{3}$

$$P\left[\frac{X=x_i}{Y=1}\right] = P\left[\frac{X=-1}{Y=1}\right] = P\left[\frac{X=-1 \cap Y=1}{P[Y=1]}\right] =$$

$X=-1,0,1$

$X=-1,0,1$

$$P\left[\frac{X=0}{Y=1}\right] = \frac{P[X=0 \cap Y=1]}{P[Y=1]} =$$

$$P\left[\frac{X=1}{Y=1}\right] = \frac{P[X=1 \cap Y=1]}{P[Y=1]} =$$

$X=-1,0,1$

$P\left(\frac{A}{B}\right)$

given

-1	0	1
0	$\frac{1}{2}$	$\frac{1}{2}$

$P\left(\frac{Y=y_i}{X=1}\right)$

y value

Topic : Question Based on Bivariate Continuous Random Variables

5 min ✓

$P[X = x_i, Y = y_j]$ = Discrete PMF

Q2. For the joint probability distribution of two random variables X and Y given below.

Find :

(i) The marginal distributions of X and Y, and

(ii) Conditional distribution of X given the value of Y=1 and that Y given the value of X = 2.

Y = 1, 2, 3, 4

$P\left[\frac{Y = y_j}{X = 2}\right]$ $P\left[\frac{X = x_i}{Y = 1}\right]$ 1, 2, 3, 4

X \ Y	1	2	3	4	Total	
1	$\frac{4}{36}$ +	$\frac{3}{36}$ +	$\frac{2}{36}$ +	$\frac{1}{36}$	$\frac{10}{36}$	$P(X=1)$
2	$\frac{1}{36}$ +	$\frac{3}{36}$ +	$\frac{3}{36}$ +	$\frac{2}{36}$	$\frac{9}{36}$	$P(X=2)$
3	$\frac{5}{36}$ +	$\frac{1}{36}$ +	$\frac{1}{36}$ +	$\frac{1}{36}$	$\frac{8}{36}$	$P(X=3)$
4	$\frac{1}{36}$ +	$\frac{2}{36}$ +	$\frac{1}{36}$ +	$\frac{5}{36}$	$\frac{9}{36}$	$P(X=4)$
	$P(Y=1)$	$P(Y=2)$	$P(Y=3)$	$P(Y=4)$		
Total	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	1	

$\boxed{\text{Do yourself}}$

	I	II	
$P\left(\frac{X=1}{Y=1}\right)$	$\frac{4}{11}$	$\frac{1}{9}$	$P\left(\frac{Y=1}{X=2}\right)$
$P\left(\frac{X=2}{Y=1}\right)$	$\frac{1}{11}$	$\frac{1}{3}$	$\frac{P(Y=2)}{P(X=2)}$
$P\left(\frac{X=3}{Y=1}\right)$	$\frac{5}{11}$	$\frac{1}{3}$	$\frac{P(Y=3)}{P(X=2)}$
$P\left(\frac{X=4}{Y=1}\right)$	$\frac{1}{11}$	$\frac{2}{9}$	$P\left(\frac{Y=3}{X=2}\right)$

$P\left(\frac{X=x_i}{Y=1}\right)$ $P\left(\frac{Y=y_j}{X=2}\right)$

Conditional prob.

Topic : Question Based on Bivariate Continuous Random Variables

Q3. The joint probability density function of a two-dimensional random variable (X, Y) is given by

$$f(x, y) = \begin{cases} 2; & 0 < x < 1,\ 0 < y < x; \\ 0, & \textit{elsewhere} \end{cases}$$

$$f(x,y) = \begin{cases} 2 & 0<x<1,\ 0<y<x \\ 0 & \text{elsewhere} \end{cases}$$

(i) Find the marginal density functions of X and Y.

(ii) Find the conditional density function of Y given X = x and conditional density function of X given Y = y.

$\boxed{\text{(iii) Check for independence of X and Y.}}$

$\rightarrow f_{XY}(x,y) = f_X(x)\, f_Y(y)$

If this is a valid joint prob. density function

$$\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} f(x,y)\,dy\,dx = 1$$

$$\Rightarrow \int_0^1\int_0^x \underline{2}\,dy\,\underline{dx} =$$ ✓ This is joint valid prob. Density function

$$\Rightarrow \int_0^1 2\,dx\left[\int_0^x dy\right]$$

$$\Rightarrow 2\int_0^1 x\,dx.$$

$$= \cancel{2}\,\frac{1}{\cancel{2}} = 1$$

$$f(x,y) = \begin{cases} 2 & \boxed{0<y<x} \\ & 0<x<1 \\ 0 & \text{elsewhere} \end{cases}$$

marginal density Function

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx.$$

conditional

$$f_{XY}\left(\frac{y}{x}\right) = \frac{f_{XY}(x,y)}{f_X(x)}$$

$$f_{XY}\left(\frac{x}{y}\right) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

