1 Verification that the Ito-Wentzell Formula works

We write a weaker version of theorem 3.1 in Krylov using notation and assumptions that are closer to our model. For $p \in [0, S]$, and $t \in [0, T]$, let:

$$\begin{aligned} dQ(p,t) &= \mu_Q(p,t)dt + \sigma_Q(p,t)\int_{s=0}^{S} b_Q(p,s,t)W(ds,dt) \qquad (1) \\ dP(t) &= \mu_P(t)dt + \sigma_P(t)\int_{s=0}^{S} b_P(s,t)W(ds,dt) \end{aligned}$$

where all stochastic processes are $\mathcal{F}_t-$adapted.

1.1 Assumption 1 [1]

i) We have $P(t) \in [0, S]$.

ii) For any $\omega \in \Omega$ and $t \in [0, T]$, the function $Q(p)$ is continuous in p

iii) For almost any $(\omega, t) \in \Omega \times [\tau, T]$

a)$\mu_Q(p)$ and $\sigma_Q(p)b_Q(p,s)$ and $\sigma_Q^2(p)$ are continuous in p

b)the following functions are continuous functions of p:

1	$\frac{1}{2}\sigma_P^2\frac{\partial^2 Q(p)}{\partial p^2} + \mu_P\frac{\partial Q(p)}{\partial p}$	$LF(x)$
2	$\sigma_P b_P(s)\frac{\partial Q(p)}{\partial p}$	$\Lambda^k F(x)$
3	$\sigma_P b_P(s)\frac{\partial \sigma_Q(p)b_Q(p,s)}{\partial p}$	$\Lambda^k H^k(x)$
4	$\sigma_P(t)^2 b_P(s)(\frac{\partial Q(p)}{\partial p})^2$	$\lvert\Lambda F(x)\rvert_{l^2}$
5	$\sigma_P^2(t)\int_0^S (b_P(s)\frac{\partial \sigma_Q(p)b_Q(p,s)}{\partial p})^2 ds$	$\lvert\Lambda H(x)\rvert_{l^2}$

iv) For $p \in [0, S]$, we have almost surely:

$$\int_0^T Q(p,t)|\mu_P(t) + \frac{1}{2}\sigma_P^2(t)| + \frac{1}{2}Q^2(p,t)\sigma_P^2(t) + |\mu_Q(p,t)| + \sigma_Q^2(p,t)dt < \infty$$

and equation (1) admits a solution.

v) We have almost surely:

$$\int_{p=0}^{S}\int_0^T Q(p,t)(|\mu_P(p,t)| + \frac{1}{2}\sigma_P^2(t))dt + \frac{1}{2}(\int_0^T Q^2(p,t)\sigma_P^2(t)dt)^{1/2}dp < \infty$$

and, for all $p \in [0, S]$,

$$\int_0^T |\mu_Q(p,t)| + |\frac{1}{2}\sigma_P^2(t)\frac{\partial^2 Q(p,t)}{\partial p^2} + \mu_P(t)\frac{\partial Q(p,t)}{\partial p}| + \left(\sigma_P(t)\frac{\partial Q(p,t)}{\partial p}\right)^2 + \sigma_Q^2(p)$$
$$+ \int_0^S (\sigma_P(t)b_P(s,t))^2[\frac{\partial}{\partial p}\sigma_Q(p,t)b_Q(p,s,t)]^2 ds dt < \infty$$

[1] We do not follow exactly the numbering of he assumptions of theorem 3.1 in Krylov. Note that ur assumptions are simpler because p is assumed bounded and we always take the L_2 norm.

1.2 Verification of Assumption 1

We study the following model:

$$dh(p,t) = a(p)(\bar{h}(p) - h(p,t))dt + \sigma_h(p) \int_{s=0}^{S} b_h(p,s) W(ds,dt) \qquad \forall p \in [0,S] \tag{2}$$

$$q(p,t) = \exp \int_{x=0}^{p} h(x,t)dx \qquad \forall p \in [0,S] \tag{3}$$

$$d\eta(t) = a_\eta(\bar{\eta} - \eta(t))dt + g(\eta(t))(\int_{s=0}^{S} \sigma_\eta(s) W(ds,dt,\omega) \tag{4}$$

$$Q(p,t) = \eta(t,\omega) \int_{x=0}^{S} q(x,t)dx - \int_{x=0}^{p} q(x,t)dx \qquad \forall p \in [0,S] \tag{5}$$

where:

- $a(p)$, $\bar{h}(p)$, $\sigma_h(p)$, and $b_h(p,s)$ are continuous functions of p, bounded on $p \in [0,S]$
- $a_\eta \geq 0$
- $\sigma_h(p)^2$ is a continuously differentiable function of p
- g is a nonnegative Lipschitz continuous function such that $g(\eta) = 0$ if $\eta \leq 0$ or $\eta \geq 1$.

