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A Liquidity Based Model for Asset Price Bubbles

Robert A. Jarrow,* Philip Protter† and Alexandre F. Roch‡

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Abstract

We provide a new liquidity based model for financial asset price bubbles that explains bubble formation and bubble bursting. The martingale approach (Cox and Hobson (2005), Jarrow et al. (2007)) to modeling price bubbles assumes that the asset's market price process is exogenous and the fundamental price, the expected future cash flows under a martingale measure, is endogenous. In contrast, we define the asset's fundamental price process exogenously and asset price bubbles are endogenously determined by market trading activity. This enables us to generate a model which explains both bubble formation and bubble bursting. In our model, the quantity impact of trading activity on the fundamental price process - liquidity risk - is what generates price bubbles. We study conditions under which asset price bubbles are consistent with no arbitrage opportunities and we relate our definition of the fundamental price process to the classical definition.

1 Introduction

The prospect that market prices diverge from their fundamental value has long been a topic of interest in economics both theoretically and empirically. Over the years, many different theories for bubble formation have been explored. Some of the main theories include large traders (see Jarrow (1992)), herd behavior (Dass et al. (2008)), heterogenous beliefs and short-sales constraints (Hong and Stein (2003), Scheinkman and Xiong (2003), Harrison and Kreps (1978)), and trading strategy admissibility conditions (Cox and Hobson (2005), Jarrow et al. (2007), (2010), Loewenstein and Willard (2000)). Empirical investigations attempting to prove or disprove the existence of various asset price bubbles are voluminous (see Jarrow et al. (2010) for a listing, as well as Bhattacharya and Yu (2008) and Camerer (1989) for surveys of the literature).

The purpose of this paper is to study the relation between asset price bubbles and market trading activity. There are two approaches to modeling bubbles in the economics literature: equilibrium

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and arbitrage-free pricing models. Equilibrium models require the specification of an economy including preferences, endowments, and a notion of a market equilibrium. The asset price process is determined endogenously in such a construct. In contrast, arbitrage-free pricing models take the asset price process as given. As such, arbitrage-free pricing models are consistent with many different equilibrium constructs. This is the benefit of using an arbitrage-free pricing model. The cost is that the economic underpinnings of the given price process are not explored in any detail. Both approaches have their uses. For the purposes of this paper, we utilize the arbitrage-free pricing methodology.

In particular, we use the liquidity risk model of Roch (2011) to analyze the formation and bursting of speculative bubbles. Liquidity risk exists when trading activity has an impact on prices. It is this quantity impact of trading activity on the market price that causes deviations from fundamental value - price bubbles - in our model. The martingale approach to modeling price bubbles (see Cox and Hobson (2005), Pal and Protter (2010), Jarrow et al. (2007), (2010), and Ekström and Tysk (2009)) assumes that the asset's market price process is exogenous and the fundamental price, the expected future cash flows under a martingale measure, is endogenous. In contrast, we define the asset's fundamental price process exogenously and asset price bubbles are endogenously determined by market trading activity including the volume of market orders, resiliency parameters, and levels of market liquidity.

More specifically, we consider a continuous-time model in which a stock is traded through a limit order book. Two types of trades are possible, namely limit orders and market orders. Limit orders provide liquidity by increasing the supply of shares available for trade, whereas market orders are executed against the best bid and ask limit orders. We define the fundamental price process as the market price process generated by trading activity against the limit order book under normal market conditions. Divergence from this fundamental market price process occurs when the resiliency of the limit order book (properly defined) is weak. It is this divergence which creates price bubbles in our model. When the resiliency is restored, the price bubble bursts.

Our model integrates both the large trader and liquidity risk models appearing in the literature. In the large trader models of Jarrow (1992), Bank and Baum (2004), the trades of a large trader have a permanent impact on the asset price process.[1] The price process returns to its equilibrium price only when the large trader liquidates his holdings. At the other end of the spectrum lies the liquidity risk model of Çetin et al. (2004), in which a competitive trader's impact on the asset price process lasts only for an instant. Here the competitive trader's transaction only affects the executed price and not the market price process itself. Our approach combines both concepts by allowing the impact of trades on prices to last longer than an instant, but not be permanent. This characteristic captures the resiliency of market prices to return to their fundamental value. As such, our model allows for richer market price dynamics than either the large trader or liquidity risk models because in our model the market price process is affected by all traders' purchases and sales relative to the their trade size.

The paper is organized as follows. In Section 2, we describe the model and discuss its three main characteristics: depth, resiliency and tightness. In Section 3, we define the fundamental and

[1]Given this is an arbitrage-free model, we concentrate on the properties of the price process itself, and not the strategic behavior of the large trader.

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market wealth processes and study the main properties of bubbles. In Section 4, we present the main example for study, and in Section 5 we give sufficient conditions for our model to be arbitrage free, and it contains our main result, Theorem 5.2.

2 The Model

We consider a liquid financial asset, called the stock, which is actively traded through a limit order book. In this setting, there are two types of trades possible, namely limit orders and market orders. A limit order is an order to buy or to sell the stock at a specific price which is not immediately executed. Limit orders provides liquidity by composing or filling the limit order book. On the other hand, impatient traders can submit market orders (also known as marketable limit orders) which are executed against the existing limit orders and thereby deplete the order book.

We assume that the stock pays dividends. We let $D = (D_t)_{t\geq 0}$ denote the cumulative dividend process with $D_0 = 0$. We define F_τ to be the terminal value of the stock with τ the liquidation time. The dividend process D and the liquidation value F_τ are the fundamental characteristics of the stock price process.

2.1 The Limit Order Book

Our starting point is the liquidity model of Roch (2011). We are given a trading horizon T with $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leq t\leq T}, \mathbb{P})$ a filtered probability space satisfying the usual conditions. We assume that the spot rate of interest is a constant, and for simplicity we use discounted price processes.

We start with a static description of the limit order book by fixing the time t. We assume that at time t all traders observe the limit order book and know the average price to pay per share for a transaction of size x via a market order. We assume that this average price per share is given by the following linear structure:

$$S_t(x) = S_t + M_t x, \quad x \in \mathbb{R} \tag{1}$$

where S and M are adapted processes that will be defined later. S_t is called the quoted (or marginal) price, it is the price per share for a purchase or sale of an infinitesimal quantity of shares ($x = 0$) at time t. Seen as a function of x, $S_t(x)$ is sometimes called the supply curve for the stock. Note that this function is completely determined by the two parameters S_t and M_t. The simplifying linearity assumption is supported by the empirical evidence in Blais and Protter (2010), especially for frequently traded and large volume stocks.

For now, Equation 1 is only a static description of the supply curve in the sense that it is written in terms of two processes S and M for which we have not yet specified the dynamics. Our goal is to define these processes in terms of the aggregate volume of market orders. This will be done in the next section.

Expression (1) gives us a way to characterize the limit order book. We represent it by the function $\rho_t(z)$ which denotes the density function for the number of shares offered at price z at time t, i.e.

$$\int_{z_1}^{z_2} \rho_t(z)dz$$

is the total number of shares offered between prices z_1 and z_2. For example, if a trader wants to buy x shares at time t through a market order then the total dollars paid for this order is

$$\int_{S_t}^{z_x} z\rho_t(z)dz$$

where z_x solves the equation

$$\int_{S_t}^{z_x} \rho_t(z)dz = x. \tag{2}$$

To understand this expression, note that a market order to buy will start at the quoted price S_t obtaining $\rho_t(S_t)dz$ shares at that price, then moving up the limit order book until the price of z_x is paid for the last $\rho_t(z_x)dz$ shares purchased. The total shares purchased is x.

Of course, the linear structure of the supply curve in expression (1) implies that its limit order book density function equals $\rho_t(z) = \frac{1}{2M_t}$. In this case $z_x = S_t + 2M_t x$ and the total dollars paid for x shares is

$$\frac{1}{2M_t}\int_{S_t}^{S_t+2M_t x} zdz = S_t x + M_t x^2 = xS_t(x). \tag{3}$$

It is well known that the limit order book, in our case the density function $\rho_t(z)$, captures the market's liquidity at any price z. Consequently, M_t in our supply curve is a measure of *illiquidity*. Indeed, the larger is M_t, the larger is the price impact realized to trade x shares (versus the quoted price of S_t).

Recall that in our market, a trade occurs when a market order is placed. As a result of this market order, limit orders in the limit order book are executed starting with the cheapest to the most expensive (in the case of a buying order) until the total number of shares ordered is reached as in expression (2). This process is characterized by the previous structure. Now let's consider what happens after the market order is executed. Once the order is executed, the respective limit orders disappear and a gap is created in the limit order book. For example, after a purchase of ΔX_t shares, the best bid price stays at S_t whereas the best ask price is now $S_t + 2M_t\Delta X_t$. This means that immediately after this trade, the limit order book density function would be 0 for prices between S_t and $S_t + 2M_t\Delta X_t$ and $\rho_{t-}(z)$ elsewhere. The zero interval in the density function for $z \in [S_t, S_t + 2M_t\Delta X_t]$ is the "gap" in the limit order book.

