Data Science and Artificial Intelligence

Probability and Statistics

Continuous Probability Distribution

Lecture No.- 01

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Topics to be Covered

Topic Continuous Probability Distribution

Topic $\boxed{\text{Uniform Distribution}}$, $\boxed{\text{Gaussian Distribution}}$

✓ Imp. M. Imp.

Uniform Distribution ⇒

Discrete Distribution (Arrival Pattern)

Uniform Distribution:

continuous Distribution (Infinite Uncountable)

→ waiting time

Throwing A Die:

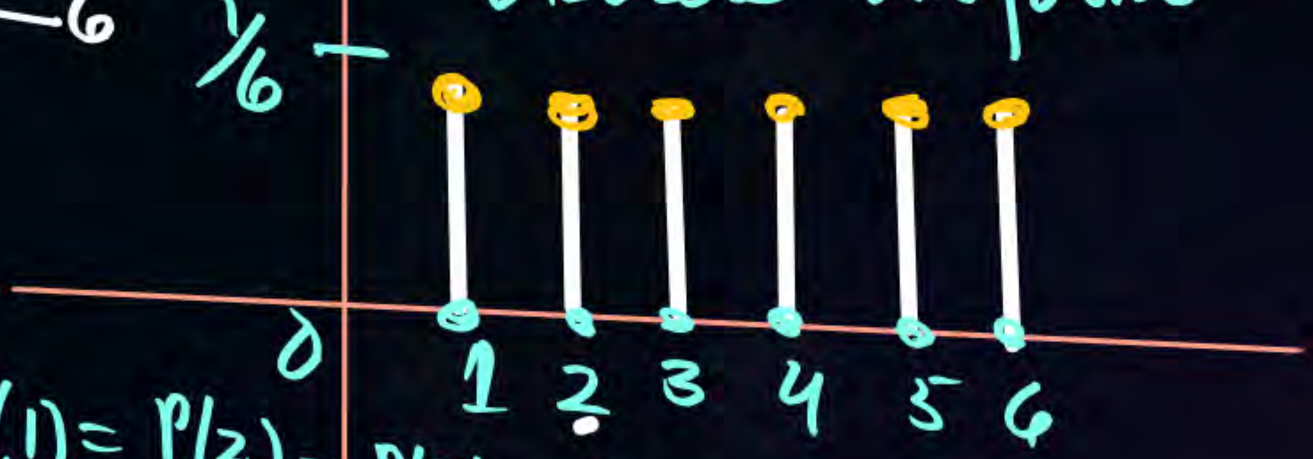
Evenspaced

$P(1) = \frac{1}{6}$ $P(2) = \frac{1}{6}$

Discrete Uniform

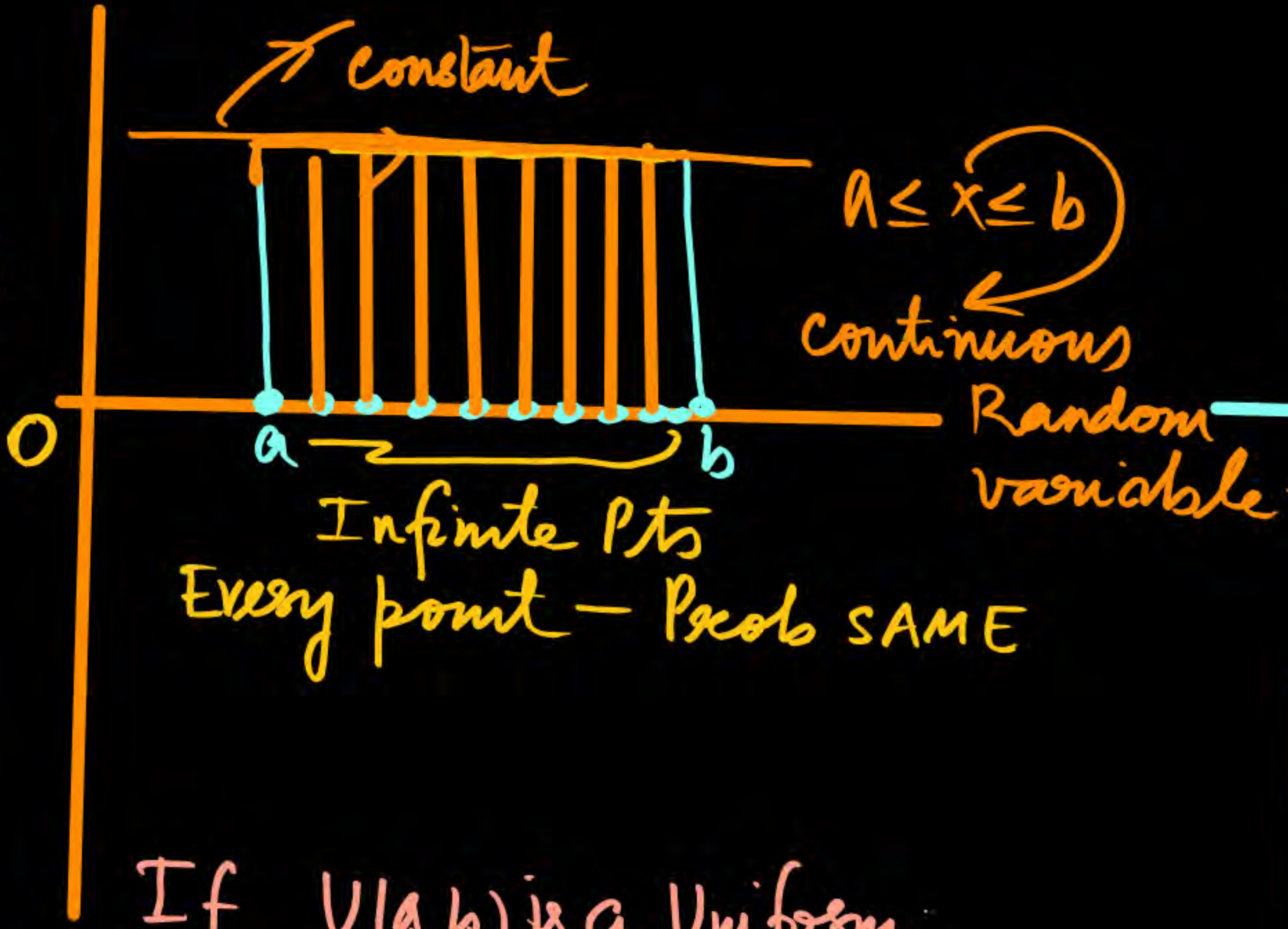
X = No. of dots

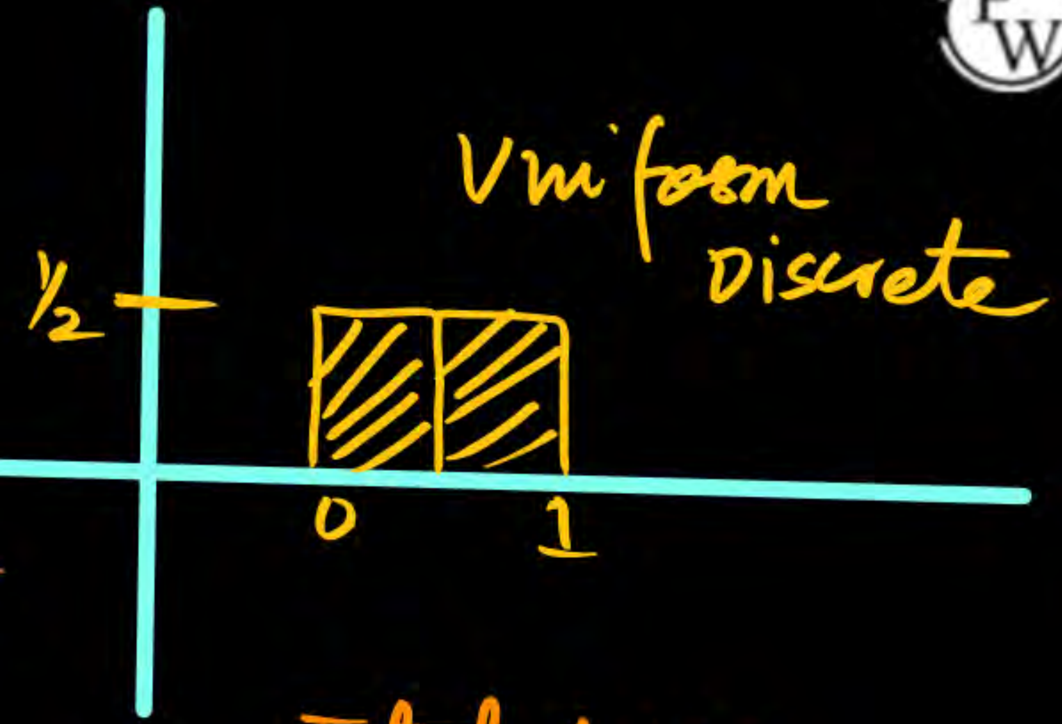
x	1	2	3	4	5	6
$P(X=x)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

