A No-Arbitrage Model Of Liquidity In Financial Markets Involving Brownian Sheets: Applications To High-Frequency Data

October 29, 2015

Ran Zhao

With Henry Schellhorn

Claremont Graduate University

Outline

Option Pricing in a Liquid Market

Model for Asset Price π at time t

$$\pi(t,\omega) = \pi(0)\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t,\omega)\right)$$

where:

ω: the state of the world ("scenario", "outcome"),

μ: the *drift*,

σ: the *volatility*,

W: Brownian Motion.

Option Pricing in a Liquid Market: what is Brownian Motion?

Random walk: equal probability at each (discrete) time-step to go up or down.

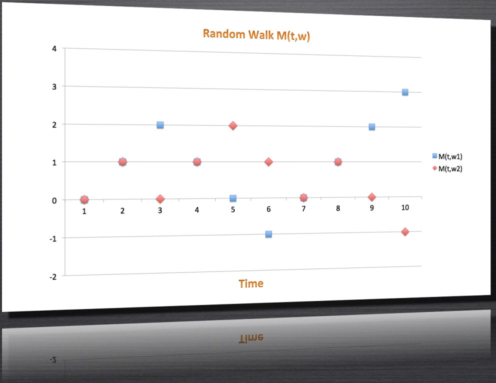

Figure: Random Walk.

Option Pricing in a Liquid Market: what is Brownian Motion?

Random walk: equal probability at each (discrete) time-step to go up or down.

Brownian Motion: accelerate time:

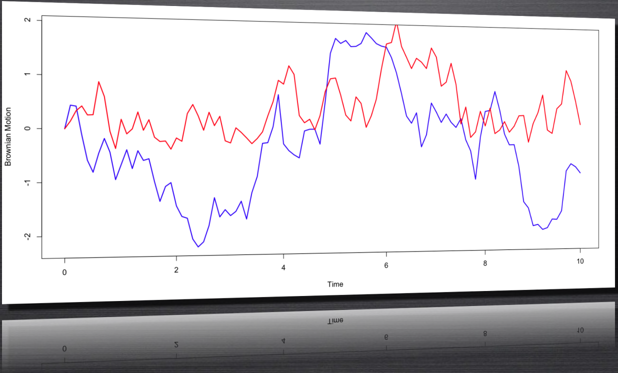

Figure: Brownian Motion.

Option Pricing in a Liquid Market: Stochastic Calculus

The Chain rule is different

- Regular calculus: with a differentiable path $x(t)$

$$\frac{d}{dt}f(x(t),t) = \frac{\partial f}{\partial x}\frac{\partial x(t)}{\partial t} + \frac{\partial f}{\partial t}$$

- Stochastic calculus: with a non-differentiable path $W(t)$

$$df(W(t),t) = \frac{\partial f}{\partial x}dW(t) + \frac{1}{2}\frac{\partial^2 f}{\partial^2 x}dt + \frac{\partial f}{\partial t}dt$$

(Ito's lemma)

Option Pricing in a Liquid Market: the Black-Scholes Formula

Call Option

A call option is a contract which gives the owner the right to buy an (underlying) stock at a future time T for a given *strike price* K.

Theorem

If there is no arbitrage, the price of the call option at time zero is:

$$C(0) = E^{\mathbb{Q}}[\max(\pi(T) - K, 0)]$$

Observation: $\mathbb{Q}$ is called the *risk-neutral* measure. It is by definition the measure where π is a *martingale*, i.e., where:

$$\pi(0) = E^{\mathbb{Q}}[\pi(t)] \quad \forall t > 0$$

Equivalently, this is the measure where $W^{\mathbb{Q}}(t) \equiv W(t) + \frac{\mu}{\sigma} t$ is Brownian motion

$$\begin{aligned} \pi(t) &= \pi(0) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right) \\ &= \pi(0) \exp\left(-\frac{\sigma^2 t}{2} + \sigma\left(W(t) + \frac{\mu}{\sigma} t\right)\right) \\ &= \pi(0) \exp\left(-\frac{\sigma^2 t}{2} + \sigma W^{\mathbb{Q}}(t)\right) \end{aligned}$$

The drift μ disappears. The option price depends only on volatility σ!

Market vs Limit Orders

A (buy) market order specifies

- how many shares a trader wants to buy,
- that he is willing to buy them at any price.

A (buy) limit order specifies

- how many shares a trader wants to buy,
- at what maximum price he is willing to buy them.

Limit order matching mechanism

Figure: Limit order matching mechanism.

Limit order matching mechanism

Figure: Limit order matching mechanism.