The next question is, what happens to this gap? Weber and Rosenow (2005) show a negative correlation between returns and the volume of incoming limit orders which suggests that investors respond to market orders by adding new limit orders in the opposite direction. An economic justification for this phenomenon is that informed traders, who can determine the fundamental value of the asset, take advantage of this quantity impact on the price by placing limit orders at the temporarily disadvantaged price. We call this the short-term resiliency effect. We model this effect by assuming that the gap in the density function partially disappears immediately after a trade.

Formally, let's temporarily introduce the process $R = (R)_{t\geq 0} \in [0,1]$ measuring the proportion of new orders to sell (resp. orders to buy) that fill the limit order book immediately after a trade to buy (resp. sell) at time t. The quantity of new limit orders to sell after a market order to buy is given by $R_t\Delta X_t$. In effect, the instantaneous impact on the supply curve due to a trade of size ΔX_t is to shift the lowest ask price $S_t + 2M_t\Delta X_t$ after the trade to

$$S_t + 2(1-R_t)M_t\Delta X_t. \tag{4}$$

Hence, the density function gap for prices $z \in [S_t, S_t + 2M_t\Delta X_t]$ after a trade is reduced to the gap for only the prices $z \in [S_t, S_t + 2(1-R_t)M_t\Delta X_t]$. If $R_t = 1$ then no gap remains. The quantity impact on the price disappears instantaneously after a trade. This is the liquidity risk model of Çetin et al. (2004). If $R_t = 0$ then the entire gap remains and the trade causes a permanent quantity impact on the price. This is thequantity impact on the price used in the large trader model of Jarrow (1992) and Bank and Baum (2004). The term $-2R_tM_t\Delta X_t$ corresponds to the adjustment due to the creation of new limit orders, the resiliency factor. To simplify the notation, for the remainder of the paper, we will use the process $\Lambda = 1 - R$ instead of R.

Note that a trade does not have an impact on limit order book density outside the gap created. In this sense, the supply curve immediately after a trade is still of the form of Equation 1, except that the marginal price S_t has been affected. In other words, the parameter M_t remains unaffected by trades. Although we are mainly interested in the behavior of the process S when studying bubbles, the entire supply curve underlies the analysis as reflect in expression (1).

So far, this is a static description of the quantity impact of a transaction on prices at a given point in time. In Section 3, we will relate the dynamics of the process S to the price impact assumptions we have made in this section, which in turn will give a full description of the dynamics of the supply curve.

2.2 Depth, Resiliency and Tightness

As stated by Kyle (1985), a reasonable liquidity model should have three dimensions: depth, resiliency and tightness. Depth is defined as the size of the order flow required to change prices a given amount. Resiliency is the degree to which the limit order book recovers from small trades. Tightness refers to the cost of turning around a position. Our model has all three dimensions.

Some resiliency has already been introduced in the model. It is characterized by the process $(\Lambda_t)_{t\geq 0}$. When $\Lambda_t = 0$, there is full resiliency since the order book recovers its previous shape after a market order of any size. On the other hand, when $\Lambda_t = 1$,the gap created by a market order at time t remains. In this case there is no resiliency. Any value of Λ_t strictly between 0 and 1 leads to a partial recovery. This partial recovery is a "short term" effect. In the next section, we will define a "long term" resiliency effect.

The depth of the order book is captured in expression (4). The impact of a trade of size ΔX_t at time t is given by $2\Lambda_tM_t\Delta X_t$. The depth, defined as the size of the order required to change the price of the asset by one unit, is thus $\frac{1}{2\Lambda_tM_t}$. Note that the resiliency is present in this expression.

Tightness, the cost of turning around a position, is related to the optimal trading strategy of an investor. In our model, the optimal trading strategy is continuous and of finite variation. This fact was first noted in the liquidity risk model of Çetin et al. (2004) and extended to the price impact case in Roch (2011). It implies that if a trader wants to buy a large block of shares ΔX_t, then she is better off dividing the order into n smaller trades of size $\frac{1}{n}\Delta X_t$ (with n large) and filling them one by one in the limit order book. This breaking up the order into smaller parts will incur a smaller liquidity cost.

To see this, note that the first trade will incur a price impact cost of $M_t\frac{1}{n}\Delta X_t$ dollars. The second trade will incur a liquidity cost of $\Lambda_tM_t\frac{1}{n}\Delta X_t$ dollars, due to the resiliency effect. Summing across all of these trades, the total cost (including the quoted price S_t) is approximately equal to

$$\begin{aligned}&\sum_{i=1}^{n}\frac{1}{n}\Delta X_t\left(S_t+2(i-1)\Lambda_t M_t\frac{1}{n}\Delta X_t+M_t\frac{1}{n}\Delta X_t\right)\\ = \quad &\Delta X_t S_t+\frac{n-1}{n}\Lambda_t M_t(\Delta X_t)^2+\frac{1}{n}M_t(\Delta X_t)^2.\end{aligned}$$

When n tends to infinity, this expression converges to $\Delta X_t S_t+\Lambda_t M_t(\Delta X_t)^2$. (A rigorous proof of this is given in Lemma 4.1 of Çetin et al. (2004) and Lemma 3.3 of Roch (2011).) In contrast, the trader pays $\Delta X_t S_t + M_t(\Delta X_t)^2$ if he makes one large block trade. Since $\Lambda_t \leq 1$, the first strategy always dominates to the second.

3 Fundamental Value and the Market Price

This section embeds the resiliency factor discussed above into the evolution of the market price of the stock S and relates it to the *fundamental value process*, denoted $F=(F_t)_{t\geq 0}$. We take the fundamental value as a primitive of the model, an exogenously given semimartingale which represents the price process observed if market orders had no quantity impact on the price, i.e. $\Lambda_t = 0$ all t a.s. This process could, for instance, be generated by an equilibrium economy which depends on preferences, endowments, and a particular market clearing mechanism. Instead, we take this process as exogenous, and consequently, it is consistent with many different equilibrium constructs. This is the standard paradigm used in arbitrage-free pricing models. Equivalently, returning to the cash flows underlying the stock, F_t represents the liquidation value of the stock if the firm is liquidated at time t. We will use both of these interpretations below.

The basic idea of our construction is that the market price of the stock S should equal its fundamental value F, except for the "short term" quantity impacts of trading activity. As discussed above, a trade of size dX_t should have a "short term" impact on the market price equal to a movement of $+2\Lambda_{t-}M_{t-}dX_t$. However, we would expect that this price impact should decay to zero in the "long term" as market prices return to fundamental values. The subtlety, of course, is that before the market price can decay back to its fundamental value, another market order may shock the price evolution again, causing another short term deviation. Depending upon their frequency, these short term deviations from fundamental value may accumulate faster then their ability to decay, thereby creating a divergence from fundamental value. This divergence generates a price bubble.

To formalize this idea, we need to introduce two additional stochastic processes. One, a process for the speed of decay, denoted $\kappa=(\kappa_t)_{t\geq 0}$. Second, we are given a semimartingale representing the signed volume of *aggregate* market orders (volume of market buy orders minus volume of market sell orders), denoted $X=\{X_t\}_{t\geq 0}$. This represents the aggregate trading activity. For example, it could be endogenously defined in terms of other processes or represent noise traders, but we do not make any such specific assumptions. We assume these two processes are observable.

The assumption we make is that each individual market order has an impact on prices as discussed in Section 2.1, so that the aggregate market orders have a short term impact given by

$2\Lambda_t M_t \Delta X_t$. Thus, we define the market price process S as the process obtained from F by adding the impact of market orders and long-term resiliency:

$$dS_t = -\kappa_t(S_t - F_t)dt + dF_t + 2\Lambda_{t-}M_{t-}dX_t \quad (t \le \tau) \tag{5}$$

where $S_0 = F_0$ and $X_0 = \Lambda_0 = \kappa_0 = 0$.

By construction, we start the market price at its fundamental value ($S_0 = F_0$). Subsequently, the evolution of the market price process equals the evolution of the fundamental value process (dF_t), except for short term deviations ($2\Lambda_{t-}M_{t-}dX_t$) caused by aggregate market orders of dX_t, which revert to zero in the long term ($-\kappa_t(S_t - F_t)dt$), with κ_t the speed of decay.

Note that the market price is defined only strictly prior to the liquidation date, at which time trading halts and the market price equals its liquidation value, i.e. $S_\tau = F_\tau$.

From the previous equation, we see that

$$d\Psi_t = 2e^{\int_0^t \kappa_u du}\Lambda_{t-}M_{t-}dX_t \quad (t \le \tau) \tag{6}$$

where

$$\Psi_t = (S_t - F_t)e^{\int_0^t \kappa_u du}.$$

S is thus well defined and given by

$$S_t = F_t + \int_0^t 2e^{-\int_s^t \kappa_u du}\Lambda_{s-}M_{s-}dX_s \quad (t \le \tau). \tag{7}$$

As seen in this expression, the market price of the stock equals its fundamental value plus a deviation. The deviation is the accumulated impact of the short term quantity impacts on the market price.

