Large Traders, Hidden Arbitrage and Complete Markets

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Abstract

This paper studies hidden arbitrage opportunities in markets where large traders affect the price process, and where the market is complete (in the classical sense). The arbitrage opportunities are "hidden" because they occur on a small set of times (typically of Lebesgue measure zero). These arbitrage opportunities occur naturally in markets where a large trader supports the price of some asset or commodity, for example corporate stock repurchase plans, government interest rate or foreign currency intervention, and price support by investment banks in IPOs. We also illustrate immediate arbitrage opportunities generated by usual market activity at specific points in time, for example the issuance date of an IPO or the inclusion date of a new stock in the S&P 500 index.

1 Introduction

For over 30 years, arbitrage opportunites and complete markets - their existence or lack thereof - has been a topic of intensive study by the financial community. This investigation lead to the famous Black-Scholes-Merton

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option pricing formula, which is claimed to have enabled the exponential growth of modern financial markets (see Jarrow [21]). The standard model for studying arbitrage opportunities and complete markets assumes a competitive setting, where all traders act as price takers - sometimes called a "small" trader economy. Also investigated, but with less intensity, are models where there exists a large trader. Typically, this literature postulates a stochastic price process, say for a stock, and the existence of a "large" trader whose trades affect the stock's price. The remaining traders are small, in the sense that their trades have no impact on the assumed price process. Given this structure, these papers investigate conditions on the price process for the non-existence of arbitrage opportunities (especially for the large trader), for market completeness, and given these conditions, they characterize the price of an option. Various different formulations have been studied (see Jarrow [19],[20], Cvitanić and Ma [6], Frey [15], Frey and Stremme [16], Platen and Schweizer [28], Cuocco and Cvitanić [3], Bank and Baum [1], Jonsson, Keppo, Ma [22]).

Our paper is another in the large trader literature. However, our paper takes a different perspective. Rather than studying conditions for the non-existence of arbitrage, we explore market situations where a large trader causes the existence of arbitrage opportunities for small traders in complete markets. The arbitrage opportunities we consider are "hidden" - almost not observable - to the small traders, or to scientists studying markets because they occur on time sets of Lebesgue measure zero. These situations can arise naturally due to company repurchase plans, government interest rate or foreign currency intervention, due to investment bank price support in the issuance of IPOs, or as the result of market manipulation trading strategies like short squeezes and bear runs by large traders. Although not central to our paper, we also illustrate immediate arbitrage opportunities that can be generated by usual market activity at specific time points; examples include the issuance date of IPOs and the inclusion date of stocks into the S&P 500 index.

An outline for this paper is as follows. Section 2 presents the model, and section 3 gives some technical results that are needed for the subsequent theorems. Section 4 gives a complete market with hidden arbitrage, while section 5 discusses bear runs. Section 6 analyzes trading strategies that take advantage of these arbitrage opportunities. Section 7 gives a complete market with immediate arbitrage, and section 8 concludes the paper.

2 The Model

We consider a simple model where $(S_t)_{t\geq 0}$ denotes a stochastic process modelling the price of a risky asset, and $(R_t)_{t\geq 0}$ denotes the value of a risk-free money market account. We assume a given filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{F})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, satisfies the "usual hypotheses" (cf Protter [27]), and S is a P semimartingale.

A trading strategy (a, b) is **self-financing** if a is predictable, b is optional, and

$$a_t S_t + b_t R_t = a_0 S_0 + b_0 R_0 + \int_0^t a_s dS_s + \int_0^t b_s dR_s, \tag{2.1}$$

all $0 \leq t \leq T$. We take $S_0 = 0$ and $R_t \equiv 1$ (thus the interest rate $r = 0$), so that $dR_t = 0$, and (2.1) becomes

$$a_t S_t + b_t = b_0 + \int_0^t a_s dS_s.$$

Note that if we are given a, we can take $b_t = b_0 - \int_0^t a_s dS_s - a_t S_t$, and (a, b) will be self-financing.

We call a contingent claim a r.v. $H \in \mathcal{F}_T$, and a contingent claim H is said to be Q-**redundant** if for a probability Q there exists a self-financing strategy (a, b) such that

$$V_t^Q = E_Q\{H|\mathcal{F}_t\} = b_0 + \int_0^t a_s dS_s, \qquad 0 \leq t \leq T.$$

Note that it is an implicit assumption that S is a Q-semimartingale.

Definition 2.1 *A market* $(S_t, R_t) = (S_t, 1)$ *is* Q-**complete** *if every* $H \in L^1(\mathcal{F}_T, dQ)$ *is* Q*-redundant.*

Classical asset pricing models begin with the reasonable assumption that there is no arbitrage. In this framework that assumption is equivalent to assuming the existence of an equivalent probability measure P^* such that S is a P^* – local martingale (in the case where S has continuous paths, at least has locally bounded paths). A standard example is the Black-Scholes model

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{2.2}$$

where B is a standard Brownian motion. Under P^* it is well-known that the market is complete.

The purpose of this paper is to provide naturally occurring or reasonable models where (2.2) may appear to be correct, but there is a "hidden" singular process present. This process is "hidden" because this singularity occurs on a time set of Lebesgue measure zero. This singular process will not affect the completeness of the model - the model appears normal - but there will nevertheless be arbitrage opportunities present.

