Financial Markets with a Large Trader: an Approach via Carmona-Nualart Integration

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Abstract

We extend the analysis of Bank and Baum (2004) of markets with a large trader by relaxing their requirement of a universal martingale measure for all primal price processes indexed by the position of the large investor. Assuming that there are no arbitrage opportunities in case the large trader employs only buy-and-hold strategies, we show that this remains true for bounded dynamic trading strategies as well. We proceed via the nonlinear stochastic integral of Carmona and Nualart (1990) which provides a crucial continuity property of the associated value process in the semimartingale topology. Moreover, we show via a case study that it may depend crucially on the market depth whether there exists a unique solution to expected utility maximization from terminal wealth. In particular, if the market depth exceeds a certain threshold, it might become optimal for the large investor to 'grab as many shares as possible'.

1 Introduction

One of the classical but unrealistic assumptions of many studies in mathematical finance is that all traders are just acting as price takers. This has been relaxed in several studies where one considers a 'large' investor who faces different prices for her transactions depending on the trade size. We focus here on the two recent papers Çetin, Jarrow and

Protter [2], and Bank and Baum [1], and refer to these also for a detailed overview of the literature dealing with the large trader problematic.

The former study postulates the existence of a stochastic supply curve governing the transactions of the large trader. Once the transaction has been settled, however, the actual price remains unchanged, hence the large trader does not move prices in this framework. Arbitrage opportunities can then be excluded by assuming the existence of a martingale measure for the point of the supply curve corresponding to the zero net trade. In contrast, in Bank and Baum [1] the prices itself are modelled as a random field indexed by the large trader's position in the asset: if the large trader changes her position at time t from ϑ to ϑ' units, the price reacts instantaneously by moving from $P(\vartheta, t)$ to $P(\vartheta', t)$. Mathematically, the resulting value process of the large investor then has to be modelled by a nonlinear stochastic integral: the integrator is affected by the strategy of the large trader. Bank and Baum [1] have chosen the Kunita integral for this purpose. Under the crucial assumption that there exists a universal martingale measure simultaneously for all primitive price processes $P(\vartheta, \cdot)$ (which in fact then turns out to be a martingale measure for all possible value processes corresponding to admissible strategies) they prove, amongst others, the absence of arbitrage for the large trader and discuss utility maximization from terminal wealth. Despite the different approaches, there are a couple of similar findings in Çetin, Jarrow and Protter [2], and Bank and Baum [1]. Let us here just single out that the authors of both papers agree in that the large trader should use 'tame' strategies, i.e. continuous strategies of finite variation. Block trades as well as highly fluctuating strategies are disadvantageous since they induce transaction costs for the large investor.

The main point of the present work is to extent the results of Bank and Baum [1] by relaxing the assumption of a universal martingale measure which would rule out an impact on the drift of the price process. Our main assumption is closer to the one used in the discrete time framework by Jarrow [5]: there are no arbitrage opportunities in the market as long as the large trader employs only elementary buy-and-hold strategies. This corresponds to requiring the existence of a martingale measure for each realizable value process given a constant stake. For every buy-and-hold strategy θ of the large trader we then construct a martingale measure Q^θ for the associated value process by concatenating the martingale measures corresponding to each position which is constant over some period of time. The main issue, however, is how to extend this concatenation procedure in case the large trader employs a dynamic trading strategy. It is here where the general non-linear stochastic integral of Carmona and Nualart [3] seems to be 'taylor-made' to resolve this problem. The key is that this integral provides a link to the semimartingale topology $\mathcal{SM}$: if we approximate a dynamic strategy by buy-and-hold strategies uniformly in probability then the associated value processes converge in $\mathcal{SM}$. This continuity property allows infinitesimally 'to glue' together the individual martingale measures which govern the dynamics of the price process for just one point in time. It results that each bounded semimartingale strategy θ induces a

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martingale measure Q^θ for the value process, from which in turn it follows easily that the large investor has no arbitrage opportunities.

We then turn to a case study which illustrates various phenomena which may occur when considering utility maximization from terminal wealth in our setting. While a general result gives a necessary condition for a strategy to be optimal, an example shows that it depends crucially on the market's depth (i.e. price impact per unit transaction) whether there exists indeed an optimum, or rather the optimal behaviour is to buy as many shares as possible. In particular, we single out a condition on the market depth which decides whether we are in the former stable situation (one could call the trader under study then rather a medium-sized trader), or whether we encounter the alternative run-away effect which leads to ultimate destabilization of the market.

2 Preliminaries. Case of buy-and-hold strategies

Our financial market consists of a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_t)$ modelling the evolution of information for the large trader. We assume that $(\mathcal{F}_t)$ fulfills the usual conditions with $\mathcal{F}_0$ being trivial, and denote by T some finite time horizon. An equivalent martingale measure for a continuous semimartingale X is a probability measure Q equivalent to P such that X is a local Q-martingale.

We now take as basic ingredient of our market a family $(P(\vartheta, \cdot))$, $\vartheta \in \mathbb{R}$, of continuous semimartingales where $P(\vartheta, \cdot)$ models the discounted price process of some risky asset given that the large trader has a constant stake of ϑ shares in that asset. The price process $S(\theta, \cdot)$ provided the large trader employs a time-varying strategy $\theta = (\theta_t)_{0 \leq t \leq T}$ is then given as

$$S(\theta, t) := P(\theta_t, t), \qquad 0 \leq t \leq T.$$

To exclude trivial arbitrage opportunities, we have to assume that each trade of the large trader induces transaction costs in a (for her) unfavourable way:

Assumption (O) We have that $\vartheta \leq \vartheta'$ implies $P(\vartheta, \cdot) \leq P(\vartheta', \cdot)$.

