Data Science and Artificial Intelligence

Probability and Statistics

Continuous Probability Distribution

Lecture No.- 06

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Topics to be Covered

- **Topic** Beta Distribution → 2
- **Topic** Gamma Distribution → 1
- **Topic** Hypergeometric Distribution
- **Topic** Problem based on Beta and Gamma Distribution

Gamma Function : Factorial function

$$\text{Gamm } \alpha = \Gamma\alpha = \int_0^\infty e^{-x} x^{\alpha-1}\, dx \quad \alpha > 0$$

Basic Properties :

(A) $\Gamma(\alpha+1) = \alpha\,\Gamma\alpha$

(B) $\Gamma\alpha = (\alpha-1)!$ $\quad \Gamma 5 = 4! = 24$

(C) $\Gamma\frac{5}{2} = \frac{3}{2} \times \frac{1}{2} \times \Gamma\frac{1}{2}$

$\Gamma\frac{1}{2} = \sqrt{\pi}$

$$\int_0^\infty \underbrace{x^3 e^{-x}\,dx}_{\text{gma function}} = \Gamma 4 = 3!$$

$$\int_0^\infty e^{-x} x^{\alpha-1}\,dx$$

$\alpha - 1 = 3$

$\alpha = 4$

$$\int_0^\infty x^7 e^{-x}\,dx = \Gamma 8 = 7!$$

$$\int_0^\infty x^{11} e^{-x}\,dx = \Gamma 12 = 11!$$

$$\int_0^\infty x^{9} e^{-x}\,dx = \Gamma 9+1 = \Gamma 10 = 9!$$

$$\boxed{\begin{aligned} &\int_0^\infty x^{n} e^{-x}\,dx = (n-1)! \\ &\int_0^\infty x^{n-1} e^{-x}\,dx = \Gamma n+1 = n! \end{aligned}}$$

$(n-1)$

$$\int_0^\infty x^{9} e^{-x}\,dx = \Gamma 9+1 = 1$$

$n - 1 = 9$

$n = 10$

$\Gamma 10 = 9!$

$$\int_0^\infty e^{-x} x^{\alpha-1}\, dx$$

$$\int_0^\infty e^{-x} x^{6}\, dx = \Gamma 7$$

$$= 6!$$

$\alpha - 1 = 6$

$\alpha = 7$

$$\int_0^\infty e^{-x} x^{\alpha-1}\, dx =$$

$\alpha - 1 = N$

$\alpha = (N+1)$

$$= \Gamma(N+1)$$

$$= \Gamma(\alpha - 1 + 1)$$

$$= \Gamma\alpha = \underbrace{(\alpha-1)!}$$

$$\int_0^\infty e^{-x} x^{9}\, dx = \Gamma 10 = (n-1)!$$

$=$

$n - 1 = 10$

$= n = \Gamma 10$

$$I = \int_0^\infty \boxed{x^6 e^{\boxed{-2x}} dx =} \longrightarrow \int_0^\infty e^{\boxed{-x}} x^{\alpha-1} dx$$

put $\quad 2x = t \quad 2\,dx = dt$

substitution method $\quad dx = \frac{dt}{2}$

$$= \int_0^\infty \left(\frac{t}{2}\right)^6 e^{-t} \frac{dt}{2}$$

$$= \frac{1}{2 \times 2^6} \int_0^\infty t^6 e^{-t} dt$$

$$= \frac{1}{2^7} \int_0^\infty t^6 e^{-t} dt = \frac{1}{2^7} \int_0^\infty t^{\alpha-1} e^{-t} dt$$

$$\Rightarrow \frac{1}{2^7} \times 6! = \frac{45}{8}$$

$$I = \int_0^\infty \sqrt{y}\, e^{-y^3}\, dy = \frac{\sqrt{\pi}}{3}$$

$$I = \int_0^\infty y^{1/2} e^{-y^3}\, dy$$

$$y^3 = t \qquad y = t^{1/3}$$

$$3y^2\, dy = dt \qquad \left.\begin{matrix} 0 = t \\ t = \infty \end{matrix}\right]$$

$$dy = \frac{dt}{3y^2} = \frac{dt}{3\left[t^{1/3}\right]^2} = \frac{dt}{3\, t^{2/3}}$$

$$= \int_0^\infty \left[t^{1/3}\right]^{\frac{1}{2}} e^{-t} \cdot \frac{dt}{3\, t^{2/3}}$$

$$= \frac{1}{3}\int_0^\infty t^{\frac{1}{6} - \frac{2}{3}} e^{-t}\, dt$$

$$\Rightarrow \frac{1}{3}\int_0^\infty t^{-\frac{9}{18}} e^{-t}\, dt$$

$$\alpha - 1 = -\frac{1}{2}$$

$$\alpha = \frac{1}{2}$$

$$\Rightarrow \frac{1}{3}\int_0^\infty t^{-1/2} e^{-t}\, dt = \Gamma\alpha \cdot \frac{1}{3} = \Gamma\tfrac{1}{2} \cdot \frac{1}{3} = \frac{\sqrt{\pi}}{3}$$

$$\int_0^\infty a^{+x^2}$$

$\Downarrow$ gamma

exp + Algebraic

$$I = \int_0^\infty 3^{-4x^2}\,dx$$

$$3^{-4x^2} = e^{\log 3^{-4x^2}}$$

$$= e^{-4x^2 \log 3}$$

$$= e^{-4x^2 \cdot \log 3}$$

$$=$$

$$\underline{\underline{t}}$$

$$\int_0^\infty e^{-4x^2 \log 3}$$

3^{-4x^2} ✓

$\downarrow$ function exp + Algebraic

$$e^{\log x} = x$$

$$I = \int_0^{\infty} e^{-4x^2 \log 3}\, dx$$

$4x^2 \log 3 = t$

both sides Diff.

