A No-Arbitrage Model Of Liquidity In Financial Markets Involving Stochastic Strings: Applications To High-Frequency Data

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Outline

Market vs Limit Orders

A (buy) market order specifies

- how many shares a trader wants to buy,
- that he is willing to buy them at any price.

A (buy) limit order specifies

- how many shares a trader wants to buy,
- at what maximum price he is willing to buy them.

Limit order matching mechanism

Figure: Limit order matching mechanism.

Limit order matching mechanism

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Limit order matching mechanism

Figure: Limit order matching mechanism.

Demand v.s. Supply

The order books contain all the information about demand and supply.

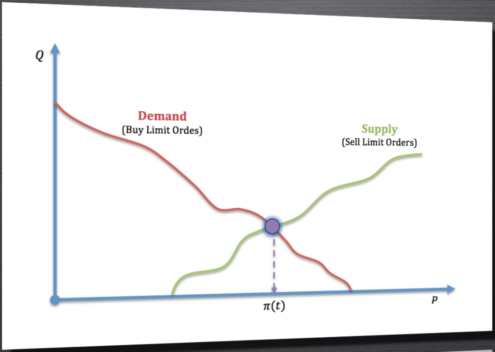

Figure: Demand v.s. Supply.

The dynamics of Limit Orders in 3D

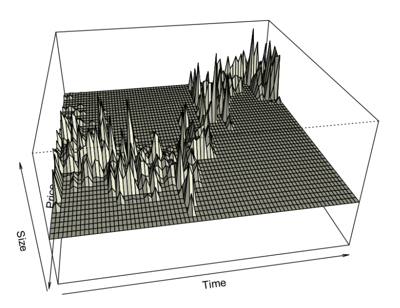

Figure: Buy Limit Orders of ORCL on April 4, 2011.

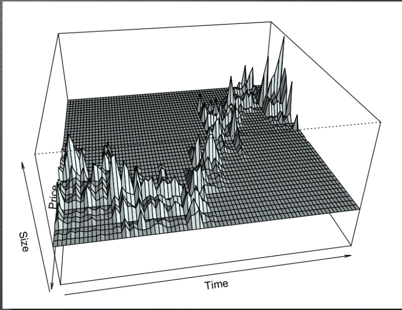

Figure: Sell Limit Orders of ORCL on April 4, 2011.

The dynamics of the Clearing Price process

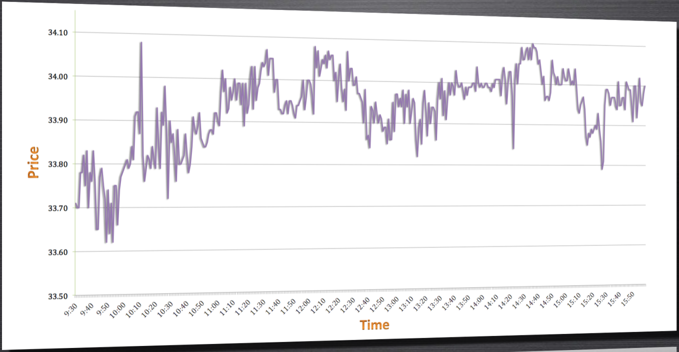

Figure: The dynamics of Oracle Corporation's Clearing Prices on April 1, 2011.

Literature Review: Liquidity Models

Market Manipulation (feedback) Models

- Jarrow (1994),
- Platen and Schweizer (1998),
- Sircar and Papanicolaou (1998),
- Frey (1998),
- Schonbucher and Wilmott (2000),
- Bank and Baum (2004).

Price-taking (competitive) Models

- Cetin, Jarrow, and Protter (2004),
- Cetin and Rogers (2006),
- Cetin, Soner, and Touzi (2009),
- Kallsen and Rheinlaender (2009),
- Gokay and Soner (2011).

