

# A NO-ARBITRAGE MODEL OF LIQUIDITY IN FINANCIAL MARKETS INVOLVING STOCHASTIC STRINGS: APPLICATIONS TO HIGH-FREQUENCY DATA



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RAN ZHAO

WITH HENRY SCHELLHORN

CLAREMONT GRADUATE UNIVERSITY

# Outline

1. Trading Limit Orders
2. Option Pricing in an Illiquid Market
3. Calibration and Simulation
4. Option Pricing Empirical Analysis



# Market vs Limit Orders

A (buy) market order specifies

- how many shares a trader wants to buy,
- that he is willing to buy them at any price.

A (buy) limit order specifies

- how many shares a trader wants to buy,
- at what maximum price he is willing to buy them.



# Limit order matching mechanism



Figure: Limit order matching mechanism.



# Limit order matching mechanism



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Figure: Limit order matching mechanism.



# Limit order matching mechanism



Figure: Limit order matching mechanism.





# Demand v.s. Supply

The order books contain all the information about demand and supply.

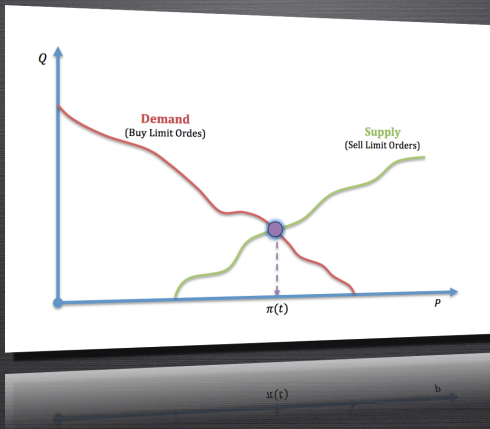


Figure: Demand v.s. Supply.



# The dynamics of Limit Orders in 3D

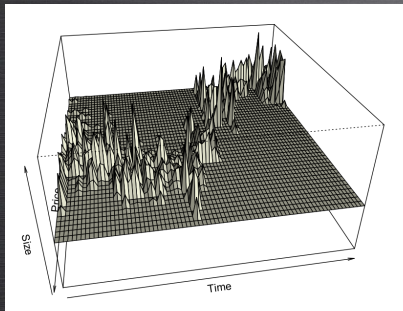


Figure: Buy Limit Orders of ORCL on April 4, 2011.

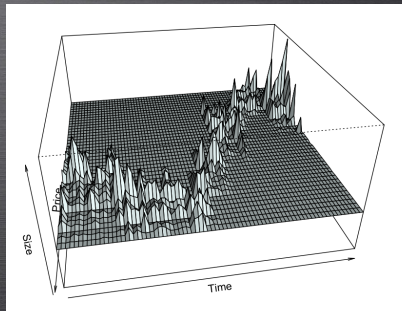


Figure: Sell Limit Orders of ORCL on April 4, 2011.

# The dynamics of the Clearing Price process

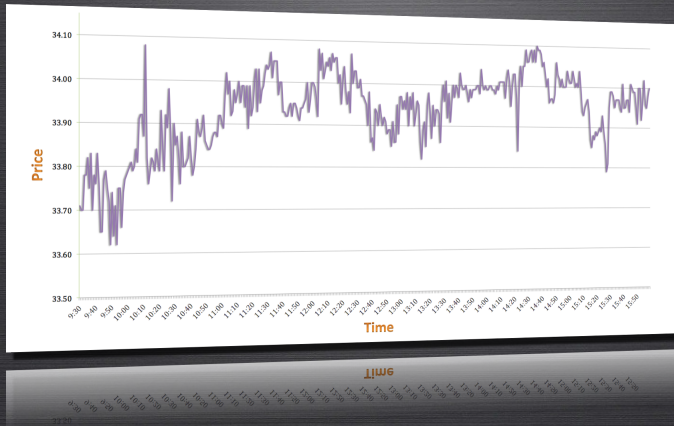


Figure: The dynamics of Oracle Corporation's Clearing Prices on April 1, 2011.



# Literature Review: Liquidity Models

## Market Manipulation (feedback) Models

- Jarrow (1994),
- Platen and Schweizer (1998),
- Sircar and Papanicolaou (1998),
- Frey (1998),
- Schonbucher and Wilmott (2000),
- Bank and Baum (2004).

## Price-taking (competitive) Models

- Cetin, Jarrow, and Protter (2004),
- Cetin and Rogers (2006),
- Cetin, Soner, and Touzi (2009),
- Kallsen and Rheinlaender (2009),
- Gokay and Soner (2011).