3.1 Bubbles

Given the market price and fundamental value process, we can now study price bubbles.

Definition 3.1 *The price bubble β is the difference between the market price and its fundamental value, i.e.*

$$\beta_t = S_t - F_t \quad t \le \tau. \tag{8}$$

Given expression (7), we have that

$$\beta_t = \int_0^t 2e^{-\int_s^t \kappa_u du}\Lambda_{s-}M_{s-}dX_s \quad t \le \tau. \tag{9}$$

As seen by this expression, the evolution of bubbles in our economy crucially depends on the fundamental value and liquidity processes $\{F, X, \Lambda, \kappa\}$. As such, our economy can have multiple bubble birth and bursting.

To see this note that, by construction, at time 0 there are no bubbles in our economy. Let $\tau_0' = \inf\{t \ge 0 : \Lambda_t > 0\}$. This is the first time after our economy begins that a quantity impact on market prices can occur - a market illiquidity arises. Next, define $\tau_0 = \inf\{t \geq \tau_0' : X_t - X_{\tau_0'} > 0\}$. This is the first time, after the market becomes illiquid, that aggregate trading activity is non-zero. Then, given our market price evolution, we see that $\tau_0 = \inf\{t \geq 0 : \beta_t > 0\}$ as well. That is, τ_0 is the first time of bubble birth - the moment when excess trading volume creates a momentum which builds upon itself.

Continuing, this bubble bursts at time $\tau_1 = \inf\{t \geq \tau_0 : \beta_t = 0\}$. Note that $\tau_1 \leq \tau$, the liquidation date. And, this process can continue. We can define $\tau_2' = \inf\{t \geq \tau_1 : \Lambda_t > 0\}$ and $\tau_2 = \inf\{t \geq \tau_2' : X_t - X_{\tau_2'} > 0\}$ to get the time of the second bubble birth for the same stock, and so forth until time τ when the bubble bursts for the last time. This process of bubble birth and bursting will be illustrated with an example in a subsequent section.

3.2 Arbitrage-Free Market Price Processes

Given the existence of bubbles in our economy, we now study the relation between bubbles and no arbitrage opportunities. To this end, we need to define two wealth processes associated with the stock price process. Recall that D is the dividend process and F_τ is the liquidation value.

Definition 3.2 *The market wealth process is defined by*

$$W_t = D_t + S_t \mathbf{1}_{\{t<\tau\}} + F_\tau \mathbf{1}_{\{t=\tau\}} \quad (0 \leq t \leq \tau) \tag{10}$$

and the fundamental wealth process by

$$W_t^F = D_t + F_t \quad (0 \leq t \leq \tau). \tag{11}$$

As indicated, the market wealth process represents the market price of the stock plus the accumulated dividends from time 0. Note that by construction, the market price S_t represents the ex-dividend value of the stock if a dividend is paid at time t, i.e. if $\Delta D_t > 0$.

It is clear from these definitions that the market value W_τ is equal to the fundamental value W_τ^F at the liquidation time. Furthermore, the difference between these two wealth processes is equal to the difference between the market price S and the fundamental value F. Hence, the difference between the market and the fundamental wealth processes equals the price bubble β.

Definition 3.3 *We denote by $\mathscr{M}_{loc}(W)$ (respectively $\mathscr{M}(W^F)$) the set of probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which W (resp. W^F) is a $\mathbb{Q}$-local martingale (resp. $\mathbb{Q}$-martingale). The measures in $\mathscr{M}_{loc}(W)$ are called equivalent local martingale measures (ELMMs).*

The correspondence between the set of ELMMs and the notion of no arbitrage is well understood (see Delbaen and Schachermayer (1998)). We apply this theory to our economy to understand when the market stock price evolution is arbitrage free.

Let $\tilde{S}$ be an arbitrary $\mathbb{R}^d$-valued càdlàg semimartingale and define

$$\begin{aligned} K(\tilde{S}) \quad = \quad \{ & \int_0^T H d\tilde{S} : H \text{ is } S\text{-integrable and} \\ & \int H d\tilde{S} \text{ is uniformly bounded from below} \}. \end{aligned}$$

The process $\tilde{S}$ satisfies No Free Lunch With Vanishing Risk (NFLVR) if and only if

$$\overline{C(\tilde{S})} \cap L^{\infty}_{+} = \{0\},$$

where $C(\tilde{S}) = \{f \in L^{\infty} : \text{ there exists a } g \in K(\tilde{S}) \text{ such that } f \leq g\}$. The following result is due to Delbaen and Schachermayer (1998):

Theorem 3.4 *The process $\tilde{S}$ satisfies the NFLVR condition if and only if there exists an equivalent probability measure under which $\tilde{S}$ is a sigma-martingale.*

We refer the reader to Delbaen and Schachermayer (1998) for a definition of sigma-martingales. It is shown in Protter (2005) that a positive sigma-martingale is in fact a local martingale. Hence we have the following corollary:

Corollary 3.5 *W satisfies the NFLVR condition if and only if $\mathscr{M}_{loc}(W)$ is non-empty.*

Our economy, therefore, has no arbitrage (satisfies NFLVR) if and only if the set of equivalent martingale measures for the market wealth process is non-empty. This is determined, as before, by the structure of the fundamental value and liquidity processes $\{F, X, \Lambda, \kappa\}$.

Using the second fundamental theorem of asset pricing (see Harrison and Pliska (1981)), our economy may be incomplete with an infinite number of local martingale measures $\mathbb{Q}$. This is due to the randomness potentially exhibited by the market price process due the four underlying processes $\{F, X, \Lambda, \kappa\}$, and the fact that we allow trading in only one stock.

We illustrate such an incomplete and NFLVR market price process in a subsequent example.

3.3 Relation to the Martingale Theory of Bubbles

In the martingale bubble literature (see Cox and Hobson (2005), Pal and Protter (2010), Jarrow et al. (2007), (2010), and Ekström and Tysk (2009)), the fundamental price process is defined as the expectation of the future cash flows under a fixed equivalent local martingale probability measure $\mathbb{Q}$. Thus, the $\mathbb{Q}$-bubble is defined as

$$\beta_t^{\mathbb{Q}} = W_t - E_{\mathbb{Q}}\left(W_{\tau} | \mathscr{F}_t\right) \tag{12}$$

for a given $\mathbb{Q} \in \mathscr{M}_{\text{loc}}(W)$, the difference between the market wealth process and its conditional expectation.

Since the market and fundamental wealth processes are equal at the liquidation date, we can rewrite this as

$$\beta_t^{\mathbb{Q}} = W_t - E_{\mathbb{Q}}\left(W_{\tau}^F | \mathscr{F}_t\right).$$

In this form, it is easy to understand the difference between the martingale defined bubble and our definition. As mentioned above, our bubble definition is equivalent to

$$\beta_t = W_t - W_t^F.$$

The two definitions coincide when the fundamental wealth process W_t^F is a $\mathbb{Q}$-martingale, and they differ otherwise. Since the NFLVR condition only guarantees that W_t is a local $\mathbb{Q}$-martingale, in general these two bubbles definitions differ. We illustrate this difference in a subsequent example.

For the case of a complete market (see Cox and Hobson (2005) and Jarrow et al. (2007)) the local martingale measure in unique. For an incomplete market, a fixed probability measure needs to be selected from the infinite collection (see Loewenstein and Willard (2000) and Jarrow et al. (2010)). Fixing such a local martingale measure $\mathbb{Q}$ for the entire trading horizon implies that the martingale defined bubble, if it exists, must exist at the start of the model. Indeed, when W is a strict $\mathbb{Q}$-local martingale, it is also a supermartingale, i.e., it satisfies $W_t \geq \mathbf{E}_{\mathbb{Q}}(W_\tau|\mathscr{F}_t)$ for all $t \leq \tau$, including time 0. Furthermore, if the bubble bursts prior to time τ, another bubble cannot be formed. These are unintuitive implications of a martingale defined bubble.

To overcome these unintuitive implications, Jarrow et al. (2010) also considered the case of an incomplete market where the equivalent martingale measure was not fixed, but could vary across time. Therein, a bubble is created when the fundamental measure changes from one where the price process is a true martingale to one where it is a strict local martingale.

A difference of our bubble definition in the case of incomplete markets is that it allows multiple bubble birth and bursting while keeping the local martingale measure fixed over the entire trading horizon.

4 An Example

To better illustrate the previous model, we consider the following example. Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space on which a Brownian motion $B = (B_{1t}, B_{2t})$ and an independent exponential random variable E are defined. Let τ_0 be a totally inaccessible stopping time with intensity process $\pi = (\pi_t)_{0\leq t\leq T}$ defined by

$$\tau_0 = \inf\{s \geq 0 : \int_0^s \pi_s ds > E\}. \tag{13}$$

Let $N_t = \mathbf{1}_{\{t\geq\tau_0\}}$ be the point process associated with this stopping time, and $\widetilde{N}_t = N_t - \int_0^{t\wedge\tau_0} \pi_s ds$ the associated martingale process.