$4 \cdot 2x\, dx = \dfrac{dt}{\log 3}$

$dx = \dfrac{dt}{\log 3 \cdot 8x}$

$4x^2 \log 3 = t$

$4x^2 = \dfrac{t}{\log 3}$

$x^2 = \dfrac{t}{4 \log 3}$

$x = \dfrac{\sqrt{t}}{2\sqrt{\log 3}}$

$$I = \int_0^{\infty} e^{-t} \frac{dt}{\log 3 \cdot 8x}$$

$$= \int_0^{\infty} e^{-t} \frac{dt}{\sqrt{\log 3} \cdot 8 \dfrac{\sqrt{t}}{2\sqrt{\log 3}}}$$

$$I = \int_0^{\infty} \frac{1}{\sqrt{\log 3}} \frac{1}{4} \frac{1}{\sqrt{t}} e^{-t}$$

$$= \frac{1}{4\sqrt{\log 3}} \underbrace{\int_0^{\infty} e^{-t} t^{-1/2}\, dt}_{\text{Gamma function}}$$

$$= \frac{1}{4\sqrt{\log 3}} \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$$

$$I = \int_0^\infty x^{\frac{1}{4}} e^{-\sqrt{x}}\, dx =$$

$\sqrt{x} = t$

$x = t^2$

$\boxed{dx = 2t\,dt}$

$$= \int_0^\infty [t^2]^{\frac{1}{4}} e^{-t} \cdot 2t\, dt$$