Net Demand Curve and Clearing Price

Definition

The net demand curve Q is a function $[0, P] \times \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}$, which value $Q(p, t, \omega)$ is equal to the difference between the quantity of shares **available** for purchase and the quantity of shares **available** for sale at price p at time t. For each p the stochastic process $Q(., t,)$ is a $\mathcal{F}_t$ adapted semimartingale.

Remark: For the clearing price to be a diffusion, the demand must be defined on a continuum of limit prices.

Remark: The net demand curve should be decreasing in p. The easiest way to do that is to model positive processes:

- $Q(0, t)$: total number of buy orders
- $q(p, t)$: density of buy orders + density of sell orders

$$Q(p, t) = Q(0, t) - \int_0^p q(y, t)dy$$

Definition

The clearing price $\pi(t)$ is a $\mathcal{F}_t-$ adapted stochastic process which satisfies market clearing:

$$Q(\pi(t), t) = 0$$

The Model

- It can be proved that the optimal strategy of a large trader is to disseminate her orders into infinitesimal orders.

  This shows that a continuous demand curve is a plausible model.

- In order to avoid arbitrage, we choose to have as many factors as limit prices

$$dQ(p,t) = \mu_Q(p,t)dt - \sigma_Q(p,t)\int_{x=0}^{S} b_q(p,x,t)W(ds,dt) \quad \text{for } 0 < p \leq S$$

Main Result: Market with a Large Trader

Main Result

Suppose in addition to our standing assumptions that

C1) The demand curve is decreasing in price and continuous in time;

C2) the volatility $\sigma_Q(p, t)$ is bounded away from zero, uniformly in p;

C3) there is no path such that $Q(S, t) \geq 0$ or $Q(0, t) \leq 0$;

C4) The market price of risk equations hold.

Then

F1) there is no arbitrage strategy,

F2) the net demand curve Q is continuous in t,

F3) the clearing price $\pi(t)$ is continuous,

F4) The $\mathbb{Q}$-measure is also a martingale measure for $\pi(t)$.

Characterization of the Risk-Neutral Measure $\mathbb{Q}$

Change of Measure

$$W^{\mathbb{Q}}(ds, dt) = W(ds, dt) + \lambda(s, t)dt$$

Goal: determine λ such that the market price of risk equations hold.

Market Price of Risk Equations

Let $\{\mu_Q, \sigma_Q, b_Q\}$ be the parameters of net demand. Let the volatility of the clearing price be:

$$\sigma_\pi(t)b_\pi(s,t) = -\frac{\sigma_Q(\pi(t),t)\int_{s=0}^{S} b_Q(\pi(t),s,t)ds}{q(\pi(t),t)}$$

Define

$$\begin{aligned}
C(\pi,t) &= \sigma_\pi(t)\int_{s=0}^{S}\frac{\partial[\sigma_Q(p,t,\omega)b_Q(p,s,t)]}{\partial p}|_{p=\pi}b_\pi(s,t)ds \\
B(\pi,t) &= \mu_Q(\pi,t) - \frac{1}{2}\frac{\partial^2 Q(p,t)}{\partial p^2}|_{p=\pi} + C(\pi,t) \\
\Sigma(\pi,s,t) &= \sigma_Q(\pi,t)b_Q(\pi,s,t)
\end{aligned}$$

The market price of risk equations are:

$$\int_{s=0}^{S}\Sigma(\pi,s)\lambda(s)ds = B(\pi) \qquad 0 \le \pi \le P$$

From a "MetaModel" to a Model

Reminder:

$$q(p,t)dp \quad = \quad \text{sum of buy and sell order quantities with limit price}$$
$$\text{in } [p, p+dp] \text{ arriving in } [0,t]$$

Plausible dynamics for q:

- positive process

  not necessarily increasing: orders can be cancelled
- mean-reverting process
- to be implemented on a computer: p and t must take discrete values

Our choice: the exponential of a (vector) Ornstein-Uhlenbeck process.

