*Mathematical Finance*, Vol. 11, No. 4 (October 2001), 447–474

# THE LIQUIDITY DISCOUNT

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This paper characterizes the liquidity discount, the difference between the market value of a trader's position and its value when liquidated. This discount occurs whenever traders face downward sloping demand curves for shares and execution lags in selling shares. This characterization enables one to modify the standard value at risk (VaR) computation to include liquidity risk.

Key Words: liquidity risk, large investors, impulse control, value at risk

## 1. INTRODUCTION

Modern finance theory is based on the competitive market paradigm (see Duffie 1992, Jarrow and Turnbull 1996). The competitive market paradigm has two implicit assumptions. The first is that security markets are perfectly elastic—that is, traders act as price takers. Price takers believe that they can buy and sell as many shares of a security as they wish without changing the price. The second is that all market orders for purchase/sales have immediate execution. Both of these assumptions are approximations, not satisfied even in the most liquid markets.[1] The absence of these conditions is sometimes labeled "liquidity risk."

Given the existence of liquidity risk, the price for one unit of a security (one round lot), called the "market" price, is different from the price received for larger purchases or sales. There is a "quantity" effect on price. The market price, therefore, can be significantly different from the liquidation price. The purpose of this paper is to quantify this difference, called the "liquidity discount."

Liquidity risk has been studied from two different perspectives in the market microstructure literature. The first perspective is that liquidity risk is due to asymmetric (private) information. Large purchases/sales reveal private information, which influences the price paid/received (see Glosten and Milgrom 1985 or Kyle 1985). Trading by informed individuals should reflect this inelastic demand. The transaction price (and the

Helpful comments are gratefully acknowledged from Jesper Christensen.

*Manuscript received September 1998; final revision received March 2001.*

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[1] For empirical investigations indirectly documenting the existence of downward sloping demand curves for security prices, see Scholes (1972), Harris and Gruel (1986), and Shleifer (1986).

bid/ask spread) reflects this liquidity risk. Execution risk is usually not considered in this branch of the literature.

The second perspective is that this liquidity risk is due to there being only a finite number of market participants, who to trade, must be induced to deviate from their optimal portfolio positions. This inducement to trade necessitates a liquidity discount (see Grossman and Miller 1988). In the standard market microstructure model with market makers, this effect is sometimes called the inventory motive (see Garman 1976 for an inventory model of market microstructure).

Of course, in actual markets, both perspectives are correct and both occur simultaneously.[2] Although of considerable conceptual importance, these market microstructure models of liquidity risk are of less practical importance for risk management procedures. This is due, in part, to the simplicity of the existing models and the complexity of solving for an equilibrium price in a more realistic and dynamic model. Consequently, to characterize the liquidity discount for its use in risk management procedures, this paper takes a partial equilibrium perspective.

We consider a trader (firm) facing an inelastic demand curve with execution delays, consistent with the above equilibrium formulations. The trader must liquidate his portfolio in a finite period of time so as to maximize his expected utility of consumption. We characterize the trader's impact on the market price—the liquidity discount—via the solution to this optimal liquidation problem.

First, we show that the trader's optimal liquidation decision problem can be characterized as a stochastic impulse control problem, whose solution is given by a system of associated quasi-variational inequalities (Bensoussan and Lions 1982). For some recent papers involving applications of impulse control to problems in finance, see Bielecki and Pliska (2000), Eastham and Hastings (1998), Korn (1998), Subramanian (2001), and the references cited therein. Two different market structures are considered: (i) a price taker (no price impact or execution delays), and (ii) a trader facing both price impact and execution delays.

For the price taker (no price impact or execution delays), the trader's optimal policy is shown to be a block sale of his entire holdings. For the trader facing price impact and execution delays, we first characterize those conditions under which the trader's optimal liquidation policy is a block sale. A necessary and sufficient condition for this is provided, called the "economies of scale in trading condition." Under this condition, we obtain an explicit analytical expression for the liquidity discount.

We then consider the optimal liquidation problem for the investor in more general settings and demonstrate the robustness of the conclusions drawn. In particular, we investigate the optimal liquidation problem for the investor with a power utility function and general price impact function and prove the existence of optimal policies and we derive an analytical characterization of the optimal policy under constant price impact and execution lag functions.

This characterization of the liquidity discount provides a simple and convenient technique for including liquidity risk into the value at risk (VaR) computations now standard in risk management procedures. This has been discussed in detail in Jarrow and Subramanian (1998).

An outline of this paper is as follows. Section 2 presents the model, Section 3 studies the optimal liquidation problem for a price taker, and Section 4 provides the solution with liquidity risk. In Section 5 the model is generalized to more realistic settings, and

[2] For an empirical investigation showing a decomposition of the bid/ask spread into its various components, see Huang and Stoll (1997).

in Section 6 a detailed investigation of the optimal liquidation problem for a trader with a power utility function is carried out. Section 7 concludes the paper.

## 2. THE MODEL

This section presents the details of the model. We consider a continuous time model with trading interval $[0, T]$, probability space $(\Omega, \mathcal{F}, Q)$, and a standard filtration $\{\mathcal{F}_t : t \in [0, \infty)\}$ generated by a single Brownian motion $\{W(t) : t \in [0, \infty)\}$ with $\mathcal{F}_\infty = \mathcal{F}$.

There is a single trader (firm) who enters at time 0 with an endowment of $S$ shares of a risky stock. This investor is "large" in the sense of Jarrow (1992); that is, his sales/purchases affect the price of the risky stock. This quantity impact could be due to either information or limited demands/supplies for the security. There are no transaction costs.

The "market price" for the risky stock is defined to be the quote for a unit purchase or sale. In an economic sense, it corresponds to the price of an infinitesimal purchase (the price a price taker would pay). Let $p(t)$ denote the market price. We assume that between trades $p(t)$ follows a geometric Brownian motion process; that is,

$$dp(t) = p(t)(\mu dt + \sigma dW(t)), \tag{2.1}$$

where $\mu$ and $\sigma > 0$ are constants.

To begin, we assume that the trader is risk neutral—his utility function is linear. This assumption is relaxed in Sections 5 and 6 to a risk-averse trader with a power utility function.

The problem faced by our trader is that he must liquidate his position of $S$ shares in the risky stock by time $T$. In this liquidation, he encounters a quantity impact on the price he receives and a time lag in the execution of his orders. Both the quantity impact and the execution lag are taken as exogenous, consistent with the partial equilibrium formulation of our problem.

The trader is allowed to liquidate his $S$ shares according to an "admissible" trading strategy. An admissible trading strategy is a collection $\{(t_i, s_i)\}$ where each $t_i$ is a bounded stopping time of the filtration $F_t$ and $s_i$ is measurable with respect to $F_{t_i}$ with

$$\begin{aligned} &s_i > 0 \text{ for all } i,\\ &t_{i+1} - t_i \geq \Delta(s_i),\\ &\lim_{i\to\infty} t_i > T \text{a.s.,} \quad \text{and} \quad \sum_i s_i 1_{\{t_i \leq T\}} \leq S. \end{aligned} \tag{2.2}$$

The quantity $\Delta(s_i)$ represents the execution lag in selling $s_i$ shares. We assume that $\Delta(s_i)$ is a deterministic, nondecreasing function in $s_i$ with $\Delta(0) \geq 0$ and $\Delta(s) < T$ for all $s \geq 0$. The nondecreasing condition implies that larger sales take more time to execute, everything else constant.

We assume that the investor is not allowed to place additional orders during the time it takes to execute an earlier one. This condition is imposed for technical reasons. In the absence of this condition, optimal policies will not always exist, and if they do, it is unlikely that they will be Markov; so that dynamic programming techniques will not apply. The consequences of relaxing this assumption are the subject of ongoing research.

To include the “quantity impact” on price, we assume that an order to sell $s_i$ shares of the stock at time $t_i$ changes the market price to

$$P\left(t_i^+\right) = c\left(s_i\right) P\left(t_i\right), \tag{2.3}$$

where $c(s_i) \leq 1$ and $t_i^+ = \lim_{\varepsilon \to 0}(t_i + \varepsilon)$.

The proportionality factor $c\left(s_i\right)$ represents the price discount due to the $s_i$ shares sold. We assume that $c\left(s_i\right)$ is a deterministic, nonnegative, nonincreasing function in $s_i$ with $c(0) = 1$. The fact that $c\left(s_i\right)$ is less than or equal to one identifies it as a negative quantity impact. The nonincreasing condition implies that larger sales generate a larger quantity impact, everything else constant. This form of the price impact function is consistent with both the asymmetric information and inventory motives in the market microstructure literature (see Garman 1976, Glosten and Milgrom 1985, Kyle 1985). We emphasize that the price impact is cumulative.

The assumption of a cumulative price impact is rather strong. An alternative structure would allow the price process to return to some “normal level” after a time delay depending on the size of the order.[3] An alternative model that incorporates this feature is the subject of ongoing research. It is interesting to observe that, in the presence of a positive drift, the assumption of an increasing execution lag $\Delta(s)$ partially allows the price process to recover to a normal level before the trade is executed.

The restriction that $c(s)$ is a deterministic function of $s$ is imposed for simplicity. It is subsequently generalized to include a dependence on time with $c(t, s)$ nonincreasing in time $t$. This implies that the longer the time $t$ to execution, the larger the price discount.

The trader’s problem can now be written. It is to choose an admissible trading strategy $\left\{(t_i, s_i)\right\}$ to maximize

$$E_{p,S,0}\left[\sum_i s_i P(t_i + \Delta(s_i)) 1_{\{t_i \leq T\}}\right], \tag{2.4a}$$

where

$$P\left(t_i+\Delta\left(s_i\right)\right)=P\left(t_i\right)c(s_i)\exp\left\{\left(\mu-\frac{1}{2}\sigma^2\right)\Delta\left(s_i\right)+\sigma\left(W\left(t_i+\Delta\left(s_i\right)\right)-W\left(t_i\right)\right)\right\} \tag{2.4b}$$

and $E_{p,S,0}(\cdot)$ is the expectation conditioned at time 0 on $P(0) = p$ with $s = S$ shares remaining to liquidate.

Note that if a trade $s_n$ is made at time $t_n = T$, then the trade is executed at time $T + \Delta(s_n)$ and there is no additional utility obtained from making a trade at time $t > T$. For later use, we compute the expected market price from selling $s_i$ shares at time $t_i + \Delta(s_i)$:

$$E_{p,S,t_i}\left(P\left(t_i + \Delta\left(s_i\right)\right)\right) = pc(s_i)\exp\left(\mu\Delta\left(s_i\right)\right), \tag{2.5}$$

where $P(t_i) = p$. This expression will enable us to more easily interpret subsequent results.

Formally, expression (2.4) describes an impulse control problem that we solve in the next section. Prior to this solution, however, we add one additional assumption. To exclude market manipulation strategies in the sense of Jarrow (1992), we assume the following condition.