We wish to point out that much simpler models of complete markets exist where arbitrage is present. For example, consider the economy with the price processes

$$S_t = e^{rt+\sigma B_t}, \text{ and } Y_t = e^{Rt+\sigma B_t}$$

with $R > r$. The difference below is that our hidden arbitrage examples more closely resemble the standard models used in the literature. Moreover, the arbitrage can arise (as we see in Section 4) by a simple and seemingly innocuous modification of the standard model.

3 Technical Results

Before presenting our key insights, we need to collect some technical results that are used in the proofs of the subsequent theorems.

Let S be a continuous semimartingale on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $S_0 = 0$ and decomposition $S_t = M_t + A_t$, with M a continuous local martingale, $M_0 = 0$, and A a continuous finite variation process with $A_0 = 0$. The beautiful theorem of Delbaen-Schachermayer (see [9] or Protter [26] for an expository treatment) states that there is "No Free Lunch with Vanishing Risk" (a technical condition related to the absence of arbitrage) if and only if there is a probability law P^* equivalent to P (that is, P and P^* have the same sets of probability zero) such that S is a local martingale (or even a σ martingale in the case where S is allowed to have jump discontinuities). Of course, versions of this theorem go back to Harrison and Kreps ([18], p. 392), although not in this form. (One can consult Darrell Duffie's book [12] for a historical perspective.) Girsanov's theorem and the Kunita-Watanabe inequality then imply that if such a P^* exists, one must have

$$dA_s << d[M,M]_s \text{ a.s.}$$

which in turn implies there must exist a predictable h such that

$$dA_s = h_s d[M, M]_s \tag{3.1}$$

(see, e.g., Protter [26] where this is explained in detail).

We do not assume *a priori* that (3.1) is satisfied. Thus we *do not assume* that there is no arbitrage.

Theorem 3.1 *Let $S_t = M_t + A_t$ be a continuous semimartingale. Then there exists h predictable, and L continuous and adapted, such that*

$$A_t = \int_0^t h_s d[M, M]_s + L_t \tag{3.2}$$

where $t \to L_t$ and $t \to [M, M]_t$ are singular measures on $\mathbb{R}_+$ a.s. Moreover the decomposition 3.2 is unique.

Proof. First let us assume A has non-decreasing paths, with $A_T \in L^1$. Then A generates a finite measure μ on $([0, T] \times \Omega, \mathcal{B}([0, T] \otimes \mathcal{F})$ such that $\mu([0]) = 0$ and $\mu([0, t]) < \infty$, and $\mu_A(X) = E\{\int_0^T X_s dA_s\}$, for any jointly measurable positive process X. By Lebesgue's decomposition theorem (cf, eg, [30], p. 149) we have

$$\mu_A = \mu_1 + \mu_2$$

where $\mu_1 << \mu_{[M,M]}$ and $\mu_2 \perp \mu_{[M,M]}$, where "$\perp$" denotes that the two measures are singular. Since $[M, M]$ is continuous and adapted it is of course predictable. Therefore if X is a positive, jointly measurable process and $\dot{X}$ is its predictable projection, we have $\mu_{[M,M]}(X) = \mu_{[M,M]}(\dot{X})$; since A is also predictable, $\mu_A(X) = \mu_A(\dot{X})$. Let X be positive and jointly measurable, and let Λ be a measurable subset of $\mathbb{R}_+ \times \Omega$ such that

$$E\left\{\int_0^\infty 1_{\Lambda^c}(s) d[M, M]_s\right\} = 0.$$

Then Λ can be taken to be a closed (in $\mathbb{R}_+$) optional set (see, eg, [11], p. 139). We have

$$\mu_1(X) = \mu_1(X 1_\Lambda), \ \mu_A(X 1_\Lambda) = \mu_1(X 1_\Lambda), \text{ and } \mu_A(X 1_\Lambda) = \mu_1(\dot{X} 1_\Lambda).$$

But

$$\mu_A(\dot{X} 1_\Lambda) = \mu_1(\dot{X} 1_\Lambda)$$

together with the above implies that

$$\mu_1(X) = \mu_1(\dot{X}).$$

Therefore μ_1 generates an increasing process C which is adapted such that $\mu_1(X) = E\{\int_0^T X_s dC_s\}$ (cf [11], p. 91). Since $\mu_1 << \mu_{[M,M]}$, we have

$$C_t = \int_0^t h_s d[M,M]_s,$$

and since both C and $[M,M]$ are optional we have h can be taken optional (cf [11], p. 109, T33). Since $[M,M]$ is continuous, one can further take h to be predictable. Thus $(\int_0^t h_s d[M,M]_s)_{t\geq 0}$ is continuous and adapted, with h predictable. The process L corresponds to the measure μ_2, hence since

$$L_t = A_t - \int_0^t h_s d[M,M]_s,$$

we have L is also continuous and adapted.

If A does not have non-decreasing paths, then μ_A is a **signed measure**, and it has a Hahn-Jordan decomposition $\mu_A = \mu_A^+ - \mu_A^-$. One now constructs two increasing processes A^+ and A^- from μ_A^+ and μ_A^- as is done, for example, in ([11], p.90, T41). One then proceeds as before with A^+ and A^-. Note that since A is continuous it follows that both A^+ and A^- can be taken to be continuous; and since A is adapted one can also take both A^+ and A^- adapted (use [11], p. 105, T26). Thus we are back in the original case.

