Data Science and Artificial Intelligence

Probability and Statistics

Continuous Probability Distribution

Lecture No.- 07

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Topics to be Covered

$\boxed{\text{Hypergeometric + Beta}}$

Moment gen. function

Topic Problem based on continuous random variable

Prob. Distribution

DPP – 45 questions – upload

DPP → bivariate Distribution } Keep Patience

Prob Dist → DPP 45 questions

→ time Bound Practice

standard Normal $N(\mu, \sigma^2) = N(0, a)$ $X[N(\mu, \sigma^2)]$ $X \sim N(0, a > 0)$

1. Let X be a normal random variable with mean 0 and variance a > 0, Calculate $P(X^2 < a)$.

A. 0.34

B. 0.42

C. 0.68

D. 0.84

$x^2 < a$

$-\sqrt{a} < x < \sqrt{a}$

$= x^2 - a < a - a$

$= x^2 - a < 0$

$= (x - \sqrt{a})(x + \sqrt{a}) < 0$

inequality

$P[X^2 < a] \Rightarrow P[-\sqrt{a} < X < \sqrt{a}]$

$\Rightarrow P\left[\frac{-\sqrt{a} - \mu}{\sqrt{a}} < \frac{X - \mu}{\sigma} < \frac{\sqrt{a} - \mu}{\sqrt{a}}\right]$

$\Rightarrow P\left[\frac{-\sqrt{a} - 0}{\sqrt{a}} < Z < \frac{\sqrt{a} - 0}{\sqrt{a}}\right]$

$= P(-1 < Z < 1)$

$= 0.68$ Ans

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{f(x)}{P(X \geq 0)}$$

2. Let X be a continuous random variable with density function.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \; for -\infty < x < \infty$$

Calculate E[X|X ≥ 0]

