Notes Introduction to Mathematical Analysis

Notation

A set is a "collection of elements"
We write $x \in A$ if x is an element of A. Otherwise, we write $x \notin A$
Given two sets A and B, A is said to be a subset of B, $A \subseteq B$, $\iff$ B contains all elements of A.
$A \subseteq B \iff A \subseteq B$ and $B \subseteq A$

Notation

$\emptyset$ denotes the set containing no elements $\rightarrow$ empty set

Remark

Given a set A
$\emptyset \subset A$

Definition

Given a set A, the power set $\mathcal{P}(A) - \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Operations on Sets

Definition

Given the sets A and B:
The union $A \cup B$ is the set of all elements that are in A or in B

$$A \cup B = \{x : x \in A \; or \; x \in B\}$$

The intersection $A \cap B$ is the set of all elements that are in A and in B

$$A \cap B = \{x : x \in A \; and \; x \in B\}$$

The relative component of A w.r.t.:
B is the set of all elements in B that are not in A

$$B \diagdown A = \{x : x \in B \; and \; x \notin A\}$$

"Sometimes" A and B are sets in a universe X.

$$A^c = X \diagdown A$$

[!ex] $A = \{a, b\}$, $B = \{b, c\}$

$$A \cup B = \{a, b, c\}$$

$$A \cap B = \{b\}$$

$$A \diagdown B = \{a\}$$

$$B \diagdown A = \{c\}$$

Remark

$A \subseteq A \cup B$, $B \subseteq A \cup B$
Proof:
$x \in A \Rightarrow x \in A \; or \; x \in B \Rightarrow x \in A \cup B$

$A \cap B \subseteq A$, $A \cap B \subseteq B$
Proof:
$x \in A \cap B \Rightarrow x \in A \; and \; x \in B \Rightarrow x \in A$

Law: Demorgan's Law

Demorgan's Law

$$i. \, (A \cup B)^c = A^c \cap B^c$$

$$ii. \, (A \cap B)^c = A^c \cup B^c$$

of i)

$x \in (A \cup B^c)$

$$\begin{aligned} &\iff x \notin A \cup B \\ &\iff x \notin A \; and \; X \notin B \\ &\iff x \in A^c \; and \; x \in B^c \\ &\iff x \in A^c \cap B^c \end{aligned}$$

Distributivity

Distributivity

$i)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$ii)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

of i)

$x \in A \cup (B \cap C)$

$$\begin{aligned} &\iff x \in A \; or \; x \in B \cap C \\ &\iff x \in A \; or \; x \in B \; and \; x \in C \\ &\iff x \in A \; or \; x \in B \; and \; x \in A \; or \; x \in C \\ &\iff x \in A \cup B \; and \; x \in A \cup C \\ &\iff x \in (A \cup B) \cap (A \cup C) \end{aligned}$$

Given two sets A and B

$$A \times B = \{(x, y) : x \in A, y \in B\}$$

Ex:

$$\begin{aligned} &A = a, b, b = \{/\} \\ &A \times B = \{(a, /), (b, /)\} \\ &B \times A = \{(/, a), (/, b)\} \end{aligned}$$

Relations and Functions

Given two sets A and B, a relation R is a subset of $A \times B$

Ex:

$$A = \{a, b\}, B = \{b, c\}$$
$$(Relations \; from \; A \rightarrow \; B)$$

$$\begin{aligned} &R_1 = \{(a, b), (a, c)\} \\ &R_2 = \{(b, b_\} \\ &R_3 = \{(a, b), (a, c), (b, c)\} \end{aligned}$$

Given a function: $f : A \rightarrow B$

i) image set: for $E \subseteq A$
$f(E) = \{y \in B \; : \; y = f(a)\}$ for some $a \in E\}$

ii) Pre-image set: for $H \subseteq B$
$f^{-1}(H) = \{x \in A \; : \; f(x) \in H\}$

Observation: $f : A \to B$

1. If $E \subseteq F \subseteq A$ then $f(E) \subseteq f(F)$
2. If $H \subseteq K(\subseteq B)$ then $f^{-1}(H) \subseteq f^{-1}(K)$

Proof

1. $y \in f(E) \Rightarrow y = f(x)$ for some $x \in E$
   since $E \subseteq F$ *then* $x \in F$
   $(\Rightarrow y = f(x)$ with $x \in F)$
   $\Rightarrow y \in f(F)$
2. $x \in f^{-1}(H) \Rightarrow f(x) \in H$
   since $H \subseteq K$ then $f(x) \in k \Rightarrow x \in f^{-1}(K)$