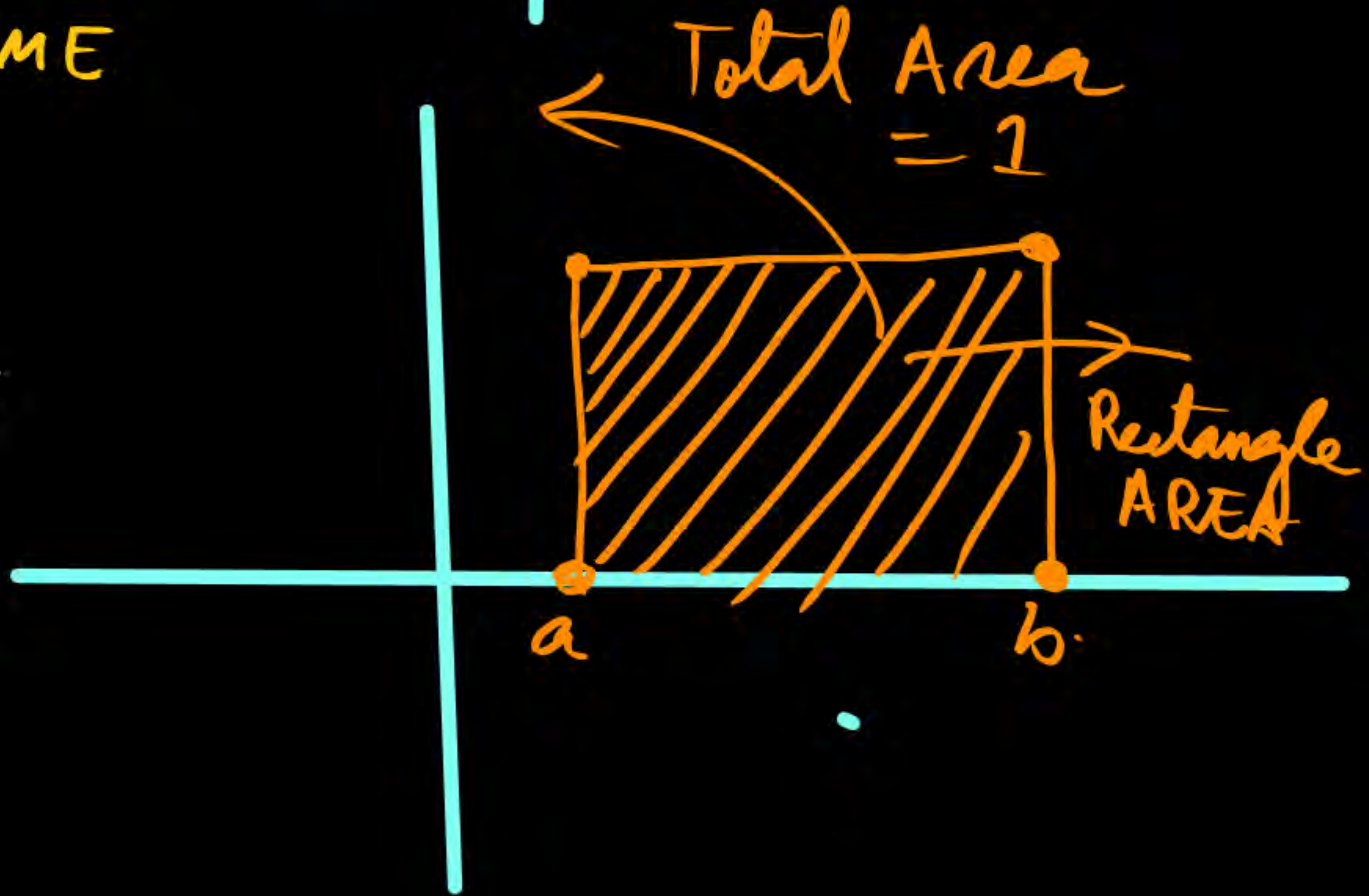


$P(1) = P(2) = P(5)$
$P(3) = P(4) = P(6) = 1/6$

Uniform
(Continuous)







If $U(a,b)$ is a Uniform continuous Distribution

Total AREA = 1

pdf $\rightarrow \int_a^b f(x)\,dx = 1$

$$f(x) = \begin{cases} K & a \le X \le b \\ 0 & \text{otherwise} \end{cases}$$

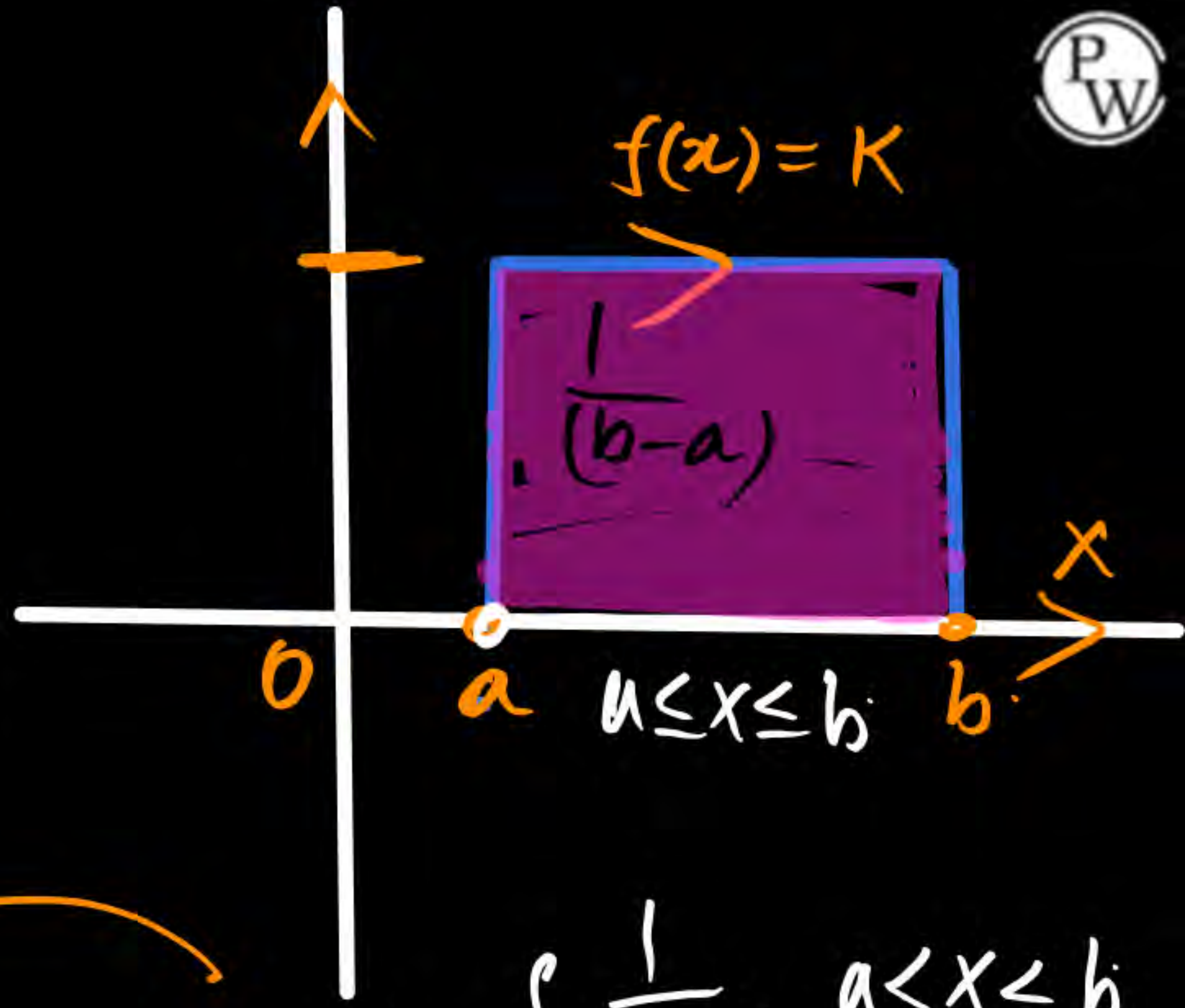
If this Probability density valid condition

$$\int_a^b f(x)\,dx = 1$$

$$= \int_a^b K\,dx = 1$$

$$\Rightarrow K(b-a) = 1$$

$$\boxed{K = \frac{1}{b-a}}$$



$$U(a,b) = \begin{cases} \frac{1}{(b-a)} & a \le X \le b \\ 0 & \text{otherwise} \end{cases}$$

Uniform continuous Dist

Two Parameters

$U(a,b)$

$$U(a,b) = \begin{cases} \frac{1}{(b-a)} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Uniform Continuous Distribution (a,b) $\rightarrow$ $a \le x \le b$

$$U(0,1) = \begin{cases} \frac{1}{1-0} & (0 \le x \le 1) \\ 0 & \text{otherwise} \end{cases}$$

$$U(0,1) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Expected value: $E[X] = \int_a^b x \cdot f(x)\, dx$

$$E[X] = \int_a^b x \cdot \frac{1}{(b-a)}\, dx$$

$$E[X] = \left[\frac{x^2}{2}\right]_a^b \frac{1}{(b-a)}$$

$$E[X] = \frac{(b^2-a^2)}{2} \cdot \frac{1}{(b-a)} = \frac{\cancel{(b-a)}(b+a)}{2} \frac{1}{\cancel{(b-a)}} = \frac{(a+b)}{2}$$

$$\boxed{\text{Expected value} = \frac{(a+b)}{2}} \rightarrow V(a,b)$$

$$\text{standard deviation} = \sqrt{\text{var}(X)}$$

$$\text{var}(X) = E[X^2] - [E[X]]^2$$

$$\Rightarrow \int_a^b x^2 f(x)\,dx - \left[\int_a^b x f(x)\,dx\right]^2$$

$$\Rightarrow \int_a^b x^2 \cdot \frac{1}{(b-a)}\,dx - \left[\frac{a+b}{2}\right]^2$$

$$E[x^2] = \int_a^b x^2 \cdot \frac{1}{(b-a)}\, dx = \frac{1}{(b-a)} \left[\frac{x^3}{3}\right]_a^b$$

$$\Rightarrow \frac{1}{(b-a)} \left[\frac{b^3-a^3}{3}\right] = \frac{1}{\cancel{(b-a)}} \frac{\cancel{(b-a)}(b^2+a^2+ab)}{3}$$

$$\boxed{E[x^2] = \frac{(a^2+b^2+ab)}{3}}$$ ✓

$$V(X) = E[x^2] - [E[x]]^2$$

$$= \frac{(a^2+b^2+ab)}{3} - \left(\frac{a+b}{2}\right)^2$$

$$V(X) = \frac{a^2+b^2+ab}{3} - \left(\frac{a^2+b^2+2ab}{4}\right)$$

$$V(X) = \frac{(b-a)^2}{12} \text{ or } \frac{(a-b)^2}{12}$$

$$S.D = \sqrt{var(x)}$$

$$= \sqrt{\frac{(b-a)^2}{12}}$$

$\underline{\underline{s^X}}$ **Moment generating function for continuous Random variable.**

$$M_X(s) = \int_{-\infty}^{\infty} e^{sx} f(x)\,dx$$

$$U(a,b) = \begin{cases} \frac{1}{(b-a)} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Uniform continuous Random var.

