

Liquidity Risk and Classical Option Pricing Theory

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Abstract

The purpose of this paper is to review the recent derivatives security research involving liquidity risk and to summarize its implications for practical risk management. The literature supports three general conclusions. The first is that the classical option price is "on average" true, even given liquidity risk. Second, it is well known that although the classical (theoretical) option hedge can not be applied as theory prescribes, its discrete approximations often provide reasonable approximations. These discrete approximations are also consistent with upward sloping supply curves. And, third, risk management measures like value-at-risk (VaR) are biased low due to the exclusion of liquidity risk. A simple adjustment for incorporating liquidity risk into standard risk measures is provided.

1 Introduction

This paper studies liquidity risk and its relation to classical option pricing theory for practical risk management applications. Classical option pricing theory is formulated under two simplifying assumptions. One assumption is that markets are frictionless and the second assumption is that markets are competitive. A frictionless market is one where there are no transactions costs, no bid/ask spreads, no taxes and no restrictions on trades or trading strategies. For example, one can trade infinitesimal amounts of a security, continuously, for as long or as short a time interval as one wishes. A competitive market is one where the available stock price quote is good for any sized purchase or sale, alternatively stated, there is no quantity impact on the price received/paid for a trade. Both assumptions are idealizations and not satisfied in reality. Both assumptions are necessary for classical option pricing theory, e.g. the Black-Scholes option pricing model. The question arises: how does classical option pricing theory change when these two simplifying assumptions are relaxed? The purpose of this paper is to answer this question, by summarizing my research that relaxes these two

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assumptions, see Jarrow [12], [13], Cetin, Jarrow, Protter [4], Cetin, Jarrow, Protter, Warachka [5], and Jarrow and Protter [15],[16]. The relaxation of these two assumptions is called *liquidity risk*.

Alternatively stated, liquidity risk is that increased volatility that arises when trading or hedging securities due to the size of the transaction itself. When markets are calm and the transaction size is small, liquidity risk is small. In contrast, when markets are experiencing a crisis or the transaction size is significant, liquidity risk is large. This quantity impact on price can be due to asymmetric information (see Kyle [19], Glosten and Milgrom [9]) or an imbalance in demand/inventory (see Grossman and Miller [10]). It implies that demand curves for securities are downward sloping and not horizontal, as in classical option pricing theory. But, classical option pricing theory has been successfully used in risk management for over 20 years (see Jarrow [14]). In light of this success, is there any reason to modify classical option pricing theory? The answer is yes! Although successfully employed in some asset classes that trade in liquid markets (e.g. equity, foreign currencies and interest rates), it works less well in others (e.g. fixed income securities and commercial loans). Understanding the changes to the classical theory due to the relaxation of this structure will enable more accurate pricing and hedging of derivative securities and more precise risk management.

Based on my research, there are three broad conclusions that one can draw with respect to liquidity risk and classical option pricing theory. The first is that the classical option price is "on average" true, even given liquidity risk. "On average" because the classical price lies between the buying and selling prices of an option in a world with liquidity risk. In some sense, the classical option price is the mid-market price. Second, it is well known that although the classical (theoretical) hedge¹ can not be applied as theory prescribes, its discrete approximations often provide reasonable approximations (see Jarrow and Turnbull [17]). These discrete approximations can, in fact, be viewed as adhoc adjustments for liquidity risk. Their usage is also consistent with upward sloping supply curves. And, third, risk management measures like value-at-risk (VaR) do need to be adjusted to reflect liquidity risk. The classical VaR computations are biased low due to the exclusion of liquidity risk, significantly underestimating the risk of a portfolio. Fortunately, simple adjustments for liquidity risk to risk measures like VaR are available.

The rest of the paper justifies, documents and explains the reasons for these three conclusions. An outline for this paper is as follows. Section 2 presents the basic model structure for handling liquidity risk. Sections 3 discusses arbitrage free markets and derivative pricing. Feasible trading strategies are studied in section 4. Section 5 investigates risk measures like VaR, and section 6 concludes.

¹This implies trading both continuously in time and in continuous increments of shares.