$$P\left[\frac{X}{X \geq 0}\right] = \frac{f(x)}{P(X \geq 0)}$$

$N(\mu, \sigma^2)$
Normal

A. 0

B. $\sqrt{\frac{2}{\pi}}$ ✓

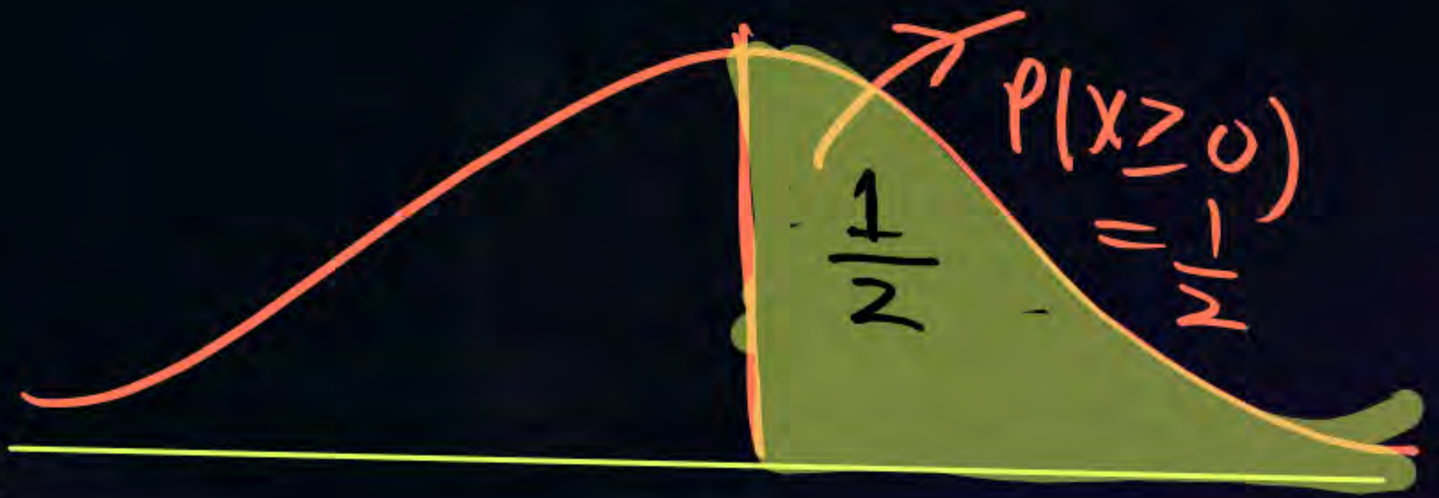
C. $\frac{1}{\sqrt{2\pi}}$

D. 1/2

$$E\left[\frac{X}{X \geq 0}\right]$$

$$P\left[\frac{X}{X \geq 0}\right] = \frac{f(x)}{\frac{1}{2}} = 2f(x)$$

cos x even cos x



$$E\left[\frac{X}{X \geq 0}\right]$$

$$= E\left[\frac{X}{X \geq 0}\right]$$

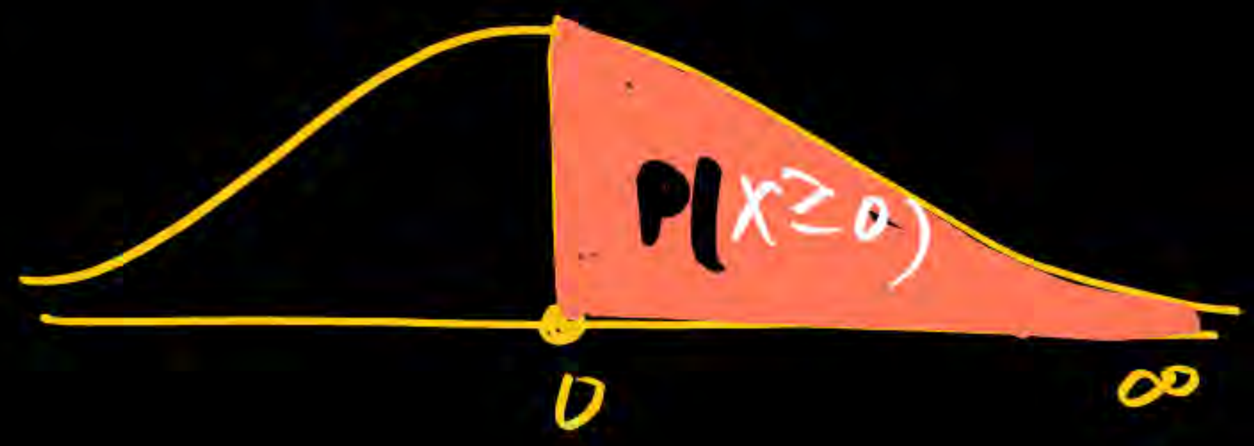
$$= \frac{\int_0^{\infty} x f(x)\, dx}{P(X \geq 0)}$$

$$= 2\int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

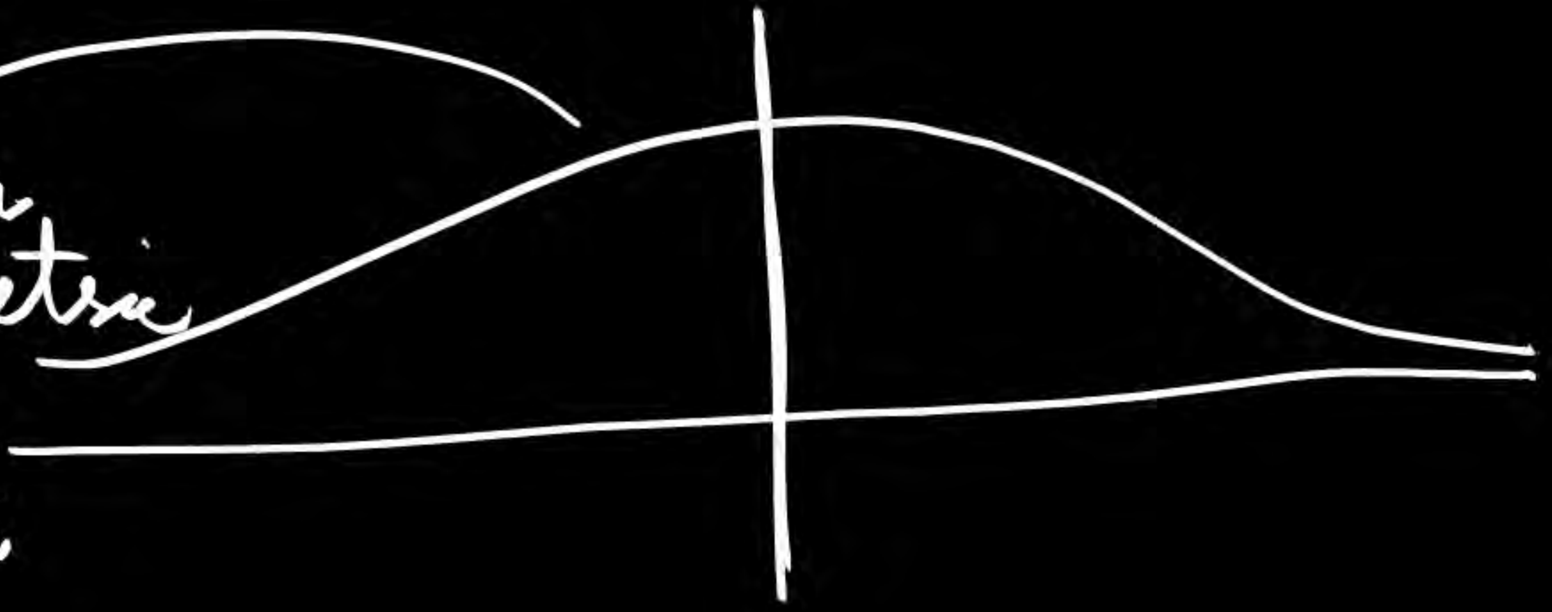
$$= 2\int_{0}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx$$

