A No-arbitrage Model of Liquidity
in Financial Markets involving Brownian Sheets

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Market vs Limit Orders

A (buy) market order specifies

- how many shares a trader wants to buy
- that he is willing to buy them at **any price**

A (buy) limit order specifies

- how many shares a trader wants to buy
- at what **maximum price** he is willing to buy them?

We call this price the *limit price*.

Research on Financial Markets

The Cetin-Jarrow-Protter Model: Market Orders

Define $S^{CJP}(t, x, \omega)$ as the stock price per share that a buyer pays

- at time t
- for an order of size x

Observe

- $S^{CJP}(t, x, \omega)$ is independent of x in the competitive model
- $\partial S^{CJP}(t, x, \omega)/\partial x \ \geq 0$ otherwise

Liquidity premium per share $\simeq S^{CJP}(t, x, \omega) - S^{CJP}(t, 0, \omega)$

Results

1. if traders place infinitesimal trades over a window of time, then there is no liquidity premium
2. the first and second fundamental theorems of asset pricing hold.

Goal of the Paper

Better characterize the volatility of the price process in a market driven by limit orders

$$dp(t) = \text{volatility} * dW^{\mathbb{Q}}$$

Inspiration:

- Heath-Jarrow-Morton model

    the drift of the forward rate is determined by volatility

- Derman-Kani model

    relations between volatilities of options prices with different strike and maturity

Plan

1. Literature Review
2. A View on the Market
3. Finite Variation Model
4. First Fundamental Theorem
5. Why the Finite Variation Model fails
6. Diffusion Model
7. Was there arbitrage on April 1st, 2011?
8. Conclusion and Future Work

Literature Review: Liquidity Models

1. Market Manipulation (feedback) Models
   - Jarrow (1994)
   - Platen and Schweizer (1998)
   - Sircar and Papanicolaou (1998)
   - Frey (1998)
   - Schonbucher and Wilmott (2000)
   - Bank and Baum (2004)
2. Price-taking (competitive) Models
   - Cetin, Jarrow, and Protter (2004)
   - Cetin and Rogers (2006)
   - Cetin, Soner, and Touzi (2009)
   - Gokay and Soner (2011)

Buy Limit Orders in 3D

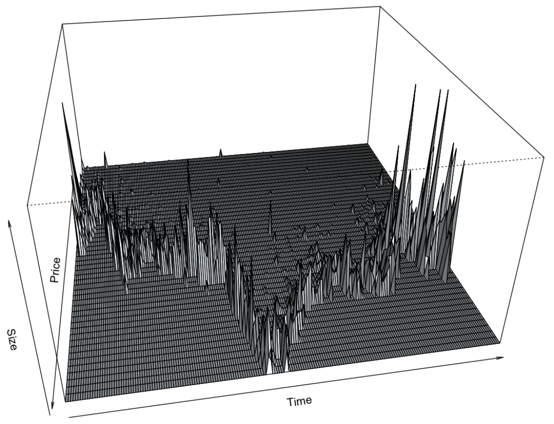


Figure: GE Buy Size on April 01, 2011 in 3D

Statistics

	Mean (minutes)	Std Deviation (minutes)
Ask Order	26.20254	91.00393
Bid Order	32.76618	103.9892

Table: Mean Lifetimes of Bid/Ask limit Orders

	Ask Orders	Bid Orders
Total (9:30-4:00)	53338	49506
1 minute	34695	34699
5 minutes	41348	41223
10 minutes	43020	43020
60 minutes	45158	45158
120 minutes	45430	45432

Table: Cumulative number of orders with lifetime less than 1, 5, 10, 60, 120 minutes

