

# A String Model of Liquidity in Financial Markets.

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## Abstract

We consider a dynamic market model where buyers and sellers submit limit orders. If at a given moment in time, the buyer is unable to complete his entire order due to the shortage of sell orders at the required limit price, the unmatched part of the order is recorded in the order book. Subsequently these buy unmatched orders may be matched with new incoming sell orders. The resulting demand curve constitutes the sole input to our model. The clearing price is then mechanically calculated using the market clearing condition. It depends on the position of the large trader. We assume a continuous model for the demand curve. We show that generically there exists an equivalent martingale measure for the clearing price, for all possible strategies of the large trader, if the driving noise is a Brownian sheet, while there may not be if the driving noise is multidimensional Brownian motion.

Another contribution of this paper is to prove that, if there exists such an equivalent martingale measure, then, under mild conditions, there is no arbitrage. We use the Ito-Wentzell formula to obtain both results. We also characterize the dynamics of the demand curve and of the clearing price in the equivalent martingale measure. We find that the volatility of the clearing price is inversely proportional to the sum of buy and sell order flow density (evaluated at the clearing price), which confirms the intuition that volatility is inversely proportional to volume. We also demonstrate that our approach is implementable. We use real order book data and simulate option prices under a particularly simple parameterization of our model.

The no-arbitrage conditions we obtain are applicable to a wide class of models, in the same way that the Heath-Jarrow-Morton conditions apply to a wide class of interest rate models.

## 1 Introduction

Most liquidity models in mathematical finance abstract the trading mechanism from the characterization of prices in the resulting market. Our viewpoint is here fundamentally different. In our model, equilibrium prices of assets are completely determined by the order flow, which is viewed as an exogenous process. We model a market of assets without specialist where every trader submits limit orders, that is, for a buy order, the buyer specifies the maximum price, or buy limit price, that he/she is willing to pay, and, for a sell order, the seller specifies the minimum price, or sell limit price, at which he/she is willing to sell <sup>1</sup>.

If at a given moment in time, the buyer is unable to complete his entire order due to the shortage of sell orders at the required limit price, the unmatched part of the order is recorded in the order book. A symmetric outcome exists in the case of incoming sell orders. Subsequently these buy unmatched orders may be matched with new incoming sell orders. We note that many electronic exchanges, such as NYSE ARCA, operate like this. Time-priority is used to break indeterminacies of a match between an incoming buyer at limit price superior to the ask price, i.e., the lowest limit price in the sell order book. As a result, the equilibrium of *clearing price* process is always defined.

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<sup>1</sup>There is no loss of generality in that statement. A buy market order can be specified in our model as a buy limit order with limit price equal to infinity. Since we model assets with only positive prices, a sell market order can be specified in our model as a sell limit order with a limit price equal to zero.

Since the matching mechanism does not add any information to the economy, all information about asset prices is included in the order flow. Whether public exchanges should or should not reveal in real-time the data contained in the order book is an important issue, which continues to preoccupy the financial markets community [29]. Our theoretical framework accommodates either viewpoint, but our empirical application is better tailored to the viewpoint that order books are public information, but that the large trader position is not known. The current blossoming of high-frequency trading activity [25, 4, 18, 3] seems to confirm our viewpoint that traders are (i) interested in understanding order book information, and (ii) trade on that information.

We do not address in this paper the issue of differential information further. The market microstructure literature (such as [1], and all following models) considers various models of trading involving uninformed traders, called noise traders, and one, or several informed traders. One of the key results of the Kyle model is that, given the information available to noise traders, the resulting price process is a martingale in the appropriate measure, whereas it may not be for the informed traders. As a consequence we do not believe that abstracting issues of differential information is limiting. The order books reflect all the public information. Public information corresponds to the filtration under which the clearing price needs to have an equivalent martingale measure, in order to avoid arbitrage. Obviously, the clearing price may not be a martingale in the aforementioned measure if the filtration is enlarged to include private information.

There are roughly two different class of models in the liquidity literature. The first class of models ([26, 27, 12, 10, 28, 23, 19, 9, 20, 21]) considers the action of a large trader who can manipulate the prices in the market. There are mainly two different types of strategies a large trader can employ to that effect. The first one is to corner the market, and then squeeze the shorts. The second one is to "front-run one's own trades". While some exchanges have rules to curtail the cornering of the market, front-running seems more difficult to ban from an exchange. It is known in discrete-time trading, that, if there is no possibility of arbitrage in periods where the large investor does not trade, then there is no market manipulation strategy.

The second class of models ([7, 6, 8, 31]) abstracts the issues of market manipulation away, and considers all traders as price-takers. In particular, [7] introduced an exogenous residual supply curve against which an investor trades. The investor trades market orders, and his/her order is matched instantaneously. As a consequence of the instantaneity, it is plausible for [7] to assume the "price effect of an order is limited to the very moment when the order is placed in the market" (dixit [19]), so that that the residual supply curve at a future time is statistically independent from the order just matched. For us however, this assumption is not convenient, since it does not explain how prices can incorporate information from the arrival of new orders. The paper by Roch [2] bridges a gap between these two classes of models, and allows for a linear impact of the large trader quantity on the demand price.

Our model belongs to the first class of models. An important difference is that we define our demand curve as quantity as a function of price, unlike most other authors, who define it as price as a function of quantity. Indeed, for our technical apparatus and our simulations to work, we found it necessary to have either compact support for the quantities or compact support for the price. Since the domain of variation of prices is better understood empirically, we decided to have compact support for the prices, which compelled us to model demand as a function of price.

We proceed in two steps to define our model. In the first step, we consider a market with atomistic traders and a large trader, and develop conditions on the net demand curve of the atomistic traders such that the large trader cannot manipulate the market. Since the large trader cannot manipulate the market, he will refrain from trading large orders, which would generate liquidity costs. Thus, the large trader is reduced to trading like an atomistic trader. Thus, in the second step, we assume that all traders are atomistic and trade in such a way that no market manipulation is possible. We then use our no-arbitrage model to price options. We point out a difference between our model and Bank and Baum (hereafter BB04) model: we do not need to consider all possible strategies of the large trader to calculate an option price, since these strategies are implicit in the model. This is an expedient feature of our model: we do not need to identify whether there is a large trader, or several large traders on the market for the option pricing formula to be plausible. We leave for future work the rigorous development a second fundamental theorem of asset pricing,

In our model all the information is contained in a Brownian sheet, which drives the dynamics of the

market net demand curve. Such a string model has already been introduced in finance to model the yield curve. Santa Clara and Sornette [30] argue that a "discontinuous [with respect to term] forward curve [...] is intuitively unlikely". We also believe that in a first step of modelling one should model demand curves as continuous in price. More importantly, a string model allows us to use the Ito-Wentzell formula, unlike, as for as our limited experience shows, a model with a large number of Brownian motions. We remind the reader that a string model allows to model any correlation structure in the net demand curve.

Our contributions are two-fold. First, we show that under a set of assumptions the large trader cannot generate arbitrage. Second, we develop a specific model that satisfies all the assumptions, and demonstrate that it is implementable. As in the early days of the Heath-Jarrow-Morton methodology, we use only historical estimation (in our case, of the order book) to fit our model and solve for the market price of risk. We expect that, should this paper meet with interest around practitioners, market-implied implementations will see the day. Unsurprisingly, we obtain a smile curve for implied volatility. We note that this particular feature is not a very strong sign of the adequacy of our approach to model asset prices, as most models that came after Black-Scholes [11] result in a smile curve for implied volatility. Although limited, our results are however encouraging. They show that a fairly demanding theoretical model can be easily implemented.

The structure of the paper is as follows. Section 2 covers preliminaries on the market mechanism and string model. In section 3 we introduce our general model, show general conditions under which there is no arbitrage; we then develop a simple version of this model, which satisfies all the conditions. Empirical analysis is done in section 4, where we describe our data set, the discretized algorithm, as well as the pricing of options.

## 2 Preliminaries

### 2.1 The Market Mechanism

A buy limit order specifies how many shares a trader wants to buy, and at what maximum price he is willing to buy them. We call this price the (buy) *limit price*. A buy limit order specifies how many shares a trader wants to buy, and at what maximum price he is willing to buy them. We call this price the (sell) *limit price*. The unmatched buy and sell orders are stored in order books, until they are either cancelled or matched with an incoming order. An incoming order is matched with the order in the opposite side of the market which has the best price. The clearing price of the transaction is equal to the limit price of the order in the book, and not of the incoming order. Partial execution is allowed, and, ties are resolved by time-priority. We give hereafter an example of the matching mechanism in discrete time, i.e., at most one order arrives at time  $t \in \{0, 1, 2, \dots\}$ .

**Example 1** Suppose that the clearing price at time 0 is any price  $P(0) \in [100, 120]$ . After clearing, that is, when  $0 < t < 1$  we suppose that the order books contain the following orders

Buy Order Book		Sell Order Book	
Price	Quantity	Price	Quantity
100	10	120	10
		130	10

At time  $t = 1$  a buy order arrives with a limit price of \$125, and a quantity of 15. The exchange matches it with the best sell order, i.e., the one with a sell limit price of \$120. However, execution is only partial, and the remainder of the buy order is placed in the order book at the limit price of \$120, resulting in the following order books:

Buy Order Book		Sell Order Book	
Price	Quantity	Price	Quantity
100	10	130	10
125	5		

The clearing price at time 1 is equal to the limit price of the sell order, i.e.:

$$P(1) = 120$$

■

This example illustrates several properties of limit order markets. First, the clearing price is always defined, and can assume any positive value<sup>2</sup>.