A={α, β, γ} B={a, b, c}

$$f(\alpha) = a, f(\beta) = b, f(\gamma) = a$$
$$f^{-1}(f(\{\alpha\})) = f^{-1}(\{a\}) = \{\alpha, \gamma\}$$

$$f(f^{-1}(\{b, c\}))$$
$$= f(\{\beta\}) = \{b\}$$

[not injective nor subjective]

Proposition

$f : A \to B$ *function*

$$1)\forall\ E \subseteq A \text{ we have } E \subseteq f^{-1}(f(E))$$
$$2)\forall\ H \subseteq B \text{ we have } f(f^{-1}(H)) \subseteq H$$

Proof

1. Let $x \in E$ then $f(x) \in f(E)$ then $x \in f^{-1}(f(E))$
2. Let $y \in f(f^{-1}(H))$ then $y = f(x)$ for some $x \in f^{-1}(H)$
   but $x \in f^{-1}(H)$ implies $f(x)[= y] \in H$

Given a function $f : A \to B$

1. We say that f is **injective** (one-to-one) iff for
   $x \neq y\ f(x) \neq f(y) \iff$ "$f(x) = f(y) \iff x = y$"
2. We say that f is **surjective** (onto) iff for every $y \in B$,
   there exists $x \in A$ s.t. $f(x) = y$
   iff $f^{-1}(\{y\}) \neq \emptyset\ \forall\ y \in B$
   iff $f(A) = B$
3. We say that f is **bijective** if it is both **injective** and **surjective**

Notation

$f : A \to B$

- A is called the domain of f
- $f(A)$ is the range of f

Proposition

1. If f is injective then $E = f^{-1}(f(E))$ for every $E \subseteq A$
   In fact,
   f is injective $\iff E = f^{-1}(f(E))\ \forall\ E \subseteq A$
2. If f is surjective then $H = f(f^{-1}(H))\forall H \subseteq B$
   In fact,
   f is surjective $\iff H = f(f^{-1}(H))$
   (ex) $\forall H \subseteq B$

Given f injective I want to show that $E = f^{-1}(f(E))$

$E \subseteq f^{-1}(f(E))$: done (independently of injectivity)

$f^{-1}(f(E)) \subseteq E$:

$$\text{Take } x \in f^{-1}(f(E)) \text{ then } f(x) \in f(E) \text{ then}$$
$$f(x) = f(x') \text{ for some } x' \in E$$
$$\text{but } f \text{ is injective then } x = x' \in E$$

$$\text{Conclusion}: \ E = f^{-1}(f(E))$$

Proof

$(\Rightarrow)$ done
$(\Leftarrow)$ Assume $f(x) = f(y)$, $x \in f^{-1}(f\{y\}) = \{y\}$
then $x = y$

Schroeder-Bernstein Theorem

Given two sets A and B

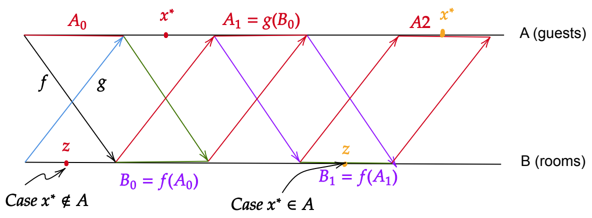
If there exists $f : A \to B$ injective and $g : B \to A$ injective then there exists a bijection $h : A \to B$

If $f : A \to B$ bijective

We use the notatoin f^{-1} to denote the inverse map $f^{-1} : B \to A$ defined as follows:

$$f^{-1}(b) = a \iff f(a) = b$$

Idea



Here we have:

$$\begin{aligned} A_0 &= A \backslash g(B) \\ A_1 &= g(f(A_0)) \\ A_2 &= g(f(A_1)) \\ &\vdots \\ A_{n+1} &= g(f(A_n)) \text{ for n=0,1,2,...} \\ &\vdots \\ A_\infty &= A_0 \cup A_1 \dots \quad \text{(guests that i moved)} \end{aligned}$$

Let $h : A \to B$

$$h(x) = \begin{cases} f(x) & x \in A_\infty \\ g^{-1}(x) & x \notin A_\infty \end{cases}$$

We show that h is injective

Assume $h(x) = h(y)$ for $x, y \in A$ "(N.T.S. $x = y$)"

Case 1: $x, y \in A_\infty$
then $h(x) = h(y) \Rightarrow f(x) = f(y)$ but f is injective then $x = y$

Case 2: $x, y \notin A_\infty$
then $h(x) = h(y) \Rightarrow g^{-1}(x) = g^{-1}(y) \Rightarrow x = y$ (by injectivity

Case 3: $x \in A_\infty$ and $y \notin A_\infty$ (or vice versa)
$h(x) = h(y)$
$\Rightarrow f(x) = g^{-1}(y)) = y$
$\Rightarrow g(f(x)) = g(g^{-1}(y)) = y$
$x \in A_{\text{inf}} = A_0 \cup A_1 \cup A_2 \cup \dots$
then $x \in A_k$ for some $k = 0, 1, 2, 3, \dots$
then $y = g(f(x) \in g(f(A_k)) = A_{k+1} \subseteq A_\infty$ then $y \in A_\infty$.

