A No-arbitrage Model of Liquidity in Financial Markets involving Brownian Sheets*

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Abstract

We consider a dynamic market model where buyers and sellers submit limit orders. If at a given moment in time, the buyer is unable to complete his entire order due to the shortage of sell orders at the required limit price, the unmatched part of the order is recorded in the order book. Subsequently these buy unmatched orders may be matched with new incoming sell orders. The resulting demand curve constitutes the sole input to our model. The clearing price is then mechanically calculated using the market clearing condition. We assume a continuous model for the demand curve. We show that generically there exists an equivalent martingale measure for the clearing price if the driving noise is a Brownian sheet, while there may not be if the driving noise is multidimensional Brownian motion.

Another contribution of this paper is to prove that, if there exists an equivalent martingale measure for the clearing price, then, under some mild assumptions, there is no arbitrage. We use the Ito-Wentzell formula to obtain both results. We also characterize the dynamics of the demand curve and of the clearing price in the equivalent martingale measure. We find that the volatility of the clearing price is inversely proportional to the sum of buy and sell order flow density (evaluated at the clearing price), which confirms the intuition that volatility is inversely proportional to volume. We also demonstrate that our approach is implementable. We use real order book data and simulate option prices under a particularly simple parameterization of our model.

The no-arbitrage conditions we obtain are applicable to a wide class of models, in the same way that the Heath-Jarrow-Morton conditions apply to a wide class of interest rate models.

1 Introduction

Most liquidity models in mathematical finance abstract the trading mechanism from the characterization of prices in the resulting market. Our viewpoint is here fundamentally different. In our model, equilibrium prices of assets are completely determined by the order flow, which is viewed as an exogenous process. We model a market of assets without specialist where every trader submits limit orders, that is, for a buy order, the buyer specifies the maximum price, or buy limit price, that he/she is willing to pay, and, for a sell order, the seller specifies the minimum price, or sell limit price, at which he/she is willing to sell [1].

If at a given moment in time, the buyer is unable to complete his entire order due to the shortage of sell orders at the required limit price, the unmatched part of the order is recorded in the order book.A symmetric outcome exists in the case of incoming sell orders. Subsequently these buy unmatched orders may be matched with new incoming sell orders.We note that many electronic exchanges, such as NYSE ARCA, operate like this. Time-priority is used to break indeterminacies of a match between an incoming

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[1] There is no loss of generality in that statement. A buy market order can be specified in our model as a buy limit order with limit price equal to infinity. Since we model assets with only positive prices, a sell market order can be specified in our model as a sell limit order with a limit price equal to zero.

buyer at limit price superior to the ask price, i.e., the lowest limit price in the sell order book. As a result, the equilibrium of *clearing price* process is always defined.

Since the matching mechanism does not add any information to the economy, all information about asset prices is included in the order flow. Whether public exchanges should or should not reveal in real-time the data contained in the order book is an important issue, which continues to preoccupy the financial markets community [WW02]. Our theoretical framework accommodates either viewpoint, but our empirical application is better tailored to the viewpoint that order books are public information. The current blossoming of high-frequency trading activity [Eng00],[BLT06], [Hau08], [SM08], [BRZ] seems to confirm our viewpoint that traders are (i) interested in understanding order book information, and (ii) trade on that information.

We do not address in this paper the issue of differential information further. The market microstructure literature (such as [Kyl85], and all following models) considers various models of trading involving uninformed traders, called noise traders, and one, or several informed traders. One of the key results of the Kyle model is that, given the information available to noise traders, the resulting price process is a martingale in the appropriate measure, whereas it may not be for the informed traders. As a consequence we do not believe that abstracting issues of differential information is limiting. The order books reflect all the public information. Public information corresponds to the filtration under which the clearing price needs to have an equivalent martingale measure, in order to avoid arbitrage. Obviously, the clearing price may not be a martingale in the aforementioned measure if the filtration is enlarged to include private information.

There are roughly two different class of models in the liquidity literature. The first class of models ([Jar92, [Jar94], [PS98a], [PS98b], [Fre98], [SW00], [BB04], [RS10]) considers the action of a large trader who can manipulate the prices in the market. There are mainly two different types of strategies a large trader can employ to that effect. The first one is to corner the market, and then squeeze the shorts. The second one is to "front-run one's own trades". While some exchanges have rules to curtail the cornering of the market, front-running seems more difficult to ban from an exchange. It is known in discrete-time trading, that, if there is no possibility of arbitrage in periods where the large investor does not trade, then there is no market manipulation strategy.

The second class of models ([CJP04], [CR07], [CST10],[GS11]) abstracts the issues of market manipulation away, and considers all traders as price-takers. In particular, [CJP04] introduced an exogenous residual supply curve against which an investor trades. The investor trades market orders, and his/her order is matched instantaneously. As a consequence of the instantaneity, it is plausible for [CJP04] to assume the "price effect of an order is limited to the very moment when the order is placed in the market" (dixit [BB04]), so that that the residual supply curve at a future time is statistically independent from the order just matched. For us however, this assumption is not convenient, since it does not explain how prices can incorporate information from the arrival of new orders.

Our model belongs to the first class of models. We define our demand curve as quantity as a function of price, unlike most other authors, who define it as price as a function of quantity. While both definitions contain the same information, we found it necessary to have a representation of the demand curve as quantity as a function of price in order to characterize the clearing price $\pi(t)$. We may thus as well start from there in order to avoid technical problems due to inverting the demand curve. In this sense, we extend the [BB04] paper, where the clearing price is not characterized, in the direction followed by [CJP04], where the clearing price is characterized.

We proceed in two steps to define our model. In the first step, we consider a market with atomistic traders and a large trader. By assumption, the net demand curve of the atomistic traders is a smooth semimartingale family. We argue that, if the large trader is rational, his net demand curve should also be a smooth semimartingale family. Since the sum of two smooth semimartingale families is also a smooth semimartingale family, the market net demand curve is also a smooth semimartingale family when the market consists of both atomistic traders and one rational large trader in the market. We do not want however to limit the applicability of our paper to such a model. Thus, in a second step, we start by assuming that the market net demand curve is a smooth semimartingale family. Alternate theoretical constructions may justify this assumption. For practitioners, only an empirical verification of this assumption is necessary for them to use the market model.

In our model all the information is contained in a Brownian sheet, which drives the dynamics of the market net demand curve. The advantage of using a Brownian sheet is that we have the same cardinality of independent sources of noise (namely, the cardinality of a real interval) as the cardinality of the set of exogenous stochastic processes, namely net demand at each limit price. Because of that reason, there exists (generically) an equivalent martingale measure for the clearing price, while there would not be any if the driving noise was Brownian motion, or multidimensional Brownian motion[2]. We use the Ito-Wentzell formula to obtain this result. The second main result in our article is to prove that, if there is an equivalent martingale measure $\mathbb{Q}$ for the clearing price $\pi(t)$, then, under some mild assumptions there is no arbitrage, as in [BB04].

We do not claim that it is indispensable to use a Brownian sheet to model an arbitrage-free limit order market. In an ingenious construction [KR09] use Carmona-Nualart [CT90] stochastic integration to weaken the assumptions in the [BB04] necessary to avoid arbitrage. They show for instance that a simple Bachelier model is arbitrage-free. Unfortunately, the result of inverting the net demand curve in the Bachelier model has singularities which prevented us from characterizing the clearing price as well as using the Ito-Wentzell formula.

We characterize the dynamics of the demand curve and of the clearing price in the measure $\mathbb{Q}$. We find that the volatility of the clearing price is inversely proportional to the sum of buy and sell order flow density (evaluated at the clearing price), which confirms the intuition that volatility is inversely proportional to volume.

We also demonstrate that our approach is implementable. Although we do not prove a second fundamental theorem of asset pricing, we naively use a special parameterization of our model to price options. As in the early days of the Heath-Jarrow-Morton methodology, we use only historical estimation (in our case, of the order book) to fit our model and solve for the market price of risk. We expect that, should this paper meet with interest around practitioners, market-implied implementations will see the day. Unsurprisingly, we obtain a smile curve for implied volatility. We note that this particular feature is not a very strong sign of the adequacy of our approach to model asset prices, as most models that came after Black-Scholes [BS73] result in a smile curve for implied volatility. Although limited, our results are however encouraging. They show that a fairly demanding theoretical model can be easily implemented.

The structure of the paper is as follows. In section 2, we introduce our model and show our main results: existence of an equivalent martingale measure for the clearing price π and absence of arbitrage. In section 3 we analyze the Bachelier model, and show how it would be difficult to make it fit our framework. In section 4, we describe our data set and the methodology we used to extract it. Finally, section 5 shows our implementation, namely a simulation of option prices under the risk-neutral measure.

