Data Science and Artificial Intelligence

Probability and Statistics

Continuous Probability Distribution

Lecture No. -05

By- Rahul Sir

Topics to be Covered

✓ exponential Distributions

Topic Problems based on exponential Distribution

Topic Uniform distribution (Discrete Distribution) ✓

Topic Problems based on Uniform Distribution ✓

Exponential Distribution

Large Amount of Loss in Single trial

λ = Parameter

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ & \text{otherwise} \end{cases}$$

$E[X] = \frac{1}{\lambda}$ $\quad V(X) = \frac{1}{\lambda^2}$ $\quad S.D = \frac{1}{\lambda}$

cdf $P(X \leq x) = 1 - e^{-\lambda x}$

$\int_0^x \lambda e^{-\lambda x}\,dx = 1 - e^{-\lambda x}$

Death prob. atmost

$P(X \geq x) = e^{-\lambda x}$

survival.

Exponential Dis μ

μ = mean

$E[X] = \mu$

$V(X) = \mu^2$

$S.D = \mu$

$f(x) = \frac{1}{\mu} e^{-\frac{1}{\mu}x}$

median $= \mu \ln 2$

$P(X \geq x) = e^{-\frac{1}{\mu}x}$

Death

$= 1 - e^{-\frac{1}{\mu}x}$

Q6. What are the mean and variance of the exponential distribution given by:

$f(x) = 3e^{-3x}, x \geq 0$

mean and var $\longrightarrow \frac{1}{\lambda}, \frac{1}{\lambda^2}$

$f(x) = 3e^{-3x}$

Compare It

$f(x) = \underline{\lambda e^{-\lambda x}}\ x \geq 0$

$\lambda = 3$

Mean $= \frac{1}{\lambda} = \frac{1}{3}$

variance $= \frac{1}{\lambda^2} = \frac{1}{9}$

S.D $= \frac{1}{3}$

Memory Less Property

$P\left[\frac{X \geq a+b}{X \geq a}\right]$

$= P[X \geq b]$

$P\left[\frac{X \geq 10+5}{X \geq 10}\right]$

$= P[X \geq 5]$

Q7. Obtain the value of $k > 0$ for which the function given by

$f(x) = 2e^{-kx}, x \geq 0$

follows an exponential distribution.

$f(x) = \lambda e^{-\lambda x}$

$\boxed{\lambda = 2}$ $K = 2$

Ans

Probability & Statistics

5 min

Q4. An institute purchases laptop from either vendor V_1 or V_2 with equal probability. The lifetimes (in years) of laptops from vendor V_1 have a U(0, 4) distribution, and the lifetimes (in years) of laptop from vendor V_2 have an $\boxed{\text{Exp }(1/2)}$ distribution. If a randomly selected laptop in the institute has lifetime more than two years, then the probability that it was supplied by vendor V_2 is