$$= 2\int_0^\infty t^{\frac{2}{4}} \cdot t\, e^{-t}\, dt$$

$$= 2\int_0^\infty t^{\frac{1}{2}+1} e^{-t}\, dt$$

$$= 2\int_0^\infty t^{3/2} e^{-t}\, dt$$

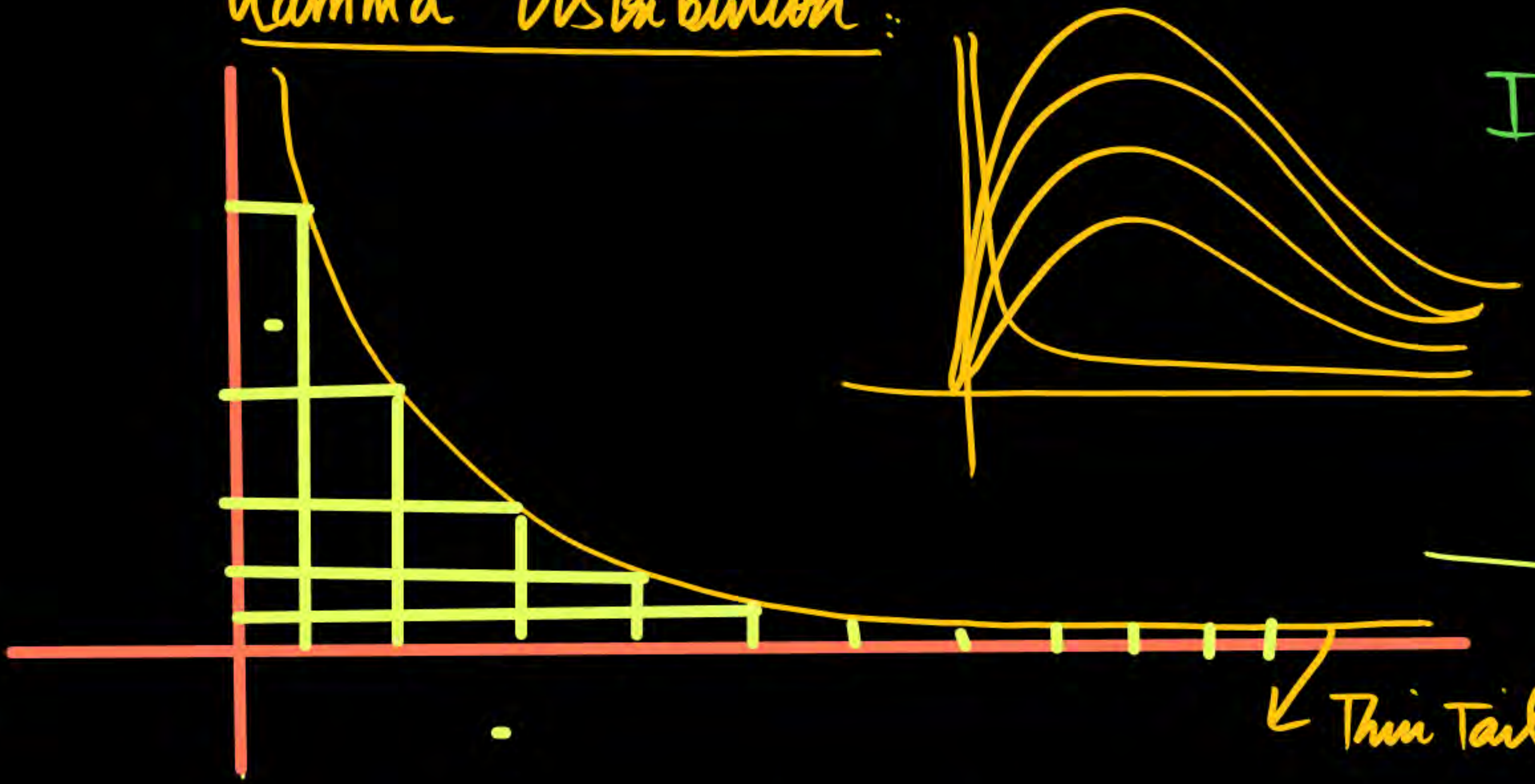
$\alpha - 1 = \frac{3}{2}$

$-\alpha = \frac{5}{2}$

$$= 2 \times \Gamma\frac{5}{2} = \cancel{2} \times \frac{3}{\cancel{2}} \times \frac{1}{2}\sqrt{\pi}$$

$$= \frac{3}{2}\sqrt{\pi}$$

Gamma Distribution:

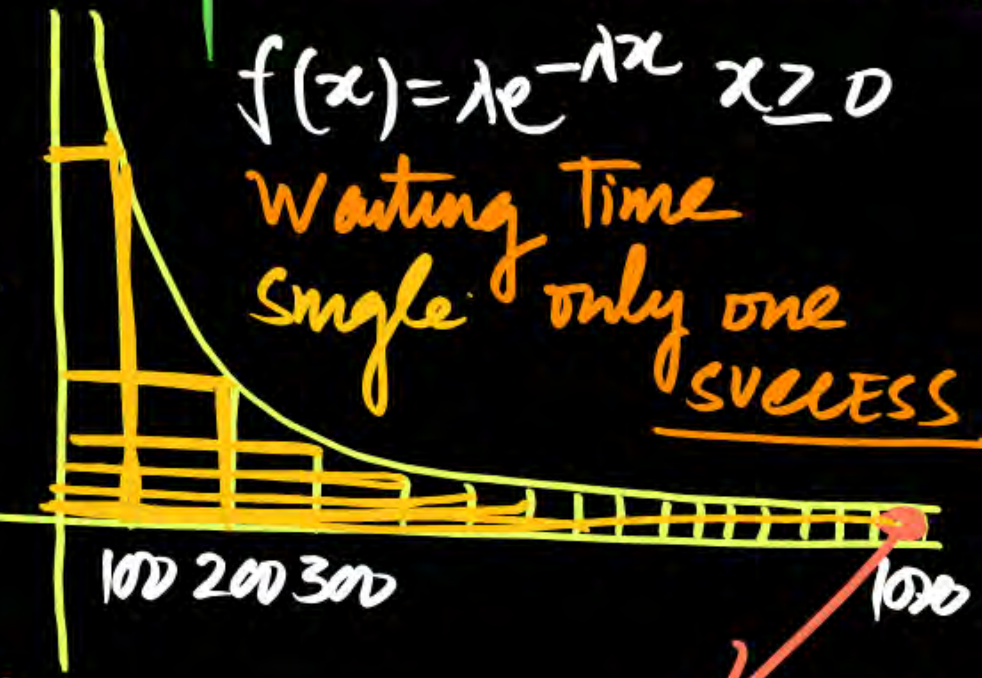


In Exponetial Distribution

$f(x) = \lambda e^{-\lambda x} \quad x \geq 0$

Waiting Time

Single only one success



Thin Tail

Positive skewed

Rate ← α # No. of success / No. of claims.

λ # Parameter of Distribution

α = No. of success

Total waiting Time OR Total claim size. α → $\alpha = 1$, $\alpha = 2$, $\alpha = 3$, $\alpha = 4$