$U \sim (a,b)$

$$M_X(s) = \int_a^b e^{sx} \cdot \frac{1}{(b-a)}\,dx$$

$$= \frac{1}{(b-a)} \int_a^b e^{sx}\,dx$$

$$= \frac{1}{(b-a)} \left[\frac{e^{sx}}{s}\right]_a^b = \frac{1}{(b-a)} \left[\frac{e^{bs} - e^{as}}{s}\right]$$

$$\boxed{M_X(s) = \frac{1}{(b-a)}\frac{\left[e^{bs} - e^{as}\right]}{s}}$$

Uniform Distribution moment generating function

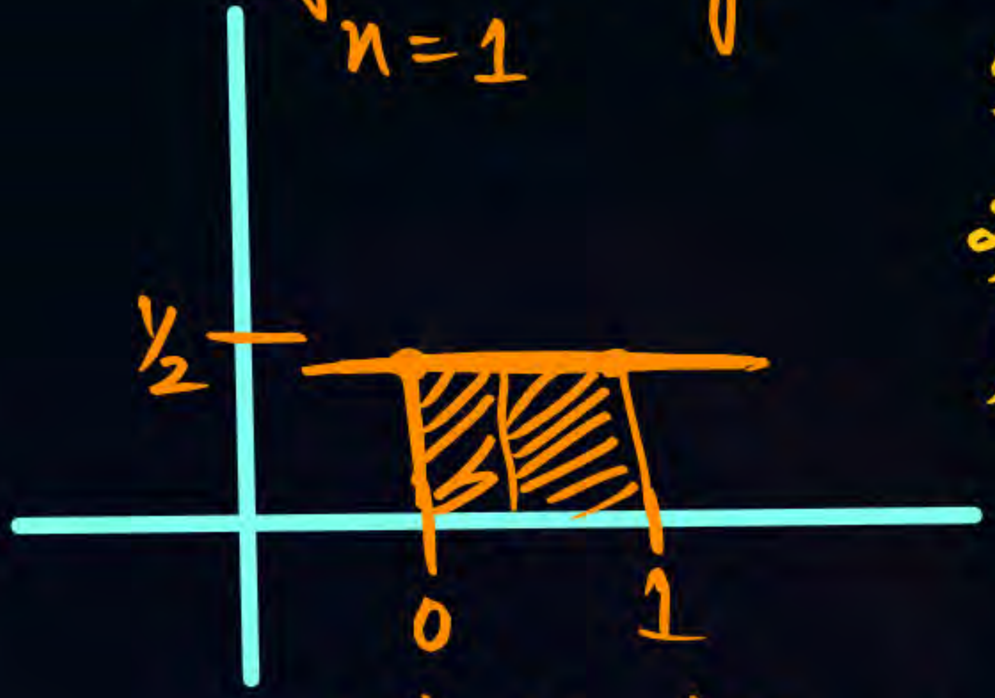
Uniform Distribution
mean, var(X), S.D, moment
MGF

Normal Distribution :

This is a continuous Random variable.

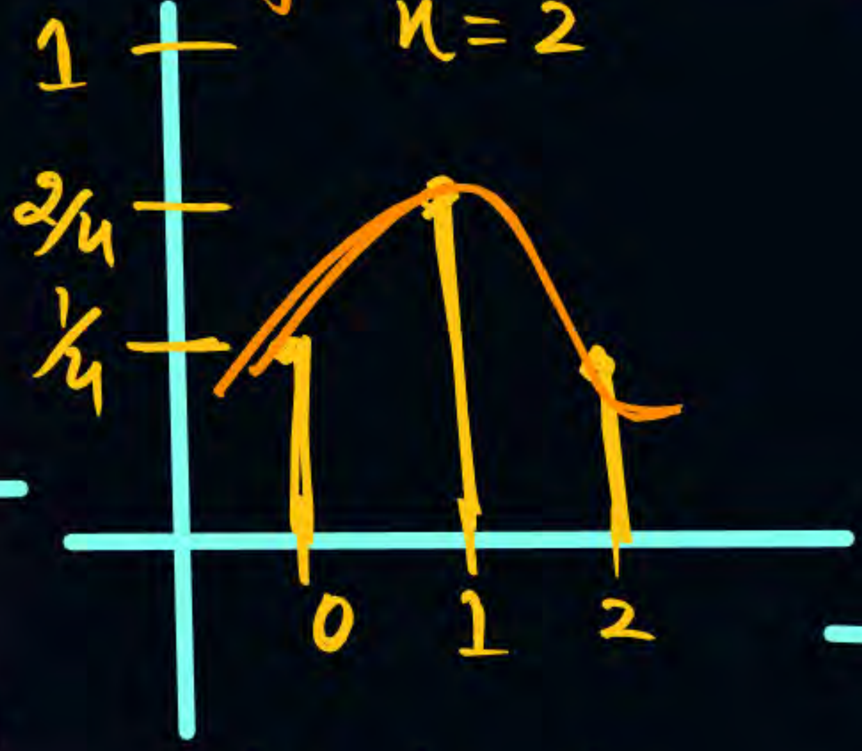
"If n is Large Number of Trials $\longrightarrow$ Normal Distribution OR Guassian Distributions

$n = 1$



X	H	T
$P(X=x)$	$\frac{1}{2}$	$\frac{1}{2}$

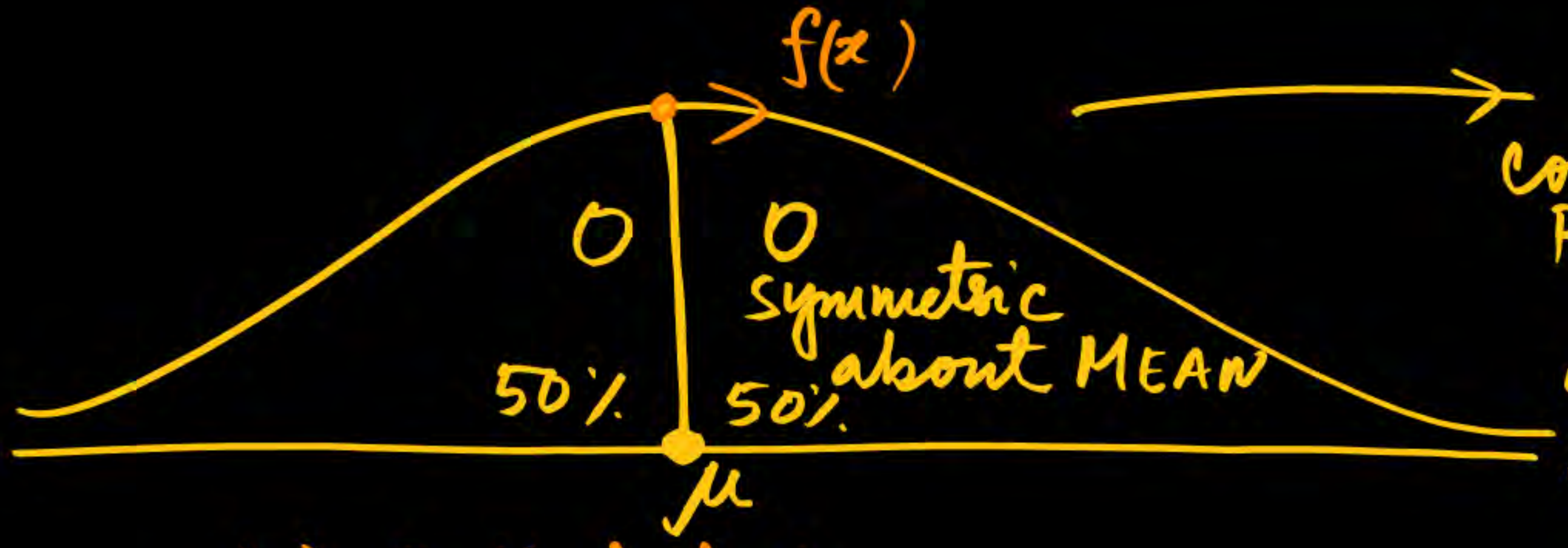
$n = 2$



X	0	1	2
$P(X=x)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

$n = 3$

X	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$



$n > 30$

SAMPLES

Continuous Random var.

✓ Guassian Distribution

✓ Normal dist.

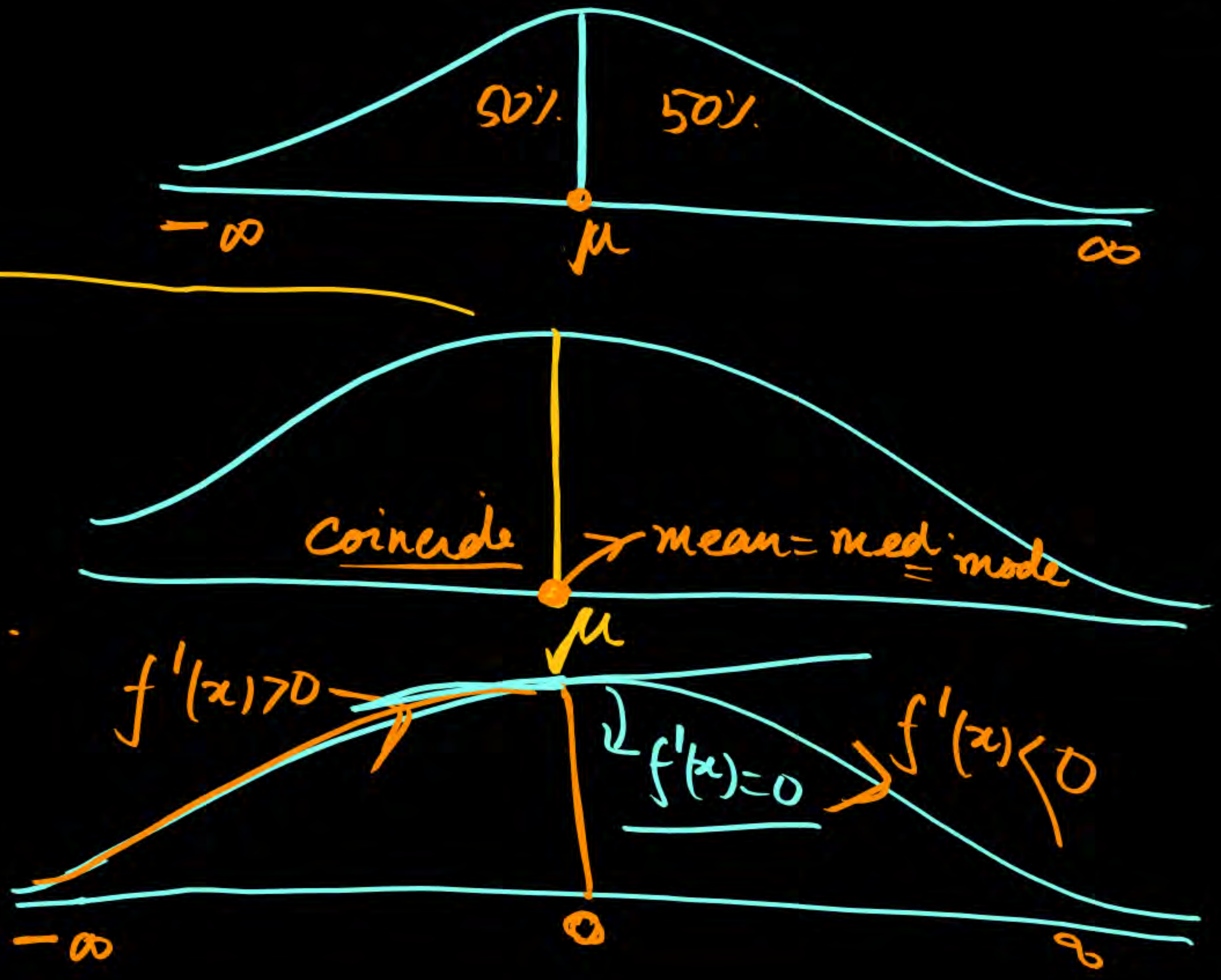
A) Bell shaped

B) This graph Never Touches or crosses. The Horizontal axis

C) Symmetric about mean.

