Data Science and Artificial Intelligence

Probability and Statistics

Bivariate Random Variable

Lecture No.- 06

By- Rahul Sir

Topics to be Covered

Volume Via Double Integrals

Topic Problems Based on Volume Via Double Integral

Topic Change the Order of Integration

Topic Bivariate Continuous Random Variables

Topic : Double Integrals

$\iint_A x^2 \, dx \, dy$ A is region

I^{st} quadrant

Q3. Illustration

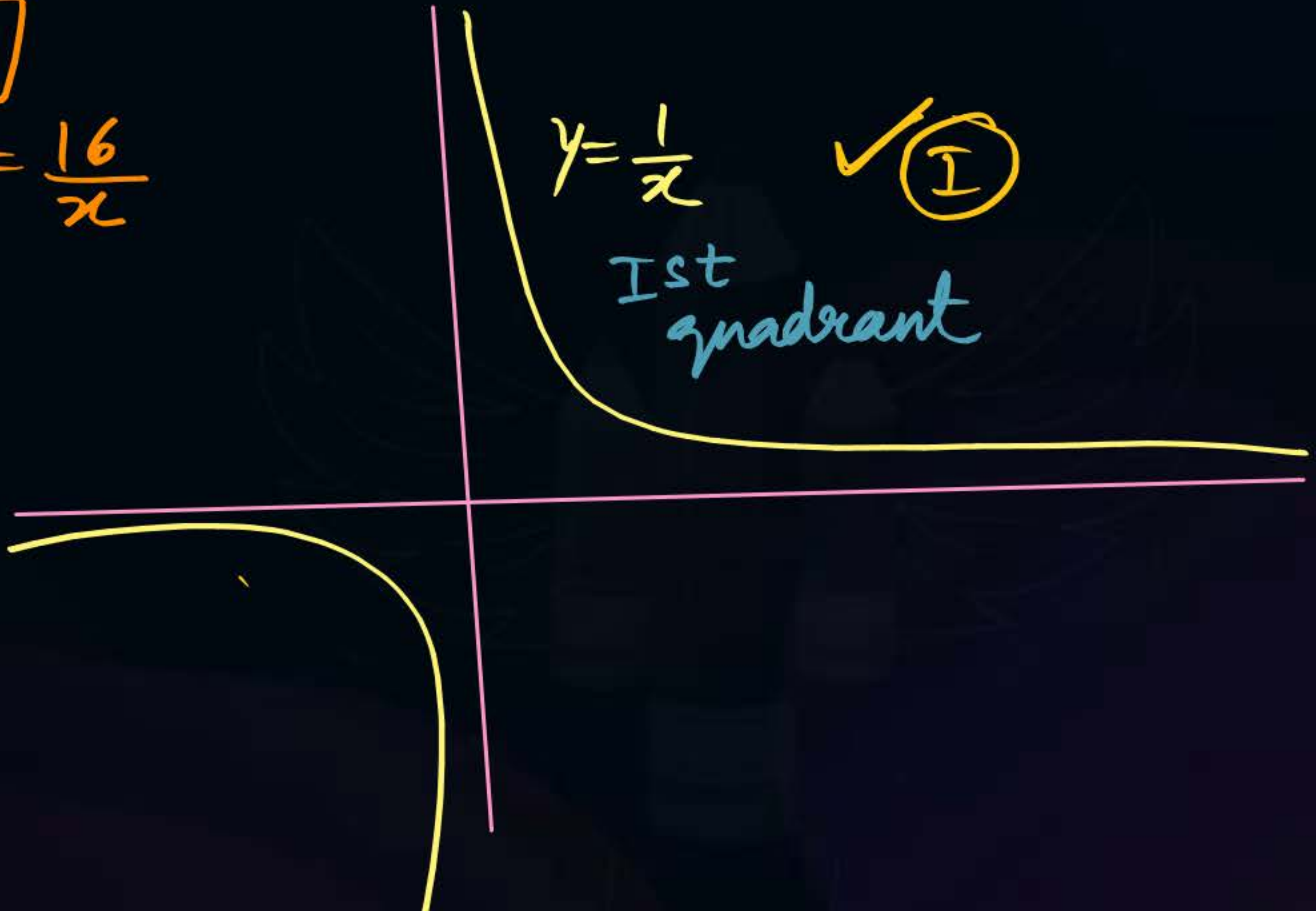
$\iint x^2 \, dx \, dy$ where A is the region in the I[st] quadrant bounded by Hyperbola