Second, it is not inconceivable that an incoming order "crosses" the order book, i.e., for the case of a buy order, that its limit price is higher than the best sell order limit price (i.e, the best ask), since the buyer does not lose a cent. Crossing the book is indeed advantageous for two reasons: first, it allows for faster execution. In our example, had the buyer submitted an order at price \$130 he would have bought the complete quantity of shares (15) that he desired, rather than waiting an indeterminate amount of time until enough sell orders arrive at his limit price. Second, suppose that several buy orders are submitted at the same time. In case the demand exceeds the supply at the best ask, the buy orders with the highest limit price are executed first. Our own data analysis shows that few orders cross the Arca book. This is consistent with the theory of optimal order book placement suggested by Rosu [15].

## 2.2 String Modeling

We now move to continuous-time. Be aware that we do not prove convergence of a discrete-time model to our continuous-time model. We take the latter as a given, plausible model of the market. We start with a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$  satisfying the usual conditions. Equalities of random variables are to be understood almost surely unless stated otherwise.

All the uncertainty is described by a one-dimensional Brownian sheet  $W(s, t)$  which generates  $\{\mathcal{F}_t\}$ . There are several approaches to construct a Brownian sheet, or, more generally stochastic partial differential equations. According to Mueller [5] "One can use either Walsh's [16] approach [...], the Hilbert space approach of Da Prato and Zabczyk [13], or Krylov's [17]  $L_p$  theory". [13] and [Mue09] note that the Hilberts space approach is at least as general as Wlsh's approach. We use a combination of the Hilbert space approach as exposed in [24], and Krylov's approach. We note that in Walsh's approach the definitions (2) and (3) do not involve infinite series, but for our purpose the approaches are equivalent.

We wish to make sense of the Brownian sheet and of its stochastic integral:

$$I_t = \int_{u=0}^t \int_{s=0}^S b(s, u) W(ds, du) \quad (1)$$

where  $W$  is a Brownian sheet and, for each  $s \in [0, S]$  the  $\mathcal{F}_t$ -adapted stochastic processes  $b(s, t)$  satisfy:

$$\int_{s=0}^S b^2(s, t) ds = 1$$

We first define  $W_t$  as a random variable on the space  $E \equiv L^2[0, S]$  such that  $W = \{W_t; t \in [0, T]\}$  is an  $E$ -valued Wiener process. If we let  $\{e_n^*; n \geq 1\}$  be a complete orthonormal basis of the dual  $H^*$  of the reproducing kernel Hilbert space  $H$  of  $E$  and define:

$$w_n(t) = e_n^*(W_t)$$

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<sup>2</sup>We do not consider markets for swaps, where the price can be negative.

we have:

$$W_t = \sum_{n=1}^{\infty} w_n(t) e_n \quad (2)$$

where  $w_n$  are independent  $\mathbb{R}$ -valued standard Wiener processes. While the series (2) is not convergent in  $E$  in  $l^2$ , we can define the following integral (see [13] p. 100)

$$W(s, t) = \sum_{n=1}^{\infty} w_n(t) \int_0^s e_n(\alpha) d\alpha \quad s \in [0, S]$$

which is our Brownian sheet. We now define a family of operators  $B_t: E \rightarrow \mathbb{R}$  for  $t \in [0, T]$  by the formula:

$$B_t f = \int_{s=0}^S b(s, t) f(s) ds$$

Note that  $B$  is a Hilbert-Schmidt operator. We can also write this expression as:

$$B_t f = \sum_{n=1}^{\infty} b^n(t) e_n^*(f)$$

for some real-valued  $\mathcal{F}_t$ -adapted stochastic processes  $b^n$  satisfying:

$$b^n(t) = e_n^*(b(\cdot, t))$$

Thus, we have the definition:

$$I_t = \int_{u=0}^t \sum_{n=1}^{\infty} b^n(u) dw_n(u) \quad (3)$$

We will use expression (1) for most of the text since it will be convenient to see  $b(s, u)W(ds, du)$  as being indexed by the continuum  $s \in [0, S]$ . However expression (3) is an equally valid expression. We now introduce a weaker version of Krylov's [17] Theorem 3.1. (the Ito-Wentzell formula), with the continuous notation. Whereas BB04 used Kunita's [14] version of the Ito-Wentzell formula, we found Krylov's version to be slightly easier to check, since it is directly developed with the Brownian sheet in mind. For the reader more familiar with Kunita's version, we note that, in a string model like ours, it is almost immediate to prove that the quadratic variation is twice continuously differentiable (see also Lemma A.2 below).

Let  $F : [0, S] \times [0, T] \times \Omega \rightarrow \mathbb{R}$  be a  $\mathcal{F}_t$ -adapted semimartingale, which we will write  $F(x, t)$ . Let  $x : [0, T] \times \Omega \rightarrow [0, S]$  be another  $\mathcal{F}_t$ -adapted semimartingale with the following characteristics:

$$\begin{aligned} dF(x, t, \omega) &= \mu_F(x, t, \omega) dt + \sigma_F(x, t, \omega) \int_{s=0}^S b_F(x, s, t, \omega) W(ds, dt, \omega) \\ dx(t, \omega) &= \mu_x(t, \omega) dt + \sigma_x(t, \omega) \int_{s=0}^S b_x(s, t, \omega) W(ds, dt, \omega) \end{aligned} \quad (4)$$

with:

$$\begin{aligned} \int_{s=0}^S b_F^2(p, s, t, \omega) ds &= 1 \\ \int_{s=0}^S b_x^2(s, t, \omega) ds &= 1 \end{aligned}$$

### 2.3 Ito-Wentzell-Krylov Theorem

We say that  $(F, \mu_F, \sigma_F, b_F, x, \mu_x, \sigma_x, b_x)$  satisfy the Ito-Wentzell-Krylov condition (hereafter *condition IWK*) if:

i) For each  $x$  the restriction of the function  $F_t(x)$  to  $\omega \times (0, T]$  is  $\mathcal{F}_T$ -measurable and  $F_0(x)$  is  $\mathcal{F}_0$ -measurable;

ii) For any  $\omega \in \Omega$  and  $t \in [0, T]$ , the function  $F(x, t)$  is continuous in  $x$

iii) For almost any  $(\omega, t) \in \Omega \times [0, T]$

a)  $\mu_F(x)$  and  $\sigma_F(x)b_F(x, s)$  and  $\sigma_F^2(x)$  are continuous in  $x$

b) the following terms are continuous functions of  $x$ :

$$1) \frac{1}{2}\sigma_x^2(t)\frac{\partial^2 F(x, t)}{\partial x^2} + \mu_x(t)\frac{\partial F(x, t)}{\partial x}$$

$$2) \sigma_x(t)b_x(s, t)\frac{\partial F(x, t)}{\partial x}$$

$$3) \sigma_x(t)b_x(s, t)\frac{\partial(\sigma_F(x, t)b_F(x, s, t))}{\partial x}$$

$$4) \sigma_x(t)^2\left(\frac{\partial F(x, t)}{\partial x}\right)^2$$

$$5) \sigma_x^2(t) \int_0^S (b_x(s, t)\frac{\partial(\sigma_F(x, t)b_F(x, s, t))}{\partial x})^2 ds$$

iv) For all  $x$  in the domain of definition of the first argument of  $F$ , we have almost surely:

$$\int_0^T F(x, t)|\mu_x(t) + \frac{1}{2}\sigma_x^2(t)| + \frac{1}{2}F^2(x, t)\sigma_x^2(t) + |\mu_F(x, t)| + \sigma_F^2(x, t)dt < \infty$$

and equation (4) admits a solution almost surely.

v) We have almost surely:

$$\int_0^S \int_0^T F(x, t)(|\mu_x(t)| + \frac{1}{2}\sigma_x^2(t))dt + \frac{1}{2}(\int_0^T F^2(x, t)\sigma_x^2(t)dt)^{1/2}dx < \infty$$

and, for all  $x$  in the domain of definition of the first argument of  $F$

$$\begin{aligned} \int_0^T |\mu_x(x, t)| + \frac{1}{2}\sigma_x^2(t)\frac{\partial^2 F(x, t)}{\partial x^2} + \mu_x(t)\frac{\partial F(x, t)}{\partial x} + \left(\sigma_x(t)\frac{\partial F(x, t)}{\partial x}\right)^2 + \sigma_F^2(x, t) \\ + \int_0^S (\sigma_x(t)b_x(s, t))^2\left[\frac{\partial}{\partial x}\sigma_F(x, t)b_F(x, s, t)\right]^2 dsdt < \infty \end{aligned}$$

**Theorem 2.1 (Ito-Wentzell-Krylov)** <sup>3</sup> Under condition IWK, for any  $t \in [0, T]$  with probability one:

$$\begin{aligned} dF(x(t), t) = & \left( \mu_F(x(t), t) + \frac{1}{2}\sigma_x^2(t)\frac{\partial^2 F(x(t), t)}{\partial x^2} + \mu_x(t)\frac{\partial F(x(t), t)}{\partial x} \right) dt \\ & + \int_0^S \sigma_x(t)b_x(s, t)\frac{\partial(\sigma_F(x(t), t)b_F(x(t), s, t))}{\partial x} dsdt \\ & + \int_0^S \sigma_F(x(t), t)b_F(x(t), s, t) + \sigma_x(t)b_x(s, t)\frac{\partial F(x(t), t)}{\partial x} W(ds, dt) \end{aligned} \quad (5)$$

## 3 A Market with Atomistic Traders and a Large Trader

**Assumption 1** (Assumption A1 in [26]). The market is frictionless.

**Assumption 2** Buy and sell limit prices can assume any real value between 0 and  $S$ . Orders can be submitted to the market at any time  $t \in [\tau, T]$  with  $\tau$  the time of beginning of trading. For convenience  $\tau < 0 < T$ , where  $t = 0$  is present time.

<sup>3</sup>This is a weaker version of theorem 3.1 in Krylov.

