Data Science and Artificial Intelligence

Probability and Statistics

Counting Techniques & Introduction to Probability

DPP Discussion Notes

By- Rahul Sir

Topics to be Covered

Topic — Questions

Topic — Discussion

Q1. If there are 6 girls and 5 boys who sit in a row. then the probability that no two boys sit together is

$P(E) = \frac{\text{fav outcome}}{\text{total outcome}}$

(a) $\frac{6!6!}{2!11!}$

(b) $\frac{7!5!}{2!11!}$

(c) $\frac{6!7!}{2!11!}$

(d) None of these

$6+5 = 11 \longrightarrow 11! = {}^{11}P_{11}$

$G_1 \quad G_2 \quad G_3 \quad G_4 \quad G_5 \quad G_6$

$6! = {}^6P_6$

X G_1 X G_2 X G_3 X G_4 X G_5 X G_5 X

1 2 3 4 5 6 7

5 Boys

7 places

${}^7P_5$

$$P(E) = \frac{6! \times {}^7P_5}{{}^{11}P_{11}}$$

$$= \frac{6! \times 7!}{2! \times 11!}$$

$\boxed{C}$

Slide 4

Q2. Three integers are chosen at random from the first 20 integers. The probability that their product is even

(a) 2/19

(b) 3/29

(c) 17/19

(d) 4/29

total no. of outcomes = ${}^{20}C_3$

Now; P(their product is odd) $\checkmark$

P(even) → ① ② ③

2 even

P(product is odd) → all the three integers must be odd.

Total ways = $\boxed{{}^{10}C_3}$

20 integers

10 odd 10 even

$P(\text{product is odd}) = \dfrac{{}^{10}C_3}{{}^{20}C_3}$

$P(\text{product is even}) = 1 - P(\text{product is odd})$

$= 1 - \dfrac{{}^{10}C_3}{{}^{20}C_3} = \dfrac{1140 - 120}{1140}$

$= \dfrac{17}{19}$

$\boxed{C}$ ✓

Slide 6

Q3. One hundred cards are numbered from 1 to 100. The probability that a randomly chosen card has a digit 5 is

1 to 100 100 cards

(a) 1/100

(b) 9/100

(c) 19/100

(d) None of these

1 to 100

1, 2, 3 digits → 100

5, 15, 25, 35, 45, 51, 52, 53, 54, 55, 56, 57, 58, 59, 65, 75, 85, 95, 50

19 outcomes

$P(E) = \frac{19}{100}$ → (c)

Q4. If the letters of word 'REGULATIONS' be arranged at random, the probability that there will be exactly 4 letters between R and E is:

(a) 6/55

(b) 3/55

(c) 49/55

(d) None of these

11 letters

Total outcomes = 11!

Now;

R _ _ _ _ E _ _ _ _ _

(1, 6) (2, 7) (3, 8) (4, 9) (5, 10) (6, 11) fav outcomes $\rightarrow 6 \times 2 \times 9!$

$$P(E) = \frac{6 \times 2 \times 9!}{11!}$$

$$= \frac{6 \times \cancel{2} \times \cancel{9!}}{11 \times \cancel{10}_{5} \times \cancel{9!}} = \frac{6}{55} \quad \boxed{a}$$

11 = 2

9 letters

9! arrange

Slide 9

Topic : Counting Techniques

Q5. In a bag there are three tickets numbered 1, 2, 3. A ticket is drawn at random and put back, and this is done four times the probability of that the sum of the numbers is even is:

(a) ~~41/81~~

(b) 39/81

(c) 40/81

(d) None of these

1, 2, 3

1 3 2

1 3 2 = 6

even

4

4 4 4 4

sum is even $\rightarrow$ 4 odd

$\rightarrow$ 4 even ✗

$\rightarrow$ 2 odd & 2 even

3 odd 1 even

even ✗

$\textcircled{X} \rightarrow$ Random variable such that X is no. of odd

$P(E) = P(X=0) + P(X=2) + P(X=4)$

$\left(\frac{1}{3}\right)^4 + \left(\frac{2}{3}\right)^4 + {}^4C_2 \times \left(\frac{1}{3}\right)^2 \times \left(\frac{2}{3}\right)^2$