[3] We thank an anonymous referee for emphasizing this point.

*Condition* 2.1 (No Manipulation: Risk-Neutral Case).

$$c(s)\exp(\mu\Delta(s)) \leq 1 \qquad \text{for all } 0 \leq s \leq S.$$

*As seen later, the contradiction of this condition implies that immediate sales would generate increased value, violating the notion of a negative quantity impact of sales on the market price. The generalization of this condition for a risk-averse investor will be provided in a later section.*

Throughout this paper, we shall also assume the following condition.

*Condition* 2.2 (No Satiation: Risk-Neutral Case).

$$sc(s)\exp(\mu\Delta(s)) \text{ is an increasing function for } s \geq 0.$$

*This condition expresses the natural requirement that if the liquidation time horizon is zero; that is, if the investor has to liquidate all his holdings immediately, it is optimal for the investor to liquidate all his holdings. This condition has a generalization to the risk-averse case.*

## 3. SOLUTION FOR A PRICE TAKER

To understand the general solution to the trader's liquidation problem, we first study the solution for a price taker where there is no liquidity risk. A price taker with no liquidity risk solves the problem given in expression (2.4) with $\Delta(s) = 0$ and $c(s) = 1$ for all $s$. That is, the trader chooses an admissible trading policy $\{(t_i, s_i)\}$ to maximize

$$E_{p,S,0}\left[\sum_i s_i P(t_i) 1_{\{t_i \leq T\}}\right] \tag{3.1}$$

where

$$P(t_i) = P(0)\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t_i + \sigma\left(W(t_i) - W(0)\right)\right\}.$$

It is possible to show that an optimal trading policy for the price taker is a block sale.

PROPOSITION 3.1 (Block Sale for a Price Taker). *Given price taking behavior and no liquidity risk* ($\Delta(s) = 0, c(s) = 1$ *for all* $s$), *the optimal trading strategy is*

*if* $\mu \leq 0$, *immediate liquidation* ($t_1 = 0, s_1 = S$), *and*
*if* $\mu > 0$, *terminal date liquidation* ($t_1 = T, s_1 = S$).

*Proof.* The proof is in the Appendix and uses the results of Propositions 4.1 and 4.1b which follow in the next section.

Let $\bar{u}(p, S, 0)$ represent the expected proceeds to the price taker from liquidation under the optimal trading strategy. Proposition 3.1 implies the following corollary.

COROLLARY 3.1 (Expected Proceeds from Liquidation to the Price Taker).

$$\bar{u}(p, S, 0) = \begin{cases} Sp & \text{if } \mu \leq 0 \\ E_{p,S,0}\left(SP(T)\right) = Spe^{\mu T} & \text{if } \mu > 0. \end{cases} \tag{3.2}$$

If $\mu \leq 0$, immediate liquidation yields the current market value of the position. If $\mu > 0$, then the expected proceeds equal the current market price times a proportionality factor that includes appreciation at the rate $\mu$.

## 4. SOLUTION WITH LIQUIDITY RISK

This section characterizes the optimal liquidation policy for expression (2.4) given there is liquidity risk. In this section, we shall assume that $\Delta(0) = 0$. All detailed proofs are contained in the Appendix. Let $u(p, s, t)$ be a bounded positive real-valued function on $R_+ \times R_+ \times [0, \infty)$ with $u(p, s, t) \equiv 0$ on $R_+ \times R_+ \times (T, \infty)$. We denote the set of all such functions by $D$.

Define the operator $M$: $D \to D$ by

$$(4.1)\qquad Mu(p, s, t) = \sup_{s', 0<s'\le s} \big\{ s'c(s') \exp(\mu\Delta(s'))p + E_{p,s,t}[u(P(t+\Delta(s')), s-s', t+\Delta(s'))] \big\}$$

for $t \in [0, T)$ and $Mu(p, s, t) \equiv 0$ on $R_+ \times R_+ \times (T, \infty)$. Note that in expression (4.1)

$$P(t+\Delta(s')) = pc(s') \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)\Delta(s') + \sigma(W(t+\Delta(s')) - W(t))\right].$$

Consider the system of quasi-variational inequalities:

$$(4.2)\qquad \begin{aligned} & u \in D;\ u \ge Mu, \\ & Lu \le 0 \text{ on } R_+ \times R_+ \times [0, T), \\ & (u - Mu)Lu = 0 \text{ on } R_+ \times R_+ \times [0, T), \end{aligned}$$

where $u(p, s, T) = sc(s)p\exp[\mu\Delta(s)]$ and $Lu \equiv u_t + \mu p u_p + \left(\frac{1}{2}\right)\sigma^2 p^2 u_{pp}$. Here $L$ is the infinitesimal generator of the price process $p(\cdot)$. The subscripts on $u_t, u_p, u_{pp}$ indicate partial derivatives.

Let $u$ be a solution to these quasi-variational inequalities. Define an impulse control policy for this function $u$ recursively as follows: Let $S > 0$ be the initial number of shares held by the investor. Set $t_0 = 0$, let $t_1 = \inf\{t \ge t_0 : u(P(t), S, t) = Mu(P(t), S, t)\}$ provided the maximum in the definition of $Mu$ is attained at some $s_1 > 0$.

We note that $t_1 \le T$ since

$$(4.3)\qquad u(P(T), S, T) = Mu(P(T), S, T) = P(T)Sc(S)\exp(\mu\Delta(S)).$$

Then, given $(t_{i-1}, s_{i-1})$, if $t_{i-1} + \Delta(s_{i-1}) \le T$ let

$$(4.4)\qquad t_i = \inf\{t \ge t_{i-1} + \Delta(s_{i-1}) \colon u(P(t), S(t_{i-1}), t) = Mu(P(t), S(t_{i-1}), t)\},$$

where $S(t_i) = S - \sum_{j=1}^{i-1} s_j > 0$ and the maximum in the definition of $Mu$ is attained at some $s_i > 0$, and if $t_{i-1} + \Delta(s_{i-1}) > T$ or $S(t_i) = 0$, let $t_i = t_{i-1} + \Delta(s_{i-1})$ and $s_i = 0$. Again, note that if $t_{i-1} + \Delta(s_{i-1}) \le T$, then $t_i \le T$ since

$$u(P(T), S(t_{i-1}), T) = Mu(P(T), S(t_{i-1}), T).$$

We can now prove the following proposition.

PROPOSITION 4.1a (Characterization of the Solution). *If $u \in D$ is a solution to the quasi-variational inequalities (4.2) where $u$ is twice continuously differentiable on $R_+ \times R_+ \times [0, T)$ and satisfies Dynkin's formula (see, e.g., Oksendal 1998) and if there exists an admissible trading policy whose value function is $u$, then this policy is optimal and $u$ is the optimal value function to expression (2.4). In particular, if $u$ generates an admissible control policy by the construction above, then this control policy is an optimal policy.*

In this paper, we also use the following proposition.

PROPOSITION 4.1b (Verification of a Solution). *If $u \in D$ is a function on $R_+ \times R_+ \times [0, T)$, left continuous at $t = T$, satisfying*

$$u \geq Mu \tag{4.5}$$
$$E[u(p(\tau_2), s, \tau_2) - u(p(\tau_1), s, \tau_1)] \leq 0,$$
$$u(p, s, T) = psc(s)\exp(\mu\Delta(s)),$$

*where $\tau_1 \leq \tau_2 \leq T$ are bounded stopping times and $u'$ is the value function of any admissible impulse control policy, then $u \geq u'$. In particular, $u \geq v$, where $v$ is the value function of the impulse control problem.*

We now investigate conditions under which the optimal policy for this trader is identical to that of the price taker—a block sale. The next proposition shows that a necessary and sufficient condition for this is the following condition.

*Condition* 4.1 (Economies of Scale in Trading).

$$\begin{aligned} &sc(s) + (S-s)c(S-s)c(s)\exp[\mu\Delta(S-s)] \\ &\quad \leq Sc(S)\exp[\mu\Delta(S)] \quad \text{for } 0 \leq s \leq S \text{ and } \mu \geq 0; \end{aligned} \tag{4.6}$$

$$\begin{aligned} &sc(s)\exp[\mu\Delta(s)] + (S-s)c(S-s)c(s)\exp[\mu\Delta(S-s)]\exp[\mu\Delta(s)] \\ &\quad \leq Sc(S)\exp[\mu\Delta(S)] \quad \text{for } 0 \leq s \leq S \text{ and } \mu < 0. \end{aligned}$$

*The right-hand side of expression (4.6) represents the price discount from liquidating all $S$ shares at a single point in time. The left-hand side represents a portion of the cumulative price discount from transacting two sales, one equal to $s$ (the first term) and the other equal to $S-s$ (the second term). Surprisingly, however, the term depending on the execution lag appears only in the second term (when $\mu > 0$), making the condition far from intuitively obvious.*

Given this economies of scale in trading condition, the following proposition holds.

PROPOSITION 4.2 (Block Liquidation Given Liquidity Risk). *A single block sale is optimal if and only if the economies of scale in trading condition holds. If $\mu > 0$, it is optimal to liquidate all holdings at the terminal date and if $\mu \leq 0$, it is optimal to liquidate all holdings immediately.*

Proposition 4.2 states that a single trade (at the terminal date if $\mu > 0$ or immediately if $\mu \leq 0$) is optimal if and only if the economies of scale in trading condition holds. The surprising aspect of Proposition 4.2 is that the economies of scale in trading condition is both necessary and sufficient.

Under the economies of scale in trading condition, the optimal liquidation policy is identical to that for the price taker. However, the expected proceeds differ, as seen in the following corollary.

COROLLARY 4.1 (Expected Proceeds from Liquidation Given Liquidity Risk). *Given the economies of scale in trading condition* (4.6),

$$u(p, S, 0) = \begin{cases} E_{p,S,0}\big(SP(\Delta(S))\big) = Spc(S)e^{\mu\Delta(S)} & \text{if } \mu \le 0 \\ E_{p,S,0}\big(SP(T+\Delta(S))\big) = Spc(S)e^{\mu T+\mu\Delta(S)} & \text{if } \mu > 0. \end{cases} \tag{4.7}$$

*Expression* (4.7) *represents the* (*present value*) *of the proceeds from liquidation. The expected proceeds, due to the no manipulation condition, are always less than those that the price taker receives. It differs by the quantity* ($c(S)e^{\mu\Delta(S)} \le 1$).

We next define the per share *liquidation value*. The per share liquidation value is the market price, $P^*$, that gives a hypothetical price taker who can execute immediately with no price impact the same expected proceeds as a trader facing liquidity risk; that is, $P^*$ such that

$$\bar{u}\big(P^*, S, 0\big) = u(p, S, 0), \tag{4.8}$$

where $\bar{u}$ is the solution to the price taker's problem given in expression (3.1).

