Asset Allocation and Liquidity Breakdowns: What if Your Broker Does not Answer the Phone?

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Running title: Asset Allocation and Liquidity Breakdowns

ABSTRACT: This paper analyzes the portfolio decision of an investor facing the threat of illiquidity. In a continuous-time setting, the efficiency loss due to illiquidity is addressed and quantified. For a logarithmic investor, we solve the portfolio problem explicitly. We show that the efficiency loss for a logarithmic investor with 30 years until the investment horizon is a significant 22.7% of current wealth if the illiquidity part of the model is calibrated to the Japanese data of the aftermath of WW II. For general utility functions, an explicit solution does not seem to be available. However, under a mild growth condition on the utility function, we show that the value function of a model in which only finitely many liquidity breakdowns can occur converges uniformly to the value function of a model with infinitely many breakdowns if the number of possible breakdowns goes to infinity. Furthermore, we show how the optimal security demands of the model with finitely many breakdowns can be used to approximate the optimal solution of the model with infinitely many breakdowns. These results are illustrated for an investor with a power utility function.

KEYWORDS: Illiquidity, Blackout Period, Portfolio Decision, Efficiency Loss, Rare Disasters

AMS SUBJECT CLASSIFICATIONS: 91B28, 93E20

JEL-CLASSIFICATION: G11

1 Introduction

Over decades the assumption that investors can trade continuously has been central to the theory of modern finance.[1] In the history of trading at the stock exchanges, there are however examples where liquidity dried out or trading has been (virtually) not possible for a number of reasons, including political turmoil, war, or hyperinflation. For instance, after World War II, the Tokyo stock exchange was closed from August 1945 until May 1949 and it reopened with a loss of 95% compared to the pre-war stock prices. Besides, the stock exchanges of European countries that had been invaded by Germany were closed down for some months. The same is true for the German stock exchanges that were closed for at least sixth months. Even the Swiss Stock Exchange closed from 10 May 1940 until 8 June 1940 and reopened with a loss of over 20%. Similarly, during the First World War several European stock exchanges were temporarily closed and even the NYSE closed for 4.5 months (31 July 1914 - 28 November 1914). Recently, the NYSE was closed for four days after the terrorist attacks of 9/11 and reopened on 17 September setting a record volume of 2.37 billion shares. The US stock market lost almost 10% of its value. This example shows that trading breaks can induce strong wishes to rebalance portfolios and may be accompanied by sharp price drops. The goal of this paper is to analyze the portfolio decision of an investor facing the threat of trading interruptions. As documented in the data, we allow for jumps in the market prices at the advent and the end of a trading interruption. The following table summarizes some examples.[2]

Exchange	Trading Break	Comment
London	07/1914 - 01/1915	WW I
New York	08/1914 - 11/1914	WW I
Zurich	05/1940 - 06/1940	WW II (Mobilization)
Frankfurt	04/1945 - 09/1945	Aftermath of WW II
Tokyo	08/1945 - 05/1949	Aftermath of WW II
New York	09/11/2001 - 09/14/2001	Terrorist Attack

Table 1: Examples for Major Trading Breaks

There are several related papers modeling liquidity effects explicitly. One strand of literature weakens the assumption that investors are price takers. In these models, trading takes place continuously, but large traders cannot trade without inducing price impacts. Papers dealing with this issue include Bank and Baum (2002) and Cetin et al. (2004), among others. A second strand of literature introduces transaction costs into the model implying that it is not optimal

[1] See, e.g., Merton (1969, 1971).

[2] See, e.g., Jorion and Goetzman (1999) or Siegel (2002). The are several other examples. For instance, Jorion and Goetzman report that in Germany, Italy, and German-occupied territories dealing in shares was subject to strict controls during WW II leading to a sharp fall in liquidity.

for investors to trade continuously. Papers in this area include Duffie and Sun (1990), Davis and Norman (1990), and Korn (1998), among others. Besides, Longstaff (2001) looks at the portfolio problem of an investor who can only implement portfolio strategies with finite variation and thus faces liquidity constraints. Schwartz and Tebaldi (2006) assume that an investor cannot trade a risky asset at all, i.e. the trading interruption is permanent. Examples are human wealth or housing. Rogers (2001) analyzes the portfolio decision of an investor who is constrained to change his strategy at discrete points in time only, although trading takes place continuously. Closely related to our paper are the papers by Kahl et al. (2003) and Longstaff (2005). Kahl et al. (2003) consider an investor's portfolio problem where the advent of a trading interruption is known and Longstaff (2005) analyzes the implications for the equilibrium prices of assets in such a setting. The problem presented in Kahl et al. (2003) can be solved recursively (firstly solve the problem in the absorbing state and then solve the problem for the non-absorbing state by using the solution of the absorbing state as boundary condition). In contrast to that, to the best of our knowledge, our paper is the first one solving a problem where both states (trading and non-trading state) are recurrent. This is mathematically more involved, since it cannot be solved recursively any more. From an economical point of view, it is also relevant, since during political turmoils or wars markets may close and reopen several times and nobody knows in advance how often this will take place. For instance, after 9/11 it was not obvious whether a second wave of terrorist attacks could soon hit the United States of America. Finally, our paper is also related to the asset pricing literature dealing with the equity premium puzzle. In particular, Rietz (1988) and recently Barro (2005) point out that the puzzle can (partly) be resolved if investors take into consideration the potential for a rare economic disaster occurring with a small probability.

Our paper contributes to the existing literature in multiple ways: From an economical point of view, the efficiency loss due to illiquidity is addressed and we are able to quantify the impact of illiquidity on an investor's portfolio decision. We show that the efficiency loss for a logarithmic investor with 30 years until the investment horizon is a significant 22.7% of current wealth if the illiquidity part of the model is calibrated to the Japanese data of the aftermath of WW II. Besides, we demonstrate that the threat of illiquidity can change the demand for risky securities tremendously. From a mathematical point of view, we are able to solve a continuous-time multi-state portfolio problem and present an (almost) closed-form solution to a system of coupled Hamilton-Jacobi-Bellman (HJB) equations if the investor has logarithmic utility. For general utility functions, an explicit solution does not seem to be available. However, under a polynomial growth condition on the utility function, we show that the value function of a model in which only finitely many liquidity breakdowns can occur converges uniformly to the value function of a model with infinitely many breakdowns if the number of possible breakdowns goes to infinity (see Theorem 3.2). Furthermore, we show how the optimal security demands of the models with finitely many breakdowns can be used to approximate the optimal solution of the model with infinitely many breakdowns. We emphasize that our convergence result is proved *without* using techniques from stochastic control theory such as HJB equations. Besides, we only require that the sequence of stopping times triggering the regime shifts satisfies a mild integrability condition. After having established the convergence result, we assume that these stopping times are the jump times of a Poisson process. Then the corresponding HJB equations are introduced and a verification result is proved.

From a practical point of view, one would also like to analyze cases where investors can trade in some stocks, while others cannot be traded. It would thus be interesting to study cases with many stocks and possible breakdowns for individual stocks (like ENRON not trading during the investigation).[3] This is a challenging problem since the dynamics of the proportions invested in stocks during a breakdown become much more involved than in our case (see Lemma 2.1). Besides, the explicitness for the case of a log-investor is lost. We thus leave this generalization as an area of future research.

The remainder of the paper is structured as follows. Section 2 describes the continuous-time framework. Section 3 introduces the investor's portfolio problem and establishes the convergence result of the value functions for general utility functions. In Section 4, the HJB equations are provided and a verification result is proved. Section 5 derives an explicit solution for a logarithmic investor if infinitely many liquidity breakdowns are possible. In Section 6, we discuss portfolio problems where only finitely many liquidity breakdowns are possible and the investor has logarithmic or power utility. Furthermore, we analyze the relations of both problems to the situations with infinitely many possible liquidity breakdowns. Section 7 illustrates our theoretical results by a numerical analysis and Section 8 concludes.

2 Continuous-Time Portfolio Dynamics with Illiquidity

In this section, we provide a model of a two-asset securities market if liquidity breakdowns can occur. It is assumed that all random variables and stochastic processes are defined on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}(t)\}_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions where $T > 0$ is a finite time horizon. One asset is a locally risk-free money market account M and the second security is a risky asset with price S. We suppose that the economy is in one of two possible liquidity regimes which we refer to as state 0 and state 1. We interpret state 0 as the normal state of the market in which trading takes place continuously and state 1 as an illiquidity state in which trading is not possible and asset prices can have different dynamics than in state 0. We assume that regime shifts from state i to state $1-i$ are triggered by a counting process $N_{i,1-i}$ as long as a given maximal number $k_0 \in \mathbb{N} \cup \{\infty\}$ of illiquidity regimes is not exceeded. The initial state is 0, and we assume that $N_{i,1-i}$ is non-explosive in that it has finitely many jumps in $[0, T]$ a.s. The current state of the market is then described by the $\{0, 1\}$-valued càdlàg process I given by

$$\begin{aligned} dI &= 1_{\{I_-=0,\ K_-<k_0\}} dN_{0,1} - 1_{\{I_-=1,\ K_-<k_0\}} dN_{1,0}, \\ dK &= 1_{\{I_-=1\}} dN_{1,0} \end{aligned}$$

with $I(t_0) = 0$ and $K(t_0) = 0$. The process K counts the number of jumps into the liquidity state since initial time $t_0 \in [0, T]$. The solutions of the above stochastic differential equations

[3] We thank an anonymous referee for mentioning this example.

(SDEs) will be denoted by I^{t_0,k_0} and K^{t_0,k_0} and we omit the superscripts if there is no ambiguity. Besides, we set $\tau_{0,1}^k = \inf\{t \in (\tau_{1,0}^{k-1}, T] : I(t) = 1\}$ as well as $\tau_{1,0}^k = \inf\{t \in (\tau_{0,1}^k, T] : I(t) = 0\}$ for $k \in \mathbb{N}$. Thus, $0 = \tau_{1,0}^0 \leq \tau_{0,1}^1 \leq \tau_{1,0}^1 \leq \tau_{0,1}^2 \leq \tau_{1,0}^2 \leq \dots$ are stopping times marking the regime shifts from one state into the other and we have

$$I^{0,k_0} = \sum_{k=1}^{k_0} 1_{[\![\tau_{0,1}^k, \tau_{1,0}^k[\![} \, .$$

The bond dynamics are given by

$$dM = M_- r_{I_-} dt, \quad M(0) = M_0$$

for constant riskless interest rates r_0, $r_1 > 0$, and the dynamics of the risky asset by

$$dS = S_-\big[(r_{I_-} + \alpha_{I_-})dt + \sigma_{I_-} dW - L_{I_-} dN_{I_-} - 1_{\{K_- < k_0\}} L_{I_-, 1-I_-} dN_{I_-,1-I_-}\big]$$

on $[\![0, T[\![$, $S(0) = S_0$, and $S(T) = \big(1 - 1_{\{I(T)=1\}}\ell\big)\, S_-(T)$. Here, $\alpha_0,\ \alpha_1 \in \mathbb{R}$ are excess returns, $\sigma_0,\ \sigma_1 \geq 0$ are volatilities, $L_0,\ L_1,\ L_{0,1},\ L_{1,0},\ \ell \in [0, 1)$ are loss rates, W is a standard Brownian motion, and $N_0,\ N_1$ are Poisson processes with constant intensities $\lambda_0,\ \lambda_1 \geq 0$. The constant ℓ models liquidation costs if at the investment horizon T the economy is in the illiquidity state. Our investor is restricted to choose self-financing strategies such that his wealth dynamics

$$dX = \varphi_- dS + [X_- - \varphi_- S_-]\, \frac{dM}{M_-}, \quad X(0) = x > 0,$$

have a unique solution X with $X(t) \geq 0$ for all $t \in [0, T]$ a.s. The càdlàg process φ denotes the number of stocks in the investor's portfolio given by

$$\varphi = \varphi_I = \{\varphi_{I(t)}(t)\}_{t\in[0,\infty)} \text{ with } \varphi_1 = \sum_{k=1}^{\infty} 1_{[\![\tau_{0,1}^k, \tau_{1,0}^k[\![}\ \varphi_0(\tau_{0,1}^k -).$$

In the liquidity state, the investor can choose his portfolio strategy φ_0 according to the above restrictions. However, when the market is illiquid, then the investor is forced to hold the portfolio that he has chosen before the liquidity breakdown. This strategy is modeled by the process φ_1. Alternatively, one can describe the investor's strategies by the wealth proportion invested in the risky asset. Therefore, we also introduce the portfolio processes π, π_0, and π_1 corresponding to φ, φ_0, and φ_1 via $\pi = \pi_I = \{\pi_{I(t)}(t)\}_{t\in[0,T]}$ with $\pi_i = \varphi_i S/X$ for $i = 0, 1$. The dynamics of π_1 are exogenously determined by the market. The wealth dynamics can then be rewritten as $dX = X_-[\pi_- \frac{dS}{S_-} + (1 - \pi_-)\frac{dM}{M_-}]$ or, more explicitly, as

$$dX = X_-\ \big[(r_{I_-} + \pi_- \alpha_{I_-})dt + \pi_- \sigma_{I_-} dW - \pi_- L_{I_-} dN_{I_-} - 1_{\{K_- < k_0\}} \pi_- L_{I_-,1-I_-} dN_{I_-,1-I_-}\big]$$

on $[\![0, T[\![$, $X(0) = x_0$, and $X(T) = \big(1 - 1_{\{I(T)=1\}} \pi_-(T)\ell\big)\, X_-(T)$. To avoid bankruptcy, short-selling is not allowed. Therefore, the class of admissible portfolio strategies consists of all càdlàg processes π_0 that take values in $[0, 1]$. As for the processes I and K, the variable X^{π_0,t_0,x_0,k_0} denotes the wealth process starting at time $t_0 \in [0, T]$ with initial value $x_0 \in (0, \infty)$. The following lemma derives an SDE for the dynamics of the investor's portfolio process in the illiquidity state and provides an explicit solution.

