Mathematical Finance, Vol. 20, No. 4 (October 2010), 597–615

THE COST OF ILLIQUIDITY AND ITS EFFECTS ON HEDGING

L. C. G. ROGERS

University of Cambridge

SURBJEET SINGH

University of Bath

Though liquidity is commonly believed to be a major effect in financial markets, there appears to be no consensus definition of what it is or how it is to be measured. In this paper, we understand liquidity as a nonlinear transaction cost incurred as a function of rate of change of portfolio. Using this definition, we obtain the optimal hedging policy for the hedging of a call option in a Black-Scholes model. This is a more challenging question than the more common studies of optimal strategy for liquidating an initial position, because our goal requires us to match a random final value. The solution we obtain reduces in the case of quadratic loss to the solution of three partial differential equations of Black-Scholes type, one of them nonlinear.

KEY WORDS: liquidity, illiquidity, option pricing, hedging, price impact.

1. INTRODUCTION

After credit risk, liquidity risk is probably the next most important risk faced by the finance industry; and yet the study of liquidity is far less advanced. This may be in part due to the fact that there is no agreed definition of what liquidity is, even in qualitative terms. Everyone would agree that the effect of illiquidity is to make it difficult or costly to trade large volumes of the underlying asset in small times, but there are different approaches to modelling this.

There is a growing literature on such effects: Bertsimas and Lo (1998), Almgren and Chriss (2001), Almgren (2003), Obizhaeva and Wang (2005), Huberman and Stanzl (2005), Schied and Schöneborn (2007), and other papers referred to therein provide a sample of what is currently known. These papers study two effects, referred to as *temporary price impact* and *permanent price impact*. Roughly speaking, the first of these arises on short time scales as the result of trading, and can be thought of as an agent having to work through the limit-order book to acquire his desired change of holding. The second is some lasting impact on price, presumed to have been caused by the fact that an agent has just traded some quantity of the asset.[1] In Almgren and Chriss (2001), the permanent price impact is fully incorporated in the (discrete) period immediately after the trade; in Obizhaeva and Wang (2005), the permanent price impact is established gradually as the order book refills the space emptied by the original trade. In all the literature cited above, the aim is to optimally unwind an initial position by some fixed

Manuscript received January 2006; final revision received December 2008.

Address correspondence to L. C. G. Rogers, Statistical Laboratory, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, UK; e-mail: L.C.G.Rogers@statslab.cam.ac.uk.

[1]The papers Frey (1988), Frey and Stremme (1997), Platen and Schweizer (1998), Papanicolaou and Sircar (1998), Schönbucher and Wilmott (2000) are studies of this effect.

time horizon, and the dynamics of the underlying in the absence of trading is always additive (i.e., a random walk in discrete time, a Brownian motion in continuous time). The present study departs from these assumptions in at least three ways:

1. We ignore permanent price impact completely,
2. The objective is to hedge a European option, and
3. The basic asset dynamic is assumed to be log Brownian.

Let us comment now on these features. The reason we ignore permanent price impact is that the proper treatment is problematic. Indeed, if the actions of a single trader will shift the price, then logically the actions of *every* trader will shift the price, and in order to understand this effect fully we would have to build a model which accounted for the behavior of all the agents in the market.[2] Permanent price impact is in any case not a feature of illiquidity costs as we define them.[3] An agent, however small, may face costs in trying to trade very rapidly; the effect is a phenomenon of *rate* of change of holding, not of *size* of change of holding, and our terminology and modeling assumptions make this clear. The illiquidity effect (i.e., the cost of trying to trade fast) is distinct from the price impact effect of a large trade, as we shall argue below in Section 2. Both can be considered as effects of supply and demand on price, but the illiquidity effect (or temporary price impact, if you prefer) arises because of the need to clear a market over a *short* time spell, whereas the permanent price impact comes from the clearing of the market over *long* periods. We can and shall consider the illiquidity costs ignoring the effects of permanent price impact, but it is nonsensical to consider the second of these and ignore the first, as is pointed out by Schönbucher and Wilmott (2000). They consider the "free round trip" phenomenon, where the large agent rapidly sells and then buys back a large amount of stock, forcing the price instantaneously to drop, and if this round trip is not costly, then the large agent could make profits by selling down-and-out calls and subsequently knocking them out by a round trip. Another problem with permanent price impact models such as Frey (1988), Frey and Stremme (1997), Platen and Schweizer (1998), Papanicolaou and Sircar (1998), and Schönbucher and Wilmott (2000) is that they typically present the solution to a hedging problem in feedback form, exhibiting the hedge as a function of time, and current stock price—but if the initial portfolio is not at the exactly correct value, it is not clear how it is to be moved to that value.

The second distinctive feature of our study is that we are trying to hedge a European option. In contrast to earlier studies where the objective is to liquidate a given initial position, the challenge here is to steer the portfolio toward a *random* endpoint, not known at earlier times, and this makes the problem a lot harder. This is evidenced by the fact that most earlier studies end up with portfolio rules which are deterministic functions of time, whereas the optimal rules which we arrive at have a genuinely interesting dependence on the underlying asset as well as time.

The third feature is that we are working with a log-Brownian asset, and this requires solution of a partial differential equation (PDE) to arrive at the answer; many of the earlier studies cited above require only the calculation of the correct moments to arrive at the solution. This should not be surprising; working out the correct hedging strategy for a European option with no illiquidity/price impact effects requires the solution of a PDE (or some equivalent method), so we must expect something comparable here.

[2]See, for example, Heritage and Rogers (2002) for a partial analysis of such a setup.
[3]As there is no agreed definition of the terms *liquidity* or *illiquidity*, we feel justified in framing our own.

Our viewpoint here is that the effect of illiquidity is a *cost*, but that illiquidity does not affect the price of the underlying asset. It affects the price at which an agent will trade the asset, however, reflecting the depth of the limit order book. The faster an agent wants to buy (sell) the asset, the deeper into the limit order book he will have to go, and higher (lower) will be the price for the later units of the asset bought (sold). However, once a rapid transaction is completed, we suppose that the limit order book quickly fills up again and that the rapid transaction has no lasting effect on the price of the underlying. What transpires is that the impact of liquidity modeled in this fashion is like a transaction cost, but not one which is *proportional* to the amounts traded, which is the assumption of the traditional proportional transaction cost model (see Magill and Constantinides 1976; Davis and Norman 1990). This modeling approach was presented in Rogers and Singh (2004); Cetin and Rogers (2007) and Singh (2005) explore further aspects of the model introduced there. Isaenko (2005) proposes a dynamic with some similarities to ours, but the way in which the illiquidity costs scale with stock price is very different.

Another related paper is that of Longstaff (2001), who proposes that the holding of stock must be a finite-variation process with bounded derivative, which may be thought of as a special case of the model we propose. Models which feature illiquidity costs include Bank and Baum (2004) and Cetin, Jarrow, and Protter (2004). However, in both these references transaction costs can be completely avoided by following a continuous trading strategy of bounded variation. This seems rather unrealistic. The paper of Bakstein and Howison (2003) shares a number of features with ours; one main difference is that it leads to permanent price impact effects, which we are trying to eliminate.