Given the economies of scale in trading condition (4.6) (so that a block sale is optimal), we have

$$P^* = pc(S)e^{\mu\Delta(S)} \le p. \tag{4.9}$$

The liquidation price is seen to be equal to the market price ($p$) times the liquidation factor ($c(S)e^{\mu\Delta(S)} \le 1$).

Using the liquidation value, we define the *liquidity discount* as

$$L(p, s) = p - P^*. \tag{4.10}$$

The liquidity discount is the difference between the market price and the liquidation value. It measures the amount of illiquidity in a market since, in the absence of either price impact or execution lags, $p = P^*$ and $L(p, s) = 0$. Furthermore, under the economies of scale in trading condition, expression (4.9) implies that

$$L(p, s) = p(1 - c(s)e^{\mu\Delta(s)}). \tag{4.11}$$

This characterizes the liquidity discount in a market where block trades represent the optimal liquidation policy.

For applications, we provide some examples of price impact ($c(s)$) and execution lag functions ($\Delta(s)$) satisfying the economies of scale in trading condition. We assume that $\mu \ge 0$ in the examples below.

EXAMPLE 4.1 (Constant Execution Lag and No Price Impact). This example has no price impact ($c(s) = 1$), but a constant execution lag ($\Delta(0) = 0$ and $\Delta(s) = \Delta > 0$ for $s > 0$). It is easy to verify that the economies of scale in trading condition is satisfied.

EXAMPLE 4.2 (Arbitrary Execution Lag and Restricted Price Impact). Let the execution lag function ($\Delta(s)$) be any increasing function in $s$ with $\Delta(0) = 0$. Let the price impact function ($c(s)$) satisfy

$$s'c(s') + (s - s')c(s - s')c(s') \le sc(s) \quad \text{for } 0 \le s' \le s. \tag{4.12}$$

For example, if the price impact is a constant for nonzero trades ($c(0) = 1$ and $c(s) = c < 1$ for $s > 0$), then this condition is satisfied.

*Proof*. Since $\Delta(s)$ is increasing,

$$\begin{aligned} s'c(s') &+ (s-s')c(s-s')c(s')\exp(\mu\Delta(s-s')) \\ &\leq s'c(s')\exp(\mu\Delta(s)) + (s-s')c(s-s')c(s')\exp(\mu\Delta(s)) \\ &\leq sc(s)\exp(\mu\Delta(s)), \end{aligned}$$

by the condition on $c(s)$. □

The next section of the paper considers generalizations of the model considered in this section to more realistic settings.

## 5. GENERALIZATIONS

In this section, we discuss generalizations to the previous economy, all of which maintain the previous conclusions.

### 5.1. Jump Diffusion Processes

This section generalizes the risky stock price process in expression (2.1) to include jumps. This generalization is especially useful where jumps include the probability of market crashes. Let the market price between trades satisfy

$$dP(t) = \mu P(t)dt + \sigma P(t)dW(t) + \int_R N(dt, dy)yP(t-), \tag{5.1}$$

where $N$ is a Poisson random measure on $R^+ \times R$ with jump rate $\lambda$ and jump size distributed $q$ on $[-1, 1]$.

Here, the market price evolves as a geometric Brownian motion except at the jump time $T_i$ of a Poisson process of rate $\lambda$. When jumps occur, the market price jumps by $Y_i P(T_i-)$ where $Y_i$ is an independent, identically distributed random variable with distribution $q$ on $[-1, 1]$.

Defining $v(dt, dy) = N(dt, dy) - \lambda dtq(dy)$, expression (5.1) can be rewritten as

$$dP(t) = (\mu + \lambda E(Y))P(t)dt + \sigma P(t)dW(t) + \int_R v(dt, dy)yP(t-). \tag{5.2}$$

Writing $\mu' \equiv \mu + \lambda E(Y)$, and defining the infinitesimal generator

$$\begin{aligned} Lu(p,s,t) = u_t + \mu p u_p + \left(\frac{1}{2}\right)\sigma^2 p^2 u_{pp} \\ + \lambda \int_R [u((1+y)p, s, t) - u(p, s, t)]q(dy), \end{aligned}$$

the results of Section 4 apply with the drift $\mu' = \mu + \lambda E(Y)$ replacing $\mu$.

## 5.2. Time Dependent Price Impact Functions

This section generalizes the model of Section 2 by letting $c(s)$ be a function of $t$. For this section, we suppose that there is no execution lag; that is, $\Delta(s) = 0$. Let

$$
\begin{aligned}
(5.3) \qquad & c(s,t) = \beta(t)\bar{c}(s) \quad \text{for } s > 0 \text{ with } c(0,t) \equiv 1, \\
& s'\bar{c}(s') + (s - s')\bar{c}(s - s')\bar{c}(s') \leq s\bar{c}(s),
\end{aligned}
$$

and with $\beta(t)$ a continuously differentiable, decreasing function initialized at $\beta(0) = 1$. Given the above, the following proposition holds.

PROPOSITION 5.1 (Block Liquidation). *If there exists a* $t_0 \in [0, T]$ *such that*

$$
\begin{aligned}
\beta(T)\exp(\mu(T-t)) &\geq 1 \quad \text{for } t \leq t_0, \\
\beta(T)\exp(\mu(T-t)) &< 1 \quad \text{for } t > t_0,
\end{aligned}
$$

*and*

$$
\frac{d\beta(t)}{dt} \leq -\mu \quad \text{for } t > t_1,
$$

*where* $t_1 = \inf\{t' \in [0,t]\colon \beta(T)\exp(\mu(T-t')) < \beta(t')\}$, *then the optimal liquidation policy involves at most one transaction in* $[0, T]$, *and can be described as follows:*

*if* $t < t_1$, *the investor liquidates all of his holdings at the terminal date, or*
*if* $t \geq t_1$, *the investor liquidates all of his holdings immediately.*

Letting $\beta(t) \equiv \exp(-vt)$ for a constant $v$, the hypothesis of Proposition 5.1 is satisfied with $t_0 = 0$ for $\mu < v$, and $t_0 = T$ for $\mu \geq v$. Then, the results of Sections 3 and 4 apply with $\mu$ replaced by $(\mu - v)$. Here the "appreciation" in the market price $(\mu)$ is reduced by the "depreciation" from waiting to sell $(v)$. This characterization is most useful in asymmetric information models where $\mu$ is determined by the "market belief" and $v$ is determined by negative "private information" leaked into the price process by other informed traders. Here, waiting to sell has a cost as the negative information is leaked into the market price.

## 5.3. Power Utility Functions

This section generalizes the model of Section 2 by allowing the trader to be risk averse with a power utility function of the form $U(x) = x^\gamma$ for $\gamma \in R^+$. This section requires the most significant modification to the previous notation.

The objective function in expression (2.4) is replaced by

$$
E_{p,S,0}\left[\sum_i U(s_i P(t_i + \Delta(s_i))) 1_{\{t_i \leq T\}}\right],
$$

which represents the expected utility of consumption for the investor.

The operator $M$ in expression (4.1) is replaced by

$$
\begin{aligned}
Mu = \sup_{0 < s' \leq s} \big\{ & p^\gamma (s')^\gamma c(s')^\gamma \exp(\alpha(\Delta(s'))) \\
& + E_{p,s,t}[u(P(t + \Delta(s')), s - s', t + \Delta(s'))] \big\}
\end{aligned}
$$

where $\alpha = (\sigma^2\gamma^2 + (2\mu - \sigma^2)\gamma)/2$.

The economies of scale in trading condition is replaced by

$$s^\gamma c(s)^\gamma + (S-s)^\gamma c(S-s)^\gamma c(S)^\gamma \exp(\alpha\Delta(S-s)) \\ \leq S^\gamma c(S)^\gamma \exp(\alpha\Delta(S)) \text{ for } \alpha \geq 0,$$

with a condition analogous to the second part of (4.6) for $\alpha < 0$.

The no-manipulation and no-satiation conditions are generalized as follows.

*Condition* 5.1 (No Manipulation: Risk-Averse Case).

$$c(s)^\gamma \exp[\alpha\Delta(s)] \leq 1.$$

*Condition* 5.2 (No Satiation: Risk-Averse Case).

$$s^\gamma c(s)^\gamma \exp(\alpha\Delta(s)) \textit{ is an increasing function on } s \geq 0.$$

Given the above, the results of Section 4 apply with $\mu$ replaced by $\gamma[2\mu+\sigma^2(\gamma-1)]/2$. For the details, see the Appendix. In addition, the following proposition can also be proved.

PROPOSITION 5.2 (Risk Aversion). *If the functions $c(s)$ and $\Delta(s)$ satisfy the economies of scale in trading condition for some $\gamma$, then they satisfy the economies of scale in trading condition for any $\gamma' > \gamma$. Thus, if it is optimal for an investor with risk aversion $\gamma$ to liquidate all of his holdings at one time, then it is optimal for an investor with greater risk aversion $\gamma'$ to do the same.*

The implication is that if the results of Section 4 apply for a risk-neutral trader, then they apply for any more risk-averse trader of the power utility type. Combined, these generalizations demonstrate the robustness of the conclusions given in Section 4.

The economies of scale in trading condition for a strictly risk-averse investor is very strong and, in particular, is not satisfied for a price taker ($c(s) \equiv 1$, $\Delta(s) \equiv 0$), or in the situation where the price impact is constant on $s > 0$ and $\Delta(s) \equiv \Delta > 0$. This implies that a block sale is not optimal in either of these two cases.[4] In fact, the value function does not even exist for a strictly risk-averse price-taker.

EXAMPLE 5.1 (Nonexistence of Value Function for a Price-Taker, i.e. $c(s) \equiv 1$ and $\Delta(s) \equiv 0$). For convenience of exposition, let us assume that $\alpha = 0$ and the price taker's initial holding is one share and the initial stock price is 1. The price taker's goal is to maximize

$$E_{p,S,0}\left[\sum_i U(s_i P(t_i))1_{\{t_i \leq T\}}\right].$$

For each positive integer $N$, consider the following admissible strategy: Let the set of trading dates be $[T/N, 2T/N, \ldots, T]$, and at each trading date the price taker liquidates $1/N$ shares of the stock. It is easy to see that the expected utility obtained from this trading policy is

$$U_N = N \cdot (1/N^\gamma) = N^{1-\gamma}.$$

If $\gamma < 1$ (the investor is strictly risk-averse), then

$$\lim_{N\to\infty} U_N = \infty.$$

Thus, the value function over the class of admissible trading strategies does not exist. This is a consequence of the fact that the time interval between successive trades for a

[4] We thank an anonymous referee for emphasizing this point.

price taker can be arbitrarily small so that there is no upper bound on the number of transactions over the time horizon.