$V_1 \rightarrow U[0, 4]$

$V_2 \rightarrow \exp\left(\frac{1}{2}\right)$ λ

A. $\boxed{\frac{2}{2+e}}$

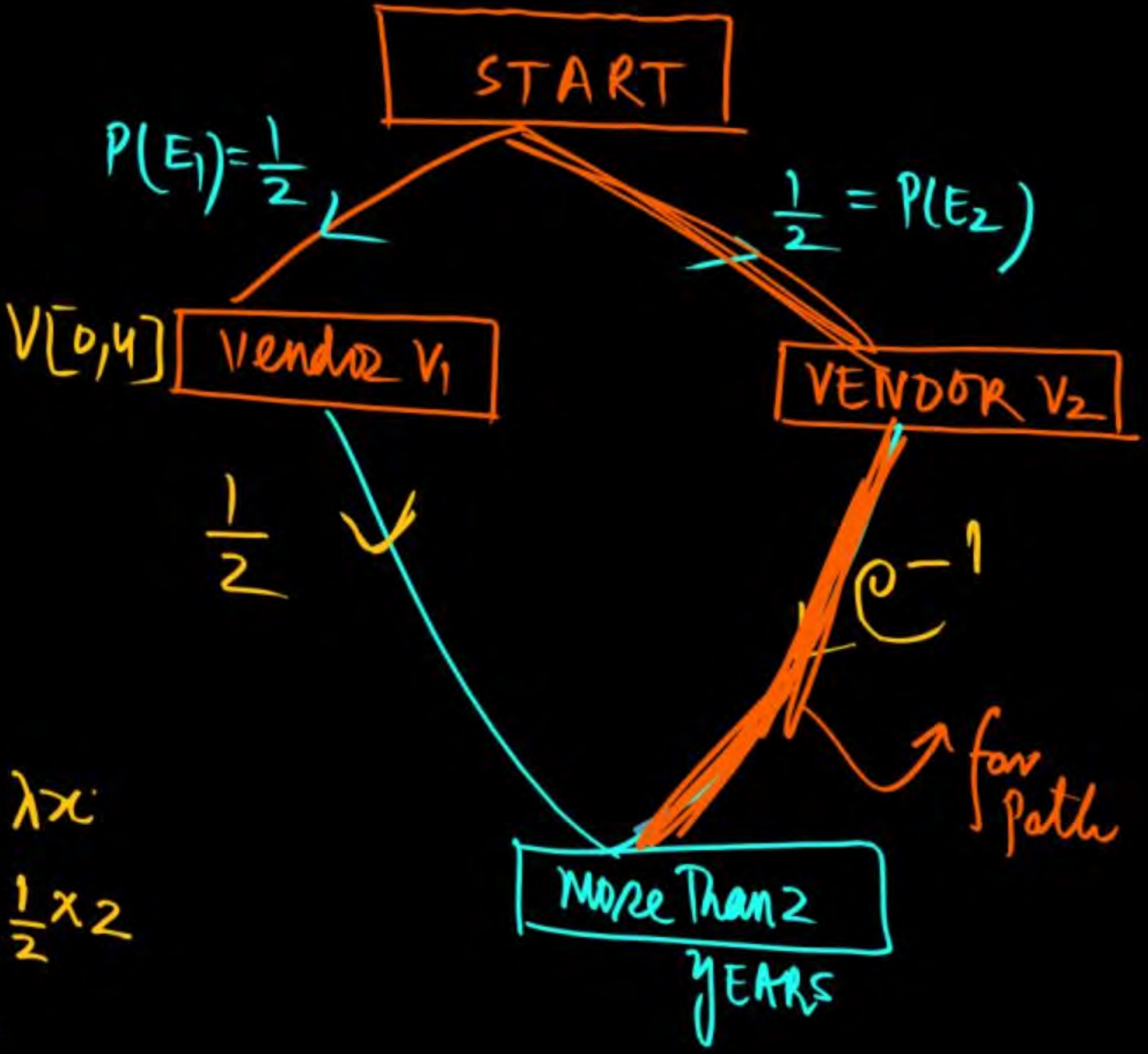
B. $\frac{1}{1+e}$

C. $\frac{1}{1+e^{-1}}$

D. $\frac{2}{2+e^{-1}}$

$$\Rightarrow \frac{e^{-1}\times\frac{1}{2}}{\frac{1}{2}\times e^{-1}+\frac{1}{2}\times\frac{1}{2}}$$

