Data Science and Artificial Intelligence

Probability and Statistics

Continuous Probability Distribution

Lecture No.- 08

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Topics to be Covered

Topic Beta Distribution

Topic Hypergeometric Distribution

Topic Problem based on Beta and Hypergeometric Distribution

Beta Distribution:

Bernoulli Trials — success (p) / failure ($1-p$) → for n Large → Binomial Distribution

Arrival = Discrete Random var.

$P(X=x) = {}^nC_x \, p^x q^{n-x}$ → Random variable

Waiting Time → How much — Time

⇓

Beta Distribution

$B(n, p) \rightarrow$ continuous form

10 coin Tosses → 7 HEADS, 3 Tails.

H H H H H H H T T T

7 HEAD 3 Tail

3H 4T
4H 3T
3H 4T
5H 2T
7H 3H

continuous form $\longrightarrow$ Bernoulli distribution

How much time wait achieve the success

prob. of success = p

prob. of failure = $(1-p)$

7 HEAD 3 Tails

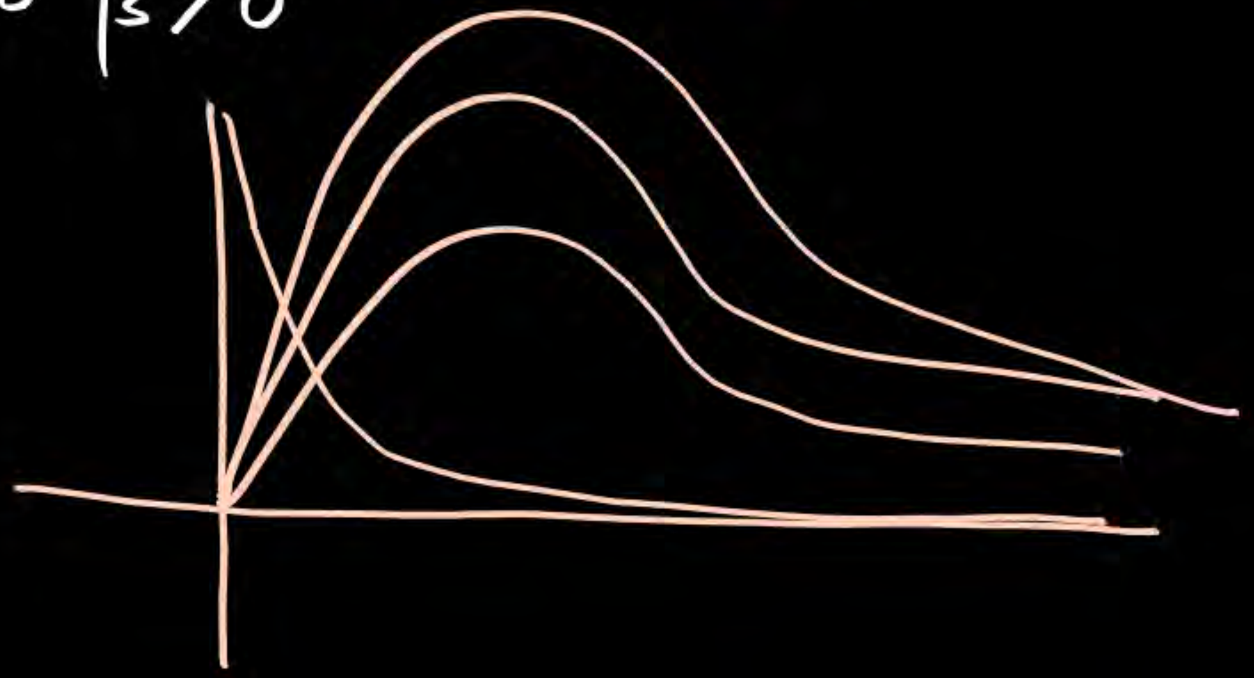
$= \underbrace{p^7}_{\text{HEAD}} \underbrace{(1-p)^3}_{\text{Tail}}$

$$f(x) = p^{\alpha-1}(1-p)^{\beta-1}$$

$\alpha > 0 \quad \beta > 0$

$$\boxed{f(x) = \underbrace{p^{\alpha-1}}_{\text{success}}\underbrace{(1-p)^{\beta-1}}_{\text{failure}}}$$