In the presence of execution lags, the argument may fail. For example, if the execution lag for nonzero trades is bounded below by a strictly positive constant, since the number of allowed trades over the time horizon would be finite, the value function may exist.

## 6. OPTIMAL LIQUIDATION POLICIES FOR POWER UTILITY FUNCTIONS

In this section, we carry out a detailed investigation of the optimal liquidation problem for a large investor with a general power utility function. We consider the situation where the execution lag is a positive constant for nonzero trades; that is,

$$\Delta(s) \equiv \Delta > 0, \quad \text{for all } s > 0,$$

the price impact function $c(s)$ is general, and the time horizon $T = N\Delta$, for some $N < \infty$.

Let the investor's utility function be given by

$$U(x) = x^{\gamma}, \qquad 0 < \gamma < 1.$$

We can now state the following propositions.

PROPOSITION 6.1 (Existence of Optimal Policies). *If the price impact function $c(s)$ is continuous on $s > 0$, an optimal policy for the investor exists and is deterministic; that is, it is completely determined at the initial time.*

We shall now derive explicit characterizations of the optimal policy when the price impact function is given by

$$c(s) \equiv c < 1 \qquad \text{for } s > 0.$$

Let us first define the set Y:

$$\mathrm{Y} = \{0, \Delta, 2\Delta, \ldots, N\Delta\}.$$

PROPOSITION 6.2 (Characterization of Optimal Policy for $\alpha \leq 0$). *If $\alpha \leq 0$, the optimal policy involves liquidating a nonzero fraction of the initial number of shares at each date in the set* Y.

PROPOSITION 6.3 (Characterization of Optimal Policy for $\alpha > 0$). *If $\alpha > 0$, there exists an integer $N' \geq 0$, such that if $N_0 = \inf(N, N')$, the investor's optimal policy is to liquidate a nonzero fraction of his holdings at the last $N_0$ dates in the set* Y. *In particular, if $N \leq N'$, the optimal policy is to liquidate a nonzero fraction of the initial number of shares at each date in the set* Y.

The above propositions apply when there is a fixed execution lag $\Delta$. If $\alpha \leq 0$, then the investor's optimal policy is to liquidate a nonzero fraction of his holdings at each date in the set Y for any $N$ independent of the index of risk aversion $\gamma$ of the investor (provided it is strictly less than one). If $\alpha > 0$, then there exists an integer $N'$ independent of $N$ and depending only on $\Delta$, the market parameters, and the index of risk aversion of the investor such that the investor's optimal policy is to liquidate a nonzero fraction of his holdings at the last $\inf(N, N')$ dates in the set Y. These propositions thus characterize the investor's optimal liquidation policy. The exact fractions liquidated at each date are given by the analytical expressions for the optimal proportions obtained in proofs of the two propositions.[5]

## 7. SUMMARY

This paper studies the problem faced by a trader who must liquidate his portfolio, given that his sales reduce the market price and that there are execution lags in processing market orders. The optimal liquidation value of the portfolio is characterized in terms of the solution to the above problem and is compared to the market price. The difference is a measure of the liquidity risk in the market and is called the liquidity discount.

The liquidity discount is characterized analytically in the presence of an economies of scale in trading condition. In the absence of the economies of scale in trading condition, an investigation of the optimal liquidation problem for an investor with a power utility function given a constant execution lag and arbitrary price impact function is studied. A complete characterization of the value function is provided and the existence of optimal policies given continuity conditions on the price impact function is proven. An explicit derivation of the optimal policies for a constant price impact function is derived.

This characterization of the liquidity discount enables one to modify standard value at risk (VaR) computations to include liquidity risk. This has been discussed in Jarrow and Subramanian (1997).

## APPENDIX

*Proof of Proposition* 3.1. Suppose $\{(t_i, s_i)\}$ is an *admissible* trading policy for the investor, so that $s_i > 0$, $s_i$ is $F_{t_i}$-measurable, $\lim_{i\to\infty} t_i > T$ a.s., and $\sum_i s_i 1_{\{t_i \le T\}} \le S$.

First suppose that $\mu > 0$. We notice that

$$\begin{aligned} E_{p,S,0}[s_i 1_{\{t_i\le T\}} P(T)] &= E_{p,S,0}[E[s_i 1_{\{t_i\le T\}} P(T)|F_{t_i}]] \\ &= E_{p,S,0}[s_i 1_{\{t_i\le T\}} E[P(T)|F_{t_i}]] \\ &= E_{p,S,0}[s_i 1_{\{t_i\le T\}} P(t_i)\exp(\mu(T-t_i))] \\ &\ge E_{p,S,0}[s_i 1_{\{t_i\le T\}} P(t_i)], \end{aligned}$$

where we have made use of the strong Markov property of geometric Brownian motion and the optional sampling theorem. Therefore,

$$\sum_i E_{p,S,0}[s_i 1_{\{t_i\le T\}} P(t_i)] \le \sum_i E_{p,S,0}[s_i 1_{\{t_i\le T\}} P(T)].$$

It follows that the investor's optimal policy is to liquidate all his holdings at the terminal date $T$.

[5] It is our conjecture (we thank Stanley Pliska for emphasizing this point) that the optimal policies exist and are deterministic for a general deterministic execution lag $\Delta(s)$ provided $\Delta(s) \ge \Delta > 0$ for $s > 0$ for some $\Delta > 0$. The proof of this conjecture is the subject of ongoing research.

Suppose $\mu < 0$. In this case we use the technique of impulse control to prove that it is optimal for the investor to liquidate all his holdings immediately.

We need to show that $u(p, s, t) = ps$ is the value function of the investor. We shall show that $u$ satisfies the quasi-variational inequalities associated with the impulse control problem and then apply the result of Proposition 4.1. It is easy to see that

$$Mu(p, s, t) = \sup_{0<s'\leq s} [ps' + u(p, s - s', t)] = ps = u(p, s, t)$$

and

$$Lu(p, s, t) = \mu p u_p = \mu ps < 0.$$

Therefore, $u$ satisfies the quasi-variational inequalities associated with the impulse control problem. Further, the admissible control policy that generates $u$ is clearly immediate liquidation of all holdings. By the result of Proposition 4.1a, $u$ is therefore the value function for the investor.

If $\mu = 0$, it is easy to see that the investor is indifferent to liquidating his holdings at any time in $[0, T]$. This completes the proof. □

*Proof of Proposition* 4.1a. Suppose $\{(t_n, s_n)\}$ is an *admissible* trading policy for the investor, so that $s_n > 0$, $t_{n+1} - t_n \geq \Delta(s_n)$, $\lim_{i\to\infty} t_n > T$ a.s., and $\sum_n s_n 1_{\{t_n\leq T\}} \leq S$. Let $s(t)$ be the share process for the investor. We now have

$$\begin{aligned}
\text{(A.1)}\quad & u(P(t_i), s(t_i), t_i) - u(P(0), s(0), 0) \\
&= u(P(t_i), s(t_i), t_i) - u(P(t_{i-1} + \Delta(s_{i-1})), s(t_i), t_{i-1} + \Delta(s_{i-1})) \\
&\quad + u(P(t_{i-1} + \Delta(s_{i-1})), s(t_i), t_{i-1} + \Delta(s_{i-1})) \\
&\quad - u(P(t_{i-1}), s(t_{i-1}), t_{i-1}) + \cdots + u(P(t_1), s(0), t_1) - u(P(0), s(0), 0).
\end{aligned}$$

We shall first show that if $\tau_1 \leq \tau_2$ are stopping times, then

$$E[u(P(\tau_2), s, \tau_2) - u(P(\tau_1), s, \tau_1)] \leq 0.$$

We first notice that

$$u(P(\tau_2), s, \tau_2) - u(P(\tau_1), s, \tau_1) \leq u(P(\tau_2 \wedge T), s, \tau_2 \wedge T) - u(P(\tau_1 \wedge T), s, \tau_1 \wedge T),$$

since $u(P(t), s, t) \equiv 0$ for $t > T$. Hence, it suffices to prove the result for $\tau_1 \leq \tau_2 \leq T$.

We obtain the result by realizing that

$$\begin{aligned}
&E_{\{P(\tau_1), s(\tau_1), \tau_1\}}[u(P(\tau_2), s, \tau_2) - u(P(\tau_1), s, \tau_1)] \\
&\quad = E_{\{P(\tau_1), s(\tau_1), \tau_1\}} \int_{\tau_1}^{\tau_2} Lu(P(r), s, r)\, dr \leq 0,
\end{aligned}$$

since $Lu \leq 0$ for $t \in [0, T)$.

By the above result, we see that

$$E_{\{P(t_{j-1}+\Delta(s_{j-1})),s(t_j),t_{j-1}+\Delta(s_{j-1}))\}}[u(P(t_j),s(t_j),t_j)\\ -u(P(t_{j-1}+\Delta(s_{j-1})),s(t_j),t_{j-1}+\Delta(s_{j-1}))]\le 0.$$

We now notice that

$$\begin{aligned}&E_{\{P(t_{j-1}),s(t_{j-1}),t_{j-1}\}}[u(P(t_{j-1}+\Delta(s_{j-1})),s(t_j),t_{j-1}+\Delta(s_{j-1}))\\ &\quad -u(P(t_{j-1}),s(t_{j-1}),t_{j-1})]\\ &\le Mu(P(t_{j-1}),s(t_{j-1}),t_{j-1})-u(P(t_{j-1}),s(t_{j-1}),t_{j-1})\\ &\quad -E_{\{P(t_{j-1}),s(t_{j-1}),t_{j-1}\}}[1_{\{t_{j-1}\le T\}}(s_{j-1}P(t_{j-1}+\Delta(s_{j-1})))].\end{aligned}$$

Using the above results and letting $i\to\infty$ in (A.1) and using the fact that $\lim_{i\to\infty}t_i>T$ a.s., we obtain

$$u(P(0),s(0),0)\ge E_{(P(0),s(0),0)}\sum_i[s_i(P(t_i'+\Delta(s_i))]1_{t_i'\le T}.$$

Therefore, $u$ dominates the expected payoff of any admissible trading strategy. If $u$ is the value function of an admissible trading strategy, then this strategy is obviously an optimal strategy. Moreover, if $u$ generates an admissible control policy by the construction described before the statement of Proposition 4.1, then all the inequalities above are replaced by equalities and this control policy is therefore an optimal policy. □

REMARK A.1. The proof of this proposition can be easily extended to the general case where the investor's utility function is $U$ and $u$ is a sufficiently regular solution to the corresponding system of quasi-variational inequalities.