Marginal density function X =

$$f_X(x) = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$\begin{cases} 0 < x < 1 \\ 0 < y < x \end{cases}$$

Horizontal strip

$$\underline{f_{XY}(x,y)} = \int_{y=-\infty}^{\infty} f_{XY}(x,y)\,dy.$$

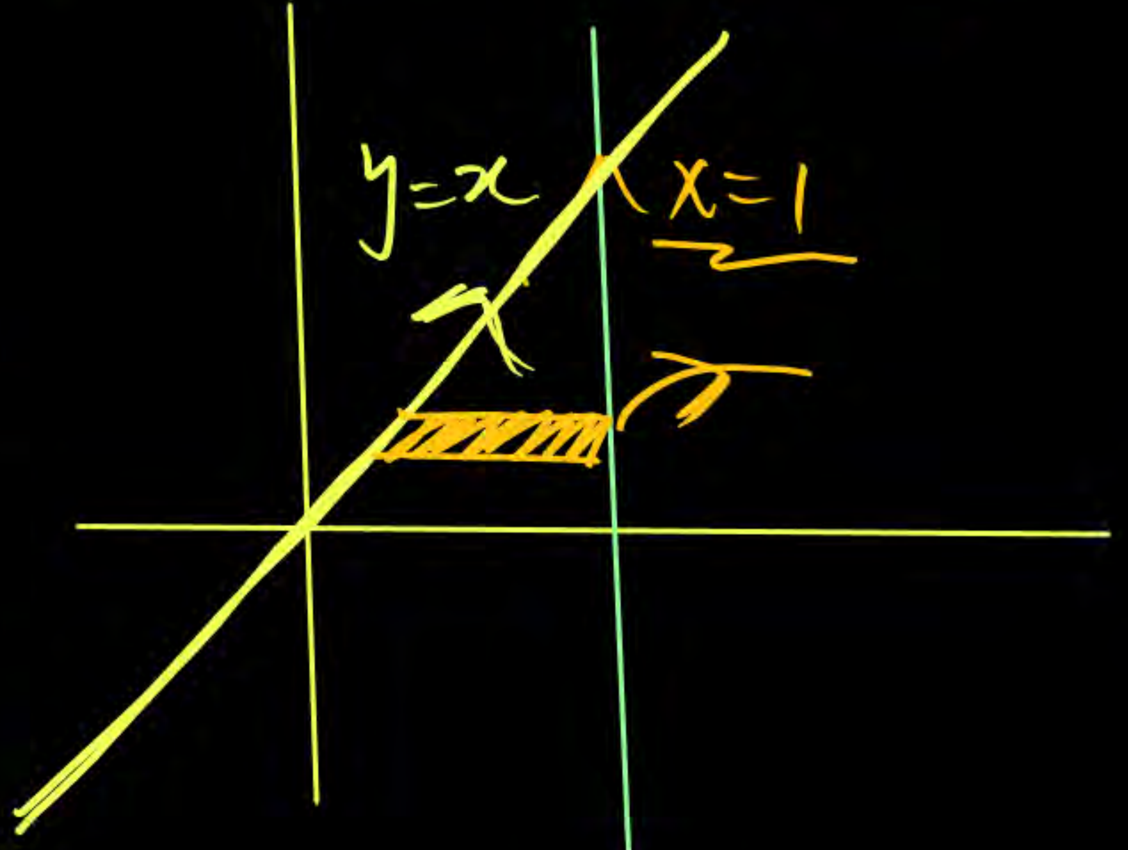
Marginal

$$\underline{f_X(x)} \Rightarrow \int_0^x 2\,dy = \underline{2x} \quad \underline{0 < x < 1}$$

marginal density function Y.