The layout of the remainder of the paper is as follows. Section 2 explains how the dynamics that we shall take as fundamental arise from consideration of the order book. Next, Section 3 deals with the hedging of a European-style option in the Black–Scholes world in the presence of illiquidity costs. The Hamilton–Jacobi–Bellman (HJB) equation that arises from the associated optimal control problem can be solved in almost closed form. The numerical solution requires us just to solve four parabolic PDEs of Black–Scholes type. We then consider approximate solutions in Section 4. Here we show that there is a simple hedging rule which approximates the optimal rule; the most important consequence of this is to derive bounds on the magnitude of the losses due to illiquidity. Next in Section 5 we explore some numerical solutions, comparing the optimal solution with the approximate solution. Finally, Section 6 concludes.

2. MODELING ILLIQUIDITY: MOTIVATION

We emphasize that the discussion of this section is intended to motivate the dynamics (2.3), (2.4) which we shall take as the starting point of the rest of the paper; those equations *are* the model—there is no question of “proving” them.

Let us start with a simple *static* model for the order book. We imagine that there is some mid-price S, and that there is an order book of quotes distributed either side of the mid-price S, with density $\rho(x)$ of quotes at relative[4] price x. If an agent wishes to acquire h units of the asset, he will have to buy up through the order book,[5] to relative price s

[4]We assume that the only effect of the value of mid-price on the problem is through proportionality; otherwise, high-priced assets would be subject to more (or less) liquidity costs than low-priced assets, which is not an effect we consider realistic.

[5]We do not take the idea of the order book too literally. The argument given would apply to an asset for which no formal order book could be observed on the market. We simply view $\rho(x)$ as the density of

defined by

$$h = \int_1^s \rho(x)\, dx,$$

which will cost

$$S \int_1^s x\rho(x)\, dx.$$

Having done this trade, the mid-price quickly[6] returns to S, so the book value of what he has just bought is hS and he will record a loss of

$$Sl(h) \equiv S \int_1^s x\rho(x)\, dx - hS = S \int_1^s (x-1)\rho(x)\, dx. \tag{2.1}$$

Notice that the same equations hold whether $s > 1$ or $s < 1$, so that $l \geq 0$, and

$$\frac{dl}{dh} = \frac{(s-1)\rho(s)}{\rho(s)} = (s-1) \geq -1$$

is increasing, and therefore the function l is convex, with slope at least -1.

EXAMPLES 2.1

(i) A natural example would be to take

$$l(x) = \frac{e^{\varepsilon x} - 1}{\varepsilon} - x, \tag{2.2}$$

where $\varepsilon > 0$ is fixed.

(ii) Taking

$$l(x) = \varepsilon\, |x|$$

models a proportional transaction cost as in Magill and Constantinides (1976) and Davis and Norman (1990).

(iii) We could take

$$\begin{aligned} l(x) &= 0 \qquad (|x| \leq a) \\ &= \infty \qquad (|x| > a) \end{aligned}$$

to model a situation where any trade up to a certain size would be allowed; this is in effect the situation in Longstaff (2001).

We extend this to a dynamic setting by first supposing that trading takes place in discrete intervals of time of length Δt, during each of which there is an order book with density $\rho(x) dx \Delta t$ at price x relative to the prevailing mid-price S_t. In particular, this means that the total available quantity of the asset which may be traded in the time period is $O(\Delta t)$. As before, an agent who wishes to acquire $h\Delta t$ units of the asset in that time period will book a loss of $S_t l(h)\Delta t$.

possible trades at relative price x, and this need not coincide with an observed order book even if there were one; quotes in the order book can be pulled as the price approaches them, and new ones inserted, so the order book has a somewhat insubstantial existence.

[6]We are therefore talking of an asset with infinite *resilience*—or resiliency.

After such a trade, parts of the order book will have been swept out; as a result, we may expect that the mid-price will have been moved by the sale/purchase. However, this is a price-impact effect, and as we explained earlier, we do not intend to model this, as this would make the price dynamics not just history dependent, but also dependent on the past trading decisions of a potentially large number of agents. One justification for this simplifying assumption is that the quotes swept out would quickly be replaced if the asset were reasonably liquid, which is the situation we are most interested in.

Given this description of the effect of liquidity in a Δt-trading-interval formulation, it is not hard to see how we are going to model the cost of illiquidity in continuous time. We shall suppose that the number H_t of units of the risky asset held at time t should be differentiable, with derivative h_t, and the wealth dynamics are summarised as

$$dw_t = r_t w_t dt + H_t(dS_t - r_t S_t dt) - S_t\, l(h_t)\, dt \tag{2.3}$$

$$dH_t = h_t dt, \tag{2.4}$$

where r_t is the riskless rate, and S_t is the asset price at time t, given exogenously, and not affected by the trades of any agents. The only part of the dynamics (2.3) which is not entirely conventional is the final term $-S_t l(h_t)dt$ representing the cost of illiquidity. Concerning the function l, we shall make the following assumption:

ASSUMPTION A. *the function l is convex and nonnegative, $l(0) = 0$.*

REMARKS.

(i) Note that we do *not* assume that $l' \geq -1$, a restriction which appears natural in view of the way we derived l from the order book description. This is because we shall presently wish to work with $l(x) = \frac{1}{2}\varepsilon x^2$ for reasons of tractability.

(ii) Note that we do *not* claim that there is some weak convergence argument which takes us from the discrete-time model for the cost of liquidity to (2.3), (2.4) as $\Delta t \downarrow 0$; there may well be, but any such argument is tangential to our purpose.

3. AN OPTIMAL CONTROL PROBLEM AND ITS SOLUTION

The introduction of illiquidity costs makes the market incomplete, so perfect replication is no longer possible. In this section, we shall study the effect of illiquidity on the hedging of a European-style option when the illiquidity costs are small.

In the Black–Scholes model, the asset dynamics are given by

$$dS_t = S_t(\sigma dW_t + \mu dt), \tag{3.1}$$

and the riskless rate r_t is assumed equal to a constant. The presence of discounting is an inessential complication to the notation, so we shall immediately assume that we are working with discounted asset prices, that is, we shall take $r = 0$. We shall also assume that $\mu = 0$, for three reasons. First, in the perfectly liquid Black–Scholes world, under the pricing measure, this is the value we would use; if illiquidity costs were small, we should be close to this situation. Second, if we are to use any value for μ other than the riskless rate $r = 0$, we would in practice have to estimate this value, which is notoriously hard to estimate with any accuracy or stability. Third, the analysis is already hard enough with this assumption. Under these assumptions, the wealth dynamics become simply

$$dw_t = H_t dS_t - S_t l(h_t)\,dt = d(H_t S_t) - S_t f(h_t)\,dt, \tag{3.2}$$

$$dS_t = \sigma S_t dW_t, \tag{3.3}$$

$$dH_t = h_t dt, \tag{3.4}$$

where $f(h) \equiv h + l(h)$.