Remark 2.1 As in the model proposed by Bank and Baum [1], trading decisions by the large trader have a permanent impact on the price process. This is unrealistic, and moreover has as unwelcome consequence that it allows for the possibility of a "free ride" (Chris Rogers), that is, the large trader can move prices without any cost, and can in particular knock out/in digital options at will. To overcome this, one may assume a certain non-zero but finite resilience effect. Extending our study to include resilience would be in fact an interesting problem for future research.

It is important to distinguish between the book value of a position and the realizable value. As discussed in Bank and Baum [1], the value a large investor can ideally achieve from terminating her position is captured in the following definition.

Definition 2.2 The *asymptotic liquidation proceeds* $L(\vartheta, t)$ from a position of ϑ shares at time t is given as

$$L(\vartheta, t) := \int_0^\vartheta P(x, t)\ dx.$$

Our main assumption is that there are no arbitrage opportunities in periods where the large investor only employs buy-and-hold strategies. A mathematical condition ensuring this (as we shall see below) can be formulated as follows:

Assumption (MM) For each $\vartheta \in \mathbb{R}$ there exist equivalent martingale measures Q^ϑ for $L(\vartheta, \cdot)$.

Example 2.3 We assume that the primal price processes $P(\vartheta, \cdot)$ are given as strong solution of the SDE

$$dP(\vartheta, t) = b^\vartheta(P(\vartheta, \cdot))\ dt + \sigma(P(\vartheta, \cdot))\ dW_t.$$

Here W is a Brownian motion, the b^ϑ are bounded Borel functions satisfying a Lipschitz condition and such that $b^\vartheta \leq b^{\vartheta'}$ for $\vartheta \leq \vartheta'$, and σ is a Borel function uniformly bounded from below by some $\varepsilon > 0$ and satisfying $|\sigma(x) - \sigma(y)|^2 \leq \rho(|x - y|)$ for some $\rho > 0$. We assume $P(\vartheta, 0) \leq P(\vartheta', 0)$ for $\vartheta \leq \vartheta'$. Now **(O)** follows from the comparison theorem for solutions of SDE's and **(MM)** follows from Novikov's criterion. Note, however, that in general there is no universal martingale measure simultaneously for all $P(\vartheta, \cdot)$, hence the analysis in Bank and Baum [1] does not apply to this situation.

The goal of this section is to investigate the situation where the large trader uses only buy-and-hold strategies in the following sense.

Definition 2.4 A *buy-and-hold strategy* θ for the large trader is a simple predictable process with representation

$$\theta(t) = \theta_{-1}\mathbf{1}_{\{0\}} + \sum_{j=0}^n \theta_j \mathbf{1}_{(\tau_j, \tau_{j+1}]}(t), \tag{2.1}$$

where $0 = \tau_0 \leq \tau_1 \leq ... \leq \tau_{n+1} = T$ is a finite sequence of $(\mathcal{F}_t)$-stopping times, θ_{-1} is some constant, and θ_j is for each $j = 0, ..., n$ a bounded $\mathcal{F}_{\tau_j}$-measurable random variable. Moreover, the stopping times τ_j and the random variables θ_j take only finitely many values. We denote the space of all buy-and-hold strategies by **S**. The book value

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process for the large trader from buy-and-hold strategies is modelled as the elementary nonlinear stochastic integral

$$\int_0^t S\left(\theta_s, ds\right) := \sum_i \left\{ P\left(\theta_{\tau_i \wedge t}, \tau_{i+1} \wedge t\right) - P\left(\theta_{\tau_i \wedge t}, \tau_i \wedge t\right) \right\}.$$

Note that $\int S\left(\theta_s, ds\right)$ is a continuous semimartingale whereas $S\left(\theta, \cdot\right)$ might have jumps. Similarly, the asymptotic liquidation proceeds from a buy-and-hold strategy are given as

$$\int_0^t L\left(\theta_s, ds\right) := \sum_i \left\{ L\left(\theta_{\tau_i \wedge t}, \tau_{i+1} \wedge t\right) - L\left(\theta_{\tau_i \wedge t}, \tau_i \wedge t\right) \right\}.$$

Proposition 2.5 Let $\theta \in \mathbf{S}$ be a buy-and-hold strategy of the large trader with representation (2.1). Under **(MM)**, there exist martingale measures Q^θ for $\int L\left(\theta, ds\right)$.

Proof. Let Z^ϑ denote the density process of the martingale measure Q^ϑ for $L\left(\vartheta, \cdot\right)$ which exists by **(MM)**. We construct the density process Z^θ of Q^θ by concatenation:

$$Z_t^\theta = \prod_{j=0}^n \frac{Z_{t \wedge \tau_{j+1}}^{\theta_j}}{Z_{t \wedge \tau_j}^{\theta_j}}.$$

To prove that Q^θ is indeed a martingale measure for $\int L\left(\theta_s, ds\right)$, we may assume by localization that the $L\left(\theta_i, \cdot\right)$ are uniformly integrable Q^{θ_i}-martingales. The assertion then follows from the fact that for any $0 \le t \le T$ and any integer $1 \le i \le n$ we have

$$E_{Q^\theta}\left[L\left(\theta_{\tau_i}, \tau_{i+1}\right) - L\left(\theta_{\tau_i}, \tau_i\right) \middle| \mathcal{F}_t \right] = L\left(\theta_{\tau_i \wedge t}, \tau_{i+1} \wedge t\right) - L\left(\theta_{\tau_i \wedge t}, \tau_i \wedge t\right)$$

which can be verified separately for each of the three cases $t \le \tau_i$, $\tau_i < t \le \tau_{i+1}$ and $\tau_{i+1} < t$. ∎

Since we work with discounted price processes, we may assume that the interest rate is fixed to $r = 0$. Assume now that at time t the large trader has β_t^θ shares in the bank account and holds θ_t shares where $\theta \in \mathbf{S}$ is assumed to be *self-financing* in the sense that $\Delta \beta_t^\theta = -\text{sign}\left(\Delta \theta_t\right) \Delta \theta_t P\left(\theta_t\right)$ if the large trader executes an order of size $\Delta \theta_t$ and zero otherwise.