Define

$$\begin{aligned} dW_t^F &= W_t^F(\mu dt + \sigma dB_{1,t}), \\ d\beta_t &= 2\Lambda_t M_t \beta_t(-\kappa dt + \alpha dB_{2,t}) + 2xW_t^F \Lambda_t M_t dN_t, \quad \text{and} \\ dW_t &= dW_t^F + d\beta_t \end{aligned} \tag{14}$$

for $t \geq 0$ where $W_0^F = W_0$, $\beta_t = 0$ for $t < \tau_0$, κ is a given positive constant, and $\beta_{\tau_0} = xW_{\tau_0}^F$ for a given $x > 0$. The processes Λ and M are assumed to be continuous and adapted to the filtration. It is clear then that $dX_t = \alpha\beta_t dB_{2,t}$ for $t > \tau_0$.

The economic interpretation of this model is straightforward. The process W^F represents the fundamental value of the stock and β the bubble. Here, τ_0 represents the time of bubble formation. Bubble birth is a surprise to the market since it is represented by an inaccessible stopping time.

At time $t < \tau_0$, the value of the asset equals its fundamental value. The magnitude of the initial bubble is generated by the aggregate trading volume at time τ_0 and it equals $xW^F_{\tau_0}$. Subsequent to this time, the aggregate trading volume evolves according to the process $dX_t = \alpha\beta_t dB_{2,t}$. Hence, the evolution of the bubble is afterwards determined by the second Brownian motion $B_{2,t}$ embedded in the trading activity, and the resiliency and liquidity processes Λ_t and M_t, respectively.

4.1 Bubble Bursting

Consider the following measure of liquidity: the *integrated effective liquidity process* $M^* = (M^*_t)_{0\le t\le \tau}$ defined by

$$M^*_t = 4\int_0^t \Lambda_s^2 M_s^2 ds \quad (0 \le t \le \tau). \tag{15}$$

When M^*_t is small the market is liquid. When M^*_t is large, the market is less liquid.

Recall that the bubble β bursts at time $\tau_1 = \inf\{t \ge \tau_0 : \beta_t = 0\}$. The following proposition proves that this price bubble bursts when the stock becomes infinitely illiquid, i.e. when the integrated liquidity process diverges to infinity.

Define $\widetilde{\tau}_1 = \inf\{t \ge \tau_0 : M^*_t = \infty\}$.

Proposition 4.1 $\widetilde{\tau}_1 = \tau_1$ $\mathbb{P}$- *a.s.*

Proof. Recall that the bubble can be represented as

$$\beta_t = \beta_{\tau_0} \exp\left(-\kappa\int_{\tau_0}^t 2\Lambda_s M_s ds - 2\int_{\tau_0}^t \Lambda_s^2 M_s^2 \alpha^2 ds + 2\int_{\tau_0}^t \Lambda_s M_s \alpha dB_{2,s}\right)$$

for $t \ge \tau_0$. Define $T(t) = \inf(s \ge \tau_0 : M^*_s > t)$, and $\mathscr{G}_t = \mathscr{F}_{T(t)}$. Then, it is clear that $\widetilde{\tau}_1 = \lim_{t\to\infty} T(t)$. Furthermore, $B'_t = 2\int_{\tau_0}^{T(t)} \Lambda_s M_s dB_{2,s}$ defines a $\mathbb{G}$-Brownian motion and

$$\beta_{T(t)} = \beta_{\tau_0} \exp\left(-2\kappa\int_{\tau_0}^{T(t)} \Lambda_s M_s ds - \frac{|\alpha|^2 t}{2} + \alpha B'_t\right).$$

Letting $t \to \infty$, we find that $\beta_{T(t)} \to 0$ by the Law of the Iterated Logarithm, or equivalently $\beta_t \to 0$ as $t \to \widetilde{\tau}_1$. Note that $M^*_t < \infty$ implies $\int_{\tau_0}^t 2\Lambda_s M_s ds < \infty$. Hence $\beta_t > 0$ for any $t < \widetilde{\tau}_1$, and we deduce that $\tau_1 = \widetilde{\tau}_1$ $\mathbb{P}$-a.s. ■

This proposition yields the intuitive result that a bubble bursts the first time the market becomes infinitely illiquid. Intuitively, the dampening effect becomes infinitely large, so that the market price instantly reverts to fundamental value, regardless of the randomness in trading volume.

4.2 The Set of ELMMs

It is interesting to study the set of ELMMs for this market price evolution. When an ELMM exists, the market is arbitrage free. The set of ELMMs are characterized by the following version of Girsanov's Theorem.

Proposition 4.2 *Let $\mathbb{Q} \in \mathscr{M}_{loc}(W)$. Then the density process $L_t = \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathscr{F}_t}$ is given by the stochastic exponential*

$$\mathscr{E}(-\int bdB - \int (1-Y)d\widetilde{N}) \tag{16}$$

in which $\widetilde{N}_t = N_t - \int_0^{t\wedge\tau_0} \pi_s ds$, b is a predictable process and $Y = (Y_t)_{t\geq 0} > 0$ is a strictly positive predictable process. Furthermore, the following equality holds

$$0 = W_t^F(\mu - \sigma b_{1,t}) - 2\beta_t M_t \mathbf{1}_{t\geq\tau_0}(\kappa + \alpha b_{2,t}) + 2\pi_t x W_t^F M_t Y_t \mathbf{1}_{t<\tau_0} \tag{17}$$

for all $0 \leq t \leq \tau$.

Proof. The process L is a $\mathbb{P}$-martingale. By the predictable martingale representation property of the pair $(B, \widetilde{N})$, there exists predictable processes b and Y such that

$$dL_t = -L_{t-}(b_t dB_t + (1-Y_t)d\widetilde{N}_t).$$

By Girsanov's theorem (see Jacod and Shiryaev (1987) Theorem III.3.11), the processes

$$B - \int \frac{1}{L_{t-}} d\langle B, L\rangle_t = B + \int b_t dt$$

and

$$\widetilde{N} - \int \frac{1}{L_{t-}} d\langle \widetilde{N}, L\rangle_t = N - \int Y_t \pi_t \mathbf{1}_{t\leq\tau_0} dt$$

are $\mathbb{Q}$-martingales. Equation 17 follows from the fact that W is a $\mathbb{Q}$-local martingale. ■

The previous proposition gives a necessary condition for an equivalent measure to be an ELMM. Candidates for ELMM need to satisfy Equation 17, however, it is not true that all processes b satisfying 17 give rise to an ELMM defined in terms of the associated density process L of Equation 16. Indeed, a necessary condition for $\mathbb{Q}$ to be an equivalent measure is that L is a martingale. This is clearly not satisfied for all processes b. On the other hand, the general result of Theorem 5.2 in the following section can be used to obtain sufficient conditions for the set of ELMMs to be non-empty in this example. We find that Condition (i) of the theorem is satisfied if $\sigma > 0$, and Condition (ii) is satisfied if $\mu \neq 0$.

Expression (17) can be broken down into two cases:

$$\sigma b_{1,t} = \mu + 2\pi_t x M_t Y_t, \text{ for } t < \tau_0 \text{ and} \tag{18}$$

$$\sigma b_{1,t} = \mu - 2\frac{\beta_t M_t}{W_t^F}(\kappa + \alpha b_{2,t}), \text{ for } t \geq \tau_0. \tag{19}$$

Let $B^{\mathbb{Q}}$ denote the $\mathbb{Q}$-Brownian motion given by $B_t^{\mathbb{Q}} = B_t + \int_0^t b_s ds$. We have the following representation of the fundamental wealth process:

$$\frac{dW_t^F}{W_t^F} = \widetilde{\mu}_t dt + \sigma dB_{1,t}^{\mathbb{Q}} \text{ for all } t \leq \tau,$$

in which

$$\widetilde{\mu}_t = \begin{cases} -\pi_t x Y_t, & \text{if } t < \tau_0; \\ 2\frac{\beta_t}{W_t^F} M_t(\kappa + \alpha b_{2,t}), & \text{if } t \geq \tau_0. \end{cases}$$

Furthermore, $\widetilde{\mu}_t < 0$ on $[0, \tau_0)$ for all choices of $\mathbb{Q}$ since $Y > 0$.

Using this characterization of the ELMMs, we can study the martingale defined bubble in the context of our model. The next proposition is the first step in this analysis.