We shall consider the hedging of a European option which expires at time T and pays $G(S_T)$ at that time. The Black–Scholes value of the option at time $t < T$ is $q(t, S_t)$, where q solves the Black–Scholes initial-value PDE

$$\mathcal{L}q = 0, \quad q(T, \cdot) = G(\cdot), \tag{3.5}$$

where

$$\mathcal{L} \equiv \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \frac{\partial}{\partial t}. \tag{3.6}$$

In the absence of illiquidity costs, the unique time-0 price of this option would be $q(0, S_0)$, and given that initial wealth, the option could be perfectly replicated by using the self-financing portfolio which holds $\theta(t, S_t) \equiv q_S(t, S_t)$ units of the stock at time t.

Once illiquidity costs are introduced, as modelled in Section 2, it is not possible to hold the Black–Scholes portfolio $H_t = \theta(t, S_t)$ at all times t, because this is a process of infinite variation. We must therefore choose a portfolio H which optimizes some criterion, which must be chosen to express our two objectives, namely, to finish up close to the payoff $G(S_T)$ of the option and not to incur large illiquidity costs on the way. We must also specify what assumption will be made about the portfolio at time T, which will typically hold nonzero amounts of the asset. For this, we shall simply suppose that the asset can be sold at spot with no illiquidity losses; in practice, any money owing on the option would be paid out of cash, and the position in the asset could be unwound sufficiently slowly that no significant illiquidity costs would be incurred. As we shall see, once we introduce (at (3.13)) a loss function l scaled by the small parameter ε, the illiquidity costs incurred by hedging are $O(\sqrt{\varepsilon})$, but the illiquidity costs incurred by trading out of position H_T at time T are in expectation

$$\varepsilon S_T H_T^2 / 2\tau$$

if we are allowed time τ to complete the unwind. Thus the unwind costs are $O(\varepsilon)$, much smaller than the illiquidity costs incurred by hedging, justifying our decision to neglect them. We could in any case just add this term into the objective, and the mathematics is not altered in any substantive way.

Suppose we hold H_0 units of the asset at time 0 and x_0 in cash. By following the (differentiable adapted) hedging portfolio process $(H_t)_{0 \le t \le T}$, the value at time T of the hedge so constructed will be

$$H_0 S_0 + x_0 + \int_0^T H_t \, dS_t \equiv \xi,$$

and we will have incurred illiquidity costs

$$\int_0^T S_t l(h_t)\, dt$$

on the way. We therefore propose to minimize the objective

(3.7)

$$\begin{aligned}\Phi_0 &= \frac{1}{2}E(\xi - G(S_T))^2 + E\int_0^T S_t l(h_t)\, dt \\ &= \frac{1}{2}(x_0 + H_0 S_0 - q(0, S_0))^2 + E\int_0^T \frac{1}{2}(H_t - \theta(t, S_t))^2\sigma^2 S_t^2\, dt + E\int_0^T S_t l(h_t)\, dt \\ &\equiv \frac{1}{2}(x_0 + H_0 S_0 - q(0, S_0))^2 + \Phi,\end{aligned}$$

which penalizes both the illiquidity costs incurred, and the mean-squared hedging error. Other criteria could be considered: for example, in Rogers and Singh (2004) the utility-indifference price was used; or we might instead try to minimize $E(\xi - \int_0^T S_t l(h_t)\, dt - G(S_T))^2$, the squared L^2-norm of the amount by which we miss the desired terminal wealth. However, the criterion we use here has the twin virtues of simplicity and reasonable tractability and balances low illiquidity costs against poor replication. Of course, a different linear combination of the two cost terms would be treated in exactly the same way.

TECHNICAL REMARKS.

(i) We require a polynomial bound on the Black–Scholes portfolio θ; we shall assume that

$$\text{(3.8)} \quad \text{for some } \gamma > 0, C > 0: \quad |\theta(t, S)| \le C(1 + S^\gamma) \quad \forall t \in [0, T], S > 0.$$

Notice that this implies that

$$E\int_0^T \theta(t, S_t)^2 S_t^2\, dt < \infty. \tag{3.9}$$

(ii) We likewise have to restrict slightly the class of possible controls, and we shall suppose always that $h \in \mathcal{H}$, where

$$\mathcal{H} \equiv \left\{h : E\int_0^T H_u^2 S_u^2\, du < \infty\right\}. \tag{3.10}$$

This simply requires that our attempt at replicating an L^2 contingent claim should also be restricted to lie in L^2, a completely natural condition.

Because the values of initial cash x_0 and holding of asset H_0 will usually be given, our goal is to minimize the objective Φ. For this, we define the value function

(3.11)

$$V(t, H, S) \equiv \inf_{h\in\mathcal{H}} E\left[\int_t^T \frac{1}{2}(H_u - \theta(u, S_u))^2\sigma^2 S_u^2\, du + \int_t^T S_u l(h_u) du \,\middle|\, H_t = H, S_t = S\right],$$

and a familiar argument shows that V solves the HJB equation

$$\inf_{h\in\mathcal{H}}\left[\ V_t + hV_H + \frac{1}{2}\sigma^2 S^2 V_{SS} + \frac{1}{2}\sigma^2 S^2(H-\theta(t,S))^2 + Sl(h)\ \right] = 0. \tag{3.12}$$

Typically, it is impossible to get very far with an HJB equation; we have to content ourselves with results on existence and uniqueness or estimates on the solution or numerical solution. However, in this case, because of the simple assumed form of the objective, we can get a long way if we suppose that

$$l(h) = \frac{1}{2}\,\varepsilon h^2, \tag{3.13}$$

where ε will be thought of as a small parameter.[7] The minimization in (3.12) can be carried out explicitly, by taking

$$h = -\frac{V_H}{\varepsilon S}, \tag{3.14}$$

giving the nonlinear second-order PDE

$$\mathcal{L}V + \frac{1}{2}\sigma^2 S^2(H-\theta)^2 - \frac{V_H^2}{2\varepsilon S} = 0, \tag{3.15}$$

where we shall omit the arguments of $\theta(t, S)$ unless there is need to state them explicitly. Now the assumed quadratic form of the objective and illiquidity loss function leads us to guess that (3.15) may be solved by a *quadratic* function

$$V(t, H, S) = a(t, S)H^2 + b(t, S)H + c(t, S), \tag{3.16}$$

and this guess will turn out to be correct. Substituting this form into (3.15) gives a quadratic in H, and equating the coefficients to zero gives us three equations for a, b, and c

$$\mathcal{L}a + \frac{1}{2}\sigma^2 S^2 - \frac{2a^2}{\varepsilon S} = 0, \tag{3.17}$$

$$\mathcal{L}b - \sigma^2 S^2\theta - \frac{2ab}{\varepsilon S} = 0, \tag{3.18}$$

$$\mathcal{L}c + \frac{1}{2}\sigma^2 S^2\theta^2 - \frac{b^2}{2\varepsilon S} = 0. \tag{3.19}$$

The first equation (3.17) is a nonlinear PDE, but given the solution to that, it is then straightforward to solve (3.18), then (3.19). We therefore focus on (3.17) and show that it has a unique nonnegative solution satisfying suitable boundedness criteria.