As discussed in Bank and Baum [1], the *real wealth process* $V(\theta)$ which results asymptotically from liquidating the large traders's position in ever smaller packages and as the duration of liquidation tends to zero is given as

$$V_t(\theta) = \beta_t^\theta + L\left(\theta_t, t\right). \tag{2.2}$$

By Lemma 3.2 of Bank and Baum [1] (this can at this stage also be seen by a straight-forward calculation), the dynamics of $V(\theta)$ is given for any self-financing strategy $\theta \in \mathbf{S}$ as

$$V_t(\theta) - V_{0-}(\theta) = \int_0^t L(\theta_{s-}, ds) - \text{sign}(\Delta\theta_s) \sum_{0 < s \le t} \int_{\theta_{s-}}^{\theta_s} \{P(\theta_s, s) - P(x, s)\} dx. \quad (2.3)$$

It follows from **(O)** that the sum on the right-hand side of (2.3) is always positive, it can be thus seen as transaction costs inflicted from the block orders occuring at times θ_i's. If we focus on admissible strategies θ in the sense that $\int L(\theta_-, ds)$ is bounded from below, it follows from Proposition 2.5 by a standard argument that there are no arbitrage opportunities for the large trader.

3   No arbitrage with dynamic strategies

In this section, **(O)** and **(MM)** are always supposed to hold, without further mentioning. We refer to Carmona and Nualart [3] for the notion of a *nonlinear L^1-strong integrator* and its properties. In particular, to show that a given random field is such an integrator one can use Theorems II.4.2 and II.4.3 of Carmona and Nualart [3]. Our concrete examples always have this property. Since by **(MM)** there exists for each $L(\vartheta, \cdot)$, $\vartheta \in \mathbb{R}$, an equivalent martingale measure Q^ϑ, a standard argument based on Girsanov's theorem and the Kunita-Watanabe inequality reveals that the continuous semimartingales $L(\vartheta, \cdot)$ have a decomposition as

$$L(\vartheta, \cdot) = M^\vartheta + \int \lambda^\vartheta d\left[M^\vartheta\right]. \quad (3.1)$$

Here M^ϑ denotes the local martingale part and λ^ϑ is some predictable process. We now impose the following conditions on the random field L. Here and in the sequel, we denote for a semimartingale S with $L(S)$ the S-integrable processes.

Assumption (RF) (i) $L = (L(\vartheta, \cdot))_{\vartheta \in \mathbb{R}}$ is a nonlinear L^1-strong integrator

(ii) L is a $\mathcal{C}^{(2)}$- (in ϑ) valued process

(iii) $\partial L(\vartheta, \cdot)/\partial\vartheta$ is also a nonlinear L^1-strong integrator

(iv) there exists an increasing continuous adapted process B such that for all $(\vartheta, \vartheta') \in \mathbb{R} \times \mathbb{R}$ we have that $\left[M^\vartheta, M^{\vartheta'}\right]$ is equivalent to B in the sense that there exist processes $\mu^{\vartheta, \vartheta'} \in L(B)$ such that $\left[M^\vartheta, M^{\vartheta'}\right] = \int \mu^{\vartheta, \vartheta'} dB$ and we have $\mu^\vartheta = \mu^{\vartheta, \vartheta} \ge \varepsilon > 0$, uniformly in ϑ.

In particular, under **(RF)** we have an Itô-Wentzell formula at our disposal (Carmona and Nualart [3], Theorem III.3.3): for every continuous semimartingale θ we get

$$
\begin{aligned}
L\left(\theta_t, t\right) - L\left(\theta_0, 0\right) &= \int_0^t L\left(\theta_s, ds\right) + \int_0^t L'\left(\theta_s, s\right) d\theta_s + \left[\int_0^{\cdot} L'\left(\theta_s, ds\right), \theta\right]_t \quad (3.2) \\
&\quad + \frac{1}{2} \int_0^t L''\left(\theta_s, s\right) d\left[\theta\right]_s .
\end{aligned}
$$

Following the analysis of Bank and Baum [1], block trades are disadvantageous for the large trader, so we restrict ourselves to consider only continuous semimartingale strategies θ. Moreover, although Bank and Baum [1] work with the Kunita integral instead of the Carmona-Nualart integral (it is beyond the scope of this paper to compare these two notions), the part of their study which is just based on the Itô-Wentzell formula translates one-to-one into our setting as well. We moreover assume that θ is *self-financing* in the sense of Bank and Baum [1], i.e. the large trader's bank account holdings β_t^θ follow the dynamics

$$
\beta_t^\theta = \beta_{0-} - \int_0^t S\left(\theta_s, s\right) d\theta_s - \left[S\left(\theta, .\right), \theta\right]_t .
$$

By Lemma 3.2 of Bank and Baum [1], the real wealth $V\left(\theta\right)$ (as in (2.2)) follows then for any such self-financing strategy θ the dynamics

$$
V_t(\theta) - V_0 = \int_0^t L\left(\theta_s, ds\right) - \frac{1}{2} \int_0^t S'\left(\theta_s, s\right) d\left[\theta\right]_s^c . \quad (3.3)
$$

The last term on the right-hand side can be interpreted as transaction costs from highly fluctuating strategies (note that $S' \geq 0$ because of Assumption **(O)**). They are absent if the large trader employs only *tame strategies,* i.e. continuous strategies of bounded variation.

