# A No-Arbitrage Model Of Liquidity In Financial Markets Involving Stochastic Strings: Applications To High-Frequency Data

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# Outline

# Market vs Limit Orders

A (buy) market order specifies

- how many shares a trader wants to buy,
- that he is willing to buy them at any price.

A (buy) limit order specifies

- how many shares a trader wants to buy,
- at what maximum price he is willing to buy them.

# Limit order matching mechanism


Figure: Limit order matching mechanism.

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Figure: Limit order matching mechanism.

# Demand v.s. Supply

The order books contain all the information about demand and supply.

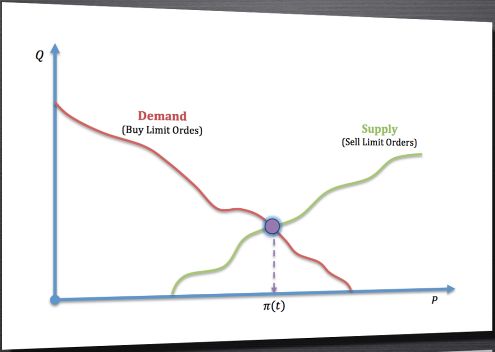


Figure: Demand v.s. Supply.

# The dynamics of Limit Orders in 3D

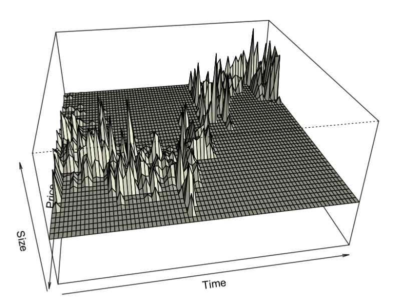


Figure: Buy Limit Orders of ORCL on April 4, 2011.

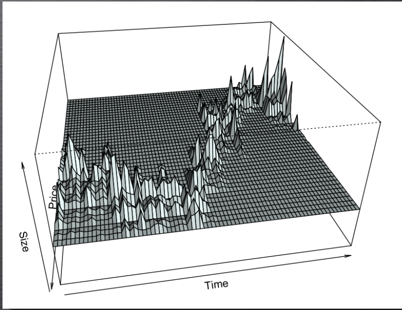


Figure: Sell Limit Orders of ORCL on April 4, 2011.

# The dynamics of the Clearing Price process

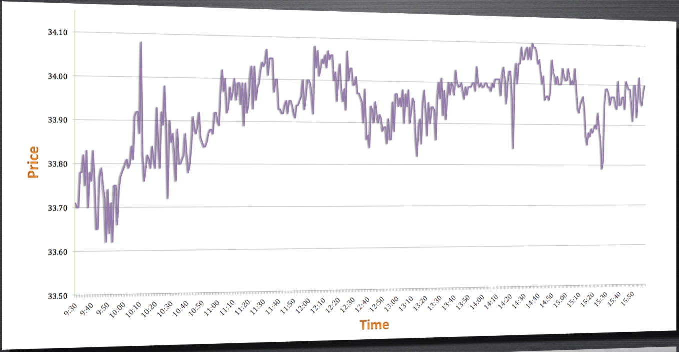


Figure: The dynamics of Oracle Corporation's Clearing Prices on April 1, 2011.

# Literature Review: Liquidity Models

Market Manipulation (feedback) Models

- Jarrow (1994),
- Platen and Schweizer (1998),
- Sircar and Papanicolaou (1998),
- Frey (1998),
- Schonbucher and Wilmott (2000),
- Bank and Baum (2004).

Price-taking (competitive) Models

- Cetin, Jarrow, and Protter (2004),
- Cetin and Rogers (2006),
- Cetin, Soner, and Touzi (2009),
- Kallsen and Rheinlaender (2009),
- Gokay and Soner (2011).

# Net Demand Curve and Clearing Price

**Definition**

The net demand curve $Q$ is a function $[0, P] \times \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}$, which value $Q(p, t, \omega)$ is equal to the difference between the quantity of shares **available** for purchase and the quantity of shares **available** for sale at price $p$ at time $t$. For each $p$ the stochastic process $Q(., t, )$ is a $\mathcal{F}_t$ adapted semimartingale.

Remark: For the clearing price to be a diffusion, the demand must be defined on a continuum of limit prices.