2 The Basic Model

To study liquidity risk in option pricing theory, we need to consider two assets: a default free or riskless money market account and an underlying security, called a stock. The money market account earns continuously compounded interest at the default free spot rate, denoted r_t at time t . The account starts with a dollar invested at time 0, $B(0) = 1$, and grows to $B(T) = e^{\int_0^T r_s ds}$ at time T . For simplicity, we assume that there is no liquidity risk associated with investments in the money market account. In contrast, the stock has liquidity risk. This can be captured via a supply curve for the stock that depends on the size of a trade. Let $S(t, x)$ be the stock price paid/received per share at time t for a trade of size x .² If $x > 0$ then the stock is purchased. If $x < 0$, the stock is sold. The zeroth trade $x = 0$ is special. It represents the marginal trade (an infinitesimal purchase or sale). On this supply curve, therefore, the price $S(t, 0)$ corresponds to the stock price in the classical theory.

We assume that the supply curve is upward sloping, that is, as x increases, $S(t, x)$ increases. The larger the purchase order, the higher the average price paid per share. An upward sloping supply curve for shares is consistent with asymmetric information in stock markets (see Kyle [19], Glosten and Milgrom [9]). Indeed, the market responds to potential informed trading by selling at higher prices and buying at lower ones. An upward sloping supply curve is graphed in Figure 1 at time 0. The time 0 supply curve is shown on the left of the diagram.

Although most practitioners will accept an upward sloping supply curve as self-evident, there has been some debate on this issue in the academic literature, called the price pressure hypothesis (see Scholes [20], Shleifer [21], Harris and Gruel [11]). The term "price pressure" captures the notion that buying or selling exerts either upward or downward pressure on the price, resulting in a price change. The evidence supports the price pressure hypothesis. A recent empirical validation for an upward sloping supply curve can be found in Çetin, Jarrow, Protter and Warachka [5].

One of their tests is reproduced here for emphasis in Table 1. Using the TAQ database, Çetin, Jarrow, Protter and Warachka chose five well known companies trading on the NYSE with varying degrees of liquidity: General Electric (GE), International Business Machines (IBM), Federal Express (FDX), Reebok (RBK) and Barnes & Noble (BKS). The time period covered was a four year period with 1,011 trading days, from January 3, 1995 to December 31, 1998. The left side of Table 1 gives the implications of an upward supply curve when both time and information are fixed. The price inequalities indicate the ordering of transaction prices given different size trades. For example, sales should transact at lower prices than do purchases; and small buys should have lower prices than do large buys. To approximate this set of inequalities, they look at a sequence of transactions keeping the change in time small (the time between

² All technical details regarding the model structure (smoothness conditions, integrability conditions, etc.) are delegated to the references.

consecutive transactions). Despite the noise introduced by a small change in time (and information), the hypothesized inequalities are validated in all cases. This Table supports the upward sloping supply curve formulation.

The supply curve is also stochastic. It fluctuates randomly through time, its shape changing, although always remaining upward sloping. When markets are calm, the curve will be more horizontal. When markets are hectic, the curve will be more upward tilted. A possible evolution for the supply curve is given in Figure 1. The supply curve at time t represents a more liquid market than at time 0 because it's less upward sloping. As depicted, the marginal stock price's evolution is graphed as well between time 0 and time t . This evolution is what is normally modeled in the classical theory. For example, in the Black Scholes model, $S(t, 0)$ follows a geometric Brownian motion. The difference here is that when studying liquidity risk, the entire supply curve and its stochastic fluctuations across time need to be modeled.

There is an important implicit assumption in this model formulation that needs to be highlighted. In the evolution of the supply curve (as illustrated in Figure 1), the impact of the trade size on the price process is *temporary*. That is, the future evolution of the price process for $t > 0$ does not depend on the trade size executed at time 0. This is a reasonable hypothesis. The alternative is that the price process for $t > 0$ depends explicitly on the trade size executed at time 0. This happens, for example, when there are large traders whose transactions permanently change the dynamics of the stock price process. In this situation, market manipulation is possible, and option pricing theory changes dramatically (see Bank and Baum [1], Cvitanic and Ma [7], and Jarrow [12],[13]). Market manipulation manifests a breakdown of financial markets, and this extreme form of liquidity risk is not discussed further herein, but left to the cited references. Suffice it to say that the three conclusions mentioned in the introduction with respect to liquidity risk fail in these extreme market conditions. This paper only studies liquidity risk in well-functioning markets.