Definition 3.1 The *space of admissible strategies* Θ consists of continuous and bounded semimartingales θ where $\theta \in \overline{\mathbf{S}}$. Here the closure is taken with respect to uniform convergence in probability (*ucp*). By Proposition II.1.1 of Carmona and Nualart [3], $\overline{\mathbf{S}}$ can be identified with the space of adapted processes with paths which are left continuous and have right limits.

For a given $\theta \in \Theta$, we now choose an approximating sequence $\left(\theta_n\right) \subset \mathbf{S}$ of buy-and-hold strategies such that $\theta_n \to \theta$ in *ucp*. By Proposition 2.5 there exist equivalent martingale measures Q_n for the stochastic integral processes $\int L\left(\theta_n, ds\right)$, so that their semimartingale decomposition is of the form

$$
\int L\left(\theta_n, ds\right) = M^n + \int \lambda^{(n)} d\left[M^n\right] \quad (3.4)
$$

where the M^n's are continuous local P-martingales and the $\lambda^{(n)}$'s are predictable processes.

Proposition 3.2 Under **(RF)**, there exists a continuous local martingale M and a predictable process λ such that

$$\int L\left(\theta, ds\right) = M + \int \lambda \, d\left[M\right].$$

Proof. By **(RF)** (i), L is a nonlinear L^1-strong integrator. This implies by Proposition II.3.1 of Carmona and Nualart [3] that

$$\int L\left(\theta_n, ds\right) \to \int L\left(\theta, ds\right) \qquad \text{in } \mathcal{SM} \tag{3.5}$$

where $\mathcal{SM}$ denotes the semimartingale topology. Since by Remarque IV.3 of Mémin [7] projection onto the summands in the semimartingale decomposition is continuous in $\mathcal{SM}$ for the continuous semimartingales $\int L\left(\theta_n, ds\right)$ we get from (3.4), (3.5) that

$$M^n \to M \qquad \text{in } \mathcal{SM} \tag{3.6}$$

where M is some continuous local martingale by Mémin [7], Theorem IV.5. It follows that also

$$\left[M^n\right] \to \left[M\right] \qquad \text{in } \mathcal{SM}. \tag{3.7}$$

Moreover,

$$\int \lambda^{(n)} \, d\left[M^n\right] \text{ converges in } \mathcal{SM}. \tag{3.8}$$

By **(RF)** (iv) we may write

$$\left[M^n\right] = \int \mu^n \, dB$$

for some predictable processes $\mu^n \geq \varepsilon$. We get from (3.7) that

$$\int \mu^n \, dB \to \int \mu \, dB \qquad \text{in } \mathcal{SM} \tag{3.9}$$

for some predictable process $\mu \geq \varepsilon$ since the space of all integrals $\int \phi \, dB$ where ϕ is B-integrable is closed in $\mathcal{SM}$ by Corollaire III.4 of Mémin [7]. By the same device it follows from (3.8) that there exists $\eta \in L\left(B\right)$ such that

$$\int \lambda^{(n)} \mu^n \, dB \to \int \eta \, dB \qquad \text{in } \mathcal{SM},$$

and, setting $\lambda := \eta \mu^{-1}$, we get

$$\int \lambda^{(n)} \, d\left[M^n\right] \to \int \lambda \, d\left[M\right] \qquad \text{in } \mathcal{SM}. \tag{3.10}$$

∎

Assumption (UB) (i) There exists $\Lambda \in L(M)$ such that $\sup_\vartheta \left| \lambda^\vartheta \right| \leq \Lambda$.

(ii) For the λ^ϑ's in (3.1) we have that $\sup_{0 \leq t \leq T} \left| \lambda_t^\vartheta \right|$ is bounded in $L^p(P)$ for some $p > 1$, uniformly in ϑ.

For example, if the $L(\vartheta, \cdot)$ are given as geometric Brownian motions with constants b^ϑ and σ,

$$L(\vartheta, t) = \exp\left(\left(b^\vartheta - \frac{1}{2}\sigma^2 \right) t + \sigma W_t \right),$$

then $\lambda^\vartheta = b^\vartheta \sigma^{-2} (L(\vartheta, \cdot))^{-1}$, and **(UB)** is fulfilled if the b^ϑ are uniformly bounded.

Remark 3.3 (UB) implies that $\sup_n \left| \lambda^{(n)} \right| \leq \Lambda \in L(M)$, and that $\sup_{0 \leq t \leq T} \left| \lambda_t^{(n)} \right|$ is bounded in $L^p(P)$, uniformly in n.

