A No-arbitrage Model of Liquidity in Financial Markets involving Brownian Sheets

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Market vs Limit Orders

A (buy) market order specifies

- how many shares a trader wants to buy
- that he is willing to buy them at *any price*.

A (buy) limit order specifies

- how many shares a trader wants to buy
- at what *maximum price* he is willing to buy them?

Example

At $t = 0$, order books look like

Buy Order Book	
Price	Quantity
100	10

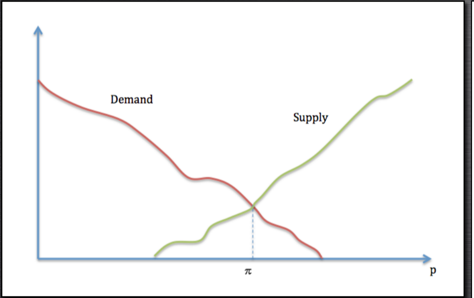
Sell Order Book	
Price	Quantity
120	10
130	10

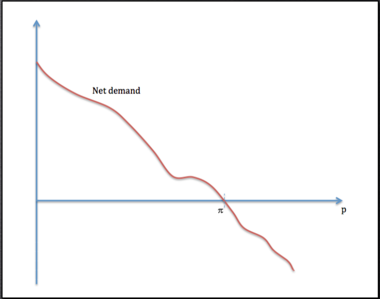
When $t > 0$, new order: buy 15 at a limit price of 125 $\longrightarrow$ clearing price $\pi(t) = 120$

Buy Order Book	
Price	Quantity
100	10
125	5

Sell Order Book	
Price	Quantity
130	10

Example





Goal of the Paper

Better characterize the volatility of the price process in a market driven by limit orders

$$dp(t) = \text{volatility } * dW^{\mathbb{Q}}$$

Inspiration:

- Heath-Jarrow-Morton model

  the drift of the forward rate is determined by volatility
- Derman-Kani model

  relations between volatilities of options prices with different strike and maturity

Literature Review: Liquidity Models

1. Market Manipulation (feedback) Models
    - Jarrow (1994)
    - Platen and Schweizer (1998)
    - Sircar and Papanicolaou (1998)
    - Frey (1998)
    - Schonbucher and Wilmott (2000)
    - Bank and Baum (2004)
2. Price-taking (competitive) Models
    - Cetin, Jarrow, and Protter (2004)
    - Cetin and Rogers (2006)
    - Cetin, Soner, and Touzi (2009)
    - Gokay and Soner (2011)

Model

Filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ where $\mathcal{F}_t$ is generated by a Brownian sheet $W(s,t)$.

Assumption 1

Buy and sell limit prices can assume any real value between 0 and P. They are usually denoted by p. Orders can be submitted to the market at any time $t \in \mathbb{R}^+$.

Definition

The net demand curve Q is a function $[0, P] \times \mathbb{R}^+ \times \Omega \to \mathbb{R}$, which value $Q(p, t, \omega)$ is equal to the difference between the quantity of shares **available** for purchase and the quantity of shares **available** for sale at price p at time t. For each p the stochastic process $Q(., t,)$ is a$\mathcal{F}_t$ adapted semimartingale.

Remark: The net demand curve is decreasing in p.

A Market with Atomistic Traders I

Assumption 2

There is a continuum of atomistic buyers and sellers who trade on the market. The resulting net demand curve Q is twice differentiable in price p and continuous in t. We assume:

$$\begin{aligned} Q(0,t) &> 0 > Q(P,t) \\ \frac{\partial Q}{\partial p}|_p &< 0 \quad \text{for } 0 < p < M \end{aligned}$$

To ensure that Q is decreasing, we define

$$Q(p,t) = Q(0,t) - \int_0^p q(y,t)dy$$

with:

$$\begin{aligned} dQ(0,t) &= \mu_Q(0,t)dt - \sigma_Q(0,t)\int_s b_q(0,s,t)W(ds,dt) \\ dq(p,t) &= \mu_q(p,t)dt + \sigma_q(p,t)\int_{s=0}^{P} b_q(p,s,t)W(ds,dt) \quad 0 < p \leq P \\ q(0,t) &= 0 \end{aligned}$$

A Market with Atomistic Traders I (*cont.*)

Comments on the Net Demand Curve:

