Math 504, 10/31

Continuing where we left off last time, let M, N be left R-modules. Last time we constructed a projective resolution of M: $\cdots P_n \cdots \to P_0 \to M \to 0$; applying $\hom(-, N)$ and omitting the first term, we have a sequence $0 \to \hom(P_0, N) \to \hom(P_1, N) \to \cdots$ which is such that if we let d_i denote the map from $\hom(P_i, N)$ to $\hom(P_{i+1}, N)$, then $d_{i+1}d_i = 0$. We call such a sequence C a *cochain complex*. We may form the quotient K_i/I_{i-1} of the kernel K_i of d_i by the image I_{i-1} of d_{i-1}; the elements of K_i are called *i-cocycles* and those of I_{i-1} *$(i-1)$-coboundaries*. The quotient $K_i/Ii - 1$ is then called the ith *cohomology group $H^i(C)$ of C*; in this particular setting it is denoted $\mathrm{Ext}^i_R(M, N)$ and called the ith Ext group of M and N (as R-modules). (This group is only an abelian group, not an R-module.) Here "Ext" should be thought of as standing for "extension"; it turns out that $\mathrm{Ext}^1_R(M, N)$ measures extensions of M by N, that is, all short exact sequences $0 \to N \to P \to M \to 0$ of R-modules (up to an equivalence defined later). We call the functors $\mathrm{Ext}^i_R(-, N)$(*higher (right) derived functors of* hom, since the functor $\hom(-, N)$ is left but not right exact; we will later see that a short exact sequence of R-modules gives rise to a long exact sequence whose first three nonzero terms are hom groups and whose remaining terms are Ext groups. It also turns out that the groups $\mathrm{Ext}^i_R(M, N)$ do not depend on our choice of projective resolution of M. We now give some examples.

Suppose first that $R = \mathbf{Z}, M = \mathbf{Z}_n$. Then a projective resolution of M is given by $0 \to \cdots \to 0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_n \to 0$, where the map from $\mathbf{Z}$ to $\mathbf{Z}$ is multiplication by n; in effect this is a finite resolution. Taking Ext groups, we find that $\mathrm{Ext}^0_{\mathbf{Z}}(M, N)$ consists of all homomorphisms from $\mathbf{Z}$ to N vanishing on multiples of n, or equivalently homomorphisms from $\mathbf{Z}_n$ to N, while $\mathrm{Ext}^1_{\mathbf{Z}}(M, N)$ is $\hom(\mathbf{Z}, N)$ modulo $\hom(n\mathbf{Z}, N)$, which is isomorphic to N/nN. The higher Ext groups $\mathrm{Ext}^i_{\mathbf{Z}}(M, N)$ are 0 (for $i \geq 2$). For a more interesting, but more complicated example, take $R = \mathbf{Z}_n, M = \mathbf{Z}_d$, where d is a divisor of n, say $n = dm$. Now a projective resolution of M is given by $\cdots \mathbf{Z}_n \to \mathbf{Z}_n \to \cdots \to Z_n \to Z_d \to 0$, where the maps from one copy of $\mathbf{Z}_n$ to the next are alternately given by multiplying by m and multiplying by d, the rightmost map from $\mathbf{Z}_n$ to $\mathbf{Z}_n$ is multiplication by d, and the map from $\mathbf{Z}_n$ to $\mathbf{Z}_d$is the canonical one (thought of as multiplication by m). Here we have $\mathrm{Ext}^0_{\mathbf{Z}_n}(\mathbf{Z}_d, N) = \hom(\mathbf{Z}_d, N)$ (as before; this is a general fact holding for left modules over any ring R), but now the other Ext groups toggle: $\mathrm{Ext}^i_{\mathbf{Z}_n}(\mathbf{Z}_d, N) = {}_mM/dM$ if i is odd, where ${}_mM$ denotes $\{x \in M : mx = 0\}$, while $\mathrm{Ext}^i_{\mathbf{Z}_n}(\mathbf{Z}_d, N) = {}_dM/mM$ if i is even. Many Ext groups exhibit this periodic behavior; many others vanish in high degrees, as we saw in the first example. We also see from these two examples that $\mathrm{Ext}_R(M, N)$ depends on R as well as M and N, as indicated by the notation.

More generally, as noted above, $\mathrm{Ext}^0_R(M, N) \cong \hom_R(M, N)$ for any R-modules M, N and any ring R. If R is a PID and M is finitely generated, then we have seen that M is the quotient of R^n for some n be a free submodule which is the column span of an $n \times n$ matrix A over R. If we make A act on N^n via left multiplication (regarding N^n as consisting of column vectors over N), then $\mathrm{Ext}^1_R(M,N) \cong n^n/AN^n$, while the Ext groups $\mathrm{Ext}^i_R(M,N)$ are 0 for $i \geq 2$.

Why don't the Ext groups depend on the choice of projective resolution of M? To answer this, we begin by noting that, given an R-module map f from M to N and projective resolutions$\{P_i\}$ and $\{Q_i\}$ of M, N, respectively, an easy inductive argument yields maps $f_i : P_i \to Q_i$ making the obvious diagram combining the two resolutions commute; applying $\hom_R(-, P)$ to this diagram, we get a map from $\mathrm{Ext}^i_R(M,P)$ to $\mathrm{Ext}^i_R(N,P)$ (so that $\mathrm{Ext}^i_R(-,P)$ is indeed a covariant functor). Using something you will construct in homework called a sl cochain homotopy, you will show that the induced map on Ext groups is always 0 if $f = 0$; whence it will follow that any two projective resolutions of M give rise to isomorphic Ext groups with a fixed module N.