Finally if $A \notin L^1$, we can let $T^n = \inf\{t > 0 : \int_0^t |dA_s| \geq n\}$, where $\int |dA_s|$ denotes the total variation process of A. Since $(\int_0^t |dA_s|)_{t\geq 0}$ is also continuous, we have $A_{t\wedge T_n}$ is of bounded total variation. We obtain the unique decomposition

$$A_{t\wedge T^n} = \int_0^t h_s^n d[M,M]_s + L_t^n$$

on $[0,T^n]$, each n, and this gives the unique decomposition on all of $\mathbb{R}_+ \times \Omega$. ■

By Theorem 3.1 we now know that we can write the semimartingale $S = (S_t)_{t\geq 0}$ in the form

$$S_t = M_t + \int_0^t h_s d[M,M]_s + L_t \tag{3.3}$$

where h is predictable and L is adapted, continuous, and $t \to L_t(\omega)$ is singular with respect to $t \to [M, M]_t(\omega)$.

Theorem 3.2 *Suppose there exists a unique P^* equivalent to P such that*

$$N_t = M_t + \int_0^t h_s d[M, M]_s \tag{3.4}$$

is a P^*-local martingale. Then the market $(S, 1)$ is P^*-complete.

Proof. Let $H \in L^1(\mathcal{F}_T, dP^*)$ represent a contingent claim. Since P^* is unique we have martingale representation (cf, eg, [27]):

$$V_t = E^*\{H|\mathcal{F}_t\} = E^*\{H\} + \int_0^t j_s dN_s$$

for some predictable process j. Let us modify j by taking

$$\tilde{j}_s = j_s 1_{\{supp(d[M,M])\}}(s).$$

Then we still have

$$V_t = E^*\{H|\mathcal{F}_t\} = E^*\{H\} + \int_0^t \tilde{j}_s dN_s$$

where $\tilde{j}$ is optional (since N is continuous $\tilde{j}$ can be optional without loss). Finally we note that $\int_0^t \tilde{j}_s dL_s = 0$, hence

$$V_t = E^*\{H|\mathcal{F}_t\} = E^*\{H\} + \int_0^t \tilde{j}_s dS_s,$$

since $S_t = N_t + L_t$,

$$= b_0 + \int_0^t \tilde{j}_s dS_s,$$

and taking $b_t = b_0 + \int_0^t \tilde{j}_s dS_s - \tilde{j}_t S_t$ we have $(\tilde{j}_t, b_t)$ is a self-financing strategy.
■

We wish to mention that Karatzas and Shreve ([25]) have already pointed out (p.327) that the presence of a singular process such as L leads to arbitrage.

It is perhaps worthwhile to note at this point that a simple application of well known criteria (see for example Protter [27]) give sufficient conditions for the existence of the measure P^* in Theorem (2.2) (but not the uniqueness).

Theorem 3.3 *Let $S = (S_t)_{t\geq 0}$ have a decomposition (3.3). Suppose further that $\int_0^t h_s^2 d[M,M]_s < \infty$, so that we can set $X_t = -\int_0^t h_s dM_s$. Suppose that one of*

(i) $\sup_\tau E\{\exp(\frac{1}{2} X_\tau)\} < \infty$, *where $\tau \leq T$ is a stopping time;*

(ii)$E\{\exp(\frac{1}{2} [X,X]_T)\} < \infty$

holds. Then there exists P^* equivalent to P such that $N_t = M_t + \int_0^t h_s d[M,M]_s$ is a P^*-local martingale $(0 \leq t \leq T)$.

Proof. Condition (i) is known as Kazamaki's criterion. It is sufficient to show that $\mathcal{E}(X)$ is a uniformly integrable martingale, where $\mathcal{E}(X)$ denotes the stochastic exponential of X. (See, e.g., [27]). Condition (i) moreover implies condition (ii) (see again [27]), which is known as Novikov's criterion, and thus it too ensures that $\mathcal{E}(X)$ is a uniformly integrable martingale. We set

$$dP^* = \mathcal{E}(X)_T dP,$$

and let $Z_t = \mathcal{E}(X)_t$, so that Z satisfies the equation

$$Z_t = 1 + \int_0^t Z_s dX_s.$$

By Girsanov's theorem (cf. eg. [27]) we have

$$M_t - \int_0^t \frac{1}{Z_s}\, d[Z,M]_s$$

is a P^*-local martingale. However

$$\begin{aligned} \int_0^t \frac{1}{Z_s}\, d[Z,M]_s &= \int_0^t \frac{1}{Z_s}\, Z_s d[X,M]_s \\ &= -\int_0^t \frac{1}{Z_s}\, Z_s h_s d[M.M]_s \\ &= -\int_0^t h_s d[M,M]_s, \end{aligned}$$

hence we conclude $M_t + \int_0^t h_s d[M,M]_s = N_t$ is a P^*-local martingale. ■

The preceding implies

Theorem 3.4 *Suppose $S = (S_t)_{t\geq 0}$ is given in (3.3) with L non-trivial, and N given in (3.4). Suppose further there exists a unique P^* equivalent to P such that N is a P^*-local martingale. Then the market $(S, 1)$ is P^*-complete and there is arbitrage.*

Proof. We know from Theorem 3.2 that the market is P^*-complete. Let

$$a_s = 1_{\{supp(d[M,M])\}^c}(s).$$

and let $H = L_T \in \mathcal{F}_T$. Assume without loss of generality that $L_T \in L^1(dP^*)$. (If not, we can use stopping times to ensure that $L_{T\wedge\tau}$, where τ is a stopping time, is in $b\mathcal{F}_T$.) Then by Theorem 3.2 there exists a self-financing strategy $(\tilde{j}, b)$ such that

$$H = L_T = E^*\{L_T\} + \int_0^T \tilde{j}_s dS_s.$$

However we also have

$$L_T = 0 + \int_0^T a_s dL_s.$$

Moreover we have $\int_0^t a_s dN_s = 0$, $0 \leq t \leq T$, by construction of the process a. Hence

$$H = L_T = 0 + \int_0^T a_s dS_s,$$

which is an arbitrage opportunity. ∎

4 A Complete Market with Hidden Arbitrage

We begin with an elementary model to illustrate the concept. Let

$$dX_t = \sigma dB_t + b(t, X_t)dt, \tag{4.1}$$

where σ is a constant. Let us assume that b is bounded; then we can find P^* such that X is a P^*-Brownian motion.