2 Preliminaries

2.1 The Market Mechanism

A buy limit order specifies how many shares a trader wants to buy, and at what maximum price he is willing to buy them. We call this price the (buy) *limit price*. A buy limit order specifies how many shares a trader wants to buy, and at what maximum price he is willing to buy them. We call this price the (sell) *limit price*. Both buy and sell limit prices are denoted by p, and should not be confused with the clearing price, which is denoted by $\pi(t)$. The unmatched buy and sell orders are stored in order books, until they are either cancelled or matched with an incoming order. An incoming order is matched with the order in the opposite side of the market which has the best price. The clearing price of the transaction is equal to the limit price of the order in the book, and not of the incoming order. Partial execution is allowed, and, ties are resolved by time-priority. We give herafter an example of the matching mechanism in discrete time, i.e., at most one

[2]This is analogous to traditional market models in mathematical finance. Equality of the number of risk factors and number of securities is (generically) necessary to obtain an equivalent martingale measure. In our liquidity model, each point on the net demand curve is analogous to an individual security.

order arrives at time $t \in \{0, 1, 2, ..\}$.

Example

Suppose that the clearing price at time 0 is any price $\pi(0) \in [100, 120]$. After clearing, that is, when $0 < t < 1$ we suppose that the order books contain the following orders

Buy Order Book	
Price	Quantity
100	10

Sell Order Book	
Price	Quantity
120	10
130	10

At time $t = 1$ a buy order arrives with a limit price of \$125, and a quantity of 15. The exchange matches it with the best sell order, i.e., the one with a sell limit price of \$120. However, execution is only partial, and the remainder of the buy order is placed in the order book at the limit price of \$120, resulting in the following order books:

Buy Order Book	
Price	Quantity
100	10
125	5

Sell Order Book	
Price	Quantity
130	10

The clearing price at time 1 is equal to the limit price of the sell order, i.e.:

$$\pi(1) = 120$$

■

This example illustrates several properties of limit order markets. First, the clearing price is always defined, and can assume any positive value[3].

Second, it is not inconceivable that an incoming order "crosses" the order book, i.e., for the case of a buy order, that its limit price is higher than the best sell order limit price (i.e, the best ask), since the buyer does not lose a cent. Crossing the book is indeed advantageous for two reasons: first, it allows for faster execution. In our example, had the buyer submitted an order at price \$130 he would have bought the complete quantity of shares (15) that he desired, rather than waiting an indeterminate amount of time until enough sell arrives arrive until enough sell arrives arrive at his limit price. Second, suppose that several buy orders are submitted at the same time. In case the demand exceeds the supply at the best ask, the buy orders with the highest limit price are executed first. Our own data analysis (see section 4) shows that few orders cross the Arca book. This is consistent with the theory of optimal order book placement suggested by Rosu [Ros09].

2.2 The Brownian Sheet

We now move to continuous-time. Be aware that we do not prove convergence of a discrete-time model to our continuous-time model. We take the latter as a given, plausible model of the market. We start with a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ satisfying the usual conditions. All the uncertainty is described by a one-dimensional Brownian sheet $W(t, s)$ which generates $\{\mathcal{F}_t\}$.There are several approaches to construct a Brownian sheet, or, more generally stochastic partial differential equations. According to Mueller [Mue09] "One can use either Walsh's [Wal86] approach [..], the Hilbert space approach of Da Prato and Zabczyk [DPZ92], or Krylov's [Kry06] L_p theory". [DPZ92] and [Mue09] note that the approach is at least as general as the [Wal86] approach. We use the Hilbert space approach as exposed in [CT06].

We wish to make sense of the Brownian sheet and of its stochastic integral:

[3]We do not consider markets for swaps, where the price can be negative.

$$I_t = \int_{u=0}^{t} \int_{s=0}^{S} b(s,u) W(ds,du) \tag{1}$$

where W is a Brownian sheet and, for each $s \in [0,S]$ the $\mathcal{F}_t-$adapted stochastic process $b(s,t)$ satisfies:

$$\int_{s=0}^{S} b^2(s,t) ds = 1$$

We first define W_t as a random variable on the space $E \equiv L^2[0,S]$ such that $W = \{W_t; t \in [0,T]\}$ is an $E-$ valued Wiener process. If we let $\{e_n^*; n \geq 1\}$ be a complete orthonormal basis of the dual $H^*$of the reproducing kernel Hilbert space H of E and define:

$$w_n(t) = e_n^*(W_t)$$

we have:

$$W_t = \sum_{n=1}^{\infty} w_n(t) e_n \tag{2}$$

where w_n are independent $\mathbb{R}$-valued standard Wiener processes. While the series (2) is not convergent in E in l^2, we can define the following integral (see [DPZ92] p. 100)

$$W(s,t) = \sum_{n=1}^{\infty} w_n(t) \int_0^s e_n(\alpha) d\alpha \quad s \in [0,S]$$

which is our Brownian sheet. We now define a family of operators B_t:$E \to \mathbb{R}$ for $t \in [0,T]$ by the formula:

$$B_t f = \int_{s=0}^{S} b(s,t) f(s) ds$$

Since B is a Hilbert-Schmidt operator, we can also write this expression as:

$$B_t f = \sum_{n=1}^{\infty} b^n(t) e_n^*(f)$$

for some real-valued $\mathcal{F}_t-$adapted stochastic processes b^n satisfying:

$$b^n(t) = e_n^*(b(.,t))$$

Thus, we have the definition:

$$I_t = \int_{u=0}^{t} \sum_{n=1}^{\infty} b^n(u) dw_n(u) \tag{3}$$

We will use expression (1) for most of the text since it will be convenient to see $b(s,u)W(ds,du)$ as being indexed by the continuum $s \in [0,S]$. However expression (3) is an equally valid expression and will allow us to use directly the Ito-Wentzell formula as developed in [Kry09]. Finally, we note that in Walsh's approach the definitions (2) and (3) do not involve infinite series, but for our purpose the approaches are equivalent. For a rigorous comparison of the two approaches, we refer the reader to [DQ11].

2.3 A Market with Atomistic Traders and a Large Trader

Assumption 1 (Assumption A1 in [Ja92]). The market is frictionless.

Assumption 2 Buy and sell limit prices can assume any real value between 0 and S. Orders can be submitted to the market at any time $t \in \mathbb{R}^+$.

Notation Buy and sell limit prices are usually denoted by p. The market clearing price is denoted by π.

Definitions: the net demand curve Q_A of the atomistic traders is a function $[0,S] \times \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}$ which value $Q_A(p,t,\omega)$ is equal to the difference between the quantity of shares **submitted** for purchase (at price lower than or equal to p) and the quantity of shares **submitted** for sale (at price larger than or equal to p) between time.zero and time t by atomistic traders. The same definition holds for the the net demand curve Q_L of the large trader

Remark: We do not define precisely what we mean by atomistic traders. We view them as a continuum of traders. Each of them submits an infinitesimally small orders at a price p, such that there is no concentration of orders at any particular price, as we formulate more clearly below, in contradistinction with the large trader.

Assumption 3 For each t the net demand curve Q_A is twice-differentiable in the first argument.

Remark: For a fixed price p the total quantity of shares submitted for purchase does not need to be increasing in time, as some orders may be cancelled. However, it has to be a positive value.

Assumption 4: The net demand curve Q_A is strictly decreasing in its first argument, for all p except $p = 0$ and $p = S$.

Remark The fact that net demand curves are decreasing is an immediate consequence of our market mechanism (see above), and the fact that order quantities are by definition positive. Indeed, the number of shares of available buy orders is decreasing in price, while the number of shares is of available sell orders is increasing. The net demand, being the difference of buy and sell orders, is a decreasing function of price. Assumption 4 is similar to assumption A3 in [Ja92] or assumption 2 in [BB04].

Assumption 5: W suppose that the $\mathbb{R}$-valued $\mathcal{F}_t$-adapted processes coefficients $\mu_{Q_A}(p,.)$, $\sigma_{Q_A}(p,.)$ and $b_{Q_A}(p,s,.)$, when evaluated at any time $t \in [0,T]$, belong to $L^2(\mathbb{P})$. The net demand curve of the atomistic traders Q_A satisfies:

$$
\begin{aligned}
dQ_A(p,t,\omega) &= \mu_{Q_A}(p,t,\omega)dt + \sigma_{Q_A}(p,t,\omega)\int_{s=0}^{S} b_{Q_A}(p,s,t,\omega)W(ds,dt,\omega) \qquad 0 \le p \le S && (4)\\
Q_A(p,0) &= Q_{A,0}(p) > 0 \qquad 0 \le p \le S && (5)\\
\int_{s=0}^{S} b^2_{Q_A}(p,s,t,\omega) &= 1 \qquad 0 \le p \le S && (6)
\end{aligned}
$$

Remark: We do not provide more specific assumptions on the coefficients μ_{Q_A},σ_{Q_A},b_{Q_A} as we will define later a more specific model which will satisfy all our assumptions, including remark **. For the moment, we note that, by theorem 4.12 in [DPZ92], each $Q_A(p,t.,.)$ is a $\mathcal{F}_t$-adapted square-integrable semimartingale.