Model Framework

We define

$$
\begin{aligned}
dh(p,t,\omega) &= [-a_h(p)h(p,t,\omega)+\mu_h(p)]dt + \int_{s=0}^{S}\sigma_h(p,s)W(ds,dt,\omega) \\
q(p,t,\omega) &= \exp\left(\int_{x=0}^{p} h(x,y,\omega)dx\right) && (Density) \\
dQ(p,t,\omega) &= \int_{x=0}^{S} q(x,t,\omega)dx\eta(t,\omega) - \int_{x=0}^{p} q(x,t,\omega)dx && \text{(Net Demand)} \\
d\eta(t,\omega) &= a_\eta(\bar{\eta}-\eta(t,\omega))dt \\
&+ \sqrt{\eta(t,\omega)(1-\eta(t,\omega))}\int_{s=0}^{S}\sigma_\eta(s)W(ds,dt,\omega) && \text{(Demand Volume Ratio)}
\end{aligned}
$$

Reasoning of Modeling Framework

- The reason of modeling demand but not demand/supply separately

  Smooth semi-martingale assumption holds for net demand curve

- The reason of modeling the ratio of demand and volume η process

  To guarantee the robustness of the estimation: $Q(0) > 0$ and $Q(S) < 0$

- The reason for Q to be twice differentiable

  Need to apply Ito-Wentzell formula.

- The reason for q to be positive

  Guarantee that the net demand is downward sloping (decreases with price).

- The reason for using Stochastic strings

  Rule out arbitrage opportunity

NYSE Arcabook Data

Industry	Exchange	Ticker	Firm
Energy	NYSE	CVX	Chevron Corporation
	NYSE	XOM	Exxon Mobil Corporation
Financial Banks	NYSE	JPM	JPMorgan Chase & Co.
	NYSE	WFC	Wells Fargo & Company
Materials and Mining	NYSE	ABX	Barrick Gold Corporation
	NYSE	FCX	Freeport-McMoRan Copper & Gold Inc.
Technology	NASDAQ	CSCO	Cisco Systems, Inc.
	NASDAQ	MSFT	Microsoft Corporation
	NASDAQ	ORCL	Oracle Corporation

Table: NYSE Arcabook data selection.

Simulation Process

Market Calibration

- Estimate excess demand $\bar{Q}(p,t)$ from p and t from the high frequency data;
- Calculate the $\eta(t)$ process by $\frac{\bar{Q}(0,t)}{\bar{Q}(0,t)-\bar{Q}(S,t)}$;
- Calculate process q and h. Estimate the variance-covariance matrix of h and η;
- Apply Cholesky decomposition to the correlation matrix, where $\bar{R}dt = Corr[\frac{dh_i}{h_i}, \frac{dh_j}{h_j}]$

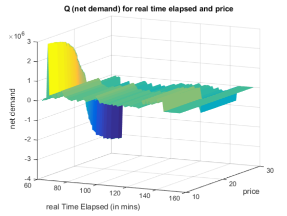

Figure: The net demand of GE as of 04/01/2011

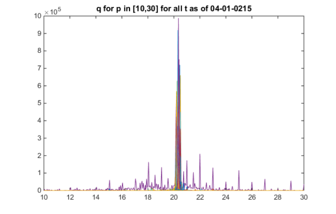

Figure: The q process of GE as of 04/01/2011

Simulation Process

Excess Demand Simulation

(The process of the simulation is to demonstrate that the calibrated parameters in framework are market consistent)

- Randomly generate n paths for h;
- Use h to integrate for q process;
- Simulate η process using the correlation estimated in market calibration process;
- Combine h, q and η to simulate excess demand Q.

Simulation Result

GE stock net demand curve with 04/01/2015 market data calibrated:

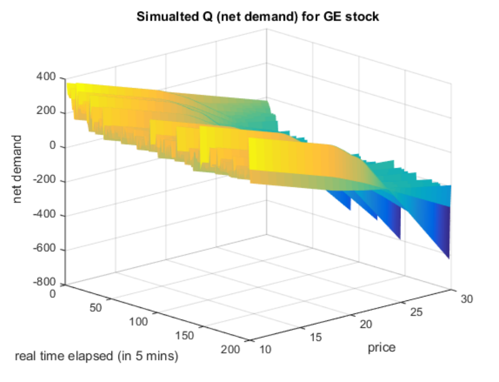

Figure: Simulated net demand Q for GE using 04/01/2015 order book calibration.