# Net Demand Curve and Clearing Price

## Definition

The net demand curve  $Q$  is a function  $[0, P] \times \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}$ , which value  $Q(p, t, \omega)$  is equal to the difference between the quantity of shares **available** for purchase and the quantity of shares **available** for sale at price  $p$  at time  $t$ . For each  $p$  the stochastic process  $Q(\cdot, t, \cdot)$  is a  $\mathcal{F}_t$  adapted semimartingale.

**Remark:** For the clearing price to be a diffusion, the demand must be defined on a continuum of limit prices.

**Remark:** The net demand curve should be decreasing in  $p$ . The easiest way to do that is to model positive processes:

- $Q(0, t)$ : total number of buy orders
- $q(p, t)$ : density of buy orders + density of sell orders

$$Q(p, t) = Q(0, t) - \int_0^p q(y, t) dy$$

## Definition

The clearing price  $\pi(t)$  is a  $\mathcal{F}_t$ - adapted stochastic process which satisfies market clearing:

$$Q(\pi(t), t) = 0$$



# The Model

- It can be proved that the optimal strategy of a large trader is to disseminate her orders into infinitesimal orders.

This shows that a continuous demand curve is a plausible model.

- In order to avoid arbitrage, we choose to have as many factors as limit prices

$$dQ(p, t) = \mu_Q(p, t)dt - \sigma_Q(p, t) \int_{x=0}^S b_q(p, x, t)W(ds, dt) \text{ for } 0 < p \leq S$$



# Main Result: Market with a Large Trader

## Main Result

*Suppose in addition to our standing assumptions that*

- C1) The demand curve is decreasing in price and continuous in time;*
- C2) the volatility  $\sigma_Q(p, t)$  is bounded away from zero, uniformly in  $p$ ;*
- C3) there is no path such that  $Q(S, t) \geq 0$  or  $Q(0, t) \leq 0$ ;*
- C4) The market price of risk equations hold.*

*Then*

- F1) there is no arbitrage strategy,*
- F2) the net demand curve  $Q$  is continuous in  $t$ ,*
- F3) the clearing price  $\pi(t)$  is continuous,*
- F4) The  $\mathbb{Q}$ -measure is also a martingale measure for  $\pi(t)$ .*



# Characterization of the Risk-Neutral Measure $\mathbb{Q}$

## Change of Measure

$$W^{\mathbb{Q}}(ds, dt) = W(ds, dt) + \lambda(s, t)dt$$

**Goal:** determine  $\lambda$  such that the market price of risk equations hold.





# Market Price of Risk Equations

Let  $\{\mu_Q, \sigma_Q, b_Q\}$  be the parameters of net demand. Let the volatility of the clearing price be:

$$\sigma_\pi(t)b_\pi(s, t) = -\frac{\sigma_Q(\pi(t), t) \int_{s=0}^S b_Q(\pi(t), s, t) ds}{q(\pi(t), t)}$$

Define

$$\begin{aligned} C(\pi, t) &= \sigma_\pi(t) \int_{s=0}^S \frac{\partial[\sigma_Q(p, t, \omega)b_Q(p, s, t)]}{\partial p} \Big|_{p=\pi} b_\pi(s, t) ds \\ B(\pi, t) &= \mu_Q(\pi, t) - \frac{1}{2} \frac{\partial^2 Q(p, t)}{\partial p^2} \Big|_{p=\pi} + C(\pi, t) \\ \Sigma(\pi, s, t) &= \sigma_Q(\pi, t)b_Q(\pi, s, t) \end{aligned}$$

The market price of risk equations are:

$$\int_{s=0}^S \Sigma(\pi, s)\lambda(s)ds = B(\pi) \quad 0 \leq \pi \leq P$$



# From a "MetaModel" to a Model

Reminder:

$$q(p, t)dp = \text{sum of buy and sell order quantities with limit price} \\ \text{in } [p, p + dp] \text{ arriving in } [0, t]$$

Plausible dynamics for  $q$ :

- positive process  
not necessarily increasing: orders can be cancelled
- mean-reverting process
- to be implemented on a computer:  $p$  and  $t$  must take discrete values