Definition the market net demand curve Q is defined by:

$$
Q = Q_A + Q_L \tag{7}
$$

Definition The clearing price $\pi(t)$ is a $\mathcal{F}_t-$ adapted stochastic process which satisfies:

$$Q(\pi(t), t) = 0 \tag{8}$$

when there is a solution to (8), or is otherwise defined by continuation, i.e., $\pi(t)$ is equal to the value of π at the latest time $s < t$ for which there was a solution to $Q(\pi(s), s) = 0$. We show in the next section a particular model where there is always a solution to (8).

Definition: The position of the large trader is denoted by $\theta(t)$.

Remark: The position of the trader changes with the clearing price, as well as with the net demand curve:

$$\begin{aligned} d\theta(t) &= d(Q_L(\pi(t), t)) && (9) \\ &= -d(Q_A(\pi(t), t)) && (10) \end{aligned}$$

Definition: Suppose that $\theta(t)$ is equal to a constant ϑ. The price $P_A(\vartheta, t, \omega)$ available on the market when the large trader position is ϑ is defined for each t by

$$Q_A(P_A(\vartheta, t, \omega), t, \omega) = \vartheta \quad \vartheta \in [Q_A(\infty, t, \omega), Q_A(0, t, \omega)]$$

and otherwise by continuation.

Definition: As in [KR09] we denote by $S_A(\theta, t)$ the price available on the market when the large trader follows a time-varying strategy $\theta(t)$. In other terms:

$$S_A(\theta, t) = P_A(\theta(t), t)$$

Remark: The clearing price satisfies:

$$\pi(t) = \int_0^t S_A(\theta, ds)$$

Definition [BB04] The asymptotic liquidation proceeeds of the large trader, denoted by L is equal to:

$$L(\vartheta, t) = \int_0^{\vartheta} P_A(x, t) dx$$

Definition define β^θ

Definition The real wealth process achieved by a self-financing trading strategy θ is given by

$$V^\theta(t) = \beta^\theta(t) + L(\theta(t), t)$$

Fact: (lemma 3.2 in [BB04]) Provided that the $L(\theta)$ is a smooth family of semimartingales (in the sense of [BB04]), for any self-financing semimartingale strategy θ, the dynamics of the real wealth process V^θ are given by:

$$V^\theta(t) - V^\theta(0_-) = \int_0^t L(\theta(u_-), du) - \frac{1}{2} \int_0^t P'(\theta(u_-), u) d[\theta, \theta]_s^c - \sum_{0 \le u \le t} \int_{\theta(u_-)}^{\theta(u)} \{P(\theta(u), u) - P(x, u)\} dx \tag{11}$$

Definition An arbitrage (strategy) is a self-financing trading strategy θ such that $V^\theta(0_-) = 0$ and:

$$\begin{aligned} P(V^\theta(t) &> 0) > 0 \\ P(V^\theta(t) &\geq 0) \geq 0 \end{aligned}$$

Lemma 1: Let:

$$\begin{aligned} \mu_P(x,t) &= -\frac{\mu_{Q_A}(P_A(x,t),t) + \frac{1}{2}\frac{\partial^2 Q_A}{\partial p^2}(P_A(x,t),t)\sigma_P^2(x,t) + \frac{\partial \sigma_{Q_A}}{\partial p}(P_A(x,t),t)\sigma_P(t)}{\frac{\partial Q_A}{\partial p}(P_A(x,t),t)} \\ \sigma_P(x,t) &= \frac{\sigma_{Q_A}(P_A(x,t),t)}{\frac{\partial Q_A}{\partial p}(P_A(x,t),t)} \\ b_P(x,s,t) &= -b_{Q_A}(P_A(x,t),s,t) \end{aligned}$$

Then the family $P_A(x)$ is a smooth family of semimartingales (in the sense of definition 2.2 in [BB04]), and satisfies:

$$dP_A(x,t) = \mu_P(x,t)dt + \sigma_P(x,t)\int_0^S b_P(x,s,t)W(ds,dt) \tag{12}$$

Proof of lemma:
By definition:

$$Q_A(P(x,t),t) = x \tag{13}$$

We suppose that (12) holds and apply the Ito-Wentzell formula (see e.g., Krylov (2009)) to both sides of (13):

$$\begin{aligned} \mu_{Q_A}(P(x,t),t) + \frac{\partial Q_A}{\partial p}(P(x,t),t)\mu_P(x,t) + \frac{1}{2}\frac{\partial^2 Q_A}{\partial p^2}(P(x,t),t)\sigma_P^2(x,t) + \frac{\partial \sigma_{Q_A}}{\partial p}(P(x,t),t)\sigma_P(x,t) &= 0 \\ \sigma_{Q_A}(P(x,t),t)b_{Q_A}(P(x,t),s,t) + \frac{\partial Q_A}{\partial p}(P(x,t),t)\sigma_P(x,t)b_P(x,s,t) &= 0 \end{aligned}$$

Since, by assumption $\frac{\partial Q_A}{\partial p} < 0$, the lemma follows.

Lemma 2 If the large trader is rational [4] the net demand curve Q_L and the clearing price π are continuous in t.

Proof The family of semimartingales $L(\vartheta)$ is smooth. We skip the proof of this fact since it is identical (modulo replacing Q_A by Q) to the proof that $L_2(\vartheta)$ is smooth, which figures in the proof of theorem 1. Lemma 1 shows then that $P_A(x,t)$ is well-defined, and thus the dynamics of the wealth process V^θ are given by (11). The latter equation shows that only tame strategies for θ are rational, that is, $[\theta,\theta]^c = 0$, and θ is continuous. By (9)

$$\begin{aligned} 0 &= \theta(t_+) - \theta(t) \\ &= -Q_A(\pi(t_+),t_+) + Q_A(\pi(t),t) \\ &= -Q_A(\pi(t_+),t_+) + Q_A(\pi(t_+),t) \\ &\quad -Q_A(\pi(t_+),t) + Q_A(\pi(t),t) \end{aligned}$$

[4] In the sense that he prefers more to less.

Since Q_A is continuous

$$Q_A(\pi(t_+), t_+) - Q_A(\pi(t_+), t) = 0$$

Thus

$$Q_A(\pi(t_+), t) - Q_A(\pi(t), t) = 0$$

Since Q_A is strictly decreasing π is continuous. Using (9):

$$\begin{aligned} 0 &= \theta(t_+) - \theta(t) \\ &= Q_L(\pi(t_+), t_+) - Q_L(\pi(t), t) \\ &= Q_L(\pi(t_+), t_+) - Q_L(\pi(t_+), t) \\ &\quad + Q_L(\pi(t_+), t) - Q_L(\pi(t), t) \end{aligned}$$

Since π is continuous:

$$0 = Q_L(\pi(t_+), t_+) - Q_L(\pi(t_+), t)$$

Thus Q_L is continuous in time. ■

Lemma 3: If the large trader is rational Q_L is twice-differentiable in p.

Proof: for Ran. You may want to add something to the hypothesis of rationality.
Since $Q = Q_L + Q_A$, we argue that the market net demand curve Q should be continuous in t and twice differentiable in p.

2.4 Market Model

As argued in the introduction, we suggest a market model with the same features as a market model with only atomistic traders. The market net demand curve Q satisfies:

$$\begin{aligned} dQ(p,t,\omega) &= \mu_Q(p,t,\omega)dt + \sigma_Q(p,t,\omega)\int_{s=0}^{S} b_Q(p,s,t,\omega)W(ds,dt,\omega) \qquad 0 \le p \le S && (14) \\ Q(p,0) &= Q_0(p) > 0 \qquad 0 \le p \le S && (15) \\ \int_{s=0}^{S} b_Q^2(p,s,t,\omega) &= 1 \qquad 0 \le p \le S && (16) \end{aligned}$$

From now on, we suppose that assumptions 1 to 5 apply where Q_A is replaced by Q . For convenience, we assume that there is always a solution to (8). The next example shows that it is possible to sastify all these assumptions.