Given trading in the stock and a money market account, we also need to briefly discuss the meaning of a portfolio's "value." The word "value" is in quotes because when prices depend on trade sizes, there is no unique value for a portfolio. To see this, consider a portfolio consisting of n shares of the stock and m shares of the money market account at time 0. The value of the portfolio is uniquely determined by the time 0 stock price, but which stock price should be selected from the supply curve? There are an infinite number of choices available, each corresponding to a particular point on the supply curve. Some economic meaningful selections are apparent. For example, the value of the portfolio if liquidated at time 0 is

$$\text{liquidation value: } nS(0, -n) + mB(0). \quad (1)$$

Note that in the stock price, the price for selling n shares is used. Another choice is the *marked-to-market value*, defined by using the price from the marginal trade (zero trade size), i.e.

$$\text{marked-to-market value: } nS(0, 0) + mB(0). \quad (2)$$

The marked-to-market value represents the portfolio's value held in place, and it also corresponds to the value of the portfolio in the classical model. Given that the supply curve is upward sloping, we have that $S(0, -n) < S(0, 0)$. This implies that the liquidation value is strictly less than the marked-to-market value of the portfolio.

3 Arbitrage Pricing Theory

To obtain a theory of arbitrage pricing, analogous to the classical case, one needs to impose more structure on the supply curve. In particular, it is very important for the theory to understand the shape of the supply curve near the zeroth trade size ($x = 0$). Unfortunately, due to the discreteness of shares traded in actual markets (units), the shape of the supply curve near zero can never be observed, and is an abstraction. Nonetheless, to proceed, we need to assume a particular structure and investigate its implications. If the implications are counter intuitive, other alternative structures can be subsequently imposed. Continuing, the simpler the structure, the better. In their initial model, Cetin, Jarrow, Protter [4] assume that the supply curve is continuous (and twice differentiable) at the origin, so that standard calculus type methods can be applied. This is the structure we will discuss below (and represented in Figure 1).

In the classical theory of option pricing, the logical steps in the development of the theory are as follows: (step 1) the notion of an arbitrage opportunity is defined, (step 2) a characterization of an arbitrage free market is obtained in terms of an equivalent martingale measure (alternatively called a risk neutral measure), (step 3) a complete market is defined, (step 4) in a complete market, the equivalent martingale measure is used to price an option, and (5) the option's delta (or hedge ratio) is determined from the option's price formula. In our structure with liquidity risk, we will follow these same five steps.

3.1 Step 1 (Definition Arbitrage)

As in the classical theory, an arbitrage opportunity is defined to be portfolio that starts with zero value (investment), the portfolio has no intermediate cash flows (or if so, they are all non-negative), and the portfolio is liquidated at some future date T with a non-negative value with probability one, and a strictly positive value with positive probability. This is the proverbial "free lunch." The only change in the liquidity risk model from the classical definition is that instead of using the marked-to-market value of the portfolio at time T , one needs to use the liquidation value, expression (1). The liquidation value implies that all liquidity costs of entering and selling a position are accounted for.

3.2 Step 2 (Characterization Result)

Cetin, Jarrow, Protter [4] show that to guarantee that a market is arbitrage free, one only needs to consider the classical case, and examine the properties of

the marginal stock price process $S(t, 0)$. In particular, they prove the following result.

Theorem 1 *If there exists an equivalent probability Q such that $S(t, 0)/B(t)$ is a martingale, then the market is arbitrage free.*

The intuition for the theorem is straightforward. If there are no arbitrage opportunities when trading with zero liquidity impact (at the zeroth trade), then trading with liquidity can create no arbitrage opportunities that otherwise did not exist. Indeed, the trade size impact on the price always works against the trader, decreasing his returns, and decreasing any potential payoffs.