$= \frac{41}{81} \rightarrow \textcircled{a}$

1, $\textcircled{2}$, 3 $\rightarrow$ P(getting even) $= \frac{1}{3}$

P(getting odd) $= 2/3$

$P(X=0) = \left(\frac{1}{3}\right)^4$

$P(X=2) = \left(\frac{2}{3}\right)^2 \times \left(\frac{1}{3}\right)^2 \times {}^4C_2$

$P(X=4) = \left(\frac{2}{3}\right)^4$

Topic : Counting Techniques

Q6. A pack of cards consists of 15 cards numbered 1 to 15. Three cards are drawn at random with replacement.
Then, the probability of getting 2 odd and one even numbered card is:

(a) 348/1125

(b) 398/1125

(c) 448/1125

(d) 498/1125

15 cards

Case 1: odd even odd

Case 2: odd odd even

Case 3: even odd odd

odd cards = 1, 3, 5, 7, 9, 11, 13, 15

$P(\text{odd}) = \frac{8}{15}$

$P(\text{even}) = \frac{7}{15}$

Case 1:- $P(A)$:- $\frac{8}{15} \times \frac{7}{15} \times \frac{8}{15}$

Case 2 !- $P(B)$!- $\frac{8}{15} \times \frac{8}{15} \times \frac{7}{15}$

Case 3 !- $P(C)$:- $\frac{7}{15} \times \frac{8}{15} \times \frac{8}{15}$

$P(E) = P(A) + P(B) + P(C)$

$= \frac{\cancel{3} \times 8 \times 8 \times 7}{\cancel{15}_{5} \times 15 \times 15}$

$= \frac{448}{1125}$

$\longrightarrow$ (C) =

Q7. Three persons A, B and C are to speak at a function along with five others. If they all speak in random order, the probability that A speaks before B and B speaks before C is:

(a) 3/8

(b) 1/6

(c) 3/5

(d) None of these

A B C P_1 P_2 P_3 P_4 P_5

A B C → specified order

group

P_1, ABC, P_2, P_3, P_4, P_5

P_1, P_2, ABC, ...

total ways of speaking $\underline{\underline{{}^8C_3}}$ ✓

Now, the 5 persons remaining

$= \boxed{\underline{\underline{5!}}}$ ✓

(c) (3) persons

↓

have been converted to a $\underline{\underline{\text{group}}}$

$$\boxed{P(E) = \frac{5! \times {}^8C_3}{8!} = \frac{1}{3!} = \underline{\underline{\frac{1}{6}}}}$$

(b)

Topic : Counting Techniques

Q8. An elevator starts with m passengers and stops at n floors $(m \leq n)$ the probability that no two passengers alight at same floor is:

(a) $\cancel{\dfrac{{}^{n}P_{m}}{m^{n}}}$

(b) $\dfrac{{}^{n}P_{m}}{n^{m}}$

(c) $\dfrac{{}^{n}C_{m}}{m^{n}}$

(d) $\dfrac{{}^{n}C_{m}}{n^{m}}$

persons

2

3

2

1

$m \leq n$

Total outcomes $\rightarrow$

1 m x

2 m x

m passengers & n floors

Total outcomes $= \underline{\underline{m^n}}$

favourable outcomes $\rightarrow$

m passengers & n floors $\boxed{m \le n}$

$\boxed{{}^nC_m} \times m!$

$= \underline{\underline{{}^nC_m}} \times m!$

$P(E) = \dfrac{{}^nC_m \times m!}{m^n} = \dfrac{{}^nP_m}{m^n}$ (a) ✓

Slide 17

Q9. A and B play a game where each is asked to select a number from 1 to 25. If the two numbers match both of them win a prize. The probability that they will not win a prize in a single trial is:

(a) 1/25

(b) 24/25

(c) 2/25

(d) None of these

$P(\bar{E}) = 1 - P(E)$

$P(E) = P($winning a prize$)$

A B

25 numbers

25 ways

25 ways

1 total outcomes $= 25 \times 25 = 625$

favourable outcomes = (1,1) (2,2) (3,3)

— — — (25, 25)

= 25 ways

$$P(E) = \frac{25}{25 \times 25} = \frac{1}{25}$$

$$P(\bar{E}) = 1 - P(E) = 1 - \frac{1}{25} = \frac{24}{25}$$ ✓ (b)

Q10. Fifteen persons among whom are A and B, sit down randomly at round table. The probability that there are 4 persons between A and B is:

(a) $\frac{9!}{14!}$

(b) $\frac{10!}{14!}$

(c) $\frac{9!}{15!}$

(d) ~~None of these~~

round table

circular pert^n

No. of ways of arranging

$= (n-1)!$

$= (15-1)!$

$= 14!$

A & B → 4 person

A _ _ _ _ B → $^{13}C_4 \times 4! \times 9! \times 2!$

total outcomes

$= 14!$

$$P(E) = \frac{^{13}C_4 \times 4! \times 9! \times 2!}{14!} = \frac{2}{14} = \frac{1}{7}$$

Q11. The probability that the 13th day of a randomly chosen month is a second Saturday is:

(a) 1/7

(b) 1/12

(c) ~~1/84~~

(d) 19/84

Probability of choosing a month $= \frac{1}{12}$.

13th day → saturday

6th day → saterday

Probability for 6th day to be a saturday $= \frac{1}{7}$

Req^n probability $= \frac{1}{7} \times \frac{1}{12} = \frac{1}{84}$

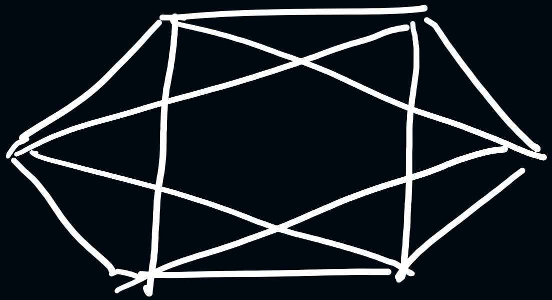
Q12. Three of the six vertices of a regular hexagon are chosen at random. The probability that the triangle with three vertices is equilateral, is:

(a) 1/2

(b) 1/5

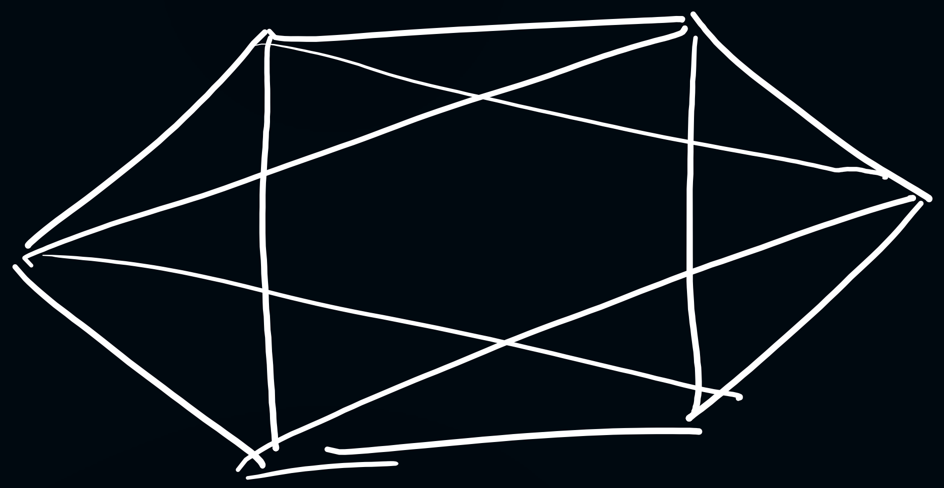
(c) 1/10

(d) 1/20



ways of choosing 3 vertices out of 6

vertices = ${}^6C_3$ = 20



hexagon 3 vertices triangle only 2 eq^{n} triangles

favourable = 2

$P(E) = \frac{2}{20} = \frac{1}{10} \longrightarrow$ (c)