Example

Let the constants $a_h(p), a_\eta > 0$, $\eta_L^0, \bar{\eta} \in (0,1)$, and $h^0(p) > 0$.for all $p \in [0,S]$. Let $\mu_h : \mathbb{R}^3 \to \mathbb{R}$, $\sigma_h : \mathbb{R}^4 \to \mathbb{R}$, $\sigma_\eta : \mathbb{R}^3 \to \mathbb{R}$. be measurable functions. Suppose that there exists a constant C such that or each $p \in [0,S]$, each $t \in [0,T]$ and h_1, h_2:

$$(\mu_h(h_1,p,t)h_1 - \mu_h(h_2,p,t)h_2)^2 + \int_0^S (\sigma_h(h_1,p,s,t)h_1 - \sigma_h(h_2,p,s,t)h_2)^2 ds \le C(h_1 - h_2)^2$$

ADD second condition

We define:

$$dh(p,t,\omega) = (-a_h(p) + \mu_h(h(p,t,\omega),p,t)dt + \int_{s=0}^{S} \sigma_h(h(p,t,\omega),p,s,t)W(ds,dt,\omega) \qquad \forall p \in [0,S] \tag{17}$$

$$q(p,t,\omega) = \exp \int_{x=0}^{p} h(x,t,\omega)dx \qquad \forall p \in [0,S] \tag{18}$$

$$Q(p,t,\omega) = \int_{x=0}^{S} q(x,t,\omega)dx\eta(t,\omega) - \int_{x=0}^{p} q(x,t,\omega)dx \qquad \forall p \in [0,S] \tag{19}$$

$$d\eta(t,\omega) = a_\eta(\bar{\eta} - \eta(t,\omega))dt + \tag{20}$$

$$+\sqrt{\eta(t,\omega)(1-\eta(t,\omega))} \int_{s=0}^{S} \sigma_\eta(s)W(ds,dt,\omega) \tag{21}$$

with initial conditions:

$$h(p,0,\omega) = h^0(p) \qquad \forall p \in [0,S] \tag{22}$$

$$\eta(t) = \eta^0 \tag{23}$$

Remark Equation (**??**) is a trivial case of the stochastic evolution equation considered for instance in [CT06] p.130, and, as such, has a weak solution. To be a fully general stochastic evolution equation, one could make the coefficients $\mu_h(.,p,t)$ and $\sigma_h(.,p,s,t)$ be functionals of h rather than making them functions of $h(p,t,\omega)$. We refrain from this generalization in order to get quickly to the main point.

Remark (Pif) The processes $h(p,.)$ are all strictly positive. The processes $q(p,.)$ are then all strictly positive. Since $f(p)$ is not necesarily monotonous, $q(p,.)$ is not necessarily increasing or decreasing in p. Equation (19) shows that Q is strictly decreasing in p, thus satisfying assumption (**). Also, we note that Q is twice differentiable in p. The process η has been constructed in such a way that it oscillates between zero and one[5]. With some conditions on the parameters, we can also ensure that, with probability one, it never reaches zero or one. Thus, for each t, there exists a value $\pi(t) \in (0,S)$ such that $Q(\pi(t),t) = 0$ almost surely.

Homework (for Ran) Calculate the infinitesimal parameters μ_Q, σ_Q and b_Q in the model (17)(20).

2.4.1 No-Arbitrage Conditions

Since our goal is to determine (provided it exists) a measure $\mathbb{Q}$ such that π as defined in (8) is a martingale, we must define π as a $\mathbb{P}$-semimartingale. By the martingale representation theorem (theorem 4.1 in [CT06]), there exists $\mathcal{F}_t$-adapted real-valued processes μ_π, σ_π, $b_\pi(s,.)$:

$$d\pi(t) = \mu_\pi(t)dt + \sigma_\pi(t) \int_s b_\pi(s,t)W^{\mathbb{Q}}(ds,dt)$$

Besides, we choose $b_\pi(s,.)$ such that, for each $t \in [0,T]$:

$$\int_0^S b_\pi^2(s,t)ds = 1$$

Remark As explained in remark (pif), the net demand curves are constructed such that there is always a solution to the market clearing equation (8). We will show in theorem 1 that σ_π is strictly positive $\mathbb{P}$-as.

[5]We note that any model of a correlation process can be used to model η_A.

Definitions The market price of risk is a collection of $\mathcal{F}_t$- adapted stochastic processes $\lambda(s,)$ for $s \in [0, S]$. We define the $\mathbb{Q}$-measure as the measure where the process $W^{\mathbb{Q}}$ is a Brownian sheet, where:

$$W^{\mathbb{Q}}(ds, dt) = W(ds, dt) + \lambda(s, t)dt \tag{24}$$

Definitions We also define the following processes:

$$\begin{aligned}
C(\pi, t, \omega) &= \sigma_\pi(t, \omega) \int_{s=0}^{S} \frac{\partial[\sigma_Q(p, t, \omega) b_Q(p, s, t, \omega)]}{\partial p}|_{p=\pi} b_\pi(s, t, \omega) ds \\
B(\pi, t, \omega) &= \mu_Q(\pi, t, \omega) - \frac{1}{2} h(f(\pi), t, \omega) \frac{df}{dp}|_{\pi=p} (\sigma_\pi(t, \omega))^2 + C(\pi, t, \omega) \\
\Sigma(\pi, s, t, \omega) &= \sigma_Q(\pi, t, \omega) b(\pi, s, t, \omega)
\end{aligned}$$

Definition The market price of risk equations consist of the uncountably infinite system of equations:

$$\int_{s=0}^{S} \Sigma(\pi, s, t, \omega) \lambda(s, t, \omega) ds = B(\pi, t, \omega) \qquad 0 \leq \pi \leq S \tag{25}$$

Assumption 10: there exists $\mathbb{P}$-a.s. a solution to the market price of risk equations (25) that satisfies Novikov's criterion.

We now introduce a second large trader, and show that the large trader does not have an arbitrage opportunity.

Remark: the dynamics of the (hypothetical) second large trader do not enter the specification of the net demand curve in assumption 9. In other words, the market does not trade based on the second large trader information. This is consistent with assumption (A4) in [Jar92].

Definition: We call Θ the set of admissible strategies of the large trader. Clearly:

$$\Theta = \{\theta(t, \omega) | Q(S, t, \omega) \leq \theta(t, \omega) \leq Q(0, t, \omega)\}$$

The following theorem shows that a market with two (uncoordinated) large traders does not admit arbitrage. If the first large trader does not trade (i.e., $Q_L = 0$), the same theorem proves that a market with one large trader is arbitrage-free.

Assumption 11:

The following definition will be useful in the proof of theorem 1 as well as in the example following the theorem. It can be skipped at first reading.

Definition The set $\mathcal{T}(\vartheta)$ is the set of times where:

$$Q(0, t, \omega) \leq \vartheta \leq Q(S, t, \omega)$$

We also define τ_- by:

$$\tau_-(t) = \begin{cases} 0 \text{ if } s \in \mathcal{T}(\vartheta) \ \forall s \in [0, t] \\ \inf\{\tau \leq t | \tau \in \mathcal{T}(\vartheta); s \in \mathcal{T}(\vartheta) \ \forall s \in [\tau, t]\} \end{cases}$$

Theorem 1 Suppose assumptions 7, 9 and 10 hold. Suppose that a new rational large trader with strategy $\theta \in \Theta$ comes to the market, and that strategies involving uncross orders do not generate arbitrage. Then there is no arbitrage. Besides the volatility of the market price is given by:

$$\sigma_\pi(t) b_\pi(s,t) = -\frac{\sigma_Q(\pi(t),t) \int_{s=0}^S b_Q(\pi(t),s,t) ds}{q(\pi(t),t)} \tag{26}$$

Proof

Market clears if $Q(\pi(t),t) = 0$. We noticed in remark (**) that the price π_π such that $Q_A(\pi_\pi(t),t) = 0$ belongs to $(0,S)$ almost surely. The same observation holds in our model for $\pi(t)$ which is restricted to belong to $(0,S)$ almost surely. Thus the differential $d\pi$ is well-defined. We use the Ito-Wentzell formula [Kry09] to calculate $dQ(\pi(t),t)$ and set $dQ(\pi(t),t) = 0$. Equating the drift part gives:

$$\begin{aligned} &\mu_Q(\pi(t),t)dt + \sigma_Q(\pi(t),t) \int_{s=0}^S b_Q(\pi(t),s,t) W(ds,dt) \qquad (27) \\ &-q(\pi(t),t) \left(\mu_\pi(t)dt + \sigma_\pi(t) \int_s b_\pi(s,t) W(ds,dt) \right) - \frac{1}{2} h(f(\pi(t)),t)(\sigma_\pi(t))^2 dt + C(\pi(t),t)dt = 0 \end{aligned}$$

We equate the volatility terms to zero above and find (26). Introducing (24) in (27) results in:

$$\begin{aligned} &\mu_Q(\pi(t),t) - \sigma_Q(\pi(t),t) \int_{s=0}^S b_Q(\pi(t),s,t)\lambda(s,t) ds \qquad (28) \\ &+q(\pi(t),t) \left(\mu_\pi(t) - \sigma_\pi(t) \int_{s=0}^S b_\pi(s,t)\lambda(s,t) ds \right) - \frac{1}{2} h(f(\pi(t)),t) \frac{df}{dp}|_{p=\pi(t)} (\sigma_\pi(t))^2 + C(\pi(t),t) = 0 \end{aligned}$$

By definition of the measure $\mathbb{Q}$, the drift of π in this measure is zero, i.e.:

$$\mu_\pi(t) - \sigma_\pi(t) \int_{s=0}^S b_\pi(s,t)\lambda(s,t) ds = 0$$

Thus, we must have:

$$\begin{aligned} &\int_{s=0}^S \Sigma(\pi(t,\omega),s,t,\omega)\lambda(s,t,\omega) ds = \qquad (29) \\ &\mu_Q(\pi(t,\omega),t,\omega) - \frac{1}{2} h(f(\pi(t,\omega)),t,\omega)(\sigma_\pi(t,\omega))^2 + C(\pi(t,\omega),t,\omega) \\ &= B(\pi(t,\omega),t,\omega) \qquad (30) \end{aligned}$$

which we may call the *stochastic market price of risk* equation. Observe that, for each outcome ω, it is only one equation, as opposed to the market price of risk equations, which are an uncontably infinite system of equations for each outcome. Of course, the stochastic market price of risk equations are satisfied if the market price of risk equations are satisfied.

