Milan J. Math. Vol. 77 (2009) 205–244
DOI 10.1007/s00032-009-0100-0
Published online September 11, 2009


Milan Journal of Mathematics

# The Numerical Approximation of Stochastic Partial Differential Equations

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**Abstract.** The numerical solution of stochastic partial differential equations (SPDEs) is at a stage of development roughly similar to that of stochastic ordinary differential equations (SODEs) in the 1970s, when stochastic Taylor schemes based on an iterated application of the Itô formula were introduced and used to derive higher order numerical schemes. An Itô formula in the generality needed for Taylor expansions of the solution of a SPDE is however not available. Nevertheless, it was shown recently how stochastic Taylor expansions for the solution of a SPDE can be derived from the mild form representation of the SPDE, which avoid the need of an Itô formula. A brief review of the literature is given here and the new stochastic Taylor expansions are discussed along with numerical schemes that are based on them. Both strong and pathwise convergence are considered.

**Mathematics Subject Classification (2000).** Primary 60H35; Secondary 60H15.

**Keywords.** Stochastic partial differential equations, stochastic ordinary differential equations, stochastic Taylor expansions, higher order numerical schemes, strong convergence, pathwise convergence.

This work has been supported by the DFG project "Pathwise numerics and dynamics of stochastic evolution equations" and the spanish Ministerio de Educación y Ciencia project MTM2005-01412.

# 1. Introduction

The numerical approximation of stochastic partial differential equations (SPDEs), by which we mean stochastic evolution equations of the parabolic or hyperbolic type, encounters all of the difficulties that arise in the numerical solution of both deterministic PDEs and finite dimensional stochastic ordinary differential equations (SODEs) as well as many more due to the infinite dimensional nature of the driving noise processes. The state of development of numerical schemes for SPDEs compares with that for SODEs in the early 1970s and most of the numerical schemes that have been proposed to date have a low order of convergence, especially in terms of an overall computational effort.

The break through for SODEs started with the Milstein scheme and continued with the systematic derivation of stochastic Taylor expansions and the numerical schemes based on them. These stochastic Taylor schemes are based on an iterated application of the Itô formula. The crucial point is that the multiple stochastic integrals which they contain provide more information about the noise processes within discretization subintervals and this allows an approximation of higher order to be obtained. This theory is presented in detail in the monographs Kloeden & Platen [76] and Milstein [97] and is briefly sketched in Kloeden [74].

An Itô formula in the generality needed for Taylor expansions of the solution of a SPDE of evolutionary type is however not available. Nevertheless, it has recently been shown that Taylor expansions for the solution of such a SPDE, which avoid the need of an Itô formula, can be constructed by taking advantage of the mild form representation of the solution. Such expansions are robust with respect to the noise, i.e. hold for other types of stochastic processes with Hölder continuous sample paths such as fractional Brownian motion.

The rest of this article is organized as follows. While stochastic Taylor expansions for SODEs and the numerical schemes based on them are briefly reviewed in Section 2, several approximation results for SPDEs in the literature are sketched in Section 3. In Section 4, a recent numerical scheme for SPDEs with additive noise with an in comparison to classical numerical schemes such as the linear-implicit Euler scheme higher order convergence rate is considered. This scheme is the simplest numerical scheme based on the new Taylor expansions for SPDEs, which are then reviewed in Section 5. Further numerical schemes based on these Taylor expansions for SPDEs are considered in the final section.

## 2. SODEs reconsidered

For simplicity, we consider a scalar Itô stochastic ordinary differential equation (SODE)

$$dX_t = f(t, X_t)\, dt + g(t, X_t)\, dW_t, \tag{2.1}$$

with a standard scalar Wiener process $W_t$, $t \geq 0$. The SODE (2.1) is, in fact, only a symbolic representation for the stochastic integral equation

$$X_t = X_{t_0} + \int_{t_0}^{t} f(s, X_s)\, ds + \int_{t_0}^{t} g(s, X_s)\, dW_s. \tag{2.2}$$

The simplest numerical scheme for the SODE (2.1) is the *Euler-Maruyama scheme* given by

$$Y_{n+1} = Y_n + f(t_n, Y_n) \int_{t_n}^{t_{n+1}} ds + g(t_n, Y_n) \int_{t_n}^{t_{n+1}} dW_s, \tag{2.3}$$

where one usually writes

$$\Delta_n = \int_{t_n}^{t_{n+1}} ds, \qquad \Delta W_n = \int_{t_n}^{t_{n+1}} dW_s$$

for $n = 0, 1, \ldots, M_T - 1$ and where $t_0 < t_1 < \ldots < t_{M_T} = T$ with $M_T \in \mathbb{N}$ is an arbitrary partition of $[t_0, T]$. Depending on whether the realizations or only their probability distributions are required to be close, one distinguishes between strong and weak convergence of numerical schemes for such SODEs, respectively. Consider a fixed interval $[t_0, T]$ and let $\Delta$ be the maximum step size of any partition of $[t_0, T]$. Then a numerical scheme is said to converge with *strong order* $\gamma$ if, for every sufficiently small $\Delta$,

$$\mathbb{E}\left(\left|X_T - Y_{M_T}^{(\Delta)}\right|\right) \leq C_T\, \Delta^{\gamma} \tag{2.4}$$

and with *weak order* $\beta$ if

$$\left|\mathbb{E}\left(p(X_T)\right) - \mathbb{E}\left(p(Y_{M_T}^{(\Delta)})\right)\right| \leq C_{p,T}\, \Delta^{\beta} \tag{2.5}$$

for each polynomial $p$. In contrast to numerical weak convergence as presented in equation (2.5), we recall that a sequence of real valued random variables $(Z_n)_{n\in\mathbb{N}}$ converges weakly to a real valued random variable $Z$, if

$$\lim_{n\to\infty} \mathbb{E}\left[\varphi(Z_n)\right] = \mathbb{E}\left[\varphi(Z)\right]$$

holds for every continuous and globally bounded function $\varphi : \mathbb{R} \to \mathbb{R}$. The largest possible values of $\gamma$ in (2.4) and $\beta$ in (2.5) give the corresponding

strong and weak orders, respectively, of the scheme for a whole class of stochastic differential equations, e.g. with sufficiently often continuously differentiable coefficient functions. For example, the Euler-Maruyama scheme has strong order $\gamma = \frac{1}{2}$ and weak order $\beta = 1$, while the *Milstein scheme*

$$\begin{aligned} Y_{n+1} = Y_n + f(t_n, Y_n) \int_{t_n}^{t_{n+1}} ds + g(t_n, Y_n) \int_{t_n}^{t_{n+1}} dW_s \\ + g(t_n, Y_n) \frac{\partial g}{\partial x}(t_n, Y_n) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_u \, dW_s \end{aligned} \tag{2.6}$$

has strong order $\gamma = 1$ and weak order $\beta = 1$. (Note the these convergence orders may be better for specific SODEs within the given class, e.g. the Euler-Maruyama scheme has strong order $\gamma = 1$ for SODEs with additive noise, since it then coincides with the Milstein scheme).

The Euler-Maruyama scheme is derived by essentially evaluating the integrals in the stochastic integral equation (2.2) over an interval $[t_n, t_{n+1}]$ with the integrand functions evaluated at the left end point $t_n$, or equivalently, first expanding the integrands in the integral equation (2.2) over the interval $[t_n, t_{n+1}]$ with the Itô formula, which, for a scalar valued function $U(t, x)$ of the solution $X_t$ of the SODE (2.1), is given by

$$U(t, X_t) = U(t_0, X_{t_0}) + \int_{t_0}^{t} L^0 U(s, X_s) \, ds + \int_{t_0}^{t} L^1 U(s, X_s) \, dW_s, \tag{2.7}$$

with the differential operators $L^0$ and $L^1$ defined by

$$L^0 = \frac{\partial}{\partial t} + f \frac{\partial}{\partial x} + \frac{1}{2} g^2 \frac{\partial^2}{\partial x^2}, \qquad L^1 = g \frac{\partial}{\partial x}, \tag{2.8}$$

and then discarding the remainder, i.e.

$$X_{t_{n+1}} = X_{t_n} + f(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} ds + g(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} dW_s + R_1(t_{n+1}, t_n)$$

with the remainder

$$\begin{aligned} R_1(t_{n+1}, t_n) = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^0 f(u, X_u) \, du \, ds + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^1 f(u, X_u) \, dW_u \, ds \\ + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^0 g(u, X_u) \, du \, dW_s \\ + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^1 g(u, X_u) \, dW_u \, dW_s. \end{aligned}$$

The next simplest strong Taylor scheme, the Milstein scheme, is derived in a similar way after first expanding the integrand $L^1 g$ in the remainder $R_1$ with the Itô formula, to obtain

$$X_{t_{n+1}} = X_{t_n} + f(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} ds + g(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} dW_s + L^1 g(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_u \, dW_s + R_2(t_{n+1}, t_n)$$

with the remainder

$$\begin{aligned} R_2(t_{n+1}, t_n) = & \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^0 f(u, X_u) \, du \, ds + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^1 f(u, X_u) \, dW_u \, ds \\ & + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} L^0 g(u, X_u) \, du \, dW_s \\ & + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{u} L^0 L^1 g(v, X_v) \, dv \, dW_u \, dW_s \\ & + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{u} L^1 L^1 g(v, X_v) \, dW_v \, dW_u \, dW_s. \end{aligned}$$

General stochastic Taylor schemes can be formulated compactly using hierarchical sets of multiply indices with iterated multiply stochastic integrals and iterated application of the differential operators to the coefficient function. This theory is sketched in Kloeden [74] and given in detail in Kloeden & Platen [76]. The above procedure already indicates the general pattern: the higher order schemes achieve their higher order through the inclusion of multiple stochastic integral terms; the coefficients of the scheme involve partial derivatives of the SODE coefficient functions; a scheme may have different strong and weak orders of convergence; and, the possible orders for strong schemes increase by a fraction $\frac{1}{2}$, taking values $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, ..., whereas possible orders for weak schemes are whole numbers 1, 2, 3, ....