Example

$t = 0$. Order books look like

Buy Order Book	
Price	Quantity
100	10

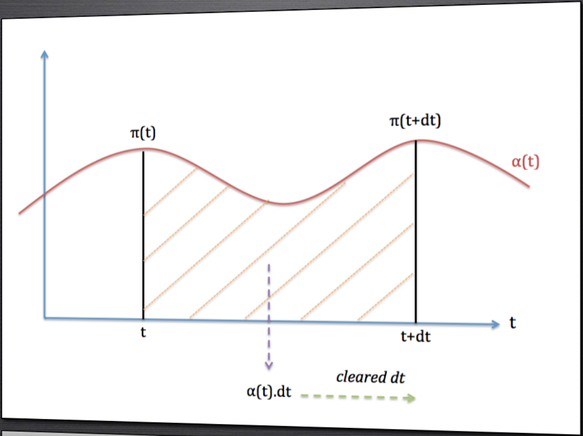
Sell Order Book	
Price	Quantity
120	10
130	10

$t > 0$ New order: buy 15 at a limit price of 125 $\longrightarrow$ clearing price $\pi(t) = 120$

Buy Order Book	
Price	Quantity
100	10
125	5

Sell Order Book	
Price	Quantity
130	5

Model: Timeline



Finite Variation Model

Buy order rate density:

$$\alpha(p,t) = \frac{\text{qty of orders with limit price } \in [p, p+dp] \text{ arriving in } [t, t+dt]}{dp\ dt}$$

Similarly for sell orders, we can integrate this rate density in two directions

$$\begin{array}{ccc} \text{rate density} & \overset{\int dt}{\longrightarrow} & \text{density} \\ \downarrow\ \int_\pi dp & & \downarrow\ \int_\pi dp \\ \text{rate} & \overset{\int dt}{\longrightarrow} & \text{quantity} \end{array}$$

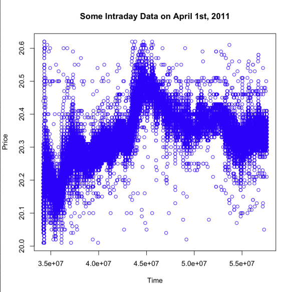


Figure: GE Limit Prices from NYSE on April 1st, 2011

Finite Variation Model

For buy orders, this means:

$$\begin{array}{ccc} \alpha(p,t) & \overset{\int dt}{\rightarrow} & a(p,t) \\ \downarrow \ \int_\pi dp & & \downarrow \ \int_\pi dp \\ \mathcal{D}(\pi,t) & \overset{\int dt}{\rightarrow} & D(\pi,t) \end{array}$$

In other terms:

$$\begin{aligned} \mathcal{D}(\pi,t) &= \int_\pi^\infty \alpha(p,t)dp \\ a(p,t) &= \int \alpha(p,s)ds \\ D(\pi,t) &= \int_0^t \mathcal{D}(\pi,s)ds \end{aligned}$$

And similarly for sell orders

$$\begin{aligned} \mathcal{S}(\pi,t) &= \int_0^\pi \nu(p,t)dp \\ v(p,t) &= \int_0^t \nu(p,s)ds \\ S(\pi,t) &= \int_0^\pi \mathcal{S}(\pi,s)ds \end{aligned}$$

Finite Variation Model

- Rate densities α and ν are positive processes in t
- For fixed π the densities $\mathcal{D}$ and S are increasing processes in t
- For $\pi(t)$ not fixed $\mathcal{D}(\pi(t), t)$ and $\mathcal{S}(\pi(t), t)$ are not necessarily increasing in t

Correlation (assume $p_1 \neq p_2$)

- $\alpha(p_1, t)$ is not completely correlated with $\alpha(p_2, t)$
- $\nu(p_1, t)$ is not completely correlated with $\nu(p_2, t)$
- $\alpha(p_1, t)$ is not completely correlated with $\nu(p_2, t)$

Need two Brownian Sheets $W_1^P(\cdot, \cdot)$ and $W_2^P(\cdot, \cdot)$. in the physical measure.

