Data Science and Artificial Intelligence

Probability and Statistics

Random Variable

Lecture No.- 04

By- Rahul Sir

Recap of Previous Lecture

Topic

Expectation of Random Variables (One Dimensional)

Central

✓ Integration

1.30 Hrs.

Tools: Integration

Double Integrals

Change the order.

volume via slap.

Slide 2

Topics to be Covered

Topic Problems Based on Expectation

Topic : Expectation of Random Variables

$X = \boxed{0, 1, 2}$ (Discrete Random var.

Q1. A machine produces 0, 1 or 2 defective pieces in a day with associated probability of 1/6, 2/3 and 1/6, respectively. Then mean value and the variance of the number of defective pieces produced by

✓ Make A Table

A. 1 and 1/3

B. 1/3 and 1

C. 1 and 4/3

D. 1/3 and 4/3

	x_1	x_2	x_3
X	0	1	2
$P(X=x_i)$	$1/6$	$2/3$	$1/6$
	p_1	p_2	p_3

mean value

$$E[X] = 0 \times \frac{1}{6} + 1 \times \frac{2}{3} + 2 \times \frac{1}{6}$$

$$\mu = \frac{2}{3} + \frac{1}{3} = 1$$

$E[X] = 1$

✓ $Var(X) > 0$

✓ $Var(X) = E[X^2] - [E[X]]^2$

$$E[X^2] = 0^2 \times \frac{1}{6} + 1^2 \times \frac{2}{3} + (2)^2 \times \frac{1}{6}$$

$$= 0 \times \frac{1}{6} + 1 \times \frac{2}{3} + \frac{4}{6} = \frac{2}{3} + \frac{4}{6}$$

$$= \frac{8}{6} = \frac{2}{3} \times \frac{2}{2} = \frac{4}{3} \qquad = \frac{12+12}{18} = \frac{24}{18} = \frac{4}{3}$$

$$V(X) = E[X^2] - \left[E[X]\right]^2$$

$$\Rightarrow \frac{4}{3} - (1)^2$$

$$\Rightarrow \frac{4}{3} - (1)$$

$$\boxed{\sigma_X^2 = \frac{1}{3}}$$

Standard deviation

$$= \sqrt{\text{variance}}$$

$$= \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$

Topic : Expectation of Random Variables

Q2. In the following table, x is a discrete random variable and p(x) is the probability density.

The standard deviation of x is:

X	1 x_1	2 x_2	3 x_3
P(X)	0.3 p_1	0.6 p_2	0.1 p_3

A. 0.18

B. 0.36

C. 0.54

D. 0.6

$E[x] = \mu = \text{mean}$

$= 1 \times 0.3 + 2 \times 0.6 + 3 \times 0.1$

$= 0.3 + 1.2 + 0.3$

$= 1.8$

$S.D = \sqrt{\text{variance}}$

$= \sqrt{E[x^2] - \{E[x]\}^2}$

$E[x^2] = (1)^2 0.3 + (2)^2 0.6 + (3)^2 \times 0.1$

$= 0.3 + 2.4 + 0.9$

$= 3.6$

$S.D = \sqrt{3.6 - (1.8)^2} = \sqrt{3.6 - 3.24} = \sqrt{0.36}$

Slide 5

Topic : Expectation of Random Variables

$P(X<a) = \int_0^a f(x)\,dx$

Q3. A random variable X has probability density function $f(x)$ as given below:

X is a continuous Random variable.

$$f(x) = \begin{cases} \boxed{a+bx} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$0<x<0.5$

If the expected value $\boxed{E[X] = 2/3}$ then Pr [X < 0.5] is ____.