This theorem is an important insight. It implies that all the classical stock price processes can still be employed in the analysis of liquidity risk, but they now represent the zeroth point on the supply curve's evolution. For example, an extended Black-Scholes economy is given by the supply curve

$$S(t, x) = S(t, 0)e^{\alpha x} \quad (3)$$

where $\alpha > 0$ is a constant and $S(t, 0)$ is a geometric Brownian motion. That is,

$$dS(t, 0) = \mu S(t, 0)dt + \sigma S(t, 0)dW_t \quad (4)$$

where $\mu, \sigma > 0$ are constants and W_t is a standard Brownian motion. The supply curve in expression (3) depends on the trade size x in an exponential manner. Indeed, as x increases, the purchase price increases by the proportion $e^{\alpha x}$. This simple supply curve, as a first approximation, appears to be consistent with the data (see Cetin, Jarrow, Protter, Warachka [5]). A typical value of α lies in the set $[.00005, .00015]$, i.e. between 0.5 and 1.5 basis points per transaction is a typical quantity impact on the price (see Cetin, Jarrow, Protter, Warachka [5], Table 1).

Since we know that a geometric Brownian motion process in the classical case admits no arbitrage, this theorem tells us that the supply curve extension admits no arbitrage as well. We will return to this example numerous times in the subsequent text to illustrate the relevant insights.

3.3 Step 3 (Definition Market Completeness)

As in the classical case, a complete market is defined to be a market where (dynamically) trading in the stock and money market account can reproduce the payoff to any derivative security at some future date. The same definition applies with supply curves, although in attempting to reproduce the payoff to a derivative security, the actual trade size determined price must be used. Otherwise, the definition is identical.

Unfortunately, one can show that in the presence of a supply curve, if the market was complete in the classical case (for the marginal price process $S(t, 0)$), it will *not* be complete with liquidity costs. The reason is that part of the portfolio's value evaporates due to the liquidity cost in the attempt to replicate the option.

But, recall that in the classical case, the completeness result for the price process requires that the replicating strategy often involves continuous trading of infinitesimal quantities of a stock in an erratic fashion³. Of course, in practice, following such a replicating strategy is impossible. And, only approximating trading strategies can be employed that involve trading at discrete time intervals (see Jarrow and Turnbull [17] for a more detailed explanation). Thus, in the classical model, the best we can really hope for (in practice) is an approximately complete market. That is, a market that can approximately reproduce the payoff to any derivative security at some future date.

It turns out that under the supply curve formulation, Cetin, Jarrow, Protter [4] prove the following theorem.

Theorem 2 *Given the existence of an equivalent martingale measure (as in Theorem 1), if it is unique, then the market is approximately complete.*

In the classical case, if the equivalent martingale measure is unique, then the market is complete. Here, almost the same result holds when applied to the marginal stock price process $S(t, 0)$. The difference is that one only gets an approximately complete market. The result follows because there always exist trading strategies, involving quick trading of small quantities, that incur very little price impact costs (since one is trading nearly zero shares at all times). By reducing the size of each trade, but accumulating the same aggregate quantity by trading more quickly, one can get arbitrarily close to the no liquidity cost case.

Again, the importance of this theorem for applications is that all of the classical stock price process results can still be employed in the analysis of liquidity risk. Indeed, if the classical stock price process is consistent with a complete market, then it will imply an approximately complete market given liquidity costs. For example, returning to the extended Black-Scholes economy presented above in expression (3), since we know in the classical case that it implies a complete market, this theorem tells us that the supply curve extension implies an approximately complete market as well.

3.4 Step 4 (Pricing Options)

Just as in the classical case, one can show that in an approximately complete market, the value of an option is its discounted expected payoff using the martingale measure in taking the expectation. The martingale measure adjusts that statistical (or actual) probabilities to account for risk (see Jarrow and Turnbull [17] for a proof of these statements). For concreteness, let C_T represent the payoff to an option at time T . For example, if the option is a European call with strike price K and maturity T , then $C_T = \max[S(T, 0) - K, 0]$. In the payoff of this European call, the marginal stock price is used (at the zeroth trade size). The reason is that (as explained in step 3) the replicating trading

³By "erratic" I mean similar to the path mapped out by a Brownian motion.

strategy avoids (nearly) all liquidity costs. This is reflected in the option's payoff by setting the trade size $x = 0$. For more discussion on this point, see Cetin, Jarrow, Protter [4].

Given this payoff, the price of the option at time 0 is given by

$$C_0 = E(C_T e^{-\int_0^T r_s ds}) \quad (5)$$

where $E(\cdot)$ represents expectation under the martingale measure. This is the standard formula used to price options in the classical case.