$$= \frac{2}{2+e} \quad \text{Ans}$$



SERVIVAL

Vendor V_2

$$P(X\geq 2) = e^{-\lambda x}$$

$$= e^{-\frac{1}{2}\times 2}$$

$$= e^{-1}$$

$V_1 \rightarrow [0,4]$

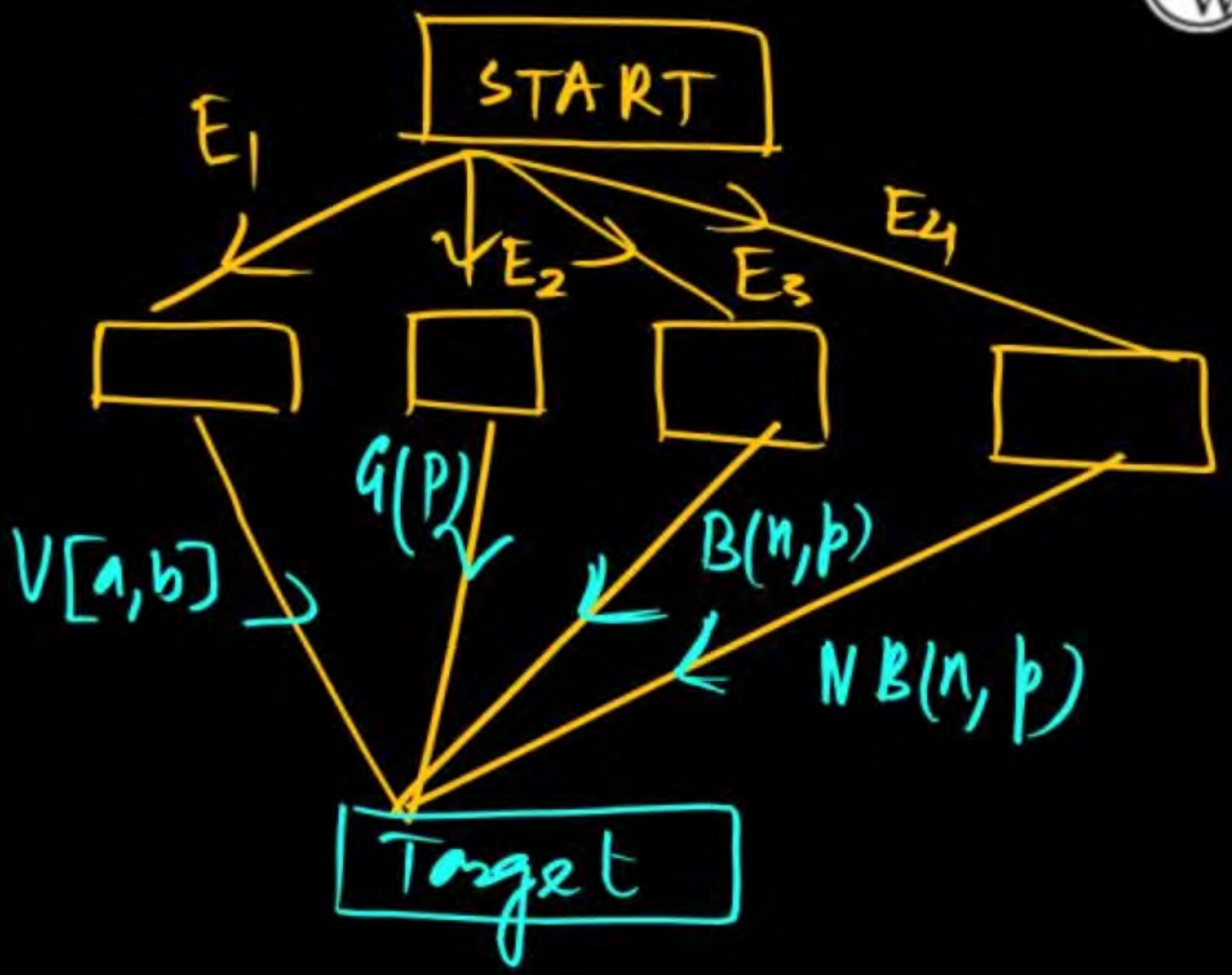
$V_2 \rightarrow \exp\left(\frac{1}{2}\right)$

$$f(x) = \frac{1}{4-0}$$

$$f(x) = \frac{1}{4}$$

$$P(X\geq 2) = \int_2^4 \frac{1}{4}\,dx$$

$$= \frac{1}{4}[4-2] = \frac{1}{2}$$



START
E1
E2
E3
E4
V[a,b]
G(p)
B(n,p)
NB(n,p)
Target

✓ Ans: $\frac{1}{4e^{3/2}+1}$

Q5. Let X_1 and X_2 denotes the lifetimes (in months) of bulbs produced at factories F_1 and F_2, respectively. The random variable X_1 and X_2 are Exp(1/8) and Exp(1/2) respectively. A shop procures 80% of its supply of bulbs from factory F_1 and 20% from factory F_2. A randomly selected bulb from the shop is put on test and is found to be working after 4 months. What is the probability that it was procured from factory F_2?

(B): Let X be a Gamma $(4, \lambda)$, $\lambda > 0$, random variable, Find the value of λ that minimizes E(Y), where $Y = X + \frac{3}{4X}$