✓ Symmetric about mean

✓ MEAN = median = mode = μ



Normal/Guassain

If this is a conti. RV

Then Prob. Density Function

$$\boxed{f(x)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

$\sigma > 0$

$-\infty \le \mu \le \infty$

$-\infty \le X \le \infty$

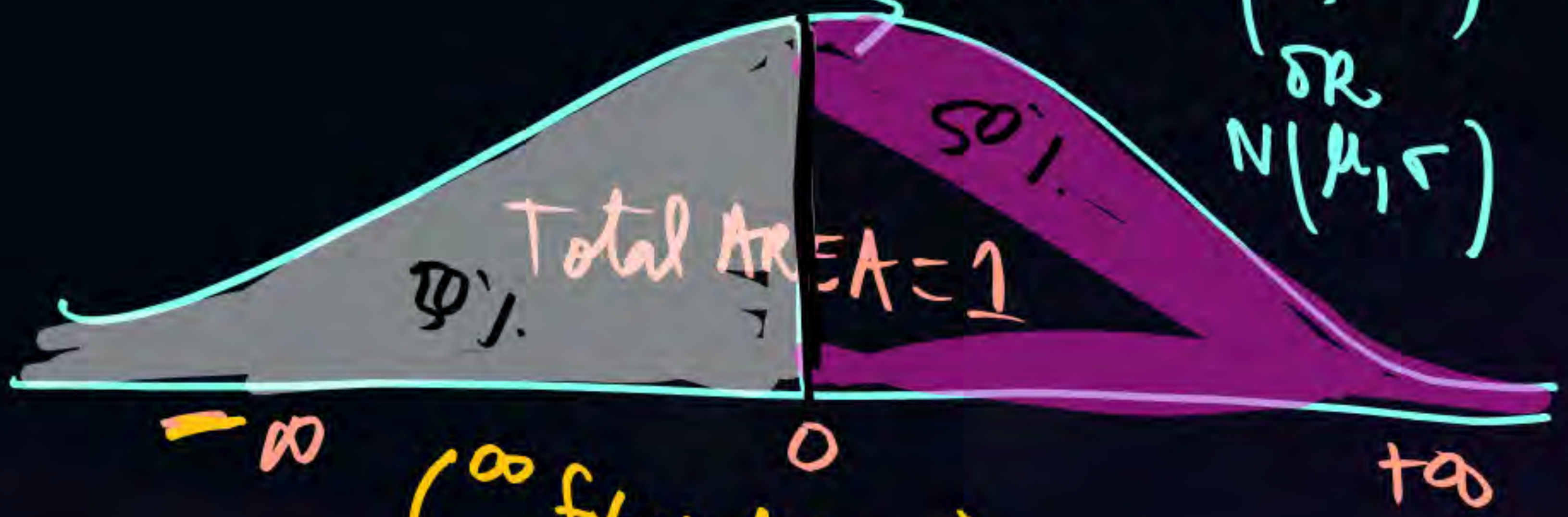
$$N(\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

OR

$N(\mu,\sigma)$ OR $N(\mu_x,\sigma_x^2)$

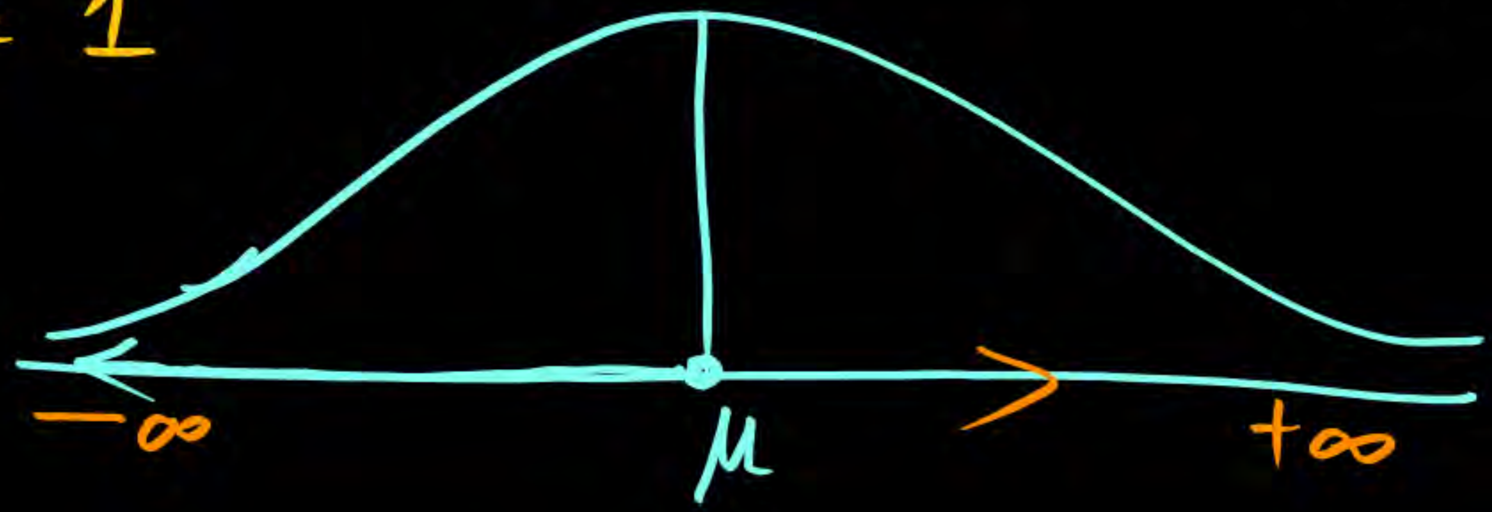
mean + var(x)
+ S.D.

$f(x) = N(\mu,\sigma^2)$
OR
$N(\mu,\sigma)$



$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$$



$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Put $\frac{x-\mu}{\sigma\sqrt{2}} = t$

both sides Differentiate It w.r.t to x

$$\frac{dx}{\sigma\sqrt{2}} - 0 = dt$$

$$\boxed{dx = \sigma\sqrt{2}\,dt}$$

$\frac{x-\mu}{\sigma\sqrt{2}}=t$ $\quad \infty = t$ $\quad -\infty = t$

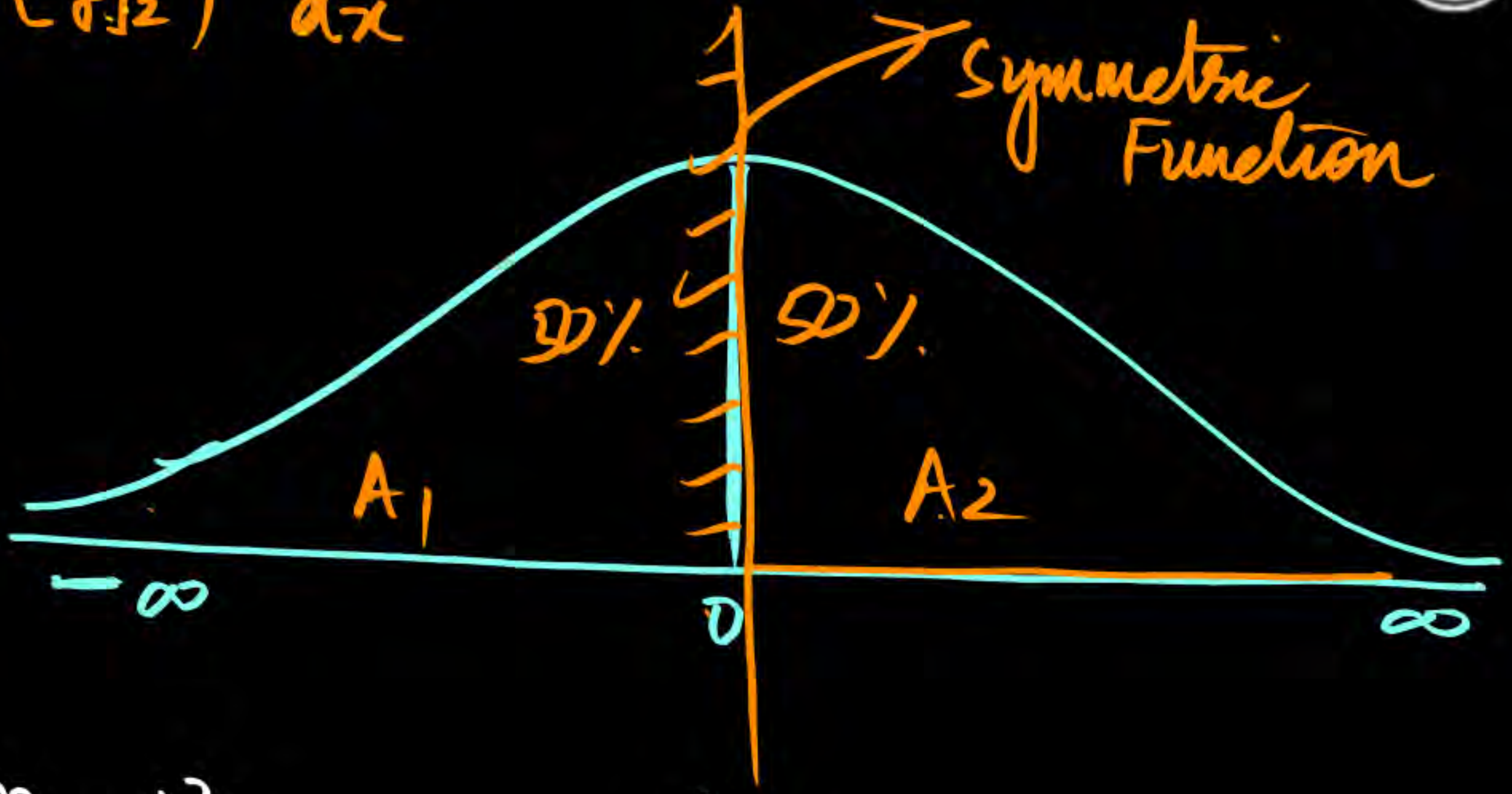
$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)^2} dx$$