**Notation 1** Buy and sell limit prices are usually denoted by  $p$ .

**Definition 1** The net demand curve  $Q$  is a function  $[0, S] \times [\tau, T] \times \Omega \rightarrow \mathbb{R}$  which value  $Q(p, t, \omega)$  is equal to the difference between the (total) quantity of shares **submitted** for purchase (at price lower than or equal to  $p$ ) and the (total) quantity of shares **submitted** for sale (at price larger than or equal to  $p$ ) between time zero and time  $t$  by atomistic traders.

**Remark 1** We do not define precisely what we mean by atomistic traders. We view them as a continuum of traders. Each of them submits an infinitesimally small order at a price  $p$ , such that there is no concentration of orders at any particular price, as we formulate more clearly below, in contradistinction with the large trader.

**Remark 2** Obviously when the order quantity is unity for every order, the (total) quantity of shares submitted for purchase is equal to the (total) number of buy orders. The term "total" refers to the fact that we accumulate buy order quantities that arrived to the exchange since the "beginning of trading" time  $\tau < 0$  until a time  $t \geq 0$ . Henceforth, we will not refer to the time of beginning of trading  $\tau$ . In numerical implementations, different values of  $\tau$  will result in different parameters for the model.

**Remark 3** For a fixed price  $p$  the total quantity of shares submitted for purchase (or for sale) does not need to be increasing in time, as some orders may be cancelled. Even in the absence of cancellation, the absolute value of net demand does not need to be increasing in time. Suppose for instance that at time  $t_1$  and price  $p_1$  the net demand is  $Q(p_1, t_1) > 0$ , i.e., there are buy orders in the order books, but no sell order. Arrival of a sell order with quantity  $\delta$ , with  $0 < \delta < Q(p_1, t_1)$  at time  $t_2 > t_1$  at a price equal to or less than  $p_1$  results in a modification of the net demand curve to  $Q(p_1, t_2) = Q(p_1, t_1) - \delta$ . In other terms orders matched results in a decrease in the absolute value of the net demand curve.

**Assumption 3** The net demand curve  $Q$  is strictly decreasing in its first argument, for all  $p$  except  $p = 0$  and  $p = S$ .

**Remark 4** The fact that net demand curves are decreasing is an immediate consequence of our market mechanism (see above), and the fact that order quantities are by definition positive. Indeed, the number of shares of available buy orders is decreasing in price, while the number of shares of available sell orders is increasing. The net demand, being the difference of buy and sell orders, is a decreasing function of price. Assumption 4 is similar to Assumption A3 in [26] or Assumption 2 in [19].

**Assumption 4** For each  $t$  the net demand curve  $Q$  is twice continuously differentiable in the first argument.

**Assumption 5** The primitives (characteristics) of the model are  $\mathbb{R}$ -valued  $\mathcal{F}_t$ -adapted processes coefficients  $\mu_Q(p, \cdot)$ ,  $\sigma_Q(p, \cdot)$  and  $b_Q(p, s, \cdot)$ .

$$dQ(p, t, \omega) = \mu_Q(p, t, \omega)dt + \sigma_Q(p, t, \omega) \int_{s=0}^S b_Q(p, s, t, \omega)W(ds, dt, \omega) \quad 0 \leq p \leq S \quad (6)$$

$$Q(p, 0) = Q_0(p) > 0 \quad 0 \leq p \leq S \quad (7)$$

$$\int_{s=0}^S b_Q^2(p, s, t, \omega) = 1 \quad 0 \leq p \leq S \quad (8)$$

**Definition 2** The position of the large trader (accumulated since beginning of trading) is denoted by the predictable process  $\theta(t)$ . It is also called the large trader strategy. We define  $\Theta$  to be the set of admissible strategies for the large trader.

**Remark 5** The large trader may have accumulated shares before the beginning of trading, but this will not impact the model. Without loss of generality, we make the following assumption.

**Assumption 6** *The large trader cannot buy or sell more than what is available on the market, and his holdings are bounded by a constant  $K$ . Namely,*

$$\Theta = \{\theta(t, \omega) \mid \max(Q(S, t, \omega), -K) \leq \theta(t, \omega) \leq \min(Q(0, t, \omega), K)\}$$

We will add a further condition on  $\Theta$  later (see below).

### 3.1 Conditions for No-Arbitrage

In order to reuse the no-arbitrage conditions presented in BB04 we need to express our model in terms of prices and not in terms of quantities.

**Definition 3** *The mapping  $S: \Theta \times [0, T] \times \Omega \rightarrow [0, S]$  is the price feedback function. Namely,  $S(\theta, t, \omega)$  is the price faced at time  $t$  by the large trader when his trading strategy is  $\theta(u)$  for  $0 \leq u \leq t$ .*

Since at time  $t$  the large trader's position  $\theta(t)$  results from acquiring shares from the atomistic traders, we have the market clearing equation:

$$Q(S(\theta, t, \omega), t, \omega) = \theta(t, \omega) \quad (9)$$

**Remark 6** *Observe that, in this model, atomistic traders are myopic in the sense that their net demand curve  $Q(p, t, \omega)$  does not depend on  $\theta(t, \omega)$ . As a consequence, as the following easy lemma will show, the trading strategy of the large trader affects asset prices only via its instantaneous value.*

**Lemma 3.1** *There exists a mapping  $P: [\tau, T] \times [0, S] \times \Omega \rightarrow [0, S]$  such that,  $\mathbb{P}$ -almost surely:*

$$S(\theta, t, \omega) = P(\theta(t, \omega), t, \omega) \quad (10)$$

**Sketch of proof of Lemma 3.1** We fix  $t, \omega$ . With obvious notation, (9) can be rewritten:

$$Q^{t, \omega}(S^{t, \omega}(\theta)) = \theta^{t, \omega}$$

Since  $Q$  is strictly decreasing in the first argument,  $Q^{t, \omega}$  is invertible. Thus

$$S(\theta, t, \omega) = (Q^{t, \omega})^{-1}(\theta^{t, \omega})$$

is a function of  $\theta^{t, \omega}$  only. ■

**Definition 4** *We call  $\Xi(t, \omega)$  the set of all  $x$  such that  $x = \theta(t, \omega)$  and  $\theta \in \Theta$ . Let  $x \in \Xi(t, \omega)$ . We call  $P(x, t)$  the price available on the market at time  $t$  when the large trader's position  $\theta(t, \omega) = x$ . By definition it satisfies Equation (10).*

**Assumption 7** *We assume continuous semimartingale dynamics for  $\theta$ :*

$$d\theta(t) = \mu_\theta(t)dt + \sigma_\theta(t) \int_0^S b_\theta(s, t)W(ds, dt)$$

where  $\mu_\theta$ ,  $\sigma_\theta$  and  $b_\theta$  are bounded in  $L^2(\mathbb{R})$ .

**Remark 7** *BB04 assume only that  $\theta$  is a RCLL semimartingale. The remark under Lemma 3.2 of BB04 shows that discontinuous large trader strategies are suboptimal. Thus, for the sake of clarity we refrain from introducing discontinuous strategies. We note however that we could, like them, extend the Ito-Wentzell formula to RCLL semimartingales.*



We now assume semimartingale dynamics for the price process  $P(x, t)$ . We will verify shortly in lemma 3.2 that this assumption is plausible. For the moment, the martingale representation theorem (Theorem 4.1 in [24]) shows the existence of  $\mathcal{F}_t$ -adapted processes  $\sigma_P(x, t)$  and  $b_P(x, s, t)$  such that:

$$dP(x, t) = \mu_P(x, t)dt + \sigma_P(x, t) \int_0^S b_P(x, s, t)W(ds, dt) \quad (11)$$

$$\int_0^S b_P^2(x, s, t)ds = 1 \quad (12)$$

**Definition 5** For notational convenience, we introduce the cross-variation in the Ito-Wentzell formula:

$$C(P(x, t), t, \omega) = -\frac{\sigma_Q(P(x, t), t)}{\frac{\partial Q}{\partial p}(P(x, t), t)} \int_{s=0}^S \frac{\partial[\sigma_Q(P(x, t), t, \omega)b_Q(P(x, t), s, t, \omega)]}{\partial p} b_Q(P(x, t), s, t, \omega)ds$$

**Assumption 8** We suppose that for each  $x \in \Xi(t)$  condition IWK holds for  $(Q, \mu_Q, \sigma_Q, b_Q, P(x, \cdot), \mu_P(x, \cdot), \sigma_P(x, \cdot), b_P(x, \cdot))$ .

**Lemma 3.2** Under Assumption 8:

$$\mu_P(x, t) = -\frac{\mu_Q(P(x, t), t) + \frac{1}{2} \frac{\partial^2 Q}{\partial p^2}(P(x, t), t) \sigma_P^2(x, t) + C(P(x, t), x, t)}{\frac{\partial Q}{\partial p}(P(x, t), t)} \quad (13)$$

$$\sigma_P(x, t) = \frac{\sigma_Q(P(x, t), t)}{\frac{\partial Q}{\partial p}(P(x, t), t)} \quad (14)$$

$$b_P(x, s, t) = -b_Q(P(x, t), s, t) \quad (15)$$

**Proof of Lemma 3.2:**

We rewrite the market clearing equation (9) as:

$$Q(P(x, t), t) = x \quad (16)$$

We apply (5) to the left side:

$$\begin{aligned} dQ(P(x, t), t) &= \mu_Q(P(x, t), t)dt + \sigma_Q(P(x, t), t) \int_0^S b_Q(P(x, t), s, t)W(ds, dt) \\ &+ \frac{\partial Q}{\partial p}(P(x, t), t) \left( \mu_P(x, t)dt + \sigma_P(x, t) \int_0^S b_P(x, s, t)W(ds, dt) \right) + \frac{1}{2} \frac{\partial^2 Q}{\partial p^2}(\sigma_P(x, t))^2 dt + \tilde{C}(P(x, t), x, t)dt \end{aligned} \quad (17)$$

where:

$$\tilde{C}(P(x, t), x, t) = \sigma_P(x, t) \int_{s=0}^S \frac{\partial[\sigma_Q(P(x, t), t)b_Q(P(x, t), s, t)]}{\partial p} b_P(x, s, t)ds$$

The differential of the right side of (16) is clearly zero. Setting the volatility part equal to zero yields, for each  $s \in [0, S]$ :

$$\sigma_Q(P(x, t), t)b_Q(P(x, t), s, t) + \frac{\partial Q}{\partial p}(P(x, t), t)\sigma_P(x, t)b_P(x, s, t) = 0$$

Thus  $\tilde{C}(P(x, t), x, t) = C(P(x, t), t)$ . Setting the drift part equal to zero, we obtain

$$\mu_Q(P(x, t), t) + \frac{\partial Q}{\partial p}(P(x, t), t)\mu_P(x, t) + \frac{1}{2} \frac{\partial^2 Q}{\partial p^2}(P(x, t), t)\sigma_P^2(x, t) + C(P(x, t), x, t) = 0 \quad (18)$$

Since, by assumption  $\frac{\partial Q}{\partial p} < 0$ , the expression (13) is well-defined. ■

We model liquidity like BB04. Suppose an investor is forced at time zero to liquidate his stock position  $\vartheta$ . As the authors say "it is more convenient to split the sell order into small packages, which would then be sold one after the other over a small time period. In the limit, as the packages become ever smaller and as the duration for liquidation tends to zero [...]" we obtain the following characterization of the asymptotic liquidation proceeds.