$$= \sqrt{\frac{2}{\pi}}$$ **Ans**

$P(X \geq 0)$



even symmetric about origin



$P[x^2 < a] = -\sqrt{a} \le x \le \sqrt{a}$ $N(\mu, \sigma^2)$ $\mu = 1$, $\sigma^2 = 4$

4. If X has a normal distribution with mean 1 and variance 4, then $P(X^2 - 2X \le 8) = ?$

A. 0.13

B. 0.43

C. 0.75

D. 0.86

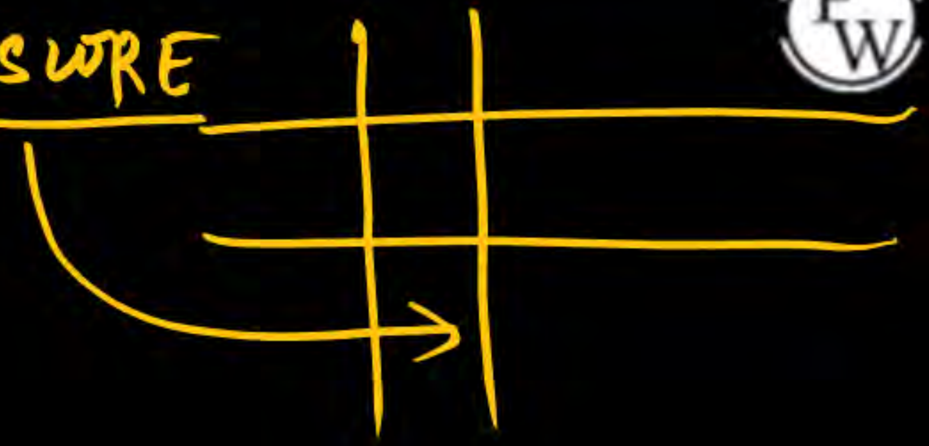
$P[\underbrace{(x^2 - 2x) \le 8}]$

$= P[(x^2 - 2x + 1) \le 8 + 1]$

$= P[(x-1)^2 \le 9]$

$= P[-3 \le x - 1 \le 3]$

Z SCORE



$$= P[-3 \le X \le 3]$$

$$= P[-2 \le X \le 4]$$

$$= P\left[\frac{-2-\mu}{\sigma} \le \frac{X-\mu}{\sigma} \le \frac{4-\mu}{\sigma}\right]$$

$$= P\left[\frac{-2-1}{2} \le Z \le \frac{4-1}{2}\right]$$

$$= P\left[-\frac{3}{2} \le Z \le \frac{3}{2}\right]$$

$$= P[-1.5 \le Z \le 1.5] = \underline{0.86}$$

Use Z-table

6. Let X follows normal distribution with mean 5 and variance 1.5. Then the mean of $Y = 2X^2 + 3$ is

A. 16

B. 5

C. 57.6

D. 56

Mean $= \mu$

var $= 1.5$

$Y = (2X^2 + 3)$

$E[Y] = 2E[X^2] + 3$

$= 2[\text{var}(X) + (E[X])^2] + 3$

$= 2[1.5 + (5)^2] + 3$

$= 56$

$\text{var}(X) = E[X^2] - (E[X])^2$

$\text{var}(X) + (E[X])^2 = E[X^2]$

8. $f(x) = \begin{cases} \theta e^{-x\theta} & x > 0 \\ 0 & otherwise \end{cases}$

Find the n-th moment of X, where n is a non – negative integer (assuming that θ >0).