Continuing our European call option with strike price K and maturity T example, let us assume that the stock price supply curve evolves according to the extended Black-Scholes economy as in expression (3) above, and that the spot rate of interest is a constant, i.e. $r_t = r$ for all t . Then, the pricing formula becomes:

$$\begin{aligned} C_0 &= E(\max[S(T, 0) - K, 0]e^{-rT}) \\ &= S(0, 0)N(h(0)) - Ke^{-rT}N(h(0) - \sigma\sqrt{T}) \end{aligned} \quad (6)$$

where $\sigma > 0$ is the stock's volatility, $N(\cdot)$ is the standard cumulative normal distribution function, and

$$h(t) \equiv \frac{\log S(t, 0) - \log K + r(T - t)}{\sigma\sqrt{T - t}} + \frac{\sigma}{2}\sqrt{T - t}.$$

This is the standard Black Scholes formula, but in a world with liquidity risk!

3.5 Step 5 (Replication)

In the classical case, the replicating portfolio can often be obtained from the valuation formula by taking its first partial derivative with respect to the underlying stock's time 0 price. For example, with respect to the classical Black-Scholes formula in expression (6) above, the hedge ratio is the option's delta and it is given by

$$\Delta_t = \frac{\partial C_t}{\partial S(t, 0)} = N(h(t)) \quad (7)$$

for an arbitrary time t .

This represents the number of shares of the stock to hold at time t to replicate the payoff to the European call option with strike price K and maturity T . The holdings in the stock must be changed continuously in time according to expression (7), buying and selling infinitesimal shares of the stock to maintain this hedge ratio.

In the situation with an upward sloping supply curve, this procedure for determining the replicating portfolio is almost the same. The difference is that the classical replicating strategy will often be too erratic, and a smoothing of the replicating strategy will need to be employed to reduce price impact costs. The exact smoothing procedure is detailed in Cetin, Jarrow, Protter [4]. Consequently, the classical hedge (appropriately smoothed) provides an approximate replicating strategy for the option. For the Black Scholes extended

economy that we have been discussing, the smoothed hedge ratio for any time t is given by

$$\Delta_t = 1_{[\frac{1}{n}, T - \frac{1}{n})}(t)n \int_{(t - \frac{1}{n})^+}^t N(h(u))du, \text{ if } 0 \leq t \leq T - \frac{1}{n} \quad (8)$$

$$\Delta_t = (nT\Delta_{(T - \frac{1}{n})} - n\Delta_{(T - \frac{1}{n})}t), \text{ if } T - \frac{1}{n} \leq t \leq T.$$

where $1_{[\frac{1}{n}, T - \frac{1}{n})}(t)$ is an indicator function for the time set $[\frac{1}{n}, T - \frac{1}{n})$ and n is the step size in the approximating procedure. As evidenced in this expression, the smoothing procedure is accomplished by taking an integral (an averaging operation). The approximation improves as $n \rightarrow \infty$.

In summary, as just documented, the classical approach almost applies. But, there is a potential problem with the smoothed trading strategy. Just as in the classical case, it usually involves continuous trading of infinitesimal quantities of the stock's shares, which is impossible in practice. And, just as in the classical case, to make the hedging theory consistent with practice we need to restrict ourselves to more realistic trading strategies. This is the subject of the next section.

4 Feasible Replicating Strategies

The implication of Cetin, Jarrow, Protter [4] is that the classical option price must hold in the extended supply curve model. Taken further, this implies that option markets should exhibit no quantity impact on prices, no bid/ask spreads, even if the underlying stock price curve does! This is a counter-intuitive implication of the model, directly due to the ability to trade continuously in time in infinitesimal quantities. Just as in the classical model, if one removes continuous trading strategies, and only admits discrete trading, then this implication changes.

Cetin, Jarrow, Protter, Warachka [5] explore this refinement. Cetin, Jarrow, Protter, Warachka reexamine the upward sloping supply curve model considering only discrete trading strategies. Discrete trading strategies only allow a finite number of trades (at random times) in any finite time interval. As one might expect, the situation becomes analogous to a market with transaction costs. Fixed (or proportionate) transaction costs, due to their existence, preclude the existence of continuous trading strategies, otherwise transaction costs would become infinite in any finite time (see Barles and Soner [2], Cvitanic and Karatzas [6], Cvitanic, Pham, Touze [8], Jouini and Kallal [18], Soner, Shreve and Cvitanic [22], and Jarrow and Protter [16]). Hence, many of the insights from the transaction cost literature can now be directly applied to Cetin, Jarrow, Protter, Warachka's market.