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= \int_y^1 2\,dx = \underline{2(1-y)}$$

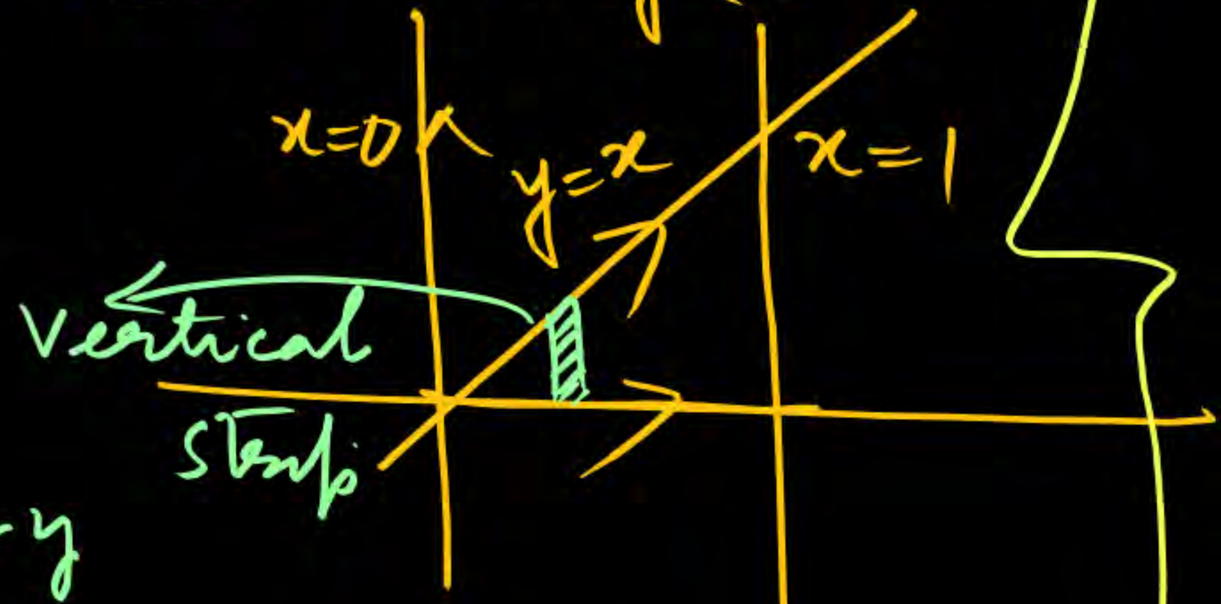


Horizontal allowed

y to 1

$x > y$

$0 < x < 1$
$0 < y < x$

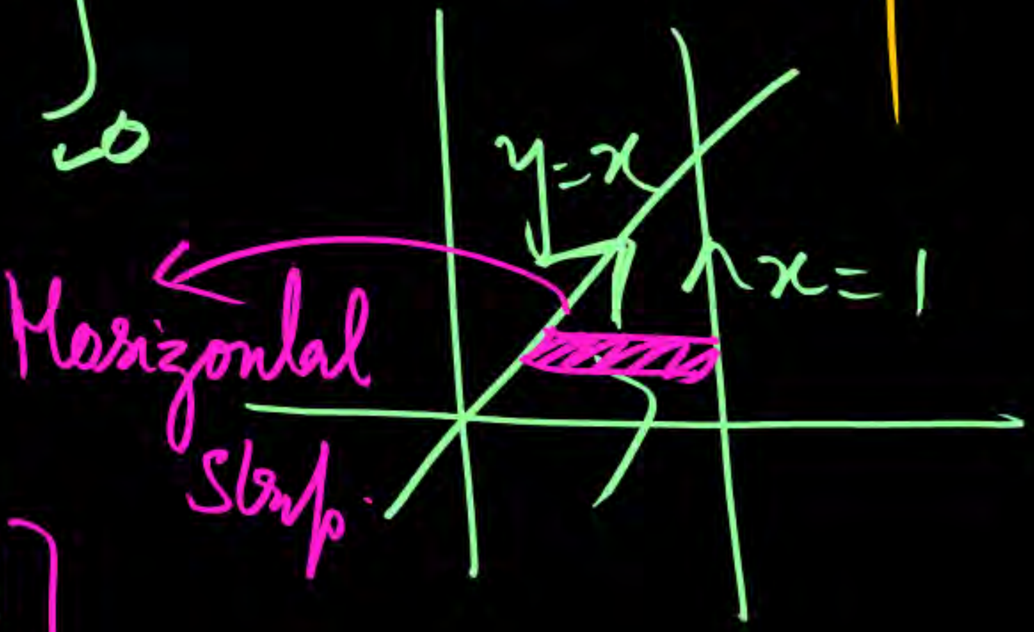


$\int_0^y$

$\int_y^1 2\,dy = 2(1-y)$

$y < x < 1$

$0 < x < 1$
$0 < y < x$



$$f_{XY}\left(\frac{y}{x}\right) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x} \qquad \underline{0<x<1}$$

$$f_{XY}\left(\frac{x}{y}\right) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{(1-y)} \qquad \underline{y<x<1}$$

both are independent

$$\underline{f_{XY}(x,y)} = f_X(x)\, f_Y(y)$$

marginal Density of Y

marginal density of X

Not Independent

$$\underline{2 \neq 4x(1-y)}$$

$$= 2x \cdot 2(1-y)$$
$$= 4x(1-y)$$

Q4. The joint p.d.f. of two random variables X and Y is given by:

$$f(x, y) = \begin{cases} k; & x^2 \le y \le x,\ 0 < x < 1 \\ 0; & \text{otherwise} \end{cases}$$