*Proof of Proposition* 4.1b. The proof of this proposition follows exactly along the lines of the proof of the first part of Proposition 4.1a. We notice that

$$E_{\{P(0),\,S,\,0\}}[u(P(t_j),s(t_j),t_j)-u(P(t_{j-1}+\Delta(s_{j-1})),s(t_j),t_{j-1}+\Delta(s_{j-1}))]\le 0$$

by the hypothesis of the proposition. The rest of the proof proceeds exactly as in the proof of the first part of the previous proposition and we arrive at the conclusion that

$$u(P(0),s(0),0)\ge E_{(P(0),s(0),0)}\sum_i[s_iP(t_i+\Delta(s_i))1_{\{t_i\le T\}}].$$

This completes the proof. □

*Proof of Proposition* 4.2. Define $\Gamma=R^+$. Let $u_0$ be the function satisfying

$$Lu_0=0,\,u_0(p,s,T)=E_{p,s,T}[sc(s)P(T+\Delta(s))].$$

We can solve the above problem analytically to obtain

$$u_0(p, s, t) = psc(s)\exp[\mu\Delta(s)]\exp[\mu(T-t)] \text{ on } R_+ \times \Gamma \times [0, T].$$

Since $Lu_0 = 0$ and $u_0$ satisfies Dynkin's formula, it is also easy to see that

$$\begin{aligned} &Mu_0(p, s, t) \\ &= \sup_{s',\, s' \le s,\, s-s' \in \Gamma} \left\{ \begin{array}{l} ps'c(s')\exp[\mu(\Delta(s')]+ \\ 1_{\{t+\Delta(s') \le T\}} pc(s')(s-s')c(s-s')\exp[\mu\Delta(s-s')]\exp[\mu(T-t)] \end{array} \right\}. \end{aligned}$$

The term $1_{\{t+\Delta(s')\le T\}}$appears in the second term in the supremum above because the investor obtains no additional utility from making trades after the time horizon $T$.

We shall first prove that if $\mu > 0$ and $c(s)$ and $\Delta(s)$ satisfy the condition in the statement of the proposition, liquidation of all holdings at the terminal date $T$ is an optimal policy.

We shall prove that $u_0(p, s, t) \ge Mu_0(p, s, t)$ for all $(p, s, t) \in R_+ \times \Gamma \times [0, T)$. We need to show that

$$\begin{aligned} \text{(A.2)} \qquad & psc(s)\exp[\mu\Delta(s)]\exp[\mu(T-t)] \\ &\quad \ge ps'c(s')\exp[\mu(\Delta(s')] + 1_{\{t+\Delta(s')\le T\}} pc(s')(s-s')c(s-s') \\ &\qquad \times \exp[\mu\Delta(s-s')]\exp[\mu(T-t)]. \end{aligned}$$

If $t + \Delta(s') > T$, we see that $psc(s)\exp[\mu\Delta(s)]\exp[\mu(T-t)] \ge ps'c(s')\exp[\mu(\Delta(s')]$ since $psc(s)\exp(\mu\Delta(s)) \ge ps'c(s')\exp(\mu(\Delta(s'))$ by the no-satiation condition.

Suppose $t + \Delta(s') \le T$. Rearranging inequality (A.2), we need to show that

$$\begin{aligned} &[sc(s)\exp(\mu\Delta(s)) - (s-s')c(s')c(s-s')\exp(\mu\Delta(s-s'))]\exp(\mu(T-t)) \\ &\qquad \ge s'c(s')\exp(\mu\Delta(s')). \end{aligned}$$

Since $t \le T - \Delta(s')$, the inequality above is true if the following inequality is true:

$$\begin{aligned} &[sc(s)\exp(\mu\Delta(s)) - (s-s')c(s')c(s-s')\exp(\mu\Delta(s-s'))]\exp(\mu\Delta(s')) \\ &\qquad \ge s'c(s')\exp(\mu\Delta(s')) \end{aligned}$$

or

$$s'c(s') + (s-s')c(s-s')c(s')\exp(\mu\Delta(s-s')) \le sc(s)\exp(\mu\Delta(s)),$$

which is exactly the hypothesis of the proposition. Therefore, $u_0(p, s, t) \ge Mu_0(p, s, t)$. Since $Lu_0 = 0$, we see that $u_0(p, s, t)$ satisfies the conditions of Proposition 4.1a. Hence, $u = u_0$, and block liquidation at the terminal date is an optimal policy for the investor.

In the case $\mu \le 0$, we show that $u(p, s, t) = psc(s)\exp(\mu(\Delta(s))$ satisfies the hypotheses of Proposition 4.1a. This follows from the fact that $Lu = \mu psc(s) \le 0$ since $\mu \le 0$ and

$$Mu = \sup_{0 \le s' \le s,\, s-s' \in \Gamma} [ps'c(s')\exp(\mu(\Delta(s'))) + E_{p,s,t}u(p(t+\Delta(s')), s-s', t+\Delta(s'))].$$

Therefore,

$$\begin{aligned} Mu = \sup_{0 \le s' \le s,\, s-s' \in \Gamma} & [ps'c(s')\exp(\mu(\Delta(s'))) + 1_{\{t+\Delta(s')\le T\}} p(s-s')c(s')c(s-s') \\ & \times \exp(\mu(\Delta(s') + \Delta(s-s')))]. \end{aligned}$$

If $t+\Delta(s') > T$ then $u = psc(s)\exp(\mu\Delta(s)) \ge ps'c(s')\exp(\mu\Delta(s'))$ by the no-satiation condition.

Suppose $t + \Delta(s') \leq T$. By the economies of scale in trading condition,

$$ps'c(s')\exp(\mu\Delta(s')) + p(s-s')c(s-s')c(s')\exp(\mu\Delta(s-s'))\exp(\mu\Delta(s'))$$
$$\leq psc(s)\exp(\mu\Delta(s)).$$

It follows that $u \geq Mu$ and, therefore, $u$ satisfies the hypotheses of Proposition 4.1a. Since a control policy that generates $u$ is immediate liquidation, this completes the proof for the case $\mu \leq 0$.

We shall now prove the "only if" part of the proposition. It is easy to see that if a single block sale is optimal, it must be at the terminal date if $\mu > 0$ or immediate if $\mu \leq 0$. We need to show that if $c(s)$ and $\Delta(s)$ do not satisfy the inequality in the statement of the proposition and $\mu > 0$, terminal date liquidation is not an optimal policy. The proof is by contradiction. If the optimal policy involves no intermediate transactions, then the value function is clearly $u_0$. This implies that

$$u_0 \geq Mu_0 \quad \text{for all } t \in [0, T].$$

Therefore,

$$\begin{aligned} &psc(s)\exp[\mu\Delta(s)]\exp[\mu(T-t)] \\ &\quad\geq ps'c(s')\exp[\mu\Delta(s')] + 1_{\{t+\Delta(s')\leq T\}}pc(s')(s-s')c(s-s') \\ &\qquad\times\exp[\mu\Delta(s-s')]\exp[\mu(T-t)] \end{aligned}$$

for all $t \in [0, T]$ and for all $s' \in [0, S]$.

Suppose $t + \Delta(s') \leq T$. We have

$$[sc(s)\exp(\mu\Delta(s)) - (s-s')c(s')c(s-s')\exp(\mu\Delta(s-s'))]\exp(\mu(T-t))$$
$$\geq s'c(s')\exp(\mu\Delta(s'))$$

for all $t \in [0, T - \Delta(s')]$. Setting $t = T - \Delta(s')$ above, we see that

$$s'c(s') + (s-s')c(s-s')c(s')\exp(\mu\Delta(s-s')) \leq sc(s)\exp(\mu\Delta(s)).$$

We have assumed throughout the paper that $\Delta(s) < T$ for all $s \geq 0$. Therefore, $c(s)$ and $\Delta(s)$ satisfy the economies of scale in trading condition, which contradicts our hypothesis. Thus, $u_0$ cannot be the value function of the impulse control problem; that is, the optimal strategy involves at least one transaction in $[0, T)$.

A similar argument shows that if $\mu < 0$, the policy of immediate liquidation of all holdings is not optimal. This completes the proof of the proposition. □

*Proof of Proposition* 5.1. For the proof, define $d\beta(t)/dt = \beta'(t)$. Let $t_0$ and $t_1$ be as in the statement of the proposition. Since $\beta(T)\exp(\mu T) \geq \beta(0) = 1$ from the hypothesis that $t_0$ exists, it follows that $0 \leq t_1 < T$ and hence $t_1$ exists. Since $\beta(t) \leq 1$ for $t \in [0, T]$, it also follows that $t_0 \leq t_1$.

As previously, we can define the system of quasi-variational inequalities associated with the impulse control problem:

$$Lu \leq 0, u \geq Mu, (Lu)(u - Mu) = 0, u(p, s, T) = psc(s, T) = ps\bar{c}(s)\beta(T),$$

where

$$Mu = \sup_{0<s'\leq s} \big[ps'\bar{c}(s')\beta(t) + u[c(s,t)p, s - s', t]\big].$$

As before, we attempt to solve this system of quasi-variational inequalities by iterating upon the obstacle:

$$u_0(p, s, t) = ps\bar{c}(s)\beta(T)\exp(\mu(T-t))$$

and

$$Mu_0 = \sup_{s'}[ps'\bar{c}(s')\beta(t) + p(s - s')\bar{c}(s')\bar{c}(s - s')\beta(t)\beta(T)\exp(\mu(T-t))].$$

It is easy to show that

$$Mu_0 = ps\bar{c}(s)\beta(t)\beta(T)\exp(\mu(T-t)) \quad \text{for } t \leq t_0$$

and

$$Mu_0 = ps\bar{c}(s)\beta(t) \quad \text{for } t > t_0.$$

Define the function $v(p, s, t)$ as follows:

$$v = u_0 \quad \text{for } t \leq t_1 \quad \text{and} \quad v = ps\bar{c}(s)\beta(t) \quad \text{for } t > t_1.$$

Trivially, $Mv = Mu_0$ for $t \leq t_1$. For $t > t_1$,

$$Mv = \sup_{s'}\lfloor ps'\bar{c}(s')\beta(t) + p(s - s')\bar{c}(s')\bar{c}(s - s')\beta(t)^2\rfloor.$$

By the “inequality” condition on $\bar{c}(s)$

$$s'\bar{c}(s')\beta(t) + (s - s')\bar{c}(s')\bar{c}(s - s')\beta(t) \leq s\bar{c}(s)\beta(t).$$

Since $\beta(t) \leq 1$, it easily follows that

$$s'\bar{c}(s')\beta(t) + (s - s')\bar{c}(s')\bar{c}(s - s')\beta(t)^2 \leq s\bar{c}(s)\beta(t).$$

In the above, equality is attained for $s' = s$. Therefore,

$$Mv = ps\bar{c}(s)\beta(t), \qquad t \geq t_1$$

Let us now show that

$$v \geq Mv, \qquad Lv \leq 0,$$

except at $t = t_1$ where $Lv$ is not defined.

