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We now introduce a new kind of variety, significantly different from an affine variety. The motivation is that (despite the Nullstellensatz) affine n-space K^n is not large enough to solve problems involving polynomials in a uniform way; we need to add certain points at infinity. More precisely, start with the coordinate ring $S = K[x_1, \ldots, x_{n+1}]$ of K^{n+1} and observe (as mentioned last quarter) that $S = \oplus_m S_m$ is a graded ring, where S_m consists of all homogeneous polynomials in S of (total) degree m. If $p \in S$, write p_m for its homogeneous component of degree m, so that for example $(x^2y + xy^2 + xy)_3 = x^2y + xy^2$. Call an ideal I of S *homogeneous* or *graded* if it is the direct sum of its projections I_m to S_m for $m \geq 0$. The variety V of such an ideal will then be a *cone*, that is, it will contain kv for any $k \in K$ whenever it contains $v) \in K^n$. We now define a new space $\mathbf{P}^n$, called *projective n-space*, which many of you will already have seen in manifolds; it is obtained from K^{n+1} by deleting the origin and identifying $v \in K^{n+1}, v \neq 0$ with kv for $k \in K, k \neq 0$. Given a homogeneous polynomial in S, it does not make sense to evaluate it at an element v of $\mathbf{P}^n$, but it does make sense to ask whether or not it vanishes there. Accordingly, given a radical homogeneous ideal $I \neq (x_1, \ldots, x_{n+1})$ of S, we define its (projective) variety V as the set of its common zeros in P^n, relying on context to distinguish between varieties in K^{n+1} and those in $\mathbf{P}^n$; we have to omit the ideal $(x_1, \ldots, x_{n+1})$ since its only common zero is the origin, which has been deleted from $\mathbf{P}^n$. (This ideal is therefore sometimes called the irrelevant ideal.) Then the Nullstellensatz and a simple Vandermonde determinant argument establish the *homogeneous Nullstellensatz*, which states the map $I \to V(I)$ sets up a 1-1 inclusion-reversing correspondence between relevant radical homogeneous ideals in S and projective varieties; we also extend the Zariski topology to $\mathbf{P}^n$ by decreeing that the closed sets are exactly the projective varieties $V(I)$ of relevant homogeneous ideals I. (Note that the radical of a homogeneous ideal is homogeneous, and that a homogeneous ideal I is prime if and only if it contains one of the factors f or g of a product fg of homogeneous polynomials whenever it contains this product). Every projective variety $V = V(I)$ has a *homogeneous coordinate ring* S/I attached to it, but now the elements of S/I are not well-defined functions on V. Nevertheless the ratio f/g of nonzero homogeneous polynomials in S/I of the same degree *is* well defined as a function on V; the K-algebra consisting of all such ratios is called the *rational function field* $K(V)$ of V, if V is irreducible in the usual sense that I is prime. (If V is irreducible and affine rather than projective, then we use the same notation $K(V)$ and the same name "rational function field" for the full quotient field of the coordinate ring $K[v]$.)

Now a point $(x_1, \ldots, x_{n+1})$ in $\mathbf{P}^n$ with $x_1 \neq 0$ is equivalent to a unique point $(1, y_1, \ldots, y_n)$ for some $(y_1, \ldots, y_n) \in K^n$; note that $(y_1, \ldots, y_n)$ can even be the origin. Hence there is a (principal) open subset U_1 of $\mathbf{P}^n$ and a natural bijection ϕ_1 between it and affine n-space K^n; even before we have defined the notion of morphism on a projective variety in general, we decree that ϕ_1 is an isomorphism of varieties and the subring $K[x_2/x_1, \ldots, x_{n+1}/x_1]$ of the quotient field of S is the coordinate ring $K[U_1]$ of U_1. In a similar manner, we define the affine open subset U_i of $\mathbf{P}^n$ as the set of nonzeros of the coordinate function x_i for $1 \leq i \leq n+1$; its coordinate ring $K[U_i]$ is the subring $K[x_1/x_i, \ldots, x_{n+1}/x_i]$ of the quotient field of S. Then a morphism between projective subvarieties V, W of $\mathbf{P}^n, \mathbf{P}^m$, respectively, is a Zariski-continuous map π from V to W such that the restriction of π to each affine open subset $V \cap U_i$ of V maps it into the affine open subset $W \cap U_j$ of W for some j and is a morphism of affine varieties when so restricted. Thus such morphisms, unlike those between affine varieties, need not be given by a uniform formula (valid over all of V). The simplest example is the subvariety V of $\mathbf{P}^2$ defined by the homogeneous equation $xz - y^2 = 0$. A point on this variety takes the form (a^2, ab, b^2) with $a, b \in K^2, a, b$ not both 0, and a and b unique up to multiplication by the same nonzero scalar. We can map this point to $(a^2, ab) \in \mathbf{P}^1$ if $a \neq 0$ and to $(ab, b^2) \in \mathbf{P}^1$ if $b \neq 0$; note that these formulas agree in $\mathbf{P}^1$ whenever both are defined. This map is then an isomorphism from V to $\mathbf{P}^1$; its inverse sends $(a, b) \in \mathbf{P}^1$ to (a^2, ab, b^2). Note also that the homogeneous coordinate rings $K[x, y]$ and $K[x^2, xy, y^2] \subset K[x, y]$ of $\mathbf{P}^1$ and V are *not* isomorphic, although their rational function fields (*not* the same as their quotient fields) are isomorphic.

Thus $\mathbf{P}^n$ can be thought of as the overlapping union of $n+1$ copies of K^n. It can also be thought of as the disjoint union of copies of $K^n, K^{n-1}, \ldots, K^0$, by letting Z_i be the complement of U_i and then taking the affine varieties $U_1, Z_1 \cap U_2, Z_1 \cap Z_2 \cap U_3$, and so on. Thus the copies of $K^{n-1}, \ldots, K^0$ consist exactly of the points "at infinity" that one must add to K^n to get $\mathbf{P}^n$.