Lemma 2.1 (Portfolio Dynamics in Illiquidity)**.** *For every $k \in \mathbb{N}$, the dynamics of the portfolio process π on the stochastic interval $[\![\tau_{0,1}^k, \tau_{1,0}^k[\![$ are given by*

$$d\pi = \pi_-(1-\pi_-)\left[(\alpha_1 - \pi_-\sigma_1^2)dt + \sigma_1 dW - \frac{L_1}{1-\pi_- L_1} dN_1\right]$$

with $\pi(\tau_{0,1}^k) = \frac{\pi(\tau_{0,1}^k-)(1-L_{0,1})}{1-\pi(\tau_{0,1}^k-)L_{0,1}}$. This SDE has the closed-form solution $\pi = \frac{1}{1+Z}$ where

$$dZ = Z_-\left[(\sigma_1^2 - \alpha_1)dt - \sigma_1 dW + \frac{L_1}{1-L_1} dN_1\right], \quad Z(\tau_{0,1}^k) = \frac{1}{\pi(\tau_{0,1}^k)} - 1.$$

Proof of Lemma 2.1. The assertion follows from Itô's formula.[4] □

Remark. The previous lemma shows in particular that π takes values in $[0,1]$ only, since Z is a stochastic exponential and $\frac{L_1}{1-L_1} > -1$. Furthermore, note that Z is a geometric Brownian motion if $L_1 = 0$. ◇

3 Portfolio Problem with Illiquidity and Convergence

We now study the portfolio for an investor trading in the market described in the previous section. It is assumed that the investor maximizes expected utility from terminal wealth with respect to a concave non-decreasing utility function $U: (0,\infty) \to \mathbb{R}$, and thus he wishes to

$$\text{determine } \pi_0^* \text{ such that } \mathbb{E}\left[U\left(X^{\pi_0^*,0,x_0,k_0}(T)\right)\right] = \max_{\pi_0} \mathbb{E}\left[U\left(X^{\pi_0,0,x_0,k_0}(T)\right)\right],$$

where $x_0 \in (0,\infty)$ denotes his initial wealth. Firstly we remark that, for an admissible strategy π_0, the solution to the wealth equation is explicitly given by

$$\begin{aligned} X^{\pi_0,t_0,x_0,k_0}(t) &= x_0 \exp\left(\int_{t_0}^t r_{I_-} + \pi_-\alpha_{I_-} - \tfrac{1}{2}\pi_-^2\sigma_{I_-}^2\, ds + \int_{t_0}^t \pi_-\sigma_{I_-} dW\right) \\ &\quad \prod_{[t_0,t]} \left(1-\pi_- L_{I_-}\right)^{\Delta N_{I_-}} \left(1 - 1_{\{K_- < k_0\}}\pi_- L_{I_-,1-I_-}\right)^{\Delta N_{I_-,1-I_-}} \end{aligned}$$

for all $t \in [t_0,T)$ and arbitrary $t_0 \in [0,T]$, $x_0 \in (0,\infty)$, $k_0 \in \mathbb{N} \cup \{\infty\}$. This implies the following

Lemma 3.1 (Moments of the Wealth Process)**.** *If $\mathbb{E}\left[\beta^{N_{i,1-i}(T)}\right] < \infty$ for all $\beta \in (0,\infty)$ and $i = 0,1$, then for any $\kappa > 0$ there exists $C_\kappa \in (0,\infty)$ such that for all $t_0 \in [0,T]$ and $x_0 \in (0,\infty)$*

$$\sup_{\pi_0,\, k_0 \in \mathbb{N}\cup\{\infty\}} \mathbb{E}\left[\sup_{t\in[t_0,T]} \left(1 + X^{\pi_0,t_0,x_0,k_0}(t) + \frac{1}{X^{\pi_0,t_0,x_0,k_0}(t)}\right)^\kappa\right] \le C_\kappa \left(1 + x_0 + \frac{1}{x_0}\right)^\kappa.$$

Proof of Lemma 3.1. For $\kappa \in \mathbb{R}$, we set

$$M_\kappa = \sup_{\pi_0,\, k_0\in\mathbb{N}\cup\{\infty\}} \mathbb{E}\left[\sup_{t\in[t_0,T]} \exp\left(\kappa \int_{t_0}^t r_{I_-^{t_0,k_0}} + \pi_-\alpha_{I_-^{t_0,k_0}} - \tfrac{1}{2}\pi_-^2\sigma_{I_-^{t_0,k_0}}^2 ds + \kappa\int_{t_0}^t \pi_-\sigma_{I_-^{t_0,k_0}} dW\right)\right].$$

[4] A detailed proof is available from the authors upon request.

If $\kappa > 0$, $t_0 \in [0,T]$, $x_0 \in (0,\infty)$, $k_0 \in \mathbb{N} \cup \{\infty\}$, and π_0 is an admissible strategy, then the above explicit solution yields $\mathbb{E}[\sup_{t\in[t_0,T]} X^{\pi_0,t_0,x_0,k_0}(t)^{\kappa}] \leq x_0^{\kappa} M_\kappa$, By Cauchy's inequality, we then obtain

$$\mathbb{E}\left[\sup_{t\in[t_0,T]} X^{\pi_0,t_0,x_0,k_0}(t)^{-\kappa}\right] \leq x_0^{-\kappa}(1-\ell)^{-\kappa} M_{-2\kappa}^{\frac{1}{2}}$$

$$\mathbb{E}\left[\sup_{t\in[t_0,T]} \prod_{[t_0,t]} \left(1 - L_{I_-^{t_0,k_0}}\right)^{-2\kappa\Delta N_{I_-^{t_0,k_0}}} \left(1 - L_{I_-^{t_0,k_0},1-I_-^{t_0,k_0}}\right)^{-2\kappa\Delta N_{I_-^{t_0,k_0},1-I_-^{t_0,k_0}}}\right]^{\frac{1}{2}},$$

where

$$\begin{aligned}
&\sup_{t\in[t_0,T]} \prod_{[t_0,t]} \left(1 - L_{I_-^{t_0,k_0}}\right)^{-2\kappa\Delta N_{I_-^{t_0,k_0}}} \left(1 - L_{I_-^{t_0,k_0},1-I_-^{t_0,k_0}}\right)^{-2\kappa\Delta N_{I_-^{t_0,k_0},1-I_-^{t_0,k_0}}} \\
\leq\ & (1-L_0)^{-2\kappa N_0(T)}(1-L_1)^{-2\kappa N_1(T)}(1-L_{0,1})^{-2\kappa N_{0,1}(T)}(1-L_{1,0})^{-2\kappa N_{1,0}(T)};
\end{aligned}$$

the quantity on the right is integrable due to our assumption on $N_{i,1-i}$. The desired conclusion will thus follow from the fact that $M_\kappa < \infty$ for all $\kappa \in \mathbb{R}$; to show this, note that

$$\begin{aligned}
M_\kappa &= \sup_{\pi_0,\, k_0\in\mathbb{N}\cup\{\infty\}} \mathbb{E}\Bigg[\sup_{t\in[t_0,T]} \exp\Bigg(\kappa \int_{t_0}^t r_{I_-^{t_0,k_0}} + \pi_-\alpha_{I_-^{t_0,k_0}} - \tfrac{1}{2}\pi_-^2\sigma_{I_-^{t_0,k_0}}^2 + \tfrac{1}{2}\kappa\pi_-^2\sigma_{I_-^{t_0,k_0}}^2 ds \\
&\qquad + \kappa \int_{t_0}^t \pi_-\sigma_{I_-^{t_0,k_0}} dW - \tfrac{1}{2}\int_{t_0}^t \kappa^2\pi_-^2\sigma_{I_-^{t_0,k_0}}^2 ds\Bigg)\Bigg] \\
&\leq e^{\rho_\infty T} \sup_{\pi_0,\, k_0\in\mathbb{N}\cup\{\infty\}} \mathbb{E}\left[\sup_{t\in[t_0,T]} \exp\left(\int_{t_0}^t \kappa\pi_-\sigma_{I_-^{t_0,k_0}} dW - \frac{1}{2}\int_{t_0}^t \kappa^2\pi_-^2\sigma_{I_-^{t_0,k_0}}^2 ds\right)\right],
\end{aligned}$$

since the process $\kappa|r_{I_-^{t_0,k_0}} + \pi_-\alpha_{I_-^{t_0,k_0}} - \frac{1}{2}\pi_-^2\sigma_{I_-^{t_0,k_0}}^2 + \frac{1}{2}\kappa\pi_-^2\sigma_{I_-^{t_0,k_0}}^2|$ is bounded by a constant $\rho_\infty \in (0,\infty)$ that is independent of π_0, t_0, and k_0. Recall that $\pi_{I_-^{t_0,k_0}}$ is $[0,1]$-valued by the remark following Lemma 2.1. Next, let π_0 be an arbitrary admissible strategy, and let $t_0 \in [0,T]$, $k_0 \in \mathbb{N} \cup \{\infty\}$. Writing $\varrho = \kappa\pi_-\sigma_{I_-^{t_0,k_0}}$, it follows that ϱ is bounded by $\varrho_\infty \in (0,\infty)$, a constant independent of π_0, t_0, and k_0. Therefore, by the Novikov condition, the exponential $\exp\left(\int_{t_0}^{\cdot} \varrho dW - \frac{1}{2}\int_{t_0}^{\cdot} \varrho^2 ds\right)$ is a martingale and consequently

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[t_0,T]} \exp\left(\int_{t_0}^t \varrho dW - \frac{1}{2}\int_{t_0}^t \varrho^2 ds\right)\right] \leq \left(\mathbb{E}\left[\sup_{t\in[t_0,T]} \exp\left(\int_{t_0}^t \varrho dW - \frac{1}{2}\int_{t_0}^t \varrho^2 ds\right)^2\right]\right)^{\frac{1}{2}} \\
&\leq 4\left(\mathbb{E}\left[\exp\left(2\int_{t_0}^T \varrho dW - \int_{t_0}^T \varrho^2 ds\right)\right]\right)^{\frac{1}{2}} \\
&\leq 4\left(\mathbb{E}\left[\exp\left(\int_{t_0}^T 2\varrho dW - \tfrac{1}{2}\int_{t_0}^T (2\varrho)^2 ds\right)\exp\left(2\int_{t_0}^T \varrho^2 ds\right)\right]\right)^{\frac{1}{2}} \leq 4e^{\varrho_\infty^2 T} < \infty
\end{aligned}$$

by Doob's L^2-inequality. This gives the desired result. □

Returning to the investor's portfolio problem, we define the corresponding value function

$$V:\ [0,T]\times(0,\infty)\times(\mathbb{N}\cup\{\infty\}) \to \mathbb{R}, \quad V(t_0,x_0,k_0) = \sup_{\pi_0} \mathbb{E}\left[U\left(X^{\pi_0,t_0,x_0,k_0}(T)\right)\right].$$

By considering the strategy $\pi_0 = 0$, i.e. a pure bond investment, and applying the previous lemma together with Jensen's inequality, we obtain the following lower and upper bounds:

$$x_0 \le V(t_0, x_0, k_0) \le U\left(C_1(1 + x_0 + \tfrac{1}{x_0})\right).$$

In particular, the value function is finite. We thus obtain the following convergence result.

Theorem 3.2 (Convergence of the Value Functions)**.** *Suppose that* $\mathbb{E}\left[\beta^{N_{i,1-i}(T)}\right] < \infty$ *for all* $\beta \in (0, \infty)$ *and* $i = 0, 1$, *and that the investor's utility function* U *is polynomially bounded at* 0, *i.e. that there exist* $\kappa > 0$, $\rho > 0$ *and* $\delta > 0$ *such that*

$$|U(x)| \le \rho \left(1 + \frac{1}{x}\right)^{\kappa} \text{ for all } x \in (0, \delta).$$

Then the value function of the investor's portfolio problem satisfies

$$\lim_{k_0 \to \infty} \sup_{t_0 \in [0,T],\, x_0 \in K} |V(t_0, x_0, k_0) - V(t_0, x_0, \infty)| = 0$$

for any compact subset K *of* $(0, \infty)$.

