Math 505, 3/3

We conclude the new material in the course by exploring the behavior of dimension in more detail in the projective setting, but before we do this we digress to generalize to finitely generated modules M over Noetherian rings A our earlier recipe attaching finitely many prime ideals of A to any ideal I of it (the minimal primes over I). So let M be such a module. The set of annihilator ideals $A_m = \{x \in A : xm = 0\}$ as m runs through the nonzero elements of M must have a largest element, say A_{m_1}, which we claim must be a prime ideal. Indeed, if $xym_1 = 0$ for some $x, y \in A$ but $xm_1, ym_1 \neq 0$, then A_{ym_1} is strictly larger than A_{m_1}, a contradiction. Then the submodule Am_1 of M generated by m_1 takes the form A/P_1 for some prime ideal P_1. Moding out by Am_1 and repeating this procedure, we find that the quotient $M' = M/Am_1$ admits an element m_2 whose annihilator P_2 is another prime ideal. Moding M' out by Am_2 and continuing, we get a chain $M_0 = 0 \subset M_1 \subset M_2, \subset \cdots$ of submodules of M such that eauch quotient $M_i/M_{i-1} \cong A/P_i$ for some prime ideal P_i of A. But there are no strictly increasing chains of submodules of M, so the above chain of submodules terminates at some $M_n = M$. Now the ideals $P_1, \ldots, P_n$ arising from the chain are not uniquely determined, but if P is one of them and we localize M at P (letting M_P consist of all formal fractions m/s with $m \in M, s \in A, s \notin P$, decreeing that $m/s = n/t$ if there is $u \notin P$ with $u(tm - ns) = 0$ in M) then we find that all quotients A/P_i disappear under this operation if P_i is not contained in P, while if $P_i = P$ then the localization is the fraction field of A/P, which is also the quotient of the localized ring A_P at its maximal ideal PA_P. The upshot is that *for any chain $M_0 = 0 \subset \cdots \subset M_n = M$ of submodules of M as above with $M_i/M_{i-1} \cong A/P_i, P_i$ prime, then the ideals P among $P_1, \ldots, P_n$ not containing any others are uniquely determined, each along with the number $\mu_P(M)$ of times it occurs as a P_i (and in fact $\mu_P(M)$ is the length of M_P over the local ring A_P).* We call the minimal ideals among the P_i the *associated primes* of M. Now if A and M both happen to be graded, then we can carry out the above construction considering only *homogeneous* elements of M throughout and observing under this restriction all quotients of M retain a grading. Thus we arrive at a finite collection of graded prime ideals $P_1, \ldots, P_n$ attached to M whose minimal elements P together with their multiplicities $\mu_P(M)$ are uniquely determined by M.

Now if $V \subset \mathbf{P}^n$ is a projective variety and $M = S/I$ is its homogeneous coordinate ring, then M is in particular a graded S-module, whence by the machinery developed last week it has a Hilbert polynomial p_M which is such that if $m \in \mathbf{Z}$ is sufficiently large, then $p_M(m)$ equals the dimension over K of the mth graded piece M_m of M. The degree d of p_M is then the dimension of V (since we have seen that the same ideal I, viewed as an ordinary radical ideal in $K[x_1, \ldots, x_{n+1}]$ has as its variety the affine cone $C(V)$, which has dimension $d+1$). We also saw last week that the leading coefficient of p_M equals $r/d!$ with r a positive integer; we call it the *degree* of V; unlike the dimension of V, this number depends on the way V is embedded in projective space and not just on V itself. If V is reducible with irreducible components $V_1, \ldots, V_r$, corresponding to the prime ideals $P_1, \dots, P_r$ containing I, then the discussion in the above paragraph shows that the degree of V is given by $\sum_i \mu_{P_i}(S/I)d_i$, where d_i is the degree of the variety corresponding to P_i, since here the P_i are exactly the minimal primes containing I. Now the Hilbert polynomial $p_S(m)$ of S itself is easily computed to be the binomial coefficient $\binom{m+n}{n}$, so $\mathbf{P}^n$ itself has degree 1. A hypersurface H in $\mathbf{P}^n$ defined by a single homogeneous polynomial of degree d has Hilbert polynomial $p_S(m) - p_S(m-d)$ whence it has degree d as well. The union of two varieties X_1, X_2 of the same dimension d such that $X_1 \cap X_2$ has dimension less than d has degree the sum of the degrees of X_1 and X_2. Finally, suppose we take the intersection $Y \cap H$ of an irreducible variety $Y \subset \mathbf{P}^n$ of dimension d and a hypersurface H not containing it. We have seen that the irreducible components $Z_1, \dots, Z_r$ of $Y \cap H$ all have dimension $d-1$; defining the *intersection multiplicity* $i(Y, H; Z_j)$ to be the length $\mu_{P_j}(S/(I_Y + I_H)$, where $I_Y I_H$ are the ideals corresponding to Y, H and each P_j is the prime ideal corresponding to Z_j, then

$$\sum_{j=1}^{r} i(Y, H; Z_j) = d_Y d_H$$

where d_Y, d_H are the respective degrees of Y and H. In particular, *two curves in $\mathbf{P}^2$, defined by homogeneous polynomials of degrees d, e, possibly reducible but having no irreducible component in common, intersect in exactly de points if these are counted with appropriate multiplicities*. Thus curves in $\mathbf{P}^2$, in stark contrast to curves in K^2, intersect in a very nice and uniform way.