In such a market, one can not exactly or even approximately replicate an option. The market is incomplete. This is due to the fact that transaction costs evaporate value from a portfolio. Consequently, one seeks to determine

the cheapest buying price and the largest selling price for an option. If the optimal super- or sub- replicating strategy can be determined, then one gets a relationship between the buying and selling prices, and the classical price:

$$C_0^{sell} \leq C_0^{classical} \leq C_0^{buy}. \quad (9)$$

The classical price lies between the buying and selling prices obtainable by super- and sub- replication. In this sense, the classical option price is the "average" of the buying and selling prices. This statement provides the justification for the first conclusion contained in the introduction.

To get a sense for the percentage magnitudes of the liquidity cost band around the classical price, Table 2 contains some results from Cetin, Jarrow, Protter, Warachka [5] where they document the liquidity costs associated with the extended Black-Scholes model given in expression (3) above given only discrete trading is allowed. As a percentage of the option's price, liquidity costs usually are less than 100 basis points.

Unfortunately, the optimal strategy for buying (or selling) is often difficult to obtain, because it involves solving a complex dynamic programming problem. For this reason, practical replicating strategies often need to be used instead. For example, the standard Black Scholes delta hedge, implemented once a day (instead of continuously in time), is one such possibility. Cetin, Jarrow, Protter, Warachka show that this strategy provides a reasonable approximation to the option's optimal buying or selling strategy. As such, these practical hedging strategies used by the industry can be viewed as adjustments to the classical model for handling liquidity risk. This observation serves as the justification for the second conclusion drawn in the introduction, i.e. that the discrete trading strategies used in practice serve as reasonable approximations to the optimal trading strategy given liquidity costs. The third conclusion is justified in the next section.

5 Risk Management Measures

The classical risk measures, like value at risk (VaR), do not explicitly incorporate liquidity risk into the calculation. Given the liquidity risk model introduced above, a simple and robust adjustment for liquidity risk is now readily available as detailed in Jarrow and Protter [15].

As discussed in section 3 above, trading continuously and in infinitesimal amounts enables one to avoid all liquidity costs. In actual markets, this corresponds to slow and deliberate selling of the portfolio's assets. But, when computing risk measures for risk management, one needs to be conservative. The worst case scenario is a crisis situation, where one has to liquidate assets immediately. The idea is that if the market is declining quickly, then one does not have the luxury to sell assets slowly (continuously) in "small" quantities until the entire position is liquidated. In a crisis situation, the supply curve formulation provides the relevant value to be used, the liquidation value of the portfolio from expression (1) above.

The liquidation value of a portfolio can be determined by estimating each stock's supply curve's stochastic process, and knowing the size of each stock position. Supply curve estimation is not a difficult exercise, see Çetin, Jarrow, Protter and Warachka [5]. For example, in the extended Black-Scholes model of expression (3) above, a simple time series regression can be employed. Given time series observations of transaction prices and trade sizes $(S(\tau_i, x_{\tau_i}), \tau_i)_{i=1}^T$, it can be shown that the regression equation is

$$\ln \left(\frac{S(\tau_{i+1}, x_{\tau_{i+1}})}{S(\tau_i, x_{\tau_i})} \right) = \alpha [x_{\tau_{i+1}} - x_{\tau_i}] + \mu [\tau_{i+1} - \tau_i] + \sigma \epsilon_{\tau_{i+1}, \tau_i}. \quad (10)$$

The error $\epsilon_{\tau_{i+1}, \tau_i}$ equals $\epsilon \sqrt{\tau_{i+1} - \tau_i}$ with ϵ being distributed $\mathcal{N}(0, 1)$.⁴ Then, in computing VaR one can explicitly include the liquidity discount parameter α to determine the portfolio value for immediate liquidation.

The bias in the classical VaR computation can be easily understood. The classical VaR computation uses the marked-to-market value of the portfolio given in expression (2). In contrast, the VaR computation including liquidity risk should use the liquidation value given in expression (1). As noted earlier, the liquidation value in expression (1) is always less than the marked-to-market value in expression (2). This implies that the classical VaR computation will be biased low, indicating less risk in the portfolio than actually exists. This liquidity risk adjustment to VaR based on the portfolio's liquidation value, rather than its marked-to-market value, yields the basis for the third conclusion stated in the introduction.

