Math 504, 11/14

We now look at a very interesting family of finite groups that deserves to be much better known (yet does not seem to have a name in the standard literature); I will call groups in this family Clifford groups. The nth of these, call it G_n, is generated by n elements $a_1, \ldots, a_n$ subject to the relations $a(i^2 = \epsilon, \epsilon^2 = 1, a_i a_j = \epsilon a_j a_i$ if $i \neq j$. Thus the elements of G_n are exactly the products $a_{i_1} \cdots a_{i_k}$, where $i_1 < \cdots < i_k$, together with $\epsilon a_{i_1} \cdots a_{i_k}$, and the order of C_n is 2^{n+1}. Its center Z_n is just $\{1, \epsilon\}$ if n is even, but is $\{1, \epsilon, a_1 \cdots a_n, \epsilon a_1 \cdots a_n\}$ if n is odd. If n is congruent to 1 mod 4, then Z_n is cyclic and generated by $a_1 \cdots a_n$; if n is congruent to 3 mod 4, then Z_n is the product of two cyclic groups of order 2. Hence C_n has $2^n + 1$ conjugacy classes if n is even (two of size 1, $2^n - 1$ of size 2) but $2^n + 2$ conjugacy classes if n is odd. The easiest (but most boring) representations of G_n are those in which ϵ acts trivially; the quotient group $G_n / < \epsilon >$ is the product of n copies of $\mathbf{Z}_2$, so G_n has 2^n representations of degree one (for all n, both even and odd). There is only one irreducible representation V_n of C_n remaining if n is even, which must have degree $2^{n/2}$ (so that the sum of the squares of the irreducible degrees equals 2^{n+1}, the order of G_n). If n is odd, there are two irreducible representations remaining, each of degree $2^{(n-1)/2}$; to see that they must have the same degree, observe that C_n is generated by the copy of G_{n-1} inside it and the central element $a_1 \cdots a_n$, so the two remaining representations are isomorphic to V_{n-1} as representations of G_{n-1}, with $a_1 \cdots a_n$ acting by $\pm i$ if n is congruent to 1 mod 4, or ± 1 if n is congruent to 3 mod 4. Now one can form the quotient of the group algebra $\mathbf{C}G_n$ by the ideal generated by $\epsilon + 1$; this quotient is generated (now as an algebra over $\mathbf{C}$) by $a_1, \ldots, a_n$ subject to the relations $a_i^2 = -1, a_i a_j = -a_j a_i$ if $i \neq j$. This quotient C_n is well known in the literature and is called the *Clifford algebra* (over $\mathbf{C}$). It is isomorphic to a single matrix ring $M_{2^{n/2}}(\mathbf{C})$ if n is even and the direct sum of two isomorphic matrix rings $M_{2^{(n-1)/2}}$ if n is odd. Its representation(s) V_n as above (one of them if n is even, two if n is odd) is/are called *half-spin representations* and play an important role in physics. Although C_n is not a group under multiplication, it admits a remarkable multiplicative subgroup, generated by all linear combinations $\sum z_i a_i$ with $z_i \in \mathbf{C}, \sum z_i^2 = 1$. This is called the *pin group* $\mathrm{Pin}_n(\mathbf{C})$ and turns out to be a double cover of the orthogonal group $O_n(\mathbf{C})$; likewise products of evenly many combinations of the above type form a subgroup of index 2 in $\mathrm{Pin}_n(\mathbf{C})$ denoted $\mathrm{Spin}_n(\mathbf{C})$ and called the *spin group*. It is a simply connected double cover of $SO_n(\mathbf{C})$. The pin and spin groups have analogues over $\mathbf{R}$ as well, defined by restricting the coefficients z_i as above to the real numbers; these too are important in physics.