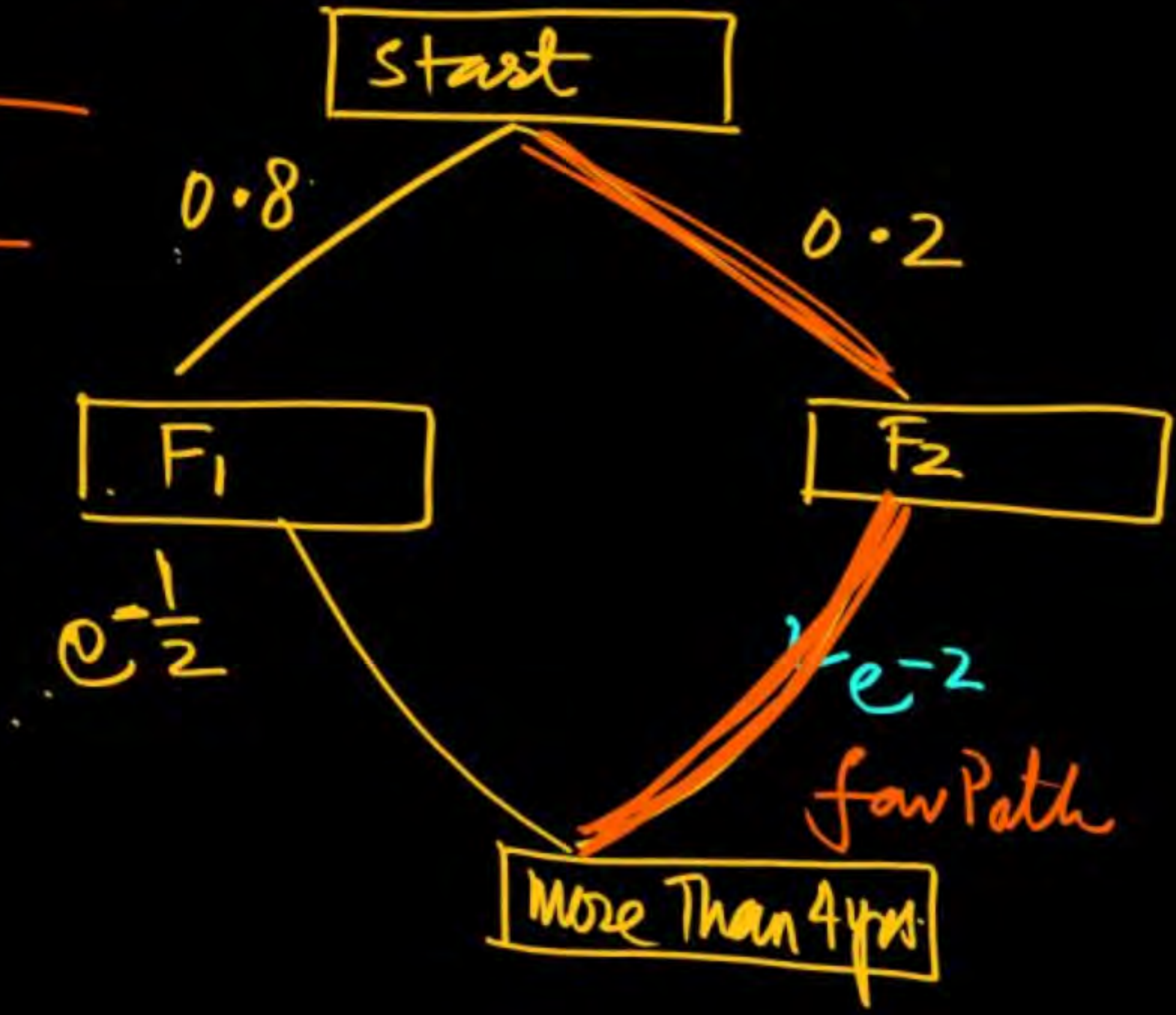
Random variable x_1, x_2

$x_1 \rightarrow \exp\left(\frac{1}{8}\right)$, $x_2 \rightarrow \exp\left(\frac{1}{2}\right)$

$$= \frac{0.2 \times e^{-2}}{0.2e^{-2} + 0.8e^{-1/2}}$$

$$= \frac{1}{4e^{3/2} + 1}$$

Ans



$X_1 \rightarrow F_1$ factory

$\rightarrow \exp\left(\frac{1}{8}\right)$

$P(X>4) = P(\text{at least } 4)$

$= e^{-\lambda x}$

$= e^{-\lambda \cdot 4}$

$= e^{-\frac{1}{8} \times 4} = e^{-\frac{1}{2}}$

$P(X>4) = e^{-\frac{1}{2} \times 4}$

$= e^{-2}$

$\checkmark$ $\boxed{15}$

$\lambda = \frac{1}{20}$

Q8. Suppose that accidents occur in a factory at a rate of $\lambda = \frac{1}{20}$ per working day. Suppose in the factory six days (from Monday to Saturday) are working. Suppose we begin observing the occurrence of accidents at the starting of work on Monday. Let X be the number of days until the first accident occurs. Find the probability that

(i) First week is accident free $= e^{-0.25}$

$e^{-\frac{7}{20}} - e^{-\frac{9}{20}}$ Ans

(ii) First accident occurs any time from starting of working day on Tuesday in second week till end of working day on Wednesday in the same week. $= 0.067$