$P[X<0.5] = \underline{0 \text{ to } 1}$

NO

$\Rightarrow \underbrace{\int_0^{0.5} (a+bx)\,dx}$

a and b Are constants

a, b

If f(x) is valid pdf

$\int_0^1 f(x)\,dx = \text{Total area} = 1$

$\int_0^1 (a+bx)\,dx = 1$ —— 1

$a + \frac{b}{2} = 1$

$$E[X] = \frac{2}{3}$$

If X is continuous Random Variable.

$$\int_0^1 x f(x)\, dx = \frac{2}{3}$$

$$\Rightarrow \int_0^1 x \cdot (a+bx)\, dx = \frac{2}{3}$$

$$\Rightarrow \int_0^1 ax\, dx + \int_0^1 bx^2\, dx = \frac{2}{3}$$

$$= \left[a\frac{x^2}{2}\right]_0^1 + \left[\frac{bx^3}{3}\right]_0^1 = 1$$

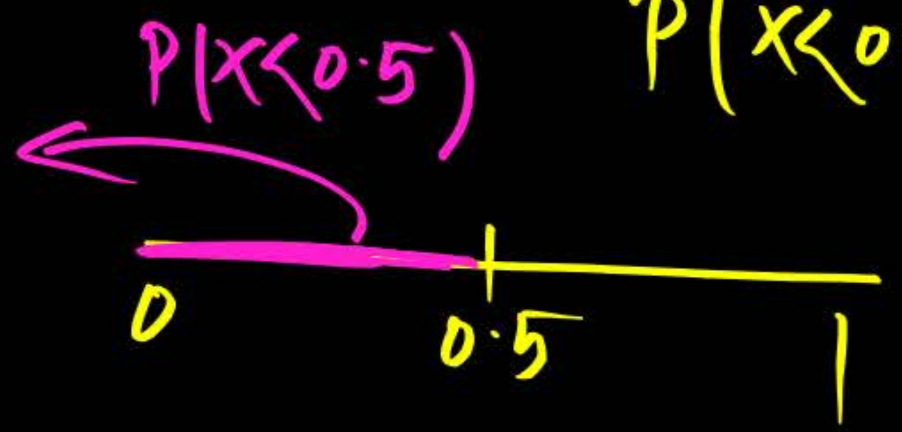
$$\Rightarrow \frac{a}{2} + \frac{b}{3} = \frac{2}{3} \quad -(2)$$

Equation (1) and (2)

$$a + \frac{b}{2} = 2/3$$

$$\frac{a}{2} + \frac{b}{3} = 1$$

Solve The equation $a = 0$ $b = 2$



$$P(X<0.5) = \int_0^{0.5} (a+bx)\,dx$$

$a=0 \quad b=2$

$$= \int_0^{0.5} (0+2x)\,dx$$

$$= \int_0^{0.5} 2x\,dx$$

$$= 2\int_0^{0.5} x\,dx$$

$$= \underline{0.25}$$

90%

Topic : Expectation of Random Variables

Q4. Consider the following probability mass function (p.m.f) of a random variable X.

Discrete Random variable

$$p(x,q)=\begin{cases} q & \text{if } x=0 \\ 1-q & \text{if } x=1 \\ 0 & \text{otherwise} \end{cases}$$

If q = 0.4, the variance of X is_____.

X	0	1
$P(X=x_i)$	q	$1-q$

$q=0.4$

$Var = E[X^2]-[E[X]]^2$

$Var(X) = (0)^2 q + (1)^2(1-q) - (0\times q + 1\times 1-q)^2$

$= (1-q) - [(1-q)]^2$

$= (1-0.4) - [(1-0.4)]^2$

$= 0.6 - (0.6)^2$

$= 0.6 - 0.36$

$= 0.24$

Topic : Expectation of Random Variables

Q5. Each of the nine words in the sentence "The Quick brown fox jumps over the lazy dog" is written on a separate price of paper. These nine pieces of paper are kept in a box. One of the pieces is drawn at random for the box. The expected length of the word drawn is_____.

(The answer should be rounded to one decimal place)

The quick brown fox jumps over The lazy dog

3 5 4 5 3 5 3 4 3

expected length

$X = 3, 4, 5$

one of those

3 letter word

4 letter word

5 letter word

$X = 3, 4, 5$

X	3	4	5
$P(X = x_i)$	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{3}{9}$

$$E[X] = 3 \times \frac{4}{9} + 4 \times \frac{2}{9} + 5 \times \frac{3}{9}$$

$$= \frac{35}{9} = \underline{3.88}$$

The	quick	Brown	fox	jumps	over	The	Lazy	dog
3	5	5	3	5	4	3	4	3

Topic : Expectation of Random Variables

Q6. The variance of the random variable X with probability density function

$f(x) = \frac{1}{2}|x|e^{-|x|}$ is ______.