Proposition 4.3 *For any $\mathbb{Q} \in \mathscr{M}_{loc}(W)$ such that $b_2 \leq -\frac{\kappa}{\alpha}$, the process W is a strict $\mathbb{Q}$-local martingale.*

Proof. Let $\mathbb{Q} \in \mathscr{M}_{\text{loc}}(W)$ such that $b_2 \leq -\frac{\kappa}{\alpha}$. When $b_2 \leq -\frac{\kappa}{\alpha}$, the drift $\widetilde{\mu}_t$ is strictly less than zero on $[0, \tau_0 \wedge \tau]$ and less or equal to zero on $[\tau_0 \wedge \tau, \tau]$. On the interval $[0, \tau_0 \wedge \tau]$, the process W satisfies

$$\begin{aligned} dW_t = dW_t^F + xW_t^F M_t dN_t &= \widetilde{\mu}_t W_t^F dt + \sigma W_t^F dB_{1,t}^{\mathbb{Q}} + xW_t^F M_t dN_t \\ &= xW_t^F dM_t \widetilde{M}_t + \sigma W_t^F dB_{1,t}^{\mathbb{Q}} \\ &= xW_{t-} M_t d\widetilde{M}_t + \sigma W_{t-} dB_{1,t}^{\mathbb{Q}} \end{aligned}$$

in which $\widetilde{M}_t = N_t - \int_0^t \pi_s Y_s ds$. Since $\widetilde{M}$ and $B^{\mathbb{Q}}$ are $\mathbb{Q}$-martingales and W is positive, $\mathbf{E}_{\mathbb{Q}}(W_{\tau_0 \wedge \tau}) \leq W_0$. Furthermore, if $\tau_0 < \tau$, then $W_{\tau_0 \wedge \tau} = (1+x) W_{\tau_0}^F > W_{\tau_0}^F = W_{\tau_0 \wedge \tau}^F$. On the other hand, if $\tau_0 \geq \tau$, then $W_{\tau_0 \wedge \tau} = W_{\tau_0 \wedge \tau}^F$. Hence, $W_{\tau_0 \wedge \tau} \geq W_{\tau_0 \wedge \tau}^F$ and $W_{\tau_0 \wedge \tau} > W_{\tau_0 \wedge \tau}^F$ with positive probability since $\mathbb{Q}(\tau_0 < \tau) > 0$. On the other hand, $\widetilde{\mu}_t \leq 0$ on $[\tau_0 \wedge \tau, \tau]$, hence

$$E_{\mathbb{Q}}(W_\tau^F) = E_{\mathbb{Q}}(E_{\mathbb{Q}}(W_\tau^F | \mathscr{F}_{\tau_0 \wedge \tau})) \leq E_{\mathbb{Q}}(W_{\tau_0 \wedge \tau}^F) < E_{\mathbb{Q}}(W_{\tau_0 \wedge \tau}) \leq W_0.$$

By definition, W is a $\mathbb{Q}$-local martingale, so it will be a strict local martingale if $E_{\mathbb{Q}}(W_\tau) < W_0$. Thus, it suffices to recall that $W_\tau^F = W_\tau$, or, equivalently, that $\beta_\tau = 0$ to obtain the desired result. ■

For the case $b_2 = -\frac{\kappa}{\alpha}$ from this proposition, we can determine the magnitude of the martingale defined fundamental value. If $b_2 = -\frac{\kappa}{\alpha}$, then

$$\begin{aligned} \mathbf{E}_{\mathbb{Q}}(W_\tau^F | \mathscr{F}_t) &= \mathbf{E}_{\mathbb{Q}}(\mathbf{E}_{\mathbb{Q}}(W_\tau^F | \mathscr{F}_{\tau_0 \wedge \tau}) | \mathscr{F}_t) \\ &= \mathbf{E}_{\mathbb{Q}}(W_{\tau_0 \wedge \tau}^F | \mathscr{F}_t) \\ &= \mathbf{E}_{\mathbb{Q}}\left(W_t^F \mathscr{E}\left(\sigma B_1^{\mathbb{Q}}\right) \exp\left(-\int_t^{\tau_0 \wedge \tau} x \pi_s Y_s ds \right) \Big| \mathscr{F}_t \right) \\ &= W_t^F \mathbf{E}_{\mathbb{Q}}\left(\exp\left(-\int_t^{\tau_0 \wedge \tau} x \pi_s Y_s ds \right) \Big| \mathscr{F}_t \right) \end{aligned}$$

on $\{t < \tau_0\}$, since $W_t = W_t^F$ on $\{t < \tau_0\}$. In the simplest case, if $\mathbb{P}(\tau_0 > \tau) = 0$, then

$$\begin{aligned}
\mathbf{E}_{\mathbb{Q}}(W_\tau^F|\mathscr{F}_t) &= W_t^F \frac{1}{1+x}\mathbf{E}_{\mathbb{Q}}\left((1+x1_{\tau_0\wedge\tau\geq\tau_0})\exp\left(-\int_t^{\tau_0\wedge\tau} x\pi_s Y_s ds\right)\Big|\mathscr{F}_t\right) \\
&= W_t \frac{1}{1+x}\mathbf{E}_{\mathbb{Q}}\left(\frac{\mathscr{E}(x\widetilde{M})_{\tau_0\wedge\tau}}{\mathscr{E}(x\widetilde{M})_t}\Big|\mathscr{F}_t\right) \\
&= W_t \frac{1}{1+x}
\end{aligned}$$

on $\{t < \tau_0\}$.

In this case, the martingale defined $\mathbb{Q}-$bubble is

$$\beta_t^{\mathbb{Q}} = \begin{cases} \frac{x}{1+x} W_t & t \leq \tau_0 \\ \beta_t & t > \tau_0 \end{cases} .$$

Under our definition, the bubble is zero before τ_0, and equals β_t for $t > \tau_0$. Hence the two bubbles differ before time τ_0. Furthermore, if $\mathbb{P}(\tau_0 > \tau) > 0$, $\beta_t^{\mathbb{Q}}$ is smaller and it decreases in size as the likelihood of the event $\{\tau_0 < \tau\}$ decreases. In other words, the size of $\beta_t^{\mathbb{Q}}$ before τ_0 is proportional to the likelihood of the event $\{\tau_0 < \tau\}$.

If $b_2 > -\frac{\kappa}{\alpha}$ then $\beta_t^{\mathbb{Q}}$ is smaller than the one described in the above paragraph and $\beta_t^{\mathbb{Q}}$ is larger if $b_2 < -\frac{\kappa}{\alpha}$.

So far, only one bubble is modeled in this example, but many other future bubbles could easily be added. Indeed, it suffices to define a counting process

$$N_t = \sum_{i \geq 0} 1_{\tau_{2i} \leq t}$$

in which $\tau_0, \tau_2, \tau_4, \ldots$ are totally inaccessible stopping times such that $\tau_i > \tau_{i-1}$ for $i \geq 1$ and $\tau_j = \inf\{\tau_{j-1} < t < \tau : \beta_t = 0\}$ for $j = 1, 3, 5, \ldots$ In this case, a bubble is created at time τ_i and it bursts at time τ_{i+1} for i even, under our definition. On the other hand, the martingale-defined bubble bursts in this case at time $\sup_{i\geq 0} \tau_{2i+1}$, typically equal to τ.

It is interesting to study the set of probability measures where $Y = 0$. If $Y = 0$, then we have the limiting case where the associated probability measure, denoted $\overline{\mathbb{Q}}$, is not an ELMM because it is not equivalent to $\mathbb{P}$. The reason is that $\overline{\mathbb{Q}}(\tau_0 < \tau) = 0$ whereas $\mathbb{P}(\tau_0 < \tau) > 0$. Yet surprisingly, the expected liquidation value of the stock under this measure is equivalent to our fundamental value, i.e.

$$\mathbf{E}_{\overline{\mathbb{Q}}}(W_\tau^F|\mathscr{F}_t) = W_t^F$$

for $t < \tau_0$ (and, as before, the bubble has a size β_t for $t \geq \tau_0$.). Furthermore, $\tau_0 < \tau$ $\overline{\mathbb{Q}}$-a.s. Hence $\mathbf{E}_{\overline{\mathbb{Q}}}(W_\tau^F|\mathscr{F}_t) = W_t^F$ for all t, $\overline{\mathbb{Q}}$-a.s.

Nevertheless, it is not true that $\mathbf{E}_{\overline{\mathbb{Q}}}(W_\tau^F|\mathscr{F}_t) = W_t^F$ for all t, $\mathbb{P}$-a.s. For this last identity to hold, we need $\overline{\mathbb{Q}} \in \mathscr{M}(W^F)$. In the above example, this only happens when $b_1 = \frac{\mu}{\sigma}$. In this case, expression (17) holds if $Y = 0$ and $b_2 = -\frac{\kappa}{\alpha}$.

As illustrated by this example, under any of these ELMMs, the martingale defined fundamental value differs from our fundamental value. The reason is that when the market price contains a bubble in our sense, the ELMM is overcompensating for the possible formation of future bubbles in our model. This, in turn, will be reflected in the martingale defined fundamental value. To understand why this happens, one needs to recall that a local martingale is a true martingale up to a sequence $\{\tau_n\}_{n\geq 1}$ of stopping times converging to infinity. By definition, we know that the market price today is equal to the expected value of the market price at time τ_n. However, as time τ_n is always prior to the bursting of the bubble, the market price at time τ_n is inflated by the bubble. Hence, if the stock does not have a bubble (as defined in our model) at time 0, then the ELMMs reflect a larger discounting factor to make the future bubble-inflated market prices equal to the time 0 value.