Limit order matching mechanism

Figure: Limit order matching mechanism.

Limit order matching mechanism

Figure: Limit order matching mechanism.

Demand v.s. Supply

The order books contain all the information about demand and supply.

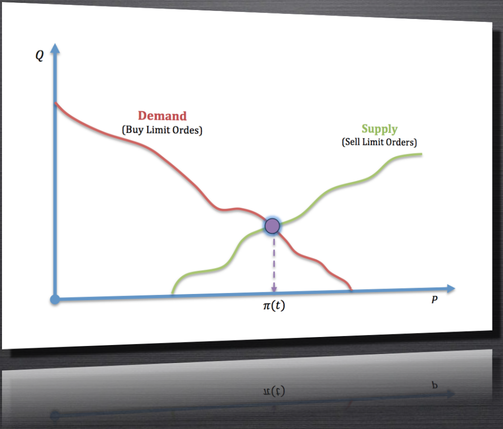

Figure: Demand v.s. Supply.

The dynamics of Limit Orders in 3D

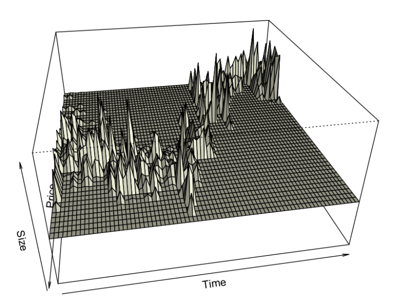

Figure: Buy Limit Orders of ORCL on April 4, 2011.

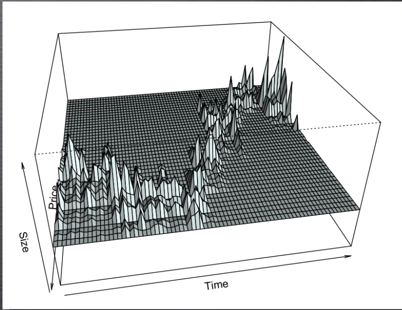

Figure: Sell Limit Orders of ORCL on April 4, 2011.

The dynamics of the Clearing Price process

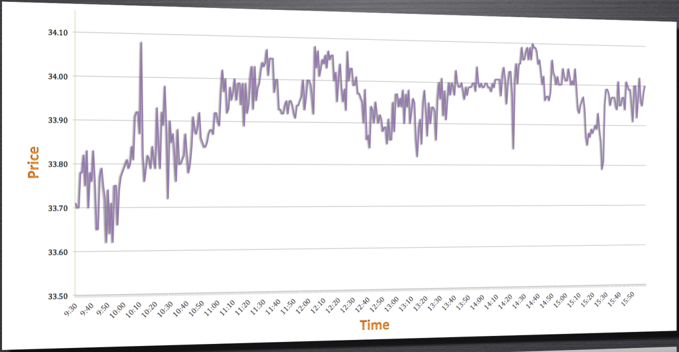

Figure: The dynamics of Oracle Corporation's Clearing Prices on April 1, 2011.

High-Frequency Trading

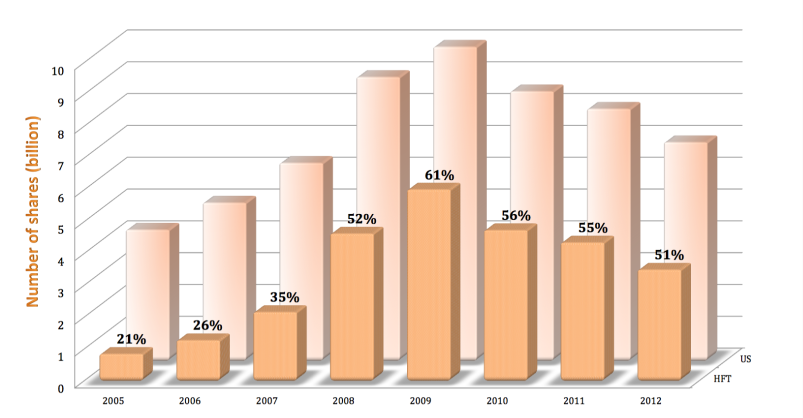

Figure: Average daily trading volume by HFT firms in all U.S. stocks (2005-2012).

Source: Tabb Group, Rosenblatt Securities, The New York Times and Agarwal (2012).

Literature Review: Liquidity Models

Market Manipulation (feedback) Models

- Jarrow (1994),
- Platen and Schweizer (1998),
- Sircar and Papanicolaou (1998),
- Frey (1998),
- Schonbucher and Wilmott (2000),
- Bank and Baum (2004).