[7]Although convex and nonnegative, this choice of l does not have derivative bounded below by –1. If ε is very small, whatever C^2 function l we may choose will look locally like this choice, so we expect that the conclusions we obtain here will apply widely.

LEMMA 3.1. *There is a unique nonnegative solution a to (3.17) which satisfies the boundedness condition*

$$\sup_{S>0,\,0\le t\le T}\left|\frac{a(t,S)}{S^2}\right|<\infty \tag{3.20}$$

and the terminal conditions $a(T,\cdot)=b(T,\cdot)=c(T,\cdot)=0$.

REMARK. The point of the bound (3.20) is that any solution of (3.17) which satisfies this bound will be given by a Feynman–Kac representation

$$a(t,S)=\frac{1}{2}E\left[\int_t^T \exp\left\{-\int_t^u 2a(v,S_v)\,\frac{dv}{\varepsilon S_v}\right\}\sigma^2S_u^2\,du\,\middle|\,S_t=S\right]. \tag{3.21}$$

Proof. We shall construct an approximating sequence of functions which converge to a solution a, and then we shall establish uniqueness. Given a nonnegative function $\alpha:[0,T]\times\mathbb{R}\to\mathbb{R}^+$, we define the function $\Psi(\alpha):[0,T]\times\mathbb{R}\to\mathbb{R}^+$ by

$$\Psi(\alpha)(t,S)=\frac{1}{2}E\left[\int_t^T \exp\left\{-\int_t^u 2\alpha(v,S_v)\,\frac{dv}{\varepsilon S_v}\right\}\sigma^2S_u^2\,du\,\middle|\,S_t=S\right]. \tag{3.22}$$

Then it is clear from (3.21) that a solution to (3.17) satisfying the boundedness condition (3.20) is a function a such that $\Psi(a)=a$. Notice that Ψ is decreasing: $a\ge\tilde a\Rightarrow\Psi(a)\le\Psi(\tilde a)$.

We define a sequence of approximations $a^{(n)}$ to a solution by the recipe

$$a^{(0)}\equiv 0,\qquad a^{(n+1)}=\Psi(a^{(n)})\quad(n\ge 0).$$

Notice immediately that $\sup|a^{(1)}(t,S)/S^2|$ is bounded and that

$$0=a^{(0)}\le a^{(2)}\le a^{(1)}.$$

Hence

$$a^{(1)}\ge a^{(3)}\ge a^{(2)},$$
$$a^{(2)}\le a^{(4)}\le a^{(3)},$$

and so on: $a^{(2n)}\le a^{(2n+2)}\le a^{(2n+1)}$, and $a^{(2n+1)}\ge a^{(2n+3)}\ge a^{(2n+2)}$ for all $n\ge 0$. Thus the sequence $a^{(2n)}$ increases to a limit $\underline{a}$, and the sequence $a^{(2n+1)}$ decreases to a limit $\bar a$, and we see that

$$\underline{a}\ \le\ \bar a.$$

In the limit, we have (from (3.22), by the Monotone Convergence Theorem) that $\Psi(\underline{a})=\bar a$, and $\Psi(\bar a)=\underline{a}$. Thus $\underline{a},\bar a$ satisfy

$$\mathcal{L}\bar a+\frac{1}{2}\sigma^2S^2-\frac{2\underline{a}\bar a}{\varepsilon S}=0,$$
$$\mathcal{L}\underline{a}+\frac{1}{2}\sigma^2S^2-\frac{2\bar a\underline{a}}{\varepsilon S}=0.$$

Hence the difference $f = \bar{a} - \underline{a} \geq 0$ satisfies the PDE $\mathcal{L}f = 0$; using the bound (4.8), which applies to $\bar{a}$, we deduce that $f \equiv 0$, and so $a = \bar{a} = \underline{a}$ is a solution.

As for uniqueness, if $\tilde{a}$ is any other nonnegative solution to $\Psi(\tilde{a}) = \tilde{a}$, we have

$$\tilde{a} = \Psi(\tilde{a}) \leq \Psi(0) = \Psi(a^{(0)}) = a^{(1)},$$

from which we learn that $\tilde{a} = \Psi(\tilde{a}) \geq \Psi(a^{(1)}) = a^{(2)}$. Continuing, we find that $\tilde{a} \geq a^{(2n)}$ and $\tilde{a} \leq a^{(2n+1)}$ for any $n \geq 0$, so that $\tilde{a} = \bar{a} = \underline{a}$. □

We summarize the preceding results as follows.

THEOREM 3.1. *Assuming (3.8), the value function V for the problem* $\min \Phi$ *is of the form (3.16), where a, b, and c are the unique solutions to (3.17), (3.18), (3.19) satisfying bounds of the form (3.8).*

Proof. Notice that

$$\begin{aligned} V(t, H_t, S_t) &\leq \sigma^2 E\left[\int_t^T \left(H_u^2 + \theta(u, S_u)^2\right) S_u^2 \, du \mid \mathcal{F}_t\right] \\ &\leq \sigma^2 E\left[\int_0^T \left(H_u^2 + \theta(u, S_u)^2\right) S_u^2 \, du \middle| \mathcal{F}_t\right] \end{aligned}$$

and so for any H for which $E\int_0^T H_u^2 S_u^2 \, du < \infty$, the process $V(t, H_t, S_t)$ is dominated by a uniformly integrable martingale.

In view of the bound (3.8) on θ, the Feynman–Kac representation of the solution to the PDE (3.18) is well defined (the integrals converge), and the solution b again satisfies a bound of the form (3.8). The same conclusion holds for c. Thus solving the PDEs (3.17), (3.18), (3.19), gives us a function V defined by (3.16) which solves the PDE (3.15). From this, whatever control h is used, the process

$$Y_t \equiv \int_0^t \frac{1}{2}(H_u - \theta(u, S_u))^2 \sigma^2 S_u^2 du + \int_0^t S_u l(h_u) \, du + V(t, H_t, S_t)$$

is expressed (by Itô's formula) as a local martingale plus a nondecreasing process. Using a stopping time $\tau \leq T$ which reduces the local martingale, we learn that

$$V(0, H_0, S_0) \leq E\left[\int_0^\tau \frac{1}{2}(H_u - \theta(u, S_u))^2 \sigma^2 S_u^2 \, du + \int_0^\tau S_u l(h_u) \, du + V(\tau, H_\tau, S_\tau)\right].$$

Now we let $\tau \uparrow T$; the integrals converge monotonically, and the final term $V(\tau, H_\tau, S_\tau)$ tends to 0 almost surely, and in view of the uniformly integrable bound, also in L^1. □

REMARKS.

