# Math 505, 2/10

A couple of further remarks about the variety $V \subset K^2$ defined last time, with equation $x^2 - y^3 = 0$: its coordinate ring $K[x,y]/(x^2 - y^3)$ is almost, but not quite, a Dedekind domain. It is Noetherian of dimension 1, but is not integrally closed, as $(x/y)^2 = y$ in its quotient field, so that $x/y$ is integral over this ring but not in it. Later we will see that $V$ is bad for another reason: it has exactly one singular point, at $(0,0)$, which is such that its tangent space is two-dimensional, though $V$ itself has dimension one.

Now let $V, W$ be affine varieties. We investigate the situation in which the coordinate ring $K[V]$ is a finitely generated integral extension of $K[W]$ in more detail; still more generally, let $A, B$ be any two commutative rings with $B$ a finitely generated integral extension of $A$. Let $M$ be a maximal ideal of $A$. Then *there is at least one but only finitely many maximal ideals $N$ of $B$ such that $N \cap A = M$; we say that there are only finitely many ideals of $B$ lying over $M$ in this situation. To see this, let $S$ be the complement of $M$ in $A$. Then the localization $B' = S^{-1}B$ is again a finitely generated integral extension of $A' = S^{-1}A$; it is integral because if $b \in B, s \in S$, and if $b^n = \sum_i a_i b^i$ with the $a_i$ in $A$, then $(b/s)^n = \sum_i (a_i/s^{n-i})(b/s)^i$. Given any maximal ideal $N'$ of $B'$, its intersection $M'$ with $A'$ must be maximal, since $B'/N'$ is integral over $A'/M'$; but $MA'$ is the only maximal ideal of $A'$, so we must have $M' = MA'$. Now $N'$ corresponds to a prime ideal of $B'$ not meeting $S$, whose intersection with $A$ is $M$; since $B/N$ is integral over $A/M$, we deduce that $N$ is maximal in $B$, as desired. If $B$ is generated over $A$ by $b_1, \ldots, b_n$, satisfying monic polynomials $f_1, \ldots, f_n$ with coefficients in $A$, then passing to the quotient $B/N$ of $B$ by any maximal ideal lying over $M$, we see that $B/N$ embeds into the splitting field $S$ of $A/M$ of the product $f_1 \cdots f_n$ of the $f_i$ (with coefficients reduced mod $M$). As there can only be finitely many such embeddings, there are only finitely many possibilities for $B/N$ and accordingly only finitely many maximal ideals $N$ lying over $M$; of course there are no inclusions among any two such ideals $N$, since both are maximal. The same reasoning shows more generally that any prime ideal $P$ of $A$ has at least one but only finitely many prime ideals $Q$ of $B$ lying over it, and none of these is contained in another (since $PA_P$ is the only maximal ideal of $A_P$). Since the coordinate ring $K[V]$ of any variety $V$ is a finitely generated integral extension of $K[y_1, \ldots, y_m]$ for some $m$, this proves our earlier claim that $V$ is a ramified finite cover of $K^m$ in this situation. The ramification arises since the number of maximal ideals $N$ of $K[V]$ lying over a fixed one $M$ in $K[y_1, \ldots, y_m]$ can vary with $M$.*

*On the other hand, if the commutative ring $B$ is the polynomial ring $A[x]$ in one variable over $A$, then given any prime ideal $P$ of $A$ there are two prime ideals $Q_1, Q_2$ of $B$ lying over $P$ with $Q_1$ properly contained in $Q_2$, but there are never three such ideals forming a strictly ascending chain. Indeed, moding out by $P$ and then localizing by all elements of $A$ not in $P$, we reduce to the case where $A = K$ is a field and $P = 0$; then the result is obvious since $K[x]$ is a PID. Later we will see that given any irreducible variety $V$ (whose coordinate ring $K[V]$ is thus an integral domain), any two saturated chains of* *prime ideals in $K[V]$, that is any two strictly ascending chains $P_1 \subset \cdots \subset P_m, Q_1 \subset \cdots \subset Q_n$ of prime ideals in $K[V]$ with no prime ideals properly between any two consecutive $P_i$ or $Q_j$, have the same length $m = n$. This length is called the dimension of the variety $V$ and it is $n$ if $K[V]$ is a finitely generated integral extension of a polynomial ring $K[y_1, \ldots, y_n]$. If $V$ is not irreducible, then its dimension is defined to b e the maximum dimension of any of its irreducible components. For example, the dimension of the variety $V(xz, yz)$ of the ideal $(xz, yz)$ in $K[x, y, z]$ is 2, since this variety is the union of the plane $x = 0$ and the line $y = z = 0$; we see from this example that the irreducible components of a variety need not have the same dimension, even if these components overlap. If the components do have the same dimension then we call the variety equidimensional; we will see that this holds under fairly general circumstances. (For example, the components of the $(n-1)$-dimensional variety defined by the equation $p = 0$ with $p \in K[x_1, \ldots, x_n]$ are defined by the equations $p_i = 0$, where the $p_i$ are the irreducible factors of $p$; each has dimension $n-1$; later we will see that the same thing happens when we intersect an irreducible variety $V$ with a hypersurface (defined by the vanishing of a single polynomial)).*

*Still more generally, given any commutative ring $A$ we define its (Krull) dimension to be the largest $n$ such that there is a strictly ascending chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of prime ideals in $A$, or $\infty$ if arbitrarily long such chains exist. Then it turns out the dimension even of a Noetherian ring can be infinite, but if we fix a prime ideal $P$ of the Noetherian ring $A$, then strictly ascending chains of prime ideals ending in $P$ are bounded in length (equivalently, the dimension of the localization $A_P$ is always finite; or there are no infinite strictly descending chains of prime ideals in $A$).*