$dx = \sigma\sqrt{2}\,dt$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2} dt\,\cancel{\sigma}\sqrt{2}$$

$$= \frac{1\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2} dt\,\cancel{\sigma}\sqrt{2}$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2} dt = \frac{2}{\sqrt{\pi}}\int_{0}^{\infty} e^{-t^2} dt$$



$t^2 = p$

$2t\,dt = dp$

$dt = \frac{dp}{2\sqrt{p}}$

$$= \frac{2}{\sqrt{\pi}}\int_{0}^{\infty} e^{-p}\,\frac{dp}{2\sqrt{p}}$$

$$= \frac{1\cancel{2}}{\cancel{2}\sqrt{\pi}}\int_{0}^{\infty} e^{-p}\,p^{-1/2}\,dp = \frac{1}{\sqrt{\pi}}\int_{0}^{\infty} e^{-p}\,p^{-1/2}\,dp = \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\pi}}\times\sqrt{\pi} = 1$$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad f(x) \text{ is even function } f(-x) = f(x)$$

$$= 0 \quad f(x) \text{ is odd function } f(-x) = -f(x)$$

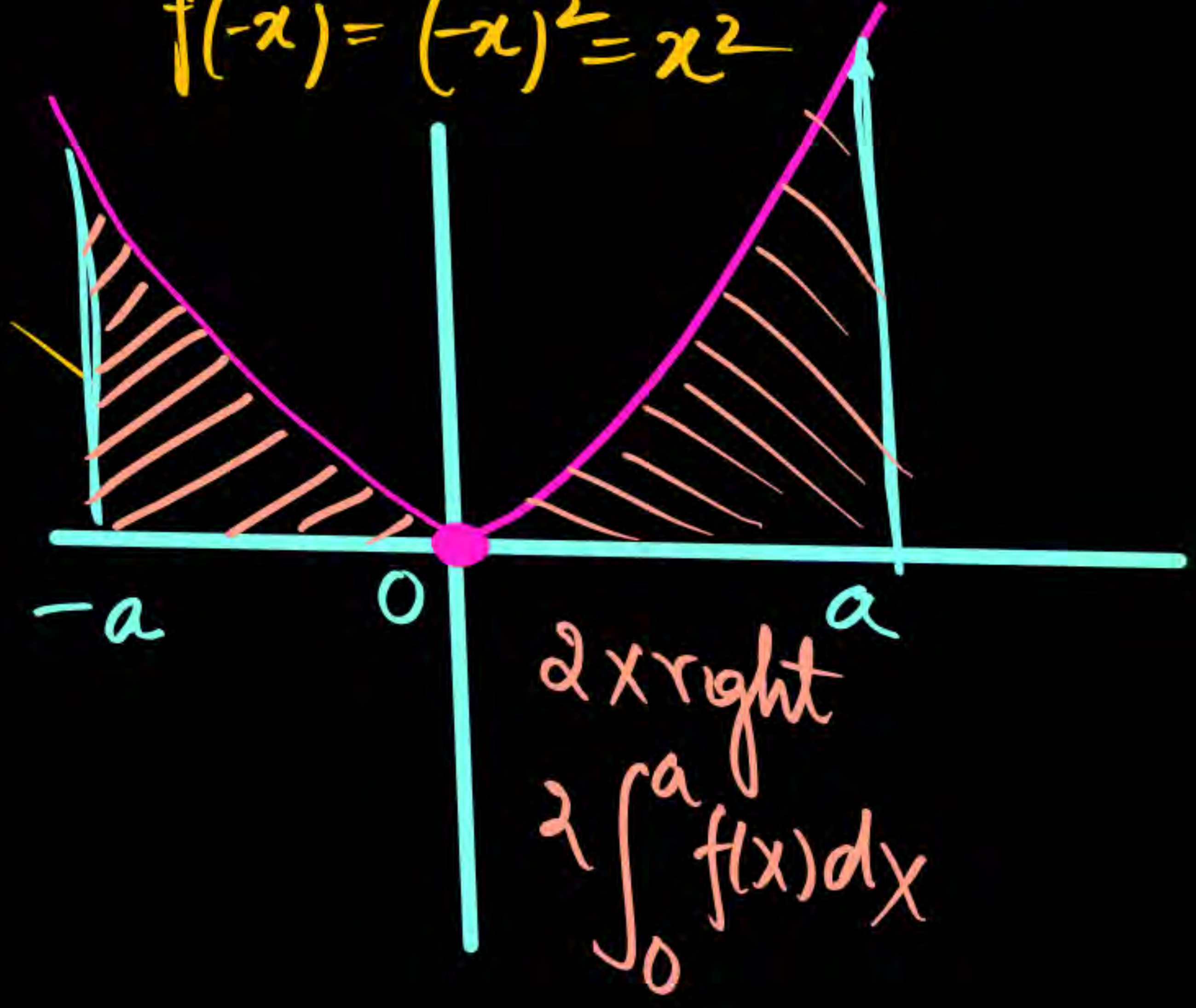
$f(x) = y = x^2$

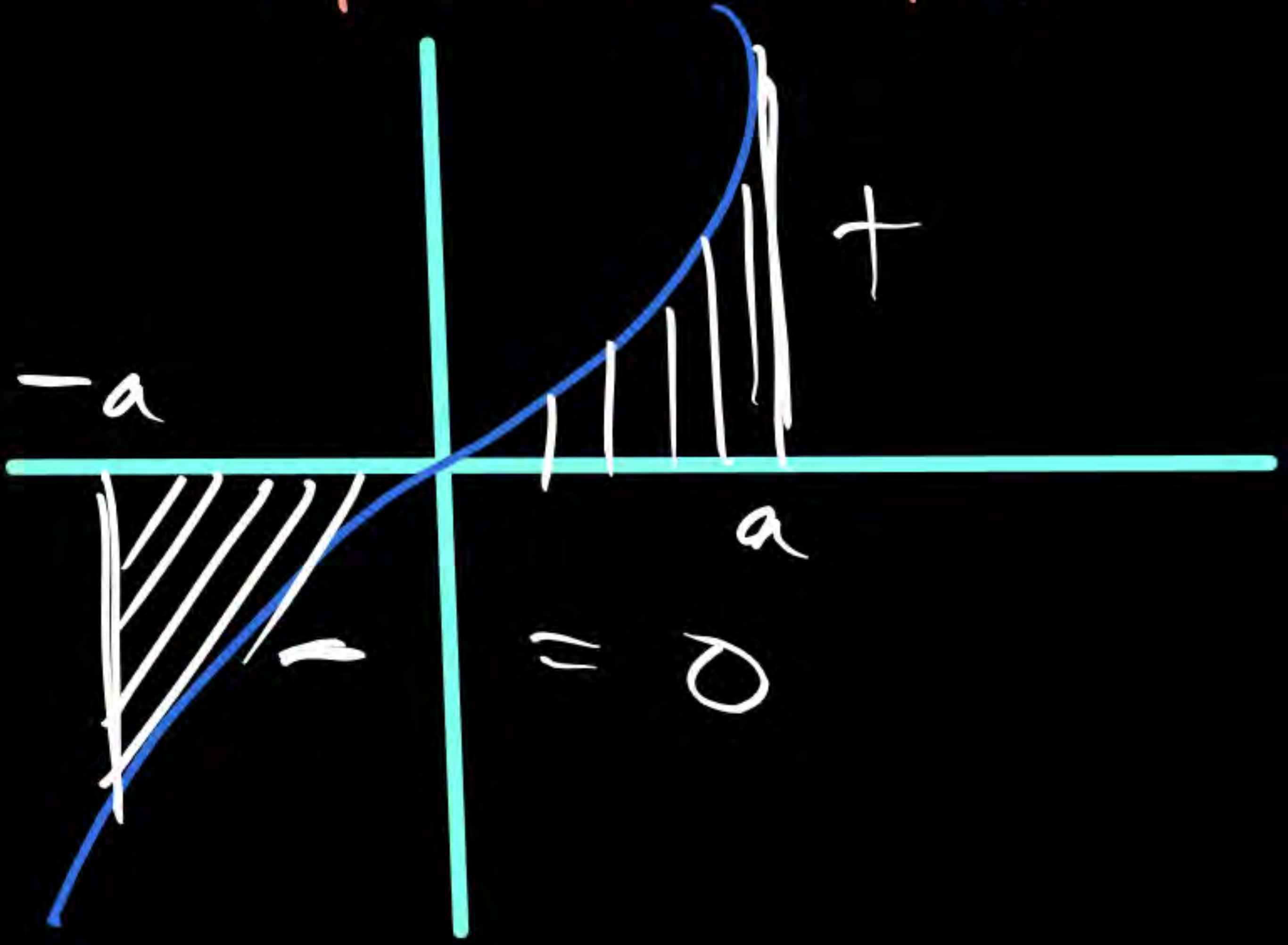
This function even/odd

$f(-x) = (-x)^2 = x^2$

$y = x^3 = (-x)^3 = -x^3$

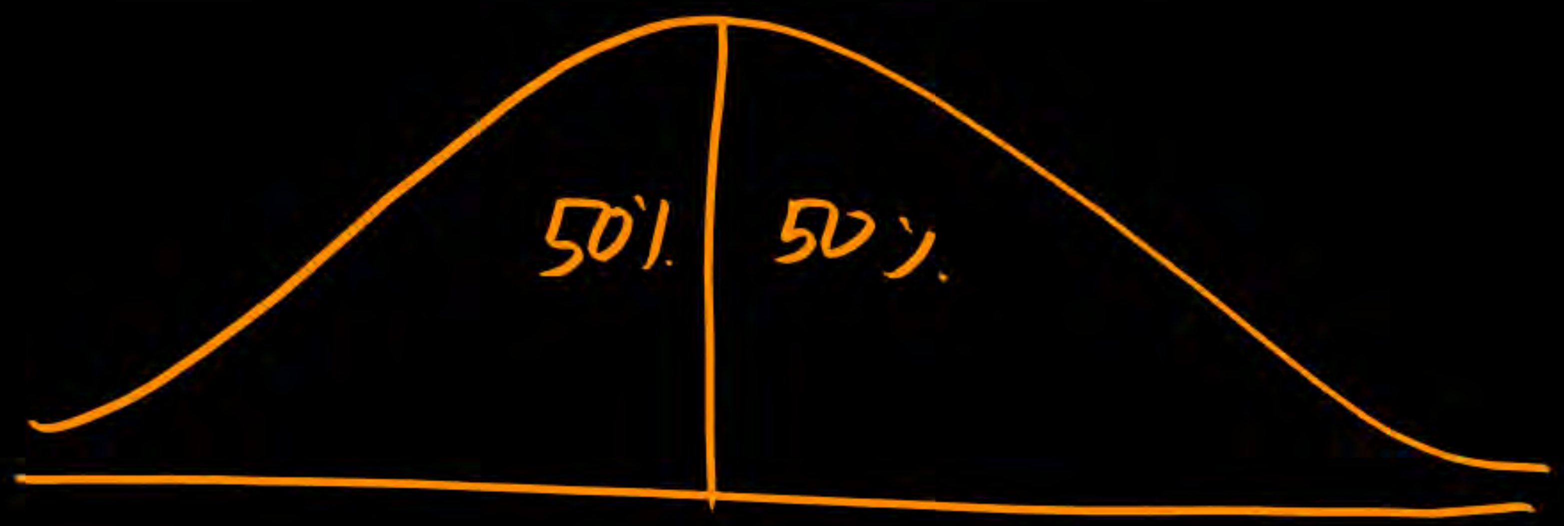
odd function $= -f(x)$





$$\int_{-\infty}^{\infty} N(\mu,\sigma^2)\,dx = 1 \quad \text{OR} \quad \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = 1$$

$$\begin{cases} \displaystyle\int_{-\infty}^{0} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \frac{1}{2} \\[2ex] \displaystyle\int_{0}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \frac{1}{2} \end{cases}$$