$xy = 16$ step ① $\boxed{xy = 16}$

$x = 8$ $x = 8$ $y = \frac{16}{x}$

$x = y$ $x = y$

$y = 0$ $y = 0$



Slide 3

$x = \frac{16}{x}$

$x^2 = 16$

$x = \pm 4$ ✓ Ist

$x = 4$

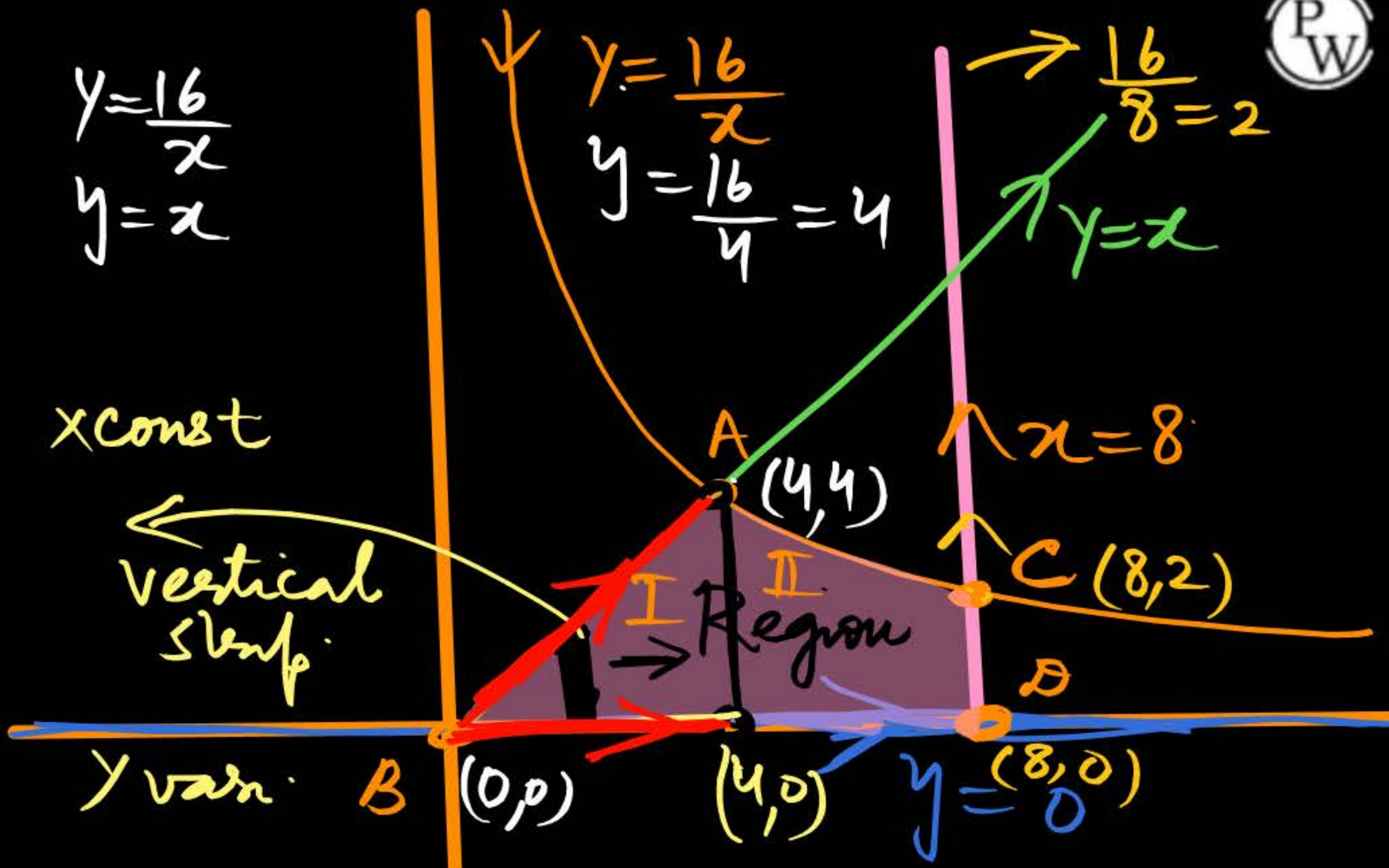
$y = \frac{16}{x}$

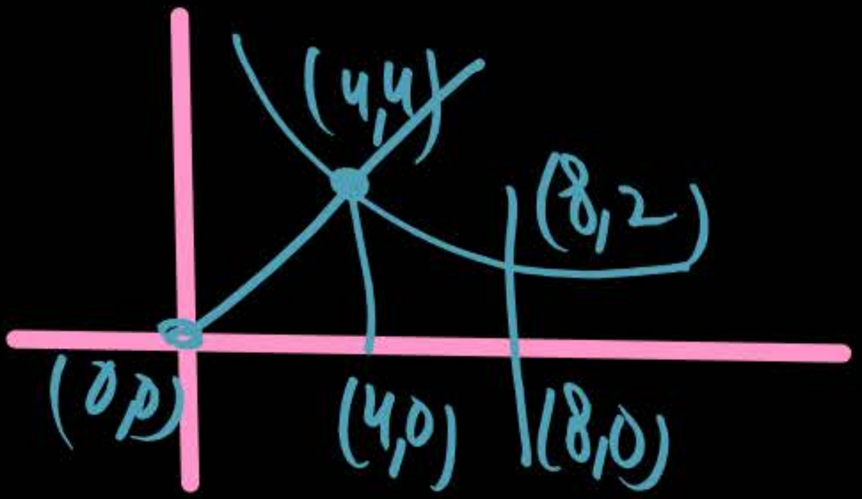
$y = x$

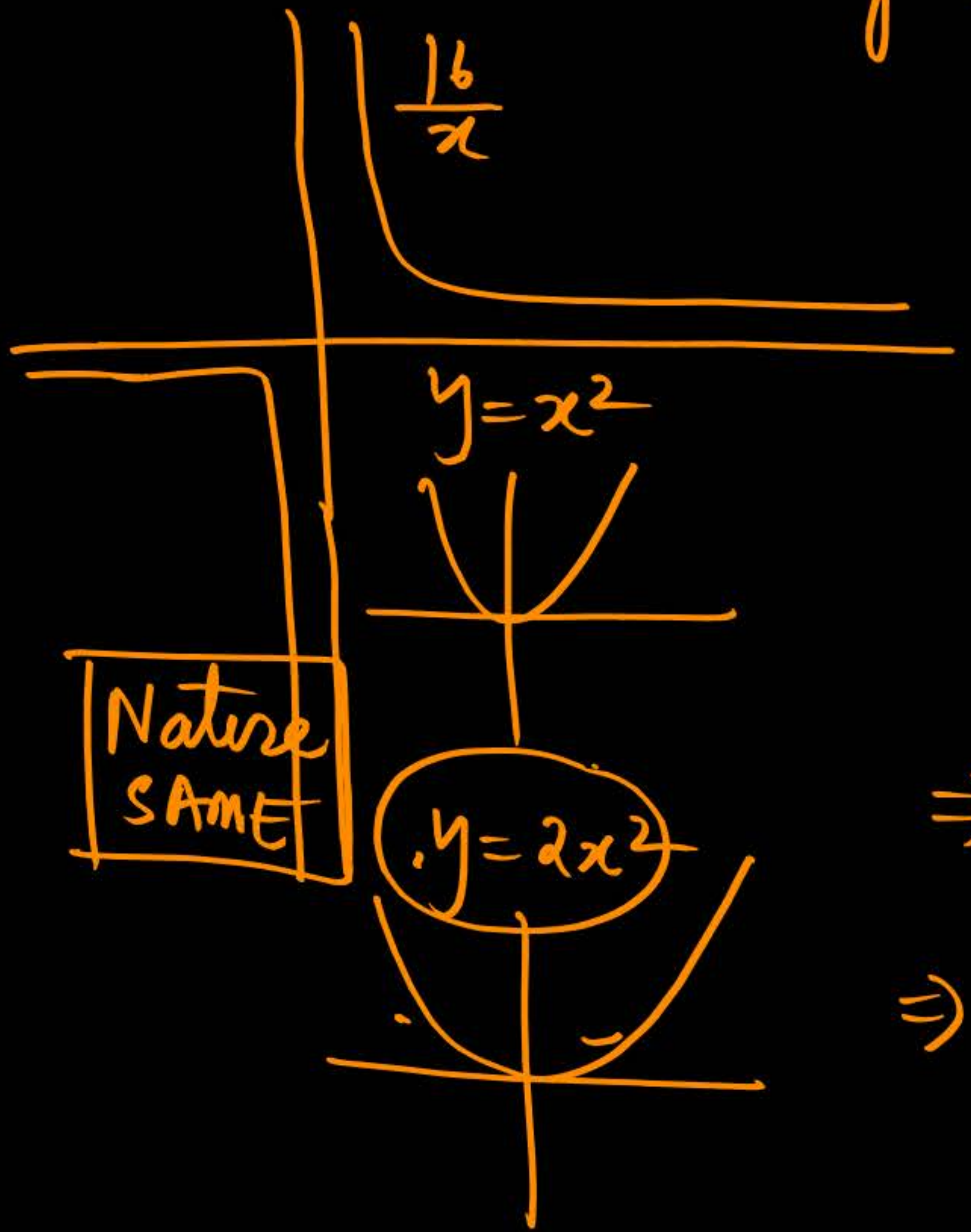
$$\int_0^4 \int_0^x x^2\, dy\, dx$$

$$= \int_{x=0}^{4} \int_{y=0}^{x} x^2\, dy\, dx$$

$$= \int_0^4 x^2\, dx \left[\int_0^x dy\right] = \int_0^4 x^2 \cdot x\, dx = 4^3 = 64$$







$$\text{II}^{nd} \text{ region} = \int_{x=4}^{8}\int_{y=0}^{\frac{16}{x}} x^2\,dy\,dx$$

$$\Rightarrow \int_4^8 x^2\,dx\left[\int_{y=0}^{\frac{16}{x}} dy\right]$$

$$\Rightarrow \int_4^8 x^{\cancel{2}}\,dx\left[\frac{16}{\cancel{x}}\right]$$

$$\Rightarrow 16\int_4^8 x\,dx$$

$$\Rightarrow 16\left[\frac{8^2}{2}-\frac{4^2}{2}\right] = 16\left[32-8\right] = \underline{16\times 24} = 384$$

Total region

$= 64+384$

$= \underline{448}$ **Ans**

Topic : Double Integrals

$f(x,y) = e^x$ $\left.\begin{matrix} x=y \\ x=0 \\ y=1 \end{matrix}\right\}$

Q1. Illustration

The volume enclosed by the surface f(x, y) = e^x over the triangle bounded by lines