Q13. The probability that out of 10 persons, all born in April, at least two have the same birthday is:

(a) $\dfrac{{}^{30}C_{10}}{(30)^{10}}$

(b) $1-\dfrac{{}^{30}C_{10}}{30!}$

(c) $\cancel{\dfrac{(30)^{10}-{}^{30}C_{10}}{(30)^{10}}}$

(d) None of these

P(none of them have the same birthday)

total $= (30)^{10}$

favourable outcomes $= {}^{30}P_{10}$

P(none of them have same birthday) $= \dfrac{{}^{30}P_{10}}{(30)^{10}}$

P(atleast two have same birthday)

$= 1 - P$(none of them have same birthday)

$= 1 - \dfrac{{}^{30}P_{10}}{(30)^{10}}$

Q14. If A and B are two events, the probability that exactly one of them occurs is given by:

(a) $P(A)+P(B)-2P(A\cap B)$

(b) $P(A\cap\bar{B})+P(\bar{A}\cap B)$

(c) $P(A\cup B)-P(A\cap B)$

(d) $P(\bar{A})+P(\bar{B})-2P(\bar{A}\cap\bar{B})$

$P\{(A\cap\bar{B})\cap(\bar{A}\cap B)\}$
$=P\{\phi\}=0$

$P\{(A\cap\bar{B})\cup(\bar{A}\cap B)\}$

$P(A\cap\bar{B})+P(\bar{A}\cap B)$

$P(A)-P(A\cap B)+P(B)-P(A\cap B)$

$P(A)+P(B)-2P(A\cap B)$

$$P(A)+P(B)-2P(A\cap B)$$

$$P(A)+P(B)-P(A\cap B)-P(A\cap B)$$

$$P(A\cup B)-\underline{\underline{P(A\cap B)}}$$

$$\underbrace{P(A\cap\bar{B})}+\underbrace{P(\bar{A}\cap B)}$$

$$=1-P(\bar{A})+1-P(\bar{B})-2\{1-P(\bar{A}\cup\bar{B})\}$$

$$=2P(\bar{A}\cup\bar{B})-P(\bar{A})-P(\bar{B})$$

$$=\{2P(\bar{A})+2P(\bar{B})-2P(\bar{A}\cap\bar{B})\}-P(\bar{A})-P(\bar{B})$$

$$=P(\bar{A})+P(\bar{B})-2P(\bar{A}\cap\bar{B})$$

Q15. If A and B are events at the same experiments with P(A) = 0.2, P(B) = 0.5, then maximum value of P (A' ∩ B).

(a) 1/4

(b) 1/2 0.5

(c) 1/8

(d) 1/16

$P(A' \cap B)$ when either $A \subseteq B$ or $B \subseteq A$

$P(A) = 0.2$

$P(B) = 0.5$

$A \subseteq B$

$P(A' \cap B)$

$P(A') = 1 - 0.2 = 0.8$

$\boxed{B \subseteq A'}$

Max value consider $B \subseteq A'$

$P(B) = 0.5$

$P(A') = 0.8$

Max value $P(A' \cap B) = 0.5$ only

$\boxed{B \subseteq A'}$

(b)

Q16. The probabilities that a student passes in mathematics, physics and chemistry are m, p and c respectively. Of these subjects, a student has a 75% chance of passing in at least one, a 50% chance of passing in at least one, 50% chance of passing in at least two and a 40% chance of passing in exactly two subjects. Which of the following relations are true?