To illustrate these computations, let us again consider the extended Black-Scholes economy of expression (3). Let us consider a portfolio of N stocks $i = 1, \dots, N$ with share holdings denoted by n_i in the i th stock. We denote the i th stock price by

$$S_i(t, x) = S_i(t, 0)e^{\alpha_i x}. \quad (11)$$

The marked-to-market value of the portfolio at time T is

$$\sum_{i=1}^N n_i S_i(T, 0).$$

The time T liquidation value is

$$\sum_{i=1}^N n_i S_i(T, -n_i) = \sum_{i=1}^N n_i S_i(T, 0)e^{-\alpha_i n_i}.$$

When computing VaR either analytically or via a simulation, the adjustment to the realization of $S_i(T, 0)$ is given by the term $e^{-\alpha_i n_i}$. This is an easy computation.

⁴This is the regression equation used to obtain the typical values for α reported earlier.

6 Conclusion

This paper reviews the recent literature on liquidity risk for its practical use in risk management. The literature supports three general conclusions. The first is that the classical option price is "on average" true, even given liquidity risk. Second, it is well known that although the classical (theoretical) option hedge can not be applied as theory prescribes, its discrete approximations often provide reasonable approximations (see Jarrow and Turnbull [17]). These discrete approximations are also consistent with upward sloping supply curves. And, third, risk management measures like value-at-risk (VaR) are biased low due to the exclusion of liquidity risk. Fortunately, simple adjustments for liquidity risk to risk measures like VaR are readily available.

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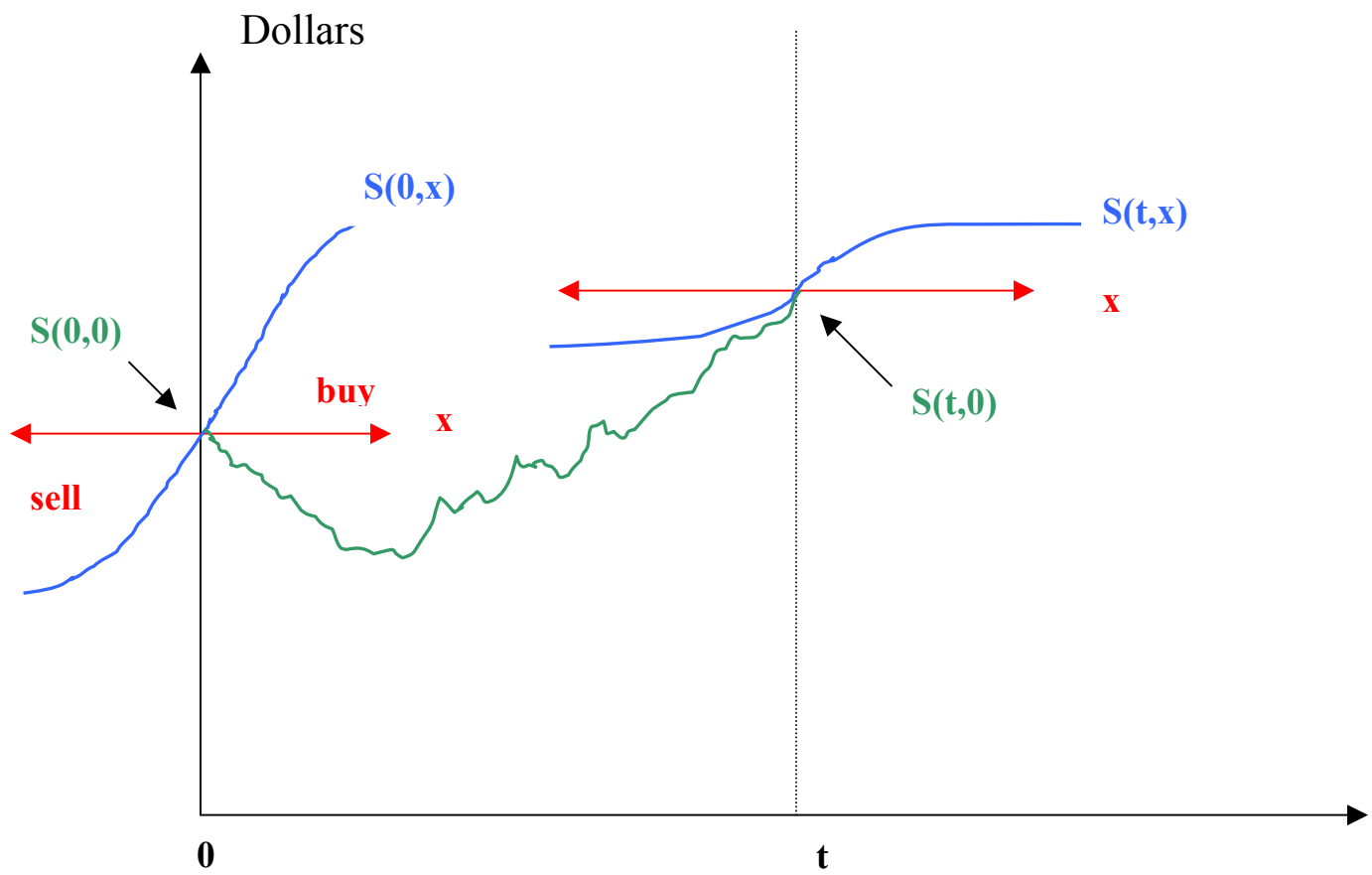