Proof of Theorem 3.2. For any utility function U, we have the concavity estimate $U(x) \le \theta(x - 1)$ for some $\theta \in \mathbb{R}$, i.e. $\theta = U'(1)$ if U is differentiable, so by our assumption on U

$$|U(x)| \le \varrho \left(1 + x + \frac{1}{x}\right)^{\kappa} \text{ for all } x \in (0, \infty),$$

for suitably chosen $\kappa > 1$ and $\varrho > 0$. Thus, due to Lemma 3.1, compactness of K and our assumption on $N_{i,1-i}$, the family

$$\left\{U\left(X^{\pi_0,t_0,x_0,k_0}(T)\right)\right\}_{\pi_0,\, t_0 \in [0,T],\, x_0 \in K,\, k_0 \in \mathbb{N} \cup \{\infty\}} \text{ is uniformly integrable.}$$

Moreover, it is clear that

$$\sup_{\pi_0,\, t_0 \in [0,T],\, x_0 \in K} \left|U(X^{\pi_0,t_0,x_0,\infty}(T)) - U(X^{\pi_0,t_0,x_0,k_0}(T))\right| \to 0 \text{ in probability as } k_0 \to \infty$$

since

$$\begin{aligned} & \mathbb{P}\left(X^{\pi_0,t_0,x_0,\infty}(T) \ne X^{\pi_0,t_0,x_0,k_0}(T) \text{ for some admissible } \pi_0,\ t_0 \in [0,T],\ x_0 \in K\right) \\ \le \quad & \mathbb{P}\left(K^{t_0,x_0,k_0}(T) = k_0 \text{ for some } t_0 \in [0,T],\ x_0 \in K\right) \le \mathbb{P}(N_{1,0}(T) \ge k_0) \to 0 \text{ as } k_0 \to \infty. \end{aligned}$$

To prove convergence, we fix some $\varepsilon > 0$ and choose $\hat{t}_0 \in [0,T]$, $\hat{x}_0 \in K$ such that

$$\sup_{t_0 \in [0,T],\, x_0 \in K} |V(t_0, x_0, k_0) - V(t_0, x_0, \infty)| \le \left|V(\hat{t}_0, \hat{x}_0, k_0) - V(\hat{t}_0, \hat{x}_0, \infty)\right| + \tfrac{\varepsilon}{2}.$$

For the moment, assume that $V(\hat{t}_0, \hat{x}_0, k_0) - V(\hat{t}_0, \hat{x}_0, \infty) \ge 0$. Then let $\hat{\pi}$ be an admissible strategy such that

$$V(\hat{t}_0, \hat{x}_0, k_0) - \mathbb{E}\left[U(X^{\hat{\pi},\hat{t}_0,\hat{x}_0,k_0}(T))\right] \le \tfrac{\varepsilon}{2}.$$

Thus we have

$$
\begin{aligned}
&\sup_{t_0\in[0,T],\,x_0\in K} |V(t_0,x_0,k_0)-V(t_0,x_0,\infty)| \le V(\hat t_0,\hat x_0,k_0)-V(\hat t_0,\hat x_0,\infty)+\tfrac{\varepsilon}{2}\\
\le\quad & \mathbb{E}\left[U(X^{\hat\pi,\hat t_0,\hat x_0,k_0}(T))\right]-V(\hat t_0,\hat x_0,\infty)+\varepsilon \le \mathbb{E}\left[U(X^{\hat\pi,\hat t_0,\hat x_0,k_0}(T))\right]-\mathbb{E}\left[U(X^{\hat\pi,\hat t_0,\hat x_0,\infty}(T))\right]+\varepsilon\\
\le\quad & \mathbb{E}\left[\sup_{\pi_0,\,t_0\in[0,T],\,x_0\in K}\left|U(X^{\pi_0,t_0,x_0,k_0}(T))-U(X^{\pi_0,t_0,x_0,\infty}(T))\right|\right]+\varepsilon.
\end{aligned}
$$

Applying an analogous argument in the case when $V(\hat t_0,\hat x_0,k_0)-V(\hat t_0,\hat x_0,\infty)\le 0$, we see that the latter inequality continues to hold. Since $\varepsilon>0$ is arbitrary, we obtain

$$
\begin{aligned}
&\sup_{t_0\in[0,T],\,x_0\in K} |V(t_0,x_0,k_0)-V(t_0,x_0,\infty)|\\
\le\quad & \mathbb{E}\left[\sup_{\pi_0,\,t_0\in[0,T],\,x_0\in K}\left|U(X^{\pi_0,t_0,x_0,k_0}(T))-U(X^{\pi_0,t_0,x_0,\infty}(T))\right|\right],
\end{aligned}
$$

so that

$$
\sup_{t_0\in[0,T],\,x_0\in K} |V(t_0,x_0,k_0)-V(t_0,x_0,\infty)|\to 0 \text{ as } k_0\to\infty
$$

by the observations made at the beginning of the proof. □

Remark. The previous result shows that the investor's portfolio problem with possibly infinitely many liquidity breakdowns can be suitably approximated by an investment problem with finitely many jumps. Moreover, due to the uniformity of convergence, the optimal strategies of problems with sufficiently many breakdowns perform arbitrarily well in the case with infinitely many breakdowns. The following corollary makes this precise. ◇

Corollary 3.3 (Approximatively Optimal Strategies)**.** *For fixed $\varepsilon>0$ and for any $t_0\in[0,T]$ and $x_0\in(0,\infty)$ there exists a $\hat k_0\in\mathbb{N}$ such that for any admissible $\varepsilon/3$-optimal strategy $\hat\pi_0$ for $V(t_0,x_0,\hat k_0)$ we have*

$$
\left|\mathbb{E}\left[U(X^{\hat\pi_0,t_0,x_0,\infty}(T))\right]-V(t_0,x_0,\infty)\right|\le\varepsilon. \tag{1}
$$

Besides, if the investor's utility function U is of the form $U(x)=\frac{1}{\gamma}x^\gamma$, then it follows that the initial wealth x_k required to achieve the given indirect utility $V(t_0,x_0,\infty)$ in the model with finitely many jumps satisfies

$$
x_k=\left(\frac{\gamma V(t_0,x_0,\infty)}{V(t_0,1,k)}\right)^{\frac{1}{\gamma}}\to x_0 \text{ as } k\to\infty. \tag{2}
$$

Proof of Corollary 3.3. Given some $\varepsilon>0$, for any $t_0\in[0,T]$ and $x_0\in(0,\infty)$ we can choose $\hat k_0\in\mathbb{N}$ such that

$$
\sup_{\pi_0}\left|\mathbb{E}\left[U(X^{\pi_0,t_0,x_0,\hat k_0}(T))\right]-\mathbb{E}\left[U(X^{\pi_0,t_0,x_0,\infty}(T))\right]\right|\le\tfrac{\varepsilon}{3} \text{ and } \left|V(t_0,x_0,\hat k_0)-V(t_0,x_0,\infty)\right|\le\tfrac{\varepsilon}{3}.
$$

Furthermore, we can choose an admissible strategy $\hat\pi_0$ with

$$
\left|\mathbb{E}\left[U(X^{\hat\pi_0,t_0,x_0,\hat k_0}(T))\right]-V(t_0,x_0,\hat k_0)\right|\le\tfrac{\varepsilon}{3}.
$$

This leads to (1). Besides, if $U(x)=\frac{1}{\gamma}x^\gamma$, then the relation

$$V(t_0,x_0,k_0)=\sup_{\pi_0}\mathbb{E}\left[U(X^{\pi_0,t_0,x_0,k_0}(T))\right]=\tfrac{1}{\gamma}x_0^\gamma\sup_{\pi_0}\mathbb{E}\left[U(X^{\pi_0,t_0,1,k_0}(T))\right]=\tfrac{1}{\gamma}x_0^\gamma V(t_0,1,k_0)$$

implies (2). $\square$

Remarks. We wish to stress that, without additional assumptions, the optimal strategies do not have to converge. In one of the following sections, we will however present examples in which convergence can be proved (see Corollaries 6.2 and 6.9). $\diamond$

4 HJB Equations and Verification Theorem

We now investigate the optimal portfolio problem applying dynamic programming techniques. To obtain Markovian dynamics, we shall henceforth assume that the regime shift process $N_{i,1-i}$ is a Poisson process with intensity $\lambda_{i,1-i}\geq 0$ for $i=0,1$ so that the integrability condition of Lemma 3.1 and Theorem 3.2 is satisfied. Consider the optimal investment problem presented in the previous sections, and suppose that there are at most $k_0\in\mathbb{N}\cup\{\infty\}$ liquidity breakdowns. Then a collection

$$\{J^{0,k_0},J^{1,k_0},J^{0,k_0-1},J^{1,k_0-1},\dots,J^{0,1},J^{1,1},J^{0,0}\},$$

where $J^{0,k}$ is a $\mathrm{C}^{1,2}$-function on $[0,T]\times(0,\infty)$ and $J^{1,k}$ is a $\mathrm{C}^{1,2,2}$-function on $[0,T]\times(0,\infty)\times[0,1]$, is said to be a solution to the HJB equations of the portfolio problem if the following partial differential equations are satisfied.

$$\begin{aligned}
0 &= \sup_{\pi\in[0,1]}\Big\{J_t^{0,0}(t,x)+x(r_0+\alpha_0\pi)J_x^{0,0}(t,x)+\tfrac{1}{2}x^2\pi^2\sigma_0^2J_{x,x}^{0,0}(t,x)\\
&\quad +\lambda_0\left[J^{0,0}(t,x(1-\pi L_0))-J^{0,0}(t,x)\right]\Big\}\\
0 &= \sup_{\pi\in[0,1]}\Big\{J_t^{0,k}(t,x)+x(r_0+\alpha_0\pi)J_x^{0,k}(t,x)+\tfrac{1}{2}x^2\pi^2\sigma_0^2J_{x,x}^{0,k}(t,x)\\
&\quad +\lambda_0\left[J^{0,k}(t,x(1-\pi L_0))-J^{0,k}(t,x)\right]\\
&\quad +\lambda_{0,1}\left[J^{1,k}\left(t,x(1-\pi L_{0,1}),\tfrac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right)-J^{0,k}(t,x)\right]\Big\}\\
0 &= J_t^{1,k}(t,x,\pi)+x(r_1+\alpha_1\pi)J_x^{1,k}(t,x,\pi)+\tfrac{1}{2}x^2\pi^2\sigma_1^2J_{x,x}^{1,k}(t,x,\pi)\\
&\quad +x\pi^2(1-\pi)\sigma_1^2J_{x,\pi}^{1,k}(t,x,\pi)+\pi(1-\pi)(\alpha_1-\sigma_1^2\pi)J_\pi^{1,k}(t,x,\pi)\\
&\quad +\tfrac{1}{2}\pi^2(1-\pi)^2\sigma_1^2J_{\pi,\pi}^{1,k}(t,x,\pi)+\lambda_1\left[J^{1,k}\left(t,x(1-\pi L_1),\tfrac{\pi(1-L_1)}{1-\pi L_1}\right)-J^{1,k}(t,x,\pi)\right]\\
&\quad +\lambda_{1,0}\left[J^{0,k-1}(t,x(1-\pi L_{1,0}))-J^{1,k}(t,x,\pi)\right]
\end{aligned}$$

with boundary conditions $J^{0,k}(T,x)=U(x)$, $J^{1,k}(T,x,\pi)=U(x(1-\ell\pi))$ for all $x\in(0,\infty)$ and $\pi\in[0,1]$. If $k_0=\infty$, then a solution to the HJB equations simply consists of a pair $\{J^{0,\infty},J^{1,\infty}\}$, and the system above reduces to a pair of equations with $J^{0,\infty-1}=J^{0,\infty}$, etc. Note that this system can be solved iteratively if $k_0 < \infty$, whereas it does not decouple when $k_0 = \infty$. Given a solution $\{J^{0,k_0}, J^{1,k_0}, J^{0,k_0-1}, J^{1,k_0-1}, \ldots, J^{0,1}, J^{1,1}, J^{0,0}\}$ of the HJB equations, to simplify notation, we set $H^{1,k}(t,x,\pi) = 0$ and

$$\begin{aligned} H^{0,0}(t,x,\pi) &= J_t^{0,0}(t,x) + x(r_0+\alpha_0\pi)J_x^{0,0}(t,x) + \tfrac{1}{2}x^2\pi^2\sigma_0^2 J_{x,x}^{0,0}(t,x) \\ &\quad + \lambda_0\left[J^{0,0}(t,x(1-\pi L_0)) - J^{0,0}(t,x)\right] \\ H^{0,k}(t,x,\pi) &= J_t^{0,k}(t,x) + x(r_0+\alpha_0\pi)J_x^{0,k}(t,x) + \tfrac{1}{2}x^2\pi^2\sigma_0^2 J_{x,x}^{0,k}(t,x) \\ &\quad + \lambda_0\left[J^{0,k}(t,x(1-\pi L_0)) - J^{0,k}(t,x)\right] \\ &\quad + \lambda_{0,1}\left[J^{1,k}\left(t, x(1-\pi L_{0,1}), \tfrac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right) - J^{0,k}(t,x)\right] \end{aligned}$$

for $t \in [0,T]$, $x \in (0,\infty)$, and $\pi \in [0,1]$. We now show that $J^{0,k}$ corresponds to the value function of the optimal investment problem with k regime shifts outstanding.

Theorem 4.1 (Verification Theorem). *Let $\{J^{0,k_0}, J^{1,k_0}, J^{0,k_0-1}, J^{1,k_0-1}, \ldots, J^{0,1}, J^{1,1}, J^{0,0}\}$ be a solution of the HJB equations associated to the optimal investment problem with at most $k_0 \in \mathbb{N} \cup \{\infty\}$ regime shifts, and assume moreover that for each $i = 0,1$ and $k = 1,\ldots,k_0$ the functions $J^{i,k}$, $J_x^{i,k}$, $J_\pi^{i,k}$ and $J^{0,0}$, $J_x^{0,0}$, $J_\pi^{0,0}$ are polynomially bounded at 0 and ∞ uniformly with respect to $t \in [0,T]$ and $\pi \in [0,1]$. Then*

$$V(t_0,x_0,k_0) \le J^{0,k_0}(t_0,x_0) \text{ for all } t_0 \in [0,T] \text{ and } x_0 \in (0,\infty).$$

Moreover, if there are continuous functions $\psi_k :\ [0,T]\times(0,\infty) \to [0,1]$ such that

$$\psi_k(t,x) \in \arg\max_{\pi\in[0,1]} H^{0,k}(t,x,\pi) \text{ for each } k = 0,\ldots,k_0,$$

then $V(t_0,x_0,k_0) = J^{0,k_0}(t_0,x_0)$ for all $t_0 \in [0,T]$ and $x_0 \in (0,\infty)$, and the optimally controlled process X^* and the optimal strategy π_0^* satisfy $\pi_0^* = \psi_{k_0-K_-}(\cdot, X^*)$.