The proofs of convergence rates in the monographs [76, 97] use the assumption that the coefficient functions $f$ and $g$ of the SODE are globally Lipschitz continuous. This assumption, however, is not satisfied in many basic and important applications, for example with polynomial coefficients such as

$$dX_t = X_t \, (1 - X_t) \, dt + X_t \, dW_t \qquad \text{or} \qquad dX_t = -X_t^3 \, dt + dW_t \tag{2.9}$$

or with square-root coefficients such as in the Cox-Ingersoll-Ross volatility model

$$dV_t = \kappa\,(\vartheta - V_t)\;dt + \mu\sqrt{V_t}\,dW_t, \tag{2.10}$$

which requires $V_t \geq 0$ (see also [76] for more of such examples and their role in applied sciences). Several results in the last decade occupy the problem of numerical approximations of nonlinear SODEs with non-globally Lipschitz coefficients. In particular, strong convergence of implicit schemes for SODEs with polynomial coefficients such as (2.9) was handled by Higham et al [56] and Mattingly et al [94] using dissipativity and ergodic arguments. Additionally, Milstein & Tretjakov [98] analyzed weak convergence of a modified explicit Euler scheme for such SODEs. Recently, it was shown in [59] that the classical explicit Euler-Maruyama scheme (and other simple explicit schemes such as Milstein's scheme) indeed fail to converge strongly and numerical weakly for SODEs with super-linearly growing coefficients of the form (2.9). Nevertheless pathwise convergence, i.e.

$$\mathbb{P}\left[\left\{\omega\in\Omega\;\middle|\;\sup_{n=0,\ldots,M_T}\left|X_{t_n}(\omega)-Y_n^{(\Delta)}(\omega)\right|\longrightarrow 0\;\text{ as }\;\Delta\to 0\right\}\right]=1$$

of explicit schemes such as the Euler-Maruyama scheme does hold for (2.9) and (2.10) and other SODEs under local assumptions on the coefficients of the SODE only (see [36] and [69]). We also refer to [59] for a more detailed overview of numerical approximations for SODEs with non-globally Lipschitz coefficients.

## 3. Stochastic partial differential equations

To fix ideas we consider parabolic stochastic partial differential equations with a Dirichlet boundary condition on a bounded domain $\mathcal{D}$ in $\mathbb{R}^d$ of the form

$$dX_t = [AX_t + f(X_t)]\,dt + g(X_t)\,dW_t. \tag{3.1}$$

Later we will give specific assumptions on the coefficients, but for now we will simply assume that the eigenvalues $\lambda_j$ and the corresponding eigenfunctions $\phi_j \in H_0^{1,2}(\mathcal{D}) = H^2(\mathcal{D}) \cap H_0^1(\mathcal{D})$ of the operator $-A$ , i.e., with

$$-A\phi_j = \lambda_j\phi_j, \qquad j = 1, 2, \ldots,$$

form an orthonormal basis in $L_2(\mathcal{D})$ with $\lambda_j \to \infty$ as $j \to \infty$. We will also assume that $W_t$ is a standard scalar Wiener process and let $|\cdot|_{L^2(\mathcal{D})}$ be the norm of $L^2(\mathcal{D})$.

Projecting the SPDE (3.1) onto the $N$–dimensional subspace $\mathcal{X}_N$ of $L_2(\mathcal{D})$ spanned by $\{\phi_1, \ldots, \phi_N\}$ gives an $N$-dimensional Itô-Galerkin SODE in $\mathbb{R}^N$ of the form

$$dX_t^N = \left[A_N X_t^N + f_N(X_t^N)\right]\, dt + g_N(X_t^N)\, dW_t \tag{3.2}$$

for $N \in \mathbb{N}$ with $f_N = P_N f\big|_{\mathcal{X}_N}$ and $g_N = P_N g\big|_{\mathcal{X}_N}$, where $f$ and $g$ are now interpreted as mappings of $L_2(\mathcal{D})$ or $H_0^{1,2}(\mathcal{D})$ into itself, where $P_N$ is the projection of $L_2(\mathcal{D})$ or $H_0^{1,2}(\mathcal{D})$ onto $\mathcal{X}_N$, while $A_N = P_N A\big|_{\mathcal{X}_N}$ is the diagonal matrix $\operatorname{diag}[\lambda_j, \ldots, \lambda_N]$.

### 3.1. Some early results

Grecksch & Kloeden [34] showed in 1996 that the combined truncation and global discretization error for an strong order $\gamma$ stochastic Taylor scheme applied to (3.2) with constant time-step $\Delta$, which we denote by $Y_k^{(N,\Delta)}$, $k = 0, 1, \ldots, M_T$, has the form

$$\sup_{k=0,1,\ldots,M_T} \mathbb{E}\left(\left|X(k\Delta) - Y_k^{(N,\Delta)}\right|_{L_2(\mathcal{D})}\right) \le C_T \left(\lambda_{N+1}^{-1/2} + \lambda_N^{\lfloor \gamma + \frac{1}{2} \rfloor + 1} \Delta^\gamma\right), \tag{3.3}$$

where $\lfloor x \rfloor$ denotes the integer part of the real number $x$ and the constant $C_T$ depends on the initial value and bounds on the coefficient functions $f$ and $g$ of the SPDE (3.1) as well as on the length of the time interval $[0, T]$ under consideration. Since $\lambda_j \to \infty$ as $j \to \infty$, a very small time-step is needed in high dimensions, which is needed for convergence, i.e. the Itô-Galerkin SODE (3.2) is stiff and explicit schemes such as strong stochastic Taylor schemes are not really appropriate. Obviously, an implicit scheme should be used here, but the special structure of the SODE (3.2) allows one to use a simpler linear-implicit scheme, i.e., it is the matrix $A_N$ in the linear part of the drift coefficient that causes the troublesome growth with respect to the eigenvalues, so only this part of the drift coefficient needs to be made implicit (we refer to the monographs [48] and [119] for a detailed handling of such stiff problems in the deterministic case). For example consider the *linear-implicit Euler scheme* for the SODE (3.2), i.e.

$$Y_{n+1}^{(N)} = Y_n^{(N)} + A_N Y_{n+1}^{(N)}\, \Delta_n + f_N(Y_n^{(N)})\Delta_n + g_N(y_n^{(N)})\, \Delta W_n, \tag{3.4}$$

which is easily solved for $Y_{n+1}^{(N,\Delta)}$ because its matrix $I_N - A_N \Delta_n$ is diagonal. Kloeden & Shott [77] showed that for a linear-implicit strong order $\gamma$

stochastic Taylor scheme the combined error has the form

$$\sup_{k=0,1,\ldots,M_T} \mathbb{E}\left(\left|X(k\Delta) - Y_k^{(N,\Delta)}\right|_{L_2(\mathcal{D})}\right) \leq C_T\left(\lambda_{N+1}^{-1/2} + \Delta^{\gamma}\right), \tag{3.5}$$

so the time-step can be chosen independently of the dimension $N$ of the Itô-Galerkin SODE (3.2).

The above results are of limited use because $W_t$ is only one-dimensional and the proofs of the convergence of Taylor schemes for SODE in the monographs [76, 97] assume that partial derivatives of the coefficient functions of the Galerkin SODE are uniformly bounded on $\mathbb{R}^N$.

### 3.2. Other results

There are now a large number of papers in the literature on the numerical approximation of SPDEs. For the readers convenience we give the following incomplete list of references, which we subdivide into categories.

First of all, the most used strategy for numerical approximations of SPDEs is to discretize a SPDE spatially via finite differences, finite elements or spectral methods and temporally via numerical schemes similar to classical Runge-Kutta schemes for stiff ODEs such as the linear-implicit Euler scheme or the Crank-Nicolson scheme. Such methods were used in [4, 1, 2, 18, 22, 23, 27, 35, 37, 31, 42, 43, 45, 46, 49, 50, 53, 54, 79, 71, 73, 78, 75, 83, 84, 85, 95, 96, 106, 103, 107, 112, 113, 114, 115, 108, 121, 122, 123, 125, 127, 128, 129, 130]. Beside this prevalent strategy also the splitting up approach for temporal discretizations (see [8, 60, 38, 39, 40, 89]), the Wiener chaos expansion (see [57, 87, 86, 90, 88, 92]) and the particle method (see [80, 13, 14, 11, 12, 15]) are frequently used ways to construct numerical approximations for SPDEs. Numerical approximations based on the new Taylor expansions for SPDEs, which are reviewed in Section 5 in this article, are considered in [61, 62, 63, 64, 67, 68]. Lower bounds for the numerical approximation of SPDEs were studied in [17, 100, 101, 102]. In contrast to the above cited results for the numerical approximation of SPDEs of evolutionary type, we also mention [5, 9, 41, 93] for the numerical approximation of SPDEs of elliptic type.

Much of the literature is concerned with a semilinear stochastic heat equation with additive space-time white noise $W_t$, $t \geq 0$, on the one-dimensional domain $[0, 1]$ over the time interval $[0, T]$ with $T > 0$, i.e.

$$dX_t = \left[\frac{\partial^2}{\partial x^2} X_t + f(X_t)\right] dt + dW_t \tag{3.6}$$

for $t \in [0,T]$ and $x \in [0,1]$ with appropriate boundary conditions. The key question arising in the numerical analysis of such SPDEs is then the number of arithmetical operations and the number of random variables (computational effort) needed to compute a numerical approximation of the solution of such a SPDE for a given accuracy. This is usually measured via an overall convergence rate, which combines the two components of the error bound due to the temporal and the spatial discretization. More precisely, let

$$dX_t^N = [A_N X_t + f_N(X_t)]\,dt + dW_t^N \tag{3.7}$$

for $t \in [0,T]$ be the corresponding $N$-dimensional Galerkin SODE of the SPDE (3.6) with $A_N$, $f_N$ and $W^N$ appropriate for $N \in \mathbb{N}$. Additionally, let $Y_k^{N,M}$, $k = 0,1,\ldots,M$, be a time-discretization of the SODE (3.7), where $M \in \mathbb{N}$ times steps of the size $\frac{T}{M}$ where used. Then (depending on the concrete scheme), oftentimes up to a constant $K = N \cdot M$ arithmetical operations and random variables are needed to compute $Y_k^{N,M}$, $k = 0,1,\ldots,M$. If the scheme has an error bound of the form

$$\sup_{k=0,\ldots,M} \left( \mathbb{E} \left| X_{\frac{kT}{M}} - Y_k^{(N,M)} \right|_{L^2(0,1)}^2 \right)^{\frac{1}{2}} \leq C_T \left( \frac{1}{N^\alpha} + \frac{1}{M^\beta} \right) \tag{3.8}$$

for $N, M \in \mathbb{N}$ with $\alpha$, $\beta > 0$, then the optimal overall rate is $\dfrac{\alpha\beta}{\alpha+\beta}$ with respect to the computational effort, i.e.

$$\sup_{k=0,\ldots,M} \left( \mathbb{E} \left| X_{\frac{kT}{M}} - Y_k^{(N,M)} \right|_{L^2(0,1)}^2 \right)^{\frac{1}{2}} \leq C_T \cdot K^{-\frac{\alpha\beta}{\alpha+\beta}}$$

for $N = K^{\frac{\beta}{\alpha+\beta}}$ and $M = K^{\frac{\alpha}{\alpha+\beta}}$. For example, if $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$, then we obtain the overall convergence rate $\frac{1}{6}$ with respect to the computational effort.

