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Continuing, we recall that A is a Noetherian local ring with maximal ideal M and residue field $K = A/M$ and N is a finitely generated A-module. We saw last time that the lengths of the respective quotients $A/M^n, N/M^nN$ as A-modules are polynomials p_A, p_N of the respective degrees $d(A), d(N)$ for sufficiently large n and that these degrees are unchanged if the power M^n is replaced by the ideal Q_n, where Q lies between some power M^k of M and M itself and the isequence (Q_n) is a stable Q-filtration. We also saw that $d(A)$ is bounded above by s whenever M (or Q) is generated by s elements. Now let N' be a submodule of N and define a filtration (N'_n) of N' via $N'_n = N' \cap N_n$, then (N'_n) is a Q-filtration admitting the increasing chain of submodules $(N_n") = (N'_0 + \ldots + N'_n + QN_n + Q^N_n + \ldots)$, which must terminate, forcing the induced filtration (N'_n) to be stable. Now let $x \in A, x$ not a zero-divisor in N. Then $N' = xN \cong N$; setting $\bar{N} = N/xN$ and $N'_n = N' \cap Q^nN$ as above, we have an exact sequence $0 \to N'/N'_n \to N/Q^nN \to \bar{N}/Q^n\bar{N} \to 0$, whence $p'_N(n) - p_N(n) + p_{\bar{N}}(n) = 0$ for all sufficiently large n. Since p'_N, p_N have the same degree and leading term, we see that $p_{\bar{N}}$ has degree at most $d(N)-1$. In particular, $d(A/(x)) \leq d(A)-1$ *if A is Noetherian local and x is not a zero-divisor in A*. Now we can show that $\dim A \leq d(A)$ *for any Noetherian local ring A*. To prove this we need a very useful result called *Nakayama's Lemma*: if N is a finitely generated module over a local ring A with $N = MN, M$ the maximal ideal of A, then $N = 0$. Indeed, note first that any $x \in A$ of the form $1 + m$ for some $m \in M$ is a unit, since any non-unit lies in a maximal ideal, but no such x can lie in M. If N is finitely generated but $N \neq 0$, let $n_1, \ldots, n_r$ be a minimal set of generators; then we must have $n_1 = m_1n_1 + \ldots + m_rn_r$ for some $m_i \in M$, whence we can divide by $1 - m_1$ and solve for m_1 in terms of the remaining m_i, contradicting minimality of $m_1, \ldots, m_r$ as a generating set for N. Now we prove $\dim A \leq d(A)$ by induction on $d(A)$. If $d(A) = 0$, then the length of A/M^n is a constant for large enough n, forcing $M^N = M^{N+1}$ for large N; since M is finitely generated, Nakayama says that $M^N = 0$; but then any prime ideal, containing 0, must contain M and M is the only prime ideal, implying that $\dim A = 0$. If $d = d(A) > 0$ and $P_1 \subset \cdots P_r$ is a strictly ascending chain of prime ideals in A, then choose $x \in P_1, x \notin P_0$, and let x' be the image of x in $A' = A/P_0$. Then $d(A'/(x') \leq d(A') - 1$; but the unique maximal ideal M' of A' (the image of M) is such that $A'/(M')^n$ is a homomorphic image of $A?M^n$ for all n, so that $d(A') \leq d(A)$ and $d(A'/(x')) \leq d(A) - 1$. Now the images of $P_1, \ldots, P_r$ form a strict chain of prime ideals in A', whence $r \leq d(A)$ and $\dim A \leq d(A)$, as desired. In particular, $\dim A$ *is finite for any Noetherian local ring A*, Next, given any Noetherian local ring of dimension d, we saw last time how to construct $x_1, \ldots, x_d \in A$ such that the only prime ideal of A containing the x_i is M (the unique prime ideal of height d), so $\dim A \geq \delta(A)$, *where $\delta(A)$ is the minimum number of generators of any primary ideal with radical M*. Since the radical $r(I)$ of any ideal I in A is finitely generated, say by $x_1, \ldots, x_m$ with $x_i^{n_i} \in I$, we see that any product of $n_1 + \ldots + n_k$ combinations of the x_i lies in I, so I contains a power of its radical. Hence *the ideal $(x_1, \ldots, x_d)$ constructed* *above (where d is the height of M), contains a power M^k of M and may be chosen as Q in the recipes above for computing $d(A)$. We finally arrive at our central result: for any Noetherian local ring A, we have $d(A) = \dim A = \delta(A)$ and all three quantities are finite. In particular, if $S = K[x_1, \ldots, x_n]$ with K a field, then $\dim S = n$, since $d(S) = n$ (the Hilbert series of S as a graded S-module is $(1-t)^{-n}$). If $A = S/P$ with P a prime ideal and if $x \in A, x \neq 0$, then x is not a zero-divisor in A, whence any minimal prime ideal over (x) has height 1. If K is algebraically closed, then this says that the intersection of an irreducible affine variety V of dimension d and a hypersurface H not containing it has all irreducible components of dimension $d-1$. In fact, given any irreducible varieties V, W in K^n with $V \subset W$ and $\dim V \leq \dim W - 2$, there is an irreducible variety V' strictly between V and W and having dimension one less than that of W: given the prime ideals P_V, P_W of V, W, respectively, then S/P_W has no zero divisors, so choose a nonzero x lying in the image of P_V in this quotient and look at a minimal prime ideal over (x) contained in this image. Hence any two saturated chains of prime ideals in S, or of irreducible varieties in K^n, have the same length n, as claimed previously; we also see for any irreducible variety V that the dimension of the localization $K[V]_v$ of the coordinate ring $K[V]$ at any point $v \in V$ agrees with that of V. Given any irreducible projective variety $V \subset P^n$, defined by the homogeneous ideal $I \in S$, its affine cone $C(V) \subset K^{n+1}$ (defined by the same ideal) has dimension one larger, since we can compute the dimension of V as the maximum length of any strict chain $I \subset \cdots \subset I_d$ of homogeneous prime ideals in S whose last term I_d is necessarily properly contained in the irrelevant ideal $S' = (x_1, \ldots, x_n)$, and then construct a corresponding chain to compute $\dim C(V)$ by adding S' to the first chain.*