Lipschitz continuity ensures that(3) has a solution. A possible form (which is not Lipschitz continuous, but can be mollified into one) is:

$$g(\eta) = \sigma_\eta \sqrt{\eta(1-\eta)}$$

1.2.1 Solution and Properties

The solution of (2) is:

$$h(x,t) = h_{0,x} \exp(-a(x)t) + \bar{h}(x)(1 - \exp(-a(x)t)) + \tag{6}$$

$$\int_0^t \exp(-a(x)(t-u)) \int_0^S b_h(p,u) W(ds,u)) du \tag{7}$$

As stated above, (4) has a solution, thus (5) is fully defined. Differentiating (5), we have:

$$\frac{\partial Q}{\partial p} = \exp\left(\int_{x=0}^{p} h(x,t)dx\right) \tag{8}$$

$$\frac{\partial^2 Q(p)}{\partial p^2} = -h(p) \exp\left(\int_{x=0}^{p} h(x)dx\right) \tag{9}$$

As for the infinitesimal parameters, let:

$$
\begin{aligned}
f(p,t) &= \int_{x=0}^{p} h(x,t)\exp(\int_{x=0}^{x} h(x,t)dx)a(p)(\bar{h}(p)-x)dx \\
&\frac{1}{2}\int_{x=0}^{p} h^2(x,t)\exp(\int_{x=0}^{x} h(x,t)dx)\sigma_h^2(x)dx + \\
&\frac{1}{2}(h(p,t)\exp(\int_{x=0}^{p} h(x,t)dx)\sigma_h^2(p)| - h(0,t)\sigma_h^2(0)) \\
&\frac{1}{2}\int_{x=0}^{p} h(x,t)\exp(\int_{y=0}^{x} h(y,t)dx)[h(x,t)\sigma_h^2(x) + \frac{d}{dx}\sigma_h^2(x)]dx
\end{aligned} \tag{10}
$$

By Ito's lemma, the drift of $Q(p,t)$ is:

$$
\begin{aligned}
\mu_Q(p,t) &= \eta(t)f(S,t) - f(p,t) + \\
&a_\eta(\bar{\eta}-\eta(t))\int_{x=0}^{S} \exp(\int_{y=0}^{x} h(y,t)dy)dx + \\
&g(\eta(t))\int_{x=0}^{S} h(x,t)\exp(\int_{y=0}^{x} h(y,t)dy)\sigma_h(x)dx
\end{aligned} \tag{11}
$$

while the volatility of $Q(p,t)$ is:

$$
\begin{aligned}
\sigma_Q(p,t)b_Q(p,s,t) &= g(\eta(t))(Q(S,t)-Q(0,t)) + \\
&\eta\int_{x=0}^{S} h(x,t)\exp(\int_{y=0}^{x} h(y,t)dy)\sigma_h(x)b_h(x,s)dx - \\
&\int_{x=0}^{p} h(x,t)\exp(\int_{y=0}^{x} h(y)dy)\sigma_h(x)b_h(x,s)dx
\end{aligned} \tag{12}
$$

1.2.2 Verification of all assumptions.

Assumption (i) (boundedness of $P(t)$) is verified in the main text. It is clear that, in our model Q is twice-differentiable in p, thus assumption (ii) (continuity of Q in p) is satisfied.

Verification of assumption iii.a) Drift: The advantage of taking a string is that, since $b_h(p,s)$ is continuous in p, then $\int_{s=0}^{S} b_h(p,s)W(ds,dt)$ is also continuous in p. Thus $h(.,t)$ is continuous. By (8) and (9), it is clear that both $\frac{\partial Q}{\partial p}$ and $\frac{\partial^2 Q}{\partial p^2}$ are continuous functions of p. By continuity $a(p)$ and of $\frac{d}{dp}\sigma_h^2(p)$ the process $f(p,t)$ is differentiable in p. Inspection of (11) shows that μ_Q is continuous in p.

Volatility: We need only investigate the third row of (12), which is clearly a continuous function of p. So is $\sigma_Q(p,t)$.

Verification of assumption iii.b) From (12), we calculate

$$\frac{\partial[\sigma_Q(p,t)b_Q(p,s,t)]}{\partial p} = -h(p,t,\omega)\exp(\int_{x=0}^{p} h(x,t)dx)\sigma_h(p,t)b_h(p,s,t) \quad (13)$$

which is clearly a continuous function of p. by (8) and (9), $\frac{\partial Q}{\partial p}$ and $\frac{\partial^2 Q}{\partial p^2}$ are continuous functions of p. This shows that $\frac{1}{2}\sigma_P^2(t)\frac{\partial^2 Q(p,t)}{\partial p^2} + \mu_P(t)\frac{\partial Q(p,t)}{\partial p}$ as well as $\sigma_P(t)b_P(t,s)\frac{\partial Q(p)}{\partial p}$ and $\sigma_P(t)b_P(t,s)\frac{\partial\sigma_Q(p,t)b_Q(p,s,t)}{\partial p}$ are continuous functions of p.Continuity of $\sigma_P(t)^2 b_P(s)(\frac{\partial Q(p)}{\partial p})^2$ and $\sigma_P^2(t)\int_0^S (b_P(s)\frac{\partial\sigma_Q(p)b_Q(p,s)}{\partial p})^2 ds$ follows trivially.