$I = \iint f(x,y)\,dy\,dx$

$x = y$

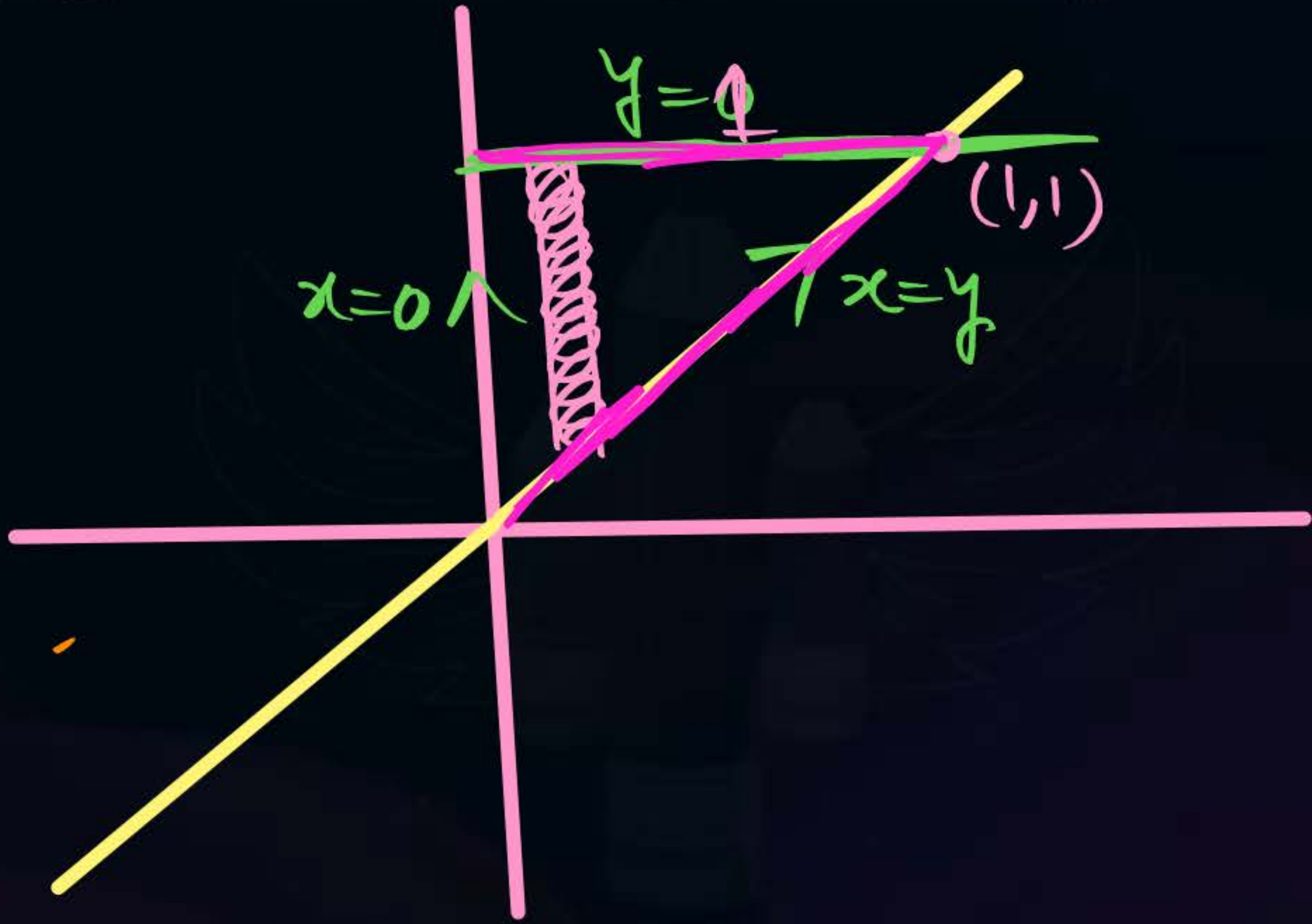
$\Rightarrow \int_{x=0}^{1}\int_{y=x}^{1} e^x\,dy\,dx$

$x = 0$

$y = 1$ in the xy plane

$\Rightarrow \int_0^1 e^x dx \int_{y=x}^{1} dy$

$\Rightarrow \int_0^1 e^x dx\,[1-x)$



$$= \int_0^1 e^x (1-x)\, dx$$

$$\Rightarrow \int_0^1 e^x\, dx - \int_0^1 x e^x\, dx$$

$$\int x e^x = (x-1)e^x$$

$$\Rightarrow \left[e^x\right]_0^1 - \left[(x-1)e^x\right]_0^1$$

$$\Rightarrow (e^1 - e^0) - \left[\underbrace{(1-1)e^1}_{} - (0-1)e^0\right]$$

$$\Rightarrow (e-1) - (0+1)$$

$$\Rightarrow e-1-1$$

$$\Rightarrow e-2 \quad \text{Ans}$$

Topic : Double Integrals

Q2. $\int\int (x^2 + y^2)\, dx\, dy$ *over the region bounded by* $y = x^2$ & $y^2 = x$

A. 6/32

B. 6/35

C. 6/30

D. 1

$$\iint (x^2+y^2)\,dx\,dy$$

$y = x^2,\ y^2 = x$

$y^2 = x^4$

$x = x^4$

$x^4 - x = 0$

$x(x^3-1) = 0$

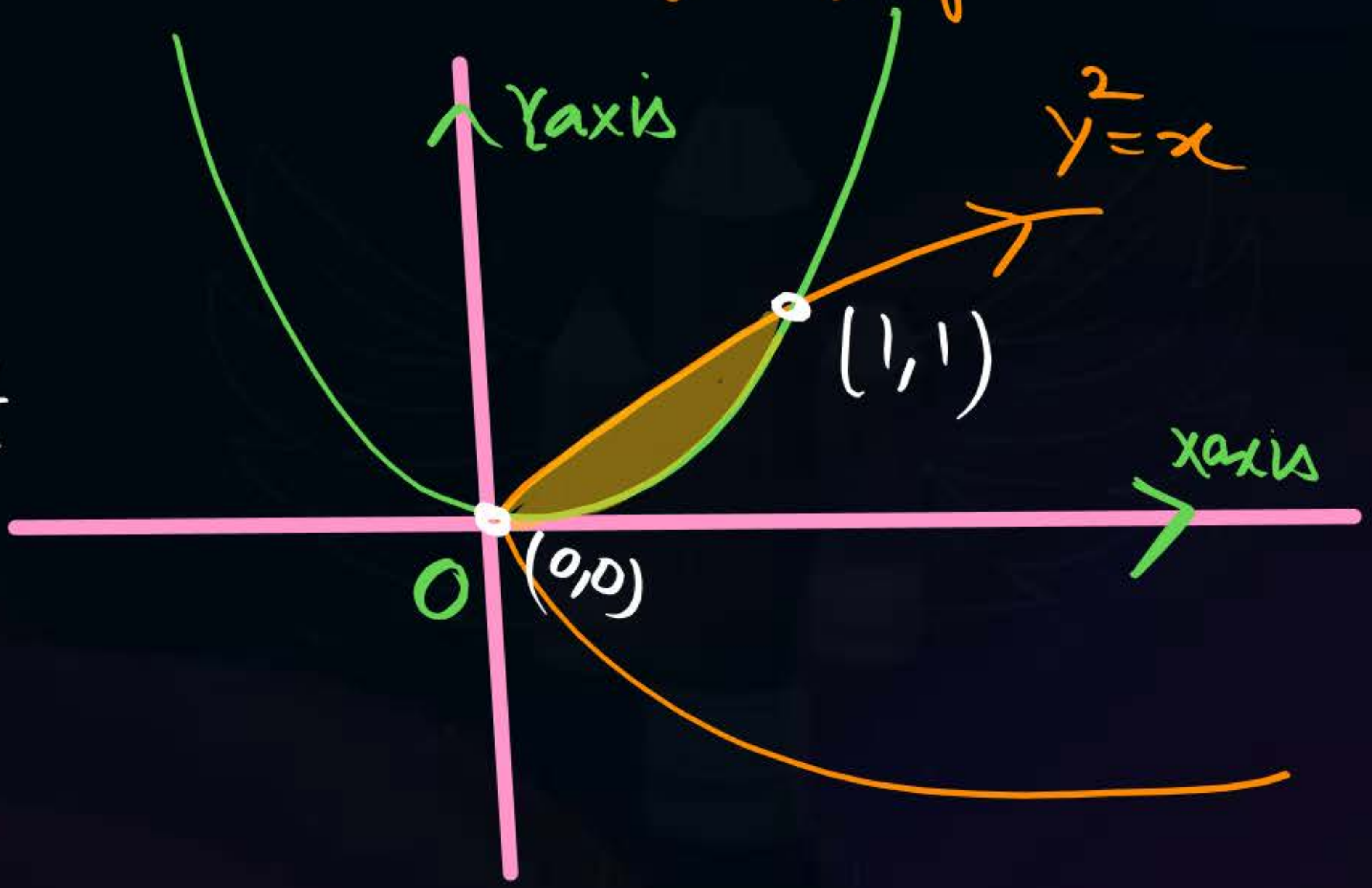
$x = 0 \quad x^3 - 1 = 0$

$x^3 = 1$

$$\left[\begin{array}{l} x^3 = 1 \\ x = 1, \\ x = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2} \\ x = -\frac{1}{2} - \frac{i\sqrt{3}}{2} \end{array}\right.$$