- $Q(0,t)$ is the total buy order quantity
- $q(p)dp$ is the total buy and sell order quantity with limit price in $[p, p+dp]$
- both $Q(0,t)$ and $q(p,t)$ must be modelled as strictly positive processes
- it could happen that, for some t, then $Q(P,t) > 0$ so that no clearing price exists
- The equation for the factor loadings is:

$$\int_{s=0}^{S} b_q^2(p,s,t)ds = 1 \qquad \forall p, t$$

- to ensure twice-differentiability in p the coefficients must take a specific form; for instance:

$$dq(p,t) = \int_{s=0}^{p} (p-s)W(ds,dt)$$

A Market with Atomistic Traders II

Definition

The clearing price $\pi(t)$ is a $\mathcal{F}_t$−adapted stochastic process which satisfies market clearing

$$Q(\pi(t), t) = 0$$

If market clearing cannot be satisfied, it is defined by continuation.

$$d\pi(t) = \begin{cases} \mu_\pi(t)dt + \sigma_\pi(t)\int_{s=0}^{P} b_\pi(s,t)W(ds,dt) & \text{if } Q(P,t) < 0 \\ 0 & \phantom{\text{if }} Q(P,t) \geq 0 \end{cases}$$

Assumption 3

There are no (non-liquidity) transaction costs.

A Market with Atomistic Traders III

Definition

A limit order *crosses the market* if

- either it is a buy order with limit price $p \geq \pi(t)$
- or it is a sell order with limit price $p \leq \pi(t)$

We call these orders *cross orders*. All the other orders are called *uncross orders*.

Definition

Asymptotic liquidation proceeds $L(\vartheta, t)$: see Bank and Baum (2004)

Definition

Let $\sigma_Q^2(p, dt)dt$ be the instantaneous variance of $dQ(p, t)$

A Market with Atomistic Traders III (*cont.*)

Theorem

Suppose that:

C1) For strategies involving only cross orders, Jarrow's (1994) discrete time conditions for absence of market manipulation strategy hold

C2) No arbitrage strategy involves uncross orders.

C3)

$$\sigma_Q(p,t) \geq \varepsilon > 0 \qquad \forall p,t$$

C4) There is no path such that $Q(P,t) \geq 0$

Then:

1. There exists at least one martingale measure $\mathbb{Q}$ for $\int L(\vartheta, dt)$
2. There is no arbitrage strategy
3. Any such measure $\mathbb{Q}$ is also a martingale measure for $\pi(t)$
4. The clearing price $\pi(t)$ is continuous

A Market with Atomistic Traders and a Large Trader I

Main Difference

- The large trader's strategy is not necessarily continuous in time a priori.
- Need to modify the definition of the net demand curves from "available" to "submitted".

Definition

The net demand curves of the large (atomistic) trader(s) Q_L (Q_A) is a function $[0, P] \times \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}$ which value $Q_L(p, t, \omega)$ $(Q_A(p, t, \omega))$ is equal to the difference between the quantity of shares **submitted** for purchase and the quantity of shares **submitted** for sale at price p at time t. For each p the stochastic processes $Q_L(., t,)$ and $Q_A(., t,)$ are $\mathcal{F}_t$ adapted semimartingales.

Assumption

Both Q_L and Q_A are twice differentiable in p. Only Q_A is assumed to be continuous in t.

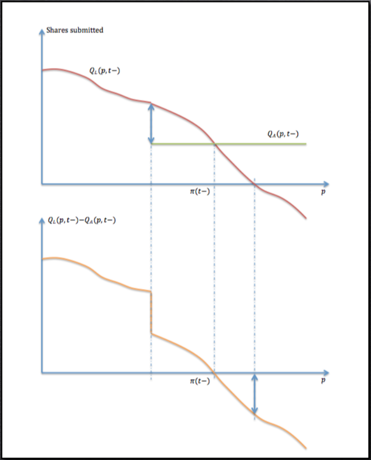
Fact: The (total) net demand curve satisfies:

$$Q = Q_L + Q_A$$

For simplicity we assume

$$Q(P) < 0 \qquad Q(0) > 0$$

A Market with Atomistic Traders and a Large Trader II



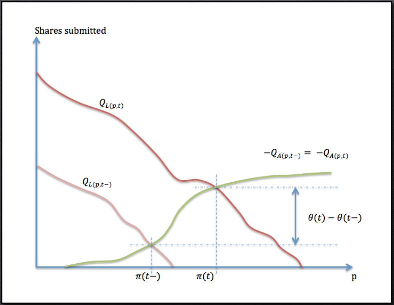
Remarks

- If there were no order cancellation, both Q_L and Q_A would be increasing in t.
- The net demand Q satisfies the same definition as before ("...shares **available**... by time t") since the trade volume from Q^A cancels the trade volume from Q^L.