Price-taking (competitive) Models

- Cetin, Jarrow, and Protter (2004),
- Cetin and Rogers (2006),
- Cetin, Soner, and Touzi (2009),
- Kallsen and Rheinlaender (2009),
- Gokay and Soner (2011).

Net Demand Curve and Clearing Price

Definition

The net demand curve Q is a function $[0, P] \times \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}$, which value $Q(p, t, \omega)$ is equal to the difference between the quantity of shares **available** for purchase and the quantity of shares **available** for sale at price p at time t. For each p the stochastic process $Q(., t,)$ is a $\mathcal{F}_t$ adapted semimartingale.

Remark: If we use Brownian motion to model demand, the net demand curve must be defined on a continuum of limit prices. Indeed the clearing price will be a diffusion (range $= \mathbb{R}^+$). Since it must fall on an existing limit price, the demand must be defined on a continuum of limit prices.

Remark: The net demand curve should be decreasing in p. The easiest way to do that is to model positive processes:

- $Q(0, t)$: total number of buy orders
- $q(p, t)$: density of buy orders + density of sell orders

$$Q(p, t) = Q(0, t) - \int_0^p q(y, t) dy$$

Definition

The clearing price $\pi(t)$ is a $\mathcal{F}_t-$ adapted stochastic process which satisfies market clearing:

$$Q(\pi(t), t) = 0$$

The Model

- It can be proved that the optimal strategy of a large trader is to disseminate her orders into infinitesimal orders.

  This shows that a continuous demand curve is a plausible model.

- For the clearing price to be a martingale, it is (generically) necessary to have as many "sources of information" as possible limit price values:

  since the set of possible limit price values has to be a continuous range we introduce the Brownian sheet $W(t, s)$.

- There is correlation among net demand at different limit prices

$$dQ(0,t) = \mu_Q(0,t)dt - \sigma_Q(0,t)\int_s b_q(0,s,t)W(0,dt), \qquad Q(0,0) = Q_0(0)$$

$$dq(p,t) = \mu_q(p,t)dt + \sigma_q(p,t)\int_s b_q(p,s,t)W(ds,dt), \qquad q(p,0) = Q_0(p) \text{ for } 0 < p \leq S$$

$$q(0,t) = 0$$

Main Result: Market with a Large Trader

Main Result

Suppose in addition to our standing assumptions that

C1) for self-financing strategies involving only immediate orders, (Jarrow, 1994)'s discrete-time conditions for absence of market manipulation strategy hold,

C2) no arbitrage strategy involves wait orders,

C3) the volatility $\sigma_{Q_A}(p, t)$ is bounded away from zero, uniformly in p,

C4) there is no path such that $Q(S, t) \geq 0$ or $Q(0, t) \leq 0$.

Then

F1) there exists at least one martingale measure $\mathbb{Q}$ for $\int L_L(\vartheta, dt)$,

F2) there is no arbitrage strategy,

F3) the net demand curve Q is continuous in t,

F4) the clearing price $\pi(t)$ is continuous,

F5) any such measure $\mathbb{Q}$ is also a martingale measure for $\pi(t)$.

Characterization of the Risk-Neutral Measure $\mathbb{Q}$

Standing assumption: There is no path such that $Q(P, t) \geq 0$

Change of Measure

In the $\mathbb{Q}$-measure the process $W^{\mathbb{Q}}$ is a Brownian sheet, where:

$$W^{\mathbb{Q}}(ds, dt) = W(ds, dt) + \lambda(s, t)dt$$

Goal: determine λ such that π is a $\mathbb{Q}$-martingale.

Market Price of Risk Equations

Define

$$
\begin{aligned}
C(\pi, t) &= -\sigma_\pi(t)\left(\frac{\partial}{\partial p}\left(\sigma_q(0,t)\int_s b_q(0,s,t)b_\pi(s,t)ds\right) + \sigma_q(\pi,t)\int_s b_q(\pi,s,t)b_\pi(s,t)ds\right), \\
b(\pi, t) &= -\mu_Q(0,t) + \int_0^\pi \mu_q(p,t)dpdt + \frac{1}{2}\frac{\partial q}{\partial p}(\pi,t)(\sigma_\pi(t))^2 - C(\pi,t), \\
\Sigma(\pi, s, t) &= \int_0^\pi \sigma_q(p,t)b_q(p,s,t)ds.
\end{aligned}
$$

The market price of risk equations are:

$$
\int_{s=0}^{P} \Sigma(\pi,s,t)\lambda(s,t)ds = b(\pi,t) \qquad 0 \le \pi \le P
$$

Theorem

Suppose all the previous assumptions hold. In addition, suppose that the market price of risk equations have a unique solution. Then there is no arbitrage.