region Bounded via

$y = x^2,\ y^2 = x$



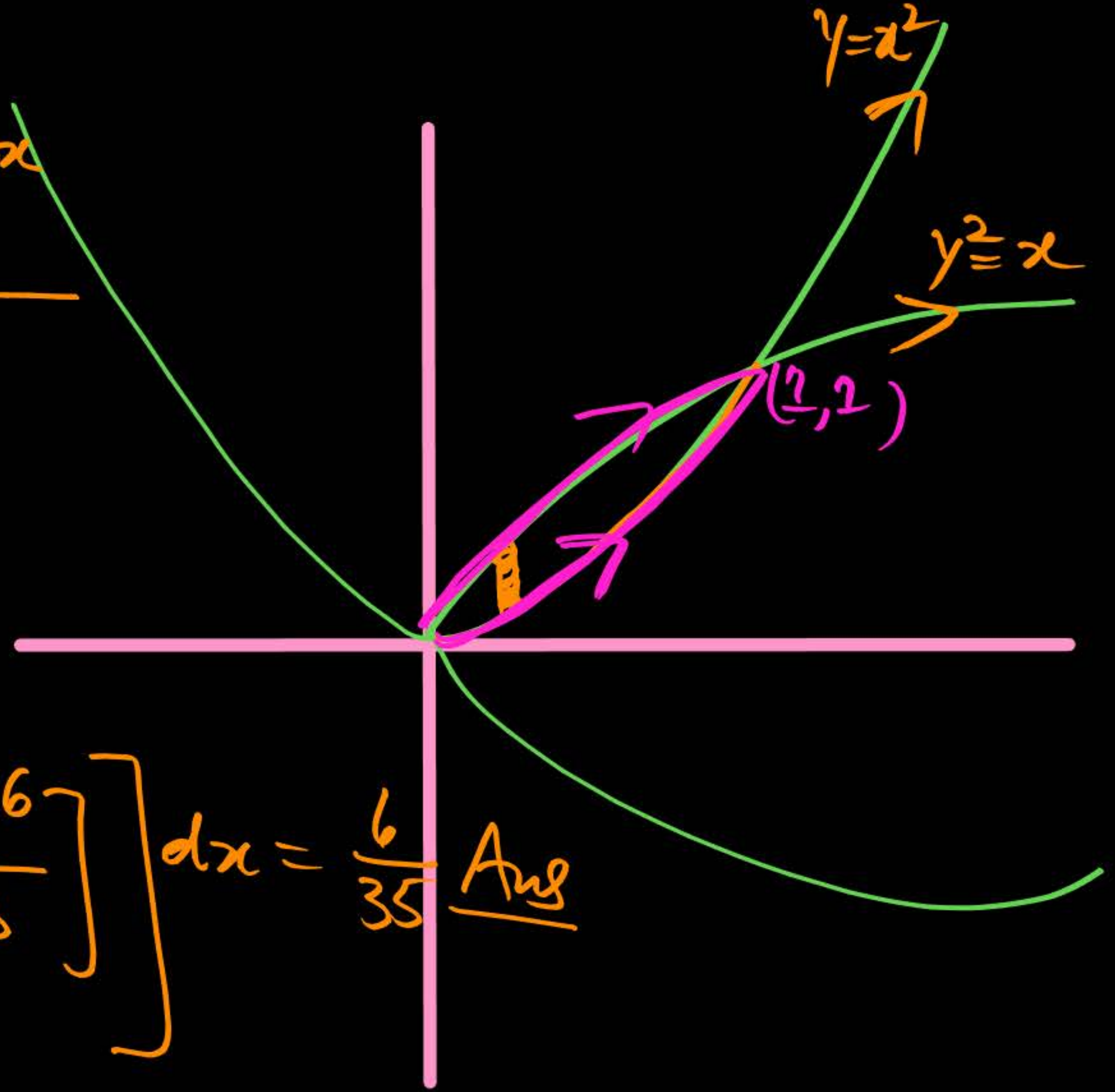
$$= \iint (x^2+y^2)\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2+y^2)\,dy\,dx$$

$$\Rightarrow \int_0^1 dx \left[\int_{x^2}^{\sqrt{x}} (x^2+y^2)\,dy \right]$$

$$\Rightarrow \int_0^1 dx \left[x^2 y + \frac{y^3}{3} \right]_{x^2}^{\sqrt{x}}$$

$$\Rightarrow \int_0^1 \left[\left[x^2\sqrt{x} + \frac{(\sqrt{x})^3}{3} \right] - \left[x^2 \cdot x^2 + \frac{x^6}{3} \right] \right] dx = \frac{6}{35} \ \underline{\text{Ans}}$$



Topic : Double Integrals

Q10. The solution of $\int_1^a \int_1^b \frac{dxdy}{xy}$ is

Do yourself

(a) ln (ab)

(b) ln (a/b)

(c) ln (a) + ln (b)

(d) ln (a) ln (b)