$$\Gamma\alpha = \int_0^\infty e^{-\lambda x} x^{\alpha-1}\, dx$$

λ is Parameter

α # No. of Claims

$$\lambda x = t$$

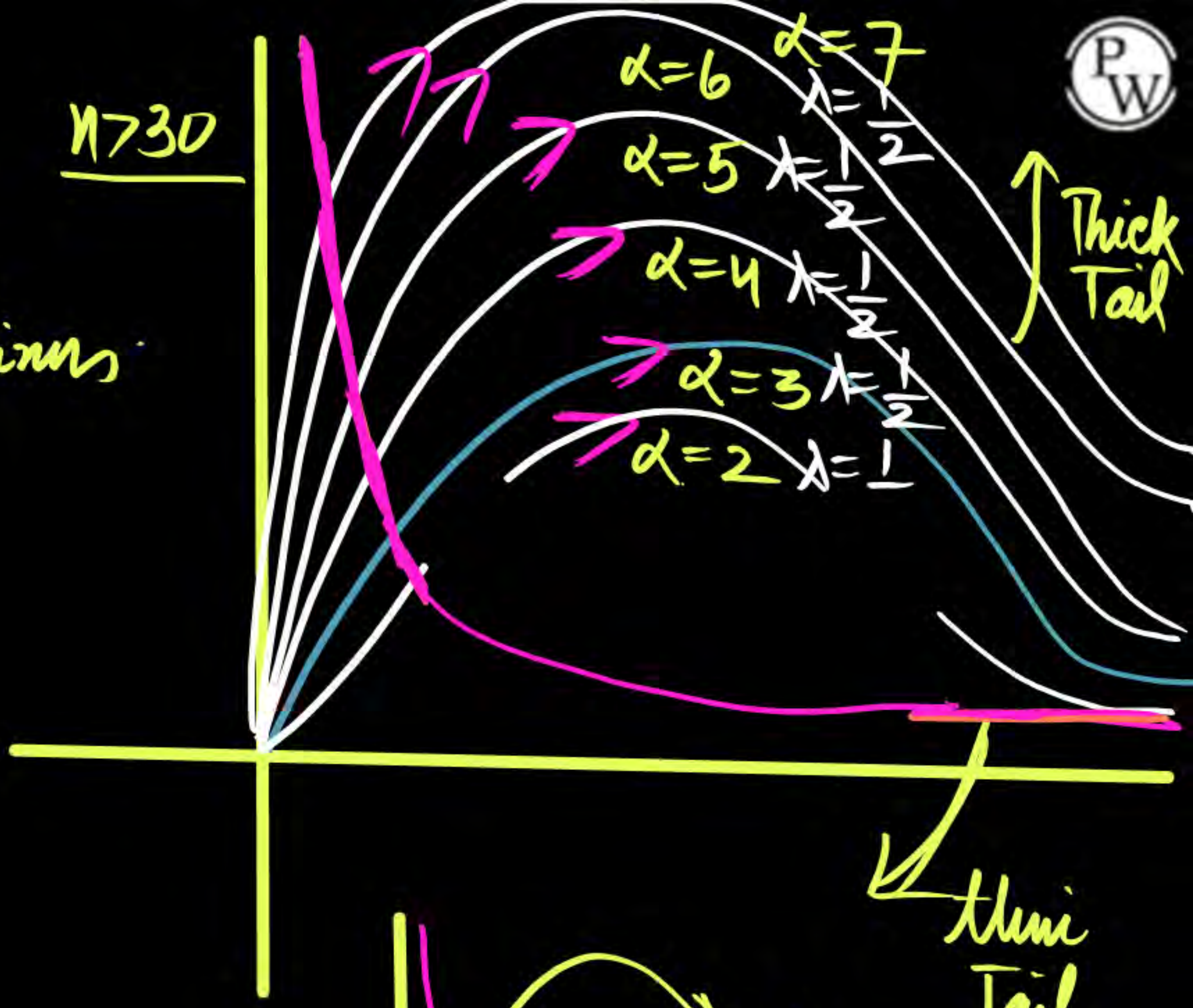
$$x = \frac{t}{\lambda} \qquad dx = \frac{dt}{\lambda}$$

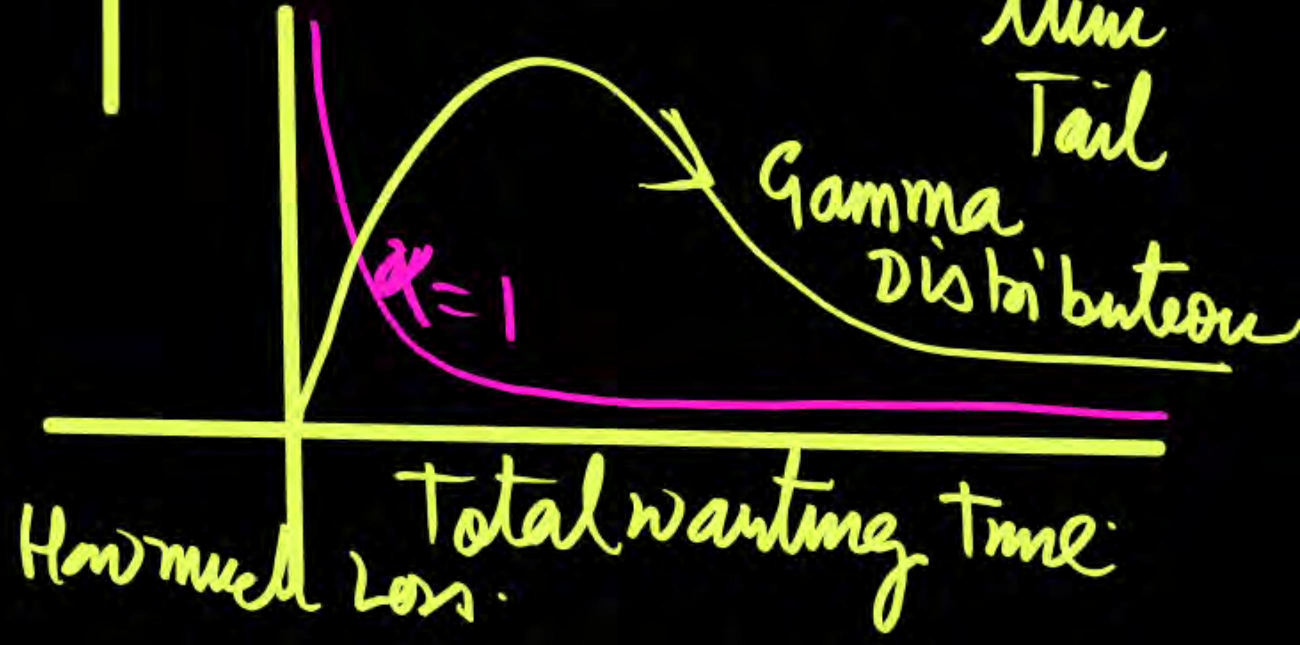
$$\Gamma\alpha = \int_0^\infty e^{-t} \left(\frac{t}{\lambda}\right)^{\alpha-1} \cdot \frac{dt}{\lambda}$$