Remark: The net demand curve should be decreasing in $p$. The easiest way to do that is to model positive processes:

- $Q(0, t)$: total number of buy orders
- $q(p, t)$: density of buy orders + density of sell orders

$$Q(p, t) = Q(0, t) - \int_0^p q(y, t)dy$$

**Definition**

The clearing price $\pi(t)$ is a $\mathcal{F}_t-$ adapted stochastic process which satisfies market clearing:

$$Q(\pi(t), t) = 0$$

# The Model

- It can be proved that the optimal strategy of a large trader is to disseminate her orders into infinitesimal orders.

  This shows that a continuous demand curve is a plausible model.

- In order to avoid arbitrage, we choose to have as many factors as limit prices

$$dQ(p,t) = \mu_Q(p,t)dt - \sigma_Q(p,t)\int_{x=0}^{S} b_q(p,x,t)W(ds,dt) \quad \text{for } 0 < p \leq S$$

# Main Result: Market with a Large Trader

## Main Result

*Suppose in addition to our standing assumptions that*

*C1) The demand curve is decreasing in price and continuous in time;*

*C2) the volatility $\sigma_Q(p, t)$ is bounded away from zero, uniformly in $p$;*

*C3) there is no path such that $Q(S, t) \geq 0$ or $Q(0, t) \leq 0$;*

*C4) The market price of risk equations hold.*

*Then*

*F1) there is no arbitrage strategy,*

*F2) the net demand curve $Q$ is continuous in $t$,*

*F3) the clearing price $\pi(t)$ is continuous,*

*F4) The $\mathbb{Q}$-measure is also a martingale measure for $\pi(t)$.*

# Characterization of the Risk-Neutral Measure $\mathbb{Q}$

**Change of Measure**

$$W^{\mathbb{Q}}(ds, dt) = W(ds, dt) + \lambda(s, t)dt$$

**Goal:** determine $\lambda$ such that the market price of risk equations hold.

# Market Price of Risk Equations

Let $\{\mu_Q, \sigma_Q, b_Q\}$ be the parameters of net demand. Let the volatility of the clearing price be:

$$\sigma_\pi(t) b_\pi(s,t) = -\frac{\sigma_Q(\pi(t),t)\int_{s=0}^{S} b_Q(\pi(t),s,t)ds}{q(\pi(t),t)}$$

Define

$$\begin{aligned}
C(\pi,t) &= \sigma_\pi(t)\int_{s=0}^{S}\frac{\partial[\sigma_Q(p,t,\omega)b_Q(p,s,t)]}{\partial p}|_{p=\pi} b_\pi(s,t)ds \\
B(\pi,t) &= \mu_Q(\pi,t) - \frac{1}{2}\frac{\partial^2 Q(p,t)}{\partial p^2}|_{p=\pi} + C(\pi,t) \\
\Sigma(\pi,s,t) &= \sigma_Q(\pi,t) b_Q(\pi,s,t)
\end{aligned}$$

The market price of risk equations are:

$$\int_{s=0}^{S}\Sigma(\pi,s)\lambda(s)ds = B(\pi) \qquad 0 \le \pi \le P$$

# From a "MetaModel" to a Model

Reminder:

$$q(p,t)dp = \text{sum of buy and sell order quantities with limit price in } [p, p+dp] \text{ arriving in } [0,t]$$

Plausible dynamics for $q$:

- positive process

  not necessarily increasing: orders can be cancelled
- mean-reverting process
- to be implemented on a computer: $p$ and $t$ must take discrete values

Our choice: the exponential of a (vector) Ornstein-Uhlenbeck process.