Slide

Topic : Double Integrals

Q11. $\boxed{I = \int_{y=0}^{2} \int_{x=y/2}^{1} e^{x^2}\, dx\,dy}$

e^{x^2} is Not Integrable

$I = \int_{y=0}^{2} \left[\int_{x=\frac{y}{2}}^{1} e^{x^2} dx \right] dy$

Inner

outer

dy dx ⇄ dx dy

Area $2\times3 = 3\times2$

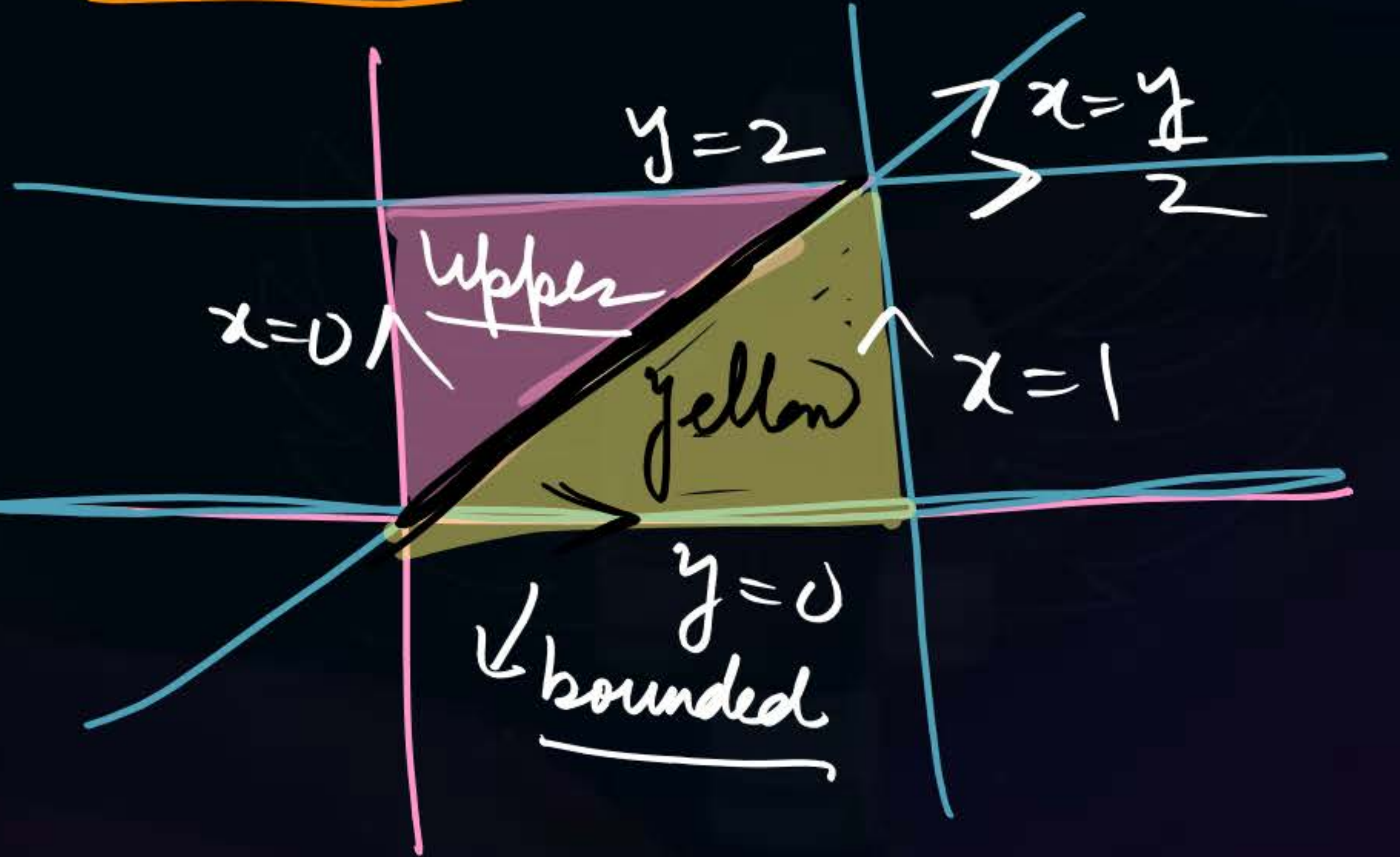
Not Integrable

$I = \int_{y=0}^{2} \int_{x=\frac{y}{2}}^{1} e^{x^2} dx\,dy$

Limit $\begin{cases} y=0 & x=\frac{y}{2} \\ y=2 & x=1 \end{cases}$

constant variable

Horizontal Strip



Slide

Horizontal strip $\rightleftharpoons$ vertical strip

$dy\,dx \rightleftharpoons dx\,dy$

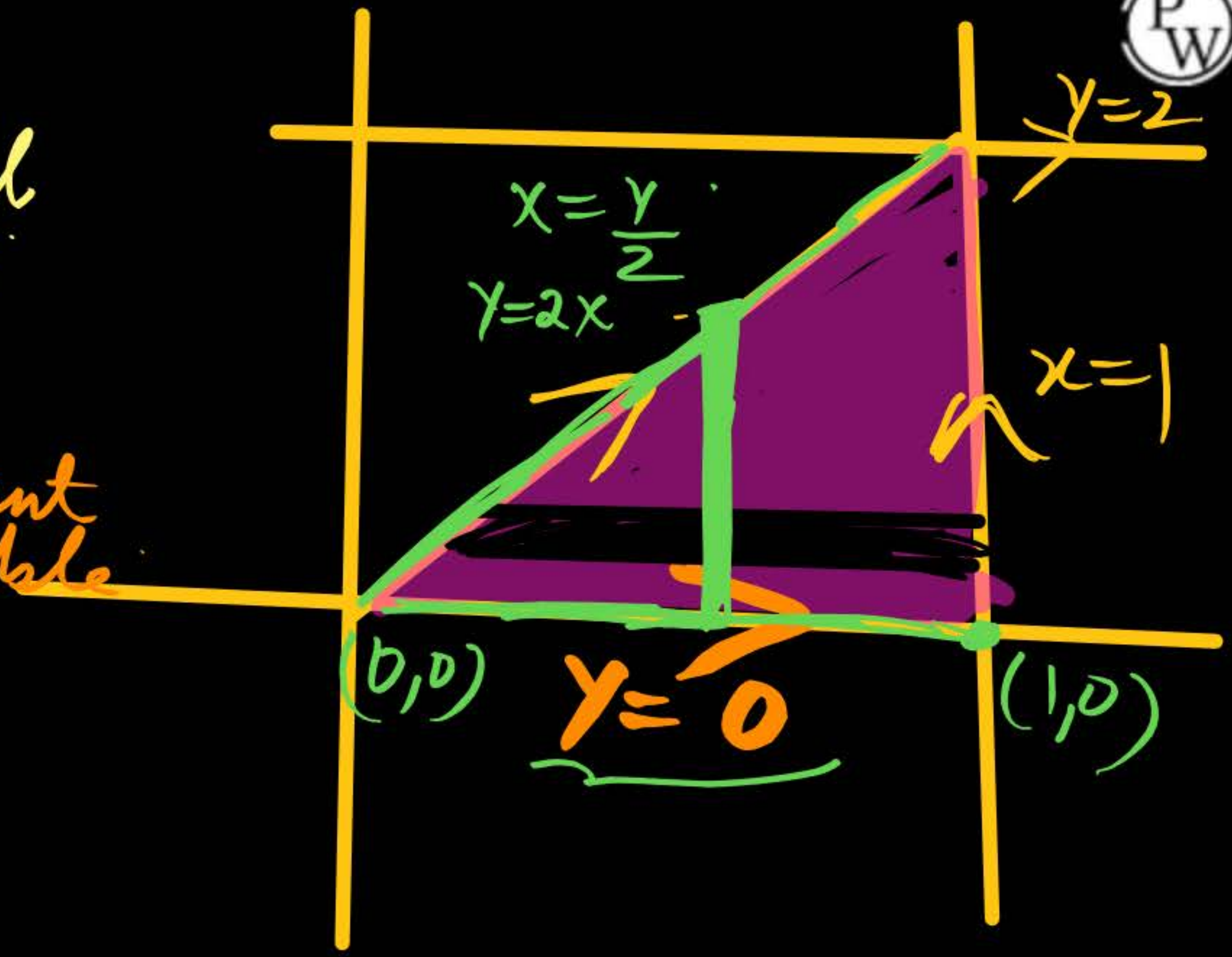
If x constant, y variable $\rightleftharpoons$ y constant, x variable

$$\Rightarrow \int_{x=0}^{1}\int_{y=0}^{2x} e^{x^2}\,dy\,dx$$

$x^2 = t$

$2x\,dx = dt$

$$\Rightarrow \int_0^1 e^{x^2}dx\int_{y=0}^{2x}dy = \int_0^1 e^{x^2}[2x] = \int_0^1 e^{x^2}2x\,dx = \int_0^1 e^t\,dt = e-1$$



CHANGE THE order of Integration : ✓ Plot The Curve (Limit)

✓ Check - Horizontal strip / vertical strip.

✓ Horizontal strip → vertical strip.

Horizontal strip: y constant, x variable.

Vertical strip: x constant, y variable.

✓ $dy\,dx \rightleftharpoons dx\,dy$.

Topic : Double Integrals

Q12. Changing the order of integration in double integral

$$I = \int_{x=0}^{8} \int_{y=\frac{x}{4}}^{2} f(x,y)\,dy\,dx$$

leads to $I = \int_r^s \int_p^q f(x,y)\,dx\,dy$. What is q?

$$I = \int_{x=0}^{8} \int_{y=\frac{x}{4}}^{2} f(x,y)\,dy\,dx$$

CHANGE The order

leads to

$$\boxed{I = \int_r^s \int_p^q f(x,y)\,dx\,dy}$$

What is The value of q;