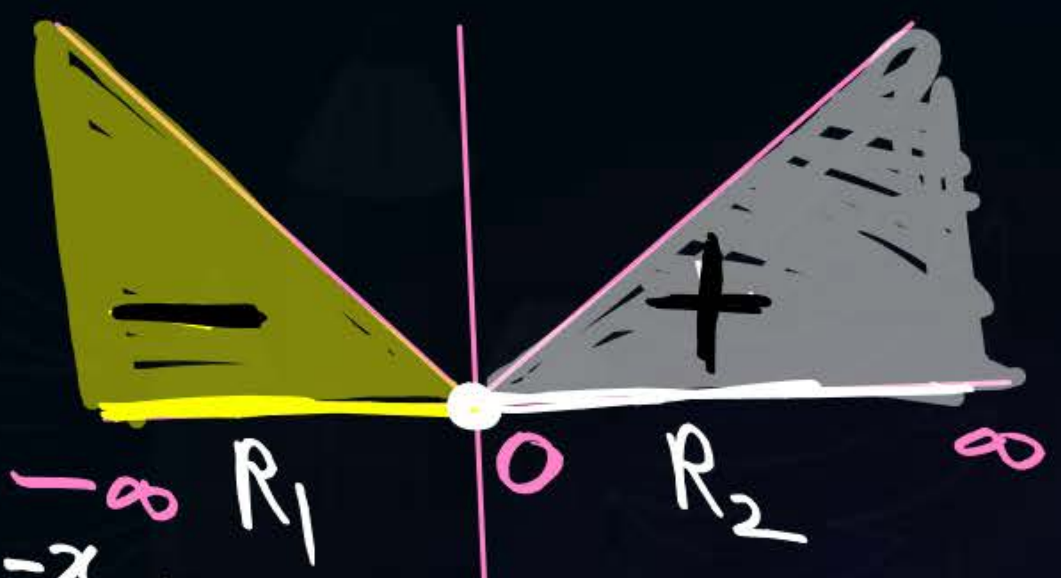
$\mathrm{var}(X) = E[X^2] - [E[X]]^2$

$f(x) = \frac{1}{2}|x|e^{-|x|}$

$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$= \int_{-\infty}^{0} x^2 f(x)\,dx + \int_{0}^{\infty} x^2 f(x)\,dx$

$\Rightarrow \int_{-\infty}^{0} x^2 \cdot \frac{1}{2}(-x)e^{-(-x)}\,dx + \int_{0}^{\infty} x^2 \frac{1}{2} x e^{-x}\,dx$



Slide 9

Do yourself

$$= \int_{-\infty}^{0} -\frac{x^3}{2} e^{x}\, dx + \int_{0}^{\infty} \frac{x^3}{2} \cdot e^{-x}\, dx = 6$$

[Tricks: Parts]

$e^{-\infty} = 0$

$e^{\infty} = \infty$

$$E[X] = \int_{-\infty}^{\infty} x \cdot \frac{1}{2}|x| e^{-|x|}\, dx$$

$$= \int_{-\infty}^{0} x \frac{1}{2}(-x) e^{-(-x)}\, dx + \int_{0}^{\infty} x \cdot \frac{1}{2}(+x) e^{-x}\, dx$$

$$= \int_{-\infty}^{0} -\frac{x^2}{2} e^{x}\, dx + \int_{0}^{\infty} \frac{x^2}{2} e^{-x}\, dx$$