Remark. Given that $J^{i,k}(t,x,\pi) = f^{i,k}(t,\pi)U(x)$ or $J^{i,k}(t,x,\pi) = f^{i,k}(t,\pi) + U(x)$, then the polynomial growth assumption is satisfied if U and U' are polynomially bounded at 0 and $f^{i,k}$ and $f_\pi^{i,k}$ are bounded. This is for instance the case for power or log utility. $\diamond$

Proof of Theorem 4.1. Given an admissible strategy π_0, $t_0 \in [0,T]$, and $x_0 \in (0,\infty)$, consider the process $J(t) = J^{I(t),k_0-K(t)}(t,X(t),\pi(t))$ for all $t \in [t_0,T]$, where the upper indices π_0, t_0, x_0, k_0 are omitted for notational convenience and, by ignoring the third coordinate, $J^{0,k}$ is interpreted as a function defined on $[0,T]\times(0,\infty)\times[0,1]$. Applying Itô's formula and using

Lemma 2.1, we obtain

$$\begin{aligned}
dJ &= J_t^{I_-,k_0-K_-}(.,X_-,\pi_-)dt + J_x^{I_-,k_0-K_-}(.,X_-,\pi_-)X_-\left[(r_{I_-}+\alpha_{I_-}\pi_-)dt+\sigma_{I_-}\pi_- dW\right] \\
&\quad +\tfrac{1}{2}J_{x,x}^{I_-,k_0-K_-}(.,X_-,\pi_-)X_-^2\sigma_{I_-}^2\pi_-^2\,dt \\
&\quad +1_{\{I_-=1\}}\Big\{J_\pi^{1,k_0-K_-}(.,X_-,\pi_-)\pi_-(1-\pi_-)\left[(\alpha_1-\sigma_1^2\pi_-)dt+\sigma_1 dW\right] \\
&\qquad +\tfrac{1}{2}J_{\pi,\pi}^{1,k_0-K_-}(.,X_-,\pi_-)\pi_-^2(1-\pi_-)^2\sigma_1^2 dt + J_{x,\pi}^{1,k_0-K_-}(.,X_-,\pi_-)X_-\sigma_1^2\pi_-^2(1-\pi_-)dt\Big\} \\
&\quad +\left[J^{I_-,k_0-K_-}\left(.,(1-\pi_-L_{I_-})X_-,\tfrac{\pi_-(1-L_{I_-})}{1-\pi_-L_{I_-}}\right)-J^{I_-,k_0-K_-}(.,X_-,\pi_-)\right]dN_{I_-} \\
&\quad +1_{\{I_-=0,\,K_-<k_0\}}\left[J^{1,k_0-K_-}\left(.,(1-\pi_-L_{0,1})X_-,\tfrac{\pi_-(1-L_{0,1})}{1-\pi_-L_{0,1}}\right)-J^{0,k_0-K_-}(.,X_-)\right]dN_{0,1} \\
&\quad +1_{\{I_-=1\}}\left[J^{0,k_0-K_--1}(.,(1-\pi_-L_{1,0})X_-)-J^{1,k_0-K_-}(.,X_-,\pi_-)\right]dN_{1,0} \\
&= H^{I_-,k_0-K_-}(.,X_-,\pi_-)dt + J_x^{I_-,k_0-K_-}(.,X_-,\pi_-)X_-\sigma_{I_-}\pi_- dW \\
&\quad +1_{\{I_-=1\}}J_\pi^{1,k_0-K_-}(.,X_-,\pi_-)\pi_-(1-\pi_-)\sigma_1 dW \\
&\quad +\left[J^{I_-,k_0-K_-}\left(.,(1-\pi_-L_{I_-})X_-,\tfrac{\pi_-(1-L_{I_-})}{1-\pi_-L_{I_-}}\right)-J^{I_-,k_0-K_-}(.,X_-,\pi_-)\right]d\tilde{N}_{I_-} \\
&\quad +1_{\{I_-=0,\,K_-<k_0\}}\left[J^{1,k_0-K_-}\left(.,(1-\pi_-L_{0,1})X_-,\tfrac{\pi_-(1-L_{0,1})}{1-\pi_-L_{0,1}}\right)-J^{0,k_0-K_-}(.,X_-)\right]d\tilde{N}_{0,1} \\
&\quad +1_{\{I_-=1\}}\left[J^{0,k_0-K_--1}(.,(1-\pi_-L_{1,0})X_-)-J^{1,k_0-K_-}(.,X_-,\pi_-)\right]d\tilde{N}_{1,0}
\end{aligned}$$

on $[\![t_0,T[\![$, where $\tilde{N}_0,\tilde{N}_1,\tilde{N}_{0,1},\tilde{N}_{1,0}$ denote the compensated Poisson processes associated with $N_0,N_1,N_{0,1},N_{1,0}$. Due to our polynomial growth assumption and Lemma 3.1, the stochastic differentials of the local martingales in the above identity are, in fact, stochastic differentials of martingales. Therefore, by taking expectations and using the boundary conditions of the HJB equations, we arrive at

$$\mathbb{E}\left[U\left(X^{\pi_0,t_0,x_0,k_0}(T)\right)\right]=\mathbb{E}\left[J(T-)\right]=J^{0,k_0}(t_0,x_0)+\mathbb{E}\left[\int_{t_0}^T H^{I_-,k_0-K_-}(t,X_-(t),\pi_-(t))dt\right].$$

Since π_0, t_0, and x_0 are arbitrary, we conclude that $V(t_0,x_0,k_0)\le J^{0,k_0}(t_0,x_0)$ for all $t_0\in[0,T]$ and $x_0\in(0,\infty)$. Now, if $\psi_k:\ [0,T]\times(0,\infty)\to[0,1]$ is a continuous function such that

$$\psi_k(t,x)\in\arg\max_{\pi\in[0,1]}H^{0,k}(t,x,\pi)\text{ for each }k=0,\dots,k_0,$$

then the family $\{\psi_k\}_{k=0,\dots,k_0}$ defines an optimal feedback strategy in the sense that the SDE

$$\begin{aligned}
dX &= X_-\big[\left(r_{I_-}+\psi_{k_0-K_-}(\cdot,X_-)\alpha_{I_-}\right)dt+\psi_{k_0-K_-}(\cdot,X_-)\sigma_{I_-}dW \\
&\quad -\psi_{k_0-K_-}(\cdot,X_-)L_{I_-}dN_{I_-}-1_{\{K_-<k_0\}}\psi_{k_0-K_-}(\cdot,X_-)-L_{I_-,1-I_-}dN_{I_-,1-I_-}\big]
\end{aligned}$$

on $[\![t_0,T[\![$, $X(t_0)=x_0$, and $X(T)=\left(1-1_{\{I(T)=1\}}\psi_{k_0-K_-(T)}(T,X_-(T))\ell\right)X_-(T)$, admits a solution X^{ψ,t_0,x_0,k_0}, and the strategy $\pi_0^*=\psi_{k_0-K_-}\left(\cdot,X^{\psi,t_0,x_0,k_0}\right)$ is admissible and optimal for the investor's portfolio problem. Of course, in this case we have $\mathbb{E}\left[U\left(X^{\pi_0^*,t_0,x_0,k_0}(T)\right)\right]=V(t_0,x_0,k_0)=J^{0,k_0}(t_0,x_0)$. □

5 Infinitely Many Liquidity Breakdowns and Log Utility

In this section, we solve the portfolio problem with infinitely many liquidity breakdowns for $U(x) = \ln(x)$. In order to apply the above verification theorem, we conjecture

$$J^0(t,x) = J^{0,\infty}(t,x) = \ln(x) + f^0(t) \text{ and } J^1(t,x,\pi) = J^{1,\infty}(t,x,\pi) = \ln(x) + f^1(t,\pi)$$

for a C^1-function f^0 on $[0,T]$ with $f^0(T) = 0$ and a $C^{1,2}$-function f^1 on $[0,T] \times [0,1]$ satisfying $f^1(T,\pi) = \ln(1-\ell\pi)$ for all $\pi \in [0,1]$. Furthermore, we set $H^0 = H^{0,\infty}$ and $H^1 = H^{1,\infty}$. Then the HJB equations read

$$0 = \sup_{\pi\in[0,1]} \left\{ f_t^0(t) + g^0(\pi) + \lambda_{0,1}\left[f^1\left(t, \tfrac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right) - f^0(t) \right] \right\} \tag{3}$$

$$\begin{aligned} 0 = {} & f_t^1(t,\pi) - \lambda_{1,0} f^1(t,\pi) + \pi(1-\pi)(\alpha_1 - \sigma_1^2\pi) f_\pi^1(t,\pi) + \tfrac{1}{2}\pi^2(1-\pi)^2\sigma_1^2 f_{\pi,\pi}^1(t,\pi) \\ & + \lambda_1\left[f^1\left(t, \tfrac{\pi(1-L_1)}{1-\pi L_1}\right) - f^1(t,\pi)\right] + g^1(\pi) + \lambda_{1,0} f^0(t), \end{aligned} \tag{4}$$

where g^j is given by $g^j(\pi) = r_j + \alpha_j\pi - \frac{1}{2}\pi^2\sigma_j^2 + \lambda_j \ln(1-\pi L_j) + \lambda_{j,1-j}\ln(1-\pi L_{j,1-j})$ on $[0,1]$, $j = 0,1$. Equation (3) leads to the first-order condition

$$0 = \alpha_0 - \sigma_0^2\pi - \lambda_0\frac{L_0}{1-\pi L_0} - \lambda_{0,1}\frac{L_{0,1}}{1-\pi L_{0,1}} + \lambda_{0,1} f_\pi^1\left(t, \tfrac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right)\frac{1-L_{0,1}}{(1-\pi L_{0,1})^2} \tag{5}$$

for the optimal stock proportion in state 0. Note that the solution of the first-order condition is a deterministic function of time (if it exists).

Proposition 5.1 (Indirect Utility in Illiquidity). *For a C^1-function $f^0 : [0,T] \to \mathbb{R}$, consider the function $f^1 : [0,T]\times[0,1] \to \mathbb{R}$ defined via the stochastic representation*

$$f^1(t,\pi) = \int_t^T \left(\lambda_{1,0} f^0(s) + \mathbb{E}\left[g^1(\tilde\pi(s))\right]\right) e^{-\lambda_{1,0}(s-t)} ds + \mathbb{E}\left[\ln(1-\tilde\pi(T)\ell)\right] e^{-\lambda_{1,0}(T-t)},$$

where $\tilde\pi$ is given by $\tilde\pi(s) = \frac{\pi}{\pi+(1-\pi)Z(s)}$ with $dZ = Z_-[(\sigma_1^2-\alpha_1)ds - \sigma_1 dW + \frac{L_1}{1-L_1}dN_1]$, $Z(t) = 1$.
(i) Then f^1 is of class $C^{1,2}$ on $[0,T]\times[0,1]$ with

$$\begin{aligned} f_\pi^1(t,\pi) &= \int_t^T \mathbb{E}\left[\tfrac{\partial\tilde\pi(s)}{\partial\pi} g_\pi^1(\tilde\pi(s))\right] e^{-\lambda_{1,0}(s-t)} ds - \mathbb{E}\left[\frac{\partial\tilde\pi(T)}{\partial\pi}\frac{\ell}{1-\tilde\pi(T)\ell}\right] e^{-\lambda_{1,0}(T-t)}, \\ f_{\pi,\pi}^1(t,\pi) &= \int_t^T \mathbb{E}\left[\tfrac{\partial^2\tilde\pi(s)}{\partial\pi^2} g_\pi^1(\tilde\pi(s)) + \left(\tfrac{\partial\tilde\pi(s)}{\partial\pi}\right)^2 g_{\pi,\pi}^1(\tilde\pi(s))\right] e^{-\lambda_{1,0}(s-t)} ds \\ &\quad -\mathbb{E}\left[\tfrac{\partial^2\tilde\pi(T)}{\partial\pi^2}\tfrac{\ell}{1-\ell\tilde\pi(T)} + \left(\tfrac{\partial\tilde\pi(T)}{\partial\pi}\right)^2 \tfrac{\ell^2}{(1-\ell\tilde\pi(T))^2}\right] e^{-\lambda_{1,0}(T-t)}, \end{aligned}$$

where $\frac{\partial\tilde\pi(s)}{\partial\pi} = \frac{Z(s)}{(\pi+(1-\pi)Z(s))^2}$ denotes the derivative of $\tilde\pi(s)$ w.r.t. the initial value π of the process $\tilde\pi$.

(ii) f^1 solves the second HJB equation.

In particular, f_π^1 does not depend on f^0, and thus the first-order condition (5) provides an algebraic equation for the optimal stock proportion π.