From a "MetaModel" to a Model

Reminder:

$$q(p,t)dp \;=\; \text{sum of buy and sell order quantities with limit price in } [p, p+dp] \text{ arriving in } [0,t]$$

Plausible dynamics for q:

- positive process

  not necessarily increasing: orders can be cancelled
- mean-reverting process
- to be implemented on a computer: p and t must take discrete values
- the relative curve, i.e., the two-argument curve $\tilde{q}(.,.,t)$ where $\tilde{q}(p-\pi(t), p, t) = q(p,t)$ can be well fitted as a function of the first argument only

Our choice: the exponential of a (vector) Ornstein-Uhlenbeck process.

NYSE Arcabook Data

Industry	Exchange	Ticker	Firm
Energy	NYSE	CVX	Chevron Corporation
	NYSE	XOM	Exxon Mobil Corporation
Financial Banks	NYSE	JPM	JPMorgan Chase & Co.
	NYSE	WFC	Wells Fargo & Company
Materials and Mining	NYSE	ABX	Barrick Gold Corporation
	NYSE	FCX	Freeport-McMoRan Copper & Gold Inc.
Technology	NASDAQ	CSCO	Cisco Systems, Inc.
	NASDAQ	MSFT	Microsoft Corporation
	NASDAQ	ORCL	Oracle Corporation

Table: NYSE Arcabook data selection.

Parameter Estimation

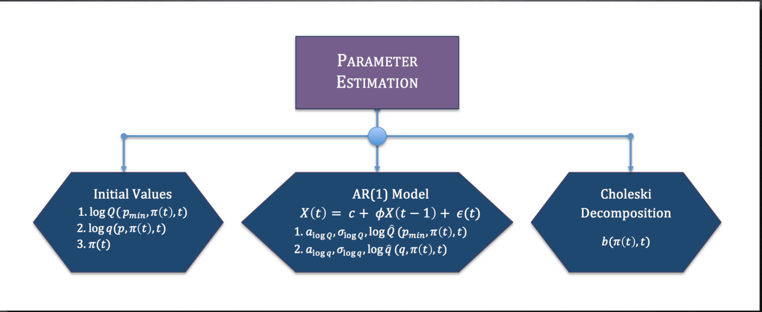

Figure: Parameter Estimation

Volatility Smile (ORCL)

Strike Price	Implied Volatility
33.77	22.66%
33.80	21.05%
33.83	19.40%
33.87	17.71%
33.90	15.94%
33.93	14.10%
33.97	12.16%
34.00	10.11%
34.04	9.35%
34.07	10.26%
34.10	11.03%
34.14	11.70%
34.17	12.25%
34.20	12.71%
34.24	13.06%
34.27	13.25%
34.31	13.34%
34.34	13.32%
34.37	13.23%
34.41	12.99%

Figure: Simulated v.s. real-time volatility smile of ORCL on April 4, 2011. *Source: Bloomberg*

Volatility Smile (ABX)

Strike Price	Implied Volatility
50.98	23.64%
51.03	21.98%
51.08	20.27%
51.13	18.51%
51.18	16.70%
51.23	14.84%
51.29	12.93%
51.34	11.69%
51.39	12.24%
51.44	12.75%
51.49	13.20%
51.54	13.58%
51.59	13.92%
51.64	14.18%
51.69	14.37%
51.74	14.46%
51.79	14.54%
51.85	14.55%
51.90	14.46%
51.95	14.39%

Figure: Simulated v.s. real-time volatility smile of ABX on April 4, 2011. *Source: Bloomberg*

Volatility Smile (CSCO)

Strike Price	Implied Volatility
16.88	13.30%
16.90	12.17%
16.92	11.02%
16.93	9.84%
16.95	8.63%
16.97	7.37%
16.99	6.04%
17.00	4.63%
17.02	4.16%
17.04	4.84%
17.05	5.28%
17.07	5.53%
17.09	5.68%
17.10	5.73%
17.12	5.79%
17.14	5.85%

Figure: Simulated v.s. real-time volatility smile of CSCO on April 4, 2011. *Source: Bloomberg*