Slide

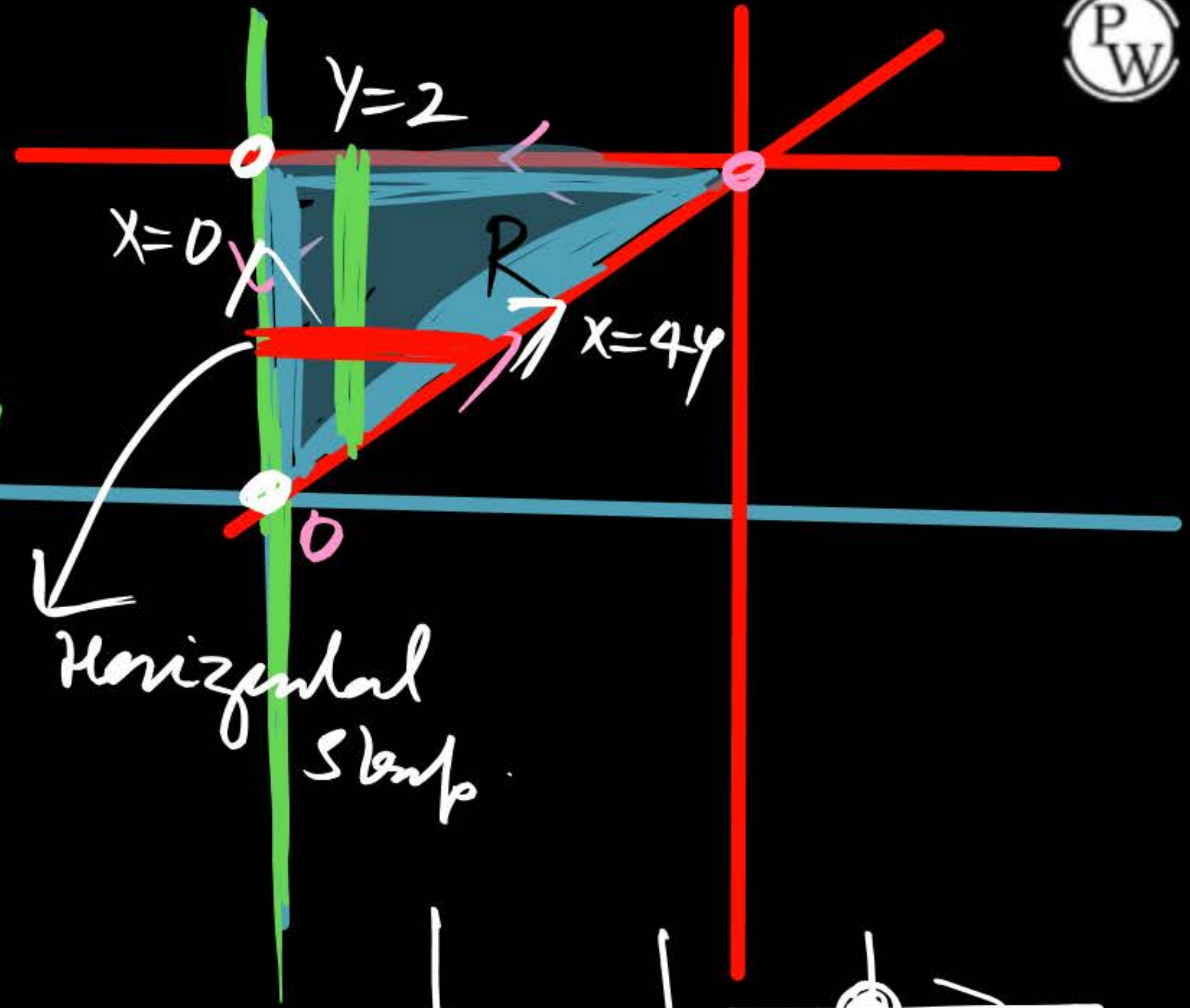
$x = 0 \;|\; y = \frac{x}{4}$

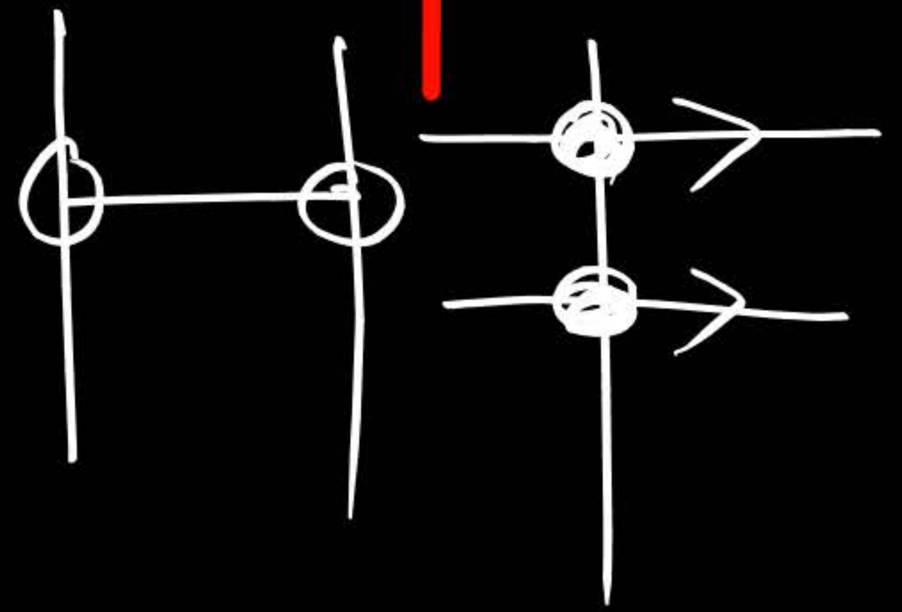
$x = 8 \;|\; y = 2$

X constant. Y variable (Vertical Strip)

$$= \int_{y=0}^{2} \int_{0}^{4y} f(x,y)\,dx\,dy$$

$$\boxed{x = 4y}$$





Topic : Double Integrals

Q13. $\int\int \cos(x+y)\,dxdy$ over the region enclosed by $y = x,\ y = \pi, x = 0$ → Doyourself

$\boxed{Ans = 2}$

Slide

Topic : Double Integrals

Q14. The value of double integral $\int_{x=0}^{\pi}\int_{y=0}^{x}\frac{\sin y}{(\pi-y)}dydx$

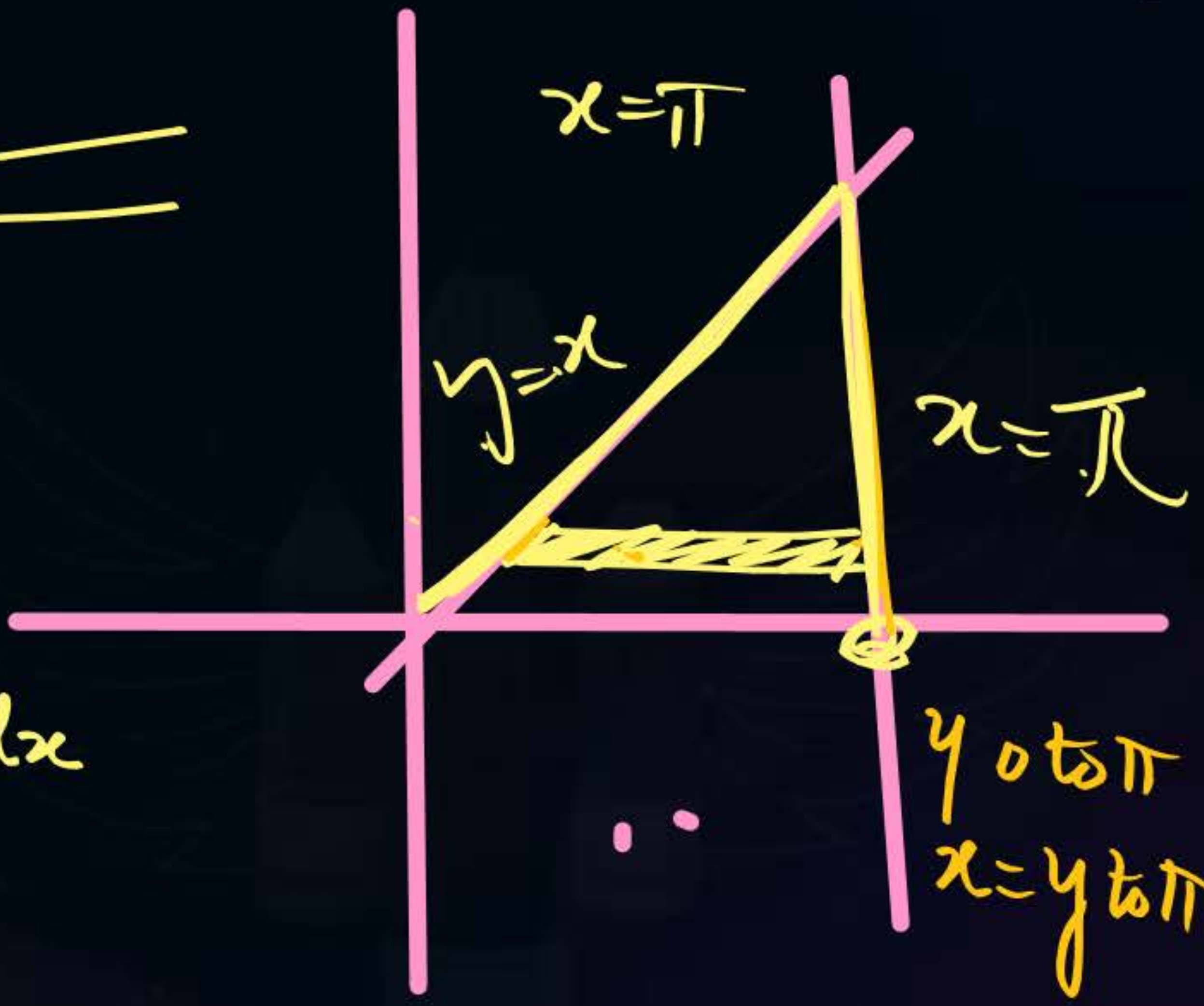
$x=0\ , x=\pi$
$y=0, \ y=x$

✓ Step Ⓐ Plot The Limit

Change.