Proposition 3.4 Under assumptions **(RF)** and **(UB)**, we have along a subsequence

$$\int \lambda^{(n)} \, dM^n \quad \to \quad \int \lambda \, dM \qquad \text{in } \mathcal{SM}, \tag{3.11}$$

$$\int \left(\lambda^{(n)} \right)^2 d[M^n] \quad \to \quad \int \lambda^2 \, d[M] \qquad \text{in } \mathcal{SM}. \tag{3.12}$$

Proof. We estimate

$$\left\| \int \lambda \, dM - \int \lambda^{(n)} \, dM^n \right\|_{\mathcal{SM}}$$
$$\leq \quad \left\| \int \left(\lambda - \lambda^{(n)} \right) dM \right\|_{\mathcal{SM}} + \left\| \int \lambda^{(n)} \, d(M - M^n) \right\|_{\mathcal{SM}}.$$

As for the first term on the right-hand side, note that due to (3.9), (3.10), we may assume that along a subsequence

$$\lambda^{(n)} \to \lambda \qquad P \times dB - \text{a.s.}$$

This follows since for finite variation processes, convergence in the semimartingale topology is just convergence in total variation on compact sets. We now get with Remark 3.3 from the dominated convergence theorem for stochastic integrals as in Theorem III.5 of Mémin [7] that along a subsequence

$$\int \lambda^{(n)} \, dM \to \int \lambda \, dM \qquad \text{in } \mathcal{SM}.$$

For the second term we shall use Emery's characterization of the semimartingale topology via (pre-)local convergence in $\mathcal{H}^p$, $p \geq 1$, see Theorem 2 of Emery [4]. According to that result, since $M^n \to M$ in $\mathcal{SM}$ there exists a sequence (T^m) of stopping times with $T^m \nearrow \infty$ such that for all m we have for $q > 1$ with $1/p + 1/q = 1$

$$(M^n)^{T_m} \to M^{T_m} \qquad \text{in } \mathcal{H}^q(P). \tag{3.13}$$

To prove that $\int \lambda^{(n)} d(M - M^n) \to 0$ in $\mathcal{SM}$ it suffices to show that locally with respect to (T^m),

$$\int \lambda^{(n)} d(M - M^n) \to 0 \qquad \text{in } \mathcal{H}^1(P).$$

This follows from the uniform boundedness of the L^p-norms of $\sup_t \left| \lambda_t^{(n)} \right|$ and (3.13) since for all m

$$E\left[\left(\int_0^{T_m} \left| \lambda_t^{(n)} \right|^2 d[M - M^n]_t \right)^{\frac{1}{2}} \right] \leq E\left[\sup_{0 \leq t \leq T} \left| \lambda_t^{(n)} \right| [M - M^n]_{T_m}^{\frac{1}{2}} \right]$$

$$\leq \text{const.} E^{\frac{1}{q}} \left[[M - M^n]_{T_m}^{\frac{q}{2}} \right] \to 0.$$

We have therefore shown (3.11) which implies in turn (3.12). ∎

The assertion of Proposition 3.2 is a necessary condition for the existence of an equivalent martingale measure for $\int L(\theta, ds)$: the finite variation part of its semimartingale decomposition is absolutely continuous with respect to the quadratic variation of the local martingale part. A possible candidate martingale measure is now given via the density process

$$\mathcal{E}\left(- \int \lambda \, dM \right) = \exp\left(- \int \lambda \, dM - \frac{1}{2} \int \lambda^2 \, d[M] \right). \tag{3.14}$$

However, as this stochastic exponential is a priori a supermartingale, we need an additional condition to ensure it is a martingale.

Assumption (UI) (i) For each n, the stochastic exponential $Z^n = \mathcal{E}\left(- \int \lambda^{(n)} \, dM^n \right)$ is a true martingale, hence Z_T^n gives the density of a martingale measure for $\int L(\theta_n, ds)$.

(ii) The sequence (Z^n) is uniformly integrable.

For example, **(UI)** is satisfied if there exists some finite constant K such that

$$\sup_n \int_0^T \left(\lambda_t^{(n)} \right)^2 d[M^n]_t \leq K \qquad P\text{-a.s.}$$

Theorem 3.5 Under assumptions **(RF)**, **(UB)**, and **(UI)**, for each $\theta \in \Theta$ there exists a martingale measure for $\int L(\theta, ds)$ with density process $\mathcal{E}\left(-\int \lambda \, dM\right)$.

Proof. It follows from Proposition 4 of Emery [4] that the composition of $\mathcal{C}^2$-functions with semimartingales is a continuous operation with respect to the semimartingale topology. Therefore, the formula (3.14) together with (3.11), (3.12) implies that along a subsequence

$$Z^n = \mathcal{E}\left(-\int \lambda^{(n)} \, dM^n\right) \to \mathcal{E}\left(-\int \lambda \, dM\right) =: Z \qquad \text{in } \mathcal{SM}.$$

This in turn gives that along a subsequence we have for the final values

$$Z_T^n \to Z_T \qquad P\text{-a.s.}$$

and we conclude by **(UI)** that Z_T is the density of a probability measure. ∎

Consider now again the dynamics (3.3) of the real wealth process, and let $\theta \in \Theta$ be such that $\int L(\theta, ds)$ is bounded from below (the transaction costs term in (3.3) can be avoided by the large trader by using only tame strategies). An *arbitrage opportunity* is an admissible strategy such that we have for the associated real wealth process $V(\theta)$ that $V_0(\theta) \leq 0$, $V_T(\theta) \geq 0$ P-a.s., and $P(V_T(\theta) > 0) > 0$. By Theorem 3.5, there exists a probability measure Q^θ such that $\int L(\theta, ds)$ is a Q^θ-local martingale, hence a supermartingale. It follows now from the dynamics (3.3) of the real wealth process that $E_{Q^\theta}[V_T(\theta)] \leq V_0(\theta)$ which, as Q^θ is equivalent to P, excludes arbitrage opportunities for the large trader.

4 Portfolio optimization for a large trader: a necessary condition

We fix some finite time horizon $T > 0$ and consider strategies defined on $[0, T]$. As we have seen, strategies $\theta \in \Theta$ which are not of finite variation lead to transaction costs for the large trader and can be discarded for convenience (the proof of the main result of this section would work for Θ instead of Φ as well, with only minor modifications).

