Math 504, 10/24

Last time we explored the functor $\mathrm{Hom}_R(M,-)$ sending any R-module N to the set of R-linear maps from M to N. We saw that this functor preserves short exact sequences (and in fact exact sequences generally) if and only if M is a projective R-module; in turn this holds if and only if M is a direct summand of a free R-module. We now look at the functor $\mathrm{Hom}_R(-,M)$ obtained by fixing M as the range rather than the domain. The first thing to note is that an R-module map $f : N \to P$ now gives rise to map $\mathrm{Hom}_R(P,M) \to \mathrm{Hom}_R(N,M)$ in the *other direction*, from P to N, by composition with f; accordingly we say that the functor $\mathrm{Hom}_R(-,M)$ is *contravariant*. Given a short exact sequence $0 \to A \to B \to C \to 0$ we find that the induced sequence $0 \to \mathrm{Hom}_R(C,M) \to \mathrm{Hom}_R(B,M) \to \mathrm{Hom}_R(A,M)$ is exact (for example, any nonzero map from C to M pulls back to a nonzero one from B to M, since the map from B to C is surjective), but again, as with $\mathrm{Hom}_R(M,-)$, there can be trouble on the right-hand end: not every map from A to M extends to a map from B to M. If every map from A to M does extend to B, for any choice of R-modules A, B with A injecting into B, then we call M *injective*.

Unfortunately it is considerably more difficult to decide which R-modules are injective than it was for projective modules. Consider first a special case. Suppose that the module M is such that every R-module map from an *ideal* I of R into M extends to a map from R into M. Let A be a submodule of B and suppose we have a map f from A into M that we want to extend to B. Pick $x \in B, x \notin A$. The set of all $r \in R$ with $rx \in A$ is an ideal I of R and we have a map from I to M sending $i \in I$ to $f(ix)$. Extending this map to R, we now have an extension of f to all of Rx, which by linearity extends f to the submodule $A + Rx$ properly containing A. Iterating this process many times (this requires the axiom of choice, to be discussed later), we extend f to all of B, as desired. So we are reduced to asking when maps f from ideals I of R to M extend to R. This question is easiest to answer whence R is a PID, for then every ideal I is principal. If $I = (i)$, then extending f to R amounts to deciding what $f(1)$ should be; the requirement on $f(1)$ is exactly that $if(1) = f(i)$. Thus in this case the key property of M that we need is that *for every nonzero* $i \in R$ *and* $m \in M$, *there is* $m_i \in M$ *with* $im_i = m$ (note that this property arose in connection with the problem in the last homework set to show that the direct product of countably many copies of $\mathbf{Z}$ is not free over $\mathbf{Z}$. Note also that *every* $m \in M$ must satisfy this requirement, since given any $m \in M$ and nonzero $i \in R$, we get a map from the principal ideal (i) of R to M sending any multiple ri to rm.) We say that M is *divisible* if it has this property. Thus *over a PID, a module is injective if and only if it is divisible.* We are forced to specialize to PIDs to get an easy criterion for injectivity; for projectivity, by contrast, our results apply over any commutative ring, In homework for this week you will show that any $\mathbf{Z}$-module M injects into an injective one. We will later generalize this result to modules over any ring R, after we introduce the analogue of projectivity and injectivity for tensor products and establish a relation between tensor products and homomorphisms. The corresponding result for projective modules is that *every R-module (for any R) is a quotient of a projective one*; this is easy to see since we already know that any R-module is a quotient of a free one and free modules are projective.

Divisible modules over integral domains are interesting in their own right, furnishing a stark contrast to finitely generated modules over PIDs. It is obvious for example that $\mathbf{Q}$ is a divisible $\mathbf{Z}$-module, but so is the quotient $\mathbf{Q}/\mathbf{Z}$ (this quotient is a $\mathbf{Z}$-module, not a ring; in particular, an injective $\mathbf{Z}$-module is *not* necessarily a $\mathbf{Q}$-module,) More generally, any quotient of a divisible R-module is again divisible (but a quotient of an injective module need not be injective in general). For a more exotic example fix a prime number p and look at $\mathbf{Z}[1/p]/\mathbf{Z}$; here $\mathbf{Z}[1/p]$ is the subring of $\mathbf{Q}$ generated by $\mathbf{Z}$ and $1/p$. This last quotient is isomorphic to $\mathbf{Q}/\mathbf{Z}_{(p)}$, where $\mathbf{Z}_{(p)}$ consists of all $a/b \in \mathbf{Q}$ with p not dividing the integer b. We will see more of $\mathbf{Z}_{(p)}$, which is a ring as well as a $\mathbf{Z}$-module, in the spring.