Incomplete function

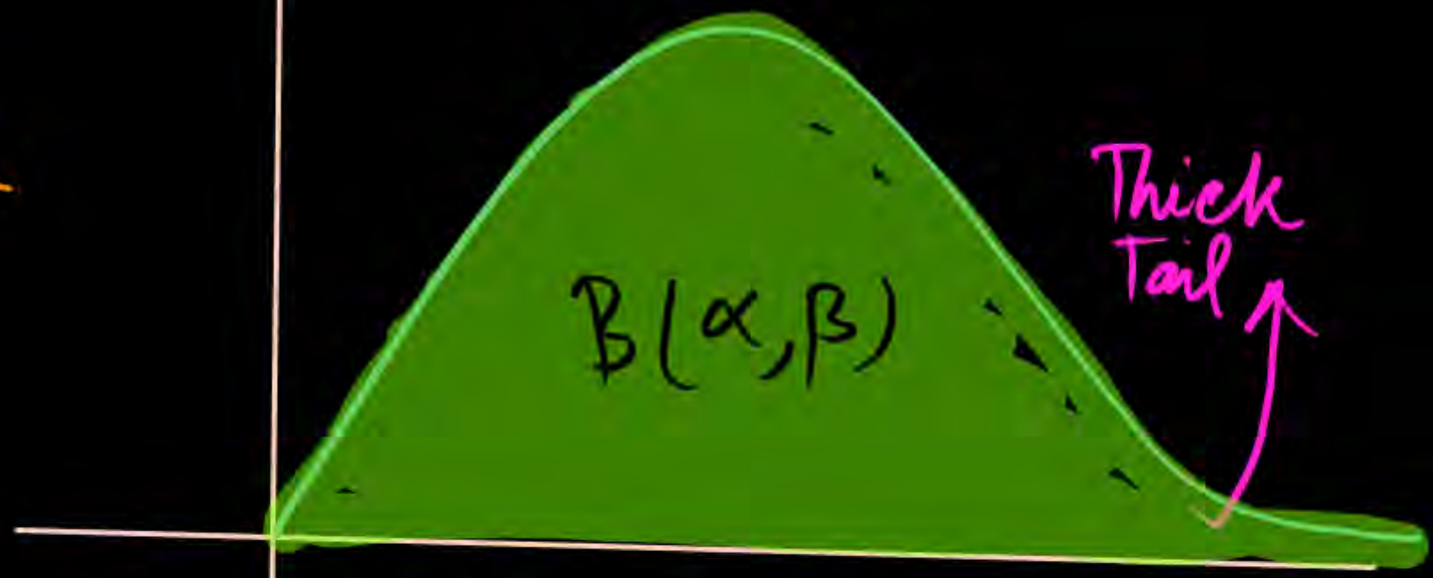


$f(x) = p^{\alpha-1}(1-p)^{\beta-1} \longrightarrow$ prob function

$B(\alpha,\beta)$ = Beta function

Beta function $= \int_0^1 (x)^{\alpha-1}(1-x)^{\beta-1}\,dx$

$$\frac{\Gamma\alpha\,\Gamma\beta}{\Gamma(\alpha+\beta)} = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,d$$



$$\int_0^1 x^3(1-x)^4\,dx = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

$\alpha-1=3 \quad \alpha=4$

$\beta-1=4 \quad \beta=5$

$= B(4,5) =$

$$\frac{\Gamma 4\,\Gamma 5}{\Gamma(4+5)} = \frac{3!\times 4!}{8!}$$

$$\int_0^1 x^5(1-x)^2\,dx$$

$$= \underline{B(6,3)}$$

pdf : Beta distribution =

$$B(\alpha, \beta) = \frac{\Gamma\alpha \, \Gamma\beta}{\Gamma\alpha+\beta}$$

$$f(x) = \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

$$\boxed{f(x) = \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1}}$$

→ Beta Distribution

If $\alpha \Rightarrow 1$

geometric means. f.