Characterization of the Risk-Neutral Measure $\mathbb{Q}$

Change of Measure

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Goal: determine λ such that the market price of risk equations hold.

Market Price of Risk Equations

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\Sigma(\pi,s,t) &= \sigma_Q(\pi,t) b_Q(\pi,s,t)
\end{aligned}$$

The market price of risk equations are:

$$\int_{s=0}^{S} \Sigma(\pi,s)\lambda(s)ds = B(\pi) \qquad 0 \le \pi \le P$$

Volatility Smile (ORCL)

Strike Price	Implied Volatility
33.77	22.66%
33.80	21.05%
33.83	19.40%
33.87	17.71%
33.90	15.94%
33.93	14.10%
33.97	12.16%
34.00	10.11%
34.04	9.35%
34.07	10.26%
34.10	11.03%
34.14	11.70%
34.17	12.25%
34.20	12.71%
34.24	13.06%
34.27	13.25%
34.31	13.34%
34.34	13.32%
34.37	13.23%
34.41	12.99%

Figure: Simulated v.s. real-time volatility smile of ORCL on April 4, 2011. *Source: Bloomberg*

Volatility Smile (ABX)

Strike Price	Implied Volatility
50.98	23.64%
51.03	21.98%
51.08	20.27%
51.13	18.51%
51.18	16.70%
51.23	14.84%
51.29	12.93%
51.34	11.69%
51.39	12.24%
51.44	12.75%
51.49	13.20%
51.54	13.58%
51.59	13.92%
51.64	14.18%
51.69	14.37%
51.74	14.46%
51.79	14.54%
51.85	14.55%
51.90	14.46%
51.95	14.39%

Figure: Simulated v.s. real-time volatility smile of ABX on April 4, 2011. *Source: Bloomberg*

Volatility Smile (CSCO)

Strike Price	Implied Volatility
16.88	13.30%
16.90	12.17%
16.92	11.02%
16.93	9.84%
16.95	8.63%
16.97	7.37%
16.99	6.04%
17.00	4.63%
17.02	4.16%
17.04	4.84%
17.05	5.28%
17.07	5.53%
17.09	5.68%
17.10	5.73%
17.12	5.79%
17.14	5.85%

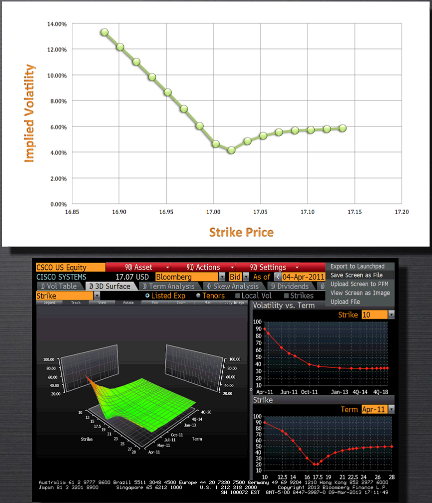

Figure: Simulated v.s. real-time volatility smile of CSCO on April 4, 2011. *Source: Bloomberg*

Conclusions

1. We developed a liquidity model with stands between
   - traditional no-arbitrage (option pricing) models and
   - financial economics models.
2. This model uses Ito-Wentzell's formula and Girsanov's theorem for Brownian sheets.
3. We give conditions for no-arbitrage, which allows us to price options
4. We specified a model
   - with positive demand density,
   - with mean-reversion,
   - where parameters are centered on the clearing price.
5. We calibrate the model parameters from high frequency stock data of GE as of 04-01-2015. The simulated net demand curve shows a downward sloping shape, as expected.
6. The model generates an implied volatility smile which matches the observed smile much better than the traditional Black-Scholes model.