(a) $p+m+c=\frac{19}{20}$

~~(b)~~ $\cancel{p+m+c=\frac{27}{20}}$

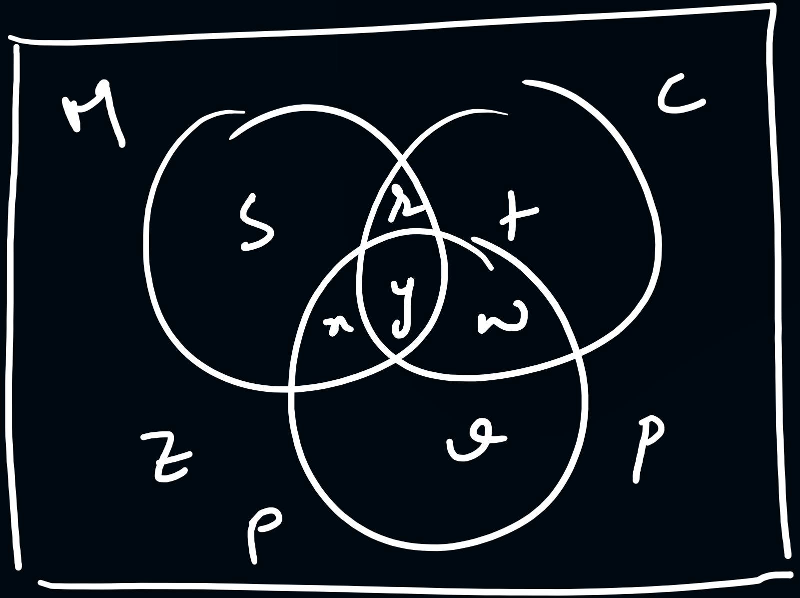
~~(c)~~ $\cancel{pmc=\frac{1}{10}}$

(d) $pmc=\frac{1}{4}$

$P(M) = m$

$P(P) = p$

$P(C) = c$



let s, t, v, V, w, y & z are probabilities of regions.

$s+t+v = 0.25$

$m+c+p = s+t+v+2(r+w+x)+3y$

$= 0.25 + 2\times 0.4 + 3\times 0.1$

$= 1.35 = \frac{135}{100} = \frac{27}{20}$

Sub^n (ii) & (iii)

$y = pmc$

$= 0.1 = \frac{1}{10}$

y common

Subtr (i) & (ii)

$s+t+v = 0.25$

Q17. A coin is tossed n times. The probability of getting at least one head is greater than that of getting at least two tails by 5/32. Then n is:

(a) 5

(b) 10

(c) 15

(d) None of these

using Binomial distribution –

$^nC_m\, p^m q^n$

$p \rightarrow$ probability of success

$q \rightarrow$ probability of failure

$n \rightarrow$ No. of trials

$m \rightarrow$ No. of success

$P(\text{head}) = \frac{1}{2}$

$P(\text{tail}) = \frac{1}{2}$

$$\boxed{\begin{aligned} P(A) &= 1 - P(\text{No head}) \\ &= 1 - {}^{n}C_{0}\left(\tfrac{1}{2}\right)^{n} \times \left(\tfrac{1}{2}\right)^{0} \end{aligned}}$$

$$\begin{aligned} P(B) &= 1 - P(\text{no tail}) - P(1 \text{ tail}) \\ &= 1 - {}^{n}C_{0}\left(\tfrac{1}{2}\right)^{0}\left(\tfrac{1}{2}\right)^{n} - {}^{n}C_{1}\left(\tfrac{1}{2}\right)^{1}\left(\tfrac{1}{2}\right)^{n-1} \\ &= 1 - \left(\tfrac{1}{2}\right)^{n} - n\left(\tfrac{1}{2}\right)^{n} \\ &= 1 - (n+1) \times \left(\tfrac{1}{2}\right)^{n} \end{aligned}$$