$$= \int_0^\infty e^{-t} \frac{t^{\alpha-1}}{\lambda^{\alpha-1}} \cdot \frac{dt}{\lambda}$$

$$= \frac{1}{\lambda^\alpha} \int_0^\infty e^{-t} t^{\alpha-1} dt \Rightarrow$$

$$= \frac{1}{\lambda^\alpha} \int_0^\infty e^{-t} t^{\alpha-1} dt = \frac{\Gamma\alpha}{\lambda^\alpha}$$





$$\int_0^\infty e^{-\lambda x} x^{\alpha-1}\, dx = \frac{\Gamma\alpha}{\lambda^\alpha}$$

$$\Rightarrow \frac{\lambda^\alpha}{\Gamma\alpha}\int_0^\infty e^{-\lambda x} x^{\alpha-1}\, dx = 1$$

Gamma Distribution

$$\boxed{f(x) = \frac{\lambda^\alpha}{\Gamma\alpha}\int_0^\infty e^{-\lambda x} x^{\alpha-1}\, dx} = 1$$

If $\alpha = 1$ (means only one success)

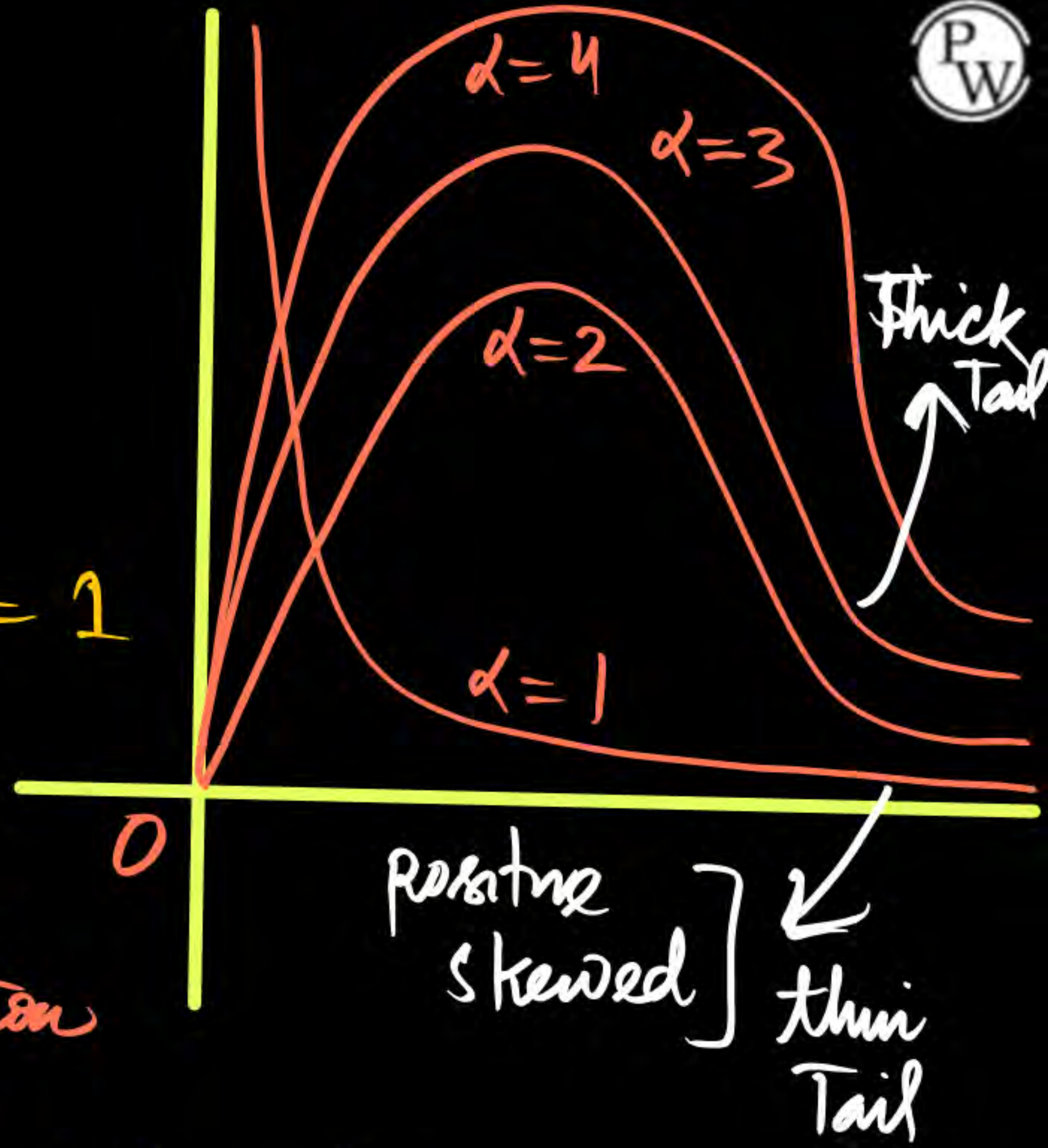
waiting Time

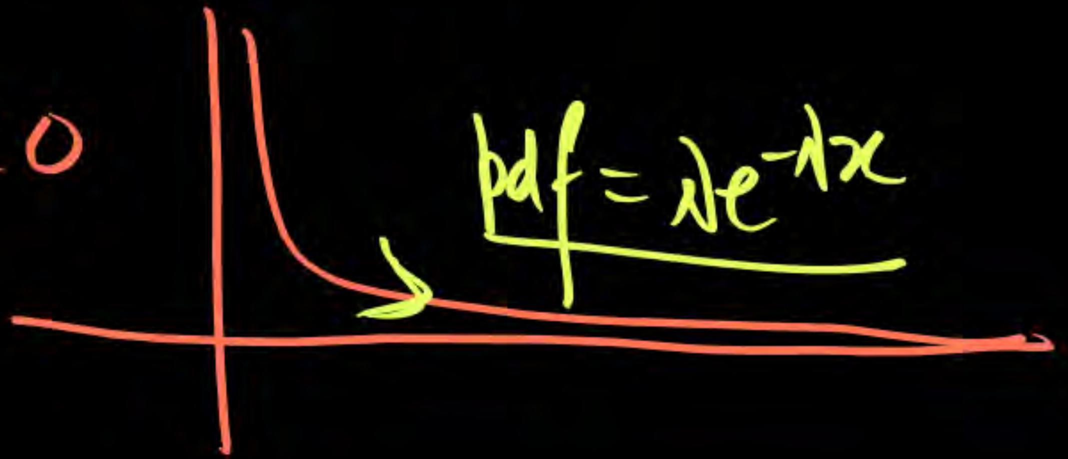
you get The exponetial Distribution

$$f(x) = \int_0^\infty \lambda e^{-\lambda x}\, dx \quad x \geq 0$$

$\exp(\lambda)$ $\quad \alpha = 1$

$\exp(\lambda)$





If this is a valid pdf $\int_0^\infty f(x)\,dx = 1$ α # No. of claims

λ = parameter
$\lambda > 0$

pdf

$$\boxed{f(x) = \frac{\lambda^\alpha}{\Gamma\alpha} e^{-\lambda x} x^{\alpha-1}}$$

$$\text{mean} = \int_0^\infty x \frac{\lambda^\alpha}{\Gamma\alpha} e^{-\lambda x} x^{\alpha-1}\,dx = \frac{\alpha}{\lambda}$$