(i) Notice that the PDE for a does not depend on the derivative to be hedged; the only effect of this is on b and c. This is perhaps not so surprising, because a controls the loss if H is very large; if H is very large, the most important thing is to get H back to somewhere near the Black–Scholes hedge, and it does not matter very much whether the Black–Scholes hedge is 4 or –65.

(ii) Our analysis tells us little about the magnitude of the value function, but we shall see in the next section that (roughly speaking) the magnitude of V is $O(\sqrt{\varepsilon})$.

4. ESTIMATING THE OPTIMAL SOLUTION

The route we take here is to suppose that the illiquidity losses are small and find an approximation to the optimal hedging policy. This will then be used to bound the optimal solution. We still suppose that $l(h) = \frac{1}{2}\varepsilon h^2$.

For small ε, if we may assume that the term $\mathcal{L}V$ in the HJB equation (3.15) will be small, then the only way that the whole expression can be zero is if

$$\frac{V_H}{\sqrt{\varepsilon S}} \doteqdot \sigma S(H-\theta).$$

In view of (3.14) this leads us to consider the following candidate for a good control:

$$\bar{h} \equiv -\sigma\sqrt{\frac{S}{\varepsilon}}\,(H-\theta). \tag{4.1}$$

REMARKS.

(i) There is at this point no need to justify the assumption that we can neglect $\mathcal{L}V$; this was only used to suggest the form (4.1) for the control. This control is suboptimal, and so using it will provide an upper bound for the value of the original problem. We find that the resulting bounds are effective.

(ii) Observe that this control $\bar{h}$, a choice for the rate of change of the holding H of the asset, has natural properties: it will pull H toward the Black–Scholes hedge θ, and will pull more strongly as ε gets smaller. For large S, it is more important to get the hedging number of units of asset correct, because this corresponds to a larger sum invested in the asset. Thus we find that $\bar{h}$ pulls H harder to θ for large S, but not proportional to S.

(iii) We are going to estimate the value of using $h = \bar{h}$ and show that this is small in a sense to be made precise in Theorem 4.1. This analysis gives an estimate of the liquidity costs (and the mishedging costs) incurred, so it gives a bound for the "liquidity premium" to be charged for selling this option written on a slightly illiquid asset.

Let v denote the value of using the policy $\bar{h}$: this solves the PDE

$$\mathcal{L}v - \frac{\sigma S(H-\theta)}{\sqrt{\varepsilon S}}\, v_H + \sigma^2 S^2 (H-\theta)^2 = 0, \quad v(T,\cdot,\cdot) = 0. \tag{4.2}$$

There is the probabilistic (Feynman–Kac) representation of the solution as

$$v(t, H, S) = E\left[\int_t^T \sigma^2 S_u^2 (H_u - \theta(u, S_u))^2 du \,\middle|\, H_t = H, S_t = S\right], \tag{4.3}$$

where the expectation (4.3) is calculated under the assumed dynamics (3.2), (3.4), taking $h = \bar{h}$. The Ornstein–Uhlenbeck dynamics of H under control $\bar{h}$ imply a solution of the form

$$v(t, H, S) = a(t, S)(H - \theta(t, S))^2 + b(t, S)(H - \theta(t, S)) + c(t, S) \tag{4.4}$$

for functions a, b, and c to be determined. Some routine calculations lead to the following equations for the three unknowns:

$$0 = \mathcal{L}a - 2\sigma\sqrt{\frac{S}{\varepsilon}}\,a + \sigma^2 S^2, \tag{4.5}$$

$$0 = \mathcal{L}b - \sigma\sqrt{\frac{S}{\varepsilon}}b + 2\sigma^2 S(a - Sa_S)\theta_S, \tag{4.6}$$

$$0 = \mathcal{L}c + \sigma^2 S\theta_S(b - Sb_S) + \sigma^2 S^2 a\theta_S^2. \tag{4.7}$$

THEOREM 4.1. *Assume that θ_S and θ_{SS} are uniformly bounded. Then the functions a, b and c determining v through (4.4) satisfy the bounds*

$$a(t, S) \le k\varepsilon^2(z + z^{3/2}); \tag{4.8}$$

$$|b(t, S)| \le k\varepsilon^3(z + z^2); \tag{4.9}$$

$$c(t, S) \le k\varepsilon^4 z^{5/2}(1 + z + \varepsilon(z^{3/2} + z^2)) \tag{4.10}$$

for some positive constant k, where $z \equiv S/\varepsilon$.

REMARKS.

(i) Notice a consequence of the bounds of Theorem 4.1: for fixed $S > 0$ and fixed H,

$$V(t, H, S) \le v(t, H, S) \le \sqrt{\varepsilon}\kappa(t, S),$$

so we have roughly speaking that $V(t, H, S) = O(\sqrt{\varepsilon})$.

(ii) The assumption of uniform bounds on θ_S and θ_{SS} is quite likely far too strong, but it is sufficient for the required result.

The proof of Theorem 4.1 involves various estimations which are relegated to the Appendix.

5. NUMERICAL SOLUTION

In this section, we discuss the numerical solution to the problem and present some results. Our test problem is the hedging of a standard European call with strike 1; not all the boundedness hypotheses which we imposed to prove our results hold for this example, but this does not stop us computing the solution. The numerical results we obtain are consistent with the theory, suggesting that the hypotheses may be relaxed somewhat.

We first solve the original problem (3.15) using the form (3.16) of the solution. This requires us to solve the three coupled PDEs (3.17), (3.18), and (3.19). For this, we worked with the variable $y = \log S$, and solved a finite-difference scheme by Crank–Nicolson.