For $t \leq t_1$, $Lv = Lu_0 = 0$. For $t > t_1$, $Lv = (\mu + \beta'(t))ps\bar{c}(s) \leq 0$. For $t \leq t_0$, it is easy to see that

$$v = ps\bar{c}(s)\beta(T)\exp(\mu(T-t)) \geq Mv = ps\bar{c}(s)\beta(t)\beta(T)\exp(\mu(T-t)),$$

since $\beta(t) \leq 1$. For $t_0 < t \leq t_1$,

$$v = ps\bar{c}(s)\beta(T)\exp(\mu(T-t)) \geq Mv = ps\bar{c}(s)\beta(t),$$

since $\beta(T)\exp(\mu(T-t)) \geq \beta(t)$ for $t_0 < t \leq t_1$ by the definition of $t_1$. For $t > t_1$,

$$v = ps\bar{c}(s)\beta(t) = Mv = ps\bar{c}(s)\beta(t).$$

Further, $v(p,s,T) = u_0(p,s,T) = ps\bar{c}(s)\beta(T)$. We have therefore shown that

$$v \geq Mv \quad \text{and} \quad v(T) = u_0(T).$$

Since $Lv \leq 0$, except at $t = t_1$, we can apply Ito's Lemma and use the continuity of $v$ at $t_1$ to conclude that

$$E[v(p(\tau_2), s, \tau_2) - v(p(\tau_1), s, \tau_1)] \leq 0,$$

where $\tau_1 \leq \tau_2 \leq T$ are stopping times.

Moreover, it is easy to see that the value function $v$ is generated by the policy of no liquidation until the terminal date for $t < t_1$ and immediate liquidation for $t \geq t_1$. Since this policy is clearly admissible, we can use the result of Proposition 4.1b to conclude that the policy is optimal and $v$ is the value function. This completes the proof of the proposition. □

PROPOSITION A.1. *For an investor with a power utility function $U(x) \equiv x^\gamma$, the optimal trading policy is a block sale if and only if the functions $\Delta(s)$ and $c(s)$ satisfy the following condition:*

$$s'^{\gamma}c(s')^\gamma + (s-s')^\gamma c(s-s')^\gamma c(s')^\gamma \exp(\alpha\Delta(s-s')) \leq s^\gamma c(s)^\gamma \exp(\alpha\Delta(s)),$$

*where $0 \leq s' \leq s$ and $\alpha = (\sigma^2\gamma^2 + (2\mu - \sigma^2)\gamma)/2$.*

*Proof.* Let us consider the case $\alpha > 0$. (The proof for the case $\alpha \leq 0$ is similar). We proceed exactly as in the proof of Proposition 4.2. Let $u_0$ be the function satisfying

$$Lu_0 = 0, u_0(p,s,T) = E_{p,s,T}[U(sc(s)P(T + \Delta(s)))].$$

It is not difficult to see that

$$u_0(p,s,t) = p^\gamma s^\gamma c(s)^\gamma \exp(\alpha\Delta(s))\exp(\alpha(T-t)) \quad \text{on } R_+ \times R_+ \times [0,T],$$

where $\alpha$ is as in the statement of the proposition. Translating the previous to this setting, it is easy to see that

$$Mu_0 = \sup_{s'}\left[\begin{array}{l} p^\gamma c(s')^\gamma s'^{\gamma} \exp(\alpha\Delta(s'))+ \\ 1_{\{t+\Delta(s')\leq T\}} p^\gamma (s-s')^\gamma c(s')^\gamma c(s-s')^\gamma \exp(\alpha\Delta(s-s'))\exp(\alpha(T-t)) \end{array}\right].$$

The remainder of the proof follows exactly along the lines of the proof of Proposition 4.2 with $\alpha$ replacing $\mu$. We therefore see that $u_0$ is the value function of the impulse control problem if and only if $c(s)$ and $\Delta(s)$ satisfy the condition in the hypothesis of the proposition. The case $\alpha \leq 0$ can be dealt with exactly like the case $\mu \leq 0$ in Proposition 4.2. This completes the proof. □

*Proof of Proposition* 5.2. Suppose $c(s)$ and $\Delta(s)$ satisfy the $\gamma$-economies of scale in trading condition for some $\gamma > 0$. Therefore,

$$\begin{aligned} &s'^{\gamma} c(s')^{\gamma} \exp((\gamma^2 - \gamma)\sigma^2(-\Delta(s))/2) + (s - s')^{\gamma} c(s')^{\gamma} c(s - s')^{\gamma} \\ &\quad \times \exp(\mu\gamma\Delta(s - s')) \exp((\gamma^2 - \gamma)\sigma^2(\Delta(s - s') - \Delta(s))/2) \\ &\le s^{\gamma} c(s)^{\gamma} \exp(\mu\gamma\Delta(s)). \end{aligned}$$

We now use the following familiar result:

If $a, b, c \in R^+$ and $a + b \le c$, then $a^r + b^r \le c^r$ for any $r \ge 1$.

Using the above result in the previous inequality with $r = \gamma'/\gamma \ge 1$, we have

$$\begin{aligned} &s'^{\gamma'} c(s')^{\gamma'} \exp((\gamma\gamma' - \gamma')\sigma^2(-\Delta(s))/2) + (s - s')^{\gamma'} c(s')^{\gamma'} c(s - s')^{\gamma'} \\ &\quad \times \exp(\mu\gamma'\Delta(s - s')) \exp((\gamma\gamma' - \gamma')\sigma^2(\Delta(s - s') - \Delta(s))/2) \\ &\le s^{\gamma'} c(s)^{\gamma'} \exp(\mu\gamma'\Delta(s)). \end{aligned}$$

Since $\gamma' \ge \gamma'^2 - \gamma' \ge \gamma\gamma' - \gamma'$. Since $s' \le s$ and $\Delta(s)$ is a monotonic increasing function of $s$,

$$\Delta(s - s') - \Delta(s) \le 0.$$

Therefore,

$$\exp((\gamma^2 - \gamma')\sigma^2(\Delta(s - s') - \Delta(s))/2) \le \exp((\gamma\gamma' - \gamma')\sigma^2(\Delta(s - s') - \Delta(s))/2)$$

and the same inequality is true replacing $s - s'$ by 0. Therefore, we now see that

$$\begin{aligned} &s'^{\gamma'} c(s')^{\gamma'} \exp((\gamma'^2 - \gamma')\sigma^2(-\Delta(s))/2) + (s - s')^{\gamma'} c(s')^{\gamma'} c(s - s')^{\gamma} \\ &\quad \times \exp(\mu\gamma'\Delta(s - s')) \exp((\gamma'^2 - \gamma')\sigma^2(\Delta(s - s') - \Delta(s))/2) \\ &\le s^{\gamma'} c(s)^{\gamma'} \exp(\mu\gamma'\Delta(s)). \end{aligned}$$

Multiplying both sides of the above inequality by $\exp((\gamma'^2 - \gamma')\sigma^2\Delta(s)/2)$, we see now that $c(s)$ and $\Delta(s)$ satisfy the $\gamma'$-economies of scale in trading condition. This completes the proof. □

*Proof of Proposition* 6.1. The outline of the proof is as follows. We shall define a function $v(p, s, t)$ to be the value function over the class of trading strategies that are completely deterministic—that is, the trading instants and the number of shares liquidated are deterministic and, in particular, independent of the stock price. More precisely, if $t$ is the initial time and $s > 0$ is the initial number of shares held by the investor, we consider the class of trading strategies $\Gamma(t, s) = [\{t_i, s_i\}]$, where

$$\begin{aligned} &t \le t_i \le T \text{ are deterministic times} \\ &t_{i+1} - t_i \ge \Delta, \\ &s_i > 0, \\ &\sum_i s_i \le s, \end{aligned}$$

where $s_i$ is nonrandom for each $i$. Note that since $T = N\Delta$, the number of trading instants cannot be greater than $N + 1$.

We then show the existence of an optimal policy within this class of trading policies (which are clearly admissible). We then prove that the function $v(p, s, t)$ satisfies the hypotheses of Proposition 4.1b. which would imply that it is the value function of the original optimal liquidation problem for the investor and that the optimal policy for the investor is completely deterministic.

In Proposition A.2, below, we prove the existence of an optimal policy in the class $\Gamma(t, s)$. The function $v(p, s, t)$ is clearly separable; that is, it is of the form

$$v(p, s, t) = p^\gamma f(s, t),$$

where $f(\cdot)$ is a deterministic function. It remains to show that $v(p, s, t)$ satisfies the hypotheses of Proposition 4.1b. First, we shall show that $v(p, s, t) \geq Mv(p, s, t)$, where

$$Mv(p, s, t) = \sup_{0<s'\leq s} [p^\gamma (s')^\gamma c(s')^\gamma \exp(\alpha\Delta) + E_{p,s,t} v(p(t+\Delta), s - s', t + \Delta)].$$

From the characterization of $v(p, s, t)$ we have

$$Mv(p, s, t) = \sup_{0<s'\leq s} [p^\gamma (s')^\gamma c(s')^\gamma \exp(\alpha\Delta) + p^\gamma c(s')^\gamma \exp(\alpha\Delta) f(t + \Delta, s - s')].$$

The trading policy where the investor liquidates $s' > 0$ shares at time $t$ and then follows the optimal policy in the class $\Gamma(t + \Delta, s - s')$ clearly belongs to the class $\Gamma(t, s)$ and the maximum expected utility of such a strategy over all $s' > 0$ is exactly $Mv(p, s, t)$. By the definition of $v(p, s, t)$, it follows that $v(p, s, t) \geq Mv(p, s, t)$.

Next, we shall show that

$$E[v(p(t_2), s, t_2) - v(p(t_1), s, t_1)|F_{t_1}] \leq 0,$$

where $t_1 \leq t_2 \leq T$ are deterministic times. The policy where the investor starts out with $s$ shares at time $t_1$, does not liquidate any shares until time $t_2$, and then follows the optimal policy in the class $\Gamma(t_2, s)$ clearly belongs to the class $\Gamma(t_1, s)$, and the expected utility of this strategy is exactly $E_{(p(t_1), s, t_1)}[v(p(t_2), s, t_2)]$. By the definition of $v(p(t_1), s, t_1)$ it follows that

$$E[v(p(t_2), s, t_2) - v(p(t_1), s, t_1)|F_{t_1}] \leq 0.$$

Therefore, $v(p(t), s, t)$ is a supermartingale. In the following Lemmas A.1 and A.2, we show that the value function $v(p, s, \cdot)$ is piecewise continuous; that is, it is continuous on the intervals $(l\Delta, (l + 1)\Delta], 0 \leq l < N$, and that $v(p(t+), s, t+) - v(p(t), s, t) \leq 0, t \in [0, N\Delta)$.