Proof of Proposition 5.1. (i) The explicit representation

$$
\begin{aligned}
f^1(t,\pi) &= \sum_{n=0}^{\infty}\int_t^T e^{-\lambda_{1,0}(s-t)}p_n(t,s)(\lambda_{1,0}f^0(s)+\int_{-\infty}^{\infty}g^1(\tilde{\pi}_n(t,s,u))\psi_{s-t}(u)du)ds \\
&\quad e^{-\lambda_{1,0}(T-t)}\sum_{n=0}^{\infty}p_n(t,T)\int_{-\infty}^{\infty}\ln(1-\ell\tilde{\pi}_n(t,T,u))\psi_{T-t}(u)du,
\end{aligned}
$$

where

$$
\begin{aligned}
p_n(t,s) &= \mathbb{P}(N_1(s-t)=n)=\frac{e^{-\lambda_1(s-t)}(\lambda_1(s-t))^n}{n!}, \\
\psi_r(u) &= \frac{1}{\sqrt{2\pi r}}e^{-\frac{u^2}{2r}}, \\
\tilde{\pi}_n(t,s,u) &= \frac{\pi}{\pi+(1-\pi)z_n(t,s,u)}, \\
z_n(t,s,u) &= \frac{e^{(\frac{1}{2}\sigma_1^2-\alpha_1)(s-t)-\sigma_1 u}}{(1-L_1)^n}.
\end{aligned}
$$

implies that f^1 is continuously differentiable with respect to t. Let $(t,\pi)\in[0,T]\times[0,1]$ and let $s\in[t,T]$. We have $|g_\pi^1(\pi)|\leq|\alpha_1|+\sigma_1^2+\lambda_1\frac{L_1}{1-L_1}+\lambda_{1,0}\frac{L_{1,0}}{1-L_{1,0}}$ and $\frac{\partial\tilde{\pi}(s)}{\partial\pi}\leq\frac{Z(s)}{(Z(s)\wedge 1)^2}$ and therefore, by the remark following Lemma 2.1, we obtain

$$
\left|\frac{\partial\tilde{\pi}(\pi,s,\omega)}{\partial\pi}g_\pi^1(\tilde{\pi}(\pi,s,\omega))\right|\leq\frac{Z(s,\omega)}{(Z(s,\omega)\wedge 1)^2}\left(|\alpha_1|+\sigma_1^2+\lambda_1\frac{L_1}{1-L_1}+\lambda_{1,0}\frac{L_{1,0}}{1-L_{1,0}}\right) \tag{6}
$$

for all $(s,\omega)\in[t,T]\times\Omega$. Furthermore we have

$$
\left|\frac{\partial\tilde{\pi}(\pi,T,\omega)}{\partial\pi}\frac{\ell}{1-\ell\tilde{\pi}(\pi,T,\omega)}\right|F\leq\frac{Z(T,\omega)}{(Z(T,\omega)\wedge 1)^2}\frac{\ell}{1-\ell} \tag{7}
$$

for all $\omega\in\Omega$. Therefore, the discounted left-hand sides of (6) and (7) are uniformly bounded in π by integrable functions and we have

$$
f_\pi^1(t,\pi)=\int_t^T\mathbb{E}\left[\frac{\partial\tilde{\pi}(s)}{\partial\pi}g_\pi^1(\tilde{\pi}(s))\right]e^{-\lambda_{1,0}(s-t)}ds-\mathbb{E}\left[\frac{\partial\tilde{\pi}(T)}{\partial\pi}\frac{\ell}{1-\tilde{\pi}(T)\ell}\right]e^{-\lambda_{1,0}(T-t)}.
$$

Again, by the remark following Lemma 2.1, we have

$$
\left|\frac{\partial}{\partial\pi}\left\{\frac{\partial\tilde{\pi}(\pi,s,\omega)}{\partial\pi}g_\pi^1(\tilde{\pi}(\pi,s,\omega))\right\}\right|\leq c_1\frac{Z(s,\omega)}{(Z(s,\omega)\wedge 1)^3}|1-Z(s,\omega)|+c_2\frac{Z(s,\omega)^2}{(Z(s,\omega)\wedge 1)^4} \tag{8}
$$

for all $(s,\omega)\in[t,T]\times\Omega$ and

$$
\left|\frac{\partial}{\partial\pi}\left\{\frac{\partial\tilde{\pi}(\pi,T,\omega)}{\partial\pi}\frac{\ell}{1-\ell\tilde{\pi}(\pi,T,\omega)}\right\}\right|\leq 2c_3\frac{Z(s,\omega)}{(Z(s,\omega)\wedge 1)^3}|1-Z(s,\omega)|+c_3^2\frac{Z(s,\omega)^2}{(Z(s,\omega)\wedge 1)^4} \tag{9}
$$

for all $\omega\in\Omega$, where $c_1=2(|\alpha_1|+\sigma_1^2+\lambda_1\frac{L_1}{1-L_1}+\lambda_{1,0}\frac{L_{1,0}}{1-L_{1,0}})$, $c_2=\sigma_1^2+\lambda_1\frac{L_1}{(1-L_1)^2}+\lambda_{1,0}\frac{L_{1,0}}{(1-L_{1,0})^2}$, and $c_3=\frac{\ell}{1-\ell}$. Thus, the discounted left-hand sides of (8) and (9) are uniformly bounded in π by integrable functions and we may thus interchange differentiating and integrating. This yields

$$
\begin{aligned}
f_{\pi,\pi}^1(t,\pi) &= \int_t^T\mathbb{E}\left[\frac{\partial^2\tilde{\pi}(s)}{\partial\pi^2}g_\pi^1(\tilde{\pi}(s))+\left(\frac{\partial\tilde{\pi}(s)}{\partial\pi}\right)^2g_{\pi,\pi}^1(\tilde{\pi}(s))\right]e^{-\lambda_{1,0}(s-t)}ds \\
&\quad -\mathbb{E}\left[\frac{\partial^2\tilde{\pi}(T)}{\partial\pi^2}\frac{\ell}{1-\ell\tilde{\pi}(T)}+\left(\frac{\partial\tilde{\pi}(T)}{\partial\pi}\right)^2\frac{\ell^2}{(1-\ell\tilde{\pi}(T))^2}\right]e^{-\lambda_{1,0}(T-t)}.
\end{aligned}
$$

(ii) The assertion follows by the Feynman-Kac formula. □

One central motivation for modeling the randomness of stock dynamics via Brownian motions is that continuous trading activity of market participants creates this kind of dynamics.[5] In state 1, however, trading is interrupted and thus it seems reasonable to set the diffusion term in state 1 to zero. Besides, we think of state 1 as a regime where the economy is hit by an extreme event such as a war or a political turmoil. Consequently, it may also be plausible to assume that $\alpha_1 \leq 0$. As the following proposition shows, these assumptions together with (10) are sufficient to ensure the existence of a unique smooth solution of the investor's portfolio problem.

Proposition 5.2 (Optimal Portfolio Choice)**.** *Assume that $\alpha_1 \leq 0$ and $\sigma_1 = 0$.*

(i) The function f^1 defined above is decreasing and concave, i.e. the derivatives f^1_π and $f^1_{\pi,\pi}$ are non-positive.

(ii) If for each $t \in [0,T]$ there exists a $\pi^*(t) \in [0,1]$ such that $\pi^*(t)$ is a solution to the first-order condition (5), then $\pi^*: [0,T] \to [0,1]$ is uniquely determined and of class C^1. Moreover, $\pi^*(t) = \arg\max_{\pi\in[0,1]} H^0(t,\pi)$ for all $t \in [0,T]$.

(iii) A solution in (ii) exists if for all $t \in [0,T]$

$$\begin{aligned} \alpha_0 - \lambda_0 L_0 - \lambda_{0,1} L_{0,1} + \lambda_{0,1} f^1_\pi(t,0)(1-L_{0,1}) &\geq 0, \\ \alpha_0 - \sigma_0^2 - \lambda_0 \frac{L_0}{1-L_0} - \lambda_{0,1}\frac{L_{0,1}}{1-L_{0,1}} + \lambda_{0,1} f^1_\pi(t,1)\frac{1}{1-L_{0,1}} &\leq 0. \end{aligned} \tag{10}$$

Remark. Condition (10) can be rewritten more explicitly as

$$\begin{aligned} 0 \;\leq\;& \alpha_0 - \lambda_0 L_0 - \lambda_{0,1} L_{0,1} - \lambda_{0,1}(1-L_{0,1})\ell \mathbb{E}[1/Z(T)] e^{-\lambda_{1,0}(T-t)} \\ &+ \lambda_{0,1}(1-L_{0,1})(\alpha_1 - \lambda_1 L_1 - \lambda_{1,0} L_{1,0}) \int_t^T \mathbb{E}[1/Z(s)] e^{-\lambda_{1,0}(s-t)} ds, \\ 0 \;\geq\;& \alpha_0 - \sigma_0^2 - \lambda_0 \frac{L_0}{1-L_0} - \lambda_{0,1}\frac{L_{0,1}}{1-L_{0,1}} - \lambda_{0,1}\frac{1}{1-L_{0,1}}\frac{\ell}{1-\ell}\mathbb{E}[Z(T)] e^{-\lambda_{1,0}(T-t)} \\ &+ \lambda_{0,1}\frac{1}{1-L_{0,1}}\Big(\alpha_1 - \sigma_1^2 - \lambda_1 \frac{L_1}{1-L_1} - \lambda_{1,0}\frac{L_{1,0}}{1-L_{1,0}}\Big) \int_t^T \mathbb{E}[Z(s)] e^{-\lambda_{1,0}(s-t)} ds. \end{aligned}$$

◇

Proof of Proposition 5.2. (i) and (ii). Let $(t,\pi) \in [0,T] \times [0,1]$ and let $s \in [t,T]$. We have $g^1_\pi(\pi) \leq 0$ and $\frac{\partial \tilde\pi(s)}{\partial \pi} \geq 0$. Thus, by the remark following Lemma 2.1 and Proposition 5.1 (i), we find $f^1_\pi \leq 0$. Since $Z(s) = \frac{e^{(\frac12 \sigma_1^2 - \alpha_1)(s-t) - \sigma_1(W(s)-W(t))}}{(1-L_1)^{(N_1(s)-N_1(t))}} = \frac{e^{-\alpha_1(s-t)}}{(1-L_1)^{N_1(s)-N_1(t)}} \geq 1$, we have $\frac{\partial^2 \tilde\pi(s)}{\partial \pi^2} = -2\frac{Z(s)(1-Z(s))}{(\pi+(1-\pi)Z(s))^3} \geq 0$. Furthermore, $g^1_{\pi,\pi}(\pi) = -\lambda_{1,0}\frac{L_{1,0}^2}{(1-\pi L_{1,0})^2} - \lambda_1 \frac{L_1^2}{(1-\pi L_1)^2} \leq 0$ and consequently

$$\begin{aligned} \frac{\partial}{\partial \pi}\left\{\frac{\partial \tilde\pi(s)}{\partial \pi} g^1_\pi(\tilde\pi(s))\right\} &= \frac{\partial^2 \tilde\pi(s)}{\partial \pi^2} g^1_\pi(\tilde\pi(s)) + \left(\frac{\partial \tilde\pi(s)}{\partial \pi}\right)^2 g^1_{\pi,\pi}(\tilde\pi(s)) \leq 0 \\ \frac{\partial}{\partial \pi}\left\{\frac{\partial \tilde\pi(T)}{\partial \pi}\frac{\ell}{1-\ell\tilde\pi(T)}\right\} &= \frac{\partial^2 \tilde\pi(T)}{\partial \pi^2}\frac{\ell}{1-\ell\tilde\pi(T)} + \left(\frac{\partial \tilde\pi(T)}{\partial \pi}\right)^2 \frac{\ell^2}{(1-\ell\tilde\pi(T))^2} \geq 0. \end{aligned}$$

[5] See, e.g., Foellmer and Schweizer (1994) and the references therein.

Thus, by Proposition 5.1 (i), we have $f^1_{\pi,\pi}(t,\pi) \le 0$. Taking the derivative with respect to π of the right hand side of the first-order condition (5), we get

$$\begin{aligned} 0 \quad \ge \quad & -\sigma_0^2 - \lambda_{0,1}\frac{L_{0,1}^2}{(1-\pi L_{0,1})^2} - \lambda_0\frac{L_0^2}{(1-\pi L_0)^2} + \lambda_{0,1} f^1_{\pi\pi}\left(t, \frac{(1-L_{0,1})\pi}{1-L_{0,1}\pi}\right)\frac{(1-L_{0,1})^2}{(1-L_{0,1}\pi)^4} \\ & +2\lambda_{0,1} f^1_{\pi}\left(t, \frac{(1-L_{0,1})\pi}{1-L_{0,1}\pi}\right)\frac{L_{0,1}(1-L_{0,1})}{(1-L_{0,1}\pi)^3}. \end{aligned}$$

Thus, the solution of the first-order condition (5) is unique. Furthermore, by the implicit function theorem, we conclude that for a solution π^* as detailed in (ii), the mapping π^* is continuously differentiable and maximizes the HJB equation (3).

(iii) Under our assumptions, the right hand side of the first-order condition (5) is continuous and decreasing in π. Therefore, by the intermediate value theorem, the claim follows. □

Note that the requirements $\alpha_1 \le 0$ and $\sigma_1 = 0$ are not necessary for the claim in the previous proposition to hold. They however imply that f^1_π and $f^1_{\pi\pi}$ are non-positive, which is sufficient to prove the claim. Besides, we remark that (10) is satisfied for reasonable choices of α_0. However, if α_0 is "too large" or "too small", then it can happen that this condition is not satisfied. For instance, if $\alpha_0 < 0$, i.e. an investment in stocks is strictly dominated by an investment in bonds, then the optimal number of stocks is zero. This is a corner solution and (10) excludes these kinds of degenerated cases. The following proposition provides a representation of the value function in state 0.