Definition 4.1 The space Φ consists of all tame strategies $\theta \in \Theta$.

We want to solve the utility maximization problem

$$\sup_{\theta \in \Phi} E\left[u\left(V_T(\theta)\right)\right]. \tag{4.1}$$

Here u is some concave, monotone increasing and differentiable utility function defined on the whole real line which is bounded from above with $u(0) = 0$. The initial value V_0 is kept fixed and constant.

We will give a necessary criterion for an optimal solution to the general problem 4.1.

Proposition 4.2 Assume **(RF)** and let $\theta \in \Phi$ be an optimizer of problem (4.1). Define a probability measure Q by (with normalizing constant q)

$$\frac{dQ}{dP} = \frac{u'(V_T(\theta))}{q}. \tag{4.2}$$

Then $M := \int S(\theta, ds)$ is a local Q-martingale.

Proof. By stopping, we may assume that M is bounded. Assume now that there is a set $A \in \mathcal{F}_{t_1}$ such that for $0 < t_1 \le t_2 < T$ we have

$$E_Q\left[\mathbf{1}_A\left(M_{t_2} - M_{t_1}\right)\right] > 0.$$

We define for small $\varepsilon > 0$ such that $(t_1 - \varepsilon, t_2 + \varepsilon) \in [0, T]$ a strategy ψ^ε by letting $\psi_t^\varepsilon = \varepsilon \mathbf{1}_A$ for $t \in [t_1, t_2]$, $\psi_t^\varepsilon = 0$ for $t \notin (t_1 - \varepsilon, t_2 + \varepsilon)$, and define ψ_t^ε for $t \in (t_1 - \varepsilon, t_1) \cup (t_2, t_2 + \varepsilon)$ by linear interpolation. It follows that for $\theta \in \Phi$ the restriction of $\theta + \psi^\varepsilon$ to $[0, T]$ is then in Φ as well. By the Itô-Wentzell formula, which we may apply due to **(RF)**,

$$
\begin{aligned}
&V_T\left(\theta + \psi^\varepsilon\right) - V_T\left(\theta\right) \\
={}& \int_0^T L\left(\theta + \psi^\varepsilon, ds\right) - \int_0^T L\left(\theta, ds\right) \\
={}& L\left(\theta_T + \psi_T^\varepsilon, T\right) - L\left(\theta_T, T\right) - L\left(\theta_0 + \psi_0^\varepsilon, 0\right) + L\left(\theta_0, 0\right) \\
& - \int_0^T \left(S\left(\theta_s + \psi_s^\varepsilon, s\right) - S\left(\theta_s, s\right)\right) d\theta_s \\
& - \int_0^T S\left(\theta_s + \psi_s^\varepsilon, s\right) d\psi_s^\varepsilon \\
={}& -\mathbf{1}_A \int_{t_1}^{t_2} \left(S\left(\theta_s + \varepsilon, s\right) - S\left(\theta_s, s\right)\right) d\theta_s \\
& -\mathbf{1}_A \int_{t_1-\varepsilon}^{t_1} S\left(\theta_s + \psi_s^\varepsilon, s\right) ds + \mathbf{1}_A \int_{t_2}^{t_2+\varepsilon} S\left(\theta_s + \psi_s^\varepsilon, s\right) ds.
\end{aligned}
$$

Due to **(RF)** it is possible to differentiate under the integral, and by using again Itô-Wentzell we reach (on the set A)

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(V_T\left(\theta + \psi^\varepsilon\right) - V_T\left(\theta\right)\right) \\
={}& -\int_{t_1}^{t_2} S'\left(\theta_s, s\right) d\theta_s + S\left(\theta_{t_2}, t_2\right) - S\left(\theta_{t_1}, t_1\right) \\
={}& \int_{t_1}^{t_2} S\left(\theta_s, ds\right).
\end{aligned}
$$

For small $\varepsilon > 0$ we therefore have by assumption

$$E\left[u'(V_T(\theta))(V_T(\theta + \psi^\varepsilon) - V_T(\theta))\right] > 0. \qquad (4.3)$$

On the other hand, by optimality of θ and the mean value theorem we get

$$\begin{aligned}
0 &\geq E[u(V_T(\theta + \psi^\varepsilon)] - E[u(V_T(\theta))] \\
&= E[u'\left(\xi^{(\varepsilon)}\right)(V_T(\theta + \psi^\varepsilon) - V_T(\theta))]
\end{aligned}$$

for some random variable

$$\xi^{(\varepsilon)} \in \left(\min\left(V_T(\theta + \psi^\varepsilon), V_T(\theta)\right), \max\left(V_T(\theta + \psi^\varepsilon), V_T(\theta)\right)\right).$$

As $V_T(\theta + \psi^\varepsilon) \to V_T(\theta)$ a.s. for $\varepsilon \to 0$, we have $u'\left(\xi^{(\varepsilon)}\right) \to u'(V_T(\theta))$ and get by Fatou's lemma that

$$\begin{aligned}
E\left[u'(V_T(\theta))(V_T(\theta + \psi^\varepsilon) - V_T(\theta))\right] &\leq \liminf_{\varepsilon \to 0} E[u'\left(\xi^{(\varepsilon)}\right)(V_T(\theta + \psi^\varepsilon) - V_T(\theta))] \\
&\leq 0
\end{aligned}$$

which is a contradiction to (4.3). The case $E_Q\left[\mathbf{1}_A(M_{t_2} - M_{t_1})\right] < 0$ is treated analogously. Finally, the martingale property extends to $t_1 = 0$ and $t_2 = T$ by dominated convergence since M is continuous and bounded. $\blacksquare$