Finite Variation Model: Rate Densities

$$d\alpha(p,t) = \mu^{\alpha}(p,t)dt + \sigma^{\alpha}(p,t)\int_{s=0}^{\infty} b_1^{\alpha}(s,t)W_1^P(ds,dt) + b_2^{\alpha}(s,t)W_2^P(ds,dt)$$
$$d\nu(p,t) = \mu^{\nu}(p,t)dt + \sigma^{\nu}(p,t)\int_{s=0}^{\infty} b_1^{\nu}(s,t)W_1^P(ds,dt) + b_2^{\nu}(s,t)W_2^P(ds,dt)$$

where

$$\int (b_1^{\alpha}(p,t))^2 + (b_2^{\alpha}(p,t))^2 dp = 1$$
$$\int (b_1^{\nu}(p,t))^2 + (b_2^{\nu}(p,t))^2 dp = 1$$

Finite Variation Model: Clearing Price

Definition

The *clearing price* $\pi(t)$ *at time* t is the price at which the total demand for the traded asset equals the total supply of the traded asset.

$$D(\pi(t), t) = S(\pi(t), t)$$

Or, in differential notation

$$dD(\pi(t), t) = dS(\pi(t), t)$$

In CJP notation

$$\pi(t) = S^{CJP}(0, t)$$

First Fundamental Theorem

Two securities:

- stock
- money market account: interest rate is zero

Focus on one investor:

- number of shares held: $X(t)$
- starts with a number of shares $X(0) = X_0$
- value of his money market account: $Y(t)$ with $Y(0) = 0$

An arbitrage is a self-financing trading strategy such that:

$$X(\tau) = 0, \qquad P(Y(\tau) > 0) > 0, \qquad P(Y(\tau) \geq 0) = 1$$

Theorem

Suppose that there is a measure where π is a martingale, then there is no arbitrage.

Sketch of Proof

Cumulated number of orders matched at price p by time t: $N(p,t)$.

- Demand $D^U(p,t)$ at price p and time t

$$D^U(p,t) = D(p,t) - N(p,t)$$

- Supply $S^U(p,t)$ at price p and time t

$$S^U(p,t) = S(p,t) - N(p,t)$$

The net supply curve $Q(p,t)$ is the difference between S^U and D^U

$$Q(p,t) = S^U(p,t) - D^U(p,t) = S(p,t) - D(p,t)$$

Observe:

$$\frac{\partial Q(p,t)}{\partial p} \geq 0$$

Sketch of Proof: Identification of S^{CJP}

Fix p. Suppose that at time t a trader submits an order with quantity x.

If the order is matched instantaneously (meaning $p \geq \pi(t)$), the total price paid will be:

$$xS^{CJP}(x,t) = \int_{\pi(t)}^{Q^{-1}(x;t)} y \frac{\partial Q(y,t)}{\partial y} dy$$

where $Q^{-1}(x;t)$ is the inverse of Q for fixed t

$$Q(Q^{-1}(x;t),t) = x$$

Clearly:

$$\int_{\pi(t)}^{Q^{-1}(x;t)} y \frac{\partial Q(y,t)}{\partial y} dy = xQ^{-1}(x;t) - \int_{\pi(t)}^{Q^{-1}(x;t)} Q(y,t) du$$

Sketch of Proof: Identification of S^{CJP} $(con.t)$

Thus

$$S^{CJP}(x,t) \leq Q^{-1}(x;t)$$

Thus:

$$\begin{aligned} x(S^{CJP}(x+\Delta x,t) - S^{CJP}(x,t)) \;&=\; \left(Q^{-1}(x;t) - S^{CJP}(x;t) + x\frac{\partial Q(p,t)}{\partial p}|_{p=Q(x,t)}\right)\Delta x \;+ \\ &\left(\frac{1}{2}\frac{\partial Q^{-1}(x;t)}{\partial x}|_x - \frac{\partial S^{CJP}(x+\Delta x,t)}{\partial x}|_x\right)\Delta x^2 \end{aligned}$$

Dividing by $x\Delta x$:

$$\frac{\partial S^{CJP}(x,t)}{\partial x} \geq 0$$

Sketch of Proof

If the order is not matched instantaneously, let $\theta(t) > t$ be the time at which the order is matched. Cost per share $S^{CJP}(x,t)$ is $\mathcal{F}_{\theta(t)}$-adapted

We can show that:

$$\begin{aligned} \frac{\partial E_t[S^{CJP}(x,t]}{\partial x} &\geq 0 \\ E_t[S^{CJP}(\Delta X_t, t) - S^{CJP}(0,t)] &\geq 0 \end{aligned}$$

The liquidity costs by time t are:

$$L(t) = \sum_{0 \leq u \leq t} \Delta X_u (S^{CJP}(u, \Delta X_u) - S^{CJP}(u,0)) + \int_0^t \frac{\partial S^{CJP}}{\partial x}(u,0) d[X,X]_u^c$$

Consequence:

$$E^{\mathbb{Q}}[L(\tau)] \geq 0$$

Fact: if $E^{\mathbb{Q}}[L(\tau)] \geq 0$, there is no arbitrage.