Proposition 5.3 (Indirect Utility in Liquidity)**.** *Suppose that there exists a continuous function $\pi^* : [0,T] \to [0,1]$ such that $\pi^*(t) \in \arg\max_{\pi\in[0,1]} H^0(t,\pi)$ for all $t \in [0,T]$. Consider the function $f^0 : [0,T] \to \mathbb{R}$ given by*

$$f^0(t) = \frac{\lambda_{0,1}}{\lambda_{0,1}+\lambda_{1,0}} e^{(\lambda_{0,1}+\lambda_{1,0})t}\int_t^T F(s)e^{-\lambda_{0,1}s}ds + \frac{\lambda_{1,0}}{\lambda_{0,1}+\lambda_{1,0}}\int_t^T F(s)e^{\lambda_{1,0}s}ds,$$

where $F(t) = g^0(\pi^*(t))e^{-\lambda_{1,0}t} + \lambda_{0,1}\int_t^T \mathbb{E}[g^1(\tilde\pi^{t,\hat\pi_0(t)}(s))]e^{-\lambda_{1,0}s}ds + \lambda_{0,1}\mathbb{E}[\ln(1-\tilde\pi^{t,\hat\pi_0(t)}(T)\ell)]e^{-\lambda_{1,0}T}$ and $\hat\pi_0(t) = \frac{(1-L_{0,1})\pi^*(t)}{1-L_{0,1}\pi^*(t)}$. Then f^0 is of class C^1 and solves the HJB equation (3).

Proof of Proposition 5.3. The function F is continuous and therefore f^0 is continuously differentiable. Recall that the Hamilton-Jacobi-Bellman equation for state 0 is given by

$$0 = f^0_t(t) + g^0(\pi^*(t)) + \lambda_{0,1}f^1(t,\hat\pi_0(t)) - \lambda_{0,1}f^0(t).$$

By Proposition 5.1, we have

$$\begin{aligned} f^1(t,\hat\pi_0(t)) \quad = \quad & \lambda_{1,0}\int_t^T f^0(s)e^{-\lambda_{1,0}(s-t)}ds + \int_t^T \mathbb{E}[g^1(\tilde\pi^{t,\hat\pi_0(t)}(s))]e^{-\lambda_{1,0}(s-t)}ds \\ & +\mathbb{E}[\ln(1-\ell\tilde\pi^{t,\hat\pi_0(t)}(T))]e^{-\lambda_{1,0}(T-t)}, \end{aligned}$$

which yields the following integro-differential equation for f^0

$$\begin{aligned} 0 \quad = \quad & f^0_t(t)e^{-\lambda_{1,0}t} + g^0(\pi^*(t))e^{-\lambda_{1,0}t} + \lambda_{0,1}\lambda_{1,0}\int_t^T f^0(s)e^{-\lambda_{1,0}s}ds - \lambda_{0,1}f^0(t)e^{-\lambda_{1,0}t} \\ & +\lambda_{0,1}\int_t^T \mathbb{E}[g^1(\tilde\pi^{t,\hat\pi_0(t)}(s))]e^{-\lambda_{1,0}s}ds + \lambda_{0,1}\mathbb{E}[\ln(1-\ell\tilde\pi^{t,\hat\pi_0(t)}(T))]e^{-\lambda_{1,0}T}. \end{aligned}$$

Substituting $H(t) = \int_t^T f^0(s)e^{-\lambda_{1,0}s}ds$ into the equation above, we get

$$\begin{aligned}0 &= -H_{t,t}(t) - (\lambda_{1,0} - \lambda_{0,1})H_t(t) + \lambda_{0,1}\lambda_{1,0}H(t) + g^0(\pi^*(t))e^{-\lambda_{1,0}t} \\ &\quad + \lambda_{0,1}\int_t^T \mathbb{E}[g^1(\tilde{\pi}^{t,\hat{\pi}_0(t)}(s))]e^{-\lambda_{1,0}s}ds + \lambda_{0,1}\mathbb{E}[\ln(1-\ell\tilde{\pi}^{t,\hat{\pi}_0(t)}(T))]e^{-\lambda_{1,0}T}.\end{aligned}$$

Eventually, setting

$$F(t) = g^0(\pi^*(t))e^{-\lambda_{1,0}t} + \lambda_{0,1}\int_t^T \mathbb{E}[g^1(\tilde{\pi}^{t,\hat{\pi}_0(t)}(s))]e^{-\lambda_{1,0}s}ds + \lambda_{0,1}\mathbb{E}[\ln(1-\ell\tilde{\pi}^{t,\hat{\pi}_0(t)}(T))]e^{-\lambda_{1,0}T}$$

leads to the following second order linear inhomogeneous differential equation

$$H_{t,t}(t) + (\lambda_{1,0} - \lambda_{0,1})H_t(t) - \lambda_{0,1}\lambda_{1,0}H(t) = F(t) \tag{11}$$

with the constraints $H(T) = 0$, $H_t(T) = 0$. The characteristic equation $\mu^2 + (\lambda_{1,0} - \lambda_{0,1})\mu - \lambda_{0,1}\lambda_{1,0} = 0$ has the two roots $\mu_1 = \lambda_{0,1}$ and $\mu_2 = -\lambda_{1,0}$. Thus, the exponential conjecture yields the pair $u^1(t) = e^{\lambda_{0,1}t}$ and $u^2(t) = e^{-\lambda_{1,0}t}$ of fundamental solutions for the homogeneous differential equation. By the method of variation of constants, a particular solution of the differential equation (11) is given by $w(t) = u^1(t)\int_t^T \frac{F(s)u^2(s)}{W(s)}ds - u^2(t)\int_t^T \frac{F(s)u^1(s)}{W(s)}ds$, where the Wronskian determinant W is given by $W(s) = -(\lambda_{0,1}+\lambda_{1,0})e^{(\lambda_{0,1}-\lambda_{1,0})s}$. Note that we have $w(T) = 0$ and $w_t(T) = 0$. Thus, the unique solution of the constraint differential equation (11) is given by the particular solution w, i.e $H(t) = \int_t^T \frac{F(s)}{\lambda_{0,1}+\lambda_{1,0}}e^{\lambda_{1,0}(s-t)}ds - \int_t^T \frac{F(s)}{\lambda_{0,1}+\lambda_{1,0}}e^{-\lambda_{0,1}(s-t)}ds$. Differentiating H we obtain

$$\begin{aligned}H_t(t) &= u_t^1(t)\int_t^T \frac{F(s)u^2(s)}{W(s)}ds - u_t^2(t)\int_t^T \frac{F(s)u^1(s)}{W(s)}ds \\ &= -\lambda_{0,1}\int_t^T \frac{F(s)}{\lambda_{0,1}+\lambda_{1,0}}e^{-\lambda_{0,1}(s-t)}ds - \lambda_{1,0}\int_t^T \frac{F(s)}{\lambda_{0,1}+\lambda_{1,0}}e^{\lambda_{1,0}(s-t)}ds\end{aligned}$$

Further, by the definition of H, we have $H_t(t) = -f^0(t)e^{-\lambda_{1,0}t}$ and thus f^0 is given by

$$f^0(t) = \lambda_{0,1}e^{\lambda_{1,0}t}\int_t^T \frac{F(s)}{\lambda_{0,1}+\lambda_{1,0}}e^{-\lambda_{0,1}(s-t)}ds + \lambda_{1,0}\int_t^T \frac{F(s)}{\lambda_{0,1}+\lambda_{1,0}}e^{\lambda_{1,0}s}ds.$$

□

There is a special case where the integrals in the above representation of f^0 can be calculated explicitly. Namely, if $\sigma_1 = \alpha_1 = L_1 = L_0 = L_{0,1} = 0$, the first order condition simplifies into $0 = \alpha_0 - \sigma_0^2\pi + \lambda_{0,1}((e^{-\lambda_{1,0}(T-t)} - 1)\frac{L_{1,0}}{1-L_{1,0}\pi} - e^{-\lambda_{1,0}(T-t)}\frac{\ell}{1-\ell\pi})$. If, in addition, $\ell = L_{1,0}$, we get $0 = \alpha_0 - \sigma_0^2\pi + \lambda_{0,1}\frac{L_{1,0}}{1-L_{1,0}\pi}$, and the function f^0 is given by

$$\begin{aligned}f^0(t) &= \frac{\lambda_{0,1}}{\lambda_{0,1}+\lambda_{1,0}}e^{(\lambda_{0,1}+\lambda_{1,0})t}\Big((g^0(\pi^*) + \frac{\lambda_{0,1}}{\lambda_{1,0}}g^1(\pi^*))\frac{1}{\lambda_{0,1}+\lambda_{1,0}}(e^{-(\lambda_{0,1}+\lambda_{1,0})t} - e^{-(\lambda_{0,1}+\lambda_{1,0})T}) \\ &\quad + (\ln(1-L_{1,0}\pi^*) - \frac{1}{\lambda_{1,0}}g^1(\pi^*))e^{-\lambda_{1,0}T}(e^{-\lambda_{0,1}t} - e^{-\lambda_{0,1}T})\Big) \\ &\quad + \frac{\lambda_{1,0}}{\lambda_{0,1}+\lambda_{1,0}}\Big((\lambda_{0,1}\ln(1-L_{1,0}\pi^*) - \frac{\lambda_{0,1}}{\lambda_{1,0}}g^1(\pi^*))e^{-\lambda_{1,0}T}\frac{1}{\lambda_{1,0}}(e^{\lambda_{1,0}T} - e^{\lambda_{1,0}t}) \\ &\quad + (g^0(\pi^*) + \frac{\lambda_{0,1}}{\lambda_{1,0}}g^1(\pi^*))(T-t)\Big),\end{aligned}$$

where $g^0(\pi) = r_0 + \alpha_0\pi - \frac{1}{2}\pi^2\sigma_0^2$ and $g^1(\pi) = r_1 + \lambda_{1,0}\ln(1 - L_{1,0}\pi)$, and π^* is the solution of the first-order condition. The following theorem summarizes our results in this section.

Theorem 5.4 (Solution of the Portfolio Problem)**.** *Consider the portfolio problem with infinitely many possible liquidity breakdowns for an investor with $U(x) = \ln(x)$. Suppose that there exists a continuous function $\pi^* : [0, T] \to [0, 1]$ such that $\pi^*(t) \in \arg\max_{\pi\in[0,1]} H^0(t, \pi)$ for all $t \in [0, T]$. Then the value function is given by $V(t_0, x_0) = \ln(x_0) + f^0(t)$ for $t \in [0, T],\ x_0 \in (0, \infty)$ and the optimal strategy is given by π^*.*

Proof of Theorem 5.4. Since $|\ln(x)| \leq \frac{1}{x}$ for $x \in (0, 1)$, the assertion follows immediately from the Verification Theorem 4.1 and Propositions 5.1 and 5.3. □

6 Finitely Many Liquidity Breakdowns

In this section, we will summarize the results for an investor's portfolio problem when only finitely many regime shifts between state 0 and state 1 are possible. We will analyze the problems of investors with logarithmic and power utility functions and, for instance, provide convergence results of the optimal portfolio strategies.

6.1 Logarithmic Utility

Firstly, we assume that $U(x) = \ln(x)$. Since only finitely many breakdowns are possible, the portfolio problem can be solved recursively. Note that $J^{0,0}$ is given by

$$J^{0,0}(t, x) = \ln(x) + f^{0,0}(t) = \ln(x) + (r_0 + \alpha_0\pi^* - \frac{1}{2}(\pi^*)^2\sigma_0^2 + \lambda_0\ln(1 - \pi^*L_0))(T - t)$$

where $0 = \alpha_0 - \sigma_0^2\pi^* - \lambda_0\frac{L_0}{1-\pi^*L_0}$. As in the previous section, for $k_0 \in \mathbb{N}$, we conjecture

$$J^{0,k_0}(t, x) = \ln(x) + f^{0,k_0}(t) \text{ and } J^{1,k_0}(t, x, \pi) = \ln(x) + f^{1,k_0}(t, \pi)$$

for a C^1-function f^{0,k_0} on $[0, T]$ with $f^{0,k_0}(T) = 0$ and a $C^{1,2}$-function f^{1,k_0} on $[0, T] \times [0, 1]$ satisfying $f^{1,k_0}(T, \pi) = \ln(1 - \ell\pi)$ for all $\pi \in [0, 1]$. The HJB equations read

$$\begin{aligned} 0 &= \sup_{\pi\in[0,1]} \left\{ f_t^{0,k_0}(t) + g^0(\pi) + \lambda_{0,1}\left[f^{1,k_0}\left(t, \tfrac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right) - f^{0,k_0}(t)\right]\right\} \\ 0 &= f_t^{1,k_0}(t, \pi) - \lambda_{1,0}f^{1,k_0}(t, \pi) + \pi(1 - \pi)(\alpha_1 - \sigma_1^2\pi)f_\pi^{1,k_0}(t, \pi) + \frac{1}{2}\pi^2(1 - \pi)^2\sigma_1^2 f_{\pi,\pi}^{1,k_0}(t, \pi) \\ &\quad +\lambda_1\left[f^{1,k_0}\left(t, \tfrac{\pi(1-L_1)}{1-\pi L_1}\right) - f^{1,k_0}(t, \pi)\right] + g^1(\pi) + \lambda_{1,0}f^{0,k_0-1}(t), \end{aligned}$$

with g^0 and g^1 as in the previous section. The first equation leads to the following first-order condition for the optimal stock proportion in state 0:

$$0 \;=\; \alpha_0 - \sigma_0^2\pi - \lambda_0\frac{L_0}{1 - \pi L_0} - \lambda_{0,1}\frac{L_{0,1}}{1 - \pi L_{0,1}} + \lambda_{0,1}f_\pi^{1,k_0}\left(t, \tfrac{\pi(1-L_{0,1})}{1-\pi L_{0,1}}\right)\frac{1 - L_{0,1}}{(1 - \pi L_{0,1})^2}. \quad (12)$$

Note that the solution of the first-order condition is a deterministic function of time given such a solution exists.

