Math 504, 11/04

Now let V be a (finite-dimensional) G-module with submodule W. We now make a fundamental assumption on our basefield K, namely that it has characteristic 0, and show that W necessarily has a complement in V. To do this, let W' be any vector space complement of W in V and let $\pi : V \to W$ be the linear map that is the identity on W and 0 on W'. Replacing π by $\pi' = (1/n)\sum_g g\pi g^{-1}$, where n is the order of G, we see that $g\pi' g^{-1} = \pi'$ for all $g \in G$, so that π' is a G-homomorphism, which is still the identity on W. Its kernel will then be the G-module complement U to W that we are looking for. Iterating this result we see that *every G-module over a field of characteristic 0, or more generally of characteristic not dividing the order of G, is a direct sum of irreducible modules*; another way to express this result is to say that every G-module is *completely reducible* or *semisimple*. This result, known as *Maschke's Theorem*, fails over every field K whose characteristic p does divide the order of G. To see this, note that G permutes the basis elements g of KG, we see that the sum $\sum k_g$ of the coefficients k_g of any element of KG is preserved by G, whence the subspace $S = \{v = \sum k_g g \in KG : \sum k_g = 0\}$ is a G-submodule such that G acts trivially on KG/S; but one easily checks that the only elements $\sum k_g g$ in G on which G acts trivially have $k_g = k_h$ for all $g, h \in G$. Thus if the characteristic of K does not divide the order of G, the K-subspace spanned by $\sum g$ is a G-stable complement of S in KG, but if the characteristic of K does divide the order of G, then S has no G-stable complement in KG.

Returning to the case where the characteristic of K does not divide the order of G, we now know that the group ring KG is semisimple as a left module over itself, whence all results from the last two problems of this week's homework apply to it: KG is the direct sum of finitely many minimal two-sided ideals I and each I is the direct sum of finitely many minimal left ideals L_i, any two them isomorphic (say to L) as left I-modules. Now bring Schur's Lemma into the picture: the ring I' of I-homomorphisms from any L_i to itself is a division ring D that is independent of i. Regard D as acting on L_i on the *right*, so that the product xy of two such homomorphisms x, y is taken to be the composition of x and y in that order. The ring R of I-homomorphisms from all of I to itself as a left module is then the ring $M_n(D)$ of $n \times n$ matrices over D, where n is the number of minimal left ideals L_i. This is because any I-homomorphism π is completely determined by the projection $p_j(\pi(L_i)$ of the image $\pi(L_i)$ to L_j and this projection defines an isomorphism from L to itself, which is given by an element of D. But any left module homomorphism from I to itself is given by *right* multiplication by an element of I (this is true of any ring). Hence the ring I' of all such homomorphisms is isomorphic to I as an abelian group, but the product xy of two elements of it equals the product yx in I. We deduce that *each I is isomorphic to the ring $M_n(D')$ of $n \times n$ matrices over D', the division ring obtained from D by replacing the product xy of two elements of it by the product yx in D. Up to isomorphism, the only simple left I-module is D^n, the space of column vectors of length n over D.* This property holds of any ring R such that every left R-module is projective; in that generality the above result is called the *Artin-Wedderburn Theorem*. In our current setting, we see that KG must be a finite direct sum of such matrix rings $M_{n_i}(D_i)$, where in addition each division ring D_i is finite-dimensional over K with K in its center. If now we further assume that K is algebraically closed, the only division ring D_i with K in its center that is finite-dimensional over K is K itself, since any element x in such a ring is algebraic over K, whence it must lie in K. We conclude that *in particular, the complex group algebra* $\mathbf{C}G$ *of any finite group* G *is a direct sum of matrix rings over* $\mathbf{C}$. A left module over such ring is just the column vectors over $\mathbf{C}$ of the same size as one of its matrix ring factors, withe the other matrix factors acting by 0. Thus G *has only finitely many inequivalent irreducible representations and the sum of the squares of their dimensions equals the order of* G (since the only irreducible module over $M_n(\mathbf{C})$ is $\mathbf{C}^n$ and the dimension of $M_n(\mathbf{C})$ over $\mathbf{C}$ is n^2). We will determine the number of inequivalent representations of G later.

For now let's look at another example. Consider again the group G of quaternion units. We know from last time that G has a two-dimensional representation over $\mathbf{C}$, namely the ring H of quaternions, which one easily checks is irreducible. Besides this G has four inequivalent 1-dimensional representations, each trivially itrreducible; in all of them $-1 \in G$ acts trivially, while each of i and j acts by 1 or -1 independently. As the order 8 of G is the sum $4+1+1+1+1$ of the squares of the dimensions of the irreducible representations found so far, we have found all of them. By way of an interesting contrast, look at the group algebra $\mathbf{R}G$ of this same group over the field $\mathbf{R}$ of real numbers. This time H remains irreducible, but is now four-dimensional over $\mathbf{R}$, while the remaining irreducible representations found above of course remain irreducible. Thus $\mathbf{R}G$, instead of being a sum of matrix rings over $\mathbf{R}$ or $\mathbf{C}$, is sum of four 1×1 matrix rings over $\mathbf{R}$ plus another such ring over H (it turns out that H is itself a division ring). This division ring can arise in real group algebras, but never in complex ones.