$$f(x) = \frac{1}{B(\alpha,\beta)} p^{\alpha-1}(1-p)^{\beta-1}$$

If $\alpha = 1, \beta = 1$

$$f(x) = \frac{1}{B(1,1)} p^{1-1}(1-p)^{1-1}$$

$$= \frac{1}{\frac{\Gamma 1 \, \Gamma 1}{\Gamma 1+1}} p^{0}(1-p)^{0}$$

$$f(x) = \underline{1}$$

$$f(x) = \text{constant } \frac{1}{1-0} \qquad \underline{0 < x < 1}$$

$\boxed{\text{Uniform Distribution}}$

In Beta distribution

$$\boxed{f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}}$$

$$B(\alpha,\beta) = \frac{\Gamma\alpha\,\Gamma\beta}{\Gamma(\alpha+\beta)}$$

$$\left.\begin{aligned}\Gamma\alpha &= (\alpha-1)!\\ \Gamma\beta &= (\beta-1)!\end{aligned}\right]\text{ Gamma function}$$

$$\text{MEAN} = E[X] = \mu = \frac{\alpha}{(\alpha+\beta)}$$

$$\text{variance } V(X) = \sigma_x^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Standard deviation

$$\sigma_x = \sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

समय खर्च हो रहा है।

waiting Time

$\Rightarrow$

$$\frac{\beta^2}{(1+q)^3}$$

Beta distribution (SECOND Type) :-

Beta function $= B(\alpha,\beta) = \int_0^\infty \frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}}\,dx = \frac{\Gamma\alpha\,\Gamma\beta}{\Gamma(\alpha+\beta)}$

$$\boxed{f(x) = \frac{1}{B(\alpha,\beta)} \frac{x^{\alpha-1}}{(1+x)^{\alpha+\beta}} \quad \alpha>0,\ \beta>0}$$

mean $= \frac{\alpha}{(\beta-1)}$

variance $= \frac{\alpha(\alpha+\beta-1)}{(\beta-1)^2(\beta-2)}$

S.D $= \sqrt{\text{variance}}$

$$\Gamma\frac{3}{2} = \frac{1}{2}\Gamma\frac{1}{2}$$

Q3. Using beta function, prove that

$$\int_0^\infty \frac{x^3}{(1+x)^{\frac{13}{2}}}\,dx = \frac{64}{15015}$$

Using Beta Function

$B(\alpha,\beta) =$

Compare It

$$\int_0^\infty \frac{x^3}{(1+x)^{13/2}}\,dx = \frac{64}{15015}$$

$$B(\alpha,\beta) = \int_0^\infty \frac{x^3}{(1+x)^{13/2}}\,dx$$

$\alpha - 1 = 3.$

$\alpha = 4$

$B\left(4,\frac{5}{2}\right)$

$\alpha+\beta = \frac{13}{2}$

$\alpha = 4$

$4+\beta = \frac{13}{2}$

$\beta = \frac{13}{2} - 4 = \frac{5}{2}$

$$B\left(4, \frac{5}{2}\right) = \frac{\Gamma 4 \, \Gamma \frac{5}{2}}{\Gamma\left(4 + \frac{5}{2}\right)} = \boxed{\frac{3! \times \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\cancel{\left(\Gamma \frac{1}{2}\right)}}{\frac{11}{2} \times \frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \cancel{\Gamma \frac{1}{2}}}}$$

$$= \frac{\cancel{6}^{\,3} \times \frac{3}{\cancel{4}\,2}}{\frac{11}{2} \times \frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}}$$

$$= \frac{\cancel{\frac{9}{2}}}{\frac{11}{2} \times \cancel{\frac{9}{2}} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2}}$$

$$\Rightarrow \frac{2^5}{11 \times 7 \times 5 \times 3 \times 1} = \frac{32}{1155}$$