Horizontal strip

$$=\int_{y=0}^{\pi}\int_{x=y}^{\pi}\frac{\sin y}{(\pi-y)}\,dy\,dx$$



y 0 to π
$x=y$ to π

Slide

$$= \int_{y=0}^{\pi} \int_{x=y}^{\pi} \frac{\sin y}{(\pi - y)}\, dy\, dx = \int_{0}^{\pi} dy \int_{y}^{\pi} \frac{\sin y}{(\pi - y)}\, dx$$

$$\int_{y=0}^{\pi} dy \int_{x=y}^{\pi} \frac{\sin y}{(\pi - y)}\, dx$$

$$\Rightarrow \int_{0}^{\pi} dy \int_{y}^{\pi} \left[\frac{1}{(\pi - y)} \sin y\, dx \right]$$

$$= \int_{0}^{\pi} dy\, \frac{1}{(\pi - y)} \sin y \left[x \right]_{y}^{\pi}$$

$$= \int_{0}^{\pi} dy\, \frac{1}{\cancel{(\pi - y)}} \sin y \left[\cancel{\pi - y} \right]$$

$$= \int_{0}^{\pi} \sin y\, dy$$

$$= \left[-\cos y \right]_{0}^{\pi} = -\cos\pi + \cos 0 = -(-1) + 1 = 2$$

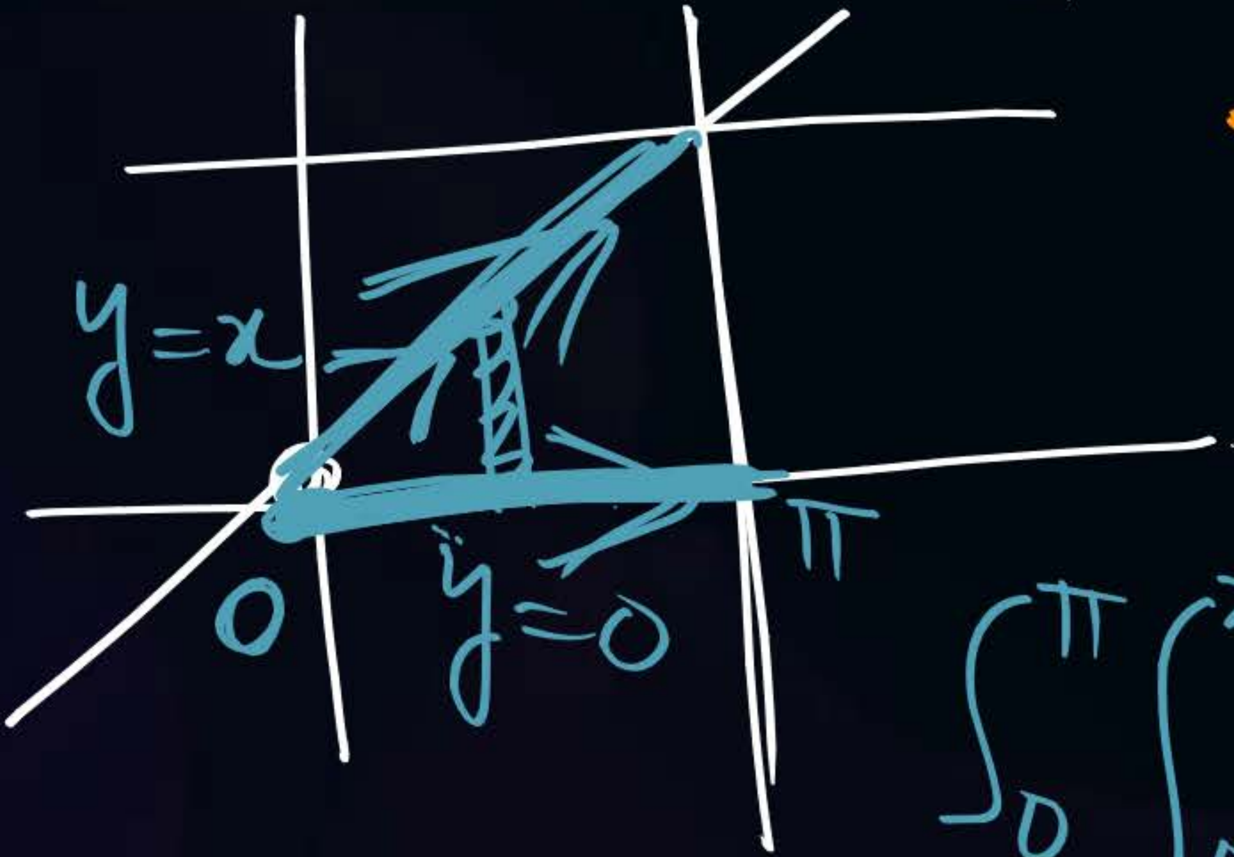
Topic : Double Integrals

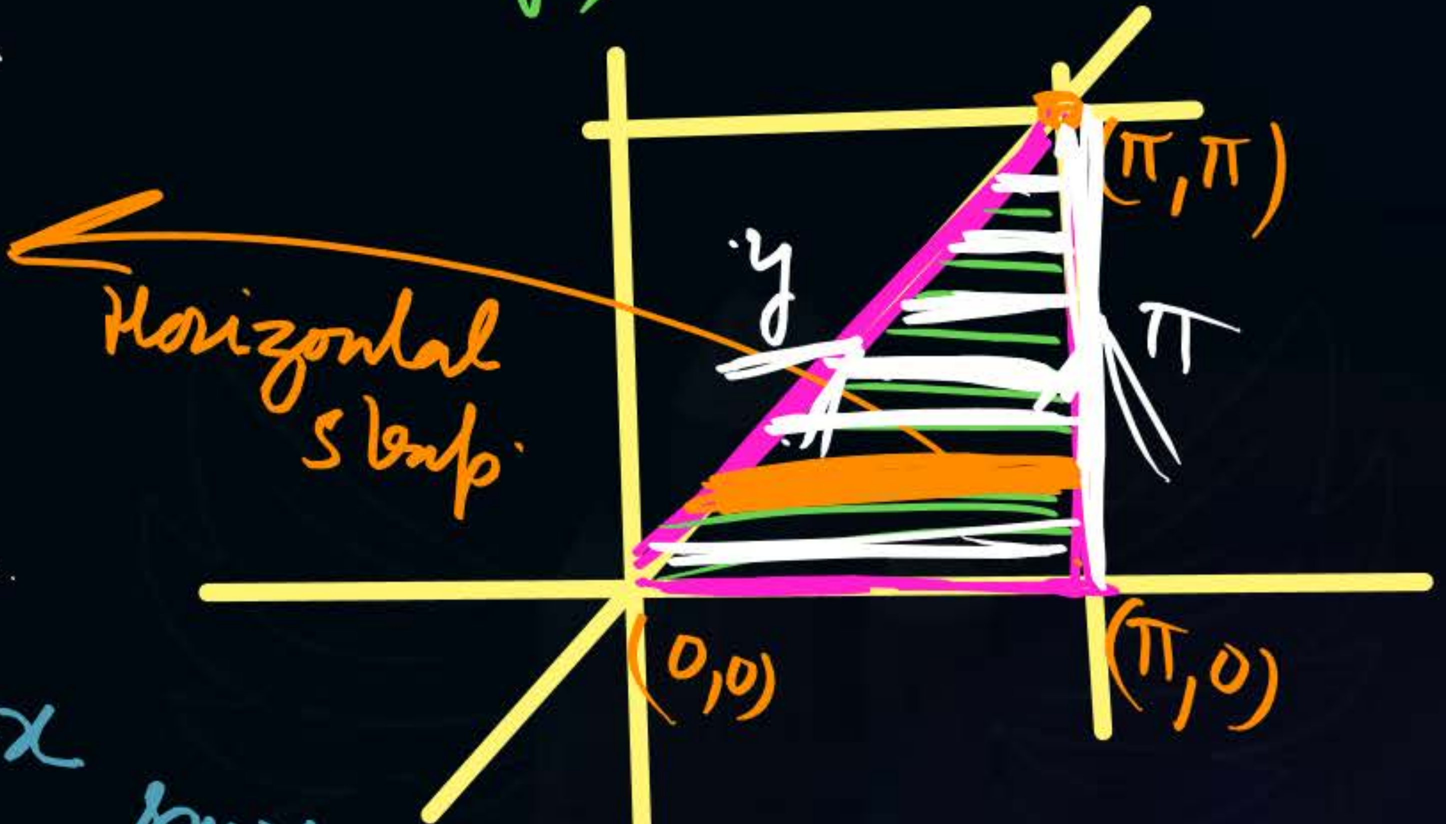
Q15. The value of the integral $\int_{y=0}^{\pi}\int_{x=y}^{\pi}\frac{\sin x}{x}\,dx\,dy$ is equal to

