# Math 504, 10/21

We now look at sequences of modules linked by homomorphisms. Given a sequence of $R$-modules $\cdots \to M \to N \to P \to \cdots$, each one sent to the next by an $R$-module map, we say that it is *exact at $N$* if the image of the ($R$-module) map from $M$ to $N$ equals the kernel of the map from $N$ to $P$. The sequence is *exact* if it is exact at all of its terms, except possibly those, if any, on the extreme ends. The most important kind of exact sequence is a *short exact sequence* $0 \to N \to M \to P \to 0$; note that such a sequence is exact if and only if the map $f$ from $N$ to $M$ is injective, the map $g$ from $M$ to $P$ is surjective, and the image of $f$ equals the kernel of $g$ (so that $P \cong M/N$, identifying $N$ with its image under $f$ in $M$). Such sequences are the basic object of study of *homological algebra*, which plays a key role in algebraic topology; one of the aims of this course is give an introduction to this subject. The short exact sequence $0 \to N \to M \to P \to 0$ is said to be *split* if there is a submodule $P'$ of its middle term $M$ mapping isomorphically onto $P$ via the map from $M$ to $P$, so that $M$ is the direct sum of the image of $N$ in it and $P'$. We will be looking at important operations, called *functors*, which take $R$-modules to $R$-modules; our first such operation fixes an $R$-module $M$ and sends an arbitrary $R$-module $N$ to the set $\mathrm{Hom}_R(M, N)$ of $R$-module maps from $M$ to $N$; note that $\mathrm{Hom}_R(M, N)$ is also an $R$-module in a natural way. Given a sequence $M_1 \to M_2 \to M_3 \to \cdots$ of $R$-modules and our fixed $R$-module $M$, we get a sequence $\mathrm{Hom}_R(M, M_1) \to \mathrm{Hom}_R(M, M_2) \to \cdots$ by composition; accordingly, we call the functor sending each $M_i$ to $\mathrm{Hom}_R(M, M_i)$ (often denoted $\mathrm{Hom}_R(M, --)$) *covariant*, since the directions of the arrows are preserved. (If they were reversed instead, as they will be in some subsequent examples, then we call the functor *contravariant*). Our first question is whether the covariant functor $\mathrm{Hom}_R(M, --)$ sends one short exact sequence to another one. Given a short exact sequence $0 \to A \to B \to C \to 0$, our functor yields the sequence $0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to 0$. Here the map $f$ from $\mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B)$ is indeed injective, since $A$ embeds in $B$, and the map from $\mathrm{Hom}_R(M, B)$ to $\mathrm{Hom}_R(M, C)$ does indeed have as kernel the image of $\mathrm{Hom}_R(M, A)$ in $\mathrm{Hom}_R(M, B)$, but the map from $\mathrm{Hom}_R(M, B)$ to $\mathrm{Hom}_R(M, C)$ need *not* be surjective; for example, if $R = \mathbf{Z}, M = \mathbf{Z}_2$, and $A, B, C$ are respectively $\mathbf{Z}, \mathbf{Z}$, and $\mathbf{Z}_2$, with the map from $A$ to $B$ sending an integer $x$ to $2x$, then the map from $\mathrm{Hom}(\mathbf{Z}_2, \mathbf{Z})$ to $\mathrm{Hom}(\mathbf{Z}_2, \mathbf{Z}_2)$ is not surjective, as there are no nonzero homomorphisms from $\mathbf{Z}_2$ to $\mathbf{Z}$, but there is a nonzero homomorphism from $\mathbf{Z}_2$ to $\mathbf{Z}_2$. Thus the first four terms of our original exact sequence $0 \to A \to B \to C \to 0$ remain exact after applying $\mathrm{Hom}_R(M, --)$, but the full sequence does not. We summarize this situation by saying that the functor $\mathrm{Hom}_R(M, --)$ is *left exact*, but not *right exact*. On the other hand, there are some modules $P$ over some rings $R$ for which this functor is *exact*, preserving the full short exact sequence we started out with. This will happen if and only if any $R$-module map $f$ from $P$ to another module $N$ will always *lift* in the sense that, given a module $M$ and a surjection $g$ from another $R$-module $M$ onto $N$, there is a map $f'$ from $P$ to $N$ with $f = gf'$. We call such $R$-modules *projective* (over $R$); unfortunately this has nothing to do with projective geometry or projective space (or the projective special linear group $PSL_n$) that arise in other contexts. For example, if $P$ is free over $R$, then the lift $f'$ always exists: let $p_1, p_2, \ldots$ be a basis of $P$ and choose preimages $m_1, m_2, \ldots$ of the images $f(p_i)$ in $M$, so that $g(m_i) = f(p_i)$, and then take $f'$ to be the unique $R$-module map from $P$ to $M$ sending each $p_i$ to $m_i$. More generally, if $P$ is a direct summand of a free module $F$, so that $F \cong P \oplus Q$, then again the lift $f'$ exists: extend $f$ to a map from $F$ to $N$ by decreeing that it be 0 on $Q$, and then lift it as above. In fact, it turns out that *the projective $R$-modules are exactly the direct summands of free $R$-modules*; to see this, let $P$ be projective over $R$ and let $F$ be a free module surjecting onto $P$; we know that such an $F$ exists. Then the identity map from $P$ to itself must lift to $F$, so that there is a map $f : P \to F$ such that composing $f$ with the surjection from $F$ to $P$ is the identity on $P$. This says exactly that $F$ is the direct sum of the image $f(P)$ in it and the kernel of the surjection from $F$ to $P$, so that $P$ is (isomorphic to) a direct summand of a free module, as claimed. Note that although the free module $R^n$ is given to us as the direct sum of $n$ copies of $R$, there could be other ways to write $R^n$ as a direct sum and accordingly projective $R$-modules that are not free. The simplest example occurs when $R = \mathbf{Z}_6$: here $R$ decomposes as the direct sum of its submodules generated by $3, 2$, which are isomorphic to $\mathbf{Z}_2, \mathbf{Z}_3$, respectively. Hence $\mathbf{Z}_2$ and $\mathbf{Z}_3$ are projective over $\mathbf{Z}_6$ (but clearly not free).