Q4. Obtain mean and variance for the beta distribution whose density is given by $f(x) = \frac{60x^2}{(1+x)^7}, 0 < x < \infty$

$f(x) = \frac{60x^2}{(1+x)^7} \quad 0<x<\infty$

$f(x) = \frac{60x^2}{(1+x)^7} = 60\,\beta(3,4)$

$\alpha = 3$

$\beta = 4$

$\frac{60\,x^{\alpha-1}}{(1+x)^{\alpha+\beta}}$

$\alpha - 1 = 2$

$\alpha = 2+1 = 3$

$\alpha = 3$

$\alpha + \beta = 7$

$3 + \beta = 7$

$\beta = 7 - 3 = 4$

mean $= 1$

$var(X) = 1$

$E[X] = \frac{\alpha}{(\beta-1)} = \frac{3}{(4-1)} = \frac{3}{3} = 1$

$V(X) = \frac{\alpha(\alpha+\beta-1)}{(\beta-1)^2(\beta-2)}$

$= \frac{3(3+4-1)}{(3)^2(2)}$

$= \frac{3(6)}{9 \times 2} = 1$

$\int_0^1 x^2(1-x)^3 dx$

$\int_0^1 60x^2(1-x)^3 dx = 1$

Beta Function

$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = B(\alpha,\beta)$

Q5. Using beta function, prove that

$\int 60x^2(1-x)^3$

$$\int_0^1 60x^2(1-x)^3\,dx = 1$$

$60\int_0^1 x^2(1-x)^3 dx$

$B(\alpha,\beta) = \dfrac{\Gamma\alpha\,\Gamma\beta}{\Gamma(\alpha+\beta)}$

$\alpha-1=2$

$\alpha = 2+1 = 3$

$\beta-1=3$

$\beta = 3+1 = 4$

$B(3,4) = \dfrac{\Gamma 3\,\Gamma 4}{\Gamma 7}$

$= \dfrac{2\times 6}{\underset{60}{\cancel{120}}\times\cancel{6}} = \dfrac{1}{60}$

$60\times B(3,4) = 60\times B(3,4) = \cancel{60}\times\dfrac{1}{\cancel{60}}$

$= 1$

$f(x) = Kx^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}$
$0 < x < 1$
is a Beta Distribution of First Kind.

Q6. Determine the constant k such that the function $f(x) = kx^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}$. $0 < x < 1$, is a beta distribution of first kind, also find its mean and variance.

mean =
var =

$$\begin{cases} K = \dfrac{2}{\pi} \\ E[X] = \mu = \dfrac{1}{4} \\ \sigma_X^2 = V(X) = \dfrac{1}{16} \end{cases}$$

$$f(x) = Kx^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}$$ — Beta distribution of First kind

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}$$

$\alpha - 1 = -\frac{1}{2}$ $\quad \alpha = -\frac{1}{2}+1 = \frac{1}{2}$

$\beta - 1 = \frac{1}{2}$ $\quad \beta = \frac{3}{2}$

If this is valid pdf

$$\frac{1}{B(\alpha,\beta)}\int_0^1 Kx^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}\,dx = 1$$

$$\frac{1}{B\left(\frac{1}{2},\frac{3}{2}\right)} K = 1$$

$\Gamma 2 = 1$

$$\frac{\Gamma\left(\frac{3}{2}+\frac{3}{2}\right)}{\Gamma\frac{1}{2}\,\Gamma\frac{3}{2}} K = 1$$

$$\frac{\Gamma 2}{\sqrt{\pi}\times\sqrt{\pi}\times\frac{1}{2}} K = 1$$

$$\frac{1}{\pi\times\frac{1}{2}} K = 1$$

$$\frac{2}{\pi} = K$$

$$K = \frac{2}{\pi}$$

$$\text{mean} = \frac{\alpha}{\alpha+\beta} = \frac{\frac{1}{2}}{\frac{1}{2}+\frac{3}{2}} = \frac{1}{4}$$

$$\text{Variance} = \frac{1}{16}$$

Hyper geometric Distribution :