5 Properties of the Equivalent Martingale Measure Sets

This section studies the properties of the equivalent martingale measure sets. To facilitate this analysis, we assume that the liquidity and volume processes Λ and X are chosen such that bubble birth, τ_0, is a totally inaccessible stopping time with an intensity process $\pi = (\pi_t)_{t\geq 0}$. We let $N_t = \mathbf{1}_{\{t\geq\tau_0\}}$ denote its point process and $\widetilde{N}_t = N_t - \int_0^{t\wedge\tau_0} \pi_s ds$ its associated martingale. As a first approximation, it reasonable to assume that at the time of bubble birth, i.e. the moment when N_t becomes nonzero, $[W^F, N] = 0$. Note that this is automatically satisfied in the example of the previous section since W^F is continuous. It is clear from the definition of S (see expression (7) that

$$dS_t = \xi_t \pi_t dt + dF_t + \xi_t d\widetilde{N}_t \;\text{ for }\; t \leq \tau_0$$

for some predictable process ξ measuring the size of the jump of S at the time τ_0 due to a jump in the excess volume process X.

The following theorem generalizes the observation in the previous example that the local martingale measures of W are incompatible with those of W^F.

Proposition 5.1 *Suppose $\beta \geq 0$ and $\mathbb{P}(0 < \tau_0 < \tau) > 0$. Then $\mathscr{M}_{loc}(W) \cap \mathscr{M}(W^F) = \emptyset$.*

Proof. If at time 0 the asset does not have a bubble (i.e. $W_0 = W_0^F$), then the difference between the two wealth processes (the bubble) is not a local martingale because it becomes positive with positive probability, i.e. $E(\beta_t|\mathscr{F}_0) > 0 = \beta_0$ on the event $\{\tau_0 > 0\}$. On the other hand, the fundamental wealth process is a true martingale under any measure $\mathbb{Q} \in \mathscr{M}(W^F)$. As a result, it is impossible for the market wealth process, which is the sum of the fundamental wealth process and the bubble, to be a $\mathbb{Q}$-local martingale for any $\mathbb{Q} \in \mathscr{M}(W^F)$, otherwise the bubble would also be one. ■

An important corollary is that the process W^F is not a $\mathbb{Q}$-martingale wealth process for any $\mathbb{Q} \in \mathscr{M}_{\text{loc}}(W)$. And, therefore, the martingale defined bubble $\beta^{\mathbb{Q}}$ will differ from our bubble β (see expression (12) and the following discussion).

We say the NFLVR condition holds after bubble birth if the price process W admits no free lunch with vanishing risk on the interval $[\tau_0, \tau]$, or equivalently, there exists an equivalent probability measure under which $(W_t)_{\tau_0\leq t\leq\tau}$ is a local martingale. Conditions for NFLVR in the presence of bubbles is well understood in the literature. However, it is not true in general that the transition from a state in which the asset does not have a bubble to a state in which it does will not create arbitrage opportunities. (Consider for instance the case of a fixed constant time τ_0.) The following theorem gives sufficient conditions for the NFLVR condition to be satisfied in our setting.

Let $W^F = \mathfrak{M} + \mathfrak{A}$ be the Doob-Meyer decomposition of W^F in which $\mathfrak{M}$ is a $\mathbb{P}$-local martingale and $\mathfrak{A}$ is a process with finite variation. We denote by W^{Fc} the continuous part of W^F.

Theorem 5.2 *Suppose $\mathscr{M}(W^F)$ is non-empty and the NFLVR condition holds after bubble birth. Condition (i), (ii) or (iii) below is sufficient for $\mathscr{M}_{loc}(W)$ to be non-empty.*

(i) $\Delta S_{\tau_0} = 0$

(ii) $d\langle N,N\rangle_t \ll d\langle W^{Fc}, W^{Fc}\rangle_t$ *and* $\mathscr{E}\left(\frac{d\langle N,N\rangle}{d\langle W^{Fc},W^{Fc}\rangle} \cdot W^{Fc}\right)$ *is a $\mathbb{Q}$-martingale for some $\mathbb{Q} \in \mathscr{M}(W^F)$,*

(iii) $d\mathfrak{A}_t \ll dt$ *and* $\frac{d\mathfrak{A}_t}{dt} \neq 0$.

Remark 1 *$\langle W^{Fc}, W^{Fc}\rangle$ always exists since W^{Fc} is continuous.*

Proof. Let $\mathbb{Q} \in \mathscr{M}(W^F)$ and define $Z_t = E\left(\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathscr{F}_t\right)$. Suppose condition (ii) or (iii) are satisfied. We start by showing that there exists a measure $\mathbb{Q}' \in \mathscr{M}(W^F)$ under which $\widetilde{N}$ is a martingale. (Recall that $\widetilde{N}$ is a $\mathbb{P}$-martingale and that the change of measure from $\mathbb{P}$ to $\mathbb{Q}$ potentially changes the distribution of N.)

By Theorem III.3.24 of Jacod and Shiryaev (1987), there exists a positive predictable process Y such that $\int Y_t \pi_t dt$ is the $\mathbb{Q}$-compensator of N.

Consider the process Z', solution of

$$dZ'_t = Z'_{t-}\left(\frac{dZ_t}{Z_{t-}} - (Y_t - 1)d\widetilde{N}_t\right)$$

with $Z'_0 = 1$. Since Z and $\widetilde{N}$ are $\mathbb{P}$-martingales, so is Z'. Define the measure $\mathbb{Q}'$ by $Z'_\tau = \frac{d\mathbb{Q}'}{d\mathbb{P}}$. Then, there exists a positive predictable process Y' such that the compensator of N is $\int Y'_t \pi_t dt$. However, since the jump size of Z at time τ_0 is $Z_{\tau_0 -}(Y_{\tau_0} - 1)$, Z' does not jump at time τ_0. Thus $Y' = 1$ and $\widetilde{N}$ is a $\mathbb{Q}'$ martingale. Furthermore, since $[W^F, N] = 0$, the compensator of W^F is the same under $\mathbb{Q}$ and $\mathbb{Q}'$, i.e. W^F is also a $\mathbb{Q}'$ local martingale.

Suppose Condition (ii) is satisfied. Define $\mathbb{Q}''$ by $Z''_t = \mathbf{E}\left(\frac{d\mathbb{Q}''}{d\mathbb{Q}'}|\mathscr{F}_t\right)$ in which

$$dZ''_t = -Z''_{t-}\xi_t \frac{d\langle N,N\rangle_t}{d\langle W^{Fc}, W^{Fc}\rangle_t} d{W^{F}}^c_t$$

and $Z''_0 = 1$. Let $W^F = \mathfrak{M}'' + \mathfrak{A}''$ be the Doob-Meyer decomposition of W^F under $\mathbb{Q}''$. By Theorem III.3.24 of Jacod and Shiryaev (1987), there exists a predictable process $b = (b_t)_{0 \leq t \leq \tau}$ such that

$$\begin{aligned} d\langle (Z''^c, W^{Fc}\rangle_t &= b_t d\langle W^{Fc}, W^{Fc}\rangle_t \text{ and} \\ \mathfrak{A}'' &= \int b_t d\langle W^{Fc}, W^{Fc}\rangle_t \end{aligned}$$

Furthermore, by definition of Z'',

$$d\langle (Z''^c, W^{F^c}\rangle_t \quad = \quad -\xi_t d\langle N, N\rangle_t = -\xi_t \pi_t dt$$

and since it is a continuous process, $\widetilde{N}$ remains a martingale under $\mathbb{Q}''$. As a result, the Doob-Meyer decomposition of $W^F + \xi \cdot N$ under $\mathbb{Q}''$ is

$$(\mathfrak{M}'' + \xi \cdot \widetilde{N}) + \left(\mathfrak{A}'' + \int \xi_t \pi_t dt\right) \quad = \quad \mathfrak{M}'' + \xi \cdot \widetilde{N},$$

hence $W^F + \xi \cdot N$ is a $\mathbb{Q}''$ local martingale.

Suppose Condition (iii) is satisfied. Let $a_t = \frac{d\mathfrak{A}_t}{dt}$. By assumption, $a_t \neq 0$ for all t.

Let $y > 0$, $l_t = y(1 \wedge |a_t|) \frac{1}{\xi_t \vee 1} \frac{1}{\pi_t \vee 1}$ and define $\mathbb{P}'$ by $Z''_\tau = \frac{d\mathbb{P}'}{d\mathbb{P}}$ in which

$$dZ''_t = Z''_t (l_t - 1) d\widetilde{N}_t$$

and $Z''_0 = 1$. Then, $N' := \widetilde{N} - \int \frac{1}{Z''_{s-}} d\langle \widetilde{N}, Z''\rangle_s = N - \int l_s \pi_s ds$ is a $\mathbb{P}'$-martingale. Note that $\mathfrak{M}$ is also $\mathbb{P}'$-local martingale since $\langle \mathfrak{M}, Z''\rangle = 0$. As a result, we can find, as shown before, a measure $\mathbb{Q}'$ under which $\mathfrak{M}$ is a local martingale and N' is a martingale.