While we are not in a position to discuss sufficent conditions for optimality rigorously, let us briefly give an informal argument. We define for each strategy $\theta \in \Phi$ a probability measure R by

$$\frac{dR}{dP} = \frac{u''(V_T(\theta))}{r}$$

where r is a normalizing constant. Formally calculating the second derivative leads us to propose the following sufficient criterion: an admissible strategy θ fulfilling the necessary criterion in Proposition 4.2 is optimal (at least locally) if

$$\left\langle\int S(\theta, dt)\right\rangle^R + \frac{q}{r}A^{\int S'(\theta, dt), Q} \text{ is an increasing process.} \qquad (4.4)$$

Here the predictable compensator of the quadratic variation of $\int S(\theta, dt)$ is calculated wrt. R, and $A^{\int S'(\theta, dt), Q}$ is the predictable finite variation part in the canonical decomposition wrt. Q of the special semimartingale $\int S'(\theta, dt)$. Note that in case of exponential utility $u(x) = 1 - \exp(-\alpha x)$, $\alpha > 0$, we have $R = Q$ and $q/r = -1/\alpha$. A formal statement and proof of this claim is an open problem.

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5   A case study: the illiquid Bachelier model

We consider now the following market model which we do not claim to be particularly realistic, but which, however, allows for an explicit solution and thereby illustrates certain phenomena which may occur in illiquid markets. It is a modified Bachelier model where the drift of the asset price is positively influenced by the engagement of a large investor. One can think of this positive influence being caused by momentum traders who react to the signal given by the large investor increasing her stake. The primitive family (P_ϑ) is given as

$$P(\vartheta, t) = P_0 + (\mu + \kappa\vartheta) t + \sigma W_t,$$

or, in differential notation,

$$P(\vartheta, dt) = (\mu + \kappa\vartheta)\ dt + \sigma\ dW_t,$$

where W is a Brownian motion and μ, κ, σ are positive parameters. The filtration is supposed to be the smallest one fulfilling the usual conditions and containing the one generated by W. Furthermore, we restrict our discussion of portfolio optimization to the choice of exponential utility function $u(x) = 1 - \exp(-\alpha x)$, $\alpha > 0$.

Remark 5.1 This model is not included in the model class studied by Kühn [6] whose assumption 2.1 ('Largeness is not favourable') implies that the drift is non-increasing in ϑ.

According to Proposition 4.2, a necessary condition for an admissible strategy $\theta \in \Phi$ to be a candidate for the optimal solution of problem (4.1) is that $\int S(\theta, ds)$ is a local Q-martingale where Q is defined as in (4.2). The liquidation process for such a strategy θ is given as

$$L(\theta, t) = \int_0^{\theta_t} P(x, t)\ dx = \theta_t\left(P_0 + \mu t + \sigma W_t\right) + \theta_t^2 \frac{\kappa}{2} t.$$

According to the Itô-Wentzell formula (3.2) we get for the terminal value of the value process

$$V_T(\theta) = V_0 + \int_0^T \left(\mu\,\theta_t + \frac{\kappa}{2}\theta_t^2\right)\ dt + \int_0^T \sigma\theta_t\ dW_t$$

and hence, for some generic constant c which may vary from line to line,

$$\begin{aligned}
\frac{dQ}{dP} &=\ c\exp\left(-\alpha V_T(\theta)\right) \\
&=\ c\exp\left(-\int_0^T \alpha\sigma\theta_t\ dW_t - \int_0^T \left(\alpha\mu\theta_t + \alpha\frac{\kappa}{2}\theta_t^2\right)\ dt\right).
\end{aligned}$$

As

$$\int S(\theta, dt) = S_0 + \int (\mu + \kappa\theta_t)\, dt + \sigma W_t,$$

we get by Girsanov and our choice of filtration that the only possible candidate for a martingale measure for $\int S(\theta, dt)$ is the one with density

$$\frac{dQ}{dP} = \exp\left(-\int_0^T \frac{\mu + \kappa\theta_t}{\sigma}\, dW_t - \frac{1}{2}\int_0^T \left(\frac{\mu + \kappa\theta_t}{\sigma}\right)^2 dt\right).$$

By Novikov's criterion, which we may apply since $\theta \in \Phi$, hence bounded, dQ/dP is indeed the density of a martingale measure. Equating the two expressions for dQ/dP, we get

$$0 = c - \int_0^T \left(\frac{\mu}{\sigma} + \theta_t\left(\frac{\kappa}{\sigma} - \alpha\sigma\right)\right) dW_t + \int_0^T \theta_t\left(\frac{\kappa}{2}\theta_t + \mu\right)\left(\alpha - \frac{\kappa}{\sigma^2}\right) dt.$$

