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Now we get to become little kids again and blow things up. More precisely, given a commutative Noetherian ring A and an ideal I of A, we have defined $B_I(A)$, the blowup algebra of I in A, to be the graded direct sum $\oplus_{i\geq 0} I^i$, where I^0 is taken to be A. The associated graded ring $G_I(A)$ identifies with $B_I(A)/IB_I(A)$. In a similar way, if M is a finitely generated A-module equipped with an I-filtration $F = (M_i)$, then $B_F(M)$, the *blowup module of F in M* is defined to be the graded $B_I(A)$-module $\oplus_{i\geq 0} M_i$. This module is finitely generated if and only if the filtration F is stable. The main case of interest in algebraic geometry occurs when A is the coordinate ring of an affine variety V and I is the ideal (not necessarily maximal) corresponding to a subvariety W. For example, take V to be K^n and W the origin, so that $A = S = K[x_1, \ldots, x_n], M = (x_1, \ldots, x_n)$. Then the variety $\mathbf{B}$ corresponding to $B_M(S)$ may be described via affine coordinates $x_1, \ldots, x_n$ and projective coordinates $y_1, \ldots, y_n$ (so that the y_i are not allowed to be simultaneously 0 and we may replace $y_1, \ldots, y_n$ by $ky_1, \ldots, ky_n$ for any $k \in K^*$). The defining equations are then $x_i y_j = x_j y_i$ for all $i \neq j$. We then have a projection map $\phi : \mathbf{B} \to K^n$ sending $(x_1, \ldots, x_n, y_1, \ldots y_n)$ to $(x_1, \ldots, x_n)$ whose fiber over any point $v \neq 0$ in K^n is a single point in $\mathbf{P}^{n-1}$, but whose fiber over 0 is all of $\mathbf{P}^{n-1}$. In effect, we have replaced the origin in K^n by a copy of $\mathbf{P}^{n-1}$ (this is why we call the operation "blowing up"). More generally, let V be a subvariety of K^n having the origin 0 as a singular point. Then the closure of the inverse image $\phi^{-1}(V - 0)$ in any of the affine open pieces covering $\mathbf{B}$ (where $V - 0$ denotes V with 0 removed) is called the *blowup* or *strict transform* of V. Revisiting once again our favorite example, the variety V defined by $x^2 = y^3$, we find that a dense subset of its blowup is defined by the two equations $x^2 = y^3, x = ty$, and the full blowup is given by the now familiar parametric equations $y = t^2, x = t^3, t = t$ for all $t \in K$. It is isomorphic to the affine line K^1; note that the singularity at the origin has disappeared. (The alternative of imposing the equation $y = ux$ rather than $x = ty$ leads to a more complicated realization of the same variety K^1.) For a more interesting example, consider the variety W defined by the equation $y^2 = x^2(x + 1)$, or equivalently by $y^2 - x^2 - x^3 = 0$. Imposing the equation $y = tx$, we get $t^2x^2 - x^2 - x^3 = x^2(t^2 - 1 - x) = 0$, which factors, having the two solutions $x = 0$ or $t^2 = x+1$. What happens here is that the singular point $(0, 0)$ in the original variety has a two-dimensional tangent space; more precisely, we regard the lines $y = x$ and $y = -x$ as defining the two directions of that space, since $y^2 - x^2 = (y - x)(y + x)$ is the homogeneous term of least degree in the equation defining W. The singular point $(0, 0)$ of W in effect splits up into two points $(0, -1)$ and $(0, 1)$, interpreting the coordinates of these points as values of x and t respectively. Both points are now nonsingular on the strict transform and each has only one of the tangent directions that were originally present at $(0, 0)$. In another example in homework, you will blow up the variety defined by $y^3 = x^5$; in this case $(0, 0)$ is still a singular point in the blowup, but one further blowup resolves the singularity.

In general the preimage of the "bad subset" W of V in the blowup of V

corresponds to the algebra $G_I(A)$, so that it is a projective variety with $G_I(A)$ as its homogeneous coordinate ring. Suppose that A is the coordinate ring of an affine variety V containing 0, so that the corresponding ideal J contains the augmentation ideal $M = (x_1, \ldots, x_n)$ generated by the variables. For each $f \in J$, define $\mathrm{in}(f)$, the *initial form* of f, by letting m be the largest index with $f \in M^m$ and then taking $\mathrm{in}(f)$ to be the image of f in M^m/N^{m+1}; you should think of $\mathrm{in}(f)$ as the sum of the homogeneous terms of f of least degree. The ideal consisting of all $\mathrm{in}(f)$ as f runs through J defines the so-called *tangent cone* of V; its coordinate ring is $G_M(A)$.

We conclude with some further remarks about minimal sets of generators for prime ideals. We have seen that, given a prime ideal P of height d in a Noetherian ring A, there are $x_1, \ldots, x_d \in P$ such that P is one of the finitely many minimal primes $P_1, \ldots, P_n$ over $J = (x_1, \ldots, x_d)$. By choosing x_{d+1} to lie in P but none of the other minimal primes P_i we get a new ideal $J' = (x_1, \ldots, x_{d+1}$ such that the only minimal prime over J' is P, whence the radical of J' must be P. Thus *any prime ideal of height d can be generated up to taking radicals by $d+1$ elements*, It is an open question, even for ideals in polynomial rings, whether a prime ideal of height d is the radical of an ideal generated by d elements. Perhaps the most famous example is the ideal of the *rational quartic curve* in $\mathbf{P}^3$, which may be described as the kernel of the map $K(t_0, t_1, t_2, t_3] \to Ks, t$ sending t_0 to s^4, t_1 to s^3t, t_2 to st^3, and t_3 to t^4. Remarkably enough, it is known in all positive characteristics that this ideal is the radical of an ideal generated by two elements (which two depends on the characteristic), but this is not known in characteristic zero! (In geometric terms, the problem is not that the quartic curve is intrinsically complicated; it is after all isomorphic to $\mathbf{P}^1$, but it sits in $\mathbf{P}^3$ in a complicated way.)