$P(A) = 1 - \left(\frac{1}{2}\right)^{n}$

$P(B) = 1 - \left(\frac{1}{2}\right)^{n} - \left(\frac{1}{2}\right)^{n-1}$

$P(A) - P(B) = \frac{5}{32}$

$n\left(\frac{1}{2}\right)^{n} = \frac{5}{32}$

$\boxed{n = 5}$

Slide 35

Q18. Suppose that
Box-I contains 8 red, 3 blue and 5 green balls,
Box-II contains 24 red, 9 blue and 15 green balls,
Box-III contains 1 blue, 12 green and 3 yellow balls,
Box-IV contains 10 green, 16 orange and 6 white balls.
A ball is chosen randomly from Box-I; call this ball b.
If b is red then a ball is chosen randomly from Box-II, if b is blue then a ball is chosen randomly from Box- III, and if b is green then a ball is chosen randomly from Box-IV. The conditional probability of the event 'one of the chosen balls is white' given that the event 'at least one of the chosen balls is green' has happened, is equal to

(a) 15/256
(b) 3/16
(c) 5/52
(d) 1/8

Box I :- Red $\xrightarrow{\text{Box 2}}$ green $\longrightarrow$ $\frac{15}{48}$

Blue $\xrightarrow{\text{Box 3}}$ green $\longrightarrow$ $\frac{12}{16}$ =

green $\xrightarrow{\text{IV}}$ white $\longrightarrow$ $= \frac{6}{32}$

Start

E_1 E_2 E_3

A

$P\left(\frac{E_2}{A}\right) = \dfrac{P(E_2) \cdot P\left(\frac{A}{E_2}\right)}{\sum_{i=1}^{3} P(E_i) \times P\left(\frac{A}{E_i}\right)}$

$P(R) = \frac{8}{16}$ ✓

$P(B) = \frac{3}{16}$ ✓

$P(W) = \frac{5}{16}$ ✓

A :- one ball white

B :- at least 1 is green.

$$P(A/B) = \frac{\frac{5}{16} \times \frac{6}{32}}{\frac{5}{16} \times \frac{6}{32} + \frac{3}{16} \times \frac{12}{16} + \frac{1}{2} \times \frac{15}{48}} = \frac{5}{62}$$

Slide 37

Q19. Consider three sets $E_1 = \{1, 2, 3\}$, $F_1 = \{1, 3, 4\}$ and $G_1 = \{2, 3, 4, 5\}$. Two elements are chosen at random, without replacement, from the set E_1, and let S_1 denote the set of these chosen elements. Let $E_2 = E_1 - S_1$ and $F_2 = F_1 \cup S_1$. Now two elements are chosen at random, without replacement, from the set F_2 and let S_2 denote the set of these chosen elements. Let $G_2 = G_1 \cup S_2$. Finally, two elements are chosen at random, without replacement, from the set G_2 and let S_3 denote the set of these chosen elements. Let $E_3 = E_2 \cup S_3$. Given that $E_1 = E_3$, let p be the conditional probability of the event $S_1 = \{1, 2\}$. Then the value of p is

(a) 1/5
(b) 3/5
(c) 1/2
(d) 2/5

$$P\left(\frac{W_1}{G_{12}}\right) = \frac{P\left(\frac{W}{G}\right)}{P(G)} \longrightarrow \text{Baye's theorem}$$

$$= \frac{\frac{5}{16}\cdot\frac{6}{32}}{\frac{5}{16}\cdot 1 + \frac{6}{16}\cdot\frac{15}{48} + \frac{3}{16}\cdot\frac{12}{16}}$$

$$= \frac{15}{80+40+36} = \frac{5}{52}.$$

Slide 39

Q20. Three randomly chosen nonnegative integers x, y and z are found to satisfy the equation x + y + z =10. Then the probability that z is even,