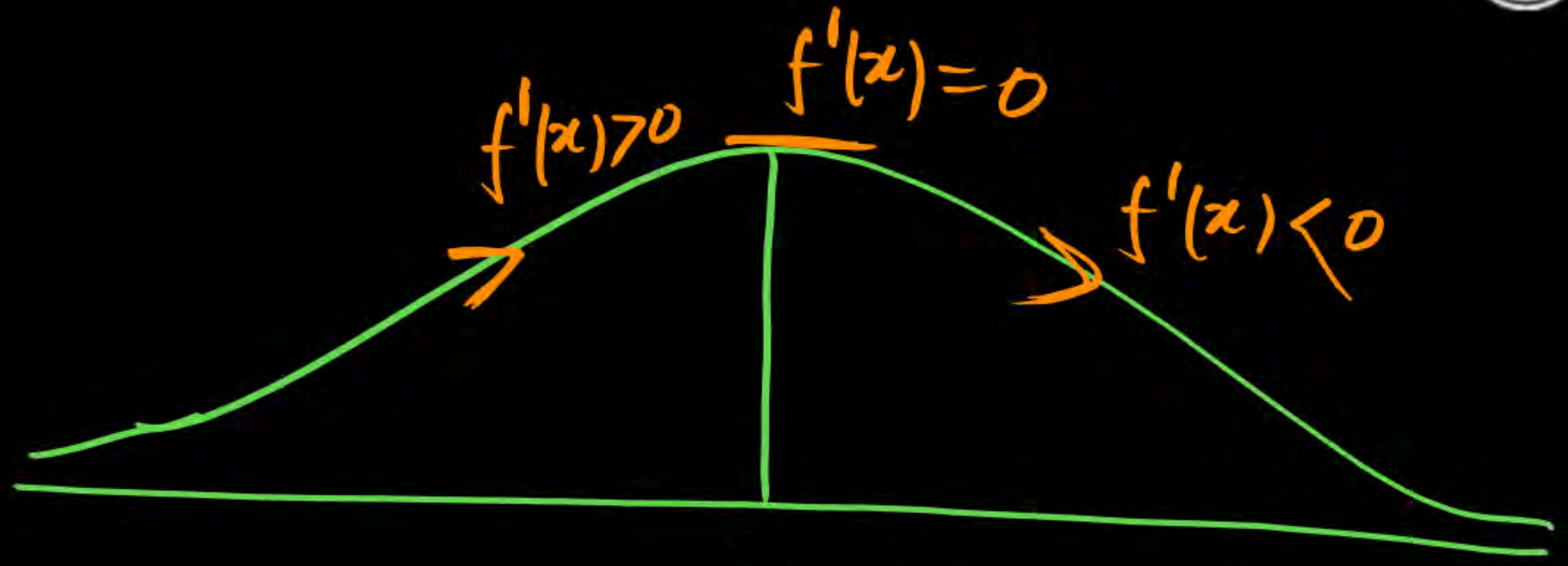
$f(x) = N(\mu, \sigma^2)$

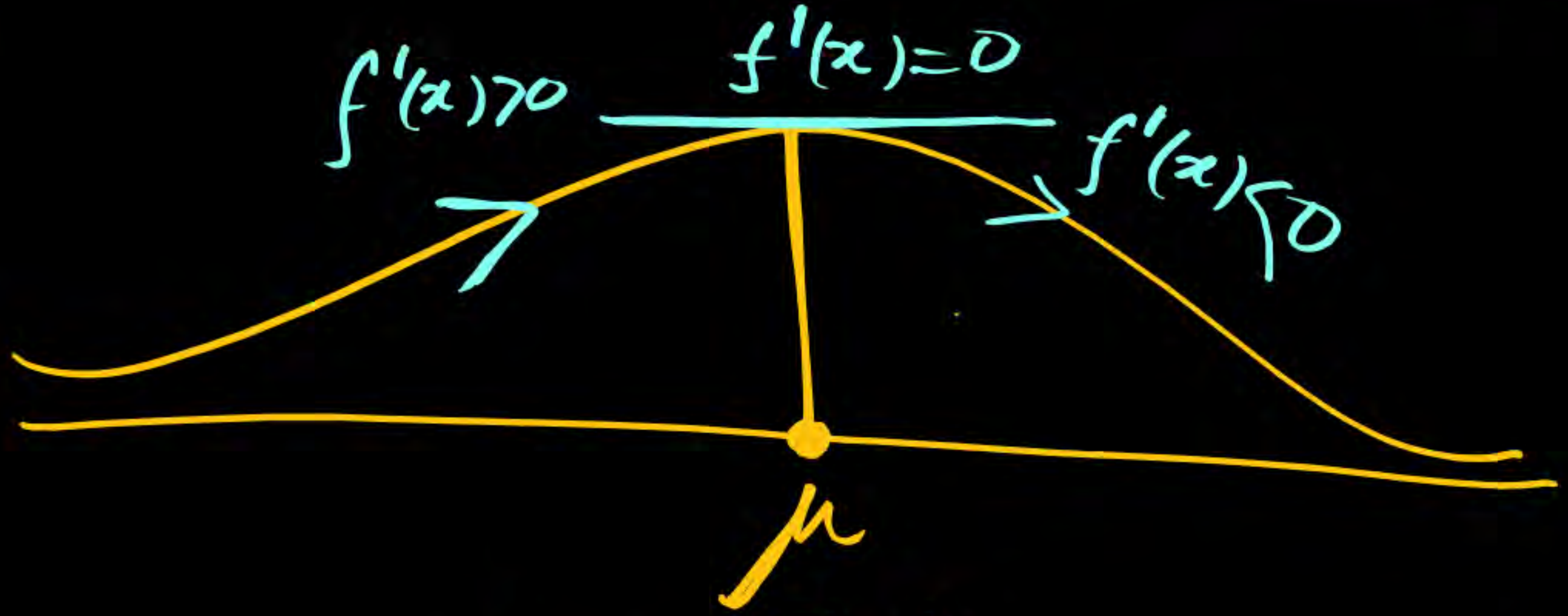
$Var(X) = \sigma^2$ MEAN $= \mu$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$f'(x) = 0$ $\frac{dy}{dx} = 0$

$\boxed{x = \mu}$



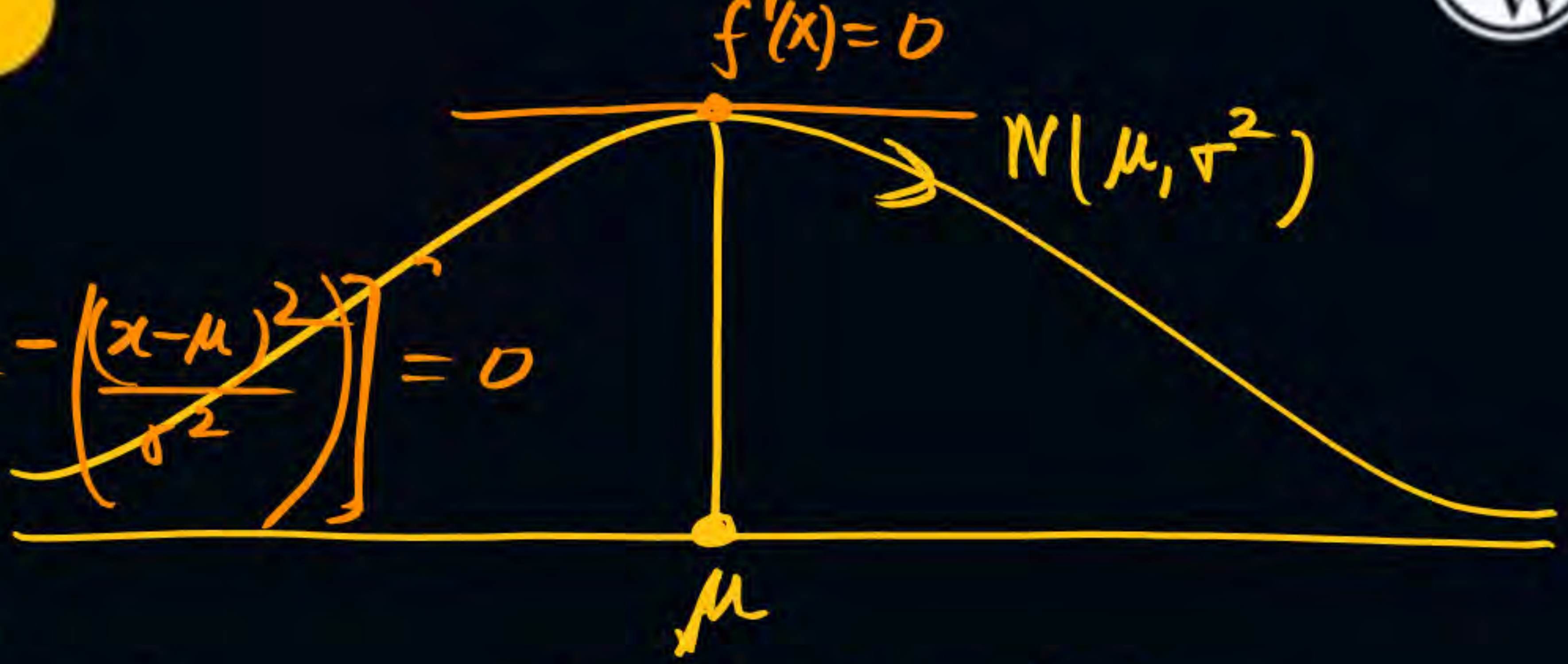


$x = \mu$

Sym: About mean

$$N(\mu,\sigma^2)=f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$f''(x)=\frac{-1}{\sigma^3\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\left[1-\left(\frac{(x-\mu)^2}{\sigma^2}\right)\right]=0$$



$$\frac{-1}{\sigma^3\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\underbrace{\left[1-\left(\frac{(x-\mu)^2}{\sigma^2}\right)\right]}=0$$