The following papers are representative of many others in the literature dealing with this equation. Gyöngy & Nualart [45] (see also [46]) introduced an implicit numerical scheme for the SPDE (3.6) in 1995 and showed that it converges uniform in probability to the exact solution without giving a rate. In 1998 and 1999 Gyöngy [35, 37] also applied finite differences to the SPDE (3.6) and then used several temporal implicit and explicit schemes, in particular, the linear-implicit Euler scheme. He showed that these schemes converge with order $\frac{1}{2}$ in the space and with order $\frac{1}{4}$ in time (assuming a smooth initial value) and, hence, he obtained an overall convergence rate

of $\frac{1}{6}$ with respect to the computational effort in space and time. Also in 1999 Shardlow [121] applied finite differences to the SPDE (3.6) to obtain a spatial discretization which he then discretized in time with a $\theta$-method. This had an overall convergence rate $\frac{1}{6}$ with respect to the computational effort.

In a seminal paper Davie & Gaines [17] showed in 2000 that any numerical scheme applied to the SPDE (3.6) with $f = 0$ which uses only equidistant values of the noise $W_t$, $t \in [0, T]$, cannot converge faster than the rate of $\frac{1}{6}$ with respect to the computational effort. Additionally, they showed that this is also a lower bound for the convergence rate in the multiplicative noise case. In this case we also refer to the result of Müller-Gronbach & Ritter [100] (see also [101]). In the linear case, i.e. the SPDE (3.6) with $f = 0$, the overall convergence rate $\frac{1}{6}$ can be improved by non-equidistant evaluations of the noise process, which is shown in [102].

Higher order rates than $\frac{1}{6}$ for nonlinear SPDEs of the form (3.6) were obtained for smoother types of noise. For example, in 2003 Hausenblas [50] applied the linear-implicit and explicit Euler scheme and the Crank-Nicholson scheme to the SPDE (3.6) driven by an infinite dimensional noise process. For trace-class noise, she obtained the order $\frac{1}{4}$ with respect to the computational effort, but in the case of space-time white noise the convergence rate was no better than the Davie-Gaines barrier rate $\frac{1}{6}$. Similarly, in 2004 Lord & Rougemont [84] discretized the Galerkin-SODE (3.7) in time with the numerical scheme

$$Y_{k+1}^{(N,M)} = e^{A_N \frac{T}{M}} \left( Y_k^{(N,M)} + \frac{T}{M} \cdot f_N \left( Y_k^{(N,M)} \right) + W^N_{\frac{(k+1)T}{M}} - W^N_{\frac{kT}{M}} \right) \quad (3.9)$$

for $k = 0, 1, \ldots, M-1$, which they showed to be useful when the noise is very smooth in space, in particular with Gévrey regularity. In the case of space-time white noise the scheme (3.9) converges also at Davie-Gaines barrier rate $\frac{1}{6}$.

## 4. A numerical scheme of higher order

Davie & Gaines [17], page 129, remarked that it may be possible to improve the convergence rate by using suitable linear functionals of the noise. This suggestion was used in [67], where a parabolic SPDE with additive noise

$$dX_t = [AX_t + F(X_t)]\, dt + dW_t, \qquad X_0 = x_0 \quad (4.1)$$

for $t \in [0,T]$ with $T > 0$ fixed in a Hilbert space $(H, |\cdot|)$ is considered. There $A : D(A) \subset H \to H$ is an in general unbounded linear operator (for example the Laplacian with Dirichlet boundary conditions), $F : H \to H$ is a smooth function and $W_t$, $t \in [0,T]$, is a cylindrical Wiener process with respect to a normal filtration $\mathcal{F}_t$, $t \in [0,T]$, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (see Section 4.2 for a precise description of the assumptions used). The SPDE (4.1) can be interpreted in the mild sense, i.e. as satisfying the integral equation

$$X_t = e^{At}x_0 + \int_0^t e^{A(t-s)}F(X_s)\,ds + \int_0^t e^{A(t-s)}\,dW_s \tag{4.2}$$

for $t \in [0,T]$. Additionally, the solution of the $N$-dimensional Itô-Galerkin SODE in the space $H_N := P_N H$ (or, equivalently, in $\mathbb{R}^N$)

$$dX_t^N = \left(A_N X_t^N + F_N(X_t^N)\right)\,dt + dW_t^N, \tag{4.3}$$

has an analogous "mild" representation

$$X_t^N = e^{A_N t}x_0^N + \int_0^t e^{A_N(t-s)}F_N(X_s^N)\,ds + \int_0^t e^{A_N(t-s)}\,dW_s^N \tag{4.4}$$

for $t \in [0,T]$. This motivated what is in [67] called the *exponential Euler scheme* and given by

$$\begin{aligned} Y_{k+1}^{(N,M)} &= e^{A_N\Delta}Y_k^{(N,M)} + A_N^{-1}\left(e^{A_N\Delta} - I\right)f_N(Y_k^{(N,M)}) \\ &\quad + \int_{t_k}^{t_{k+1}} e^{A_N(t_{k+1}-s)}\,dW_s^N \end{aligned} \tag{4.5}$$

with time-step $\Delta = \frac{T}{M}$ for $M \in \mathbb{N}$ and discretization times $t_k = k\Delta$ for $k = 0, 1, \ldots, M$. This scheme is easier to simulate than it may seem on the first sight. More precisely, denoting the components of $Y_k^{(N,M)}$ and $F_N$ by

$$Y_{k,i}^{(N,M)} = \left\langle e_i, Y_k^{(N,M)} \right\rangle, \qquad F_N^i = \langle e_i, F_N \rangle, \qquad i = 1, \ldots, N,$$

the numerical scheme (4.5) can be rewritten as

$$\begin{aligned} Y_{k+1,1}^{(N,M)} &= e^{-\lambda_1\Delta}Y_{k,1}^{(N,M)} + \frac{(1-e^{-\lambda_1\Delta})}{\lambda_1}F_N^1(Y_k^{(N,M)}) + \sqrt{\frac{q_1}{2\lambda_1}(1-e^{-2\lambda_1\Delta})}R_k^1 \\ &\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ Y_{k+1,N}^{(N,M)} &= e^{-\lambda_N\Delta}Y_{k,N}^{(N,M)} + \frac{(1-e^{-\lambda_N\Delta})}{\lambda_N}F_N^N(Y_k^{(N,M)}) + \sqrt{\frac{q_N}{2\lambda_N}(1-e^{-2\lambda_N\Delta})}R_k^N, \end{aligned}$$

where the $R_k^i$ for $i = 1, \ldots, N$ and $k = 0, 1, \ldots, M-1$ are independent, standard normally distributed random variables.

**Theorem 4.1.** [67] *Suppose that Assumptions* 4.2–4.5 *below are satisfied. Then, there is a constant* $C_T > 0$ *such that*

$$\sup_{k=0,\ldots,M} \left( \mathbb{E} \left| X_{\frac{kT}{M}} - Y_k^{(N,M)} \right|^2 \right)^{\frac{1}{2}} \leq C_T \left( (\lambda_N)^{-\gamma} + \frac{(1+\log(M))}{M} \right) \quad (4.6)$$

*holds for all* $N, M \in \mathbb{N}$, *where* $X_t$, $t \in [0,T]$, *is the solution of SPDE* (4.1), $Y_k^{(N,M)}$, $k = 0, 1, \ldots, M$, $N, M \in \mathbb{N}$, *are the numerical solutions given by* (4.5) *and* $\lambda_N$, $N \in \mathbb{N}$, *and* $\gamma > 0$ *are given in Assumption* 4.2 *and Assumption* 4.3 *below.*

In fact, the exponential Euler scheme (4.5) converges in time with a strong order $1-\varepsilon$ for an arbitrarily small $\varepsilon > 0$, since $1 + \log(M)$ can be estimated by $M^{\varepsilon}$, so $\frac{\log(M)}{M} \approx \frac{1}{M^{(1-\varepsilon)}}$. Importantly, the error coefficient $C_T$ does not depend on the dimension $N$ of the Itô-Galerkin SODE.

An essential point is that the integral $\int_{t_k}^{t_{k+1}} e^{A_N(t_{k+1}-s)}\, dW_s^N$ includes more information about the noise on the discretization interval. Such additional information was the key to the higher order of stochastic Taylor schemes for SODEs, but is included there in terms of simple multiple stochastic integrals rather than integrals weighted by an exponential integrand.