**Definition 6** *The asymptotic liquidation proceeds of the large trader, denoted by  $L$  are equal to:*

$$L(\vartheta, t) = \int_0^\vartheta P(x, t) dx$$

The large trader *holdings in the bank account* are denoted by  $\beta_t^\theta$ . The *realizable wealth* of the large trader achieved by a trading strategy  $\theta$  up until time  $t$  is denoted by  $V_t^\theta$ , with:

$$V_t^\theta = \beta_t^\theta + L(\theta_t, t)$$

We recall the definition of a stochastic integral with respect to a semimartingale  $L(\vartheta, ds)$  by starting from the definition:

$$\int_0^t L(\theta_u, du) = \sum_i L(\theta_{s_i \wedge t}, s_{i+1} \wedge t) - L(\theta_{s_i \wedge t}, s_i \wedge t)$$

for simple integrands of the form  $\theta = \sum_i \theta_i 1_{(s_i, s_{i+1}]}$  with  $0 \leq s_0 \leq \dots \leq s_n \leq T$  and  $\theta_i \in L^0(\mathcal{F}_{s_i})$ . For more details, we refer the reader to remark 2.4 in BB04.

We now permute the roles of quantity and price in the application of the Ito-Wentzell formula. To this effect, we need the following assumption.

**Assumption 9** *We suppose that condition IWK holds for  $(P, \mu_P, \sigma_P, b_P, \theta, \mu_\theta, \sigma_\theta, b_\theta)$ .*

The following fact implies that, if  $\int_0^t L(\theta_u, du)$  is a local martingale under an equivalent martingale measure  $\mathbb{Q}$ , then the realizable wealth under  $\mathbb{Q}$  will be a supermartingale, since the two terms to the right-hand side of Equation (19) are nonpositive.

**Fact 3.3** (Lemma 3.2 in BB04) *For any self-financing semimartingale strategy  $\theta$ , under Assumption 9, the dynamics of the real wealth process  $V^\theta$  are given by:*

$$V^\theta(t) - V^\theta(0_-) = \int_0^t L(\theta_{u-}, du) - \frac{1}{2} \int_0^t P'(\theta_u, u) d[\theta, \theta]_u \quad (19)$$

We now investigate conditions for no-arbitrage in the model, but we first provide motivation for its definition. The payoff of a derivative may depend on the investor's strategy. Thus, in order to price derivatives a market model should not have arbitrage, whatever the trading strategy of the large trader.

**Definition 7** An arbitrage strategy is a self-financing trading strategy  $\theta \in \Theta$  such that  $V^\theta(0_-) = 0$  and:

$$\begin{aligned} P(V^\theta(T) > 0) &> 0 \\ P(V^\theta(T) \geq 0) &= 1 \end{aligned}$$

A market model has arbitrage if there exists an arbitrage strategy  $\theta \in \Theta$ . The market price of risk is a collection of  $\mathcal{F}_t$ -adapted stochastic processes  $\lambda(s, \cdot)$  for  $s \in [0, S]$ .

**Definition 8** The  $\mathbb{Q}$ -measure as the measure where the process  $W^\mathbb{Q}$  is a Brownian sheet, where:

$$W^\mathbb{Q}(ds, dt) = W(ds, dt) + \lambda(s, t)dt \quad (20)$$

For notational convenience, we define the following processes, for  $x \in \Xi(t, \omega)$ :

$$\begin{aligned} B(P(x, t), t) &= \mu_Q(P(x, t), t, \omega) - \frac{1}{2} \frac{\partial^2 Q}{\partial p^2}(P(x, t), t) (\sigma_P(x, t))^2 + C(P(x, t), x, t, \omega) \\ \Sigma(P(x, t), s, t, \omega) &= \sigma_Q(P(x, t), t, \omega) b_Q(P(x, t), s, t, \omega) \end{aligned} \quad (21)$$

The market price of risk equations consist of the uncountably infinite system of equations:

$$\int_{s=0}^S \Sigma(p, s, t, \omega) \lambda(s, t, \omega) ds = B(p, t, \omega) \quad 0 \leq p \leq S \quad (22)$$

**Assumption 10** Novikov's condition holds for the family of processes  $\lambda(s, \cdot)$ , that is,

$$E[\exp(\int_0^T \int_0^S \lambda(s, t) W(ds, dt) - \frac{1}{2} \int_0^T \int_0^S \lambda^2(s, t) ds dt)] = 1 \quad (23)$$

**Theorem 3.4** Suppose Assumptions 1 to 10 hold. If the market price of risk equations (22) are satisfied, the market model has no arbitrage.

#### Proof of Theorem 3.4

We investigate conditions under which, for all  $x \in \Xi(t, \omega)$  there exists a local martingale measure  $\mathbb{Q}$  for  $P(x, t)$ . As in Lemma 3.2, we calculate the differential of  $Q(P(x, t), t)$ . We introduce (20) into (17) and obtain:

$$\begin{aligned} dQ(P(x, t), t) &= \mu_Q(P(x, t), t) dt - \sigma_Q(P(x, t), t) \int_{s=0}^S b_Q(P(x, t), s, t) \lambda(s, t) ds dt \\ &\quad + \sigma_Q(P(x, t), t) \int_{s=0}^S b_Q(P(x, t), s, t) W^\mathbb{Q}(ds, dt) \\ &\quad + \frac{\partial Q}{\partial p}(P(x, t), t) \left( \mu_P(x, t) - \sigma_P(x, t) \int_{s=0}^S b_P(x, s, t) \lambda(s, t) ds \right) dt \\ &\quad + \frac{\partial Q}{\partial p}(P(x, t), t) \sigma_P(x, t) \int_{s=0}^S b_P(P(x, t), s, t) W^\mathbb{Q}(ds, dt) \\ &\quad + \frac{1}{2} \frac{\partial^2 Q}{\partial p^2}(\sigma_P(x, t))^2 dt + C(P(x, t), t) dt \end{aligned} \quad (24)$$

By definition the drift of  $P(x, t)$  in the measure  $\mathbb{Q}$  is zero. We thus obtain:

$$\mu_P(x, t) - \sigma_P(x, t) \int_{s=0}^S b_P(x, s, t) \lambda(s, t) ds = 0 \quad (25)$$

We set  $dQ(P(x, t), t) = 0$  as in Lemma 3.2. We insert (25) in (24) and set the drift part equal to zero:

$$\begin{aligned} \mu_Q(P(x, t), t) - \sigma_Q(P(x, t), t) \int_{s=0}^S b_Q(P(x, t), s, t) \lambda(s, t) ds \\ + \frac{1}{2} \frac{\partial^2 Q}{\partial p^2} (\sigma_P(x, t))^2 + C(P(x, t), t) = 0 \end{aligned} \quad (26)$$

Setting the volatility in (24) equal to zero, we obtain (14) and (15), and thus (26) can be rewritten:

$$\sigma_Q(P(x, t), t) \int_s b_Q(P(x, t), s, t) \lambda(s, t) ds = \mu_Q(P(x, t), t) + \frac{1}{2} \frac{\partial^2 Q}{\partial p^2} (\sigma_Q(P(x, t), t))^2 dt + C(P(x, t), t) dt \quad (27)$$

This set of equations must hold for each value  $P(x, t)$ . Since  $P(x, t) \in [0, S]$ , existence of a solution to the market price of risk equations are sufficient for (27) to hold. Under Assumption 10, we can assume that Girsanov's theorem holds (see, e.g. Theorem 10.14 in DPZ), so that there exists a local martingale measure  $\mathbb{Q}$  for  $P(x, t)$ . By assumption RAN we can invoke (19) as well as Theorem 3.3 in BB04. ■

What happens when the market price of risk equations are not satisfied? Then, generically, there is no equivalent martingale measure  $\mathbb{Q}$  such that, for all  $\theta \in \Theta$  the real wealth process  $V^\theta$  is a local martingale. The first fundamental theorem shows that no arbitrage for small investors is essentially equivalent to the existence of this equivalent martingale measure (BB04 p.8). More precisely, if there is no free lunch with vanishing risk (NFLVR), then there should exist an equivalent martingale measure  $\mathbb{Q}$ . To our knowledge, no rigorous proof of this result exists in the literature, so we content ourselves with the following generic example, to show that a market model with only two Brownian motions admits arbitrage.

