Math 505, 2/8

For the rest of the course let K be an algebraically closed field. Given varieties V, W, lying in K^n, K^m, respectively, we need to say what mappings are allowed between V and W. A natural choice is the polynomial maps (sending $v \in V$ to $w = (f_1(v), \ldots, f_m(v)$ for some $f_i \in K[x_1 \ldots, x_n]$); these are called *morphisms*. Any choice of the f_i will define a morphism from K^n to K^m, but in order to define a map from V to W, it must be the case that if v is a common zero of the (radical) ideal I corresponding to V, then w must be a common zero of the (radical) ideal J corresponding to W. We now observe that the f_i also define a unique homomorphism from the polynomial ring $K[y_1, \ldots, y_m]$ to $K[x_1, \ldots, x_n]$, sending y_i to f_i, and that every such homomorphism takes this form for unique $f_1, \ldots, f_m$; it will induce a map from V to W if and only if it takes the ideal J in $K[y_1, \ldots, y_m]$ to I, or equivalently it induces a well-defined map from $K[W] = K[y_1, \ldots, h_m]/J$ to $K[V] = K[x_1, \ldots, x_n]/I$. We deduce that *there is a natural 1-1 correspondence between morphisms from V to W and algebra homomorphisms from $K[W]$ to $K[V]$* (so that the map sending V to its coordinate ring $K[V]$ is a contravariant functor, in the language we used last quarter). In particular, our morphism from V to W is an isomorphism (i.e. has an inverse which is also a morphism) if and only if the homomorphism from $K[W]$ to $K[V]$ is an algebra isomorphism.

The Noether normalization theorem that we used to prove the weak Hilbert Nullstellensatz furnishes some especially interesting examples of morphisms. We have shown that the coordinate ring $K[V]$ of any variety V is a (finitely generated) integral extension of some polynomial ring $K[x_1, \ldots, x_m]$ over K, so that there is a natural inclusion of $K[x_1, \ldots, x_m]$ in $K[V]$; going backwards to the corresponding morphism, we see that *there is a surjective morphism π from V to the affine space K^m*, which we will later see has finite fibers (i.e. the inverse image $\pi^{-1}(v)$ of any $v \in K^m$ is finite. (We will also see that the integer m here is uniquely determined by V and is naturally enough called its *dimension*.) We call π a *ramified finite cover*; it is analogous to the covering maps one studies in topological manifolds, but is less well behaved and in particular does *not* define a local homeomorphism between any neighborhoods in V and K^m. For example, look at the variety W in K^2 consisting of the zeros of the single polynomial $x^2 - y^3$. There are two obvious surjective morphisms from W to the line K^1, given by the projections π_1, π_2 onto the first and second coordinates, respectively. The first map is generically a triple cover; for any $x \neq 0$ there will be three distinct $y \in K$ with $x^2 = y^3$, but for $x = 0$ there is only one such y. Likewise, for any $y \neq 0$ there are generically two distinct $x \in K$ with $x^2 = y^3$, but for $y = 0$ there is only one such x. It is because the fibers have different sizes that we call such a cover ramified (=branched). There is another very interesting algebra map, this time from $K[W]$ to $K[x]$, which you will define in homework for this week. The corresponding morphism is bijective but not an isomorphism, since its inverse is not a morphism. As another example, look at the variety V, again in K^2, defined by the equation $xy = 1$. This variety again admits projections π_1, π_2 to the first and second coordinates, but this time the π_i are not surjective. Accordingly the corresponding algebra maps, sending $K[x], K[y]$ respectively to their canonical images in $K[V] \cong K[x, y]/(xy - 1) \cong K[x, x^{-1}]$, a localization of $K[x]$, does *not* realize $K[V]$ as an integral extension of either image (though $K[V]$ is in fact an integral extension of the polynomial ring $K[z]$ for a different embedding of $K[z]$ in $K[V]$, as you will work out in another homework problem). We can use these maps to give the image K^* of π_1 or π_2 the structure of an affine variety, as follows; note that K^* is not a closed subset of K and thus not as it stands affine. More generally, let W be any affine variety in K^n and W' its intersection with a (Zariski-)open subset U of K^n. Call an element f/g of the quotient field of $K[W]$ *regular on* W' if the denominator g never vanishes at any point of W'; denote by $K[W']$ the algebra of regular functions on W'. In practice we can restrict to the case where U is principal open; that is, it is the set of nonzeros of a single polynomial f; then $K[W']$ is just the localization $K[W]_f$ of $K[W]$ by powers of f. This is a finitely generated K-algebra, isomorphic to the coordinate ring of the variety $V = \{(x_1, \ldots, x_{n+1} \in K^{n+1} : (x_1, \ldots, x_n) \in W, f(x_1, \ldots, x_n)x_{n+1} = 1\}$; accordingly we declare that W' is an affine variety isomorphic to V with the morphism from V' to W' given by projection onto the first n coordinates.

Still more generally, we call any intersection $V \cap U$ of an affine variety with an open subset a *quasi-affine* variety; in general it is not isomorphic to any affine variety. We will look at another class of naturally occurring non-affine varieties later (the projective ones).