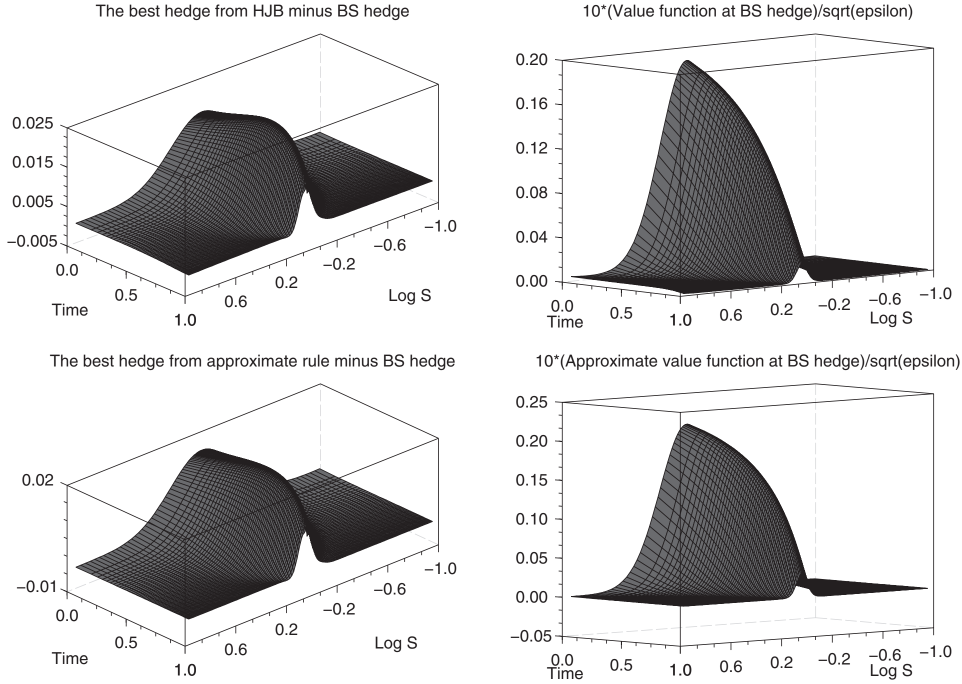

FIGURE 5.1. Numerical results, $\sigma = 0.25$, $\varepsilon = 0.006\%$.

The computations were quick and accurate, and we would recommend the use of this algorithm in practice.

For comparison,[8] the approximate solution from Section 4 was calculated, using the good policy (4.1), and the value function for that policy is found. While the analysis of Section 4 used the form

$$v(t, H, S) = a(t, S)(H - \theta(t, S))^2 + b(t, S)(H - \theta(t, S)) + c(t, S)$$

for the solution, which was necessary to establish bounds on the solution, what we did numerically was to take the solution in the equivalent form

$$v(t, H, S) = a(t, S)H^2 + b(t, S)H + c(t, S)$$

and solve for that instead. The equations we obtain are slightly different, but this time all are linear and so may be solved easily

$$0 = \mathcal{L}a - 2\sigma\sqrt{\frac{S}{\varepsilon}}\,a + \sigma^2 S^2, \tag{5.1}$$

$$0 = \mathcal{L}b - \sigma\sqrt{\frac{S}{\varepsilon}}(b - 2a\theta) - 2\sigma^2 s^2\theta, \tag{5.2}$$

[8].. and only for comparison; the computational effort involved in calculating the approximate solution is much the same as the computational effort required to calculate the true solution, so in practice we advocate the calculation of the true solution.

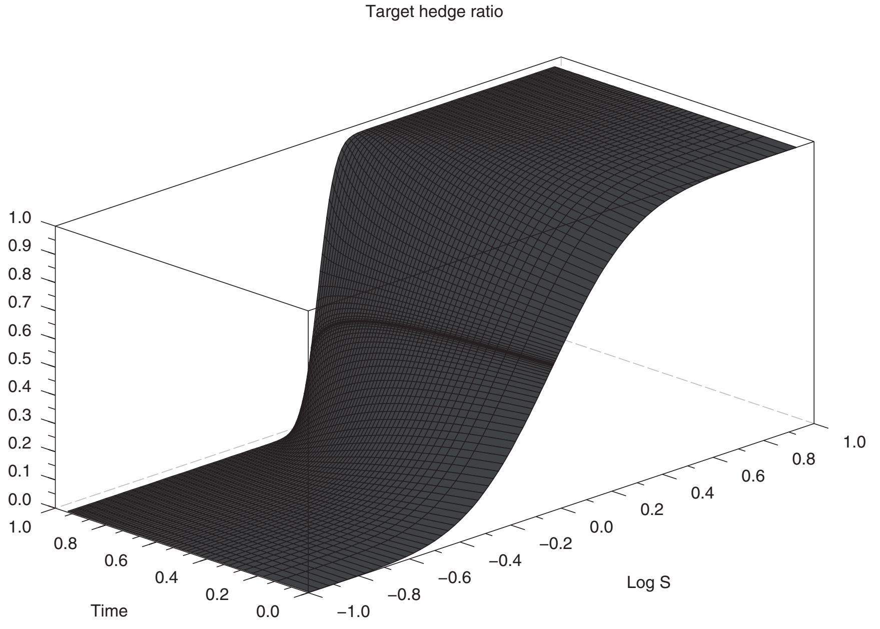

FIGURE 5.2. Target hedge ratio, $\sigma = 0.25$, $\varepsilon = 0.006\%$.

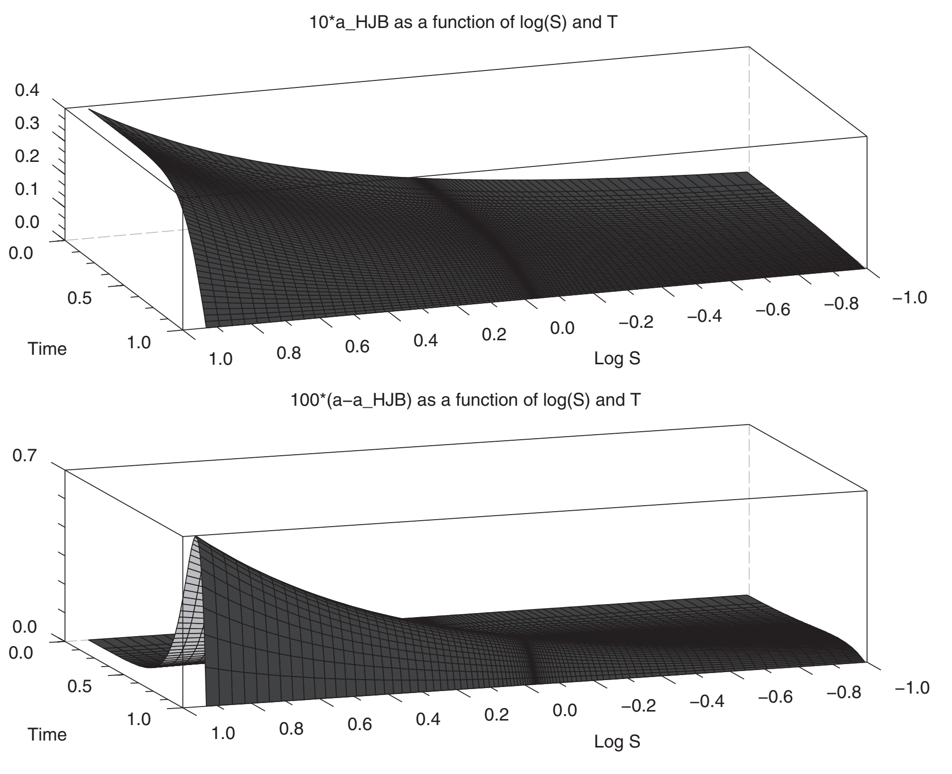

FIGURE 5.3. Plot of a, $\sigma = 0.25$, $\varepsilon = 0.006\%$.

$$0 = \mathcal{L}c + \sigma\sqrt{\frac{S}{\varepsilon}}\theta b + \sigma^2 S^2 \theta^2. \tag{5.3}$$

The next question is what values we should take for the parameters of the problem. We suppose that $\sigma = 0.25$, but for ε the value we take should reflect realistic levels of liquidity. We asked a quant with experience of trading equities by how much he would expect the buying price to rise if one attempted to purchase 1% of the stock of a major company in one day, and his reply was "A 'dawn raid' of 10% or so of the shares will probably propel the market 15% higher. I would say 1.5% for a quiet 1% purchase."[9] From this, we deduce the value $\varepsilon = 0.006$.