### 3.2 Liquidity Arbitrage with a Finite Number of Brownian Motions: an Example

Consider two stocks with price  $P^{(1)}$  and  $P^{(2)}$ . The large trader has a position  $\theta^{(1)}$  in stock 1 and  $\theta^{(2)}$  in stock 2. These stocks are correlated, so we assume a simple model with two Brownian motions,  $W_1$  and  $W_2$ . It should be clear that the same reasoning applies to a model with a finite number of Brownian motions. We assume that the drift or volatility coefficient depends on  $x$ , except one, the volatility coefficient of the second price with respect to the second Brownian motion. We have then:

$$\begin{aligned} dP^{(1)}(x, t, \omega) &= \mu_P^{(1)}(\omega) dt + \sigma_{P,1}^{(1)}(\omega) dW_1(t, \omega) + \sigma_{P,2}^{(1)}(\omega) dW_2(t, \omega) \\ dP^{(2)}(x, t, \omega) &= \mu_P^{(2)}(\omega) dt + \sigma_{P,1}^{(2)}(\omega) dW_1(t, \omega) + \sigma_{P,2}^{(2)}(\omega) x dW_2(t, \omega) \end{aligned}$$

We assume regularity of the stochastic matrix:

$$\begin{bmatrix} \sigma_{P,1}^{(1)} & \sigma_{P,2}^{(1)} \\ \sigma_{P,1}^{(2)} & \sigma_{P,2}^{(2)} \end{bmatrix}$$

and that the process  $\bar{\theta}$  defined in (31),(32) has finite variation. Then, by Fubini's theorem for stochastic integral (cf Protter 1990, Theorem IV.46 in [22]),

$$\begin{aligned} \int_0^t L(\theta, du) &= \int_0^t \theta^{(1)}(u) \mu_P^{(1)} + \int_0^t \theta^{(1)}(u) \mu_P^{(1)} du \\ &\quad + \int_0^t S_1(u) dW_1(u) + \int_0^t S_2(u) dW_2(u) \end{aligned} \quad (28)$$

where the diffusion coefficient of the proceeds of a fast liquidation strategy are:

$$S_1 = \sigma_{P,1}^{(1)} \theta^{(1)} + \sigma_{P,1}^{(2)} \theta^{(2)} \quad (29)$$

$$S_2 = \sigma_{P,2}^{(1)} \theta^{(1)} + \sigma_{P,2}^{(2)} (\theta^{(2)})^2 / 2 \quad (30)$$

We fix time  $s$  and set  $S_1$  and  $S_2$  to zero. Eliminating  $\theta^{(1)}$  yields:

$$\sigma_{P,2}^{(1)} \sigma_{P,1}^{(2)} \theta^{(2)} - \sigma_{P,1}^{(1)} \sigma_{P,2}^{(2)} (\theta^{(2)})^2 / 2 = 0$$

Apart from  $\theta = 0$ , an obvious solution is  $\theta = \bar{\theta}$ , where:

$$\bar{\theta}^{(1)} = -\frac{\sigma_{P,1}^{(2)}}{\sigma_{P,1}^{(1)}} \bar{\theta}^{(2)} \quad (31)$$

$$\bar{\theta}^{(2)} = 2 \frac{\sigma_{P,2}^{(1)} \sigma_{P,1}^{(2)}}{\sigma_{P,2}^{(2)} \sigma_{P,1}^{(1)}} \quad (32)$$

By the assumption of finite variation of  $\bar{\theta}$ , we have:

$$\begin{aligned} V^{\bar{\theta}}(t) - V^{\bar{\theta}}(0_-) &= \int_0^t L(\bar{\theta}, du) \\ &= \int_0^t \mu_P^{(1)} \bar{\theta}^{(1)}(u) + \mu_P^{(2)} \bar{\theta}^{(2)}(u) du \\ &= \int_0^t \bar{\theta}^{(2)}(u) \left( -\frac{\sigma_{P,1}^{(2)}}{\sigma_{P,1}^{(1)}} \mu_P^{(1)} + \mu_P^{(2)} \right) du \end{aligned}$$

Thus, if  $\mu_P^{(2)} > \frac{\sigma_{P,1}^{(2)}}{\sigma_{P,1}^{(1)}} \mu_P^{(1)}$ , then:

$$V^{\bar{\theta}}(t) - V^{\bar{\theta}}(0_-) \geq 0$$

### 3.3 A Market Model Satisfying all Assumptions

We define a "log demand delta-density" string  $h$ , a demand density string  $q$ , and a net demand ratio process  $\eta$  followingly in order to define net demand  $Q$ :

$$h(x, t) = h_{0,x} \exp(-a(x)t) + \bar{h}(x)(1 - \exp(-a(x)t)) + \quad (33)$$

$$\begin{aligned} &\int_{u=0}^t \exp(-a(x)(t-u)) \sigma_h(x) \int_0^S b_h(x, s) W(ds, u) du \\ q(p, t) &= \exp\left(\int_{x=0}^p h(x, t) dx\right) \quad \forall p \in [0, S] \end{aligned} \quad (34)$$

$$d\eta(t) = a_\eta(\bar{\eta} - \eta(t)) dt + g(\eta(t)) \int_0^S b_\eta(u) W(ds, dt) \quad (35)$$

$$Q(p, t) = \eta(t, \omega) \int_{x=0}^S q(x, t) dx - \int_{x=0}^p q(x, t) dx \quad \forall p \in [0, S] \quad (36)$$

with initial condition:

$$\eta(0) = \eta^0 \in (0, 1) \quad (37)$$

where:

- $a, \bar{h}, \sigma_h$ , and  $b_h(\cdot, s)$  (for  $s \in [0, S]$ ) are continuously differentiable functions of  $p$ , bounded on  $p \in [0, S]$
- $a_\eta \geq 0, \bar{\eta} \in (0, 1)$
- $g$  is a nonnegative Lipschitz continuous function such that  $g(\eta) = 0$  if  $\eta \leq 0$  or  $\eta \geq 1$ .

Let  $0 < \delta_1 < \delta_2 < S$ . We also assume that  $b_\eta(s) = 0$  for  $\delta_2 \leq s \leq S$  and  $b_h(x, s) = 0$  for  $0 \leq s \leq \delta_1$ , and  $x \in [0, S]$ , so that:

$$\int_{\delta_1}^S b_h^2(p, s) ds = 1 \quad \forall p \in [0, S] \quad (38)$$

$$\int_0^{\delta_2} b_\eta^2(s) ds = 1 \quad (39)$$

and that the bounded linear operator  $B_h : E \rightarrow E$  given by

$$(B_h f)(p) = \int_0^S b_h(p, u) f(u) du$$

has full rank.

**Remark 8** *Lipschitz continuity ensures that (34) has a solution. A possible form (which is not Lipschitz continuous, but can be mollified into one) is:*

$$g(\eta) = \sigma_\eta \sqrt{\eta(1-\eta)}$$

**Remark 9** *To make it twice differentiable in price, we build  $Q$  from the ground up, by first defining the log demand delta-density  $h$ , then integrating with respect to price to obtain the logarithm of the demand density  $q$ . The demand density string  $q$  is then positive. While any well-behaved positive function applied to  $\int_{x=0}^p h(x, t) dx$  in (34) would make  $q$  positive, our assumptions are easier to check if we assume a simple exponential. We introduced the net demand ratio  $\eta(t)$  as an exogeneous variable so that the clearing price is always in  $[0, S]$ . Observe that, by evaluating (36) at  $p = 0$  and  $p = S$ , we can solve for  $\eta$ :*

$$\eta(t) = \frac{Q(0, t)}{Q(0, t) - Q(S, t)} \quad (40)$$

In other words it is the ratio of the quantity of all shares available for purchase over the quantity of shares available for purchase or for sale, and, as such should lie between zero and one. This is indeed the case for the double-barrier process in (35). Examination of (36) shows then that, if  $P(x, t)$  satisfies (17), then it does not reach barrier 0 and  $S$  almost surely. We write  $Q(p, t)$  as:

$$Q(p, t) = Q(0, t) - \int_{x=0}^p f(x, t) dx$$

for some  $f$ . Thus equation (36) becomes:

$$Q(0, t) - \int_{x=0}^p f(x, t) dx = \frac{Q(0, t)}{\int_{x=0}^S f(x, t) dx} \int_{x=0}^S q(x, t) dx - \int_{x=0}^S q(x, t) dx$$

Thus  $f = q$  and the process  $q$  represents indeed the density of demand. Equation (36) shows that  $Q$  is strictly decreasing in  $p$ , thus satisfying Assumption 3.

**Remark 10** *The reason why the support for the factor loadings in (38) and (39) is not  $[0, S]$  is that we would like to have easy conditions for the market price of risk equations to hold. We chose them to overlap, so that we can build correlation between  $\eta$  and  $q$ . The assumption that  $B_{h,p}$  has full rank is quite innocuous in practice. We will see in the empirical section that it was always verified.*

**Proposition 3.5** *Under assumption RAN Our model satisfies Assumptions 4, 8, 9, and 10. Also, the market price of risk equations (22) hold.*

We can thus conclude that there is no arbitrage in our model.

## 4 Empirical Analysis

As mentioned in the introduction, we argue that in equilibrium the large trader is statistically indistinguishable from an atomistic trader. The clearing price  $\pi(t)$  on the exchange will thus be

$$\pi(t) \equiv P(0, t)$$

so that  $Q(\pi(t), t) = 0$ . In this section, we calibrate the market model and then simulate it in the risk-neutral measure to price options. We use empirical high frequency data of a magnitude of a millisecond. Then the calibrated parameters are utilized in the discretized simulation model in both the physical and risk-neutral measures. When simulating the net demand curve  $Q$  in the risk neutral measure, the market price of risk equation is solved. As expected, we could not find any case where the market price of risk equation does not have a solution.