### 4.1. Numerical results

To illustrate Theorem 4.1 consider the semilinear stochastic heat equation (3.6) on the one-dimensional domain $(0,1)$ with $f(X) = \frac{1}{2}X$ and with Dirichlet boundary conditions, i.e.

$$dX_t = \left[ \frac{\partial^2}{\partial x^2} X_t + \frac{1}{2} X_t \right] dt + dW_t, \qquad X_t(0) = X_t(1) = 0 \quad (4.7)$$

for $x \in (0,1)$ and $t \geq 0$ and where the initial value satisfies

$$X_0(x) = \sum_{n=1}^{\infty} \frac{\sqrt{2}}{n} \sin(n\pi x)$$

for $x \in (0,1)$. This choice will fulfill the assumptions used in Theorem 4.1. The linear-implicit Euler scheme is given by

$$\tilde{Y}_{k+1}^{(N,M)} = (I - \Delta A_N)^{-1} \left( \tilde{Y}_k^{(N,M)} + \Delta \cdot f(\tilde{Y}_k^{(N,M)}) + W_{t_{k+1}}^N - W_{t_k}^N \right),$$

the Lord-Rougemont scheme is given by

$$\bar{Y}_{k+1}^{(N,M)} = e^{A_N \Delta} \left( \bar{Y}_k^{(N,M)} + \Delta \cdot f(\bar{Y}_k^{(N,M)}) + W_{t_{k+1}}^N - W_{t_k}^N \right),$$

and the exponential Euler scheme is given by

$$Y_{k+1}^{(N,M)} = e^{A_N \Delta} Y_k^{(N,M)} + A_N^{-1} \left( e^{A_N \Delta} - I \right) f(Y_k^{(N,M)}) + \int_{t_k}^{t_{k+1}} e^{A_N (t_{k+1} - s)} \, dW_s^N$$

for $k = 0, 1, \ldots, M-1$ and $N, M \in \mathbb{N}$. From Figure 1 we see that the linear-implicit Euler and Lord-Rougemont schemes clearly converge with the rate $\frac{1}{6}$, while the exponential Euler scheme converges with the rate $\frac{1}{3}$. Hence, all three schemes converge with their theoretically predicted order (see [67] for details).

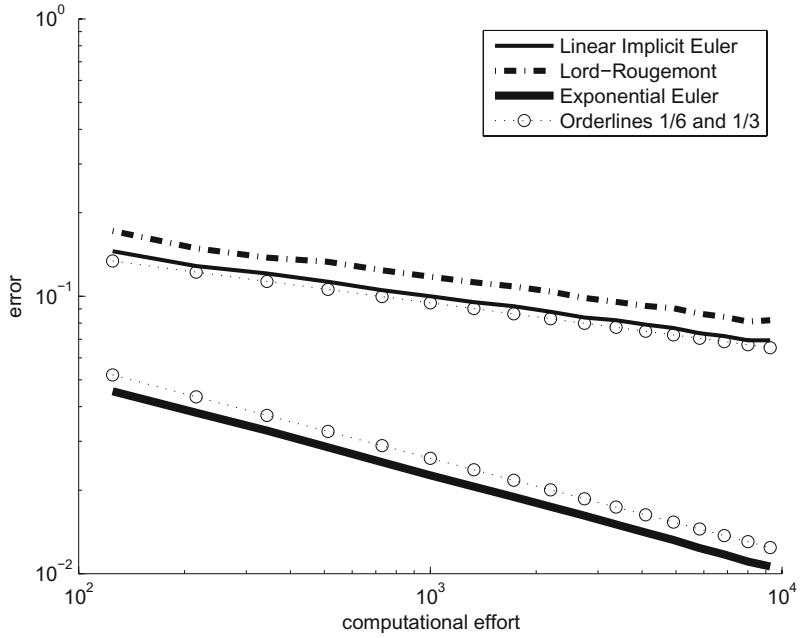


FIGURE 1. Mean square error vs. up to a constant the computational effort as log-log plot for the SPDE (4.7).

### 4.2. The assumptions in Theorem 4.1

In Theorem 4.1 above the following assumptions are used.

**Assumption 4.2 (Linear Operator $A$).** *There exist a sequence of real eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ and an orthonormal basis $e_n \in H$, $n \in \mathbb{N}$, of eigenfunctions of $A$ such that the linear operator $A : D(A) \subset H \to H$ is given by*

$$Av = \sum_{n=1}^{\infty} -\lambda_n \langle e_n, v \rangle e_n$$

*for all $v \in D(A)$ with $D(A) = \left\{ v \in H \big| \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle e_n, v \rangle|^2 < \infty \right\}$.*

**Assumption 4.3 (Cylindrical Wiener process $W$).** *There exist a sequence $q_n \geq 0$, $n \in \mathbb{N}$, of real numbers, a real number $\gamma \in (0,1)$ with $\sum_{n=1}^{\infty} (\lambda_n)^{(2\gamma-1)} q_n < \infty$ and independent scalar standard $\mathcal{F}$-Brownian motions $\beta_t^n$, $t \in [0,T]$, $n \in \mathbb{N}$, such that the cylindrical Wiener process $W_t$, $t \in [0,T]$, is formally given by*

$$W_t = \sum_{n=1}^{\infty} \sqrt{q_n}\, e_n\, \beta_t^n \tag{4.8}$$

*for every $t \in [0,T]$.*

**Assumption 4.4 (Nonlinearity $F$).** *The nonlinearity $F : H \to H$ is two times continuously Fréchet differentiable and its derivatives satisfy*

$$\left| F'(x) - F'(y) \right| \leq L |x - y|, \qquad \left| (-A)^{(-r)} F'(x) (-A)^r v \right| \leq L |v|$$

*for all $x, y \in H$, $v \in D((-A)^r)$ and $r = 0, \frac{1}{2}, 1$, and*

$$\left| A^{-1} F''(x)(v,w) \right| \leq L \left| (-A)^{-\frac{1}{2}} v \right| \left| (-A)^{-\frac{1}{2}} w \right|$$

*for all $v, w, x \in H$, where $L > 0$ is a positive constant.*

Here and below $D\left((-A)^r\right)$ for $r \in \mathbb{R}$ are the domains of (fractional) powers of the linear operator $-A : D(A) \subset H \to H$ given by Assumption 4.2.

**Assumption 4.5 (Initial value $x_0$).** *Let $x_0 : \Omega \to D\left((-A)^\gamma\right)$ be a $\mathcal{F}_0/\mathcal{B}(H)$-measurable mapping, which satisfies $\mathbb{E}\left|(-A)^\gamma x_0\right|^4 < \infty$, where $\gamma > 0$ is given in Assumption 4.3.*

Under Assumptions 4.2-4.5 the SPDE (4.1) has an up to modifications unique mild solution $X_t$, $t \in [0,T]$, where $X_t$, $t \in [0,T]$, is a predictable stochastic process in $D\left((-A)^\gamma\right)$, which satisfies equation (4.2) (see Da Prato & Zabczyk [16] or Prévot & Röckner [105]). Since Assumption (A3) also

applies to $F_N$, the Itô-Galerkin SODE (4.3) also has a (up to modifications) unique solution on $[0,T]$, which satisfies (4.4). Moreover, the above series (4.8) for the cylindrical Wiener process may not converge in $H$, but in another space into which $H$ can be continuously embedded. Thus, our formalism allows us to consider space-time white noise (in one-dimensional domains) as well as trace class noise.

Theorem 4.1 has several serious shortcomings. Firstly, we need to know the eigenvalues and eigenfunctions of the operator $A$, which are rarely known except in very simple domains. Finite element methods are a possible way around this difficulty. More serious, Assumption 4.4 on the nonlinearity $F$ is very restrictive and excludes Nemytskii operators for functions like

$$f(x) = \frac{x}{1+x^2}, \qquad f(x) = x - x^3, \qquad x \in \mathbb{R},$$

which are important in applications, e.g. stochastic reaction-diffusion equations, see Section 5.1. (Here, we understand $F$ as the corresponding Nemytskii operator of the function $f : \mathbb{R} \to \mathbb{R}$.) One reason for this is the assumed Fréchet differentiability of the function $F$ when considered as a mapping between the Hilbert-space $H = L^2(0,1)$ and also the assumed boundedness of the derivatives of $F$ in the sense of Assumption 4.4. This restriction is weakened with the order $\frac{1}{2}$ in time in [61], where also nonlinearities such as $f(x) = \frac{x}{1+x^2}$ are considered, by the observation that the solution process takes in fact values not only in $H$ but in a smaller subspace $V$, e.g. the space of continuous functions, on which $F$ is indeed Fréchet differentable. The other problem is the global Lipschitz estimate on $F$. This difficulty also arises in the case of finite dimensional SODEs, which it can be overcome by using pathwise convergence rather than strong convergence as mentioned in Section 2. In particular, the results for SODEs [69] have been generalized to SPDEs [64], where also polynomial nonlinearities of the form $f(x) = x - x^3$ are considered.

## 5. Stochastic Taylor expansions in Banach spaces

Taylor expansions of solutions of SPDEs in Banach spaces are the basis for deriving higher order numerical schemes for SPDEs, just as for SODEs. There is however a major difficulty for SPDEs: Although the SPDE (4.1) is driven by a Wiener process, the solution process is with respect to the space $H$ in general not a semi-martingale any more (see [33] for a clear discussion of that problem) and a general Itô formula does not exist for its solutions

(except in the case of an Itô formula with real valued test functions, see [33] and also Theorem 4.2.5 in [105]). Hence stochastic Taylor expansions for the solution of the SPDE (4.1) cannot be derived as in [76] for the solution of a finite dimensional SODE via an iterated application of Itô's formula.

To overcome this problem we thus need to derive Taylor expansions for SPDEs, which avoid the need of an Itô formula. In this section we will outline how this can be done following [62]. We concentrate here on the case of additive noise and refer to the above named thesis for the general case of non-additive noise (see also [63]).

Throughout this section we assume that the following setting and the following assumptions are fulfilled. Fix $T > 0$ and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. In addition, let $(V, \|\cdot\|)$ be a separable $\mathbb{R}$-Banach space and by $(L(V), \|\cdot\|)$ we denote the $\mathbb{R}$-Banach space of all bounded linear operators from $V$ to $V$. For a topological space $W$ we denote by $\mathcal{B}(W)$ the Borel $\sigma$-algebra of $W$. We call a mapping $Y : [0,T] \times \Omega \to V$ a *stochastic process* if the mappings

$$Y_t : \Omega \to V, \qquad \omega \mapsto Y_t(\omega) := Y(t,\omega), \qquad \omega \in \Omega$$

are $\mathcal{F}/\mathcal{B}(V)$-measurable for every $t \in [0,T]$. We say that a stochastic process $Y : [0,T] \times \Omega \to V$ has *continuous sample paths*, if the mappings

$$[0,T] \to V, \qquad t \mapsto Y_t(\omega), \qquad t \in [0,T]$$

are continuous for every $\omega \in \Omega$. In addition, we say that a stochastic process $Y : [0,T] \times \Omega \to V$ has *$r$-Hölder continuous sample paths*, if

$$\sup_{0 \le t_1 < t_2 \le T} \frac{\|Y_{t_2}(\omega) - Y_{t_1}(\omega)\|}{(t_1 - t_2)^r} < \infty$$

holds for every $\omega \in \Omega$ with $r \in (0,1]$.