We now change gears slightly and investigate the entries in character tables (and certain related complex numbers) in more detail. Call a complex number z an *algebraic integer* if it satisfies a *monic* polynomial with integral coefficients; this is a stronger than being algebraic (over $\mathbf{Q}$). Equivalently, z is an algebraic integer if the subring $\mathbf{Z}[z]$ of $\mathbf{C}$ generated by $\mathbf{Z}$ and z is finitely generated as a $\mathbf{Z}$-module, so that if z is an algebraic integer, so is every element of $\mathbf{Z}[z]$. Now it is easy to see that if x and y are algebraic integers, then the subring $\mathbf{Z}[x, y]$

generated by both x and y is generated as a $\mathbf{Z}$-module by finitely many products $x^i y^j$, whence all elements of $\mathbf{Z}[x, y]$ are again algebraic integers and in particular $x + y, x - y, xy$ are. On the other hand, *a rational number r/s is an algebraic integer if and only if it is an integer*, for if we write down a monic polynomial over $\mathbf{Z}$ satisfied by r/s and clear the denominators, we find by looking at the greatest common divisor of the terms that $s = \pm 1$. Now the entries $\chi(g)$ in the character table of a finite group G, being sums of roots of 1 in $\mathbf{C}$, must be algebraic integers, so in particular no nonintegral rational number can occur in a character table.

A beautiful consequence for the symmetric group S_n is that *the entries in its character table are all integers*. To see this, note first that S_n has a very special group-theoretic property: the order of any element g is the least common multiple m of the lengths of the cycles in its cycle decomposition, whence any power g^a with $\gcd(a, m) = 1$ is a product of cycles of the same lengths and so is conjugate to g. Now for any representation π of S_n, the matrix $\pi(g)$ will have all eigenvalues m-th roots of 1 in $\mathbf{C}$, whence they will lie in the field $F_m = \mathbf{Q}[e^{2\pi i/m}]$ generated by all these m-th roots, as will the trace $\chi(g)$ of $\pi(g)$. Passing from g to g^a (with $\gcd(a, m) = 1$), we replace every eigenvalue of $\pi(g)$ by its a-th power, and yet the trace must remain the same, since g^a is conjugate to g. Now we will see next term that for any such a there is an automorphism of F_m mapping $e^{2\pi i/m}$ to its a-th power, and likewise for any power of $e^{2\pi i/m}$. The set of all such automorphisms of F_m is a group under composition, called the Galois group of F_m, and th e only elements of F_m fixed by every element of this group are those of $\mathbf{Q}$. Hence we must have $\chi(g) \in \mathbf{Q}$; but now since $\chi(g)$ must be an algebraic integer, we must in fact have $\chi(g) \in \mathbf{Z}$, as claimed.

There is another very nice result proved using algebraic integers. Recall the sums $S_g = \sum_{g \in C_g} g$ in $\mathbf{C}G$ introduced earlier, where C_g is the conjugacy class of G. We saw before that S_g is central in $\mathbf{C}G$, so acts as a complex scalar $z_{g,\pi}$ on any irreducible representation π of G. Now we can say something about z_g. Note that the product $S_g S'_g$ of any two S_g's is again central in $\mathbf{C}G$ and has only nonnegative integers as coefficients of group elements. It follows that $S_g S'_g$ is a nonnegative integral combination of S_h's (as h runs over G) and the same holds of $z_{g,\pi} z_{g',\pi}$ and the $z_{h,\pi}$'s. This says that the subring $\mathbf{Z}[z_{g,\pi} : g \in G]$ is finitely generated as a $\mathbf{Z}$-module, whence every $z_{g,\pi}$ must be an algebraic integer. Its value must be $c_g \chi(g)/\chi(1)$ where χ is the character of π and c_g is the size of the conjugacy class C_g, whence $c_g \chi(g)/\chi(1)$ *is an algebraic integer*. Summing over one element of every conjugacy class of G and using the orthonormality of the irreducible characters, we deduce that the ratio $|G|/\chi(1)$ is an integer for every irreducible character χ: *the degree of any irreducible representation divides the order of the group*. Thus it was no coincidence that the representations of the Clifford group G_n all have degrees that are powers of 2.