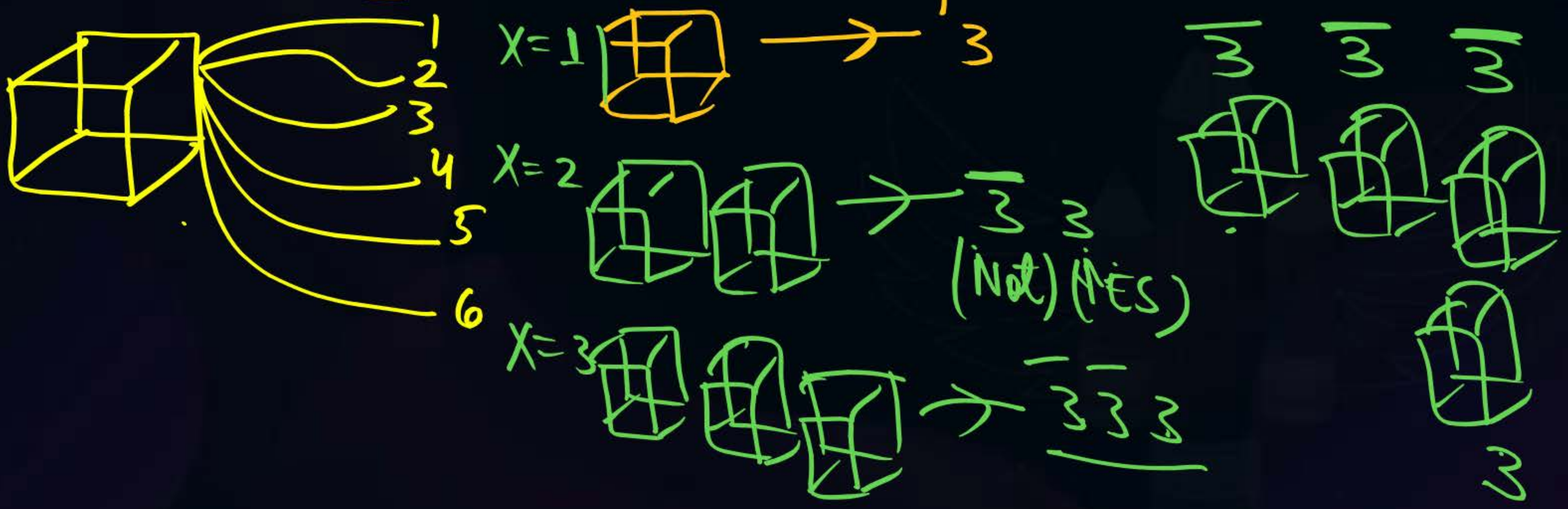
$$= 0$$

$$Var(X) = E[X^2] - [E[X]]^2$$

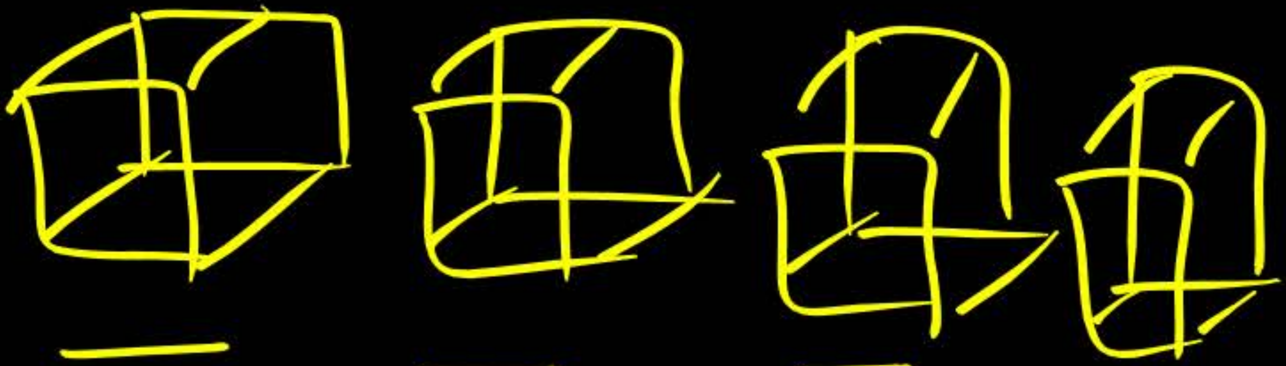
$$= 6 - 0 = 6$$

Q7. A fair die with faces {1, 2, 3, 4, 5, 6} is thrown repeatedly till '3' is observed for the first time. Let X denote the number of times the dice is thrown. The expected value of X is______.