**Assumption 5.1 (Semigroup $S$).** *Suppose that $S : [0,\infty) \to L(V)$ is a mapping, which satisfies*

$$S_0 = I, \quad S_{t_1} S_{t_2} = S_{(t_1+t_2)}, \quad \sup_{0 \le t \le T} \|S_t\| < \infty, \quad \sup_{0 \le s < t \le T} \left( \frac{\|S_t - S_s\| \cdot s}{(t-s)} \right) < \infty$$

*for every $t_1, t_2 \in [0,\infty)$, where $I$ is the identity on $V$.*

**Assumption 5.2 (Nonlinearity $F$).** *Let $F : V \to V$ be an infinitely often Fréchet differentiable mapping.*

**Assumption 5.3 (Stochastic process $O$).** *Let $O : [0,T] \times \Omega \to V$ be a stochastic process with $\theta$-Hölder continuous sample paths, where $\theta \in (0,1)$.*

**Assumption 5.4 (Initial value $x_0$).** *Let $x_0 : \Omega \to V$ be a $\mathcal{F}/\mathcal{B}(V)$-measurable mapping with $\sup_{t\in(0,T]} \frac{1}{t} \|S_t x_0(\omega) - x_0(\omega)\| < \infty$ for every $\omega \in \Omega$.*

**Assumption 5.5 (Existence of a solution).** *Assume that there is a stochastic process $X : [0,T] \times \Omega \to V$ with continuous sample paths, which fulfills*

$$X_t(\omega) = S_t x_0(\omega) + \int_0^t S_{(t-s)} F(X_s(\omega))\, ds + O_t(\omega) \tag{5.1}$$

*for every $t \in [0,T]$ and every $\omega \in \Omega$.*

We refer to Section 2.1 in [62] for a detailed discussion of the assumptions used. We just mention here that the solution process $X : [0,T] \times \Omega \to V$ given in Assumption 5.5 is pathwise unique under Assumptions 5.1-5.5 (see Lemma 1 in [62]). Finally, we also consider the following simple example.

### 5.1. A stochastic reaction diffusion equation

A prominent choice of the setting introduced above is the following. Let $V = C([0,1], \mathbb{R})$ be the $\mathbb{R}$-Banach space of continuous functions from $[0,1]$ to $\mathbb{R}$ equipped with the supremum norm. Additionally, let $S : [0,\infty) \to L(V)$ be the semigroup generated by the Laplacian with Dirichlet boundary conditions in $V$, i.e.

$$(S_0 v)(x) = v(x), \quad (S_t v)(x) = \sum_{n=1}^{\infty} 2\, e^{-\pi^2 n^2 t} \left( \int_0^1 \sin(n\pi s)\, v(s)\, ds \right) \sin(n\pi x)$$

for every $t \in (0,\infty)$, $x \in [0,1]$ and every $v \in V$. Moreover, let $F : V \to V$ be a Nemytskii operator given by $(F(v))(x) = v(x) - (v(x))^3$ for every $x \in [0,1]$ and every $v \in V$. Let the stochastic process $O : [0,T] \times \Omega \to V$ be given by

$$\mathbb{P}\left[ O_t(x) = \sum_{n=1}^{\infty} \sqrt{2} \left( \int_0^t e^{-\pi^2 n^2 (t-s)} d\beta_s^n \right) \sin(n\pi x) \right] = 1$$

for every $t \in [0,T]$ and every $x \in [0,1]$, where $\beta^n : [0,T] \times \Omega \to \mathbb{R}$, $n \in \mathbb{N}$, is a sequence of independent standard Brownian motions with continuous sample paths. For the initial value $x_0 : \Omega \to V$ we choose $(x_0(\omega))(x) = \sin(\pi x)$ for all $\omega \in \Omega$ and all $x \in [0,1]$. The SPDE (5.1) then reduces to

$$dX_t = \left[ \frac{\partial^2}{\partial x^2} X_t + X_t - X_t^3 \right] dt + dW_t, \quad X_t(0) = X_t(1) = 0, \quad X_0(x) = \sin(\pi x)$$

for $t \in [0,T]$ and $x \in [0,1]$, where $W_t$, $t \in [0,T]$, is a cylindrical $I$-Wiener process. In this case it can be seen in Section 2.2 in [62] that Assumptions 5.1-5.5 are fulfilled for every $\theta \in (0, \frac{1}{4})$. We also refer to Section 2.2 in [62] for further examples of Assumptions 5.1-5.5.

### 5.2. Integral operators and notation

In this subsection we present the notation and the basic idea behind the derivation of Taylor expansions. Henceforth we fix $t_0 \in [0,T)$ and denote by $\mathcal{C}$ the set of all stochastic processes $Y : \Omega \times [t_0,T] \to V$ with continuous sample paths. We write

$$\Delta X_t := X_t - X_{t_0}, \qquad \Delta O_t := O_t - O_{t_0}, \qquad \Delta t := t - t_0$$

for $t \in [t_0,T] \subset [0,T]$, thus $\Delta X$ (respectively $\Delta O$) denotes the stochastic process $\Delta X_t$ (respectively $\Delta O_t$), $t \in [t_0,T]$, in $\mathcal{C}$. In the next step, we use the following formula for the increment $\Delta X$ of the solution $X$. First of all, we have

$$\begin{aligned} X_t &= S_t x_0 + \int_0^t S_{(t-s)} F(X_s) ds + O_t \\ &= S_{\Delta t}\left(S_{t_0} x_0\right) + \int_0^{t_0} S_{(t-s)} F(X_s) ds + \int_{t_0}^t S_{(t-s)} F(X_s) ds + O_t \\ &= S_{\Delta t}\left(S_{t_0} x_0 + \int_0^{t_0} S_{(t_0-s)} F(X_s) ds\right) + \int_{t_0}^t S_{(t-s)} F(X_s) ds + O_t \\ &= S_{\Delta t}\left(X_{t_0} - O_{t_0}\right) + \int_{t_0}^t S_{(t-s)} F(X_s) ds + O_t \end{aligned}$$

for every $t \in [t_0,T]$. Therefore, we obtain

$$\begin{aligned} \Delta X_t &= \left(S_{\Delta t}\left(X_{t_0} - O_{t_0}\right) + \int_{t_0}^t S_{(t-s)} F(X_s) ds + O_t\right) - X_{t_0} \\ &= \left(S_{\Delta t} - I\right)\left(X_{t_0} - O_{t_0}\right) + \int_{t_0}^t S_{(t-s)} F(X_s) ds + \Delta O_t \end{aligned} \tag{5.2}$$

for all $t \in [t_0,T]$. Formula (5.2) will be the starting point for deriving Taylor expansions for the solution process $X$ respectively its increment $\Delta X$. In the next step, we introduce some integral operators and an expression relating

them. We define the stochastic processes $I^0, I^0_* \in \mathcal{C}$ by

$$\begin{aligned} I^0(t) &:= (S_{\Delta t} - I)(X_{t_0} - O_{t_0}) + \int_{t_0}^{t} S_{(t-s)} F(X_{t_0})\, ds \\ &= (S_{\Delta t} - I)(X_{t_0} - O_{t_0}) + \left( \int_0^{\Delta t} S_s\, ds \right) F(X_{t_0}) \end{aligned}$$

and

$$I^0_*(t) := (S_{\Delta t} - I)(X_{t_0} - O_{t_0}) + \int_{t_0}^{t} S_{(t-s)} F(X_s)\, ds$$

for each $t \in [t_0, T]$. Given $i \in \mathbb{N}$, we define the $i$-multilinear symmetric mappings $I^i : \mathcal{C}^i := \underbrace{\mathcal{C} \times \cdots \times \mathcal{C}}_{i\text{-times}} \to \mathcal{C}$ and $I^i_* : \mathcal{C}^i \to \mathcal{C}$ by

$$I^i[g_1, \ldots, g_i](t) := \frac{1}{i!} \int_{t_0}^{t} S_{(t-s)} F^{(i)}(X_{t_0})(g_1(s), \ldots, g_i(s))\, ds$$

and

$$\begin{aligned} &I^i_*[g_1, \ldots, g_i](t) \\ &:= \int_{t_0}^{t} S_{(t-s)} \left( \int_0^1 F^{(i)}(X_{t_0} + r\Delta X_s)(g_1(s), \ldots, g_i(s)) \frac{(1-r)^{(i-1)}}{(i-1)!} dr \right) ds \end{aligned}$$

for all $t \in [t_0, T]$ and all $g_1, \ldots, g_i \in \mathcal{C}$. One immediately checks that the stochastic processes $I^0, I^0_*$ and the mappings $I^i, I^i_*$ are well defined. Here the subscript $*$ labels certain integrals with time dependent values of the solutions in the integrand. While $I^0$ and $I^i[g_1, \ldots, g_i]$ are expressions containing only a constant value of the SPDE solution (5.1).

The mild solution $X$ of the SPDE (5.1) also satisfies

$$\Delta X_t = (S_{\Delta t} - I)(X_{t_0} - O_{t_0}) + \int_{t_0}^{t} S_{(t-s)} F(X_s) ds + \Delta O_t \tag{5.3}$$

(see equation (5.2)) or, in terms of the above integral operators,

$$\Delta X_t = I^0_*(t) + \Delta O_t$$

for every $t \in [t_0, T]$, which we can write symbolically in the space $\mathcal{C}$ as

$$\Delta X = I^0_* + \Delta O. \tag{5.4}$$

The stochastic processes $I^i[g_1, \ldots, g_i]$ for $g_1, \ldots, g_i \in \mathcal{C}$, $i \in \mathbb{N}_0$ only depend on the solution at time $t = t_0$. These terms are therefore useful approximations for the solution $X_t$, $t \in [t_0, T]$. However, the stochastic processes

$I^i_*[g_1,\dots,g_i]$ for $g_1,\dots,g_i \in \mathcal{C}$, $i \in \mathbb{N}_0$ depend on the solution path $X_t$, $t \in [t_0,T]$. Therefore, we need a further expansion for these processes. For this we will use the important formula

$$\begin{aligned} I^0_* &= I^0 + I^1_*[\Delta X] \\ &= I^0 + I^1_*[I^0_*] + I^1_*[\Delta O], \end{aligned} \tag{5.5}$$

which is an immediate consequence of integration by parts and equation (5.4), and more generally the iterated formula

$$\begin{aligned} I^i_*[g_1,\dots,g_i] &= I^i[g_1,\dots,g_i] + I^{(i+1)}_*[\Delta X, g_1,\dots,g_i] \\ &= I^i_1[g_1,\dots,g_i] + I^{(i+1)}_*[I^0_*, g_1,\dots,g_i] + I^{(i+1)}_*[\Delta O, g_1,\dots,g_i] \end{aligned} \tag{5.6}$$

for every $g_1,\dots,g_i \in \mathcal{C}$ and every $i \in \mathbb{N}$ (see Lemma 15 in [62] for a proof of the both equations above).