We now prove that there is no arbitrage. Denote by $\pi^\vartheta(t,\omega)$ the clearing price obtained when the (second) large trader tries to maintain a fixed position ϑ , acquired at time $t=0$. The fixed strategy $\theta(t) = \vartheta$ may not belong to Θ for all $t \in [0,T]$. Indeed, at some time $\tau \in [0,T]$ the large trader is obliged [6] to change his

[6] The large trader may be obliged to liquidate his long position $\vartheta > 0$ if (part of) this long position consists of borrowed shared. The large trader may be forced to liquidate his short position $\vartheta > 0$ in case of a short squeeze.

position to either $Q(0,\tau)$ if $\vartheta > Q(0,\tau)$ or to $Q(S,0)$ if $\vartheta < Q(S,0)$. We thus need to modify slightly the argument in [BB04]. Define by $P_{A,2}(\vartheta,t)$ the price of the second large trader when his strategy is ϑ.

Rather than proving that $P_{A,2}(\vartheta,t)$ is a martingale, we want to prove that for $t \in \mathcal{T}(\vartheta)$:

$$P_{A,2}(\vartheta,s) = E^{\mathbb{Q}}[P_{A,2}(\vartheta,t)|\mathcal{F}_t] \quad \forall s \leq t,\ s \in [\tau_-(t)]$$

If this is the case, we say that $P_{A,2}$ is a *piecewise martingale.* At every time $t \in \mathcal{T}(\vartheta)$ the market clearing equations are:

$$Q(\pi^{\vartheta}(t,\omega),t,\omega) = \vartheta$$

Application of the Ito-Wentzell formula results in exactly the same relations (27) as before, with $\pi^{\vartheta}(t,\omega)$ replacing $\pi(t,\omega)$ (which we could also denote by $\pi^0(t,\omega)$). The dynamics of the net demand curve $Q(p,t,\omega)$ are unchanged. Suppose that the market price of risk equations (25) are satisfied. Then, for any ϑ and $t \in \mathcal{T}(\vartheta)$, and ω:

$$\int_{s=0}^{S} \Sigma(\pi^{\vartheta}(t,\omega),s,t,\omega)\lambda(s,t,\omega)ds = B(\pi^{\vartheta}(t,\omega),t,\omega)$$

Thus, for any ϑ, π^{ϑ} is a $\mathbb{Q}-$piecewise martingale. Since this trader trades at the clearing price, we have:

$$P_{A,2}(\vartheta,t) = \begin{cases} \pi^{\theta}(t) & t \in \mathcal{T}(\vartheta) \\ 0 & \text{if } t \notin \mathcal{T}(\vartheta), \vartheta < 0 \\ S & t \notin \mathcal{T}(\vartheta), \vartheta > 0 \end{cases}$$

Define:

$$L_2(\vartheta,t) = \int_0^{\vartheta} P_{A,2}(x,t)dx$$

We now show that the family of piecewise martingales $L_2(\vartheta)$ is smooth in the sense of definition 2.2 in [BB04].Let σ_{π}^{ϑ} be the volatility of π^{ϑ} and b_{π}^{ϑ} the corresponding factor loadings. Relation (26) can be extended to show that:

$$\sigma_{\pi}^{\vartheta}(t)b_{\pi}^{\vartheta}(s,t) = \frac{\sigma_Q(\pi^{\vartheta}(t),t)\int_{s=0}^{S} b_Q(\pi^{\vartheta}(t),s,t)ds}{q(\pi^{\vartheta}(t),t)}$$

Since $\sigma_Q(\pi^{\vartheta}(t),t)$ and $q(\pi^{\vartheta}(t),t)$ are strictly positive, the covariation of $L_{A,2}(\vartheta)$ and $L_{A,2}(\vartheta')$ satisfies 2.2iii in [BB04]. Since Q is twice differentiable in p, it is not hard to see that π^{ϑ} is also twice-differentiable in ϑ. Since q is differentiable in π^{ϑ}, the volatility $\sigma_{\pi}^{\vartheta}(t)b_{\pi}^{\vartheta}(s,t)$ is differentiable in ϑ. Thus the covariation of $L_{A,2}(\vartheta)$ and $L_{A,2}(\vartheta')$ is twice differentiable in ϑ. Also assumptions 2 and 3 in [BB04] are satisfied. We then invoke lemma 3.2 in [BB04], with a slight modification. Examination of the proof of lemma 3.2 in [BB04] shows that it is not necessary for $P_{A,2}(\vartheta)$ to be a smooth family of martingales. Only $L_{A,2}$ needs to be a smooth family of $\mathbb{Q}-$piecewise martingales for that lemma to hold provided that $\theta \in \Theta$. Thus, if $\theta \in \Theta$, we conclude that the real wealth process V_2^{θ} of the second large trader is a $\mathbb{Q}-$ supermartingale. ■

One may wonder when the market price of risk equations are satisfied in our model. The main problem with a continuous model is that the market price of risk may become infinite and thus the market price of risk may have to be interpreted in the sense of distributions. This is generically not a problem in practice, as we will show in the empirical section, where we discretize the market price of risk equations in both dimensions p and s.

Definition: there is no ε-arbitrage for a large trader with price per share $P_2(\vartheta,t)$ if the family $P_2(\vartheta,t)$ is such that

$$E[(P_2(\vartheta,t)-E[P_2(\vartheta,s)|\mathcal{F}_t])^2]\leq\varepsilon \quad \forall t\in\mathcal{T}(\vartheta)$$

Example 1: We show that it is possible to find a bounded market price of risk such that there is no ε-arbitrage. The smaller ε the larger the market price of risk. For each t and ω we define an approximation $B_\delta(\pi,t,\omega)$ of $B(\pi,t,\omega)$ i.e. such that

- the error $E[(B_\delta(\pi,t)-B(\pi,t))^2]\leq\delta$ uniformly in t and π
- $B_\delta(\pi,t,\omega)$ is twice differentiable in π

We show briefly how this can be done in the following simplified model. Let δ be the Dirac delta function. Define:

$$\begin{aligned}
a(p) &= \mu_h(h,p,t)=0\\
\sigma_h(h,p,t) &= \sigma_h(p)\\
b_h(h,p,s,t) &= \delta(p-s)\\
f(p) &= p
\end{aligned}$$

Thus:

$$\begin{aligned}
h(p,t) &= h_0\exp(-\frac{1}{2}\sigma^2(p)t+\sigma(p)W(p,t)) \qquad (31)\\
B(0,t) &= 0
\end{aligned}$$

$$\begin{aligned}
C(\pi,t,\omega) &= \sigma_\pi(t,\omega)\int_{s=0}^{S}\frac{\partial[\sigma_Q(p,t,\omega)b_Q(p,s,t,\omega)]}{\partial p}|_{p=\pi}b_\pi(s,t,\omega)ds\\
B(\pi,t,\omega) &= \mu_Q(\pi,t,\omega)-\frac{1}{2}h(f(\pi),t,\omega)\frac{df}{dp}|_{\pi=p}(\sigma_\pi(t,\omega))^2+C(\pi,t,\omega)
\end{aligned}$$

The only part of $B(\pi,t)$ which is not differentiable is $h(f(\pi),t,\omega)$. Let us fix t. Observe then by Walsh's construction the process $W(.,t)$ is a Brownian motion "in the variable s" (i.e., in the filtration generated by $W(s,t)$ for all $s\in[0,S]$). By the Karhunen-Loeve theorem, that Brownian motion $W(.,t)$ can be approximated by a truncated series of differentiable wavelets. Thus there exists an approach to construct a function $B_\delta(\pi,t,\omega)$ which is differentiable in π, and, more importantly, where the error between B_δ and B can be controlled, i.e., given δ one can decide the number of terms in the truncated series. The same techniques can be applied to construct a B_δ twice-differentiable.

The solution $\lambda_\delta(s,t,\omega)$ to the smoothed market price of risk equations

$$\int_{s=0}^{S}\Sigma(\pi,s,t,\omega)\lambda_\delta(s,t,\omega)ds=B_\delta(\pi,t,\omega)$$

is then:

$$\lambda_\delta(s,t,\omega) = \frac{1}{\sigma_h(s)h(s,t,\omega)}\frac{\partial^2}{\partial s^2}B_\delta(s,t,\omega) + \frac{2}{S^2}[s\frac{\partial}{\partial s}B_\delta(s,t,\omega)|_{s=0} + B_\delta(0,t)]$$

It satisfies Novikov's criterion. Is it true????