Model Framework

We define

$$\begin{aligned}
\frac{dh(p,t,\omega)}{h(p,t,\omega)} &= [-a_h(p) + \mu_h(h(p,t,\omega),p,t)]dt + \int_{s=0}^{S} \sigma_h(h(p,t,\omega),p,s,t)W(ds,dt,\omega) \\
q(p,t,\omega) &= \exp\left(\int_{x=0}^{p} h(x,y,\omega)dx\right) \\
dQ(p,t,\omega) &= \int_{x=0}^{S} q(x,t,\omega)dx\eta(t,\omega) - \int_{x=0}^{p} q(x,t,\omega)dx \\
d\eta(t,\omega) &= a_\eta(\bar{\eta} - \eta(t,\omega))dt + \sqrt{\eta(t,\omega)(1-\eta(t,\omega))}\int_{s=0}^{S} \sigma_\eta(s)W(ds,dt,\omega)
\end{aligned}$$

Reasoning of Modeling Framework

- The reason of modeling demand but not demand/supply separately

  Smooth semi-martingale assumption holds for net demand curve

- The reason of modeling η process

  Make sure Q can be positive or negative

- The reason for Q to be twice differentiable

  Ensure that the larger trader is rational, and thus the net demand curve is continuous

- The reason for q to be positive

  Guarantee that the net demand is downward sloping (decreases with price).

- The reason for using Brownian sheet

  Rule out arbitrage opportunity

Simulation Process

Market Calibration

- Estimate excess demand $\bar{Q}(p,t)$ from p and t from the high frequency data;
- Calculate the $\eta(t)$ process by $\frac{\bar{Q}(0,t)}{\bar{Q}(0,t)-\bar{Q}(S,t)}$;
- Calculate process q and h. Estimate the variance-covariance matrix of h and η;
- Apply Cholesky decomposition to the correlation matrix, where $\bar{R}dt = Corr[\frac{dh_i}{h_i}, \frac{dh_j}{h_j}]$

(net demand).png (net demand).png

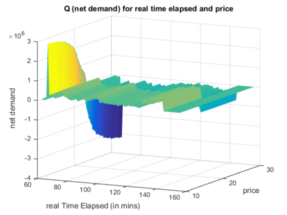

Figure: The net demand of GE as of 04/01/2011

(diff of Q).png (diff of Q).png

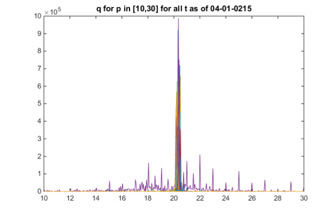

Figure: The q process of GE as of 04/01/2011

Simulation Process

Excess Demand Simulation

(The process of the simulation is to demonstrate that the calibrated parameters in framework are market consistent)

- Randomly generate n paths for h;
- Use h to integrate for q process;
- Simulate η process using the correlation estimated in market calibration process;
- Combine h, q and η to simulate excess demand Q.

(net demand).png (net demand).png

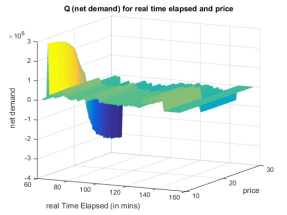

Figure: The net demand of GE as of 04/01/2011

(diff of Q).png (diff of Q).png

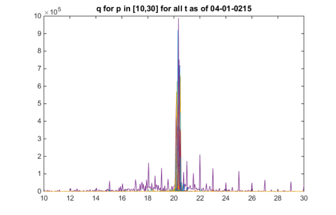

Figure: The q process of GE as of 04/01/2011

Simulation Result

GE stock net demand curve with 04/01/2015 market data calibrated:

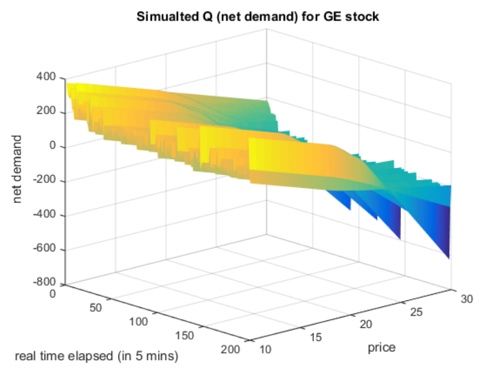

Figure: Q (net demand) for GE.

Conclusions

1. We developed a liquidity model with stands between
   - traditional no-arbitrage (option pricing) models and
   - financial economics models.
2. This model uses Ito-Wentzell's formula and Girsanov's theorem for Brownian sheets.
3. We give conditions for no-arbitrage, which allows us to price options
4. We specified a model
   - with positive demand density,
   - with mean-reversion,
   - where parameters are centered on the clearing price.
5. The model generates an implied volatility smile which matches the observed smile much better than the traditional Black-Scholes model.
6. We calibrate the model parameters from high frequency stock data of GE as of 04-01-2015. The simulated net demand curve shows a downward sloping shape, as expected.