### 5.3. Derivation of simple Taylor expansions

To derive a further expansion of equation (5.4) we insert formula (5.5) to the stochastic process $I^0_*$, i.e.,

$$I^0_* = I^0 + I^1_*[I^0_*] + I^1_*[\Delta O]$$

into equation (5.4) to obtain

$$\Delta X = \left(I^0 + I^1_*[I^0_*] + I^1_*[\Delta O]\right) + \Delta O,$$

which can also be written as

$$\Delta X = I^0 + \Delta O + \left(I^1_*[I^0_*] + I^1_*[\Delta O]\right). \tag{5.7}$$

In [62] it is shown that the double integral terms $I^1_*[I^0_*]$ and $I^1_*[\Delta O]$ are sufficient small. Hence, we obtain the approximation

$$\Delta X \approx I^0 + \Delta O, \tag{5.8}$$

or, using the definition of the stochastic processes $I^0$,

$$\Delta X_t \approx (S_{\Delta t} - I)(X_{t_0} - O_{t_0}) + \left(\int_{t_0}^{t} S_{(t-s)}\, ds\right) F(X_{t_0}) + \Delta O_t$$

for $t \in [t_0,T]$. Hence

$$X_t \approx S_{\Delta t} X_{t_0} + \left(\int_0^{\Delta t} S_s\, ds\right) F(X_{t_0}) + (O_t - S_{\Delta t} O_{t_0}), \qquad t \in [t_0,T] \tag{5.9}$$

is an approximation for the solution of SPDE (5.1). Since the right hand side of equation (5.9) depends on the solution only at time $t_0$, it is the first

non trivial Taylor approximation of the solution of the SPDE (5.1). The remainder terms $I_*^1[I_*^0]$ and $I_*^1[\Delta O]$ of this approximation can be estimated by

$$\left\| I_*^1[I_*^0](t) + I_*^1[\Delta O](t) \right\| \leq C(\Delta t)^{(1+\theta)}$$

with a $\mathcal{F}/\mathcal{B}([0,\infty))$-measurable mapping $C : \Omega \to [0,\infty)$ and where $\theta \in (0,1)$ is given in Assumption 5.3 (see Theorem 1 in [62]). We write $Y_t = O\left((\Delta t)^r\right)$ with $r > 0$ for a stochastic process $Y \in \mathcal{C}$ if

$$\sup_{t \in (t_0,T]} \frac{\|Y_t(\omega)\|}{(t-t_0)^r} < \infty$$

holds for all $\omega \in \Omega$. Therefore, we obtain

$$X_t - \left( S_{\Delta t} X_{t_0} + \left( \int_0^{\Delta t} S_s \, ds \right) F(X_{t_0}) + (O_t - S_{\Delta t} O_{t_0}) \right) = O\left( (\Delta t)^{(1+\theta)} \right),$$

or

$$X_t = S_{\Delta t} X_{t_0} + \left( \int_0^{\Delta t} S_s \, ds \right) F(X_{t_0}) + (O_t - S_{\Delta t} O_{t_0}) + O\left( (\Delta t)^{(1+\theta)} \right). \tag{5.10}$$

The approximation (5.9) thus has order $1+\theta$ in the above strong respectively pathwise sense. It plays an analogous role to the simplest strong Taylor expansion in [76] on which the Euler-Maruyama scheme for finite dimensional SODEs is based and was used in [67] to derive the exponential Euler scheme for the SPDE (4.1) (see Section 4). We note that the Euler-Maruyama scheme in [76] approximates the solution of a SODE with additive noise locally with order $\frac{3}{2}$. Here in the case of commutative trace class noise e.g. in one dimension, one has $\theta = \frac{1}{2} - \varepsilon$ for an arbitrarily small $\varepsilon \in \left(0, \frac{1}{2}\right)$ (see example (2.29) in [62]). Therefore the exponential Euler scheme for the SPDE (5.1) in [67] also approximates the solution locally with order $\frac{3}{2} - \varepsilon$, while other schemes in use, in particular the linear-implicit Euler scheme or the linear-implicit Crank-Nicholson scheme, approximate the solution locally with order $\frac{1}{2}$ instead of order $\frac{3}{2}$ as in the finite dimensional case. Therefore the Taylor approximation (5.10) attains the classical order of the Euler approximation for finite dimensional SODEs.

### 5.4. Higher order Taylor expansions

Further expansions of the remainder terms in a Taylor expansion give a Taylor expansion of higher order. To illustrate this, we will expand the term $I_*^1[\Delta O]$ in equation (5.7). From (5.5) and (5.6) we have

$$I_*^1[\Delta O] = I^1[\Delta O] + I_*^2[I_*^0, \Delta O] + I_*^2[\Delta O, \Delta O],$$

which we insert into equation (5.7) to obtain

$$\Delta X = \left(I^0 + \Delta O + I^1[\Delta O]\right) + R,$$

where the remainder term $R \in \mathcal{C}$ is given by

$$R = I^1_*[I^0_*] + I^2_*[I^0_*, \Delta O] + I^2_*[\Delta O, \Delta O].$$

From Theorem 1 in [62] we get $R = O\left((\Delta t)^{(1+\min(1,2\theta))}\right)$. Thus we have

$$\Delta X = I^0 + \Delta O + I^1[\Delta O] + O\left((\Delta t)^{(1+\min(1,2\theta))}\right),$$

which can also be written as

$$\begin{aligned} X_t &= S_{\Delta t} X_{t_0} + \left(\int_0^{\Delta t} S_s \, ds\right) F(X_{t_0}) + (O_t - S_{\Delta t} O_{t_0}) \\ &\quad + \int_{t_0}^{t} S_{(t-s)} F'(X_{t_0}) \Delta O_s \, ds + O\left((\Delta t)^{(1+\min(1,2\theta))}\right). \end{aligned}$$

This approximation is of order $1 + \min(1, 2\theta)$.

By iterating this idea we can construct further Taylor expansions. In particular Taylor expansions of arbitrarily high orders can be achieved. See [62] for details of how this can done in a systematic way using appropriate stochastic trees and woods to express things in a compact way (instead of the hierarchical sets of multi-indices used in the stochastic Taylor expansions for SODEs in Kloeden & Platen [76]).

### 5.5. Further Taylor approximations for SPDEs

Here we present further examples of the Taylor approximations and state their convergence orders in dependence of the parameter $\theta \in (0, 1)$. Note e.g. that $\theta = \frac{1}{4} - \varepsilon$ for every arbitrarily small $\varepsilon \in \left(0, \frac{1}{4}\right)$ in our example in Section 5.1. We present these Taylor approximations here in a brief way and refer to [62] for derivations and proofs of these estimates.

**5.5.1. Taylor approximation of order 1.** First of all, we have

$$X_t = X_{t_0} + \Delta O_t + O\left(\Delta t\right) \tag{5.11}$$

for a Taylor approximation of order 1.

**5.5.2. Taylor approximation of order $1+\theta$.** In the next step, we conside the Taylor approximation

$$X_t = S_{\Delta t}X_{t_0} + \left(\int_0^{\Delta t} S_s\,ds\right) F(X_{t_0}) + (O_t - S_{\Delta t}O_{t_0}) + O\left((\Delta t)^{(1+\theta)}\right), \tag{5.12}$$

which is of order $1+\theta$. This example corresponds to the exponential Eule scheme, which was already discussed in Section 4 and Section 5.3.

**5.5.3. Taylor approximation of order $1+\min(1,2\theta)$.** Here, we have

$$\begin{aligned} X_t = S_{\Delta t}X_{t_0} &+ \left(\int_0^{\Delta t} S_s\,ds\right) F(X_{t_0}) + (O_t - S_{\Delta t}O_{t_0}) \\ &+ \int_{t_0}^{t} S_{(t-s)}F'(X_{t_0})\Delta O_s\,ds + O\left((\Delta t)^{(1+\min(1,2\theta))}\right). \end{aligned} \tag{5.13}$$

This example corresponds to the Taylor approximation introduced in Sec tion 5.4.

**5.5.4. Taylor approximation of order $1+\min(1,3\theta)$.** In the next step, w obtain

$$\begin{aligned} X_t = S_{\Delta t}X_{t_0} &+ \left(\int_0^{\Delta t} S_s\,ds\right) F(X_{t_0}) + (O_t - S_{\Delta t}O_{t_0}) \\ &+ \int_{t_0}^{t} S_{(t-s)}F'(X_{t_0})\Delta O_s\,ds + \frac{1}{2}\int_{t_0}^{t} S_{(t-s)}F''(X_{t_0})\,(\Delta O_s, \Delta O_s)\,ds \\ &+ O\left((\Delta t)^{(1+\min(1,3\theta))}\right) \end{aligned} \tag{5.14}$$

for a Taylor approximation of order $1+\min(1,3\theta)$.