Remark This example can be generalized to any differentiable function f but the data handling becomes heavy.

2.4.2 When Arbitrage Occurs

An obvious corollary to theorem 1 is that, if the market price of risk equations (25) are not satisfied, then there may be some arbitrage. Generically, it is necessary to use a Brownian sheet to model the net demand curve. Let us replace the Brownian sheet by a collection W_i of independent standard Brownian motions, for $i = 0, ..[S]$.Suppose that one replaces (17) by:

$$\frac{dh(p,t,\omega)}{h(p,t,\omega)} = (-a_h(p) + \mu_h(h(p,t,\omega),p,t)dt + \sum_{i=1}^{[S]} \sigma_h(h(p,t,\omega),p,s,t)dW_i(t) \qquad \forall p \in [0,S]$$

Then the market price of risk equations take the form:

$$\Sigma_{i=0}^{[S]} \Sigma_{discr}(\pi,i,t,\omega)\lambda(i,t,\omega) = B_{discr}(\pi,t,\omega) \quad \forall \pi \in [0,S] \tag{32}$$

for some collection of stochastic processes $\Sigma_{discr}(\pi, i)$ and $B_{discr}(\pi)$. The system (32) is generically not soluble since it has only $[S]+1$ equations for an uncountably infinite amount of unknowns. Practically, one overcomes this problem by discretizing (32) in the direction π and conducting Monte Carlo simulation. However, if we compare (after discretization in the direction π) the right hand side of (32) and the right handside of (25), we see that the former is much more irregular than the latter. Thus the market price of risk in the (discretized) multidimensional Brownian motion model is much more volatile and numerically unstable than in the (discretized) Brownian sheet model. We believe that, based on economic considerations, a market price of risk should be fairly stable. Because of that reason, and because of numerical instability, we do not advocate the use of a multidimensional Brownian motion.

A more interesting question is whether arbitrage may exist in a model with a finite number of Brownian motions. As may be expected from the proof of theorem 1, the answer is positive. A second question is how to construct such an arbitrage strategy. For simplicity, we examine the following model, with only one Brownian motion. It should be clear that the same reasoning applies to a model with a finite number of Brownian motions.

We call $C(t)$ the price of a derivative written on the clearing price determined by the pricing rule:

$$C(t) = E$$

We assume that C does not depend on the position θ of the large trader, and can thus write:

$$\begin{aligned} dC(t,\omega) &= \mu_C(t,\omega)dt + \sigma_C(t,\omega)dW(t,\omega) \\ dQ(t,\omega) &= \mu_Q(p,t,\omega)dt + \sigma_Q(p,t,\omega)dW(t,\omega) \end{aligned}$$

for $\mathcal{F}_t$ −adapted processes μ_C and σ_C.

Proposition: Suppose that μ_C is bounded above and that σ_C is positive, bounded away from zero. Then there is an arbitrage.

Proof: We determine the dynamics of the price $P(\theta)$ faced by the large trader, namely the coefficients μ_P and σ_P of:

$$dP(\theta,t,\omega) = \mu_P(\theta,t,\omega)dt + \sigma_P(\theta,t,\omega)dW(t,\omega)$$

We reuse lemma1 and find:

$$
\begin{aligned}
\mu_P(\theta,t) &= -\frac{\mu_Q(P(\theta,t),t)+\frac{1}{2}\frac{\partial^2 Q}{\partial p^2}(P(\theta,t),t)\sigma_P^2(\theta,t)+\frac{\partial\sigma_Q}{\partial p}(P(\theta,t),t)\sigma_P(\theta,t)}{\frac{\partial Q}{\partial p}(P(\theta,t),t)} \\
\sigma_P(\theta,t) &= \frac{\sigma_Q(P(\theta,t),t)}{\frac{\partial Q}{\partial p}(P(\theta,t),t)}
\end{aligned}
$$

We now assume that θ is a smooth trajectory. Letting $N(t)$ be the number of derivatives held by the large investor, we have:

$$
dV^\theta(t) = \left(\int_0^{\theta(t)} \mu_P(x,t)dx + N(t)\mu_C(t)\right)dt + \left(\int_0^{\theta(t)} \sigma_P(x,t)dx + N(t)\sigma_C(t)\right)dW(t)
$$

Setting:

$$
N(t) = -\frac{\int_0^{\theta(t)} \sigma_P(x,t)dx}{\sigma_C(t)}
$$

We have:

$$
dV^\theta(t) = \left(\int_0^{\theta(t)} \mu_P(x,t)dx + N(t)\mu_C(t)\right)dt
$$

For this to be positive, we need:

$$
\int_0^{\theta(t)} \left(\mu_P(x,t)dx - \frac{\mu_C(t)}{\sigma_C(t)}\sigma_P(x,t)\right)dx > 0
$$

Clearly this is positive if (but not only if):

$$
\mu_P(\theta(t),t) - \frac{\mu_C(t)}{\sigma_C(t)}\sigma_P(\theta(t),t) > 0
$$

By definition of μ_P and σ_P we obtain:

$$
\mu_Q(P(\theta,t),t) + \frac{1}{2}\frac{\partial^2 Q}{\partial p^2}(P(\theta,t),t)\sigma_P^2(\theta,t) + \frac{\partial \sigma_Q}{\partial p}(P(\theta,t),t)\sigma_P(\theta,t) - \frac{\mu_C(t)}{\sigma_C(t)}\sigma_P(\theta,t),t) > 0
$$

Or:

$$
\mu_Q(P(\theta,t),t) + \sigma_P(\theta,t)(\frac{1}{2}\frac{\partial^2 Q}{\partial p^2}(P(\theta,t),t)\sigma_P(\theta,t) + \frac{\partial \sigma_Q}{\partial p}(P(\theta,t),t) - \frac{\mu_C(t)}{\sigma_C(t)}) > 0
$$

Suppose $\mu_Q = 0$. Then this equation is solved if

$$
\frac{1}{2}\frac{\partial^2 Q}{\partial p^2}(P(\theta,t),t)\sigma_P(\theta,t) + \frac{\partial \sigma_Q}{\partial p}(P(\theta,t),t) - K(t) = 0
$$

where $K(t) = \varepsilon + \frac{\mu_C(t)}{\sigma_C(t)}$. For simplicity, we write:

$$
\sigma_P(\theta,t) = \frac{\sigma_Q(P(\theta,t),t)}{\frac{\partial Q}{\partial p}(P(\theta,t),t)}
$$

and the equation becomes:

$$
\frac{1}{2}\frac{\partial^2 Q}{\partial p^2}(P,t)\frac{\sigma_Q(P,t)}{\frac{\partial Q}{\partial p}(P,t)} + \frac{\partial \sigma_Q}{\partial p}(P,t) - K(t) = 0 \tag{33}
$$

We suppose $K > 0$. Suppose there exists a point where

$$\begin{aligned} O(1) &= \frac{\partial^2 Q}{\partial p^2}(P,t) < 0 \\ \frac{\partial Q}{\partial p}(P,t) &= -\varepsilon \\ \sigma_Q(P,t) &= 1 \end{aligned}$$

Then the equation is satisfied. For instance:

$$\begin{aligned} Q(p) &= (C - \frac{1}{2}p^2)\exp(W(t)) \\ \frac{\partial Q}{\partial p}(P,t) &= -p\exp(W(t)) \\ \frac{\partial^2 Q}{\partial p^2}(P,t) &= -\exp(W(t)) \end{aligned}$$

Equation (33) becomes:

$$\frac{1}{2}\frac{1}{p} - K(t) = 0$$

which is satisfied for any finite value of $K(t)$. Suppose now that $0 < K(t) < M$. Then any value of p works. Thus

$$p = \max(\frac{1}{2K(t)}, 0)$$

The strategy is of course:

$$\theta(t) = \begin{cases} Q(\frac{1}{2K(t)}) & \text{if } K(t) > 0 \\ \text{anything} & \quad else \end{cases}$$

Example 2 (probably useless)
If we replace (31) in example 1 by:

$$h(p,t) = h_0 \exp(-\frac{1}{2}\sigma^2(i)t + \sigma(i)W_i(t)) \text{ if } i-1 < p \leq i$$

and discretize $h(p,t)$ again in the π direction, to obtain:

$$h_i(t) = h_0 \exp(-\frac{1}{2}\sigma^2(i)t + \sigma(i)W_i(t)) \qquad i \in [0, 1, .., [S]]$$

The resulting vector h will be much more irregular than in the Brownian sheet case (i.e., with $W_i(t)$ replaced by $W(i,t)$, namely the Brownian sheet evaluated at i).