Suppose that X represents the stock price of a firm that was recently spun off from a large company, and that this large company offers a share purchase or sell offer at a fixed price c for a limited time. This is a situation of price impact by a large trader - the large company.

If the price crosses the level c, one expects a fraction α of the investors to buy or sell the stock accordingly. We assume α is the same fraction in each case (buy or sell). Let $\{0 = t < t_1 < \cdots < t_n = T\}$ be a partition of the time interval $[0, T]$, and suppose the stock is observed at these times.

If $X_{t_i} < c$ and $X_{t_{i+1}} > c$, then a sale of proportion α occurs with an assumed price impact of $\alpha(X_{t_{i+1}} - c) = \alpha|X_{t_{i+1}} - c|$. If $X_{t_i} > c$ and $X_{t_{i+1}} < c$, then a purchase of proportion α occurs with a price impact of $\alpha(c - X_{t_{i+1}}) = \alpha|X_{t_{i+1}} - c|$.

The cumulative effect on the stock price X by such a repurchase plan will be

$$\sum_{i=1}^{n-1} \alpha|X_{t_{i+1}} - c| 1_{\{sign(X_{t_{i+1}} - c) \neq sign(X_{t_i} - c)\}}.$$

Let αL_t^c denote the limit:

$$\begin{aligned} \alpha L_t^c &= \lim_{\|\pi^n\| \to 0} \alpha \sum_{t_i \in \pi^n} |X_{t_{i+1}} - c| \\ &\qquad 1_{\{sign(X_{t_{i+1}} - c) \neq sign(X_{t_i} - c)\}}. \end{aligned} \tag{4.2}$$

The impact of this repurchase plan yields a "new" price process:

$$dY_t = \sigma B_t + b(t, X_t)dt + d\alpha L_t^c. \tag{4.3}$$

As shown in the previous section, this admits hidden arbitrage. This example will be complete if we can show that the limit exists.

Theorem 4.1 *The process αL_t^c exists as a limit in u.c.p (uniform convergence on compact time sets, in probability). Moreover, $t \to L_t^c$ is singular with respect to dt, a.s., and is the local time of X at the level c.*

Proof. Since b is assumed bounded, by Girsanov's theorem there exists P^* equivalent to P such that X is a P^* continuous local martingale; hence X is Brownian motion by Lévy's theorem. Let us take $\sigma = 1$, so that X is standard Brownian motion. Then by [17], page 160, we have the limit (4.3) exists in ucp, and L^c is the local time of X at level c. ■

Theorem 4.2 *Let X and Y be given by (4.1) and (4.3), where $B_0 = 0$. Then the natural filtrations of X and Y are the same. That is,*

$$\sigma(X_s; s \leq t) = \sigma(Y_s; s \leq t),$$

completed under P.

Proof. We first note that changing to the equivalent measure P^*, it suffices to establish that

$$U_t = B_t + \alpha L_t^c; \qquad t \geq 0$$

and $(B_t)_{t\geq 0}$ have the same filtrations, where $B_0 = 0$. This fact is not at all obvious, and was first proved in [8] for $|\alpha| \geq 16$ and then proved to be true for all $\alpha \in \mathbb{R}$ in [14], in the case where $c = 0$. We can easily deduce the case for general c from the case $c = 0$. Indeed, let

$$\begin{aligned} T &= \inf\{t > 0 : U_t = c\} \\ &= \inf\{t > 0 : B_t = c\}. \end{aligned}$$

We have

$$\begin{aligned} U_{T+t} - c &= B_{T+t} + \alpha L_{T+t}^c - c \\ &= \beta_t + \alpha L_t^0 \end{aligned}$$

where β_t is $B_{T+t} - c$, a Brownian motion starting at 0. Thus if $V_t = U_{T+t} - c$, we have

$$\sigma(V_s; s \geq 0) = \sigma(\beta_s; s \geq 0)$$

by the result in [14]. The result now follows. ■

Theorem 4.3 *Let Y given by*

$$dY_t = \sigma B_t + b(t, X_t)dt + d\alpha L_t^c. \tag{4.4}$$

be a stock price. Then Y gives rise to a complete market with hidden arbitrage.