Figure 5.1 shows the optimal solution, and the approximate optimal solution, displaying for each the difference between the best hedge (defined to be the value of H which minimizes the value function) and the Black–Scholes hedge and also displaying the value function taking $H = \theta(t, S)$, scaled by $10/\sqrt{\varepsilon}$ to keep things $O(1)$. We see that qualitatively there is little difference between the optimal solution and the approximate optimal solution from Section 4. Notice that deep in or out of the money, the value function evaluated at the Black–Scholes hedge is very small, as one would expect, because at these levels the hedge is correctly set, at either 0 or 1 units of the stock, and there is little chance that it will need to be changed by very much. Near the money, we expect more changes to be required, and we see that the minimized cost of liquidity is higher. The illiquid hedges are seen to be greater than the Black–Scholes hedge, probably because of convexity of the payoff.

Figure 5.2 displays the target hedge ratio which the hedger is trying to steer toward. As would be expected from the results displayed in Figure 5.1, this looks very much like the Black–Scholes hedge ratio.

Figure 5.3 gives the other part of the hedging solution, namely the strength $a(t, S)$ of adjusting back to the target level, again as a function of (t, S). The upper panel shows how the strength of adjustment increases with the underlying price and decreases with time-to-go, which are entirely plausible properties. The lower panel shows how the true value of a differs from the value of a calculated in the estimation of Section 4. Notice that in places this difference is quite substantial relative to the true value of a, which may go some way to explaining why the values plotted in Figure 5.1, though comparable, are not very close.

6. CONCLUSIONS

This paper has presented a model for the effects of illiquidity and explored some of its consequences for the hedging of European-style options. We emphasized the distinction between price-impact effects and illiquidity effects, an important distinction that is sometimes blurred in the literature. Focussing on illiquidity allows us to suppose that hedging decisions do *not* affect price and therefore allow us to analyze the effect of illiquidity on a small agent who is required to hedge an option. We pose an optimization problem for this agent, which can be solved in almost closed form, requiring the numerical solution of three parabolic PDEs, one of them nonlinear. A near-optimal solution provides

[9]Interestingly, the scaling expressed in this reply matches the conclusions of our model. When asked to expand on the time scales involved, the trader remarked that a dawn raid might happen over quite a short time scale, perhaps half an hour.

a good approximation, and useful bounds on the cost of illiquidity, which we deduce is $O(\sqrt{\varepsilon})$.

APPENDIX

Proof of Theorem 4.1. Throughout the proof, k will denote a positive constant whose (relatively unimportant) value changes from place to place.

The first step is to change to the variable $z \equiv S/\varepsilon$ in equations (4.5), (4.6), and (4.7), transforming them to

$$0 = \mathcal{L}\alpha - 2\sigma\sqrt{z}\,\alpha + \sigma^2\varepsilon^2 z^2, \tag{A.1}$$

$$0 = \mathcal{L}\beta - \sigma\sqrt{z}\beta + 2\sigma^2\varepsilon z(\alpha - z\alpha_z)\theta_S, \tag{A.2}$$

$$0 = \mathcal{L}\gamma + \sigma^2\varepsilon z\theta_S(\beta - z\beta_z) + \sigma^2\varepsilon^2 z^2\alpha\theta_S^2. \tag{A.3}$$

Here, we write $\alpha(t, z) = a(t, S)$, $\beta(t, z) = b(t, S)$, $\gamma(t, z) = c(t, S)$, and by slight abuse of notation we set

$$\mathcal{L} \equiv \frac{1}{2}\sigma^2 z^2 \frac{\partial^2}{\partial z^2} + \frac{\partial}{\partial t}.$$

Remarkably, there is an *explicit* solution to (A.1):

$$\alpha^{(0)}(t, z) = \alpha^{(0)}(z) = \varepsilon^2\left(\frac{3\sigma^2}{32}\,z + \frac{\sigma}{2}z^{3/2}\right), \tag{A.4}$$

though this does not satisfy the boundary condition $\alpha(T, \cdot) = 0$. The solution to (4.5) which *does* satisfy the boundary condition is expressed as

$$\alpha(t, S) = \alpha^{(0)}(S) - \bar{\alpha}(t, S),$$

where $\bar{\alpha}$ solves

$$\mathcal{L}\bar{\alpha} - 2\sigma\sqrt{z}\,\bar{\alpha} = 0, \quad \bar{\alpha}(T, z) = \alpha^{(0)}(z).$$

The function $\bar{\alpha}$ therefore has the Feynman–Kac representation as

$$\bar{\alpha}(t, z) = E\big[\exp(-A_{t,T})\, a^{(0)}(z_T) \mid H_t = H, z_t = z\big]. \tag{A.5}$$

where we define

$$A_t \equiv \int_0^t 2\sigma\sqrt{z_u}\, du, \quad A_{t,s} \equiv A_s - A_t. \tag{A.6}$$

Elementary estimation of (A.5) using (A.4) gives us the uniform bound (4.8).

We therefore know that a is *small*, in the precise sense given by (4.8). The next step is to show that β is also comparably small, and for this we again use the Feynman–Kac representation of the solution to (4.6)

$$\beta(t, S) = E\left[\int_t^T \exp\left(-\frac{1}{2}A_{t,u}\right)\, q(u, z_u)\, du \mid H_t = H, z_t = z\right], \tag{A.7}$$

where

$$q(u, z) \equiv 2\sigma^2 \varepsilon z(\alpha - z\alpha_z)(u, z)\,\theta_S(u, S). \tag{A.8}$$

From (A.5) we deduce that

$$z\bar{\alpha}_z = E_t\left[e^{-A_{t,T}}\left\{z_T\,\alpha_z^{(0)}(z_T) - \frac{1}{2}A_{t,T}\,\alpha^{(0)}(z_T)\right\}\right],$$

and hence using the boundedness of θ we have a bound

$$|q(u, z)| \le k\varepsilon^3 z(z + z^{3/2})$$

for some constant $k > 0$. We may therefore majorise β by the solution to the equation

$$\mathcal{L}g - \sigma\sqrt{z}g + k\varepsilon^3 z(z + z^{3/2}) = 0, \tag{A.9}$$

together with the boundary condition $g(T, \cdot) = 0$. Now one solution to (A.9) can be found explicitly:

$$g^{(0)}(t, z) = k\varepsilon^3\sigma^{-1}\left(\frac{3\sigma}{8}(1+\sigma)z + (1+\sigma)z^{3/2} + z^2\right).$$