# Model Framework

We define

$$
\begin{aligned}
dh(p,t,\omega) &= [-a_h(p)h(p,t,\omega)+\mu_h(p)]dt + \int_{s=0}^{S}\sigma_h(p,s)W(ds,dt,\omega) \\
q(p,t,\omega) &= \exp\left(\int_{x=0}^{p} h(x,y,\omega)dx\right) && (Density) \\
dQ(p,t,\omega) &= \int_{x=0}^{S} q(x,t,\omega)dx\eta(t,\omega) - \int_{x=0}^{p} q(x,t,\omega)dx && \text{(Net Demand)} \\
d\eta(t,\omega) &= a_\eta(\bar{\eta}-\eta(t,\omega))dt \\
&+ \sqrt{\eta(t,\omega)(1-\eta(t,\omega))}\int_{s=0}^{S}\sigma_\eta(s)W(ds,dt,\omega) && \text{(Demand Volume Ratio)}
\end{aligned}
$$

# Reasoning of Modeling Framework

- The reason of modeling demand but not demand/supply separately

  Smooth semi-martingale assumption holds for net demand curve

- The reason of modeling the ratio of demand and volume $\eta$ process

  To guarantee the robustness of the estimation: $Q(0) > 0$ and $Q(S) < 0$

- The reason for $Q$ to be twice differentiable

  Need to apply Ito-Wentzell formula.

- The reason for $q$ to be positive

  Guarantee that the net demand is downward sloping (decreases with price).

- The reason for using Stochastic strings

  Rule out arbitrage opportunity

# NYSE Arcabook Data

| Industry | Exchange | Ticker | Firm |
|---|---|---|---|
| Energy | NYSE | CVX | Chevron Corporation |
| | NYSE | XOM | Exxon Mobil Corporation |
| Financial Banks | NYSE | JPM | JPMorgan Chase & Co. |
| | NYSE | WFC | Wells Fargo & Company |
| Materials and Mining | NYSE | ABX | Barrick Gold Corporation |
| | NYSE | FCX | Freeport-McMoRan Copper & Gold Inc. |
| Technology | NASDAQ | CSCO | Cisco Systems, Inc. |
| | NASDAQ | MSFT | Microsoft Corporation |
| | NASDAQ | ORCL | Oracle Corporation |

Table: NYSE Arcabook data selection.

# Simulation Process

Market Calibration

- Estimate excess demand $\bar{Q}(p,t)$ from $p$ and $t$ from the high frequency data;
- Calculate the $\eta(t)$ process by $\frac{\bar{Q}(0,t)}{\bar{Q}(0,t)-\bar{Q}(S,t)}$;
- Calculate process $q$ and $h$. Estimate the variance-covariance matrix of $h$ and $\eta$;
- Apply Cholesky decomposition to the correlation matrix, where $\bar{R}dt = Corr[\frac{dh_i}{h_i}, \frac{dh_j}{h_j}]$

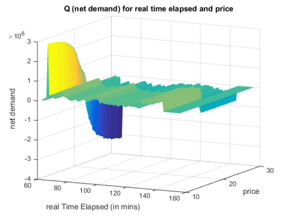


Figure: The net demand of GE as of 04/01/2011

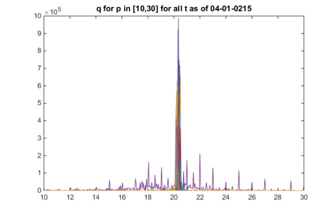


Figure: The q process of GE as of 04/01/2011

# Simulation Process

Excess Demand Simulation

(The process of the simulation is to demonstrate that the calibrated parameters in framework are market consistent)

- Randomly generate $n$ paths for $h$;
- Use $h$ to integrate for $q$ process;
- Simulate $\eta$ process using the correlation estimated in market calibration process;
- Combine $h$, $q$ and $\eta$ to simulate excess demand $Q$.

# Simulation Result

GE stock net demand curve with 04/01/2015 market data calibrated:

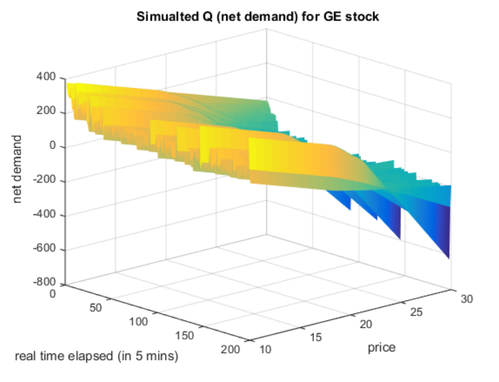


Figure: Simulated net demand Q for GE using 04/01/2015 order book calibration.