**Our choice:** the exponential of a (vector) Ornstein-Uhlenbeck process.



# Model Framework

We define

$$dh(p, t, \omega) = [-a_h(p)h(p, t, \omega) + \mu_h(p)]dt + \int_{s=0}^S \sigma_h(p, s)W(ds, dt, \omega)$$

$$q(p, t, \omega) = \exp\left(\int_{x=0}^p h(x, y, \omega)dx\right) \quad (\text{Density})$$

$$dQ(p, t, \omega) = \int_{x=0}^S q(x, t, \omega)dx\eta(t, \omega) - \int_{x=0}^p q(x, t, \omega)dx \quad (\text{Net Demand})$$

$$\begin{aligned} d\eta(t, \omega) &= a_\eta(\bar{\eta} - \eta(t, \omega))dt \\ &+ \sqrt{\eta(t, \omega)(1 - \eta(t, \omega))} \int_{s=0}^S \sigma_\eta(s)W(ds, dt, \omega) \quad (\text{Demand Volume Ratio}) \end{aligned}$$



# Reasoning of Modeling Framework

- The reason of modeling demand but not demand/supply separately  
Smooth semi-martingale assumption holds for net demand curve
- The reason of modeling the ratio of demand and volume  $\eta$  process  
To guarantee the robustness of the estimation:  $Q(0) > 0$  and  $Q(S) < 0$
- The reason for  $Q$  to be twice differentiable  
Need to apply Ito-Wentzell formula.
- The reason for  $q$  to be positive  
Guarantee that the net demand is downward sloping (decreases with price).
- The reason for using Stochastic strings  
Rule out arbitrage opportunity



# NYSE Arcabook Data

Industry	Exchange	Ticker	Firm
Energy	NYSE	CVX	Chevron Corporation
	NYSE	XOM	Exxon Mobil Corporation
Financial Banks	NYSE	JPM	JPMorgan Chase & Co.
	NYSE	WFC	Wells Fargo & Company
Materials and Mining	NYSE	ABX	Barrick Gold Corporation
	NYSE	FCX	Freeport-McMoRan Copper & Gold Inc.
Technology	NASDAQ	CSCO	Cisco Systems, Inc.
	NASDAQ	MSFT	Microsoft Corporation
	NASDAQ	ORCL	Oracle Corporation

Table: NYSE Arcabook data selection.



# Simulation Process

## Market Calibration

- Estimate excess demand  $\bar{Q}(p, t)$  from  $p$  and  $t$  from the high frequency data;
- Calculate the  $\eta(t)$  process by  $\frac{\bar{Q}(0, t)}{\bar{Q}(0, t) - \bar{Q}(S, t)}$ ;
- Calculate process  $q$  and  $h$ . Estimate the variance-covariance matrix of  $h$  and  $\eta$ ;
- Apply Cholesky decomposition to the correlation matrix, where  $\bar{R}dt = \text{Corr}[\frac{dh_i}{h_i}, \frac{dh_j}{h_j}]$

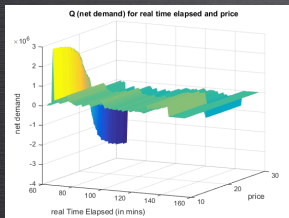


Figure: The net demand of GE as of 04/01/2011

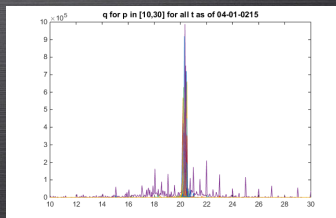


Figure: The q process of GE as of 04/01/2011



# Simulation Process

## Excess Demand Simulation

(The process of the simulation is to demonstrate that the calibrated parameters in framework are market consistent)

- Randomly generate  $n$  paths for  $h$ ;
- Use  $h$  to integrate for  $q$  process;
- Simulate  $\eta$  process using the correlation estimated in market calibration process;
- Combine  $h$ ,  $q$  and  $\eta$  to simulate excess demand  $Q$ .





## Simulation Result

GE stock net demand curve with 04/01/2015 market data calibrated:

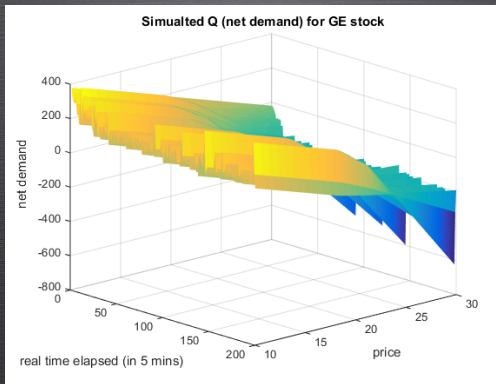


Figure: Simulated net demand  $Q$  for GE using 04/01/2015 order book calibration.