We can then apply the standard arguments used in the proof of Doob's optional sampling theorem (see, e.g., Elliott 1982) to conclude that

$$E[v(p(\tau_2), s, \tau_2) - v(p(\tau_1), s, \tau_1)] \leq 0,$$

where $\tau_1 \leq \tau_2 \leq T$ are bounded stopping times of the filtration. This completes the proof of the proposition. □

PROPOSITION A.2. *If $c(s)$ is a continuous function on $s > 0$, then an optimal policy in the class of deterministic strategies $\Gamma(t, s)$ exists.*

*Proof.* By definition,

$$v(p,s,t) = \sup_{\{t_i,s_i\}\in\Gamma(t,s)} \left[ p^\gamma \sum_i (s_i)^\gamma \exp(\alpha(t_i + \Delta - t)) \prod_{j=1}^{i} c(s_j)^\gamma \right]$$

$$= \exp(-\alpha t) \sup_{\{t_i,s_i\}\in\Gamma(t,s)} \left[ p^\gamma \sum_i (s_i)^\gamma \exp(\alpha(t_i + \Delta)) \prod_{j=1}^{i} c(s_j)^\gamma \right].$$

Note that $v(p,s,t)$ clearly exists since the expression being maximized is uniformly bounded above as $0 < s_i \le s$ and $c(s_i) \le 1$.

CASE 1: $\alpha \ge 0$. *It is easy to see that the expression being maximized in the definition of* $v(p,s,t)$ *is not decreased if the trading instants* $\{t_i\}$ *belong to the set* $\{l\Delta, (l+1)\times \Delta, \dots, N\Delta\}$, *where* $l = \inf\{\alpha \in Z_+ \colon l\Delta \ge t\}$; *that is, the investor delays his nonzero trades as much as possible.*

CASE 2: $\alpha < 0$. *In this case, the expression being maximized in the definition of* $v(p,s,t)$ *is not decreased if the trading instants* $\{t_i\}$ *belong to the set* $\{t, t+\Delta, \dots, t+m\Delta\}$, *where* $m = \sup\{\beta \in Z_+ \colon t + \beta\Delta \le N\Delta\}$; *that is, the investor executes his nonzero trades as soon as possible.*

Without loss of generality, let us assume that $t = 0$. By the preceding arguments, it is sufficient to consider the situation where the trading instants $\{t_i\} \subseteq \{l\Delta; l \in Z_+\}$. Define

$$A(s) = \left\{ \mathbf{s}' \equiv (s'_0, s'_1, \dots, s'_N) \in R^N_+ : \sum_{i=0}^{N} s'_i \le s \right\}.$$

We interpret $s'_i$ as the number of shares the investor decides to liquidate at time $i\Delta$ and therefore actually liquidates at time $(i+1)\Delta$. Then,

$$v(p,s,0) = p^\gamma \sup_{s' \in A(s)} \left[ \sum_{i=0}^{N} (s'_i)^\gamma \exp((i+1)\alpha\Delta) \prod_{j=1}^{i} c(s'_j)^\gamma \right].$$

Note that some of the coordinates of $s'$ may be zero, but these terms do not contribute to the summation above.

First we note that the supremum above exists since the function being maximized is bounded above as $c(s) \le 1$ for all $s \ge 0$.

If $c(s)$ is right continuous at $s = 0$, then the supremum is the maximum of a continuous function on a compact set. The maximum is therefore attained and an optimal liquidation policy exists.

It remains to consider the case where $c(s)$ has a jump discontinuity $\delta > 0$ at $s = 0$. Let us denote the function being maximized as $f(\mathbf{s}')$ and its maximum on $A(s)$ by $MAX$. The reader will note that $f(\mathbf{s}')$ is not a continuous function of $\mathbf{s}'$ on $A(s)$, but it is a continuous function of $\mathbf{s}'$ on the interior of $A(s)$ denoted by $\mathrm{int}(A(s))$ and is also continuous on the relative interior of any face $\overline{A(s)}$ of the polytope $A(s)$ where

$$\overline{A(s)} = \{(s_0, \dots, s_N) \in A(s) : s_{i_1}, \dots, s_{i_j} > 0; s_k = 0, k \in \{0,1,\dots,N\} \backslash \{i_1, \dots, i_j\}\}$$

and where $\{i_1, \dots, i_j\} \subseteq \{0, 1, \dots, N\}$.

By compactness there exists a sequence $\mathbf{s}^n$ such that $f(\mathbf{s}^n) \to MAX$ and $\mathbf{s}^n \to \mathbf{s}^\infty \in A(s)$. If $\mathbf{s}^\infty \in \mathrm{int}(A(s))$, then $f(\mathbf{s}^\infty) = MAX$ since $f(\cdot)$ is continuous on $\mathrm{int}(A(s))$. Therefore, $\mathbf{s}^\infty$ is an optimal liquidation policy for the investor.

Suppose $\mathbf{s}^\infty \in bd(A(s))$. Without loss of generality, let us assume that

$$\mathbf{s}^\infty \equiv (s_0^\infty, \ldots, 0, s_{k+1}^\infty, \ldots, s_N^\infty),$$

where the $k$th coordinate is zero and the others are strictly positive. The following arguments can easily be extended to the general case where any finite proper subset of the coordinates are equal to zero, but the notation becomes more cumbersome.

We shall prove that $f(\mathbf{s}^\infty) \geq MAX$, which implies that $f(\mathbf{s}^\infty) = MAX$ and that $\mathbf{s}^\infty$ is an optimal policy. By choosing a subsequence and relabeling if necessary, we can choose the sequence $\mathbf{s}^n$ so that $s_k^n$ decreases to zero and $f(\mathbf{s}^n)$ increases to *MAX*.

If $\exists N_0 < \infty$ such that $s_k^n = 0$ for all $n \geq N_0$, then the continuity of $f(\cdot)$ on the relative interior of each face of the convex polytope $A(s)$ would imply that $f(\mathbf{s}^\infty) = MAX$ and that $\mathbf{s}^\infty$ is an optimal policy.

Therefore it only remains to consider the case where $\mathbf{s}^n \in R_{++}^N$ for each $n$. Since $s_i^n \to s_i^\infty$ for each $i$ and $c(s)$ is continuous on $s > 0$ and strictly positive, we can choose $R$ such that

$$f(s_0^n, \ldots, 0, \ldots, s_N^n) \geq f(s_0^n, \ldots, s_k^n, \ldots, s_N^n) \qquad \text{for } n \geq R.$$

This is true because

$$f(s_0^n, \ldots, s_k^n, \ldots, s_N^n) = \sum_{i=0}^{N} (s_i^n)^\gamma \exp((i+1)\alpha\Delta) \prod_{j=0}^{i} c(s_j^n)^\gamma$$

and

$$f(s_0^n, \ldots, 0, \ldots, s_N^n) = \sum_{i=0,\, i\neq k}^{N} (s_i^n)^\gamma \exp((i+1)\alpha\Delta) \prod_{j=0,\, j\neq k}^{i} c(s_j^n)^\gamma.$$

Therefore,

$$\begin{aligned} &f(s_0^n, \ldots, 0, \ldots, s_N^n) - f(s_0^n, \ldots, s_k^n, \ldots, s_N^n) \\ &\quad = -(s_k^n)^\gamma \exp((k+1)\alpha\Delta) \prod_{j=0}^{k} c(s_j^n)^\gamma + \sum_{i=k+1}^{N} (s_i^n)^\gamma \exp((i+1)\alpha\Delta)(1 - c(s_k^n)^\gamma) \prod_{j=0,\, j\neq k}^{i} c(s_j^n)^\gamma. \end{aligned}$$

Since $1 - c(s_k^n)^\gamma \geq 1 - (1-\delta)^\gamma$ for all $n$(due to the jump discontinuity $\delta$ of $c(s)$ at $s = 0$), we see that

$$\begin{aligned} &\sum_{i=k+1}^{N} (s_i^n)^\gamma \exp((i+1)\alpha\Delta)(1 - c(s_k^n)^\gamma) \prod_{j=0,\, j\neq k}^{i} c(s_j^n)^\gamma \\ &\qquad \geq \sum_{i=k+1}^{N} (s_i^n)^\gamma \exp((i+1)\alpha\Delta)(1 - (1-\delta)^\gamma) \prod_{j=0,\, j\neq k}^{i} c(s_j^n)^\gamma. \end{aligned}$$

Since $s_i^n > 0$ for each $i$ and all $n, s_k^n \to 0, s_i^n \to s_i^\infty > 0$ for $i \neq k$, and $c(s)$ is continuous on $s > 0$,

$$\begin{aligned} &\sum_{i=k+1}^{N} (s_i^n)^\gamma \exp((i+1)\alpha\Delta)(1 - (1-\delta)^\gamma) \prod_{j=0,\, j\neq k}^{i} c(s_j^n)^\gamma \\ &\qquad \to \sum_{i=k+1}^{N} (s_i^\infty)^\gamma \exp((i+1)\alpha\Delta)(1 - (1-\delta)^\gamma) \prod_{j=0,\, j\neq k}^{i} c(s_j^\infty)^\gamma > 0. \end{aligned}$$

and

$$-(s_k^n)^\gamma \exp((k+1)\alpha\Delta) \prod_{j=0}^{k} c(s_j^n)^\gamma \to 0.$$

Therefore we can choose $R$ sufficiently large so that

$$f(s_0^n, \ldots, 0, \ldots, s_N^n) \geq f(s_0^n, \ldots, s_k^n, \ldots, s_N^n) \qquad \text{for } n \geq R.$$

Since $\mathbf{s}^n \to \mathbf{s}^\infty$, $(s_0^n, \ldots, 0, \ldots, s_N^n) \to \mathbf{s}^\infty$. Therefore, $f(s_0^n, \ldots, 0, \ldots, s_N^n) \to f(\mathbf{s}^\infty)$ due to the continuity of $f(\cdot)$ on the relative interior of the face of $A(s)$ defined by $s_k^n = 0$. Since $f(s_0^n, \ldots, s_k^n, \ldots, s_N^n) \uparrow MAX$, and $f(s_0^n, \ldots, 0, \ldots, s_N^n) \geq f(s_0^n, \ldots, s_k^n, \ldots, s_N^n)$ for $n \geq R$, we conclude that $f(\mathbf{s}^\infty) \geq MAX$. Therefore, $f(\mathbf{s}^\infty) = MAX$ and $\mathbf{s}^\infty$ is an optimal liquidation policy. This completes the proof of the proposition. □

We shall now prove two lemmas that were used in the proof of Proposition 6.1.