$f(x,y) = \begin{cases} K \\ x^2 \le y \le x \\ 0 < x < 1 \end{cases}$

$\int_{x=0}^{1} \int_{x^2}^{x} K \, dy \, dx = 1$

$\boxed{K = 6}$ ✓

Find : (i) k, ✓ $f_X(x), f_Y(y)$

(ii) $\boxed{\text{The marginal p.d.f.'s of X and Y.}}$

(iii) The conditional p.d.f of X given Y = y and of Y given X = x.

$$\begin{cases} f_X(x) = 6(x - x^2) & 0 < x < 1 \\ f_Y(y) = 6(\sqrt{y} - y) & 0 < y < 1 \end{cases}$$

$$\int_0^1 \int_{x^2}^{x} K\,dy\,dx = 1 \Longrightarrow \text{It is a Valid pdf}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\,dx = 1$$

$$\boxed{K = 6}$$

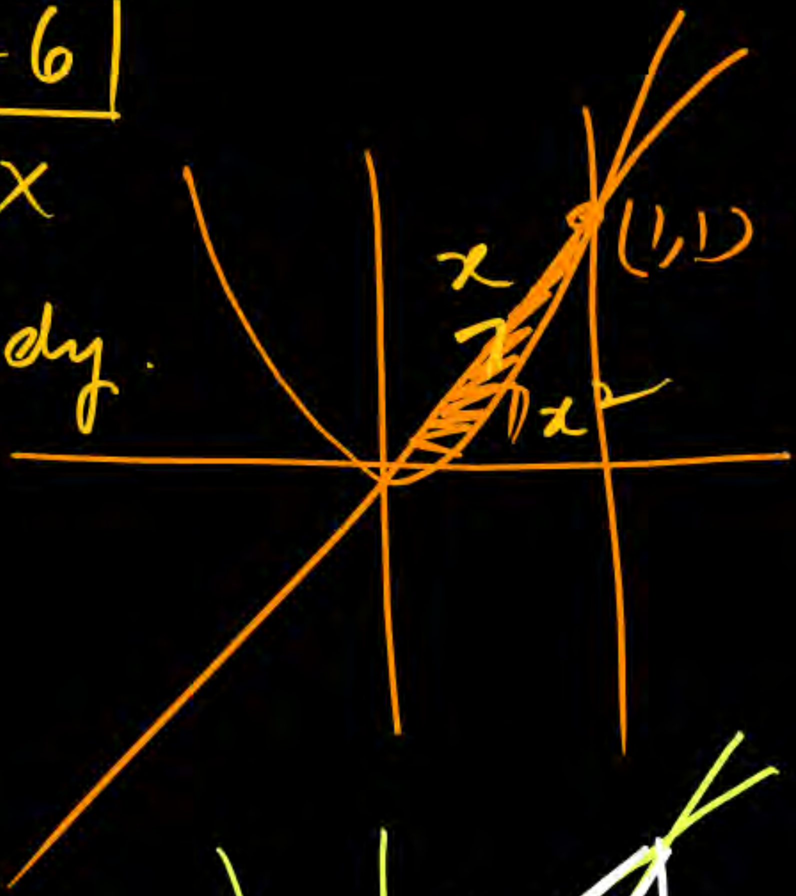
marginal pdf of X