Contradiction!
We conclude that h is injective.

It remains to show that h is surjective.
Let $z \in B, x^* = g(z)$

Case $x^* \notin A_\infty$: $h(x^*) = g^{-1}(x^*) = g^{-1}(g(z)) = z$
Case $x^* = g(z)$: Notice that $x^* = g(z) \in g(B)$ and $A_0 \cap g(B) = \emptyset$
Then $x^* \notin A_0$ so$x^* \in A_1 \cup A_2 \cup \dots$
Then $x^* \in A_k$ for some $k = 1, 2, 3 \dots$
$(A_k == g(f(A_{k-1})))$
Then $x^* = g(f(x'))$ for some $x' \in A_{k-1}$

We have
$x^* = g(f(x')) = g(z)$ but g is injective
then $z = f(x')$
but $x' \in A_{k-1} \subseteq A_\infty$ then $h(x') = f(x')$
$\Rightarrow z = h(x')$

Orders and Peanu Axioms

Definition

Given a set S, a (strict total) order denoted by "$<$" is a relation from A to A to satisfy the following:

1. Given $x, y \in A$ we have exactly one of the following:
$$x < y \text{ or } y < x \text{ or } x = y \text{ (Comparability)}$$
2. If $x < y$ and $y < z$ then
$$x < z$$

Ex

$A = a, b$
$<= \{(a, a), (a, b)\}$ not a strict order since we have $a < a$
$A = a, b, c$
$<_1= \{(a, b), (a, c)\}$ not an order since b and c are not com
$<_2= \{(a, b), (a, c), (b, c)\}$
$<_3= \{(a, b), (b, c), (c, a)\}$ not an order bc $a < b$ and $b < c$

Given a set A and an order $<$ on A, we say that $(A, <)$ is an ordered set (totally)

Given an ordered set $(A, <)$

"$a \leq b$" means $a < b$ or $a = b$
"$a \geq b$" means $b \leq a$

The set on natural numbers $\mathbb{N}$ is defined by the following axioms:

$(P.1)$ $1 \in \mathbb{N}$

$(P.2)$ If $n \in \mathbb{N}$, the successor $n + 1 \in \mathbb{N}$

$(P.3)$ If $n + 1 = M + 1$ then $n = m$ (no two elements have the sa

$(P.4)$ 1 is not the successor of any element.

$(P.5)$ Mathematical induction says:

If $S \subseteq \mathbb{N}$ such that:
$i)1 \in S$
$ii)$if $k \in S$ then $k + 1 \in S$
Then $S = \mathbb{N}$

P.5: Induction:

Assume I have a set of elements $P(n)$

$$S = \{n \in \mathbb{N} : P(n) \text{ is true }\}$$

If:
Base case: $P(1)$ true $(1 \in S)$

Inductive step: "$P(n)$ true" $\Rightarrow$ "$P(n + 1)$ true" ($n \in S$ then $n + 1 \in S$)
Then from P.5 $S = \mathbb{N}$ $P(n)$ is true for every $n \in \mathbb{N}$.

Def

$m, n \in \mathbb{N}$

$$m + n = \underbrace{(((n+1)+1)+1)+ \ldots +1)}_{\text{m times}}$$

$$m.n = \underbrace{m + m + m + m + m + \ldots + m}_{\text{n times}}$$

Order
We say that

$m < n$ if and only if n is "a successor" of m
$n = m + k$ for some $k \in \mathbb{N}$

$(\mathbb{N}, <)$ is an order

Induction Example

$$2^n (n+1)! \quad \forall \quad n \in \mathbb{N}$$

$n! = 1.2.3. \ldots .n$

$P(n) = \{2^n \leq (n+1)!\}$

Base case: n=1

$$2^1 = 2 \qquad 2^1(1+1)!$$
$$(1+1)! = 2! = 2.1 = 2 \qquad \text{verifies}$$

Inductive step: Assume $P(k)$ is true for some $k \in \mathbb{N}$

$2^k \leq (k+1)!$ I want to show tha
$P(k+1)$ is true i.e. $2^{k+1} \leq (k+1+1)$

$$2^{k+1} = 2^k.2 \leq \quad (k+1)!(2) \leq (k+1)!(k+2) = (k+2)$$

Then $P(n)$ is true for every $n \in \mathbb{N}$

Strong Induction

If $S \subseteq \mathbb{N}$ s.t.
i) $1 \in S$
ii) If $1, 2, 3, \ldots k \in S$ then $k+1 \in S$
Then $S = \mathbb{N}$