Proof. Recall that we take the interest rate $r = 0$ so that $R_t = 1$. We can find P^* via Girsanov's theorem such that Y is a P^* - local martingale, hence a Brownian motion by Lévy's theorem. Next note that the filtrations in question are equal: that is, $\sigma(Y_s; s \leq t) = \sigma(X_s; s \leq t)$, where each is assumed completed with the P^*-null sets of $\mathcal{F}$ by Theorem 4.2, hence we have martingale representation for X even using the filtration of Y. We conclude by applying Theorem 3.4. ■

Of course, the arithmetic Brownian motion model of expression (4.3) does not respect limited liability of the stock. However, it can easily be generalized to the standard Black-Scholes paradigm:

$$\frac{dX_t}{X_t} = \sigma dB_t + \mu dt;\ X_0 = 1, \tag{4.5}$$

where σ and μ are constants. Then of course $X_t = e^{\sigma B_t - \frac{1}{2}(\sigma^2 - 2\mu)t}$, known as geometric Brownian motion, and the filtration of X equals that of the Brownian motion B. The geometric Brownian motion X inherits the market completeness of B: we change to P^* such that X is a local martingale, and then if H is a contingent claim in $L^2(\mathcal{F}_T, dP^*)$ we have

$$E^*\{H|\mathcal{F}_t\} = E^*\{H\} + \int_0^t h_s d\beta_s$$

where $\beta_t = B_t + \frac{\mu}{\sigma}t$ is a P^* Brownian motion. Now X satisfies

$$dX_t = \sigma X_t d\beta_t$$

and since $X > 0$ a.s. (all t), we have

$$E^*\{H|\mathcal{F}_t\} = E^*\{H\} + \int_0^t h_s \frac{1}{\sigma X_s}\, dX_s,$$

and we can conclude that the market is complete for contingent claims in $L^2(dP^*)$.

We now use a slightly different procedure to derive our model with hidden arbitrage. This procedure is reminiscent of the derivation used by Carr and Jarrow to resolve the Stop-Loss Start-Gain paradox (see [4]). Again, suppose there is a share purchase or sell offer at a given price c for a limited time. Suppose that a group of people acting in concert or a large trader watches the stock price X of (4.5) continuously, and when it crosses the threshold c to reach the price $c + \epsilon$, a fraction α buys the stock at price \$c with a resultant price impact of $\alpha(c + \epsilon - c) = \alpha\epsilon$. If the price decreases across c to $c - \epsilon$, then a fraction α sells the stock, again with a price impact of $\alpha\epsilon$. Thus if U_t^ϵ

represents the number of upcrossings of X from c to $c+\epsilon$ before time t, and D_t^ϵ represents the number of downcrossings from c to $c-\epsilon$, we have

$$\lim_{\epsilon \downarrow 0} \alpha\epsilon(U_t^\epsilon + D_t^\epsilon) = \alpha L_t^0(X),$$

as is shown in El Karoui [13], Theorem 3, page 69. Thus, the new stock price becomes

$$dY_t = \sigma X_t dB_t + \mu X_t dt + \alpha L_t^c(X) \tag{4.6}$$

where X solves (4.5). The previous theory gives us that $(Y, 1)$ is a P^* complete market. The only issue is a question of filtrations. We have that $(Y, 1)$ is P^* complete under $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, the filtration of X. It would be preferable to show it is also complete for the filtration of Y. In our first example this result was trivial since the two filtrations were equal. It turns out they are here as well, again thanks to a theorem of M. Emery and E. Perkins. Indeed, since X is a P^*-continuous local martingale, we need only to apply the Theorem on page 386 of [14] and the argument of Theorem 4.2.

Although the previous example concentrated on a share repurchase plan of a large company, a similar analysis applies whenever there is artificial price support by a large trader. For example, the same analysis applies to **government intervention in interest rate markets**. Here, the government acts as the large trader. In this situation, let $X = (X_t)_{t\geq 0}$ denote a stochastic process giving the price of the 30 year benchmark Treasury bonds (let's pretend they are still being issued). Suppose, for example, that the government wants interest rates to remain low and pursues a fiscal policy of stabilization. Rates depend on bonds: higher prices mean lower rates. The government may support the price by buying bonds when they fall below a threshold price, call it c. The preceding argument, using only one side of it, leads to a change in price from X_t to $X_t + \frac{\alpha}{2} L_t^c(X)$.

A third example occurs with the government acting as a large trader with respect to **foreign exchange intervention**. Le Baron [23] has a cogent empirical study of this showing such intervention leads to "predictive value."

A fourth example concerns **IPO's (Initial Public Offerings)**. These are often supported in price (that is, given a floor) by a sponsoring investment bank or consortium of investment banks, thus acting in concert as a large trader. This has been detailed by Cornelli and Goldreich [5].[1]

5 Bear Runs

As a final example of a large trader market with hidden arbitrage opportunities, we consider "bear runs." A "Bear Run" refers to a competition between a large trader who is pushing up the price of an asset (stock) by buying it, usually on margin, and another large trader who is pushing the price down by shorting the stock, again usually on margin. As the price varies, strain will be put on the resources of the two large traders, and the one who is forced to give up first will lose. That is, it is a **game of wealth**. One player can win by exhausting the financial resources of his opponent, or – more typically – he can bring in a third party, such as a government regulating bureau. A History of the New York Stock Exchange, published in 1905 (see [29]), gives true anecdotes of some of these "market battles" in its Chapter XIII.

