Math 504, 10/28

We now point out a fundamental connection between tensor products and homomorphisms, which indicates why the tensor product functor is right exact while the Hom functors are left exact. Assume first that R is commutative and that A, B, C are R-modules. Then *there is an isomorphism*

$\mathrm{Hom}_R(B \otimes_R A, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C))$

sending a homomorphism f on the left to the homomorphism mapping $a \in A$ to the map g defined by $g(b) = f(b \otimes a)$. We express this relationship by saying that *the tensor product functor is the left adjoint of the Hom functor, or that the Hom functor is the right adjoint of the tensor product functor*, since moving across the comma converts Hom to $\otimes$. We need an extension of this to show that every left module over a possibly noncommutative ring R embeds in an injective one. Let S be another ring, let A be a left R-module, let C be a left S-module, and finally let B be an (S, R) bimodule, so that B is simuntaneously a left S-module and a right R-module and we have $(sb)r = s(br)$ for all $r \in R, s \in S, b \in B$. As noted in the last lecture,'/ $\mathrm{Hom}_R(B, C)$ is then a left R-module (making R act on the domain rather than the range). Now our extension reads

$\mathrm{Hom}_S(B \otimes_R A, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C))$

and again a homomorphism f on the left is sent to the homomorphism taking $a \in A$ to the map sending $b \in B$ to $f(b \otimes a)$. We apply this map in the special case where our ring S is the integers $\mathbf{Z}, B = R, A = N$ is a left R-module, and $C = Q$ is an injective $\mathbf{Z}$-module containing the left R-module M (whose existence you showed in homework this week). Then we have

$\mathrm{Hom}_{\mathbf{Z}}(R \otimes_R N, Q) \cong \mathrm{Hom}_{\mathbf{Z}}(N, Q) \cong \mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbf{Z}}(R, Q))$

and since the functor taking N to $\mathrm{Hom}_{\mathbf{Z}}(N, Q)$ is exact, so too is the functor taking N to $\mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbf{Z}}(R, Q))$. Hence $\mathrm{Hom}_{\mathbf{Z}}(R, Q)$ is an injective left R-module. It contains $\mathrm{Hom}_R(R, M) \cong M$, as desired. Whew! This argument is a real workout in manipulating the formalism surrounding Hom and the tensor product, but it is well worth studying in detail, as ultimately such formalism is much more efficient than applying the definition of injective module directly. To summarize again, it shows that *any left module over any ring R embeds in an injective one*. Given a left R-module M, there is in fact in a precise sense a unique smallest injective R-module containing M, called its *injective hull*, but I will not pursue its construction here (you can read about it in §10.5 of Dummit and Foote). The corresponding object on the projective side is called the *projective cover*, but it does not exist in the category of general R-modules, though it does in certain restricted subcategories of this one. Another example of the interplay between projectivity and injectivity is the observation that *direct sums of porjective modules are projective, while direct products of injective modules are injective*; this holds because given any collection of R-modules M_i and another R-module M, any set of homomorphisms from M_i to M (one for each i) induces a unique homomorphism from the direct sum of the M_i to M, while any set of homomorphisms from M to M_i induces one from M to the direct product of the M_i. In the language of category theory, direct sums are called products, while direct products are called coproducts.

We now investigate in more detail the behavior of the Hom (also denoted hom) functors for non-projective and non-injective modules, seeking to measure the failure of exactness of this functor in a precise way. Let R be a not necessarily commutative ring (always with 1) and left M be a left R-module. We have seen that there is a projective R-module P_0 and a surjective homomorphism d_0 from P_0 onto M. If this map is not injective, there is another projective module P_1 and a surjective map d_1 from P_1 onto the kernel of d_0. Continuing, we get an infinite sequence of projective modules $P_0, P_1, P_2, \ldots$ and a set of maps d_i taking P_i to P_{i-1} for $i > 0$, such that the corresponding sequence $\cdots P_i \to P_{i-1} \to \cdots \to P_0 \to M$ is exact. We call such a sequence a *projective resolution* of M. Given another R-module N, applying a Hom functor yields a sequence $0 \to \hom(P_0, N) \to \hom(P_1, N) \to \cdots$ (we omit the $\hom(M, N)$ term), which is not exact in general, but is such that composing the maps given by two consecutive arrows always gives 0 (or equivalently the image of every map lies in the kernel of the next one). Such a sequence is called a *cochain complex*; the quotient of the kernel of the ith map in it (starting from $\hom(P_0, N)$ by the image of the preceding map is called its ith *cohomology group*. We will continue with this setup next time.