# Characterization of the Risk-Neutral Measure $\mathbb{Q}$

## Change of Measure

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**Goal:** determine  $\lambda$  such that the market price of risk equations hold.



# Market Price of Risk Equations

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Define

$$\begin{aligned} C(\pi, t) &= \sigma_\pi(t) \int_{s=0}^S \frac{\partial[\sigma_Q(p, t, \omega)b_Q(p, s, t)]}{\partial p} \Big|_{p=\pi} b_\pi(s, t) ds \\ B(\pi, t) &= \mu_Q(\pi, t) - \frac{1}{2} \frac{\partial^2 Q(p, t)}{\partial p^2} \Big|_{p=\pi} + C(\pi, t) \\ \Sigma(\pi, s, t) &= \sigma_Q(\pi, t)b_Q(\pi, s, t) \end{aligned}$$

The market price of risk equations are:

$$\int_{s=0}^S \Sigma(\pi, s)\lambda(s)ds = B(\pi) \quad 0 \leq \pi \leq P$$



# Volatility Smile (ORCL)

Strike Price	Implied Volatility
33.77	22.66%
33.80	21.05%
33.83	19.40%
33.87	17.71%
33.90	15.94%
33.93	14.10%
33.97	12.16%
34.00	10.11%
34.04	9.35%
34.07	10.26%
34.10	11.03%
34.14	11.70%
34.17	12.25%
34.20	12.71%
34.24	13.06%
34.27	13.25%
34.31	13.34%
34.34	13.32%
34.37	13.23%
34.41	12.99%

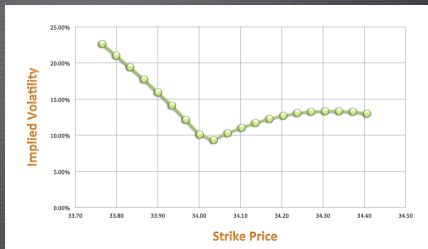


Figure: Simulated vs. real-time volatility smile of ORCL on April 4, 2011. Source: Bloomberg



# Volatility Smile (ABX)

Strike Price	Implied Volatility
50.98	23.64%
51.03	21.98%
51.08	20.27%
51.13	18.51%
51.18	16.70%
51.23	14.84%
51.29	12.93%
51.34	11.69%
51.39	12.24%
51.44	12.75%
51.49	13.20%
51.54	13.58%
51.59	13.92%
51.64	14.18%
51.69	14.37%
51.74	14.46%
51.79	14.54%
51.85	14.55%
51.90	14.46%
51.95	14.39%



Figure: Simulated vs. real-time volatility smile of ABX on April 4, 2011. Source: Bloomberg



# Volatility Smile (CSCO)

Strike Price	Implied Volatility
16.88	13.30%
16.90	12.17%
16.92	11.02%
16.93	9.84%
16.95	8.63%
16.97	7.37%
16.99	6.04%
17.00	4.63%
17.02	4.16%
17.04	4.84%
17.05	5.28%
17.07	5.53%
17.09	5.68%
17.10	5.73%
17.12	5.79%
17.14	5.85%

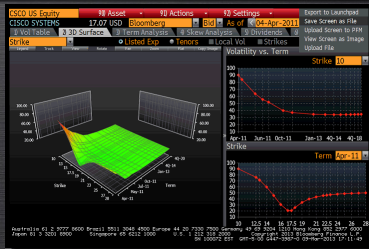
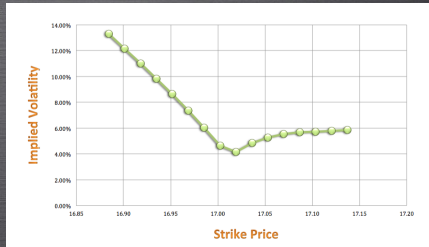


Figure: Simulated vs. real-time volatility smile of CSCO on April 4, 2011. Source: Bloomberg





# Conclusions

1. We developed a liquidity model with stands between
  - traditional no-arbitrage (option pricing) models and
  - financial economics models.
2. This model uses Ito-Wentzell's formula and Girsanov's theorem for Brownian sheets.
3. We give conditions for no-arbitrage, which allows us to price options
4. We specified a model
  - with positive demand density,
  - with mean-reversion,
  - where parameters are centered on the clearing price.
5. We calibrate the model parameters from high frequency stock data of GE as of 04-01-2015.  
The simulated net demand curve shows a downward sloping shape, as expected.
6. The model generates an implied volatility smile which matches the observed smile much better than the traditional Black-Scholes model.