Why the Finite Variation Model fails

We write

$$d\pi(t) = \mu^\pi(t)dt + \sigma^\pi(t)\int_0^{2M} b_1^\pi(p,t)W_1^{\mathbb{Q}}(dp,dt) + b_2^\pi(p,t)W_2^{\mathbb{Q}}(dp,dt)$$

Ito-Wentzell formula:

$$\begin{aligned}
dS(\pi(t),t) &= \mathcal{S}(\pi(t),t)dt + \frac{\partial S}{\partial \pi}d\pi + \frac{1}{2}\frac{\partial^2 S}{\partial \pi^2}d\pi d\pi \\
&= \mathcal{S}(\pi(t),t)dt + v(\pi(t),t)d\pi + \frac{1}{2}a_p d\pi d\pi \\
&= \mathcal{S}(\pi(t),t)dt + \frac{1}{2}\nu_p(\pi(t),t)(\sigma^\pi(t))^2 dt + \nu(\pi(t),t)\mu^\pi(t)dt \\
&\quad + \nu(\pi(t),t)\sigma^\pi(t)\int b_1^\pi(s,t)W_1^Q(ds,dt) + b_2^\pi(s,t)W_2^Q(ds,dt)
\end{aligned}$$

Likewise

$$\begin{aligned}
dD(\pi(t),t) &= \mathcal{D}(\pi(t),t)dt + \frac{\partial \mathcal{D}}{\partial \pi}d\pi + \frac{1}{2}\frac{\partial^2 \mathcal{D}}{\partial \pi^2}d\pi d\pi \\
&= \mathcal{D}(\pi(t),t)dt - a(\pi(t),t)d\pi - \frac{1}{2}a_p d\pi d\pi \\
&= \mathcal{D}(\pi(t),t)dt - \frac{1}{2}a_p(\pi(t),t)(\sigma^\pi(t))^2 dt - a(\pi(t),t)\mu^\pi(t)dt \\
&\quad - a(\pi(t),t)\sigma^\pi(t)\int b_1^\pi(s,t)W_1(ds,dt) + b_2^\pi(s,t)W_2(ds,dt)
\end{aligned}$$

Why the Finite Variation Model fails

Equating the volatility part:

$$\nu(\pi(t), t) = -a(\pi(t), t) = 0$$

In other terms:

$$\int_0^t \nu(\pi(t), s) ds = \int_0^t \alpha(\pi(t), s) ds = 0$$

However, $\pi(t)$ can assume any value. This means that the order rate densities (defined positive) have to be zero!

Diffusion Model

For simplicity, assume:

- $\mathbb{Q}$ exists (it will turn out to be the case)
- only one Brownian sheet

Allow traders to cancel orders.

$$da(p,t) = \mu^{a,\mathbb{Q}}(p,t)dt + \sigma^a(p,t)\int_{s=0}^{\infty} b^a(p,s,t)W^{\mathbb{Q}}(ds,dt)$$
$$dv(p,t) = \mu^{v,\mathbb{Q}}(p,t)dt + \sigma^v(p,t)\int_{s=0}^{\infty} b^v(p,s,t)W^{\mathbb{Q}}(ds,dt)$$

Problem: cannot cancel orders that have already been matched.

Solution: introduce density of total number of orders matched at price p: $n(p,t)$

Need:

$$\mu^a(p,t) \geq 0 \qquad \mu^\nu(p,t) \geq 0$$

and

$$\sigma^a(p,t) = 0 \quad \text{if } a(p,t) \leq n(p,t) \qquad \sigma^v(p,t) \geq 0 \quad \text{if } v(p,t) \leq n(p,t)$$

This introduces feedback!