$e^{-\frac{7}{20}} - e^{-9/20}$ $\quad P(7 \le X \le 9) = P(X \le 9) - P(X \le 7)$

$1-[P(X\le 5)]$

P(First week No Accident occur]

$P(X>6) = \cancel{1}-(\cancel{1}-e^{-\lambda x})$

$= e^{-\lambda x} = e^{-\frac{1}{20}\times 5} = e^{-\frac{1}{4}}$

1st Week
[Mon
Tue
Wed
Thurs
Fri
SAT]

1 Week 6 days.

$\lambda = \frac{1}{20}$

$P(X>6) = P(X=7)+P(8)+P(9)+P(10)+ - -$

IInd week
{ Mon
Tue.
Wed
Thurs.
fri
SAT

$P(X>6) = 1-P(X\le 5)$

$= \cancel{1}-(\cancel{1}-e^{-\lambda x})$

$= e^{-\lambda x} = e^{-\frac{1}{20}\times 5} = e^{-\frac{1}{4}} = e^{-0.25} = \underline{0.7738}$

$$P(X \geq 6) = P(X=6) + P(X=7) + P(X=8) + P(X=9) + \cdots$$

$$P(X > 6) = 1 - P(X \leq 5)$$

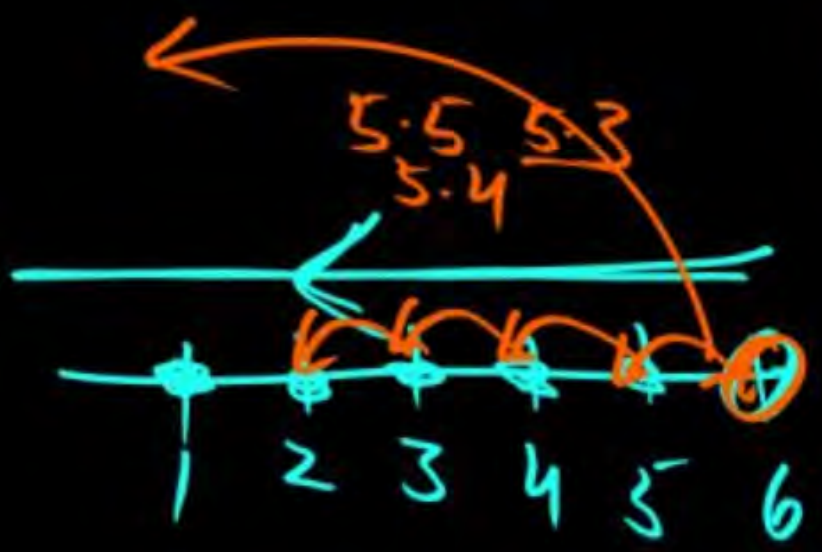
$$= 1 - (1 - e^{-\lambda x})$$

$$= e^{-\lambda x} = e^{-\frac{1}{20} \times 5} = e^{-\frac{1}{4}}$$ **Ans**

$P(X > 6$

$\rightarrow P(X > 6)$

$= 1 - P(X \leq 5)$



Q9. Telephone calls arrive at a switchboard following an exponential distribution with parameter $\lambda = 12$ per hour. If we are at the switchboard, what is the probability that the waiting time for a call is

(i) At least 15 minutes $= e^{-3}$

(ii) Not more than 10 minutes. $= 1 - e^{-2}$

$$P\left(X \geq \frac{15}{60}\right) = P\left(X \geq \frac{1}{4}\right) = e^{-12 \times \frac{1}{4}} = e^{-3}$$

$$P\left(X \leq \frac{10}{60}\right) = P\left(X < \frac{1}{6}\right) = 1 - e^{-12 \times \frac{1}{6}} = 1 - e^{-2}$$

$f(x) = Ae^{-x}$
$0 \le x < \infty$
MEAN
Variance

Q10. Show that for the exponential distribution:

$f(x) = Ae^{-x}$, $0 \le x < \infty$, mean and variance are equal.