(Discrete Random variable)

Bernoulli Trials — SUCCESS (p), failure (q) = $(1-p)$

$\xrightarrow{n \text{ Large}}$ ${}^nC_r p^r q^{n-r}$

r = No. of SUCCES

✓ Binomial Distribution $\xrightarrow[\text{with replacement}]{}$ $P(X=r) = {}^nC_r p^r q^{n-r}$

$P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$; $P(R) = \frac{4}{5}$, $P(G) = \frac{1}{5}$

without replacement :

M Items

DEF	Non defective

$\underbrace{\hphantom{DEF \quad Non defective}}_{\text{Bernoulli Trials}}$

M Items — DEF (m_1 Items), ${}^{m_1}C_x$; Non DEF (m_2 Items), ${}^{m_2}C_{n-x}$

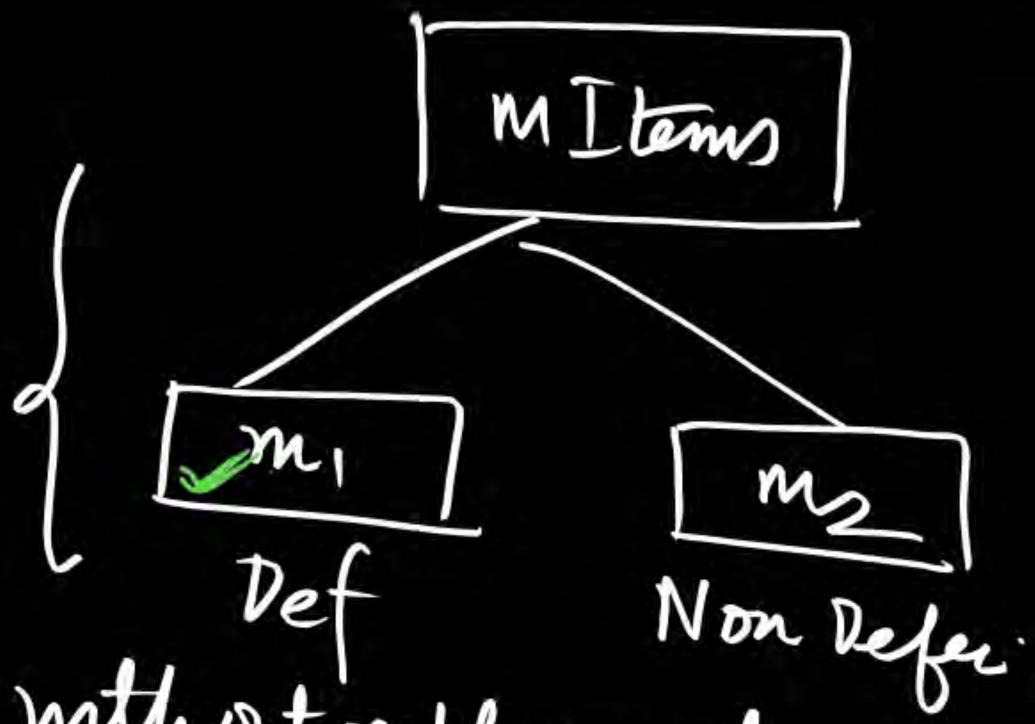
$X = \underline{\underline{x}}$ Defective Items

Prob for

P[X = x Defective]

$$\boxed{P[X=x \text{ defective}] = \frac{{}^{m_1}C_x \cdot {}^{m_2}C_{(n-x)}}{{}^{m}C_x}}$$