A simple model would be for the long player to buy whenever the stock falls below a level c, and for the short player to sell whenever the stock rises above a level d. Let τ and ν represent the random (stopping) times when the long and short players abandon, respectively. Suppose the stock price without the Bear Run activity follows a Black-Scholes model:

$$dX_t = \sigma X_t dB_t + \mu X_t dt.$$

The model changes by the Bear Run activity to:

$$dY_t = \sigma X_t dB_t + \mu X_t dt + \alpha L^c_{t\wedge\tau}(X) - \beta L^d_{t\wedge\tau}$$

and, for example, if $\tau < \nu$, then the short seller will "win" the game, and the long buyer will have a loss of

$$\alpha L^c_{t\wedge\tau}(X) + \beta L^d_{t\wedge\nu}(X).$$

A more sophisticated strategy might be for the two players to buy and sell at different levels, even randomly chosen ones. In this case let $c(t)$ denote the curve such that the buyer adds to his holdings if the asset price falls below

[1] Another example, in commodities prices, could be price supports for various agricultural commodities by the U. S. government.

$c(t)$ at time t. The curve $c(t)$ can be randomly chosen; for simplicity let us assume it has continuous paths of finite variation on compacts. Analogously, the seller has an activity curve, also possibly random, $d(t)$.

If X is the Black-Scholes model, then $X_t - c(t)$ and $X_t - d(t)$ are both continuous semimartingales with local times. Using the limit theorems established in [13], the new price model then becomes

$$dY_t = \sigma X_t dB_t + \mu X_t dr + \alpha L^0_{t\wedge\tau}(X - c(\cdot)) - \beta L^0_{t\wedge\nu}(X - d(\cdot)).$$

We can extend this a bit to curves $c(\cdot)$ and $d(\cdot)$ that behave like Brownian curves, instead of only curves with paths of finite variation. We would then use the limit theorems established in [17].

6 Arbitrage Strategies with These Local Time Models

The fact that price support by a large trader generates arbitrage opportunities for small traders was proven in the previous sections. Here, we understand the trading strategies that exploit these opportunities. It is interesting to note that the trading strategy to take advantage of these arbitrage opportunities differ according to the duration of the price support.

Case 1: "Permanent Intervention"

We work on a time interval $[0, T]$, where T is a fixed, non random time horizon. In this case the arbitrage strategy is to **trade with the large trader**. Let X denote the usual Black-Scholes model when there is no large trader arbitrage inducing activity:

$$dX_t = \sigma X_t dB_t + \mu X_t dt \tag{6.1}$$

and let Y denote the corresponding new asset price incorporating the large trader activity:

$$dY_t = \sigma X_t dB_t + \mu X_t dt + \alpha L^c_t(X) \tag{6.2}$$

It is easy to produce strategies that give arbitrage profits by constructing trades with the large trader. For example, the trading strategy could be:

$$h_s = 1_{supp(dL_s)}(s)$$

which, when followed, produces a value V_T at time T of

$$\begin{aligned} V_T &= 0 + \int_0^T h_s dY_s \\ &= \int_0^T h_s dX_s + \int_0^T h_s \frac{\alpha}{2} dL_s \\ &= \int_0^T h_s \frac{\alpha}{2} dL_s \\ & \quad \frac{\alpha}{2} L_T. \end{aligned} \tag{6.3}$$

Of course, any positive strategy on the support of dL will produce positive arbitrage profits in this case.

Case 2: "Transient Intervention"

In this case we assume the large trader intervention lasts for a random amount of time. The end time, denoted by a stopping time τ, is considered unknown by the investor. Moreover $\tau \leq T$. In this case the strategy given for Case 1 *does not work* in the sense that it does not produce arbitrage:

$$\begin{aligned} V_T &= 0 + \int_0^T h_s dY_s \\ &= \int_0^T h_s dX_s + \int_0^\tau h_s \frac{\alpha}{2} dL_s - h_\tau \frac{\alpha}{2} L_\tau \\ &= \frac{\alpha}{2} L_\tau - \frac{\alpha}{2} L_\tau \\ &= 0. \end{aligned} \tag{6.4}$$

Instead we employ a strategy going against the large trader. Denote our strategy by j_s. Suppose $j = -h$. Then we get

$$\int_0^\tau (-j_s) d\frac{\alpha}{2} L_s - (-j_\tau) \frac{\alpha}{2} L\tau = \int_0^\tau (-j_s) d\frac{\alpha}{2} L_s + j_\tau \frac{\alpha}{2} L\tau.$$

Thus if $j_s = 1_{supp(dL_s)}(s)$, we just get 0. However if we take $j_s = \frac{\alpha}{2} L_s 1_{supp(dL_s)}(s)$, then we get

$$
\begin{aligned}
V_T &= 0 + \int_0^T j_s dY_s \\
&= \int_0^T j_s dX_s + \int_0^\tau \frac{\alpha}{2} L_s \frac{\alpha}{2} dL_s + \frac{\alpha}{2} L_\tau \frac{\alpha}{2} L_\tau \\
&= -\frac{\alpha^2}{4} \frac{L_\tau^2}{2} + \frac{\alpha^2}{4} L_\tau^2 \\
&= \frac{\alpha^2}{4} L_\tau^2 \\
&> 0. \qquad (6.5)
\end{aligned}
$$

Note that if H is nonnegative off the support of dL, and also increasing, then we get a loss, and certainly not an arbitrage profit. But if H is decreasing and nonnegative off the support of dL, then we still do get arbitrage profits.

This is the type of trading strategy that George Soros allegedly used when trading foreign currencies in the 1992 against a managed fixed exchange rate by the Bank of England. When the Bank of England finally gave up its support of British sterling, Soros made massive profits. (See, e.g., [24].)