If this is a valid pdf $= \int_0^\infty Ae^{-x}dx = 1$

$\boxed{A = 1}$

$f(x) = 1e^{-x}$

$\rightarrow$ compare with the original function $f(x) = \lambda e^{-\lambda x}$

mean $= \frac{1}{\lambda}$ $\quad$ var$(x) = \frac{1}{\lambda^2}$

mean $= \frac{1}{1} = 1$ $\quad$ $v(x) = 1$

S.D $= 1$

Uniform Distribution (Discrete)

(A) all Prob sum equivalent 1

(B) Random vari are evenly spaced

Throwing A Die

X = No. of dots

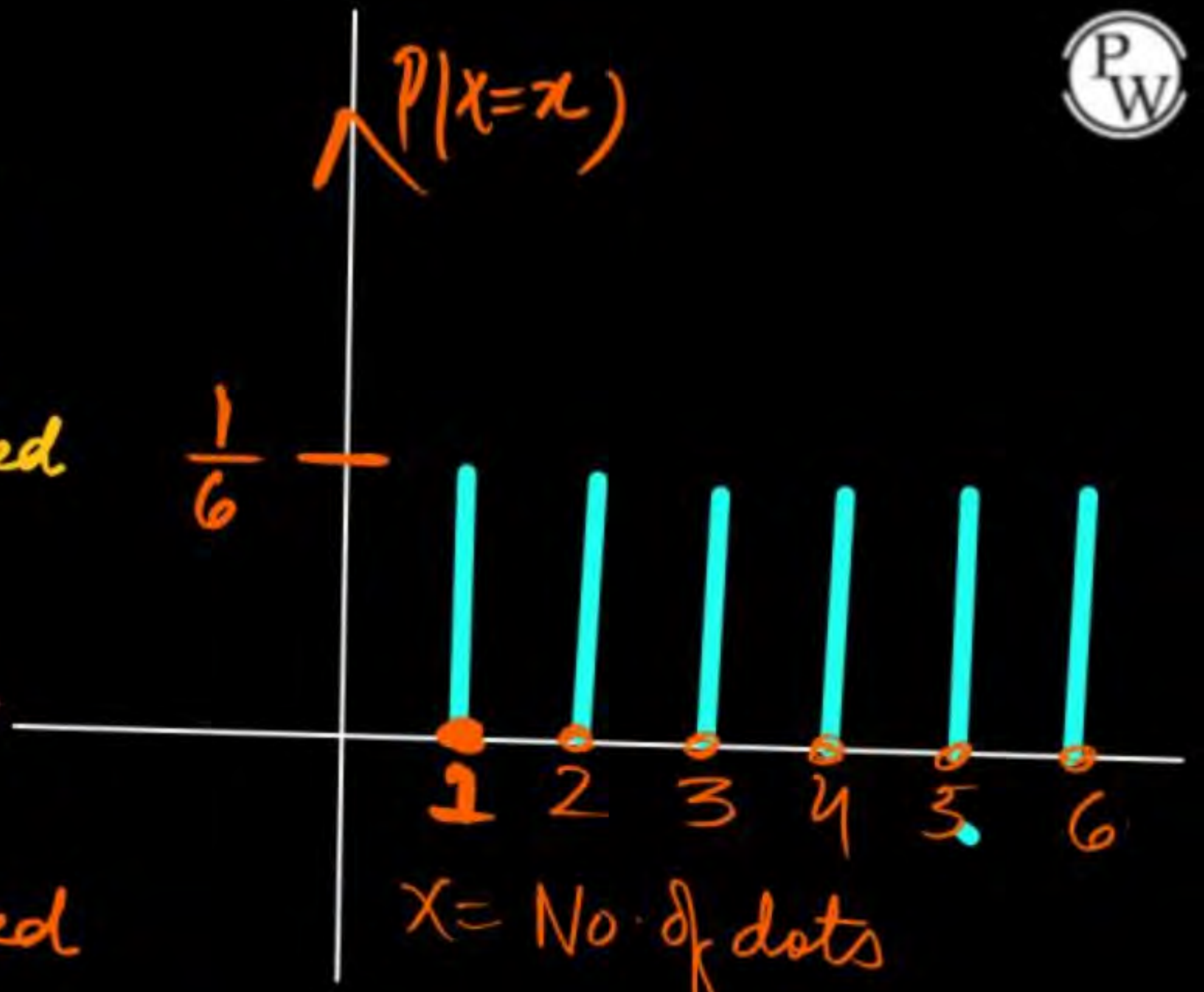
X = 1, 2, 3, 4, 5, 6

X	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Discrete Uniform

A) $\sum P = 1$

B) evenly spaced



(2) Tossing A Single coin

X = 0, 1 → evenly spaced

$\sum P = 1$

$P[X=0] = \frac{1}{2}$

$P(X=1] = \frac{1}{2}$

$P[X=x] = \frac{1}{3}$

$X = \underbrace{0, 9}, \underbrace{9, 18}, \underbrace{18, 27}, \underbrace{27, 36} \ldots$