Hypergeometric / without replacement



without replacement

$$P(\text{defective}) = \frac{{}^{m_1}C_x \times {}^{m_2}C_{n-x}}{{}^{m}C_x}$$

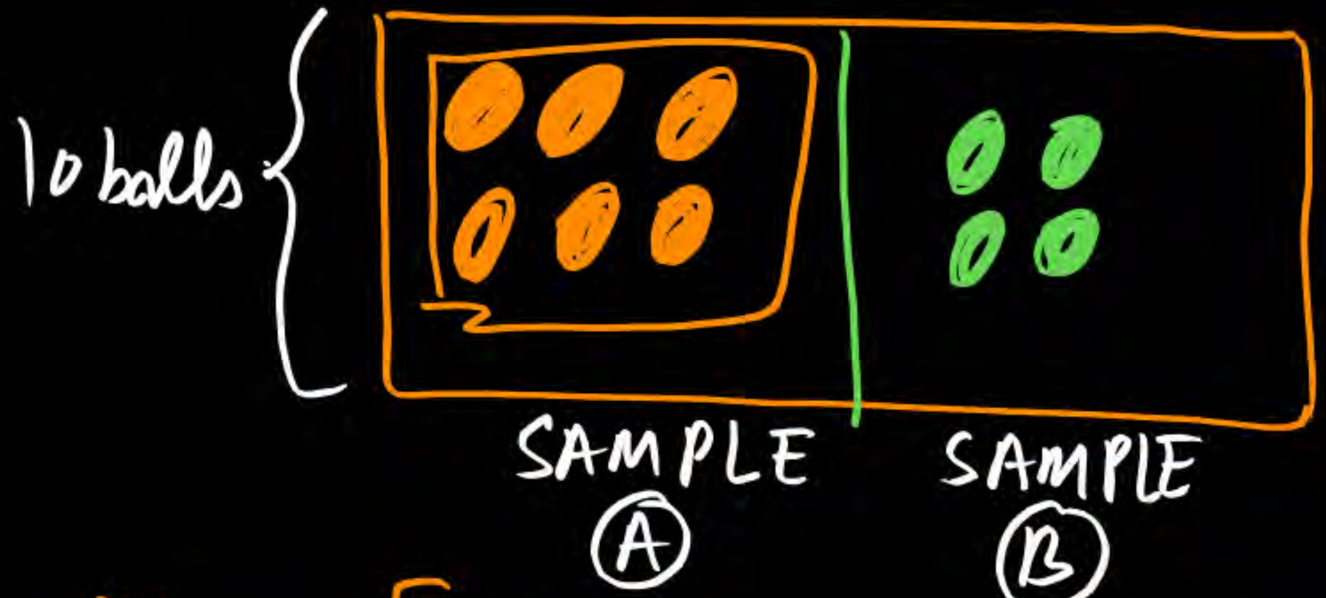
✓

26 Items

12 DEF	13 Non Defective

3 Items Are choosing.

2 Def 1 Non $= \dfrac{{}^{12}C_2 \; {}^{13}C_{3-2}}{{}^{26}C_3}$



m Items [m_1 Items + m_2 Items

x × $(n-x)$

$$\frac{{}^{m_1}C_x \times {}^{m_2}C_{n-x}}{{}^{m}C_n}$$

P(2 orange + 1 green)

$$\frac{{}^{6}C_2 \times {}^{4}C_1}{{}^{10}C_3}$$

Restrications

$$\frac{{}^{500}C_3 \times {}^{500}C_2}{{}^{1000}C_5}$$

$\boxed{\underline{1000}}$

500	500
S	A

Q7. A jury of 5 members is drawn at random from a voter's list of 100 persons, out of which 60 are non-graduates and 40 are graduates. What is the probability that the jury will consist of 3 graduates?

60 Non graduate	40 graduate
n_1	n_2

$$\frac{{}^{40}C_3 \times {}^{60}C_2}{{}^{100}C_5} = \underline{\text{Ans}}$$

$$\frac{{}^{n_2}C_2 \times {}^{n_1}C_3}{{}^{n}C_5}$$

M.W

Q8. Let us suppose that in a lake there are N fish. A catch of 500 fish (all at the same time) is made and these fish are returned alive into the lake after making each with a red spot. After two days, assuming that during this time these 'marked' fish have been distributed themselves 'at random' in the lake and there is no change in the total number of fish, a fresh catch of 400 fish (again, all at once) is made. What is the probability that of these 400 fish, 100 will be having red spots.

[redacted]

THANK - YOU

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