The solution to (A.9) with the required boundary condition is related to $g^{(0)}$ as α is related to $\alpha^{(0)}$; we deduce similarly the bounds

$$|\beta(t, z)| \le g(t, z) \le k\varepsilon^3(z + z^2).$$

Our last task is to show that γ is small in some suitable sense, and in view of Feynman–Kac representation of γ as

$$\gamma(t, z) = E\left[\int_t^T \exp\left(-\frac{1}{2}A_{t,u}\right)\,\psi(u, z_u)\,du \mid H_t = H, z_t = z\right], \tag{A.10}$$

where

$$\psi(u, z) = \sigma^2\varepsilon z\theta_S(\beta - z\beta_z) + \sigma^2\varepsilon^2 z^2\alpha\theta_S^2, \tag{A.11}$$

the task is equivalently to show that ψ is suitably small. We already have bounds on α and β, but we need bounds on zb_z. For this, notice that

$$\mathcal{L}(z\beta_z) = z\frac{\partial}{\partial z}(\mathcal{L}\beta)$$

and so

$$\begin{aligned}
\mathcal{L}(\beta - z\beta_z) &= z^2\frac{\partial}{\partial z}\left(-\frac{1}{z}\mathcal{L}\beta\right)\\
&= z^2\frac{\partial}{\partial S}\left(-\frac{\sigma}{\sqrt{z}}\beta + 2\sigma^2\varepsilon(\alpha - z\alpha_z)\theta_S\right)\\
&= \sigma\sqrt{z}(\beta - z\beta_z) - \frac{\sigma}{2}\sqrt{z}\,\beta + 2\sigma^2\varepsilon^2 z^2(\alpha - z\alpha_z)\,\theta_{SS} - 2\sigma^2\varepsilon z^3\alpha_z\theta_S.
\end{aligned}$$

Rearranging this gives a PDE for $(\beta - z\beta_z)$:

$$\mathcal{L}(\beta - z\beta_z) - \sigma\sqrt{z}(\beta - z\beta_z) = -\frac{\sigma}{2}\sqrt{z}\,\beta + 2\sigma^2\varepsilon^2 z^2(\alpha - z\alpha_z)\,\theta_{SS} - 2\sigma^2\varepsilon z^3\alpha_{zz}\theta_S.$$

We need a bound on α_{zz}, which is derived from the Feynman–Kac representation of α; we deduce that

$$|z^2\alpha_{zz}| \leq \varepsilon^2(z + z^{3/2}),$$

and hence

$$\begin{aligned}|\beta - z\beta_z| &\leq k\{\varepsilon^3 z^{3/2}(1+z) + \varepsilon^4(z^{7/2} + z^3)\}\\ &\leq k\varepsilon^3 z^{3/2}[1 + z + \varepsilon(z^{3/2} + z^2)].\end{aligned}$$

Using again the Feynman–Kac representation of the solution to (A.3), we deduce that

$$\begin{aligned}|\gamma| &\leq k\varepsilon^4 z^{5/2}[1 + z + \varepsilon(z^{3/2} + z^2)] + k\varepsilon^4 z^3(1 + \sqrt{z})\\ &\leq k\varepsilon^4 z^{5/2}(1 + z + \varepsilon(z^{3/2} + z^2))\end{aligned}$$

as required. □

REFERENCES

ALMGREN, R. (2003): Optimal Execution with Non-Linear Impact Functions and Trading-Enhanced Risk, *Appl. Math. Finance* 10, 1–18.

ALMGREN, R., and N. CHRISS (2001): Optimal Execution of Portfolio Transactions, *J. Risk* 3, 5–39.

BAKSTEIN, D., and S. HOWISON (2003): *A Non-Arbitrage Liquidity Model with Observable Parameters for Derivatives*, Oxford, UK: Oxford University Press.

BANK, P., and D. BAUM (2004): Hedging and Portfolio Optimization in Financial Markets with a Large Trader, *Math. Finance* 14, 1–18.

BERTSIMAS, D., and A. W. LO (1998): Optimal Control of Execution costs, *J. Finan. Markets* 1, 1–50.

CETIN, U., R. A. JARROW, and P. PROTTER (2004): Liquidity Risk and Arbitrage Pricing Theory, *Finance Stoch.* 8, 311–341.

CETIN, U., and L. C. G. ROGERS (2007): Modelling Liquidity Effects in Discrete Time, *Math. Finance* 17, 15–29.

DAVIS, M. H. A., and A. R. NORMAN (1990): Portfolio Selection with Transaction Costs, *Math. Operat. Res.* 15, 676–713.

FREY, R. (1988): Perfect Option Hedging for a Large Trader, *Finance Stoch*. 2, 115–141.

FREY, R., and A. STREMME (1997): Market Volatility and Feedback Effects from Dynamic Hedging, *Math. Finance* 7, 351–374.

HERITAGE, J. P., and L. C. G. ROGERS (2002): Large Investors, Takeovers, and the Rule of Law, *Monte Carlo Methods Applicat* 8, 357–370.

HUBERMAN, G., and W. STANZL (2005): Optimal Liquidity Trading, *Rev. Finance* 9, 165–200.

HULL, J. C. (1993): *Options, Futures and Other Derivative Securities*, 2nd ed. Englewood Cliffs, NJ: Prentice-Hall.

ISAENKO, S. (2005): Portfolio Choice Under Convex Transaction Costs. Technical report, Concordia University.

JARROW, R. A. (1992): Market Manipulation, Bubbles, Corners, and Short Squeezes, *J. Finan. Quant. Anal.* 27, 311–336.

LONGSTAFF, F. A. (2001): Optimal Portfolio Choice and the Valuation of Illiquid Securities, *Rev. Finan. Stud.* 14, 407–431.

MAGILL, M. J. P., and G. M. CONSTANTINIDES (1976): Portfolio Selection with Transaction Costs, *J. Econ. Theory* 13, 245–263.

OBIZHAEVA, A., and J. WANG (2005): Optimal Trading Strategy and Supply/Demand Dynamics. Technical report, MIT.

PAPANICOLAOU, G., and K. R. SIRCAR (1998): General Black–Scholes Models Accounting for Increased Market Volatility from Hedging Strategies, *Appl. Math. Finance* 5, 45–82.

PLATEN, E., and M. SCHWEIZER (1998): On Feedback Effects from Hedging Derivatives, *Math. Finance* 8, 67–84.

ROGERS, L. C. G., and S. SINGH (2004): Modelling Liquidity and its Effects on Price. Technical report, Cambridge University.

SCHIED, A., and T. SCHÖNEBORN (2007): Optimal Portfolio Liquidation for Cara Investors. Technical report, Cornell University.

SCHÖNBUCHER, P. J., and P. WILMOTT (2000): The Feedback Effect of Hedging in Illiquid Markets, *SIAM J. Appl. Math.* 61, 232–272.

SINGH, S. (2005): *Modelling Liquidity and the Valuation of American Options using the Dual Method*. PhD. thesis, University of Bath.