Point of Inflection

$$\boxed{\frac{d^2y}{dx^2}=0}$$

$$1-\frac{(x-\mu)^2}{\sigma^2}=0$$

$$\therefore\ \boxed{x=\mu\pm\sigma}$$

$$1=\frac{(x-\mu)^2}{\sigma^2} \qquad \frac{x-\mu}{\sigma}=\pm1 \qquad x=\mu\pm\sigma$$

$X = \mu \pm \sigma$

Again Differentiation

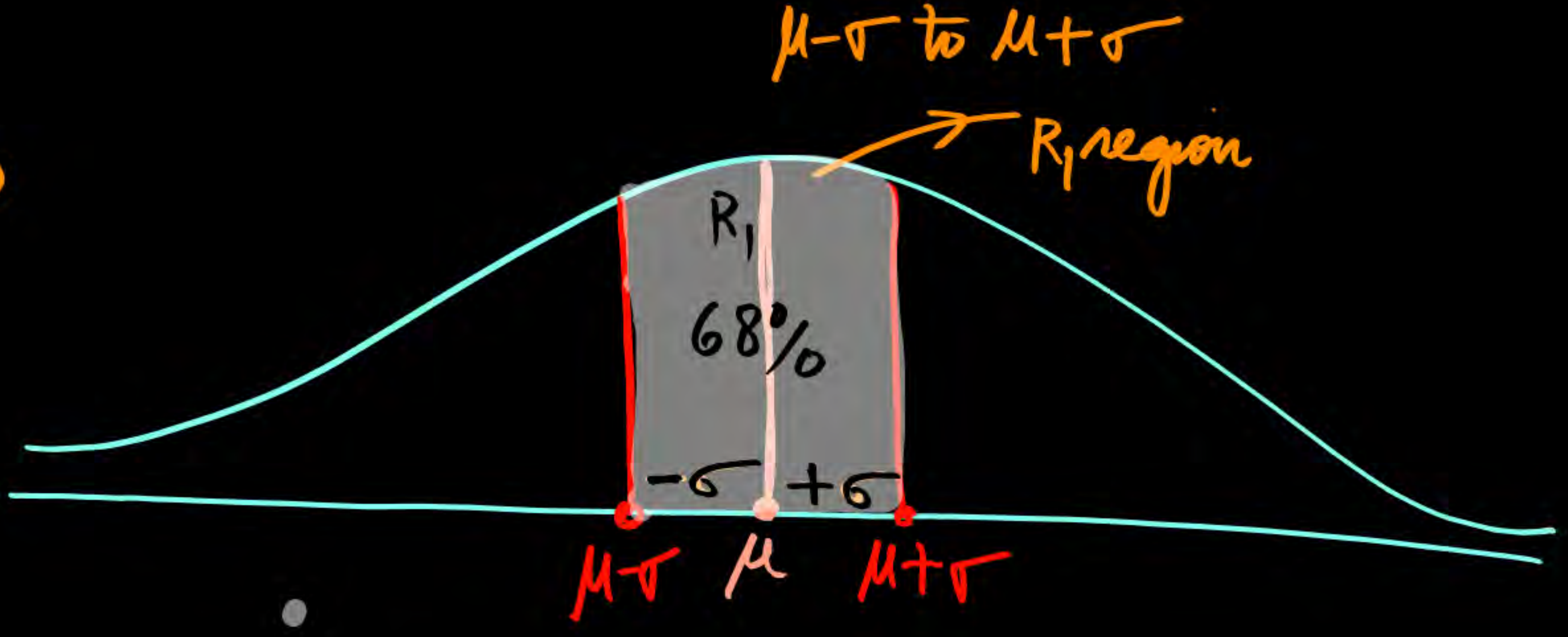
$$\frac{d^3y}{dx^3} = 0$$

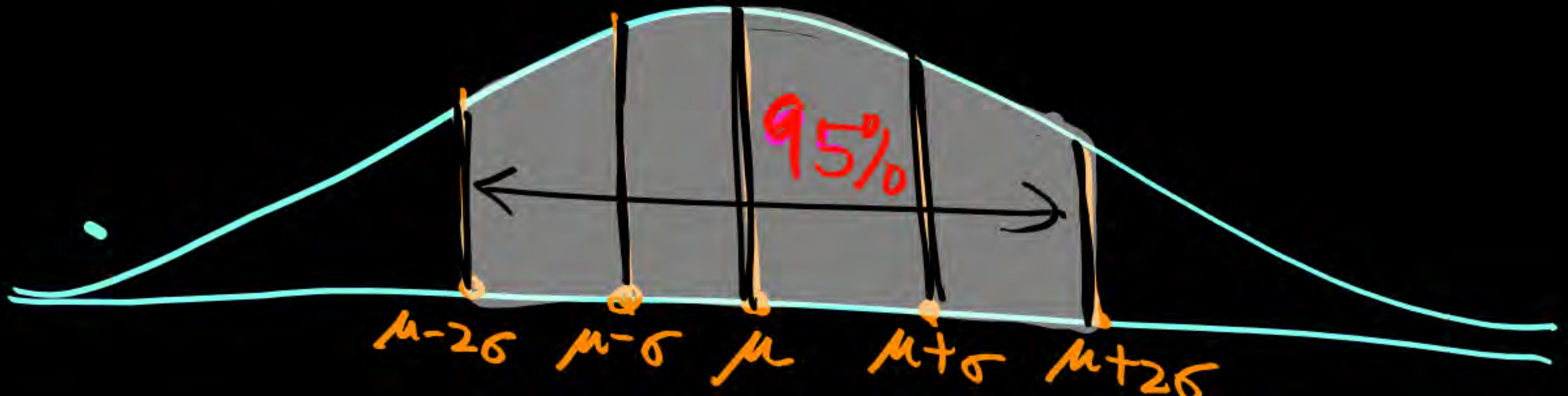
$X = \mu \pm 2\sigma$

Again Diff.

$$\boxed{x = \mu \pm 3\sigma}$$

$$\left.\begin{aligned} X &= \mu \pm \sigma \\ &= \mu \pm 2\sigma \\ &= \mu \pm 3\sigma \end{aligned}\right]$$