LEMMA A.1. *If $\alpha \geq 0$, the function $v(p, s, \cdot)$ is piecewise continuous—that is, it is continuous on $((l-1)\Delta, l\Delta]$ and $v(p, s, t+) - v(p, s, t) \leq 0$.*

*Proof.* We recall that

$$v(p, s, t) = \sup_{\{t_i, s_i\} \in \Gamma} \Big[ p^\gamma \sum_i (s_i)^\gamma \exp(\alpha(t_i + \Delta - t)) \prod_{j=1}^{i} c(s_j)^\gamma \Big], \tag{A.3}$$

where the trading instants $\{t_i\}$ belong to the set $\{l\Delta, (l+1)\Delta, \ldots, N\Delta\}$, where

$$l = \inf\{n \in Z_+ : n\Delta \geq t\}.$$

Firstly, we easily notice that $v(p, s, \cdot)$ is a nonincreasing function since $\exp(-\alpha t)$ is a nonincreasing function of $t$ for $\alpha \geq 0$. It follows that

$$\lim_{t_2 \to t_1+} [v(p, s, t_2) - v(p, s, t_1)] \leq 0.$$

It follows that $v(p, s, t_1+) - v(p, s, t_1) \leq 0$.

We shall now prove the piecewise continuity of $v(p, s, \cdot)$. Fix $t \in ((l-1)\Delta, l\Delta)$. By the arguments in the proof of the previous proposition, we know that the supremum in the definition of $v(p, s, t)$ is attained for some strategy $\{t_i, s_i\}$ where $s_i > 0$ for each $i$ and the trading instants $\{t_i\}$ belong to the set $\{l\Delta, (l+1)\Delta, \ldots, N\Delta\}$. Therefore,

$$\begin{aligned} v(p, s, t) &= \sup_{\{t_i, s_i\} \in \Gamma} \Big[ p^\gamma \sum_i (s_i)^\gamma \exp(\alpha(t_i + \Delta - t)) \prod_{j=1}^{i} c(s_j)^\gamma \Big] \\ &= p^\gamma \exp(-\alpha t) \sup_{t_i \in \{l\Delta, \ldots, N\Delta\}} \sup_{s_i} \Big[ \sum_i (s_i)^\gamma \exp(\alpha(t_i + \Delta)) \prod_{j=1}^{i} c(s_j)^\gamma \Big]. \end{aligned}$$

We notice that for $t \in ((l-1)\Delta, l\Delta]$, the expression being maximized is independent of $t$. Therefore, we easily conclude that $v(p, s, t)$ is a continuous function for $t \in ((l-1)\Delta, l\Delta)$, is left continuous at $t = l\Delta$, and the optimal policy is the same for $t \in ((l-1)\Delta, l\Delta]$; that is, the trading instants and number of shares liquidated are the same. This completes the proof of the lemma. □

LEMMA A.2. *If $\alpha < 0$, the function $v(p, s, \cdot)$ is piecewise constant; that is, it is constant on $((l-1)\Delta, l\Delta]$ and $v(p, s, t+) - v(p, s, t) \le 0$.*

*Proof.* We recall that

$$v(p, s, t) = \sup_{\{t_i, s_i\} \in \Gamma} \left[ p^\gamma \sum_i (s_i)^\gamma \exp(\alpha(t_i + \Delta - t)) \prod_{j=1}^{i} c(s_j)^\gamma \right], \tag{A.4}$$

where the trading instants $\{t_i\}$ belong to the set $\{t, t+\Delta, t+2\Delta, \ldots, t+m(t)\Delta\}$, where

$$m(t) = \sup\{n \in Z_+ \colon t + n\Delta \le T\}. \tag{A.5}$$

We can therefore rewrite expression (A.4) as

$$v(p, s, t) = \sup_{\{t_i, s_i\} \in \Gamma} \left[ p^\gamma \sum_i (s_i)^\gamma \exp(\alpha m_i \Delta) \prod_{j=1}^{i} c(s_j)^\gamma \right], \tag{A.6}$$

where $\{m_i\} \subseteq \{1, 2, \ldots, m(t) + 1\}$. Since $m(t)$ given by (A.5) is constant for $t \in ((l-1)\Delta, l\Delta]$, (A.6) tells us that $v(p, s, t)$ is constant for $t \in ((l-1)\Delta, l\Delta]$. It also easily follows from (A.6) that $v(p, s, t_2) \le v(p, s, t_1)$ for $t_2 \ge t_1$ since $m(t_2) \le m(t_1)$. Therefore, $v(p, s, t+) - v(p, s, t) \le 0$. This completes the proof of the lemma. □

By the results just obtained, we can conclude that the investor's value function $u(p, s, 0)$ is given by

$$u(p, s, 0) = p^\gamma \sup_{\mathbf{s}' \in A(s)} \left[ \sum_{i=0}^{N} (s_i')^\gamma \exp((i+1)\alpha\Delta) \prod_{j=0}^{i} c(s_j')^\gamma \right]. \tag{A.7}$$

We shall now derive explicit characterizations of the optimal policy when the price impact function is given by

$$c(s) \equiv c < 1 \qquad \text{for } s > 0.$$

*Proof of Proposition* 6.2. We first search for a maximum (if it exists) on $\operatorname{int}(A(s))$. Setting $\delta = c^\gamma \exp(\alpha\Delta)$, we rewrite the supremum in (A.7) as

$$\sup_{x'} \left[ p^\gamma s^\gamma \sum_{i=0}^{N} (x_i')^\gamma \delta^{i+1} \right]$$

where $x_i' = s_i'/s$, so that

$$\sum_{i=0}^{N} x_i' = 1.$$

Using standard constrained optimization techniques, we see that there exists a Lagrange multiplier $\tau_N$ such that the optimal proportions $\mathbf{x} \equiv (x_0, \ldots, x_N)$ satisfy

$$\delta^{i+1} x_i^{\gamma-1} = \tau_N, \qquad \text{for } i = 0, \ldots, N.$$

Since $\sum_{i=0}^{N} x_i = 1$, we obtain $\tau_N$ as a function of $\delta$. If we set

$$\rho_N^{-1} = \sum_{i=0}^{N} \frac{1}{\delta^{(i+1)/(\gamma-1)}},$$

we see that $\tau = \rho_N^{\gamma-1}$ and we can check that the value of the maximum is $p^\gamma s^\gamma \tau_N$.

We notice that $0 < x_i < 1$ for all $i$ and therefore this is a local maximum in the interior of $A(s)$. In order to determine whether this is a global maximum on $A(s)$, we need to compare this maximum with the local maxima on $bd(A(s))$. Each of these maxima can be obtained exactly as above, since each face of $bd(A(s))$ is defined as follows:

$$\overline{A(s)} = \left\{(s_0, \ldots, s_N)\colon \sum_{i=0}^{N} s_i = s; s_{i_0}, s_{i_2}, \ldots, s_{i_j} > 0\right\},$$

where $\{i_0, \ldots, i_j\} \subseteq \{0, 1, \ldots, N\}, i_0 < i_1 < \cdots < i_j$, and the other coordinates are equal to zero.

Hence, we can disregard the coordinates that are equal to zero and the problem becomes similar to the one considered above, with the control policies defined by the nonzero coordinates above. We write the supremum in (A.7) as

$$\sup_{x'}\left[p^\gamma s^\gamma \sum_{k=0}^{j}(x'_{i_k})^\gamma c^{(k+1)\gamma} \exp((i_k + 1)\alpha\Delta)\right]. \tag{A.8}$$

Since $\alpha \le 0$, it is easy to see that the expression being maximized above is not decreased if $i_k = k$; that is, among all faces with $j + 1$ nonzero coordinates it is enough to consider those where the first $j+1$ coordinates are nonzero. We now rewrite the above as

$$\sup_{x'}\left[p^\gamma s^\gamma \sum_{k=0}^{j}(x'_k)^\gamma \delta^{(k+1)}\right].$$

We see that there exists a Lagrange multiplier $\tau_j$ such that the optimal proportions $\mathbf{x} \equiv (x_0, \ldots, x_j)$ satisfy

$$\delta^{i+1} x_i^{\gamma-1} = \tau_j, \qquad \text{for } i = 0, \ldots, j.$$

Since $\sum_{i=0}^{j} x_i = 1$, we obtain $\tau_j$ as a function of $\delta$. If we set

$$\rho_j^{-1} = \sum_{i=0}^{j} \frac{1}{\delta^{(i+1)/(\gamma-1)}},$$

then $\tau_j = \rho_j^{\gamma-1}$ and the value of the maximum is $p^\gamma s^\gamma \tau_j$.

Since $0 < x_i < 1$, for $i = 0, \ldots, j$, the maximum obtained above is a local maximum in the face under consideration. By considering the maximum obtained above as a function of $j$, it is easy to see that it is an increasing function of $j$. It follows that the local maximum in $\text{int}(A(s))$ is, in fact, the global maximum in $A(s)$.

Therefore, the investor's optimal policy is to liquidate a nonzero fraction of his initial number of shares at each trading date. This completes the proof. □

*Proof of Proposition* 6.3. Since $\alpha > 0$, one can see that the expression being maximized in (A.7) is not decreased if $i_k = N - k + j$; that is, among all faces with $j + 1$ nonzero coordinates, it is enough to consider those where the last $j + 1$ coordinates are nonzero. It is now easy to check that the supremum in (A.7) can be rewritten as

$$\sup_{x'}\left[p^\gamma s^\gamma \exp((N - j)\alpha\Delta) \sum_{k=0}^{j}(x'_k)^\gamma \delta^{k+1}\right].$$

Using the results obtained in the case $\alpha \le 0$, we see that the maximum above is equal to

$$p^\gamma s^\gamma \exp(N\alpha\Delta) \exp(-j\alpha\Delta)\rho_j^{\gamma-1} = p^\gamma s^\gamma \exp(N\alpha\Delta) M_j,$$

where $M_j = \exp(-j\alpha\Delta)\rho_j^{\gamma-1}$. In this expression, $M_j$ does not depend on $N$ for a fixed $\Delta$.

It is easy to check that for a fixed $\Delta$, $M_0 = \delta$ and $\lim_{j\to\infty} M_j = 0$. Therefore, $M_j$ has a maximum on $[0, \infty)$ and one can check that this maximum occurs at a unique $j_0 \in [0, \infty)$; that is, $M_j$ is increasing on $[0, j_0]$ and decreasing on $[j_0, \infty)$. Let $N_1 = floor(j_0)$ and $N_2 = \text{ceil}(j_0)$. Then,

$$
\begin{aligned}
N' &= N_1, \quad \text{if } M_{N_1} > M_{N_2}, \\
N' &= N_2, \quad \text{otherwise},
\end{aligned}
$$

satisfy the conditions of the proposition; in other words, for a given $N$, the maximum on the face $\overline{A(s)}$ defined by

$$
\overline{A(s)} = \left\{(s_0, \ldots, s_N)\colon s_0 = s_1 = \cdots = s_{N-N_0-1} = 0;\, s_{N-N_0}, \ldots, s_N > 0;\, \sum_{i=0}^{N} s_i = s\right\}
$$

is, in fact, the global maximum on $A(s)$. This completes the proof. □

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