### 4.1 Data

The high frequency trading data are collected from the NYSE Arcabook limit orders for April 2011. Historical NYSE Arcabook data provide information of the complete limit order book (LOB) from NYSE, NYSE Arca, NYSE MKT, NASDAQ and the ArcaEdge platforms from 3:30 a.m. to 8:00 p.m. ET under high speed of latencies (less than 5 milliseconds). Each limit order contains the unique reference number, the time stamp in seconds and milliseconds, the limit price in U.S. dollars, the quantity in number of shares, and the trading type (“B”: buy or “S”:sell).

All the limit order book records are categorized into three groups: “A” Add, “M” Modified and “D” Deleted. For the market liquidity model, we consider the net demand of the stock, which is captured by summing added records (“A”) with modified (“M”) adjustment and subtracting the deleted orders. To be specific, within a certain partitioned time period, the added orders would be updated by the modified orders, if applicable and the orders occur in the same partitioned period.

In this section, we use the time series of Apple Inc. stock (AAPL) as of April 1st 2011 for calibration.

### 4.2 Discretized Model

The price variable  $p$  belongs to the set  $\{0, \Delta p, 2\Delta p, \dots, I\Delta p\}$  where  $I = 401$  and  $\Delta p = \$1.91$ . This corresponds for our data set to a winsorization at the 99% quantile of the price for all the limit orders within the trading day. The variable  $s$  will belong to the same set, so we set  $\Delta s = \Delta p$ . The variable  $t$  will be a multiple of  $\Delta t$ , for  $\Delta t = 15$  minutes. For any function  $f$  of real-valued variables  $p = k\Delta p$ ,  $s = i\Delta s$  and  $t = j\Delta t$  we streamline the notation by writing:

$$\begin{aligned} f(k, i, j) &\leftarrow f(k\Delta p, i\Delta s, j\Delta t) \\ \Delta f(k, i, j) &= f(k, i, j+1) - f(k, i, j) \end{aligned}$$

We denote observed quantities by  $\hat{f}(t)$  and simulated quantities by  $f(t)$ . We discretize the model (33) to (36), and simplify it by setting most of the drifts equal to zero. Let  $\{z(i, j)\}$  be a collection of standard normal random variates where  $0 \leq i \leq I$ , and the values of  $j$  will be clear from context. After initializing all stochastic processes, we use the Euler scheme:

$$\Delta h(k, j) = \sigma_h(k) \sum_{i=2}^I b_h(k, i) z(i, j) \sqrt{\Delta s \Delta t} \quad 2 \leq k \leq I \quad (41)$$

$$\Delta q(1, j) = \sigma_q(1) \sum_{i=2}^I b_q(1, i) z(i, j) \sqrt{\Delta s \Delta t} \quad (42)$$

$$\Delta \eta(j) = a_\eta(\bar{\eta} - \eta(j)) \Delta t + \sigma_\eta \sqrt{\eta(j)(1 - \eta(j))} \sum_{i=0}^I b_\eta(i) z(i, j) \sqrt{\Delta s \Delta t} \quad (43)$$

We then complete the model by setting, for  $2 \leq k \leq I$ :

$$\begin{aligned} q(k, j) &= q(1, j) \exp \left( \sum_{i=2}^k h(i, j) \Delta p \right) \\ Q(0, j) &= \frac{\eta(j)}{2 - \eta(j)} \sum_{i=1}^I q(i, j) \Delta p \\ Q(k, j) &= [Q(0, j) + \sum_{i=1}^I q(i, j) \Delta p] \eta(j) - [Q(0, j) + \sum_{i=1}^k q(i, j) \Delta p] \end{aligned}$$

### 4.3 Calibration Methodology

The number of observations is equal to  $N = 32$ . Observation times take values in the set  $\{-N\Delta t, -(N-1)\Delta t, \dots, 0\}$ . Since each time step represents an interval of 15 minutes, we use all the dates between April 1st 9 AM and April 1st 5PM, which covers the regular trading hours during the day. The net demand  $Q$  at price  $p$  is calculated, by definition as:

$$\begin{aligned} \hat{Q}(k, \cdot) &= \text{quantity of shares available for purchase at a price } > k\Delta p \\ &\quad - \text{quantity of shares available for sale at a price } < k\Delta p \end{aligned}$$

A typical net demand  $\hat{Q}$  surface as a function of price and time for AAPL stock on April 1st, 2011 is shown in Figure 1.

After the net demand surface  $\hat{Q}$  is calculated, the net demand density  $\hat{q}$  is calculated by taking the first order difference of  $\hat{Q}(p)$ :

$$\hat{q}(k, j) = -\frac{\hat{Q}(k, j) - \hat{Q}(k-1, j)}{\Delta p}, \quad 1 \leq k \leq I$$

Discretization of (34) yields:

$$\hat{h}(k, j) = \frac{1}{\Delta p} (\log(\hat{q}(k, j)) - \log \hat{q}(k-1, j)), \quad 2 \leq k \leq I$$





Figure 1: The calculated net demand surface  $Q$  by price and time, from market data for AAPL as of April 1st, 2011.

The net demand elasticity  $\hat{\eta}$  is calculated by (40). We estimate the instantaneous variances by the method of moments:

$$\begin{aligned}\hat{\sigma}_{\eta}^2 &= \frac{1}{N\Delta t} \sum_{j=-N}^{-1} \frac{\Delta \hat{\eta}(j)^2}{\hat{\eta}(j)(1 - \hat{\eta}(j))} \\ \hat{\sigma}_q^2(1) &= \frac{1}{N\Delta t} \sum_{j=-N}^{-1} \Delta \hat{q}(1, j)^2 \\ \hat{\sigma}_h^2(k) &= \frac{1}{N\Delta t} \sum_{j=-N}^{-1} \Delta \hat{h}(k, j)^2 \quad k = 2, \dots, S\end{aligned}$$

We do the same for the instantaneous covariances:

$$\begin{aligned}\hat{s}_{\eta, q}^2(1) &= \frac{1}{N\Delta t} \sum_{j=-N}^{-1} \Delta \hat{\eta}(j) \Delta \hat{q}(1, j) \\ \hat{s}_{\eta, h}^2(k) &= \frac{1}{N\Delta t} \sum_{j=-N}^{-1} \frac{\Delta \hat{\eta}(j) \Delta \hat{h}(k, j)}{\sqrt{\hat{\eta}(j)(1 - \hat{\eta}(j))}} \quad k = 2, \dots, S \\ \hat{s}_{q, h}^2(1, k) &= \frac{1}{N\Delta t} \sum_{j=-N}^{-1} \Delta \hat{q}(1, j) \Delta \hat{h}(k, j) \quad k = 2, \dots, S \\ \hat{s}_h^2(k_1, k_2) &= \frac{1}{N\Delta t} \sum_{j=-N}^{-1} \Delta \hat{h}(k_1, j) \Delta \hat{h}(k_2, j) \quad k_1, k_2 = 2, \dots, S\end{aligned}$$

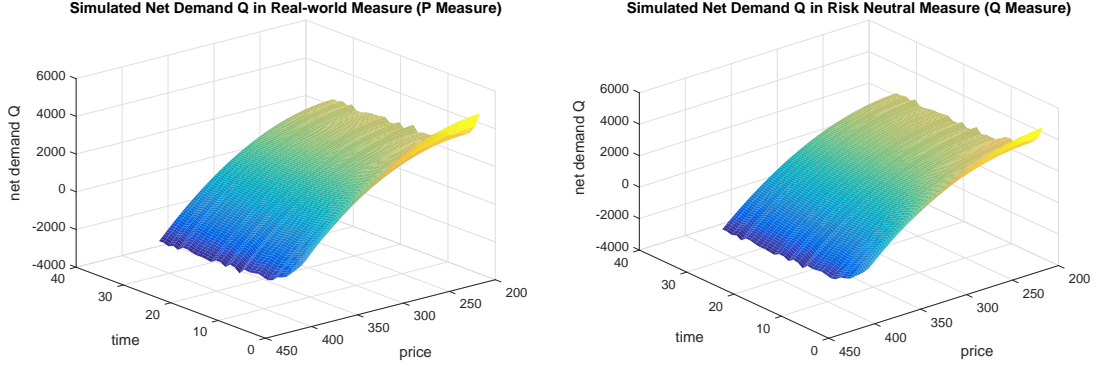


Figure 2: The simulated net demand surface  $Q(p,t)$  under physical measure (left) and risk neutral measure (right).

We organize the instantaneous variance-covariance matrix as:

$$V = \begin{pmatrix} \hat{\sigma}_\eta^2 & \hat{s}_{\eta,q}^2(1) & \hat{s}_{\eta,h}^2(2) & \cdots & \hat{s}_{\eta,h}^2(I) \\ \hat{s}_{\eta,q}^2(1) & \hat{\sigma}_q^2(1) & \hat{s}_{q,h}^2(1,2) & \cdots & \hat{s}_{q,h}^2(1,I) \\ \hat{s}_{\eta,h}^2(2) & \hat{s}_{q,h}^2(1,2) & \hat{\sigma}_h^2(2) & \cdots & \hat{s}_h^2(2,I) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \hat{s}_{\eta,h}^2(I) & \hat{s}_{q,h}^2(1,I) & \hat{s}_h^2(2,I) & \cdots & \hat{\sigma}_h^2(I) \end{pmatrix}$$

The factor loading matrix  $B$  is the square root of the correlation matrix corresponding to  $V$ . It is made of the following coefficients:

$$\hat{B} = \begin{pmatrix} \hat{b}_\eta(0) & 0 & \cdots & \cdots & \cdots \\ \hat{b}_\eta(1) & \hat{b}_q(1,1) & \ddots & & \vdots \\ & \vdots & \hat{b}_h(2,1) & \ddots & \vdots \\ & \vdots & \vdots & \ddots & 0 \\ \hat{b}_\eta(I) & \hat{b}_q(1,I) & \hat{b}_h(I,1) & \cdots & \hat{b}_h(I,I) \end{pmatrix}$$

We use Cholesky decomposition to calculate  $\hat{B}$ .