✓ R_1 region

✓ $\mu-\sigma$ to $\mu+\sigma = 68\%$

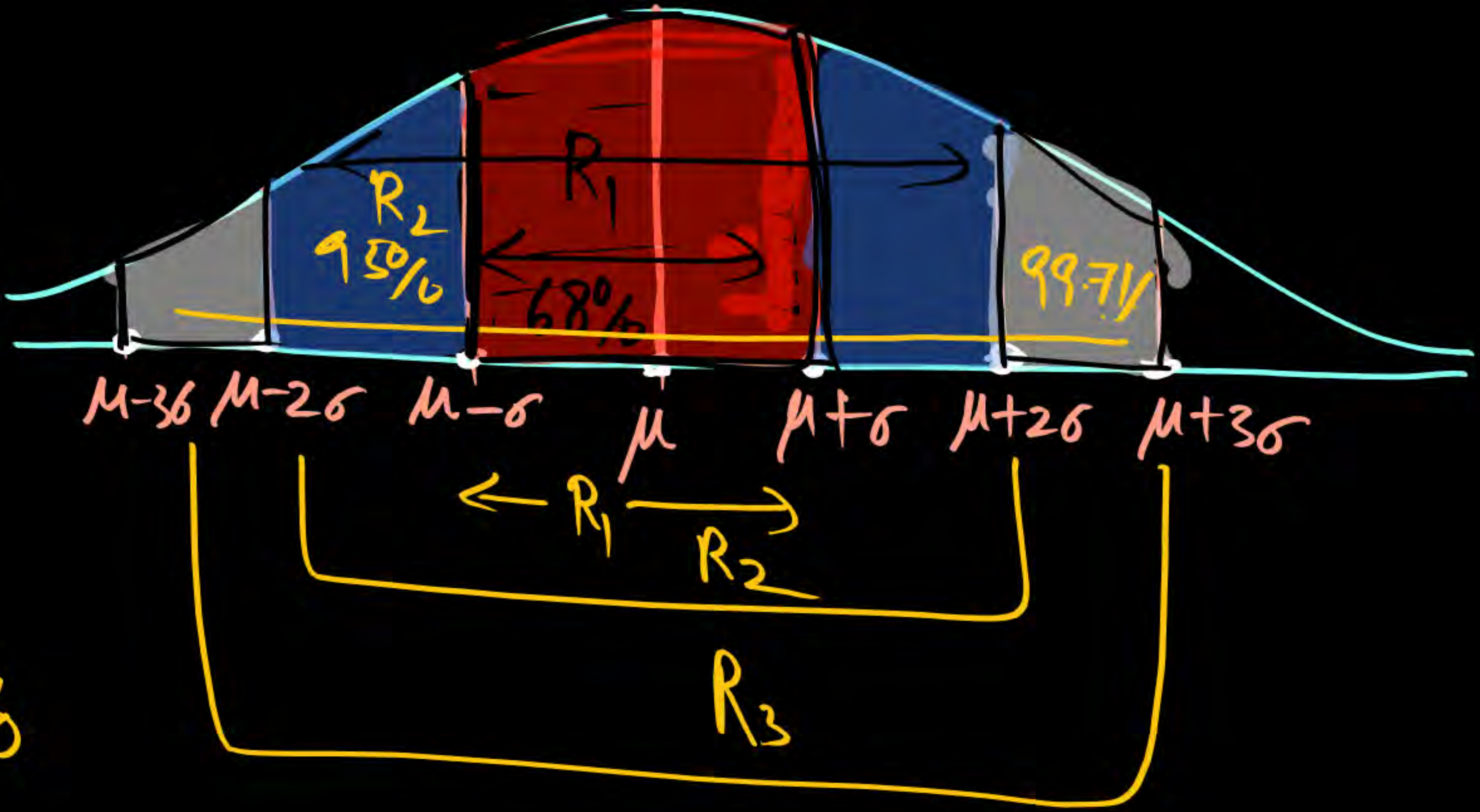
✓ $\mu-2\sigma$ to $\mu+2\sigma = 95\%$

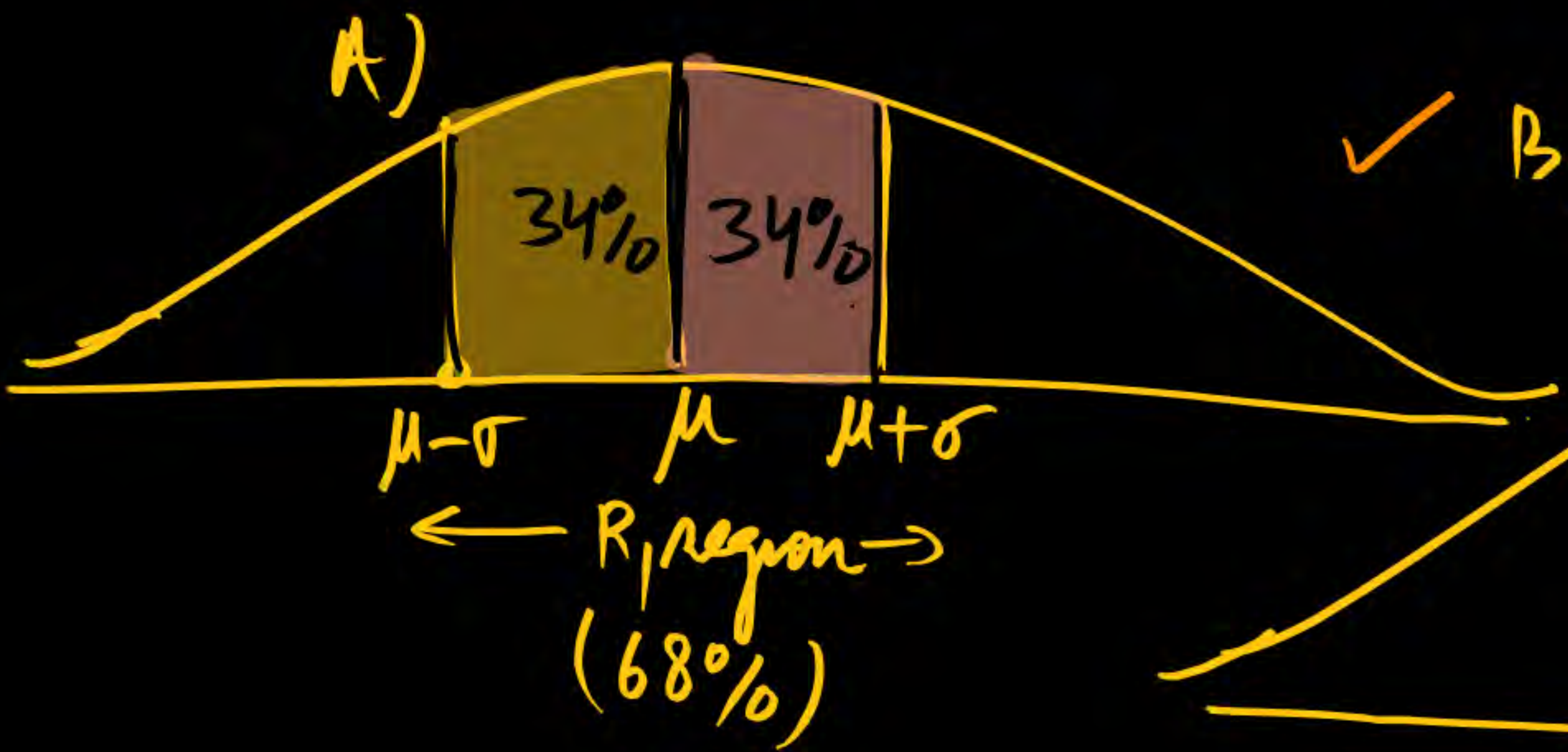
R_2 region

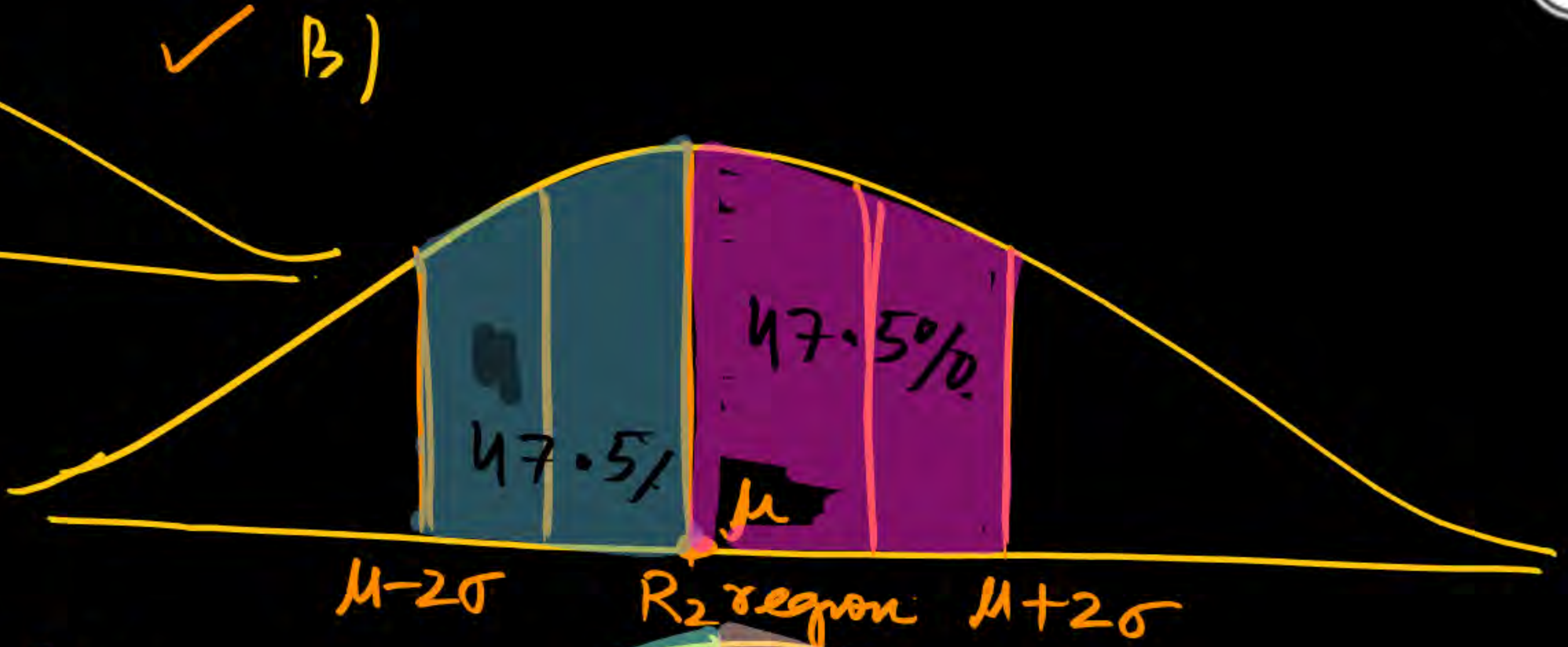
✓ R_3 region.

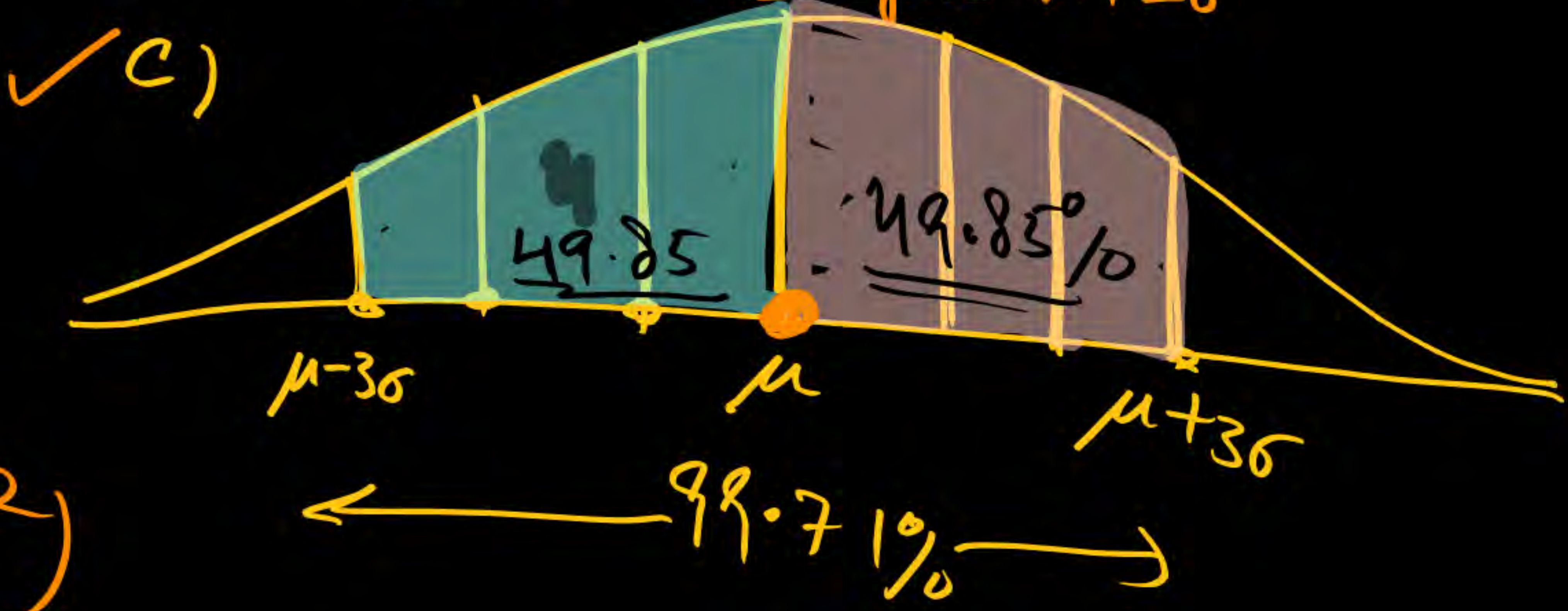
$\mu-3\sigma$ to $\mu+3\sigma$

$= 99.71\%$









MEAN = μ

median = μ

mode = μ

S.D = σ

var(X) = σ^2

$N(\mu, \sigma^2)$

THANK - YOU

Slide 13