Math 505, 2/22

We continue to develop the theory of dimension, digressing for a moment to introduce a measure of the rate of growth of the graded subspaces of a graded module; we will then construct such a module from any Noetherian local ring (thus also from any localization at a maximal ideal of any Noetherian ring). So let $A = \oplus_0^\infty A_n$ be a Noetherian graded ring with $A_0 = k$ a field (not assumed to be algebraically closed), so that $A_n A_m \subset A_{n+m}$. Then the ideal $A_+ = \oplus_1^\infty A_n$ is finitely generated by homogeneous elements, say by $x_1 \in A_{k_1}, \ldots, x_s \in A_{k_s}$. An easy induction then shows that the nth graded piece A_n lies in the k-algebra generated by the x_i for all n, so that A is a finitely generated k-algebra. Let $M = \oplus_0^\infty M_n$ be a finitely generated graded A-module, so that $A_n M_m \subset M_{n+m}$ and M is generated by $m_1 \in M_{r_1}, \ldots, m_t \in M_{r_t}$. Then the nth graded piece M_n is spanned over k by monomials of the appropriate degree in the x_i times generators m_j, whence M_n is finite-dimensional over k. Form the *generating function* of the sequence $\{\dim M_n\}$ of the dimensions of the M_n; that is, the power series $P(M,t) = \sum_i \dim M_i t^i$; we call this the *Hilbert series* or *Poincaré series* of M. Then $P(M,t)$ *takes the form* $f(t)/\prod_{i=1}^s (1 - t^{k_i}$, *for some* $f \in \mathbf{Z}[t]$. We prove this by induction on s. If $s = 0$, then $A_n = 0$ for all $n > 0$, whence $A = A_0 = k$ and M is a finite-dimensional vector space over k. In this case $M_n = 0$ for all large n and $f(t)$ is a polynomial, as desired. Now suppose that $s > 0$ and the theorem is true for $s-1$. Multiplication by x_s gives an A-module homomorphism from M to itself sending M_n to M_{n+k_s}, whence we get an exact sequence

$$0 \to K_n \to M_n \to M_{n+k_s} \to L_{n+k_s} \to 0$$

for all n; the direct sums $K.L$ of the K_n, L_n, respectively, are then finitely generated graded A-modules (being respectively a submodule and a quotient of M) sent to 0 by x_s, whence the induction hypothesis applies to them. Taking dimensions over k and using the additivity of dimension in exact sequences, we get $\dim K_n - \dim M_n + \dim M_{n+k_s} - \dim L_{n+k_s} = 0$ for all nonnegative n and then $(1 - t^{k_s})P(M,t) = P(L,t) - t^{k_s}P(K,t) + g(t)$, where $g(t)$ is a polynomial over $\mathbf{Z}$ of degree at most k_s. The inductive hypothesis then yields the desired result. The order of the pole of $P(M,t)$ at $t = 1$ is denoted $d(M)$; although this quantity seems to be a million miles from chains of prime ideals, we will define it for any Noetherian local ring and eventually relate it to the Krull dimension of that ring. In case all k_i happen to equal 1 (the main case of interest for us), we can refine this result: *for sufficiently large* n *the dimension* d_n *of* M_n *is a polynomial in* n, *called the Hilbert polynomial of* M, *of degree* $d(M) - 1$. Indeed, we have $d_n =$coefficient of t^n in $f(t)/(1-t)^s$ in this case; cancelling a suitable power of $1 - t$, we may assume that $s = d = d(M)$ and $f(1) \neq 0$. Write $f(t) = \sum_{k=0}^N a_k t^k$; then the binomial theorem gives

$$(1-t)^{-d} = \sum_0^\infty \binom{d+k-1}{d-1} t^k$$

whence

$$d_n = \sum_k a_k \binom{d+n-k-1}{d-1}$$

for $n \geq N$; the right side is a polynomial in n of degree $d-1$ and leading coefficient $(\sum a_k)/(d-1)! \neq 0$, as desired. Thus for example if $A = k[x_1, \ldots, x_n]$ has the standard grading, with $A_0 = k$, and $M = A$, then $d(M) = n$ (the Hilbert series of M is $(1-t)^{-n}$). Returning to the exact sequence above and replacing x_s there by any $x \in A_k$ which is not a zero-divisor (in the sense that $xm = 0$ implies $m = 0$, for any $m \in M$, we see that $K = 0$ and $d(L) = d(M/xM) = d(M) - 1$.