Diffusion Model

Define the instantaneous correlations between the order flow and the price:

$$C^a(\pi(t),t)dt = \int b^a(\pi(t),s,t)b^\pi(s,t)ds \qquad C^v(\pi(t),t)dt = \int b^v(\pi(t),s,t)b^\pi(s,t)ds$$

$$\mu^{D,\mathbb{Q}}(p,t) = \int_{y=p}^{\infty} \mu^{a,\mathbb{Q}}(y,t)dy \qquad \mu^{S,\mathbb{Q}}(p,t) = \int_{y=0}^{p} \mu^{v,\mathbb{Q}}(y,t)dy$$

Apply Ito-Wentzell formula:

$$\begin{aligned} dD(\pi(t),t) &= \mu^D(\pi(t),t)dt + \sigma^D(\pi(t),t)\int_{s=0}^{\infty} b^D(\pi(t),s,t)W^{\mathbb{Q}}(ds,dt) \\ &\quad -a(\pi(t),t)d\pi(t) - \frac{1}{2}a_p(\pi(t),t)d\pi(t)d\pi(t) - \sigma^a(\pi(t),t)C^a(\pi(t),t)dt \end{aligned}$$

Equate the drift:

$$\begin{aligned} \mu^{D,\mathbb{Q}}(\pi(t),t) - \mu^{S,\mathbb{Q}}(\pi(t),t) &= \frac{1}{2}a_p(\pi(t),t)(\sigma^\pi(t))^2 + \sigma^a(\pi(t),t)C^a(\pi(t),t) + \\ &\quad \frac{1}{2}v_p(\pi(t),t)(\sigma^\pi(t))^2 + \sigma^v(\pi(t),t)C^v(\pi(t),t) \end{aligned}$$

Equate the volatility:

$$(\sigma^\pi(t))^2 = \int \left(\frac{\sigma^D(\pi(t),t)b^D(\pi(t),s,t) - \sigma^S(\pi(t),t)b^S(\pi(t),s,t)}{a(\pi(t),t) + v(\pi(t),t)} \right)^2 ds$$

Volatility is well-defined since $a(\pi(t),t) + v(\pi(t),t) > 0$

Diffusion Model

Let $\lambda(s,t)$ be the market price of risk. For simplicity, drop argument t. Define:

$$\Lambda^D(\pi) = \int_{s=0}^{\infty} \lambda(s) b^D(\pi, s) ds \qquad \Lambda^S(\pi) = \int_{s=0}^{\infty} \lambda(s) b^S(\pi, s) ds$$

The market price of risk equations are:

$$\mu^{D,\mathbb{Q}}(\pi) - \mu^{S,\mathbb{Q}}(\pi) = \mu^{D,\mathbb{P}}(\pi) - \mu^{S,\mathbb{P}}(\pi) + \sigma^D(\pi)\Lambda^D(\pi) + \sigma^S(\pi)\Lambda^S(\pi)$$

Since this holds for each π we can solve the market price of risk equations "generically".

Proposition

"Generically" there exists a unique risk-neutral measure.

Was there Arbitrage on April 1st, 2011?

1. Source: NYSE Arca on April 1st, 2011
2. Daily trading time: Select data during the trading day (9:30 AM - 4:00 PM EST).
3. New adding orders: Select new adding orders only, defined as type "A" in the NYSE ArcaBook, version 1.5a, revised in October 13, 2010.
4. Price range:
   - Choose a selected price range [*lowest limit price*, *highest limit price*]
   - Consider more than 90% of the original data will be good enough to present the original data without loss of generality.

Buy Limit Orders

Figure: Histogram of GE Buy Limit Prices

Bin	Frequency	Percentage
0.00	0	0.00%
10.00	91	0.19%
15.00	496	1.04%
16.00	305	0.64%
17.00	419	0.88%
18.00	855	1.80%
19.00	1165	2.45%
19.99	685	1.44%
20.49	43570	91.56%
Total	**47586**	**100%**

Buy Limit Orders

Obtain 50 intervals by choosing one cent as an increment, where 20.00 is the lowest buy limit price and 20.49 is the highest buy limit price.