Lemma Let $Y_i(s,t)$ and $Z_i(s,t)$ be collections of random variables, for $i = 1,...,n$, $0 \leq s \leq S$, and $0 \leq t \leq T$. Suppose $E[\int_0^S \int_0^T \sum_{i=1}^n Y_i(s,t)Z_i(s,t)ds] < \infty$. A sufficient condition for

$$P(\int_0^S \int_0^T \sum_{i=1}^n Y_i(s,t)Z_i(s,t)ds < \infty) = 1$$

is that for all s,t and $i = 1,..,n$

$$\begin{aligned} E[Y_i(s,t)^8] &< \infty \\ E[Z_i(s,t)^8] &< \infty \end{aligned}$$

Verification of assumptions iv and v Clearly $\int_{y=0}^{p} h(y,t)dy$ is normal, thus $q(p,t)$ is lognormal, and has thus finite moments. For simplicity, we drop the argument t. By the lemma above, we can break the argument into verifying finiteness of the 16th moments of the following random variables:

1. **Net demand** $Q(p)$. By Jensen's inequality:

   $$E[(\int_0^S q(x)dx)^{16}] \leq \int_0^S E[\exp(16\int_0^x h(y)dy)]ds$$

   which is bounded. By (5)

   $$-\int_{x=0}^{S} q(x)dx \leq Q(p) \leq \int_{x=0}^{S} q(x)dx$$

   Thus $Q(p)$ is bounded.

2. **Volatility of net demand** σ_Q :Observe that $\sigma_h(x,s)$ is bounded. We apply Jensen's inequality again, and it is sufficient to see that the following

term is bounded:

$$E[(\int_{x=0}^{S} h(x)\exp(\int_{y=0}^{x} h(y)dy)dx)^{16}] = E[(\int_{x=0}^{S} \frac{d}{dx}\exp(\int_{y=0}^{x} h(y)dy)dx)^{16}] =$$
$$E[\exp(\int_{y=0}^{S} h(y)dy)dx)^{16}]$$

Thus $\sigma_Q(p)$ is bounded.

3. **Drift of net demand** μ_Q.Applying Jensen's inequality and the Cauchy-Schwartz inequalities to (11), it is sufficient to verify the finiteness of the appropriate moments of:

   $f(S)$, $\int_{x=0}^{S}\exp(\int_{y=0}^{x} h(y)dy)dx$, and $\int_{x=0}^{S} h(x)\exp(\int_{y=0}^{x} h(y)dy)\sigma_h(x)dx$. For $f(S)$, the same type of development applies, and $f(S)$ has finite appropriate moment because $\sigma_h^2(x)$ is differentiable.

4. **Price drift** μ_P.We refer the reader to (*) in lemma 1 in the main text. Since the numerator has finite 16th moment, it is sufficient, by the Cauchy-Schwartz inequality, to prove boundedness of:

$$E[(\frac{1}{\partial Q/\partial p})^{16}] = E[\exp\left(-16\int_{x=0}^{p} h(x,t)dx\right)]$$

   which is obvious.

5. **Price volatility** $\sigma_P(t)$:We use the same argument as above, since

$$\sigma_P = \frac{\sigma_Q}{\partial Q/\partial p}$$

6. **First derivative of net demand** $\frac{\partial Q(p)}{\partial p}$ Finiteness of the moment is clear, since the random variable in (8) is lognormal.

7. **Second Derivative of Net Demand** $\frac{\partial Q^2(p)}{\partial p^2}$

   Using Cauchy-Schwartz on (9), we see that this has finite appropriate moment.

8. **Derivative of Net Demand Volatility** $\frac{\partial \sigma_Q(p) b_Q(p,s)}{\partial p}$

   We reuse (13), and use the same logic as above.

1.2.3 Proof of lemma

Let

$$X_i(s,t) = Y_i(s,t)Z_i(s,t)$$

By Chebyshev's inequality, it is sufficient to prove that:

$$E[(\int_0^S \int_0^T \sum_{i=1}^n X_i(s,t)ds)^2] < \infty$$

Since:

$$E[(\int_0^S \int_0^T \sum_{i=1}^n X_i(s,t)ds)^2] \leq S^2T^2 \max_{\substack{0\leq s,s'\leq S,\\ 0\leq t,t'\leq T}} E[(\sum_{i=1}^n X_i(s,t))(\sum_{j=1}^n X_j(s',t'))]$$

Applying Jensen's inequality twice, and then the Cauchy-Schwartz inequality:

$$\begin{aligned} E[(\sum_{i=1}^n X_i(s,t) \sum_{j=1}^n X_i(s',t'))^2] &\leq n^2 \sum_{i=1}^n \sum_{j=1}^n E[X_i(s,t)^2 X_j(s',t')^2] \\ &\leq n^2 \sum_{i=1}^n \sum_{j=1}^n (E[X_i(s,t)^4]E[X_j(s',t')^4])^{1/2} \end{aligned}$$

We conclude by the Cauchy-Schwartz inequality:

$$E[X_i(s,t)^4] \leq (E[Y_i(s,t)^8]E[Z_i(s,t)^8])^{1/2}$$