We now show how to define $d(A), d(N)$ for any Noetherian local ring A and any finitely generated A-module N. Let M be the unique maximal ideal of A. Form the *associated graded ring* $G = G(A) = \oplus_{i\geq 0} G_i = \oplus_{i\geq 0}(M^i/M^{i+1})$ in which addition is defined componentwise and the product of $xG_i, y \in G_j$ is obtained by taking the image $\bar{s}\bar{y}$ in G_{i+j}, where $\bar{x}, \overline{(y)}$ are any two preimages of x, y in M^i, M^j, respectively; one easily checks that this does not depend on the choice of $\bar{x}$ or $\bar{y}$. Similarly define $G(N)$ to be the direct sum of the quotients $G_{N,i} = M^i N/M^{i+1}N$, making this into a G-module in the obvious way. Setting $K = A/M$, we see immediately that G is a K-algebra; if $m_1, \ldots, m_s$ generate M as an ideal, then the images of the m_i in G_1 generate G as a K-algebra, whence G is Noetherian (and in fact a quotient of a polynomial ring, such as we have been working with in algebraic geometry). Similarly $G(N)$ is a finitely generated graded G-module (generated by any set of generators of N). Defining $d(G), d(G(N)$ as above, we then denote these quantities by $d(A), d(N)$, respectively. Then *the sum of the K-dimensions of the G_i for $0 \leq i \leq n$ is a polynomial $g(n)$ for sufficiently large n of degree $d(A)$*, since the difference between the sum up to $n+1$ and the sum up to n is a polynomial of degree $d(A) - 1$ for sufficiently large n. Similarly the sum of the K-dimensions of the $G_{N,i}$ for $0 \leq i \leq n$ is a polynomial $g_N(n)$ of degree $d(N)$ for sufficiently large n. More generally, we could replace M here by any ideal Q lying between some power M^k of M and M, replacing the dimension of Q^n/Q^{n+1} over K (which does not make sense) by the length of this quotient as an A-module, which equals the sum of the K-dimensions of $Q^n/MQ^n, MQ^n/M^2Q^n, \ldots$, the sequence of quotients stopping after at most k steps since $M^k \subset Q$. The leading coefficients of $g(n), g_N((n)$ then depend on the choice of Q, but its degree $d(A)$ does not. Even more generally, we could replace the powers Q^n of Q here by any *stable Q-filtration*, that is, by any sequence (Q_n) of ideals such that $Q_0 = A, QQ_i \subset Q_{i+1}$ for all i and $Q_{i+1} = QQ_i$ for all sufficiently large i (defining the "K-dimension" of Q_i/Q_{i+1} as above for Q^i/Q^{i+1}, for then $Q_n \subset Q^n$ for all n and $Q^n \subset Q_{n-N}$ for n sufficiently large (where N is a fixed index chosen so that $Q_{i+1} = QQ_i$ for $i \geq N$, whence the polynomials g, g' attached to the powers Q^n on the one hand and the ideals Q_n on the other are such that $g(n) \leq g'(n+n_0), g'(n) \leq g'(n+n_0)$ for a fixed index n_0 and all sufficiently large n, and g, g' have the same degree and leading coefficient.

We close here by recording two results that will be needed in homework this week (which we are not quite ready to prove now): *let V, W be affine varieties in*

K^n *of respective dimensions* r, s*; then any component of* $V \cap W$ *has dimension at least* $r+s-n$. For projective varieties V, W in $\mathbf{P}^n$, we have a stronger result: *under the same hypotheses, every component of* $V \cap W$ *has dimension at least* $r+s-n$ *and in addition* $V \cap W$ *is nonempty whenever* $r+s-n \geq 0$.