Math 504, 11/07

Let G be a finite group. We have now seen that $R = \mathbf{C}G$, the complex group algebra of G, is isomorphic as a ring to a direct sum of matrix rings $M_{n_i}(\mathbf{C})$ over $\mathbf{C}$. Up to isomorphism, every irreducible left R-module takes the form $\mathbf{C}^{n_i}$ for some i, where the summand $M_{n_i}(\mathbf{C})$ of R acts on $\mathbf{C}^{n_i}$ by left multiplication, while the other matrix ring summands act by 0. We call R the *regular representation of* G and we observe (as in the last lecture) that each irreducible G-module $M = \mathbf{C}^{n_i}$ appears exactly $\dim M = n_i$ times in R. Now we want to compute how many nonisomorphic irreducible modules R has. We do so in a rather sneaky way, by computing in two different ways the dimension of the center C of R over $\mathbf{C}$. On the one hand, any element of C acts (as an R-bimodule homomorphism) by a complex scalar on the minimal two-sided ideals of R, which are irreducible as R-bimodules, so C is a direct sum of copies of $\mathbf{C}$, one for each ideal summand of R, or equivalently one for every distinct irreducible representation of G. On the other hand, a direct examination of a typical element $x = sum_g k_g g$ of R shows that it lies in C if and only if $hx = xh$ for all $h \in G$, or if and only if $hxh^{-1} = x$ for all $h \in G$, or if and only if $k_g = k_{g'}$ whenever the group elements g, g' are conjugate in G. Hence the dimension of C over $\mathbf{C}$ equals the number of conjugacy classes in G, and this is the number of distinct irreducible G-modules.

In particular, if $G = A$ is abelian, say of order n, then A has n conjugacy classes, each consisting of a single element, and accordingly n distinct irreducible representations, each necessarily of dimension 1, since the sum of the squares of their dimensions must be n, the order of A. Once we know the irreducible representations of A all have dimension 1, it is easy to find all of them. Write A as the direct product of cyclic groups, say of orders $n_1, \ldots, n_m$, with respective generators $g_1, \ldots, g_m$. Then each g_i must act by a complex n_ith root of 1, say r_i, on any 1-dimensional module; we have n_i choices for r_i and accordingly $\prod n_i = n$ choices for an irreducible representation of A; so we have all of them. As another example, if $G = S_3$, the symmetric group on three letters, then we know that G has a two-dimensional irreducible representation (realized by looking at the action of G on an equilateral triangle centered at $(0,0)$ in $\mathbf{C}^2$ via symmetries of this triangle. This representation is irreducible as there is no line in $\mathbf{C}^2$ that it preserves. Besides this representation we have the trivial one on $\mathbf{C}$, where all elements of G act trivially, and the sign representation on $\mathbf{C}$, where even permutations act by 1, odd ones by -1. The sum of the squares of the dimensions of these representations is $4+1+1=6$, so again we have all of the irreducible representations. There are three of them, matching the number of conjugacy classes in G.

Now an obvious question is how to construct irreducible representations of G, particularly if G is large. This is difficult to do directly, attaching a possibly large matrix $\pi(g)$ to every element of G, but fortunately the essential information in $\pi(g)$ can be distilled down to a single number, namely its trace (the sum of its diagonal entries, or of its eigenvalues). One might wonder why we do not use the determinant $\det(\pi(g))$ instead, as a more famous number attached to a matrix, but it turns out for most π that $\det(\pi(g)) = 1$ for all $g \in G$, so that it does not give useful information about π. Accordingly, we call the trace tr $\pi(g)$ the *character* of π, regarded as a complex-valued function on G, and denote it by χ_π. Since the matrices $\pi(g)$ have finite order in $GL_n(\mathbf{C})$ for some n, each is diagonalizable with eigenvalues all mth roots of 1 in $\mathbf{C}$ for some m (in fact we may take $m = |G|$, the order of G). A consequence is that $\chi_\pi(g^{-1} = \overline{\chi_\pi(g)}$, the complex conjugate of $\chi_\pi(g)$, since the eigenvalues of $\pi(g^{-1})$ are the inverses of the eigenvalues of $\pi(g)$, and each such eigenvalue lies on the unit circle in $\mathbf{C}$. (In fact, since the map $g \to \pi(g^{-1})^t$ is a homomorphism of G into $GL_n(\mathbf{C})$ whenever π is, where the superscript t denotes transpose, we see that $\overline{\chi_\pi}$ is the character of a representation whenever χ_π is.) Furthermore, since the trace of two similar matrices is always the same, the function χ_π is constant on conjugacy classes of G. The number of such classes, as we saw above, equals the number of distinct irreducible representations of G; we will see later that the characters of χ_π of the distinct irreducible representations of G form a basis for the space of all complex-valued functions on G that are constant on conjugacy classes.

Let V, V' be distinct irreducible representations of G, of respective dimensions say n and m, and let π, π' be the corresponding homomorphisms from G to $GL_n, (\mathbf{C}), GL_m(\mathbf{C})$. Then the only G-homomorphism from V to V' is the 0 map; but the other hand the proof of Maschke's Theorem shows that if μ is any $\mathbf{C}$-linear map from V to V', then $(1/|G|)\sum_{g\in G} \pi'(g)\mu\pi(g^{-1})$ is a G-homomorphism f and so must be 0. Taking μ to have matrix having 1 as its ij-entry and 0s everywhere else, for indices i, j with $1 \le i \le m, 1 \le j \le m$, we deduce that $\sum_g \pi(g)_{jj}\overline{\pi'(g)}_{ii} = 0$ (by looking at the ij-entry of the matrix of f), where $\pi(g)_{ij}$ is the ij-entry of the matrix $\pi(g)$; summing over all indices i, j, we deduce that $\sum_g \chi_\pi(g)\overline{\chi_{\pi'}(g)} = 0$. We will deduce further relations among the characters χ_π next time.