A Market with Atomistic Traders and a Large Trader II

Definition

A trading strategy θ is a predictable process representing the number of shares the large trader has acquired using only cross orders.



A Market with Atomistic Traders and a Large Trader II

Theorem

Suppose that:

C1) For strategies involving only cross orders, Jarrow's (1994) discrete time conditions for absence of market manipulation strategy hold

C2) No arbitrage strategy involves uncross orders.

C3)

$$\sigma_Q(p,t) \geq \varepsilon > 0 \qquad \forall p, t$$

C4) There is no path such that $Q(P,t) \geq 0$

Then:

1. There exists at least one martingale measure $\mathbb{Q}$ for $\int L(\vartheta, dt)$.
2. There is no arbitrage strategy.
3. Any such measure $\mathbb{Q}$ is also a martingale measure for $\pi(t)$.
4. The clearing price $\pi(t)$ is continuous.
5. The net demand curve Q is continuous in t.

Proof

1. We verify that the Kallsen and Rheinlaender (2009) conditions hold. Thus $\int L(\vartheta, dt)$ is a $\mathbb{Q}$-martingale, and no market manipulation exists that involves cross orders.
2. We prove that tame strategies (θ continuous in t) are optimal for the large trader so that, first Q is continuous in t., and, second, cross orders are traded only at the clearing price $\pi(t)$.

Characterization of the Risk-Neutral Measure II

1) Set

$$d\pi(t) = \sigma_\pi(t) \int_{s=0}^{P} b_\pi(s,t) W^{\mathbb{Q}}(ds, dt)$$

2) Market clears if $Q(p(t), t) = 0$

3) Use the Ito-Wentzell formula to calculate $dQ(p(t), t)$ and set $dQ(p(t), t) = 0$:

$$\begin{aligned}\mu_Q(0,t)dt - \int_{0+}^{\pi(t)} \mu_q(p,t)dpdt - \int_0^{\pi(t)} \sigma_q(p,t) \int_s b_q(p,s,t) W(ds,dt)dp \\ -q(\pi(t),t)\sigma_\pi(t) \int_s b_\pi(s,t) W^{\mathbb{Q}}(ds,dt) - \frac{1}{2}\frac{\partial q}{\partial p}(\pi(t),t)(\sigma_\pi(t))^2 dt + C(\pi(t),t)dt = 0\end{aligned}$$

where

$$C(\pi,t) = -\sigma_\pi(t) \left(\frac{\partial}{\partial p} \left(\sigma_Q(0,t) \int_s b_q(0,s,t) b_\pi(s,t) ds \right) + \sigma_q(\pi(t),t) \int_s b_q(\pi,s,t) b_\pi(s,t) ds \right)$$

4) Isolate the volatility terms:

$$\sigma_\pi(t) b_\pi(s,t) = -\frac{\int_0^{\pi(t)} \sigma_q(p,t) b_q(p,s,t) dp}{q(\pi(t),t)}$$

Remark: price volatility is inversely proportional to the number of buy and sell order density at the clearing price.

Characterization of the Risk-Neutral Measure III

5) The equation for the drift is:

$$\int_s \int_0^{\pi(t)} \sigma_q(p,t) b_q(p,s,t) dp \lambda(s,t) ds =$$

$$-\mu_Q(0,t) + \int_{0+}^{\pi(t)} \mu_q(p,t) dp dt + \frac{1}{2}\frac{\partial q}{\partial p}(\pi(t),t)(\sigma_\pi(t))^2 - C(\pi(t),t)$$

Then

$$\begin{aligned} b(\pi,t) &= -\mu_Q(0,t) + \int_0^{\pi} \mu_q(p,t) dp dt + \frac{1}{2}\frac{\partial q}{\partial p}(\pi,t)(\sigma_\pi(t))^2 - C(\pi,t) \\ \Sigma(\pi,s,t) &= \int_0^{\pi} \sigma_q(p,t) b_q(p,s,t) ds \end{aligned}$$

The market price of risk equations are:

$$\int_{s=0}^{P} \Sigma(\pi,s,t)\lambda(s,t) ds = b(\pi,t) \qquad 0 \le \pi \le P$$

Characterization of the Risk-Neutral Measure III

Theorem

Suppose all the previous assumptions hold. In addition, suppose that the market price of risk equations have a unique solution. Then there is no arbitrage.