$$\boxed{\text{mean} = E[X] = \frac{\alpha}{\lambda}}$$

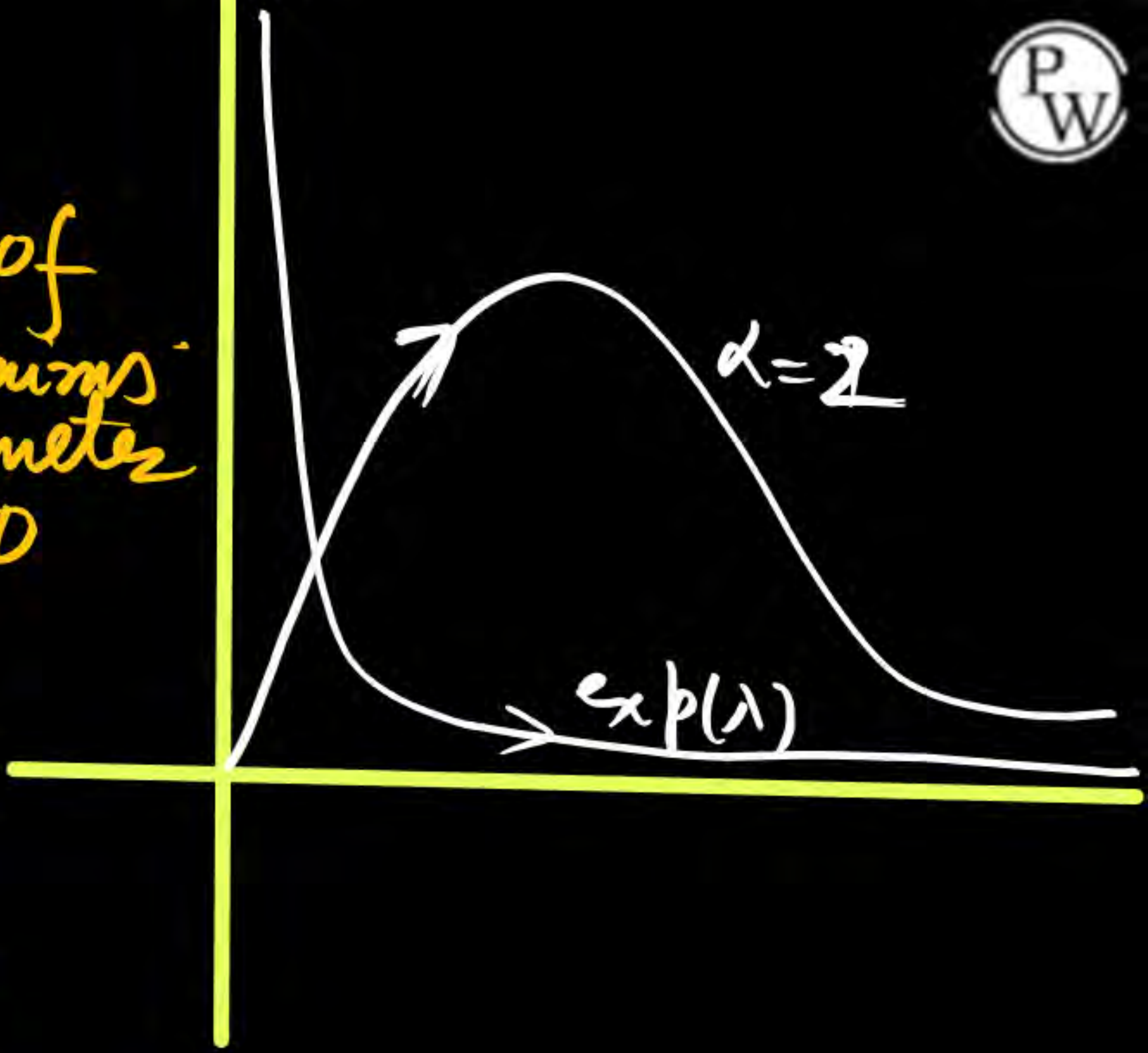
If this is $\exp(\lambda)$

$E[X] = \frac{1}{\lambda}$ $\alpha = 1$

$$\boxed{\text{variance} = \frac{\alpha}{\lambda^2}}$$

$V(X) = \frac{\alpha}{\lambda^2} = \frac{1}{\lambda^2}$ $\alpha = 1$

$$\text{Standard deviation} = \sqrt{\frac{\alpha}{\lambda^2}}$$



moment generating function

$$f(x) = \frac{\lambda^{\alpha}}{\Gamma\alpha} e^{-\lambda x} x^{\alpha-1}$$

α # No. of claims

λ # Parameter $\lambda > 0$

$$M_G(F) = \Pi_X(s) = \int_0^{\infty} e^{sx} \cdot \frac{\lambda^{\alpha}}{\Gamma\alpha} x e^{-\lambda x} x^{\alpha-1} \, dx = \left[\frac{\lambda}{(\lambda - s)}\right]^{\alpha}$$