In the simulation, we use  $a\eta = 0.9$  and  $\bar{\eta} = 0.6$ .

#### 4.4 Simulation - Real-world Measure ( $\mathbb{P}$ Measure)

The simulated net demand curve under physical measure using the above methods is shown in left graph of Figure 2.

#### 4.5 Simulation - Risk-Neutral Measure ( $\mathbb{Q}$ Measure)

Let  $k^*(j)$  be the index of the approximate clearing price, i.e., the value of  $k$  that minimizes  $|Q(k,j)|$ .

We first need to solve the market price of risk equations. Expressions for  $\mu_Q(k, j)$  and  $\sigma_Q(k, j)$  and  $b_Q(k, i, j)$  are obtained by discretization of equations 41 and 42 in the appendix. Remember that:

$$\Sigma_Q(k, i, j) = \sigma_Q(k, j)b_Q(k, i, j)$$

We calculate:

$$\begin{aligned}\sigma_\pi(j)b_\pi(i, j) &= -\frac{\Sigma_Q(k^*(j), i, j)}{q(k^*(j), j)} \\ C(k, j) &= \sigma_\pi(j) \sum_{i=0}^I \frac{\Sigma_Q(k^*(j), i, j) - \Sigma_Q(k^*(j) - 1, i, j)}{\Delta p} b_\pi(i, j) \\ B(k, j) &= \mu_Q(k, j) - \frac{1}{2} \frac{Q(k+1, j) - 2Q(k, j) + Q(k-1, j)}{\Delta p^2} + C(k, j)\end{aligned}$$

The market price of risk equations are, for each scenario and each time  $j$ :

$$\sum_{i=0}^I \Sigma_Q(k, i, j) \lambda(i, j) = B(k, j) \quad k = 0, \dots, I \quad (44)$$

Now let  $\{z^Q(i, j)\}$  be a collection of standard normal random variates. It remains only to make the substitution:

$$z(i, j) = z^Q(i, j) \sqrt{\Delta t \Delta s} - \lambda(i, j) \sqrt{\Delta s}$$

in our model (41) to (43) to obtain the risk-neutral model.

The simulated net demand surfaces under risk neutral measure is shown in the right graph of Figure 2. The range of the difference on net demand (over time and price) between  $\mathbb{P}$ -measure simulation and  $\mathbb{Q}$ -measure simulation is  $[-50.22, 66.10]$  in the case shown in the simulation.

## 4.6 Option Pricing

With the simulations under risk neutral measure, the corresponding call/put option price,  $c(\pi, \omega)$ , with strike  $K$  at given time  $t$  is

$$\begin{aligned}c(K) &= \frac{1}{N} \sum_{\omega} \max(\pi(t, \omega) - K, 0) \\ p(K) &= \frac{1}{N} \sum_{\omega} \max(K - \pi(t, \omega), 0) \quad \omega = 1, 2, \dots, N\end{aligned}$$

where  $\pi$  is the clearing price at simulation time  $t$ . With the option prices, the implied volatility for different strike levels are calculated by solving

$$p(K) = BS(\pi, K, r, T, \sigma(K))$$

where  $BS$  is the Black-Scholes function,  $\pi$  is the clearing price,  $K$  is the strike level,  $r$  is the interest rate,  $T$  is time to maturity and  $p$  is the put option price corresponding to the strike  $K$ . The implied volatility  $\sigma(K)$  is being solved.

The implied volatility smile generated from the equity options at simulation period is shown in Figure 3 (right graph). The implied volatilities by simulations under risk neutral measure are plotted as blue line,

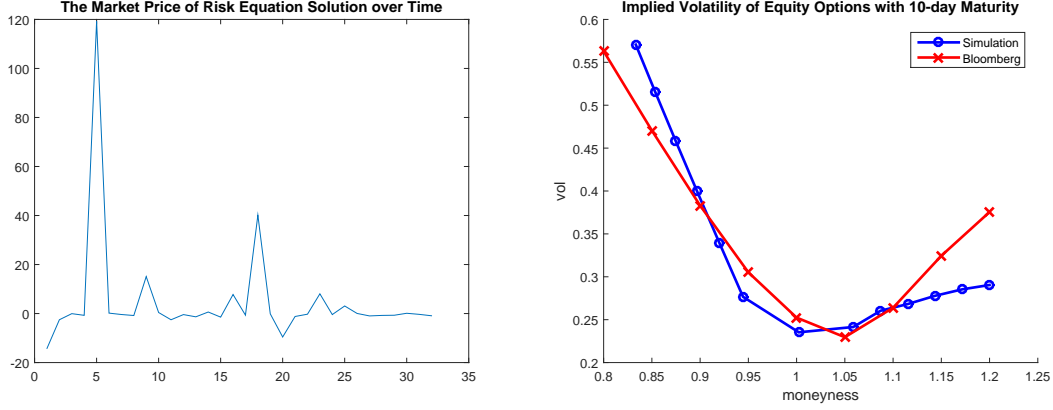


Figure 3: The left graph shows the solution ( $\lambda$ ) of market price of risk equation at clearing price over time. The right graph plots implied volatility levels from equity options with maturity of 10 days.

which are compared to the real historical implied volatility levels from Bloomberg (historical market quotes). The interest rate level is selected as 0.301% at real historical level, and the dividend yield is set to 0%. The maturity of the option is 60 days. The simulation is evaluated under 500 scenarios.

The at-the-money simulated implied volatility is within 5% error range of the Bloomberg quote, at 25% level. For low strike options the simulation results tend to be steeper than the Bloomberg volatility, and the percentage difference between simulation results and Bloomberg volatilities is within  $\pm 10\%$ . As for the high strike options, the simulation results are flatter than Bloomberg volatilities.

The left graph in Figure 3 plots the solution of market price of risk equation 44 ( $\lambda(i, j)$ ) over time, evaluated on at-the-money price levels. In our simulation process, the market price of risk equation is always solvable through matrix inversion.

## A Proof of Proposition

To prove boundedness in IWK, we will make use of the following elementary lemma, which is a direct application of the Chebyshev, Cauchy, and Jensen inequalities.

**Lemma A.1** *Let  $Y_i(s, t)$  and  $Z_i(s, t)$  be collections of random variables, for  $i = 1, \dots, n$ ,  $0 \leq s \leq S$ , and  $0 \leq t \leq T$ . Suppose  $|E[\int_0^S \int_0^T \sum_{i=1}^n Y_i(s, t) Z_i(s, t) ds]| < \infty$ . A sufficient condition for*

$$P(\int_0^S \int_0^T \sum_{i=1}^n Y_i(s, t) Z_i(s, t) ds < \infty) = 1$$

*is that for all  $s, t$  and  $i = 1, \dots, n$*

$$\begin{aligned} E[Y_i(s, t)^8] &< \infty \\ E[Z_i(s, t)^8] &< \infty \end{aligned}$$

In order to prove that some stochastic processes are continuously differentiable with respect to a parameter  $x$ , we have the following lemma.

**Lemma A.2** Suppose  $\int_0^S b(x, s, t, \omega)^2 dx = 1$ . Let

$$F(x, t, \omega) = \int_0^t \int_0^S b(x, s, t, \omega) W(ds, dt, \omega)$$

If  $b(\cdot, s, t, \omega)$  is continuously differentiable, then so is  $F(\cdot, t, \omega)$

**Sketch of Proof** We write the quadratic variation of  $F$  as:

$$[F(x), F(x)]_t = \int_0^t G(x, S, u) du$$

where, by a slight abuse of notation, for  $s \in [0, S]$ :

$$G(x, s, u) = \int_0^s b^2(x, s', u) W(ds', u)$$

Omitting the parameter  $u$ ,  $G(x, \cdot)$  has quadratic variation in the  $s$  direction given by:

$$[G(x), G(x)]_s = \int_0^s b^2(x, s') ds'$$

Since  $[G(x), G(x)]_s$  is continuously differentiable in  $s$ , by theorem 3.1.2 in Kunita  $G(x, s, t)$  is differentiable in  $x$ . Thus  $[F(x), F(x)]_t$  is continuously differentiable in  $p$ . Applying 3.1.2 in Kunita again, the result obtains.

## A.1 Preliminaries

Differentiating (36), we have:

$$\frac{\partial Q(p, t)}{\partial p} = -\exp\left(\int_{x=0}^p h(x, t) dx\right) \quad (45)$$

$$\frac{\partial^2 Q(p, t)}{\partial p^2} = -h(p, t) \exp\left(\int_{x=0}^p h(x, t) dx\right) \quad (46)$$

As for the infinitesimal parameters of  $Q$ , let  $f(p, t)$  be the drift of  $\int_{x=0}^p q(x, t) dx$ . By lemma A.2,

$$\begin{aligned} f(p, t) &= \int_{x=0}^p h(x, t) \exp\left(\int_{y=0}^x h(y, t) dy\right) a(p)(\bar{h}(p) - x) dx \\ &\quad + \frac{1}{2} \int_{x=0}^p h^2(x, t) \exp\left(\int_{y=0}^x h(y, t) dy\right) \sigma_h^2(x) dx + \\ &\quad + \frac{1}{2} (h(p, t) \exp\left(\int_{x=0}^p h(x, t) dx\right) \sigma_h^2(p) - h(0, t) \sigma_h^2(0)) + \\ &\quad + \frac{1}{2} \int_{x=0}^p h(x, t) \exp\left(\int_{y=0}^x h(y, t) dy\right) [h(x, t) \sigma_h^2(x) + \frac{d}{dx} \sigma_h^2(x)] dx \end{aligned} \quad (47)$$

By Ito's lemma, the drift of  $Q(p, t)$  is:

$$\begin{aligned} \mu_Q(p, t) &= \eta(t) f(S, t) - f(p, t) + \\ &\quad + a_\eta(\bar{\eta} - \eta(t)) \int_{x=0}^S \exp\left(\int_{y=0}^x h(y, t) dy\right) dx + \\ &\quad + g(\eta(t)) \rho_{h, \eta}(x) \int_{x=0}^S h(x, t) \exp\left(\int_{y=0}^x h(y, t) dy\right) \sigma_h(x) dx \end{aligned} \quad (48)$$

where  $\rho_{h,\eta}(x)$  is the instantaneous correlation of  $h(x)$  and  $\eta$ . The volatility of  $Q(p, t)$  is:

$$\begin{aligned}\Sigma(p, s, t) = & g(\eta(t))b_\eta(s)(Q(S, t) - Q(0, t)) + \\ & \eta(t) \int_{x=0}^S h(x, t) \exp\left(\int_{y=0}^x h(y, t) dy\right) \sigma_h(x) b_h(x, s) dx - \\ & \int_{x=0}^p h(x, t) \exp\left(\int_{y=0}^x h(y, t) dy\right) \sigma_h(x) b_h(x, s) dx\end{aligned}\tag{49}$$

## A.2 Verification of Assumptions 4 and 8

It is clear from (46), and lemma 999 that  $Q$  is twice continuously differentiable in  $p$ . Thus assumption 4 is satisfied as well as IWK (ii) (continuity of  $Q$  in  $p$ )

Assumption IWK (i) (boundedness of  $P(t)$ ) is verified in remark RAN.