Proof

Under theorems 1 and 2, there exist measures under which $\int L(\vartheta, dt)$ is a martingale. Under each of these measures $\pi(t)$ must be a local martingale. Uniqueness of the measure under which π is a local martingale ensures that this is the only measure under which $\int L(\vartheta, dt)$ is a martingale. In other words, there is no need to characterize it further.

Buy Limit Orders

We obtained 50 price intervals by choosing one cent as an increment, where 20.00 is the lowest buy limit price and 20.49 is the highest buy limit price; and obtained 195 time intervals by dividing the trading time period from 9:30 AM to 4:00 PM into *two-minute* intervals.

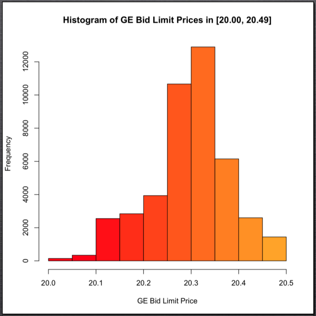


Figure: The histogram of GE buy limit prices in [20.00, 20.49]

Buy Limit Orders in 3D

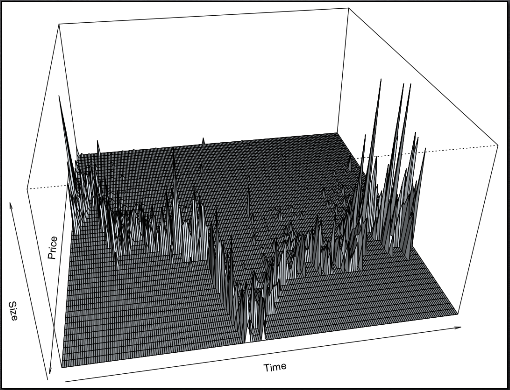


Figure: GE buy size on April 1^{st}, 2011 (*unit of time = two minutes*)

Sell Limit Orders

Similarly we obtained 50 price intervals by choosing one cent as an increment, where 20.13 is the lowest sell limit price and 20.62 is the highest sell limit price; and obtained 195 *two-minute* time intervals from 9:30 AM to 4:00 PM.

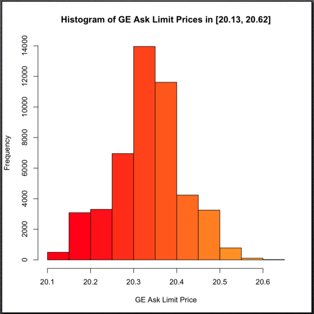


Figure: The histogram of GE sell limit prices in [20.13, 20.62]

Sell Limit Orders in 3D

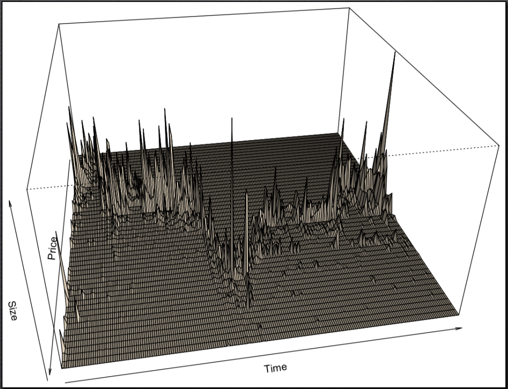


Figure: GE sell size on April 1st, 2011 (*unit of time = two minutes*)