7 A Complete Market with Immediate Arbitrage

The preceding models study hidden arbitrage opportunities generated by a large trader acting to support the price of some asset. These arbitrage opportunities involve what might be called "traditional arbitrage": one has a strategy to follow, which leads, at time T, to a nonnegative gain with positive probability of a positive gain, while one has started with nothing. We now study what we call "immediate" arbitrage opportunities.

In the preceding, we made the key assumption that if

$$S_t = M_t + \int_0^t h_s d[M, M,]_s + L_t,$$

then there existed a unique probability measure P^* equivalent to P such that

$$N_t = M_t + \int_0^t h_s d[M, M]_s$$

is a P^* local martingale. However if

$$\int_0^\epsilon H_s^2 d[M,M]_s = \infty \text{ a.s.},$$

then no such P^* exists! (See [10].) In this case one has **immediate arbitrage**.

Definition 7.1 *A semimartingale price process S admits* **immediate arbitrage** *at a stopping time τ with $P(\tau < \infty) > 0$ if there is an S-integrable strategy H such that $H_t = H_t 1_{\{t>\tau\}}$ and $\int_0^t H_s dS_s > 0$ for all $t > \tau$.*

An example of a price process S that admits immediate arbitrage is the process

$$S_t = B_t + \sqrt{t} \tag{7.1}$$

which satisfies the (trivial) stochastic differential equation

$$dS_t = dB_t + \frac{1}{2\sqrt{t}}\, dt. \tag{7.2}$$

Following [10] (p.935), one takes

$$H_t = \frac{1}{\sqrt{t}(\ln t)^2}, \quad t > 0;$$

Let

$$\begin{aligned}\tau &= \inf\{t > 0 : \int_0^t H_s dS_s = 0\} \\ \tau_n &= \tau \wedge \frac{1}{n} \\ J_t &= \sum_{n=1}^\infty \alpha_n H_t 1_{\{t \le \tau_n\}}.\end{aligned}$$

where α_n is a sequence converging to 0 at an appropriately fast speed. Then one obtains immediate arbitrage at $t = 0$.

We remark that this example does not involve a singular process (as in Sections 2 and 3), and moreover even has the form $B_t + \int_0^t h_s d[B,B]_s$ where $h_s = \frac{1}{2\sqrt{s}}$, and where $\int_0^\epsilon h_s^2 d[B,B]_s = \int_0^\epsilon \frac{1}{4s} ds = \infty$ a.s.

In Section 1 we defined a market to be Q-complete if every claim in $L^1(\mathcal{F}_T, dQ)$ were redundant. We wish to relax this definition a bit.

Definition 7.2 *A market* $(S_t, R_t) = (S_t, 1)$ *is* $(\mathbf{Q}, \mathcal{H})$**-complete** *if every random variable* H *in a class* $\mathcal{H}$ *of random variables is redundant.*

We note that if $\mathcal{H} = L^1(\mathcal{F}_T, dQ)$, then Definitions 2.1 and 7.2 agree.

Theorem 7.3 *Let* S *be as given in (7.1), and define*

$$\mathcal{H} = \{H \in \mathcal{F}_1 : \lim_{\epsilon \downarrow 0} \ E\{He^{-\frac{1}{2}\int_\epsilon^1 \frac{1}{\sqrt{s}} dB_s - \frac{1}{8}\int_\epsilon^1 \frac{1}{s} ds}\} \text{ exists}\}.$$

Then $(S_t, 1)$ *is* $(P, \mathcal{H})$*-complete.*

Proof. Let $H \in \mathcal{H}$ and $\epsilon > 0$. Then H has a unique decomposition

$$H = \Lambda_\epsilon + \int_\epsilon^1 h_s^\epsilon dS_s \tag{7.3}$$

where $\Lambda_\epsilon \in \mathcal{F}_\epsilon$, and h^ϵ is predictable. Indeed, let $S_t^\epsilon = S_t - S_{t\wedge\epsilon}$. Then we let $dQ^\epsilon = Z^\epsilon dP$, where

$$Z^\epsilon = \exp\left(-\frac{1}{2}\int_\epsilon^1 \frac{1}{\sqrt{u}}\, dB_u - \frac{1}{8}\int_\epsilon^1 \frac{1}{u}\, du\right).$$

By Girsanov and Lévy's theorems S^ϵ is a Q^ϵ-Brownian motion on $[\epsilon, 1]$. Let E^ϵ denote expectation under the probability measure Q^ϵ. We have

$$H = E^\epsilon\{H|\mathcal{F}_\epsilon\} + (H - E\{H|\mathcal{F}_\epsilon\}),$$

and using martingale representation under Q^ϵ, we get

$$\begin{aligned} H &= \Lambda_\epsilon + \int_\epsilon^1 h_s^\epsilon dS_s^\epsilon \\ &= \Lambda_\epsilon + \int_\epsilon^1 h_s^\epsilon dS_s, \end{aligned}$$

and we have 7.3. The uniqueness of h_s^ϵ comes from the following observation: For a Brownian motion β, if

$$\int_a^b h_s d\beta_s = \int_a^b j_s d\beta_s,$$

then $h = j\ dsdP$ - a.e. on $[a,b]$. We can see this by computing the quadratic variation of the local martingale

$$\int h_s d\beta_s - \int j_s d\beta_s$$

on $[a,b]$ to get:

$$\left[\int hd\beta - \int jd\beta, \int hd\beta - \int jd\beta\right]_a^b = \int_a^b (h_s - j_s)^2 ds = 0 \quad a.s.$$

(Here $[M,M]_a^b$ denotes $[M,M]_b - [M,M]_a$.)