Now define $\widetilde{Z}$ as the solution of

$$\frac{d\widetilde{Z}_t}{\widetilde{Z}_{t-}} = \left(1 + \frac{l_t \xi_t \pi_t}{a_t}\right) \frac{dZ'_t}{Z'_{t-}}$$

and $\widetilde{Z}_0 = 1$. Since $1 + \frac{l_t \xi_t \pi_t}{a_t} = 1 + y \frac{1 \wedge |a_t|}{a_t} \frac{\xi_t}{\xi_t \vee 1} \frac{\pi_t}{\pi_t \vee 1}$ is bounded by $1 + y$ and Z' is a $\mathbb{P}'$-martingale, $\widetilde{Z}$ is also a $\mathbb{P}'$ martingale. Define $\widetilde{\mathbb{Q}}$ from $\widetilde{Z}_t = E\left(\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}'} | \mathscr{F}_t\right)$. Then, $\widetilde{\mathbb{Q}}$ is equivalent to $\mathbb{P}$. Furthermore, it is easy to see that N' is a $\widetilde{\mathbb{Q}}$-martingale since it is a $\mathbb{Q}'$-martingale.

By Theorem III.3.24 of Jacod and Shiryaev (1987), there exists a predictable process $b = (b_t)_{0 \leq t \leq \tau}$ and a positive predictable function Y on $\Omega \times [0, \infty) \times [0, \infty)$ such that

$$\begin{aligned} 0 \quad &= \quad \mathfrak{A} + b \cdot \langle \mathfrak{M}^c, \mathfrak{M}^c\rangle + (Y - 1) * \nu \qquad (20) \\ \text{and } \langle (Z'^c, \mathfrak{M}^c\rangle \quad &= \quad bZ'_- \cdot \langle \mathfrak{M}^c, \mathfrak{M}^c\rangle \end{aligned}$$

in which ν is the compensator of the jump measure of $\mathfrak{M}$. Furthermore, there exists a predictable process $\widetilde{b} = (\widetilde{b}_t)_{0 \leq t \leq \tau}$ and a positive predictable function $\widetilde{Y}$ on $\Omega \times [0, \infty) \times [0, \infty)$ such that the $\widetilde{\mathbb{Q}}$-compensator of W^F is

$$\begin{aligned} \widetilde{\mathfrak{A}} \quad &= \quad \mathfrak{A} + \widetilde{b} \cdot \langle \mathfrak{M}^c, \mathfrak{M}^c\rangle + (\widetilde{Y} - 1) * \nu \qquad (21) \\ \text{and } \langle \widetilde{Z}^c, \mathfrak{M}^c\rangle \quad &= \quad \widetilde{b}\widetilde{Z}_- \cdot \langle \mathfrak{M}^c, \mathfrak{M}^c\rangle. \end{aligned}$$

From the definition of $\widetilde{Z}$, we see that

$$\begin{aligned} \widetilde{b}\widetilde{Z}_- \cdot \langle \mathfrak{M}^c, \mathfrak{M}^c\rangle \quad &= \quad \langle \widetilde{Z}^c, \mathfrak{M}^c\rangle \\ &= \quad \left(1 + \frac{l\xi\pi}{a}\right) \frac{\widetilde{Z}_-}{Z'_-} \cdot \langle (Z'^c, \mathfrak{M}^c\rangle \\ &= \quad \left(1 + \frac{l\xi\pi}{a}\right) \widetilde{Z}_- b \cdot \langle \mathfrak{M}^c, \mathfrak{M}^c\rangle. \end{aligned}$$

It is then clear that $\widetilde{b} = \left(1 + \frac{l\xi\pi}{a}\right) b$. Moreover, $\widetilde{Y} - 1 = \left(1 + \frac{l\xi\pi}{a}\right)(Y-1)$ since $1 + \frac{l\xi\pi}{a}$ is predictable. Comparing Equations 20 and 21, we find that $\widetilde{\mathfrak{A}} = \mathfrak{A} - \left(1 + \frac{l\xi\pi}{a}\right) \cdot \mathfrak{A} = -\int l_t \xi_t \pi_t dt$.

As a result, we find that $W^F + \int l_t \xi_t \pi_t dt = W^F + (\xi \cdot N - \xi \cdot N')$ is a $\widetilde{\mathbb{Q}}$-local martingale. Hence, $W^F + \xi \cdot N$ is a $\widetilde{\mathbb{Q}}$-local martingale.

So far, we have shown that under Condition (i), (ii) or (iii) there exists an equivalent local martingale measure, say $\mathbb{Q}^1$, for the process W stopped at τ_0. By assumption, there exists also an equivalent local martingale measure $\mathbb{Q}^2$ for the process $\bar{W} := (W - W_{\tau_0})\mathbf{1}_{\{\cdot \geq \tau_0\}}$. Clearly, $W = W^{\tau_0} + \bar{W}$, in which W^{τ_0} is the process W stopped at τ_0. Let Z^1 and Z^2 denote the corresponding change of measure processes with respect to $\mathbb{P}$. Define

$$Z_t = Z^1_{t\wedge\tau_0} \frac{Z^2_t}{Z^2_{t\wedge\tau_0}} \quad (0 \leq t \leq \tau)$$

and let $\mathbb{Q}$ be given by the change of measure $Z_\tau = \frac{d\mathbb{Q}}{d\mathbb{P}}$. We start by showing that Z is a $\mathbb{P}$-uniformly integrable martingale, so that $\mathbb{Q}$ is well defined equivalent probability measure. Indeed, for $s \leq t$,

$$\begin{aligned}
\mathbf{E}(Z_t|\mathscr{F}_s) &= \mathbf{E}\left(Z_t\mathbf{1}_{\{\tau_0\leq s\}} + Z_t\mathbf{1}_{\{s<\tau_0<t\}} + Z_t\mathbf{1}_{\{s\leq t\leq\tau_0\}}|\mathscr{F}_s\right) \\
&= \mathbf{E}\left(\frac{Z^1_{\tau_0}}{Z^2_{\tau_0}} Z^2_t \mathbf{1}_{\{\tau_0\leq s\}} + \frac{Z^1_{\tau_0}}{Z^2_{\tau_0}} Z^2_t \mathbf{1}_{\{s<\tau_0<t\}} + Z^1_t \mathbf{1}_{\{s\leq t\leq\tau_0\}}|\mathscr{F}_s\right) \\
&= \frac{Z^1_{\tau_0}}{Z^2_{\tau_0}} \mathbf{1}_{\{\tau_0\leq s\}} \mathbf{E}\left(Z^2_t|\mathscr{F}_s\right) + \mathbf{E}\left(\mathbf{E}\left(\frac{Z^1_{\tau_0}}{Z^2_{\tau_0}} Z^2_t \mathbf{1}_{\{s<\tau_0<t\}} + Z^1_t \mathbf{1}_{\{s\leq t\leq\tau_0\}}|\mathscr{F}_{\tau_0}\right)|\mathscr{F}_s\right) \\
&= \frac{Z^1_{\tau_0}}{Z^2_{\tau_0}} \mathbf{1}_{\{\tau_0\leq s\}} Z^2_s + \mathbf{E}\left(Z^1_{\tau_0}\mathbf{1}_{\{s<\tau_0<t\}} + Z^1_t\mathbf{1}_{\{s\leq t\leq\tau_0\}}|\mathscr{F}_s\right) \\
&= Z_s\mathbf{1}_{\{\tau_0\leq s\}} + \mathbf{E}\left(Z^1_{\tau_0\wedge t}\mathbf{1}_{\{s<\tau_0<t\}} + Z^1_{\tau_0\wedge t}\mathbf{1}_{\{s\leq t\leq\tau_0\}}|\mathscr{F}_s\right) \\
&= Z_s\mathbf{1}_{\{\tau_0\leq s\}} + \mathbf{E}\left(Z^1_{\tau_0\wedge t}\mathbf{1}_{\{s<\tau_0\wedge t\}}|\mathscr{F}_s\right) \\
&= Z_s\mathbf{1}_{\{\tau_0\leq s\}} + Z^1_s\mathbf{1}_{\{s<\tau_0\}} \\
&= Z_s.
\end{aligned}$$

Let $\{\tau^1_n\}_{n\geq 1}$ be a sequence of stopping time converging to ∞ such that $W^{\tau_0\wedge\tau^1_n}$ is a $\mathbb{Q}^1$ martingale for each n. Similarly, let $\{\tau^2_n\}_{n\geq 1}$ be the sequence of stopping times for $\bar{W}$. Let $\tau^3_n = \tau^1_n \wedge \tau^2_n$