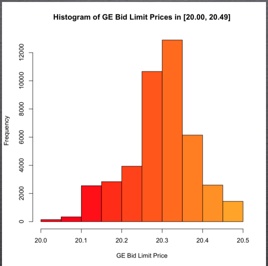


Figure: Histogram of GE Buy Limit Prices in [20.00, 20.49]

Obtain 195 2-minute intervals by dividing the time period from 9:00 AM to 4:00 PM into 2-minute intervals. Finally, we create a 3D matrix with x as "Price", y as "Time" and z as "Size".

Buy Limit Orders in 3D

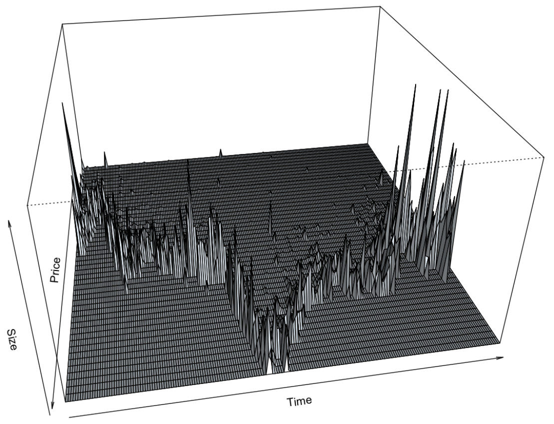
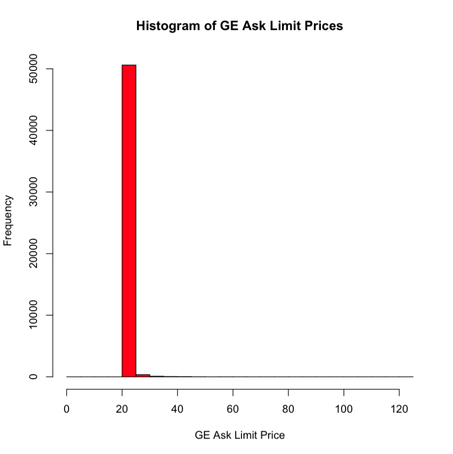


Figure: GE Buy Size on April 01, 2011 in 3D

Ask Limit Orders



Bin	Frequency	Percentage
20.12	0	0.00%
20.62	47818	93.47%
21.00	584	1.14%
22.00	1102	2.15%
23.00	544	1.06%
24.00	261	0.51%
25.00	297	0.58%
26.00	118	0.23%
27.00	65	0.13%
28.00	56	0.11%
29.00	38	0.07%
30.00	78	0.15%
35.00	94	0.18%
40.00	50	0.10%
45.00	35	0.07%
122.00	17	0.03%
Total	**51157**	**100%**

Figure: Histogram of GE Ask Limit Prices

Ask Limit Orders

Similarly obtain 50 intervals by choosing one cent as an increment, where 20.13 is the lowest sell limit price and 20.62 is the highest sell limit price.

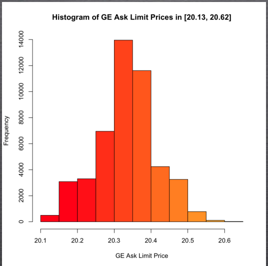


Figure: Histogram of GE Ask Limit Prices in [20.13, 20.62]

With 195 *2-minute* intervals, we create a 3D matrix with x as "Price", y as "Time" and z as "Size".

Ask Limit Orders in 3D

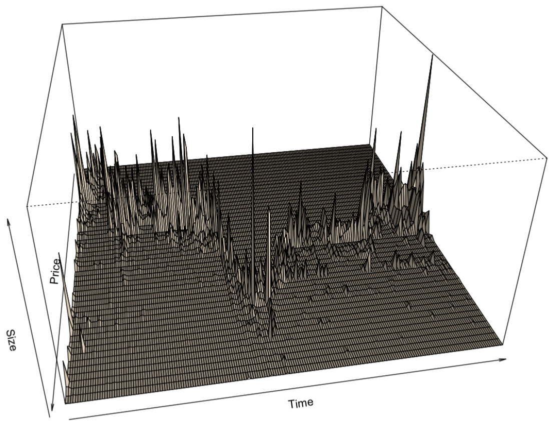
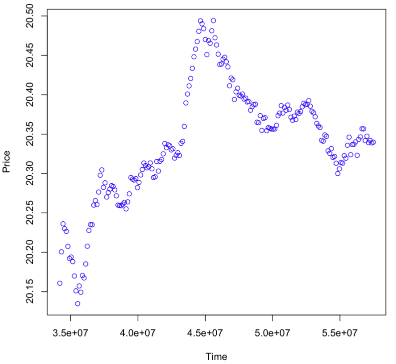