(a) 36/55

(b) 6/11

(c) 1/2

(d) 5/11

$z = 0, 2, 4, 6, 8, 10$

$x + y + z = 10$

$x, y \in \{0, 1, 2, 3, \ldots 10\}$

$x + y = 10 - z$

Case 1 :- $z = 0$

$x + y = 10$

$^{10+2-1}C_{2-1} = {}^{11}C_1$

Case 2 : $z = 2$

$x + y = 8$

$S_1 = \{1, 2\}$

$E_2 = \{3\}$

$S_3 = \{1, 2\}$

$$\text{probability} = \frac{1}{3} \times \frac{3}{6} \times \frac{{}^2C_2}{{}^5C_2}$$

$$= \frac{1}{60} \text{ — } (1)$$

$S_1 = \{1, 3\}$

$E_2 = \{2\}$

$F_2 = \{1, 3, 4\}$

$G_2 = \{1, 2, 3, 4, 5\}$

Probability

$$= \frac{1}{3} \times \frac{2}{3} \times \frac{1}{10}$$

$$= \frac{1}{45}$$

if $S_1 = \{2, 3\}$

$E_2 = \{1\}$

$F_2 = \{1, 2, 3, 4\}$

Prob $= \frac{1}{3} \times (S_2$ contain $1 = \frac{1}{2}) \times \frac{1}{10}$

$$= \frac{1}{3} \times \frac{1}{2} \times \frac{1}{10} \text{ — } (iii)$$

Q21. There are three bags B_1, B_2 and B_3. The bag B_1 contains 5 red and 5 green balls, B_2 contains 3 red and 5 green balls, and B_3 contains 5 red and 3 green balls. Bags B_1, B_2 and B_3 have probabilities 3/10, 3/10 and 4/10 respectively of being chosen. A bag is selected at random, and a ball is chosen at random from the bag. Then which of the following options is/are correct?

(a) Probability that the chosen ball is green, given that the selected bag is B_3, equals 3/8

(b) Probability that the chosen ball is green equals 39/80

(c) Probability that the selected bag is B_3, given that the chosen ball is green, equals 5/13

(d) Probability that the selected bag is B_3 and the chosen ball is green equals 3/10

$prob = \frac{1}{3} \times [S_2 \text{ does not contain } 1 = \frac{1}{2}] \times \frac{1}{6}$

$= \frac{1}{3} \times \frac{1}{2} \times \frac{1}{6} = \frac{1}{36}$ ✓

$$\text{Req}^n \text{ probability} = \frac{\frac{3}{60}}{\frac{3}{60} + \frac{2}{30} + \frac{1}{12} + \frac{1}{20}}$$ ✓

Topic : Counting Techniques

Q22. Consider an experiment of tossing a coin repeatedly until the outcomes of two consecutive tosses are same. If the probability of a random toss resulting in head is 1/3, then the probability that the experiment stops with head is

(a) 1/3

(b) 5/21

(c) 4/21

(d) 2/7

$P(H) = \frac{1}{3}$ $\quad P(T) = \frac{2}{3}$

$P(E) = P(HH) + P(THH) + P(HTHH) \cdots$

$= \frac{1}{3^2} + \frac{2}{3^3} + \frac{2}{3^4} + \frac{4}{3^5} + \frac{4}{3^6} \cdots$

$= \left(\frac{1}{3^2} + \frac{2}{3^4} + \frac{4}{3^6} \cdots\right) + \left(\frac{2}{3^3} + \frac{4}{3^5} + \frac{8}{3^7} \cdots\right)$

$P(E) = \frac{1}{7} + \frac{2}{21} = \frac{5}{21}$

$B_1 \to 5$ Red balls $\quad$ 1 green

$B_2 \to 3$ Red balls $\quad$ 5 green

$B_3 \to 5$ Red $\quad$ 3 green

B_1, B_2, B_3 have probabilities;

(A) $\to$ green

(B) $\to$ Bag is B_3

$$P(A/B_3) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{15}{16} \quad (a) \times$$

P(green ball is chosen)

$$\frac{5}{10} \times \frac{3}{10} + \frac{5}{8} \cdot \frac{3}{10} + \frac{3}{8} \cdot \frac{4}{10}$$

$$\frac{15}{100} + \frac{15+12}{80} = \frac{15}{100} + \frac{27}{80} = \frac{39}{80}$$

THANK - YOU