**5.5.5. Taylor approximation of order $1+\min(1,4\theta)$.** A Taylor approxima tion of order $1+\min(1,4\theta)$ is given by

$$\begin{aligned} X_t = S_{\Delta t}X_{t_0} &+ \left(\int_0^{\Delta t} S_s\,ds\right) F(X_{t_0}) + (O_t - S_{\Delta t}O_{t_0}) \\ &+ \int_{t_0}^{t} S_{(t-s)}F'(X_{t_0})\Delta O_s\,ds + \frac{1}{2}\int_{t_0}^{t} S_{(t-s)}F''(X_{t_0})\,(\Delta O_s, \Delta O_s)\,ds \\ &+ \frac{1}{6}\int_{t_0}^{t} S_{(t-s)}F^{(3)}(X_{t_0})\,(\Delta O_s, \Delta O_s, \Delta O_s)\,ds + O\left((\Delta t)^{(1+\min(1,4\theta))}\right) \end{aligned} \tag{5.15}$$

**5.5.6. Taylor approximation of order** $1 + \min(1 + \theta, 3\theta)$**.** Furthermore, the Taylor approximation

$$\begin{aligned} X_t &= S_{\Delta t} X_{t_0} + \left( \int_0^{\Delta t} S_s \, ds \right) F(X_{t_0}) + (O_t - S_{\Delta t} O_{t_0}) \\ &+ \int_{t_0}^{t} S_{(t-s)} F'(X_{t_0}) \left( (S_{\Delta s} - I)(X_{t_0} - O_{t_0}) + \left( \int_0^{\Delta s} S_u \, du \right) F(X_{t_0}) \right) ds \\ &+ \int_{t_0}^{t} S_{(t-s)} F'(X_{t_0}) \Delta O_s \, ds + \frac{1}{2} \int_{t_0}^{t} S_{(t-s)} F''(X_{t_0}) (\Delta O_s, \Delta O_s) \, ds \qquad (5.16) \\ &+ O\left( (\Delta t)^{(1+\min(1+\theta, 3\theta))} \right) \end{aligned}$$

is of order $1 + \min(1 + \theta, 3\theta)$.

**5.5.7. Taylor approximation of order** $1 + \min(1 + \theta, 4\theta)$**.** We have

$$\begin{aligned} X_t &= S_{\Delta t} X_{t_0} + \left( \int_0^{\Delta t} S_s \, ds \right) F(X_{t_0}) + (O_t - S_{\Delta t} O_{t_0}) \\ &+ \int_{t_0}^{t} S_{(t-s)} F'(X_{t_0}) \left( (S_{\Delta s} - I)(X_{t_0} - O_{t_0}) + \left( \int_0^{\Delta s} S_u \, du \right) F(X_{t_0}) \right) ds \\ &+ \int_{t_0}^{t} S_{(t-s)} F'(X_{t_0}) \Delta O_s \, ds + \frac{1}{2} \int_{t_0}^{t} S_{(t-s)} F''(X_{t_0}) (\Delta O_s, \Delta O_s) \, ds \qquad (5.17) \\ &+ \frac{1}{6} \int_{t_0}^{t} S_{(t-s)} F^{(3)}(X_{t_0}) (\Delta O_s, \Delta O_s, \Delta O_s) \, ds + O\left( (\Delta t)^{(1+\min(1+\theta, 4\theta))} \right). \end{aligned}$$

**5.5.8. Taylor approximation of order $1 + \min(1 + \theta, 5\theta)$.** The Taylor approximation

$$
\begin{aligned}
X_t &= S_{\Delta t} X_{t_0} + \left( \int_0^{\Delta t} S_s \, ds \right) F(X_{t_0}) + (O_t - S_{\Delta t} O_{t_0}) \\
&+ \int_{t_0}^{t} S_{(t-s)} F'(X_{t_0}) \left( (S_{\Delta s} - I)(X_{t_0} - O_{t_0}) + \left( \int_0^{\Delta s} S_u \, du \right) F(X_{t_0}) \right) ds \\
&+ \int_{t_0}^{t} S_{(t-s)} F'(X_{t_0}) \Delta O_s \, ds + \frac{1}{2} \int_{t_0}^{t} S_{(t-s)} F''(X_{t_0}) (\Delta O_s, \Delta O_s) \, ds \qquad (5.18) \\
&+ \frac{1}{6} \int_{t_0}^{t} S_{(t-s)} F^{(3)}(X_{t_0}) (\Delta O_s, \Delta O_s, \Delta O_s) \, ds \\
&+ \frac{1}{24} \int_{t_0}^{t} S_{(t-s)} F^{(4)}(X_{t_0}) (\Delta O_s, \Delta O_s, \Delta O_s, \Delta O_s) \, ds \\
&+ O\left( (\Delta t)^{(1+\min(1+\theta, 5\theta))} \right)
\end{aligned}
$$

is of order $1 + \min(1 + \theta, 5\theta)$.

**5.5.9. Taylor approximation of order $1 + \min(1 + \theta, 6\theta)$.** A Taylor approximation of order $1 + \min(1 + \theta, 6\theta)$ is given by

$$
\begin{aligned}
X_t &= S_{\Delta t} X_{t_0} + \left( \int_0^{\Delta t} S_s \, ds \right) F(X_{t_0}) + (O_t - S_{\Delta t} O_{t_0}) \\
&+ \int_{t_0}^{t} S_{(t-s)} F'(X_{t_0}) \left( (S_{\Delta s} - I)(X_{t_0} - O_{t_0}) + \left( \int_0^{\Delta s} S_u \, du \right) F(X_{t_0}) \right) ds \\
&+ \int_{t_0}^{t} S_{(t-s)} F'(X_{t_0}) \Delta O_s \, ds + \frac{1}{2} \int_{t_0}^{t} S_{(t-s)} F''(X_{t_0}) (\Delta O_s, \Delta O_s) \, ds \\
&+ \frac{1}{6} \int_{t_0}^{t} S_{(t-s)} F^{(3)}(X_{t_0}) (\Delta O_s, \Delta O_s, \Delta O_s) \, ds \qquad (5.19) \\
&+ \frac{1}{24} \int_{t_0}^{t} S_{(t-s)} F^{(4)}(X_{t_0}) (\Delta O_s, \Delta O_s, \Delta O_s, \Delta O_s) \, ds \\
&+ \frac{1}{120} \int_{t_0}^{t} S_{(t-s)} F^{(5)}(X_{t_0}) (\Delta O_s, \Delta O_s, \Delta O_s, \Delta O_s, \Delta O_s) \, ds \\
&+ O\left( (\Delta t)^{(1+\min(1+\theta, 6\theta))} \right).
\end{aligned}
$$

**5.5.10. Taylor approximation of order $1 + \min(2, 1 + 2\theta, 6\theta)$.** Finally, we obtain

$$
\begin{aligned}
X_t &= S_{\Delta t} X_{t_0} + \left( \int_0^{\Delta t} S_s \, ds \right) F(X_{t_0}) + (O_t - S_{\Delta t} O_{t_0}) \\
&+ \int_{t_0}^{t} S_{(t-s)} F'(X_{t_0}) \left( (S_{\Delta s} - I)(X_{t_0} - O_{t_0}) + \left( \int_0^{\Delta s} S_s \, ds \right) F(X_{t_0}) \right) ds \\
&+ \int_{t_0}^{t} S_{(t-s)} F'(X_{t_0}) \Delta O_s \, ds + \frac{1}{2} \int_{t_0}^{t} S_{(t-s)} F''(X_{t_0}) (\Delta O_s, \Delta O_s) \, ds \\
&+ \frac{1}{6} \int_{t_0}^{t} S_{(t-s)} F^{(3)}(X_{t_0}) (\Delta O_s, \Delta O_s, \Delta O_s) \, ds \\
&+ \frac{1}{24} \int_{t_0}^{t} S_{(t-s)} F^{(4)}(X_{t_0}) (\Delta O_s, \Delta O_s, \Delta O_s, \Delta O_s) \, ds \qquad (5.20) \\
&+ \frac{1}{120} \int_{t_0}^{t} S_{(t-s)} F^{(5)}(X_{t_0}) (\Delta O_s, \Delta O_s, \Delta O_s, \Delta O_s, \Delta O_s) \, ds \\
&+ \int_{t_0}^{t} S_{(t-s)} F''(X_{t_0}) \left[ (S_{\Delta s} - I)(X_{t_0} - O_{t_0}) + \left( \int_0^{\Delta s} S_s \, ds \right) F(X_{t_0}), \Delta O_s \right] ds \\
&+ \int_{t_0}^{t} S_{(t-s)} F'(X_{t_0}) \left( \int_{t_0}^{s} S_{(s-u)} F'(X_{t_0}) \Delta O_u \, du \right) ds \\
&+ O\left( (\Delta t)^{(1+\min(2,1+2\theta,6\theta))} \right).
\end{aligned}
$$

# 6. Numerical schemes based on the Taylor expansions for SPDEs

This section is based on Section 5 in Chapter 2 in [62]. To consider numerical approximations of the SPDE (5.1), we have to discretize both, the time interval $[0, T]$ and also the infinite dimensional space $V$. While the time interval $[0, T]$ is discretized via the Taylor expansions above, the space discretization is introduced in the following assumption, which is motivated by the SODE (3.2).

**Assumption 6.1 (Projection operators).** *For every natural number $N \in \mathbb{N}$, let $P_N : V \to V$ be a bounded linear operator with $(P_N)^2 = P_N$ and $P_N S_t = S_t P_N$ for every $t \in [0, \infty)$.*

For every $N \in \mathbb{N}$, we denote by $V_N := P_N(V) = \mathrm{im}(P_N) \subset V$ the image of $P_N$ in $V$. Since, $P_N : V \to V$ is linear, $(V_N, \|\cdot\|)$ is a normed $\mathbb{R}$-vector subspace of $V$ for every $N \in \mathbb{N}$. In addition, $V_N$ is separable (see e.g. Exercise I.4.26 in [126]) and its Borel $\sigma$-algebra satisfies $\mathcal{B}(V_N) = \mathcal{B}(V) \cap V_N$ (see for instance Corollary 1.84 in [72]) for every $N \in \mathbb{N}$. Moreover, due to Assumption 6.1, we have $P_N(v) = v$ for every $v \in V_N$ and every $N \in \mathbb{N}$ (see also Section IV.6 in [126]). Of course, the spaces $V_N$ are invariant with respect to the semigroup $S_t$, i.e. $S_t(V_N) \subset V_N$ for every $t \in [0, \infty)$ and every $N \in \mathbb{N}$ due to Assumption 6.1 (see also I.5.12 in [28]). Usually the spaces $V_N$ will be finite dimensional (as in (3.2)) and then $P_N$ projects the infinite dimensional SPDE (5.1) down into the finite dimensional space $V_N$, where we can do numerical computations.

