Math 504, 10/19

Continuing with tensor products, let us return for a moment to finitely generated modules M over a PID R and show how we can use tensor products to recover the uniqueness of the decomposition of M as a direct sum of quotients of R. Indeed, let K be the field of fractions of R, consisting by definition of all formal fractions a/b with $a, b \in R, b \neq 0$. Then $K \otimes_R R/(d) = 0$ for all $d \neq 0$ in R: any tensor $k \otimes x = (k/d)d \otimes x = (k/d) \otimes dx = 0$. On the other hand, $K \otimes_R R \cong K$: the isomorphism sends $k \otimes r$ to kr (the map sending (k, r) to kr is clearly bilinear out of $K \times R$, and induces an isomorphism from the tensor product to K. Thus tensoring M with K replaces each copy of R in M with K, so the number of copies of R in M cannot depend on the choice of decomposition of M. As for the torsion submodule T of M, it too can be analyzed via tensor products instead of by quotients as we did in class last week: $R/(d) \otimes_R R/(e) \cong R/(c)$, where c is a gcd of d and e (you will prove a special case of this in homework), whence by working with various powers p^k of irreducible elements in R you can show that the number of copies of $R/(p^k)$ occurring in M, or in its torsion submodule, is independent of the choice of decomposition of M.

The construction of the tensor product extends in a natural way to any finite set $M_1, \ldots, M_k$ of R-modules: multinear maps from $M_1 \times \cdots \times M_k$ to an R-module P correspond bijectively to R-linear maps from the tensor product $M_1 \otimes \cdots \otimes M_k$ to P. Here I leave the definitions of "multilinear" and tensor product in this setting to you. In the special case of copies of the *same* module M there are a couple of important related constructions called symmetric and exterior powers. A bilinear map f from $M \times M$ to P is called *symmetric* if $f(m, n) = f(n, m)$ for all $m, n \in M$. Such maps are in bijection to linear maps, not from the tensor product $M \otimes M$ to P, but from the quotient of it by the submodule generated by all differences $m \otimes n - n \otimes m$ as m, n range over M; the resulting quotient is denoted S^2M and is called the *symmetric square* of M. The elements of S^2M are usually denoted in the same way as elements of $M \otimes M$ (the latter is sometimes denoted T^2M and called the *tensor square* of M), but it is understood that an element $m \otimes n$ of S^2M, called a symmetric tensor, is identified with $n \otimes m$. In a similar manner, higher symmetric powers S^kM are defined as quotients of $T^kM = M \otimes \cdots \otimes M$ by the submodule generated by suitable differences and the elements of S^kM are again called symmetric tensors. A similar but often more useful construction results if we instead look at *alternating* bilinear maps f from $M \times M$ to P, satisfying $f(m, m) = 0, f(m, n) = -f(n, m)$ for all $m, n \in M$. Here one should replace S^2M by the quotient of $M \otimes M$ by the submodule generated by all sums $m \otimes n + n \otimes m$ and tensors $m \otimes m$; the resulting module is denoted by $\bigwedge^2 M$. In a similar way one defines alternating multilinear maps from $M \times \cdots \times M$ to P and there is a bijection between them and linear maps from the *exterior power* $\bigwedge^k M$ to P. As for tensor products, we find that both symmetric and exterior powers of free R-modules are free, but this time their ranks behave very differently. In fact, if M is free of rank n, then in particular $\bigwedge^n M$ is free of rank 1 and $\bigwedge^k M = 0$ if $k > n$! (You will work out the formula for the rank of an arbitrary power $\bigwedge^k R^n$ in homework for next week and exhibit a basis for it.) That $\bigwedge^n R^n \cong R$ is often expressed by saying that *the determinant is the only alternating multilinear function from the columns of an $n \times n$ matrix over R to R, up to scalar multiple.*

Taking the direct sum of all the tensor powers of M (with the 0th power being by definition the base ring R) we get the *tensor algebra TM of M*, so called because it admits a natural multiplication, taking the product of $x_1 \otimes \cdots \otimes x_n$ and $y_1 \otimes \cdots \otimes y_m$ to be $x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_n$. In a similar manner one obtains the *symmetric* and *exterior* algebras of M, denoted respectively by SM and $\bigwedge M$. If $R = K$ is a field and M is finite-dimensional over it, say with basis $x_1, \ldots, x_n$, then the symmetric algebra SM may be identified in a natural way with the ring of polynomials $K[x_1, \ldots, x_n]$ over K in the variables $x_1, \ldots, x_n$, while the exterior algebra $\bigwedge M$ is finite-dimensional over K, in fact of dimension 2^n.