Figure 1: Stochastic Supply Curve $S(t,x)$

Table 1 - Summary of order flow and transaction prices. Recorded below are the relationships between consecutive transaction prices as a function of order flow. The first two rows impose no constraints on the magnitude of the transaction. The next four rows have small (large) trades defined as those less than or equal to (greater than) 10 lots.

Trade Type and Hypothesis			Company Ticker				
First Trade(x1)	Second Trade (x2)	Transaction Hypothesis	GE	IBM	FDX	BKS	RBK
Sell	Buy	$S(t1, x1) \leq S(t2, x2)$	100.00%	100.00%	100.00%	100.00%	100.00%
Buy	Sell	$S(t1, x1) \geq S(t2, x2)$	100.00%	100.00%	100.00%	100.00%	100.00%
Small Buy	Large Buy	$S(t1, x1) \leq S(t2, x2)$	98.77%	98.58%	98.46%	98.58%	99.05%
Small Sell	Large Sell	$S(t1, x1) \geq S(t2, x2)$	98.10%	97.99%	98.61%	98.43%	98.95%
Large Buy	Small Buy	$S(t1, x1) \geq S(t2, x2)$	93.60%	90.28%	80.59%	80.53%	83.88%
Large Sell	Small Sell	$S(t1, x1) \leq S(t2, x2)$	90.33%	87.33%	78.87%	78.67%	81.89%

Source: Cetin, Jarrow, Protter, Warachka [5, Table 2].

Table 2- Summary of the liquidity costs for 10 options, each on 100 shares. For at-the-options, the strike price equals the initial stock price. The stock price is increased (decreased) by \$5 for in-the-money (out-of-the-money) options.

Source: Cetin, Jarrow, Protter, Warachka [5, Table 3].

Option Characteristics			Costs Associated with Replicating Portfolio for 10 Options			
Company Name	Option Moneyess	Option Price with $a = 0$	Liquidity Cost (at $t = 0$)	Liquidity Cost (after $t = 0$)	Total Liquidity Cost	Percentage Impact
GE	In	548.91	19.31	5.77	25.08	0.46
	At	177.94	9.59	10.06	19.65	1.10
	Out	46.94	0.86	0.82	1.68	0.36
IBM	In	730.56	8.61	3.46	12.07	0.17
	At	362.91	4.81	4.46	9.27	0.26
	Out	229.90	2.44	2.29	4.73	0.21
FDX	In	562.42	15.70	4.96	20.66	0.37
	At	188.42	8.46	8.09	16.55	0.88
	Out	56.07	0.92	0.87	1.79	0.32
BKS	In	517.87	23.39	4.52	27.91	0.54
	At	133.50	11.07	10.18	21.25	1.59
	Out	12.25	0.12	0.11	0.23	0.19
RBK	In	513.40	20.02	3.67	23.69	0.46
	At	129.33	9.16	8.44	17.60	1.36
	Out	8.36	0.05	0.05	0.10	0.12