

## Math 505, 2/6

For the rest of the course let  $K$  be an algebraically closed field. Given varieties  $V, W$ , lying in  $K^n, K^m$ , respectively, we need to say what mappings are allowed between  $V$  and  $W$ . A natural choice is the polynomial maps (sending  $v \in V$  to  $w = (f_1(v), \dots, f_m(v))$  for some  $f_i \in K[x_1, \dots, x_n]$ ); these are called *morphisms*. Any choice of the  $f_i$  will define a morphism from  $K^n$  to  $K^m$ , but in order to define a map from  $V$  to  $W$ , it must be the case that if  $v$  is a common zero of the (radical) ideal  $I$  corresponding to  $V$ , then  $w$  must be a common zero of the (radical) ideal  $J$  corresponding to  $W$ . We now observe that the  $f_i$  also define a unique homomorphism from the polynomial ring  $K[y_1, \dots, y_m]$  to  $K[x_1, \dots, x_n]$ , sending  $y_i$  to  $f_i$ , and that every such homomorphism takes this form for unique  $f_1, \dots, f_m$ ; it will induce a map from  $V$  to  $W$  if and only if it takes the ideal  $J$  in  $K[y_1, \dots, y_m]$  to  $I$ , or equivalently it induces a well-defined map from  $K[W] = K[y_1, \dots, y_m]/J$  to  $K[V] = K[x_1, \dots, x_n]/I$ . We deduce that *there is a natural 1-1 correspondence between morphisms from  $V$  to  $W$  and algebra homomorphisms from  $K[W]$  to  $K[V]$*  (so that the map sending  $V$  to its coordinate ring  $K[V]$  is a contravariant functor, in the language we used last quarter). In particular, our morphism from  $V$  to  $W$  is an isomorphism (i.e. has an inverse which is also a morphism) if and only if the homomorphism from  $K[W]$  to  $K[V]$  is an algebra isomorphism.

The Noether normalization theorem that we used to prove the weak Hilbert Nullstellensatz furnishes some especially interesting examples of morphisms. We have shown that the coordinate ring  $K[V]$  of any variety  $V$  is a (finitely generated) integral extension of some polynomial ring  $K[x_1, \dots, x_m]$  over  $K$ , so that there is a natural inclusion of  $K[x_1, \dots, x_m]$  in  $K[V]$ ; going backwards to the corresponding morphism, we see that *there is a surjective morphism  $\pi$  from  $V$  to the affine space  $K^m$* , which we will later see has finite fibers (i.e. the inverse image  $\pi^{-1}(v)$  of any  $v \in K^m$  is finite. (We will also see that the integer  $m$  here is uniquely determined by  $V$  and is naturally enough called its *dimension*.) We call  $\pi$  a *ramified finite cover*; it is analogous to the covering maps one studies in topological manifolds, but is less well behaved and in particular does *not* define a local homeomorphism between any neighborhoods in  $V$  and  $K^m$ . For example, look at the variety  $W$  in  $K^2$  consisting of the zeros of the single polynomial  $x^2 - y^3$ . There are two obvious surjective morphisms from  $W$  to the line  $K^1$ , given by the projections  $\pi_1, \pi_2$  onto the first and second coordinates, respectively. The first map is generically a triple cover; for any  $x \neq 0$  there will be three distinct  $y \in K$  with  $x^2 = y^3$ , but for  $x = 0$  there is only one such  $y$ . Likewise, for any  $y \neq 0$  there are generically two distinct  $x \in K$  with  $x^2 = y^3$ , but for  $y = 0$  there is only one such  $x$ . It is because the fibers have different sizes that we call such a cover ramified (=branched, in some sense). There is another very interesting algebra map, this time from  $K[W]$  to  $K[x]$ , which you will define in homework for this week. The corresponding morphism is bijective but not an isomorphism, since its inverse is not a morphism. As another example, look at the variety  $V$ , again in  $K^2$ , defined by the equation  $xy = 1$ . This variety again admits projections  $\pi_1, \pi_2$  to the first

and second coordinates, but this time the  $\pi_i$  are not surjective. Accordingly the corresponding algebra maps, sending  $K[x], K[y]$  respectively to their canonical images in  $K[V] \cong K[x, y]/(xy - 1) \cong K[x, x^{-1}]$ , a localization of  $K[x]$ , does *not* realize  $K[V]$  as an integral extension of either image (though  $K[V]$  is in fact an integral extension of the polynomial ring  $K[z]$  for a different embedding of  $K[z]$  in  $K[V]$ , as you will work out in another homework problem). We use these maps later to give the image  $K^*$  of  $\pi_1$  or  $\pi_2$  the structure of an affine variety; note that  $K^*$  is *not*  $V(I)$  for any ideal  $I$  of  $K[x]$ . More generally, let  $V$  be any affine variety and  $f \in K[V], f \neq 0$ . The set  $V_f = \{v \in V : f(v) \neq 0\}$  is then a Zariski-open (not closed) subset of  $V$  and as such not an affine algebraic variety by our definition. Nevertheless we (now) call  $V_f$  an affine variety and attach to it the coordinate ring  $K[V]_f$  (the localization of  $K[V]$  by all powers of  $f$ ; this is still a finitely generated  $K$ -algebra. We regard  $V_f$  as isomorphic to the affine algebraic variety with coordinate ring  $K[V]_f$ ; this justifies calling it affine.