Math 505, 2/15

Continuing with both affine and projective varieties, we generalize the notion of morphism to that of rational map: given irreducible varieties V, W a *rational map* π from V to W is a morphism from a nonempty open subset U of V to W, with two such defined on open subsets U, U' being identified if they agree on the intersection $U \cap U'$. Thus a rational map differs from a morphism in that it need not be everywhere defined; in fact the restriction of any morphism to any open subset of its domain is a rational map. Note that the notion of rational map and the related notions below have no counterparts in manifold theory; a crucial point here is that nonempty open subsets of irreducible varieties, unlike those of manifolds, are always dense. A rational map from V to W is called *dominant* if its range is dense in W; note that this property does not change if the map is replaced by an equivalent one. Dominant rational maps from a variety V to another one W correspond bijectively to homomorphisms $K(W) \to K(V)$ of their rational function fields. If two rational maps $\pi : V \to W$ and $\phi : W \to V$ are such that their composition in either order is identified with the identity map, then we say that V and W are *birationally equivalent* or just *birational*. The simplest example of a pair of nonisomorphic birtaional varieties is an old friend from last week, namely the affine variety V defined by the equation $x^2 = y^3$ in K^2 and the affine line K^1; here the respective coordinate rings are $K[x,y]/(x^2 - y^3) \cong K[t^3, t^2]$ and $K[t]$. As you saw in homework last week, these rings are not isomorphic and accordingly V is not isomorphic to K^1, but the rings become isomorphic after each is localized by powers of t. Correspondingly we have a birational map from K^1 to V defined on K^* by sending t to the pair (t^3, t^2); its inverse is defined on the complement C of the origin in V by sending (x, y) to y/x; this is a legitimate morphism on C because its coordinate ring picks up the function x^{-1} that was not present in $K[V]$. In fact C is isomorphic as a variety to K^*, and indeed two varieties are birational if and only if they have isomorphic open subsets, or equivalently if and only if they ave isomorphic rational function fields; in particular K^n and $\mathbf{P}^n$ are birational. Assuming for simplicity that the field K has characteristic 0 (though the following result holds in general), we know that the rational function field $K(V)$ of any variety (affine, quasi-affine, projective, or quasi-projective, the last denoting the intersection of a projective variety and an open subset of projective space) is finitely generated as a field extension of K, so that there are finitely many elements $x_1 \ldots, x_n$ of $K(V)$ that are algebraically independent over K and $K(V)$ is a finite extension of the rational function field $K_n = K(x_1, \ldots, x_n)$. By the Primitive Element Theorem from Galois theory, there is a single element y generating $K(V)$ as a K_n-algebra; if p is its minimal polynomial over K_n, then we can clear denominators in p to get a single irreducible polynomial q in $n+1$ variables over K such that $q(x_1, \ldots, x_n, y) = 0$ and then our variety V is birational to the affine hypersurface in K^{n+1} defined by the equation $q = 0$.

We now want to extend a basic notion from manifold theory to algebraic geometry, namely the definition of tangent space to a variety at a point. Let $V \subset K^n$ be any affine variety, v a point of V, and M_v the maximal ideal of the coordinate ring $K[V]$ corresponding to the point v. Then the *tangent space* $T_v(V)$ of V at v is the dual of the K-vector space M_v/M_v^2; that is, it consists of the K-linear maps from the quotient M_v/M_v^2 to $K \cong K[V]/M_v$. (This definition parallels one you have seen or will see shortly in manifolds.) Recall that a morphism f from V to another affine variety W induces a K-algebra homomorphism $f^* : K[W] \to K[V]$ mapping $M_{f(v)} \subset K[W]$ for $v \in V$ to M_v, thus also $M_{f(v)}^2$ to M_v^2. It follows that f^* induces a K-linear map called the *differential* of f and denoted by *df* from $T_v(V)$ to $T_f(v)W$ (exactly as in manifolds). Now it turns out that we can compute the dimension of the tangent space $T_v(V)$ of V at v in the same way as we would for manifolds. Let I be the ideal of V, generated by the polynomials $f_1, \ldots, f_m$. Form the Jacobian matrix J_v whose ij-th entry is the partial derivative $\partial x_i/\partial x_j$ evaluated at v (computed via the usual formal rules; we do not need limits). *Then the corank of this matrix (that is, n minus its rank), equals the dimension of $T_v(V)$; in particular, the corank of J_v does not depend on the choice of generators of I. To see this, observe that the map sending any $f \in S = K[x_1, \ldots, x_n]$ to its gradient evaluated at v vanishes on M^2 (by the product rule), where M is the maximal ideal of S corresponding to v. It induces a vector space isomorphism between M/M^2 and K^n. Its value on a linear combination of the f_i is a K-linear combination of the rows of J_v (whose coefficients are those of the original linear combination evaluated at v). Hence the corank of J_v matches the dimension of $T_v(V)$, as desired. Now this corank will generically (more precisely, on an open set, defined by the nonvanishing of one of a finite collection of determinants) take a certain value; we call such points regular or nonsingular. At other points, called singular it will take a higher value. Thus the dimension of $T_v(V)$ takes a certain value on the set of regular points of V, a nonempty open set, and a higher value elsewhere. It is natural to expect (and correct) that this value is none other than the dimension of V itself (assuming V is irreducible). We will prove this next time and then digress to discuss the notion of dimension in a more general context.For now, we mention a couple of examples. The variety $V \subset K^2$ with equation $x^2 - y^3 = 0$ has exactly one singular point, at $(0,0)$; there the tangent space is two-dimensional, while it is one-dimensional elsewhere. Similarly, the variety $W \subset K^3$ defined by $xz - y^2 = 0$ has exactly one singular point, at the origin. The projective variety defined by the same equation has no singular points, as $(0,0)$ is not a point in $\mathbf{P}^2$, and indeed we have observed that this last variety is isomorphic to $\mathbf{P}^1$.*