Proposition 6.1 (Indirect Utility in Illiquidity). *Let $k_0 \in \mathbb{N}$ and let g^1, $\tilde{\pi}$ as in Section 5. Given a C^1-function $f^{0,k_0-1} : [0,T] \to \mathbb{R}$, consider the function $f^{1,k_0} : [0,T] \times [0,1] \to \mathbb{R}$ defined via the stochastic representation*

$$f^{1,k_0}(t,\pi) \quad = \quad \int_t^T \left(\lambda_{1,0} f^{0,k_0-1}(s) + \mathbb{E}\left[g^1(\tilde{\pi}(s))\right]\right) e^{-\lambda_{1,0}(s-t)} ds + \mathbb{E}\left[\ln(1-\tilde{\pi}(T)\ell)\right] e^{-\lambda_{1,0}(T-t)}.$$

(i) Then f^{1,k_0} is of class $\mathrm{C}^{1,2}$ on $[0,T] \times [0,1]$ with

$$f_\pi^{1,k_0}(t,\pi) \quad = \quad \int_t^T \mathbb{E}\left[\frac{\partial \tilde{\pi}(s)}{\partial \pi} g_\pi^1(\tilde{\pi}(s))\right] e^{-\lambda_{1,0}(s-t)} ds - \mathbb{E}\left[\frac{\partial \tilde{\pi}(T)}{\partial \pi} \frac{\ell}{1-\tilde{\pi}(T)\ell}\right] e^{-\lambda_{1,0}(T-t)}.$$

(ii) f^{1,k_0} solves the second HJB equation.

Proof of Proposition 6.1. Analogously to the proof of Proposition 5.1. □

Corollary 6.2 (k_0-Invariance of Optimal Portfolio Strategy). *Let $k_0 \in \mathbb{N}$ and let f^{1,k_0} as in the previous proposition. The first-order condition (12) coincides with the first-order condition (5) when infinitely many liquidity breakdowns are possible.*

Remark. By Corollary 6.2, the convergence of the optimal strategies for $k_0 \to \infty$ is trivial if the investor has logarithmic utility. ⋄

Proof of Corollary 6.2. The function f_π^{1,k_0} does not depend on the number of possible regime shifts k_0, and we have $f_\pi^{1,k_0} = f_\pi^1$ where f_π^1 is given in Proposition 5.1. □

In general, a logarithmic investor makes his investment decisions myopically. If liquidity breakdowns are possible, then he adjusts his portfolio decision to take the threat of illiquidity into account. However, by the previous corollary, he remains myopic in the sense that he disregards the number of possible breakdowns.

Proposition 6.3 (Indirect Utility in Liquidity). *Let $k_0 \in \mathbb{N}$ and let g^0, $\hat{\pi}_0$ as in Section 5. Suppose that there exists a continuous function $\pi^* : [0,T] \to [0,1]$ such that $\pi^*(t) \in \arg\max_{\pi \in [0,1]} H^0(t,\pi)$ for all $t \in [0,T]$. Given a $C^{1,2}$-function $f^{1,k_0} : [0,T] \times [0,1] \to \mathbb{R}$, consider the function $f^{0,k_0} : [0,T] \to \mathbb{R}$ defined via*

$$f^{0,k_0}(t) \quad = \quad \int_t^T \left(\lambda_{0,1} f^{1,k_0}(s, \hat{\pi}_0(s)) + g^0(\pi^*(s))\right) e^{-\lambda_{0,1}(s-t)} ds.$$

Then f^{0,k_0} is of class C^1 on $[0,T]$, and f^{0,k_0} solves the first HJB equation stated above.

Proof of Proposition 6.3. It is clear that f^{0,k_0} is of class C^1 on $[0,T]$, and the second claim follows by differentiating f^{0,k_0} with respect to t. □

Collecting the above results and applying the Verification Theorem 4.1 yields

Theorem 6.4 (Solution of the Portfolio Problem). *For $U(x) = \ln(x)$ we consider the portfolio problem with $k_0 \in \mathbb{N}$ possible regime shifts. Suppose that there exists a continuous function $\pi^* : [0,T] \to [0,1]$ such that $\pi^*(t) \in \arg\max_{\pi \in [0,1]} H^0(t,\pi)$ for all $t \in [0,T]$. Then the value function is given by $V(t_0, x_0, k_0) = \ln(x_0) + f^{0,k_0}(t)$ for $t \in [0,T]$, $x_0 \in (0,\infty)$ and π^* is the optimal portfolio strategy.*

6.2 Power Utility

In this subsection, we consider an economy where only finitely many regime shifts between state 0 and state 1 are possible and where $U(x) = x^\gamma/\gamma$ with $\gamma \neq 0$. As in the previous subsection, this problem can be solved recursively. We assume that $L_0 = L_1 = L_{0,1} = 0$ and that $\sigma_1 = 0$. Note that $J^{0,0}$ is given by $J^{0,0}(t,x) = \frac{x^\gamma}{\gamma} f^{0,0}(t) = \frac{x^\gamma}{\gamma}\exp(\gamma(r_0 + \frac{1}{2}\frac{\alpha_0^2}{(1-\gamma)\sigma_0^2})(T-t))$ and the optimal stock proportion is given by $\pi^* = \alpha_0/((1-\gamma)\sigma_0^2)$. For $k_0 \in \mathbb{N}$ we conjecture

$$J^{0,k_0}(t,x) = \frac{x^\gamma}{\gamma} f^{0,k_0}(t) \text{ and } J^{1,k_0}(t,x,\pi) = \frac{x^\gamma}{\gamma} f^{1,k_0}(t,\pi)$$

for a C^1-function f^{0,k_0} on $[0,T]$ with $f^{0,k_0}(T) = 1$ and a $\mathrm{C}^{1,2}$-function f^{1,k_0} on $[0,T]\times[0,1]$ satisfying $f^{1,k_0}(T,\pi) = (1-\ell\pi)^\gamma$ for all $\pi \in [0,1]$. The HJB equations read

$$\begin{aligned} 0 &= \sup_{\pi\in[0,1]} \left\{ \gamma^{-1}\Big(f_t^{0,k_0}(t) - d^0(\pi) f^{0,k_0}(t) + \lambda_{0,1} f^{1,k_0}(t,\pi) \Big) \right\} \\ 0 &= f_t^{1,k_0}(t,\pi) - d^1(\pi) f^{1,k_0}(t,\pi) + \pi(1-\pi)\alpha_1 f_\pi^{1,k_0}(t,\pi) + \lambda_{1,0}(1-\pi L_{1,0})^\gamma f^{0,k_0-1}(t), \end{aligned}$$

where d^0 and d^1 are given by $d^0(\pi) = \lambda_{0,1} - \gamma(r_0 + \pi\alpha_0) + \frac{1}{2}\gamma(1-\gamma)\pi^2\sigma_0^2$ and $d^1(\pi) = \lambda_{1,0} - \gamma(r_1 + \alpha_1\pi)$ on $[0,1]$. The first HJB equation leads to the first-order condition

$$0 = \gamma\alpha_0 f^{0,k_0}(t) - \gamma(1-\gamma)\pi\sigma_0^2 f^{0,k_0}(t) + \lambda_{0,1} f_\pi^{1,k_0}(t,\pi). \tag{13}$$

for the optimal stock proportion in state 0. As before, the solution to the first-order condition is a deterministic function of time given such a solution exists.

Proposition 6.5 (Indirect Utility in Illiquidity)**.** *Let $k_0 \in \mathbb{N}$ and let $f^{0,k_0-1} : [0,T] \to \mathbb{R}$ be a given function which is of class C^1 on $[0,T]$. Consider the function $f^{1,k_0} : [0,T]\times[0,1] \to \mathbb{R}$ defined via*

$$\begin{aligned} f^{1,k_0}(t,\pi) &= \lambda_{1,0}\int_t^T e^{(\gamma r_1 - \lambda_{1,0})(s-t)} \Big(1 + \pi[e^{\alpha_1(s-t)}(1-L_{1,0}) - 1]\Big)^\gamma f^{0,k_0-1}(s)ds \\ &\quad + e^{(\gamma r_1 - \lambda_{1,0})(T-t)} \Big(1 + \pi[e^{\alpha_1(T-t)}(1-\ell) - 1]\Big)^\gamma. \end{aligned}$$

Then f^{1,k_0} is of class $\mathrm{C}^{1,2}$ on $[0,T]\times[0,1]$ and f^{1,k_0} solves the second HJB equation.

Proof of Proposition 6.5. Since f^{0,k_0-1} is continuously differentiable, it obvious that f^{1,k_0} is of class $C^{1,2}$. The second assertion follows by differentiation with respect to t and π. □

Proposition 6.6 (Indirect Utility in Liquidity)**.** *Let $k_0 \in \mathbb{N}$ and suppose that there exists a continuous function $\pi^* : [0,T] \to [0,1]$ such that $\pi^*(t) \in \arg\max_{\pi\in[0,1]} H^{0,k_0}(t,\pi)$ for all $t \in [0,T]$. Given a $C^{1,2}$-function $f^{1,k_0} : [0,T]\times[0,1] \to \mathbb{R}$, consider the function $f^{0,k_0} : [0,T] \to \mathbb{R}$ defined via*

$$f^{0,k_0}(t) = \lambda_{0,1}\int_t^T e^{-\int_t^v d^0(\pi^*(u))du} f^{1,k_0}(v,\pi^*(v))dv + e^{-\int_t^T d^0(\pi^*(u))du}.$$

Then f^{0,k_0} is of class C^1 on $[0,T]$, and f^{0,k_0} solves the first HJB equation stated above.

Proof of Proposition 6.6. Analogously to the proof of Proposition 6.3. □

Collecting the above results, by the Verification Theorem 4.1, we obtain

Theorem 6.7 (Solution of the Portfolio Problem)**.** *For $U(x) = x^\gamma/\gamma$, $\gamma \neq 0$, we consider the portfolio problem with $k_0 \in \mathbb{N}$ possible regime shifts and assume that there exists a continuous function $\pi^* : [0,T] \to [0,1]$ such that $\pi^*(t) \in \arg\max_{\pi\in[0,1]} H^{0,k_0}(t,\pi)$ for all $t \in [0,T]$. Then the value function of the portfolio problem is given by $V(t_0,x_0,k_0) = \frac{x^\gamma}{\gamma} f^{0,k_0}(t)$ for $t \in [0,T]$, $x_0 \in (0,\infty)$ and π^* is the optimal portfolio strategy.*

Finally, we will derive a convergence result for the optimal strategies. In the case of infinitely many possible liquidity breakdowns, i.e. $k_0 = \infty$, there is an analogue to the representation of $f^{1,\infty}$ in Proposition 6.5:

$$\begin{aligned} f^{1,\infty}(t,\pi) &= \lambda_{1,0} \int_t^T e^{(\gamma r_1 - \lambda_{1,0})(s-t)} \Big(1 + \pi[e^{\alpha_1(s-t)}(1-L_{1,0}) - 1]\Big)^\gamma f^{0,\infty}(s)ds \\ &\quad + e^{(\gamma r_1 - \lambda_{1,0})(T-t)} \Big(1 + \pi[e^{\alpha_1(T-t)}(1-l) - 1]\Big)^\gamma . \end{aligned}$$

Proposition 6.8. *The sequence $(f_\pi^{1,k})_{k\in\mathbb{N}}$ converges to $f_\pi^{1,\infty}$ uniformly on $[0,T]\times[0,1]$ in the space of continuous functions.*

Proof of Proposition 6.8. Let $k \in \mathbb{N}$. Since we may interchange differentiating and integrating in the representations of $f_\pi^{1,k}$ and $f_\pi^{1,\infty}$, we have

$$\begin{aligned} &\sup_{(t,\pi)\in[0,T]\times[0,1]} |f_\pi^{1,k}(t,\pi) - f_\pi^{1,\infty}(t,\pi)| \\ &= \lambda_{1,0} \sup_{(t,\pi)\in[0,T]\times[0,1]} \Big|\, \gamma \int_t^T e^{(\gamma r_1 - \lambda_{1,0})(s-t)} \Big(1 + \pi[e^{\alpha_1(s-t)}(1-L_{1,0}) - 1]\Big)^{\gamma-1} \\ &\qquad [e^{\alpha_1(s-t)}(1-L_{1,0}) - 1](f^{0,k-1}(s) - f^{0,\infty}(s))ds \,\Big| \\ &\leq \lambda_{1,0}\, T \sup_{s\in[0,T]} |f^{0,k-1}(s) - f^{0,\infty}(s)| \\ &\quad \sup_{\pi\in[0,1],\, s,t\in[0,T]} \Big|\, \gamma e^{(\gamma r_1 - \lambda_{1,0})(s-t)} \Big(1 + \pi[e^{\alpha_1(s-t)}(1-L_{1,0}) - 1]\Big)^{\gamma-1} [e^{\alpha_1(s-t)}(1-L_{1,0}) - 1] \,\Big|. \end{aligned}$$

Since $L_{1,0} < 1$ the second supremum is finite. Thus, the assertion follows from Theorem 3.2. □

In the case of $k_0 = \infty$, the first-order condition (13) becomes

$$0 = \gamma\alpha_0 f^{0,\infty}(t) - \gamma(1-\gamma)\pi\sigma_0^2 f^{0,\infty}(t) + \lambda_{0,1} f_\pi^{1,\infty}(t,\pi).$$

Corollary 6.9 (Convergence of $\pi_{k_0}^*$)**.** *Let $t \in [0,T]$ and suppose that every first-order condition (13) has a solution $\pi_{k_0}^*(t) \in [0,1]$ for each $k_0 \in \mathbb{N}\cup\{\infty\}$, where the solution for $k_0 = \infty$ has to be unique. Then $\pi_k^*(t) \to \pi_\infty^*(t)$ for $k \to \infty$.*