Figure: GE Ask Size on April 01, 2011 in 3D

Clearing Price $\pi(t)$

Obtain $\pi(t)$ by matching the ID of type "C" orders with the ID of the bid/ask orders.




```
> GE.clrPi
  [1] 20.16073 20.20056 20.23613 20.22996 20.22650 20.20745 20.19216 20.19391
  [9] 20.18813 20.16987 20.15110 20.13476 20.15725 20.14929 20.17054 20.16737
 [17] 20.18515 20.20772 20.22779 20.23514 20.23494 20.25990 20.26567 20.26059
 [25] 20.27649 20.29782 20.30460 20.28240 20.28827 20.27011 20.27593 20.28024
 [33] 20.28456 20.28345 20.27917 20.27166 20.25988 20.25933 20.25909 20.26118
 [41] 20.26342 20.25507 20.26396 20.27415 20.29496 20.29286 20.29171 20.29355
 [49] 20.28206 20.28874 20.29820 20.30506 20.31331 20.31006 20.30720 20.30854
 [57] 20.31342 20.30594 20.29486 20.29585 20.31525 20.30313 20.31553 20.31778
 [65] 20.32498 20.33810 20.33267 20.33640 20.33518 20.32999 20.33142 20.31962
 [73] 20.32310 20.32635 20.32289 20.33829 20.34088 20.35978 20.38962 20.40092
 [81] 20.41122 20.42064 20.43357 20.44819 20.45799 20.46751 20.48054 20.49384
 [89] 20.49000 20.48391 20.47016 20.45099 20.46887 20.46529 20.48117 20.49414
 [97] 20.47238 20.46326 20.45147 20.43831 20.43906 20.44516 20.44755 20.44185
[105] 20.43542 20.41135 20.42122 20.41902 20.39397 20.40370 20.40825 20.39930
[113] 20.39844 20.40092 20.39455 20.39573 20.39084 20.39118 20.38043 20.38471
[121] 20.38743 20.38784 20.36511 20.36443 20.37348 20.35474 20.36995 20.37094
[129] 20.35439 20.35821 20.35746 20.35652 20.35636 20.35655 20.36112 20.37343
[137] 20.37669 20.38619 20.37674 20.38407 20.38040 20.38694 20.38125 20.37154
[145] 20.36766 20.37294 20.36861 20.37801 20.37635 20.37866 20.38474 20.38915
[153] 20.38736 20.38780 20.39255 20.38552 20.37902 20.37710 20.37206 20.36366
[161] 20.36032 20.35838 20.34225 20.34106 20.34899 20.34729 20.32881 20.32559
[169] 20.33157 20.32047 20.32174 20.31305 20.29998 20.30593 20.31407 20.31317
[177] 20.32259 20.31946 20.33590 20.34619 20.32391 20.33669 20.33710 20.34009
[185] 20.32316 20.34334 20.34646 20.35682 20.35671 20.34170 20.34741 20.33928
[193] 20.34204 20.33886 20.33984
```

GE Clearing Prices $\pi(t)$ in [20.00, 20.62]

Clearing Price $\pi(t)$

Based on the above methodology, the opening price is 20.16 vs. 20.14; the high price is 20.49 vs. 20.50; the low price is 20.13 vs. 20.12; and the closing price is 20.34 vs. 20.34.

Figure: GE Prices on April 1st, 2011 from Yahoo! Finance.