$$\Pi_X(s) = \left(\frac{\lambda}{\lambda - s}\right)^{\alpha}$$

If $\boxed{\alpha = 1}$ Then $\Pi_X(s) = \frac{\lambda}{(\lambda - s)}$ $\quad \alpha = 1$

exponetial distribution

$\lambda = 1$ customer $\qquad$ $\lambda = 1$ customer Per minute

Q1. Suppose that on an average 1 customer per minutes arrive at a shop. What is the probability that the shopkeeper will wait more than 5 minutes before

(i) Both of the first two customer arrive, and

(ii) The first customer arrive?

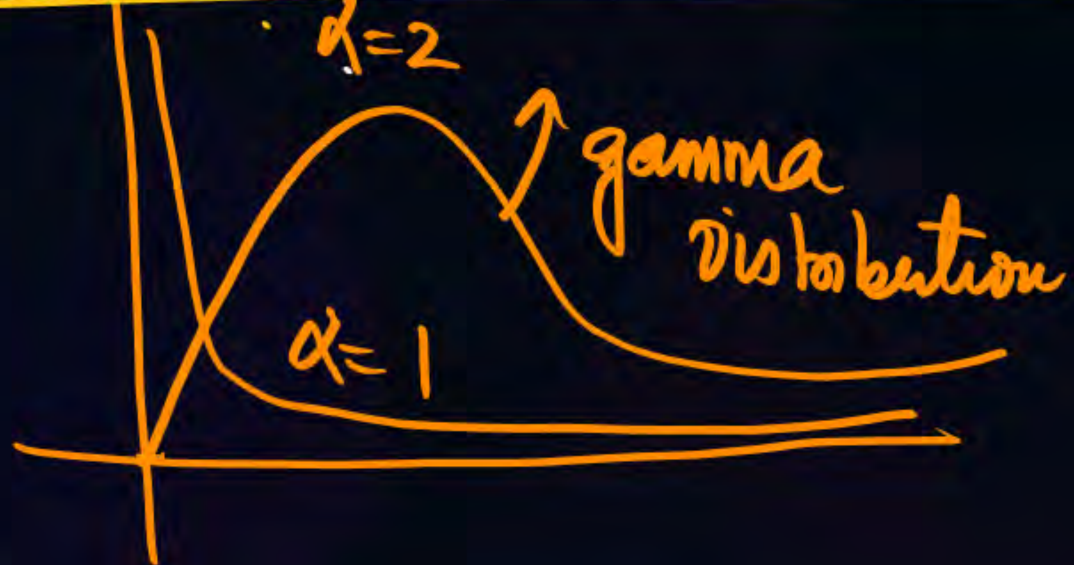
Assume that waiting times follows gamma distribution.

