Math 504, 10/21

We now look at sequences of modules linked by homomorphisms. Given a sequence of R-modules $\cdots \to M \to N \to P \to \cdots$, each one sent to the next by an R-module map, we say that it is *exact at* N if the image of the (R-module) map from M to N equals the kernel of the map from N to P. The sequence is *exact* if it is exact at all of its terms, except possibly those, if any, on the extreme ends. The most important kind of exact sequence is a *short exact sequence* $0 \to N \to M \to P \to 0$; note that such a sequence is exact if and only if the map f from N to M is injective, the map g from M to P is surjective, and the image of f equals the kernel of g (so that $P \cong M/N$, identifying N with its image under f in M). Such sequences are the basic object of study of *homological algebra*, which plays a key role in algebraic topology; one of the aims of this course is give an introduction to this subject. The short exact sequence $0 \to N \to M \to P \to 0$ is said to be *split* if there is a submodule P' of its middle term M mapping isomorphically onto P via the map from M to P, so that M is the direct sum of the image of N in it and P'. We will be looking at important operations, called *functors*, which take R-modules to R-modules; our first such operation fixes an R-module M and sends an arbitrary R-module N to the set $\mathrm{Hom}_R(M,N)$ of R-module maps from M to N; note that $\mathrm{Hom}_R(M,N)$ is also an R-module in a natural way. Given a sequence $M_1 \to M_2 \to M_3 \to \cdots$ of R-modules and our fixed R-module M, we get a sequence $\mathrm{Hom}_R(M,M_1) \to \mathrm{Hom}_R(M,M_2) \to \cdots$ by composition; accordingly, we call the functor sending each M_i to $\mathrm{Hom}_R(M,M_i)$ (often denoted $\mathrm{Hom}_R(M,-)$) *covariant*, since the directions of the arrows are preserved. (If they were reversed instead, as they will be in some subsequent examples, then we call the functor *contravariant*). Our first question is whether the covariant functor $\mathrm{Hom}_R(M,-)$ sends one short exact sequence to another one. Given a short exact sequence $0 \to A \to B \to C \to 0$, our functor yields the sequence $0 \to \mathrm{Hom}_R(M,A) \to \mathrm{Hom}_R(M,B) \to \mathrm{Hom}_R(M,C) \to 0$. Here the map f from $\mathrm{Hom}_R(M,A) \to \mathrm{Hom}_R(M,B)$ is indeed injective, since A embeds in B, and the map from $\mathrm{Hom}_R(M,B)$ to $\mathrm{Hom}_R(M,C)$ does indeed have as kernel the image of $\mathrm{Hom}_R(M,A)$ in $\mathrm{Hom}_R(M,B)$, but the map from $\mathrm{Hom}_R(M,B)$ to $\mathrm{Hom}_R(M,C)$ need *not* be surjective; for example, if $R = \mathbf{Z}, M = \mathbf{Z}_2$, and A, B, C are respectively $\mathbf{Z}, \mathbf{Z}$, and $\mathbf{Z}_2$, with the map from A to B sending an integer x to $2x$, then the map from $\mathrm{Hom}(\mathbf{Z}_2, \mathbf{Z})$ to $\mathrm{Hom}(\mathbf{Z}_2, \mathbf{Z}_2)$ is not surjective, as there are no nonzero homomorphisms from $\mathbf{Z}_2$ to $\mathbf{Z}$, but there is a nonzero homomorphism from $\mathbf{Z}_2$ to $\mathbf{Z}_2$. Thus the first four terms of our original exact sequence $0 \to A \to B \to C \to 0$ remain exact after applying $\mathrm{Hom}_R(M,-)$, but the full sequence does not. We summarize this situation by saying that the functor $\mathrm{Hom}_R(M,-)$ is *left exact*, but not *right exact*. On the other hand, there are some modules P over some rings R for which this functor is *exact*, preserving the full short exact sequence we started out with. This will happen if and only if any R-module map f from P to another module N will always *lift* in the sense that, given a module M and a surjection g from another R-module M onto N, there is a map f' from P to N with $f = gf'$. We call such R-modules *projective* (over R); unfortunately this has nothing to do with projective geometry or projective space (or the projective special linear group PSL_n) that arise in other contexts. For example, if P is free over R, then the lift f' always exists: let $p_1, p_2, \ldots$ be a basis of P and choose preimages $m_1, m_2, \ldots$ of the images $f(p_i)$ in M, so that $g(m_i) = f(p_i)$, and then take f' to be the unique R-module map from P to M sending each p_i to m_i. More generally, if P is a direct summand of a free module F, so that $F \cong P \oplus Q$, then again the lift f' exists: extend f to a map from F to N by decreeing that it be 0 on Q, and then lift it as above. In fact, it turns out that *the projective R-modules are exactly the direct summands of free R-modules*; to see this, let P be projective over R and let F be a free module surjecting onto P; we know that such an F exists. Then the identity map from P to itself must lift to F, so that there is a map $f : P \to F$ such that composing f with the surjection from F to P is the identity on P. This says exactly that F is the direct sum of the image $f(P)$ in it and the kernel of the surjection from F to P, so that P is (isomorphic to) a direct summand of a free module, as claimed. Note that although the free module R^n is given to us as the direct sum of n copies of R, there could be other ways to write R^n as a direct sum and accordingly projective R-modules that are not free. The simplest example occurs when $R = \mathbf{Z}_6$: here R decomposes as the direct sum of its submodules generated by $3, 2$, which are isomorphic to $\mathbf{Z}_2, \mathbf{Z}_3$, respectively. Hence $\mathbf{Z}_2$ and $\mathbf{Z}_3$ are projective over $\mathbf{Z}_6$ (but clearly not free).