In terms of statement.
Given a statement $P(n) \ \in \mathbb{N}$

Base case: $P(1)$ is true

Inductive step:
If $P(1), P(2), \ldots, P(k)$ are true then $P(k+1)$ is true

Conclusion: $P(n)$ is true $\forall\, n \in \mathbb{N}$

Proof

Take
$T = n \in S : \{1, 2, \ldots, n\} \subseteq S \ (T \subseteq S \subseteq \mathbb{N})$

Base case: $1 \in T$ since by hypothesis $1 \in S$, so $\{1\} \subseteq S$

Inductive step: Assume $k \in T$, then $\{1, 2, \ldots .k\} \subseteq S$
Then by hypothesis (ii) $\quad k+1 \in S$
then $\{1, 2, \ldots ., k, k+1\} \subseteq S$
$\Rightarrow k+1 \in T$
by induction $T = \mathbb{N}$ and $S = \mathbb{N}$

Exercise

Given $m, n \in \mathbb{N}m$ we say that n divides m, $n|mm$ iff there exists $k \in \mathbb{N}$ s.t. $m = kn$
Ex: $2|4, 2|16$ since $16 = 2 \times 8$

We say that $p \geq 2$ is "prime number if and only if the only divisers of p are 1 and p".
ex: 16 isn't prime but 17 is

Theorem

If $n \leq 2$ then n can be factorized into prime numbers i.e. we can write

$$n = p_1 p_2 p_3 \ldots p_k \text{ with } p_1, p_2, p_3, \ldots, p_k \text{ primes}$$

Proof

$n = 2$: 2 is prime
Inductive step: Assume the statement is true for $2, 3, 4, \ldots, k$
Need to show it for $k+1$
Case 1: $k+1$ is prime. $k+1 = k+1$
Case 2: $k+1$ is not prime. *then*
$k+1 = m.n$ with $2 \leq m, n \leq b$

But $P(m)$ and $P(n)$ are true then

$m = p_1 p_2 \ldots p_k$ prime then $k+1 = p_1 \ldots p_k\, q_1$
$n = q_1 q_2 \ldots q_s$

Then $S = \mathbb{N}$

Proposition: Well-ordering principle

If S is a nonempty subset of $\mathbb{N}$ then S has a least element.
i.e. there exists $n \in S$ s.t.

$$n \leq x \text{ for every } x \in S$$

We write $n = min(S)$

Proof

We will show it with strong induction.
Take $P(n) = \{n \in S \Rightarrow S \text{ has a least element}\}$
i) Base case: Is $P(1)$ true?
If $1 \in S$ then 1 is the smallest element of S by (p.4)
ii) Inductive step: Assume $P(1), P(2), P(3), \ldots, P(k)$ are true
We need to show that $P(k+1)$ is true
Assume $k+1 \in S$

Case 1: $\{1, 2, \ldots, k\} \cap S = \emptyset$
Case 2: $\{1, 2, \ldots, k\} \cap S \neq \emptyset$
s.t. $p \in S$ but $P(p)$ is true then S has a least el

Conclusion: $P(n)$ is true for every $n \in \mathbb{N}$
but $S \neq \emptyset$ i.e. $\exists\, n_0 \in S$ and $P(n_0)$ is true then S has a least element.

Theorem

Induction $\iff$ Strong Induction $\iff$ Well-ordering principle

Proof

We showed
Induction $\Rightarrow$ Strong Induction $\Rightarrow$ Well-ordering principle
We show that
Well-ordering principle $\Rightarrow$ Induction

Let $S \subseteq \mathbb{N}$ s.t.
i) $i \in S$
ii) $k \in S \Rightarrow k+1 \in S$
I need to prove that $S = \mathbb{N}$

Assume S is a proper subset of $\mathbb{N}$ i.e. $S \subset \mathbb{N}$
Then $\mathbb{N} \setminus S \neq \emptyset$ then by the Well-ordering principle, $\mathbb{N}$ has a least element

$$p = min(\mathbb{N} \setminus S)$$

$p \neq 1$ since $1 \in S$ then $\exists\, k \in \mathbb{N}$ s.t.

$k+1 = p\,,\ k < p$

so $k \notin \mathbb{N} \setminus S$ then $k \in S$

by the inductive hypothesis, $k+1 \in S$

$\mathbb{N} \setminus S = \emptyset\,,\ S = \mathbb{N}$

Contradiction!