$P(X>5) = ?$ $\quad \alpha = 2 \quad \lambda = 1$

$$P(X>5) = \int_5^\infty \frac{\lambda^\alpha}{\Gamma\alpha} e^{-\lambda x} x^{\alpha-1}\,dx$$

$$= \int_5^\infty e^{-2x} x^{2-1}\,dx$$

$$= \int_5^\infty x e^{-x}\,dx \Rightarrow \boxed{1 - \frac{10}{e}}$$



b) The first customer arrive

$P(X>5) = ?$ $\alpha = 1$ $\lambda = 1$

$$= \int_5^\infty \frac{\lambda^\alpha}{\Gamma\alpha} e^{-\lambda x} x^{\alpha-1}\, dx$$

$$= \int_5^\infty e^{-x} x^0\, dx$$

$$= \boxed{\int_5^\infty e^{-x}\, dx = e^{-5}}$$

$P(X>5) = 1-(1-e^{-\lambda x})$

$= e^{-\lambda x}$ $\left.\begin{matrix}\lambda = 1 \\ x = 5\end{matrix}\right]$

$= e^{-5}$ Ans.

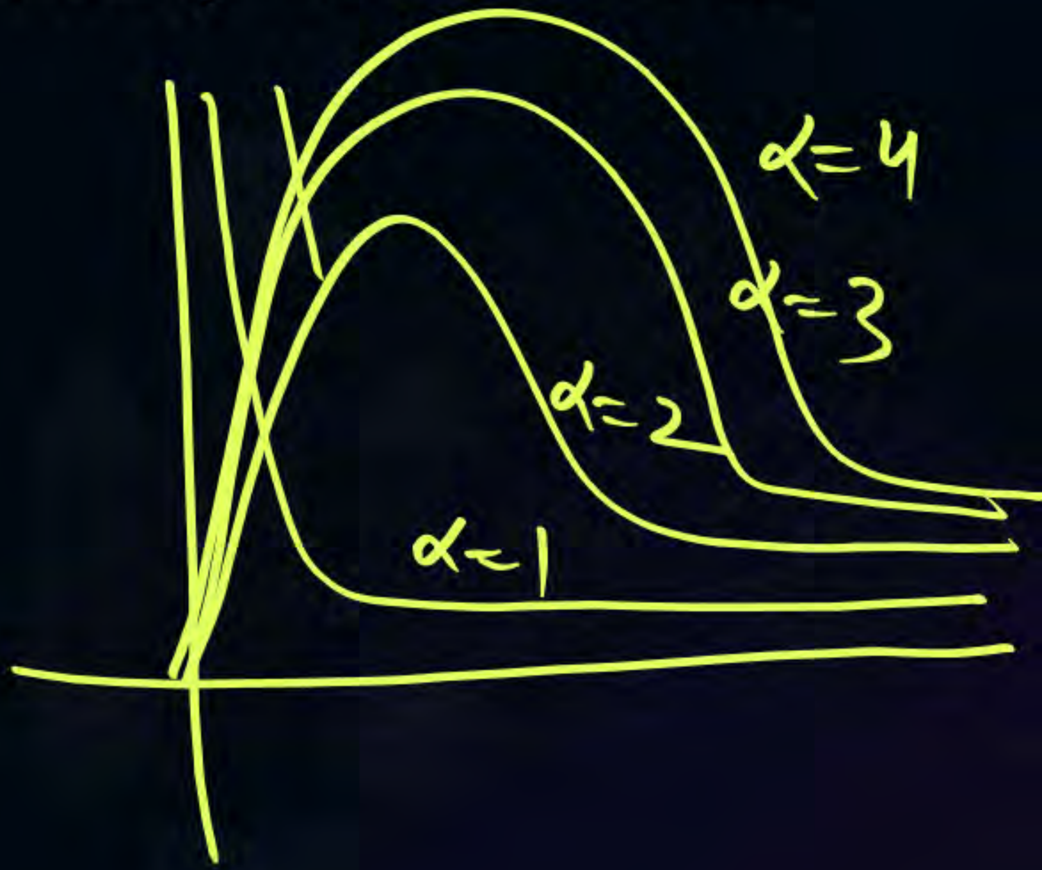
$\lambda = 2$ per minute

Q2. Telephone calls arrive at a switchboard at an average rate of 2 per minute. Let X denotes the waiting time in minutes until the 4th call arrives and follows gamma distribution. Write the probability density function of X. Also find its mean and variance.

$\begin{cases} \alpha = 4 \text{ call} \\ \lambda = 2 \end{cases}$ $f(x) = \frac{\lambda^\alpha}{\Gamma\alpha} e^{-\lambda x} x^{\alpha-1}$

mean $= \frac{\alpha}{\lambda} = \frac{4}{2} = 2$

variance $= \frac{\alpha}{\lambda^2} = \frac{4}{2^2} = 1$



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Distribution

Arrival — Ber, Bio, GEO, Poisson, Uniform, Negative Binomial, Hypergeo.

Waiting: Normal + Gamma + exp(λ) + Standard Normal + Uniform conti + + Beta

SAMPLING — CHI square
t –
F –

Weekend + Revise

THANK - YOU