Set Date Range

Start Date: Apr 1 2011 Eg. Jan 1, 2010

End Date: Apr 1 2011

Daily

Weekly

Monthly

Dividends Only

Get Prices

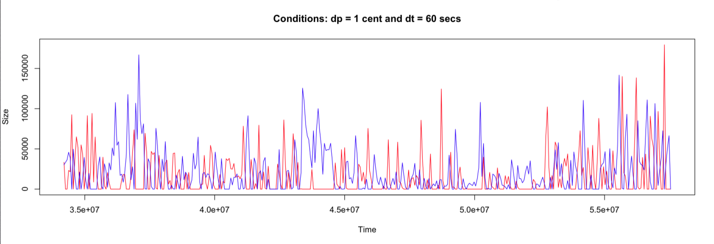
First | Previous | Next | Last

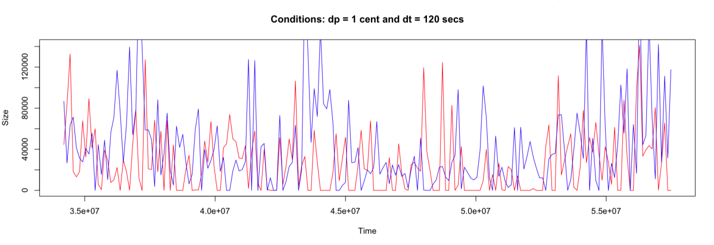
Prices

Date	Open	High	Low	Close	Volume	Adj Close*
Apr 1, 2011	20.14	20.50	20.12	20.34	48,400,000	19.79

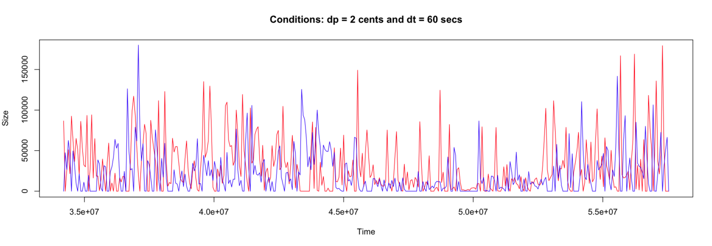
* Close price adjusted for dividends and splits.

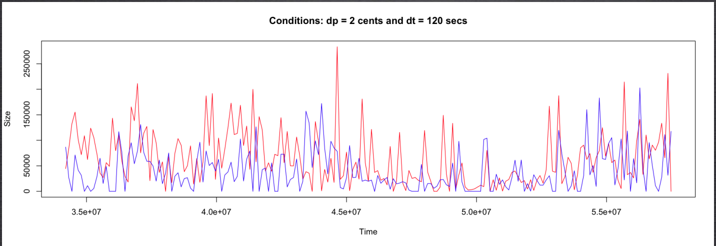
Plot $\mathcal{D}(\pi(t),t)$ in *blue* and $\mathcal{S}(\pi(t),t)$ in *red*



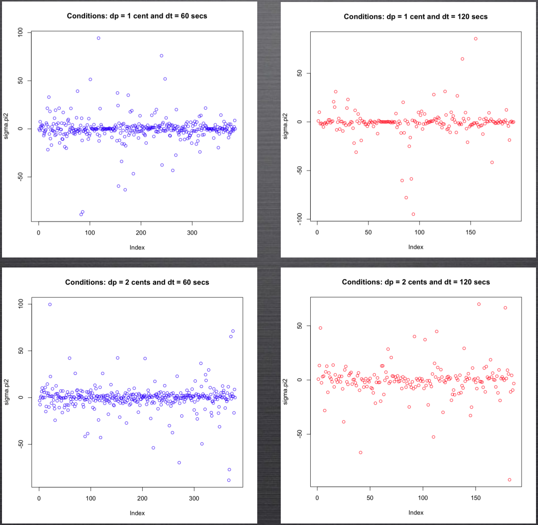


Plot $\mathcal{D}(\pi(t),t)$ in *blue* and $\mathcal{S}(\pi(t),t)$ in *red*





Plot $\sigma^{\pi}(t)^2$



Conclusion and Future Work

1. Second Fundamental Theorem
2. Convergence of a Binomial Model to This Model
3. What about large trader(s)?
4. Identify Arbitrage Strategy
5. Option Pricing