$(n \geq 1)$. It suffices to show that $Z_t W_t^{\tau_n^3}$ is a $\mathbb{Q}$ martingale for each n. For $s < t$, $n \geq 1$,

$$\begin{aligned}
\mathbf{E}\left(Z_t W_t^{\tau_n^3} | \mathscr{F}_s\right) &= \mathbf{E}\left(Z_t W_t^{\tau_n^3} \mathbf{1}_{\{\tau_0 \leq s\}} + Z_t W_t^{\tau_n^3} \mathbf{1}_{\{s < \tau_0 < t\}} + Z_t W_t^{\tau_n^3} \mathbf{1}_{\{s \leq t \leq \tau_0\}} | \mathscr{F}_s\right) \\
&= \frac{Z^1_{\tau_0}}{Z^2_{\tau_0}} \mathbf{1}_{\{\tau_0 \leq s\}} \mathbf{E}\left(Z^2_t (W^{\tau_0 \wedge \tau_n^3}_{\tau_0} + \bar{W}^{\tau_n^3}_t) | \mathscr{F}_s\right) \\
&\quad + \mathbf{E}\left(\mathbf{E}\left(\frac{Z^1_{\tau_0}}{Z^2_{\tau_0}} Z^2_t (W^{\tau_0 \wedge \tau_n^3}_{\tau_0} + \bar{W}^{\tau_n^3}_t) \mathbf{1}_{\{s < \tau_0 < t\}} + Z^1_t W_t^{\tau_0 \wedge \tau_n^3} \mathbf{1}_{\{s \leq t \leq \tau_0\}} | \mathscr{F}_{\tau_0}\right) | \mathscr{F}_s\right) \\
&= \frac{Z^1_{\tau_0}}{Z^2_{\tau_0}} \mathbf{1}_{\{\tau_0 \leq s\}} Z^2_s (W^{\tau_0 \wedge \tau_n^3}_{\tau_0} + \bar{W}^{\tau_n^3}_s) \\
&\quad + \mathbf{E}\left(\frac{Z^1_{\tau_0}}{Z^2_{\tau_0}} Z^2_{\tau_0} (W^{\tau_0 \wedge \tau_n^3}_{\tau_0} + \bar{W}^{\tau_n^3}_{\tau_0}) \mathbf{1}_{\{s < \tau_0 < t\}} + Z^1_t W_t^{\tau_0 \wedge \tau_n^3} \mathbf{1}_{\{s \leq t \leq \tau_0\}} | \mathscr{F}_s\right) \\
&= Z_s W_s^{\tau_n^3} \mathbf{1}_{\{\tau_0 \leq s\}} + \mathbf{E}\left(Z^1_{\tau_0 \wedge t} W^{\tau_0 \wedge \tau_n^3}_{\tau_0 \wedge t} \mathbf{1}_{\{s < \tau_0 \wedge t\}} | \mathscr{F}_s\right) \\
&= Z_s W_s^{\tau_n^3} \mathbf{1}_{\{\tau_0 \leq s\}} + Z^1_s W_s^{\tau_n^3} \mathbf{1}_{\{s < \tau_0 \wedge t\}} \\
&= Z_s W_s^{\tau_n^3}.
\end{aligned}$$

■

To understand these sufficient conditions, consider the market wealth process W up to the time of bubble birth, i.e. $W^{\tau_0} = W^{F\,\tau_0} + \xi \cdot N = W^{F\,\tau_0} + \xi \cdot \widetilde{N} + \int \xi_t \pi_t dt$. When $\mathscr{M}(W^F)$ is nonempty, we know that W^F can be transformed into a local martingale with the proper change of measure. This measure change also keeps $\widetilde{N}$ a martingale (as seen in the first part of the proof of the previous theorem). To make W a local martingale, we need to "absorb" the drift $\xi_t \pi_t dt$ into W^F (it cannot be "absorbed" by $\xi \cdot \widetilde{N}$ unless we send the intensity of τ_0 to zero, in which case the resulting measure is no longer equivalent to $\mathbb{P}$). Conditions (i) and (ii) of Theorem 5.2 gives sufficient conditions under which this is possible. Condition (iii) uses the fact that the term $\xi_t \pi_t dt$ is absolutely continuous with respect to the drift term $\mathfrak{A}$ of the process W^F. Here we can adjust the size and the sign of the drift of W^F to make it cancel this term with the proper change of measure. Alternatively, if the drift of W^F is not always nonzero (i.e. the second condition is not satisfied), the first condition can be used and implies that the term $\xi_t \pi_t dt$ can be absorbed by creating a drift for the process W^{F^c} which cancels out this term.

In the example of Section 4, $\langle W^{F^c}, W^{F^c} \rangle = \int \sigma^2 (W_t^F)^2 dt$ and $\mathfrak{A} = \int \mu W_t^F dt$. Hence, Condition (i) is satisfied if $\sigma > 0$, and Condition (ii) is satisfied if $\mu \neq 0$.

The measure $\overline{\mathbb{Q}}$ introduced in the example is neither in $\mathscr{M}(W^F)$ nor in $\mathscr{M}_{\text{loc}}(W)$. However it is equivalent to the measure $\widetilde{\mathbb{P}}$ defined by

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\Big|_{\mathscr{F}_t} = \mathscr{E}(-\widetilde{N})_t \quad (t \leq T).$$

Under $\widetilde{\mathbb{P}}$, $\tau_0 = \infty$ a.s., so $W^F \equiv W$. In particular, if we define $\widetilde{\mathscr{M}}(W^F)$ (resp. $\widetilde{\mathscr{M}}_{\text{loc}}(W)$) as the set of probability measures equivalent to $\widetilde{\mathbb{P}}$ under which W^F (resp. W) is a local martingale then

$\overline{\mathbb{Q}} \in \widetilde{\mathscr{M}}(W^F) = \widetilde{\mathscr{M}}_{\text{loc}}(W)$. Furthermore, the measures in this set are limit points of $\mathscr{M}_{\text{loc}}(W)$ and $\mathscr{M}(W^F)$ in the following sense:

Proposition 5.3 *$\widetilde{\mathscr{M}}(W^F) \neq \emptyset$. Furthermore, if $\mathscr{M}_{loc}(W) \neq \emptyset$, then for any $\widetilde{\mathbb{Q}} \in \widetilde{\mathscr{M}}(W^F)$ there exist sequences $(\mathbb{Q}_n^1)_{n\geq 1} \subset \mathscr{M}(W^F)$ and $(\mathbb{Q}_n^2)_{n\geq 1} \subset \mathscr{M}_{loc}(W)$ such that*

$$\frac{d\mathbb{Q}_n^1}{d\mathbb{P}} \to \frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}} \text{ and } \frac{d\mathbb{Q}_n^2}{d\mathbb{P}} \to \frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}} \text{ a.s.}$$

as $n \to \infty$.

Proof. Let $\mathbb{Q} \in \mathscr{M}(W^F)$ and define $\widetilde{\mathbb{Q}}$ by

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = \mathscr{E}(-\widetilde{N})_\tau.$$

Then, since $[N, W^F] = 0$, $[\mathscr{E}(-\widetilde{N}), W^F] = 0$ and W^F is a $\widetilde{\mathbb{Q}}$-martingale. It is also clear that $\widetilde{\mathbb{Q}}$ is equivalent to $\widetilde{\mathbb{P}}$. Let $\mathbb{Q}' \in \mathscr{M}_{\text{loc}}(W)$, and define $\mathbb{Q}_n^2 = \frac{1}{n}\mathbb{Q}' + (1-\frac{1}{n})\widetilde{\mathbb{Q}}$. Then, since W is both a $\mathbb{Q}'$ and a $\widetilde{\mathbb{Q}}$-local martingale, we deduce that it is also a $\mathbb{Q}_n^2$-local martingale. Furthermore, $\mathbb{Q}_n^2$ is equivalent to $\mathbb{P}$ since $\mathbb{Q}'$ is equivalent to $\mathbb{P}$ and $\widetilde{\mathbb{Q}} \ll \mathbb{P}$. Hence, $\mathbb{Q}_n^2 \in \mathscr{M}_{\text{loc}}(W)$. Clearly, $\frac{d\mathbb{Q}_n^2}{d\mathbb{P}} \to \frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}}$ a.s. when $n \to \infty$. The same construction is used for the sequence $(\mathbb{Q}_n^1)_{n\geq 1} \subset \mathscr{M}(W^F)$. ■

6 Conclusion

This paper provides a new liquidity based model for financial asset price bubbles that explains bubble formation and bubble bursting. The martingale approach to modeling price bubbles assumes that the asset's market price process is exogenous and the fundamental price, the expected future cash flows under a martingale measure, is endogenous. In contrast, we define the asset's fundamental price process exogenously and asset price bubbles are endogenously determined by market trading activity. This enables us to generate a model which explains both bubble formation and bubble bursting. In our model, the quantity impact of trading activity on the fundamental price process - liquidity risk - is what generates price bubbles. We study conditions under which asset price bubbles are consistent with no arbitrage opportunities and we relate our definition of the fundamental price process to the classical definition.

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