X = No. of Die Thrown

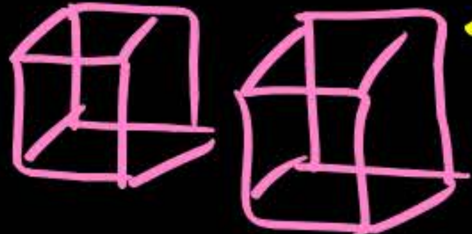


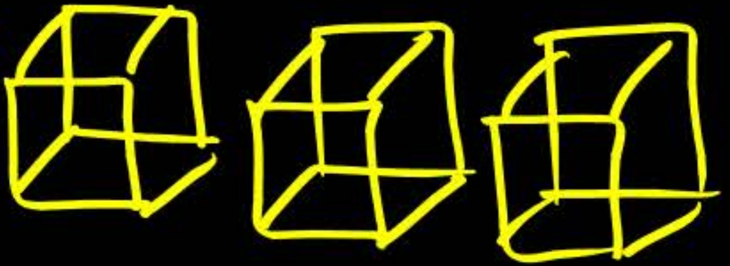
X = No.  of times Are Thrown $\longrightarrow 3 = \frac{1}{6}$



$\bar{3}\quad \bar{3}\quad \bar{3}\quad 3$

$X=4 = \frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}$ ✓

$X=2$  $\bar{3}\cdot 3 = \frac{5}{6}\times\frac{1}{6}$

$X=3$ 

$\bar{3}\quad \bar{3}\quad 3$

$= \frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}$

$X=5$ 

$\bar{3}\quad \bar{3}\quad \bar{3}\quad \bar{3}\quad 3$

$\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}$

X is a Discrete Random variable. Infinite countable

X	1	2	3	4	5
$P(X=x_i)$	$\frac{1}{6}$	$\frac{5}{6}\times\frac{1}{6}$	$\left(\frac{5}{6}\right)^2\cdot\frac{1}{6}$	$\left(\frac{5}{6}\right)^3\frac{1}{6}$	$\left(\frac{5}{6}\right)^4\cdot\frac{1}{6}$

$$E[X] = \boxed{1\cdot\frac{1}{6}} + 2\boxed{\left(\frac{5}{6}\right)}\left(\frac{1}{6}\right) + 3\boxed{\left(\frac{5}{6}\right)^2}\frac{1}{6} + 4\boxed{\left(\frac{5}{6}\right)^3}\frac{1}{6} + \cdots$$

A.

$$\frac{5}{6}E[X] = \quad + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + 2\left(\frac{5}{6}\right)^2\frac{1}{6} + 3\left(\frac{5}{6}\right)^3\frac{1}{6} + \cdots$$

Whole SERIES multiply with common ratio G.P.

SHIFT

$$\frac{1}{6}E[X] = 1\cdot\frac{1}{6} + \left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{5}{6}\right)^2\frac{1}{6} + \left(\frac{5}{6}\right)^3\frac{1}{6} + \cdots$$

$$\frac{1}{6}E[X] = \frac{1}{6} + \left(\frac{5}{6}\right)\frac{1}{6} + \left(\frac{5}{6}\right)^2\frac{1}{6} + \left(\frac{5}{6}\right)^3\frac{1}{6} + \cdots$$

$$\frac{1}{6}E[X] = \frac{1}{6} + \frac{1}{6}\Big[\underbrace{\frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \cdots}_{\text{Infinite Terms of G.P.}}$$

$$\frac{5}{6}, \left(\frac{5}{6}\right)^2, \left(\frac{5}{6}\right)^3, \left(\frac{5}{6}\right)^4 \cdots$$

$T_1 \quad T_2 \quad T_3 \quad T_4$

$\frac{T_2}{T_1}$ = common ratio

G.P.

$$(5/6)^2 / (5/6) = (5/6)$$

$$\Rightarrow \frac{1}{6}E[X] = \frac{1}{6} + \frac{1}{6}\left[\frac{5/6}{1 - 5/6}\right]$$

$$\Rightarrow \frac{1}{6}E[X] = \frac{1}{6} + \frac{1}{6}\left[\frac{5/6}{1/6}\right]$$

$$a + ar + ar^2 + ar^3 + \cdots$$

$$S_\infty = \frac{a}{(1-r)} = \frac{\text{first term}}{1 - \text{common ratio}}$$

$$\boxed{E[X] = 6}$$

$$\boxed{\text{expected value} = 6}$$

Topic : Expectation of Random Variables

Q8. A player tosses two unbiased coins. He wins Rs 5 if 2 heads appear, Rs 2 if one head appears and Rs 1 if no head appears. Find the expected value of the amount won by him.

SAME QUESTION

Q9. Two cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the expected value for the number of aces.