We assume now that $\kappa \neq \alpha\sigma^2$ (the case where equality holds will be dealt later with). Setting

$$
\begin{aligned}
\vartheta^* \quad &:= \quad \frac{\mu}{\alpha\sigma^2 - \kappa}, \\
\mu_t^* \quad &:= \quad \frac{\alpha\sigma^2 - \kappa}{\sigma^2}\left(\frac{\kappa}{2}\theta_t^2 + \mu\theta_t - \frac{\kappa}{2}(\vartheta^*)^2 - \mu\vartheta^*\right), \\
\sigma_t^* \quad &= \quad \frac{\mu}{\sigma} + \theta_t\left(\frac{\kappa}{\sigma} - \alpha\sigma\right),
\end{aligned}
$$

we can reformulate this equation, by collecting all constant terms into the generic constant c, as

$$0 = c + \int_0^T \mu_t^*\, dt - \int_0^T \sigma_t^*\, dW_t. \tag{5.1}$$

We will now show that this equation will only be solved by $\sigma^* \equiv 0$ (modulo some Lebesgue zero set), which implies that θ equals a.s. the constant strategy ϑ^*. For this we consider the measure $\widehat{Q}$ with density wrt. P given as the stochastic exponential

$$\frac{d\widehat{Q}}{dP} = \mathcal{E}\left(-\int \frac{\mu^*}{\sigma^*}\, dW\right)_T = \mathcal{E}\left(-\int \frac{\kappa(\theta + \vartheta^*) + 2\mu}{2\sigma}\, dW\right)_T$$

where we set $\frac{0}{0} := 0$. The key point is again that $\theta \in \Phi$ is an admissible strategy, hence in particular bounded. This implies by Novikov that the stochastic exponential has P-expectation equal to one, and therefore $\widehat{Q}$ is a probability measure. By Girsanov,

$$\widehat{W} = W + \int \frac{\mu^*}{\sigma^*}\, dt$$

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is a $\widehat{Q}$-Brownian motion and (5.1) implies that

$$0 = c - \int_0^T \sigma_t^* \, d\widehat{W}_t. \tag{5.2}$$

Since θ is bounded, σ^* must be bounded as well, and therefore the integral process $\int \sigma^* \, d\widehat{W}$ is a true $\widehat{Q}$-martingale with constant expectation equal to zero. By (5.2), this martingale even vanishes identically which in turn yields that σ^* is indeed zero a.s. Summing up, the unique strategy fulfilling the necessary condition for optimality is the constant strategy ϑ^*.

It turns out that the qualitative behaviour of the solution depends crucially on whether the so-called *stability condition*

$$\kappa < \alpha\sigma^2 \tag{5.3}$$

is fulfilled or not. This condition corresponds to the formally sufficient condition (4.4). The large investor achieves by implementing the constant strategy ϑ an expected exponential utility (given initial wealth 0, say) of

$$1 - \exp\left(-\alpha\vartheta\mu T + \frac{\alpha\vartheta^2 T}{2}\left(\alpha\sigma^2 - \kappa\right)\right). \tag{5.4}$$

In case she chooses the candidate strategy ϑ^* she gains an expected utility of

$$1 - \exp\left(-\frac{\alpha\mu^2 T}{2\left(\alpha\sigma^2 - \kappa\right)}\right).$$

(i) instable regime $\kappa > \alpha\sigma^2$: In this case the strategy ϑ^* performs worst among all constant strategies, while the expected utility grows with $|\vartheta|$ up to the maximum value. This can be interpreted in that the impact of the strategy on the drift is so substantial that the large investor buys as many shares as possible.

(ii) stable regime $\kappa < \alpha\sigma^2$: The strategy ϑ^* performs best under all constant strategies. Moreover, it is the only strategy among all admissible ones which fulfills the necessary optimality condition. However, it has to be said that at present we are not able to verify that ϑ^* is indeed the optimal strategy and have to leave it as topic for future research (however, see the heuristic discussion at the end of last section).

We now want to compare in this stable regime the expexted utility of the large trader with the optimal utility in the classical Merton problem. In the situation of the Merton problem where we face a hypothetical small investor with the same utility function and initial wealth 0, and with given price process $S(\theta, \cdot)$, we can calculate the optimal strategy by substituting in the above calculations 0 for κ, and $\mu + \kappa\theta$ for μ. Given that the large trader is present in the market and behaves rationally, i.e. chooses the

constant strategy ϑ^*, it results that the small trader would choose a constant strategy as well, namely

$$\psi = \psi\left(\vartheta^*\right) = \frac{\mu + \kappa\vartheta^*}{\alpha\sigma^2} = \vartheta^*.$$

His expected utility in that case would be

$$1 - \exp\left(-\frac{\alpha^2\left(\vartheta^*\right)^2\sigma^2 T}{2}\right) = 1 - \exp\left(-\frac{\alpha\mu^2 T}{2\left(\alpha\sigma^2 - \kappa\right)}\frac{\alpha\sigma^2}{\left(\alpha\sigma^2 - \kappa\right)}\right).$$

Therefore, the small investor would implement the same strategy as the large one, but would achieve a higher expected utility.

If there is no large trader around, which corresponds to the case $\theta = 0$, the small trader would hold an optimal portfolio of

$$\psi\left(0\right) = \frac{\mu}{\alpha\sigma^2}$$

stocks, and his expected utility in that case is

$$1 - \exp\left(-\frac{\mu^2 T}{2\sigma^2}\right) = 1 - \exp\left(-\frac{\alpha\mu^2 T}{2\left(\alpha\sigma^2 - \kappa\right)}\frac{\left(\alpha\sigma^2 - \kappa\right)}{\alpha\sigma^2}\right).$$

Hence the small investor purchases less stocks than a large investor would do, and he achieves, in contrast to the case studied in Bank and Baum [1] only a lower expected utility, compared to the large investor. The only exception would be the case $\mu = 0$, however, then a large trader could not benefit from the fact that her actions could enlarge the drift of the price process and thereby change a martingale into a submartingale.

(iii) critical case $\kappa = \alpha\sigma^2$: it follows from (5.4) that the result depends on μ. In case $\mu = 0$, all constant strategies perform equally as the investor always gets the expected utility of the zero strategy. For $\mu \neq 0$ she can like in the instable regime achieve expected utility arbitrarily close to the maximum value of one, however now her stake has to have the right sign, depending on the sign of μ.

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