For Ran: show mathematically that convergence (from discrete to continuous in the sense of ε-arbitrage) of the market price of ris is much slower in example 2 than in example 1. Or, show it by numerical simulations. This would require using a statistical test that a (discrete time) process is a martingale −> look in the literature.

3 The Bachelier Market

We resolve in this section an apparent contradiction. Our message is that a Brownian sheet is in theory generically indispensable (and, in practice, better) to obtain a market without arbitrage and to model the clearing price π. However, the Bachelier market as considered by [KR09] consists of only one Brownian motion, and is arbitrage free. This contradiction is resolved when one realizes that [KR09] do not obtain a clearing price π for the Bachelier market. Note that we do not claim that it is impossible to find a clearing price in the Bachelier market. We just highlight the fact that, when inverted in order to obtain a clearing price, the Bachelier market is not smooth enough to apply the technical tools that we used in this paper.

The primitives for the Bachelier market are:

$$P_A(\vartheta, t) = P_{A_0} + (\mu + \kappa\vartheta)\, t + \sigma W(t)$$

We can obtain the large trader net demand curve by solving:

$$P_A(Q_A(p,t), t) = p \tag{34}$$

Assuming that:

$$dQ_A(p,t) = \mu_{Q_A}(p,t,\omega)dt + \sigma_{Q_A}(p,t,\omega)dW(t)$$

Differentiating (34), and using formally the Ito-Wentzell formula for the left hand side, one obtains:

$$\begin{aligned} \mu_{Q_A}(p,t,\omega) &= -\frac{Q(p,t,\omega)}{t} \\ \sigma_{Q_A}(p,t,\omega) &= -\frac{\sigma}{t} \end{aligned}$$

This model is not smooth in time. Indeed, it starts with an infinite drift and volatility.

4 Simulations

In this section, we calibrate the market model of net demand curve Q. We use empirical height frequency data of a magnitude of a millisecond. Then the calibrated parameters are utilized in the discretized simulation model on both physical measure and risk neutral measure. When simulating the net demand curve Q in risk neutral measure, the market price of risk equation is solved. We could not find any case where the market price of risk equation does not have solution.

4.1 Data

The high frequency trading data are collected from the NYSE Arcabook limit orders for April 2011. Historical NYSE Arcabook data provide information of the complete limit order book (LOB) from NYSE, NYSE Arca, NYSE MKT, NASDAQ and the ArcaEdge platforms from 3:30 a.m. to 8:00 p.m. ET under the high speed of latencies (less than 5 milliseconds). Each limit order contains the unique reference number, the time stamp in seconds and milliseconds, the limit price in U.S. dollars, the quantity in number of shares, and the trading type (B: buy or S:sell).

All the limit order book records are categorized into three groups: "A" Add, "M" Modified and "D" Deleted. For the market liquidity model, we consider the net demand of the stock, which is captured by summing added records ("A") with modified ("M") adjustment and subtracting. To be specific, within a certain partitioned time period, the added orders would be updated by the modified orders, if applicable and the orders occur in the same partitioned period. Then cancel off the the corresponding record with deleted

Figure 1: The calculated net demand surface Q by price and time, from market data for AAPL as of April 1st, 2011.

order. The records after modifications and deletion are considered to be effective when calculating the net demand curve.

In this section, we take the dynamics of Apple Inc. stock (AAPL) as of April 1st 2011 as example to illustrate the calibration and simulation processes.

4.2 Calibration Methodology

The calibration process starts with the discretized version of the model, with the partitioning on both time t and price p. Within a particular time partition, the net demand Q at price p is calculated, by definition, as the the difference between the total number of buy orders with limit order price less than p and the total number of sell orders with limit order price greater than p. That is,

$$\hat{Q}(p) = \#\text{of buy order with limit order price} > p - \#\text{of sell order with limit order price} < p$$

Note that $\hat{Q}(0)$ measures the total number of buy orders on the market, whereas $\hat{Q}(S)$ measures the total number of sell order on the market. And $\hat{Q}(\pi) = 0$. A typical net demand $\hat{Q}$ surface with two dimension as price and time for AAPL stock on April 1st, 2011 is shown in Figure 1.

After the net demand surface $\hat{Q}$ is calculated, the net demand sensitivity $\hat{q}$ is calculated by taking the first order difference of $\hat{Q}(p)$, where $p = 0, 1, \ldots, S$. That is,

$$\hat{q}(p,t) = -\frac{\hat{Q}(p,t) - \hat{Q}(p - \Delta p, t)}{\Delta p}$$

where $p = 1, 2, \ldots, S$. And $\Delta p = \$1.91$. By this definition, $\hat{q}(p,t)$ is no less than 0, since $\hat{Q}(p,t) \geq \hat{Q}(p - \delta p, t)$ if $\delta p \geq 0$. The discretization of $q(p,t,\omega) = \exp \int_{x=0}^{p} h(x,t,\omega)dx$ yields

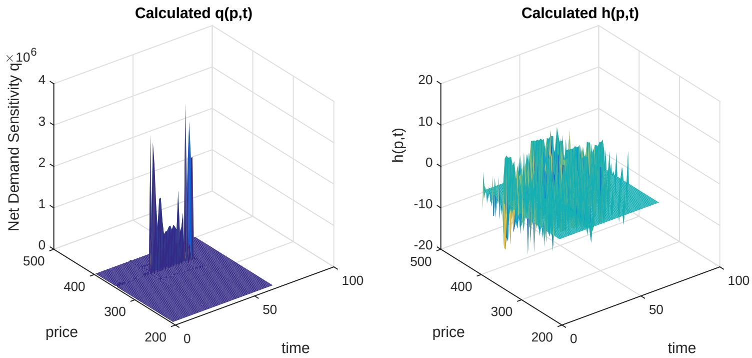

Figure 2: The calculated q(p,t) and h(p,t) from market data for AAPL as of April 1st, 2011.

$$\hat{h}(p,t) = \log(\hat{q}(p,t)) - \log \hat{q}(p - \Delta p, t)$$

where δp represents the price partition. The net demand elasticity $\hat{\eta}$ is calculated by

$$\hat{\eta}(t) = \frac{\hat{Q}(0,t)}{\hat{Q}(0,t) - \hat{Q}(S,t)}$$

and $0 \leq \hat{\eta}(t) \leq 1$, since $\hat{Q}(0,t) > 0$ and $\hat{Q}(S,t) < 0$.

Now we have all the values of $\hat{Q}(p,t)$ with $p = 0, 1, \ldots, S$ and $\hat{q}(p,t)$ with $p = 1, 2, \ldots, S$ and $\hat{h}(p,t)$ with $p = 2, 3, \ldots, S$. Let the H matrix be

$$H = \begin{pmatrix} \hat{q}(1,0) & \Delta\hat{h}_{2,0} & \cdots & \Delta\hat{h}_{S,0} & \frac{\Delta\hat{\eta}_0}{\sqrt{\hat{\eta}_0(1-\hat{\eta}_0)}} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \hat{q}(1,T) & \Delta\hat{h}_{2,T} & \cdots & \Delta\hat{h}_{S,T} & \frac{\Delta\hat{\eta}_T}{\sqrt{\hat{\eta}_T(1-\hat{\eta}_T)}} \end{pmatrix}$$

We calculate the instantaneous volatilities σ_h and σ_η by the method of moments. We use Cholesky decomposition with respect to calculate $B_H = Corr^{1/2}(H)$, where $b_{\eta,h_p} = 0$ except for $b_{\eta,h_S} = 1$. That is, $B_H B_H^T = Corr(H)$, and

$$B_H = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ b_{q,2} & 1 & 0 & \cdots & 0 \\ \vdots & b_{h_{2,1}} & 1 & 0 & \vdots \\ b_{q,S} & \vdots & \ddots & 1 & 0 \\ b_{q,\eta} & b_{h_{2,\eta}} & \cdots & b_{h_{S,\eta}} & 1 \end{pmatrix}$$