Clearing Price $\pi(t)$

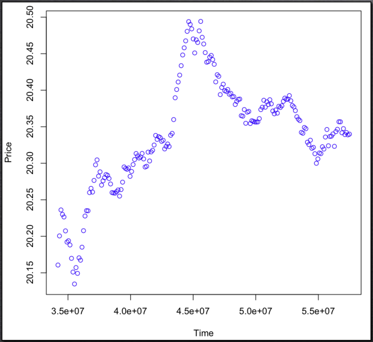


Figure: The plot of GE clearing prices from 9:30 AM to 4:00 PM (*unit of time = two minutes counted as milliseconds*)

```
> GE.clrPi
  [1] 20.16073 20.20056 20.23613 20.22996 20.22650 20.20745 20.19216 20.19391
  [9] 20.18813 20.16987 20.15110 20.13476 20.15725 20.14929 20.17054 20.16737
 [17] 20.18515 20.20772 20.22779 20.23514 20.23494 20.25990 20.26567 20.26059
 [25] 20.27649 20.29782 20.30460 20.28240 20.28827 20.27011 20.27593 20.28024
 [33] 20.28456 20.28345 20.27917 20.27166 20.25988 20.25933 20.25909 20.26118
 [41] 20.26342 20.25507 20.26396 20.27415 20.29496 20.29286 20.29171 20.29355
 [49] 20.28206 20.28874 20.29820 20.30506 20.31331 20.31006 20.30720 20.30854
 [57] 20.31342 20.30594 20.29486 20.29585 20.31525 20.30313 20.31553 20.31778
 [65] 20.32498 20.33810 20.33267 20.33640 20.33518 20.32999 20.33142 20.31962
 [73] 20.32310 20.32635 20.32289 20.33829 20.34088 20.35978 20.38962 20.40092
 [81] 20.41122 20.42064 20.43357 20.44819 20.45799 20.46751 20.48054 20.49384
 [89] 20.49000 20.48391 20.47016 20.45099 20.46887 20.46529 20.48117 20.49414
 [97] 20.47238 20.46326 20.45147 20.43831 20.43906 20.44516 20.44755 20.44185
[105] 20.43542 20.41135 20.42122 20.41902 20.39397 20.40370 20.40825 20.39930
[113] 20.39844 20.40092 20.39455 20.39573 20.39084 20.39118 20.38043 20.38471
[121] 20.38743 20.38784 20.36511 20.36443 20.37348 20.35474 20.36995 20.37094
[129] 20.35439 20.35821 20.35746 20.35652 20.35636 20.35655 20.36112 20.37343
[137] 20.37669 20.38619 20.37674 20.38407 20.38040 20.38694 20.38125 20.37154
[145] 20.36766 20.37294 20.36861 20.37801 20.37635 20.37866 20.38474 20.38915
[153] 20.38736 20.38780 20.39255 20.38552 20.37902 20.37710 20.37206 20.36366
[161] 20.36032 20.35838 20.34225 20.34106 20.34899 20.34729 20.32881 20.32559
[169] 20.33157 20.32047 20.32174 20.31305 20.29998 20.30593 20.31407 20.31317
[177] 20.32259 20.31946 20.33590 20.34619 20.32391 20.33669 20.33710 20.34009
[185] 20.32316 20.34334 20.34646 20.35682 20.35671 20.34170 20.34741 20.33928
[193] 20.34204 20.33886 20.33984
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Figure: The value of GE clearing prices for each two-minute interval from 9:30 AM to 4:00 PM.

Smile Curve Resulting from our Model II

We fitted a particular model to ARCA book data.

1. Source: NYSE ARCA
2. Stock: General Electric Company (GE)
3. Date: April 1^{st}, 2011 from 9:30 AM to 4:00 PM EST
4. Select new adding orders only, defined as type "A" in the NYSE ArcaBook

This model is based on relative prices, i.e., instead of fitting $Q(p, t)$ we fit

$$\tilde{Q}(p - \pi(t), t) = Q(p, t)$$

Smile Curve Resulting from our Model II

We approximated by simulation $E^{\mathbb{Q}}[\max(\pi(T) - K, 0)]$ for various values of K. The resulting smile curve is:

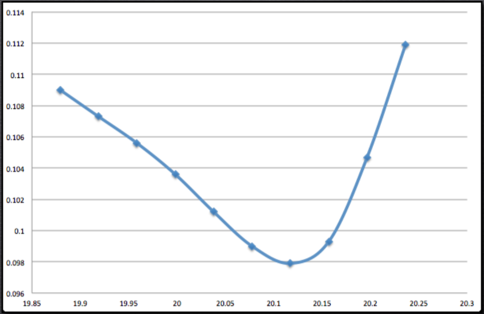

Figure: Implied volatility – Call option with 1 week expiration

Conclusion

Using empirical order book data, we construct a model of the clearing price.

Future Work:

1. Specification of some more particular Models
2. Comparison with other Models
3. Convergence of a Binomial Model to this Model