$$I = \int_{y=0}^{\pi}\int_{x=y}^{\pi}\frac{\sin x}{x}\,dx\,dy$$

$y=0,\ y=\pi$

$x=y,\ x=\pi$

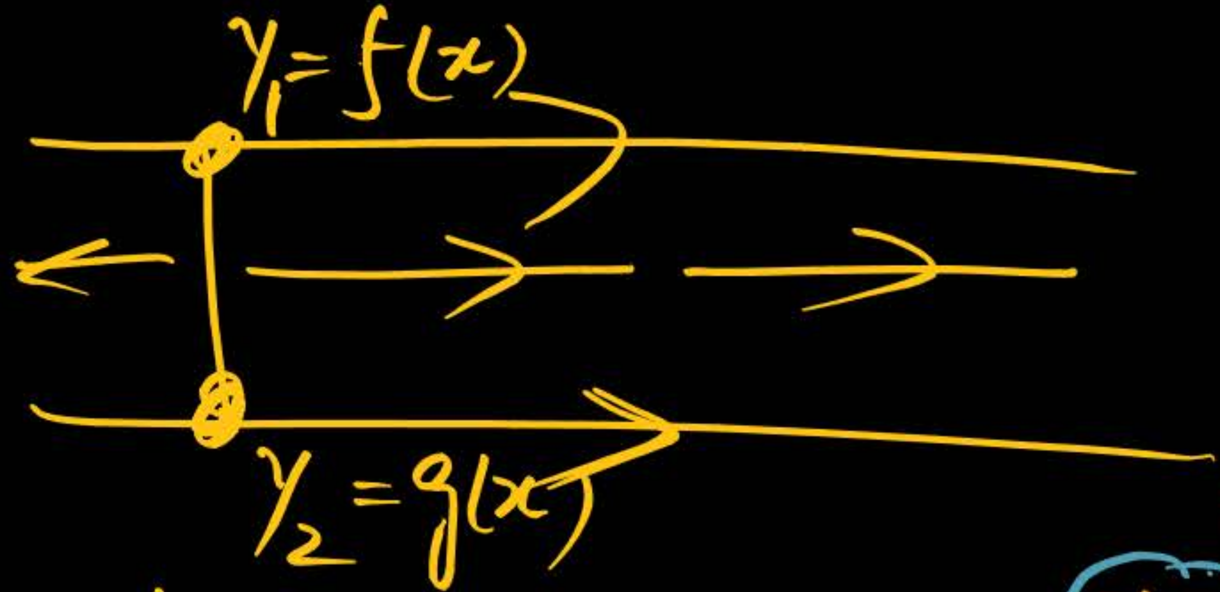




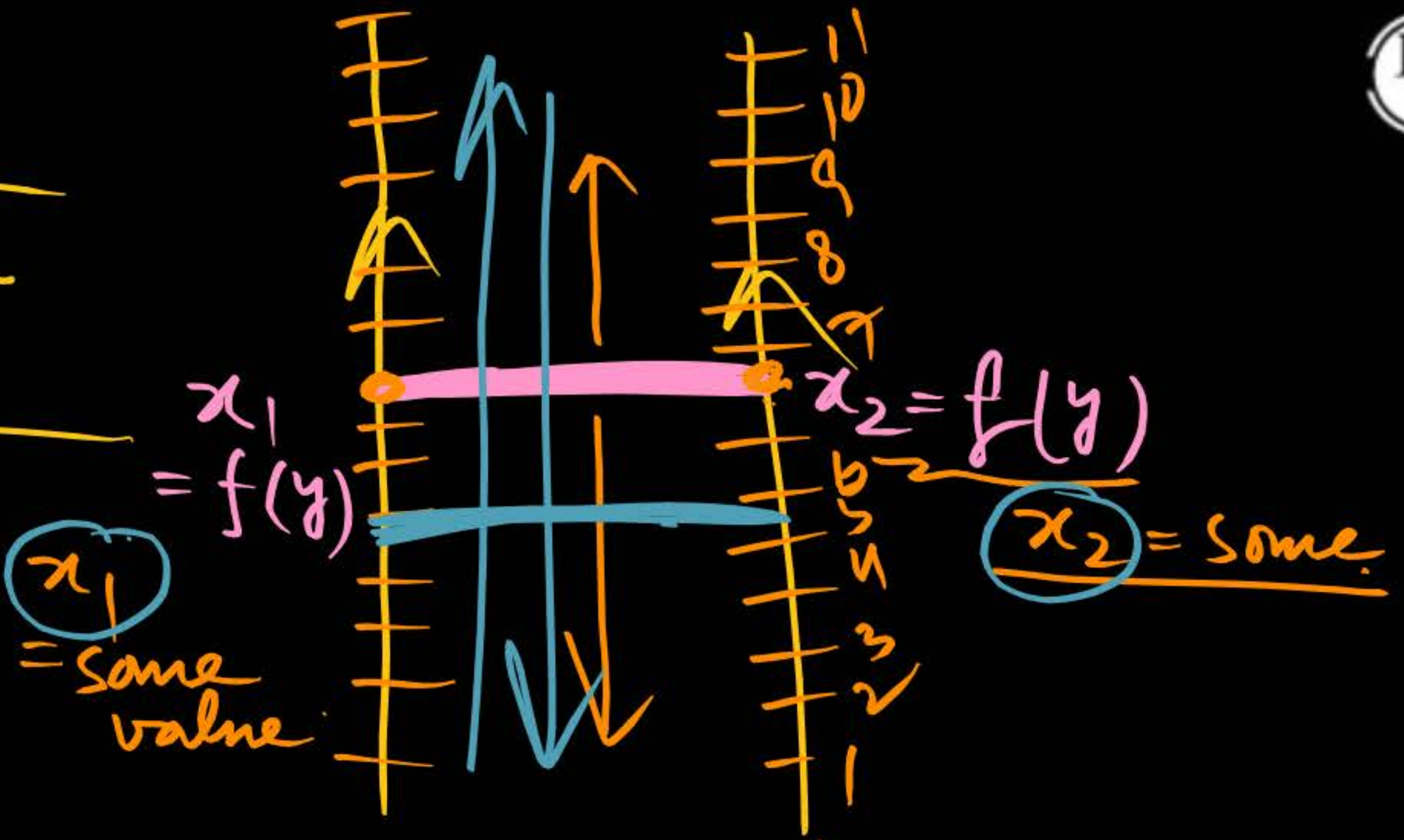
$$\int_{x=0}^{\pi}\int_{y=0}^{x}\frac{\sin x}{x}\,dy\,dx$$

Slide

#

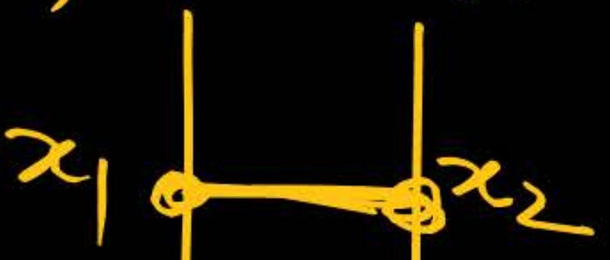


$$= \int_{x=a}^{b} \int_{y_2 = g(x)}^{y_1 = f(x)} f(x, y)\, dy\, dx$$



y axis

y constant



$$\int_{a}^{b} \int_{f(y)}^{g(y)} f(x, y)\, dy\, dx$$

Topic : Double Integrals

Q16. The value of integral $\int_0^2\int_0^x e^{x+y}\,dx\,dy$ is

(a) $\frac{1}{2}(e-1)$

(b) $\frac{1}{2}(e^2-1)^2$

(c) $\frac{1}{2}(e^2-e)$

(d) $\frac{1}{2}\left(e-\frac{1}{e}\right)^2$

$I=\int_0^2\int_0^x e^{x+y}\,dx\,dy$ is

$I=\int_0^2\int_0^x e^x e^y\,dx\,dy$

$=\left[\int_0^2 e^x\,dx\right]\left[\int_0^x e^y\,dy\right]$

$\Rightarrow \int_0^2 e^x\left[e^y\right]_0^x dx$

$\Rightarrow \int_0^2 e^x\left[e^x-e^0\right]dx$

$=\int_0^2\left[e^{2x}-e^x\right]dx$

$\Rightarrow \left[\frac{e^{2x}}{2}-e^x\right]_0^2$

$\Rightarrow \left[\frac{e^4}{2}-e^2\right]-\left[\frac{e^0}{2}-e^0\right]$

$\Rightarrow \left(\frac{e^4-2e^2}{2}\right)-\left[\frac{1}{2}-1\right]$

$\Rightarrow \frac{e^4-2e^2}{2}-\left(-\frac{1}{2}\right)$

$=\frac{e^4-2e^2}{2}+\frac{1}{2}$

Slide

Topic : Double Integrals

Do yourself

Q17. The double integral $\int_0^a \int_0^y f(x,y)\,dx\,dy$ is equivalent to

(a) $\int_0^x \int_0^y f(x,y)\,dx\,dy$

(b) $\int_0^a \int_x^y f(x,y)\,dx\,dy$

(c) $\int_0^a \int_x^a f(x,y)\,dy\,dx$

(d) $\int_0^a \int_0^a f(x,y)\,dx\,dy$

Slide

THANK - YOU