~~X~~ X Uniform Discrete Distribution

Prob. Mass function

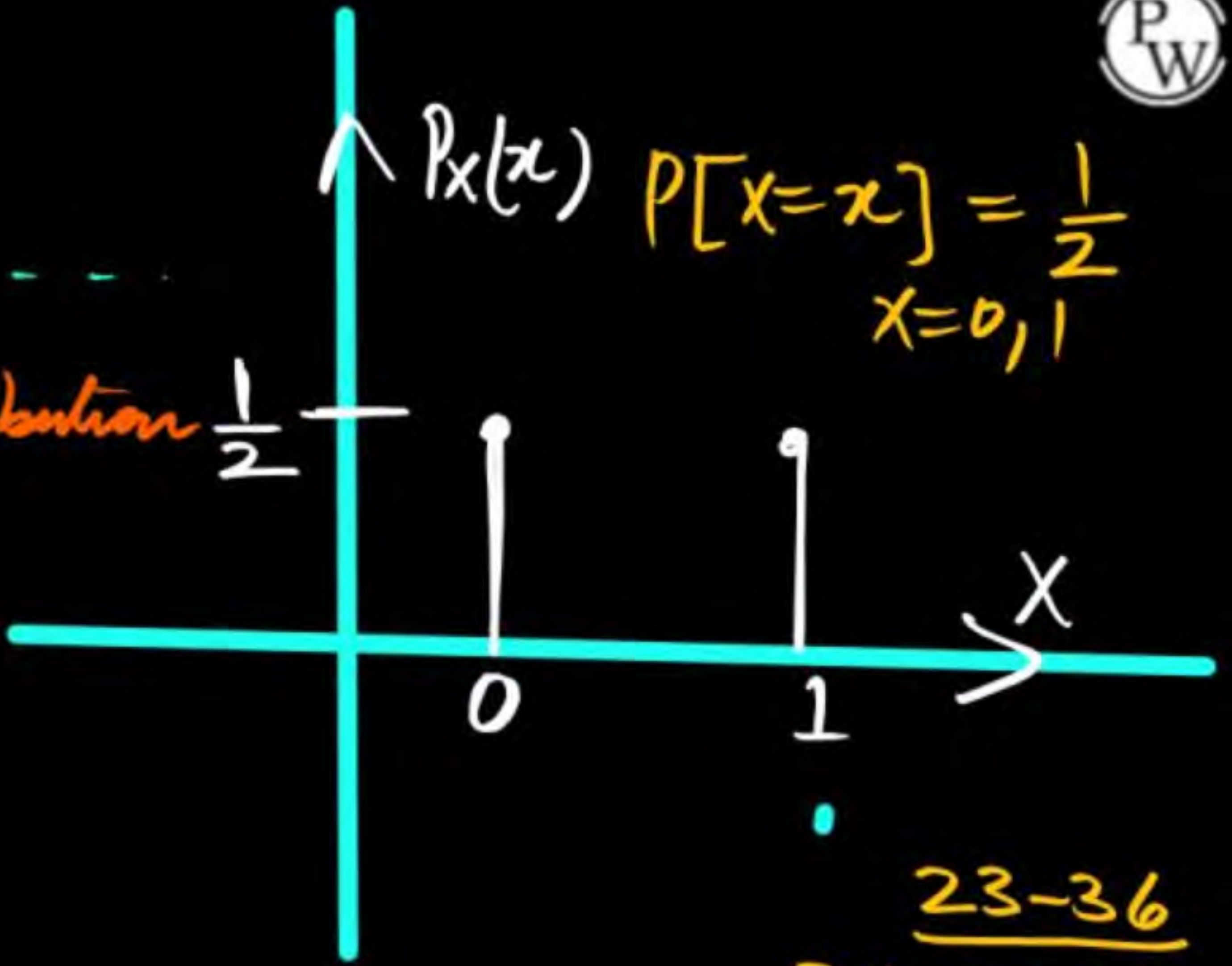
$$PMF = \frac{1}{\text{possible No. of outcomes}}$$