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_{-\infty}^{\infty} 6\,dy$$

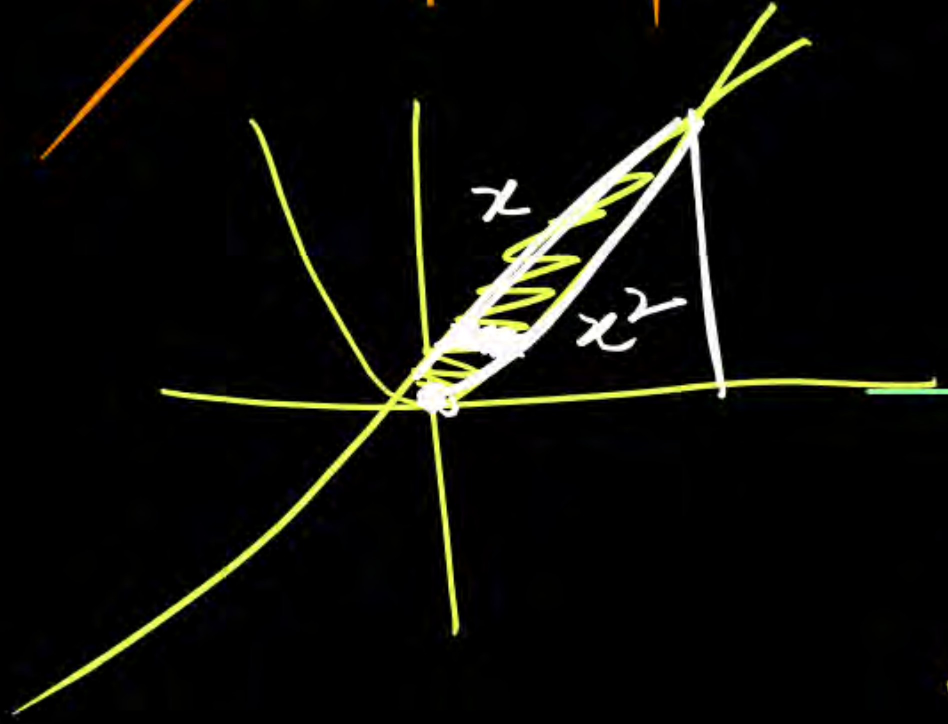
$$= \int_{x^2}^{x} 6\,dy$$

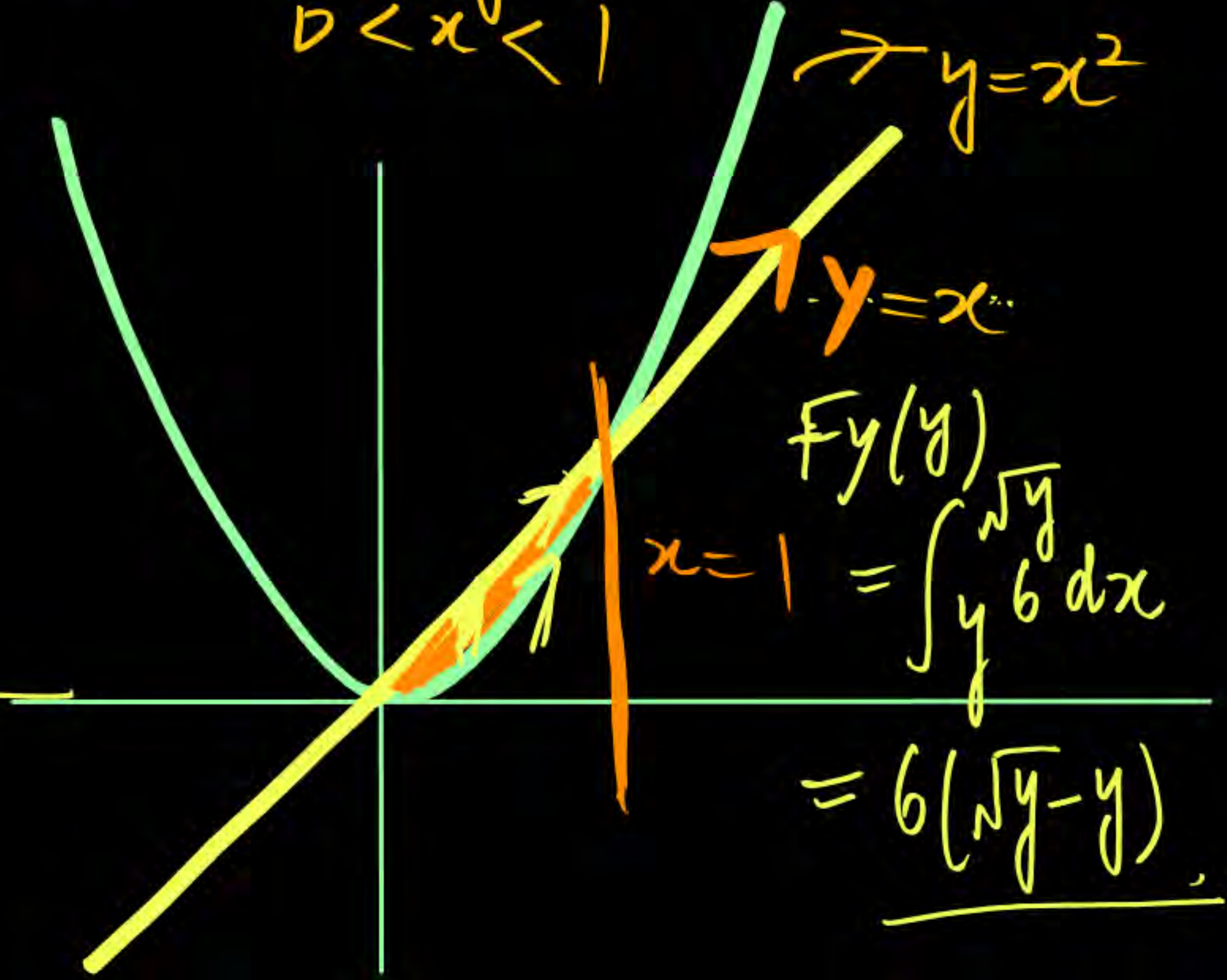
$$\boxed{f_X(x) = 6(x - x^2)}$$



$y = x^2$ $\quad x^2 \le y \le x$

$0 < x < 1$



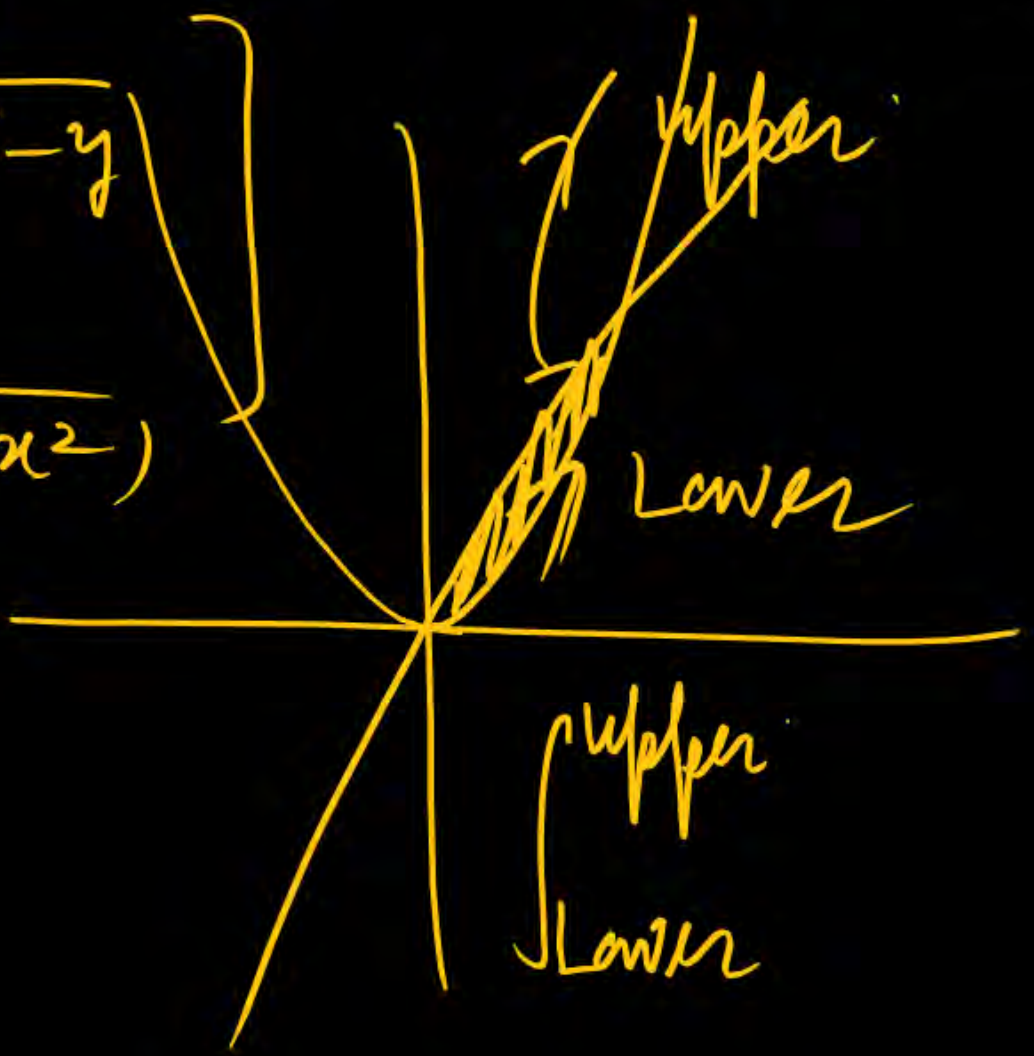


