Math 505, 2/3

We now know that every proper ideal I of $R = K[x_1, \ldots, x_n]$ has a nonempty variety $V(I)$ of common zeros, but the map from I to $V(I)$ is not 1-1, even if $n = 1$. The best we can do in that case is to send the ideal I generated by the product of *distinct* linear polynomials $x - a_1, \ldots, x - a_m$ to the finite subset $\{a_1, \ldots, a_m\}$ of K; this is a bijection between certain ideals of R and all (affine algebraic) varieties. To define the higher-dimensional analogue of begin generated by a product of distinct linear polynomials, we need a general ring-theoretic notion Given any ideal I in a commutative ring A, its *radical* $\sqrt{I}$ consists by definition of all $x \in A$ such that $x^n \in I$ for some n; commutativity of A and the binomial theorem guarantee that $\sqrt{I}$ is indeed an ideal and $\sqrt{\sqrt{I}} = \sqrt{I}$. We call the ideal I *radical* if it equals its own radical. Now we can finally specify precisely which ideals I of R we will focus on, namely the radical ones. Then the strong form of the Nullstellensatz proved last time states that *if $f \in R$ is such that $f(v) = 0$ for all $v \in V(I)$, then $f \in \sqrt{I}$; thus the map $I \to V(I)$ implements an inclusion-reversing bijection between proper radical ideals of R and varieties in K^n*. To prove this, we need a fact about radicals of ideals in general commutative rings: *the radical of an ideal I in a ring A is the intersection of all prime ideals containing I*. Indeed, any prime ideal P containing I contains any element of its radical, by definition of prime ideal; conversely, if no power of x lies in I, let J be an ideal of A maximal with respect to exclusion of all powers of x. Then J is prime, for any ideals J_1, J_2 properly larger than J contain powers of x, say x^{n_1}, x^{n_2}, but then $J_1 J_2$ contains $x^{n_1+n_2}$ and so does not lie in J. Returning now to the polynomial ring R, suppose that no power of $f \in R$ lies in the ideal I and let P be a prime ideal of R containing I but not f. Then P generates a proper prime ideal PR_f in R_f, the localization of R at all powers of f; enlarging this to a maximal ideal M of R_f, we know that the quotient of R_f by M must be isomorphic to K, whence we get a point $(a_1, \ldots, a_n)$ in K^n at which all polynomials in I vanish but f does not, as desired. (Note also that the variety $V(I)$ of any ideal coincides with the variety $V(\sqrt{I})$ of its radical, so we don't lose any varieties by restricting to radical ideals.)

Now that we finally know that our map $I \to V(I)$ is a bijection if the ideal I is suitably restricted, we can record some simple properties of this map. Clearly the variety V of a sum $\sum I_i$ ideals (even an infinite sum) is the intersection of the varieties $V(I_i)$ of the I_i, while the variety $V(I_1 \cdots I_m)$ is the union of the $V(I_i)$. Likewise the variety $V(I_1 \cap \cdots \cap I_m)$ of the intersection of the I_i is the union of the $V(I_i)$; in fact the intersection of the I_i is exactly the radical of their product, if the I_i are radical. This says exactly that the varieties $V(I)$ of all ideals in R (including R itself) obey the axioms for the *closed* sets in a topology on K^n, which we call the *Zariski topology*; the open sets in this topology are of course the complements of the closed ones. If $K = \mathbb{C}$, then every Zariski-open subset is also open in the Euclidean topology; but the converse is quite false; in a nutshell, the difference is that nonempty Zariski-open subsets of $\mathbb{C}^n$ have to be very "fat", in the sense that they are dense in both the Zariski and Euclidean topologies. For $n = 1$, the Zariski-closed subsets of K are precisely the finite subsets together with K itself; in general, points in K^n are always closed in the Zariski topology, but this topology is not Hausdorff. It has a very special property with no counterpart in the Euclidean topology, namely that K^n is *Noetherian*: there are no strictly descending infinite chains of closed subsets (since there are no strictly ascending chains of ideals in R, as you will prove in homework next week). It follows that every closed set in this topology is a finite union of *irreducible* closed sets, none of these being the union of two closed proper subsets. (If there were any closed subsets that are not finite unions of irreducible sets, there would be a smallest such closed set S, which would have to be reducible; but then both of the proper subsets of S would be finite unions of irreducible sets and so S would be also, a contradiction.) The irreducible subvarieties whose union is a given variety are called its *irreducible components*; they are unique if we impose the natural requirement that none of them be contained in another. They behave somewhat like connected components of a topological space, but are *not* disjoint, in general; for example, the variety defined by the equation $xy = 0$ in K^2 is the union of the coordinate axes, which intersect at the origin. More generally, the variety $V(p)$ of a nonzero principal ideal (p) in R has as its irreducible components $V(p_i)$, where the p_i are the unique monic irreducible factors of p in R (recall that R is a unique factorization domain). Since we have seen that the variety of the intersection of radical ideals is the union of the varieties of the ideals, we deduce that *every radical ideal in R is a finite intersection of prime ideals, where these are unique if none of them is allowed to contain another*. We also see that *a variety $V(I)$ is irreducible if and only if its coordinate ring is an integral domain, or equivalently if and only if $\sqrt{I}$ is prime*. In the literature varieties are often taken to be irreducible by definition.