Proof of Corollary 6.9. Let $(\pi_{k_l}^*(t))_{l\in\mathbb{N}}$ be a subsequence of $(\pi_k^*(t))_{k\in\mathbb{N}}$. Since $\pi_k^*(t) \in [0,1]$ for each $k \in \mathbb{N}$, there exists a subsequence $(\pi_{k_{l_m}}^*(t))_{m\in\mathbb{N}}$ which converges. By Theorem 3.2, Proposition 6.8 and the first-order condition (13), we then obtain

$$0 = \gamma\alpha_0 f^{0,\infty}(t) - \gamma(1-\gamma) \lim_{m\to\infty} \pi^*_{k_{l_m}}(t)\sigma_0^2 f^{0,\infty}(t) + \lambda_{0,1} f_\pi^{1,\infty}(t, \lim_{m\to\infty} \pi^*_{k_{l_m}}(t)).$$

Thus, we have shown that each subsequence of $(\pi_k^*(t))_{k\in\mathbb{N}}$ has another subsequence which converges towards $\pi^*_\infty(t)$. □

Remark. Similar as in Proposition 5.2, the solution of the first-order condition corresponds to the optimal strategy if, for instance, $\alpha_1 \leq 0$. In this case, Corollary 6.9 implies that the optimal strategies converge. ◇

7 Numerical Illustrations

Firstly, we wish to illustrate the convergence of the value functions and strategies in the markets with finitely many breakdowns to the corresponding objects in the market in which infinitely many liquidity breakdowns are possible. In contrast to the Japanese case that we will discuss below, we consider a situation where a liquidity breakdown is more likely and choose $\lambda_{0,1} = 0.2$, i.e. on average a liquidity breakdown occurs every five years. Furthermore, we assume that the average duration of a liquidity breakdown is one month, i.e. $\lambda_{1,0} = 12$, and that $r_0 = r_1 = 0.03$, $\alpha_0 = 0.08$, $\alpha_1 = -0.03$, $\sigma_0 = 0.25$, and $L_{1,0} = \ell = 0.3$. The other parameters are assumed to be zero. This example is similar to the 9/11 case where the New York stock exchange was closed for a week and reopened with a loss of 10%. To get more pronounced effects, we use higher loss rates and a longer average duration of the liquidity breakdowns. The investor is assumed to have a power utility function with $\gamma = -3$. Figure 1 depicts the convergence of the strategies and the (non-wealth dependent parts of) the value functions, f^{0,k_0}. As can be seen from the figure, the value functions converge extremely fast. The strategies also converge to an almost straight line that intersects the y-axis around 0.061. The upper line corresponds to the optimal strategy if at most one liquidity breakdown can occur, the second upper line to the optimal strategy if at most two breakdowns can occur and so on. These results illustrate the theoretical results of Theorem 3.2 and Corollary 6.9. The figure also depicts the percentages Δx by which the initial capital can be reduced in order to get the same utility as in models where trading is allowed in both states. It can be seen that this percentage also converges if the number of possible breakdowns increases. This is because Δx is a function of the value functions that converge. To end this example, we wish to remark that we have found similar convergence behaviors in all numerical experiments that we have analyzed, but that are not included in the paper. We have also analyzed situations where $L_{1,0}$ is random. Alternatively, one can set the loss rate $L_{1,0}$ to zero, but use a higher α_1 to model a continuous depreciation of the stock in the illiquidity state. In any case, we got similar convergence results. The optimal strategies however can vary significantly.

[INSERT FIGURE 1 ABOUT HERE]

$\lambda_{0,1}$	$\lambda_{1,0}$	$L_{1,0}$	ℓ	T	$\pi^*_{0,illiq}(\%)$	$\pi^*_{0,liq}$ (%)	Δx (%)
0.01	0.3	0.5	0.5	10	66.34 (16.69)	80 (20)	4.72 (1.20)
				30	66.26 (16.66)	80 (20)	13.64 (3.36)
				50	66.26 (16.66)	80 (20)	21.74 (5.99)
0.01	0.3	0.5	0	30	66.26 (16.66)	80 (20)	12.48 (3.30)
0.01	0.3	0.9	0.9	30	52.28 (13.65)	80 (20)	22.71 (6.11)
0.01	1	0.5	0.5	30	67.39 (17.00)	80 (20)	13.18 (3.42)
0.02	0.3	0.5	0.5	30	54.98 (13.84)	80 (20)	22.55 (6.20)
0.02	0.3	0.9	0.9	30	36.47 (9.44)	80 (20)	32.67 (9.34)

Table 2: We consider investors maximizing expected logarithmic utility or power utility ($\gamma = -3$) from terminal wealth. The values for the power utility case are the values in brackets. The investment horizon is given by T (in years). The market consists of a bond with interest rate $r_0 = r_1 = 0.03$ and a stock which has a state dependent volatility $\sigma_0 = 0.25$ and $\sigma_1 = 0$ as well as a state dependent excess return $\alpha_0 = 0.05$ and $\alpha_1 = -r_1$. The variable $\pi^*_{0,illiq}$ denotes the time-0 optimal stock demand of an investor who is not able to trade in state 1, whereas $\pi^*_{0,liq}$ denotes the time-0 demand of an investor who can trade in both states. In any case, short selling is forbidden. Δx denotes the percentage of initial wealth that an investor who cannot trade in state 1 would be willing to give up if he were able to trade in both states. The illiquidity parameters are calibrated to mimic situations as in Japan after WW II. For instance, the parametrization $\lambda_{0,1} = 0.01$, $\lambda_{1,0} = 0.3$, $L_{0,1} = 0$, and $L_{1,0} = 0.9$ implies that, on average, once in a century the illiquidity state is reached and, on average, this state is left after 3.33 years triggering a stock price loss of 90%. In this particular case, for an an investment horizon of $T = 30$ years, a logarithmic investor would be willing to give up 22.7% of his initial wealth. The remaining parameters are zero: $\lambda_0 = \lambda_1 = L_0 = L_1 = L_{0,1} = t = 0$.

Secondly, we wish to analyze an important example for a major trading break that happened in the aftermath of World War II in Japan. At that time, the Tokyo Stock Exchange was shut down for almost four years reopening with a loss of more than 90%. Rietz (1988) and Barro (2005), among others, emphasize that these kinds of events can have a significant impact on security prices in an economy. For this reason, we wish to quantify the investor's utility gain expressed in terms of his initial capital when he is able to trade even in the illiquidity state. More precisely, we calculate the percentage of initial capital that a log investor and a power utility investor ($\gamma = -3$) would be willing to give up in order to be able to trade in all states. Due to our results above, we can approximate the case of $\gamma = -3$ by a model where only finitely many jumps are possible. For the log investor, we use our explicit solutions. Since $\lambda_{0,1}$ is small, it is sufficient to consider a model where at most four jumps into the illiquidity state can occur, i.e. $k_0 = 4$. For k_0 greater than 4, the results are virtually identical. Table 2 summarizes our numerical results for different parameterizations of the model.

We assume that the stock dynamics follow a diffusion process in state 0 and are deterministic in state 1, since $\sigma_0 = 0.25$, $L_0 = 0$ and $\sigma_1 = 0$, $L_1 = 0$. However, when leaving state 1 the stock loses a fraction of its value, i.e. $L_{1,0} > 0$. The parameters $\lambda_{0,1}$ and $\lambda_{1,0}$ are chosen in order to mimic situations such as in Japan after World War II. For instance, the parametrization $\lambda_{0,1} = 0.01$, $\lambda_{1,0} = 0.3$, $L_{0,1} = 0$, and $L_{1,0} = 0.9$ implies that, on average, once in a century the illiquidity state is reached and, on average, this state is left after 3.33 years triggering a stock price decrease of 90%. In this particular case, a log investor with a horizon of $T = 30$ years would be willing to give up 22% of his initial wealth. This is due to the fact that an investor who is able to trade can avoid the loss that is triggered by a jump from state 1 to state 0. He will sell his stocks once the economy is in state 1 and thus use the money market account as a "safe harbor". If the investor cannot trade, then he will not be able to avoid this loss. For this reason, he invests considerably less of his wealth into the risky asset. The column labeled by $\pi^*_{0,illiq}$ contains the optimal time-0 stock demands in state 0 (liquidity) when the investment horizon is T and trading is not possible in state 1. If trading is allowed in both states, then the the optimal stock demand in the liquidity state is $\pi^*_{0,liq} = 80\%$ for a log investor. As mentioned above, we have $\pi^*_{1,liq} = 0$. As $\lambda_{0,1}$ is much smaller than $\lambda_{1,0}$, it is likely that at time T the economy is in state 0. Therefore, setting $\ell = 0$ has only a small impact on the percental change of initial capital, which can be seen in the fourth line of Table 2. However, if the loss rate $L_{1,0}$ increases from 50% to 90%, then the percental change of initial capital increases significantly. Increasing $\lambda_{1,0}$ to 1 results in a small change indicating that the effect of illiquidity is small if the investor does not suffer additional losses. The percental change of the initial capital, however, strongly depends on the intensity $\lambda_{0,1}$ modeling the probability that the exchange is closed.

8 Conclusion

This paper studies the portfolio decision of an investor facing the threat of illiquidity. Illiquidity is understood as a state in which the investor is not able to trade at all. Calibrating the illiquidity part of the dynamics of the risky asset to the Japanese data of the aftermath of WW II, it is shown that this threat has a significant effect on the investor's portfolio decision and that the efficiency loss is remarkable 22.7% of current wealth if the investment horizon is 30 years and the investor has logarithmic utility. To obtain these results, we solve the corresponding control problem explicitly, which means that we derive the solution to a system of coupled HJB equations. For investors with arbitrary utility functions we show that a model with infinitely many liquidity breakdowns can be approximated by a model in which only finitely many breakdowns are possible. We illustrate this result for an investor with a power utility function. Our paper also contributes to the literature dealing with the equity premium puzzle, since we introduce a model that is able to address the time dimension of an economic crisis in which trading is not possible. We remark, however, that our model is of partial equilibrium type and thus our numerical results should be viewed as suggestive rather than definitive. One possible direction for future research might be to study a general equilibrium model with multiple agents.

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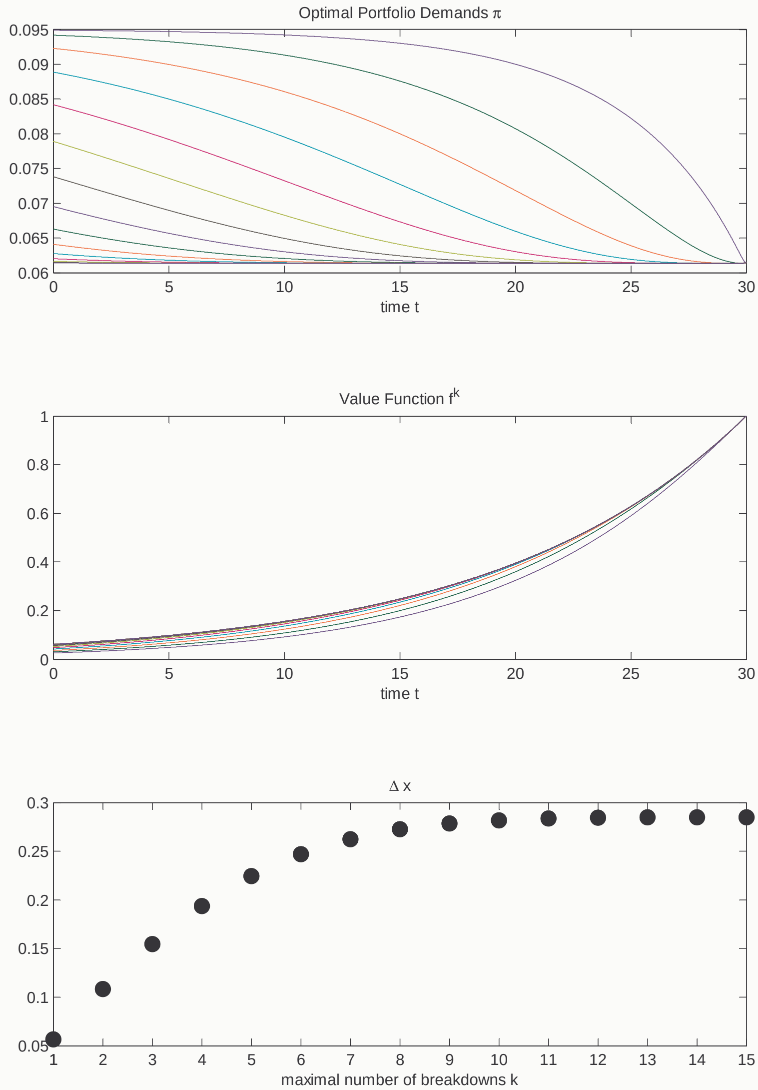

Figure 1: The figure illustrates the convergence of the strategies and the value function in the markets with finitely many breakdowns to the corresponding objects in the market in which infinitely many liquidity breakdowns are possible. We choose $\gamma = -3$, $\lambda_{0,1} = 0.2$, $\lambda_{1,0} = 12$, $r_0 = r_1 = 0.03$, $\alpha_0 = 0.08$, $\alpha_1 = -0.03$, $\sigma_0 = 0.25$, and $L_{1,0} = \ell = 0.3$. The other parameters are zero. The figure also depicts the percentages Δx by which the initial capital can be reduced in order to get the same utility as in models where trading is allowed in both states.