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## Change of Measure

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\end{aligned}$$

The market price of risk equations are:

$$\int_{s=0}^{S}\Sigma(\pi,s)\lambda(s)ds = B(\pi) \qquad 0 \le \pi \le P$$

# Volatility Smile (ORCL)

| Strike Price | Implied Volatility |
|---|---|
| 33.77 | 22.66% |
| 33.80 | 21.05% |
| 33.83 | 19.40% |
| 33.87 | 17.71% |
| 33.90 | 15.94% |
| 33.93 | 14.10% |
| 33.97 | 12.16% |
| 34.00 | 10.11% |
| 34.04 | 9.35% |
| 34.07 | 10.26% |
| 34.10 | 11.03% |
| 34.14 | 11.70% |
| 34.17 | 12.25% |
| 34.20 | 12.71% |
| 34.24 | 13.06% |
| 34.27 | 13.25% |
| 34.31 | 13.34% |
| 34.34 | 13.32% |
| 34.37 | 13.23% |
| 34.41 | 12.99% |



Figure: Simulated v.s. real-time volatility smile of ORCL on April 4, 2011. *Source: Bloomberg*

# Volatility Smile (ABX)

| Strike Price | Implied Volatility |
|---|---|
| 50.98 | 23.64% |
| 51.03 | 21.98% |
| 51.08 | 20.27% |
| 51.13 | 18.51% |
| 51.18 | 16.70% |
| 51.23 | 14.84% |
| 51.29 | 12.93% |
| 51.34 | 11.69% |
| 51.39 | 12.24% |
| 51.44 | 12.75% |
| 51.49 | 13.20% |
| 51.54 | 13.58% |
| 51.59 | 13.92% |
| 51.64 | 14.18% |
| 51.69 | 14.37% |
| 51.74 | 14.46% |
| 51.79 | 14.54% |
| 51.85 | 14.55% |
| 51.90 | 14.46% |
| 51.95 | 14.39% |



Figure: Simulated v.s. real-time volatility smile of ABX on April 4, 2011. *Source: Bloomberg*

# Volatility Smile (CSCO)

| Strike Price | Implied Volatility |
|---|---|
| 16.88 | 13.30% |
| 16.90 | 12.17% |
| 16.92 | 11.02% |
| 16.93 | 9.84% |
| 16.95 | 8.63% |
| 16.97 | 7.37% |
| 16.99 | 6.04% |
| 17.00 | 4.63% |
| 17.02 | 4.16% |
| 17.04 | 4.84% |
| 17.05 | 5.28% |
| 17.07 | 5.53% |
| 17.09 | 5.68% |
| 17.10 | 5.73% |
| 17.12 | 5.79% |
| 17.14 | 5.85% |



Figure: Simulated v.s. real-time volatility smile of CSCO on April 4, 2011. *Source: Bloomberg*

# Volatility Smile (AAPL)

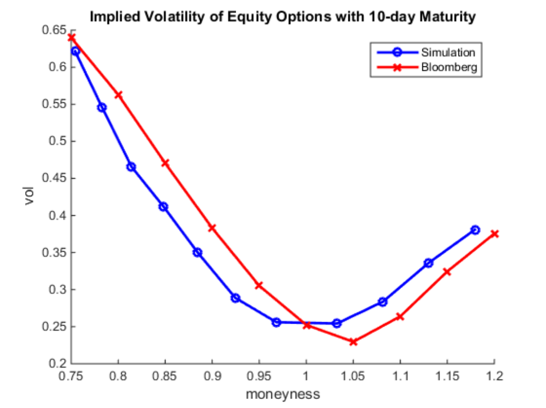


Figure: Simulated v.s. real-time volatility smile of AAPL on April 1, 2011. *Source: Bloomberg*

# Conclusions

1. We developed a liquidity model with stands between
   - traditional no-arbitrage (option pricing) models and
   - financial economics models.
2. This model uses Ito-Wentzell's formula and Girsanov's theorem for Brownian sheets.
3. We give conditions for no-arbitrage, which allows us to price options
4. We specified a model
   - with positive demand density,
   - with mean-reversion,
   - where parameters are centered on the clearing price.
5. We calibrate the model parameters from high frequency stock data of GE as of 04-01-2015. The simulated net demand curve shows a downward sloping shape, as expected.
6. The model generates an implied volatility smile which matches the observed smile much better than the traditional Black-Scholes model.