### A.2.1 Verification of IWK iii.a)

Drift: As seen above  $h(p, t)$  is continuous in  $p$ . By continuity of  $a(p)$  and  $\frac{d}{dp}\sigma_h^2(p)$  the process  $f(p, t)$  is differentiable in  $p$ . Inspection of (48) shows that  $\mu_Q$  is continuous in  $p$ .

Volatility: We need only investigate the third row of (49), which is clearly a continuous function of  $p$ . So is  $\sigma_Q(p, t)$ .

### A.2.2 Verification of IWK iii.b)

Continuity of terms (1) and (2) obvious. From (49), we calculate

$$\frac{\partial \Sigma(p, s, t)}{\partial p} = -h(p, t) \exp\left(\int_{x=0}^p h(x, t) dx\right) \sigma_h(p, t) b_h(p, s, t)\tag{50}$$

which is clearly a continuous function of  $p$ , thus proving continuity of term (3). Continuity of terms (4) and (5) follows trivially.

### A.2.3 Verification of IWK iv and v

Clearly  $\int_{y=0}^p h(y, t) dy$  is normal, thus  $q(p, t)$  is lognormal, and has thus finite moments. For simplicity, we drop the argument  $t$ . By Lemma A.2, we can break the argument into verifying finiteness of the 16th moments of the following random variables:

1. **Net demand**  $Q(p)$ . By Jensen's inequality:

$$E\left[\left(\int_0^S q(x) dx\right)^{16}\right] \leq \int_0^S E\left[\exp\left(16 \int_0^x h(y) dy\right)\right] ds$$

which is bounded. By (36)

$$-\int_{x=0}^S q(x) dx \leq Q(p) \leq \int_{x=0}^S q(x) dx$$

Thus  $E[Q(p)^{16}]$  is bounded.

2. **Volatility of net demand**  $\sigma_Q$ . Observe that  $\sigma_h(x, s)$  is bounded. Since  $g(\eta)$  and  $\eta$  are bounded, it suffices to look at the last term of (49), to which we apply Jensen's inequality:

$$E\left[\left(\int_{x=0}^S h(x) \exp\left(\int_{y=0}^x h(y) dy\right) dx\right)^{16}\right] = E\left[\left(\int_{x=0}^S \frac{d}{dx} \exp\left(\int_{y=0}^x h(y) dy\right) dx\right)^{16}\right] = E\left[\exp\left(\int_{y=0}^S h(y) dy\right) dx\right]^{16}$$

Thus  $E[\sigma_Q(p)^{16}]$  is bounded.

3. **Drift of net demand**  $\mu_Q$ . Applying Jensen and Cauchy inequalities to (48), it is sufficient to verify the finiteness of the 16th moments of:  
 $f(S)$ ,  $\int_{x=0}^S \exp(\int_{y=0}^x h(y) dy) dx$ , and  $\int_{x=0}^S h(x) \exp(\int_{y=0}^x h(y) dy) \sigma_h(x) dx$ . The same type of development as before applies to the first 3 terms of  $f(S)$ .  $E[f(S)^{16}]$  is bounded because  $\sigma_h^2(x)$  is differentiable.
4. **Price drift**  $\mu_P$ . We refer the reader to (13) in the main text. Since the numerator has finite 16th moment, it is sufficient, by the Cauchy-Schwartz inequality, to prove boundedness of:

$$E\left[\left(\frac{1}{\partial Q / \partial p}\right)^{16}\right] = E\left[\exp\left(-16 \int_{x=0}^p h(x, t) dx\right)\right]$$

which is clear.

5. **Price volatility**  $\sigma_P$ . We use the same argument as above, since

$$\sigma_P = \frac{\sigma_Q}{\partial Q / \partial p}$$

6. **First derivative of net demand**  $\frac{\partial Q(p)}{\partial p}$ . Finiteness of the moment is clear, since the random variable in (45) is lognormal.
7. **Second Derivative of Net Demand**  $\frac{\partial^2 Q(p)}{\partial p^2}$ . Using Cauchy-Schwartz on (46), we see that this has finite 16th moment.
8. **Derivative of Net Demand Volatility**  $\frac{\partial \Sigma(p, s)}{\partial p}$ . We reuse (50), and use the same logic as above.

### A.3 Verification of Assumption 9

Since  $Q(., t)$  is bijective and continuous, so is its inverse  $P(., t)$ . Thus IWK (ii) is verified. We rewrite (45) as:

$$\frac{\partial Q(P(x, t), t)}{\partial p} = -\exp\left(\int_{x=0}^{P(x, t)} h(y, t) dy\right)$$

Thus:

$$\frac{\partial P(x, t)}{\partial x} = -\exp\left(\int_{x=0}^{-P(x, t)} h(y, t) dy\right) \quad (51)$$

$$\frac{\partial^2 P(x, t)}{\partial x^2} = P(x, t) \exp\left(\int_{x=0}^{-P(x, t)} h(y, t) dy\right) \quad (52)$$

### A.3.1 Verification of IWK iii.a)

Immediate from (13),(14) and (15). Continuity of the characteristics of  $Q$  was proved under assumption 7. The result follows from the continuity of  $P$  as a function of  $x$ .

### A.3.2 Verification of IWK iii.b)

Continuity of terms (1) and (2) obvious follows from (51) and (52).. From (14), we calculate

$$\frac{\partial[\Sigma(x, s, t)]}{\partial x} = -\frac{\partial}{\partial x} \left( \frac{\Sigma(x, s, t)}{\frac{\partial Q}{\partial p}(P(x, t))} \right) \quad (53)$$

and continuity of term (3) follows from the differentiability of  $P$  (see (50) and (51)).

### A.3.3 Verification of IWK iv and v

For simplicity, we drop the argument  $t$ . By Lemma A.2, we can break the argument into verifying finiteness of the 16th moments of the following random variables:

1. **Price**  $P(x)$ . By definition it is bounded.
2. **Volatility of price**  $\sigma_P$ . Obvious from the proof of Assumption 8.
3. **Price drift**  $\mu_P$ . Obvious from the proof of Assumption 8.
4. **Strategy drift**  $\mu_\theta$ . By Assumption 7.
5. **Strategy volatility**  $\sigma_\theta$ . By Assumption 7.
6. **First derivative of price**  $\frac{\partial P}{\partial x}$ . From (51).
7. **Second derivative of price**  $\frac{\partial^2 P}{\partial x^2}$ . From (52).
8. **Derivative of Price Volatility**  $\frac{\partial[\sigma_P(x, t)b_P(x, s, t)]}{\partial x}$ . From (53) and the fact that  $\frac{\partial[\Sigma(x, s, t)]}{\partial x}$  has finite 8th moment (see above).

## A.4 Verification of Assumption 10 and Equation (22)

We first verify that  $B(p, t)$  defined (21) in is differentiable in  $L^2(\mathbb{P})$ . The term  $\frac{\partial^2 Q}{\partial p^2}$  is differentiable in  $p$  by Lemma A.2. A similar line of proof allows us to conclude (see also the proof of assumption 1) that  $\mu_Q(p, t)$  and  $C(p, t)$  are also differentiable. Thus  $B(p, t)$  is differentiable. We now differentiate (22) w.r.t.  $p$  to obtain:

$$\int_{\delta_1}^S h(p) \exp\left(\int_{y=0}^p h(y) dy\right) \sigma_h(p) b_h(p, s) \lambda(s) ds = \frac{\partial B(p)}{\partial p}$$

Since  $h(p)$  is different from zero by probability one, we have, for  $p \in [0, S]$ :

$$\int_{\delta_1}^S b_h(p, s) \lambda(s) ds = \frac{\frac{\partial B(p)}{\partial p}}{h(p, t) \exp\left(\int_{y=0}^p h(y) dy\right) \sigma_h(p)} \quad (54)$$

where the right-hand side is a.s. bounded (see the proof of Assumption 1). By the assumption that  $B_h$  has full rank,  $\lambda(s)$  is determined by (54) for  $s \in [\delta_1, S]$ , and is a.s. bounded. We can then specify  $\lambda(s)$  for  $s \in [0, \delta_1]$  by solving the single equation:



$$\int_{s=0}^{\delta_1} \Sigma(0, s) \lambda(s) ds = B(0) - \int_{s=\delta_1}^S \Sigma(0, s) \lambda(s) ds \quad (55)$$

Since the right-hand side is a.s. bounded, it is possible to find a solution  $\lambda(s)$  to (55) for  $s \in [0, \delta_2]$  which is a.s. bounded. By Theorem 7.19 in RAN, Novikov's criterion holds.

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