52 cards.

Aces.

Two cards.

Expected value

No Ace

(Ace ∩ 0) + (0 ∩ A)

2 Aces.

4 Aces

52 cards

2 cards

$\rightarrow$ No Ace $\qquad$ 0 Ace

$\rightarrow (A+0)+(0+A)$ $\qquad$ 1 Ace

$\rightarrow$ 2 Aces $\qquad$ 2 Ace

X is a Discrete Random variable.

$X = 0, 1, 2$

$P[X=0] = P[X = \text{No Aces}]$

4 Aces
52 cards

$\rightarrow \frac{48}{52} \times \frac{48}{52}$

$= \frac{144}{169}$

$P[X=1] = P(\text{one Ace and other cards})$

$P(A)P(0) + P(0)P(A)$

$P(A \cap 0) + P(0 \cap A)$

$= \frac{4}{52} \times \frac{48}{52} + \frac{48}{52} \times \frac{4}{52}$

$= \frac{1}{13} \times \frac{12}{13} + \frac{1}{13} \times \frac{12}{13} = \frac{24}{169}$

$$P[X=2\text{ Aces}] = \frac{4}{52} \times \frac{4}{52} = \frac{1}{169}$$

X	0	1	2
$P(X=x_i)$	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

$$E[X] = 0\times\frac{144}{169} + 1\times\frac{24}{169} + 2\times\frac{1}{169}$$

$$= \frac{26}{169} \quad \underline{\underline{\text{Ans}}}$$

$$= \frac{2}{13}$$

Topic : Expectation of Random Variables

Q10. If it rains, a rain coat dealer can earn Rs 500 per day. If it is a dry day, he can lose Rs 100 per day. What is his expectation, if the probability of rain is 0.4?

X = Rainy day, Dry day

X	500 Per Day	Dry Day (−100)
$P(X=x_i)$	0.4	0.6

X	500	−100
$P(X=x_i)$	0.4	1−0.4 = 0.6

$E[X]$

$= 500 \times 0.4 - 100 \times 0.6$

$= 140$ Ans

Topic : Expectation of Random Variables

Q11. You toss a fair coin. If the outcome is head, you win Rs 100; if the outcome is tail, you win nothing. What is the expected amount won by you?

X	$100\,(H)$	$0\,(T)$
$P(X=x_i)$	$\frac{1}{2}$	$\frac{1}{2}$

$$E[X] = 100 \times \frac{1}{2} + 0 \times \frac{1}{2} = 50$$

AGP Hint

Do yourself

Q12. A fair coin is tossed until a tail appears. What is the expectation of number of tosses?

$\boxed{Ans = 2}$

Topic : Expectation of Random Variables

$(A-B)^3 = A^3 - B^3 - 3AB(A-B)$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + $

Q13. The distribution of a continuous random variable X is defined by

$$f(x)=\begin{cases} x^3 & , \quad 0<x\le 1 \\ (2-x)^3, & 1<x\le 2 \\ 0 & , \quad \text{elsewhere} \end{cases}$$

$= \frac{1}{2}$ Ans.

$E[X] = \int_0^1 x(x^3)\,dx + \int_1^2 x(2-x)^3\,dx$

$\Rightarrow \frac{1}{2}$

Obtain the expected value of X.

Expected value $= \int_{-\infty}^{\infty} x f(x)\,dx$ OR $\int_a^b x f(x)\,dx$

Topic : Expectation of Random Variables

Q14. For a continuous distribution, whose probability density function is given by:

$$f(x) = \frac{3x}{4}(2-x), 0 \le x \le 2,$$

find the expected value of X.

$f(x) = \frac{3x}{4}(2-x)$

$0 \le x \le 2$

Expected value $= \int_0^2 x \left(\frac{3x}{4}\right)(2-x)$

$\boxed{E[X] = 1}$

Topic : Expectation of Random Variables

Q15. Given the following probability distribution

Do yourself

X	–2	–1	0	1	2
p(X)	0.15	0.30	0	0.30	0.25

Find (i) $E(X) =$

(ii) $E(2X + 3) = 2E[X] + 3$

(iii) $E(X^2) =$

(IV) $E(4X - 5)$ $4E[X] - 5$

THANK - YOU