$$F_Y(y) = \int_y^{\sqrt{y}} 6\,dx$$

$$= 6(\sqrt{y} - y)$$

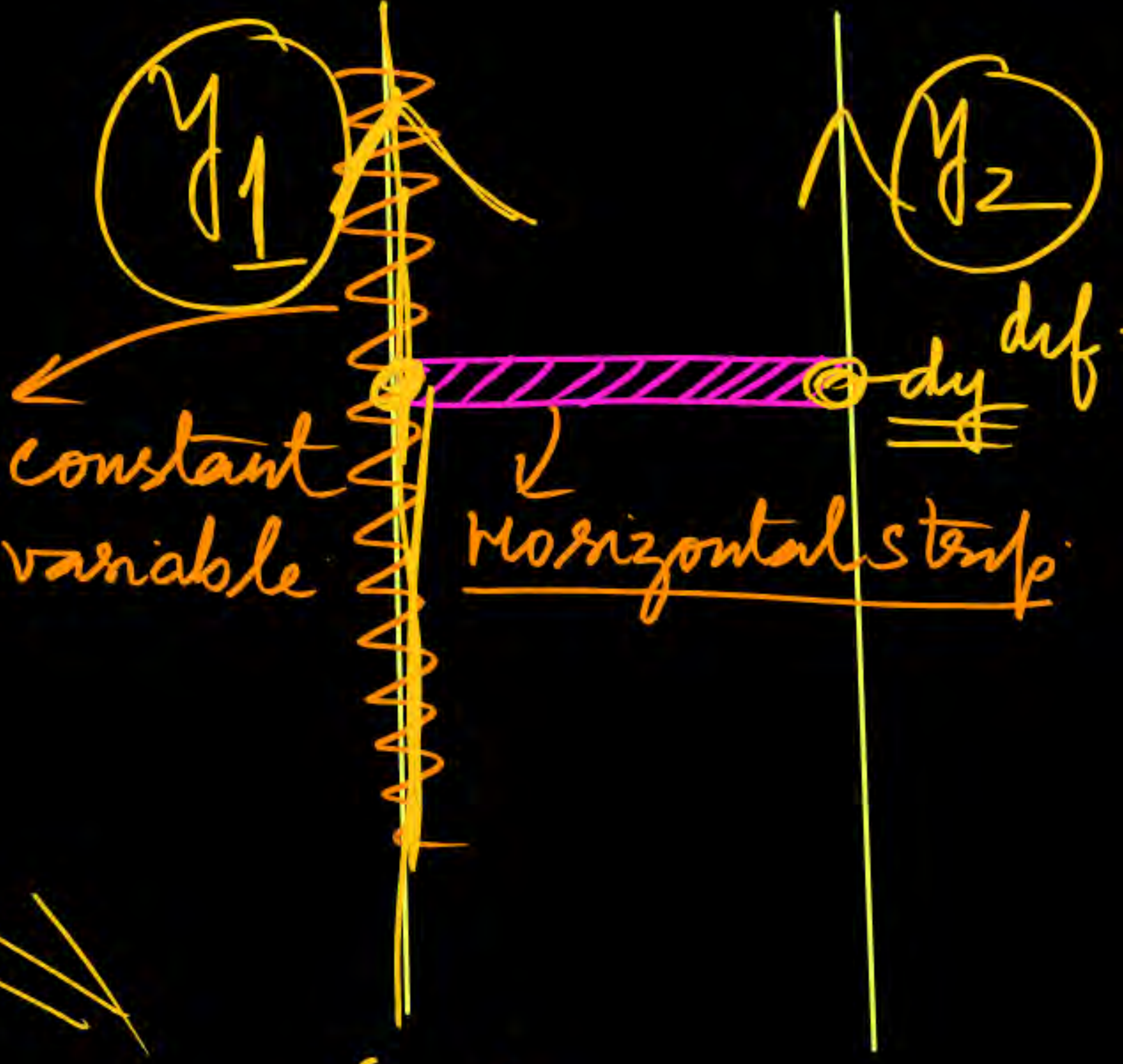
$$f_{X|Y}\left(\frac{x}{y}\right) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1}{\sqrt{y}-y}$$

$$f_{Y|X}\left(\frac{y}{x}\right) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{1}{(x-x^2)}$$



