Math 505, 3/1

We now explore the behavior of dimension in more detail in the projective setting, but before we do this we digress to generalize to finitely generated modules M over Noetherian rings A our earlier recipe attaching finitely many prime ideals of A to any ideal I of it (the minimal primes over I). So let M be such a module. The set of annihilator ideals $A_m = \{x \in A : xm = 0\}$ as m runs through teh nonzero elements of M must have a largest element, say A_{m_1}, which we claim must be a prime ideal. Indeed, if $xym_1 = 0$ for some $x, y \in A$ but $xm_1, ym_1 \neq 0$, then A_{ym_1} is strictly larger than A_{m_1}, a contradiction. Then the submodule Am_1 of M generated by m_1 takes the form A/P_1 for some prime ideal P_1. Moding out by Am_1 and repeating this procedure, we find that the quotient $M' = M/Am_1$ admits an element m_2 whose annihilator P_2 is another prime ideal. Moding M' out by Am_2 and continuing, we get a chain $M_0 = 0 \subset M_1 \subset M_2, \subset \cdots$ of submodules of M such that eauch quotient $M_i/M_{i-1} \cong A/P_i$ for some prime ideal P_i of A. But there are no strictly increasing chains of submodules of M, so the above chain of submodules terminates at some $M_n = M$. Now the ideals $P_1, \ldots, P_n$ arising from the chain are not uniquely determined, but if P is one of them and we localize M at P (letting M_P consist of all formal fractions m/s with $m \in M, s \in A, s \notin P$, decreeing that $m/s = n/t$ if there is $u \notin P$ with $u(tm - ns) = 0$ in M) then we find that all quotients A/P_i disappear under this operation if P_i is not contained in P, while if $P_i = P$ then the localization is the fraction field of A/P, which is also the quotient of the localized ring A_P at its maximal ideal PA_P. The upshot is that *for any chain $M_0 = 0 \subset \cdots \subset M_n = M$ of submodules of M as above with $M_i/M_{i-1} \cong A/P_i, P_i$ prime, then the ideals P among $P_1, \ldots, P_n$ not containing any others are uniquely determined, each along with the number $\mu_P(M)$ of times it occurs as a P_i (and in fact $\mu_P(M)$ is the length of M_P over the local ring A_P).* We call the minimal ideals among the P_i the *associated primes* of M. Now if A and M both happen to be graded, then we can carry out the above construction considering only *homogeneous* elements of M throughout and observing under this restriction all quotients of M retain the grading. Thus we arrive at a finite collection of graded prime ideals $P_1, \ldots, P_n$ attached to M whose minimal elements P together with their multiplicities $\mu_P(M)$ are uniquely determined by M.

Now if $V \subset \mathbb{P}^n$ is a projective variety and $M = S/I$ is its homogeneous coordinate ring, then M is in particular a graded S-module, whence by the machinery developed last week it has a Hilbert polynomial p_M which is such that if $m \in \mathbb{Z}$ is sufficiently large, then $p_M(m)$ equals the dimension over K of the mth graded piece M_m of M. The degree d of p_M is then the dimension of V (since we have seen that the same ideal I, viewed as an ordinary radical ideal in $K[x_1, \ldots, x_{n+1}]$ has as its variety the affine cone $C(V)$, which has dimension $d+1$). We also saw last week that the leading coefficient of p_M equals $r/d!$ with r a positive integer; we call it the *degree* of V; unlike the dimension of V, this number depends on the way V is embedded in projective space and not just on V itself. If V is reducible with irreducible components $V_1, \ldots, V_r$, corresponding to the prime ideals $P_1, \ldots, P_r$ containing I, then the discussion in the above paragraph shows that the degree of V is given by $\sum_i \mu_{P_i}(S/I)d_i$, where d_i is the degree of the variety corresponding to P_i, since here the P_i are exactly the minimal primes containing I. Now the Hilbert polynomial $p_S(m)$ of S itself is easily computed to be the binomial coefficient $\binom{m+n}{n}$, so $\mathbb{P}^n$ itself has degree 1. A hypersurface H in $\mathbb{P}^n$ defined by a single homogeneous polynomial of degree d has Hilbert polynomial $p_S(m) - p_S(m-d)$ whence it has degree d as well. The union of two varieties X_1, X_2 of the same dimension d such that $X_1 \cap X_2$ has dimension less than d has degree the sum of the degrees of X_1 and X_2. Finally, suppose we take the intersection $Y \cap H$ of an irreducible variety $Y \subset \mathbb{P}^n$ of dimension d and a hypersurface H not containing it. We have seen that the irreducible components $Z_1, \ldots, Z_r$ of $Y \cap H$ all have dimension $d-1$; defining the *intersection multiplicity* $i(Y, H; Z_j)$ to be the length $\mu_{P_j}(S/(I_Y + I_H)$, where $I_y.I_h$ are the ideals corresponding to Y, H and each P_j is the prime ideal corresponding to Z_j, then

$$\sum_{j=1}^{r} i(Y, H; Z_j) = d_Y d_H$$

where d_Y, d_H are the respective degrees of Y and H. In particular, *two curves in $\mathbb{P}^2$, defined by homogeneous polynomials of degrees d, e, possibly reducible but having no irreducible component in common, intersect in exactly de points if these are counted with appropriate multiplicities*. Thus curves in $\mathbb{P}^2$, in stark contrast to curves in K^2, intersect in a very nice and uniform way.