Math 505, 2/10

A couple of further remarks about the variety $V \subset K^2$ defined last time, with equation $x^2 - y^3 = 0$: its coordinate ring $K[x,y]/(x^2 - y^3)$ is almost, but not quite, a Dedekind domain. It is Noetherian of dimension 1, but is not integrally closed, as $(x/y)^2 = y$ in its quotient field, so that x/y is integral over this ring but not in it. Later we will see that V is bad for another reason: it has exactly one singular point, at $(0,0)$, which is such that its tangent space is two-dimensional, though V itself has dimension one.

Now let V, W be affine varieties. We investigate the situation in which the coordinate ring $K[V]$ is a finitely generated integral extension of $K[W]$ in more detail; still more generally, let A, B be any two commutative rings with B a finitely generated integral extension of A. Let M be a maximal ideal of A. Then *there is at least one but only finitely many maximal ideals N of B such that $N \cap A = M$; we say that there are only finitely many ideals of B lying over M in this situation. To see this, let S be the complement of M in A. Then the localization $B' = S^{-1}B$ is again a finitely generated integral extension of $A' = S^{-1}A$; it is integral because if $b \in B, s \in S$, and if $b^n = \sum_i a_i b^i$ with the a_i in A, then $(b/s)^n = \sum_i (a_i/s^{n-i})(b/s)^i$. Given any maximal ideal N' of B', its intersection M' with A' must be maximal, since B'/N' is integral over A'/M'; but MA' is the only maximal ideal of A', so we must have $M' = MA'$. Now N' corresponds to a prime ideal of B' not meeting S, whose intersection with A is M; since B/N is integral over A/M, we deduce that N is maximal in B, as desired. If B is generated over A by $b_1, \ldots, b_n$, satisfying monic polynomials $f_1, \ldots, f_n$ with coefficients in A, then passing to the quotient B/N of B by any maximal ideal lying over M, we see that B/N embeds into the splitting field S of A/M of the product $f_1 \cdots f_n$ of the f_i (with coefficients reduced mod M). As there can only be finitely many such embeddings, there are only finitely many possibilities for B/N and accordingly only finitely many maximal ideals N lying over M; of course there are no inclusions among any two such ideals N, since both are maximal. The same reasoning shows more generally that any prime ideal P of A has at least one but only finitely many prime ideals Q of B lying over it, and none of these is contained in another (since PA_P is the only maximal ideal of A_P). Since the coordinate ring $K[V]$ of any variety V is a finitely generated integral extension of $K[y_1, \ldots, y_m]$ for some m, this proves our earlier claim that V is a ramified finite cover of K^m in this situation. The ramification arises since the number of maximal ideals N of $K[V]$ lying over a fixed one M in $K[y_1, \ldots, y_m]$ can vary with M.*

On the other hand, if the commutative ring B is the polynomial ring $A[x]$ in one variable over A, then given any prime ideal P of A there are two prime ideals Q_1, Q_2 of B lying over P with Q_1 properly contained in Q_2, but there are never three such ideals forming a strictly ascending chain. Indeed, moding out by P and then localizing by all elements of A not in P, we reduce to the case where $A = K$ is a field and $P = 0$; then the result is obvious since $K[x]$ is a PID. Later we will see that given any irreducible variety V (whose coordinate ring $K[V]$ is thus an integral domain), any two saturated chains of *prime ideals in $K[V]$, that is any two strictly ascending chains $P_1 \subset \cdots \subset P_m, Q_1 \subset \cdots \subset Q_n$ of prime ideals in $K[V]$ with no prime ideals properly between any two consecutive P_i or Q_j, have the same length $m = n$. This length is called the dimension of the variety V and it is n if $K[V]$ is a finitely generated integral extension of a polynomial ring $K[y_1, \ldots, y_n]$. If V is not irreducible, then its dimension is defined to b e the maximum dimension of any of its irreducible components. For example, the dimension of the variety $V(xz, yz)$ of the ideal (xz, yz) in $K[x, y, z]$ is 2, since this variety is the union of the plane $x = 0$ and the line $y = z = 0$; we see from this example that the irreducible components of a variety need not have the same dimension, even if these components overlap. If the components do have the same dimension then we call the variety equidimensional; we will see that this holds under fairly general circumstances. (For example, the components of the $(n-1)$-dimensional variety defined by the equation $p = 0$ with $p \in K[x_1, \ldots, x_n]$ are defined by the equations $p_i = 0$, where the p_i are the irreducible factors of p; each has dimension $n-1$; later we will see that the same thing happens when we intersect an irreducible variety V with a hypersurface (defined by the vanishing of a single polynomial)).*

Still more generally, given any commutative ring A we define its (Krull) dimension to be the largest n such that there is a strictly ascending chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of prime ideals in A, or ∞ if arbitrarily long such chains exist. Then it turns out the dimension even of a Noetherian ring can be infinite, but if we fix a prime ideal P of the Noetherian ring A, then strictly ascending chains of prime ideals ending in P are bounded in length (equivalently, the dimension of the localization A_P is always finite; or there are no infinite strictly descending chains of prime ideals in A).