$f(x) = \theta . e^{-x\theta} \quad x > 0$

$\exp(\theta)$

$\left\{ \frac{n!}{\theta^n} = E[X^n] \right.$

n^{th} moment

$E[X^n] = ✓$

$$E[X^n] = \int_0^\infty x^n \cdot f(x)\, dx$$

$$= \int_0^\infty \underbrace{x^n \cdot \theta e^{-x\theta}\, d\theta}_{\text{Gamma function}} = \frac{(n-1)!}{\theta^n} \text{ or } \frac{\overline{|n}}{\theta^n}$$

$$E[X^n] = \frac{d^n}{ds^n}\left[\Pi_X(s)\right]_{s=0}$$

$$f(x) = \begin{cases} 5e^{-5x} & x > 0 \\ 0 & otherwise \end{cases}$$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & otherwise \end{cases}$$

9. $f(x) = \begin{cases} 5e^{-5x} & x > 0 \\ 0 & otherwise \end{cases}$

Find the moment generating function of X and use it to find the first and second moments of X, and variance of X.

$\exp(\lambda)$

Moment generating $\Pi_X(s) = \frac{\lambda}{(\lambda - s)} = \frac{5}{(5-s)} = \Pi_X(s)$

Expect = First moment $E[X] = \frac{d}{ds}\left[\frac{5}{(5-s)}\right]_{s=0} = \frac{1}{5}$ Ans

mean

Average

$V(X) = \frac{2}{25} - \frac{1}{25} = \frac{1}{25}$

SECOND moment $E[X^2] = \frac{d^2}{ds^2}\left[\frac{5}{5-s}\right]_{s=0} = \frac{2}{25}$

10. Let X be a continuous random variable with density function

$M_X(t)$ = moment generating

$$f(x) = \begin{cases} be^{-bx} & x > 0 \\ 0 & otherwise \end{cases}$$

$f(x) = \begin{cases} be^{-bx} & x > 0 \\ 0 & \text{otherwise} \end{cases}$

exp(b)

Where b > 0. If $\boxed{M(t)}$ is the moment generating function of X, then what is $M_x(-6b)$?

$$\left[M_X(t) = \atop -6b \right] \Rightarrow$$

$$M_X(t) = \frac{\lambda}{\lambda - t}$$

$$M_X(t) = \frac{b}{b-t}$$

$$M_X(t) = \frac{b}{b + 6b} = \frac{\cancel{b}}{7\cancel{b}} = \frac{1}{7}$$

$\sum_{x=0}^{\infty} s^x p(x)$ $e^{sx} f(x)$ Dis ✓

11. The moment generating function for a random variable X is:

$$M_X(t) = \frac{1}{8} + \frac{1}{4}e^t + \frac{5}{8}e^{2t}$$

Calculate P [X ≥ 1].

$$M_X(t) = \frac{1}{8} + \frac{1}{4}e^t + \frac{5}{8}e^{2t}$$

$$= P(X=0) + \underline{P(X=1)}e^t + P(X=2)e^{2t}$$

$$P(X \geq 1) = P(X=1) + P(X=2)$$

$$= \frac{1}{4} + \frac{5}{8} = \frac{2}{8} + \frac{5}{8} = \frac{7}{8}$$

A. 1/8

B. 3/8

C. 5/8

D. 7/8

$f(x) = (k+1)x^2$

$0 < x < 1$

$\int_0^1 (k+1)x^2 = 1$

$k = \checkmark$

12. If $f(x) = (k + 1)x^2$ for $0 < x < 1$, find the moment generating function of X.

A. $\frac{e^t(6+6t+3t^2)}{t^3}$

B. $\frac{e^t(6-6t+3t^2)}{t^3}$

C. $\boxed{\frac{e^t(6+6t+3t^2)}{t^3} - \frac{6}{t^3}}$

D. $\frac{e^t(6-6t+3t^2)}{t^3} - \frac{6}{t^3}$

Moment generating function

$k=2$

$= \int_0^1 e^{sx} \cdot f(x)\, dx$

$s=t$ $= \int_0^1 e^{tx} \cdot 3x^2\, dx$

Moment generating function $= \int_0^1 e^{tx} \cdot 3x^2 \, dx$

short cut

$I = \int e^{tx} \cdot 3x^2 \, dx$

one function ↙ algebraic

Integration Parts

D(algebraic)

