Math 504, 9/30

We continue with the inequality derived last time for a finite subgroup G of order N of SO_3, having three orbits of poles, consisting of points stabilized by r_1, r_2, r_3 rotations, respectively; then

$$(1/r_1) + (1/r_2) + (1/r_3) = 1 + (2/N) > 1$$

Assuming as we may that $r_1 \leq r_2 \leq r_3$, and recalling that the r_i are integers at least equal to 2, it is easy to see that the only solutions for (r_1, r_2, r_3) are $(2, 2, M)$ [with $N = 2M$], $(2, 3, 3)$ [with $N = 12$], $(2, 3, 4)$ [with $N = 24$], and $(2, 3, 5)$ [with $N = 60$]. Each triple corresponds to a uniquely determined group, once the set of poles has been specified, and in all cases the group is the full group of symmetries (in SO_3) of a familiar geometric object. More precisely, the triple $(2, 2, M)$ corresponds to the dihedral group of order $N = 2M$ of symmetries of a regular M-gon; the poles are the vertices of the M-gon, the midpoints of its sides (rescaled so as to have length 1), and two points at distance 1 from the plane of the M-gon, one above this plane (and directly over the center of the M-gon), and one below this plane. The triple $(2, 3, 3)$ corresponds to the symmetry group T of a regular tetrahedron; the poles are its vertices, the midpoints of its edges, (rescaled as always to have length 1), and the centers of its faces (rescaled). The other two triples $(2, 3, 4)$ and $(2, 3, 5)$ correspond to the respective symmetry groups O, I of a regular octahedron and icosahedron, with poles specified as for the tetrahedron. (A cube has the same symmetry group as an inscribed octahedron; similarly a dodecahedron has the same symmetry group as an inscribed icosahedron.) The group T is isomorphic to the alternating group A_4; the orientation-preserving symmetries of a tetrahedron act by the even permutations of its vertices. The group O is isomorphic to the symmetric group S_4; here the orientation-preserving symmetries act by all permutations of the four pairs of opposite faces. Finally, and most subtly, I is isomorphic to the alternating group A_5: any dodecahedron has exactly 5 inscribed cubes with vertices among its vertices; the orientation-preserving symmetries act by all even permutations of these 5 cubes. For more details see section 6.12 of Artin's book Algebra, on reserve in the Math Library. We will see these groups later, defined in a particularly striking way by generators and relations.

Return now to a general finite group G. We noted in the last lecture that an action of G on a set S is equivalent to a homomorphism from G to the group Perm(S) of permutations of S. We are particularly interested in the case where this homomorphism is 1-1, so that G is realized as (isomorphic to) a subgroup of Perm(S). To this end, we note that if s is in S and has stabilizer H, then the stabilizer of another element gs inits orbit is just the conjugate gHg^{-1} of H by the element g. The kernel of the homomorphism from G to the permutation group Perm(G·s) of just this orbit is thus the intersection of the conjugates of H. By studying the subgroups of a given group G, we can thus decide whether G is isomorphic to a subgroup of Perm(S) for various choices of S. For example, if G is dihedral of order 8, then it has four cyclic subgroups generated by reflections, forming two conjugacy classes. The intersection of the cyclic subgroups in each conjugacy class is trivial, whence we get a *faithful* action (with trivial kernel) of G on a four-element set, namely the set of left cosets of any of these subgroups. (Of course, we already knew that G acts faithfully on the four vertices of a square, but we just deduced this property of G by looking at G alone.) By contrast, if G is the quaternion group of order 8, consisting of $\pm 1, \pm i, \pm j$, and $\pm k$, with $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$, then G has only one cyclic subgroup or oder 2, generated by -1, and indeed every nontrivial subgroup of G contains -1. It follows that G does not admit a faithful transitive action on any 4-element set; with a bit more work one sees that G does not admit a faithful action on any set of size less than 8.

We conclude with a statement of the Sylow theorems for finite groups, which you will prove next week in homework. Let p be a prime number and G a finite group of order $p^m n$, where p does not divide n. Then the number n_p of subgroups of G of order p^m (called p-Sylow subgroups) is congruent to 1 mod p and divides n; in particular there is always at least one such subgroup. In addition any two such subgroups are conjugate in G.