Next we let $\gamma < \epsilon$. Then using 7.3:

$$\begin{aligned} H &= \Lambda_\gamma + \int_\gamma^1 h_s^\gamma dS_s \\ &= \Lambda_\gamma + \int_\gamma^\epsilon h_s^\gamma dS_s + \int_\epsilon^1 h_s^\gamma dS_s. \end{aligned}$$

Since $\Lambda_\gamma + \int_\gamma^\epsilon h_s^\gamma dS_s \in \mathcal{F}_\epsilon$, by the uniqueness of 7.3 we conclude $h^\gamma = h^\epsilon$ on $[\epsilon, 1]$, $dsdP$-a.e. Let us now write

$$H = E_\epsilon\{H\} + \Lambda_\epsilon + \int_\epsilon^1 h_s^\epsilon dS_s, \tag{7.4}$$

where we have replaced the original Λ_ϵ of 7.3 with $\Lambda_\epsilon - E\{\Lambda_\epsilon\}$. Let $\alpha_n = E_{\frac{1}{n}}\{H\}$, and 7.4 becomes

$$H = \alpha_n + \Lambda_{\frac{1}{n}} + \int_0^1 h_s 1_{[\frac{1}{n},1]}^{(s)} dS_s; \tag{7.5}$$

we have defined h by taking h equal to $h_{\frac{1}{n}}$ on $[\frac{1}{n}, 1]$ for each n as n tends to ∞. Then since $H \in \mathcal{H}$ we have

$$H = \lim_{n\to\infty} \alpha_n + \int_0^1 h_s dS_s,$$

where $\lim_{n\to\infty} \alpha_n = \alpha$ exists and is finite. Thus $H = \alpha + \int_0^1 h_s dS_s$, and we have that the market $(S_t, 1)$ is $(P, \mathcal{H})$-complete. ■

Theorem 7.4 *The complete market $(S_t, 1)$ of Theorem 7.3 has immediate arbitrage.*

Proof. We have already seen that $H_t = \int_0^t \frac{1}{\sqrt{s}(\ln s)^2}\, dS_s$ leads to immediate arbitrage. Thus it suffices to show that $H \in \mathcal{H}$. Let $h_s = \frac{1}{\sqrt{s}(\ln s)^2}$, and let

$$Z_\epsilon = e^{(-\frac{1}{2}\int_\epsilon^t \frac{1}{\sqrt{s}(\ln s)^2} dB_s - \frac{1}{8}\int_\epsilon^t \frac{1}{s(\ln s)^4}\, ds)}.$$

Then

$$\begin{aligned}
\lim_{\epsilon\to 0} E\{HZ_\epsilon\} &= lim_{\epsilon\to 0}\ E\{H(1 - \int_\epsilon^t Z_s^2 \frac{1}{2\sqrt{s}}\, dB_s)\} \\
&= E\{H\} - \lim_{\epsilon\to 0}\ E\{H \int_\epsilon^t Z_s^\epsilon \frac{1}{2\sqrt{s}}\, dB_s\}.
\end{aligned}$$

However

$$H_t = \int_0^t \frac{1}{\sqrt{s}(\ln s)^2}\, dB_s + \int_0^t \frac{1}{2s(\ln s)^2}\, ds,$$

hence

$$\begin{aligned}
E\{H_t \int_\epsilon^t Z_s^\epsilon \frac{1}{2\sqrt{s}}\, dB_s\} &= E\{\int_\epsilon^t \frac{1}{2s(\ln s)^2}\, Z_s^\epsilon ds\} + 0 \\
&= \int_\epsilon^t \frac{1}{2s(\ln s)^2}\, E\{Z_s^\epsilon\} ds \\
&= \int_\epsilon^t \frac{1}{2s(\ln s)^2}\, ds,
\end{aligned}$$

thus

$$\begin{aligned}
\lim_{\epsilon\to 0}\ E\{H_t \int_\epsilon^t Z_s^\epsilon \frac{1}{2\sqrt{s}}\, dB_s\} &= \lim_{\epsilon\to 0} \int_\epsilon^t \frac{1}{2s(\ln s)^2}\, ds \\
&= \lim_{\epsilon\to 0} \frac{1}{2}\, \frac{1}{\ln s}\Big|_\epsilon^t \\
&= \frac{1}{2}(\ln t)^{-1};
\end{aligned}$$

thus $H_t \in \mathcal{H}$ and we are done. ∎

An example of naturally occurring immediate arbitrage is the **underpricing of IPO's** (Initial Public Offerings.) The underpricing leads to a vertical slope for the drift at time $t = 0$, the offering time. This kind of arbitrage has been investigated by the U.S. Securities and Exchange Commission, as reported in the Wall Street Journal. (See [2] for a discussion of "fad" buying of IPO's.). Another example of immediate arbitrage would be on the date of inclusion of a stock in a widely followed stock index like the S&P 500. On the day of the stock's inclusion into the index, all mutual funds that are indexed to the portfolio buy the stock. This would generate an immediate arbitrage opportunity of the type discussed above (see Chen, Noronha and Singal [7] for documentation of abnormal returns of this type). The preceding model of $dS_t = dB_t + \frac{1}{2\sqrt{t}}dt$ is an example of how these types of immediate arbitrages can be modelled.

8 Conclusion

This paper studies naturally occurring examples of markets with large traders where the markets are complete, but arbitrage opportunities exist. These arbitrage opportunities can be "hidden" or "immediate."

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