To this end we truncate the semigroup $S$, the nonlinearity $F$, the stochastic process $O$ and the initial value $x_0$, by mappings $S^N : [0, \infty) \to L(V_N)$ and $F^N : V_N \to V_N$, by a stochastic process $O^N : [0, T] \times \Omega \to V_N \subset V$ and a $\mathcal{F}/\mathcal{B}(V_N)$-measurable mapping $x_0 : \Omega \to V_N$ defined by

$$S_t^N(v) := S_t(v), \qquad F_N(v) := P_N\left(F(v)\right)$$

and

$$O_s^N(\omega) = P_N\left(O_s(\omega)\right), \qquad x_0^N(\omega) := P_N\left(x_0(\omega)\right)$$

for all $v \in V_N$, $t \in [0, \infty)$, $s \in [0, T]$, $\omega \in \Omega$ and each $N \in \mathbb{N}$. Here $(L(V_N), \|\cdot\|)$ is the normed $\mathbb{R}$-vector space of all bounded linear operators from $V_N$ to $V_N$ for every $N \in \mathbb{N}$. We now present some properties of these truncated mappings. First of all, note that $S^N$ satisfies

$$S_0^N = I, \;\; S_{t_1}^N S_{t_1}^N = S_{(t_1+t_2)}^N, \;\; \sup_{0 \le t \le T} \left\| S_t^N \right\| < \infty, \;\; \sup_{0 \le s < t \le T} \frac{\left\| S_t^N - S_s^N \right\| s}{(t-s)} < \infty$$

for every $t_1, t_2 \in [0, \infty)$ and every $N \in \mathbb{N}$ due to Assumption 5.1. Moreover, $F_N : V_N \to V_N$ is infinitely often Fréchet differentiable with

$$F_N^{(n)}(v_0)(v_1, \ldots, v_n) = P_N\left(F^{(n)}(v_0)(v_1, \ldots, v_n)\right)$$

for every $v_0, \ldots, v_n \in V_N$, $n \in \mathbb{N}$ and every $N \in \mathbb{N}$ due to Assumption 5.2. Finally, $O^N : [0, T] \times \Omega \to V_N$ has $\theta$-Hölder continuous sample paths for every $N \in \mathbb{N}$, where $\theta \in (0, 1)$ is given in Assumption 5.3.

Then, we introduce one-step numerical schemes in the spaces $V_N$, $N \in \mathbb{N}$, based on the Taylor approximations in the previous section to approximate the solution process $X_t$, $t \in [0, T]$, of the SPDE (5.1). These

schemes will always be denoted by $\mathcal{F}/\mathcal{B}(V_N)$-measurable mappings

$$Y_k^{(N,M)} : \Omega \to V_N \qquad \text{with} \qquad Y_0^{(N,M)} := x_0^N + O_0^N$$

for $k = 0, 1, \ldots, M$ and $N, M \in \mathbb{N}$. Here $N \in \mathbb{N}$ will be the accuracy of the spatial discretization and $M \in \mathbb{N}$ will be the number of times steps on the interval $[0, T]$, where we use time steps of size $h = T/M$. In that sense, we will have $X_{kh} \approx Y_k^{N,M}$ for every $k = 0, 1, \ldots, M$ and $N, M \in \mathbb{N}$. Note that we choose the discretization of the interval $[0, T]$ just for simplicity equidistant and in principle an arbitrary non-equidistant discretization could be considered. Furthermore, we have $X_0 = x_0 + O_0$ for the exact solution $X$ of the SPDE (5.1). In that sense the initial value is $x_0 + O_0$, which can also be seen in the definition of $Y_0^{N,M}$ above.

**The exponential Euler scheme.** Since the global convergence order of the Taylor approximation (5.11) is too low to be consistent, we begin with the Taylor approximation (5.12). We define the numerical scheme

$$Y_{k+1}^{(N,M)} = S_h^N Y_k^{(N,M)} + \left( \int_0^h S_s^N \, ds \right) F_N \left( Y_k^{(N,M)} \right) + \left( O_{(k+1)h}^N - S_h^N O_{kh}^N \right) \tag{6.1}$$

for every $k = 0, 1, \ldots, M-1$ and every $N, M \in \mathbb{N}$.

First of all, we show, that the $\mathcal{F}/\mathcal{B}(V_N)$-measurable mappings $Y_k^{(N,M)}$, $k = 0, 1, \ldots, M$, $N, M \in \mathbb{N}$, are well defined via (6.1), although $V_N$ with $N \in \mathbb{N}$ is in general not complete. The integral $\int_0^h S_s^N \, ds \in L(V_N, V)$ is understood as a $L(V_N, V)$-valued Bochner integral for every $N, M \in \mathbb{N}$. Here $L(V_N, V)$ with $N \in \mathbb{N}$ is the $\mathbb{R}$-Banach space of all bounded linear operators from $V_N$ to $V$, which is indeed complete (see e.g. Theorem II.I.4 (b) in [126]). Due to Proposition A.2.2 in [105], we obtain

$$P_N \left( \int_0^h S_s^N \, ds \right) = \int_0^h P_N \left( S_s^N \right) ds = \int_0^h S_s^N ds$$

for every $N, M \in \mathbb{N}$, which shows that $\int_0^h S_s^N ds$ is in fact in $L(V_N) = L(V_N, V_N)$ for every $N, M \in \mathbb{N}$ and hence the numerical approximations in equation (6.1) are well defined.

The numerical scheme (6.1) is the simplest Taylor scheme for SPDEs with additive noise obtained via the Taylor expansions above. It is nothing else than the exponential Euler scheme considered in Section 4.

**A further Taylor scheme for SPDEs.** With the Taylor approximation (5.13), we obtain the approximation scheme

$$Y_{k+1}^{(N,M)} = S_h^N Y_k^{(N,M)} + \left(\int_0^h S_s^N \, ds\right) F_N\left(Y_k^{(N,M)}\right) + \left(O_{(k+1)h}^N - S_h^N O_{kh}^N\right) + \int_{kh}^{(k+1)h} S_{((k+1)h-s)}^N F_N'\left(Y_k^{(N,M)}\right)\left(O_s^N - O_{kh}^N\right) ds \tag{6.2}$$

for every $k = 0, 1, \ldots, M-1$ and every $N, M \in \mathbb{N}$. In a similar way as above, one can also show that the numerical approximations (6.2) are well defined.

**A Runge-Kutta scheme for SPDEs.** In principle, we can proceed with the next Taylor approximations and obtain numerical schemes of higher order. These schemes however would be of limited practical use due to cost and difficulty of computing the higher iterated integrals as well as the higher order derivatives in the Taylor approximations. Therefore, we follow a different approach and derive a derivative free numerical scheme with simple integrals, a so called *Runge-Kutta scheme for SPDEs*. We would like to mention, that this way is the usual procedure for numerical schemes for differential equations: Taylor expansions and their corresponding Taylor schemes provide the underlying theory for deriving numerical schemes, but are rarely implemented in practice. Instead of these Taylor schemes other numerical schemes, which are easier to compute but still depend on the Taylor expansions, respectively Taylor schemes, such as Runge-Kutta schemes or multi-step schemes, are used.

To derive a Runge-Kutta scheme for SPDEs, we consider for example the Taylor approximation in equation (5.13) (see also the Taylor scheme above) from $kh$ to $(k+1)h$ and obtain

$$\begin{aligned}
X_{(k+1)h} &\approx S_h X_{kh} + \left( \int_0^h S_s \, ds \right) F(X_{kh}) + \left( O_{(k+1)h} - S_h O_{kh} \right) \\
&\quad + \int_{kh}^{(k+1)h} S_{((k+1)h-s)} F'(X_{kh}) \left( O_s - O_{kh} \right) ds \\
&\approx S_h \left( X_{kh} + h \, F(X_{kh}) + \int_{kh}^{(k+1)h} F'(X_{kh}) \left( O_s - O_{kh} \right) ds \right) \\
&\quad + \left( O_{(k+1)h} - S_h O_{kh} \right) \\
&= S_h \left( X_{kh} + h \, F(X_{kh}) + h \, F'(X_{kh}) \left( \frac{1}{h} \int_{kh}^{(k+1)h} \left( O_s - O_{kh} \right) ds \right) \right) \\
&\quad + \left( O_{(k+1)h} - S_h O_{kh} \right)
\end{aligned}$$

for every $k = 0, 1, \ldots, M-1$ and every $M \in \mathbb{N}$. Now, the expression in the last but one line is the classical Taylor approximation of

$$\begin{aligned}
&F \left( X_{kh} + \frac{1}{h} \int_{kh}^{(k+1)h} \left( O_s - O_{kh} \right) ds \right) \\
&\qquad \approx F(X_{kh}) + F'(X_{kh}) \left( \frac{1}{h} \int_{kh}^{(k+1)h} \left( O_s - O_{kh} \right) ds \right)
\end{aligned}$$

for every $k = 0, 1, \ldots, M-1$ and every $M \in \mathbb{N}$. Therefore, we obtain

$$\begin{aligned}
X_{(k+1)h} \approx S_h \left( X_{kh} + h \, F \left( X_{kh} + \frac{1}{h} \int_{kh}^{(k+1)h} \left( O_s - O_{kh} \right) ds \right) \right) \\
+ \left( O_{(k+1)h} - S_h O_{kh} \right),
\end{aligned}$$

which indicates the numerical one-step scheme

$$\begin{aligned}
Y_{k+1}^{(N,M)} = S_h^N \left( Y_k^{(N,M)} + h \, F_N \left( Y_k^{(N,M)} + \frac{1}{h} \int_{kh}^{(k+1)h} \left( O_s^N - O_{kh}^N \right) ds \right) \right) \\
+ \left( O_{(k+1)h}^N - S_h^N O_{kh}^N \right) \quad (6.3)
\end{aligned}$$

for every $k = 0, 1, \ldots, M-1$ and every $N, M \in \mathbb{N}$, which is also well defined. This numerical scheme was originally introduced and analyzed in [61] (see also [64] for the analysis of that scheme under weaker assumptions). On

the one hand this scheme is based on a higher order Taylor approximation of the solution, which yields a higher order convergence rate than classical numerical schemes, which can be seen in the above named references. On the other hand, this scheme is as the exponential Euler scheme very simple to simulate and therefore has not the disadvantage of the Taylor scheme in equation (6.2) (see Section 4 in [61] for details). Finally, we believe that this is just the beginning of numerical schemes derived via these Taylor expansions and much more schemes could be considered.

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Lecture held by P. Kloeden in the *Seminario Matematico e Fisico di Milano* on September 29, 2008

Received: January 20, 2009.