4.3 Simulation Methodology

We first simulate the $h(p,t)$ process using the calibrated parameters over simulated time. For simplicity we remove the "hat" sign on the parameters. We also assume the h and q processes have zero drift. For a function of x(t), define $\Delta X(t) = X(t + \Delta t) - X(t)$. Then

$$\begin{aligned}\Delta h(p,t) &= h_p(t+\Delta t) - h_p(t) \\ &= \sigma_{h_p} \sum_{i=2}^{S} b_{h_{p,i}} z(i,t)\sqrt{\Delta t \Delta p} \\ &= \sigma_{h_p} \sum_{i=2}^{S} b_{h_{p,i}} z(i,t)\sqrt{\Delta t \Delta p} \qquad p = 2,3,\ldots,S\end{aligned}$$

and $h(0,t) = h(1,t) = 0$ for computational convenience. The time step Δt is 15 minutes in this case. The boundary condition of the $q(p,t)$ process, $q(1,t)$, is simulated by

$$q(1,t) = \sigma_{q_0} \sum_{i=2}^{S} b_{q_{0,i}} z(i,t)\sqrt{\Delta t \Delta p}$$

where $z(i,t)$ is a random normally distributed number. The rest of the numerical $q(p,t)$ values are

$$q(p,t) = q(1,t) \exp \sum_{i=2}^{p} h_i(t) \qquad p = 2,3,\ldots,S$$

Then the boundary condition of the net demand $Q(0,t)$ is calculated as

$$Q(0,t) = \frac{\eta(t)}{2-\eta(t)} \sum_{p=1}^{S} q(p)$$

which is consistent with the intuition that $Q(0,t) \geq Q(1,t)$ given $q(1,t) \geq 0$. And the rest of the $Q(p,t)$ is

$$Q(p,t) = [Q(0,t) + \sum_{i=1}^{S} q(i,t)]\eta(t) - [Q(0,t) + \sum_{i=1}^{p} q(i,t)] \qquad p = 1,2,\ldots,S$$

where

$$\begin{aligned}\Delta\eta(t) &= \eta(t+\delta t) - \eta(t) \\ &= a_\eta(\bar{\eta} - \eta(t))\Delta t + \sigma_\eta \sqrt{\eta(t)(1-\eta(t))} \sum_{i=2}^{S} b_{\eta,h_p} z(i,t)\sqrt{\Delta t} \\ &= a_\eta(\bar{\eta} - \eta(t))\Delta t + \sigma_\eta \sqrt{\eta(t)(1-\eta(t))} z(S,t)\sqrt{\Delta t}\end{aligned}$$

4.4 Simulation Results in the Physical Measure

The simulated net demand curve under physical measure using the above methods is shown in Figure 3.

4.5 Simulation Results in the Risk Neutral Measure

Changing from physical measure simulation to P measure simulate requires the following adjustments on $h(p,t)$ process simulation.

$$\Delta h(p,t) = \sigma_{h_p} \Sigma_{i=2}^{S} b_{h_{p,i}} [z(i,t)\sqrt{\Delta t \Delta p} - \lambda(i)\sqrt{\Delta p}] \qquad p = 2,3,\ldots,S$$

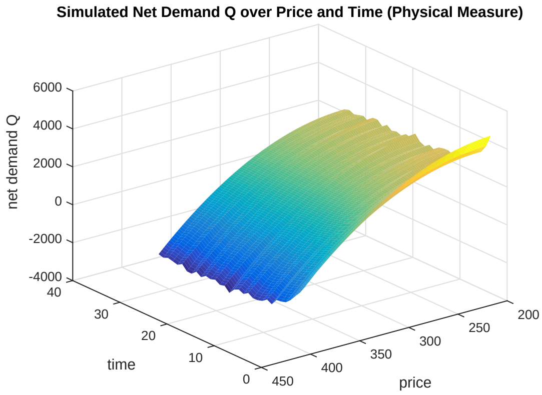

Figure 3: The simulated net demand surface Q(p,t) under physical measure.

To solve for $\lambda(i)$, let $\{\mu_Q, a_Q, \sigma_Q, b_Q\}$ be the parameters of net demand. Let the volatility of the clearing price be:

$$\sigma_\pi(t,\omega)b_\pi(s,t,\omega) = -\frac{\sigma_Q(\pi(t),t)b_Q(s,t,\omega)}{q(\pi(t),t,\omega)}$$

Define:

$$\begin{aligned}
C(\pi,t) &= \sigma_\pi(t)\sum_{s=0}^{S}\frac{\Delta[\sigma_Q(p,t,\omega)b_Q(p,s,t)]}{\Delta p}|_{p=\pi}b_\pi(s,t) \\
&= \sum_{s=0}^{S}\frac{\Delta[\sigma_Q(p,t,\omega)b_Q(p,s,t)]}{\Delta p}|_{p=\pi}\sigma_\pi(t)b_\pi(s,t) \\
&= -\sum_{s_2=0}^{S}\frac{\Delta[\sigma_Q(p,t,\omega)b_Q(p,s_2,t)]}{\Delta p}|_{p=\pi}\frac{\sigma_Q(\pi(t),t)\sum_{s=0}^{S}b_Q(\pi(t),s,t)}{q(\pi(t),t)} \\
B(\pi,t) &= \mu_Q(\pi,t) - \frac{1}{2}\frac{Q(p+1,t)-2Q(p,t)+Q(p-1,t)}{\Delta p^2}|_{p=\pi} + C(\pi,t) \\
\Sigma(\pi,s,t) &= \sigma_Q(\pi,t)b_Q(\pi,s,t)
\end{aligned}$$

The M.P.R. are, for each t and ω to find $\lambda(s)$ that satisfy:

$$\sum_{s=0}^{S}\Sigma(\pi,s)\lambda(s) = B(\pi)$$

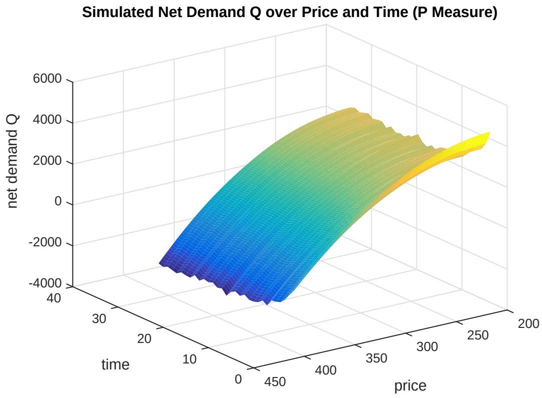

Figure 4: The simulated net demand surface Q(p,t) under risk neutral measure.

The value of $\sigma_Q(p,t)b_Q(p,s,t)$ is given by $(p > 2)$:

$$\left(Q(0,t)\sigma_{Q_0}b_{Q_0,s} + q(1,t)\sigma_{q_1}b_{q_1,s} + \sum_{y=2}^{S}\exp(\sum_{i=2}^{y-1}h_i(t))\sum_{x=2}^{y-1}\sigma_{h_x}b_{h_{x,y}}\right)\eta(t) +$$
$$\left([Q(0,t) + \sum_{x=1}^{S}q(x,t)]\sigma_\eta\sqrt{\eta(t)(1-\eta(t))}\right)b_{\eta,s} +$$
$$\left(Q(0,t)\sigma_{Q_0}b_{Q_0,s} + q(1,t)\sigma_{q_1}b_{q_1,s} + \sum_{y=2}^{p}\exp(\sum_{i=2}^{y-1}h_i(t))\sum_{x=2}^{y-1}\sigma_{h_x}b_{h_{x,y}}\right)$$

The simulated net demand curve under risk neutral measure using the above methods is shown in Figure 4.

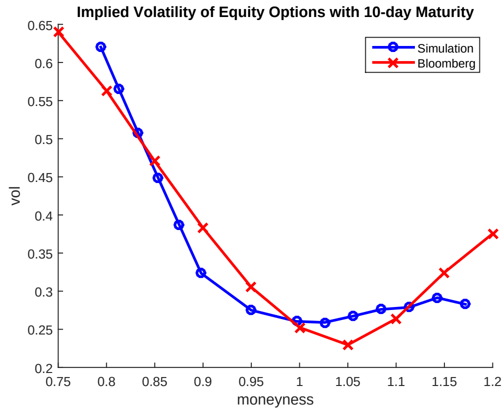

Figure 5: The implied volatility levels from equity options with maturity of 10 days.

4.6 Option Pricing under Risk Neutral Measure

With the simulations under risk neutral measure, the corresponding call/put option price, $c(\pi, \omega)$, with strike K at given time t is

$$\begin{aligned} c(K) &= \frac{1}{N} \sum_{\omega} \max\left(\pi(t,\omega) - K, 0\right) \\ p(K) &= \frac{1}{N} \sum_{\omega} \max\left(K - \pi(t,\omega), 0\right) \qquad \omega = 1, 2, \ldots, N \end{aligned}$$

where π is the clearing price at simulation time t. With the option prices, the implied volatility for different strike levels are calculated by solving

$$p(K) = BS(\pi, K, r, T, \sigma(K))$$

where π is the clearing price, K is the strike level, r is the interest rate, T is time to maturity and p is the put option price corresponding to the strike K. The implied volatility $\sigma(K)$ is being solved.

The implied volatility smile generated from the equity options at simulation period is shown in Figure 5. The implied volatilities by simulations under risk neutral measure are plotted as blue line, which are compared to the real historical implied volatility levels from Bloomberg. The interest rate level is selected as 0.301%, and the dividend yield is set to 0%. The maturity of the option is in 10 days. The simulation is evaluated under 500 scenarios.

The at-the-money simulated implied volatility is lined up with the Bloomberg quote, at 25% level. For low strike options the simulation results tend to underestimate the implied volatility from 5% to 10%. For high strike options the simulation results tend to overestimate the implied volatility from 5% to 7%.