Math 504, 11/02

Continuing from last time, we have learned that $\mathrm{Ext}^i_R(--,N)$ is a contravariant functor from left R-modules to abelian groups, while $\mathrm{Ext}^i_R(M,--)$ is a covariant functor from R-modules to abelian groups. Attached to any short exact sequence $0 \to A \to B \to C \to 0$ of left R-modules and another R-module M, it turns out that we now get a *long* exact sequence $0 \to \hom_R(C,M) \to \hom_R(B,M \to \hom_R(A,M) \to \mathrm{Ext}^1_R(C,M) \to \mathrm{Ext}^1_R(B,M) \to \mathrm{Ext}^1_R(A,M) \to \mathrm{Ext}^2_R((C,M) \to \mathrm{Ext}^2_R(B,M) \to \cdots$ which fills out the subshort exact sequence of its first four terms that we saw earlier with Ext groups. We have a similar long exact sequence obtained by reversing the order of A,B,C and inserting the M as the first argument in the hom and Ext groups. Finally, starting with the projective resolution $\{P_i\}$ of M we can tensor with a fixed *right* R-module N to obtain a chain complex (just like a cochain complex, but this time ending rather than starting with 0) $\{P_i \otimes_R N\}$ of abelian groups, whose homology (so-called rather than cohomology, this being a chain complex rather than a cochain complex) groups are called Tor groups and denoted $\mathrm{Tor}^R_i(M,N)$. These are again independent of the choice of projective resolution of M and the functors $\mathrm{Tor}^R_i(M,--)$ and $\mathrm{Tor}^R_i(--,N)$ are both covariant (from R-modules to abelian groups). As $\otimes_R$ is now right but not left exact, the version of the long exact sequence attached to the short exact sequence $0 \to A \to B \to C \to 0$ is now $\cdots \mathrm{Tor}^R_1(M,A) \to \mathrm{Tor}^R_1(M,B) \to \mathrm{Tor}^R_1(M,C) \to M \otimes_R A \to M \otimes B \to M \otimes C \to 0$ where A,B,C are left R-modules and M is a right R-module. I hope to prove the existence and develop more properties of these long exact sequences later; for now I move on to (I hope) easier material.

Let G be a group. I will be discussing G-modules (vector spaces V over a field K equipped with a *linear* G-action, so that $g \cdot (v_1 + v_2) = g \cdot v_1 + g \cdot v_2$ and $g \cdot (kv) = kg \cdot v$, for all $g \in G, v_1, v_2, v \in V, k \in K$. I usually write gv instead of $g\dot{v}$. When I discussed group actions on finite sets earlier, I observed that an action of G on a finite set S is equivalent to a homomorphism from G to the group $\mathrm{Perm}(S)$ of all permutations of S; in a similar manner, given a G-module V we get a homomorphism π from G to the general linear group $GL(V)$ of all $1-1$ linear maps from V onto itself. I will assume henceforth that V is finite-dimensional over K and that G is finite, though I may look at a few examples where G is infinite later. Either the vector space V or the homomorphism π is often called a *representation* of G (as it represents the abstract elements of G by concrete square matrices). Let's look at a couple of examples. If G is the group of quaternion units, let H be the quaternions (real linear combinations of $1,i,j,k$, where $\pm i,j,k$ multiply in the same way as for the quaternion units). Then H ic a vector space over the complex numbers $\mathbf{C}$, where the complex scalars act on H by *right* multiplication. Then G acts on H by left multiplication. Fixing the basis i,j of H, we find that the matrices I,J by which i,j act on H are given by $\begin{pmatrix} i & 0//0 & -i// \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, respectively. Another way to make G act linearly, this time on $\mathbf{C}$, is to decree that $\pm 1, \pm i$ act trivially, while $\pm j, \pm k$ act by -1.

A G-homomorphism between two G-modules V, W is a K-linear map π from V to W such that $\pi g(v) = g\pi(v)$ (i.e. π commutes with the action of G). If π is an isomorphism (in the usual sense of being $1-1$ and onto) then we call the modules (or representations) V and W *equivalent*. The G-module V is called *simple* or *irreducible* if its only submodules (in the obvious sense) are 0 and V. The key result in the study of simple R-modules, where R is a ring, is Schur's Lemma (which you will prove in homework this week); in order to apply it in our setting, we need to realize our G-modules V as modules over a suitable ring. To that end, we from the *group algebra* KG, consisting by definition of all finite formal sums $\sum_{g\in G} k_g g$, where the k_g lie in G. It is clear (by linearity) what $(\sum_g k_g g)v$ should be, for any $v \in V, k_g \in K$, namely $\sum_g k_g g v$, making this definition, we realize V as a left KG-module as desired. (Note that KG is not commutative as a ring unless G is abelian as a group.) Clearly the KG-submodules of V are the same as the G-submodules, so V is irreducible over G if and only if it is so over KG. The nicest behavior occurs when $K = \mathbf{C}$, the complex numbers, or more generally any algebraically closed field of characteristic 0. Here for example if G is cyclic of order n, then we get a family of irreducible one-dimensional representations of G by decreeing that a fixed generator g of G act by the complex scalar $e^{2\pi ik/n}$, where k lies between 0 and $n-1$. This family turns out to account for all the irreducible representations of G; note that these would not be available to us if we worked over $\mathbf{R}$, as crucial n-th roots of 1 would be missing.