$$\text{Prob. Mass function} = \frac{1}{(b-a+1)}$$

$$\boxed{PMF = \frac{1}{(b-a+1)}}$$

$P[X=x] = \frac{1}{2}$

$x = 0, 1$



23–36

36 − 23 + 1

1, 2, 3, 4, 5, 6 =

Initial No — final No

= final − initial + 1

EXPECTED value of X

$$\boxed{\mu = E[X] = \frac{(a+b)}{2}}$$

$$Var(X) = E[X^2] - [E[X]]^2$$

$$\boxed{V(X) = \frac{(b-a+1)^2 - 1}{12}}$$

$$\boxed{\text{Standard deviation} = \sqrt{\frac{(b-a+1)^2 - 1}{12}}}$$

Moment generating function $\pi_X(s) = \frac{1}{(b-a+1)}\left[\frac{e^{as} - e^{(b+1)s}}{1-e^s}\right]$

$$PMF = \frac{1}{(b-a+1)}$$

$a, a+1, a+2, \ldots, b-2, b-1, b$

Total Possible $= b-a+1$

Average $= \frac{\text{Initial} + \text{final}}{2}$

Discrete

Q1. A discrete random variable X follows uniform distribution over the values 7, 12, 15, 20, 23, 25, 30. Then $P(X > 18 \mid X < 26)$ equals

A. $\frac{1}{2}$

B. $\frac{4}{7}$

C. $\frac{2}{3}$

D. $\frac{3}{4}$

$$P\left[\frac{X>18}{X<26}\right] = \frac{P(X>18 \wedge X<26)}{P(X<26)}$$

$$= \frac{P(X=20)+P(X=23)+P(X=25)}{P(X=7)+P(X=12)+P(X=15)+P(X=20)+P(X=23)+P(X=25)}$$

$$= \frac{\frac{1}{7}+\frac{1}{7}+\frac{1}{7}}{\frac{6}{7}} = \frac{\cancel{3/7}}{6/\cancel{7}} = \frac{3}{6}$$

0	1	0	1	0	1
↑	↑	↑	↑	↑	↑
7	12	15	20	23,	25

30

↓

Q10. Suppose X is uniformly distributed on $\{-3, -2, -1, 0, 1, 2, 3\}$. Then $P(X^2 = 9)$ is

A. $\frac{1}{7}$

B. $\frac{2}{7}$

C. $\frac{3}{7}$

D. $\frac{2}{5}$

$\frac{1}{7}$ $\frac{1}{7}$ $\frac{1}{7}$ $\frac{1}{7}$ $\frac{1}{7}$

on $[-3, -2, -1, 0, 1, 2, 3]$

$$P[X^2 = 9] = P[X = \pm 3]$$
$$= P[X = 3] + P[X = -3]$$
$$= \frac{1}{7} + \frac{1}{7}$$
$$= \frac{2}{7}$$

Probability Distribution

Q11. Suppose x is uniformly distributed on $\{-\theta, -\theta + 1 \ldots\ldots 0 \ldots\ldots \theta - 1, \theta\}$. Then V(X) is

A. $\frac{(2\theta)^2}{12}$

B. $\frac{\theta^2}{12}$

C. $\frac{(2\theta+1)^2-1}{12}$

D. $\frac{(2\theta+1)^2}{12}$

$\underset{a}{-\theta}, -\theta+1, \ldots 0 \ldots \theta-1, \underset{b}{\theta}$

$V(x) =$

$$V(x) = \frac{(b-a+1)^2-1}{12}$$

$$= \frac{(\theta+\theta+1)^2-1}{12}$$

$$= \frac{(2\theta+1)^2-1}{12} \checkmark$$

Probability Distribution

$\frac{1}{n}$ $\frac{n+1}{2}$ $\frac{n^2-1}{12}$

Q12. X has a discrete uniform distribution on the integers 0, 1, 2, ... n and Y has a discrete uniform distribution on the integers 1, 2, 3, ... , n

Find Var[X] - Var[Y].

A. $\frac{2n+1}{12}$

B. $\frac{1}{12}$

C. 0

D. $-\frac{1}{12}$

$0, 1, 2 \ldots n$ X

$1, 2, 3 \ldots n$ Y

$V(X)$ $V(Y)$

$var(X) - var(Y) =$

$$\boxed{V(x) = \frac{(b-a+1)^2 - 1}{12}}$$

$$V(X) = \frac{(n-0+1)^2-1}{12} = \frac{(n+1)^2-1}{12}$$

$$V(y) = \frac{(n-\cancel{1+1})^2-1}{12} = \frac{n^2-1}{12}$$

$$var(x) - var(y) = \frac{(n+1)^2-1}{12} - \frac{(n^2-1)}{12} = \frac{2n+1}{12} \text{ Ans}$$

Revision one time

Distribution

✓ GAMMA Dis

✓ Problem - continuous

+ Hyper geometric

THANK - YOU