→ vertical strip

→ Y variable
X constant



Y constant
X variable

horizontal strip

dif $-(y_1 - y_2)$
OR
$(y_2 - y_1) \times dx$

$$\int \int f(x,y)\,dy\,dx$$

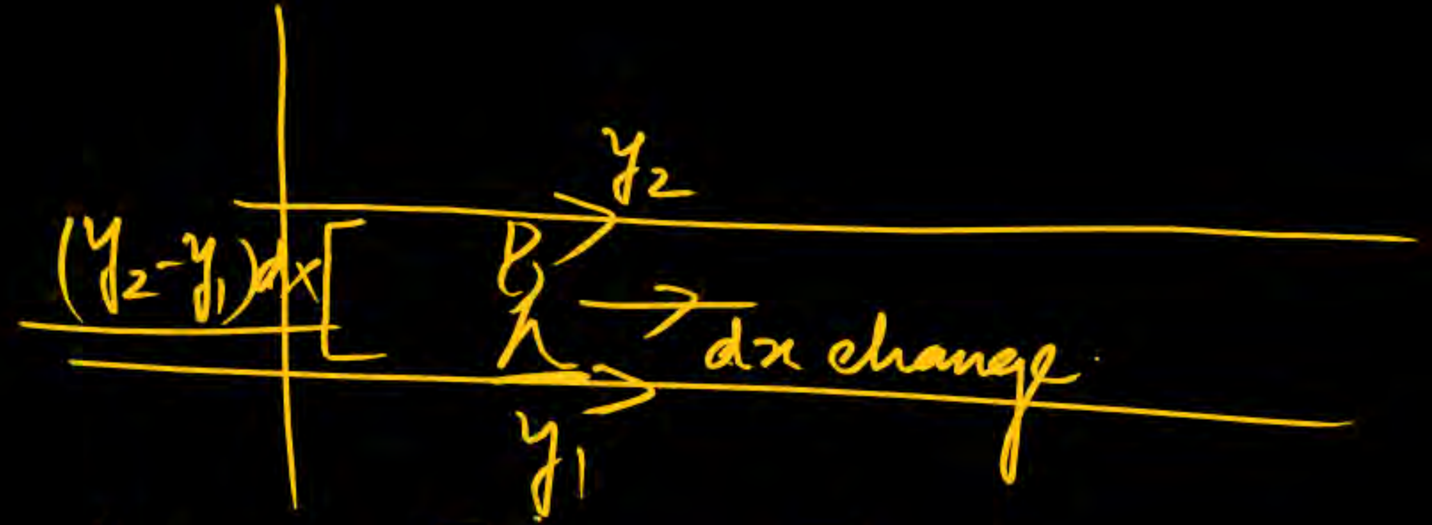
x const y variable

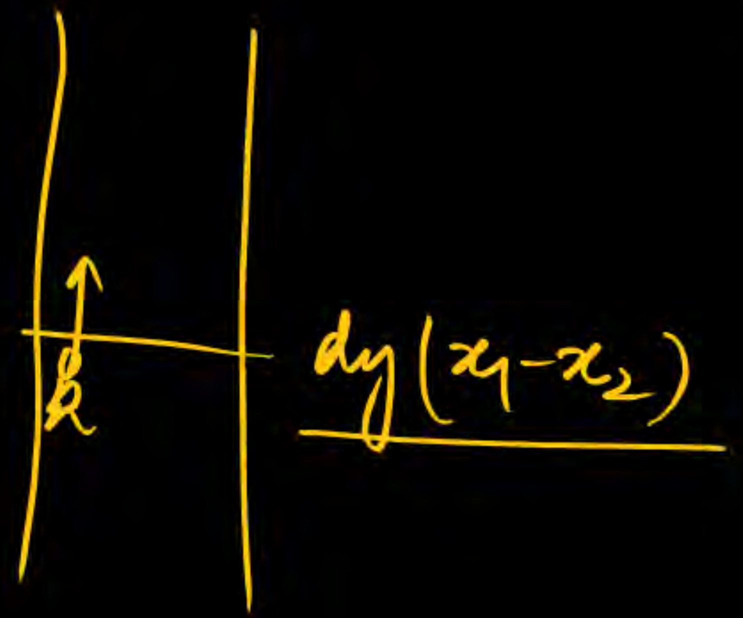
$(y_1 - y_2)\,dx$

$(x_1 - x_2)\,dy$

$$\int \int f(x,y)\,dx\,dy$$

y constant x = variable





$x \longrightarrow$ 

$y \longrightarrow$ variable

 $\longrightarrow$ x variable
y constant

THANK - YOU