D	I
$3x^2$	e^{tx}
$6x$	$\frac{e^{tx}}{t}$
6	$\frac{e^{tx}}{t^2}$
0	$\frac{e^{tx}}{t^3}$

becomes zero

$$= \left[3x^2 \frac{e^{tx}}{t} - 6x \frac{e^{tx}}{t^2} + \frac{6e^{tx}}{t^3}\right]_0^1$$

$$= e^t \left[\frac{6+6t+3t^2}{t^3}\right] - \frac{6}{t^3}$$

$\int x^2 \sin x$

$=$

x^2	$+\sin x$
$2x$	$-\cos x$
2	$-\sin x$
0	$\cos x$

13. If the moment generating function of the random variable X is $M_X(t) = \frac{1}{1+t}$, Find $\boxed{\text{the third moment of X about the point X = 2.}}$

A. 1/3

B. 2/3

C. 3/2

D. -38

$E[X^3]$ = Third moment

Third moment about the Point $x=2$

$\underbrace{E[[X-2]^3]}_{\text{about The Pt 2}} = -38$

about The Point a

$E[(X-a)^n]$

$a^3 - b^3 - 3ab(a-b)$

$M_X(t) = \frac{1}{(t+1)}$

$E[X^3] = \frac{d^3}{dt^3}\left[\frac{1}{t+1}\right]$

$E[X^3] = ?$

$E[X^2] = ?$

$E[X] = ?$

$$\left.\begin{aligned} E[X] &= -1 \\ E[X^2] &= +2 \\ E[X^3] &= 6 \end{aligned}\right\}$$

$$M_X(t) = \frac{1}{(1+t)}$$

$$M_X'(t) = \frac{-1}{(1+t)^2} = -1$$

$$M_X''(t) = \frac{-2\times -1}{(1+t)^3} = +2$$

$$M_X'''(t) = \frac{-6}{(1+t)^4} = -6$$

$$E[(X-2)^3] = E\left[X^3 - 8 - 6X(X-2)\right]$$

$$= E\left[X^3 - 8 - 6X^2 + 12X\right]$$

$$= E[X^3] - 8 - 6E[X^2] + 12E[X]$$

$$= -6 - 8 - 6\times 2 + 12\times -1$$

$$= -6 - 8 - 12 - 12$$

$$= \underline{-38}$$

14. Let X be a discrete random variable with the moment generative function

$\Pi_X(t)$ $\boxed{M_X(t) = e^{0.5(e^t-1)}}$, $t \in \mathbb{R}$

→ Poisson Distribution $\boxed{\lambda = 0.5}$

$M_X(t) = e^{0.5(e^t-1)}$

$P(X \le 1) =$ ✓

Then P (X ≤ 1)equals

$P(X \le 1) = \underline{P(X=0)} + \underline{P(X=1)}$

$= e^{-\lambda} + \lambda e^{-\lambda}$

$= e^{-0.5} + 0.5e^{-0.5}$

$= \frac{3}{2}e^{-0.5}$ Ans

A. $e^{-\frac{1}{2}}$

B. $\frac{3}{2}e^{-\frac{1}{2}}$

C. $\frac{1}{2}e^{-\frac{1}{2}}$

D. $e^{-(e-1)/2}$

$(q+ps)^n \quad q=\frac{5}{6} \quad p=\frac{1}{6} \quad n=3$

15. Let X be a random variable with the moment generating function

$$\underbrace{M_X(t)=\frac{1}{216}(5+e^t)^3}_{\text{Binomial}}, t \in \mathbb{R}$$

$M_X(t)=\left(\frac{5}{6}+\frac{1}{6}e^t\right)^3$

$t \in R$

Then P (X > 1) equals

A. 2/27

B. 1/27

C. 1/12

D. 2/9

$P(X \geq 1) = 1 - P(X=0) - P(X=1)$

$= 1 - \left[{}^3C_0\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{3-0} - {}^3C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{3-1}\right]$

$= 1 - 3\times\frac{5^3}{6^3} - 3\times\frac{1}{6}\times\frac{5^2}{6^3} = \frac{2}{27}$

THANK - YOU